PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201. TITLE, &c.

TITLE, CONTENTS, INDEX, &c.

A, VOL. 201.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN

1903.

A 345. 26.8.03

Price Sixpence.

PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON.

SERIES A

CONTAINING PAPERS OF A MATHEMATICAL OR PHYSICAL CHARACTER.

VOL. 201.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,

Printers in Ordinary to His Majesty.

AUGUST, 1903.

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LIST OF ILLUSTRATIONS.

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But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

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PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 1–35.

EXPERIMENTAL RESEARCHES ON DRAWN STEEL

BY

J. REGINALD ASHWORTH, M.Sc. (Vict.).

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON BERLIN.

1903.

A 331. 11.3.03

Price Three Shillings.

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PHILOSOPHICAL TRANSACTIONS.

I. *Experimental Researches on Drawn Steel.*

By J. REGINALD ASHWORTH, *M.Sc.* (*Vict.*).

Communicated by Professor A. SCHUSTER, *F.R.S.*

Received January 30,—Read March 6, 1902. Received in revised form August 12, 1902.

PART I.

THE INFLUENCE OF CHANGES OF TEMPERATURE ON MAGNETISM.

1. THE experiments which are included in this part are the outcome of a former investigation, and relate chiefly to the influence of drawing on the magnetism of steel wires and its changes with moderate fluctuations of temperature. The effect of alterations of temperature on the residual magnetism of steel was examined many years ago by WIEDEMANN. His experiments, which have often been repeated, show that on heating a magnet to the temperature of steam much of the magnetism disappears, but that on cooling part of the magnetism so lost is restored; at each repetition of the heating and cooling the permanent loss becomes less and less, and ultimately the magnetic intensity fluctuates between two definite values, higher and lower intensities corresponding to lower and higher temperatures respectively. The

change of intensity in this cyclic state is nearly a linear function of the temperature, and the relation is

$$I_{t'} = I_t \{1 + \alpha(t' - t)\},$$

where $I_{t'}$ and I_t are the magnetic intensities at the higher and lower temperatures t' and t, and α is a coefficient which, in general, is negative.

For a given range of temperature* the irreversible part of the change may be expressed by the equation

$$I_f = I_i(1 + \beta),$$

where I_i and I_f are initial and final intensities. Hitherto, almost without exception, β, for residual magnetism, has been found to be negative, that is to say, there is a permanent loss of magnetism as the result of repetitions of heating and cooling.

The magnitude of both α and β varies considerably, but the conditions which determine the magnitude have not been exhaustively examined. Some of these conditions are investigated here, and it will be shown that under certain well-defined circumstances the coefficients α and β may change sign.

2. In a former paper† it was proved that the dimension ratio of a magnet governing its demagnetising factor controls to a large extent the magnitude and even the sign of the temperature coefficient. The experiments then made were carried out on pianoforte drawn steel in the commercial state, but they have now been extended to steel in other conditions, and the results are given in Table I., from which Diagrams I. and II. are plotted.‡

* In the experiments in this paper the range of temperature is from 14° to 100° C.

† 'Roy. Soc. Proc.,' vol. 62, p. 210.

‡ Throughout this paper all numerical results are expressed in c.g.s. units and in degrees Centigrade.

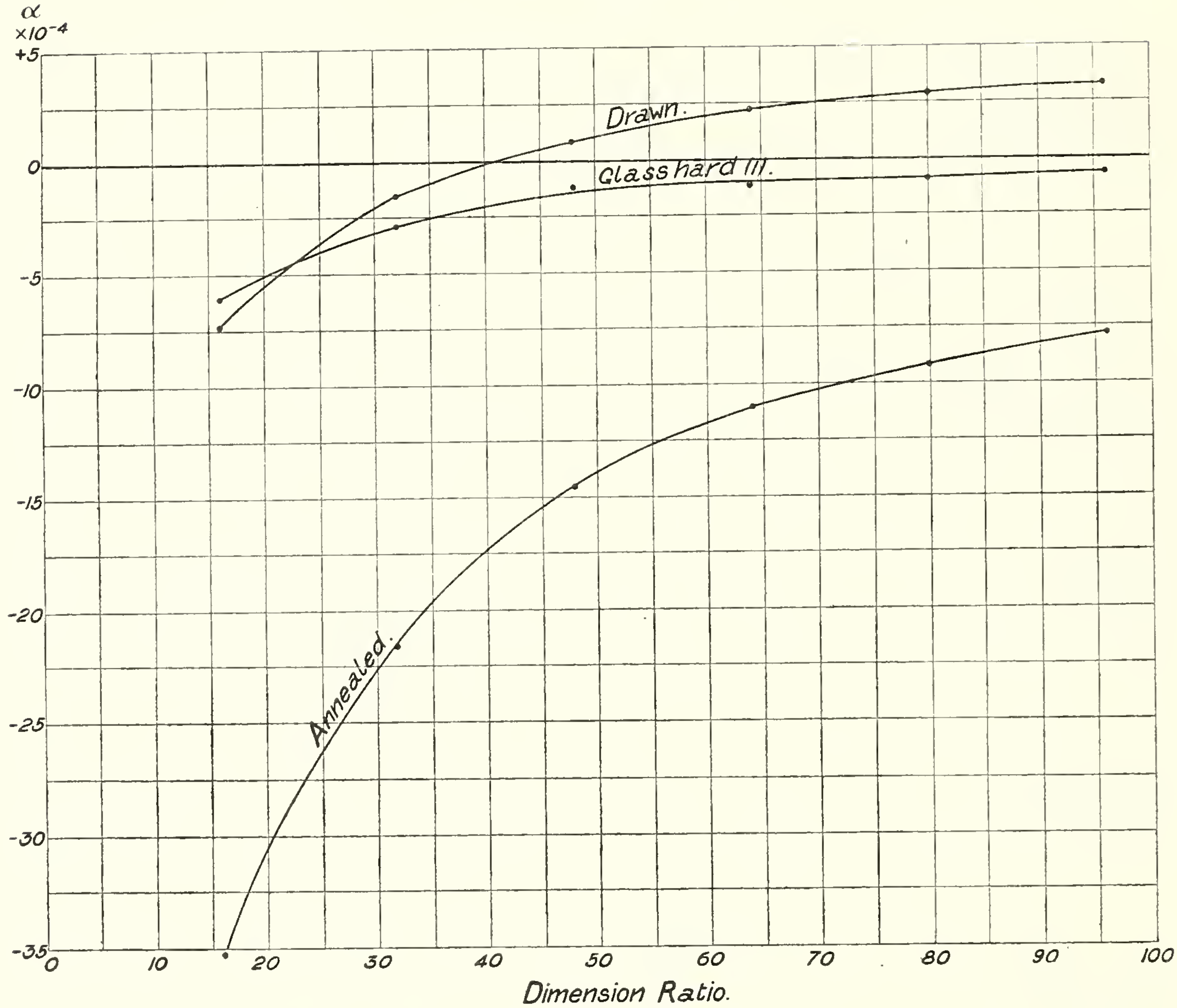
Table I.—Relation of Dimension Ratio to Residual Magnetic Intensity and its Temperature Coefficient in Steel Wire.

H 30 Piano Wire.

Length, in centims.	3	6	9	12	15	18
Dimension ratio	16	32	48	64	80	96
Demagnetising factor	0·1120	0·0340	0·0177	0·0106	0·0069	0·0040
Annealed.						
Intensity initially, *i.e.*, before heating and cooling	160	161	276	387	461	525
Intensity finally, *i.e.*, after heating and cooling	36	103	196	290	363	431
Temperature coefficient (α) × 10^{-4}	−35·3	−21·6	−14·5	−11·0	−9·2	−7·8
Permanent loss (β)	− 0·78	− 0·36	− 0·29	− 0·25	−0·22	−0·18
Glass Hard.						
Intensity initially	252	483	551	602	570	580
Intensity finally	216	437	520	562	556	569
Temperature coefficient (α) × 10^{-4}	−6·03	−2·28	−1·17	−1·22	−0·97	−0·55
Permanent loss (β)	−0·15	−0·09	−0·06	−0·07	−0·03	−0·02
Glass Hard, Remagnetised.						
Intensity initially	299	524	608	655	681	659
Intensity finally	286	508	594	643	668	646
Temperature coefficient (α) × 10^{-4}	−5·26	−2·31	−1·47	−1·05	−0·91	−0·82
Permanent loss (β)	−0·04	−0·03	−0·02	−0·02	−0·02	−0·02
Cold Drawn.						
Intensity initially	137	313	483	602	683	727
Intensity finally	79	204	378	514	595	637
Temperature coefficient (α) × 10^{-4}	−7·32	−1·51	+0·84	+2·25	+2·96	+3·17
Permanent loss (β)	−0·40	−0·35	−0·22	−0·15	−0·13	−0·12

Diagram I.

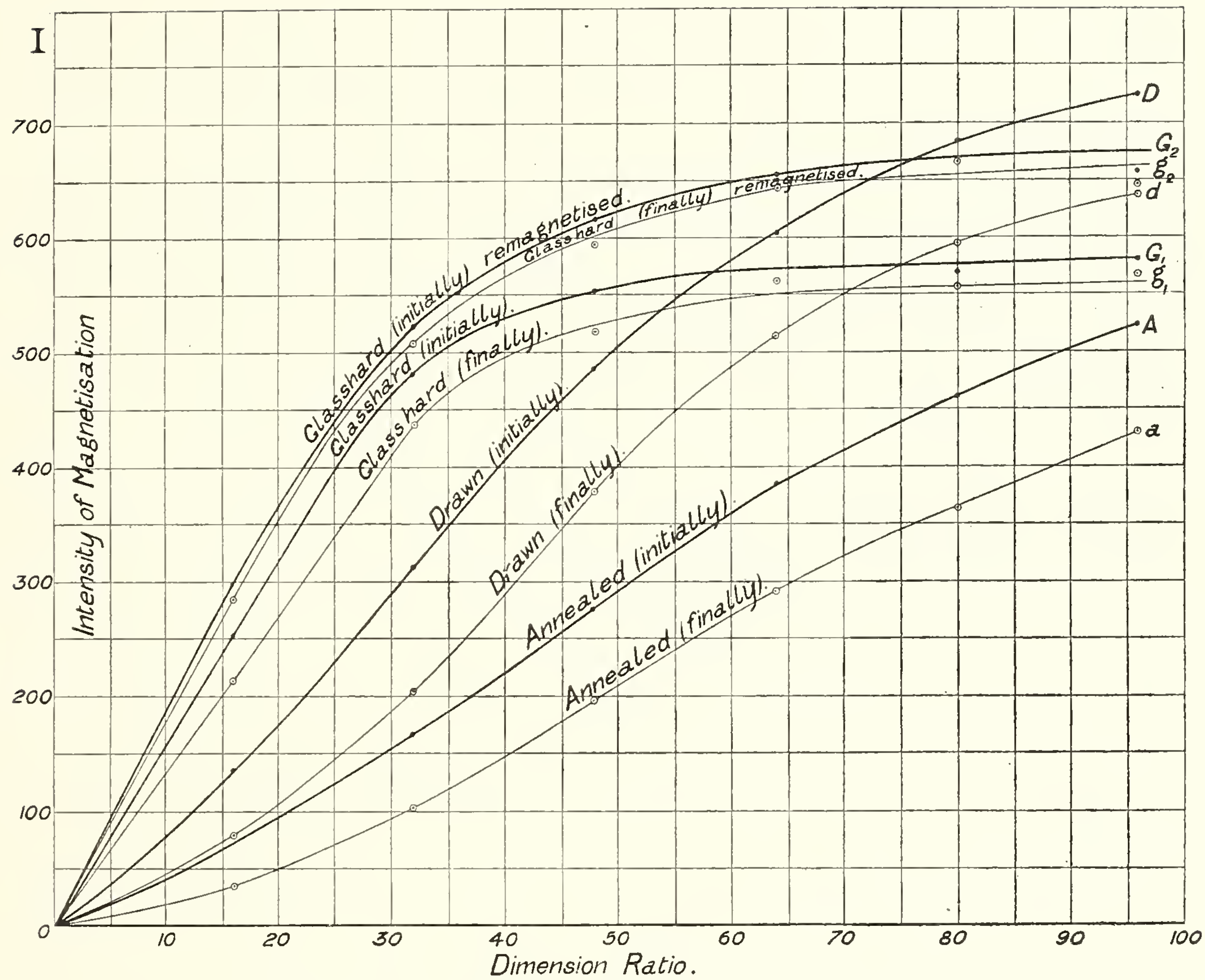
Piano Steel Wire.



The relation of magnetic temperature coefficient (α) to dimension ratio.

Diagram II.

Piano Steel Wire.



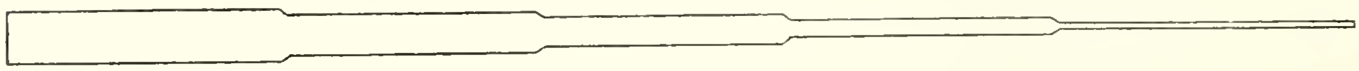
The relation of residual magnetic intensity to dimension ratio.

It was also shown, and it can be seen from the table and diagram, that a rise of temperature in the cyclic state increases the magnetic intensity, and a fall of temperature diminishes it in the case of a magnet of large dimension ratio constructed of drawn steel of the kind supplied for pianofortes,* an effect contrary to what is ordinarily observed.

The results of these experiments, and of others which need not here be introduced, upon a number of steels of different chemical composition, as well as the fact that stretching simply does not produce any marked change in the temperature coefficient, strengthen the conjecture that the abnormal effect is due to repeated drawing.

* Knitting needle and pinion wires do not yield this effect, as they are treated differently to music wire in the process of drawing.

3. In order definitely to test this conclusion, it was necessary to procure samples of wires drawn down finer and finer from one original piece, and Messrs. W. SMITH and SONS, of Warrington, kindly undertook to supply them. The first delivery which came to hand was drawn successively from a wire about 0·159 centim. in diameter, the appearance of the wire being like this :—



The material of the whole was one and the same, and the only difference between one part and another was the amount of traction which had been applied. Lengths were cut off from every stage in the drawing, so that each piece was 100 times longer than its diameter, and all were separately magnetised between the poles of a powerful electromagnet and then immediately examined for magnetic properties. The coefficient which, at the first, is incremental (marked in Table II. with the positive sign*), becomes not more so, but less so as the drawing proceeds, and is finally only half as large as at the beginning, and it seems as if extreme traction might ultimately reduce it to zero. This unexpected result indicates that, if drawing produces the abnormal effect, there must be some stage earlier than the first of this series where a maximum incremental coefficient would be developed. It became necessary again to apply to Messrs. W. SMITH and SONS to prepare for me a complete set of wires drawn successively as before, but beginning now at the rough

TABLE II.—Residual Magnetic Intensity and Temperature Coefficient for Successive Amounts of Traction.

Fine Piano Wire.

No.	Diameter.	Dimension ratio.	Residual intensity initially.	Temperature coefficient, α.
	centims.			$\times 10^{-4}$.
7*a*	0·159	100	608	+2·66
8*a*	0·134	100	677	+2·39
9*a*	0·118	100	690	+2·54
10*a*	0·106	100	785	+2·27
11*a*	0·091	100	866	+2·32
12*a*	0·089	100	915	+2·25
13*a*	0·067	100	970	+1·22

* In a previous paper on this subject ('Roy. Soc. Proc.,' vol. 62, p. 210) an opposite convention was employed in regard to the sign, following an older usage.

rod as received from the rolling mill, and after some delay this set of samples was received. The whole series comprised twelve stages in the manufacture of fine wire as follows:–

4. (1) *The Rolled Rod.*—This is produced from a billet of good Sheffield steel containing less than 1 per cent. of carbon. It is passed whilst hot successively through a number of rolls until its diameter is about 0·5 centim.; the hot rod is finally coiled in a heap, and so cools quickly in the open air.

(2) *Rod Annealed.*—The rod as received from the rolling mill is now annealed by enclosing it in pots from which air is excluded; these pots are heated in a furnace to a bright red heat, at which point the firing is stayed and the fire is allowed to die out. This operation occupies 24 hours.

(3) *Rod Hard Drawn.*—In this stage the annealed rod is forcibly drawn through a perforated plate, which at once reduces the sectional area by about 50 per cent.; the rod now becomes hard.

(4) *Rod Tempered.*—The process to which the rod is next submitted is sometimes called "patenting" or "improving." It is carried out in different ways by different manufacturers, but in these wires it consisted in heating uniformly to a bright red heat in absence of air, and afterwards cooling slowly in a special chamber at a moderate temperature.

(5), (6), (7), (8) *Wire Cold Drawn.*—The tempered wire is drawn through smaller and smaller holes in the draw plate, the sectional area being reduced each time by about 40 per cent. of its preceding value; the diameter in these specimens is in this way diminished to 0·137 centim. at the 8th stage.

(9), (10), (11), (12) The succeeding wires are now all drawn from No. 8 directly and do not pass through every intermediate hole; thus No. 10 is not No. 9 drawn one hole smaller, but is No. 8 reduced at once by a single drawing, and similarly for the others, except, perhaps, No. 12, which probably passed through stage 9 or 10.

These facts relating to the drawing of the last three stages are worthy of notice, as the results of a number of experiments on these wires show some irregularity in the progressive change of their physical properties in the final stages, and the method of drawing in these stages may in part account for the irregularity.

5. The samples illustrating the twelve stages were not long enough to allow the thickest of them to be made more than 50 diameters long, and this fixed the dimension ratio for the whole series, but an additional series, from No. 5 upwards, was cut to a dimension ratio of 100. All the wires were magnetised in the same way between the poles of a powerful electromagnet, and then immediately examined for magnetic intensity, and its changes under variations of temperature, with the apparatus formerly described.* The results, which are given in Table III., and plotted in Diagram III., disclose several interesting facts; thus rolling hot and

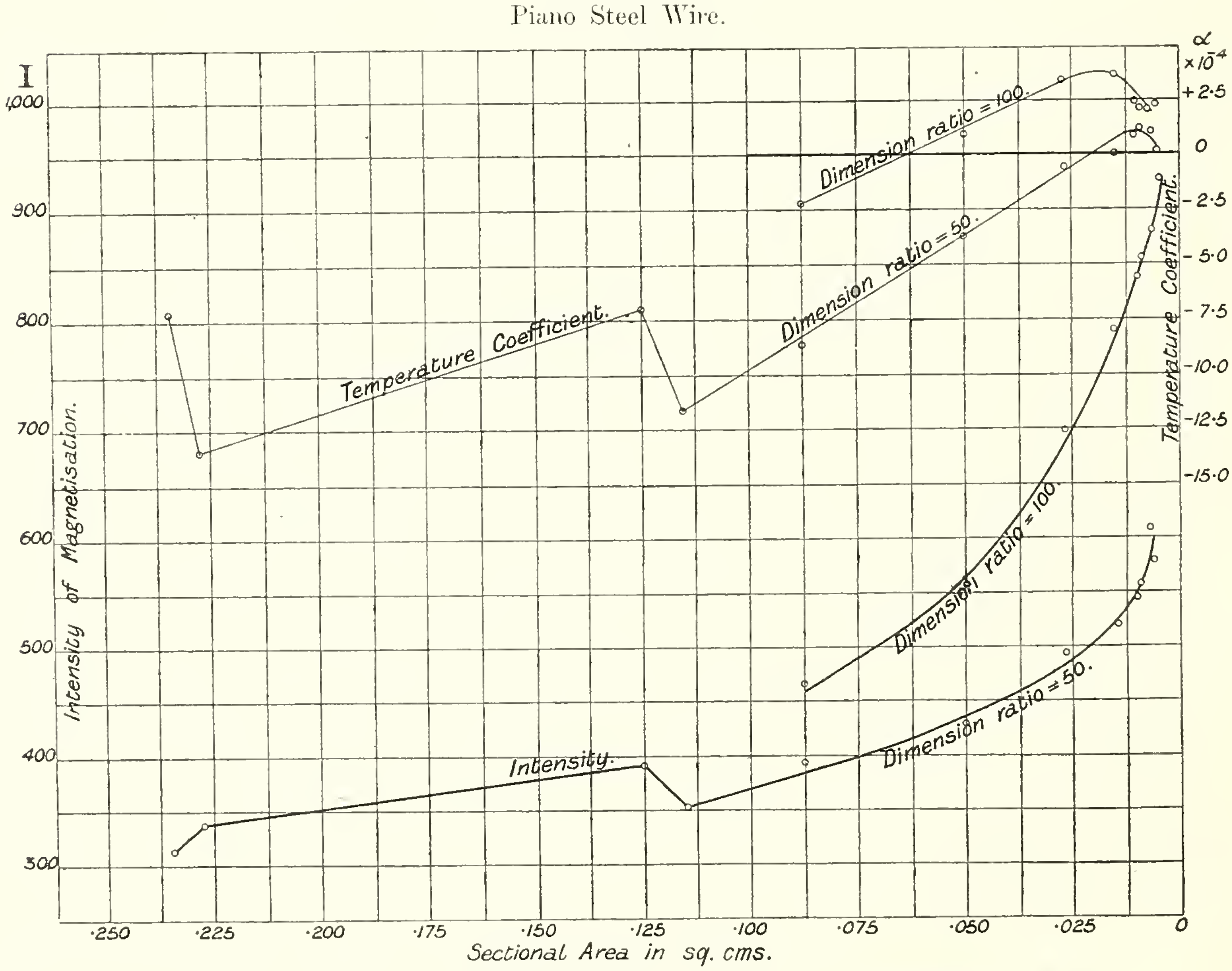
* 'Roy. Soc. Proc.,' vol. 62, p. 210.

Table III.—Residual Magnetic Intensity, Temperature Coefficient, and Permanent Loss at Successive Stages in the Drawing of a Steel Rod to Fine Wire.

No.	Condition.	Diameter.	Dimension ratio = 50. Residual intensity initially.	Temperature coefficient, α.	Permanent loss, β.	Dimension ratio = 100. Residual intensity initially.	Temperature coefficient, α.	Permanent loss, β.
		centims.		$\times 10^{-4}$			$\times 10^{-4}$	
1	Hot rolled .	0·545	315	− 7·11	−0·110	—	—	—
2	Annealed .	0·539	339	−12·62	−0·300	—	—	—
3	Hard drawn	0·399	393	− 6·93	−0·084	—	—	—
4	Tempered .	0·381	352	−11·66	−0·321	—	—	—
5	Cold drawn	0·333	396	− 8·70	−0·292	469	−2·35	−0·076
6	,,	0·253	424	− 3·78	−0·284	551	+0·85	−0·180
7	,,	0·185	497	− 0·61	−0·248	701	+3·43	−0·105
8	,,	0·137	519	− 0·19	−0·207	789	+3·82	−0·032
9	,,	0·121	543	+ 0·75	−0·192	840	+2·28	−0·066
10	,,	0·109	557	+ 1·33	−0·203	857	+1·88	−0·041
11	,,	0·099	613	+ 1·25	−0·155	881	+1·99	−0·016
12	,,	0·089	574	± 0·00	−0·074	930	+2·24	−0·028

Diagram III.

Piano Steel Wire.



The relation of residual magnetic intensity and its temperature coefficient to drawing.

drawing cold both tend to diminish the magnitude of a negative coefficient, whilst drawing the wire several times after tempering, and without re-annealing, completely reverses the sign of the coefficient, which then becomes positive. But extreme drawing bends the curve again to the zero line, and, in the case of the twelfth wire, 50 diameters long, the coefficient actually becomes zero; thus the curve for a dimension ratio of 50 cuts the zero line twice, namely, between the 8th and 9th stages and at the 12th stage.

The diminution of the positive coefficient observed in the experiments on the first set of wires received from Messrs. W. SMITH and SONS is now explained, for it is evident that that series must have commenced beyond the final bend in the curve.

This bend also marks a distinct change in other properties of the wire, for steel wire when drawn too far loses the qualities of strength, elasticity, and electrical conductivity* which moderate drawing confers to a high degree.

The tempering or patenting process in the 4th stage does not seem to be essential to the production, by subsequent drawing, of an incremental coefficient; for if the curve belonging to the larger dimension ratio were continued backwards, following the same path as its companion curve, it nearly, if not quite, reaches the zero line at the 3rd stage, where the wire is drawn *after* annealing, but *before* tempering, and if the experiments had been made on endless wires it is certain that the zero line would have been crossed at the 3rd stage. Hence the production of a positive temperature coefficient is entirely due to cold drawing, if not carried to an extreme stage.

6. In Table III. the initial residual intensities have been calculated as the magnetic moment per unit volume, the volume being obtained from the mass and density. From the figures given in the table, or more clearly from the curve given in Diagram III., it is seen that the intensity steadily mounts upwards as the drawing proceeds. The maximum intensity reached is about 930 units; altogether the residual intensity, after magnetisation, has been increased from 469 at the first drawing to 930 at the last, for magnets 100 diameters long, an increment of no less than 100 per cent. Thus the magnetic properties of steel can be modified to an extraordinary degree by the simple operation of cold drawing through successive holes.

7. Nevertheless, considerable skill and judgment are required in conducting the operation of drawing, if the peculiar qualities which piano wire possesses are to be developed in the highest degree. One fact in connection with the process of manufacture, which may be mentioned here because of its physical interest, is that a wire after drawing through one hole draws more satisfactorily through the next if given a *period of rest* between the operations, and the longer the period of rest, extending even to many weeks, the more satisfactory is the subsequent drawing.†

* Part II., §§ 2 and 4.

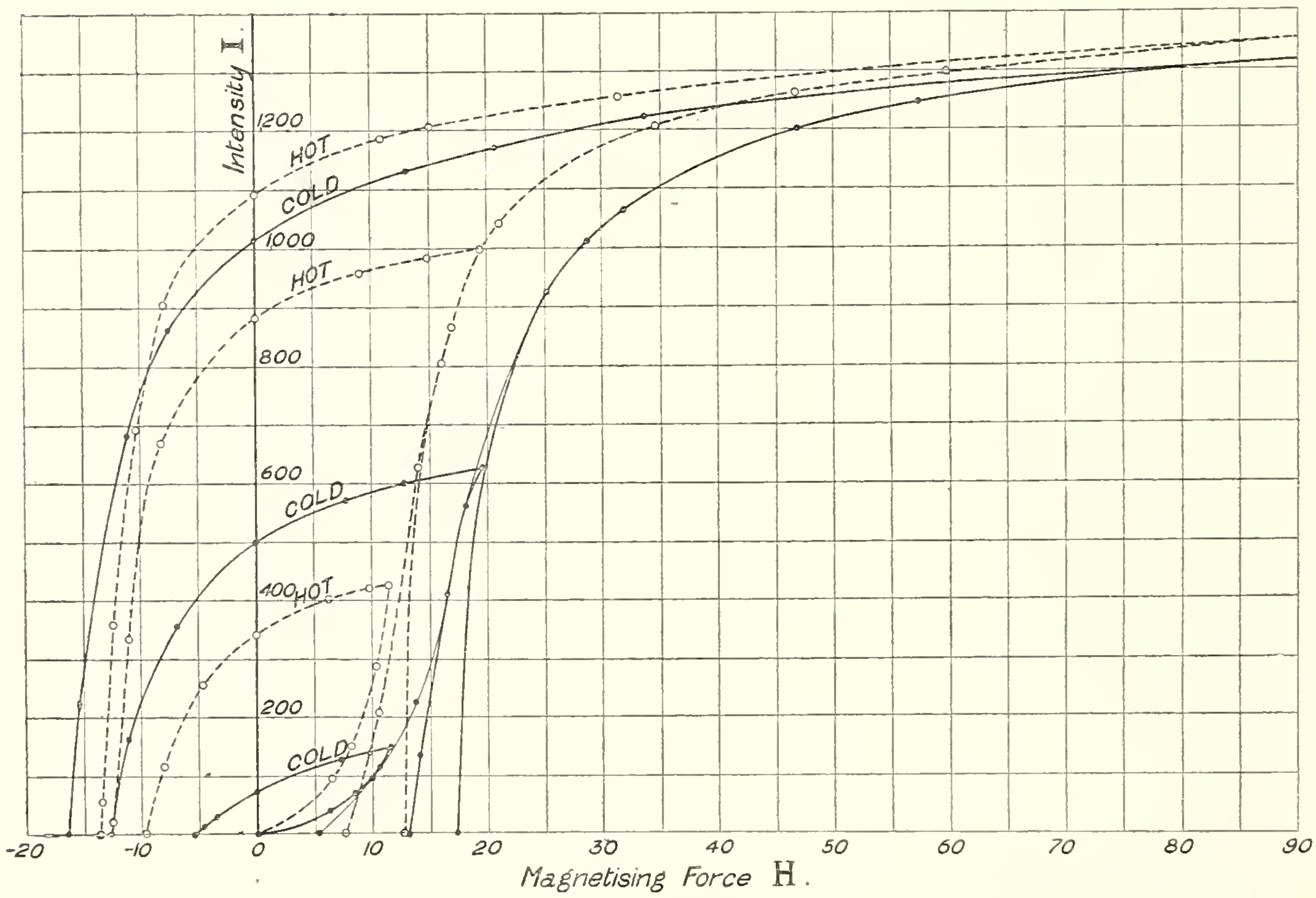
† TOMLINSON remarks that rest after strain diminishes internal friction, 'Roy. Soc. Phil. Trans.,' vol. 177, Part II., p. 835; also vol. 174, p. 5; *vide* EWING, 'Roy. Soc. Proc.,' vol. 30, p. 510.

8. With the object of comparing the magnetisation curves of drawn steel wire when cold and when hot, and also of determining the relation of the temperature coefficient to the intensity, another series of experiments was undertaken. A wire, No. H 30,* was selected of the same diameter and gauge as one upon which a number of experiments had previously been made;† it was 0·187 centim. in diameter, and its length was 85·3 centims., so that the dimension ratio was 456 and the demagnetising factor negligibly small. The wire was fixed in a glass tube in an upright solenoid, 33·2 centims. from the magnetometer, and the usual arrangements were adopted for tracing the curve of magnetisation according to the one-pole method of EWING. In all the experiments the vertical component of the earth's force was neutralized.

After some preliminary heatings and coolings the wire was carried through a series of graded magnetic cycles at air temperature; then, after demagnetisation, a current of steam was passed and maintained through the tube, and the same series of graded cycles was repeated, the forces used at either temperature being exactly alike. It will be seen from Diagram IV. that the susceptibility of the hot wire when the force

Diagram IV.

Piano Steel Wire.



Cyclic magnetisations of piano wire at 14° and at 100°

* Wires marked with the letter H were kindly supplied by Mr. W. D. HOUGHTON, of Warrington.

† 'Roy. Soc. Proc.,' vol. 62, p. 210.

is about 11 or 12 units is nearly three times greater than the susceptibility of the wire cold, and throughout, up to the maximum force employed, the hot curve always lies above the cold curve. On the application of a demagnetising force, after maximum induction, the hot curve droops faster than the cold curve, and crosses it when $-$ H is about 9 or 10 units, and for this force the intensity is the same whether the wire be hot or cold.

9. This suggests that if a suitable demagnetising force be applied the temperature coefficient would be zero, and previous experiments* have verified that a wire of this kind does yield a zero coefficient with an appropriate self-demagnetising force. But it is not possible to calculate with precision, from curves of induction, the requisite demagnetising factor, and consequently the dimension ratio to which such a wire must be cut, in order that the coefficient may be zero, because of complexities in magnetic behaviour arising in part from the irreversible effects of changes of temperature. Nevertheless, an estimate can be made, for, in order that the point of intersection of the curves should lie on the ordinate of no external force, the curves must be sheared by an amount equivalent to about 10 units of force, and as the corresponding intensity is about 800, this would mean that the force per unit of intensity, that is to say the demagnetising factor, must be 0·0125, and in the case of a cylinder the dimension ratio for this factor is nearly 59. Experiment shows, however, that a piece of this kind of wire has an approximately zero coefficient when it is 8 centims. long, and thus the dimension ratio for this condition is 43 instead of 59.

10. Again, the irreversible changes which occur on heating and cooling do not allow the temperature coefficient to be simply calculated from the difference of the hot and cold residual intensities which are left after removal of the magnetising force. Experiments were made on the wire in two ways. Beginning with the wire hot, the intensity fell from 1294 to 1139 during a series of heatings and coolings, and the coefficient was + 0·000358. Then, repeating the experiment, but beginning with the wire cold, the intensity fell from 1204 to 1126, and the coefficient was + 0·000327, a value not very different from the former, but less than half the coefficient calculated from the difference of the hot and cold intensities left immediately after the withdrawal of the magnetising force.

Nevertheless, the relation of these hot and cold curves throws light on the fact that the temperature coefficient of a magnet of drawn steel is positive or negative, according as the demagnetising factor is below or above a certain value, a result which has been fully established (*vide* Table I.).

11. In another series of experiments the temperature coefficient was determined at different stages of the curve, the wire being kept during heating and cooling *under a constant force*. The wire, the same one as before, was, in the first place, demagnetised carefully, and then a field of 11·0 units was applied and maintained; the intensity at 16° was 120·7, but on passing steam through the tube the intensity

* 'Roy. Soc. Proc.,' vol. 62, p. 210.

immediately rose to 400, more than three times the former amount, and the subsequent alternations of temperature slowly augmented the intensity until a final value was approached at 477; the coefficient then was + 0·000731. Raised another step by a force of 19·1 units, the intensity became 724, and, after heating and cooling a few times, it rose to 993; the coefficient now was + 0·000466. And lastly, with a force of 76·5, the intensity was 1311 initially, and after heating and cooling 1316, the coefficient being + 0·000206. The enormous growth of magnetism under changes of temperature in the earlier stages of magnetisation, and the insignificant increment in the final state, concurrently with the large coefficient when the susceptibility is large and its diminution as the susceptibility becomes less, are the features here principally to notice.

Applying next a small negative force of − 7·19 units, the intensity dropped finally to 745, and the coefficient was then + 0·000229; and it might be anticipated from the position of the point of intersection of the hot and cold curves already examined, that a slightly larger demagnetising force ought to annul the positive coefficient, and that a still larger demagnetising force ought to yield a negative coefficient.

A force of − 11·23 units was next applied and maintained, which, while less than the coercive force for the material hot or cold, was greater than the force at the point where the curves cross. With this force we get the following result:—

Intensity.		
Cold.	Hot.	
+618		Here the south pole was upwards.
	+259	
+227		
	+145	
+134		
	+ 69	
+ 65		
	+ 8	
+ 10		Magnetism reversed.
	− 38	The north pole is now upwards.
− 34		
	− 80	
− 74		
	−116	
−107		
	−151	
−141		

Here it is evident that the force applied has been too large to allow a cyclic state to be established before reversal of the magnetism sets in, yet, in accordance with anticipation, up to the point where the original magnetism is entirely removed the hot value of the intensity intermediate between any two cold values is always less than the mean of the cold values, a result which corresponds to a negative coefficient; later on, after the reversal, the hot is larger than the mean of the adjacent cold values, a result which corresponds to a positive coefficient, and this positive coefficient will, no doubt, continue up to maximum induction.

Both before and after reversal heating appears to produce much larger effects than cooling.

Under alternate applications of heat and cold, the coercive force is greatly lessened, and now lies at less than 11·2, instead of its former value 13·5 hot and 17·0 cold.

12. In the next place the effects of alternate changes of temperature on *residual* magnetism were studied.

The same wire was again employed, and, starting from the demagnetised state, a small force was applied, and the residual magnetism subjected to heating and cooling. Then stronger forces were successively applied and, in the same way, at each step the coefficient was determined; after the maximum had been reached, part of the magnetism was removed, a step at a time, and again the temperature coefficient was examined at each stage. Table IV. is an abstract of the results so obtained.

TABLE IV.—Effects of Heating and Cooling on Residual Magnetism for Progressive Magnetisation.

H 30 Piano Wire.

Magnetising force, H.	Residual intensity. Initially, I_i.	Residual intensity. Finally, I_f.	Temperature coefficient, α.	Permanent change, β.
			$\times 10^{-4}$	$\times 10^{-2}$
10·32	107	91	+ 3·10	− 15·0
19·52	616	572	+ 4·53	− 7·0
23·33	838	794	+ 3·59	− 5·0
(About 100)	1022	967	+ 2·82	− 5·0
(Small negative force)	837	840	+ 4·34	+ 0·4
− 10·54	762	775	+ 4·42	+ 2·0
− 15·03	270	312	+ 3·66	+ 16·0
− 15·93	114	157	+ 1·88	+ 38·0
—	21	64	− 2·90	+ 204·0
—	− 12	+ 30	− 4·70	—
—	− 62	− 20	+ 25·00	− 224·0

From this table the influence of the intensity on the magnitude and sign of the coefficients α and β can be traced.

In the first place, the coefficient rises to an early maximum and then falls to a minimum at the highest intensity; on the return path the coefficient again attains a maximum, which occurs at a higher intensity than before, and after this it continually grows less, and ultimately changes sign and becomes negative. The rise and fall suggest some connexion with susceptibility, for which there is evidence in another group of experiments described later on.

In the second place, when the magnetic intensity is rising, the final intensity, after a series of heatings and coolings, is always less than initially, and β is therefore negative, but on gradually removing successive fractions of the magnetism, then, at each stage, heatings and coolings produce a *gain* of magnetism, and the loss or gain is always much greater the smaller the intensity; indeed, the gain becomes very large at low intensities on the downward path, and at last, when the reversed magnetic field has been increased so far as to leave a small reversed residual magnetic intensity, then heatings and coolings clear this out and restore a small magnetisation of the original kind.

These results could be obtained repeatedly; thus, in the following example, which is a typical one, after magnetising to saturation a reversed force left a residual of 109·2, which was augmented by heating and cooling to 167·2 with a positive coefficient of $+ 3{\cdot}30 \times 10^{-4}$, a further application of reversed force left a residual of 15·2 in the opposite direction, and then as follows:—

Cold.	Hot.	
−15·2		
	+11·4	Direction of original magnetisation restored.
+19·0		
	+19·0	
+23·0		
	+22·2	
+24·9		
	+24·3	
+26·2		The coefficient is negative and equal to −0·00085.
	+24·7	
+26·8		

Here there is a change in the direction of the magnetisation due to heating and cooling and a coefficient which is now negative. But the negative coefficient is found to become less negative when the intensity thus restored is greater, as the following table shows:—

Final intensity after reversal.	Coefficient.
+23·8	−0·00120
+26·8	−0·00085
+30·4	−0·00047
+63·6	−0·00029

and no doubt a zero coefficient would be reached for a still higher intensity than 63·6.

If, however, the magnetic force which has been applied is strong enough to leave a residual intensity which, after heatings and coolings, still remains in the reverse direction to the original, the positive coefficient is again established.

13. In the experiments recorded above the intensity was raised a step at a time, after each series of heatings and coolings, and a suspicion might be entertained that the magnetic state at any stage had been seriously disturbed by the heatings and coolings at a preceding stage. The next series of experiments was undertaken to test this question, and accordingly, after any series of heatings and coolings, the wire was completely demagnetised by reversals before the magnetisation was carried a grade higher. Beginning at a low intensity, the magnetisation was thus carried to its highest value by easy steps. In returning, the wire was magnetised strongly and then a small reversed force applied, enough to remove some of the residual magnetism; after heating and cooling, the wire was demagnetised, carried again to its highest intensity, and a larger fraction of the residual magnetism removed, heating and cooling repeated, and so on. In this way the following table was constructed (Table V.).

TABLE V.—Effects of Heating and Cooling on Residual Magnetism with Demagnetisation between each Step. Relation of Intensity to Temperature Coefficient.

H 30 Piano Wire.

Magnetising force, H.	Induced intensity, I.	Residual intensity.		Temperature coefficient, α.	Permanent change, β.	Susceptibility, $\kappa = I/H$.
		Initially, I_i.	Finally, I_f.			
				$\times 10^{-4}$	$\times 10^{-2}$	
8·75	70	25	12	+3·05	−50·8	8·2
12·79	181	112	86	+4·07	−23·1	14·1
15·26	288	247	212	+4·98	−13·9	18·8
18·18	500	476	432	+5·15	− 9·4	27·5
22·44	785	702	653	+5·06	− 7·0	35·0
41·07	1141	915	839	+4·66	− 6·5	27·7
101·20	1334	1028	961	+3·27	− 6·5	13·2
− 8·53	—	810	813	+4·45	+ 0·4	—
− 13·69	—	516	535	+4·37	+ 5·8	—
− 15·03	—	291	331	+4·16	+13·8	—
− 17·05	—	− 17	+ 39	−4·10	—	—

The same general features as before again exhibit themselves, namely, a rise and fall of the temperature coefficient as the intensity proceeds either to a maximum or proceeds to a minimum, the largest value of α occurring earlier for increasing than for decreasing intensities; a loss, β negative, so long as the applied force has been positive and a gain when the force has been negative, the magnitude of the gain or loss being greater the less the intensity; a reversal of the direction of the magnetisation at a low intensity by the operation of changes of temperature; and also a change in the sign of α. But the magnitude of the temperature coefficient and of the irreversible change is decidedly larger in this table than in the former one. Hence, for the production of a magnet of constant intensity constructed of drawn steel wire, it would appear to be advantageous to magnetise step by step, heating and cooling at each step without intermediate demagnetisations up to maximum intensity, and then to remove a small fraction of the magnetism by a reversed force; β is then at its least value.* On the other hand, this gives a larger value for α than if no reversed force had been applied.

14. In order to determine how far these results are due to drawing, it is necessary to have a comparison with similar experiments performed on an iron wire, and this was subsequently done. The iron wire was that which is supplied in commerce as such, but probably it borders on very mild steel; it was carefully annealed, and then submitted to a cycle of magnetisation, at first cold and afterwards hot. The curves of magnetisation intersect in this material when the intensity, for rising forces, is about 800 units; at higher intensities the susceptibility is less hot than cold, and the hot residual lies below the cold residual,† accordingly subsequent heating is found to diminish and cooling to increase the residual magnetism. The coercive force is about 4·0 units cold and a little less when hot.

In the next place the wire, after demagnetisation by reversal, was submitted to a very small force and the force withdrawn, then a series of heatings and coolings was applied, and the permanent loss and coefficient were calculated in the usual way. Again, after demagnetisation, a stronger force was applied and withdrawn and a series of heatings and coolings executed. Repeating these operations a step higher each time and demagnetising between each step, we get the result exhibited in Table VI.

* *Vide* HOOKHAM, 'Journal Inst. Electr. Engineers,' vol. 18, p. 688.

† BAUR, 'Wied Ann.,' vol. 11, 1880; EWING, 'Phil. Trans.,' vol. 176, Part II., p. 637.

Table VI.—Effects of Heating and Cooling on Residual Magnetism with Demagnetisation between each Step. Relation of Intensity to Temperature Coefficient.

Annealed Iron Wire.

Magnetising force, H.	Induced intensity, I.	Residual intensity.		Temperature coefficient, α.	Permanent change, β.	Susceptibility, $\kappa = I/H$.
		Initially, before heating and cooling, I_i.	Finally, after heating and cooling, I_f.			
				$\times 10^{-4}$	$\times 10^{-2}$	
3·03	115	78	75	−2·14	− 3·67	37·9
4·93	346	279	275	−1·49	− 1·57	70·2
6·01	616	527	518	−1·24	− 1·71	102·5
8·53	824	704	690	−1·45	− 2·02	96·6
14·13	1027	855	837	−1·45	− 2·12	72·6
40·70	1248	952	931	−1·40	− 2·14	30·6
87·50	1346	962	941	−1·45	− 2·19	15·4
− 2·69	733	750	752	−1·44	+ 0·35	—
− 3·14	677	699	702	−1·28	+ 0·38	—
− 3·81	218	271	276	−1·68	+ 1·70	—
—	39	100	104	−1·68	+ 4·62	—
− 4·26	− 92	− 24	− 16	+1·63	−35·00	—

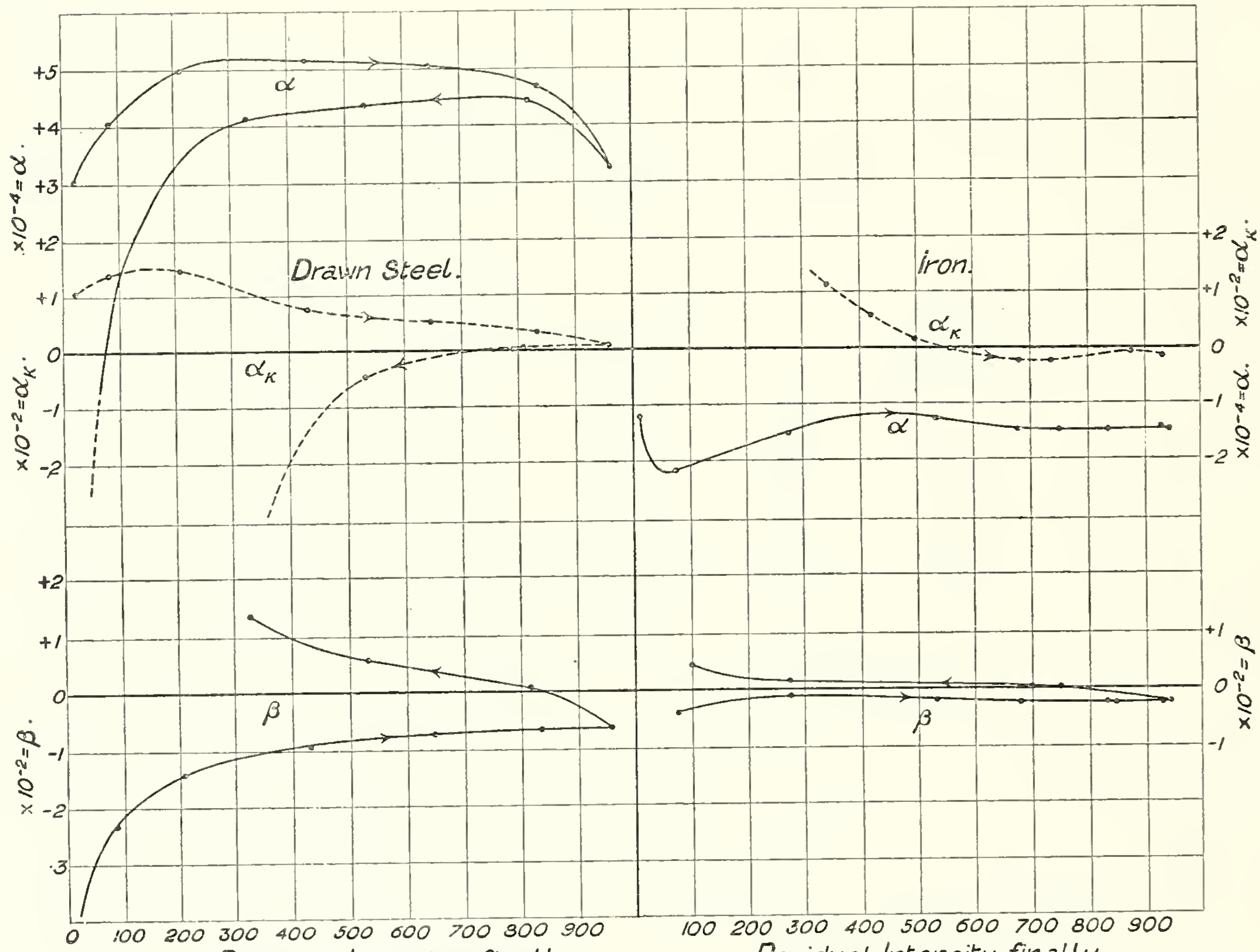
Here the temperature coefficient is negative throughout for rising forces; as the intensity progresses it diminishes to a minimum value when I is about 518 and then, increasing again, becomes nearly constant. For reversed forces the coefficient diminishes to another low value at about 700 units (a higher intensity than before), and then again increases, but if the reversed force is just sufficient to leave a small inverse magnetisation, the coefficient changes sign and becomes *positive.* It appears then that in iron similar features present themselves in the relation of α to I as in drawn steel, but in iron the coefficient is in general negative, and in drawn steel positive, and in each the sign of the coefficient can be reversed when a small inverse magnetisation succeeds a strong direct magnetisation.

The permanent change, β, is larger for low intensities than for high ones, and fluctuates in sympathy with α; for reversed forces it changes sign, that is to say, there is a gain of magnetism due to cyclic temperature changes as there was with drawn sheet. Throughout it will be noticed that in soft iron β, as well as α, is much smaller than in drawn steel. The table shows that a very long soft iron wire may have a temperature coefficient of less than −0·00015 per degree centigrade; also, that if a small reversed force be applied after magnetisation to saturation, no *loss* of residual magnetism takes place due to heating and cooling, but the intensity tends to increase.

15. The experiments which have been narrated, show that the sign and the

magnitude of the temperature coefficient may be inferred from the disposition of the hot and cold curves of magnetisation. This is still further confirmed by reference to Diagram V., where the temperature coefficient of drawn steel, for both ascending

Diagram V.



The relation of temperature coefficients (α) and (α_κ) and permanent change of magnetism (β) to residual intensity.

and descending values of magnetisation, is plotted against the residual intensities left finally after heating and cooling. The broken curve traced underneath is the change of residual magnetism per unit per degree of temperature calculated from curves of residual magnetism when the wire was at 16° and when it was at 100°. This is marked α_κ (as it corresponds to the temperature coefficient of susceptibility for residual magnetism), and is traced for both ascending and descending intensities. The distinctive features of one curve are reproduced in the other, although, as might be expected, owing to irreversible changes, the curves are not an accurate fit. Thus, the zero coefficient experimentally found is displaced largely to the left of the zero position in the calculated curve.

If the iron curves, traced in the same way, are examined, the distinctive features of the one will be found also reproduced in the other, but with a large displacement of one relatively to the other. Thus there is a small value of the negative coefficient in the calculated curve between intensities of 800 and 900, and a maximum about an intensity of 700; the counterpart to these occur in the experimental curve between intensities of 400 and 500, and at less than 100 respectively. Still more interesting is the large displacement of the zero coefficient. In the α_κ curve the zero state occurs about an intensity of 560, and in the α curve probably at an intensity of only a few units. Although I have attempted to obtain experimentally the zero temperature coefficient in iron, I have not succeeded unmistakably, partly because of the general difficulty of working at very low intensities, and partly because of the special difficulty of clearing out all traces of pre-existing magnetisation, which is a very necessary precaution, and of operating in a field of no force. But I have ascertained that at a very feeble intensity the negative coefficient becomes decidedly less negative, and that the curve tends towards zero at some extremely low magnetisation.

EWING, however, has obtained at a very early stage in the magnetisation of iron a positive, and at a higher but still a very low intensity a zero coefficient.* Although his experiment was not performed on residual magnetism alone, as the vertical component of the earth's force was always in operation in such a way as to tend to increase the magnetisation, and at low intensities its effect would be considerable, yet there is little doubt that at some very low residual intensity iron yields a zero coefficient. There is thus a satisfactory correspondence between the results calculated from the relation of the curves of residual intensity when hot and when cold and the results experimentally found for the temperature coefficient of residual magnetism. Every magnet therefore may have a positive, a negative, or a zero coefficient, unless the hot and cold curves happen to be coincident throughout.

The changes which take place in the magnitude of the permanent loss and gain of magnetism due to a series of heatings and coolings, are shown graphically for both drawn steel and iron on the lower part of Diagram V.

16. The experiments I have selected for description throw light, I think, on many of the numerous results which have been published on the effects of cyclic changes of temperature on magnetism,† and also afford some guiding rules for the construction of magnets of high permanence and with small temperature coefficients.

* 'Roy. Soc. Phil. Trans.,' vol. 176, p. 633.

† References to papers on the "Influence of Changes of Temperature on Magnetism" and allied subjects:—

FARADAY, 'Phil. Mag.,' vol. 8, p. 177, 1836; KATER, 'Roy. Soc. Phil. Trans.,' 1821; BARLOW and BONNYCASTLE, 'Roy. Soc. Phil. Trans.,' 1822; RIESS and MOSER, 'Pogg. Ann.,' vol. 17, p. 425, 1829; KUPFFER, 'Kastner's Archiv,' vol. 6; LAMONT'S 'Magnetismus'; SCORESBY, 'Edin. Phil. Trans.,' vol. 9, p. 254; WIEDEMANN, 'Pogg. Ann.,' vol. 103, p. 563, 1858; MAURITIUS, 'Pogg. Ann.,' 1863, 'Phil. Mag.,' 1864; GORE, 'Phil. Mag.,' 1869 and 1870; GORDON and NEWALL, 'Phil. Mag.,' vol. 42, p. 335, 1871; WHIPPLE, 'Roy. Soc. Proc.,' 1877; ROWLAND, 'Phil. Mag.,' vol. 48, p. 321, 1874; FAVÉ, 'C.R.,' vol. 82, p. 276, 1876; GAUGAIN, 'C.R.,' vol. 80, p. 297, vol. 82, p. 685, vol. 83, p. 896, vol. 85, pp. 219, When the magnet is short relatively to its thickness, the self-demagnetising force will have so preponderating an influence, that it will be advisable, in order to reduce its effect to a minimum, to choose a material of small susceptibility; a hard steel is thus preferred. If, however, the magnet is long and thin, attention must be paid chiefly to the quality and treatment of the material, so that it may develop that condition in which the hot and cold curves of its magnetisation are separated as little as possible.

And further, it has been shown that drawing influences the disposition of the hot and cold curves in such a way that it affords an effective method of regulating the magnitude and sign of the temperature coefficient.

Time Tests

On the Constancy of Magnets with Negligible Coefficients.

17. At the conclusion of a previous paper* reciting experiments upon the construction of magnets with zero temperature coefficients, a brief note was added on the question of the constancy of the zero state with lapse of time. This is obviously important in the application of such magnets to the work of an observatory, and it has therefore received some attention.

In May, 1897, a magnet was constructed of a piece of H 30 wire, 0·187 centim in diameter, and from a previous series of experiments upon this kind of wire it was calculated that it should be cut to a length of 8 centims., having a dimension ratio of 42·6, in order to yield a zero coefficient. It was then magnetised and heated and cooled about twenty times so as to reduce the magnetism to a settled state. The intensity before heating and cooling was 474·4 c.g.s. units, calculating this here, as elsewhere, as the magnetic moment per unit volume and, after heating and cooling, the intensity was 28·1 per cent. less; the coefficient, α, was very small and positive, its value being $+ 0·000015$ per degree centigrade. This magnet was tested at intervals for the next three years, and its history is given in Table VII., and also in the diagram constructed from this table (Diagram VI.). After the first test it

615, 1014, and vol. 86, p. 536; JAMIN and GAUGAIN, 'C.R.,' 1876, 'Phil. Mag.,' 1876; POLONI, 'Wied. Beibl.,' 1878; WASSMUTH, 'Wien. Ber.,' 1880–02; BAUR, 'Wied. Ann.,' vol. 11, 1880; BROWN, 'Phil. Mag.,' vol. 23, pp. 293, 420, 1887; CHEESMAN, 'Wied. Ann.,' vol. 15, p. 204, vol. 16, p. 712; BARUS and STROUHAL, 'Wied. Ann.,' vol. 20, p. 662, 1883; 'Bulletin U.S. Geol. Survey,' No. 14, 1885; BARUS, 'Phil. Mag.,' Nov., 1888; GRAY, 'Phil. Mag.,' vol. 6, p. 321, 1878; BOSANQUET, 'Phil. Mag.,' vol. 19, p. 57, 1885; CANCANI, 'Atti della R. Acc. dei Lincei' (4), vol. 3, p. 501, 1887; MORRIS, D. K., 'Phil. Mag.,' vol. 44, p. 213, 1897; DURWARD, A., 'Am. Journal of Sci.,' April, 1898; PIERCE, B. O., 'Am. Journal of Sci.,' May, 1898; EWING, 'Roy. Soc. Phil. Trans.,' vol. 176, 1885; also CANTON and HALLSTRÖM, COULOMB, HANSTEEN, CHRISTIE, LLOYD, CHREE, &c.

* 'Roy. Soc. Proc.,' vol. 62, p. 210.

Table VII.—Influence of Time on the Temperature Coefficient and Residual Magnetic Intensity of Drawn Steel.

H 30 Piano Wire, 8 centims. long.

Date.	Residual intensity. Initially.	Residual intensity. Finally.	Temperature coefficient, α.	Remarks.
			$\times 10^{-4}$	
1897.				
May 21	474	341	+0·15	Magnetised for the first time.
„ 26	—	—	—	Boiled for 2 hours.
June 1	—	—	—	„ „ $1\frac{1}{2}$ „
„ 9	265	264	+0·15	
July 5	255	255	−0·14	
September 10	240	240	−0·40	
October 18	222	—	−0·40	
1898.				
July 11	222	222	−0·47	
„ 11	475	349	+0·05	Remagnetised.
„ 14	341	338	+0·12	Boiled for $4\frac{1}{2}$ hours.
„ 17	—	—	—	„ „ 12 „
„ 18	333	330	±0·00	
October 10	324	320	+0·09	
„ 14	310	307	+0·07	
1899.				
June 16	322	322	+0·61	
October 30	317	316	+0·58	
1900.				
March 29	312	311	+0·42	

Second Sample.

H 30 Piano Wire, 8 centims. long.

Date.	Residual intensity. Initially.	Residual intensity. Finally.	Temperature coefficient, α.	Remarks.
1898.				
October 14	474	—	—	Magnetised first time. Heated and cooled 12 times.
„ 21	368	364	+0·33	
„ 22	—	—	—	Remagnetised. Boiled for 3 hours. Heated and cooled 12 times.
November 9	345	321	+0·31	
„ 14	335	334	+0·32	
1899.				
June 16	341	337	+0·76	
November 3	332	331	+0·65	
1900.				
April 9	331	327	+0·59	

was boiled for three and a half hours in two stages, and a week later the coefficient was exactly the same as before. It was now laid aside for a month, when the coefficient was found to have changed from +0·000015 to −0·000014, and this again slowly altered for twelve months, and became finally nearly steady at − 0·000047, which is less than a half of 1 per cent. for 100° C. The intensity had also changed, at first quickly, but latterly very slowly. Here it is interesting to notice that at any stage heating and cooling through a range of 80° or 90° diminishes the intensity in the later tests to a very trifling extent, but the undisturbed action of time produces a very slow but steady diminution to the final limit; and this recalls the circumstance mentioned previously, how the action of time alone alters the molecular structure of steel so that its drawing qualities are greatly improved (§ 7).

The magnet was now remagnetised (July 11th, 1898), which immediately raised the intensity from 221·7 to 474·8, a value practically identical with its initial intensity; after a series of heatings and coolings, with boiling at intervals, the magnet was laid aside, its coefficient then (October 14th) being + 0·000007, and the intensity 306·9. At the same time another piece of H 30 wire was cut so as to be 8 centims. long, magnetised and tested. Its intensity initially was 474·7 and its coefficient +0·000033. After a number of heatings and coolings, remagnetisation and boiling for several hours, it was tested again, and laid aside for comparison with the former. Its coefficient was then + 0·000032, and the intensity 333·9 (Table VII.). Both these magnets were tested in June and November, 1899, and again in April, 1900. It appears that in the summer of 1899 the coefficients were larger than seven months previously, and reached a maximum of + 0·000061 and + 0·000076, and that since then they have become a little less, and now (April, 1900) stand at + 0·000042 and + 0·000059.

The intensity after heating and cooling has only fluctuated to the extent of 3 per cent.

Both these magnets are almost exactly alike in all the details of their behaviour, and it will be noticed that the second magnet when magnetised and immediately remagnetised, without any long interval between the magnetisations as with number one, arrives at once at the same condition as the first.

18. Magnets made from this wire with a larger dimension ratio would have higher intensities and yield still more constant results, but then the coefficients would depart considerably from zero, because of the too small demagnetising factor. If, however, a longer piece of the wire be taken, a part of its abnormal properties can be removed by heating to a suitable temperature and quenching, as already shown,* and thus a zero coefficient can be obtained with a smaller demagnetising factor.

Two lengths of 12 centims. each were cut from the same kind and thickness of wire as before, and were heated until just red-hot and quenched in water; they were then magnetised and repeatedly heated and cooled. The coefficient was negative and in magnitude −0·000044, and the intensity about 670 for both, the original intensity

* 'Roy. Soc. Proc.,' vol. 62, p. 215.

immediately after magnetisation and before heating and cooling being about 700. They were then remagnetised, boiled for some hours, heated and cooled many times, and the coefficient determined ; after this they were laid aside for several months. This was in November, 1898. When re-examined in June and November of 1899, and again in March, 1900, very little change had taken place in either the coefficients or the intensities, which were about $-$ 0·000025 and 700 respectively. Both magnets were almost exactly the same in their behaviour in every respect, as will be seen from Table VIII., which gives their history.

Table VIII.—Influence of Time on the Temperature Coefficient and Residual Magnetic Intensity of Drawn Steel when Tempered.

H 30 Piano Wire, 12 centims. long, heated to redness and quenched.

Date.	Residual intensity.		Temperature coefficient, α.	Remarks.
	Initially.	Finally.	$\times 10^{-4}$	
1898.				Magnetised first time and heated and cooled 12 times.
October 14. . . .	707	682	—	
„ 24. . . .	686	675	$-0 \cdot 44$	Remagnetised, boiled for $2\frac{1}{2}$ hours, and heated and cooled 12 times.
November 9 . . .	740	716	$-0 \cdot 32$	
1899.				
June 16	736	730	$-0 \cdot 26$	
November 8 . . .	718	711	$-0 \cdot 24$	
1900.				
March 29	704	700	$-0 \cdot 25$	

Second Sample.

H 30 Piano Wire, 12 centims. long, heated to redness and quenched.

Date.	Initially.	Finally.	Temperature coefficient, α.	Remarks.
1898.				Magnetised first time and heated and cooled 12 times.
October 14 . . .	700	680	—	
November 3 . . .	677	669	$-0 \cdot 45$	Remagnetised, boiled $2\frac{1}{2}$ hours, heated and cooled 12 times.
„ 9 . . .	742	728	$-0 \cdot 27$	
1899.				
June 16.	749	742	$-0 \cdot 32$	
November 8 . . .	726	721	$-0 \cdot 21$	
„ 15 . . .	—	—	—	Boiled 3 hours.
„ 15 . . .	724	720	$-0 \cdot 25$	
1900.				
April 9	709	709	$-0 \cdot 24$	

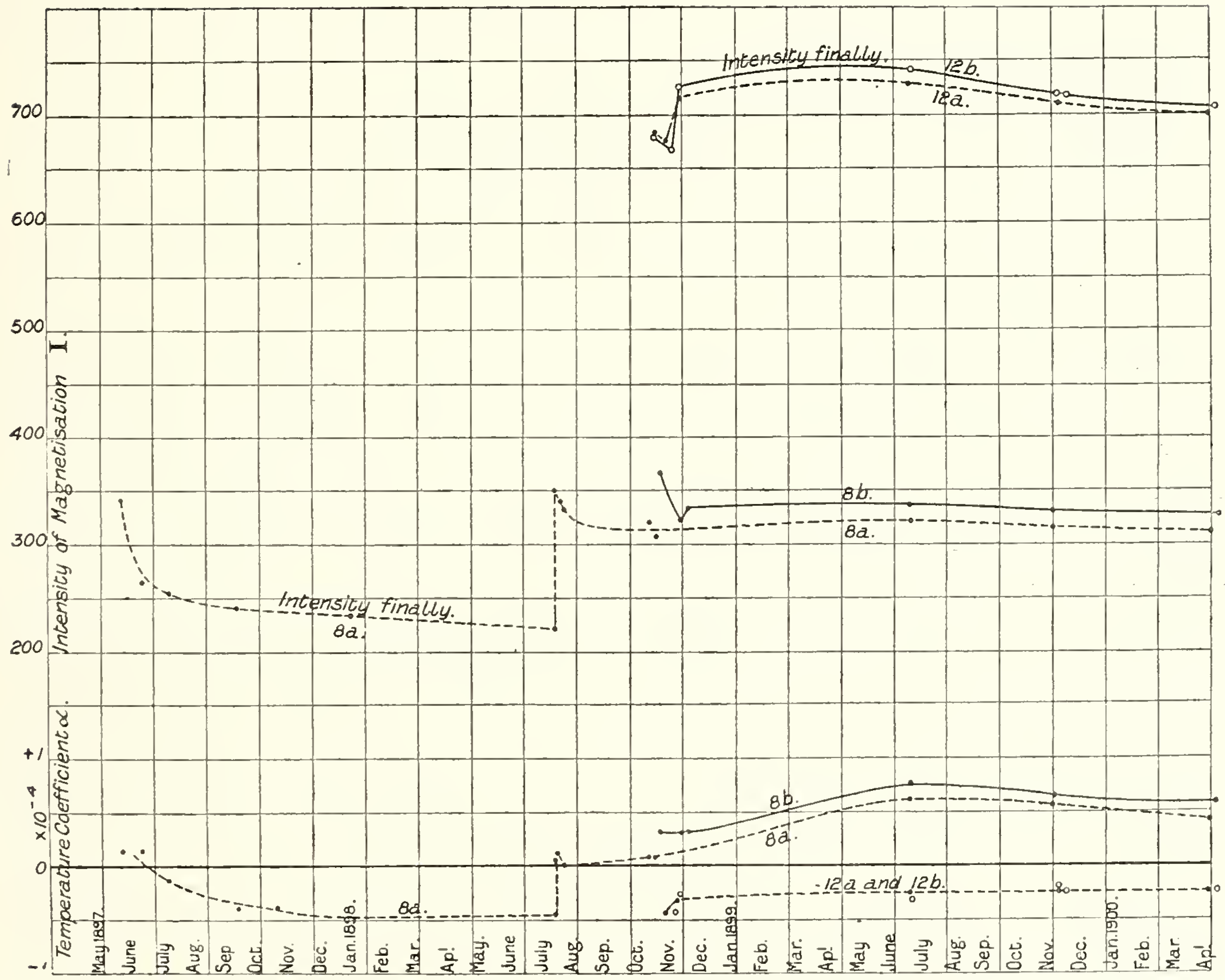
For steady values of α and I this latter method of constructing magnets is to be preferred, and with further experimental experience even the small coefficient here exhibited might yet be reduced; the high intensity, too, which is yielded by this method of producing approximately zero coefficients is another advantage. The magnetic moment of these magnets was about 227 each for a weight of 2·59 grammes.

It will be noticed that, in the magnets which have been constructed to have nearly zero coefficients by cutting them to a suitable dimension ratio, there is a tendency for the positive coefficient to grow less and become negative as the intensity declines, and *vice versâ*. This is in accord with the results already found for the relation of the hot and cold curves, where larger and smaller values of the intensities than those corresponding to the intersection of the curves give positive and negative coefficients respectively.

Indeed, in *any* magnet we should expect a change of intensity to produce a change of the coefficient if the latter is dependent on the disposition of the hot and cold curves of magnetisation, and hence the decay of magnetism with the lapse of time, or the increment of magnetism which takes place on remagnetisation, will tend to alter the magnitude of the temperature coefficient. In the latter case such effects have been noticed.*

* *Vide* CHREE, 'Roy. Soc. Proc.,' vol. 65, p. 375.

Diagram VI.



Change of magnetic intensity and temperature coefficient with lapse of time in drawn and tempered piano steel.

Nos. 8*a* and 8*b* refer to the first and second samples of piano wire in the commercial state, 8 centims. long.
Nos. 12*a* and 12*b* refer to the first and second samples of tempered piano wire, 12 centims. long.

PART II.

Resistivity, Elasticity, and Density, and the Temperature Coefficients of Resistivity and Elasticity.

1. The magnetic behaviour of repeatedly drawn steel wire led to the suggestion that some of the other physical properties of such wire would also exhibit interesting changes. Besides, it was worth while to attempt to trace broadly some connection between one property and another, since, whilst no chemical change presumably takes place by drawing, yet the physical properties might be considerably modified.

The selfsame wires which were employed in determining the change of magnetic properties, and which have been numbered (1) to (12), have been used in the experiments about to be described on resistivity, elasticity, and density, in order to remove any doubt which might be entertained that the material of any one of these specimens was not identically the same during the different tests for its several properties. This unquestionably added to the experimental difficulties, for a length or thickness suitable under one set of conditions was not so suitable under other conditions. Thus, in the determination of Young's modulus, the method of flexure had to be employed which, under other circumstances, would not have been adopted.

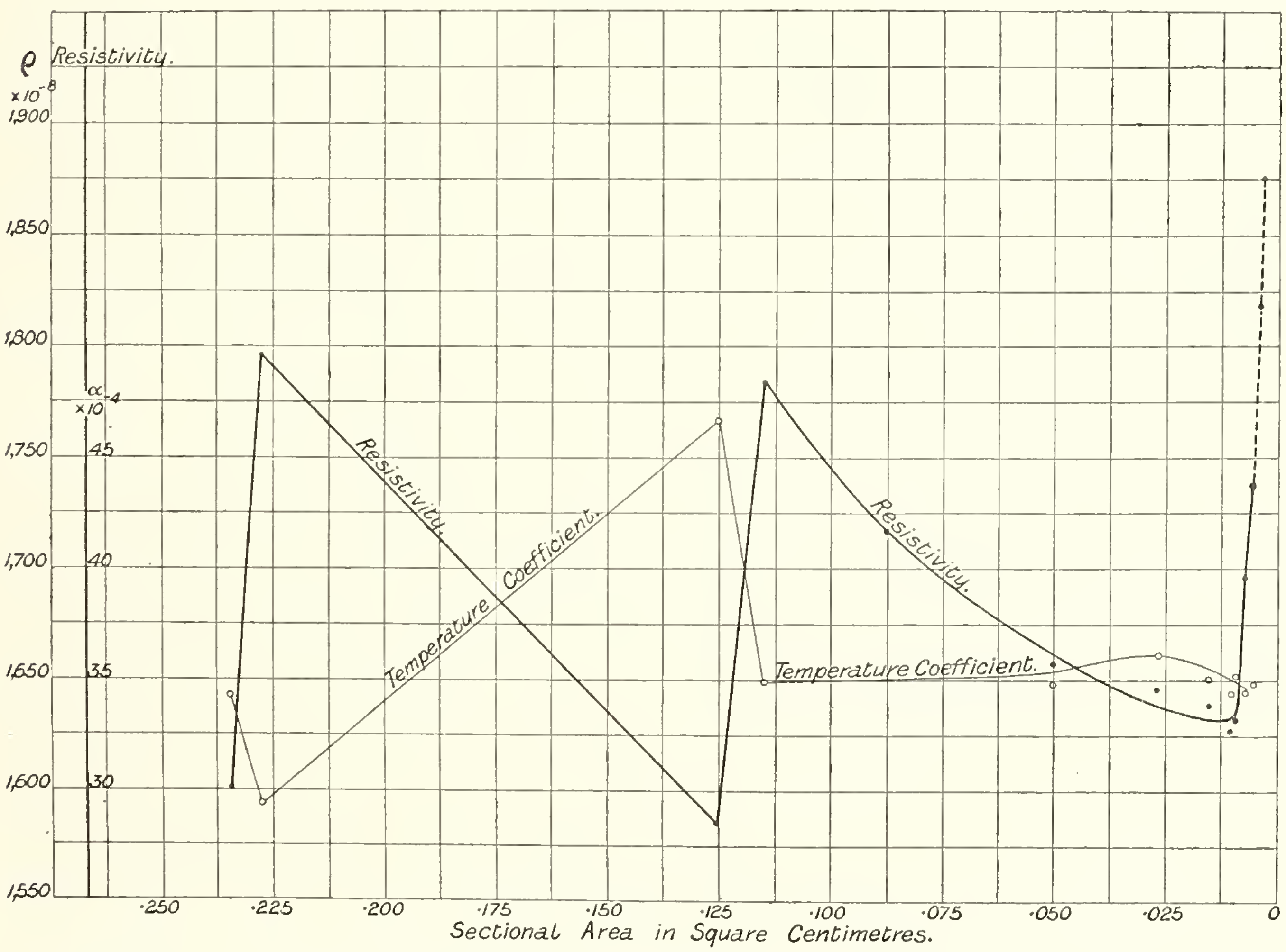
As the methods used here for the determination of resistivity, elasticity, and density are those commonly employed, it will be unnecessary to describe at length the course of the experiments, and the present part will be confined to a brief recital of results.

Resistivity.

2. The resistivity of the wires was observed by comparing the difference of potential at the ends of a known resistance with the difference of potential between two points along the wire when the same current was flowing through the wire and the known resistance. Each wire has been tested by two, and in some cases by three, independent experiments, in which as much variation was introduced as the

apparatus would allow, and the results in the table subjoined are the mean values of the experiments, which, however, differed very little among themselves. The principal source of error lay in the measurement of the diameters of the wires, for in one or two cases the wires appeared to be conical in shape, and in two cases were slightly elliptical in cross-section. By measuring the diameter at many places along the length of the wires, and confirming the results by calculation of the diameter from determinations of the density, the average sectional area has been arrived at. The numerical results are given in the third column of Table IX.,* and in Diagram VII.;

Diagram VII.



Relation of resistivity (ρ) and its temperature coefficient (α) to drawing in piano steel wires.

they are plotted as a curve, the sectional areas of the wires, which may be taken as a scale of traction, being treated as abscissæ. There is a small reduction of diameter on annealing and tempering which is due not to traction, but presumably to the production of a little oxidation, which would be rubbed off afterwards when the wire was cleaned.

The effects of annealing or tempering upon the resistivity come out in the curve

* The diameters of the wires are given in Tables II. and III., and are not repeated in Table IX.

as two prominent peaks; either of these processes increases the resistivity by about 12 per cent. upon the initial state. Hard drawing after annealing between stages (2) and (3) brings the wire down to nearly the original condition, and again, after tempering, cold drawing through two or three holes decidedly reduces resistivity, but an unmistakable increase sets in at the 11th and 12th stages. From the last point the curve has been extended by a broken line to include two more points belonging to still finer wires, Nos. 13*a* and 14*a*, which are not, however, of identically the same material as the others. Nevertheless, the broken line emphasizes the fact that the effect of extreme drawing is prejudicial to conductivity. These contrary effects of drawing, it will be remembered, were also in evidence in the curve of magnetic temperature coefficient, and in both curves the change occurs near to the 8th and 9th stages. A length of the steel wire upon which the experiments just described were carried out was subsequently made glass hard, and the resistivity in that state was 2760×10^{-8}, or about 70 per cent. higher than the wire in its state of least resistance. It is worth notice that minimum resistivity occurs in the *hard drawn* and *hot rolled conditions*, and hence the order of resistivity in an ascending scale is: hard drawn or hot rolled; annealed or tempered; glass hard.

Temperature Coefficient of Resistivity.

3. A simple modification of the apparatus described in the last section allowed the temperature coefficient of resistivity to be obtained.

The same wires were used as before. Each was fixed in a trough surrounded by a water-bath, which could be raised in temperature by the application of gas jets from about 16° to 90° C. Readings were made at intervals during the process of heating and cooling, and the usual precautions were taken for the elimination of the effects of thermo-currents.

Two independent sets of observations were taken for nearly all the specimens, and the mean results, which are given in the fourth column of Table IX., have been plotted on the same diagram as the curve of resistivity, so that the two curves may be conveniently compared.

It will be noticed that the smallest value of the temperature coefficient, namely, 0·00294, occurs when the steel is annealed, and the highest value, 0·00466, when hard drawn, and these least and greatest values coincide respectively with the largest and smallest values of the resistivity. This confirms and extends a law which BARUS has shown to be true for the iron carburets, according to which the temperature coefficient of resistivity is approximately inversely as the resistivity;* and in the

* The relation given by BARUS is $\rho(m+\alpha) = n$, where ρ is the resistivity, α the temperature coefficient, and m and n are constants. 'Bulletin U.S. Geol. Survey,' No. 14, 1885.

curves here plotted a similar relation between temperature coefficient and resistivity in general holds good, the two curves moving oppositely to each other. The change, however, in the magnitude of the temperature coefficient of the repeatedly drawn steel is small compared to the change in resistivity.

The temperature coefficient of this steel made glass hard is only 0·00177, about half the average value in the drawn state; this again is an example of BARUS' law, for the resistivity when glass hard is, as stated, 70 per cent. greater than when hard drawn.

The ascending order of the temperature coefficient is thus: glass hard; annealed; hard drawn; the inverse of resistivity.

YOUNG'S *Modulus.*

4. The elastic properties of steel are known to undergo a considerable change with drawing, and it seemed desirable to discover if longitudinal elasticity was correlated in any way with the electric and magnetic properties of these steel wires, and how it was modified by annealing, tempering, and traction. The determination of YOUNG'S modulus was effected by measuring the amount of flexure of the wires when loaded at the middle, and with the ends resting on rigid supports. The depression produced by loading was observed through a telescope supplied with a micrometer eye-piece, which allowed an exceedingly small interval to be measured with accuracy. After a reading had been taken of the fiducial mark, with no load hanging from the rod except the pan itself, weights were applied one by one, great care being exercised so that there should be a minimum of vibration; the weights were then removed, one at a time, the depression corresponding to these weights at each stage being observed through the telescope. Two, and in some cases three, independent sets of experiments of this kind were performed on each wire, and nearly always with different distances between the supports, and with different increments and amounts of load, and the mean of the several experiments was taken as the final result.

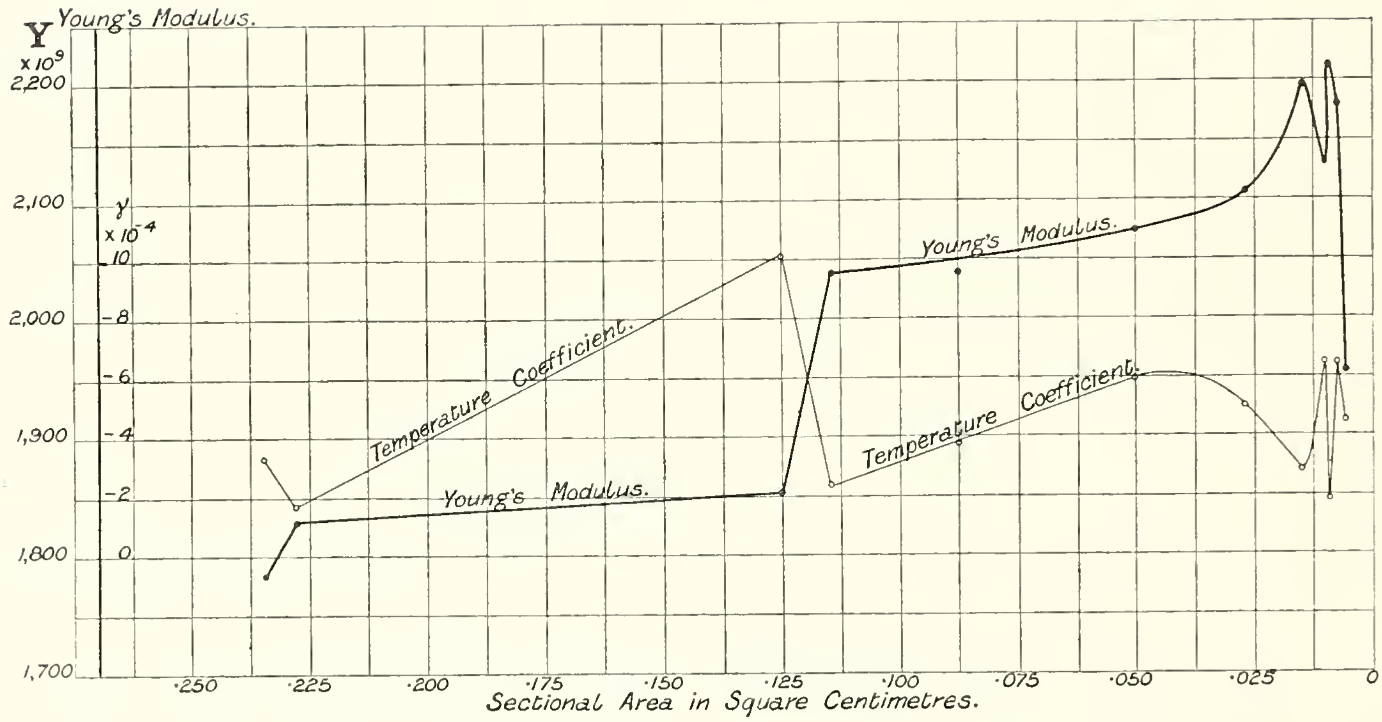
In the formula for the calculation of the modulus the fourth power of the radius appears as one of the factors, and hence an accurate value of the radius is required if errors are to be avoided. As already mentioned,* a considerable number of careful measurements of the diameters of the wires were made, and these were confirmed from determinations of the density.† The values there used have been adopted here.

The fifth column of Table IX. gives YOUNG'S modulus for the twelve specimens of steel upon which the magnetic and electric experiments already described had been carried out. Diagram VIII. exhibits this column of figures as a curve with sectional

* *Vide* § 2.

† As the deviation of any single determination of the density from its mean never exceeded $\frac{1}{4}$ per cent. in any one specimen, the maximum error on this account in the fourth power of the radius will not be more than double of this.

Diagram VIII.



Relation of YOUNG'S modulus (Y) and its temperature coefficient (γ) to drawing in piano steel wires.

areas as abscissæ and YOUNG'S modulus as ordinates; on the same diagram the curve of the temperature coefficient of the modulus is plotted, but this will be referred to afterwards.

A feature of this curve is that annealing, hard drawing, and tempering all produce an upward effect, which is continued until the last stages are reached, and then a decided drop occurs; thus, again, the initial effects of drawing are reversed by extreme traction. The increase of the modulus is very conspicuous when the wire has been tempered, the rise at this stage being nearly one half of the whole change, which, from least to greatest, is about 21 per cent. On the other hand, drawing between the 2nd and 3rd stages has only a small effect on the modulus, and the influence of successive cold drawing after tempering for at least two stages is comparatively unimportant; then, after a sharp rise at the 8th and 10th stages, with some irregularity at the 9th, there is the rapid fall to the 11th and 12th points. The irregularity in the final stages is here apparent, as in some of the other curves, and confirms the suspicion that there has probably been some departure from the even course of drawing after the 8th stage, as mentioned when the process of manufacture was described.* It seems not unlikely that it requires very skilful manipulation of the material in order to obtain maximum elasticity, and about stages (8) to (10) the wire possibly develops a critical condition.

* Part I., § 4.

Temperature Coefficient of YOUNG'S *Modulus.*

5. In determining the temperature coefficient of YOUNG'S modulus the same apparatus as before was used, with the addition of a brass tube in which slots were cut, so that the upper ends of the supports could project into the interior of it. This tube enclosed the steel wire, and another slot of smallest allowable size was cut at the centre of the tube to permit the passage of the suspension for the pan and weights. The slots near the ends were made steam-tight, but, necessarily, this could not be done for the central one, as the suspension wire which passed through had to hang freely. One end of the tube was in connection with a boiler, and this produced a supply of steam which could pass freely along the tube and escape at the other end. No special arrangement, however, was made for the cooling of the tube, the steam was simply shut off and the tube and its contents allowed to grow cold gradually. A thermometer projected into the tube, with the bulb nearly at the centre, and the temperature of the steel was taken to be the reading of the thermometer.

The apparatus thus set up was at first intended for rapid tests, to ascertain whether the *sign* of the coefficient changed at any stage, and less attention was paid to its magnitude, but later on it became possible to obtain numerical results which are worth recording. In some earlier experiments the method was tried of observing the position of the fiducial mark at air temperature, then at steam temperature, and again at air temperature, from which data the coefficient could easily have been deduced. But it was found much better to follow the plan of loading and unloading as already described for determining the modulus, carrying out the observations firstly at air temperature, secondly at steam temperature, again at air temperature, and so on alternately, and calculating the coefficient, γ, from the observed depressions, D_0 and D_t, thus:

$$\gamma = (D_t - D_0)/D_0 \cdot t.$$

By adopting this method, corrections for expansion of supports, etc., were eliminated. The formula assumes that the change in YOUNG'S modulus is linear and that there are no hysteresis-like effects, but these assumptions are, no doubt, not quite justifiable, although probably not far from the truth.

With rise of temperature the modulus was found to decrease, but on cooling it it was very seldom that it returned exactly to its original value, although, after a repetition of the experiments, the modulus was found to change from one to another of two nearly constant values; thus there is a permanent change before a cyclic state is established. Here is an example ;—

Drawn Steel Wire No. 7.

Temperature.	Depression on arbitrary scale (load increment = 100 grammes).	
Cold	145·8	—
Hot	—	147·6
Cold	144·0	—
Hot	—	146·6
Cold	144·0	—

Each of these numbers has been arrived at after a series of loadings and unloadings as described above. Since the modulus is inversely proportional to the depression, the diminution of the latter from 145·8 to 144 means a "permanent" increase of 1·25 per cent. in elasticity.*

The amount of this increase of the modulus varies from stage to stage and roughly follows the variation of the temperature coefficient, large and small values of each being associated. Thus there is a large "permanent" increase when the cyclic change is large. The average amount of the "permanent" increase of the modulus in these wires is about 2·5 per cent. for 80° change of temperature.†

The mean results of the several determinations of the temperature coefficient of YOUNG'S modulus on each wire are given in the sixth column of Table IX.

The coefficient is throughout negative, implying that, in the cyclic state, YOUNG'S modulus decreases with rise of temperature; the magnitude varies considerably, namely, from $-1·64 \times 10^{-4}$ when annealed, at the 2nd stage, to a maximum of $-10·25 \times 10^{-4}$ at the 3rd stage, when hard drawn. There is a small value again of the coefficient at the 4th stage on tempering, and, after that, the figures for the cold drawn specimens present apparently much irregularity, but if this column of figures be plotted, as in Diagram VIII., we get a curve which repeats in an inverse sense all the features of the modulus curve, so that a relation clearly exists between the modulus and its temperature coefficient. Although the figures in the table do not exhibit a simple inverse proportion between the magnitude of the coefficient and YOUNG'S modulus, yet the general statement may be made that larger and smaller values of the modulus are progressively associated respectively with smaller and larger values of its temperature coefficient. This law recalls the similar law connecting the resistivity of the iron carburets and their temperature coefficients, with this difference, that the coefficient for the modulus is negative, whereas the coefficient for resistivity is positive. The average value of the coefficient for YOUNG'S

* This "permanent" effect does not persist indefinitely, but probably disappears in a few days or hours.

† These curious effects of temperature on YOUNG'S modulus are in accordance with results published by Mr. SHAKSPEAR in a paper which appeared just after these experiments were carried out. 'Phil. Mag.,' vol. 47, p. 539; also *vide* TOMLINSON, 'Roy. Soc. Phil. Trans.,' vol. 174, Part I., p. 132.

modulus in these wires is $-$ 0·00045,* whilst for resistivity it is $+$ 0·0035 approximately.

Density.

6. When a steel rod is drawn into wire the variation of density as the drawing proceeds may not progress uniformly, for it is not unlikely that the stress required to force the rod through the draw plate may so far separate the molecules longitudinally that the lateral compression does not compensate for the extension. In this case the density will diminish, and, in short, density will depend upon the ratio of extension to compression. It is, therefore, not only interesting, but of some importance to trace the change of density at each stage of manufacture and to compare this with the change of the properties already examined.

The method adopted was to weigh a suitable length of the wire in air and afterwards in distilled water, at the same time noting the data necessary for corrections on account of density of the water, the buoyancy of the air, the weight of the suspending fibre, etc., the most important correction being the one which makes allowance for the temperature of the water differing from that of its greatest density, approximately 4°.

For the sake of confirmation an entirely independent duplicate set of observations was carried out, and the agreement between the two sets was very satisfactory, especially in the earlier specimens, which, being of greater size and mass, could be weighed with relatively higher accuracy. The results are given in the last column of Table IX. It will be seen that they end at the eleventh wire, as the twelfth was accidentally mislaid; this is particularly unfortunate, as it would have been useful to know whether No. 12 exhibited a density greater or less than the abnormally high density of No. 11.

The feature which first claims notice is the diminution of density, although only slight, which takes place when the rod is first subjected to traction, namely, between the 2nd and 3rd, and between the 4th and 5th stages.

This appears to be an illustration of the remarks at the beginning of this section. Afterwards, as the drawing progresses, there is a steady increase of density, with a large increment between the 10th and 11th stages. The entire variation of density is about 2·5 per cent., the least and greatest values lying respectively at the beginning and end of the list.

As a general rule, it has been observed that density and YOUNG's modulus in steel vary directly together, and this leads to a comparison of the present results with the curve of the modulus. The similarity is not immediately obvious, but, in both, annealing and tempering produce an upward movement, whilst the first drawing after either of these operations produces very little change, subsequent drawings, however,

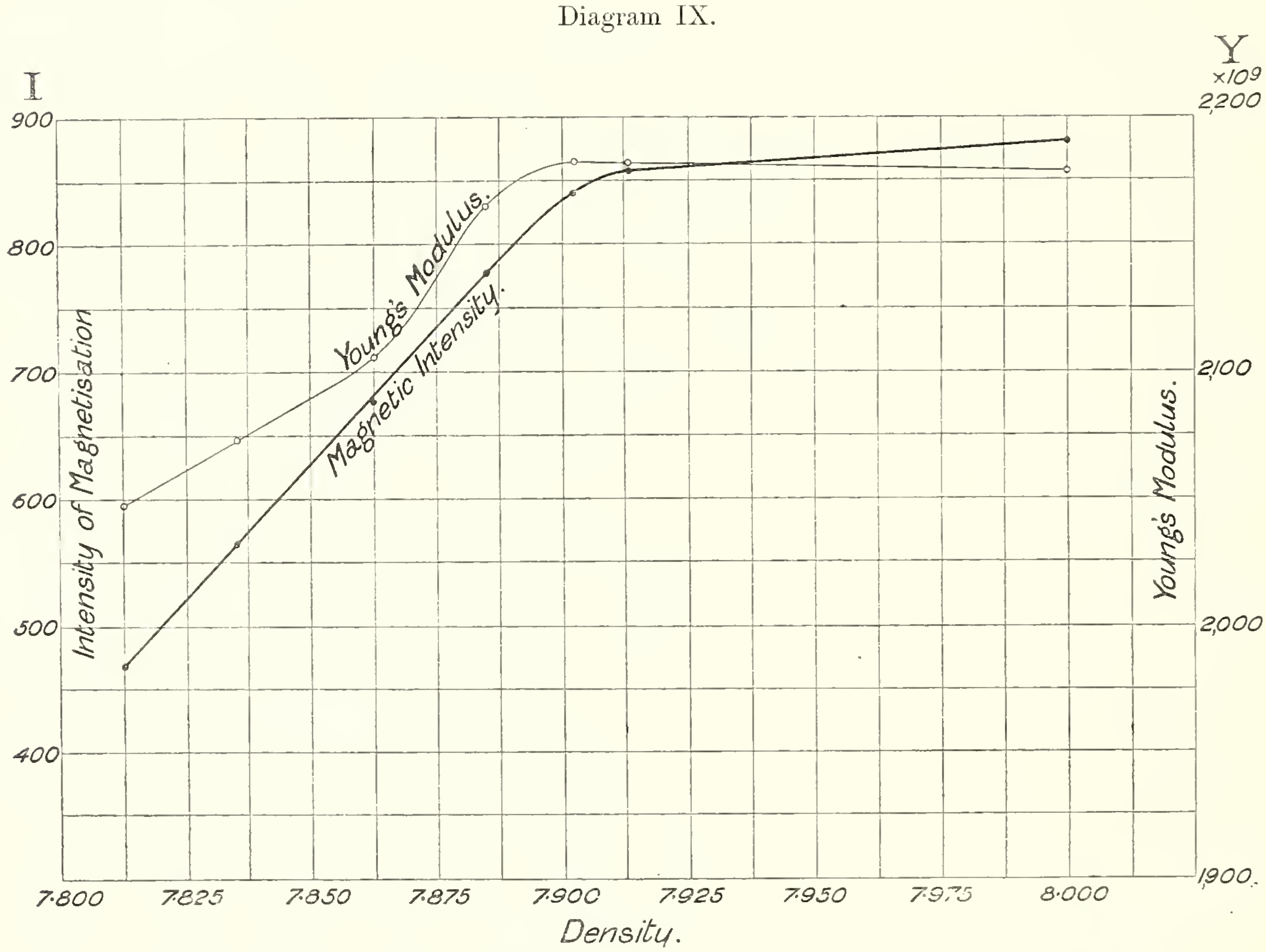
* According to STYFFE the average value for ordinary steel is 0·0003. "Strength of Iron and Steel," by KNUT STYFFE, p. 122. *Vide* also TOMLINSON, 'Roy. Soc. Phil. Trans.,' vol. 179, p. 23; vol. 174, pp. 132–133.

bending the curves upwards rapidly. But the final drop in the modulus does not appear to have a counterpart here, unless the missing No. 12 diminishes in density.

It is to be noticed that the percentage variation of YOUNG'S modulus from one end of the curve to the other is about ten times the percentage variation of the density, and also the average temperature coefficient of the modulus is about ten times the temperature coefficient of density, or cubical expansion, so that it is possible that much of the diminution of elasticity with rise of temperature may be due to thermal expansion diminishing the density.

A more obvious correspondence exists between the curve of residual magnetic intensity (dimension ratio = 100) and the curve of density, both rising continually and rapidly from the 5th point. The relationship is more easily traced when intensity is plotted against density, as in Diagram IX. Between the 5th and 9th

Diagram IX.



The relation of residual magnetic intensity and YOUNG'S modulus to density in drawn steel.

stages inclusive the ratio of the increment of magnetic moment per unit volume to the increment of mass per unit volume is nearly constantly 4100, or each molecule added per unit volume contributes directly or indirectly to the whole a magnetic moment 4100 times its mass. This is, however, a doubtful clue to even an inferior limit to the magnetic moment of a molecule, since it cannot be assumed that there is an invariable structure maintained throughout the progress of the drawing. Indeed, it is not difficult to see from the fracture of the wires that as the drawing proceeds a fibrous structure is developed for several stages after tempering, and the formation of this fibrous structure may be of importance in augmenting the magnetic intensity.

For the sake of comparison, YOUNG'S modulus has been plotted on Diagram IX., the points being taken from a smooth curve of the modulus and traction, and the curve bears out the statement that an increase of density in general improves elastic properties. It also shows that elasticity and magnetic intensity are correlated.

To complete this part of the investigation of the properties of drawn steel, it was intended to add an account of the changes which might take place in the cubical expansion of these wires, and to trace the connection of these with other changes, but although some preliminary experiments have been made, the results are not yet sufficiently advanced to be presented.

This investigation has been carried out at the Owens College, Manchester, at intervals during the last three or four years, and I am greatly indebted to Professor SCHUSTER for allowing me to avail myself of the facilities for research which the Physical Laboratory there provides.

TABLE IX.—Influence of Drawing on Resistivity, YOUNG'S Modulus, and their Temperature Coefficients, and Density.

No.	Condition.	Resistivity (at air temperature, about 16°). Ohms per centimetre cube, ρ.	Temperature coefficient of resistivity, α.	YOUNG'S modulus (at air temperature), Y.	Temperature coefficient of YOUNG'S modulus, γ.	Density. Grams. per cubic centimetre at 16°.
		$\times 10^{-5}$	$\times 10^{-3}$	$\times 10^{11}$	$\times 10^{-4}$	
1	Rolled Rod	1·601	+3·43	1·78	− 3·35	7·803
2	Annealed	1·796	+2·94	1·83	− 1·64	7·827
3	Hard drawn	1·585	+4·66	1·85	−10·2	7·815
4	Tempered	1·784	+3·49	2·04	− 2·30	7·818
5	Cold drawn	1·716	+3·55	2·04	− 3·79	7·813
6	,,	1·657	+3·48	2·08	− 5·90	7·835
7	,,	1·645	+3·61	2·11	− 4·87	7·871
8	,,	1·638	+3·51	2·20	− 2·79	7·881
9	,,	1·627	+3·43	2·13	− 6·53	7·902
10	,,	1·633	+3·52	2·21	− 1·79	7·913
11	,,	1·696	+3·44	2·18	− 6·42	8·001
12	,,	1·738	+3·48	1·95	− 4·24	—
12*a*	,,	1·739	—	—	—	—
13*a*	,,	1·772	—	—	—	—
14*a*	,,	1·876	—	—	—	—
	Glass hard	2·760	+1·77	—	—	7·740

PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 37–43.

THE SPECIFIC HEATS OF METALS AND THE RELATION OF SPECIFIC HEAT TO ATOMIC WEIGHT.—PART II.

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LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 332.

21.3.03

Price One Shilling.

INDEX SLIP.

Tilden, W. A.—The Specific Heats of Metals and the Relation of Specific Heat to Atomic Weight. Part II.
Phil. Trans., A, vol. 201, 1903, pp. 37–43.

Specific Heats of Various Metals; Changes with Temperature; Relation to Atomic Weights.
Tilden, W. A. Phil. Trans., A, vol. 201, 1903, pp. 37–43.

II. *The Specific Heats of Metals and the Relation of Specific Heat to Atomic Weight.—Part* II.

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Received December 8,—Read December 11, 1902.

In the Bakerian Lecture for 1900 (' Phil. Trans.,' A, vol. 194, p. 233) it was shown that the specific heats of very pure cobalt and nickel, when compared at temperatures from 100° C. down to the boiling-point of liquid oxygen, − 182°·5 C., steadily approach each other and together tend towards a least value which is at present unknown.

It was thought desirable to increase the number of determinations at successive points on the thermometric scale, and to extend the total range of the experiments so as to afford better data for calculation of the form of the curves. The following is an account of the results obtained.*

It has not yet been possible to arrange for the conduct of experiments at temperatures lower than − 182°·5, as this could only be done with the aid of liquid hydrogen. The temperatures above 100° C. have been obtained by the use of a bath of aniline vapour, melted fusible metal or melted lead, and were estimated by the use of a platinum resistance thermometer which was carefully calibrated, and of which the fixed points 0°, 100°, and 184° (boiling-point of aniline) were verified. The specific heats were determined in the same calorimeter and with the same precautions as described in the Bakerian Lecture.

The holder employed in the experiments at low temperatures was found equally useful in the experiments above 100°. Between air temperature and 100° the steam calorimeter was again employed.

The figures given in the following Table I. are in nearly all cases the mean values deduced from several experiments which were always closely concordant.

The total amount of heat per unit mass measured in the calorimeter is equal to the product of the mean specific heat and the range of temperature, beginning in each case at 15° C. Taking the values given in the table, this product, Q, may be plotted as an ordinate, the higher absolute temperature, t, being the abscissa. The result is

* The nickel, cobalt, and platinum employed are the pure specimens prepared for the former series of experiments. Pure silver was obtained from Messrs. Johnson and Matthey. For the aluminium I am indebted to Professor J. W. Mallet; it was described as nearly pure.

shown in fig. 1. Cobalt has been omitted, as the metal apparently undergoes some oxidation at high temperatures and the results are less regular than the rest.

TABLE I.—Mean Specific Heats.

Range of temperature.	Aluminium.	Nickel.	Cobalt.	Silver.	Platinum.
° C.					
– 182 to + 15	·1677	·0838	·0822	·0519	·0292
– 78 ,, + 15	·1984	·0975	·0939	·0550	—
+ 15 ,, + 100	—	·1084	·1030	·0558	·0315
15 ,, 185	·2189	·1101	·1047	·0561	—
15 ,, 335	·2247	—	—	—	—
15 ,, 350	—	·1186	·1087	·0576	—
15 ,, 415	—	·1227	—	—	—
15 ,, 435	·2356	·1240	·1147	·0581	·0338
15 ,, 550	—	·1240	·1209	—	—
15 ,, 630	—	·1246	·1234	—	—
0 ,, 1000	—	—	—	—	·0377*
0 ,, 1177	—	—	—	—	·0388*

The general shape of the curves is the same and the connection between Q and t may be assumed as hyperbolic :

$$Q^2 + aQ + bt^2 + ct + f = 0.$$

The values of $k, = dQ/dt$, for the specific heats at successive temperatures on the absolute scale are given in the following table :—

TABLE II.—Specific Heats.

t abs.	Aluminium.	Nickel.	Silver.	Platinum.
° C.				
100	·1226	·0575	·0467	·0275
200	·1731	·0878	·0528	·0293
300	·2053	·1054	·0558	·0311
400	·2254	·1168	·0572	·0328
500	·2384	·1233	·0581	·0344
600	·2471	·1275	·0587	·0358
700	·2531	·1301	·0590	·0372
800	—	·1321	—	·0385
900	—	·1338	—	·0397
1000	—	—	—	·0409
1100	—	—	—	·0421
1200	—	—	—	·0432
1300	—	—	—	·0442
1400	—	—	—	·0452
1500	—	—	—	·0461

One important result of the extension of the experiments to other metals is that the assumption of a constant atomic heat at the absolute zero, which seemed justified

* VIOLLE, 'Comptes Rendus' (1877), vol. 85, p. 543; also 'Phil. Mag.' [5], vol. 4, p. 318.

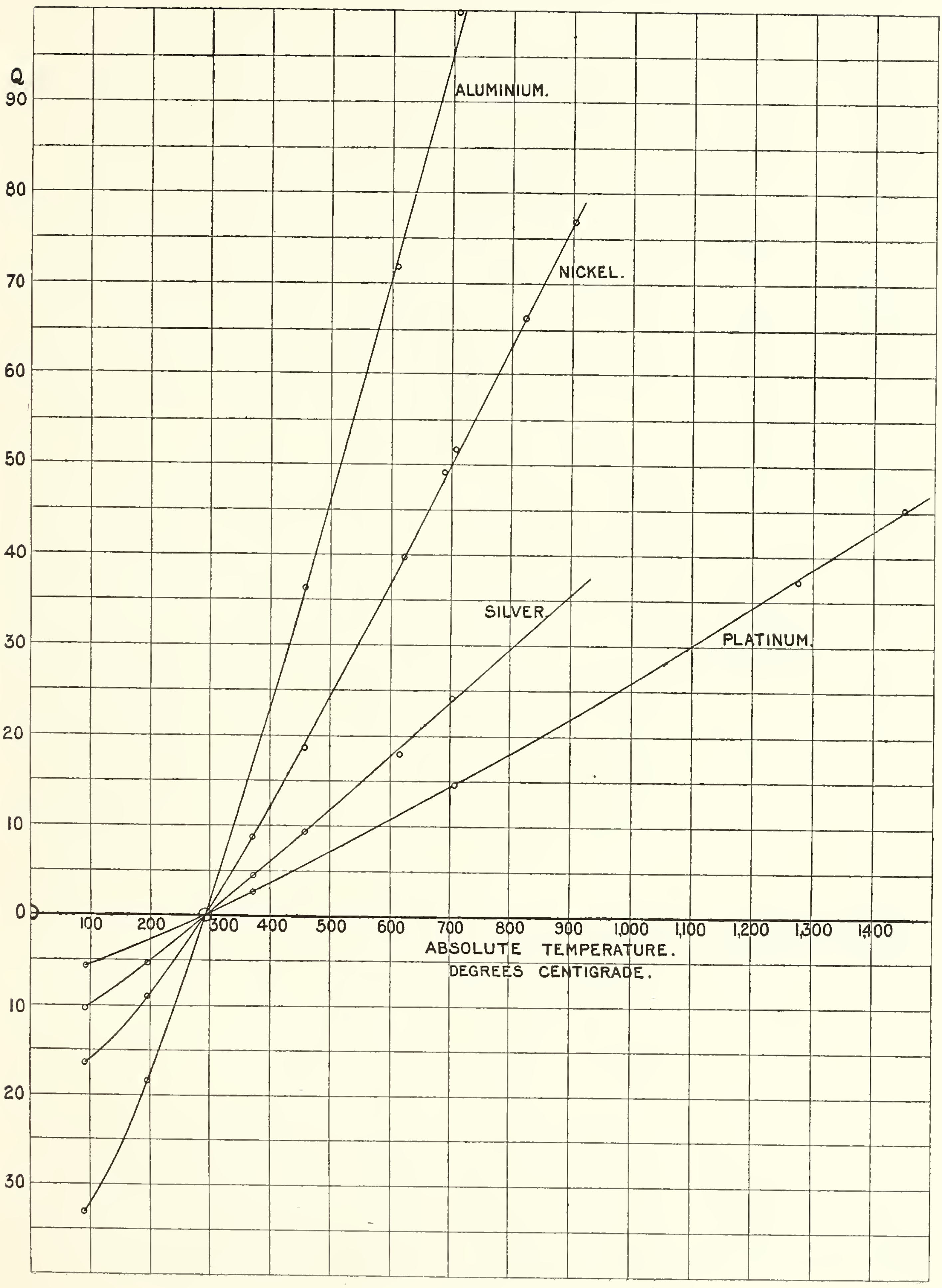
Q
90
80
70
60
50
40
30
20
10
0
10
20
30
100
200
300
400
500
600
700
800
900
1,000
1,100
1,200
1,300
1400
ALUMINIUM.
NICKEL.
SILVER.
PLATINUM.
ABSOLUTE TEMPERATURE.
DEGREES CENTIGRADE.

Fig. 1.

in the case of cobalt and nickel (see Appendix to BAKERIAN LECTURE), is apparently untenable.

Plotting the specific heats in Table II. against absolute temperatures, the curves shown in fig. 2 are obtained, from which it is obvious that unless some remarkable change in the specific heats of silver and platinum occurs below − 182° C. the curves representing *atomic* heats cannot meet at the absolute zero.

It will be observed that the influence of rise of temperature on the specific heat is in the inverse order of the atomic weights of the metals compared, being greatest in the case of aluminium and least in the case of platinum. This appears to be generally true and is supported by the experiments of BEHN ('WIEDEMANN'S Ann.,' vol. 66, p. 237). It appears, therefore, that the usual application of the law of DULONG and PETIT to the rectification of atomic weights is a rough empirical rule which, setting aside boron, carbon, silicon, and beryllium, is only available when the specific heats have been determined at comparatively low temperatures, usually, and most conveniently, between 0° and 100° C.

What mechanical properties of the metals are concerned in affecting the value of the specific heat is not known. The work done in expansion has apparently very little to do with it. Lead and platinum, for example, the atomic weights and specific heats of which are near together, have very different coefficients of expansion, that of lead being nearly ten times as great as that of platinum. I have, however, on the suggestion of Professor PERRY, thought it of some interest to determine the specific heat of the remarkable nickel steel which is said to have a smaller dilatation than that of any other metal. The sample used was found by analysis to contain 35·92 per cent. of nickel, practically 36 per cent., with ·11 of carbon and about ·30 of manganese. The mean specific heats observed at four widely separate temperatures show that there is decidedly an increase with rise of temperature to an extent about the same as in the case of nickel itself.

Range of temperature.	Mean specific heat of nickel steel.
− 182° to + 15°	·0947
15 „ 100	·1204
15 „ 360	·1245
15 „ 600	·1258

Evidence as to the cause of the difference between two such metals as nickel and silver has been sought by making comparative experiments with the two metals in the form of sulphide. These compounds were prepared by precipitation by hydrogen sulphide from solutions of the sulphate and nitrate respectively, and subsequent fusion of the dry sulphide with an excess of sulphur. If the differences between the metals are due to peculiarities of the atoms of each, similar differences would be

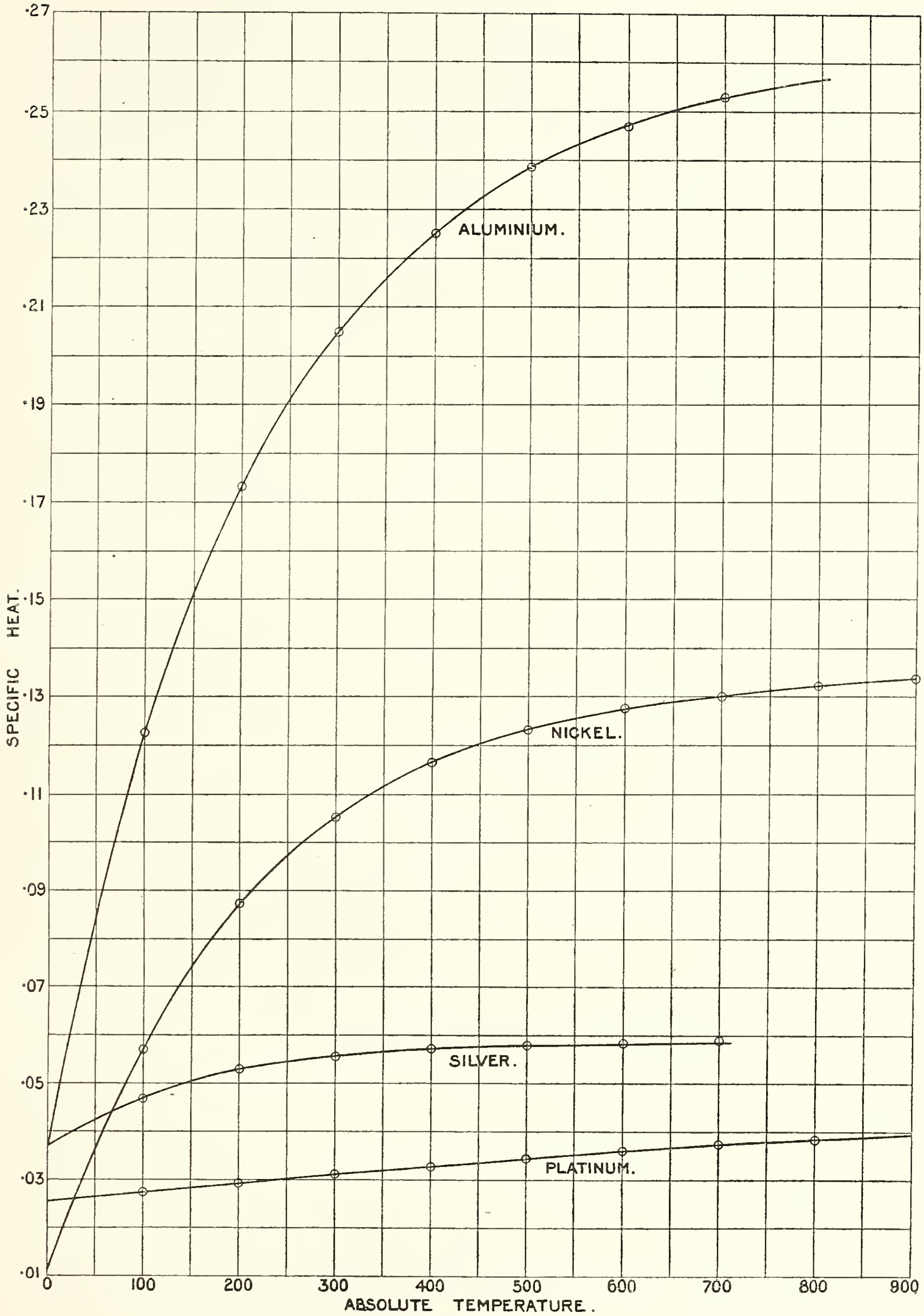
·27
·25
·23
·21
·19
·17
·15
·13
·11
·09
·07
·05
·03
·01
SPECIFIC HEAT.
ALUMINIUM.
NICKEL.
SILVER.
PLATINUM.
0
100
200
300
400
500
600
700
800
900
ABSOLUTE TEMPERATURE.

Fig. 2.

observed in the specific heats of their sulphides, in which presumably the atoms are separated. On the other hand, if the differences are due to molecular differences or to the properties of the metal in mass, somewhat different values for the specific heats might be obtained. In the result it was found that the mean specific heat of silver sulphide is less than that of nickel sulphide at all temperatures.

Range of temperature.	Mean specific heats.	
	Nickel sulphide.*	Silver sulphide.*
− 180° to + 15°	·0972	·0568
15 „ 100	·1248	·0737
15 „ 324	·1333	·0903

When these results are plotted in the same manner and on the same scale as shown for the metals in fig. 1, the two curves for the sulphides are seen to be very similar to those for the metals. See fig. 3.

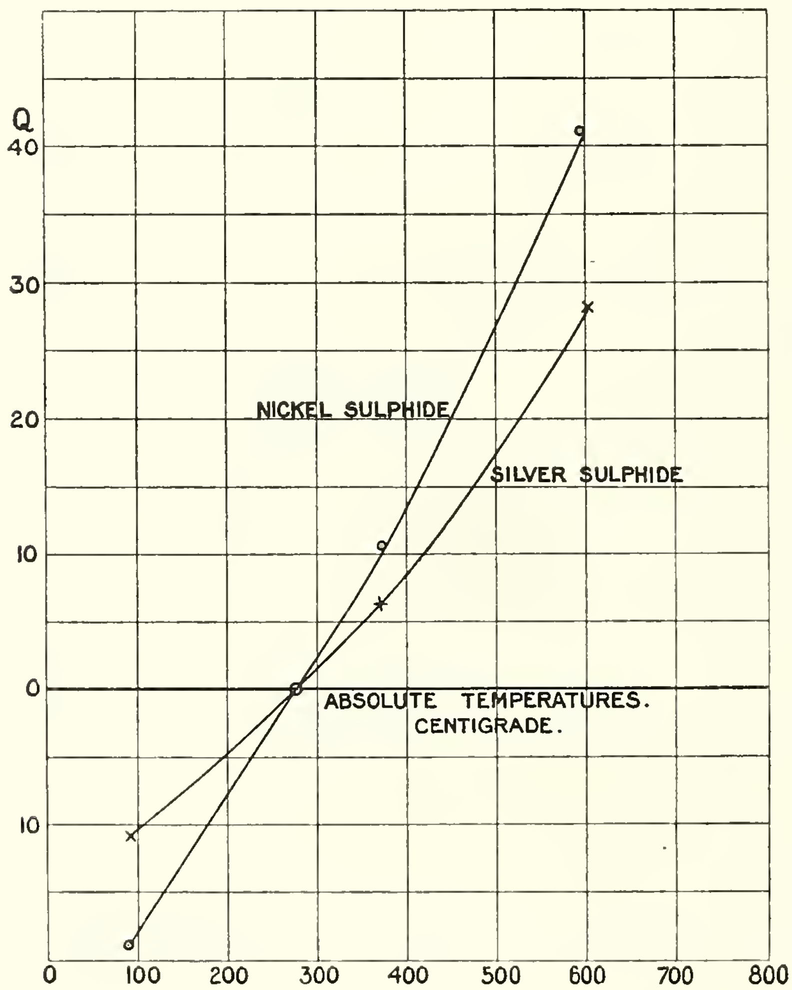

Fig. 3.

* REGNAULT found the mean specific heat of fused NiS to be ·1281 and of fused Ag_2S ·0746 between 0° and 100°.

This, however, takes no account of the sulphur in the compounds, and though the molecular heat of each may be calculated and the mean atomic heats of the two metals so obtained, the result is of little value without a knowledge of the rate at which the specific heat of sulphur increases with temperature.

In conclusion, I desire again to acknowledge the skilful assistance of Mr. SIDNEY YOUNG in the experiments. I have also to thank Mr. LEONARD BAIRSTOW, Whit. Sch., for valuable help in the calculations.

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AN EXPERIMENTAL DETERMINATION OF THE VARIATION WITH TEMPERATURE OF THE CRITICAL VELOCITY OF FLOW OF WATER IN PIPES

BY

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INDEX SLIP.

III. *An Experimental Determination of the Variation with Temperature of the Critical Velocity of Flow of Water in Pipes.*

By E. G. COKER, *M.A.* (*Cantab.*), *D.Sc.* (*Edin.*), *Assistant Professor of Civil Engineering, and* S. B. CLEMENT, *B.Sc.* (*McGill*), *Demonstrator in Civil Engineering, both of McGill University, Montreal.*

Communicated by Professor OSBORNE REYNOLDS, *F.R.S.*

Received July 16,—Read November 20, 1902.

1. *Introduction.*

THE motion of water in pipes and channels has been the subject of frequent investigation, both from the theoretical and the experimental side, and it is well known that while in some cases theory and experiment are in exact accord, yet in many others the experimental results differ widely from the calculated.

In some cases, while the theory holds for one set of conditions, it is found not to hold for conditions which at first do not appear to be fundamentally different.

A striking instance is that of the flow of a viscous liquid through a pipe of circular section, a case for which a strict mathematical solution can be obtained under certain assumed conditions of flow. Experiment shows that the theory is verified if the pipe is of capillary bore and the motion small, while if the pipe is large and the motion appreciable, there is a large discrepancy between experiment and calculation. The discrepancy is due to the assumption that the motion is stream-line, a condition of things which is true for tubes of capillary bore, but in general is not true for tubes of appreciable diameter unless the motion is below a certain limit, fixed by the size of the pipe and the physical characteristics of the liquid. Above this limit, the motion is eddying and the hydrodynamical equations no longer apply.

The change from stream-line to eddy or sinuous motion was first studied by OSBORNE REYNOLDS,* who showed that the determining factors in the case of a circular pipe depended on the dimensions of the pipe and the viscosity of the water. His results are based partly on deductions from the equations of motion for a viscous

* "An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels." 'Phil. Trans.,' 1883.

23.3.03

fluid; thus, if we take the general equations of motion for an incompressible fluid subject to no external forces, as of type

$$\frac{du}{dt} = -\frac{1}{\rho}\left\{\frac{d}{dx}(p_{xx} + \rho u^2) + \frac{d}{dy}(p_{yx} + \rho uv) + \frac{d}{dz}(p_{zx} + \rho uw)\right\}^*$$

and eliminate the pressures from the equations, we obtain the accelerations in terms of different types. Thus, if we take the middle term, viz., $-\frac{1}{\rho}\frac{d}{dy}(p_{yx} + \rho uv)$, and for p_{yx} write $\mu\left(\frac{dv}{dx} + \frac{du}{dy}\right)$, we get $-\frac{d}{dy}\left\{\frac{\mu}{\rho}\left(\frac{dv}{dx} + \frac{du}{dy}\right) + uv\right\}$. Now, since dv/dx and du/dy have the dimension of a velocity divided by a length and the other term has dimension of the square of a velocity, the relative values of these two terms are to one another as $\mu/c\rho$ to v, where c is a length, say the radius of the tube.

The equations do not show in what way the motion depends upon this relation, but it was inferred that the eddying motion must depend on some definite relation between v and $\mu/c\rho$, expressible in the form $v = k\mu/c\rho$, where k is some constant.

The experimental observations were of two kinds, the earlier depending on the device of introducing a colour band into a glass pipe and observing the velocity at which break-down of the stream-line motion occurred, and the later method depending upon the fact that stream-line motion is associated with resistance proportional to the velocity, while for eddy motion the resistance is proportional to a higher power of the velocity.

Both methods showed that the critical velocity at which stream-line motion changed to eddy motion varied directly as the viscosity, and inversely as the radius of the pipe.

Object of the Experiments.

In the experimental verification of the temperature effect upon the critical velocity a satisfactory agreement was obtained with the formula, but as the range of temperature was extremely limited, it was pointed out that "it would be desirable to make experiments at higher temperature; but there were great difficulties about this, which caused me, at all events for the time, to defer the attempt."†

It does not appear that such experiments have since been made, and although the difficulties were not estimated lightly, it seemed worth while to attempt experiments through a much larger range of temperature.

Scope of the Experiments.

Although it would be eminently satisfactory to make experiments throughout the whole range of temperature of water, yet the experimental difficulties of maintaining

* 'Phil. Trans.,' A, 1895, p. 131.

† 'Phil. Trans.,' 1883, p. 977

a uniform temperature in the pipe increase in a much greater ratio than the increase of temperature beyond, say, 50° C., and there are other difficulties, due to convection and evaporation, which made it desirable to limit the investigation, at any rate for the time, to a range of 45° to 50° C. With these limits it was found that the decrease or increase of temperature along the pipe, when thickly lagged, was inconsiderable, and the correction to be applied was therefore small and not likely to cause an appreciable error. In order to carry on experiments at a higher temperature, it would apparently be necessary to surround the experimental tube with a water-jacket maintained at the same temperature as the water in the tank, otherwise drop of temperature along the pipe would be so considerable as to seriously increase the chances of error.

Method of Experiment.

The principle of the method is the same as originally devised by OSBORNE REYNOLDS, but the manner of carrying out the work differed somewhat in detail.

The method of colour bands is unsuitable for water at a high temperature, as it is impossible to eliminate the effect of conduction and convection, and the water consequently never comes to rest; moreover, experiments by this method lead to a different form of the criterion, viz., the maximum limit at which stream-line motion is possible, while experiments on the variation in the resistance of pipes lead to the minimum criterion, viz., that at which eddies change to steady motion. This latter method is also more likely to be accurate, for the maximum velocity of stream-line motion depends upon external causes, which influence it to a remarkable extent. Experiments were made with the tank in the laboratory to discover how far stream-line motion could be carried under favourable conditions; the tank rests directly upon the ground, and after water at the temperature of the room had been allowed to stand therein for two or three days, stream-line motion in pipes could be maintained at higher velocities than that given by the upper limit formula for the critical velocity v_c, viz.:

$$v_c = \frac{1}{43 \cdot 79} \frac{f(\tau)}{D},*$$

the units being metres and degrees centigrade, a result no doubt due to the complete absence of vibration in the tank, which was founded on rock, and also the freedom of the water from sediment.

Moreover, it is easy to lower the critical velocity by subjecting the water to a disturbing cause; thus fine matter in suspension in the water will lower the critical velocity. Tapping the pipe or interposing therein a piece of wire gauze will also act likewise; in fact, the point of break-down can be varied within wide limits according to the circumstances.

Whatever be the disturbing causes, however, if stream-line motion exists, the

* 'Phil. Trans.,' 1884, p. 957.

relation of slope to velocity is a perfectly definite one at a definite temperature for the flux, being expressed by the equation

$$q = \frac{\pi r^4}{8\mu}\left(\frac{p_1 - p_2}{l}\right).*$$

If we write $\bar{v} = \frac{q}{\pi r^2}$ and $\frac{p_1 - p_2}{l} = i$, we obtain $\bar{v} = \frac{r^2}{8\mu} i$, and this relation between $\bar{v}$ and i, plotted logarithmically, is, at a definite temperature, a line inclined at 45° to the axes.

Slightly above critical velocity, it can be shown experimentally that no definite relation exists, but well above this point, where the motion is perfectly eddying, it can be shown experimentally that the relation between $\bar{v}$ and i is a perfectly definite one at a definite temperature, and is expressed by some straight line inclined at an angle $\tan^{-1} n$, where n is a constant for any particular pipe.

It therefore appears that the minimum critical velocity is the intersection of the two branches of the logarithmic homologue; and throughout this paper this point has been taken as the critical velocity for the temperature considered.

As the experiments below the critical velocity require apparatus for measuring pressures of extreme accuracy and limited range, while above the critical velocity the limit of accuracy is relatively less important and the range is large, it simplifies matters to take a series of runs at different temperatures below the critical velocity without any change, and afterwards to take runs above the critical velocity. With this method the variation of temperature during a single series is small, and the correction to a standard temperature is generally negligible.

Apparatus used in the Experiments.

The experimental tank A, fig. 1, is of cast-iron, 5 feet square in section and about

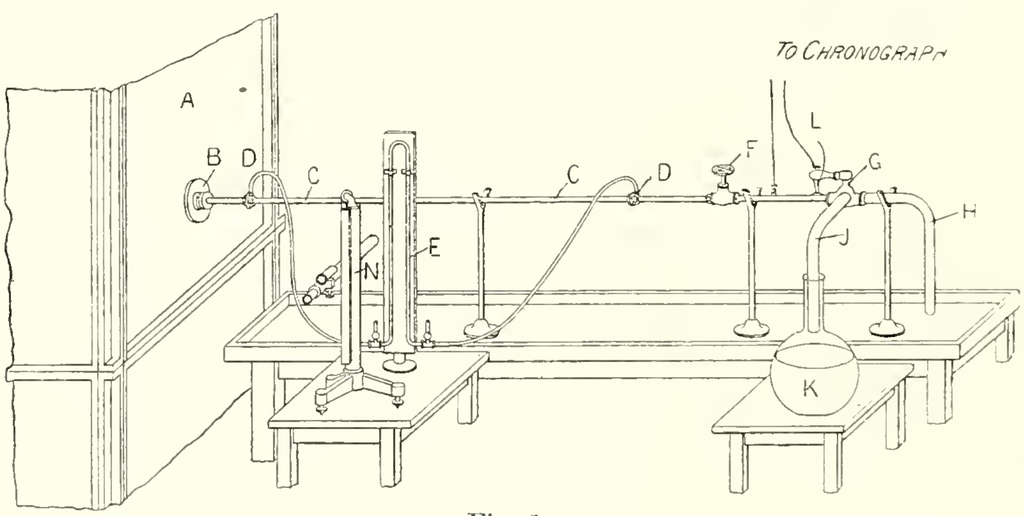

Fig. 1.

30 feet in height, its base resting upon the earth, so that the water in it is not easily disturbed by external causes. It is provided with a steam heater for the inflowing

* LAMB'S 'Hydrodynamics,' p. 521.

water, and there is a direct steam connection to the boiler room, so that steam can be blown directly into the tank. About 8 feet above the base there is an opening, B, in the middle of one side, through which the tube C was inserted, its bell-mouth being placed at the centre; and at suitable distances apart pressure chambers, D, were formed and connected up to the U-gauge E. The flow of water was controlled by a valve, F, and on the prolongation of the pipe a three-way plug valve, G, was inserted, so that the water could run to waste through the pipe H, or could be discharged, by the pipe J, into the glass flask K. The handle of this tap was provided with a flexible brass plate, L, in circuit with a chronograph, so that at the middle of its swing a circuit was completed by the contact of the brass strip with the pipe, and a record was obtained on the drum of the chronograph. This latter instrument was furnished with two pens, marking in opposite directions, one ticking seconds and the other operating at the beginning and end of each run. This arrangement tends to prevent errors in reading.

The pressure chambers were of a special design and consisted of three separate pieces, the outer one (A) of which couples the parts B and C together, leaving a continuous opening, D, which may be of any required width. In the present case, the two sides forming the slit were separated by an interval not more than $\frac{1}{200}$ inch, so as to prevent, as far as possible, any interference with stream-like flow. The part B is recessed to form a pressure chamber, P, connected to the gauge by an opening, E. The parts B and C are faced so that when drawn together by the coupling A, they form a water-tight joint at F, and the ends of the pipe are screwed into corresponding recesses in B and C. This form of pressure chamber has several advantages. The continuous opening gives an accurate mean value of the pressure, and it can be faced without any burr; moreover, it may be readily disconnected for inspection.

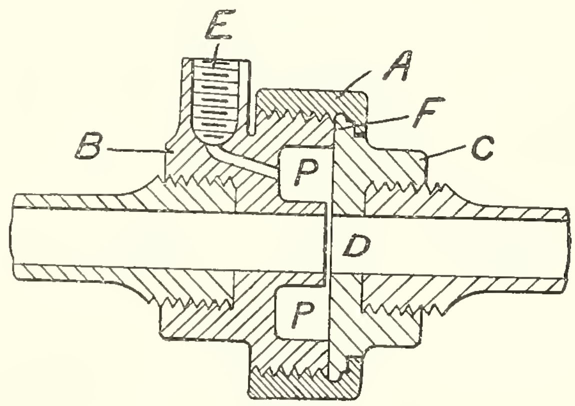

Fig. 2.

The pipe was of brass, without seam, and 6 feet in length between the pressure chambers; its mean diameter was determined by first weighing empty and then full of mercury. The mean diameter thus determined was 0·3779 inch.

The Measurement of Pressure.

The accurate determination of the pressures at the given sections of the pipe is a matter of considerable difficulty, especially at the very low differences of head required for the accurate determination of the slope of pressure at velocities below the critical velocity. At the higher pressures, a U-tube containing mercury was found to answer all requirements. Errors due to the inequalities of the tube were got rid of by measurements taken on both tubes, while a suitable correction was made for temperature. At the low velocities, considerable difficulties were experienced. The difference of heads between the two sections at the lowest velocities was about ·005 inch of mercury, and as this must be read very accurately, it became a matter of such difficulty that a new gauge was made and filled with carbon bisulphide. A number of trials were made, but it was found that the carbon bisulphide was very sluggish in action, and, unless a very considerable time was allowed between every two successive runs, its readings could not be relied upon. Another and more serious defect was the shape of the falling meniscus, which rarely assumed its proper form, so as to afford a definite measurement. This was due to the adherence of the carbon bisulphide to the glass; and in spite of repeated cleanings with different re-agents, no decided improvement was made and the gauge was abandoned. A return was made to the mercury gauge, and the cathetometer was replaced by micrometer microscopes, which had been carefully calibrated beforehand. These afforded much better results, but the observations were still irregular. Finally, the solution of the difficulty was found by turning the U-gauge upside down and imprisoning in its upper part a fixed column of air above the water in both limbs of the gauge.

At first sight this might not seem to be a good arrangement, since any small variation of temperature will affect the imprisoned volume of air considerably, but this affects both legs equally, and there is no error from this cause. A possible source of error is the creeping of air from the pipe to the gauges. This is extremely unlikely, as the air in this case must first descend. If any liberation of air occurred from the water, its effect in altering the gauge would only be momentary.

In practice, this gauge proved extremely sensitive and the readings could be repeated very accurately.

The cathetometer used in reading the heights of the liquid in the U-tubes was of a standard pattern made by the Cambridge Scientific Instrument Company and capable of reading to $\frac{1}{100}$ of a millimetre.

Stream-line Flow.

The determination of the relation of slope to pressure, for water in stream-line motion flowing through tubes of more than capillary size, is rendered somewhat difficult because of the smallness of the difference of pressure required to produce the flow. The difference may be increased by using a long length of pipe, or by using apparatus of extreme accuracy. The disposition of the permanent apparatus in the laboratory prevented the use of a pipe more than 6 feet in length between the pressure chambers; and at first considerable difficulty was experienced in obtaining consistent results, but after many trials this was accomplished. A disturbing cause, which could not be altogether avoided, was the rise or fall of the temperature of the water as it flowed along the pipe ; a fall if the temperature of the room was above the temperature of the water, and *vice versâ*. This was partly removed by covering the pipe with thick cotton-wool lagging overlaid with flannel, and in order to obtain a mean value of the temperature of the water in the pipe a long-stem thermometer was fixed in the tank and another was immersed in the outflowing water, and a mean value of these readings was taken as the true temperature in the pipe.

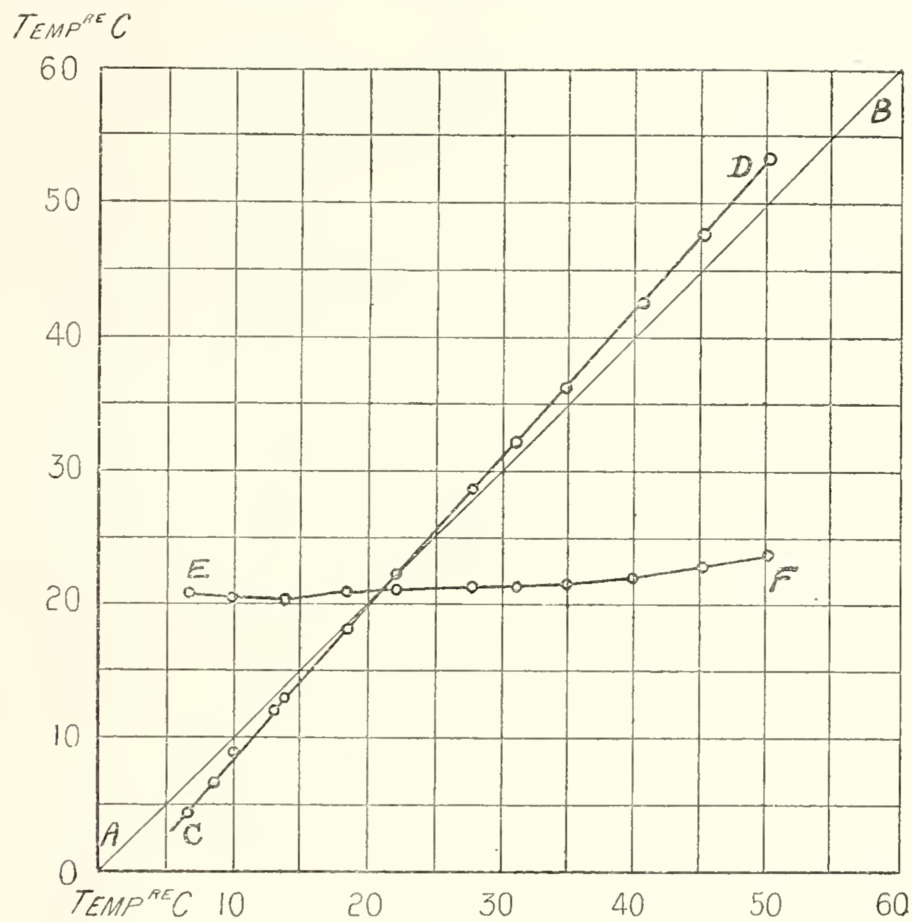

Fig. 3.

A plot of the variations obtained is shown in fig. 3, in which the line AB gives the temperature of the outflow water, CD the temperature of the tank, and EF the corresponding temperature of the room. The relation between the tank temperature and outflow temperature is shown to be practically a linear one, thereby warranting the correction.

In all, ten series of runs were made at temperatures covering the range, and the results obtained are recorded in the following table, and are shown on fig. 4.

TABLE I.

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of water.	log v, v in feet per second.	log h', h' in feet of water.
1	4·3	121·28	7·112	3·819	0·0815	$\bar{1}$·0975
2	4·3	188·84	9·402	3·209	0·0102	$\bar{1}$·0219
3	4·4	148·60	6·312	2·713	$\bar{1}$·9412	$\bar{2}$·9491
4	4·5	137·71	4·204	1·921	$\bar{1}$·7978	$\bar{2}$·7991
5	4·6	164·92	4·033	1·539	$\bar{1}$·7016	$\bar{2}$·7028
6	4·6	198·80	3·380	1·067	$\bar{1}$·5436	$\bar{2}$·5438
7	4·7	84·26	1·336	0·999	$\bar{1}$·5134	$\bar{2}$·5152
8	4·7	152·00	1·540	0·648	$\bar{1}$·3189	$\bar{2}$·3272
9	11·2	141·18	6·130	2·315	$\bar{1}$·9511	$\bar{2}$·8799
10	11·2	168·15	5·131	1·570	$\bar{1}$·7978	$\bar{2}$·7133
11	11·2	194·18	6·686	1·811	$\bar{1}$·8503	$\bar{2}$·7733
12	11·2	140·40	3·318	1·224	$\bar{1}$·6868	$\bar{2}$·6032
13	11·2	199·70	3·018	0·795	$\bar{1}$·4925	$\bar{2}$·4158
14	11·3	235·68	2·407	0·507	$\bar{1}$·3224	$\bar{2}$·2204
15	11·3	175·10	1·295	0·388	$\bar{1}$·1824	$\bar{2}$·1042
16	16·8	70·90	1·991	1·221	$\bar{1}$·7621	$\bar{2}$·6020
17	16·8	82·00	3·138	1·678	$\bar{1}$·8964	$\bar{2}$·7401
18	16·8	75·70	3·565	2·088	$\bar{1}$·9865	$\bar{2}$·8351
19	16·8	83·53	2·266	1·212	$\bar{1}$·7470	$\bar{2}$·5988
20	16·8	88·28	1·320	0·660	$\bar{1}$·4883	$\bar{2}$·3348
21	16·8	180·43	2·707	0·668	$\bar{1}$·4898	$\bar{2}$·3401
22	16·9	143·75	1·347	0·404	$\bar{1}$·2855	$\bar{2}$·1217
23	16·9	123·00	4·439	1·594	$\bar{1}$·8710	$\bar{2}$·7178
24	18·0	188·60	7·150	1·671	$\bar{1}$·8825	$\bar{2}$·7380
25	18·0	159·35	4·821	1·304	$\bar{1}$·7945	$\bar{2}$·6302
26	18·0	182·55	3·973	0·932	$\bar{1}$·6515	$\bar{2}$·4844
27	18·1	187·30	3·926	0·931	$\bar{1}$·6353	$\bar{2}$·4839
28	18·1	190·15	2·713	0·604	$\bar{1}$·4682	$\bar{2}$·2960
29	18·1	181·53	1·208	0·183	$\bar{1}$·1368	$\bar{3}$·7775
30	18·1	181·30	3·790	0·886	$\bar{1}$·6340	$\bar{2}$·4624
31	18·2	111·25	1·155	0·456	$\bar{1}$·3301	$\bar{2}$·1740
32	18·2	346·40	10·932	1·376	$\bar{1}$·8129	$\bar{2}$·6536
33	27·2	135·85	4·594	1·232	$\bar{1}$·8445	$\bar{2}$·6059
34	27·2	166·33	4·782	1·020	$\bar{1}$·7733	$\bar{2}$·5239
35	27·2	135·12	2·929	0·779	$\bar{1}$·6508	$\bar{2}$·4068
36	27·2	177·30	3·140	0·660	$\bar{1}$·5629	$\bar{2}$·3348
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38	27·1	139·78	6·300	1·674	$\bar{1}$·9684	$\bar{2}$·7391
39	27·1	83·33	4·404	2·028	0·0378	$\bar{2}$·8224
40	31·1	175·83	8·218	1·628	$\bar{1}$·9848	$\bar{2}$·7263
41	31·1	123·95	4·346	1·172	$\bar{1}$·8601	$\bar{2}$·5837
42	31·1	146·55	3·717	0·851	$\bar{1}$·7194	$\bar{2}$·4446
43	31·1	150·30	1·334	0·304	$\bar{1}$·2635	$\bar{3}$·9976

TABLE I.—*continued.*

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of water.	log v, v in feet per second.	log h', h' in feet of water.
44	34·5	157·70	4·329	0·871	$\bar{1}$·7541	$\bar{2}$·4553
45	34·5	105·60	5·563	1·713	·0372	$\bar{2}$·7490
46	34·4	96·68	3·745	1·214	$\bar{1}$·9037	$\bar{2}$·5995
47	34·4	158·10	4·122	0·767	$\bar{1}$·7317	$\bar{2}$·4001
48	34·3	222·02	3·487	0·463	$\bar{1}$·5117	$\bar{2}$·1809
49	34·2	138·63	2·105	0·458	$\bar{1}$·4969	$\bar{2}$·1762
50	34·2	301·23	3·551	0·357	$\bar{1}$·3870	$\bar{2}$·0680
51	38·0	251·10	9·849	1·179	$\bar{1}$·9098	$\bar{2}$·5866
52	38·0	141·37	6·872	1·477	·0031	$\bar{2}$·6844
53	37·9	112·45	3·893	1·027	$\bar{1}$·8556	$\bar{2}$·5265
54	37·8	265·68	6·681	0·719	$\bar{1}$·7169	$\bar{2}$·3717
55	37·8	194·45	3·518	0·497	$\bar{1}$·5738	$\bar{2}$·2114
56	37·7	162·95	2·043	0·371	$\bar{1}$·4144	$\bar{2}$·0844
57	37·7	148·73	2·318	0·442	$\bar{1}$·5089	$\bar{2}$·1604
58	42·4	140·27	5·365	1·031	$\bar{1}$·8998	$\bar{2}$·5281
59	42·4	121·07	3·532	0·779	$\bar{1}$·7820	$\bar{2}$·4064
60	42·3	130·38	2·879	0·583	$\bar{1}$·6613	$\bar{2}$·2806
61	42·3	158·38	2·720	0·474	$\bar{1}$·5520	$\bar{2}$·1907
62	42·2	229·33	3·384	0·397	$\bar{1}$·4860	$\bar{2}$·1137
63	42·2	167·63	1·825	0·294	$\bar{1}$·3541	$\bar{3}$·9832
64	42·1	124·75	4·365	0·933	$\bar{1}$·8611	$\bar{2}$·4848
65	49·5	213·65	3·673	0·417	$\bar{1}$·5539	$\bar{2}$·1351
66	49·4	268·15	7·015	0·637	$\bar{1}$·7362	$\bar{2}$·3191
67	49·3	162·93	5·766	0·836	$\bar{1}$·8745	$\bar{2}$·4372
68	49·0	303·75	1·929	0·153	$\bar{1}$·1214	$\bar{3}$·6997
69	48·9	134·80	2·383	0·423	$\bar{1}$·5660	$\bar{2}$·1413
70	48·8	162·05	3·839	0·578	$\bar{1}$·6930	$\bar{2}$·2769
71	48·7	139·88	4·692	0·794	$\bar{1}$·8440	$\bar{2}$·4148

In this table the observations are recorded in Columns 2, 3, 4 and 5, and from Columns 3 and 4 the mean values of the velocity of the water in feet per second have been calculated, and the logarithms of these quantities are given in Column 6. The observed differences of head given in Column 5 have been reduced to feet of water, h', to correspond, and the values of log h' are given in Column 7.

In most cases, owing to the large volume of water in the tank (usually not less than 300 cubic feet), the temperature remained remarkably steady during the runs forming a series, and no correction for temperature was necessary, and none was made unless the temperature differed more than 0°·1 C. In some cases, however, a much greater variation was met with, especially at the higher temperatures, and correction was necessary, not only in this series, but in the second series when the motion was eddy or sinuous. The correction factor to be applied may be obtained as follows:—

If we assume that in stream-line motion or sinuous motion the total resistance i depends on powers of the pipe radius, the kinematic viscosity, the density and the velocity, we may write, with the usual notation,

$$i = kr^x \nu^y \rho^z v^n,$$

where r = radius of the pipe,
ν = coefficient of kinematic viscosity,
ρ = density,
$\bar{v}$ = mean velocity of water along the pipe,
k = a constant.

Dimensionally this equation becomes

$$\frac{[\mathrm{M}][\mathrm{L}]}{[\mathrm{T}^2]} = [\mathrm{L}]^x \left[\frac{\mathrm{L}^2}{\mathrm{T}}\right]^y \left[\frac{\mathrm{M}}{\mathrm{L}^3}\right]^z \left[\frac{\mathrm{L}}{\mathrm{T}}\right]^n,$$

[giving the relations

$$z = 1, \qquad x + 2y - 3z + n = 1, \qquad y + n = 2,$$

and therefore

$$i = k\rho r^n \nu^{2-n} v^n.$$

For the case of stream-line motion, $n = 1$ giving

$$i = k\rho r\nu v.$$

For the case of sinuous motion n is greater than unity, and we may write the equation

$$i = k\rho \frac{\mu^{2-n}}{\rho^{2-n}} r^n v^n$$

$$= \mathrm{K}\mu^{2-n}\rho^{n-1}, \qquad \text{where } \mathrm{K} = kr^n v^n.$$

Taking logarithms, we get

$$\log i = \log \mathrm{K} + (2-n)\log\mu + (n-1)\log\rho.$$

Differentiating with v constant, we obtain

$$\frac{1}{i}\frac{di}{d\tau} = (2-n)\frac{1}{\mu}\frac{d\mu}{d\tau} + (n-1)\frac{1}{\rho}\frac{d\rho}{d\tau}.$$

Now $\mu = \dfrac{c}{1 + \alpha\tau + \beta\tau^2}$, therefore $\dfrac{1}{\mu}\dfrac{d\mu}{d\tau} = \dfrac{-(\alpha + 2\beta\tau)}{1 + \alpha\tau + \beta\tau^2}$, and $\rho = \rho_0(1 - \gamma t)$ approximately, therefore

$$\frac{1}{\rho}\frac{d\rho}{d\tau} = -\frac{\gamma}{1-\gamma\tau}.$$

Hence

$$\frac{di}{i} = -(2-n)\frac{\alpha + 2\beta\tau}{1+\alpha\tau+\beta\tau^2}\,d\tau + (1-n)\frac{\gamma}{1-\gamma\tau}\,d\tau.]^*$$

For stream-line motion $n = 1$, and the values of α and β taken from POISEUILLE'S formula are

$$\alpha = \cdot 03368, \qquad \beta = \cdot 000221.$$

The value of γ for water is approximately ·00048, and hence the second term is never important, and the first term is only important where $d\tau$ is large (and the term $\alpha + 2\beta\tau$ is not small).

For stream-line motion the correction for a difference of 1° at 10° C. is approximately ·028, while at 50° C. it is ·017.

The observations recorded in the table, corrected to a mean temperature, are plotted in fig. 4. [†The diagram shows the logarithmic homologues for stream-line motion in the pipe at ten different temperatures between 4° C. and 50° C., and these are represented by a series of straight lines equally inclined to the axes of co-ordinates. The mean temperatures to which the observations have been reduced are as follows :—

Experiments.	Reduced to a mean temperature Centigrade of	Corresponding lines on fig. 4.
1 to 8	4·5°	1
9 ,, 15	11·2	2
16 ,, 23	16·8	3
24 ,, 32	18·1	4
33 ,, 39	27·2	5
40 ,, 43	31·1	6
44 ,, 50	34·4	7
51 ,, 57	37·8	8
58 ,, 64	42·3	9
65 ,, 71	49·3	10

These lines were] found to agree closely with the formula

$$q = \frac{\pi r^4}{8\mu}\cdot\frac{p_1 - p_2}{l},\ddagger$$

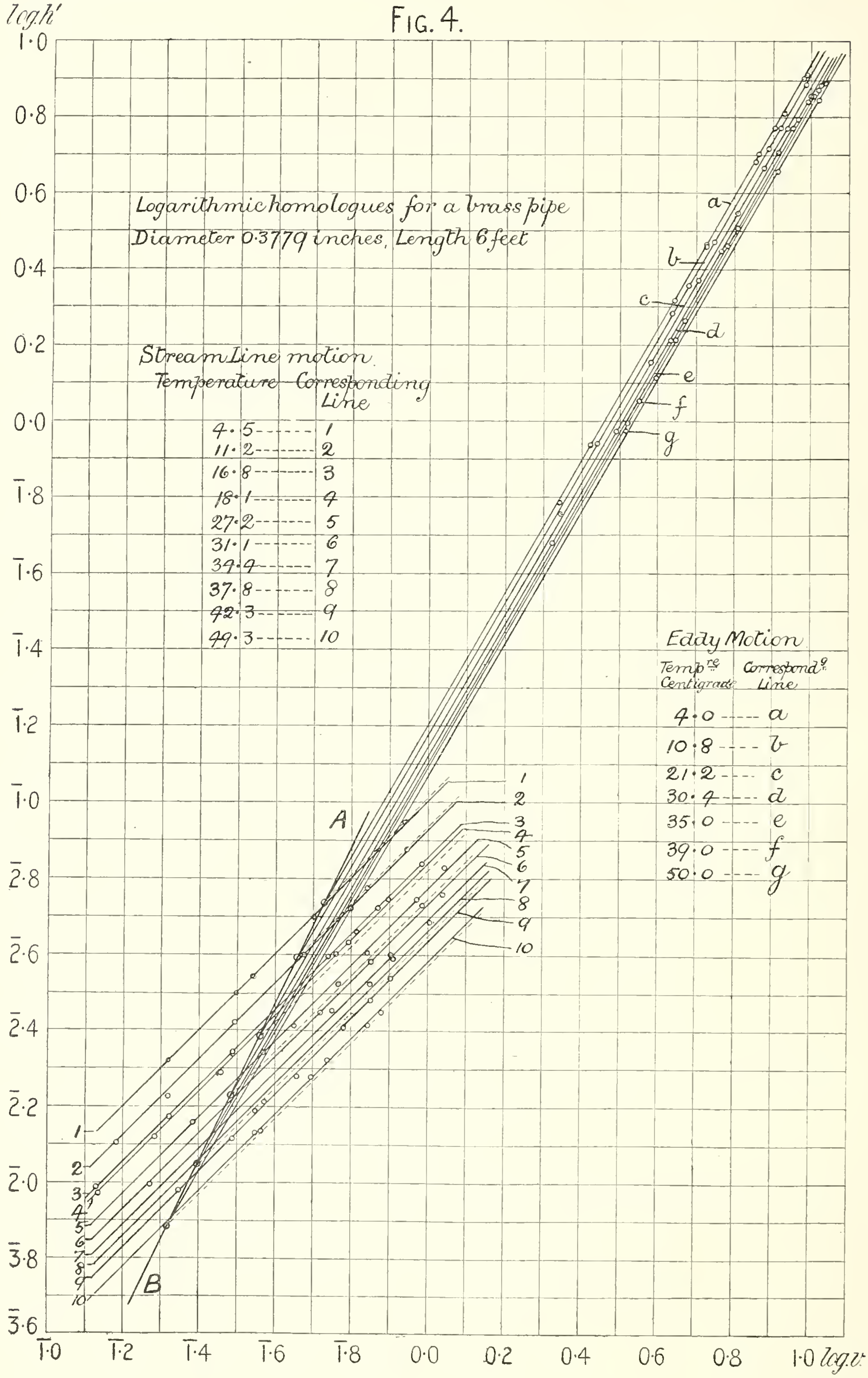
where r is the radius of the pipe, p_1 and p_2 the pressures at the ends, l the length of pipe, and μ is the coefficient of viscosity.

* Corrected Nov. 14, 1902, as pointed out by the Referee.

† Added Nov. 14, 1902.

‡ *Loc. cit. ante* (p. 48).

FIG. 4.


log h'
1·0
0·8
0·6
0·4
0·2
0·0
$\bar{1}$·8
$\bar{1}$·6
$\bar{1}$·4
$\bar{1}$·2
$\bar{1}$·0
$\bar{2}$·8
$\bar{2}$·6
$\bar{2}$·4
$\bar{2}$·2
$\bar{2}$·0
$\bar{3}$·8
$\bar{3}$·6
$\bar{1}$·0
$\bar{1}$·2
$\bar{1}$·4
$\bar{1}$·6
$\bar{1}$·8
0·0
0·2
0·4
0·6
0·8
1·0 log v.
Logarithmic homologues for a brass pipe
Diameter 0·3779 inches, Length 6 feet
Stream Line motion
Temperature Corresponding Line
4·5 1
11·2 2
16·8 3
18·1 4
27·2 5
31·1 6
39·4 7
37·8 8
42·3 9
49·3 10
Eddy Motion
Tempre Centigrade Correspondg Line
4·0 a
10·8 b
21·2 c
30·4 d
35·0 e
39·0 f
50·0 g
A
B
a
b
c
d
e
f
g
1
2
3
4
5
6
7
8
9
10

This agreement is shown clearly by fig. 5, where the intersections of the logarithmic homologues with the zero ordinate are plotted with reference to the temperature as abscissa, and are compared with the intersections determined from the equation above, taking the value of μ according to POISEUILLE's formula. At temperatures between 5° and 20° C. the agreement is close, the values at 27°·2 and 31° do not correspond very well, and there is a very fair agreement at the higher temperatures. The dotted lines on fig. 4 are the logarithmic homologues at the temperatures of 4°, 10°·8, 21°·2, 30°·4, 35°, 39° and 50° C. respectively, and these have been interpolated by aid of figs. 4 and 5, in order to determine the intersections with the homologues for eddy motion also plotted on fig. 4, and which are referred to in the next section.

The Relation of Slope to Velocity for Water in Eddy Motion.

A second series of experiments was now commenced with water in eddy motion to determine the relation between the loss of head and the velocity at a sufficient number of temperatures within the range.

It was extremely difficult to control the temperature, and so no attempt was made to obtain a series of runs with temperatures corresponding exactly to those obtained

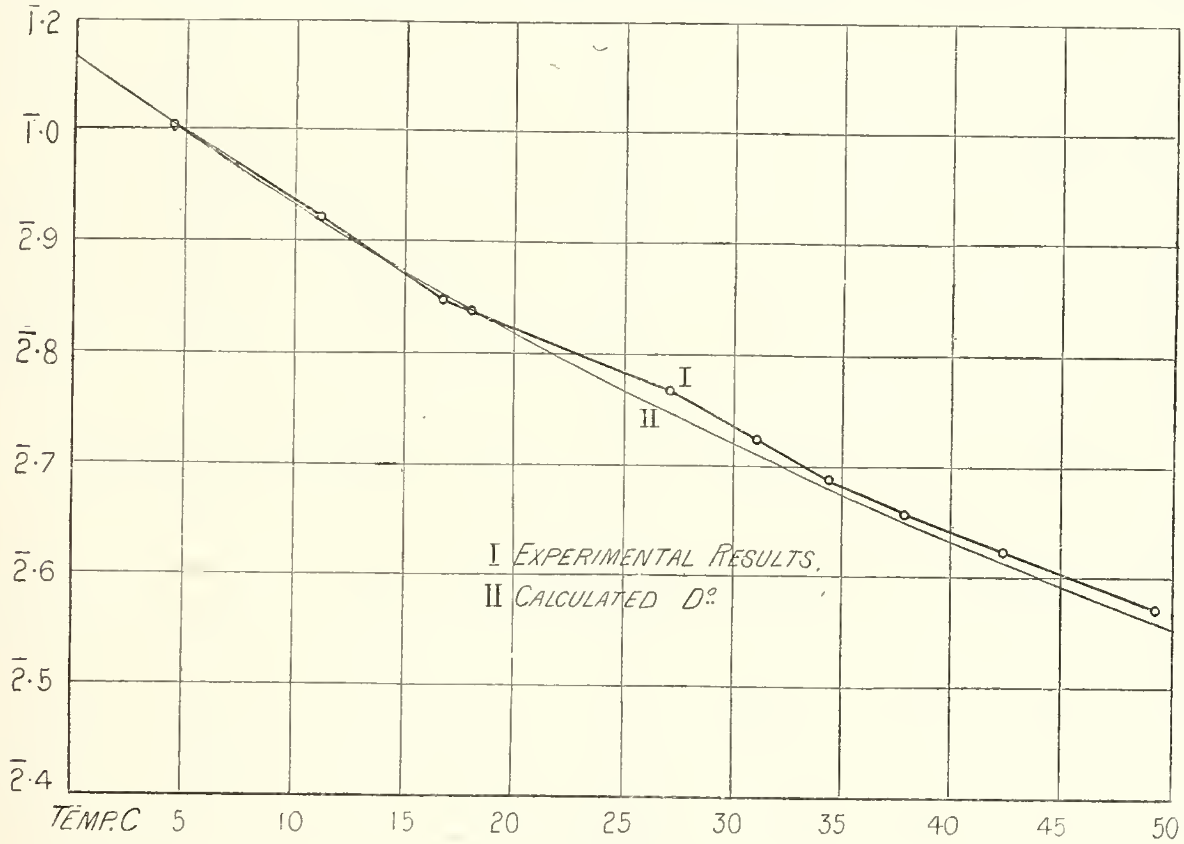


Fig. 5.

for stream-line motion, nor was this necessary, as the logarithmic homologue for stream-line motion, corresponding to a similar one for eddy motion, was obtained by interpolation from figs. 4 and 5. The observations were made under precisely the same conditions as before, except that in the pressure gauge mercury in contact with water was used instead of water in contact with air and the gauge was in its normal position with the connecting U below.

After some preliminary trials, seven complete series of runs were made, covering the range of temperature, and the observations made are recorded in Table II., and the logarithmic plots are shown on fig. 4.

As slight differences of temperature were now of much less importance, the plotted results lie much better upon the mean line, and enable the value of n to be determined with considerable accuracy.

TABLE II.

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of mercury.	log v, v in feet per second.	log h', h' in feet of water.
72	4·0	25·05	11·849	18·313	·9882	·9119
73	4·0	26·00	10·791	14·631	·9313	·8144
74	4·0	36·20	12·873	11·353	·8642	·7042
75	4·0	38·48	13·559	11·039	·8602	·6920
76	4·0	49·30	12·780	6·520	·7270	·4633
77	4·0	62·05	13·198	4·612	·6410	·3130
78	4·0	98·08	12·603	1·945	·4221	$\bar{1}$·9380
79	4·1	27·50	12·868	18·028	·9834	·9051
80	4·1	34·83	13·665	13·327	·9068	·7738
81	4·1	30·68	11·999	13·271	·9054	·7720
82	10·8	31·10	12·605	13·422	·9224	·7763
83	10·8	46·80	12·748	6·719	·7484	·4758
84	10·8	62·95	13·257	4·300	·6368	·2820
85	10·8	120·60	13·235	1·392	·3537	$\bar{1}$·7921
86	10·8	25·86	12·221	17·469	·9878	·8908
87	10·9	34·74	13·131	11·852	·8908	·7223
88	10·9	5·58	13·006	5·137	·6808	·3592
89	10·9	100·00	13·405	1·967	·4406	$\bar{1}$·9423
90	21·2	28·38	12·037	13·481	·9416	·7774
91	21·2	34·62	12·864	10·701	·8841	·6771
92	21·2	39·58	12·444	7·942	·8815	·5476
93	21·2	51·27	12·668	5·280	·7068	·3703
94	21·2	70·00	13·094	3·228	·5859	·1567
95	21·2	114·90	12·797	1·265	·3609	$\bar{1}$·7498
96	21·2	32·82	12·894	11·882	·9803	·7227
97	30·4	26·81	13·250	16·206	1·0090	·8573
98	30·4	24·66	11·960	16·145	1·0009	·8556
99	30·4	28·93	12·679	13·536	·9569	·7791
100	30·4	28·49	12·851	14·138	·9693	·7979
101	30·4	31·75	12·639	11·472	·9151	·7072
102	30·4	45·64	12·908	6·326	·7665	·4487
103	30·4	61·05	12·870	3·743	·6390	·2208
104	30·4	83·60	12·580	2·122	·4926	$\bar{1}$·9743
105	30·4	127·70	12·968	1·082	·3218	$\bar{1}$·6818
106	30·4	25·11	11·995	15·725	·9942	·8442

TABLE II.—*continued.*

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of mercury.	log v, v in feet per second.	log h', h' in feet of water.
107	35·0	23·05	11·898	17·415	1·0285	·8888
108	35·0	22·52	11·611	17·397	1·0277	·8884
109	35·0	27·90	11·854	12·703	·9440	·7518
110	35·0	42·00	13·034	7·213	·8075	·5060
111	35·0	54·95	12·508	4·231	·6725	·2743
112	34·9	73·40	12·527	2·541	·5475	·0529
113	34·9	26·10	13·370	17·220	1·0252	·8839
114	40·4	23·57	12·311	17·482	1·0343	·8905
115	39·6	25·60	10·905	11·792	·9458	·7195
116	39·2	40·50	11·869	6·526	·7834	·4624
117	38·8	56·55	12·214	3·770	·6508	·2241
118	38·5	77·14	12·308	2·228	·5191	$\bar{1}$·9957
119	38·0	24·67	12·802	17·550	1·0313	·8920
120	50·5	24·76	12·531	15·820	1·0226	·8470
121	50·5	67·20	12·771	2·933	·5972	·1151
122	50·0	27·18	12·245	12·708	·9721	·7519
123	50·0	56·48	11·870	3·340	·6410	·1716
124	50·0	32·40	12·928	10·492	·9194	·6686
125	49·8	78·40	12·346	2·147	·5178	$\bar{1}$·9796

The slopes of the lines for the different temperatures are shown in the accompanying table, and their mean value is $n = 1{\cdot}731$.

Temperature, ° C.	4·0	10·8	21·2	30·4	35·0	39·0	50·0
n	1·722	1·733	1·740	1·734	1·738	1·737	1·715

The different values obtained are no doubt due to temperature errors, and this view is confirmed when it is seen that the variations from the mean value are greater the further the temperature of the water is from the temperature of the laboratory.

As there seems no reason to suppose that the value of n varies with the temperature, the logarithmic homologues have been drawn at a mean inclination of $\tan^{-1} 1{\cdot}731$ in determining the critical velocity. The corrections to be applied for differences of temperature are now much smaller, and for the mean value of $n = 1{\cdot}731$ is found to be ·0076 for a difference of 1° from a temperature of 10° C., and ·0047 for the same difference at 50° C. The observations for each series of runs in Table II. have been plotted to a mean temperature like those for stream-line motion, the temperatures corresponding to the interpolated homologues for stream-line motion described above.

Critical Velocity.

It has been pointed out in an earlier section that no attempt was made to determine the velocity at which stream-line motion broke down, but that the intersections of the two sets of lines above and below critical velocity were used to determine the minimum critical velocity. This method of procedure amounts to the determination of the curve of intersection of two families of straight lines, whose positions are experimentally determined, and it is clear that if the points of intersection lie upon some straight line in the logarithmic plot, the variation of the critical velocity must follow the viscosity of the water linearly, while, if they do not, the law cannot be a linear one.

Fig. 4 shows the observations for stream-line flow, and the lines representing eddy motion are drawn thereon, and are produced to meet the interpolated lines for

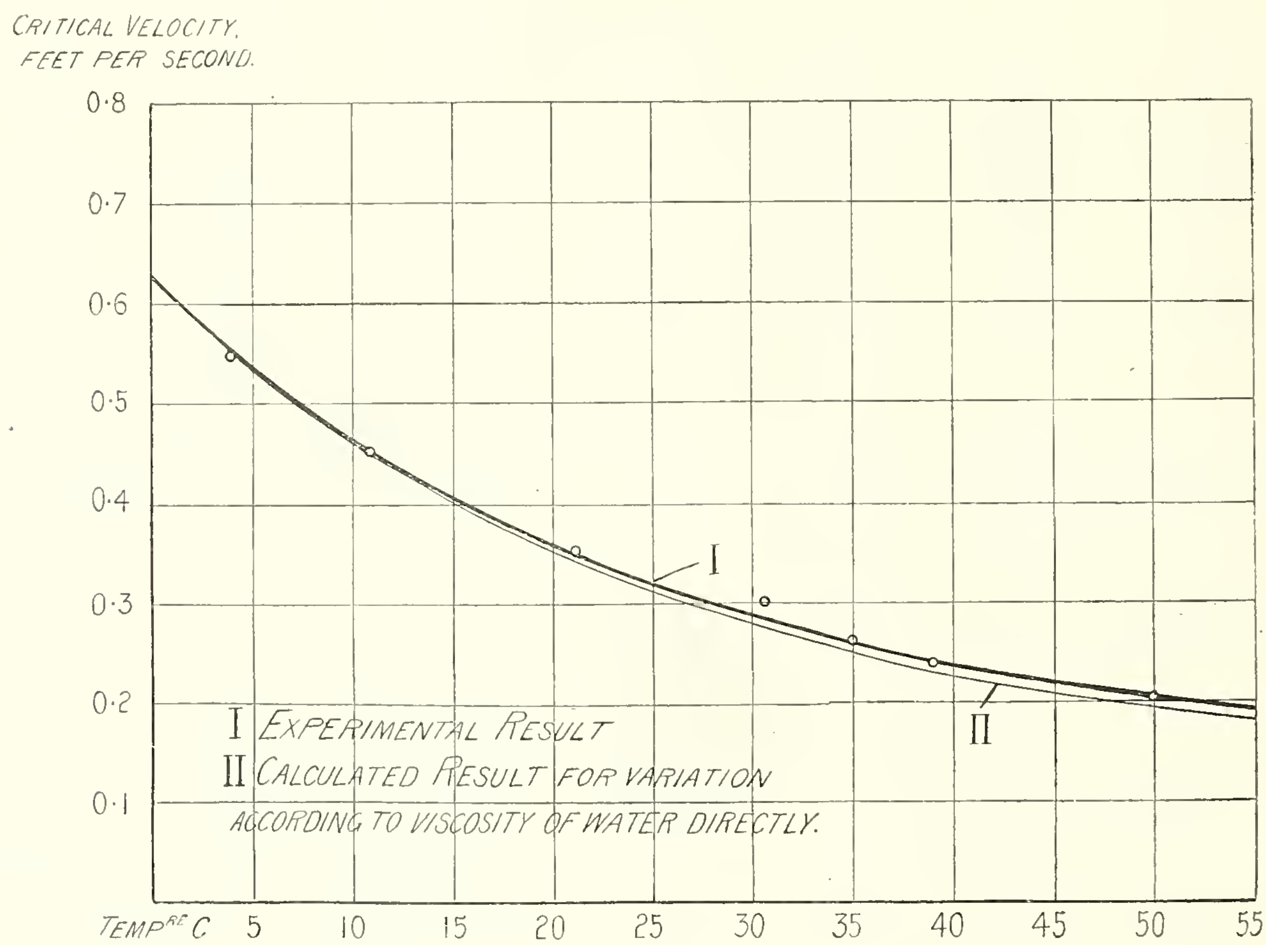


Fig. 6.

stream-line flow (shown dotted); the points of meeting are found to lie very approximately upon the straight line AB.

It is therefore apparent that these intersections vary as the viscosity, and they afford a verification of the formula.

This is brought out clearly by fig. 6, in which the velocities so found are plotted directly with respect to temperature. As will be seen, less weight is given to the observations in the neighbourhood of 30°, for the experimental results on stream-line motion are probably less accurate, as was pointed out in a previous section. The law of variation is found to be approximately

$$v_c^{-1} \propto 1 + 0{\cdot}03368\mathrm{T} + {\cdot}000156\mathrm{T}^2,$$

which agrees very closely with the variation in the viscosity of water, viz.,

$$\mu^{-1} \propto 1 + 0{\cdot}03368\mathrm{T} + {\cdot}000221\mathrm{T}^2.$$

It may therefore be concluded that for small pipes, over the range of temperature examined, the critical velocity of water varies directly as the viscosity.

In conclusion the authors desire to thank Professor BOVEY for placing the hydraulic laboratory of McGill University at their disposal, and also Dr. BARNES, who gave much assistance during the progress of the work.

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ON AN APPROXIMATE SOLUTION FOR THE BENDING OF A BEAM OF RECTANGULAR CROSS-SECTION UNDER ANY SYSTEM OF LOAD

WITH

SPECIAL REFERENCE TO POINTS OF CONCENTRATED OR DISCONTINUOUS LOADING

BY

L. N. G. FILON, B.A. (CANTAB.), M.A., B.SC. (LOND.),

KING'S COLLEGE, CAMBRIDGE, FELLOW OF UNIVERSITY COLLEGE, LONDON, AND 1851 EXHIBITION SCIENCE RESEARCH SCHOLAR.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

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INDEX SLIP.

IV. *On an Approximate Solution for the Bending of a Beam of Rectangular Cross-Section under any System of Load, with Special Reference to Points of Concentrated or Discontinuous Loading.*

By L. N. G. Filon, *B.A.* (*Cantab.*), *M.A.*, *B.Sc.* (*Lond.*), *King's College, Cambridge, Fellow of University College, London, and* 1851 *Exhibition Science Research Scholar.*

Communicated by C. Chree, *F.R.S.*

Received June 12,—Read June 19, 1902.

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Part II.—Discussion of the General Solution when the Forces on the Beam are Purely Normal and are Symmetrical about $x = 0$.

PART I.

Establishment and General Solution of the Equations of the Problem Discussed.

§ 1. *General Sketch of the Problem proposed.*

The consideration of the stresses and strains which occur in a rectangular parallelopiped of elastic material subjected to given surface forces over its six faces leads to one of the most general, as it is one of the oldest, problems in the Theory of Elasticity. Lamé, in his 'Leçons sur l'Élasticité des Corps solides,' published in 1852, describes it as "le plus difficile peut-être de la théorie mathématique de l'élasticité." In spite of repeated attempts, however, the problem remains still unsolved.

In its complete form it may be stated as follows:—

Let the origin be taken at the centre of the parallelopiped and the axes $0x$, $0y$, $0z$ parallel to its edges. Let the lengths of these edges be $2a$, $2b$, $2c$. Let u, v, w denote the displacements of any point (x, y, z) parallel to the three axes, and, following the notation of Todhunter and Pearson's 'History of Elasticity,' let $\widehat{st}$ denote the stress, parallel to s, across an elementary area perpendicular to t, then we have the six stresses

$$\left.\begin{aligned} \widehat{xx} &= \lambda\delta + 2\mu\frac{du}{dx} & \widehat{yz} &= \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ \widehat{yy} &= \lambda\delta + 2\mu\frac{dv}{dy} & \widehat{zx} &= \mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ \widehat{zz} &= \lambda\delta + 2\mu\frac{dw}{dz} & \widehat{xy} &= \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1),$$

where $\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, and λ, μ are the elastic constants of Lamé.

Also u, v, w must satisfy, inside the material, the following differential equations,

$$\left.\begin{aligned} (\lambda + \mu)\frac{d\delta}{dx} + \mu\nabla^2 u &= 0 \\ (\lambda + \mu)\frac{d\delta}{dy} + \mu\nabla^2 v &= 0 \\ (\lambda + \mu)\frac{d\delta}{dz} + \mu\nabla^2 w &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2),$$

where $\nabla^2 \equiv \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, there being no body force acting on the matter inside the block. It is required to find the values of u, v, w at each point, subject to the condition that the stress across the outer faces $x = \pm a$, $y = \pm b$, $z = \pm c$ shall be arbitrarily given at each point—regard being had, of course, to the conditions of rigid equilibrium of the block.

Since LAMÉ'S time the problem has been attacked by a large number of mathematicians, among them DE SAINT-VENANT, CLEBSCH, BOUSSINESQ, and more recently M. MATHIEU, M. RIBIÈRE and Mr. J. H. MICHELL. Although they have not been able so far to obtain the solution of the problem as stated quite generally above, they have nevertheless made great progress with various particular cases, more especially those in which some of the dimensions of the block are large compared with the rest.

Fuller references to their work and to the results obtained by them are given in the historical summary at the end of this paper.

§ 2. *Object of the Investigation.*

The object of the present investigation is to obtain the solution for the rectangular parallelopiped under an arbitrary system of surface loading in two cases, when the problem reduces to one of two dimensions, namely :—

(A) When two of the faces $z = \pm c$ of the bar are constrained to remain plane and the stress applied to the other faces is independent of z. In this case $w = 0$, u and v are functions of x and y only. If the breadth $2c$ of the beam be sufficiently large, we may relinquish the constraint along the sides altogether, and we have thus the case of a thick plate bent in a plane perpendicular to its own plane. When the plate is made indefinitely thick we have two-dimensional strain in an infinite elastic solid with a plane boundary.

(B) When we make the assumption that $\widehat{xz}$ and $\widehat{yz}$ vanish at the boundaries $z = \pm c$, while $\widehat{zz}$ is actually zero throughout. That this will be very near the truth if c is very small is quite evident, so that in any case this condition will hold for a flat beam or girder whose height is large compared with its breadth.*

But it seems not improbable that it may continue to hold approximately up to a fairly large value of c; we may remember that DE SAINT-VENANT, in his solution for flexure, assumes both $\widehat{zz}$ and $\widehat{yy}$ to be zero, in the case where his beam is unstressed except at the ends, and his solution is sufficient to satisfy all conditions. Obviously vertical pressures and tensions across the faces $y = \pm b$ must introduce important stresses $\widehat{yy}$, so that that part of DE SAINT-VENANT'S hypothesis, in the generalised problem, must go. Still it appears reasonable to suppose, on the whole, that, even for a beam where c and b are of the same order, we may, as a first approximation, retain the hypothesis $\widehat{zz} = 0$. Of course, eventually, as c increases a stress $\widehat{zz}$ must appear until when c is very large we reach the limiting case of problem (A) when this stress is sufficient to ensure the vanishing of the displacement w.

If, however, c be not too large, so that we can suppose $\widehat{zz}$ sensibly zero throughout,

* September 13, 1902. I have, since writing the above, verified that a solution for rectangular beams does exist, which fulfils *rigidly* these conditions. It is, in fact, identical with part of CLEBSCH'S solution for a thick plate.

then the *mean* values U, V taken across the breadth of the beam of the displacements u, v in the plane xy are found to satisfy two differential equations of the same form as the equations of elasticity when the displacements are independent of z and $w = 0$, with this change, that the elastic constant λ is replaced by another constant λ'. The *mean stresses* in the plane of xy are found by differentiation from U and V by similar formulæ to those giving $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ in terms of u, v for two-dimensional strain.

Now the distribution of such mean stresses inside the beam is independent of the ratio $\lambda' : \mu$. This has been shown by Mr. J. H. MICHELL ('London Mathematical Society's Proceedings,' vol. 31, pp. 100–124). It had been previously pointed out by STOKES ('Phil. Mag.,' Ser. V., vol. 32, p. 503). The equations being of the same form in problems (A) and (B), there follows this curious result, that the distribution of *stress* inside the beam, consequent upon a given distribution of stress upon the upper and lower faces (this latter distribution being uniform with regard to the breadth of the beam) is the same when this breadth is very small and when it is very large.

§ 3. *Establishment of the Equations.*

The centre of the rectangular beam being the origin, let its axis, which is supposed horizontal, be taken as axis of x. The axis of y will be vertical and the axis of z horizontal. The bounding surfaces of the beam are $x = \pm a$, $y = \pm b$, $z = \pm c$.

Using the notation explained in § 1, equations (2) may be written

$$\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\frac{d\widehat{xy}}{dx} + \frac{d\widehat{yy}}{dy} + \frac{d\widehat{yz}}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\frac{d\widehat{xz}}{dx} + \frac{d\widehat{yz}}{dy} + \frac{d\widehat{zz}}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Integrate equations (3) and (4) with regard to z from $-c$ to $+c$. Then, noting that $(\widehat{xz})_{z=\pm c}$, $(\widehat{yz})_{z=\pm c}$ are both zero, owing to the surface conditions at the side of the beam, and also that integration with regard to z and differentiation with regard to x and y are independent, we find

$$\frac{d}{dx}\left[\int_{-c}^{+c} \widehat{xx}\, dz\right] + \frac{d}{dy}\left[\int_{-c}^{+c} \widehat{xy}\, dz\right] = 0,$$

$$\frac{d}{dx}\left[\int_{-c}^{+c} \widehat{xy}\, dz\right] + \frac{d}{dy}\left[\int_{-c}^{+c} \widehat{yy}\, dz\right] = 0.$$

Now if we write $\int_{-c}^{+c} \widehat{xx}\, dz = 2c\mathrm{P}$, $\int_{-c}^{+c} \widehat{yy}\, dz = 2c\mathrm{Q}$, $\int_{-c}^{+c} \widehat{xy}\, dz = 2c\mathrm{S}$, then P, Q, S are the mean values of the two tractions and of the shear in the plane xy—taken, for any values of x, y, across the breadth of the beam. These will in future be referred to as the mean stresses and often, for shortness, as the stresses.

We obtain, therefore, the equations

$$\frac{dP}{dx} + \frac{dS}{dy} = 0 \quad . \quad . \quad . \quad . \quad (6), \qquad \frac{dS}{dx} + \frac{dQ}{dy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now consider equations (1), namely,

$$\widehat{xx} = \lambda\left(\frac{du}{dx} + \frac{dv}{dy}\right) + 2\mu\frac{du}{dx} + \lambda\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

$$\widehat{yy} = \lambda\left(\frac{du}{dx} + \frac{dv}{dy}\right) + 2\mu\frac{dv}{dy} + \lambda\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\widehat{zz} = \lambda\left(\frac{du}{dx} + \frac{dv}{dy}\right) + (\lambda + 2\mu)\frac{dw}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

$$\widehat{xy} = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

If we integrate (8), (9), and (11) with regard to z from $-c$ to $+c$, we have

$$P = \lambda\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dU}{dx} + \lambda\left(\frac{w_{+c} - w_{-c}}{2c}\right) \quad . \quad . \quad . \quad . \quad . \quad (12),$$

$$Q = \lambda\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dV}{dy} + \lambda\left(\frac{w_{+c} - w_{-c}}{2c}\right) \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$S = \mu\left(\frac{dU}{dy} + \frac{dV}{dx}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where $U = \frac{1}{2c}\int_{-c}^{+c} u\,dz$, $V = \frac{1}{2c}\int_{-c}^{+c} v\,dz$ are the *mean* displacements in the plane of xy taken across the breadth of the beam for any point (x, y). They will be referred to as the mean displacements. Besides these there is a variable $(w_{+c} - w_{-c})/2c$ which has to be eliminated somehow.

One way of doing this is by integrating (10) in the same way. We obtain

$$\frac{1}{2c}\int_{-c}^{+c}\widehat{zz}\,dz = \lambda\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + (\lambda + 2\mu)\left(\frac{w_{+c} - w_{-c}}{2c}\right).$$

Now if, as explained in the last section, $\widehat{zz}$ may be treated as small, so that its mean value across the breadth of the beam may be neglected, we have

$$\frac{w_{+c} - w_{-c}}{2c} = -\frac{\lambda}{\lambda + 2\mu}\left(\frac{dU}{dx} + \frac{dV}{dy}\right).$$

Substituting for $(w_{+c} - w_{-c})/2c$, the equations for P and Q become

$$P = \lambda'\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dU}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$Q = \lambda'\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + 2\mu\frac{dV}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

where $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$. Putting these into (6) and (7), we have

$$(\lambda' + \mu)\frac{d}{dx}\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + \mu\nabla^2 U = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$(\lambda' + \mu)\frac{d}{dy}\left(\frac{dU}{dx} + \frac{dV}{dy}\right) + \mu\nabla^2 V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

(15), (16), (17), and (18) are precisely of the same form as the stress-strain relations and the body equations of equilibrium for two-dimensional elastic strain, with the exception that λ' is written for λ. They will in fact be found to be identical with the equations satisfied by the displacements of an elastic plate under thrust in its own plane, as obviously they should be, since, when the beam is made indefinitely thin, the mean displacements U, V coincide with the actual displacements u, v.

§ 4. *General Solution of the Equations in Arbitrary Functions.*

If we write $\frac{dU}{dx} + \frac{dV}{dy} = \delta$, $x + iy = \xi$, $x - iy = \eta$, where $i = \sqrt{-1}$, so that $\frac{d}{dx} + i\frac{d}{dy} = 2\frac{d}{d\eta}$ and $\frac{d}{dx} - i\frac{d}{dy} = 2\frac{d}{d\xi}$, multiply (18) by i and add to (17), we find

$$2(\lambda' + \mu)\frac{d\delta}{d\eta} + \mu\nabla^2(U + iV) = 0.$$

Multiply (18) by i and subtract from (17)

$$2(\lambda' + \mu)\frac{d\delta}{d\xi} + \mu\nabla^2(U - iV) = 0.$$

But $\nabla^2 = 4\frac{d^2}{d\xi\, d\eta}$, and if $T = U + iV$, $W = U - iV$, then

$$\delta = \frac{dU}{dx} + \frac{dV}{dy} = \frac{d}{d\xi}(U + iV) + \frac{d}{d\eta}(U - iV)$$

$$= \frac{dT}{d\xi} + \frac{dW}{d\eta}.$$

Hence

$$(\lambda' + \mu)\frac{d}{d\eta}\left(\frac{dT}{d\xi} + \frac{dW}{d\eta}\right) + 2\mu\frac{d^2T}{d\xi\, d\eta} = 0,$$

$$(\lambda' + \mu)\frac{d}{d\xi}\left(\frac{dT}{d\xi} + \frac{dW}{d\eta}\right) + 2\mu\frac{d^2W}{d\xi\, d\eta} = 0.$$

From these, by simple integration,

$$\left.\begin{aligned}(\lambda' + 3\mu)\frac{d\mathrm{T}}{d\xi} + (\lambda' + \mu)\frac{d\mathrm{W}}{d\eta} &= \phi'(\xi)\\ (\lambda' + \mu)\frac{d\mathrm{T}}{d\xi} + (\lambda' + 3\mu)\frac{d\mathrm{W}}{d\eta} &= \chi'(\eta)\end{aligned}\right\}\text{ where } \phi'(\xi),\ \chi'(\eta) \text{ are arbitrary functions,}$$

whence

$$\frac{d\mathrm{T}}{d\xi} = \frac{1}{4\mu}\{\phi'(\xi) - \chi'(\eta)\} + \frac{1}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\},$$

$$\frac{d\mathrm{W}}{d\eta} = \frac{1}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\} - \frac{1}{4\mu}\{\phi'(\xi) - \chi'(\eta)\},$$

or

$$\mathrm{T} = \frac{1}{4\mu}[\phi(\xi) - \xi\chi'(\eta)] + \frac{1}{4(\lambda' + 2\mu)}[\phi(\xi) + \xi\chi'(\eta)] + \mathrm{F}(\eta) \quad . \quad . \quad (19),$$

$$\mathrm{W} = \frac{1}{4(\lambda' + 2\mu)}[\eta\phi'(\xi) + \chi(\eta)] - \frac{1}{4\mu}[\eta\phi'(\xi) - \chi(\eta)] + \mathrm{G}(\xi) \quad . \quad . \quad (20),$$

where $\mathrm{F}(\eta)$, $\mathrm{G}(\xi)$ are again arbitrary functions.

Hence U and V can be found almost immediately. Writing

$$\mathrm{F}(\eta) - \frac{1}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}\eta\chi'(\eta) = \mathrm{F}_1(\eta),$$

$$\mathrm{G}(\xi) - \frac{1}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}\xi\phi'(\xi) = \mathrm{G}_1(\xi),$$

we have

$$\mathrm{U} = \frac{\lambda' + 3\mu}{8\mu(\lambda' + 2\mu)}\{\phi(\xi) + \chi(\eta)\} + \frac{i}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}y(\phi'(\xi) - \chi'(\eta)) + \tfrac{1}{2}(\mathrm{F}_1(\eta) + \mathrm{G}_1(\xi)) \ . \ (21),$$

$$\mathrm{V} = \frac{i(\lambda' + 3\mu)}{8\mu(\lambda' + 2\mu)}\{\chi(\eta) - \phi(\xi)\} - \frac{1}{4\mu}\frac{\lambda' + \mu}{\lambda' + 2\mu}y(\chi'(\eta) + \phi'(\xi)) + \frac{i}{2}\{\mathrm{G}_1(\xi) - \mathrm{F}_1(\eta)\} \ . \ (22),$$

from which we obtain easily

$$\mathrm{P} = \frac{3(\lambda' + \mu)}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\} + \frac{\lambda' + \mu}{2(\lambda' + 2\mu)}iy\{\phi''(\xi) - \chi''(\eta)\} + \mu\mathrm{G}'_1(\xi) + \mu\mathrm{F}'_1(\eta),$$

$$\mathrm{Q} = \frac{\lambda' + \mu}{4(\lambda' + 2\mu)}\{\phi'(\xi) + \chi'(\eta)\} - \frac{\lambda' + \mu}{2(\lambda' + 2\mu)}iy(\phi''(\xi) - \chi''(\eta)) - \mu\mathrm{G}'_1(\xi) - \mu\mathrm{F}'_1(\eta),$$

$$\mathrm{S} = -\frac{(\lambda' + \mu)}{2(\lambda' + 2\mu)}y\{\phi''(\xi) + \chi''(\eta)\} + \tfrac{1}{4}\frac{\lambda' + \mu}{\lambda' + 2\mu}i(\phi'(\xi) - \chi'(\eta)) + \mu i\{\mathrm{G}'_1(\xi) - \mathrm{F}_1'(\eta)\},$$

and these last may be put into the simpler form

$$\mathrm{P} = \left(\frac{3}{4}\frac{\lambda' + \mu}{\lambda' + 2\mu}\frac{d}{dx} + \frac{\lambda' + \mu}{2(\lambda' + 2\mu)}y\frac{d^2}{dx\,dy}\right)\{\phi(\xi) + \chi(\eta)\} + \mu\frac{d}{dx}\{\mathrm{G}_1(\xi) + \mathrm{F}_1(\eta)\} \ . \ (23),$$

$$Q = \left(\tfrac{1}{4}\frac{\lambda'+\mu}{\lambda'+2\mu}\frac{d}{dx} - \frac{\lambda'+\mu}{2(\lambda'+2\mu)}y\frac{d^2}{dxdy}\right)\{\phi(\xi)+\chi(\eta)\} - \mu\frac{d}{dx}\{G_1(\xi)+F_1(\eta)\} \quad . \quad (24),$$

$$S = \left(-\tfrac{1}{2}\frac{\lambda'+\mu}{\lambda'+2\mu}y\frac{d^2}{dx^2} + \tfrac{1}{4}\frac{\lambda'+\mu}{\lambda'+2\mu}\frac{d}{dy}\right)\{\phi(\xi)+\chi(\eta)\} + \mu\frac{d}{dy}\{G_1(\xi)+F_1(\eta)\} \quad . \quad (25),$$

which have the advantage of not containing imaginaries if $\phi(\xi)+\chi(\eta)$, $G_1(\xi)+F_1(\eta)$ are real.

§ 5. *Solution involving Hyperbolic and Circular Functions.*

Assume now for the arbitrary functions the following typical forms

$$\phi(\xi) = A\sin m\xi + iB\cos m\xi + E\cos m\xi + iF\sin m\xi,$$

$$\chi(\eta) = A\sin m\eta - iB\cos m\eta + E\cos m\eta - iF\sin m\eta,$$

$$G_1(\xi) = C\sin m\xi + iD\cos m\xi + G\cos m\xi + iH\sin m\xi,$$

$$F_1(\eta) = C\sin m\eta - iD\cos m\eta + G\cos m\eta - iH\sin m\eta,$$

so that

$$\phi(\xi)+\chi(\eta) = 2\sin mx\,(A\cosh my + B\sinh my) + 2\cos mx\,(E\cosh my - F\sinh my),$$

$$\phi(\xi)-\chi(\eta) = 2i\cos mx\,(A\sinh my + B\cosh my) - 2i\sin mx\,(E\sinh my - F\cosh my),$$

$$G_1(\xi)+F_1(\eta) = 2\sin mx\,(C\cosh my + D\sinh my) + 2\cos mx\,(G\cosh my - H\sinh my),$$

$$G_1(\xi)-F_1(\eta) = 2i\cos mx\,(C\sinh my + D\cosh my) - 2i\sin mx\,(G\sinh my - H\cosh my).$$

Whence from (23), (24), (25) we get after some reductions

$$P = \cos mx\begin{Bmatrix}(3A'+C')\cosh my + (3B'+D')\sin my\\ +2my\,(A'\sinh my + B'\cosh my)\end{Bmatrix} + \sin mx\begin{Bmatrix}-(3E'+G')\cosh my + (3F'+H')\sinh my\\ +2my\,(-E'\sinh my + F'\cosh my)\end{Bmatrix} \quad . \quad . \quad (26),$$

$$Q = \cos mx\begin{Bmatrix}(A'-C')\cosh my + (B'-D')\sinh my\\ -2my\,(A'\sinh my + B'\cosh my)\end{Bmatrix} + \sin mx\begin{Bmatrix}-(E'-G')\cosh my + (F'-H')\sinh my\\ -2my\,(-E'\sinh my + F'\cosh my)\end{Bmatrix} \quad . \quad . \quad (27),$$

$$S = \sin mx \left\{ \begin{array}{c} (A' + C') \sinh my + (B' + D') \cosh my \\ + 2my\,(A' \cosh my + B' \sinh my) \end{array} \right\}$$

$$+ \cos mx \left\{ \begin{array}{c} (E' + G') \sinh my - (H' + F') \cosh my \\ + 2my\,(E' \cosh my - F' \sinh my) \end{array} \right\} \quad . \quad . \quad . \quad (28),$$

where $A' = \frac{m}{2}\,\frac{\lambda' + \mu}{\lambda' + 2\mu}A$, $B' = \frac{m}{2}\,\frac{\lambda' + \mu}{\lambda' + 2\mu}B$, $E' = \frac{m}{2}\,\frac{\lambda' + \mu}{\lambda' + 2\mu}E$, $F' = \frac{m}{2}\,\frac{\lambda' + \mu}{\lambda' + 2\mu}F$, $C' = 2\mu mC$, $D' = 2\mu mD$, $G' = 2\mu mG$, $H' = 2\mu mH$, and the expressions for the mean displacements come out to be

$$U = \sin mx \left[\begin{array}{l} \frac{1}{2m\mu}\left\{\frac{\lambda' + 3\mu}{\lambda' + \mu}(A' \cosh my + B' \sinh my) + C' \cosh my + D' \sinh my\right\} \\ \qquad + \frac{1}{\mu}\,y\,(A' \sinh my + B' \cosh my) \end{array} \right]$$

$$+ \cos mx \left[\begin{array}{l} \frac{1}{2m\mu}\left\{\frac{\lambda' + 3\mu}{\lambda' + \mu}(E' \cosh my - F' \sinh my) + G' \cosh my - H' \sinh my\right\} \\ \qquad + \frac{y}{\mu}\,(E' \sinh my - F' \cosh my) \end{array} \right] \quad (29).$$

$$V = \cos mx \left[\begin{array}{l} \frac{1}{2m\mu}\left\{\frac{\lambda' + 3\mu}{\lambda' + \mu}(A' \sinh my + B' \cosh my) - C' \sinh my - D' \cosh my\right\} \\ \qquad - \frac{y}{\mu}\,(A' \cosh my + B' \sinh my) \end{array} \right]$$

$$+ \sin mx \left[\begin{array}{l} \frac{1}{2m\mu}\left\{\frac{\lambda' + 3\mu}{\lambda' + \mu}(-E' \sinh my + F' \cosh my) + G' \sinh my - H' \cosh my\right\} \\ \qquad + \frac{y}{\mu}\,(E' \cosh my - F' \sinh my) \end{array} \right] (30).$$

§ 6. *Determination of the Arbitrary Constants from the Stress Conditions over the Faces $y = \pm b$.*

We shall suppose that the mean stresses Q and S are given arbitrarily over the top and bottom surfaces $y = \pm b$. Expanding these in Fourier series, we have, say:

$$\left. \begin{array}{l} [Q]_{y=+b} = \alpha_0 + \Sigma\alpha_n \cos mx + \Sigma\gamma_n \sin mx \\ [Q]_{y=-b} = \beta_0 + \Sigma\beta_n \cos mx + \Sigma\delta_n \sin mx \\ [S]_{y=+b} = \zeta_0 + \Sigma\zeta_n \cos mx + \Sigma\kappa_n \sin mx \\ [S]_{y=-b} = \theta_0 + \Sigma\theta_n \cos mx + \Sigma\nu_n \sin mx \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

where α_n, β_n, γ_n, δ_n, ζ_n, θ_n, κ_n, ν_n are known constants, and $m = n\pi/a$ where n is any positive integer.

Now, if we take expressions (27) and (28) and equate them, for $y = \pm b$, to the expressions (31), we obtain eight typical equations for the constants which, when combined in pairs, may be written in the simpler form :

$$\left.\begin{aligned}(A' - C') \cosh mb - 2mb\, A' \sinh mb &= \frac{\alpha_n + \beta_n}{2} \\ (A' + C') \sinh mb + 2mb\, A' \cosh mb &= \frac{\kappa_n - \nu_n}{2}\end{aligned}\right\} \quad \ldots \quad (32),$$

$$\left.\begin{aligned}(B' - D') \sinh mb - 2mb\, B' \cosh mb &= \frac{\alpha_n - \beta_n}{2} \\ (B' + D') \cosh mb + 2mb\, B' \sinh mb &= \frac{\kappa_n + \nu_n}{2}\end{aligned}\right\} \quad \ldots \quad (33),$$

$$\left.\begin{aligned}(E' - G') \cosh mb - 2mb\, E' \sinh mb &= -\frac{\gamma_n + \delta_n}{2} \\ (E' + G') \sinh mb + 2mb\, E' \cosh mb &= \frac{\zeta_n - \theta_n}{2}\end{aligned}\right\} \quad \ldots \quad (34),$$

$$\left.\begin{aligned}(F' - H') \sinh mb - 2mb\, F' \cosh mb &= \frac{\gamma_n - \delta_n}{2} \\ (F' + H') \cosh mb + 2mb\, F' \sinh mb &= -\frac{\zeta_n + \theta_n}{2}\end{aligned}\right\} \quad \ldots \quad (35).$$

These equations solve in pairs. We find easily

$$A' = \frac{\alpha_n + \beta_n}{2} \frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\kappa_n - \nu_n}{2} \frac{\cosh mb}{\sinh 2mb + 2mb} \quad \ldots \quad (36),$$

$$C' = -\frac{\alpha_n + \beta_n}{2} \frac{\sinh mb + 2mb \cosh mb}{\sinh 2mb + 2mb} + \frac{\kappa_n - \nu_n}{2} \frac{\cosh mb - 2mb \sinh mb}{\sinh 2mb + 2mb} \quad . \quad (37),$$

$$B' = \frac{\alpha_n - \beta_n}{2} \frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\kappa_n + \nu_n}{2} \frac{\sinh mb}{\sinh 2mb - 2mb} \quad \ldots \quad (38),$$

$$D' = -\frac{\alpha_n - \beta_n}{2} \frac{\cosh mb + 2mb \sinh mb}{\sinh 2mb - 2mb} + \frac{\kappa_n + \nu_n}{2} \frac{\sinh mb - 2mb \cosh mb}{\sinh 2mb - 2mb} \quad . \quad (39),$$

$$E' = -\frac{\gamma_n + \delta_n}{2} \frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\zeta_n - \theta_n}{2} \frac{\cosh mb}{\sinh 2mb + 2mb} \quad \ldots \quad (40),$$

$$G' = \frac{\gamma_n + \delta_n}{2} \frac{\sinh mb + 2mb \cosh mb}{\sinh 2mb + 2mb} + \frac{\zeta_n - \theta_n}{2} \frac{\cosh mb - 2mb \sinh mb}{\sinh 2mb + 2mb} \quad . \quad . \quad (41),$$

$$F' = \frac{\gamma_n - \delta_n}{2} \frac{\cosh mb}{\sinh 2mb - 2mb} - \frac{\zeta_n + \theta_n}{2} \frac{\sinh mb}{\sinh 2mb - 2mb} \quad \ldots \quad (42),$$

$$H' = -\frac{\gamma_n - \delta_n}{2} \frac{\cosh mb + 2mb \sinh mb}{\sinh 2mb - 2mb} - \frac{\zeta_n + \theta_n}{2} \frac{\sinh mb - 2mb \cosh mb}{\sinh 2mb - 2mb} \quad . \quad (43),$$

where in the above n corresponds to a positive integer.

The case where $n = 0$ has to be investigated separately.

§ 7. *Expressions for the Displacements and Stresses.*

Substituting the values of the constants found above into the equations (26)–(30), we obtain for the mean displacements U, V and for the mean stresses, P, Q, S the following values, in so far as we merely consider the terms corresponding to $n =$ a positive integer :—

$$U = \sum_1^\infty \frac{\left[\begin{array}{l}(\alpha_n + \beta_n)\left\{\dfrac{1}{\lambda' + \mu}\sinh mb - \dfrac{1}{\mu} mb \cosh mb\right\} \\ \quad + (\kappa_n - \nu_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb - \dfrac{1}{\mu} mb \sinh mb\right\}\end{array}\right]}{2m(\sinh 2mb + 2mb)} \cosh my \sin mx$$

$$+ \sum_1^\infty \frac{\left[\begin{array}{l}(\alpha_n - \beta_n)\left\{\dfrac{1}{\lambda' + \mu}\cosh mb - \dfrac{1}{\mu} mb \sinh mb\right\} \\ \quad + (\kappa_n + \nu_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb - \dfrac{1}{\mu} mb \cosh mb\right\}\end{array}\right]}{2m(\sinh 2mb - 2mb)} \sinh my \sin mx$$

$$+ \sum_1^\infty \left\{\frac{\alpha_n + \beta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\kappa_n - \nu_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb + 2mb}\right\} y \sinh my \sin mx$$

$$+ \sum_1^\infty \left\{\frac{\alpha_n - \beta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\kappa_n + \nu_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb - 2mb}\right\} y \cosh my \sin mx$$

$$+ \sum_1^\infty \frac{\left[\begin{array}{l}-(\gamma_n + \delta_n)\left\{\dfrac{1}{\lambda' + \mu}\sinh mb - \dfrac{1}{\mu} mb \cosh mb\right\} \\ \quad + (\zeta_n - \theta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb - \dfrac{1}{\mu} mb \sinh mb\right\}\end{array}\right]}{2m(\sinh 2mb + 2mb)} \cosh my \cos mx$$

$$+ \sum_1^\infty \frac{\left[\begin{array}{l}-(\gamma_n - \delta_n)\left\{\dfrac{1}{\lambda' + \mu}\cosh mb - \dfrac{1}{\mu} mb \sinh mb\right\} \\ \quad + (\zeta_n + \theta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb - \dfrac{1}{\mu} mb \cosh mb\right\}\end{array}\right]}{2m(\sinh 2mb - 2mb)} \sinh my \cos mx$$

$$+ \sum_1^\infty \left\{\frac{-(\gamma_n + \delta_n)}{2\mu}\,\frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{(\zeta_n - \theta_n)}{2\mu}\,\frac{\cosh mb}{\sinh 2mb + 2mb}\right\} y \sinh my \cos mx$$

$$+ \sum_1^\infty \left\{\frac{-(\gamma_n - \delta_n)}{2\mu}\,\frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{(\zeta_n + \theta_n)}{2\mu}\,\frac{\sinh mb}{\sinh 2mb - 2mb}\right\} y \cosh my \cos mx \qquad (44).$$

$$V = \sum_1^\infty \frac{\left[\begin{array}{l}(\alpha_n + \beta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb + \dfrac{mb}{\mu}\cosh mb\right\} \\ \quad + (\kappa_n - \nu_n)\left\{\dfrac{1}{\lambda' + \mu}\cosh mb + \dfrac{mb}{\mu}\sinh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb + 2mb)} \sinh my \cos mx$$

$$+ \sum_1^\infty \frac{\left[\begin{array}{l}(\alpha_n - \beta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb + \dfrac{mb}{\mu}\sinh mb\right\} \\ \quad + (\kappa_n + \nu_n)\left\{\dfrac{1}{\lambda' + \mu}\sinh mb + \dfrac{mb}{\mu}\cosh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb - 2mb)} \cosh my \cos mx$$

$$- \sum_1^\infty \left\{\frac{\alpha_n + \beta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\kappa_n - \nu_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb + 2mb}\right\} y \cosh my \cos mx$$

$$- \sum_1^\infty \left\{\frac{\alpha_n - \beta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\kappa_n + \nu_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb - 2mb}\right\} y \sinh my \cos mx$$

$$+ \sum_1^\infty \frac{\left[\begin{array}{l}(\gamma_n + \delta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb + \dfrac{mb}{\mu}\cosh mb\right\} \\ \quad - (\zeta_n - \theta_n)\left\{\dfrac{\cosh mb}{\lambda' + \mu} + \dfrac{mb}{\mu}\sinh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb + 2mb)} \sinh my \sin mx$$

$$+ \sum_1^\infty \frac{\left[\begin{array}{l}(\gamma_n - \delta_n)\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb + \dfrac{mb}{\mu}\sinh mb\right\} \\ \quad - (\zeta_n + \theta_n)\left\{\dfrac{\sinh mb}{\lambda' + \mu} + \dfrac{mb}{\mu}\cosh mb\right\}\end{array}\right]}{2m\,(\sinh 2mb - 2mb)} \cosh my \sinh mx$$

$$+ \sum_1^\infty \left\{-\frac{\gamma_n + \delta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb + 2mb} + \frac{\zeta_n - \theta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb + 2mb}\right\} y \cosh my \sin mx$$

$$+ \sum_1^\infty \left\{-\frac{\gamma_n - \delta_n}{2\mu}\,\frac{\cosh mb}{\sinh 2mb - 2mb} + \frac{\zeta_n + \theta_n}{2\mu}\,\frac{\sinh mb}{\sinh 2mb - 2mb}\right\} y \sinh my \sin mx \qquad (45).$$

$$P = \sum_1^\infty \frac{(\alpha_n + \beta_n)(\sinh mb - mb\cosh mb) + (\kappa_n - \nu_n)(2\cosh mb - mb\sinh mb)}{\sinh 2mb + 2mb} \cosh my \cos mx$$

$$+ \sum_1^\infty \frac{(\alpha_n - \beta_n)(\cosh mb - mb\sinh mb) + (\kappa_n + \nu_n)(2\sinh mb - mb\cosh mb)}{\sinh 2mb - 2mb} \sinh my \cos mx$$

$$+ \sum_1^\infty \frac{(\alpha_n + \beta_n)\sinh mb + (\kappa_n - \nu_n)\cosh mb}{\sinh 2mb + 2mb}\, my \sinh my \cos mx$$

$$+ \sum_1^\infty \frac{(\alpha_n - \beta_n)\cosh mb + (\kappa_n + \nu_n)\sinh mb}{\sinh 2mb - 2mb}\, my \cosh my \cos mx$$

$$+ \sum_1^\infty \frac{(\gamma_n + \delta_n)(\sinh mb - mb\cosh mb) - (\zeta_n - \theta_n)(2\cosh mb - mb\sinh mb)}{\sinh 2mb + 2mb} \cosh my \sin mx$$

$$+\sum_1^\infty \frac{(\gamma_n - \delta_n)(\cosh mb - mb \sinh mb) - (\zeta_n + \theta_n)(2 \sinh mb - mb \cosh mb)}{\sinh 2mb - 2mb} \sinh my \sin mx$$

$$+\sum_1^\infty \frac{(\gamma_n + \delta_n) \sinh mb - (\zeta_n - \theta_n) \cosh mb}{\sinh 2mb + 2mb} my \sinh my \sin mx$$

$$+\sum_1^\infty \frac{(\gamma_n - \delta_n) \cosh mb - (\zeta_n + \theta_n) \sinh mb}{\sinh 2mb - 2mb} my \cosh my \sin mx \quad . \quad . \quad . \quad . \quad (46).$$

$$.Q = \sum_1^\infty \frac{(\alpha_n + \beta_n)(\sinh mb + mb \cosh mb) + (\kappa_n - \nu_n) mb \sinh mb}{\sinh 2mb + 2mb} \cosh my \cos mx$$

$$+\sum_1^\infty \frac{(\alpha_n - \beta_n)(\cosh mb + mb \sinh mb) + (\kappa_n + \nu_n) mb \cosh mb}{\sinh 2mb - 2mb} \sinh my \cos mx$$

$$-\sum_1^\infty \frac{(\alpha_n + \beta_n) \sinh mb + (\kappa_n - \nu_n) \cosh mb}{\sinh 2mb + 2mb} my \sinh my \cos mx$$

$$-\sum_1^\infty \frac{(\alpha_n - \beta_n) \cosh mb + (\kappa_n + \nu_n) \sinh mb}{\sinh 2mb - 2mb} my \cosh my \cos mx$$

$$+\sum_1^\infty \frac{(\gamma_n + \delta_n)(\sinh mb + mb \cosh mb) - (\zeta_n - \theta_n) mb \sinh mb}{\sinh 2mb + 2mb} \cosh my \sin mx$$

$$+\sum_1^\infty \frac{(\gamma_n - \delta_n)(\cosh mb + mb \sinh mb) - (\zeta_n + \theta_n) mb \cosh mb}{\sinh 2mb - 2mb} \sinh my \sin mx$$

$$-\sum_1^\infty \frac{(\gamma_n + \delta_n) \sinh mb - (\zeta_n - \theta_n) \cosh mb}{\sinh 2mb + 2mb} my \sinh my \sin mx$$

$$-\sum_1^\infty \frac{(\gamma_n - \delta_n) \cosh mb - (\zeta_n + \theta_n) \sinh mb}{\sinh 2mb - 2mb} my \cosh my \sin mx \quad . \quad . \quad . \quad . \quad (47).$$

$$S = \sum_1^\infty \frac{-(\alpha_n + \beta_n) mb \cosh mb + (\kappa_n - \nu_n)(\cosh mb - mb \sinh mb)}{\sinh 2mb + 2mb} \sinh my \sin mx$$

$$+\sum_1^\infty \frac{-(\alpha_n - \beta_n) mb \sinh mb + (\kappa_n + \nu_n)(\sinh mb - mb \cosh mb)}{\sinh 2mb - 2mb} \cosh my \sin mx$$

$$+\sum_1^\infty \frac{(\alpha_n + \beta_n) \sinh mb + (\kappa_n - \nu_n) \cosh mb}{\sinh 2mb + 2mb} my \cosh my \sin mx$$

$$+\sum_1^\infty \frac{(\alpha_n - \beta_n) \cosh mb + (\kappa_n + \nu_n) \sinh mb}{\sinh 2mb - 2mb} my \sinh my \sin mx$$

$$+\sum_1^\infty \frac{(\gamma_n + \delta_n) mb \cosh mb + (\zeta_n - \theta_n)(\cosh mb - mb \sinh mb)}{\sinh 2mb + 2mb} \sinh my \cos mx$$

$$+\sum_1^\infty \frac{(\gamma_n - \delta_n) mb \sinh mb + (\zeta_n + \theta_n)(\sinh mb - mb \cosh mb)}{\sinh 2mb - 2mb} \cosh my \cos mx$$

$$+\sum_1^\infty \frac{-(\gamma_n + \delta_n) \sinh mb + (\zeta_n - \theta_n) \cosh mb}{\sinh 2mb + 2mb} my \cosh my \cos mx$$

$$+\sum_1^\infty \frac{-(\gamma_n - \delta_n) \cosh mb + (\zeta_n + \theta_n) \sinh mb}{\sinh 2mb - 2mb} my \sinh my \cos mx \quad . \quad . \quad . \quad . \quad (48).$$

§ 8. *Conditions at the Two Ends* $x = \pm a$.

It is, however, impossible to satisfy fully the conditions over the two ends $x = \pm a$. These would require that P and S should have given values over these ends. If, however, a is so large that, at a long distance from the ends, the effect of any self-equilibrating system of stress over these same ends may be neglected, then we need only consider *total* terminal conditions at $x = \pm a$.

These conditions will involve

(i.) The total tension $\text{T} = \int_{-b}^{b} \text{P}\, dy$ across either end.

(ii.) The total shear $\bar{\text{S}} = \int_{-b}^{b} \text{S}\, dy$ across either end.

(iii.) The bending moment $\text{M} = -\int_{-b}^{b} \text{P}y\, dy$ across either end.

I now propose to calculate the quantities T, $\bar{\text{S}}$ and M for that part of the solution which has been given in the last section.

I find, after reduction,

$$(\text{T})_{+a} = (\text{T})_{-a} = \sum_{1}^{\infty} \frac{\cos ma}{m} (\kappa_n - \nu_n) \quad \text{(49)}.$$

$$(\bar{\text{S}})_{a} = (\bar{\text{S}})_{-a} = \sum_{1}^{\infty} (\gamma_n - \delta_n) \frac{\cos ma}{m} \quad \text{(50)}.$$

$$-(\text{M})_{+a} = -(\text{M})_{-a} = \sum_{1}^{\infty} \frac{(\alpha_n - \beta_n)}{m^2} \cos ma + \sum_{1}^{\infty} \frac{b}{m} (\kappa_n + \nu_n) \cos ma \quad \text{(51)}.$$

Now we can always adjust M and T so as to be zero, for the solutions for a uniform tension and a uniform bending moment, viz.:—

$$\left.\begin{aligned} \text{U} &= \frac{\text{T}x}{2b\text{E}} - \frac{3\text{M}xy}{2b^3\text{E}} \\ \text{V} &= \frac{\eta \text{T}y}{2b\text{E}} + \frac{3\text{M}}{2b^3\text{E}} \frac{x^2 - \eta y^2}{2} \end{aligned}\right\} \quad \text{(52)}$$

(where $\eta = -\frac{1}{2}\lambda/(\lambda + \mu)$ and E is YOUNG'S Modulus), produce no stress across the faces $y = \pm b$, and therefore such solutions can always be arbitrarily superimposed. They correspond to stresses which are *transmitted* from the ends; and we shall find that it is necessary, in various cases, to add such solutions in order to satisfy the end conditions, which are not necessarily satisfied by the series merely involving circular functions.

§ 9. *Part of the Solutions Corresponding to the Terms* α_0, β_0, ζ_0, θ_0.

In the first place it is obvious, having regard to the conditions of rigid equilibrium, that if the ends $x = \pm a$ are free from stress, then α_0 must $= \beta_0$. If $\alpha_0 \neq \beta_0$ we must have a shear over the two ends in order to balance the excess of the pressure on the one side over the pressure on the other side, and this will require special investigation. The solution arising from such conditions is discussed in §§ 39–40. For the present let us confine ourselves to $\alpha_0 = \beta_0$. This corresponds to a uniform traction along the axis y and introduces the following additional terms :—

$$\left.\begin{array}{lll} U = \dfrac{\eta \alpha_0 x}{E}, & & V = \dfrac{\alpha_0 y}{E} \\ P = 0, & Q = \alpha_0, & S = 0 \end{array}\right\} \quad \ldots \ldots \ldots \quad (53).$$

Now turning to the terms in ζ_0 and θ_0, it is easy to verify that the additional terms

$$\left.\begin{array}{l} U = \dfrac{-(\lambda' + 2\mu)}{16\mu(\lambda' + \mu)}(\zeta_0 - \theta_0)\dfrac{x^2}{b} + \dfrac{3\lambda' + 4\mu}{16\mu(\lambda' + \mu)}(\zeta_0 - \theta_0)\dfrac{y^2}{b} \\ V = \dfrac{\lambda'}{8\mu(\lambda' + \mu)}(\zeta_0 - \theta_0)\dfrac{xy}{b} \\ \text{and therefore} \\ Q = 0, \qquad P = -\dfrac{\zeta_0 - \theta_0}{2b}x, \qquad S = \dfrac{\zeta_0 - \theta_0}{2b}y \end{array}\right\} \quad \ldots \quad (54),$$

satisfy the conditions that S shall have constant values over the two boundaries $y = \pm b$, these values being equal in magnitude and opposite in sign. The effect of these shears is balanced by the pressure and tension $(\zeta_0 - \theta_0)\,a/2b$ over the two ends, and the conditions of rigid equilibrium are satisfied.

Finally, if we have equal shears over the boundaries, the sign being the same (so that the external impressed forces act in *opposite* directions), the solution

$$\left.\begin{array}{lll} U = \dfrac{1}{4\mu}(\zeta_0 + \theta_0)\,y, & & V = \dfrac{1}{4\mu}(\zeta_0 + \theta_0)\,x \\ P = 0, & Q = 0, & S = \tfrac{1}{2}(\zeta_0 + \theta_0) \end{array}\right\} \quad \ldots \ldots \quad (55)$$

will satisfy all conditions over the boundaries $y = \pm b$, and will introduce over the boundaries $x = \pm a$ a system of shear necessary to maintain rigid equilibrium.

Adding together the solutions (54) and (55), we find that the conditions $Q = 0$ over $y = \pm b$, $S = \zeta_0$ over $y = + b$, $S = \theta_0$ over $y = - b$ are all satisfied.

This completes the solution of the problem proposed, with the exception of the case $\alpha_0 \neq \beta_0$, which can be reduced to the problem of a beam uniformly loaded along the top and free along the bottom, the load being taken by shears over the ends.

PART II.

Discussion of the General Solution when the Forces on the Beam are Purely Normal and are Symmetrical about $x = 0$.

§ 10. *Expressions for the Stresses and Displacements.*

If the forces are purely normal, and if the solution is to be even in x, then the γ, δ, ζ, θ, κ, ν terms disappear.

Further, we have the additional condition that, over the ends $x = \pm a$, $\mathrm{T} = 0$, $\mathrm{S} = 0$, $\mathrm{M} = 0$; by introducing suitable terms of the form (52) we can satisfy this last condition, and we finally obtain

$$\left.\begin{aligned}
\mathrm{U} &= \frac{\eta\alpha_0 x}{\mathrm{E}} - \frac{3xy}{2b^3\mathrm{E}}\sum_1^\infty(\alpha_n - \beta_n)\frac{\cos ma}{m^2} \\
&+ \sum_1^\infty \frac{(\alpha_n + \beta_n)}{2m}\frac{\left(\dfrac{1}{\lambda' + \mu}\sinh mb - \dfrac{1}{\mu} mb\cosh mb\right)}{\sinh 2mb + 2mb}\cosh my \sin mx \\
&+ \sum_1^\infty \frac{(\alpha_n - \beta_n)}{2m}\frac{\left(\dfrac{1}{\lambda' + \mu}\cosh mb - \dfrac{1}{\mu} mb\sinh mb\right)}{\sinh 2mb - 2mb}\sinh my \sin mx \\
&+ \sum_1^\infty \frac{(\alpha_n + \beta_n)}{2\mu}\frac{y\sinh mb\sinh my\sin mx}{\sinh 2mb + 2mb} + \sum_1^\infty \frac{(\alpha_n - \beta_n)}{2\mu}\frac{y\cosh mb\cosh my\sin mx}{\sinh 2mb - 2mb} \\
\mathrm{V} &= \frac{\alpha_0 y}{\mathrm{E}} + \frac{3}{2b^3\mathrm{E}}\left(\frac{x^2 - \eta y^2}{2}\right)\sum_1^\infty(\alpha_n - \beta_n)\frac{\cos ma}{m^2} + \mathrm{B} \\
&+ \sum_1^\infty\left(\frac{\alpha_n + \beta_n}{2m}\right)\frac{\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\sinh mb + \dfrac{1}{\mu} mb\cosh mb\right\}}{\sinh 2mb + 2mb}\sinh my\cos mx \\
&+ \sum_1^\infty\left(\frac{\alpha_n - \beta_n}{2m}\right)\frac{\left\{\left(\dfrac{1}{\lambda' + \mu} + \dfrac{1}{\mu}\right)\cosh mb + \dfrac{1}{\mu} mb\sinh mb\right\}}{\sinh 2mb - 2mb}\cosh my\cos mx \\
&- \sum_1^\infty\left(\frac{\alpha_n + \beta_n}{2\mu}\right)\frac{y\sinh mb\cosh my\cos mx}{\sinh 2mb + 2mb} - \sum_1^\infty\left(\frac{\alpha_n - \beta_n}{2\mu}\right)\frac{y\cosh mb\sinh my\cos mx}{\sinh 2mb - 2mb}
\end{aligned}\right\} \quad (56),$$

where B is an arbitrary constant to be determined from some condition of fixing. It merely corresponds to a total vertical displacement of the beam.

$$\left.\begin{aligned}
\mathrm{P} &= -\frac{3y}{2b^3}\sum_1^\infty(\alpha_n - \beta_n)\frac{\cos ma}{m^2} + \sum_1^\infty(\alpha_n + \beta_n)\frac{\sinh mb - mb\cosh mb}{\sinh 2mb + 2mb}\cosh my\cos mx \\
&+ \sum_1^\infty(\alpha_n - \beta_n)\frac{\cosh mb - mb\sinh mb}{\sinh 2mb - 2mb}\sinh my\cos mx \\
&+ \sum_1^\infty(\alpha_n + \beta_n)\frac{my\sinh mb\sinh my\cos mx}{\sinh 2mb + 2mb} + \sum_1^\infty(\alpha_n - \beta_n)\frac{my\cosh mb\cosh my\cos mx}{\sinh 2mb - 2mb}
\end{aligned}\right\} \quad (57),$$

$$\left.\begin{aligned}
Q &= \alpha_0 + \sum_1^\infty (\alpha_n + \beta_n) \frac{\sinh mb + mb \cosh mb}{\sinh 2mb + 2mb} \cosh my \cos mx \\
&\quad - \sum_1^\infty (\alpha_n + \beta_n) \frac{my \sinh mb \sinh my \cos mx}{\sinh 2mb + 2mb} \\
&\quad + \sum_1^\infty (\alpha_n - \beta_n) \frac{\cosh mb + mb \sinh mb}{\sinh 2mb - 2mb} \sinh my \cos mx \\
&\quad - \sum_1^\infty (\alpha_n - \beta_n) \frac{my \cosh mb \cosh my \cos mx}{\sinh 2mb - 2mb} \\
S &= - \sum_1^\infty (\alpha_n + \beta_n) \frac{mb \cosh mb \sinh my \sin mx}{\sinh 2mb + 2mb} - \sum_1^\infty (\alpha_n - \beta_n) \frac{mb \sinh mb \cosh my \sin mx}{\sinh 2mb - 2mb} \\
&\quad + \sum_1^\infty (\alpha_n + \beta_n) \frac{my \sinh mb \cosh my \sin mx}{\sinh 2mb + 2mb} + \sum_1^\infty (\alpha_n - \beta_n) \frac{my \cosh mb \sinh my \sin mx}{\sinh 2mb - 2mb}
\end{aligned}\right\} \quad (57).$$

§ 11. *Approximate Values to which the Expressions of* § 10 *lead when* "b" *is made very small.*

If b is very small compared with a, so that, even for certain fairly high values of m, mb is still small, we may expand the coefficients in (56) and (57) in powers of mb, and also we may expand $\cosh my$ and $\sinh my$ in powers of my. This is the method which has been employed by POCHHAMMER ('Crelle's Journal,' vol. 81). I have shown in a previous paper ("On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load," 'Phil. Trans.,' A, vol. 198, pp. 147–233), that such an approximation was valid provided that the original series and each of the approximate series obtained from the various terms in the expansion of the coefficients of $\cos mx$, $\sin mx$ (which expansion is supposed carried out only to a limited number of terms) are absolutely and uniformly convergent for the region considered.

Assuming that the values of α_n, β_n are such as to ensure that these conditions are satisfied, let us see what happens when, in the expressions for the displacements U and V, we neglect all terms of order greater than -1 in m.

We find

$$\begin{aligned}
U &= -\frac{3xy}{2b^3 E} \sum_1^\infty (\alpha_n - \beta_n) \frac{\cos ma}{m^2} + \sum_1^\infty \left(\frac{1}{\lambda' + \mu} - \frac{1}{\mu}\right) \frac{\alpha_n + \beta_n}{8m} \sin mx \\
&\quad + \sum_1^\infty \frac{\alpha_n - \beta_n}{2} \frac{\left\{\frac{1}{\lambda' + \mu}\left(1 + \frac{m^2 b^2}{2}\right) - \frac{1}{\mu} m^2 b^2\right\}}{\frac{4}{3} m^3 b^3 \left(1 + \frac{m^2 b^2}{5}\right)} y \left(1 + \frac{m^2 y^2}{6}\right) \sin mx \\
&\quad + \sum_1^\infty \frac{\alpha_n - \beta_n}{2\mu} \frac{\left\{1 + \frac{m^2 b^2}{2} + \frac{m^2 y^2}{2}\right\}}{\frac{4}{3} m^3 b^3 \left(1 + \frac{m^2 b^2}{5}\right)} y \sin mx
\end{aligned}$$

$$= -\frac{3xy}{2b^3\mathrm{E}}\sum_1^\infty(\alpha_n-\beta_n)\frac{\cos ma}{m^2} - \frac{\lambda'}{8\mu(\lambda'+\mu)}\sum_1^\infty\frac{\alpha_n+\beta_n}{m}\sin mx$$
$$+\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3y}{8b^3}\sum_1^\infty\frac{\alpha_n-\beta_n}{m^3}\sin mx$$
$$+\frac{3y}{8b}\left\{\frac{3\lambda'+4\mu}{6\mu(\lambda'+\mu)}\frac{y^2}{b^2}-\frac{7\lambda'+4\mu}{10\mu(\lambda'+\mu)}\right\}\sum_1^\infty\frac{\alpha_n-\beta_n}{m}\sin mx \quad . \quad . \quad . \quad . \quad . \quad (58).$$

$$\mathrm{V} = \frac{3}{2b^3\mathrm{E}}\frac{x^2-\eta y^2}{2}\sum_1^\infty(\alpha_n-\beta_n)\frac{\cos ma}{m^2} - \sum_1^\infty\frac{(\alpha_n-\beta_n)}{2\mu}\frac{y^2\cos mx}{\frac{4}{3}m^2b^3}$$
$$+\sum_1^\infty\frac{(\alpha_n-\beta_n)}{2m}\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left(1+\frac{m^2b^2}{2}\right)+\frac{1}{\mu}m^3b^2}{\frac{4}{3}m^2b^2\left(1+\frac{m^2b^2}{5}\right)}\left(1+\frac{m^2y^2}{2}\right)\cos mx$$

$$= \frac{3}{2b^3\mathrm{E}}\frac{x^2-\eta y^2}{2}\sum_1^\infty(\alpha_n-\beta_n)\frac{\cos ma}{m^2}+\frac{3}{8}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{1}{b^3}\sum_1^\infty\frac{\alpha_n-\beta_n}{m^4}\cos mx$$
$$+\frac{3}{8b}\left\{\frac{13\lambda'+16\mu}{10\mu(\lambda'+\mu)}-\frac{\lambda'}{2\mu(\lambda'+\mu)}\frac{y^2}{b^2}\right\}\sum_1^\infty\frac{(\alpha_n-\beta_n)}{m^2}\cos mx \quad . \quad . \quad . \quad . \quad . \quad (59).$$

Now $\sum_1^\infty(\alpha_n-\beta_n)\cos mx = \mathrm{L}$, where L is the difference of stress on the top and bottom, in other words, the transverse load per unit length of the beam.

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m}\sin mx = \int_0^x \mathrm{L}dx = \int_a^x \mathrm{L}dx = -\overline{\mathrm{S}},$$

where $\overline{\mathrm{S}}$ is the total shear at any section.

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos mx - \sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos ma = +\int_a^x\overline{\mathrm{S}}dx = -\mathrm{M},$$

where M is the bending moment at any section.

Integrating again:

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m^3}\sin mx - x\sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos ma = -\int_0^x\mathrm{M}dx$$

$$\sum_1^\infty\frac{\alpha_n-\beta_n}{m^4} - \sum_1^\infty\frac{\alpha_n-\beta_n}{m^4}\cos mx - \frac{x^2}{2}\sum_1^\infty\frac{\alpha_n-\beta_n}{m^2}\cos ma = -\int_0^x\left(\int_0^x\mathrm{M}dx\right)dx.$$

Also if $\overline{\mathrm{Q}}$ is the transverse tensile stress at any section $\overline{\mathrm{Q}} = \sum_1^\infty\frac{\alpha_n+\beta_n}{2}\cos mx$, and $\sum_1^\infty\frac{\alpha_n+\beta_n}{2m}\sin mx = \int_0^x\overline{\mathrm{Q}}dx.$

Substituting from the above values into the expressions (58) and (59) for U and V, we find, remembering that $\frac{1}{\lambda'+\mu}+\frac{1}{\mu}=\frac{4}{\mathrm{E}}$ and $\eta = -\lambda'/(\lambda'+2\mu)$,

$$\left.\begin{aligned} \mathrm{U} &= -\frac{3y}{2b^3\mathrm{E}}\int_0^x \mathrm{M}dx - \frac{3y}{8b}\,\bar{\mathrm{S}}\left\{\frac{3\lambda' + 4\mu}{6\mu(\lambda' + \mu)}\frac{y^2}{b^2} - \frac{7\lambda' + 4\mu}{10\mu(\lambda' + \mu)}\right\} + \frac{\eta}{\mathrm{E}}\int_0^x \bar{\mathrm{Q}}dx \\ \mathrm{V} &= \frac{3}{2b^3\mathrm{E}}\int_0^x\int_0^x \mathrm{M}dx^2 - \frac{3\mathrm{M}}{8b}\left\{\frac{13\lambda' + 16\mu}{10\mu(\lambda' + \mu)} - \frac{\lambda'}{2\mu(\lambda' + \mu)}\frac{y^2}{b^2}\right\} \end{aligned}\right\} \quad (60),$$

dropping a constant in V.

The stresses P, Q, S might be directly deduced from the equations (60) by differentiation. But here we require to be extremely careful, for, y and x being of different orders of magnitude, differentiation with regard to y will not give a term of the same order as differentiation with regard to x. The criterion to be used in this case is this: The series $\mathrm{L} = -d\bar{\mathrm{S}}/dx$ is of order 0 in m, and is therefore among the terms which we have agreed to neglect. Similarly for the series $\bar{\mathrm{Q}}$. In consequence, every time L and $\bar{\mathrm{Q}}$ appear owing to differentiation, they should be neglected if we keep the same order of approximation for the stresses as for the displacements. It will then be found that some terms disappear whose effect is felt in the displacements, as it were, by accumulation.

Keeping this rule in mind, we obtain easily

$$\left.\begin{aligned} \mathrm{P} &= -\frac{3\mathrm{M}y}{2b^3} \\ \mathrm{Q} &= 0 \\ \mathrm{S} &= \frac{3}{4b}\,\bar{\mathrm{S}}\,(b^2 - y^2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (61).$$

Now these are the stresses we should have obtained had we treated that part of the bar as *free*, but subject to a bending moment M and a total shear $\bar{\mathrm{S}}$, transmitted from a distant terminal. Hence we see that, to a first approximation the stress at each point of a bar, whatever the manner of its transverse loading, depends only upon the total bending moment at the section and upon the total shear at the section, and will be given in terms of these by the same formulæ which are valid for a *free* bar subjected to a given couple and shear at its extremities. Similar conclusions follow from the formulæ found by Professor POCHHAMMER in the paper quoted previously.

§ 12. *Analysis of the Approximate Expressions for the Displacements. Shearing Deflection.*

Now if we look at the values (60) we see easily that they are composed of three parts.

(i.) The parts $-\frac{3y}{2b^3\mathrm{E}}\int_0^x \mathrm{M}dx$ of U and $\frac{3}{2b^3\mathrm{E}}\int_0^x\int_0^x \mathrm{M}dx^2$ of V.

These are what we may call the "Euler-Bernoulli" terms. They correspond to a strain in which cross-sections originally plane remain plane, and the curvature of the elastic line is at all points proportional to the bending moment.

(ii.) The part $\int_0^x \frac{\eta \overline{Q}}{E} dx$ of U. This corresponds to the lateral contraction of the material under tensions $\overline{Q}$, and is the same as if each strip of thickness dx and height $2b$ were independently stretched.

(iii.) The terms $-\frac{3y}{8b}\overline{S}\left\{\frac{3\lambda'+4\mu}{6\mu(\lambda'+\mu)}\frac{y^2}{b^2}-\frac{7\lambda'+4\mu}{10\mu(\lambda'+\mu)}\right\}$ of U and

$$-\frac{3M}{8b}\left\{\frac{13\lambda'+16\mu}{10\mu(\lambda'+\mu)}-\frac{\lambda'}{2\mu(\lambda'+\mu)}\frac{y^2}{b^2}\right\} \text{ of V.}$$

These correspond to a distortion of the cross-sections and to a parabolic distribution of shear.

In the particular case, where the load reduces to a central isolated weight W and the two symmetrical support reactions, the additional terms (iii.) in V are of the form (omitting the constant)

$$\tfrac{3}{8}\frac{Wx}{\mu b}\left\{\frac{13\lambda'+16\mu}{20(\lambda'+\mu)}\right\}+\tfrac{3}{8}\frac{\eta W(l-x)y^2}{Eb^3} \text{ for } x>0$$

and

$$-\tfrac{3}{8}\frac{Wx}{\mu b}\left\{\frac{13\lambda'+16\mu}{20(\lambda'+\mu)}\right\}+\tfrac{3}{8}\frac{\eta W(l+x)y^2}{Eb^3} \text{ for } x<0,$$

$2l$ being the distance between the supports.

It might have been supposed that this particular problem would have been capable of solution by breaking up the beam in the middle and treating it as two inverted cantilevers, to each of which we could apply DE SAINT-VENANT'S solution. This, I believe, is often done by engineers.

Now such an attempt is, in strictness, bound to fail, because DE SAINT-VENANT'S solution implies distortion of the cross-section at the fixed end, whereas in the present problem the central cross-section of the beam must necessarily remain plane, from symmetry.

Moreover, we are left in doubt as to the condition of fixing to be adopted. Are we to suppose, with DE SAINT-VENANT, the central element of the terminal cross-section to remain vertical, or, with Professor LOVE ('Theory of Elasticity,' vol. 1, pp. 179–180), the elastic line to be horizontal at the built-in end? In the case of a cantilever the difference is quite immaterial, as it merely amounts to a rigid body displacement. But here we must remember the cantilevers are only fictitiously severed, and the above difference corresponds to an actual sharp bend of the beam in the middle.

It is interesting to compare the true solution with those obtained in this way.

If we assume DE SAINT-VENANT'S fixing condition, we find, for the additional terms in V corresponding to (iii.),

$$\tfrac{3}{8}\frac{Wx}{\mu b} + \tfrac{3}{8}\eta\frac{W(l-x)y^2}{Eb^3} \text{ for } x > 0, \text{ and}$$

$$-\tfrac{3}{8}\frac{Wx}{\mu b} + \tfrac{3}{8}\eta\frac{W(l+x)y^2}{Eb^3} \text{ for } x < 0.$$

The y^2 terms are therefore identical in this and in the true solution, but the first term which represents the additional deflection of the central axis of the beam, and which is sometimes spoken of as the shearing deflection, is less than in the true solution, being $(13\lambda' + 16\mu)/20(\lambda' + \mu)$, that is $(42\lambda + 32\mu)/(60\lambda + 40\mu)$ of that given by the double cantilever solution. This fraction comes to be ·74 for uni-constant isotropy.

If we assume what I have called LOVE'S fixing condition, the shearing deflection disappears entirely.

The true solution shows us, therefore, that it is permissible in this case to use the double cantilever as an artifice to obtain the solution, *provided* we adopt, at the section of fictitious severance, a fixing condition intermediate between those of LOVE and DE SAINT-VENANT, but nearer to the latter. In other words, a central isolated load does actually introduce a sharp bend.

§ 13. *Value of the Deflection when b is not small and the Beam is Doubly Supported.*

Suppose the beam rests on two knife-edge supports A, B (fig. i.) at a distance $2l$ apart, and a weight W is borne by another knife-edge which presses on the upper part of the beam at C.

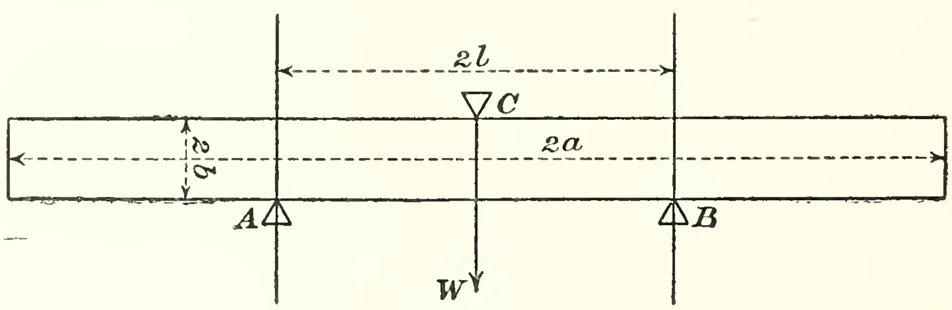

Fig. i.

Then we have $\alpha_0 = -\dfrac{W}{2a}$, $\alpha_n = -\dfrac{W}{a}$ $(n \neq 0)$, $\beta_0 = \alpha_0$, $\beta_n = -\dfrac{W}{a}\cos\dfrac{n\pi l}{a}$.

The central deflection of the elastic line (what DE SAINT-VENANT calls "la flèche de flexion") is then given by $f = V_{x=l\ y=0} - V_{x=0\ y=0}$; substituting for α's and β's in (56), we find

$$f = \frac{3}{4b^3E} l^2 \sum_1^\infty \frac{\cos n\pi}{m^2} \frac{W}{a}\left(\cos\frac{n\pi l}{a} - 1\right)$$

$$+ \sum_1^\infty \frac{W}{a}\left(\cos\frac{n\pi l}{a} - 1\right)\frac{1}{2m} \frac{\left\{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh mb + \frac{mb}{\mu}\sinh mb\right\}}{\sinh 2mb - 2mb}\left(\cos\frac{n\pi l}{a} - 1\right). \quad (62).$$

Now the first term can be evaluated. It is $\frac{3}{16}\frac{l^4}{ab^3}\frac{W}{E}$. We have therefore

$$f = \frac{3Wl^4}{16Eb^3a} + \sum_1^\infty \frac{W}{2n\pi} \frac{\left(\frac{1}{\lambda'+\mu}\cosh\frac{n\pi b}{a} + \frac{1}{\mu}\left(\cosh\frac{n\pi b}{a} + \frac{n\pi b}{a}\sinh\frac{n\pi b}{a}\right)\right)}{\left(\sinh\frac{2n\pi b}{a} - \frac{2n\pi b}{a}\right)}\left(1 - \cos\frac{n\pi l}{a}\right)^2.$$

Now let us remove the ends to infinity, that is, make a very large. This will transform the Σ above into a definite integral. It is easily seen that the term under the Σ remains finite and continuous when n is made zero; we may therefore take our limits from 0 to ∞. We then obtain, putting $n\pi b/a = u$, $\pi b/a = du$:

$$f = \int_0^\infty \frac{W}{2\pi} \frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u + \frac{u}{\mu}\sinh u}{\sinh 2u - 2u}\left(1 - \cos\frac{ul}{b}\right)^2 \frac{du}{u}$$

$$= \frac{2W}{\pi}\int_0^\infty \frac{\frac{4}{E}\cosh u + \frac{u}{\mu}\sinh u}{\sinh 2u - 2u}\left(\sin\frac{ul}{2b}\right)^4 \frac{du}{u},$$

or writing $l/2b = \lambda_0$,

$$f = \frac{2W}{\pi}\left(\frac{l}{2b}\right)^4 \int_0^\infty \frac{\left(\frac{4u^3\cosh u}{E} + \frac{u^4\sinh u}{\mu}\right)}{\sinh 2u - 2u}\left(\frac{\sin u\lambda_0}{u\lambda_0}\right)^4 du \quad . \quad . \quad . \quad . \quad (63).$$

Now $(\sin u\lambda_0/u\lambda_0)^4$ is always < 1, so that

$$f < \frac{2W}{\pi}\left(\frac{l}{2b}\right)^4 \int_0^\infty \frac{\frac{4}{E}u^3\cosh u + \frac{u^4\sinh u}{\mu}}{\sinh 2u - 2u} du,$$

and f tends to become equal to the right-hand side of the last written inequality if $l/2b$ becomes small, that is, if we make our supports close up.

The integrals $\int_0^\infty \frac{u^3\cosh u\, du}{\sinh 2u - 2u}$ and $\int_0^\infty \frac{u^4\sinh u\, du}{\sinh 2u - 2u}$ when calculated by quadratures come out to be equal to 7·22 and 24·82 respectively.

We have therefore $$f < \frac{2W}{\pi}\left(\frac{l}{2b}\right)^4\left(\frac{28\cdot 9}{E} + \frac{24\cdot 8}{\mu}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64).$$

Now if f_0 be the Euler-Bernoulli deflection, that is, the deflection calculated in the usual way by taking the curvature proportional to the bending moment and fixing, so that the elastic line is horizontal at the origin,

$$f_0 = \frac{Wl^3}{4Eb^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Comparing (64) and (65) we see that the true deflection will certainly be less than the Euler-Bernoulli deflection if $l\left(28\cdot9 + \frac{E}{\mu}24\cdot8\right) < 2\pi b$; or, $l < \cdot069\,b$, if for purposes of numerical calculation we suppose uni-constant isotropy and therefore $E = 5\mu/2$.

So that if l be less than about $\frac{1}{28}$th of the height of the beam, the correction to be applied to the Euler-Bernoulli deflection becomes negative. The critical point where, as we shorten the span, the correction passes from additive to subtractive corresponds to l slightly, but only very slightly, greater than $\cdot069\,b$, as in the neighbourhood of this value λ_0 is quite sufficiently small to make $(\sin u\lambda_0/u\lambda_0)^4 = 1$, a fair approximation for all the most important part of the range of integration of the integral in (63).

We see therefore that when we have a beam loaded in this way, with a section of symmetry constrained to remain plane, the deflection at the centre, for all spans greater than $\frac{1}{28}$th of the height, is larger than the one indicated by the Euler-Bernoulli theory. In the limit when the span is made very large, this additive correction is found to be of the same form as that given by DE SAINT-VENANT for a cantilever under special conditions of end fixing, but the coefficient is different, the correction being just under $\frac{3}{4}$ths of DE SAINT-VENANT'S value. For spans smaller than $\frac{1}{28}$th of the height the correction is negative.

§ 14. *The Doubly-supported Beam under Central Load. Expressions for the Strains and Stresses when we remove the Supports to the Two Extremities.*

Going back to the general expressions for U, V, P, Q, S given in § 10, if we have a beam as in § 13, but we make the two supports coincide with $x = \pm a$, we have

$$\alpha_0 = \beta_0 = -\frac{W}{2a}, \quad \alpha_n = -\frac{W}{a}, \quad \beta_n = -(-1)^n\frac{W}{a},$$

with the following values for the displacements and stresses :—

$$\begin{aligned} U = &-\frac{y}{\mu}\sum_1^\infty \frac{W}{a}\frac{\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a}\sin 2n\pi x/a \sinh 2n\pi y/a \\ &-\frac{y}{\mu}\frac{W}{a}\sum_0^\infty \frac{\cosh(2n+1)\pi b/a}{\sinh(4n+2)\pi b/a - (4n+2)\pi b/a}\sin(2n+1)\pi x/a\cosh(2n+1)\pi y/a \\ &-\frac{W}{a}\sum_0^\infty \frac{a}{(2n+1)\pi}\left(\frac{\left[\frac{1}{\lambda'+\mu}\cosh(2n+1)\pi b/a - \frac{1}{\mu}\frac{(2n+1)\pi b}{a}\sinh(2n+1)\pi b/a\right]}{\sinh(4n+2)\pi b/a - (4n+2)\pi b/a}\right)\sin(2n+1)\pi x/a\sinh(2n+1)\pi y/a \\ &-\frac{W}{a}\sum_1^\infty \frac{a}{2n\pi}\frac{\left(\frac{1}{\lambda'+\mu}\sinh 2n\pi b/a - \frac{1}{\mu}\frac{2n\pi b}{a}\cosh 2n\pi b/a\right)}{\sinh 4n\pi b/a + 4n\pi b/a}\sin 2n\pi x/a\cosh 2n\pi y/a \\ &-\frac{3Wa}{E\pi^2}xy\sum_0^\infty\frac{1}{(2n+1)^2} - \frac{\eta W x}{2Ea} \quad . \quad . \quad . \quad . \quad (66). \end{aligned}$$

$$
\begin{aligned}
\mathrm{V} = {} & \frac{y\mathrm{W}}{\mu a} \sum_1^\infty \frac{\sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \cosh 2n\pi y/a \\
& + \frac{y\mathrm{W}}{\mu a} \sum_0^\infty \frac{\cosh (2n+1)\pi b/a}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos (2n+1)\pi x/a \sinh (2n+1)\pi y/a \\
& - \frac{\mathrm{W}}{a} \sum_1^\infty \frac{a}{2n\pi} \frac{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right) \sinh 2n\pi b/a + \frac{1}{\mu}\frac{2n\pi b}{a} \cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \sinh 2n\pi y/a \\
& - \frac{\mathrm{W}}{a} \sum_1^\infty \frac{a}{(2n+1)\pi} \frac{\left[\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right) \cosh \overline{2n+1}\pi b/a + \frac{1}{\mu}\frac{\overline{2n+1}\pi b}{a} \sinh \overline{2n+1}\pi b/a\right]}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos (2n+1)\pi x/a \cosh (2n+1)\pi y/a \\
& + \frac{3}{2} \frac{\mathrm{W}a}{\mathrm{E}b^3\pi^2} (x^2 - \eta y^2) \sum_0^\infty \frac{1}{(2n+1)^2} - \frac{\mathrm{W}y}{\mathrm{E}a} + \mathrm{B} \quad \ldots \quad (67),
\end{aligned}
$$

where B is a constant depending on the origin from which the displacement is to be measured.

$$
\begin{aligned}
\mathrm{P} = {} & - \frac{3y\mathrm{W}a}{b^3\pi^2} \sum_0^\infty \frac{1}{(2n+1)^2} \\
& - \sum_1^\infty \frac{2\mathrm{W}}{a} \frac{\sinh 2n\pi b/a - (2n\pi b/a) \cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \cosh 2n\pi y/a \\
& - \sum_1^\infty \frac{2\mathrm{W}}{a} \frac{(2n\pi y/a) \sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \sinh 2n\pi y/a \\
& - \sum_0^\infty \frac{2\mathrm{W}}{a} \frac{\cosh \overline{2n+1}\,\pi b/a - (\overline{2n+1}\pi b/a) \sinh \overline{2n+1}\pi b/a}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos \overline{2n+1}\pi x/a \sinh \overline{2n+1}\pi y/a \\
& - \sum_0^\infty \frac{2\mathrm{W}}{a} \frac{(\overline{2n+1}\pi y/a) \cosh \overline{2n+1}\pi b/a}{\sinh (4n+2)\pi b/a - (4n+2)\pi b/a} \cos \overline{2n+1}\pi x/a \cosh \overline{2n+1}\pi y/a \quad \ldots \quad (68).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Q} = {} & - \frac{\mathrm{W}}{2a} - \sum_1^\infty \frac{2\mathrm{W}}{a} \frac{\sinh 2n\pi b/a + (2n\pi b/a) \cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \cosh 2n\pi y/a \\
& + \sum_1^\infty \frac{2\mathrm{W}}{a} \frac{(2n\pi y/a) \sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \cos 2n\pi x/a \sinh 2n\pi y/a \\
& - \sum_0^\infty \frac{2\mathrm{W}}{a} \frac{\cosh \overline{2n+1}\pi b/a + (\overline{2n+1}\pi b/a) \sinh \overline{2n+1}\pi b/a}{\sinh \overline{4n+2}\pi b/a - \overline{4n+2}\pi b/a} \cos \overline{2n+1}\pi x/a \sinh \overline{2n+1}\pi y/a \\
& + \sum_0^\infty \frac{2\mathrm{W}}{a} \frac{(\overline{2n+1}\pi y/a) \cosh \overline{2n+1}\pi b/a}{\sinh \overline{4n+2}\pi b/a - \overline{4n+2}\pi b/a} \cos \overline{2n+1}\pi x/a \cosh \overline{2n+1}\pi y/a \quad \ldots \quad (69).
\end{aligned}
$$

$$
\begin{aligned}
S = & \sum_{1}^{\infty} \frac{2W}{a} \frac{(2n\pi b/a) \cosh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \sin 2n\pi x/a \sinh 2n\pi y/a \\
& - \sum_{1}^{\infty} \frac{2W}{a} \frac{(2n\pi y/a) \sinh 2n\pi b/a}{\sinh 4n\pi b/a + 4n\pi b/a} \sin 2n\pi x/a \cosh 2n\pi y/a \\
& + \sum_{0}^{\infty} \frac{2W}{a} \frac{(\widehat{2n+1}\pi b/a) \sinh \widehat{2n+1}\pi b/a}{\sinh \widehat{4n+2}\pi b/a - \widehat{4n+2}\pi b/a} \sin \widehat{2n+1}\pi x/a \cosh \widehat{2n+1}\pi y/a \\
& - \sum_{0}^{\infty} \frac{2W}{a} \frac{(\widehat{2n+1}\pi y/a) \cosh \widehat{2n+1}\pi b/a}{\sinh \widehat{4n+2}\pi b/a - \widehat{4n+2}\pi b/a} \sin \widehat{2n+1}\pi x/a \sinh \widehat{2n+1}\pi y/a \quad . \quad . \quad . \quad . \quad (70).
\end{aligned}
$$

§ 15. *Definite Integrals to which the expressions of the last Section tend when we make "a" very large.*

If we make a very large, the Σ's in the preceding expressions will become integrals in the limit. It will be found, however, that certain terms in the last found values of U, V, P, Q, S become infinite when 0 is substituted for b/a. In these cases the sum may not be directly transformed into an integral. The reason why this occurs is that, if a be made infinite, an infinite bending moment $\frac{1}{2}Wa$ is introduced at the centre of the beam. It is this moment which produces the parts of the displacements and stresses that become infinite when a is infinite. If, however, we apply at the two ends pure couples $-\frac{1}{2}Wa$, we get rid of this infinite moment, and we have only the terms due to the local effect, which produces only finite stresses at a finite distance from the origin.

Thus, if in U we add $\frac{yW}{a} \sum_{0}^{\infty} \frac{3}{4} \frac{1}{\mu} \frac{xa^2}{b^3(2n+1)^2\pi^2}$ to the second Σ and $\frac{W}{a} \sum_{0}^{\infty} \frac{3}{4} \frac{1}{\lambda'+\mu} \frac{xya^2}{b^3(2n+1)^2\pi^2}$ to the third Σ, these Σ's remain finite even when we make $a = \infty$. We have, however, to introduce negative terms to balance those that have been added. Remembering that $\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right) = \frac{4}{E}$ and $\sum_{0}^{\infty} \frac{1}{(2n+1)^2} = \pi^2/8$, we see that the part of the series in U which becomes infinite, is $-\frac{3}{8} \frac{xyWa}{Eb^3}$, which, added to the other infinite term in the last line of (66), gives for the infinite part of U:

$$
U_0 = -\frac{3}{4} \frac{xyWa}{Eb^3}.
$$

Similarly with V. The terms which have to be added to the second and fourth Σ's to make them finite in the limit are

$$
-\frac{3y^2W}{4\mu a} \sum_{0}^{\infty} \frac{a^2}{(2n+1)^2\pi^2 b^3},
$$

$$
+\frac{W}{a} \sum_{0}^{\infty} \frac{a^4}{(2n+1)^4\pi^4 b^3} \frac{3}{4} \left[\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\left(1 + \frac{3}{10} \frac{(2n+1)^2\pi^2 b^2}{a^2}\right) + \frac{1}{\mu} \frac{(2n+1)^2\pi^2 b^2}{a^2}\right]
$$

$$
+\frac{W}{a} \sum_{0}^{\infty} \frac{a^2}{(2n+1)^2\pi^2 b^3} \frac{3}{8} (y^2 - x^2) \left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right),
$$

respectively. The first sum in the second of these expressions does not contain the variables, and therefore may be supposed taken from the constant B. The other terms, added to the first term of the last line of (67), will give for the infinite part of V

$$V_0 = \tfrac{3}{8}\frac{Wa}{Eb^3}(x^2 - \eta y^2).$$

Proceeding to deal in the same way with the stresses, we find that to ensure finiteness in the limit we must add:

(a) to the third and fourth series in P: $\dfrac{2W}{a}\sum_0^\infty \dfrac{3y}{4b^3}\dfrac{a^2}{(2n+1)^2\pi^2}$ in each case;

(b) to the third and fourth series in Q: $\dfrac{2W}{a}\sum_0^\infty \dfrac{3y}{4b^3}\dfrac{a^2}{(2n+1)^2\pi^2}$ and $-\dfrac{2W}{a}\sum_0^\infty \dfrac{3y}{4b^3}\dfrac{a^2}{(2n+1)^2\pi^2}$ respectively; the infinite part of P is then

$$P_0 = -\tfrac{3}{4}\frac{Wa}{b^3}y,$$

Q and S having no infinite parts.

If we leave out of account the parts U_0, V_0, P_0, which belong to a couple $Wa/2$ and which can be destroyed by introducing an equal and opposite couple, we find that, when a is made infinite, the displacements and stresses tend to the following limiting values:

$$\left.\begin{aligned}
U = &-\frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \left(\frac{\cosh u}{\sinh 2u - 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b} - \frac{3x}{4bu^2}\right)du \\
&-\frac{W}{2\pi}\int_0^\infty\left\{\frac{\dfrac{1}{\lambda'+\mu}\cosh u - \dfrac{1}{\mu}u\sinh u}{\sinh 2u - 2u}\frac{1}{u}\sin\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3xy\left(\dfrac{1}{\lambda'+\mu}\right)}{4b^2u^2}\right\}du \\
&-\frac{W}{2\pi}\int_0^\infty\left\{\frac{\dfrac{1}{\lambda'+\mu}\sinh u - \dfrac{1}{\mu}u\cosh u}{\sinh 2u + 2u}\right\}\frac{1}{u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
V = &\ \frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{1}{\mu}\frac{Wy}{2\pi b}\int_0^\infty\left\{\frac{\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3y}{4bu^2}\right\}du \\
&-\frac{W}{2\pi}\int_0^\infty\frac{\left\{\left(\dfrac{1}{\lambda'+\mu}+\dfrac{1}{\mu}\right)\sinh u + \dfrac{1}{\mu}u\cosh u\right\}}{\sinh 2u + 2u}\frac{1}{u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{W}{2\pi}\int_0^\infty\left\{\frac{\left(\dfrac{1}{\lambda'+\mu}+\dfrac{1}{\mu}\right)\cosh u + \dfrac{1}{\mu}u\sinh u}{\sinh 2u - 2u}\frac{1}{u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\right. \\
&\qquad\left.-\tfrac{3}{4}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{1}{u^4} - \tfrac{3}{40}\left(\frac{3}{\lambda'+\mu}+\frac{13}{\mu}\right)\frac{1}{u^2} - \tfrac{3}{8}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{y^2-x^2}{b^2u^2}\right\}du \\
&\qquad + \text{an arbitrary constant } B'
\end{aligned}\right\} \quad (71).$$

$$\left.\begin{aligned}
P = &-\frac{W}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&-\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{W}{\pi b}\int_0^\infty \left\{\frac{\cosh u - u\sinh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3}{4}\frac{y}{bu^2}\right\}du \\
&-\frac{Wy}{\pi b^2}\int_0^\infty \left\{\frac{u\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b} - \frac{3}{4u^2}\right\}du \\
Q = &-\frac{W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{W}{\pi b}\int_0^\infty \left\{\frac{\cosh u + u\sinh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3}{4}\frac{y}{bu^2}\right\}du \\
&+\frac{Wy}{\pi b^2}\int_0^\infty \left\{\frac{u\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b} - \frac{3}{4u^2}\right\}du \\
S = &\frac{W}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{W}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&-\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du
\end{aligned}\right\} \quad . \quad . \quad (72).$$

§ 16. *Consideration of the Stresses in the Neighbourhood of the Point where the Concentrated Load is Applied.*

The integrals in the expressions (71) and (72) are finite, one-valued and continuous at every point (x, y) inside the beam, such that y is numerically less than b by a finite quantity. For in this case, for large values of u, the integrand is comparable with $e^{-u(b-|y|)}$, where $|y|$ stands for the numerical value of y. If, however, $|y| = b$, or the point in question lies on the edges of the beam, the integrals are no longer necessarily convergent. In this case the expressions (71) and (72) have to be transformed.

Let us start with the stresses P, Q, S, as in their case the transformation is somewhat simpler. Further, let us consider instead of P and Q the somewhat more compactly expressed quantities

$$\left.\begin{aligned}
P + Q &= -\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&\quad - \frac{2W}{\pi b}\int_0^\infty \left\{\frac{\cosh u}{\sinh 2u - 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b} - \frac{3y}{4u^2 b}\right\}du \\
P - Q &= \frac{2W}{\pi b}\int_0^\infty \frac{u\cosh u\cosh\frac{uy}{b} - \frac{uy}{b}\sinh u\sinh\frac{uy}{b}}{\sinh 2u + 2u}\cos\frac{ux}{b}\,du \\
&\quad + \frac{2W}{\pi b}\int_0^\infty \left\{\frac{u\sinh u\sinh\frac{uy}{b} - \frac{uy}{b}\cosh u\cosh\frac{uy}{b}}{\sinh 2u - 2u}\cos\frac{ux}{b} + \frac{3}{4}\frac{y}{bu^2}\right\}du
\end{aligned}\right\} \quad (73).$$

$P - Q$ and S give the lines of principal stress and the principal stress-difference, $P + Q$ gives the compression at the point considered.

If in the values (73) and in S we write $y = b - y'$ so that we are referring our co-ordinates x, y' to the point C (fig. i.) where the concentrated load is applied, as origin, we find, on re-arranging the terms,

$$P + Q = -\frac{2W}{\pi b}\int_0^\infty e^{-\frac{uy'}{b}}\cos\frac{ux}{b}\,du$$

$$-\frac{2W}{\pi b}\int_0^\infty \left\{\left[\frac{u}{\sinh 2u - 2u} - \frac{u}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{4u^2}\right\}du$$

$$+\frac{2W}{\pi b}\int_0^\infty \left\{\frac{1}{2}\left[\frac{1 + 2u + e^{-2u}}{\sinh 2u - 2u} - \frac{1 + 2u - e^{-2u}}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{4u^2}\frac{y'}{b}\right\}du.$$

$$P - Q = \frac{2Wy'}{\pi b^2}\int_0^\infty ue^{-\frac{uy'}{b}}\cos\frac{ux}{b}\,du$$

$$-\frac{2W}{\pi b}\int_0^\infty \left\{\left[\frac{u}{\sinh 2u - 2u} - \frac{u}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{4u^2}\right\}du$$

$$+\frac{2Wy'}{\pi b^2}\int_0^\infty \left\{\frac{u}{2}\left[\frac{1 + 2u + e^{-2u}}{\sinh 2u - 2u} - \frac{1 + 2u - e^{-2u}}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{4u^2}\right\}du$$

$$-\frac{2Wy'}{\pi b^2}\int_0^\infty \left[\frac{u^2}{\sinh 2u - 2u} - \frac{u^2}{\sinh 2u + 2u}\right]\cos\frac{ux}{b}\sinh\frac{uy'}{b}\,du.$$

$$S = \frac{Wy'}{\pi b^2}\int_0^\infty ue^{-\frac{uy'}{b}}\sin\frac{ux}{b}\,du$$

$$+\frac{W}{\pi b}\int_0^\infty \left[\frac{u}{\sinh 2u - 2u} - \frac{u}{\sinh 2u + 2u}\right]\sin\frac{ux}{b}\sinh\frac{uy'}{b}\,du$$

$$-\frac{Wy'}{\pi b^2}\int_0^\infty \frac{u}{2}\left[\frac{1 + 2u + e^{-2u}}{\sinh 2u - 2u} - \frac{1 + 2u - e^{-2u}}{\sinh 2u + 2u}\right]\sin\frac{ux}{b}\sinh\frac{uy'}{b}\,du$$

$$+\frac{Wy'}{\pi b^2}\int_0^\infty \left[\frac{u^2}{\sinh 2u - 2u} - \frac{u^2}{\sinh 2u + 2u}\right]\sin\frac{ux}{b}\cosh\frac{uy'}{b}\,du.$$

The leading integrals in each case can be evaluated.

If we write $x = r' \sin \phi'$ $y' = r' \cos \phi'$, so that r' is the distance of the point considered from the point of application of the concentrated load and ϕ' is the angle which r' makes with the vertical, then :

$$\int_0^\infty e^{-\frac{uy'}{b}} \cos \frac{ux}{b}\, du = \frac{b \cos \phi'}{r'}$$

$$\int_0^\infty ue^{-\frac{uy'}{b}} \cos \frac{ux}{b}\, du = \frac{b^2 \cos 2\phi'}{r'^2}$$

$$\int_0^\infty ue^{-\frac{uy'}{b}} \sin \frac{ux}{b}\, du = \frac{b^2 \sin 2\phi'}{r'^2}$$

and we have

$$P + Q = -\frac{2W}{\pi r'} \cos \phi' - \frac{8W}{\pi b} \int_0^\infty \left\{ \frac{u^2}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \cosh \frac{uy'}{b} - \frac{3}{16u^2} \right\} du$$
$$+ \frac{8W}{\pi b} \int_0^\infty \left\{ \frac{u^2 + \frac{u}{2} + \frac{1}{8} - \frac{1}{8} e^{-4u}}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \sinh \frac{uy'}{b} - \frac{3}{16u^2} \frac{y'}{b} \right\} du \quad . \quad . \quad . \quad (74).$$

$$P - Q = \frac{2Wy'}{\pi r'^2} \cos 2\phi' - \frac{8W}{\pi b} \int_0^\infty \left\{ \frac{u^2}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \cosh \frac{uy'}{b} - \frac{3}{16u^2} \right\} du$$
$$+ \frac{8Wy'}{\pi b^2} \int_0^\infty \left\{ \frac{u^3 + \frac{u^2}{2} + \frac{u}{8} - \frac{u}{8} e^{-4u}}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \cosh \frac{uy'}{b} - \frac{3}{16u^2} \right\} du$$
$$- \frac{8Wy'}{\pi b^2} \int_0^\infty \frac{u^3}{\sinh^2 2u - 4u^2} \cos \frac{ux}{b} \sinh \frac{uy'}{b}\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

$$S = \frac{Wy'}{\pi r'^2} \sin 2\phi' + \frac{4W}{\pi b} \int_0^\infty \frac{u^2}{\sinh^2 2u - 4u^2} \sin \frac{ux}{b} \sinh \frac{uy'}{b}\, du$$
$$- \frac{4Wy'}{\pi b^2} \int_0^\infty \frac{u^3 + \frac{u^2}{2} + \frac{u}{8} - \frac{u}{8} e^{-4u}}{\sinh^2 2u - 4u^2} \sin \frac{ux}{b} \sinh \frac{uy'}{b}\, du$$
$$+ \frac{4Wy'}{\pi b^2} \int_0^\infty \frac{u^3}{\sinh^2 2u - 4u^2} \sin \frac{ux}{b} \cosh \frac{uy'}{b}\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

The expressions for the stresses therefore consist of two parts, namely, the integrated parts

$$\left.\begin{aligned} P_1 &= -\frac{W}{\pi r'} \cos \phi' + \frac{Wy'}{\pi r'^2} \cos 2\phi' = -\frac{2W}{\pi} \frac{x^2 y'}{r'^4} \\ Q_1 &= -\frac{W}{\pi r'} \cos \phi' - \frac{Wy'}{\pi r'^2} \cos 2\phi' = -\frac{2W}{\pi} \frac{y'^3}{r'^4} \\ S_1 &= \frac{Wy'}{\pi r'^2} \sin 2\phi' \qquad\qquad\quad = \frac{2W}{\pi} \frac{xy'^2}{r'^4} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (77).$$

and the parts still in the form of integrals, which we may call P_2, Q_2, S_2.

P_1, Q_1, S_1 agree with the expressions found by FLAMANT ('Comptes Rendus,' vol. 114, pp. 1465–1468) and confirmed by BOUSSINESQ ('Comptes Rendus,' vol. 114, pp. 1510–1516) for the stresses in an infinite solid due to a line of load W per unit length, in which case the problem is reduced to two dimensions. They correspond, therefore, to the stresses that would be induced in the beam by the concentrated load if the height $2b$ were made infinite.

The stresses P_2, Q_2, S_2 are regular functions of x and y throughout the beam. They nowhere become discontinuous or infinite, and they tend to zero as b is made large. They represent the correction that we have to apply to FLAMANT's and BOUSSINESQ's result as a consequence of the finite height of the beam.

BOUSSINESQ, in the paper quoted above, has made an attempt to obtain such a correction, by finding the stresses given by (77) over the lower edge of the beam, superimposing an equal and opposite system to annul these, and calculating the strains due to this last system as if the *top* boundary of the beam were removed to infinity. This corrective system, as he calls it, will now introduce extra stresses over the top of the beam. To get rid of these a corrective system of the second order is superimposed, and we may go on indefinitely in this way. The complexity of the expressions increases enormously for each system we add, and, on finding the approximation so slowly convergent that the terms of the second order were practically as important as those of the first, BOUSSINESQ threw up the method in despair, and fell back upon an empirical assumption, given by Sir GEORGE STOKES in a supplement to a paper by CARUS WILSON ('Phil. Mag.,' Series V., vol. 32, pp. 500–503), namely, that the stress system introduced by the finiteness of the height of the beam was such as to annul the stresses due to (77) at the lower boundary, and varied linearly along the vertical, giving zero stress over the upper boundary. The functions P_2, Q_2, S_2 of the present article solve the problem exactly.

§ 17. *Expansion in Integral Powers about the Point of Discontinuous Loading.*

In the integrals for P_2, Q_2, S_2 we may expand the quantities $\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}\frac{ux}{b} \times \left.\begin{matrix}\cosh\\ \sinh\end{matrix}\right\}\frac{uy'}{b}$ in series as follows :—

$$\left.\begin{aligned}
\sin\frac{ux}{b}\sinh\frac{uy'}{b} &= \sum_1^\infty\left(\frac{ur'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}\\
\sin\frac{ux}{b}\cosh\frac{uy'}{b} &= \sum_0^\infty\left(\frac{ur'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)}\\
\cos\frac{ux}{b}\cosh\frac{uy'}{b} &= \sum_0^\infty\left(\frac{ur'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}\\
\cos\frac{ux}{b}\sinh\frac{uy'}{b} &= \sum_0^\infty\left(\frac{ur'}{b}\right)^{2\nu+1}\frac{\cos\widehat{2\nu+1}\,\phi'}{(2\nu+1)!}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (78),$$

ν being an integer. Now when these values are substituted in (77) and similar formulæ, we may distribute the integral sign among the terms of the series, provided that both the original and the resulting series are absolutely and uniformly convergent. This is easily seen to hold good for the series (78), and it will be shown later, in § 18, to be true of the resulting series, providing the points considered lie inside a certain circle of convergence.

Assuming for the moment this result, we obtain from (77)

$$\left.\begin{aligned}
P_2 &= -\frac{4W}{\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu}H_{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!} - \frac{4W}{\pi b}\sum_0^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu\frac{\cos\nu\phi'}{\nu}H_\nu \\
&\quad + \frac{4Wy'}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu H_{\nu+1}\frac{\cos\nu\phi'}{\nu!} \\
Q_2 &= \frac{4W}{\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}H_{2\nu+1}\frac{\cos\widehat{2\nu+1}\phi'}{(2\nu+1)!} - \frac{4Wy'}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu H_{\nu+1}\frac{\cos\nu\phi'}{\nu!} \\
S_2 &= \frac{4W}{\pi b}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}H_{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!} - \frac{4Wy'}{\pi b^2}\sum_1^\infty(-1)^\nu\left(\frac{r'}{b}\right)^\nu H_{\nu+1}\frac{\sin\nu\phi'}{\nu!}
\end{aligned}\right\} \quad . \quad (79).$$

where

$$\left.\begin{aligned}
H_0 &= \int_0^\infty\left(\frac{u^2}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right)du \\
H_1 &= \int_0^\infty\left(\frac{u^3 + \frac{1}{2}u^2 + \frac{1}{8}u - \frac{1}{8}u\,e^{-4u}}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right)du \\
H_{2\nu} &= \int_0^\infty\frac{u^{2\nu+2}}{\sinh^2 2u - 4u^2}\,du \\
H_{2\nu+1} &= \int_0^\infty\left(\frac{u^{2\nu+3} + \frac{1}{2}u^{2\nu+2} + \frac{1}{8}u^{2\nu+1} - \frac{1}{8}u^{2\nu+1}e^{-4u}}{\sinh^2 2u - 4u^2}\right)du \\
&(\nu > 0).
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (80).$$

§ 18. *Convergency of the Series of the last Section.*

In order to justify the distribution of the integral sign over the separate terms of the series (78), we have to show that the series (79) are absolutely and uniformly convergent.

Now the series are absolutely and uniformly convergent provided that the series $\sum_0^\infty\left(\frac{r'}{b}\right)^\nu\frac{H_\nu}{\nu!}$ is absolutely and uniformly convergent. The convergency ratio of this latter series $= \underset{\nu=\infty}{L}\frac{r'}{b}\frac{H_{\nu+1}}{H_\nu}\frac{1}{\nu+1}$.

Now, in order to find the approximate value of H_ν when ν is large, let us consider the integral

$$I_n = \int_0^\infty\frac{u^n}{\sinh^2 2u - 4u^2}\,du$$

write $u = av$

$$I_n = a^{n+1}\int_0^\infty\frac{v^n\,dv}{\sinh^2 2av - 4a^2v^2} = a^{n+1}\int_0^\infty\frac{4v^n\,e^{-4av}\,dv}{1 + e^{-8av} - (2 + 16a^2v^2)\,e^{-4av}}.$$

Now let a be chosen so large that for all values of $v > \omega$, where ω is numerically less than unity,

$$(2 + 16a^2v^2)\, e^{-4av} - e^{-8av} < \epsilon,$$

where ϵ is a small, finite, assigned quantity.

We then find

$$I_n = a^{n+1}\left\{\int_0^\omega \frac{2v^n\, dv}{\sinh^2 2av - 4a^2v^2} + U_n\right\},$$

where U_n lies between $\int_\omega^\infty 4v^n\, e^{-4av}\, dv$ and $\frac{1}{1+\epsilon}\int_\omega^\infty 4v^n\, e^{-4av}\, dv$.

Now

$$\int_\omega^\infty v^n\, e^{-4av}\, dv$$

$$= \frac{n!\, e^{-4a\omega}}{(4a)^{n+1}}\left(1 + \frac{4a\omega}{1!} + \frac{(4a\omega)^2}{2!} + \ldots + \frac{(4a\omega)^n}{n!}\right)$$

$$= \frac{n!\, e^{-4a\omega}}{(4a)^{n+1}}\left(e^{4a\omega} - \left\{\frac{(4a\omega)^{n+1}}{(n+1)!} + \frac{(4a\omega)^{n+2}}{(n+2)!} + \ldots \text{ to } \infty\right\}\right)$$

$$= \frac{n!}{(4a)^{n+1}}\left(1 - e^{-4a\omega}\left\{\frac{(4a\omega)^{n+1}}{(n+1)!} + \ldots \text{ to } \infty\right\}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

Next

$$\sinh^2 2av - 4a^2v^2 > \tfrac{16}{3}a^4v^4.$$

Therefore

$$\int_0^\omega \frac{2v^n\, dv}{\sinh^2 2av - 4a^2v^2} < \int_0^\omega \frac{3}{8}\frac{v^{n-4}}{a^4}\, dv < \frac{3}{8}\frac{\omega^{n-3}}{a^4(n-3)}.$$

Now, ω being > 1, this tends to zero when n is large. Further, by making n sufficiently large, the second term in (81) is negligible compared with the first.

We then find that the most important terms in I_n lie between $\frac{1}{1+\epsilon}\frac{n!}{4^n}$ and $\frac{n!}{4^n}$. Hence when n is large we may neglect I_{n-1}, I_{n-2}, &c., compared with I_n.

Now

$$H_{2\nu} = I_{2\nu+2},$$

$$H_{2\nu+1} = I_{2\nu+3} + \tfrac{1}{2}I_{2\nu+2} + \tfrac{1}{8}I'_{2\nu+1},$$

where

$$I'_{2\nu+1} = \int_0^\infty \frac{u^{2\nu+1}(1 - e^{-4u})}{\sinh^2 2u - 4u^2}\, du < \int_0^\infty \frac{u^{2\nu+1}}{\sinh^2 2u - 4u^2}\, du < I_{2\nu+1}.$$

Therefore $H_{2\nu+1} = I_{2\nu+3}$ if we neglect all but the most important terms. Therefore in the limit $H_\nu = I_{\nu+2}$.

$$\text{Convergency ratio} = \underset{\nu=\infty}{L}\left(\frac{r'}{b}\right)\frac{I_{\nu+3}}{I_{\nu+2}}\frac{1}{\nu+1}$$

$$= \underset{\nu=\infty}{L}\left(\frac{r'}{b}\right)\frac{\left(\dfrac{1}{1+\theta\epsilon}\right)}{\left(\dfrac{1}{1+\theta'\epsilon}\right)}\frac{\nu+3}{4(\nu+1)} = \frac{r'}{4b}\frac{1+\theta'\epsilon}{1+\theta\epsilon},$$

where θ, θ' are proper fractions. If we take ϵ small enough, the convergency ratio tends to $\frac{r'}{4b}$.

The series we are dealing with are therefore absolutely and uniformly convergent inside a circle whose centre is the point where the concentrated load is applied and whose radius is twice the height of the beam.

The transformation used in the previous section was therefore justifiable for this region and the expressions (79) are real arithmetical equivalents of the stresses P_2, Q_2, S_2, which have to be superimposed upon FLAMANT and BOUSSINESQ's solutions for an infinite solid when we take into account the height of the beam. The values of the first few coefficients, calculated approximately by quadratures, were found to be as follows: $H_0 = -\cdot 2417$, $H_1 = -\cdot 0598$, $H_2 = +\cdot 2271$, $H_3 = +\cdot 3370$.

§ 19. *Transformed Expressions for the Displacements.*

If we take the expressions (71) for U and V, we may treat them exactly as we treated the expressions for P, Q, S. We then obtain, after some rather lengthy reductions, $U = U_1 + U_2$, $V = V_1 + V_2$, where

$$\left.\begin{aligned} U_1 &= \frac{1}{\mu}\frac{Wy'}{2\pi b}\int_0^\infty e^{-\frac{uy'}{b}}\sin\frac{ux}{b}\,du - \frac{W}{2\pi}\frac{1}{\lambda'+\mu}\int_0^\infty \frac{1}{\mu}e^{-\frac{uy'}{b}}\sin\frac{ux}{b}\,du \\ &= \frac{1}{\mu}\frac{Wy'}{2\pi r'}\sin\phi' - \frac{W}{2\pi}\frac{1}{\lambda'+\mu}\phi' \\ V_1 &= -\frac{1}{\mu}\frac{Wy'}{2\pi b}\int_0^\infty e^{-\frac{uy'}{b}}\cos\frac{ux}{b}\,du - \frac{W}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty \frac{\cos\frac{ux}{b}e^{-\frac{uy'}{b}} - e^{-u\beta}}{u}\,du + B_1 \\ &= -\frac{1}{\mu}\frac{W}{2\pi}\frac{y'}{r'}\cos\phi' + \frac{W}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\log\left(\frac{r'}{\beta b}\right) + B_1 \end{aligned}\right\} \quad (82).$$

$$\left.\begin{aligned} U_2 = {} & \frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty\left[\frac{u^2+\frac{u}{2}+\frac{1}{8}+\frac{1}{8}e^{-4u}}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3x}{16u^2 b}\right]du \\ & -\frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty \frac{u^2}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy'}{b}\,du \\ & -\frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty \left[\frac{u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{16}\frac{x}{u^2 b}\right]du \\ & +\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\int_0^\infty\left[\frac{u+\frac{1}{2}+\frac{1}{8}u^{-1}-\frac{1}{8}u^{-1}e^{-4u}}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{16}\frac{xy'}{u^2 b}\right]du \end{aligned}\right\} \quad (83),$$

$$\left.\begin{aligned} V_2 = &-\frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty\left(\frac{u^2}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy'}{b} - \frac{3}{16u^2}\right)du \\ &+\frac{1}{\mu}\frac{2Wy'}{\pi b}\int_0^\infty\left[\frac{u^2+\frac{1}{2}u+\frac{1}{8}-\frac{1}{8}e^{-4u}}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{16}\frac{y'}{bu^2}\right]du \\ &-\frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\int_0^\infty\left[\frac{u+\frac{1}{2}+\frac{1}{8}u^{-1}-\frac{1}{8}u^{-1}e^{-4u}}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy'}{b}\right. \\ &\qquad\qquad \left. - \frac{3}{16u^4} - \frac{3}{20u^2} - \frac{3}{32u^2}\frac{y'^2-x^2}{b^2} + \frac{e^{-u\beta}}{4u}\right]du \\ &+\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\int_0^\infty\left[\frac{u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy'}{b} - \frac{3}{16}\frac{y'}{bu^2}\right]du + B_2 \end{aligned}\right\} \quad (83),$$

where β, B_1, B_2 are arbitrary constants.

The expressions U_1, V_1 agree with those found by BOUSSINESQ in the paper referred to above ('Comptes Rendus,' vol. 114, pp. 1500–1516) for the displacements when b is made infinite. We see that U is indeterminate and V infinite at the point where the concentrated load acts.

Of course such infinite and indeterminate displacements could not occur in nature. With any real material, if it were possible to approximate to a true knife-edge, the infinite stress under the knife-edge would at once either cause the material to break, or else—and this is what must almost always occur in practice—reduce the parts in the immediate neighbourhood of the knife-edge to a plastic condition, so that in this region the equations of elasticity would no longer apply.

Hence for practical applications we have to exclude the actual line of application of the load, $r' = 0$, and a very thin cylinder surrounding it. If we do this, then all our results will be valid for points whose distance from the knife-edge is at all large compared with the radius of this thin cylinder. A notable point about the results (82) is that U_1 is independent of r' and depends only upon the angular co-ordinate of the point considered with regard to the knife-edge as origin. Hence all points lying on a plane through this knife-edge receive the same horizontal displacement

The parts U_2, V_2, of the displacements are finite, one-valued, and continuous throughout the beam and over the edges. They can be, like the stresses P_2, Q_2, S_2, expanded in series of powers of r', which are absolutely and uniformly convergent within a circle of radius $4b$.

These expansions are easily seen to be the following :

$$\left.\begin{aligned} U_2 = &-\frac{2Wy'}{\mu\pi b}\sum_1^\infty(-1)^\nu H_\nu\left(\frac{r'}{b}\right)^\nu\frac{\sin\nu\phi'}{\nu!} - \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu} \\ &+\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu-1} \\ V_2 = &-\frac{2Wy'}{\mu\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^\nu(-1)^\nu H_\nu\frac{\cos\nu\phi'}{\nu!} - \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left[H_{-1}+\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}H_{2\nu-1}\right] \\ &+\frac{2W}{\pi}\frac{1}{\lambda'+\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+)!}H_{2\nu} \end{aligned}\right\} \quad (84).$$

where H_{-1} is an arbitrary constant, and the other H's have the same meaning as before.

These equations represent the effect of the finite height of the beam upon the displacements. If in them we put $y' = 0$, $\phi' = \pi/2$, we have the alteration in the displacements over the upper surface due to the finite thickness. This gives us, retaining only the leading terms,

$$(V_2)_{y'=0} = \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\frac{H_1}{2\,!}\frac{x^2}{b^2} = -\frac{4W}{\pi E}(\cdot 0598)\frac{x^2}{b^2},$$

$$(U_2)_{y'=0} = -\frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)H_0\frac{x}{b} = \frac{1\cdot 9336 Wx}{\pi E b}.$$

giving a downward curvature at the point of discontinuous load equal to $\frac{\cdot 478W}{\pi E b^2}$ and a horizontal stretch $\frac{1\cdot 934W}{\pi E b}$. The effect of the finite thickness appears therefore to be to stiffen the beam and to decrease its curvature under the load.

§ 20. *Expansions about Other Points. Expansion about the Origin.*

The expressions (71) and (72) are capable of being expanded in many other ways. Considering only expansions in powers of the radius vector from a given point, we may write in U, V, P, Q, S : $x = X + \rho \sin\theta$, $y = Y + \rho\cos\theta$, and we shall obtain an expansion which is valid for all points which are contained between $y = \pm b$, and which lie inside a circle with centre (X, Y) passing through the point $(0, +b)$. The coefficients of $\rho^n \cos n\theta$, $\rho^n \sin n\theta$, &c., will be integrals containing X, Y.

The only expansions worth considering are those about the origin and those about the point $(0, -b)$, which is vertically below the load.

The expansions about the origin are deduced immediately from (71) and (72). They are

$$\left.\begin{aligned}
U &= -\frac{1}{\mu}\frac{Wy}{2\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin\nu\phi}{\nu\,!}F_{\nu} - \frac{W}{2\pi}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin\nu\phi}{\nu\,!}\left(\frac{1}{\lambda'+\mu}F_{\nu-1} - \frac{1}{\mu}G_{\nu}\right)\\
V &= \frac{1}{\mu}\frac{Wy}{2\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos\nu\phi}{\nu\,!}F_{\nu} - \frac{W}{2\pi}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos\nu\phi}{\nu\,!}\left\{\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)F_{\nu-1} + \frac{1}{\mu}G_{\nu}\right\} + G_0
\end{aligned}\right\} \quad (85),$$

$$\left.\begin{aligned}
P &= -\frac{W}{\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos\nu\phi}{\nu\,!}(F_{\nu} - G_{\nu+1}) - \frac{Wy}{\pi b^2}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos\nu\phi}{\nu\,!}F_{\nu+1}\\
Q &= -\frac{W}{\pi b}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos\nu\phi}{\nu\,!}(F_{\nu} + G_{\nu+1}) + \frac{Wy}{\pi b^2}\sum_{0}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\cos\nu\phi}{\nu\,!}F_{\nu+1}\\
S &= \frac{W}{\pi b}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin\nu\phi}{\nu\,!}G_{\nu+1} - \frac{Wy}{\pi b^2}\sum_{1}^{\infty}\left(\frac{r}{b}\right)^{\nu}\frac{\sin\nu\phi}{\nu\,!}F_{\nu+1}
\end{aligned}\right\} \quad . \quad . \quad (86),$$

where $y = r\cos\phi$, $x = r\sin\phi$,

$$F_1 = \int_0^\infty \left(\frac{u\cosh u}{\sinh 2u - 2u} - \frac{3}{4u^2}\right) du,$$

$$F_{2\nu+1} = \int_0^\infty \frac{u^{2\nu+1}\cosh u}{\sinh 2u - 2u}\, du \; (\nu > 0),$$

$$F_{2\nu} = \int_0^\infty \frac{u^{2\nu}\sinh u}{\sinh 2u + 2u}\, du \; (\nu \gtreqless 0),$$

$$G_{2\nu+1} = \int_0^\infty \frac{u^{2\nu+1}\cosh u}{\sinh 2u + 2u}\, du \; (\nu \gtreqless 0),$$

$$G_{2\nu} = \int_0^\infty \frac{u^{2\nu}\sinh u}{\sinh 2u - 2u}\, du \; (\nu > 0),$$

G_0 = a constant to be adjusted from the fixing conditions. The series in (85) and (86) are absolutely and uniformly convergent inside a circle centre the origin and radius b.

The first few coefficients are given by

$F_0 = \cdot 527$	$G_1 = \cdot 918$
$F_1 = \cdot 438$	$G_2 = 2\cdot 818$
$F_2 = 1\cdot 740$	$G_3 = 5\cdot 750$
$F_3 = 7\cdot 224$	$G_4 = 24\cdot 824$,

where the integrals have been obtained approximately by quadratures.

Retaining in the expressions (85), (86) only the most important terms, we find for the displacements of points on the x-axis: $U_{y=0} = \frac{W}{2\pi}\frac{x}{b}\left(\frac{1}{\mu}\cdot 918 - \frac{1}{\lambda' + \mu}\cdot 527\right)$, which is positive with x.

We have therefore a horizontal stretch equal to $\frac{W}{2\pi b}\left(\frac{1\cdot 444}{\mu} - \frac{2\cdot 108}{E}\right)$.

For uni-constant isotropy $E = 5\mu/2$, and the stretch is $\frac{W}{2bE}\left(\frac{1\cdot 503}{\pi}\right)$, or about one half the stretch due to the load W acting horizontally along the length of the beam, so as to produce a tension $W/2b$.

Similarly $V_{y=0} = G_0 + \frac{W}{2\pi}\frac{x^2}{b^2}\frac{1}{2!}\left(\frac{4F_1}{E} + \frac{G_2}{\mu}\right)$; this gives a curvature upwards equal to $\frac{W}{2\pi b^2}\left(\frac{1\cdot 753}{E} + \frac{2\cdot 818}{\mu}\right)$, *i.e.*, to the curvature that would be produced by a pure couple $\frac{Wb}{3\pi}\left(1\cdot 753 + \frac{E}{\mu}2\cdot 818\right)$, or (putting $E = 5\mu/2$) by a couple $Wb \times (\cdot 5622)$.

The stresses at points along the x-axis are

$$P_{y=0} = -\frac{W}{\pi b}\left[(F_0 - G_1) - \frac{x^2}{b^2}\frac{1}{2\,!}(F_2 - G_3)\right]$$

$$= \frac{W}{\pi b}\left[\cdot 391 - \frac{x^2}{b^2}\,2\cdot 005\right]$$

$$Q_{y=0} = -\frac{W}{\pi b}\left[1\cdot 444 - \frac{x^2}{b^2}\,3\cdot 745\right];$$

we have therefore at the origin a horizontal tension and vertical pressure. These vanish when $x = \pm\cdot 195b$ and $x = \pm\cdot 386b$ respectively, assuming that for these values of x the first two terms are a sufficient approximation, which is certainly true for $x = \cdot 195b$, but only roughly true for $x = \cdot 386b$, as it amounts to neglecting terms of order about $\frac{1}{7}$ compared with 1·44. It will, however, be sufficient for a rough estimate.

The actual stresses at the origin are :—

$P = \frac{W}{2b}(\cdot 249)$, or about $\frac{1}{4}$ of the tension due to W acting along the horizontal,

$Q = -\frac{W}{2b}(\cdot 920)$, or about $\frac{9}{10}$ths of the pressure due to W acting along the horizontal.

If we had used the expressions P_1, Q_1 which hold for an infinite solid, we should find, at the origin, $P = 0$, $Q = -\frac{2W}{\pi b} = -\frac{W}{2b}(1\cdot 273)$.

If we correct the last by STOKES' empirical rule, we have to add $-\frac{1}{2}$[0 + (stress at bottom of beam as given by the formula for an infinite solid)].

This will give $Q = -\frac{3W}{2\pi b} = -\frac{W}{2b}(\cdot 955)$. The error in the vertical stress, calculated from this amended formula, is therefore only (·035) W/2b, or only about $3\frac{1}{2}$ per cent.

With regard to the correction for the horizontal tension, BOUSSINESQ finds, for a span $2l$ and depth $2b$,

$$P = \frac{W}{2b}\left[\frac{4}{\pi} - \frac{3y'}{\pi b} + \frac{3(y'-b)}{2b^2}\,l\right],$$

where y' is measured from the point $(0, b)$ as before.

The terms $\frac{3W}{4b^3}(y'-b)\,l$ correspond to the bending moment which we have removed.

We have left therefore $P = \frac{W}{2b}\left[\frac{4}{\pi} - \frac{3y'}{\pi b}\right]$, so that, at the origin, when $y' = b$, $P = \frac{W}{2b}\frac{1}{\pi} = \frac{W}{2b}(\cdot 318)$, and this gives a tension which is greater than the actual one by only (·069) W/2b.

§ 21. *Expansions about the Point* (0, − *b*).

It appears of some interest to give the values of the displacements and stresses about the point (0, − *b*), that is, the point of the lower boundary of the beam which is vertically under the load.

The integral expressions (71), (72), (73) transform as follows, if we write $y = y'' - b$,

$$\begin{aligned} \mathrm{U} = &- \frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty \left\{\frac{2u\cosh 2u + \sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy''}{b} - \frac{3}{4}\frac{x}{bu^2}\right\}du \\ &+ \frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty \frac{2u\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}\,du \\ &+ \frac{\mathrm{W}}{2\pi}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\int_0^\infty \left[\frac{2\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy''}{b} - \frac{3}{4}\frac{x}{bu^2}\right]du \\ &- \frac{\mathrm{W}}{2\pi}\frac{1}{\lambda' + \mu}\int_0^\infty \left\{\frac{2\cosh 2u + u^{-1}\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b} - \frac{3}{4}\frac{xy''}{b^2u^2}\right\}du, \end{aligned}$$

$$\begin{aligned} \mathrm{V} = &- \frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty \left\{\frac{2u\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b} - \frac{3}{4u^2}\right\}du \\ &+ \frac{1}{\mu}\frac{\mathrm{W}y''}{2\pi b}\int_0^\infty \left\{\frac{2u\cosh 2u + \sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b} - \frac{3}{4}\frac{y''}{bu^2}\right\}du \\ &- \frac{\mathrm{W}}{2\pi}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\int_0^\infty \left\{\frac{2\cosh 2u + u^{-1}\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b}\right. \\ &\qquad\qquad \left. - \frac{3}{4u^4} - \frac{3}{5u^2} - \frac{3}{8u^2}\frac{y''^2 - x^2}{b^2}\right\}du \\ &+ \frac{\mathrm{W}}{2\pi}\left(\frac{1}{\lambda' + \mu}\right)\int_0^\infty \left\{\frac{2\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b} - \frac{3}{4u^2}\frac{y''}{b}\right\}du, \end{aligned}$$

$$\begin{aligned} \mathrm{P} + \mathrm{Q} = &\frac{2\mathrm{W}}{\pi b}\int_0^\infty \left\{\frac{2u\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b} - \frac{3}{4u^2}\right\}du \\ &- \frac{2\mathrm{W}}{\pi b}\int_0^\infty \left\{\frac{2u\cosh 2u + \sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b} - \frac{3}{4u^2}\frac{y''}{b}\right\}du, \end{aligned}$$

$$\begin{aligned} \mathrm{P} - \mathrm{Q} = &\frac{2\mathrm{W}}{\pi b}\int_0^\infty \left\{\frac{2u\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b} - \frac{3}{4u^2}\right\}du \\ &- \frac{2\mathrm{W}y''}{\pi b^2}\int_0^\infty \left\{\frac{2u^2\cosh 2u + u\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\cosh\frac{uy''}{b} - \frac{3}{4u^2}\right\}du \\ &+ \frac{2\mathrm{W}y''}{\pi b^2}\int_0^\infty \frac{2u^2\sinh 2u}{\sinh^2 2u - 4u^2}\cos\frac{ux}{b}\sinh\frac{uy''}{b}\,du, \end{aligned}$$

$$S = \frac{W}{\pi b}\int_0^\infty \frac{2u\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}\,du$$
$$+ \frac{Wy''}{\pi b^2}\int_0^\infty \frac{2u^2\sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\cosh\frac{uy''}{b}\,du$$
$$- \frac{Wy''}{\pi b^2}\int_0^\infty \frac{2u\cosh 2u + \sinh 2u}{\sinh^2 2u - 4u^2}\sin\frac{ux}{b}\sinh\frac{uy''}{b}\,du.$$

These integrals remain convergent when we put $y'' = 0$, but they are not convergent in their present form for $y'' = 2b$.

If we expand now in powers of r'', where $x = r''\sin\phi''$, $y'' = r''\cos\phi''$, we obtain the following series, which can easily be shown to be uniformly and absolutely convergent inside a circle of radius $2b$:

$$\left.\begin{aligned}
U &= \frac{1}{\mu}\frac{2Wy''}{\pi b}\sum_1^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\sin\nu\phi''}{\nu!}H'_{\nu+1} \\
&\quad - \frac{2W}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_1^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\sin\nu\phi''}{\nu!}H'_\nu + \frac{2W}{\pi}\frac{1}{\mu}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu+1}\frac{\sin\widehat{2\nu+1}\phi''}{(2\nu+1)!}H'_{2\nu+1} \\
V &= -\frac{1}{\mu}\frac{2Wy''}{\pi b}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+1} \\
&\quad - \frac{2W}{\pi}\frac{1}{\lambda'+\mu}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_\nu - \frac{2W}{\pi}\frac{1}{\mu}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\cos 2\nu\phi''}{(2\nu)!}H'_{2\nu}
\end{aligned}\right\} \quad (87),$$

$$\left.\begin{aligned}
P &= \frac{4W}{\pi b}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+1} + \frac{4W}{\pi b}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\cos 2\nu\phi''}{(2\nu)!}H'_{2\nu+1} \\
&\quad - \frac{4Wy''}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+2} \\
Q &= -\frac{4W}{\pi b}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu+1}\frac{\sin\widehat{2\nu+1}\phi''}{(2\nu+1)!}H'_{2\nu+2} + \frac{4Wy''}{\pi b^2}\sum_0^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\cos\nu\phi''}{\nu!}H'_{\nu+2} \\
S &= \frac{4W}{\pi b}\sum_1^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\sin 2\nu\phi''}{(2\nu)!}H'_{2\nu+1} - \frac{4Wy''}{\pi b^2}\sum_1^\infty(-1)^\nu\left(\frac{r''}{b}\right)^\nu\frac{\sin\nu\phi''}{\nu!}H'_{\nu+2}
\end{aligned}\right\} . (88),$$

where

H'_0 = an arbitrary constant depending upon the fixing conditions.

$$H'_1 = \int_0^\infty\left\{\frac{\frac{u}{2}\sinh 2u}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right\}du$$

$$H'_2 = \int_0^\infty\left\{\frac{\frac{1}{2}u^2\cosh 2u + \frac{1}{4}u\sinh 2u}{\sinh^2 2u - 4u^2} - \frac{3}{16u^2}\right\}du$$

$$H'_{2\nu+1} = \int_0^\infty \frac{\frac{1}{2}u^{2\nu+1}\sinh 2u}{\sinh^2 2u - 4u^2}\,du \quad (\nu > 0)$$

$$H'_{2\nu} = \int_0^\infty \frac{\frac{1}{2}u^{2\nu}\cosh 2u + \frac{1}{4}u^{2\nu-1}\sinh 2u}{\sinh^2 2u - 4u^2}\,du \quad (\nu > 1).$$

The values of the first few odd H′'s are all we shall require. They are $H'_1 = -\cdot049$, $H'_3 = +\cdot537$, $H'_5 = +1\cdot951$.

We then find, for points along the bottom edge, $y'' = 0$, $\phi'' = \pi/2$,

$$Q = 0,\ S = 0$$

$$P = \frac{8W}{\pi b}\left(-\cdot049 - \frac{x^2}{b^2}\frac{1}{2!}(\cdot537) + \frac{x^4}{b^4}\frac{1}{4!}(1\cdot951) + \ldots\right).$$

This gives therefore a horizontal pressure at the point $(0, -b)$ equal to $\frac{W}{2b}(\cdot250)$, and this pressure increases at a fairly rapid rate as we move away from the axis of y.

The stress P, obtained from BOUSSINESQ's calculation on STOKES' hypothesis, gives for the same point $P = \frac{W}{2b}\left(\frac{4}{\pi} - \frac{6}{\pi}\right) = -\frac{W}{2b}(\cdot657)$. This value is considerably too high. We gather that STOKES' hypothesis ceases to give valid results for the points in the lower half of the beam.

§ 22. *Effect of Distributing the Concentrated Load over a small Area instead of a Line.*

In all the above work we have supposed the load W concentrated upon a line perpendicular to the plane of the strain. This has led us to expressions which make the stresses, and one displacement, infinite at the line where the load is applied, and the other displacement indeterminate. In practice, however, owing to the elasticity and plasticity of the materials both of the beam and of the knife-edge, contact along a geometrical line is impossible, and the load always distributes itself over an area, small but finite.

In the present section we shall therefore consider the effect of a uniform distribution of load W per unit area (W was formerly load per unit length), extending on either side of $x = 0$, $y = b$ for a distance a'.

Every line element $Wd\xi$ of this load at distance ξ from the middle will produce a system of stresses and displacements $Pd\xi$, $Qd\xi$, $Sd\xi$, $Ud\xi$, $Vd\xi$, such as we have just been investigating, except that for x we must write $(x - \xi)$.

The stresses and displacements due to the total load are therefore $\int_{-a'}^{+a'} P(x-\xi)\,d\xi$, $\int_{-a'}^{+a'} Q(x-\xi)\,d\xi$, $\int_{-a'}^{+a'} S(x-\xi)\,d\xi$, $\int_{-a'}^{+a'} U(x-\xi)\,d\xi$, $\int_{-a'}^{+a'} V(x-\xi)\,d\xi$, $P(x-\xi)$

denoting that $x - \xi$ is substituted for x in P. Similarly for Q, &c. ; or writing $x - \xi = x'$, we have

$$P' = \int_{x-a'}^{x+a'} P(x')\,dx' \qquad U' = \int_{x-a'}^{x+a} U(x')\,dx'$$

$$Q' = \int_{x-a'}^{x+a'} Q(x')\,dx' \qquad V' = \int_{x-a'}^{x+a'} V(x')\,dx'$$

$$S' = \int_{x-a'}^{x+a'} S(x')\,dx'$$

P′, Q′, S′, U′, V′ referring to the stresses and displacements due to the uniform layer.

We can obtain in this way, at once, as many different forms for P′, Q′, S′, U′, V′ as we had for P, Q, S, U, V. The series for the latter integrate at once, for they are composed of terms of the form const. $\times\, r^n \cos n\phi$ or $r^n \sin n\phi$, or $yr^n \cos n\phi$ or $yr^n \sin n\phi$, where $r = \sqrt{x^2 + y^2}$, $\tan\phi = x/y$. We have then

$$\int r^n \sin n\phi\, dx = -\frac{1}{n+1}\, r^{n+1} \cos \widehat{n+1}\,\phi$$

$$\int r^n \cos n\phi\, dx = \frac{1}{n+1}\, r^{n+1} \sin \widehat{n+1}\,\phi.$$

The only case where this fails is when $n = -1$, and in this case it is easy to show that $\int \frac{\cos\phi}{r}\, dx = \phi$, $\int \frac{\sin\phi}{r}\, dx = \log r$.

Terms of the form ϕ and $\log r$ also occur. They can be integrated as follows :—

$$\int \phi\, dx = x\phi - y\log r,$$

$$\int \log r\, dx = x \log r - x + y\phi.$$

If we apply these formulæ, and if we call D_1 and D_2 (fig. ii.) the points $(-a', +b)$ and $(+a', +b)$, *i.e.*, the extremities of the layer of stress, and if r_1, r_2 denote the

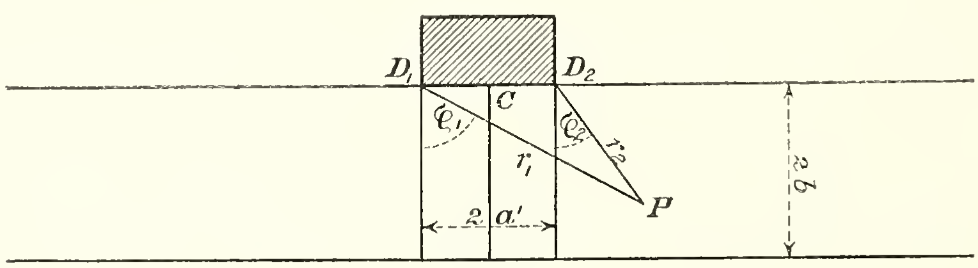

Fig. ii.

distances of any point from D_1, D_2 respectively, and if ϕ_1, ϕ_2 be the angles which r_1, r_2 make with the vertical, we find, if we start with the expressions for U, V, R, Q, S in the form (77), (79), (82), (84),

$$\left.\begin{aligned}
U' &= \frac{1}{\mu}\frac{W}{2\pi}y'\log\frac{r_1}{r_2} - \frac{W}{2\pi}\frac{1}{\lambda'+\mu}\{(a'+x)\phi_1 - (x-a')\phi_2 - y'\log(r_1/r_2)\} \\
&\quad + \frac{2Wy'}{\mu\pi}\sum_1^\infty(-1)^\nu H_\nu \frac{r_1^{\nu+1}\cos\widehat{\nu+1}\,\phi_1 - r_2^{\nu+1}\cos\widehat{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
&\quad + \frac{2Wb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty H_{2\nu}\frac{r_1^{2\nu+2}\cos\widehat{2\nu+2}\,\phi_1 - r_2^{2\nu+2}\cos\widehat{2\nu+2}\,\phi_2}{b^{2\nu+2}(2\nu+2)!} \\
&\quad - \frac{2Wb}{\pi}\frac{1}{\lambda'+\mu}\sum_1^\infty H_{2\nu-1}\frac{r_1^{2\nu+1}\cos\widehat{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\cos\widehat{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!} \\
V' &= -\frac{1}{\mu}\frac{W}{2\pi}y'(\phi_1-\phi_2) + \frac{W}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\Big\{(a'+x)\log\frac{r_1}{b\beta} \\
&\qquad - (x-a')\log\left(\frac{r_2}{b\beta}\right) + y'(\phi_1-\phi_2)\Big\} \\
&\quad + 2B_1a' - \frac{2Wy'}{\mu\pi}\sum_0^\infty(-1)^\nu H_\nu\frac{r_1^{\nu+1}\sin\widehat{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\widehat{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
&\quad - \frac{2Wb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left(2aH_{-1} + \sum_1^\infty H_{2\nu-1}\frac{r_1^{2\nu+1}\sin\widehat{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\sin\widehat{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!}\right) \\
&\quad + \frac{2Wb}{\pi}\frac{1}{\lambda'+\mu}\sum_0^\infty\frac{r_1^{2\nu+2}\sin\widehat{2\nu+2}\,\phi_1 - r_2^{2\nu+2}\sin\widehat{2\nu+2}\,\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu}
\end{aligned}\right\} \quad (89).$$

$$\left.\begin{aligned}
P' &= -\frac{W}{\pi}(\phi_1-\phi_2) + \frac{W}{2\pi}(\sin 2\phi_1 - \sin 2\phi_2) \\
&\quad - \frac{4W}{\pi}\sum_0^\infty H_{2\nu}\frac{r_1^{2\nu+1}\sin\widehat{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\sin\widehat{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!} \\
&\quad - \frac{4W}{\pi}\sum_0^\infty(-1)^\nu H_\nu\frac{r_1^{\nu+1}\sin\widehat{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\widehat{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
&\quad + \frac{4Wy'}{\pi b}\sum_0^\infty(-1)^\nu H_{\nu+1}\frac{(r_1^{\nu+1}\sin\widehat{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\widehat{\nu+1}\,\phi_2)}{b^{\nu+1}(\nu+1)!} \\
Q' &= -\frac{W}{\pi}(\phi_1-\phi_2) - \frac{W}{2\pi}(\sin 2\phi_1 - \sin 2\phi_2) \\
&\quad + \frac{4W}{\pi}\sum_0^\infty H_{2\nu+1}\frac{r_1^{2\nu+2}\sin\widehat{2\nu+2}\,\phi_1 - r_2^{2\nu+2}\sin\widehat{2\nu+2}\,\phi_2}{b^{2\nu+2}(2\nu+2)!} \\
&\quad - \frac{4Wy'}{\pi b}\sum_0^\infty(-1)^\nu H_{\nu+1}\frac{r_1^{\nu+1}\sin\widehat{\nu+1}\,\phi_1 - r_2^{\nu+1}\sin\widehat{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!} \\
S' &= -\frac{W}{2\pi}(\cos 2\phi_1 - \cos 2\phi_2) \\
&\quad - \frac{4W}{\pi}\sum_1^\infty H_{2\nu}\frac{r_1^{2\nu+1}\cos\widehat{2\nu+1}\,\phi_1 - r_2^{2\nu+1}\cos\widehat{2\nu+1}\,\phi_2}{b^{2\nu+1}(2\nu+1)!} \\
&\quad + \frac{4Wy'}{\pi b}\sum_1^\infty H_{\nu+1}(-1)^\nu\frac{r_1^{\nu+1}\cos\widehat{\nu+1}\,\phi_1 - r_2^{\nu+1}\cos\widehat{\nu+1}\,\phi_2}{b^{\nu+1}(\nu+1)!}
\end{aligned}\right\} \quad . \quad . \quad (90).$$

The expressions (89) and (90) show us that in this case the stresses and the displacements are obtained as the difference of two functions taken with the extremities of the layer as origins. The series are everywhere uniformly and absolutely convergent inside the common part of two circles of radius $4b$ described about each of these extremities as centre. It follows that if these series are to be valid anywhere, the length of the layer must not exceed $8b$. And if they are to be valid round each extremity the length of the layer must be less than $4b$. If these conditions be not fulfilled, then we have to fall back on the results (74), (75), (76), and (83) for P, Q, S, U, V. Integrating these we obtain formulæ valid over the whole beam, and these again may be expanded in powers about any point we please, as has been previously shown. The results are rather long and do not seem to present sufficient interest to justify the writing out of them at length.

Assuming $2a' < 4b$, so that the expressions (89) and (90) are valid over a region enclosing the layer of application of the load, we see that here no displacement is either infinite or discontinuous. For in the limit, both $(a' + x) \log r_1$ and $y' \log r_1$ are zero when $x = -a'$, $y' = 0$; and in like manner $(x - a') \log r_2$ and $y' \log r_2$ are zero when $x = +a'$, $y' = 0$.

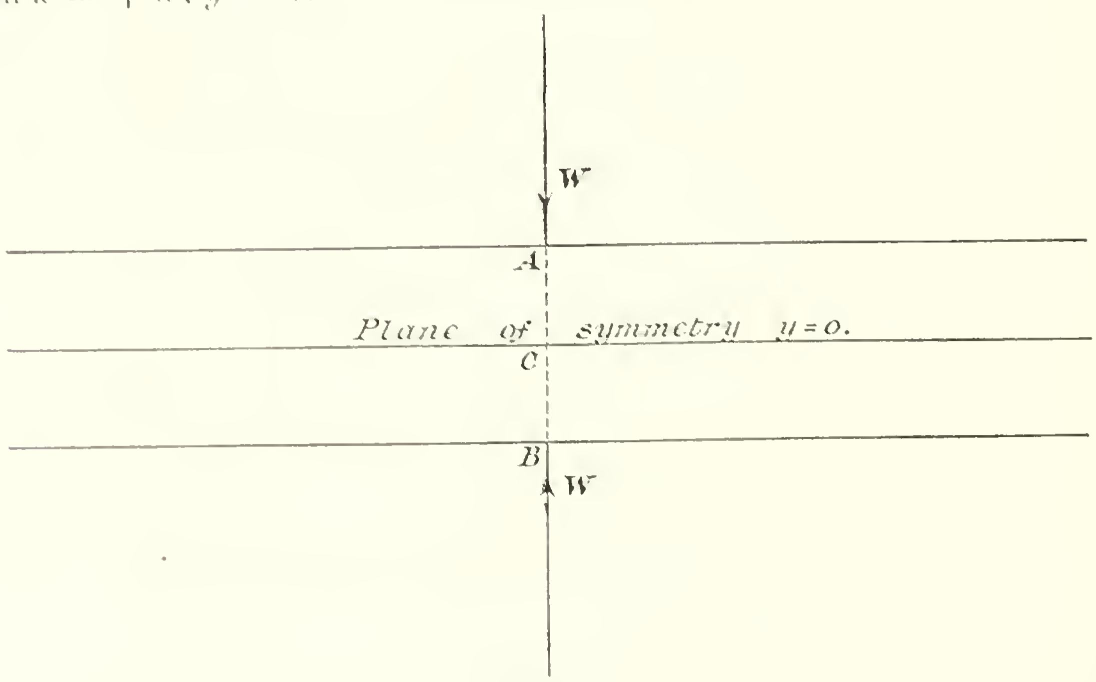

Fig. iii.

The shear S′ is continuous.

The stresses P′, Q′ however are discontinuous at the extremities of the layer. This indeed is obvious in the case of Q′, since it is one of the data of the problem. But it is curious to note that P′ is discontinuous at those points by precisely the same amount as Q′.

§ 23. *Case of a Beam under Two Equal and Opposite Loads, or Resting upon a Rigid Smooth Plane.*

If we take the solution we have obtained, turn it upside down, as it were, and superpose it to itself, we obtain the solution of the problem of an infinitely long beam gripped between two knife-edges exactly opposite each other (fig. iii.). The solution is obtained from the previous one by changing the signs of y, V and S, and then adding the new U, V, P, Q, S to the old.

I do not propose to write down fully the solution; it is easily obtained in various forms by using the several expansions which have already been given for the beam under a single concentrated load only. The parts of the stresses and displacements which become infinite at the points of loading are of exactly the same form as in the previous case.

Let us, however, consider the stresses. We easily find the following expressions:

$$\left.\begin{aligned}
P &= -\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&\quad -\frac{2W}{\pi b}\int_0^\infty \frac{uy}{b}\,\frac{\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du. \\
Q &= -\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&\quad +\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du. \\
S &= \frac{2W}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&\quad -\frac{Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (91).$$

The last written equation shows that $S = 0$ over the plane $y = 0$. Further, from considerations of symmetry $V = 0$ over this plane. Hence we may, if we choose, leave the lower part of the beam out of account altogether, and consider it as replaced by an infinite smooth rigid plane, against which the beam is pressed by a single weight, W. It then becomes of considerable interest to find out how this weight W distributes itself, after transmission through the beam, over this rigid plane.

The pressure $-Q$ on the plane corresponding to $y = 0$ is given by

$$-Q = +\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos\frac{ux}{b}\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad (92).$$

It is easy to show that this pressure tends to zero when x is large.

Integrating by parts with regard to u, we have

$$Q = \frac{2W}{\pi x}\int_0^\infty \frac{d}{du}\left(\frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\right)\sin\frac{ux}{b}\,du.$$

The integral on the right-hand side is obviously not infinite, however large x may be. Hence Q tends to zero as x tends to infinity.

We might repeat this process any finite number of times. It will be found that $\frac{\sinh u + u\cosh u}{\sinh 2u + 2u}$ being an even function of u, the integrated terms will in all cases vanish at both limits, and we obtain $Q = \frac{2Wb^n}{\pi x^{n+1}} \times$ an integral which is not infinite when x is large. Therefore we see that Q diminishes faster than any finite inverse power of x, however high. This seems to suggest an exponential law.

§ 24. *New Form of Expansion for the Pressure on the Rigid Plane.*

Consider the integral

$$I = \int_0^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u}\cos uz\, du.$$

We have

$$\frac{1}{\sinh 2u + 2u} = \frac{1}{\sinh 2u} - \frac{2u}{\sinh^2 2u} + \ldots + (-1)^r \frac{(2u)^r}{\sinh^{r+1} 2u} + \ldots$$
$$+ (-1)^{n-1}\frac{(2u)^{n-1}}{\sinh^n 2u} + (-1)^n \frac{(2u)^n}{\sinh^n 2u(\sinh 2u + 2u)}.$$

Substitute in I, we find

$$I = J_0 + J_1 + \ldots + J_r + \ldots + J_{n-1} + R_n,$$

where

$$J_r = (-1)^r \int_0^\infty \frac{(2u)^r}{\sinh^{r+1} 2u}(\sinh u + u\cosh u)\cos uz\, du,$$

$$R_n = (-1)^n \int_0^\infty \frac{(2u)^n}{\sinh^n 2u}\frac{(\sinh u + u\cosh u)}{\sinh 2u + 2u}\cos uz\, du.$$

Now

$$J_r = (-1)^r 2^{2r+1}\int_0^\infty \frac{u^r(\sinh u + u\cosh u)}{e^{2(r+1)u}(1 - e^{-4u})^{r+1}}\cos uz\, du.$$

Let us assume that in this we may expand $(1 - e^{-4u})^{r+1}$ in ascending powers of e^{-4u}. This will be justified later.

$$\frac{1}{(1 - e^{-4u})^{r+1}} = \sum_{s=0}^{s=\infty} \frac{(s+1)\ldots(s+r)}{r!} e^{-4su},$$

whence

$$J_r = (-1)^r 2^{2r}\int_0^\infty u^r \sum_{s=0}^{s=\infty}\frac{(s+1)\ldots(s+r)}{r!}\{e^{-(4s+2r+1)u} - e^{-(4s+2r+3)u}\}\cos uz\, du$$

$$+ (-1)^r 2^{2r}\int_0^\infty u^{r+1} \sum_{s=0}^{s=\infty}\frac{(s+1)\ldots(s+r)}{r!}\{e^{-(4s+2r+1)u} + e^{-(4s+2r+3)u}\}\cos uz\, du.$$

The cases r even and r odd have to be treated separately. Consider first r even and $= 2t$, and let K_r and L_r denote the first and second integrals in the last written expression for J_r. Then owing to the vanishing factor we may take the Σ in K_r as going back to $s = -t$, or, putting $s' = s + t$,

$$K_{2t} = 2^{4t} \int_0^\infty u^{2t} \sum_{s'=0}^{s'=} \frac{(s'-t+1)\dots(s'-1)s'\dots(s'+t)}{(2t)!} \{e^{-(4s'+1)u} - e^{-(4s'+3)u}\} \cos uz \, du.$$

But

$$\begin{aligned}
4s' - 4t + 4 &= (4s'+1) - (4t-3) \\
&\dots\dots\dots \\
4s' - 4 &= (4s'+1) - 5 \\
4s' &= (4s'+1) - 1 \\
4s' + 4 &= (4s'+1) + 3 \\
&\dots\dots\dots \\
4s' + 4t &= (4s'+1) + (4t-1),
\end{aligned}$$

and similarly

$$\begin{aligned}
4s' - 4t + 4 &= (4s'+3) - (4t-1) \\
&\dots\dots\dots \\
4s' - 4 &= (4s'+3) - 7 \\
4s' &= (4s'+3) - 3 \\
4s' + 4 &= (4s'+3) + 1 \\
&\dots\dots\dots \\
4s' + 4t &= (4s'+3) + (4t-3).
\end{aligned}$$

Now let $a_0, a_1, \dots a_{2t}$ be the coefficients in the product of degree $2t$

$$\{x + (4t-1)\}\{x + (4t-5)\}\dots\{x - (4t-3)\},$$

when it is expanded out, so that this product is

$$a_0 x^{2t} + a_1 x^{2t-1} + \dots + a_{2t}.$$

Then

$$\{x - (4t-1)\}\{x - (4t-5)\}\dots\{x + (4t-3)\}$$
$$= a_0 x^{2t} - a_1 x^{2t-1} + \dots + a_{2t}.$$

K_{2t} may then be written

$$\int_0^\infty \frac{u^{2t}}{(2t)!} \left[\begin{aligned}
&\sum_{s'=0}^{s'=\infty} a_0 \{(4s'+1)^{2t} e^{-\widehat{4s'+1}u} - (4s'+3)^{2t} e^{-\widehat{4s'+3}u}\} \\
&+ \sum_{s=0}^{s'=\infty} a_1 \{(4s'+1)^{2t-1} e^{-\widehat{4s'+1}u} + (4s'+3)^{2t-1} e^{-\widehat{4s'+3}u}\} \\
&+ \sum_{s'=0}^{s'=\infty} a_2 \{(4s'+1)^{2t-2} e^{-\widehat{4s'+1}u} - (4s'+3)^{2t-2} e^{-\widehat{4s'+3}u}\} \\
&+ \dots\dots\dots \\
&+ \sum_{s'=0}^{s'=\infty} a_{2t-1} \{(4s'+1) e^{-\widehat{4s'+1}u} + (4s'+3) e^{-\widehat{4s'+3}u}\} \\
&+ \sum_{s'=0}^{s'=\infty} a_{2t} \{e^{-\widehat{4s'+1}u} - e^{-\widehat{4s'+3}u}\}
\end{aligned} \right] \cos uz \, du.$$

Now if, as we have assumed, our expansion of $(1 - e^{-4u})^{-r-1}$ was justifiable, we may stop at the ν^{th} term, leaving a remainder less than an assigned quantity, provided ν be taken large enough. It will be shown in the next article that this is the case.

We may then, in the above, write for the upper limit of s' a number ν, large but finite. The series now consisting of a finite number of terms, we may distribute the integral sign, and further, we can replace u^{2t} by $(-1)^t \frac{d^{2t}}{dz^{2t}}$, since obviously each of the integrals of the type $\int_0^\infty e^{-ku} \cos uz\, du$ when $k > 0$ allows of being differentiated under the integral sign. This gives us, when the several integrals are evaluated,

$$
\begin{aligned}
\mathrm{K}_{2t} = \frac{(-1)^t}{(2t)!} \frac{d^{2t}}{dz^{2t}} \Bigg[& a_0 \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)^{2t+1}}{(4s'+1)^2+z^2} - \frac{(4s'+3)^{2t+1}}{(4s'+3)^2+z^2} \right\} \\
& + a_1 \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)^{2t}}{(4s'+1)^2+z^2} + \frac{(4s'+3)^{2t}}{(4s'+3)^2+z^2} \right\} \\
& + \cdots\cdots\cdots\cdots \\
& + a_{2t-1} \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)^2}{(4s'+1)^2+z^2} + \frac{(4s'+3)^2}{(4s+3)^2+z^2} \right\} \\
& + a_{2t} \sum_{s'=0}^{s'=\nu} \left\{ \frac{(4s'+1)}{(4s'+1)^2+z^2} - \frac{(4s'+3)}{(4s'+3)^2+z^2} \right\} \Bigg]
\end{aligned}
$$

Now writing in the above

$$(4s'+1)^2 = \{(4s'+1)^2 + z^2\} - z^2,\ (4s'+3)^2 = \{(4s'+3)^2+z^2\} - z^2,$$

and remembering that d^{2t}/dz^{2t} destroys any power of $z < 2t$, we find

$$
\begin{aligned}
\mathrm{K}_{2t} = \frac{(-1)^t}{(2t)!} \frac{d^{2t}}{dz^{2t}} \Bigg[& (a_0(-1)^t z^{2t} + a_2(-1)^{t-1} z^{2t-2} + \ldots \\
& \quad + a_{2t}) \sum_{s'=0}^{s'=\nu} \left\{ \frac{4s'+1}{(4s'+1)^2+z^2} - \frac{4s'+3}{(4s'+3)^2+z^2} \right\} \\
& + (a_1(-1)^t z^{2t} + a_3(-1)^{t-1} z^{2t-2} + \ldots \\
& \quad - a_{2t-1} z^2) \sum_{s'=0}^{s'=\nu} \left\{ \frac{1}{(4s'+1)^2+z^2} + \frac{1}{(4s'+3)^2+z^2} \right\} \Bigg]
\end{aligned}
$$

But, from CHRYSTAL'S 'Algebra,' vol. 2, p. 338,

$$\frac{\pi}{4z} \tanh \frac{\pi z}{2} = \sum_{s'=0}^{s'=\infty} \left\{ \frac{1}{(4s'+1)^2+z^2} + \frac{1}{(4s'+3)^2+z^2} \right\}$$

$$\frac{\pi}{4} \operatorname{sech} \frac{\pi z}{2} = \sum_{s'=0}^{s'=\infty} \left\{ \frac{4s'+1}{(4s'+1)^2+z^2} - \frac{4s'+3}{(4s'+3)^2+z^2} \right\}.$$

If, therefore, in our expression for K_{2t} we now allow ν to increase indefinitely, we obtain

$$\mathrm{K}_{2t} = \frac{\pi}{4} \frac{(-)^t}{(2t)!} \left\{ \frac{d^{2t}}{dz^{2t}} \left[\psi_{2t}(z) \operatorname{sech} \frac{\pi z}{2} + \chi_{2t}(z) \tanh \frac{\pi z}{2} \right] \right\},$$

where

$$\psi_{2t}(z) = a_0(-1)^t z^{2t} + a_2(-1)^{t-1} z^{2t-2} + \ldots + a_{2t}$$

$$= \tfrac{1}{2}\left[\begin{array}{l}(\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5)\ldots(\sqrt{-1}z - \widehat{4t-3}) \\ +(-\sqrt{-1}z + 4t - 1)(-\sqrt{-1}z + 4t - 5)\ldots(-\sqrt{-1}z - \widehat{4t-3})\end{array}\right]$$

$$\chi_{2t}(z) = a_1(-1)^t z^{2t-1} + \ldots - a_{2t-1}z$$

$$= \frac{\sqrt{-1}}{2}\left[\begin{array}{l}(\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5)\ldots(\sqrt{-1}z - \widehat{4t-3}) \\ -(-\sqrt{-1}z + 4t - 1)(-\sqrt{-1}z + 4t - 5)\ldots(-\sqrt{-1}z - \widehat{4t+3})\end{array}\right].$$

If we treat in a precisely similar way the second integral L_{2t}, we find

$$L_{2t} = \frac{\pi}{4}\frac{(-)^t}{(2t)!}\frac{d^{2t+1}}{dz^{2t+1}}\left(\psi_{2t}(z)\tanh\frac{\pi z}{2} - \chi_{2t}(z)\operatorname{sech}\frac{\pi z}{2}\right).$$

Coming now to the case where $r = \text{odd} = 2t + 1$, we work out K_{2t+1} and L_{2t+1} by a similar method. We consider in this case the product of degree $2t + 1$,

$$(x + 4t - 1)(x + 4t - 5)\ldots(x - \widehat{4t-3})(x - \widehat{4t+1}),$$

which we denote by

$$b_0 x^{2t+1} + b_1 x^{2t} + b_2 x^{2t-1} + \ldots + b_{2t+1}.$$

After reductions of the same type as those used for K_{2t}, we find

$$K_{2t+1} = \frac{(-1)^t}{(2t+1)!}\frac{d^{2t+1}}{dz^{2t+1}}\left[\begin{array}{l}(b_0(-1)^t z^{2t+1} + b_2(-1)^{t-1} z^{2t-1} + \ldots \\ \quad + b_{2t}z)\sum\limits_{s'=0}^{s'=\nu}\left\{\dfrac{4s'+1}{(4s'+1)^2+z^2} - \dfrac{4s'+3}{(4s'+3)^2+z^2}\right\} \\ +(b_1(-1)^t z^{2t+1} + b_3(-1)^{t-1} z^{2t-1} + \ldots \\ \quad + b_{2t+1}z)\sum\limits_{s'=0}^{s'=\nu}\left\{\dfrac{1}{(4s'+1)^2+z^2} + \dfrac{1}{(4s'+3)^2+z^2}\right\}\end{array}\right].$$

$$L_{2t+1} = \frac{(-1)^t}{(2t+1)!}\frac{d^{2t+2}}{dz^{2t+2}}\left[\begin{array}{l}(b_0(-1)^{t+1} z^{2t+2} + \ldots \\ \quad - b_{2t}z^2)\sum\limits_{s'=0}^{s'=\nu}\left\{\dfrac{1}{(4s'+1)^2+z^2} + \dfrac{1}{(4s'+3)^2+z^2}\right\} \\ +(b_1(-1)^t z^{2t} + \ldots \\ \quad + b_{2t+1})\sum\limits_{s'=0}^{s'=\nu}\left\{\dfrac{4s'+1}{(4s'+1)^2+z^2} - \dfrac{4s'+3}{(4s'+3)^2+z^2}\right\}\end{array}\right],$$

whence writing

$$b_0(-1)^t z^{2t+1} + b_2(-1)^{t-1} z^{2t-1} + \ldots + z b_{2t}$$

$$= \frac{1}{2\sqrt{-1}} \left[\begin{array}{l} (\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5) \ldots \\ \qquad (\sqrt{-1}z - \overline{4t-3})(\sqrt{-1}z - \overline{4t+1}) \\ + (\sqrt{-1}z - \overline{4t-1})(\sqrt{-1}z - \overline{4t-5}) \ldots \\ \qquad (\sqrt{-1}z + \overline{4t-3})(\sqrt{-1}z + \overline{4t+1}) \end{array} \right] = \psi_{2t+1}(z)$$

$$b_1(-1)^t z^{2t} + b_3(-1)^{t-1} z^{2t-2} + \ldots + b_{2t+1}$$

$$= \tfrac{1}{2} \left[\begin{array}{l} (\sqrt{-1}z + 4t - 1)(\sqrt{-1}z + 4t - 5) \ldots \\ \qquad (\sqrt{-1}z - \overline{4t-3})(\sqrt{-1}z - \overline{4t+1}) \\ - (\sqrt{-1}z - \overline{4t-1})(\sqrt{-1}z - \overline{4t-5}) \ldots \\ \qquad (\sqrt{-1}z + 4t-3)(\sqrt{-1}z + 4t+1) \end{array} \right] = \chi_{2t+1}(z),$$

we obtain

$$K_{2t+1} = \frac{(-1)^t}{(2t+1)!} \frac{\pi}{4} \frac{d^{2t+1}}{dz^{2t+1}} \left[\psi_{2t+1}(z) \operatorname{sech} \frac{\pi z}{2} + \chi_{2t+1}(z) \tanh \frac{\pi z}{2} \right]$$

$$L_{2t+1} = \frac{(-1)^t}{(2t+1)!} \frac{\pi}{4} \frac{d^{2t+2}}{dz^{2t+2}} \left[-\psi_{2t+1}(z) \tanh \frac{\pi z}{2} + \chi_{2t+1}(z) \operatorname{sech} \frac{\pi z}{2} \right],$$

and since $J_r = K_r + L_r$, we find that the required integral

$$I = \frac{\pi}{4} \sum_{t=0}^{t=n} \frac{(-1)^t}{(2t)!} \frac{d^{2t}}{dz^{2t}} \left\{ \begin{array}{l} \psi_{2t}(z) \operatorname{sech} \frac{\pi z}{2} + \chi_{2t}(z) \tanh \frac{\pi z}{2} \\ + \frac{d}{dz}\left(\psi_{2t}(z) \tanh \frac{\pi z}{2} - \chi_{2t}(z) \operatorname{sech} \frac{\pi z}{2} \right) \end{array} \right\}$$

$$+ \frac{\pi}{4} \sum_{t=0}^{t=n} \frac{(-1)^t}{(2t+1)!} \frac{d^{2t+1}}{dz^{2t+1}} \left\{ \begin{array}{l} \psi_{2t+1}(z) \operatorname{sech} \frac{\pi z}{2} + \chi_{2t+1}(z) \tanh \frac{\pi z}{2} \\ - \frac{d}{dz}\left(\psi_{2t+1}(z) \tanh \frac{\pi z}{2} - \chi_{2t+1}(z) \operatorname{sech} \frac{\pi z}{2} \right) \end{array} \right\}$$

$$+ R_{2n+2} \quad \ldots \ldots \quad (93).$$

§ 25. *Justification of the Procedure employed in the last Section.*

We have to show that, in the case of the integral

$$J_n = \int_0^\infty \frac{(2u)^n}{\sinh^{n+1} 2u} (\sinh u + u \cosh u) \cos uz \, du$$

$$= 2 \int_0^\infty \frac{(4u)^n e^{-\overline{2n+2}u}}{(1 - e^{-4u})^{n+1}} (\sinh u + u \cosh u) \cos uz \, du$$

we were justified in expanding $(1 - e^{-4u})^{-n-1}$ in ascending powers of e^{-4u}.

Now

$$\frac{1}{1-x} = 1 + x + \ldots + x^{n+r-1} + \frac{x^{n+r}}{1-x}.$$

Differentiate n times with regard to x,

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \ldots + \frac{r(r+1)\ldots(r+n-1)}{n!}x^{r-1} + \frac{1}{n!}\frac{d^n}{dx^n}\left(\frac{x^{n+r}}{1-x}\right).$$

The remainder is therefore

$$\frac{1}{n!}\frac{d^n}{dx^n}\frac{x^{n+r}}{(1-x)} = \frac{x^{n+r}}{(1-x)^{n+1}} + \ldots$$

$$+ \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}\frac{x^{n+r-s}}{(1-x)^{n+1-s}} + \ldots + \frac{(n+r)\ldots(r+1)}{n!}\frac{x^r}{1-x}.$$

This holds for all values of x however near to 1. Putting $x = e^{-4u}$ and substituting in J_n, we find J_n = 1st r terms of the series + a remainder term consisting of the sum of $(n+1)$ integrals of the form

$$2\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!} e^{-(6n+4r-4s+2)u} \frac{(4u)^n(\sinh u + u\cosh u)}{(1-e^{-4u})^{n+1-s}} \cos uz\, du,$$

s ranging from 0 to n, and the product $\frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}$ being replaced by unity for $s = 0$.

Now $\sinh u$ is always $< u \cosh u$: hence the general integral in the remainder (the factor multiplying $\cos uz$ in the integrand being positive throughout) is less than

$$\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}(4u)^s e^{-(6n+4r-4s+2)u}\cosh u\left(\frac{4u}{1-e^{-4u}}\right)^{n+1-s} du,$$

i.e., than

$$\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!} e^{-(4n+4r-2s)u}(4u)^s \cosh u\left(\frac{2u}{\sinh 2u}\right)^{n+1-s} du.$$

Now $\frac{2u}{\sinh 2u} < 1$ always, and $\cosh u < e^u$. The general remainder term is therefore less than

$$\int_0^\infty \frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{s!}(4u)^s e^{-(4n+4r-2s-1)u} du$$

$$< \tfrac{1}{4}\frac{(n+r)(n+r-1)\ldots(n+r-s+1)}{\left(n+r-\frac{s}{2}-\frac{1}{4}\right)^{s+1}} \text{ for } s \text{ ranging from 1 to } n.$$

For $s = 0$ the remainder term $< \frac{1}{4}\frac{1}{\left(n+r-\frac{s}{2}-\frac{1}{4}\right)}$. Thus for every value of s the value of the corresponding term in the remainder is seen to become very small of the order $1/r$ when r is made very large, n remaining finite. The value of the whole remainder is therefore also small of the order $1/r$. Consequently this remainder tends to zero as we make r large, and the series is therefore a true arithmetical equivalent of J_n.

We have still to show that a similar result holds for the expansion found for I, namely, that the integral we have called R_n tends to the limit zero when n is indefinitely increased. This we can do as follows:—

$$R_n = (-1)^n \int_0^\infty \frac{(2u)^n}{\sinh^n 2u} \frac{\sinh u + u\cosh u}{\sinh 2u + 2u} \cos uz \, du$$

is numerically less than

$$\int_0^\infty \frac{(2u)^n}{\sinh^n 2u} \frac{\sinh u + u\cosh u}{\sinh 2u + 2u} du,$$

and it is easy to show that both $(2u)^n/\sinh^n 2u$ and $(\sinh u + u\cosh u)/(\sinh 2u + 2u)$ continually decrease as u increases.

Hence, if we split up $\int_0^\infty$ into $\int_0^\omega + \int_\omega^\infty$, the first part is less than $\{\omega \times$ value of the integrand when $u = 0\}$, *i.e.*, $< \omega/2$. The second part is also less than

$$\frac{(2\omega)^n}{(\sinh^n 2\omega)} \int_\omega^\infty \frac{\sinh u + u\cosh u}{\sinh 2u + 2u} du.$$

Denoting the last integral, which is finite, by M, we have $R_n < \frac{\omega}{2} + \frac{(2\omega)^n}{(\sinh^n 2\omega)} M$ numerically.

But

$$\frac{(2\omega)^n}{\sinh^n 2\omega} < \frac{(2\omega)^n}{\left(2\omega + \frac{8\omega^3}{6}\right)^n} < \frac{1}{\left(1 + \frac{2\omega^2}{3}\right)^n} < \frac{1}{1 + \frac{2n\omega^2}{3}}.$$

Therefore $R_n < \frac{\omega}{2} + \frac{M}{1 + \frac{2n\omega^2}{3}}$.

Now if ω be chosen equal to $n^{-\frac{1}{3}}$, $R_n < \frac{1}{2n^{\frac{1}{3}}} + \frac{M}{1 + \frac{2}{3}n^{\frac{1}{3}}}$, a quantity which tends to zero when n tends to infinity. R_n itself therefore tends to zero for all values of z, so that the series (93) may be extended to infinity.

§ 26. *Deductions as to the Rapidity with which the Local Disturbances die out as we leave the neighbourhood of the Load.*

If we look at (93) and perform the differentiations, then, remembering that $\chi_{2t}(z)$ is of degree $(2t - 1)$ in z, $\chi_{2t+1}(z)$ and $\psi_{2t}(z)$ are of degree $2t$ in z, and $\psi_{2t+1}(z)$ is of degree $(2t + 1)$ in z, the only terms occurring in I will be of the form (algebraic polynomial in z) $\times \left(\operatorname{sech}\frac{\pi z}{2}\right.$ or $\operatorname{sech}^2\frac{\pi z}{2}$, or their differential coefficients$\left.\right)$. Now

$\operatorname{sech} \frac{\pi z}{2}$ and all its differential coefficients will be of order $e^{-\frac{\pi z}{2}}$ when z is large. Similarly $\operatorname{sech}^2 \frac{\pi z}{2}$ and its differential coefficients will be of order $e^{-\pi z}$ when z is large. We see, therefore, that the first n terms of the series for I will be of the form (algebraic polynomial of degree n in z) $e^{-\frac{\pi z}{2}}$ to the first approximation when z is large.

Further we have obtained an expression for the remainder R_n, which is small independently of z, for any given large value of n. We see therefore that, n being assigned, we may make z as large as we please and I will eventually tend to zero, $e^{\pi z/2}$ becoming large more rapidly than any polynomial of finite degree, if z be large enough.

Now $z = x/b$. We see therefore that, if b be small, the pressure, after a certain value of x, decreases with extreme rapidity as we get away from the neighbourhood

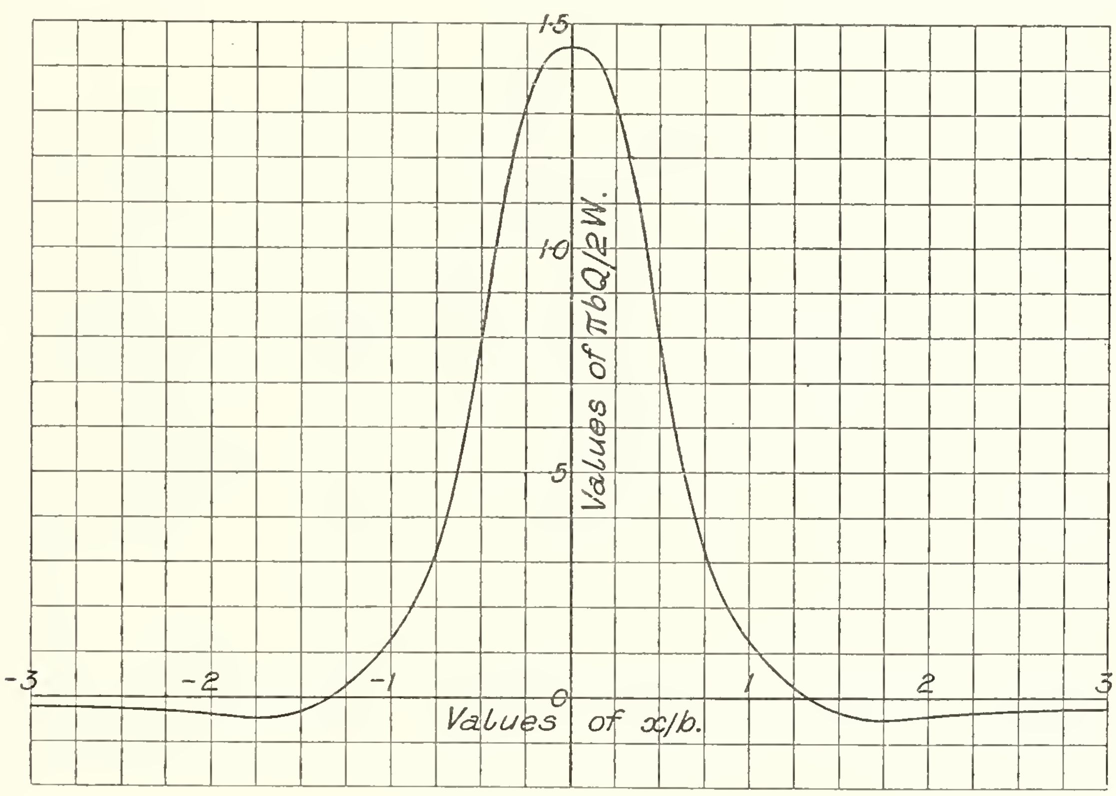

Fig. iv.

of the concentrated load, because, z being then large, even for moderate values of x the influence of the exponential term will be predominant. On the other hand, if b becomes finite, or even large, the algebraic polynomial factor will become predominant, and the decrease as we go away from the point of loading will become much less rapid. The expansion (93) gives us a link, as it were, between the case of a very thin beam, where the local effects die out according to a negative exponential of the distance along the axis, and that of an infinite solid, where they decrease as an inverse power of the distance from the point of loading.

A diagram is given in fig. iv. showing the variation of the pressure Q along the base of the elastic block where it rests on the rigid plane. The ordinates represent the ratio of Q to $2W/\pi b$—that is, the integral which has been called I. The abscissæ represent the quantities x/b. The diagram has been plotted from the following values of I, which have been calculated :—

x/b.	I.
0	1·4444
$\pi/6$	·7412
$\pi/3$	·1125
$\pi/2$	− ·0300
$2\pi/3$	− ·0252
π	− ·0036

For a value of x/b equal to 1·35 about the pressure vanishes, and is replaced by a *tension*. This is a very remarkable result, as it shows that an elastic block, acted upon by a concentrated load along a line of its upper surface transverse to its length, cannot have its whole base in contact with a smooth rigid plane on which it rests: at a certain distance from the load the body of the beam is lifted off the plane.

It would therefore appear as though the problem treated of above were impossible to realise in practice. But obviously we may superimpose any uniform pressure on the top of the beam, sufficient to make the total pressure at every point below positive. This may be done, in some cases, by the weight of the beam itself, if the weight W be not too large.

Further, the tensions required to keep the lower surface of the block horizontal are, as we may see from fig. iv., very small. If we leave them out of account, we do not sensibly disturb the distribution of the large pressure under the load, so that fig. iv. still gives us an approximation if we omit the negative part of the curve altogether.

This gives a maximum pressure just below the load equal to $(W/b) \times {}^{\cdot}920$, or rather less than the pressure due to the load W distributed uniformly over the vertical cross-section of the block. This pressure diminishes rapidly as we go away from this point, being very small at a distance from it equal to about 1·35 of the height of the block.

We cannot tell exactly, in the actual case, where the pressure will be first absolutely nil. We can form a rough estimate, however, of the dimensions of the area in contact by taking the area over which, in the solution obtained, the stress is always a pressure. This area extends to a distance of 1·35 × height of block, on either side of the vertical through the load.

Some rough experiments on a block of india-rubber lying on a wooden table have confirmed the result that the block is lifted out of contact with the table away from the load, and that the area of contact is of the above order.

PART III.

Solution for a Beam Under Asymmetrical Normal Forces: Special Case of Two Opposite Concentrated Loads not in the same Vertical Straight Line.

§ 27. *Expressions for the Displacements and Stresses in Series.*

Let us now proceed to consider what the general solution becomes in the case of a beam subject to normal forces which are now no longer restricted to be symmetrical.

In this case coefficients γ and δ come in, as well as α, β; κ, ν, ζ, θ being all zero.

Consider particularly a beam (fig. v.) subject to a downwards concentrated

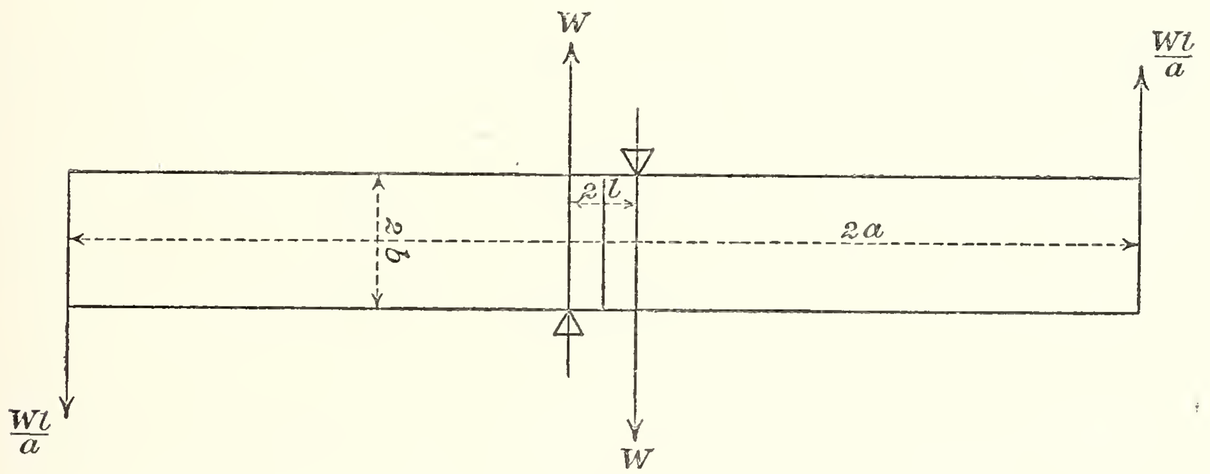

Fig. v.

load W, acting upon its upper surface at $x = l$, and an upwards concentrated load W, acting upon its lower surface at $x = -l$.

Such a system by itself is not in equilibrium. But the solution will introduce two shears over the ends, equal to $\sum_1^\infty (\gamma_n - \delta_n) \dfrac{\cos ma}{m}$ by equation (50).

In the case taken above $\alpha_0 = \beta_0 = - W/2a$, $\alpha_n = \beta_n = - \dfrac{W}{a}\cos ml, \gamma_n = -\delta_n$ $= - \dfrac{W}{a}\sin ml$, where $m = n\pi/a$, n being an integer. Hence the shears over the ends are $\sum_1^\infty (-1)^{n-1}\dfrac{2W}{n\pi}\sin\dfrac{n\pi l}{a} = \dfrac{Wl}{a}$, and these will satisfy the conditions of rigid equilibrium.

We then find the following expressions for the stresses and displacements in series :

$$
\left.
\begin{aligned}
U = &-\sum_1^\infty \frac{1}{m}\frac{W}{a}\cos ml \frac{\left\{\dfrac{1}{\lambda'+\mu}\sinh mb - \dfrac{1}{\mu} mb\cosh mb\right\}}{\sinh 2mb + 2mb}\cosh my \sin mx \\
&-\sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\cos ml \sinh mb}{\sinh 2mb + 2mb}\, y \sinh my \sin mx \\
&+\sum_1^\infty \frac{1}{m}\frac{W}{a}\sin ml \frac{\left\{\dfrac{1}{\lambda'+\mu}\cosh mb - \dfrac{1}{\mu} mb\sinh mb\right\}}{\sinh 2mb - 2mb}\sinh my \cos mx \\
&+\sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\sin ml \cosh mb}{\sinh 2mb - 2mb}\, y \cosh my \cos mx \\
&-\frac{\eta W x}{2aE} - Ay. \\
V = &-\sum_1^\infty \frac{1}{m}\frac{W}{a}\cos ml \frac{\left\{\left(\dfrac{1}{\lambda'+\mu}+\dfrac{1}{\mu}\right)\sinh mb + \dfrac{1}{\mu} mb\cosh mb\right\}}{\sinh 2mb + 2mb}\sinh my \cos mx \\
&+\sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\cos ml \sinh mb}{\sinh 2mb + 2mb}\, y \cosh my \cos mx \\
&-\sum_1^\infty \frac{1}{m}\frac{W}{a}\sin ml \frac{\left\{\left(\dfrac{1}{\lambda'+\mu}+\dfrac{1}{\mu}\right)\cosh mb + \dfrac{1}{\mu} mb\sinh mb\right\}}{\sinh 2mb - 2mb}\cosh my \sin mx \\
&+\sum_1^\infty \frac{1}{\mu}\frac{W}{a}\frac{\sin ml \cosh mb}{\sinh 2mb - 2mb}\, y \sinh my \sin mx \\
&-\frac{Wy}{2aE} + Ax.
\end{aligned}
\right\} \quad (94).
$$

$$
\left.
\begin{aligned}
P = &-\frac{2W}{a}\sum_1^\infty \cos ml \frac{\sinh mb - mb\cosh mb}{\sinh 2mb + 2mb}\cos mx \cosh my \\
&-\frac{2W}{a}\sum_1^\infty \cos ml \frac{my \sinh mb}{\sinh 2mb + 2mb}\cos mx \sinh my \\
&-\frac{2W}{a}\sum_1^\infty \sin ml \frac{\cosh mb - mb\sinh mb}{\sinh 2mb - 2mb}\sin mx \sinh my \\
&-\frac{2W}{a}\sum_1^\infty \sin ml \frac{my \cosh mb}{\sinh 2mb - 2mb}\sin mx \cosh my. \\
Q = &-\frac{W}{2a} - \frac{2W}{a}\sum_1^\infty \cos ml \frac{\sinh mb + mb\cosh mb}{\sinh 2mb + 2mb}\cos mx \cosh my \\
&+\frac{2W}{a}\sum_1^\infty \cos ml \frac{my \sinh mb}{\sinh 2mb + 2mb}\cos mx \sinh my \\
&-\frac{2W}{a}\sum_1^\infty \sin ml \frac{\cosh mb + mb\sinh mb}{\sinh 2mb - 2mb}\sin mx \sinh my \\
&+\frac{2W}{a}\sum_1^\infty \sin ml \frac{my \cosh mb}{\sinh 2mb - 2mb}\sin mx \cosh my.
\end{aligned}
\right\} \quad (95).
$$

$$\left.\begin{aligned} S = {} & \frac{2W}{a} \sum_1^\infty \cos ml \frac{mb \cosh mb}{\sinh 2mb + 2mb} \sin mx \sinh my \\ & - \frac{2W}{a} \sum_1^\infty \cos ml \frac{my \sinh mb}{\sinh 2mb + 2mb} \sin mx \cosh my \\ & - \frac{2W}{a} \sum_1^\infty \sin ml \frac{mb \sinh mb}{\sinh 2mb - 2mb} \cos mx \cosh my \\ & + \frac{2W}{a} \sum_1^\infty \sin ml \frac{my \cosh mb}{\sinh 2mb - 2mb} \cos mx \sinh my \end{aligned}\right\} \quad . \quad . \quad . \quad (96),$$

where A in the above is an arbitrary constant representing a rigid body rotation. If the conditions of fixing are that the two extremities of the horizontal axis are to remain at the same vertical height after strain, A is zero.

If, on the other hand, we fix the beam in such a way that the shears Wl/a over the ends are each allowed to produce, at the extremities of the axis, the deflection which they would produce if the bar were clamped at its middle and the deflection were calculated on the Euler-Bernoulli theory, then we find $Aa = \frac{Wla^2}{2Eb^3}$. This appears to be the more natural method of fixing. We shall, therefore, in what follows, suppose A to have this value.

§ 28. *Integral Expressions when a is made Infinite.*

When we increase the length of the bar indefinitely, it is easy to show that, if we take the last given value of A, the displacements remain finite at a finite distance and the stresses remain finite throughout—excepting, of course, at the points where the concentrated loads act.

We then obtain, as in § 15,

$$\left.\begin{aligned} U = {} & -\frac{W}{\pi}\int_0^\infty \frac{1}{u}\left(\frac{\frac{1}{\lambda'+\mu}\sinh u - \frac{1}{\mu} u \cosh u}{\sinh 2u + 2u}\right) \cos\frac{ul}{b} \cosh\frac{uy}{b} \sin\frac{ux}{b}\, du \\ & - \frac{Wy}{\mu\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u + 2u} \cos\frac{ul}{b} \sinh\frac{uy}{b} \sin\frac{ux}{b}\, du \\ & + \frac{W}{\pi}\int_0^\infty \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\cosh u - \frac{u}{\mu}\sinh u\right)}{\sinh 2u - 2u} \sin\frac{ul}{b} \sinh\frac{uy}{b} \cos\frac{ux}{b} - \left(\frac{1}{\lambda'+\mu}\right)\frac{3ly}{4u^2b^2}\right\} du \\ & + \frac{Wy}{\pi b\mu}\int_0^\infty \left(\frac{\cosh u}{\sinh 2u - 2u} \sin\frac{ul}{b} \cosh\frac{uy}{b} \cos\frac{ux}{b} - \frac{3}{4u^2}\frac{l}{b}\right) du \end{aligned}\right\} \quad (97)$$

$$\left.\begin{aligned}
V = &-\frac{W}{\pi}\int_0^\infty \frac{1}{u}\left\{\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sinh u+\frac{u}{\mu}\cosh u}{\sinh 2u+2u}\right\}\cos\frac{ul}{b}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{Wy}{\mu\pi b}\int_0^\infty \frac{\sinh u}{\sinh 2u+2u}\cos\frac{ul}{b}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&-\frac{W}{\pi}\int_0^\infty\left[\frac{1}{u}\left\{\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u+\frac{u}{\mu}\sinh u}{\sinh 2u-2u}\right\}\sin\frac{ul}{b}\cosh\frac{uy}{b}\sin\frac{ux}{b}\right. \\
&\qquad\qquad\left.-\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4u^2}\frac{lx}{b^2}\right]du \\
&+\frac{Wy}{\mu\pi b}\int_0^\infty \frac{\cosh u}{\sinh 2u-2u}\sin\frac{ul}{b}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du
\end{aligned}\right\} \quad (97).$$

$$\left.\begin{aligned}
P = &-\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u-u\cosh u}{\sinh 2u+2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&-\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u+2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2W}{\pi b}\int_0^\infty \frac{\cosh u-u\sinh u}{\sinh 2u-2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u-2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \\[1ex]
Q = &-\frac{2W}{\pi b}\int_0^\infty \frac{\sinh u+u\cosh u}{\sinh 2u+2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u+2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2W}{\pi b}\int_0^\infty \frac{\cosh u+u\sinh u}{\sinh 2u-2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&+\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u-2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \\[1ex]
S = &\ \frac{2W}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u+2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
&-\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u+2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&-\frac{2W}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u-2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
&+\frac{2Wy}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u-2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du
\end{aligned}\right\} \quad . \quad . \quad (98).$$

Now, as before, these expressions may be expanded in powers of r about the origin. In this case they will be found to have a radius of convergence $\sqrt{l^2+b^2}$. Or they may be expanded about either point of concentrated loading, when they will have a radius of convergence $2\sqrt{l^2+b^2}$, or they may be split up as follows :—

Write

$$\frac{\cosh u}{\sinh 2u + 2u} = e^{-u} + e^{-u}\frac{(1 - 4u + e^{-2u})}{2(\sinh 2u + 2u)}$$

$$\frac{\cosh u}{\sinh 2u - 2u} = e^{-u} + e^{-u}\frac{(1 + 4u + e^{-2u})}{2(\sinh 2u - 2u)}$$

$$\frac{\sinh u}{\sinh 2u + 2u} = e^{-u} + e^{-u}\frac{(-1 - 4u + e^{-2u})}{2(\sinh 2u + 2u)}$$

$$\frac{\sinh u}{\sinh 2u - 2u} = e^{-u} + e^{-u}\frac{(-1 + 4u + e^{-2u})}{2(\sinh 2u - 2u)}$$

and consider separately the parts of the integrals due to the first and second terms of the right-hand sides of the above equations.

We find, after some reductions, on writing $b - y = y'$, $y'^2 + (x-l)^2 = r_1^2$, $y - b = y''$, $y''^2 + (x+l)^2 = r_2^2$, $(x+l)/y'' = \tan\phi_2$, $(x-l)/y' = \tan\phi_1$,

$$\left.\begin{aligned}
P &= -\frac{W\cos\phi_1}{\pi r_1} - \frac{W\cos\phi_2}{\pi r_2} + \frac{Wy'}{\pi r_1^2}\cos 2\phi_1 + \frac{Wy''}{\pi r_2^2}\cos 2\phi_2 + P_2 \\
Q &= -\frac{W\cos\phi_1}{\pi r_1} - \frac{W\cos\phi_2}{\pi r_2} - \frac{Wy'}{\pi r_1^2}\cos 2\phi_1 - \frac{Wy''}{\pi r_2^2}\cos 2\phi_2 + Q_2 \\
S &= \frac{Wy'}{\pi r_1^2}\sin 2\phi_1 - \frac{Wy''}{\pi r_2^2}\sin 2\phi_2 + S_2
\end{aligned}\right\} \quad . \quad . \quad (99),$$

where

$$\left.\begin{aligned}
P_2 = {} & \frac{W}{\pi b}\int_0^\infty \frac{(1 + 5u - 4u^2 - (1-u)e^{-2u})e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
& - \frac{W}{\pi b}\int_0^\infty \frac{(1 + 5u - 4u^2 + (1-u)e^{-2u})e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& + \frac{Wy}{\pi b^2}\int_0^\infty \frac{u(1 + 4u - e^{-2u})e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& - \frac{Wy}{\pi b^2}\int_0^\infty \frac{u(1 + 4u + e^{-2u})e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du, \\
Q_2 = {} & \frac{W}{\pi b}\int_0^\infty \frac{(1 + 3u + 4u^2 - (1+u)e^{-2u})e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\
& - \frac{W}{\pi b}\int_0^\infty \frac{(1 + 3u + 4u^2 + (1+u)e^{-2u})e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& - \frac{Wy}{\pi b^2}\int_0^\infty \frac{u(1 + 4u - e^{-2u})e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \\
& + \frac{Wy}{\pi b^2}\int_0^\infty \frac{u(1 + 4u + e^{-2u})e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du,
\end{aligned}\right\} \quad . \quad (100),$$

$$\left.\begin{aligned} S_2 = {} & \frac{W}{\pi b}\int_0^\infty \frac{u(1-4u+e^{-2u})e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\sinh\frac{uy}{b}\,du \\ & -\frac{W}{\pi b}\int_0^\infty \frac{u(-1+4u+e^{-2u})e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ & +\frac{Wy}{\pi b^2}\int_0^\infty \frac{u(1+4u-e^{-2u})e^{-u}}{\sinh 2u + 2u}\cos\frac{ul}{b}\sin\frac{ux}{b}\cosh\frac{uy}{b}\,du \\ & +\frac{Wy}{\pi b^2}\int_0^\infty \frac{u(1+4u+e^{-2u})e^{-u}}{\sinh 2u - 2u}\sin\frac{ul}{b}\cos\frac{ux}{b}\sinh\frac{uy}{b}\,du \end{aligned}\right\} \quad . \ (100).$$

P_2, Q_2, S_2 are finite and continuous all over the beam. They may be expanded in powers of r about the origin, the series being convergent inside a circle of radius $\sqrt{l^2+(3b)^2}$, so that the points of concentrated loading are included. The parts of P, Q, S which become infinite at the points where the load acts are of the same form as if the beam were an infinite plate.

§ 29. *Series in Powers of r.*

We may here quote the expressions for P_2, Q_2, S_2 in powers of r. They are :

$$\begin{aligned} P_2 = {} & \frac{W}{\pi b}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}(1+5u-4u^2-(1-u)e^{-2u})e^{-u}}{\sinh 2u+2u}\cos\frac{ul}{b}\,du \\ & -\frac{W}{\pi b}\sum_1^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}(1+5u-4u^2+(1-u)e^{-2u})e^{-u}}{\sinh 2u-2u}\sin\frac{ul}{b}\,du \\ & +\frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!}\int_0^\infty \frac{u^{2\nu+2}(1+4u-e^{-2u})e^{-u}}{\sinh 2u+2u}\cos\frac{ul}{b}\,du \\ & -\frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!}\int_0^\infty \frac{u^{2\nu+2}(1+4u+e^{-2u})e^{-u}}{\sinh 2u-2u}\sin\frac{ul}{b}\,du, \end{aligned}$$

$$\begin{aligned} Q_2 = {} & \frac{W}{\pi b}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\cos 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}(1+3u+4u^2-(1+u)e^{-2u})e^{-u}}{\sinh 2u+2u}\cos\frac{ul}{b}\,du \\ & -\frac{W}{\pi b}\sum_1^\infty\left(\frac{r}{b}\right)^{2\nu}\frac{\sin 2\nu\phi}{(2\nu)!}\int_0^\infty \frac{u^{2\nu}(1+3u+4u^2+(1+u)e^{-2u})e^{-u}}{\sinh 2u-2u}\sin\frac{ul}{b}\,du \\ & -\frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!}\int_0^\infty \frac{u^{2\nu+2}(1+4u-e^{-2u})e^{-u}}{\sinh 2u+2u}\cos\frac{ul}{b}\,du \\ & +\frac{Wy}{\pi b^2}\sum_0^\infty\left(\frac{r}{b}\right)^{2\nu+1}\frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!}\int_0^\infty \frac{u^{2\nu+2}(1+4u+e^{-2u})e^{-u}}{\sinh 2u-2u}\sin\frac{ul}{b}\,du, \end{aligned}$$

$$
\begin{aligned}
S_2 = & \frac{W}{\pi b} \sum_1^\infty \left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} \int_0^\infty \frac{u^{2\nu+1}(1 - 4u + e^{-2u})\, e^{-u}}{(\sinh 2u + 2u)} \cos \frac{ul}{b}\, du \\
& + \frac{W}{\pi b} \sum_0^\infty \left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} \int_0^\infty \frac{u^{2\nu+1}(1 - 4u - e^{-2u})\, e^{-u}}{(\sinh 2u - 2u)} \sin \frac{ul}{b}\, du \\
& + \frac{Wy}{\pi b^2} \sum_0^\infty \left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \int_0^\infty \frac{u^{2\nu+2}(1 + 4u - e^{-2u})\, e^{-u}}{\sinh 2u + 2u} \cos \frac{ul}{b}\, du \\
& + \frac{Wy}{\pi b^2} \sum_0^\infty \left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \int_0^\infty \frac{u^{2\nu+2}(1 + 4u + e^{-2u})\, e^{-u}}{\sinh 2u - 2u} \sin \frac{ul}{b}\, du,
\end{aligned}
$$

where $r^2 = x^2 + y^2$, $x = y \tan \phi$.

U, V may be broken up in like manner and the parts U_2, V_2 which remain finite and continuous everywhere can be expanded in the same way.

We shall require also the series for U, V, P, Q, S in powers of r, deduced directly from the expressions (98). They are

$$
\left.
\begin{aligned}
U = & -\frac{W}{\pi} \sum_0^\infty \left(\frac{1}{\lambda' + \mu} C_{2\nu-1} - \frac{1}{\mu} C_{2\nu}\right)\left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{Wy}{\pi b\mu} \sum_1^\infty C_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} + \frac{Wy}{\pi b\mu} \sum_0^\infty S_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} \\
& + \frac{W}{\pi} \sum_0^\infty \left(\frac{1}{\lambda' + \mu} S_{2\nu-1} - \frac{1}{\mu} S_{2\nu}\right)\left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!}, \\
V = & -\frac{W}{\pi} \sum_0^\infty \left\{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right) C_{2\nu-1} + \frac{1}{\mu} C_{2\nu}\right\}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& + \frac{Wy}{\pi b\mu} \sum_0^\infty C_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} + \frac{Wy}{\pi b\mu} \sum_1^\infty S_{2\nu-1}\left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} \\
& - \frac{W}{\pi} \sum_0^\infty \left\{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right) S_{2\nu-1} + \frac{1}{\mu} S_{2\nu}\right\}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!}, \\
P = & -\frac{2W}{\pi b} \sum_0^\infty (C_{2\nu-1} - C_{2\nu})\left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} - \frac{2Wy}{\pi b^2} \sum_0^\infty C_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{2W}{\pi b} \sum_1^\infty (S_{2\nu-1} - S_{2\nu})\left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} - \frac{2Wy}{\pi b^2} \sum_0^\infty S_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!}, \\
Q = & -\frac{2W}{\pi b} \sum_0^\infty (C_{2\nu-1} + C_{2\nu})\left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} + \frac{2Wy}{\pi b^2} \sum_0^\infty C_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{2W}{\pi b} \sum_1^\infty (S_{2\nu-1} + S_{2\nu})\left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} + \frac{2Wy}{\pi b^2} \sum_0^\infty S_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin (2\nu+1)\phi}{(2\nu+1)!}, \\
S = & \frac{2W}{\pi b} \sum_1^\infty C_{2\nu}\left(\frac{r}{b}\right)^{2\nu} \frac{\sin 2\nu\phi}{(2\nu)!} - \frac{2Wy}{\pi b^2} \sum_0^\infty C_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\sin \widehat{2\nu+1}\,\phi}{(2\nu+1)!} \\
& - \frac{2W}{\pi b} \sum_0^\infty S_{2\nu}\left(\frac{r}{b}\right)^{2\nu} \frac{\cos 2\nu\phi}{(2\nu)!} + \frac{2Wy}{\pi b^2} \sum_0^\infty S_{2\nu+1}\left(\frac{r}{b}\right)^{2\nu+1} \frac{\cos \widehat{2\nu+1}\,\phi}{(2\nu+1)!},
\end{aligned}
\right\} \quad (101),
$$

where

$$\left.\begin{aligned}
C_{2\nu} &= \int_0^\infty \frac{u^{2\nu+1}\cosh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du & \nu &= 0, 1, 2, \ldots \\
C_{2\nu+1} &= \int_0^\infty \frac{u^{2\nu+2}\sinh u}{\sinh 2u + 2u}\cos\frac{ul}{b}\,du & \nu &= -1, 0, 1, 2, \ldots \\
S_{2\nu} &= \int_0^\infty \frac{u^{2\nu+1}\sinh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du & \nu &= 0, 1, 2, \ldots \\
S_{2\nu+1} &= \int_0^\infty \frac{u^{2\nu+2}\cosh u}{\sinh 2u - 2u}\sin\frac{ul}{b}\,du & \nu &= 0, 1, 2, \ldots
\end{aligned}\right\} \quad . \quad . \quad (102).$$

and

$$S_{-1} = \int_0^\infty \left\{\frac{\cosh u \sin ul/b}{\sinh 2u - 2u} - \frac{3l}{4u^2 b}\right\} du.$$

§ 30. *Distortion of the Axis of the Beam.*

If in the expression for V we write $y = 0$ we obtain the equation of the distorted form of the axis

$$V = -\frac{W}{\pi}\sum_0^\infty\left(\left\{\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right\}S_{2\nu-1} + \frac{1}{\mu}S_{2\nu}\right)(-1)^\nu\left(\frac{x}{b}\right)^{2\nu+1}\frac{1}{(2\nu+1)!}.$$

To the first approximation it is a semi-cubical parabola

$$V = -\frac{W}{\pi}\left\{\left(\frac{4}{E}S_{-1} + \frac{1}{\mu}S_0\right)\frac{x}{b} - \left(\frac{4}{E}S_1 + \frac{1}{\mu}S_2\right)\frac{x^3}{6b^3}\right\}.$$

This holds if x be small compared with b. If further we have l small compared with b, so that the two concentrated loads are applied in near parallel lines (*e.g.*, as in the case of material pressed between the edges of a pair of scissors), then we have, to the first approximation,

$$\begin{aligned}
S_{-1} &= \frac{l}{b}\int_0^\infty\left(\frac{u\cosh u}{\sinh 2u - 2u} - \frac{3}{4u^2}\right)du - \frac{l^3}{6b^3}\int_0^\infty\frac{u^3\cosh u\,du}{\sinh 2u - 2u}, \\
&= \frac{l}{b}F_1 - \tfrac{1}{6}\frac{l^3}{b^3}F_3 \text{ (see p. 99)}, \\
S_0 &= \frac{l}{b}G_2 - \tfrac{1}{6}\frac{l^3}{b^3}G_4. \\
S_1 &= \frac{l}{b}F_3 - \tfrac{1}{6}\frac{l^3}{b^3}F_5, \\
S_2 &= \frac{l}{b}G_4 - \tfrac{1}{6}\frac{l^3}{b^3}G_6.
\end{aligned}$$

The terms of order l^3/b^3 may be dropped in the coefficient of x^3/b^3, the latter quantity being already small, and we have finally

$$V = -\frac{W}{\pi E}\frac{lx}{b^2}\left[\left(4F_1 + \frac{E}{\mu}G_2\right) - \left(\tfrac{4}{6}F_3 + \frac{E}{\mu}\frac{G_4}{6}\right)\frac{l^2 + x^2}{b^2}\right],$$

or putting $E = 5\mu/2$ to simplify the arithmetic

$$V = -\frac{Wx}{Eb^2}\left[2\cdot80 - 4\cdot96\,\frac{l^2 + x^2}{b^2}\right].$$

The slope of the strained form of the axis at the origin is therefore a maximum when $2\cdot80 - 4\cdot96 \times \frac{3l^2}{b^2} = 0$, or $l/b = \cdot434$.

For such a value of l/b the approximation will not be quite valid. Still, it will be sufficient, even then, to give a rough idea of the values of the coefficients.

Assuming the formula given for V to hold for this value of l/b, we see that this greatest slope is $-\frac{W}{Eb}(\cdot810)$.

Now if the part of the beam between $x = \pm l$ were subjected to a uniform shear $W/2b$ giving the same total shear across the section, then, if the sections $x = \pm l$ were kept vertical, we should have $V = -\frac{W}{2b}\frac{x}{\mu} = -\frac{W}{Eb}x \times 1\cdot25$, if $E = 5\mu/2$. This gives a slope nearly 3/2 of the preceding one.

§ 31. *Distortion of the Cross-section $x = 0$, and Shear in that Cross-section.*

If we work out in the same way the value of U for $x = 0$ we find

$$U = \frac{W}{\pi}\sum_0^\infty \left(\frac{y}{b}\right)^{2\nu+1}\left\{\left(\frac{1}{\lambda' + \mu}S_{2\nu-1} - \frac{1}{\mu}S_{2\nu}\right)\frac{1}{(2\nu + 1)!} + \frac{1}{\mu}S_{2\nu-1}\frac{1}{(2\nu)!}\right\}.$$

If l be very small and y/b sufficiently small for 5th and higher powers to be neglected, this gives, assuming $E = 5\mu/2$ to simplify the arithmetic,

$$U = \frac{Wyl}{\pi Eb^2}\left[(4\,F_1 - 2\cdot5\,G_2) + \frac{y^2}{b^2}\left(\tfrac{3}{2}\,F_3 - \tfrac{5}{12}\,G_4\right)\right],$$

i.e.,

$$U = \frac{Wyl}{\pi Eb^2}\left[-\,5\cdot292 + \frac{y^2}{b^2}(\cdot492)\right].$$

We see, therefore, that the y^3 term is practically negligible, or, for a very large range of y, the mid-section remains sensibly plane.

For the shear in this cross-section, we have

$$S = -\frac{2W}{\pi b}\sum_0^\infty S_{2\nu}\left(\frac{y}{b}\right)^{2\nu}\frac{1}{(2\nu)!} + \frac{2W}{\pi b}\sum_0^\infty S_{2\nu+1}\left(\frac{y}{b}\right)^{2\nu+2}\frac{1}{(2\nu + 1)!}$$

or

$$S = -\frac{2W}{\pi b}S_0 - \frac{2W}{\pi b}\frac{y^2}{b^2}\left(\left(\frac{S_2}{2}\right) - S_1\right) - \text{higher terms.}$$

S is therefore a numerical minimum at the centre if $\frac{S_2}{2} - S_1 > 0$.

Now for the small values of l/b

$$S_2 = (l/b)\,G_4 - \tfrac{1}{6}(l/b)^3\,G_6,$$

$$S_1 = (l/b)\,F_3 - \tfrac{1}{6}(l/b)^3\,F_5.$$

But since $G_4 = 24{\cdot}824$, $F_3 = 7{\cdot}224$, when l/b is small $S_2 > 2S_1$, and the shear increases from the centre outwards. This is shown by the full curve (a) in fig. vi.

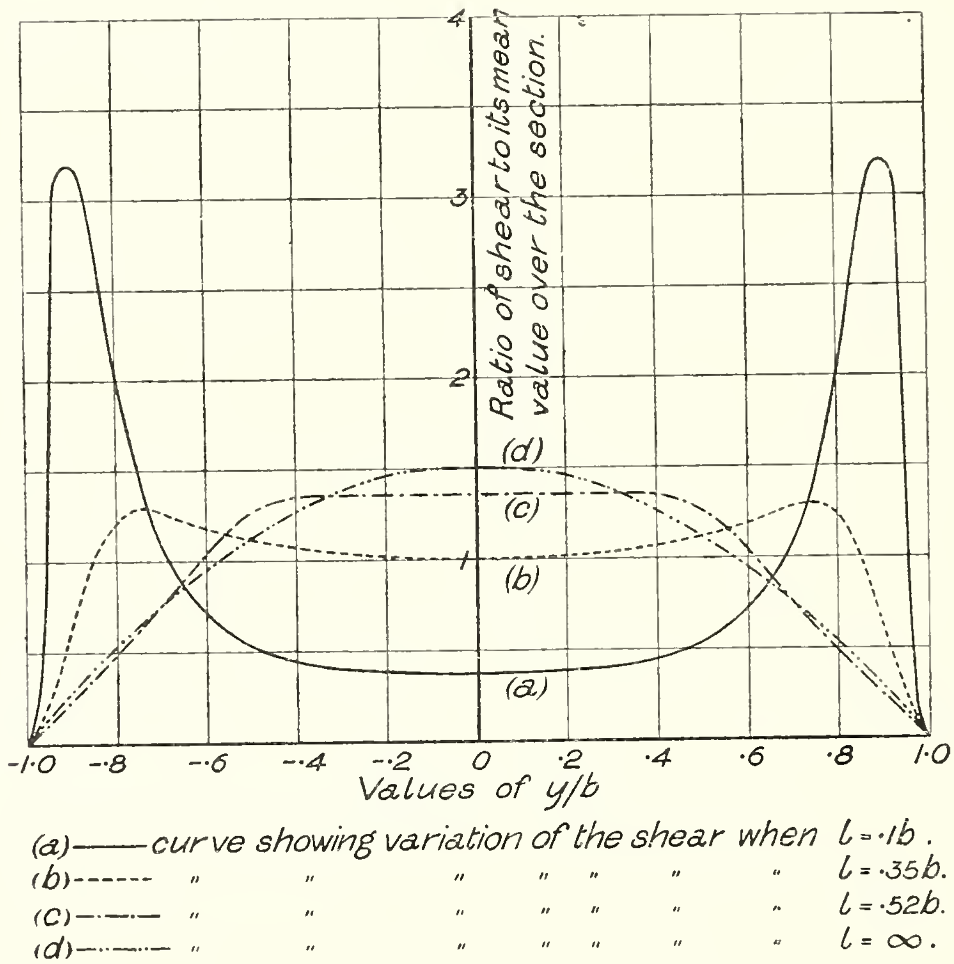

Fig. vi.

Near the edges $y = \pm b$, if l be small, the terms $\dfrac{Wy'}{\pi r_1^2}\sin 2\phi_1 - \dfrac{Wy''}{\pi r_2^2}\sin 2\phi_2$ will be the most important. Hence the shear is a minimum at the centre, increases to a high maximum corresponding to a distance from the edge equal to l approximately, and decreases down again to zero. The full curve in fig. vi. has been drawn for $l = b/10$.

As we increase l, these maxima at the sides become smaller and smaller and move towards the centre. At the same time the shear at the centre increases.

When l/b is made indefinitely large it is easily seen that S_0 and S_1 tend to the finite limit $3\pi/8$ whereas S_2 and all the others tend to zero.

Hence, for some value of l/b we must have $S_2 = 2S_1$.

If we calculate the values of S_2 and S_1 for $l/b = \pi/6, \pi/3, \pi/2$, we find

l/b.	S_1.	S_2.
$\pi/6$	1·9862	3·9475
$\pi/3$	1·2585	·3235
$\pi/2$	—	– ·0591

From these values and from the known behaviour of these functions near $l/b = 0$ and $l/b = \infty$ we can draw a rough diagram illustrating their variations. Fig. vii.

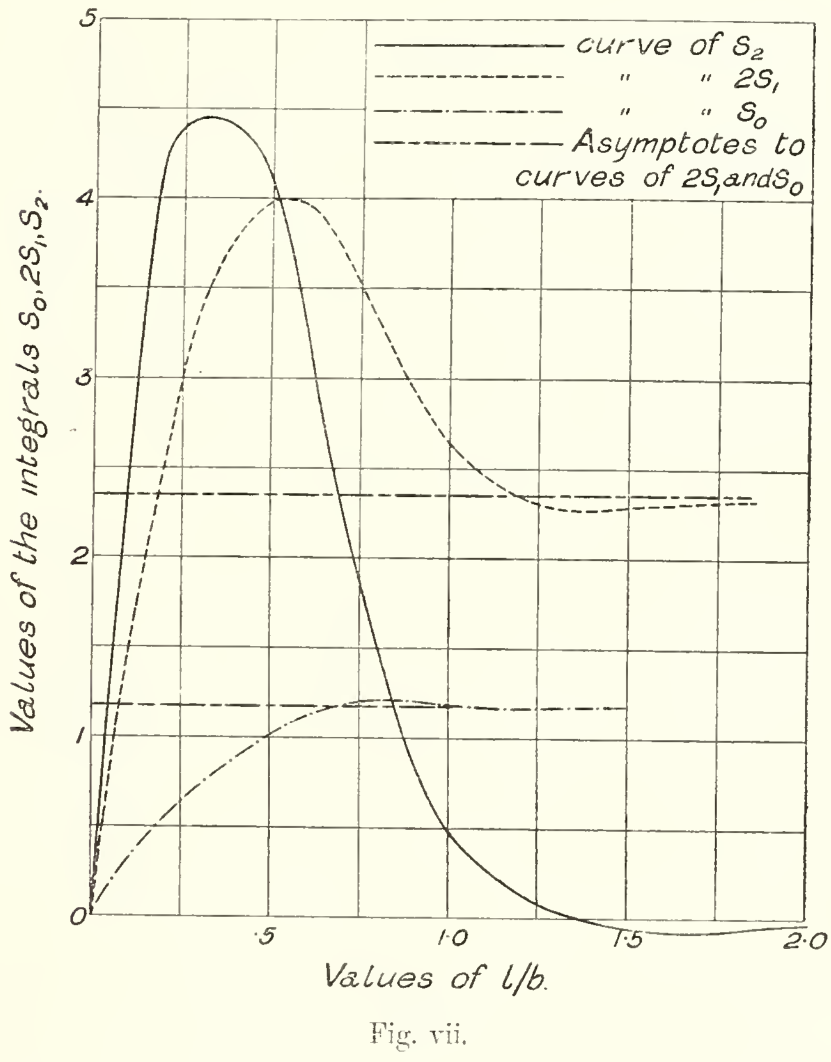

Fig. vii.

gives the curves of S_0, $2S_1$, and S_2 as we increase l. It will be seen from the figure that S_2 and $2S_1$ intersect when $l/b = ·52$ nearly.

Hence, when the arm of the couple is about half the height of the beam the shear is stationary at the centre, a horizontal straight line having contact of the third order with the curve. Curve (c), fig. vi., shows the distribution of shear, roughly sketched, for this case. It is easy to see that the centre corresponds to a maximum for the shear, for the next higher terms in the expansion of S are $-\frac{2W}{\pi b}\frac{y^4}{b^4}\left(\frac{S_4}{4!}-\frac{S_3}{3!}\right)$.

We have therefore a numerical maximum if $S_4 < 4S_3$, and a rough numerical calculation enables us to verify that this is the case.

The shear is therefore greatest at the centre, but decreases extremely slowly and remains constant over nearly half the section.

Another case of interest presents itself when the shear at the centre is exactly equal to its mean value over the section.

This occurs when $S_0 = \cdot 7854 = \pi/4$.

If we write $S_0 = (l/b)\,G_2 - \frac{1}{6}(l/b)^3\,G_4 = 2\cdot 818\; l/b - 4\cdot 138\; l^3/b^3$, we find that this roughly corresponds to $l/b = \cdot 32$.

Measured on the diagram for S_0 on fig. vii. the value of l/b corresponding to $S_0 = \pi/4$ would be about ·35. This latter value is probably the more correct, as for values of $l/b > \cdot 3$ the above approximation for S_0 is hardly sufficient.

In this case it is found that $S_2/2 - S_1 = \cdot 4$ roughly. The shear is therefore a minimum at the centre. It increases as we proceed outwards, but not very rapidly, and decreases down to zero at the edges. The curve is shown as (b) on fig. vi. The total area of the curve reckoned from a horizontal tangent at the middle point as base is zero, *i.e.*, there is as much above as below.

Finally, curve (d) on fig. vi. shows the distribution of shear when the arm of the couple is indefinitely increased. This is the parabola

$$S = -\frac{3W}{4b^3}\left(b^2 - y^2\right).$$

It is striking how very early this limiting distribution is reached. Fig. vii. already shows that the coefficients of the series reach their limiting values with great rapidity. For an arm of the couple equal to twice the height of the beam, the parabolic distribution of shear, corresponding to a long cantilever, will, at the mid-section, be practically undisturbed.

§ 32. *Practical Importance of this Problem.*

The problem which has been investigated in this part of the paper is one of considerable importance in practice. The only way in which we can apply a shearing force to materials is by means of two opposite asymmetrically situated pressures, such as we have dealt with in this case. The case of material cut through by scissors, which is frequently quoted as an example of the application of shearing stress, really corresponds to a stress-distribution of this kind. Similarly, a rivet which fastens together two plates is subjected to stress-systems of this type whenever the compound plate undergoes strain in its own plane. In nearly every modern engineering structure, such as railway bridges, &c., cases of this kind are of constant occurrence, and the strength of the structure depends, to a very great extent, upon the strength of the individual rivets. It becomes therefore a problem of the very greatest practical importance to know how the distribution of shear inside such a rivet varies with the dimensions of the rivet and with the thickness of the plates. At present our knowledge of the subject is purely empirical; and although the results of the present paper apply only to a rivet of rectangular section, and even then are only an approximation, yet they should furnish some indications which may be of value in other cases.

Another point which is illustrated by these results is the manner in which DE SAINT-VENANT'S solutions are modified, when we gradually bring the terminal systems of load closer together. We see that the modifications introduced are practically insensible at distances from the section where the load is applied which are greater than the height of the beam. This is of importance, as it tells us within which limits, in any experiment, we may assume the state of a beam to be given by one of the "uniform" solutions which only depend upon the total terminal conditions and which are transmitted without change of type.

PART IV.

SOLUTION FOR A BEAM WHOSE UPPER AND LOWER BOUNDARIES ARE ACTED UPON BY SHEARING STRESS ONLY.

§ 33. *Expressions for the Displacements and Stresses in Series and Integrals.*

Let us now consider a beam acted upon by shearing stress alone, over the boundaries $y = \pm b$. Then, in the general solution of § 7, $\alpha_n = \beta_n = \gamma_n = \delta_n = 0$.

If further we suppose the shear to reduce to a single concentrated force L at one point $(0, b)$ we have $\zeta_0 = \frac{L}{2a}$, $\zeta_n = \frac{L}{a}$, $\kappa_n = 0 = \theta_n = \nu_n$.

Putting in these values into (44), (45), (46), (47), (48), (54), and (55) we obtain

$$\left.\begin{aligned} U = & -\frac{\lambda' + 2\mu}{32\mu(\lambda' + \mu)}\frac{Lx^2}{ab} + \frac{3\lambda' + 2\mu}{32\mu(\lambda' + \mu)}\frac{Ly^2}{ab} + \frac{1}{\mu}\frac{Ly}{8a} - Ay + B \\ & + \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\cosh mb - \frac{1}{\mu}mb\sinh mb}{\sinh 2mb + 2mb}\cosh my\cos mx \\ & + \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\sinh mb - \frac{1}{\mu}mb\cosh mb}{\sinh 2mb - 2mb}\sinh my\cos mx \\ & + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\cosh mb\sinh my}{\sinh 2mb + 2mb}\cos mx + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\sinh mb\cosh my}{\sinh 2mb - 2mb}\cos mx, \end{aligned}\right\} \quad (103)$$

$$\left.\begin{aligned}
V = {} & \frac{\lambda'}{16\mu(\lambda'+\mu)}\frac{Lxy}{ab} - \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\frac{1}{\lambda'+\mu}\cosh mb + \frac{1}{\mu}mb\sinh mb}{\sinh 2mb + 2mb}\sinh my\sin mx \\
& - \sum_{n=1}^{\infty}\frac{L}{2am}\frac{\frac{1}{\lambda'+\mu}\sinh mb + \frac{1}{\mu}mb\cosh mb}{\sinh 2mb - 2mb}\cosh my\sin mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\cosh mb\cosh my}{\sinh 2mb + 2mb}\sin mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{1}{\mu}\frac{y\sinh mb\sinh my}{\sinh 2mb - 2mb}\sin mx + Ax + C, \\
P = {} & -\frac{Lx}{4ab} - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{4\cosh mb - 2mb\sinh mb}{\sinh 2mb + 2mb}\cosh my\sin mx \\
& - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{4\sinh mb - 2mb\cosh mb}{\sinh 2mb - 2mb}\sinh my\sin mx \\
& - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\cosh mb\sinh my}{\sinh 2mb + 2mb}\sin mx - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\sinh mb\cosh my}{\sinh 2mb - 2mb}\sin mx, \\
Q = {} & -\sum_{n=1}^{\infty}\frac{L}{2a}\frac{2mb\sinh mb\cosh my}{\sinh 2mb + 2mb}\sin mx - \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2mb\cosh mb\sinh my}{\sinh 2mb - 2mb}\sin mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\cosh mb\sinh my}{\sinh 2mb + 2mb}\sin mx + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\sinh mb\cosh my}{\sinh 2mb - 2mb}\sin mx, \\
S = {} & \frac{Ly}{4ab} + \frac{L}{4a} + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\cosh mb\cosh my}{\sinh 2mb + 2mb}\cos mx \\
& + \sum_{n=1}^{\infty}\frac{L}{2a}\frac{2my\sinh mb\sinh my}{\sinh 2mb - 2mb}\cos mx \\
& + \sum_{n=1}^{\infty}\frac{L}{a}\frac{\cosh mb - mb\sinh mb}{\sinh 2mb + 2mb}\sinh my\cos mx \\
& + \sum_{n=1}^{\infty}\frac{L}{a}\frac{\sinh mb - mb\cosh mb}{\sinh 2mb - 2mb}\cosh my\cos mx
\end{aligned}\right\} \quad (103),$$

where $m = n\pi/a$, and A, B, C are arbitrary constants to be determined from the fixing conditions.

Now if the fixing conditions are

(i.) That the displacement of the origin is to be zero ;

(ii.) That the extremities of the axis are to remain on the same horizontal line, then

$$C = 0,$$

$$B = -\sum_{n=1}^{\infty}\frac{L}{2am}\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh mb - \frac{1}{\mu}mb\sinh mb}{\sinh 2mb + 2mb},$$

$$A = 0 ;$$

but if we put in these values and then proceed to make a infinite, certain parts of the expressions for U and V do not give finite integrals in the limit.

This is due to the fact that the conditions of rigid equilibrium require shears $Lb/2a$ at the two ends (fig. viii.).

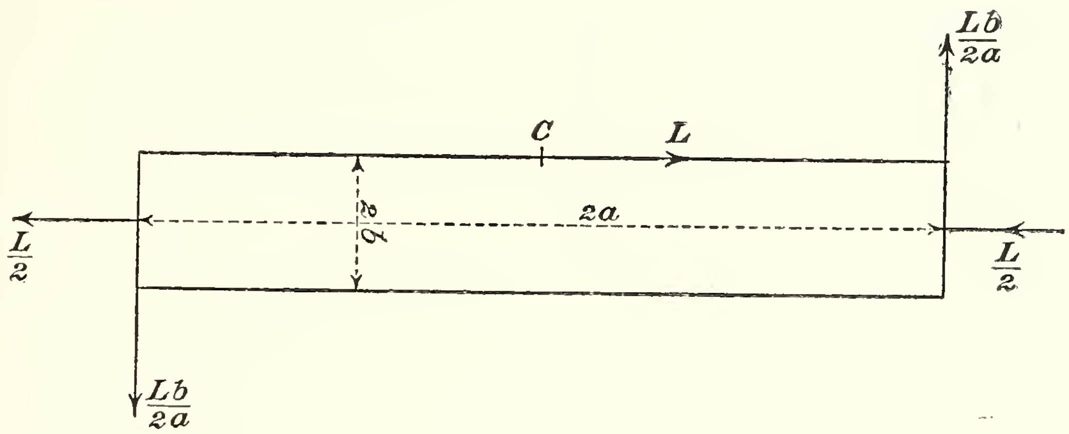

Fig. viii.

These shears $Lb/2a$ will produce a deflection due to bending alone, which, calculated from the Euler-Bernoulli formula, comes to

$$V = \frac{3L}{32ab^2}\left(ax^2 - \frac{x^3}{3}\right)\left(\frac{1}{\mu} + \frac{1}{\lambda' + \mu}\right) \quad (\text{for } x > 0),$$

and when a is made very large, this gives

$$V = \frac{3L}{32}\frac{x^2}{b^2}\left(\frac{1}{\mu} + \frac{1}{\lambda' + \mu}\right) \quad \ldots \ldots \ldots \quad (104)$$

for the bending deflection produced by the end shears at large distances x, which, however, are still finite compared with a. If, therefore, we allow the beam to bend freely under these end loads, in such a way that each of these produces its proper bending deflection and no more, the constant A must be adjusted so that, for large values of x, V tends to the value (104).

This implies that A must have an infinite part, which will exactly cancel the infinite part of V. It is easily found that the value

$$A = \left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\frac{3}{8}\frac{L}{a}\sum_{n=1}^{\infty}\frac{1}{m^2b^2} + A',$$

where A′ is finite, will introduce terms in both U and V which will make these quantities remain finite in the limit when a is infinite.

We then find, putting in for B the value found and proceeding to the limit,

$$\left.\begin{aligned}
\mathrm{U} = {} & \frac{\mathrm{L}}{2\pi}\int_0^\infty \frac{1}{u}\left[\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u - \frac{u}{\mu}\sinh u}{\sinh 2u + 2u}\left(\cosh\frac{uy}{b}\cos\frac{ux}{b} - 1\right)\right]du \\
& + \frac{\mathrm{L}}{2\pi}\int_0^\infty \frac{1}{u}\left[\frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sinh u - \frac{u}{\mu}\cosh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b} - \frac{1}{\lambda'+\mu}\frac{3y}{4ub}\right]du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\frac{\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\left[\frac{\sinh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b} - \frac{3}{4u^2}\right]du - \mathrm{A}'y, \\
\mathrm{V} = {} & -\frac{\mathrm{L}}{2\pi}\int_0^\infty \frac{1}{u}\left[\frac{\frac{1}{\lambda'+\mu}\cosh u + \frac{1}{\mu}u\sinh u}{\sinh 2u + 2u}\right]\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
& - \frac{\mathrm{L}}{2\pi}\int_0^\infty \left[\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3x}{4u^2 b}\right]du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\frac{\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
& + \frac{\mathrm{L}y}{2\pi b}\int_0^\infty \frac{1}{\mu}\frac{\sinh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du + \mathrm{A}'x
\end{aligned}\right\} \quad (105).$$

Now, when in V we put $y = 0$, we have left

$$-\frac{\mathrm{L}}{2\pi}\int_0^\infty \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u}\sin\frac{ux}{b} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3x}{4u^2 b}\right\}du + \mathrm{A}'x.$$

This integral may be written as the sum of two others,

$$-\frac{\mathrm{L}}{2\pi}\int_0^\infty \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4u^3} - \frac{1}{40u}\left(\frac{9}{\mu} - \frac{1}{\lambda'+\mu}\right)\right\}\sin\frac{ux}{b}\,du$$

$$+\frac{\mathrm{L}}{2\pi}\int_0^\infty \left\{\left[\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4}\frac{ux/b - \sin ux/b}{u^3}\right] - \frac{1}{40u}\left(\frac{9}{\mu} - \frac{1}{\lambda'+\mu}\right)\sin\frac{ux}{b}\right\}du.$$

Consider the first of these integrals, and let

$$f(u) = \left\{\frac{1}{u}\frac{\left(\frac{1}{\lambda'+\mu}\sinh u + \frac{1}{\mu}u\cosh u\right)}{\sinh 2u - 2u} - \left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\frac{3}{4u^3} - \frac{1}{40u}\left(\frac{9}{\mu} - \frac{1}{\lambda'+\mu}\right)\right\}.$$

Then $f(u)$ and its differential coefficients are finite and continuous for all values of u, and vanish for $u = \infty$. $f(u)$ itself $= 0$, when $u = 0$ and the integral $\int_0^\infty |f''(u)|\,du$ is finite, $|f''(u)|$ denoting the absolute value of $f''(u)$. It is then easy to see that $I = \int_0^\infty f(u) \sin u\xi \, du$ tends to zero as ξ tends to infinity. For, integrating by parts

$$I = \left[-\frac{1}{\xi}\cos u\xi f(u)\right]_0^\infty + \frac{1}{\xi}\int_0^\infty \cos u\xi f'(u)\, du$$

$$= \frac{1}{\xi}\int_0^\infty \cos u\xi f'(u)\, du.$$

But $\left|\int_0^\infty \cos u\xi f'(u)\, du\right| < \int_0^\infty |f'(u)|\, du <$ a finite quantity M; hence $I < \frac{M}{\xi}$,

and therefore tends to zero as ξ tends to infinity.

Hence, when x is large, V reduces to the second integral. The latter can be evaluated, and it comes to

$$\frac{3}{32}L\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\frac{x^2}{b^2} - \frac{L}{160}\left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu}\right)$$

for $x > 0$ and

$$-\frac{3L}{32}\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\frac{x^2}{b^2} + \frac{L}{160}\left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu}\right) \text{ for } x < 0.$$

The first terms correspond to the bending due to the shears at the ends.

We should therefore try to make $A'x - \frac{L}{160}\left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu}\right) = 0$ for all large values of x.

This is obviously impossible. But $A'x$ being eventually the most important term, the condition is approximately fulfilled by taking $A' = 0$. This determines U and V. We see that the effect of the isolated shear L is to *deflect* the central line of the beam through the distance $2 \times \frac{L}{160}\left(\frac{9}{\mu} - \frac{1}{\lambda' + \mu}\right)$ *away* from its line of action.

Putting $A' = 0$ in equations (105) they give us U and V. Integral expressions for the stresses are obtained in like manner. They are

$$\left.\begin{aligned} P = &-\frac{L}{\pi b}\int_0^\infty \frac{2\cosh u - u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\ &-\frac{L}{\pi b}\int_0^\infty \frac{2\sinh u - u\cosh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\ &-\frac{L}{\pi b}\int_0^\infty \frac{uy}{b}\frac{\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\ &-\frac{L}{\pi b}\int_0^\infty \frac{uy}{b}\frac{\sinh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \end{aligned}\right\} \quad . \quad . \quad . \quad (106)$$

$$\left.\begin{aligned}
Q = &-\frac{L}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&-\frac{L}{\pi b}\int_0^\infty \frac{u\cosh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&+\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&+\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du, \\
S = &\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{Ly}{\pi b^2}\int_0^\infty \frac{u\sinh u}{\sinh 2u - 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{L}{\pi b}\int_0^\infty \frac{\cosh u - u\sinh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&+\frac{L}{\pi b}\int_0^\infty \frac{\sinh u - u\cosh u}{\sinh 2u - 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du
\end{aligned}\right\} \quad . \quad . \quad (106).$$

§ 34. *Expressions for the Displacements and Stresses in Series of Powers of the Radius Vector from a Point.*

The expressions given above for U, V, P, Q, S may be transformed exactly as in §§ 16, 17, and we obtain expansions about the point $(0, b)$ where the shear is applied. Eventually, r', ϕ' having the same meaning as on p. 92, we find:

$$\left.\begin{aligned}
U = &-\frac{L}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\log\left(\frac{r'}{b}\right) - \frac{L}{2\pi\mu}\frac{y'}{r'}\cos\phi' \\
&+\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left\{D + \sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+2}\frac{\cos(2\nu+2)\phi'}{(2\nu+2)!}(H_{2\nu+1}-H_{2\nu})\right\} \\
&-\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu} \\
&-\frac{2Ly'}{\pi b}\frac{1}{\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}H_{2\nu} + \frac{2Ly'}{\pi b}\frac{1}{\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1}-H_{2\nu}), \\
V = &-\frac{L\phi'}{2\pi(\lambda'+\mu)} - \frac{L}{2\pi\mu}\frac{y'}{r'}\sin\phi' \\
&-\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu} \\
&\qquad+\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}(H_{2\nu-1}-H_{2\nu-2}) \\
&-\frac{2Ly'}{\pi b\mu}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1}-H_{2\nu}) + \frac{2Ly'}{\pi b\mu}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu}.
\end{aligned}\right\} (107),$$

$$
\left.
\begin{aligned}
P = &-\frac{2L}{\pi}\frac{\sin\phi'}{r'} + \frac{L}{\pi}\frac{y'}{r'^2}\sin 2\phi' \\
&-\frac{8L}{\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1}-H_{2\nu}) \\
&+\frac{4L}{\pi b}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu} + \frac{4Ly'}{\pi b^2}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu+2} \\
&-\frac{4Ly'}{\pi b^2}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}(H_{2\nu+1}-H_{2\nu}) \\
Q = &-\frac{L}{\pi}\frac{y'}{r'^2}\sin 2\phi' + \frac{4L}{\pi b}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}H_{2\nu} \\
&-\frac{4Ly'}{\pi b^2}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu+2} + \frac{4Ly'}{\pi b^2}\sum_1^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\sin 2\nu\phi'}{(2\nu)!}(H_{2\nu+1}-H_{2\nu}) \\
S = &\frac{L}{\pi}\frac{\cos\phi'}{r'} - \frac{L}{\pi}\frac{y'\cos 2\phi'}{r'^2} - \frac{4L}{\pi b}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}(H_{2\nu+1}-H_{2\nu}) \\
&-\frac{4Ly'}{\pi b^2}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu}\frac{\cos 2\nu\phi'}{(2\nu)!}(H_{2\nu+1}-H_{2\nu}) + \frac{4Ly'}{\pi b^2}\sum_0^\infty\left(\frac{r'}{b}\right)^{2\nu+1}\frac{\cos(2\nu+1)\phi'}{(2\nu+1)!}H_{2\nu+2}
\end{aligned}
\right\} (107),
$$

where the H's are given by equations (80) and are the same as before; and

$$
D = \int_0^\infty\left(\frac{u-\frac{1}{2}+\frac{1}{8}u^{-1}-\frac{1}{8}u^{-1}e^{-4u}}{\sinh^2 2u - 4u^2} + \frac{e^{-u}}{4u} - \frac{3}{16u^2} - \frac{1}{4u}\frac{\cosh u}{\sinh 2u + 2u}\right)du
$$

$$
+\frac{E}{4\mu}\int_0^\infty\frac{\sinh u}{4(\sinh 2u + 2u)}\,du.
$$

The leading terms in U, V, P, Q, S which precede the Σ's form what is left of this solution when b is made infinite. They give therefore the displacements and stresses due to a shear acting at an edge of an infinite plate.

They will be found to agree with the expressions obtained by BOUSSINESQ ('Comptes Rendus,' vol. 114, pp. 1465–1468) for an infinite solid, the strain being two-dimensional; provided that λ be changed into λ'.

At the point of loading itself the stresses are infinite and the displacements infinite or indeterminate.

The series in the expressions (107) are easily seen to have a radius of convergence $4b$.

The series for the shear reveals a very curious phenomenon. The terms due to the infinite plate may be written $\frac{2L}{\pi}\frac{x'^2y'}{r'^4}$. They give therefore a *positive* shear throughout, and zero shear on the axis of y. But when the other terms are taken into account, the shear at points on the axis of y is

$$
S = -\frac{4L}{\pi b}\left\{\frac{y'}{b}\,2(H_1 - H_0) - \frac{y'^2}{b^2}H_2 + \frac{y'^3}{b^3}\,\tfrac{2}{3}(H_3 - H_2) - \ldots\right\}
$$

$$
= -\frac{4L}{\pi b}\left\{\cdot 3638\,\frac{y'}{b} - \cdot 2271\,\frac{y'^2}{b^2} + \cdot 0733\,\frac{y'^3}{b^3} - \ldots\right\},
$$

which gives a *negative* shear on the axis of y, as soon as we get away from the point of loading.

It follows that there must be, on either side of the cross-section through the load, a locus of points of zero shear.

It is easy to find the approximate form of this locus in the neighbourhood of the point of loading. Retaining only the leading terms in the Σ's in the expression for the shear, we find that $S = 0$ when

$$\frac{2L}{\pi}\frac{x^2y'}{r'^4} = \frac{4L}{\pi b}\frac{y'}{b} \times 2(H_1 - H_0),$$

or

$$4(H_1 - H_0)\, r'^4 = x^2b^2, \quad i.e., \quad r'^2 \mp \frac{bx}{2\sqrt{H_1 - H_0}} = 0.$$

These are two circles passing through the point of loading and having their centres lying on the upper edge of the beam, at a distance from the point of loading equal to $\dfrac{b}{4\sqrt{H_1 - H_0}} = \cdot 587b$. These give a kind of wedge-shaped area, similar to that

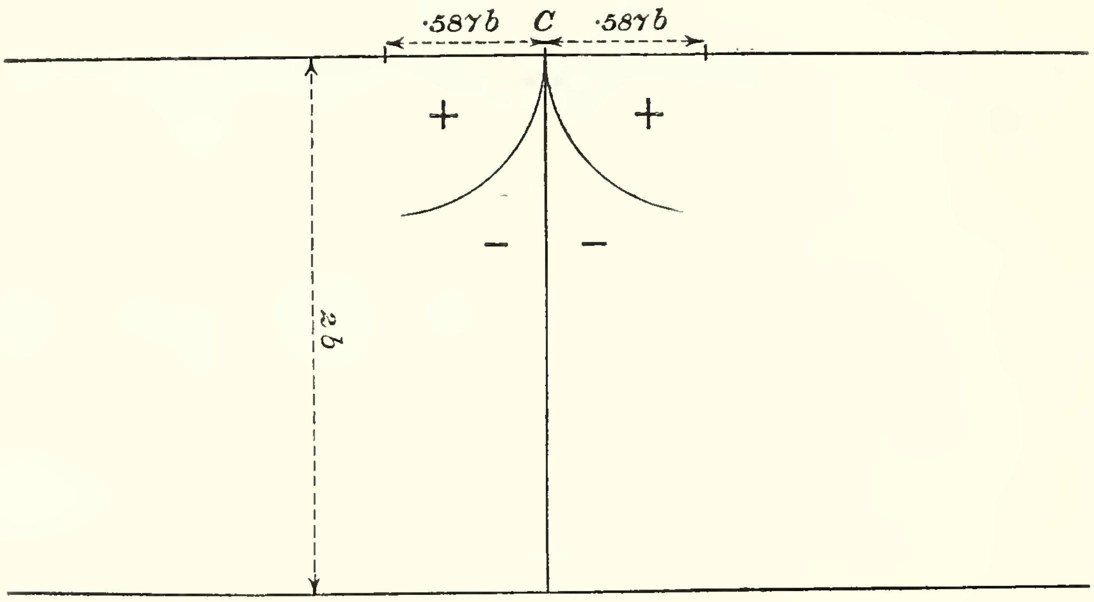

Fig. ix.

enclosed by the cusp of a caustic curve, inside which the shear is negative. This cusp is shown in fig. ix.

For higher values of y'/b this approximation will no longer hold, and the curve will deviate from the circle.

§ 35. *Distortion of the Beam.*

An interesting feature of a stress-system of this type is the distortion suffered by lines parallel to the axis of the beam.

We have already seen that at a certain distance the axis itself suffered a bodily shift, being depressed in front of the acting load and raised behind it.

The series for V in the neighbourhood of the load shows a similar phenomenon, points to the right of $x = 0$ being depressed by $\frac{L}{4}\frac{1}{\lambda' + \mu}$, and points to the left raised by the same amount.

$$(V)_{\substack{y'=0\\x>0}} = -\frac{L}{4(\lambda'+\mu)} - \frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\sum_0^\infty \left(\frac{x}{b}\right)^{2\nu+1}\frac{(-1)^\nu}{(2\nu+1)!}H_{2\nu},$$

H_0 being negative and H_2 being positive, as we go away from the load, the effect of the series is to decrease this effect, the level of the points in front of and behind the load tending to equalize itself. If we work out the series for V in the neighbourhood of the origin and of the point $(0, -b)$, we find (V) in the neighbourhood of origin

$$= -\frac{L}{2\pi}\sum_1^\infty\left(\frac{r}{b}\right)^\nu \frac{\sin\nu\phi}{\nu!}\left(\frac{G_{\nu-1}}{\lambda'+\mu} + \frac{F_\nu}{\mu}\right) + \frac{Ly}{2\pi b\mu}\sum_1^\infty\left(\frac{r}{b}\right)^\nu\frac{\sin\nu\phi}{\nu!}G_\nu,$$

where the F's and G's have the values given on p. 99, except that now

$$G_0 = \int_0^\infty\left(\frac{\sinh u}{\sinh 2u - 2u} - \frac{3}{4u^2}\right)du = -\cdot 2875.$$

Similarly (V) in neighbourhood of point $(0, -b)$

$$= -\frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right)\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi''}{(2\nu+1)!}H'_{2\nu+1}$$

$$+ \frac{2L}{\pi}\left(\frac{1}{\lambda'+\mu}\right)\sum_1^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\sin 2\nu\phi''}{(2\nu)!}(H'_{2\nu} - H'_{2\nu-1})$$

$$-\frac{2Ly''}{\pi b\mu}\sum_0^\infty\left(\frac{r''}{b}\right)^{2\nu+1}\frac{\sin(2\nu+1)\phi''}{(2\nu+1)!}(H'_{2\nu+2} - H'_{2\nu+1})$$

$$+\frac{2Ly''}{\pi b\mu}\sum_1^\infty\left(\frac{r''}{b}\right)^{2\nu}\frac{\sin 2\nu\phi''}{(2\nu)!}H'_{2\nu+1},$$

where the H′'s have the value given on page 102.

From these expressions we obtain the following values for the transverse displacements of points on the lines $y = 0$, $y = -b$:—

$$V_0 = -\frac{L}{2\pi}\sum_0^\infty\left(\frac{x}{b}\right)^{2\nu+1}\frac{(-1)^\nu}{(2\nu+1)!}\left(\frac{1}{\lambda'+\mu}G_{2\nu} + \frac{1}{\mu}F_{2\nu+1}\right)$$

$$V_{-b} = -\frac{2L}{\pi}\sum_0^\infty\left(\frac{x}{b}\right)^{2\nu+1}\frac{(-1)^\nu}{(2\nu+1)!}H'_{2\nu+1}\left(\frac{1}{\lambda'+\mu} + \frac{1}{\mu}\right).$$

So that, approximately, putting in the values of the constants,

$$V_{+b} = \frac{L}{E}\left\{ \begin{matrix} -\cdot 375\ (x+) \\ +\cdot 375\ (x-) \end{matrix} + \frac{x}{b}(\cdot 615) + \frac{x^3}{b^3}(\cdot 096) - \ldots \right\}$$

$$V_0 = \frac{L}{E}\left\{ -\frac{x}{b}(\cdot 106) + \frac{x^3}{b^3}(\cdot 591) - \ldots \right\}$$

$$V_{-b} = \frac{L}{E}\left\{ \frac{x}{b}(\cdot 125) + \frac{x^3}{b^3}(\cdot 228) - \frac{x^5}{b^5}(\cdot 041) + \ldots \right\},$$

where in the above uni-constant isotropy has been assumed to simplify the calculations, so that $\lambda' = \frac{2}{3}\mu$, $E = \frac{5}{2}\mu$.

The distortion, calculated from these formulæ, is represented on fig. x. for a range of x between $\pm \cdot 5b$. Curve (a) shows the distorted form of the upper edge, (b) that

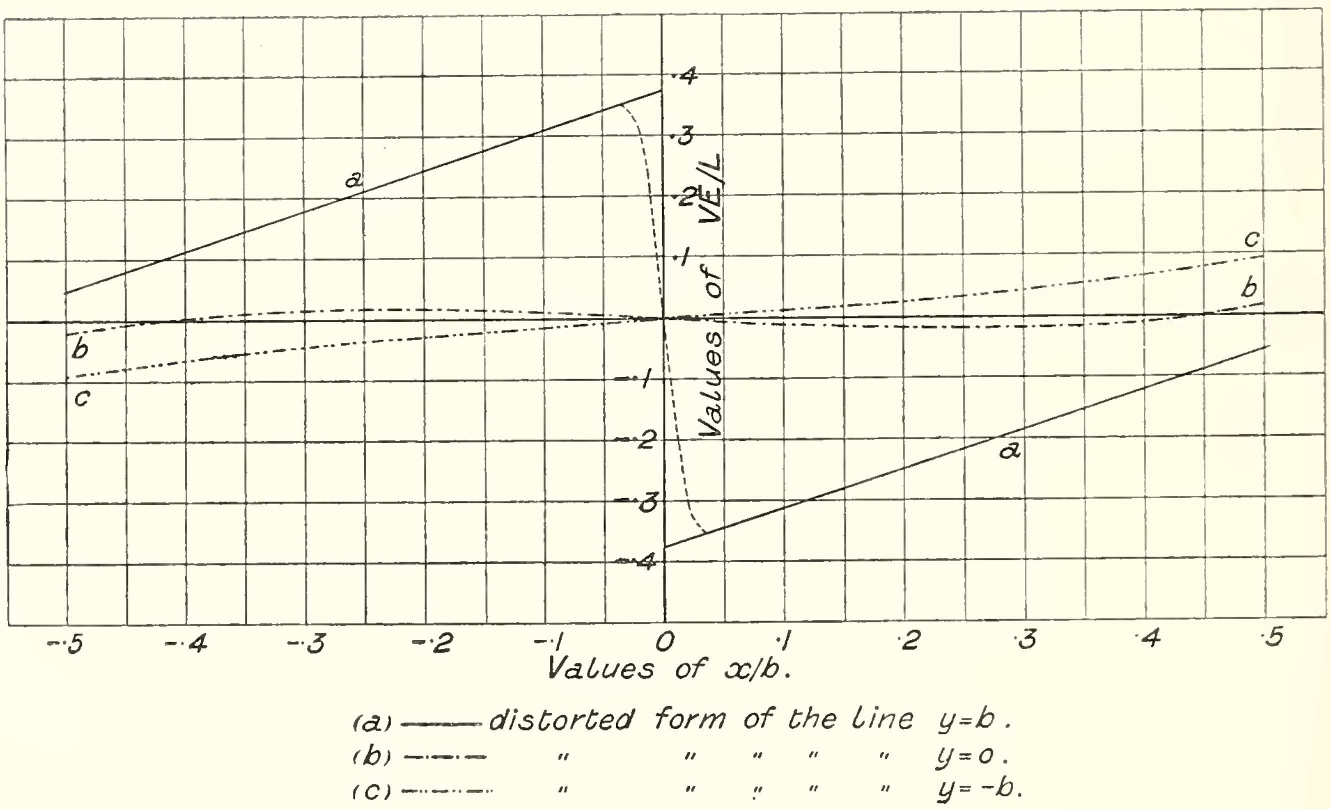

Fig. x.

of the axis, (c) that of the lower edge. With regard to (a) the limiting case, in which V is actually discontinuous, does not occur in practice. In order to get a real case, we have to take a horizontal line whose y' is very small without being actually zero. The discontinuity is then replaced by a very rapid variation, as shown by the dotted line.

The curves show that the depression produced in front of the load diminishes rapidly as we go away from the upper edge, and is even changed to a rise at the bottom of the beam. In every case, as we go away from the mid-section, the distorted lines rise to the right and fall to the left.

§ 36. *Case where the Shear is spread over an Area instead of a Line.*

As in § 22, we may consider the effect of distributing the concentrated shear over an area, instead of over a line. This is all the more important because, although we can, in practice, approximate to a line-distribution of pressure by means of a knife-edge, we cannot in the same way approximate to a line distribution of shear—shear being usually transmitted by means of projecting collars, which have a certain thickness. It is true that a thin notch might be cut into the material and an edge inserted in it which might be pulled sideways. But the cutting of such a notch would seriously weaken the material, besides altering the conditions so much as to render our solution inapplicable.

If we suppose our shear spread over a length $2a'$ of the upper edge, and if we adhere to the notation on p. 104, we find easily, L now denoting shearing force per unit area :—

$$U = -\frac{L}{2\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left\{(x+a')\log\frac{r_1}{b}-(x-a')\log\frac{r_2}{b}-2a'+y'(\phi_1-\phi_2)\right\}$$

$$-\frac{Ly'}{2\pi\mu}(\phi_1-\phi_2)-\frac{2Lb}{\pi(\lambda'+\mu)}\sum_0^\infty H_{2\nu}\frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1-r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{(2\nu+2)!\,b^{2\nu+2}}$$

$$+\frac{2Lb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\left\{\frac{D}{b}\cdot 2a'+\sum_0^\infty\frac{r_1^{2\nu+3}\sin(2\nu+3)\phi_1-r_2^{2\nu+3}\sin(2\nu+3)\phi_2}{b^{2\nu+3}(2\nu+3)!}(H_{2\nu+1}-H_{2\nu})\right\}$$

$$-\frac{2Ly'}{\pi\mu}\sum_0^\infty\frac{r_1^{2\nu+1}\sin(2\nu+1)\phi_1-r_2^{2\nu+1}\sin(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu}$$

$$+\frac{2Ly'}{\pi\mu}\sum_0^\infty\frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1-r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1}-H_{2\nu})$$

$$V = -\frac{L}{2\pi}\frac{1}{\lambda'+\mu}\left\{(x+a')\phi_1-(x-a')\phi_2-y'\log\frac{r_1}{r_2}\right\}-\frac{Ly'}{2\pi\mu}\log\frac{r_1}{r_2}$$

$$+\frac{2Lb}{\pi}\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\sum_0^\infty\frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1-r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu}$$

$$-\frac{2Lb}{\pi(\lambda'+\mu)}\sum_1^\infty\frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1-r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu-1}-H_{2\nu-2})$$

$$+\frac{2Ly'}{\pi\mu}\sum_0^\infty\frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1-r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1}-H_{2\nu})$$

$$-\frac{2Ly'}{\pi\mu}\sum_1^\infty\frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1-r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu}$$

$$P = -\frac{2L}{\pi}\log\frac{r_1}{r_2} - \frac{Ly'}{\pi}\left(\frac{\cos\phi_1}{r_1} - \frac{\cos\phi_2}{r_2}\right) + \frac{8L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1 - r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1} - H_{2\nu})$$

$$- \frac{4L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu} - \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1 - r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu+2}$$

$$+ \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu+1} - H_{2\nu}).$$

$$Q = \frac{Ly'}{\pi}\left(\frac{\cos\phi_1}{r_1} - \frac{\cos\phi_2}{r_2}\right) - \frac{4L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}H_{2\nu}$$

$$+ \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+2}\cos(2\nu+2)\phi_1 - r_2^{2\nu+2}\cos(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu+2} - \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+1}\cos(2\nu+1)\phi_1 - r_2^{2\nu+1}\cos(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu+1} - H_{2\nu}).$$

$$S = \frac{L}{\pi}(\phi_1 - \phi_2) - \frac{Ly'}{\pi}\left(\frac{\sin\phi_1}{r_1} - \frac{\sin\phi_2}{r_2}\right)$$

$$- \frac{4L}{\pi}\sum_0^\infty \frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1 - r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}(H_{2\nu+1} - H_{2\nu})$$

$$- \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+1}\sin(2\nu+1)\phi_1 - r_2^{2\nu+1}\sin(2\nu+1)\phi_2}{b^{2\nu+1}(2\nu+1)!}(H_{2\nu+1} - H_{2\nu})$$

$$+ \frac{4Ly'}{\pi b}\sum_0^\infty \frac{r_1^{2\nu+2}\sin(2\nu+2)\phi_1 - r_2^{2\nu+2}\sin(2\nu+2)\phi_2}{b^{2\nu+2}(2\nu+2)!}H_{2\nu+2}.$$

The same remarks which were made on p. 106 as to the validity of such expressions apply here. Assuming that $2a' < 4b$, we may apply these to obtain the state of things near the layer of shear and at its extremities.

Clearly the only terms where discontinuities in U, V, P, Q, S, or their differential coefficients, may be introduced are their leading terms. Let us therefore study these.

It is easily seen that $(x + a')\log r_1$ and $(x - a')\log r_2$ are finite, continuous, and one-valued throughout, tending to 0 at the points $(\mp a', 0)$. Their differential coefficients with regard to y' are likewise everywhere finite, but are indeterminate at $(\mp a', 0)$. They introduce, however, no discontinuity if we proceed along $y' = 0$.

Similarly $y'\log r_1$ and $y'\log r_2$ are everywhere continuous, finite, and one-valued, and their differential coefficients with regard to x give no discontinuity if we keep to $y' = 0$.

$y'(\phi_1 - \phi_2)$ is everywhere continuous. Its differential coefficient with regard to y is indeterminate at $(\pm a', 0)$; if we proceed along $y' = 0$ it increases by π as we pass the point $(-a', 0)$ and decreases by π as we pass the point $(+a', 0)$. The same holds with regard to $(x + a')\phi_1 - (x - a')\phi_2$ and its differential coefficient with regard to x.

Hence, as far as U and V are concerned, they are both finite, continuous, and one-valued throughout the beam. $\frac{dU}{dy'}$, $\frac{dV}{dx}$ are everywhere finite, but are indeterminate at $(\pm a', 0)$. As we proceed along $y' = 0$, $\frac{dU}{dy'}$ decreases abruptly by $\frac{L}{2}\left(\frac{1}{\lambda' + \mu} + \frac{2}{\mu}\right)$ as we pass $(-a', 0)$ and increases again by the same amount as we pass $(+a', 0)$. Similarly $\frac{dV}{dx}$ decreases by $\frac{L}{2}\frac{1}{\lambda' + \mu}$ as we pass $(-a', 0)$, and increases by the same amount as we pass $(+a', 0)$. The first of these results means an abrupt change in the angle at which the distorted cross-sections meet the horizontal, and the second shows that the distorted form of the upper edge of the beam receives a sudden inflection downwards as we enter the layer of shear, and is again suddenly inflected upwards as we emerge from it.

It has been shown in a paper by the author "On the Equilibrium of Circular Cylinders under Certain Practical Systems of Load" ('Phil. Trans.,' A, vol. 198, pp. 147–233), that a precisely similar occurrence takes place in a circular cylinder subjected to a uniform ring of shear, over a certain length of its curved surface. The law that shear depresses the parts of the surface towards which it acts appears to be a general one.

Passing on now to consider the stresses P, Q, S, we find that Q and S remain everywhere finite, but are indeterminate at the points $(\pm a', 0)$. If we keep to $y' = 0$, Q is continuously zero and S changes by L at $(\pm a', 0)$, as it should. But P not only contains a part which becomes indeterminate at $(\pm a', 0)$, it also contains a term $-\frac{2L}{\pi}\log\frac{r_1}{r_2}$ which becomes infinite at those points.

This is a result for which we had no analogue in the case of a uniform layer of *pressure*. In that problem the stresses were everywhere finite. We now see that any finite discontinuity in the shear introduces an infinite pressure or tension P in the neighbourhood of this discontinuity. This result, again, has been found to hold good for circular cylinders. It may be laid down as an absolute rule that for an engineering structure to be safe, there should never occur any discontinuity in the shearing stress across any surface inside the material or on its boundary. It is true that in most cases the stress will be relieved by plastic flow and the variation of the shear will become continuous, though rapid. But such points, especially the point *from which* the shear starts acting $(-a', 0)$, where the infinite stress is a *tension*, will remain points of weakness and danger.

§ 37. *Application of Solutions of § 33 to the Case of Tension Produced by Shearing Stress Applied to the Edges.*

In practice test-pieces for tension are usually strained by pressure applied to projecting collars, the latter transmitting this pressure to the body of the material in the shape of shear. In no case can we apply tension directly to the ends of a bar. It is therefore important to know how far the effect of the method of application of the total pull disturbs the usual solution for a uniform tension.

Let us then consider the effect of having two concentrated shears L, one as before acting at the point $(0, b)$, and another equal and parallel to the first acting at the point $(0, -b)$. By superimposing on one another two solutions of the type obtained in § 33, we get the solution required. It will be found that this solution gives a tension $L/2b$ over the left-hand extremity of the beam and a pressure $L/2b$ over the right-hand end.

If we require to have no tension over the right-hand end, and a uniform tension L/b at the left-hand end, to balance the shears, we have to introduce the uniform tension solution $P = L/2b$, $Q = 0$, $S = 0$, $U = Lx/2bE$, $V = \eta Ly/2bE$; we eventually find

$$U = -\frac{\lambda' + 2\mu}{16\mu(\lambda' + \mu)}\frac{Lx^2}{ab} + \frac{3\lambda' + 2\mu}{16\mu(\lambda' + \mu)}\frac{Ly^2}{ab} + \frac{Lx}{2bE}$$

$$+ \sum_1^\infty \frac{L}{ma}\frac{\left(\frac{1}{\lambda' + \mu} + \frac{1}{\mu}\right)\cosh mb - \frac{mb}{\mu}\sinh mb}{\sinh 2mb + 2mb}(\cosh my \cos mx - 1)$$

$$+ \sum_1^\infty \frac{L}{m\mu a}\frac{(my \cosh mb)}{\sinh 2mb + 2mb}\sinh my \cos mx$$

$$V = \frac{\lambda'}{8\mu(\lambda' + \mu)}\frac{Lxy}{ab} + \frac{L\eta y}{2bE} + \sum_1^\infty \frac{L}{m\mu a}\frac{\cosh mb}{\sinh 2mb + 2mb}my \cosh my \sin mx$$

$$- \sum_1^\infty \frac{L}{ma}\frac{\frac{1}{\lambda' + \mu}\cosh mb + \frac{1}{\mu}mb \sinh mb}{\sinh 2mb + 2mb}\sinh my \sin mx$$

$$P = \frac{L}{2ab}(a - x) - \sum_1^\infty \frac{L}{a}\frac{4 \cosh mb - 2mb \sinh mb}{\sinh 2mb + 2mb}\cosh my \sin mx$$

$$- \sum_1^\infty \frac{2L}{a}\frac{\cosh mb}{\sinh 2mb + 2mb}my \sinh my \sin mx$$

$$Q = -\sum_1^\infty \frac{L}{a}\frac{2mb \sinh mb}{\sinh 2mb + 2mb}\cosh my \sin mx$$

$$+ \sum_1^\infty \frac{2L}{a}\frac{\cosh mb}{\sinh 2mb + 2mb}my \sinh my \sin mx$$

$$S = \frac{Ly}{2ab} + \sum_1^\infty \frac{2L}{a}\frac{\cosh mb}{\sinh 2mb + 2mb}my \cosh my \cos mx$$

$$+ \sum_1^\infty \frac{2L}{a}\frac{(\cosh mb - mb \sinh mb)}{\sinh 2mb + 2mb}\sinh my \cos mx.$$

If a be made to tend towards infinity, we get the expressions:

$$
\left.\begin{aligned}
\mathrm{U} &= \frac{\mathrm{L}x}{2b\mathrm{E}} + \frac{\mathrm{L}}{\pi}\int_0^\infty \frac{1}{u} \frac{\left(\frac{1}{\lambda'+\mu}+\frac{1}{\mu}\right)\cosh u - \frac{1}{\mu} u \sinh u}{\sinh 2u + 2u}\left(\cosh\frac{uy}{b}\cos\frac{ux}{b} - 1\right) du \\
&\quad + \frac{\mathrm{L}y}{\pi b}\int_0^\infty \frac{1}{\mu}\frac{\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
\mathrm{V} &= \frac{\mathrm{L}\eta y}{2b\mathrm{E}} - \frac{\mathrm{L}}{\pi}\int_0^\infty \frac{1}{u}\frac{\frac{1}{\lambda'+\mu}\cosh u + \frac{1}{\mu} u\sinh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&\quad + \frac{\mathrm{L}y}{\pi b}\int_0^\infty \frac{1}{\mu}\frac{\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
\mathrm{P} &= \frac{\mathrm{L}}{2b} - \frac{2\mathrm{L}}{\pi b}\int_0^\infty \frac{2\cosh u - u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&\quad - \frac{2\mathrm{L}y}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
\mathrm{Q} &= -\frac{2\mathrm{L}}{\pi b}\int_0^\infty \frac{u\sinh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
&\quad + \frac{2\mathrm{L}y}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\sin\frac{ux}{b}\,du \\
\mathrm{S} &= \frac{2\mathrm{L}y}{\pi b^2}\int_0^\infty \frac{u\cosh u}{\sinh 2u + 2u}\cosh\frac{uy}{b}\cos\frac{ux}{b}\,du \\
&\quad + \frac{2\mathrm{L}}{\pi b}\int_0^\infty \frac{\cosh u - u\sinh u}{\sinh 2u + 2u}\sinh\frac{uy}{b}\cos\frac{ux}{b}\,du
\end{aligned}\right\} \quad (108).
$$

§ 38. *Correction to be Applied in this Case to the Stretch along the Edges as we approach the Points of Application of the Load.*

One of the most interesting points about a problem of this kind is to find out at what distance from the region of loading the stretch parallel to the axis takes the value it should have on the uniform tension hypothesis. In practice all measurements of Young's modulus for bars are made by observing the stretch between two points marked on the outer surface of the bar. It is of importance to know the error introduced as we bring these points closer to the places where the stress is applied.

Let us therefore see how the stretch $d\mathrm{U}/dx$ varies as we go away from the points of application of the load, keeping upon either edge of the beam. If we differentiate the expression (108) for U with regard to x and then write $y = b - y'$ and transform the expression as in § 16, we get easily

$$\frac{dU}{dx} = \frac{L}{2bE} - \frac{2L}{\pi E}\frac{x}{r'^2} + \frac{L}{\pi\mu}\frac{xy'^2}{r'^4} - \frac{2L}{E\pi b}\int_0^\infty \frac{1 - 2u + e^{-2u}}{\sinh 2u + 2u}\cosh\frac{uy'}{b}\sin\frac{ux}{b}\,du$$
$$- \frac{L}{(\lambda' + \mu)\pi b}\int_0^\infty \frac{u}{\sinh 2u + 2u}\sinh\frac{uy'}{b}\sin\frac{ux}{b}\,du - \frac{Ly'}{\pi\mu b^2}\int_0^\infty \frac{u^2}{\sinh 2u + 2u}\cosh\frac{uy'}{b}\sin\frac{ux}{b}\,du$$
$$- \frac{Ly'}{2\pi\mu b^2}\int_0^\infty \frac{1 - 2u + e^{-2u}}{\sinh 2u + 2u}\sinh\frac{uy'}{b}\sin\frac{ux}{b}\,du.$$

Putting in this $y' = 0$,

$$\left(\frac{dU}{dx}\right)_{y'=0} = \frac{L}{2bE} - \frac{2L}{\pi E}\frac{1}{x} - \frac{2L}{\pi bE}\int_0^\infty \frac{1 - 2u + e^{-2u}}{\sinh 2u + 2u}\sin\frac{ux}{b}\,du.$$

Now the last integral may be written

$$\int_0^\infty \left(\frac{1 - 2u + e^{-2u}}{\sinh 2u + 2u} + e^{-\frac{uh}{b}} - \frac{1}{2u}\right)\sin\frac{ux}{b}\,du$$
$$+ \frac{\pi}{4} - \frac{bx}{h^2 + x^2} \text{ if } x \text{ is positive,}$$

and if x is negative, then $-\pi/4$ must be written instead of $+\pi/4$. h is any positive constant.

Now the function

$$f(u) = \frac{1 - 2u + e^{-2u}}{\sinh 2u + 2u} + e^{-\frac{uh}{b}} - \frac{1}{2u}$$

is such that $f(0) = f(\infty) = 0$, $f'(\infty) = 0$ and $\int_0^\infty |f''(u)|\,du$ is finite. It follows therefore from reasoning similar to that given on p. 107 that $\int_0^\infty f(u)\sin\frac{ux}{b}\,du$ tends to zero as x increases.

Hence, if x be positively increasing $\left(\frac{dU}{dx}\right)_{y'=0}$ tends to 0, and if x be negatively increasing $\left(\frac{dU}{dx}\right)_{y'=0}$ tends to L/bE, as it should.

The values of the integral, calculated for various values of the ratio x/b, have given the following values for $\left(\frac{dU}{dx}\right)_{y'=0}$, as compared with its value for a uniform tension L/b.

x/b	$\left(\frac{dU}{dx}\right)_{y'=0} \Big/ \left(\frac{L}{bE}\right)$
$-\pi$	·997
$-2\pi/3$	·982
$-\pi/2$	·985
$-\pi/3$	1·084
$-\pi/6$	1·652
$+\pi/6$	− ·652
$+\pi/3$	− ·084
$+\pi/2$	·015
$+2\pi/3$	·018
$+\pi$	·003

We see therefore that the stretch reaches its limiting value with very great rapidity. At a distance from the point of application of the load equal to about $1\frac{1}{2}$ times the greatest breadth $2b$ of the bar the error in the stretch is only 3/1000. In fact the stretch begins to get near its limiting value at a much earlier stage than this, the error being less than 10 per cent. at a distance from the load of about half the greatest breadth.

We find therefore that in this case also the *distribution* of the load becomes practically indifferent as soon as we come to distances from the load which are of the same order of magnitude as the greatest dimension of the cross-section. As a practical rule, when accurate measurements are to be taken, it will be advisable to keep always a length varying from 1 to $1\frac{1}{2}$ times this greatest dimension between the points where the stress-system is applied and those at which measurements are taken.

PART V.

Solutions in Finite Terms; Special Application to the Case of a Beam Carrying a Uniform Load.

§ 39. *Solutions in Finite Terms.*

If in (21)–(25) of pp. 70, 71 we write

$$\left.\begin{aligned} \phi(\xi) &= \tfrac{1}{2}\frac{\lambda' + 2\mu}{\lambda' + \mu}(A_n + iB_n)\,\xi^n \\ \chi(\eta) &= \tfrac{1}{2}\frac{\lambda' + 2\mu}{\lambda' + \mu}(A_n - iB_n)\,\eta^n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

$$\left.\begin{aligned} G_1(\xi) &= \frac{1}{2\mu}(C_n + iD_n)\,\xi^n \\ F_1(\eta) &= \frac{1}{2\mu}(C_n - iD_n)\eta^n \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (109),$$

we obtain the following homogeneous solutions in x, y.

$$\left.\begin{aligned} U &= \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}(A_n u_n + B_n v_n) - \frac{ny}{4\mu}(A_n v_{n-1} - B_n u_{n-1}) + \frac{1}{2\mu}(C_n u_n + D_n v_n) \\ V &= \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}(A_n v_n - B_n u_n) - \frac{ny}{4\mu}(A_n u_{n-1} + B_n v_{n-1}) - \frac{1}{2\mu}(C_n v_n - D_n u_n) \\ P &= A_n\left(\frac{3n}{4}u_{n-1} - \frac{n(n-1)}{2}yv_{n-2}\right) + B_n\left(\frac{3n}{4}v_{n-1} + \frac{n(n-1)}{2}yu_{n-2}\right) \\ &\qquad + n(C_n u_{n-1} + D_n v_{n-1}) \\ Q &= A_n\left(\frac{n}{4}u_{n-1} + \frac{n(n-1)}{2}yv_{n-2}\right) + B_n\left(\frac{n}{4}v_{n-1} - \frac{n(n-1)}{2}yu_{n-2}\right) \\ &\qquad - n(C_n u_{n-1} + D_n v_{n-1}) \\ S &= A_n\left(-\frac{n}{4}v_{n-1} - \frac{n(n-1)}{2}yu_{n-2}\right) + B_n\left(\frac{n}{4}u_{n-1} - \frac{n(n-1)}{2}yv_{n-2}\right) \\ &\qquad - n(C_n v_{n-1} - D_n u_{n-1}) \end{aligned}\right\} \; . \; (110),$$

where u_n, v_n are the two homogeneous solutions of $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$, thus,

$$u_n = x^n - \frac{n(n-1)}{1 \, . \, 2} x^{n-2} y^2 + \ldots$$

$$v_n = nx^{n-1}y - \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3} x^{n-3} y^3 + \ldots$$

and $u_0 = 1$, $v_0 = 0$, $u_{-1} = 0$, $v_{-1} = 0$.

We may add any number of such polynomial solutions. If we take n of them, beginning with $n = 1$, and in the expressions (110) write $y = \pm b$, we find $(Q)_{+b}$, $(Q)_{-b}$, $(S)_{+b}$ and $(S)_{-b}$ each equal to algebraic polynomials in x of degree $(n - 1)$.

Also, since A_1, B_1, C_1, D_1 come in only in the form $\frac{A_1}{4} - C_1$, $\frac{B_1}{4} + D_1$, they are equivalent to only *two* constants. We have therefore $(4n - 2)$ constants free.

Now these are not enough to make Q and S coincide with any two given polynomials on the upper and lower faces of the beam. Obviously, however, the term containing x^{n-1} both in Q and S is independent of y and therefore cannot satisfy a perfectly general condition. If we make this term disappear by writing $C_n = A_n/4$, $D_n = - B_n/4$, we have now only $4n - 4$ free constants left, but our polynomials being now of the $(n-2)^{\text{th}}$ degree, we should have enough constants to be able to identify Q and S with any two given polynomials on either face of the beam.

As a matter of fact this is not so; for there are solutions, namely, those for: (i.) a uniform longitudinal tension, (ii.) a pure bending couple, (iii.) bending with constant shear, which make Q and S zero over both faces and yet do not annul all the $4n-4$ free constants. There must therefore be relations between the $4n-4$ equations giving the constants. They are not all independent and, consequently, not every system of surface stress expressible in polynomials corresponds to a solution of this type.

§ 40. *Case of* $n = 4$.

Let us see what surface conditions can be satisfied by the solutions of the fourth order.

In this case, remembering $D_4 = -B_4/4$,

$$Q = \left(\frac{A_1}{4} - C_1\right) + \left(\frac{A_2}{2} - 2C_2\right)x + \left(-\frac{B_2}{2} - 2D_2\right)y$$
$$+ \left(\frac{3A_3}{4} - 3C_3\right)x^2 + (-\tfrac{3}{2}B_3 - 6D_3)\,xy + (\tfrac{9}{4}A_3 + 3C_3)\,y^2$$
$$+ 12A_4xy^2 + (-3B_4 - 12D_4)\,x^2y + (5B_4 + 4D_4)\,y^3$$

$$S = \left(\frac{B_1}{4} + D_1\right) + \left(\frac{B_2}{2} + 2D_2\right)x + \left(-\frac{3A_2}{2} - 2C_2\right)y$$
$$+ \left(\frac{3B_3}{4} + 3D_3\right)x^2 + (-\tfrac{9}{2}A_3 - 6C_3)\,xy + (-\tfrac{15}{4}B_3 - 3D_3)\,y^2$$
$$+ (-9A_4 - 12C_4)\,x^2y - 12B_4xy^2 + (7A_4 + 4C_4)\,y^3,$$

and

$$U = \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}\left\{\begin{array}{l} A_0 + A_1x + A_2(x^2 - y^2) + A_3(x^3 - 3xy^2) + A_4(x^4 - 6x^2y^2 + y^4) \\ \quad + B_1y + 2B_2xy + B_3(3x^2y - y^3) + B_4(4x^3y - 4xy^3) \end{array}\right\}$$
$$+ \frac{1}{4\mu}\left\{\begin{array}{l} B_1y + 2B_2xy + 3B_3(x^2y - y^3) + 4B_4(x^3y - 3xy^3) \\ \quad - 2A_2y^2 - 6A_3xy^2 - 4A_4(3x^2y^2 - y^4) \end{array}\right\}$$
$$+ \frac{1}{2\mu}\left\{\begin{array}{l} C_0 + C_1x + C_2(x^2 - y^2) + C_3(x^3 - 3xy^2) + C_4(x^4 - 6x^2y^2 + y^4) \\ \quad + D_1y + 2D_2xy + D_3(3x^2y - y^3) + D_4(4x^3y - 4xy^3) \end{array}\right\}$$

$$V = \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)}\left\{\begin{array}{l} -B_0 - B_1x - B_2(x^2 - y^2) - B_3(x^3 - 3xy^2) - B_4(x^4 - 6x^2y^2 + y^4) \\ \quad + A_1y + 2A_2xy + A_3(3x^2y - y^3) + A_4(4x^3y - 4xy^3) \end{array}\right\}$$
$$- \frac{1}{4\mu}\left\{\begin{array}{l} A_1y + 2A_2xy + 3A_3(x^2y - y^3) + 4A_4(x^3y - 3xy^3) \\ \quad + 2B_2y^2 + 6B_3xy^2 + 4B_4(3x^2y^2 - y^4) \end{array}\right\}$$
$$+ \frac{1}{2\mu}\left\{\begin{array}{l} D_0 + D_1x + D_2(x^2 - y^2) + D_3(x^3 - 3xy^2) + D_4(x^4 - 6x^2y^2 + y^4 \\ \quad - C_1y - 2C_2xy - C_3(3x^2y - y^3) - C_4(4x^3y - 4xy^3) \end{array}\right\}$$

$$P = \left(\frac{3A_1}{4} + C_1\right) + x\left(\frac{3A_2}{2} + 2C_2\right) + y\left(\frac{5B_2}{2} + 2D_2\right)$$
$$+ x^2\left(\frac{9A_3}{4} + 3C_3\right) + xy\left(\tfrac{15}{2}B_3 + 6D_3\right) + y^2\left(-\tfrac{21}{4}A_3 - 3C_3\right)$$
$$+ x^3(3A_4 + 4C_4) + x^2y(15B_4 + 12D_4) + xy^2(-21A_4 - 12C_4)$$
$$+ y^3(-9B_4 - 4D_4),$$

and we notice that, in virtue of the relation $B_4 = -4D_4$ the coefficient of x^2y in Q goes out. Hence the coefficient of x^2 is the same for Q_{+b} and Q_{-b}. This alone shows that the solution is not the most general that can be got, given that the stresses on the upper and lower surfaces are quadratic functions of x.

§ 41. *Determination of the Constants for a Beam Uniformly Loaded.*

Here we have, over the upper surface $y = +b$: Q = constant $= q$ say; over $y = -b$: $Q = 0$; and over $y = \pm b$: $S = 0$.

The last two conditions imply

$$B_1 + 4D_1 + b^2\left(-\frac{15B_3}{4} - 3D_3\right) = 0 \quad \text{. . . . . . . (111).}$$

$$\frac{B_2}{2} + 2D_2 - 12B_4b^2 = 0 \quad \text{. . . . . (112).}$$

$$-\frac{3A_2}{2} - 2C_2 + b^2(7A_4 + 4C_4) = 0 \quad \text{. . . . . (113).}$$

$$-\tfrac{9}{2}A_3 - 6C_3 = 0 \quad \text{. . . . . (114).}$$

$$\frac{3B_3}{4} + 3D_3 = 0 \quad \text{. . . . . (115).}$$

$$-9A_4 - 12C_4 = 0 \quad \text{. . . . . (116).}$$

(116) and

$$4C_4 = A_4 \quad \text{. . . . . (117),}$$

give at once

$$A_4 = 0, \quad C_4 = 0 \quad \text{. . . . . (118).}$$

The conditions for Q give

$$\frac{A_1}{4} - C_1 - \left(\frac{B_2}{2} + 2D_2\right)b + b^2\left(\frac{9A_3}{4} + 3C_3\right) + b^3(5B_4 + 4D_4) = q \quad \text{. (119).}$$

$$\frac{A_1}{4} - C_1 + \left(\frac{B_2}{2} + 2D_2\right)b + b^2\left(\frac{9A_3}{4} + 3C_3\right) - b^3(5B_4 + 4D_4) = 0 \quad \text{. (120).}$$

$$-\tfrac{3}{2}B_3 - 6D_3 = 0 \quad \text{. . . . . (121).}$$

$$\frac{A_2}{2} - 2C_2 + b^2(12A_4) = 0 \quad \text{. . . . . (122).}$$

$$\frac{3A_3}{4} - 3C_3 = 0 \quad \text{. . . . . (123).}$$

$$-3B_4 - 12D_4 = 0 \quad \ldots \ldots \quad (124).$$

(124) is identically satisfied since $4D_4 = -B_4$.

Also (122), (118), (113) imply

$$A_2 = C_2 = 0 \quad \ldots \ldots \quad (125).$$

(119), (120), (112) lead to

$$\left.\begin{aligned} B_2 + 4D_2 &= -\frac{3q}{2b} \\ B_4 &= -\frac{q}{16b^3} \\ A_1 - 4C_1 &= 2q \end{aligned}\right\} \quad \ldots \ldots \quad (126).$$

Also (123), (114) imply

$$A_3 = C_3 = 0 \quad \ldots \ldots \quad (127).$$

(121), (115) are identical. Together with (111) they give

$$\left.\begin{aligned} B_3 &= \frac{1}{3b^2}(B_1 + 4D_1) \\ D_3 &= -\frac{1}{12b^2}(B_1 + 4D_1) \end{aligned}\right\} \quad \ldots \ldots \quad (128).$$

Equations (118), (124), (125), (126), (127), (128) contain the solution we require. If we substitute the values of the constants in the expressions for the displacements and stresses, we find, after some reductions :

$$\begin{aligned} U = \text{const.} &+ x\left\{A_1 \frac{\lambda' + 3\mu}{8\mu(\lambda' + \mu)} + \frac{C_1}{2\mu}\right\} + \frac{B_1 y}{E} + \frac{B_2}{E} 2xy \\ &+ \left(\frac{B_1}{4} + D_1\right)\left\{\frac{y}{2\mu} + \frac{4}{b^2}\left(\frac{x^2 y}{E} - \frac{y^3}{3}\left(\frac{1}{E} + \frac{1}{2\mu}\right)\right)\right\} \\ &- \frac{q}{4b^3}\left\{\frac{x^3 y}{E} - xy^3\left(\frac{1}{E} + \frac{1}{2\mu}\right)\right\}, \end{aligned}$$

$$\begin{aligned} V = \text{const.} &- y\left\{A_1 \frac{\lambda' - \mu}{8\mu(\lambda' + \mu)} + \frac{C_1}{2\mu}\right\} - \frac{B_1}{E} x - \frac{B_2}{E}(x^2 - \eta y^2) \\ &+ \left(\frac{B_1}{4} + D_1\right)\left\{\frac{4x^3}{3Eb^2} - \eta \frac{4xy^2}{Eb^2}\right\} + \frac{q}{16b^3}\left\{\frac{x^4}{E} - \eta \frac{6x^2y^2}{E} + y^4\left(\frac{1}{E} - \frac{1}{\mu}\right)\right\}, \end{aligned}$$

$$\begin{aligned} P = \left(\frac{3A_1}{4} + C_1\right) &+ 2B_2 y - \frac{3qy}{2b} + \frac{3xy}{b^2}\left(\frac{B_1}{4} + D_1\right) \\ &- \frac{3}{4}\frac{x^2 yq}{b^3} + \frac{qy^3}{2b^3}, \end{aligned}$$

$$Q = \frac{q}{2} + \frac{3qy}{4b} - \frac{qy^3}{4b^3},$$

$$S = \left(\frac{B_1}{4} + D_1\right)\left(1 - \frac{y^2}{b^2}\right) - \frac{3}{4}\frac{qx}{b}\left(1 - \frac{y^2}{b^2}\right).$$

In the above terms in A_1, C_1 correspond to a uniform tension along x, the terms B_1 to a rigid body rotation, the terms B_2 to a solution for a pure bending couple, and the terms $\left(\frac{B_1}{4} + D_1\right)$ to a solution for bending under a uniform shear. These various constants can be adjusted according to the conditions at the ends $x = \pm a$.

If, for instance, the total pressure over the ends and the total bending moment are to be zero, the load $2qa$ being balanced by the shear at these ends, we have

$$\frac{3A_1}{4} + C_1 = 0,$$

$$\frac{B_1}{4} + D_1 = 0,$$

$$B_2 = +\tfrac{3}{8}\frac{qa^2}{b^3} + \tfrac{3}{5}\frac{q}{b},$$

and we then have

$$\left.\begin{aligned}
P &= -\tfrac{3}{10}\frac{qy}{b} - \tfrac{3}{4}\frac{x^2yq}{b^3} + \frac{qy^3}{2b^3} + \tfrac{3}{4}\frac{a^2yq}{b^3} \\
Q &= \frac{q}{2} + \frac{3qy}{4b} - \frac{qy^3}{4b^3} \\
S &= -\tfrac{3}{4}\frac{qx}{b}\left(1 - \frac{y^2}{b^2}\right) \\
U &= +\tfrac{3}{4}\frac{qa^2xy}{Eb^3} + \tfrac{6}{5}\frac{qxy}{Eb} - \frac{q}{4b^3}\left\{\frac{x^3y}{E} - xy^3\left(\frac{1}{E} + \frac{1}{2\mu}\right)\right\} \\
V &= -\left(\tfrac{3}{8}\frac{a^2}{b^3} + \frac{3}{5b}\right)\frac{q}{E}(x^2 - \eta y^2) + \frac{q}{16b^3}\left\{\frac{x^4}{E} - \eta\frac{6x^2y^2}{E} + y^4\left(\frac{1}{E} - \frac{1}{\mu}\right)\right\}
\end{aligned}\right\} \quad (129).$$

This is the solution for a beam uniformly loaded on the top over a length $2a$ and held up by shears over its terminal cross-sections. In this way the case which occurred in the general solution, and of which the consideration was postponed in § 9, namely $\alpha_0 \neq \beta_0$, is seen to lead to a fairly simple solution in finite terms.

§ 42. *Remarks on the above Solution.*

The above values (129) for U, V, P, Q, S in the case of a beam carrying a uniform load, lead to the following remarks :—

(1) There is no "Neutral Axis" properly so-called; *i.e.*, although the tension vanishes for $y = 0$, t is not strictly proportional to y, a cubic term being introduced. But if $(a^2 - x^2)/y^2$ be large, which is the case for any beam whose length is large compared with its height, the proportional effect of the terms introduced will be small.

(2) The stress Q is not zero; that is, DE SAINT-VENANT'S assumption, that there is no stress across fibres parallel to the axis of the beam, does not hold. Indeed, it was obvious from the beginning that it would not, seeing that there is a stress Q at the upper surface, by hypothesis.

(3) The distribution of shear at each cross-section is parabolic, and is given in terms of the mean shear by the same formula which holds when the shear is uniform.

(4) $\left(\dfrac{d^2V}{dx^2}\right)_{y=0} = -\frac{3}{4}\dfrac{(a^2 - x^2)}{Eb^3} q - \dfrac{6q}{5Eb}.$

The curvature is therefore no longer exactly proportional to the bending moment, but contains an additional constant term. A similar result has been obtained by Professor KARL PEARSON and the author for beams of elliptic cross-section under their own weight ('Quarterly Journal of Mathematics,' vol. 31, p. 90). It has since been shown to hold for beams of all forms of section by Mr. J. H. MICHELL ('Quarterly Journal of Mathematics,' vol. 32).

§ 43. *Historical Summary: Remarks and Criticism.*

It may be of interest to give in this place a short sketch of the previous works on the subject, in so far as they are at present known to me.

LAMÉ, in his 'Leçons sur l'Elasticité' (p. 156 *et seq.*), discusses the general problem of the rectangular block, with the single restriction, that the surface stresses are purely normal and are even functions of the co-ordinates. He fails to determine his constants, except in the particular case where the cubical dilatation throughout the block happens to be previously known. As this condition is never satisfied in any actual problem, the solution is of comparatively little use.

DE SAINT-VENANT, in a classical memoir ('Mémoires des Savants Etrangers de l'Académie des Sciences de Paris,' vol. 14), has given solutions for the rectangular parallelopiped under torsion and flexure. These solutions correspond to the case of terminal stress-systems which are *transmitted* through an otherwise unstressed long bar.

Numerous attempts have been made to solve the problem of the rectangular elastic solid by removing one or more faces to infinity, and thus simplifying the surface conditions.

M. EMILE MATHIEU, in his treatise, 'Théorie de l'Elasticité des Corps Solides,' Paris, 1890 (see also 'Comptes Rendus,' vol. 90, pp. 1272–1274), has given a solution of the problem when it can be reduced to two dimensions. His problem is therefore practically the same as that of this paper, except that he has considered only what I have called case (A) on p. 66, and also, that the length a is not taken to be large and the distribution of stress over the faces $x = \pm a$ is given. The solution is, however, so complex in form, and the determination of the constants, by means of long and exceedingly troublesome series, so laborious, that the results defy all attempts at interpretation.

Dr. CHREE ('Roy. Soc. Proc.,' vol. 44, and 'Roy. Soc. Archives'; also 'Quarterly Journal of Mathematics,' vol. 22) has considered at length the solutions of the equations of elasticity in integral powers of x, y, z, and has applied them to the beam problem. Among other results he has obtained expressions for the terms independent of z of a form similar to (110) of this paper. Incidentally, he verifies a number of DE SAINT-VENANT'S results; but no further application is, I think, made of the two-dimensional terms.

Quite recently, Mr. J. H. MICHELL has investigated the theory of long beams under uniform load ('Quarterly Journal of Mathematics,' vol. 32, pp. 28 *et seq.*). The object appears to be to extend DE SAINT-VENANT'S researches to uniformly loaded beams. Mr. MICHELL deduces several interesting results applicable to beams in general and to the rectangular beam in particular, but, so far as I can see, he makes no claim to having obtained explicitly the complete solution in any case.

The surface conditions, however, may be thinned down still further by removing four faces to infinity, leaving only an infinite plate of finite thickness. The problem in this form has been formally solved by LAMÉ and CLAPEYRON ("Sur l'équilibre intérieur des solides homogènes"; 'Mémoires des Savants Etrangers de l'Académie des Sciences de Paris,' vol. 4, pp. 548–552). Their solution, obtained in the form of quadruple integrals, satisfies the surface stress conditions over the two infinite faces. The objections to this solution are two-fold. In the first place it is difficult of interpretation, and the integrals do not enable us to obtain a clear notion of the separate effects of the various forces applied to the plate. In the second place this solution takes no heed of the conditions at the other four limiting faces of the plate which, we should always remember, although they have been removed to a very large distance away, have not physically disappeared. Given total tensions, shears and couples, applied to the four narrow faces of the plate, will produce stresses that will be transmitted through the plate, exactly as in the case of a bent or twisted bar, and will produce a finite effect at points of the plate infinitely distant from the edges, even though the large plane surfaces should be absolutely free from stress.

In order therefore that LAMÉ and CLAPEYRON'S formulæ may correspond to a physical reality, we must superimpose on their solution another of this transmissional type, such that the total shears and total couples due to the sum of the two solutions are all zero round the contour of the plate. Now the problem of the thick elastic rectangular plate, under given *total* shears and couples round its contour, but otherwise free from stress (which is the analogue for plates of the ordinary tensional and flexural solutions for bars), is another of the unsolved problems of the theory of elasticity and, until it is solved, LAMÉ and CLAPEYRON'S solution, unless it happens of itself to satisfy the conditions of no total force at the edge—which will only be true in special cases—fails.

More recently the same problem has been attacked by M. C. RIBIÈRE in a thesis ("Sur divers Cas de la Flexion des Prismes Rectangles," Bordeaux, 1889; see also 'Comptes Rendus,' vol. 126, pp. 402–404 and 1190–1192) in which he gives a solution in a series of circular and hyperbolic functions. He takes his plate of finite dimensions and built-in (*encastrée*) at the edge. By this term he understands that the edge is constrained to remain plane and vertical, and is subject to no shearing-stress. For other terminal conditions the solution, as M. RIBIÈRE states himself, is insufficient. I find that, if the edges of the plate be removed to infinity, his solutions degenerate into LAMÉ and CLAPEYRON'S integrals, of which they therefore give the true meaning.

M. RIBIÈRE, in the same thesis, has also investigated the two-dimensional case, which has been treated of in the present paper.* I am indebted to M. RIBIÈRE for very kindly communicating to me his thesis, with which I became acquainted after my work had been completed. His solutions are of the form (26) (27) (28), and he determines his coefficients, as far as I can see, by the method used here, but does not transform his expressions further. Like LAMÉ and CHAPEYRON, he restricts his applied surface stresses to be normal and investigates only two special cases.

M. RIBIÈRE takes, as I have done, $m = n\pi/a$. This, by the way, is not absolutely necessary. Another set of solutions might be obtained by taking $m = (2n + 1)\pi/2a$. When a is made very large, as is the case in every one of the problems treated here, either set of solutions will lead to the same final form, provided the total terminal conditions are attended to. M. RIBIÈRE, on the contrary, in order to be able to evaluate his series, which become far more manageable when b/a is large, treats chiefly of cases of thick beams of very short span. Now in this case it is no longer permissible to consider merely the total conditions over the ends $x = \pm a$, and to treat the actual *distribution* over these ends as unimportant. M. RIBIÈRE gets over this difficulty by supposing his beam to be *encastré*, as defined above. The same mathematical condition of fixing is assumed by Professor POCHHAMMER ('CRELLE'S Journal,' vol. 81) when treating in a similar fashion of the beam of circular cross section.

It seems doubtful whether anything of this kind does really occur at an actual built-in end of a beam. Certainly POCHHAMMER and RIBIÈRE'S conditions do not agree with the view taken by DE SAINT-VENANT, who, in his calculation of the deflection for a cantilever, has assumed that the elastic line is not horizontal at the built-in end. In this case, however, LOVE has pointed out that the elastic line may have any small slope at the built-in end, provided we superimpose a suitable rigid body displacement. But both he and DE SAINT-VENANT agree to make the end

* Since writing the above, I find that Professor LAMB ('Proc. Lond. Math. Soc.,' vol. 21, p. 70, paper read December, 1889) has worked out the same problem in the form of a series of circular and hyperbolic functions, but he has left his results in this form, without interpreting them further, and I cannot discover that he has considered end-conditions.

sections distorted. As a matter of fact, what really happens at a built-in end is quite unknown. Under these conditions any solution which makes $U = 0$, $dV/dx = 0$ over the ends must be restricted to the case of an infinite continuous beam resting upon a series of equidistant supports, each at the same vertical height; the load carried by the beam being exactly repeated over each span. A rail under its own weight and carried on sleepers is an approximate example. In this case POCHHAMMER and RIBIÈRE'S solutions are *exact*, and it is then legitimate to make the span as small as we please.

In practice such conditions will but rarely occur, because, as is well known, any slight difference in the height of the supports, or in the manner in which the beam bears upon them, will upset the symmetry altogether.

The ultimate step in the process of thinning down the boundary conditions is taken when one of the two boundaries of the infinite plate is itself removed to infinity, leaving only one plane bounding an otherwise unlimited solid.

This problem also has been solved by LAMÉ and CLAPEYRON (*loc. cit.*) in terms of quadruple integrals. In this case the limiting conditions at infinity cease to be important, because, in a solid infinite in three dimensions, finite stresses are not transmitted undiminished from infinity, as in a rod or lamina. The solutions, in fact, will lead to stresses that become zero at infinity. This has been shown by BOUSSINESQ ('Applications des Potentiels à l'Etude de l'Equilibre et du Mouvement des Solides Elastiques,' Paris, GAUTHIER-VILLARS, 1885), who has interpreted LAMÉ and CLAPEYRON'S results, and obtained by a new method simple expressions for the stresses in an infinite solid, due to arbitrary surface forces applied to a bounding plane. The same results have been obtained by Professor CERRUTI ("Ricerche intorno all' Equilibrio de Corpi Elastici Isotropi," 'Reale Accademia dei Lincei,' vol. 13, 1881–2) in a different way.

BOUSSINESQ, on p. 280, suggests a possible application of his method to the case of two parallel planes, but he makes no attempt to follow it up.

In two papers in the 'Comptes Rendus' (vol. 94, pp. 1510–1516, and vol. 95, pp. 5–11) he has considered the case when the problem of the infinite plane may be treated as two-dimensional, and there he has tried to extend his method to two parallel planes, but had to fall back upon an assumption mathematically unjustifiable.

§ 44. *Recapitulation of Results and Conclusion.*

Looking back upon the results obtained, we see that the general solution given has enabled us to deal with all the most important statical problems connected with the elastic equilibrium of a long beam, of finite height, in so far as the approximation involved in treating them two-dimensionally is valid; and it will be valid, if the horizontal dimension of the cross-section be either very small or very great.

Incidentally the question of the effect of concentrated loads, whether in the form of pressure or of shear, has been discussed. In the case of a beam doubly supported and carrying a concentrated load in the middle, a convergent series has been obtained, giving the exact correction which the finite height of the beam makes it necessary to apply to Boussinesq's results for an infinite elastic solid.

The results of this part of the paper have been tested by experiments on glass beams, of which it is hoped to eventually publish an account, and they have been found to agree, on the whole, with observation.

The effects of pressing a block of elastic material which rests on a rigid plane, and the manner in which such pressure is transmitted to the plane have also been investigated. It has been found that the pressure on the plane is limited to a restricted area, outside which the elastic block ceases to be in contact with the plane.

The effects of shearing stress have next been considered, in particular the distortion which it produces in lines parallel to the axis of the beam. As in the case of the circular cylinder and in that of the infinite solid bounded by a plane, shear is found to depress those parts of the material towards which it acts.

It is also found that a discontinuity in the shear applied to the surface—although the shear remains finite—involves one of the other stresses becoming infinite, and so is a source of weakness and danger.

The behaviour of a beam under two concentrated loads, acting in opposite senses upon opposite faces of the beam, has been studied. The manner in which the shear across the section varies as these loads are made to approach each other has been exhibited by various diagrams. They show how rapidly the effects of the particular distribution of any total terminal load die out as we go away from the end. At a distance of the order of the height of the beam, they already begin to be negligible.

At a lesser distance than this, however, such effects may become exceedingly important. The case of rivets is instanced, and it is suggested that the results obtained here may give some information which shall be useful in this connection.

Finally a solution in finite terms is obtained for a beam which carries a uniform load. It is shown that the assumptions of the usual theory of flexure are in this case no longer true, but are approximately true only if the height be very small compared with the span. The correction to the curvature, as calculated from the usual formula, is found to be a constant.

With regard to the numerical work, the arithmetic has been checked wherever possible, and it is believed that no serious error has crept in. The values of the integrals, however, have been obtained by the use of quadrature formulæ, and these may not have given a satisfactory approximation in all cases. The three first decimal places, nevertheless, should be correct. As the numerical work was undertaken chiefly to illustrate fairly large variations and to represent them by diagrams, this accuracy appears sufficient.

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PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 157–184.

ON THE VIBRATIONS AND STABILITY OF A GRAVITATING PLANET

BY

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LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 335. 4.4.03

Price One Shilling and Sixpence.

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V. *On the Vibrations and Stability of a Gravitating Planet.*

By J. H. Jeans, *B.A., Isaac Newton Student, and Fellow of Trinity College, Cambridge.*

Communicated by Professor G. H. Darwin, *F.R.S.*

Received November 8,—Read December 4, 1902.

Introduction.

§ 1. In a former paper* I have considered the effect of gravitation as a factor tending towards instability, in the case of a spherical nebula of gas. The object of the present paper is to investigate the analogous problem in the case of a spherical planet, the planet being supposed composed of solid or fluid matter. The main question at issue is the following.

§ 2. So long as gravitation is neglected there can be no doubt as to the stability of an elastic solid; any displacement increases the potential energy, and an unstressed configuration of equilibrium is therefore necessarily stable. But when gravitation is taken into account, the gravitational energy may be either increased or decreased by a displacement from equilibrium, and if a displacement can be found which effects a decrease in the gravitational potential energy of amount sufficient to outweigh the increase in the potential of the elastic forces, then the equilibrium configuration will be unstable.

Now, in § 2 of the previous paper already referred to, it was shown that for any spherical body displacements can be found such that there is a decrease in the gravitational potential. This is sufficient to prove that a spherical configuration of equilibrium may be unstable.

In the terminology of Poincaré† it appears that on any "linear series" of spherical configurations there may be "points of bifurcation."

We must, therefore, attempt to settle the position of these points of bifurcation.

In particular, it will be of interest to examine whether a sphere of the size and material of the earth may be regarded as being anywhere in the neighbourhood of a point of bifurcation.

* "The Stability of a Spherical Nebula," 'Phil. Trans.,' A, vol. 199, p. 1.

† 'Acta. Math.,' vol. 7, p. 259.

Preliminary Approximation.

§ 3. A rough and very simple calculation will give an approximate answer to this latter question.

Let a be the radius of a sphere, which will ultimately be taken to be the earth, M its mass, and ρ_0 the mean density given by $M = \frac{4}{3}\pi\rho_0 a^3$.

Let us use the elastic constants λ, μ,* and let λ_0 be the mean value of λ. Since the sphere is supposed to be spherically symmetrical, λ, μ, and ρ will be functions of the single co-ordinate r, the distance from the centre. Imagine λ/λ_0, μ/λ_0, and ρ/ρ_0 each expressed as functions of r/a, and let $c_1, c_2, \ldots$ be the coefficients which occur in these functions, these coefficients being mere numbers and independent of the system of units in which λ, ρ, and a are measured.

Imagine a linear series of equilibrium configurations obtained by varying any one of the quantities λ_0, ρ_0, or a, while keeping the remaining two quantities and the coefficients $c_1, c_2, \ldots$ constants. The points of bifurcation on this series will occur when the varying parameter becomes equal to some definite function of the remaining quantities and of γ, the gravitational constant.

Hence, however the linear series are arrived at, the points of bifurcation will be given by an equation of the form

$$f(\gamma, \lambda_0, \rho_0, a, c_1, c_2, \ldots) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now the coefficients $c_1, c_2, \ldots$ are mere numbers, and the only way in which γ, λ_0, ρ_0, and a can be combined so as to give a mere number is through the term $\gamma\rho_0^2a^2/\lambda_0$. Hence equation (1) can be expressed in the form

$$f\left(\frac{\gamma\rho_0^2a^2}{\lambda_0}, c_1, c_2, \ldots\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

We have seen that the spherical configuration must be unstable for some values of γ, ρ_0, a, and λ (*e.g.*, it is always unstable for $\gamma\rho_0^2a^2/\lambda_0 = \infty$), hence equation (2) must have at least one real root between $\gamma\rho_0^2a^2/\lambda_0 = 0$ and $\gamma\rho_0^2a^2/\lambda_0 = \infty$. Let the lowest root be

$$\gamma\rho_0^2a^2/\lambda_0 = \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where ϕ is a function of $c_1, c_2, \ldots$; then a spherical configuration is stable so long as $\gamma\rho_0^2a^2/\lambda_0 < \phi$, and becomes unstable as soon as $\gamma\rho_0^2a^2/\lambda_0 > \phi$.

The coefficients $c_1, c_2, \ldots$ will, on the average, be comparable with unity, because λ, ρ are referred to their *mean* values; they are as likely (speaking somewhat loosely) to be above as to be below unity. Hence ϕ itself will be comparable with unity, and

* The notation is that of LOVE'S 'Theory of Elasticity.' The m, n of THOMSON and TAIT are given by

$$\lambda + \mu = m, \qquad \mu = n.$$

it is not at present possible to say whether it is more likely to be greater or less than unity.

§ 4. Now, in the case of the earth (THOMSON and TAIT, § 838), we have

$$a = 640 \times 10^6 \text{ centims.}, \qquad \rho_0 = 5 \cdot 5,$$

and the value of γ in C.G.S. units is known to be

$$\gamma = 648 \times 10^{-10}.$$

This gives for $\gamma\rho_0^2 a^2$ the value

$$\gamma\rho_0^2 a^2 = 8 \times 10^{11},$$

whence it appears that for a sphere of the size and mass of the earth the spherical configuration will be unstable unless λ_0 has a value comparable with 8×10^{11}.

Now for steel (*cf.* THOMSON and TAIT, p. 435) the values of the elastic constants in absolute units are $n = \mu = 7 \cdot 7 \times 10^{11}$, $m = \lambda + \mu = 16 \cdot 0 \times 10^{11}$, whence $\lambda = 8 \cdot 3 \times 10^{11}$. We therefore see that the critical values of the elastic constants in the case of the earth are comparable with those of steel.

The foregoing calculation is, of course, very rough, but it shows that the critical values for the earth are at least in the neighbourhood of what must be supposed to be the actual values, so that we are driven to attempting a more accurate determination of these values. If the view of the present paper is sound, this approximate equality is more than a mere coincidence ; we shall see that it could have been predicted from our hypotheses of planetary evolution.

We now attempt a rigorous mathematical investigation of certain problems which have a bearing upon the astronomical questions in hand. Those readers whose interest lies in the application of the results rather than in the processes by which they are obtained may be recommended to turn at once to § 22.

THE STABILITY OF A GRAVITATING ELASTIC SOLID.

The Equations of Small Vibrations.

§ 5. We shall begin by discussing the principal vibrations and the frequency equation of a spherically symmetrical solid. The case of a non-gravitating sphere has been fully discussed by Professor LAMB,* but the inclusion of the gravitational terms, as will be seen later, brings about a considerable complication in the analysis. The case of a gravitating but incompressible sphere has been considered by BROMWICH,† but this has no bearing on the present problem, in which the whole

* "On the Vibrations of an Elastic Sphere," 'Proc. Lond. Math. Soc.,' vol. 13, p. 189.

† "On the Influence of Gravity on Elastic Waves, &c.," 'Proc. Lond. Math. Soc.,' vol. 30, p. 98.

interest turns upon the compressibility. The solution which follows is, in its main points, very similar to that of Professor LAMB, so that I have not thought it necessary to give the steps of the argument in great detail.

From the symmetry of the solid it follows that the elastic constants λ, μ, and the density ρ, will be functions of the single co-ordinate r, the distance from the centre. Taking the centre as origin, we shall use rectangular co-ordinates, x, y, z, and shall suppose the solid to execute a small vibration, such that the displacement of the element initially at x, y, z has components, ξ, η, ζ. The component of displacement along the radius will be denoted by u and the cubical dilatation by Δ, so that

$$u = \frac{1}{r}(\xi x + \eta y + \zeta z), \qquad \Delta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}.$$

§ 6. After displacement the density at x, y, z is

$$\rho - \frac{d}{dx}(\rho\xi) - \frac{d}{dy}(\rho\eta) - \frac{d}{dz}(\rho\zeta),$$

or, since ρ is a function of r only,

$$\rho - \Delta\rho - u\frac{d\rho}{dr}.$$

Hence the gravitational potential at x, y, z is changed by displacement from V into V − E, where E is the potential of the following distribution of matter :—

(i.) A volume distribution of density

$$\Delta\rho + u\frac{d\rho}{dr} \quad \ldots \ldots \ldots \quad (4),$$

(ii.) A surface distribution of which the surface density is

$$u(\rho_0 - \rho_1) \quad \ldots \ldots \ldots \quad (5),$$

this being taken over every surface at which the density changes abruptly, the change being from ρ_0 to ρ_1 in crossing the surface in the direction of r increasing. In particular this will occur at the outer surface of the solid, the value of ρ_1 in this case being zero.*

§ 7. The potential at x, y, z after displacement being V − E, that at $x + \xi$, $y + \eta$, $z + \zeta$ will be

$$\begin{aligned} V + \xi\frac{\partial V}{\partial x} + \eta\frac{\partial V}{\partial y} + \zeta\frac{\partial V}{\partial z} + \tfrac{1}{2}\xi^2\frac{\partial^2 V}{\partial x^2} + \ldots \\ - E - \xi\frac{\partial E}{\partial x} - \eta\frac{\partial E}{\partial y} - \zeta\frac{\partial E}{\partial \zeta} - \ldots \end{aligned}$$

* In the investigations on gravitating spheres given in THOMSON and TAIT'S 'Natural Philosophy,' the course of procedure is tantamount to neglecting the volume distribution (4), and regarding E as the potential of a surface distribution (5) alone. For this reason the result obtained differs from that of the present paper.

Hence the force at $x+\xi$, $y+\eta$, $z+\zeta$ in the direction of x increasing, found by differentiating the foregoing expression with respect to ξ, is, neglecting squares of the displacements,

$$\frac{\partial V}{\partial x}+\xi\frac{\partial^2 V}{\partial x^2}+\eta\frac{\partial^2 V}{\partial x\,\partial y}+\zeta\frac{\partial^2 V}{\partial x\,\partial z}-\frac{\partial E}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Let us suppose that, in addition to its own gravitation, the sphere is acted upon by an external field of force of potential V_0, and let us, in the usual notation, denote the six stresses by P, Q, R, S, T, U. Then the equations of motion of the element at $x+\xi$, $y+\eta$, $z+\zeta$ in the displaced configuration are three of the form

$$\rho\frac{\partial^2\xi}{\partial t^2}=\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}+\rho\left(\frac{\partial W}{\partial x}+\xi\frac{\partial^2 W}{\partial x^2}+\eta\frac{\partial^2 W}{\partial x\,\partial y}+\zeta\frac{\partial^2 W}{\partial x\,\partial z}-\frac{\partial E}{\partial x}\right) \quad . \quad . \quad (7),$$

in which $W = V + V_0$, and all the terms such as $\frac{\partial W}{\partial x}$, $\frac{\partial^2 W}{\partial x^2}$, . . . are evaluated at x, y, z, but ρ, P, Q . . . are calculated in the displaced configuration at $x+\xi$, $y+\eta$, $z+\zeta$.

§ 8. Now the only case in which we have any accurate knowledge as to the values of P, Q, R, S, T, U is when the whole strain is small, *i.e.*, when W is small. In the case of the earth, V is not, in this sense, small.* The only way in which we can proceed with any certainty is, therefore, by taking $V_0 = -V$, or $W = 0$. That is to say, we must artificially annul gravitation in the equilibrium configuration, so that this equilibrium configuration may be completely unstressed, and each element of matter be in its normal state. In this case it seems justifiable to suppose both the density and rigidity to be constant throughout the sphere, and, indeed, it is only with the help of this simplification that the equations become at all manageable.

The vibrations of this system will be of two kinds. First there are "spherical" vibrations in which the displacement is purely radial at every point, so that the solid remains spherically symmetrical after displacement, and, secondly, there is the larger class of vibrations in which the displacement is not of this simple type, so that the displaced configuration is not one of spherical symmetry.

We hope, by discussing the vibrations of this system, to obtain some insight into the corresponding vibrations of a natural non-homogeneous solid, say the earth. Now it is extremely doubtful whether the spherical vibrations of our artificial system have much in common with those of the natural system, but it will be seen later that this is of no importance. We shall not be in any way concerned with these vibrations. What we shall require is a knowledge of the unsymmetrical vibrations, and this, it is hoped, can be obtained with fair accuracy from a consideration of the corresponding vibrations in the artificial case. There must be some uncertainty even in the case of unsymmetrical vibrations, and, unfortunately, this seems to be inevitable; our

* LOVE, 'Elasticity,' I., p. 220.

artificial case appears to be the only case in which the equations can be solved by ordinary analysis.

We now replace P, Q, R, S, T, U by their ordinarily assumed values, and equation (7), putting W = 0, takes the form

$$\rho \frac{d^2\xi}{dt^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 \xi - \rho \frac{\partial E}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and there are two similar equations for η, ζ.

The Principal Vibrations and Frequency Equations.

§ 9. Differentiate these three equations of motion with respect to x, y, z and add; then

$$\rho \frac{d^2\Delta}{dt^2} = (\lambda + 2\mu) \nabla^2 \Delta - \rho \nabla^2 E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Now, from the definition of E, we have, in the case in which ρ is constant,

$$\nabla^2 E = - 4\pi\rho\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and hence equation (9) becomes

$$\rho \frac{d^2\Delta}{dt^2} = (\lambda + 2\mu) \nabla^2 \Delta + 4\pi\rho^2 \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

If we suppose Δ proportional to $\cos pt$, this equation assumes the form $(\nabla^2 + \kappa^2)\Delta = 0$, where

$$\kappa^2 = \frac{\rho(p^2 + 4\pi\rho)}{\lambda + 2\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

There is, therefore, a particular solution of (11) of the form

$$\Delta = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) S_n(\theta, \phi) \cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where $S_n(\theta, \phi)$ is a surface harmonic of order n, and the general solution found by summation of solutions of this type is

$$\Delta = \Sigma\Sigma r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) S_n(\theta, \phi) (A \cos pt + B \sin pt) \quad . \quad . \quad . \quad . \quad (14),$$

where the summation extends over all possible harmonics, and over all values of κ.

It will appear later that each term in this solution can be made to satisfy the boundary conditions, and, therefore, that each term represents a normal vibration.

The vibrations may, therefore, be classified into vibrations of *order* 0, 1, 2, &c., the order being that of the harmonic which occurs in the expression for Δ. The vibrations of order $n = 0$ are the spherical vibrations already referred to.

We shall assume this provisionally, in order to avoid the continual repetition of double summation, and now proceed to evaluate ξ, η, ζ and to form the boundary equations for the simple vibration given by equation (13).

§ 10. From equation (8) it appears that the displacement ξ is given by

$$p^2\rho\xi + \mu\nabla^2\xi = -(\lambda+\mu)\frac{\partial\Delta}{\partial x} + \rho\frac{\partial \mathrm{E}}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

The solution is

$$\xi = \frac{d\phi}{dx} + \xi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

where ϕ is any solution of

$$p^2\rho\phi + \mu\nabla^2\phi = -(\lambda+\mu)\Delta + \rho\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

and ξ_0 is the most general solution of

$$p^2\rho\xi_0 + \mu\nabla^2\xi_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

It can easily be verified that a solution of equation (17) is

$$\phi = -\frac{1}{p^2}\left(\frac{\lambda+2\mu}{\rho}\Delta - \mathrm{E}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

There will be solutions for η, ζ similar to (16), but the three solutions for ξ, η, ζ must be such that

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

The left-hand member of (20) is, from (16),

$$\nabla^2\phi + \frac{d\xi_0}{dx} + \frac{d\eta_0}{dy} + \frac{d\zeta_0}{dz},$$

and from (19) and (17), $\nabla^2\phi = \Delta$. Hence (20) is satisfied if

$$\frac{d\xi_0}{dx} + \frac{d\eta_0}{dy} + \frac{d\zeta_0}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

§ 11. Write u for $\frac{x}{r}\xi + \frac{y}{r}\eta + \frac{z}{r}\zeta$ as before, and u_0 for $\frac{x}{r}\xi_0 + \frac{y}{r}\eta_0 + \frac{z}{r}\zeta_0$. Then we shall verify that the solutions for u and u_0 are

$$u = \alpha \mathrm{S}_n, \qquad u_0 = \alpha_0 \mathrm{S}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

in which α, α_0 are functions of r, as yet unknown.

Assuming these solutions, the value of E, calculated as explained in § 6, is

$$\mathrm{E} = \frac{4\pi\rho \mathrm{S}_n}{2n+1}\left\{\frac{1}{r^{n+1}}\int_0^r r^{n+\frac{3}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa r)\,dr + r^n\int_r^a r^{-n+\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa r)\,dr + \frac{r^n}{a^{n-1}}\alpha_a\right\} \quad . \quad (23),$$

where α_a denotes $(\alpha)_{r=a}$.

We can calculate the value of the integrals which occur in this expression, and the sum of the first two terms inside the curled brackets is found to be

$$\frac{2n+1}{\kappa^2} r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) - \frac{r^n}{\kappa a^{n-\frac{1}{2}}} J_{n-\frac{1}{2}}(\kappa a).$$

Hence we may write (19) in the form

$$\phi = \mathfrak{G} S_n + \frac{4\pi\rho S_n}{(2n+1)p^2} \frac{r^n}{a^{n-1}} \alpha_n,$$

where

$$\mathfrak{G} = C r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) + D r^n J_{n-\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

$$C = -\frac{\lambda + 2\mu}{p^2 \rho} + \frac{4\pi\rho}{p^2 \kappa^2} = -\frac{1}{\kappa^2},$$

$$D = -\frac{4\pi\rho}{(2n+1)\, p^2 \kappa} a^{-n+\frac{1}{2}}.$$

We now have, from equation (16),

$$\xi = \frac{d}{dx}(\mathfrak{G} S_n) + \frac{4\pi\rho}{(2n+1)p^2} \frac{d}{dx}\left[\frac{r^n \alpha_n S_n}{a^{n-1}}\right] + \xi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

and hence

$$u = \frac{d\mathfrak{G}}{dr} S_n + \frac{4\pi\rho n}{(2n+1)p^2} \frac{r^{n-1}\alpha_n S_n}{a^{n-1}} + u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

§ 12. There are three boundary-conditions to be satisfied, expressing that the normal pressure and the two tangential tractions shall vanish at every point of the free surface. As LAMB* shows, these may be represented by three symmetrical equations, to be satisfied at the surface $r = a$, each of the type

$$\lambda x \Delta + \mu \frac{d}{dx}(ru) + \mu\left(r\frac{d\xi}{dr} - \xi\right) = 0.$$

Substituting for ξ and u from (25) and (26) this becomes

$$\lambda \Delta x + \mu\left[\frac{d}{dx}\left(r\frac{d\mathfrak{G}}{dr} S_n\right) + r\frac{d}{dr}\frac{d}{dx}(\mathfrak{G} S_n) - \frac{d}{dx}(\mathfrak{G} S_n)\right]$$

$$+ \mu \frac{4\pi\rho(2n-2)}{(2n+1)p^2} \frac{d}{dx}\left[\frac{r^n \alpha_n S_n}{a^{n-1}}\right] + \mu\left[\frac{d}{dx}(ru_0) + r\frac{d\xi_0}{dr} - \xi_0\right] = 0 \quad . \quad . \quad (27).$$

§ 13. Write

$$\frac{d}{dx}(r^n S_n) = r^{n-1}\omega, \qquad \frac{d}{dx}(r^{-(n+1)} S_n) = r^{-(n+2)}\chi$$

so that the right-hand members are solid harmonics of degrees $n-1$ and $-(n+2)$: then

* LAMB, *loc. cit. ante*, p. 191.

$$xS_n = \frac{r}{2n+1}(\omega - \chi),$$

$$\frac{d}{dx}\{f(r)\,S_n\} = \frac{1}{2n+1}\left\{r^{-(n+1)}\frac{d}{dr}(r^{n+1}f)\,\omega - r^n\frac{d}{dr}(r^{-n}f)\,\chi\right\}.$$

From these identities it is clear that if the terms in (27) which do not depend on ξ_0 or u_0 are expanded in spherical harmonics, they will contain no harmonics other than ω and χ. We therefore see that the form of ξ_0 may be assumed to be

$$\xi_0 = \mathfrak{P}\omega + \mathfrak{Q}\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

where $\mathfrak{P}$ and $\mathfrak{Q}$ are functions of r. The value of u_0 is

$$u_0 = (n\mathfrak{P} - (n+1)\,\mathfrak{Q})\,S_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

whence

$$\alpha_0 = n\mathfrak{P} - (n+1)\,\mathfrak{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

§ 14. Substituting for ξ_0 in (27) and equating the coefficients of ω and χ, we obtain the two following equations which must be satisfied at $r = a$:—

$$\frac{\lambda}{\mu}\frac{r^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r)}{2n+1} + \frac{1}{2n+1}\left\{r^{-(n+1)}\frac{d}{dr}\left(r^{n+2}\frac{d\mathfrak{G}}{dr}\right) + \left(r\frac{d}{dr} - 1\right)\left(r^{-(n+1)}\frac{d}{dr}(r^{n+1}\mathfrak{G})\right)\right\}$$

$$+ \frac{4\pi\rho(2n-2)}{(2n+1)\,p^2}\left(\frac{r}{a}\right)^{n-1}\alpha_a + \frac{1}{2n+1}r^{-(n+1)}\frac{d}{dr}(r^{n+2}\alpha_0) + r\frac{d\mathfrak{P}}{dr} - \mathfrak{P} = 0 \quad . \quad (31),$$

and a second equation of a similar kind, of which the first line can be obtained from the first line of the above by writing $-(n+1)$ for n, and the second line is

$$-\frac{1}{2n+1}r^n\frac{d}{dr}(r^{-(n-1)}\alpha_0) + r\frac{d\mathfrak{Q}}{dr} - \mathfrak{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

The expression which occurs in curled brackets in (31) can be transformed into

$$2\left\{r^{-(n+1)}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^{n+1}\mathfrak{G})\right) - nr^{-(n+1)}\frac{d}{dr}(r^{n+1}\mathfrak{G})\right\} \quad . \quad . \quad . \quad . \quad (33),$$

while the corresponding expression in (32) is seen to be

$$2\left\{r^{n+2}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^{-n}\mathfrak{G})\right) + (n+1)\,r^n\frac{d}{dr}(r^{-n}\mathfrak{G})\right\} \quad . \quad . \quad . \quad . \quad (34).$$

From the value of $\mathfrak{G}$, given by equation (24),

$$\frac{d}{dr}(r^{n+1}\mathfrak{G}) = C\kappa r^{n+\frac{1}{2}}J_{n-\frac{1}{2}}(\kappa r) + (2n+1)\,Dr^{2n}J_{n-\frac{1}{2}}(\kappa a),$$

$$\frac{d}{dr}(r^{-n}\mathfrak{G}) = -\,C\kappa r^{-(n+\frac{1}{2})}J_{n+\frac{3}{2}}(\kappa r).$$

Hence expression (33) becomes

$$2\{C\kappa^2 r^{\frac{1}{2}} J_{n-\frac{3}{2}}(\kappa r) - nC\kappa r^{-\frac{1}{2}} J_{n-\frac{1}{2}}(\kappa r) + (2n+1)(n-1) D r^{n-1} J_{n-\frac{1}{2}}(\kappa a)\},$$

of which the value at $r = a$ is

$$\theta_1 \equiv 2a^{\frac{1}{2}} C\kappa^2 J_{n-\frac{3}{2}}(\kappa a) + 2[(2n+1)(n-1) D a^{n-1} - n a^{-\frac{1}{2}} C\kappa] J_{n-\frac{1}{2}}(\kappa a).$$

This is the value at $r = a$ of the term which occurs in curled brackets in equation (31). The value of the similar term in (32), namely expression (34), is seen to be

$$\theta_2 \equiv 2a^{\frac{1}{2}} C\kappa^2 J_{n+\frac{3}{2}}(\kappa a) - 2(n+1) a^{-\frac{1}{2}} C\kappa J_{n+\frac{3}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad (35).$$

Write

$$\mathfrak{U} = \theta_1 + \frac{\lambda}{\mu} a^{\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

$$\mathfrak{V} = \theta_2 + \frac{\lambda}{\mu} a^{\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

then equations (31) and (32) become, at $r = a$,

$$\mathfrak{U} + \frac{4\pi\rho(2n-2)}{p^2} \alpha_a + a^{-(n+1)} \frac{d}{dr}(r^{n+2}\alpha_0) + (2n+1)\left(r\frac{d\mathfrak{P}}{dr} - \mathfrak{P}\right) = 0 \quad . \quad (38),$$

$$\mathfrak{V} + a^n \frac{d}{dr}(r^{-(n-1)}\alpha_0) - (2n+1)\left(r\frac{d\mathfrak{Q}}{dr} - \mathfrak{Q}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

Now we have, from equation (26),

$$\alpha_a = \left(\frac{d\mathfrak{G}}{dr}\right)_{r=a} + \frac{4\pi\rho n}{(2n+1)p^2}\alpha_a + (\alpha_0)_{r=a}.$$

Write

$$c = \left[1 - \frac{4\pi\rho n}{(2n+1)p^2}\right]^{-1} \equiv \frac{p^2(2n+1)}{(2n+1)p^2 - 4\pi\rho n},$$

then this last equation becomes

$$\alpha_a = c\left(\alpha_0 + \frac{d\mathfrak{G}}{dr}\right)_{r=a}.$$

Now, at $r = a$,

$$a^{-(n+1)} \frac{d}{dr}(r^{n+2}\alpha_0) = (n+2)\alpha_0 - a\frac{d\alpha_0}{dr},$$

$$a^n \frac{d}{dr}(r^{-(n-1)}\alpha_0) = -(n+1)\alpha_0 + a\frac{d\alpha_0}{dr},$$

and equations (38) and (39) become

$$\mathfrak{U} + \frac{4\pi\rho(2n-2)c}{p^2}\left(\alpha_0 + \frac{d\mathfrak{G}}{dr}\right) + (n+2)\alpha_0 - a\frac{d\alpha_0}{dr} + (2n+1)\left(r\frac{d\mathfrak{P}}{dr} - \mathfrak{P}\right) = 0 \quad . \quad (40),$$

$$\mathfrak{V} - (n+1)\alpha_0 + a\frac{d\alpha_0}{dr} - (2n+1)\left(r\frac{d\mathfrak{Q}}{dr} - \mathfrak{Q}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

in which r must be put equal to a.

§ 15. Now ξ_0 is known to be a solution of equation (18), and hence, from equation (28), we may assume

$$\mathfrak{P} = \mathfrak{P}_0 r^{-\frac{1}{2}} J_{n-\frac{1}{2}}(hr), \qquad \mathfrak{Q} = \mathfrak{Q}_0 r^{-\frac{1}{2}} J_{n+\frac{3}{2}}(hr),$$

where

$$h^2 = p^2\rho/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

It follows that at $r = a$ we may write

$$a\frac{d\mathfrak{P}}{dr} = A\mathfrak{P}, \qquad a\frac{d\mathfrak{Q}}{dr} = B\mathfrak{Q},$$

where

$$A = a\frac{d}{da}\log\left(a^{-\frac{1}{2}} J_{n-\frac{1}{2}}(ha)\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43),$$

$$B = a\frac{d}{da}\log\left(a^{-\frac{1}{2}} J_{n+\frac{3}{2}}(ha)\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).$$

Hence, from (30), we have at $r = a$,

$$a\frac{d\alpha_0}{dr} = nA\mathfrak{P} - (n+1)B\mathfrak{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Lastly, we have from (28)

$$\xi_0 = \mathfrak{P} r^{-n-1}\frac{d}{dr}(r^n S_n) + \mathfrak{Q} r^{n+2}\frac{d}{dr}(r^{-(n+1)} S_n).$$

In order that (21) may be satisfied, we must have

$$\frac{d}{dr}(\mathfrak{P} r^{-(n-1)})\, nr^{n-1} - \frac{d}{dr}(\mathfrak{Q} r^{n+2})(n+1)\, r^{-(n+2)} = 0 \quad . \quad . \quad . \quad . \quad (46).$$

Substituting for $\mathfrak{P}$ and $\mathfrak{Q}$, we find that this is satisfied for all values of r if

$$n\mathfrak{P}_0 + (n+1)\mathfrak{Q}_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

i.e., if we have, at $r = a$,

$$\mathfrak{Q} = \theta\mathfrak{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where

$$\theta = -\frac{n}{n+1}\frac{J_{n+\frac{3}{2}}(ha)}{J_{n-\frac{1}{2}}(ha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Equations (40) and (41) now assume the forms

$$\mathfrak{U} + \frac{4\pi\rho(2n-2)e}{p^2}\frac{d\mathfrak{G}}{dr} + \mathfrak{P}\left[\frac{4\pi\rho(2n-2)e}{p^2}(n-(n+1)\theta) + (n^2-1) + (n^2+3n+2)\right.$$
$$\left. + (3n+1)A - (n+1)B\theta\right] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\mathfrak{V} + \mathfrak{P}\left[n - n^2 + (n^2+2n)\theta + nA - (3n+2)B\theta\right] = 0 \quad . \quad . \quad . \quad (51),$$

in which r must be put equal to a.

The general frequency equation may be found at once by the elimination of $\mathfrak{P}$, the values of $\mathfrak{U}$ and $\mathfrak{V}$ being given by equations (36) and (37).

Points of Bifurcation.

§ 16. The interest of the question lies in the position of the points of bifurcation; to find these we must put $p^2 = 0$ in the frequency equation. The reason why it was not possible to put $p^2 = 0$ at an earlier stage will be understood by those who have read the former paper "On the Stability of a Spherical Nebula." In the present instance it is, perhaps, sufficient to say that putting $p^2 = 0$ at an earlier stage would have led to an entirely misleading result. Upon putting $p^2 = 0$ in equations (50) and (51) we find that the two brackets multiplying $\mathfrak{P}$ vanish, and we therefore see that $\mathfrak{P}$ must be treated as an infinite quantity of the order of $1/p^2$.

Expanding these brackets as far as p^2, and then putting $p^2 = 0$, we find that the two equations become

$$x_1 - \mathfrak{P}p^2 y_1 = 0 \quad . \quad . \quad . \quad . \quad (52), \qquad x_2 + \mathfrak{P}p^2 y_2 = 0 \quad . \quad . \quad . \quad . \quad (53),$$

where

$$x_1 = \mathfrak{U} + \frac{4\pi\rho(2n-2)r}{p^2}\frac{d\mathfrak{G}}{dr}, \qquad x_2 = \mathfrak{F},$$

$$y_1 = \frac{(n-1)(2n+1)^2}{2n\pi\rho} + \frac{\rho a^2}{\mu}\left\{\frac{2(2n^2-1)}{(2n+1)(2n+3)} - \frac{3n+1}{2(n+1)}\right\}$$

$$y_2 = \frac{\rho a^2}{\mu}\frac{n}{2(2n+1)(2n+3)}.$$

The equation giving points of bifurcation is, of course,

$$x_1 y_2 + x_2 y_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

The values of x_1 and x_2 are found, after some simplification, to be

$$x_1 = \frac{\lambda}{\mu}a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a) + 2C\left\{a^{\frac{1}{2}}\kappa^2 J_{n-\frac{3}{2}}(\kappa a) - na^{-\frac{1}{2}}\kappa J_{n-\frac{1}{2}}(\kappa a) - \frac{(n-1)(2n+1)}{n}\frac{d}{da}\left(a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a)\right)\right\}$$
$$+ \frac{2(n-1)(2n+1)}{n\kappa}a^{-\frac{1}{2}}J_{n-\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

$$x_2 = \frac{\lambda}{\mu}a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a) + 2a^{\frac{1}{2}}C\kappa^2 J_{n+\frac{3}{2}}(\kappa a) - 2(n+1)a^{-\frac{1}{2}}C\kappa J_{n+\frac{3}{2}}(\kappa a) \quad . \quad . \quad (56).$$

Now, it has already been seen that $C = -\frac{1}{\kappa^2}$ (p. 164). If we substitute this value for C, write x for κa, and simplify equations (55) and (56) as far as possible, we have

$$x_1 a^{-\frac{1}{2}} = \frac{\lambda + 2\mu}{\mu}J_{n+\frac{1}{2}}(x) + \frac{2(2n+1)^2(n-1)}{nx^2}J_{n+\frac{1}{2}}(x) - \frac{2(n-1)(3n+2)}{nx}J_{n+\frac{3}{2}}(x) \quad . \quad (57),$$

$$x_2 a^{-\frac{1}{2}} = \frac{\lambda + 2\mu}{\mu}J_{n+\frac{1}{2}}(x) - \frac{2(n+2)}{x}J_{n+\frac{3}{2}}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58),$$

while the value of y_1 and y_2 may be written in the form

$$y_1 = \frac{\rho a^2}{\mu}\left\{-\frac{4n^3 + 20n^2 + 21n + 7}{(2n+1)(2n+2)(2n+3)} + \frac{2(n-1)(2n+1)^2}{nx^2}\frac{\mu}{\lambda + 2\mu}\right\} \quad . \quad . \quad (59),$$

$$y_2 = \frac{\rho a^2}{\mu}\frac{n}{2(2n+1)(2n+3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

The equation giving points of bifurcation can now be found by substituting these values in equation (54).

§ 17. This equation will have roots corresponding to the different integral values of n, $n = 0, 1, 2 \ldots$; these determine points of bifurcation such that the critical vibrations are of orders $n = 0, 1, 2 \ldots$ respectively.

Of these the points of bifurcation of zero order are of no importance. The reason is exactly similar to that given in the case of a spherical nebula (§ 28 of the paper already quoted), namely, that a point of bifurcation of order $n = 0$ does not indicate a departure from the spherical shape. We therefore will only discuss values of n different from zero.

Case of $\mu = 0$.

§ 18. Before discussing the general form assumed by equation (54), it will be well to consider the simple case of $\mu = 0$. Putting $\mu = 0$, we obtain from equations (57) and (58)

$$x_1 = x_2 = \frac{\lambda + \mu}{\mu} a^{\frac{1}{2}} J_{n+\frac{1}{2}}(x).$$

Referring to equations (52) and (53) we see that the equation giving points of bifurcation is

$$J_{n+\frac{1}{2}}(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

The lowest roots of various orders other than zero are

$n = 1,$	2,	3,	4,	5,
$x = 4\cdot49,$	5·76,	6·98,	8·18,	9·37, &c.,

the roots continually increasing with n. Thus the first point of bifurcation is given by $x = 4\cdot49$, and the critical vibration is of order $n = 1$.

Case of μ Different from Zero.

§ 19. The general equation in which μ is not put equal to zero is much more complicated than equation (61), which has just been considered. If we write u_n for $J_{n+\frac{1}{2}}(x)/J_{n-\frac{1}{2}}(x)$, it will be seen that the equation giving points of bifurcation of order n is of the form

$$u_{n+1} = \text{an algebraic function of } x \text{ and of } (\lambda + 2\mu)/\mu.$$

To obtain approximate numerical solutions, my plan has been to draw graphs of the functions u_n, and in this way obtain a graphical solution of the equations for different values of μ. There is no difficulty in drawing graphs of the functions u_n; these are trigonometrical functions, and we have

$$u_1 = \frac{1}{x} - \cot x,$$

while the successive u's are connected by the relation

$$u_{n+1} = \frac{2n+1}{x} - \frac{1}{u_n}.$$

To save space I have suppressed all details of this somewhat tedious part of the work. The results for $n = 1, 2, 3$ are given in the following table:—

LOWEST Values of x.

	$\mu = 0.$	$\mu = \frac{1}{2}\lambda.$	$\mu = \lambda.$
$n = 1$	4·49	4·2	4·0
$n = 2$	5·76	5·6	5·4
$n = 3$	6·98	6·8	6·7

For large values of n it will be found that equation (54) reduces approximately to $x_2 = 0$, and hence that for any value of μ the lowest value of x is slightly less than the corresponding value in the case in which $\mu = 0$.

The First Point of Bifurcation.

§ 20. It therefore appears that the first point of bifurcation may be safely assumed to be of order $n = 1$. The value of x for which it occurs will have some value between 4·0 and 4·49, according to the value of μ/λ. Now $x = \kappa a$, and the value of κ^2 is $4\pi\rho^2/(\lambda + 2\mu)$. Hence the first point of bifurcation is approximately given by

$$\frac{4\pi\rho^2 a^2}{\lambda + 2\mu} = \begin{cases} 4{\cdot}00^2 = 16{\cdot}00, \text{ when } \mu = \lambda, \\ 4{\cdot}49^2 = 20{\cdot}16, \text{ when } \mu = 0. \end{cases}$$

In equation (3) we supposed this point of bifurcation to be given by

$$\gamma\rho_0^2 a^2/\lambda_0 = \phi.$$

In our present analysis we have already taken $\gamma = 1$; if we take $(\lambda + 2\mu)$ to be identical with our former λ_0, we see that the actual values of ϕ are roughly

$$\phi = 1{\cdot}60, \text{ when } \mu = 0, \qquad \phi = 1{\cdot}27, \text{ when } \mu = \lambda.*$$

* It will be found that the first point of bifurcation is given, with great accuracy, by the single equation $\rho^2 a^2/(\lambda + \frac{7}{5}\mu) = 1{\cdot}6$ for all values of μ between 0 and λ. This is of interest, as showing the relative importance of λ and μ in maintaining stability. As might be foreseen, the importance of μ relatively to λ increases as n increases, and for $n = \infty$, the factor $\lambda + \frac{7}{5}\mu$ must be replaced by $\lambda + 2\mu$.

We have now found a closer approximation to the value of ϕ than that which was given in § 3, and have obtained the additional information that instability first enters through a vibration of order $n = 1$. It must, however, be borne in mind that these results are only true of the special and somewhat artificial case specified in § 8.

Comparison with the Case of a Spherical Nebula.

§ 21. It will be seen that the general argument of § 3 will apply to the case of a gaseous planet or nebula if λ be taken to mean the pressure in the gas. In this case, however, the laws of distribution of density and pressure are not independent. If the gas is in conductive equilibrium throughout, the planet or nebula must be supposed to extend to infinity, and for these conditions the criterion of stability was worked out in the former paper already referred to. Calling the elasticity of the gas κ, the first point of bifurcation was found to be reached when the function $\mathop{\mathrm{Lt}}_{r=\infty} \frac{2\pi\rho r^2}{\kappa}$ attains a certain finite value. Now $\mathop{\mathrm{Lt}}_{r=\infty} \rho$ vanishes in comparison with ρ_0, the mean density, so that writing a for the radius of the nebula, and λ_0 for the mean pressure ($\lambda_0 = \kappa\rho_0$), we have, at this first point of bifurcation

$$2\pi\rho_0^2 a^2/\lambda_0 = \infty .$$

Comparing this with the general result obtained in § 3, we see that in this extreme case the value of ϕ becomes infinite. This result is only of importance to the present investigation as showing the tendency of a concentration of density about the centre. It seems to show that as the density becomes more concentrated about the centre, the value of ϕ may be expected to increase. We are therefore led to expect that in general ϕ will have a value rather greater than that found for it upon the assumption of homogeneity of density.

Recapitulation and Discussion of Results.

§ 22. It will be well to recapitulate our results before attempting to draw any deductions from them.

We consider a spherically symmetrical mass of solid, liquid, or gaseous matter. We denote the radius of this by a, the mean density by ρ_0, and the mean value of λ by λ_0, where λ denotes an elastic constant or the pressure of the fluid, according as the matter is solid or fluid. We have seen that the stability of this dynamical system depends upon the value of the function $\gamma\rho_0^2 a^2/\lambda_0$, a pure number. When $\gamma = 0$ (*i.e.*, when we deal with artificial matter which is totally devoid of gravitation) there can be no doubt that the system is stable. We have seen that a point of bifurcation occurs when the number $\gamma\rho_0^2 a^2/\lambda_0$ has a certain value ϕ. It has not been proved in the present paper that an exchange of stabilities accompanies this point of bifurcation, but it will be seen that, with slight alterations, the proof of the exchange of stabilities for the spherical nebula, which was given in § 28 of the earlier paper, can be made to apply to the present case. Admitting this, it appears that the spherical system which is at present under discussion will be stable so long as $\gamma\rho_0^2 a^2/\lambda_0$ is less than ϕ, and becomes unstable so soon as $\gamma\rho_0^2 a^2/\lambda_0$ exceeds ϕ.

§ 23. The next question is as to the exact value of ϕ, and as to the vibration through which instability enters at the point of bifurcation. To the first part of the question we have not been able to obtain a very definite answer. This matters the less, since the numerical data which would have to be used in making any applications of our results are not themselves very definite. On the whole, the uncertainty in the value of ϕ is not much greater than the uncertainty in the value of the numerical data (or, what comes to the same thing for our present purpose, the uncertainty in our knowledge of the law of compressibility and distribution of density in the planets of our system).

The general argument of § 3 showed that ϕ must, except in extreme cases, be comparable with unity. We then examined an artificial case: a planet in which the density and elasticity were constant throughout—this system being made mechanically possible by introducing an external field of force, of amount just sufficient to annul gravitation in the equilibrium configuration. For this system ρ_0 was, of course, taken equal to ρ, the uniform density, and λ_0 was taken to be equal to $\lambda + 2\mu$ in the notation of LOVE, or $m + n$ in the notation of THOMSON and TAIT. The value of ϕ depends, of course, on the ratio μ/λ or n/m. For $\mu/\lambda = 0$ we found $\phi = 1{\cdot}6$; for $\mu/\lambda = 1$ we found $\phi = 1{\cdot}27$; for intermediate value of μ/λ we saw that the value of ϕ was intermediate between these two values.

The planets to which we wish to apply our results do not possess uniform density: it is almost certain that in every case the mean density is much greater than the surface density. The general argument of § 3 shows that there is still a point of bifurcation corresponding to a value of ϕ which is comparable with unity, but affords no evidence as to the change which a concentration of density will effect in the value of ϕ. We therefore examined a case in which there is an infinite concentration of density—the case of a spherical nebula extending to infinity—and found that in this extreme case the value of ϕ becomes infinite. It therefore seems probable that a concentration of density is attended by an increase in the value of ϕ. As a working hypothesis we shall assume for the planets of the solar system the uniform value $\phi = 2$. It must be left to the reader to form a judgment as to the amount of error involved in this assumption, but it will, perhaps, be admitted that results depending upon it will at least be right as regards order of magnitude. It will be seen later that considerable variation in the value of ϕ is possible before the astronomical evidence which we are going to bring forward is seriously invalidated.

§ 24. As regards the nature of the vibration through which instability of the spherical configuration enters, we are able to come to a more definite conclusion. In each of the cases referred to in the last section this vibration is found to be one of order $n = 1$, *i.e.*, one in which the displacement at every point is proportional to the first harmonic. This is the result which we should naturally expect—just as we expect a mass of liquid to become unstable through long surface waves sooner than through short ones. We shall, therefore, suppose it to be true of the planets in general. It is conceivable that planets could be artificially constructed for which this assumption would not be true, but, at present, since we have not a complete knowledge of the structure of the planets and are therefore compelled to make some assumptions, it seems as if the assumption just made is far and away the best to take as a working hypothesis.

Application to the Nebular Hypothesis.

Theoretical Conclusions.

§ 25. In the former paper, already referred to, the suggestion has been put forward that the instability of a nebula, sun or planet, which, upon the nebular hypothesis, is supposed ultimately to result in the ejection of a satellite, may be largely brought about by a gravitational tendency to instability of the kind we have been investigating. Let us, for the moment, take an extreme hypothesis, and imagine that this agency is the preponderating agency, and that the rotational tendency to instability may be disregarded in comparison.

Upon this hypothesis let us consider the case of an approximately spherical planet or sun which is known to have thrown off a satellite. Before the ejection of this satellite commenced, the primary mass would have an approximately spherical form, for which $\rho_0^2 a^2/\lambda_0$ would be below the critical value ϕ. When this critical value is reached, a divergence from the spherical form occurs, and a series of new processes begins. We are not now concerned with the details of these processes, but they must be supposed ultimately to result in the ejection of a satellite. It must be noticed that we are not supposing the primary to be devoid of rotation—for this would be inconsistent with the ejection of a satellite—but are supposing the rotation to be so small that the rotational tendency to instability is small in comparison with the gravitational.

If we suppose one or more satellites to have been ejected, and the primary to have regained an approximately spherical form, the new value of $\rho_0^2 a^2/\lambda_0$ must be less than ϕ. Now every satellite of which we have any knowledge, in our own system at any rate, is small in comparison with the primary. A legitimate inference seems to be that the ejection of a satellite is only a small part of the life-history of the primary. We shall not, however, need to make any assumption so definite as this, but shall suppose only that the values of ρ_0, a, λ_0 for the primary after ejection are nearly equal (*i.e.*, equal, except for a multiplying factor of, say, $1\frac{1}{4}$ or $1\frac{1}{2}$) to the values of the same quantities before the process of ejection commenced.

All this may be summed up, with sufficient accuracy for the present investigation, in the statement that when the ejection of a satellite is taking place the value of $\rho_0^2 a^2/\lambda_0$ must be nearly equal to ϕ.

Conclusions Tested by the Solar System.

§ 26. To a certain extent the solar system supplies material for testing this conclusion. Let ρ_0, a denote the present mean density and radius of any member of the system which is known to have thrown off a satellite, and let λ_0 be the mean value of λ, whether this denotes a pressure or an elastic constant, then we may write

$$\rho_0^2 a^2/\lambda_0 = \theta\phi.$$

At the time at which the satellite was ejected the same equation ought to hold with θ nearly equal to unity. Unfortunately we have no knowledge as to the changes which will have taken place in ρ_0, a, λ_0 since the ejection of the last satellite. We shall, for a rough approximation, neglect these changes altogether, take $\theta = 1$, $\phi = 2$, and examine to what extent the equation

$$\rho_0^2 a^2/\lambda_0 = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

holds for the solar system as it now stands.

It will be remembered that there are three sources of error in this equation :—

(i.) We are neglecting the effect of rotation in bringing about the ejection of a satellite.

(ii.) We are neglecting the changes which have taken place since the ejection of the last satellite.

(iii.) We are using an arbitrarily chosen value for ϕ, and applying this to every planet, while we know, from the difference in the physical constitutions of the planets, that the value of ϕ must be different for each.

These three sources of error would each be serious if we were attempting to get accurate results, but as our calculations are necessarily only of the roughest kind, we may be content to neglect them.

§ 27. In the accompanying table the masses and radii of the sun, and of those planets which possess satellites, are given in the first two columns; the value of mass/(radius)^2 is given in the third. The units are so chosen that the earth is measured by unity in each of these columns. Venus is included for the sake of comparison, although the existence of a satellite is extremely doubtful.

	Observed.			Calculated upon the hypothesis of the present paper ($\phi = 2$).
	(1) Mass.	(2) Radius.	(3) $\frac{\text{Mass}}{(\text{Radius})^2}$	(4) Coefficient λ_0. Unit = 10^{11} absolute = 10^8 grammes weight per sq. centim.
Sun	315,000	109	26	2700
Venus	0·8	1·0	0·9	3·2
Earth	1·0	1·0	1·0	4·0
Mars	0·1	0·5	0·4	·6
Jupiter	300·0	11·0	2·5	25·0
Saturn	90·0	9·0	1·1	5·0
Uranus	14·0	4·0	0·9	3·2
Neptune . . .	16·0	4·4	0·8	2·6

If our hypotheses give a fair account of the facts the numbers in this third column will be proportional to $\sqrt{\lambda_0 \phi}$. Assuming for ϕ the uniform value $\phi = 2$, we can calculate the actual values of λ_0, and these are given in the fourth column.

§ 28. Knowing nothing about the variation in λ_0, we shall be content as a preliminary hypothesis to suppose it to have the same value for each planet. Combining this with the hypotheses already formulated, we notice that $\sqrt{\lambda_0 \phi}$ ought to have the same value for each planet, as therefore ought also the function mass/(radius)^2, which is tabulated in column (3).

It will be seen at once that there is a certain amount of uniformity about the numbers in this column, but it requires some consideration to determine how much significance is to be attached to this uniformity.

Now, apart from any hypothesis as to how the solar system originated or reached its present configuration—*i.e.*, regarding the solar system as a fortuitous collection of bodies of varying sizes—we should expect the mean density to be greatest in the greatest planets. We should, therefore, expect the quantity (mass)/(radius)^2 to be more variable than the radius. In other words, we should, *à priori*, expect less uniformity in the third column than in the second. Judged by this criterion, the uniformity of the numbers in the third column would be very significant. Further, the variation in these numbers is of the kind we should expect. For instance, it is known that the density of Jupiter is very much greater near the centre than near the surface; we should accordingly expect a large value of ϕ, and therefore a large entry in the third column. The same argument would apply to the Sun, but the physical constitution of the Sun is probably so different from that of the planets that there could be no surprise at the Sun figuring as an exceptional case. Another exception is that of Mars. Part of the discrepancy might, perhaps, be attributed to the smallness of the planet, but the figure in the fourth column would seem to suggest that rotational instability must have played a large part in the creation of the Martian satellites.

If, on the other hand, we begin by regarding the planets not as a fortuitous collection of bodies, but as a series of satellites all ejected from the same primary, the case is different. For here we should expect the smaller planets to have cooled more than the heavier ones, and therefore to be at a lower temperature. Against this must be set the fact that the heavier planets will probably have the greatest concentration of density about the centre, and the greatest mean pressure. The first consideration tends to increase the value which we should expect for the mean density of the smaller planets as compared with that of the greater ones; the second consideration tends in the opposite direction. We can hardly profess to estimate the relative weights of these two considerations with any approach to accuracy; perhaps it is best to revert to the argument given in the last paragraph, while bearing in mind that the approximate equality of our numbers may become considerably less significant as soon as the question of relative temperature is taken into account.

§ 29. We now consider the evidence afforded by the absolute value of our figures. After allowing for the exceptional cases, it appears that the value of λ_0 for the earth and for most of the planets is about 4×10^{11}. In other words, if we suppose these planets suddenly to resume the molten state, while retaining their present mass and radius, the spherical form will be stable or unstable according as the mean value of $\lambda + 2\mu$ is greater or less than 4×10^{11}. In the molten state we may take $\mu = 0$, and the value $\lambda = 4 \times 10^{11}$ corresponds to a value equal to about half of that of steel, for which $\lambda = 8{\cdot}3 \times 10^{11}$. If, however, we attempt to trace the history of a planet backward in time, we cannot suppose the mass and radius kept constant: the mass may be constant, but the radius will increase. Under these conditions we find that the critical value of λ_0 will be inversely proportional to the fourth power of the radius, and will, therefore, be somewhat less than the value $\lambda = 4 \times 10^{11}$. It would be extremely difficult to form a reliable estimate of what this corrected value for λ ought to be, and equally difficult to estimate what would be the actual value of λ for molten material similar to that of which our planets must have been composed when in the molten state. Our argument is that the two values of λ are at least of the same order of magnitude, and probably equal, except for inaccuracies in our calculations.

Comparison of the Rotational and Gravitational Hypotheses.

§ 30. We may conclude this part of our work by comparing two extreme hypotheses: the first referring the phenomena of planetary evolution solely to rotational, and the second solely to gravitational instability.

Given the approximate values of λ and ρ for a planet, and the fact that this planet has thrown off a satellite, the former hypothesis leads to a certain inference as to the angular momentum of the system; the latter to an inference as to the radius of the primary. The former hypothesis leads to no inference at all as to the size of planets which are to be expected—they are as likely to be of the size of billiard balls as of the size of the planets of our system—while the latter leads to no inference as to the angular momentum of the system, but presupposes it to be small. The contention of the present paper is that the inferences which are drawn from the former hypothesis are not borne out by observations on the planets of our system, while those which are drawn from the latter are borne out as closely as could be expected. The true hypothesis must of necessity lie somewhere between the two extremes which we are comparing, and our evidence seems to show that it is much nearer to the latter (gravitational) than to the former (rotational).*

Stresses and Vibrations in the Earth.

§ 31. It has already been seen that in dealing with a gravitating sphere of the size of the earth it is necessary to take into account terms which are omitted by Lord Kelvin and others—the terms which introduce into our equations the gravitational tendency to instability.

It is of some importance to know whether the existing solution for the vibrations and displacements of the earth would be altered to an appreciable extent by the inclusion of these terms. The general frequency-equation which is given on p. 167 is too complicated for manipulation, and is, moreover, open to the objection that it does not represent the facts of the case; for, inside the earth, the strains caused by permanent gravitation cannot legitimately be treated as small.†

§ 32. Considerations of a general nature will, however, give us some insight into the question. In an imaginary earth, in which λ, μ are infinitely great, the gravitational terms will be of no importance in comparison with those representing the elastic stresses. The true solution will, therefore, become identical with the classical solution in which the gravitational terms are neglected. For smaller values of λ, μ the error will become appreciable, and if λ, μ continue to decrease this error will become infinite as soon as the first point of bifurcation is reached; for at a point of bifurcation the application of an infinitesimally small external force will produce a finite displacement in the solid. For intermediate values of λ, μ the error will be small if λ, μ are great compared with the critical values of λ, μ at the point of bifurcation, and great if λ, μ are near to these critical values.

* In addition to the inference as to the size of the planets, the hypothesis of gravitational instability leads to a further inference as to the distances of the fixed stars. This has been discussed in my former paper, "On the Stability of a Spherical Nebula" (§ 48), and here also the results seem to agree with observation as closely as could reasonably be expected.

† Chree, 'Camb. Phil. Soc. Proc.,' vol. 14, or Love, 'Elasticity,' vol. 1, p. 220.

§ 33. The most reliable evidence as to the actual values of λ, μ is to be obtained from the phenomena of earthquake propagation.* From the "time curves" given in the British Association Report presented at the 1902 meeting, there seems to be little doubt that the so-called "large-waves" are propagated merely through a thin crust on the earth's surface, while the "preliminary tremor" is propagated in a sensibly straight line through the earth itself. The average velocity of propagation is found to be about 9·7 kiloms. per second, and this is independent of the length of the path. The inference is that $(\lambda + 2\mu)/\rho$ is nearly constant throughout the earth's interior, and that its value is about $(9\cdot 7 \times 10^5)^2$ or $9\cdot 4 \times 10^{11}$. If we suppose the mean value of ρ to be 5·5, this gives for the mean value of $\lambda + 2\mu$,

$$\lambda + 2\mu = 51\cdot 7 \times 10^{11}.$$

Now, the critical mean value of $\lambda + 2\mu$ which corresponds to the first point of bifurcation has already been seen to be about 4×10^{11}. It would, therefore, appear that the error introduced in the classical solution for the displacements and stresses is appreciable, although not great—it is probably comparable with the error to which attention has already been attracted by CHREE.†

FIGURE OF THE EARTH.

Theoretical Conclusions.

§ 34. From the evidence of the last section it will be seen that there is an overwhelming probability that the values of the elastic constants of the earth are such that a state of spherical symmetry would be one of stable equilibrium.

Whether or not the earth is at present in a state of spherical symmetry is a different question; various indications and, in particular, the inequality in the distribution of land between the two hemispheres of the globe suggest that it is not so.

Now, even if the material of the earth is at the present moment of sufficient strength to maintain a spherical configuration in spite of the gravitational tendency to instability, it does not seem probable that it has always been so. Looking backwards in time we must come to a stage in which the material of the earth was plastic, and, further back still, fluid. At this time the value of λ would be much smaller than its present value, and, as already pointed out in § 29, would probably be about equal to the critical value for the planet at that period of its existence. There would, therefore, seem to be a sufficient reason for considering the possibility that the earth, at the moment at which consolidation set in, was not in a state of spherical symmetry. Let us examine some of the consequences of this conjecture.

* Professor MILNE has kindly assisted me in this question.

† *Loc. cit. ante.*

It is easy to see that enormous stresses would be set up in the interior of the earth after consolidation. An equilibrium configuration depends in general upon the compressibility of the material, and a configuration which was one of equilibrium for the compressibility which obtained at the moment of solidification would not remain so after the incompressibility and rigidity of the material had increased by cooling. If we suppose the earth to cool in an unsymmetrical configuration the stresses set up will soon become very great. In fact, Professor DARWIN has shown that the stresses which would be produced by the weights of our continents in an earth initially homogeneous (*i.e.*, by an irregularity of less than a thousandth part of the radius) would be so great that the material would be near the breaking point.*

We must therefore suppose that as the earth cools and the elastic constants change there will be a series of ruptures resulting from the stresses set up in the interior. The configuration will become approximately spherical (spheroidal if rotation is taken into account) as soon as the point of bifurcation is passed.

The fact that the ultimate configuration is reached only as the result of a long succession of ruptures puts the whole question outside the range of exact mathematical treatment. We can, however, see that the final configuration (disregarding rotation) will probably not be quite spherical, but will retain traces of the initial unsymmetrical configuration.

§ 35. Before we can attempt to decide whether or not the earth shows traces of a process such as that just described, it will be necessary to form some idea of the unsymmetrical configuration with which the process must have begun. We cannot accurately calculate the "linear series" of unsymmetrical configurations except in the immediate neighbourhood of the point of bifurcation. Near to this point the configuration is spherical except for terms proportional to the first harmonic. The free surface will, therefore, be strictly spherical, and it will, of course, be an equipotential, but its centre will not coincide with the centres of other surfaces of equal potential. If we suppose a fluid mass of this kind to solidify, and then to shrink by cooling, the shrinking being accompanied by a series of ruptures of the kind already explained, we can easily imagine that the free surface would retain an approximately spherical form, but that when the final state is reached this surface would not be quite an equipotential, and the centre of gravity would not quite coincide with the centre of figure. If water is placed on the surface of a planet of this kind, it will form a circular sea, of which the centre will be on the axis of harmonics, while the dry land will form a spherical cap.

Evidence from the Distribution of Seas and Land.

§ 36. Now this is not observed on the earth, and it could not be expected, since we have ignored all the agencies which have contributed to the figure of the earth,

* 'Phil. Trans.,' vol. 173, 1882, p. 187.

except the one with which this paper is specially concerned. The question is not whether we observe the state just described, but whether we can detect any approach to this state, and this, I believe, can be done. Professor DARWIN writes* :—

"It is well known that the earth may be divided into two hemispheres, one of which consists almost entirely of land and the other of sea. If the south of England be taken as the pole of a hemisphere, it will be found that almost the whole of the land, excepting Australia, lies in that hemisphere, whilst the antipodal hemisphere consists almost entirely of sea. This proves that the centre of gravity of the earth's mass is more remote from England than the centre of figure of the solid globe. A deformation of this kind is expressed by a surface harmonic of the first order."

§ 37. We can carry our calculations a step further. The divergence from the initial configuration is only represented by a first harmonic so long as squares of this divergence may be neglected. If these squares are taken into account, we must

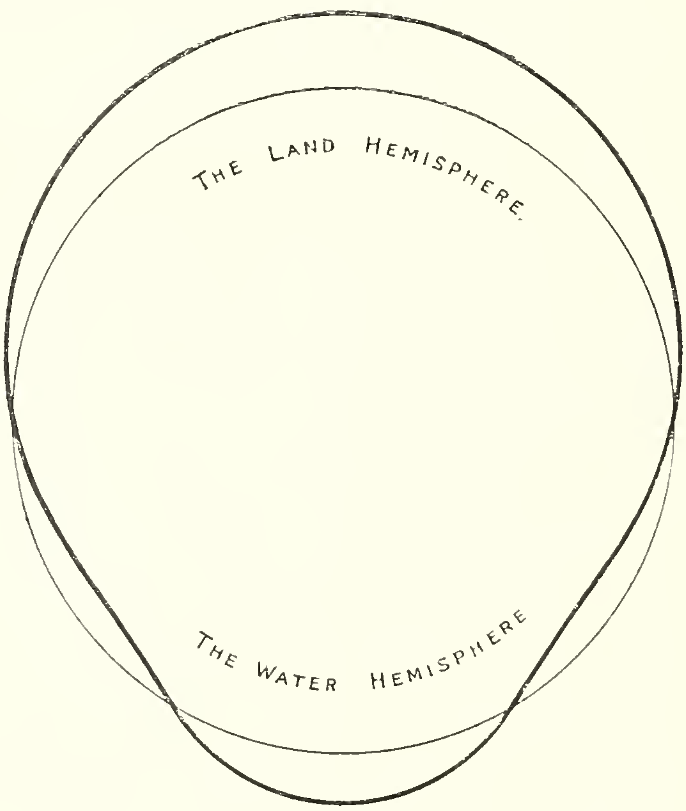

Fig. 1.

include a term proportional to the second harmonic as well as that proportional to the first. This process of successive approximation might be continued to any extent, so that a complete series of unsymmetrical configurations might be calculated in the manner explained in my former paper.† We may, however, be content to stop at the second harmonic The free surface will now be of the form

* G. H. DARWIN, 'Phil. Trans.,' vol. 173, 1882, p. 230.

† 'Phil. Trans.,' A, vol. 199, p. 41.

$$r = a_0 + a_1 P_1 + a_2 P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

and we therefore examine whether any traces of the second harmonic term can be found in the earth's surface. Now, if we take a_2 positive in equation (63), the equation is that of the pear-shaped curve which was found on p. 46 of this earlier paper. This differs from the spherical shape mainly in possessing a protuberance—the stalk end of the pear—of which the centre is on the axis of harmonics. Traces of this protuberance may, I think, perhaps be found in the Australian continent, the arrangement being that shown in fig. 1. It is true that the centre of Australia does not coincide with the antipodes of England, but the discrepancy becomes less when we take into account the enormous region of ocean shallows which lies to the east of Australia.

[*The discrepancy can be further reduced by taking the rotation of the earth into account. When the rotation of the earth was greater than at present the ellipticity of the earth's surface would be greater, and the transition from this to the present ellipticity would take place through a series of ruptures similar to those already described. The rotation (assumed small) of the pear can be allowed for by adding a term $-\beta P'_2$ to the right-hand side of equation (63), this representing a second harmonic deformation having the axis of rotation for axis of harmonics.

The present rotation of the earth can similarly be represented by a term $-\beta' P'_2$, where $\beta' < \beta$. The equation to the present surface of the sea may accordingly be taken to be

$$r = a'_0 - \beta' P'_2,$$

and hence the height above the present sea-level of the surface of the primæval rotating pear, if restored, would be

$$(a_0 - a'_0) + a_1 P_1 + a_2 P_2 - (\beta - \beta') P'_2.$$

It will be found that the effect of the rotational term $(\beta - \beta') P'_2$ is to move the theoretically predicted Australia nearer to the equator of the earth, and to change its shape from a spherical cap to a sphero-conic.]

Again, we should expect the highest land to be on the axis of harmonics, and, therefore, in or near England. Here, again, the agreement of facts with theory might be closer if we could suppose the continent, which geology shows to have existed at one time in mid-Atlantic, to be restored to its former position. But the agreement of facts with theory can only be expected to be of the roughest kind, and we must always bear in mind that our theory does not lead us to expect that the present figure of the earth will be pear-shaped, but only that it will resemble a pear disfigured by a long series of ruptures.

* Added January 3, 1903. I am indebted to the referee for suggesting this addition.

Evidence from the Distribution of Earthquake Centres.

§ 38. It can be seen that the earthquake regions of the world have a reference, as regards their distribution on the earth's surface, to this pear-shaped figure, and this, again, must be considered as evidence.

Let us first examine the facts. MILNE divides the earthquake-areas of the globe into twelve distinct regions, and a map of these is given in the 'British Association Report' for 1902.* These regions are given in the following table. The first figure denotes the number of large earthquakes which have occurred in these regions in the three years 1899–1901. The earthquakes from the three regions printed in italics were small in comparison with the others. In the last column is given the approximate latitude of the centre of each region, referred to Greenwich as pole (the latitude of Greenwich being taken to be $+ 90°$).

TABLE of Earthquake Regions.

			°				°
A	25	Alaskan	+10	G	17	Mauritian	+10
B	14	Cordillerean	0	*H*	*22*	*N.E. Atlantic*	*+75*
C	16	Antillean	+25	*I*	*3*	*N.W. ,,*	*+65*
D	12	Andean	0	*J*	*3*	*N. ,,*	*+70*
E	29	Japanese	− 5	K	14	Asiatic	+45
F	41	Javan	−25	L	2	Antarctic	[small]

Now, it will be at once noticed that for most of these regions the latitude is small. If we weight the regions according to the corresponding number of earthquakes, giving half-weight to the small earthquakes in regions H, I, J, we find as the mean of the numerical values of the latitude about 20°, whereas if the regions were distributed at random we should expect the mean latitude to be $(\frac{1}{2}\pi - 1)$ radians, or about 33°. We therefore see that the earthquake regions tend to lie near the equator of our pear. The evidence can be put in a more striking way as follows:— Exactly half of the surface of the globe is of a latitude less than 30°. The half for which the latitude is less than 30°, measured from Greenwich as pole, was responsible for 156 earthquakes; the remaining half was responsible only for 42, of which 28 were the small earthquakes from regions H, I, J. There is, therefore, no doubt that the *principal* earthquakes tend to emanate from points near to the equator of the supposed pear.

Now, if we look back to fig. 1, we see that this is equivalent to saying that earthquakes occur where the "slope" in the figure of the earth is steepest. This conclusion is the same as that to which the British Association Committee were led from a

* 'Brit. Assoc., 72nd Report,' Belfast, 1902, "Seismological Investigations," p. 4.

consideration of the actual figure of the earth, and it is that which might naturally be expected. The theory put forward in this paper may, perhaps, suggest a reason why these regions should lie approximately on a great circle of the earth, and why this great circle should approximately divide the earth into two hemispheres of sea and land.

Summary and Conclusion.

§ 39. In conclusion it may be well to summarise those parts of the paper which refer to the figure of the earth.

We saw that at the moment of solidification the earth might be either spherical (except in so far as it was deformed by its rotation) or pear-shaped. Our theoretical calculations and our knowledge of the constants of the earth at the time of solidification were not sufficiently accurate to enable us to decide which of the two alternatives is the more probable. The shape of the earth, whether spherical or pear-shaped, could not be maintained long against the enormous strains which would be set up in the earth as the process of cooling proceeded, and this shape would gradually give place to an approximately spherical shape, the change in shape being produced by a long succession of ruptures. The suggestion of this paper is that the earth, in spite of this series of ruptures, still shows traces of a pear-shaped configuration. Such a configuration would possess a single axis of symmetry, and this, it is suggested, is an axis which meets the earth's surface somewhere in the neighbourhood of England (or, possibly, some hundreds of miles to the S.W. of England). Starting from England we have in the first place a hemisphere which is practically all land; this would be the blunt end of our pear. Bounding this hemisphere we have a great circle of which England is the pole, and it is over this circle that earthquakes and volcanoes are of most frequent occurrence. If we suppose our pear contracting to a spherical shape we notice that it would probably be in the neighbourhood of its equator that the change in curvature and the relative displacements would be greatest, and hence we should expect to find earthquakes and volcanoes in greatest numbers near to this circle. Passing still further from England we come to a great region of deep seas—the Pacific Ocean, the South Atlantic Ocean, and the Indian Ocean: these may mark the place where the "waist" of the pear occurred. Lastly we come, almost at the antipodes of England, to the Australian continent and the shallow seas which lie to the east of it; these may be the remains of the stalk-end of the pear.

§ 40. It may, I am afraid, be thought that the hypotheses upon which the paper is based are too speculative and the results, consequently, too uncertain. In defence it may be said that the object of the paper is not so much to establish new doctrines as to point out possibilities, and that these possibilities seem to be of such a kind that it may be useful to keep them in mind in discussing questions connected with the figure and structure of the earth, as well as the more general questions of planetary evolution.

In conclusion I have to express my indebtedness to Professor G. H. DARWIN and Professor A. E H. LOVE for advice and assistance which I have received from them.

[NOTE.—*Added February 20th*, 1903. While the above paper was in the press, Professor W. J. SOLLAS read a paper before the Geological Society in which the Figure of the Earth was discussed from a geological standpoint. Professor SOLLAS had arrived, from an examination of the geological features of the earth, at a conclusion very similar to that to which I had been led from theoretical considerations: he had detected an axis of symmetry, other than the axis of rotation, in the earth's figure, and expressed the opinion that "the pear-shaped form, now that it was pointed out, became obvious to mere inspection: it was a geographical fact, and not a speculation."

The axis of Professor SOLLAS' pear does not, however, coincide with that which I tentatively put forward in the above paper, and the object of this note is to accept the alteration suggested by his paper. The conclusion reached in his paper is that the axis of symmetry of the pear-shaped figure passes through a point of latitude and longitude about 6° N. by 30° E. Thus Africa—the continent whose mean height above sea-level is greatest—must be taken to be the centre of the "Land Hemisphere" in fig. 1 of my paper, while the protuberance which formed the stalk of the pear is submerged in the Pacific Ocean, which now forms the "Water Hemisphere." Almost the only remaining evidence of the existence of this protuberance is the fact that the axis of the pear coincides with the earth's greatest diameter. The great circle of earthquake-centres suggested in § 38 of my paper is to be replaced by the line of Pacific folding; this approximately forms a small circle (of radius about 80°) which almost coincides with the proposed great-circle in the northern hemisphere. Further details of Professor SOLLAS' view will be found in his paper ("The Figure of the Earth," 'Quart. Journ. Geol. Soc.,' vol. lix., Part 2).

The fact that Africa is surrounded by a belt of seas, and this again by a belt of land before the Pacific is reached, points perhaps to a bodily subsidence of the blunt end of the pear, the circle of fracture having possibly been the line of Pacific folding. Such a fracture would, of course, displace the centre of gravity of the pear, and probably this would account not only for the feature just mentioned, but also for the non-appearance of the protuberance. It will be noticed that the smallness of the latitude of the extremities of the axis (6°) agrees well with the theory of planetary evolution put forward in §§ 25–30 of the present paper.]

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OF THE

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SERIES A, VOL. 201, pp. 185-204

ON THE FORMATION OF DEFINITE FIGURES BY THE DEPOSITION OF DUST

BY

W. J. RUSSELL, PH.D., F.R.S.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 336. 28.5.03

Price Two Shillings and Sixpence.

INDEX SLIP.

VI. *On the Formation of Definite Figures by the Deposition of Dust.*

By W. J. Russell, *Ph.D., F.R.S.*

Received January 29,—Read February 19, 1903.

When trying some experiments which had an object other than that described in the following communication, it was noticed that a fine powder when allowed to settle on a slightly warmed plate produced figures which were remarkably clear and definite. So striking and peculiar were these figures, and so simple were the conditions of their formation, that a careful study of them was undertaken. These figures are so clear and sharp that it is easy to obtain exact photographic records of them, an important point, for, at present, it does not seem possible to offer a simple explanation of the complicated relationships which exist between the external conditions and the figures formed. Sensitive as these figures are to outside influences, the forms they assume are very characteristic of different conditions, are perfectly constant, and are easily produced.

The general method of obtaining these figures is as follows:—The plate on which the figure is to be deposited is best supported on three pins about $1\frac{1}{2}$ to 2 inches high, and the dust most convenient to use is that made by burning magnesium ribbon. It is kindled and allowed to burn in a receiver. A circular glass dish with straight sides, about 4 inches high and 9 inches in diameter, is a convenient form of vessel to use; and if the vessel be large enough (there should be about 2 inches between the plate and the inside of the receiver); the shape and the material of this dust containing vessel is not of much consequence. After the magnesium has burnt out, this receiver is allowed to stand for a minute or so, and it is then placed over the plate on its stand and allowed to remain there for six to seven minutes. On removing it a clear and definite figure will be found to have formed on the plate.* If the plate has been a square one, then a cross consisting of four rays, each starting from a corner and meeting, but not necessarily joining, in the centre, is produced. If the corners be varnished or covered by a small piece of tinfoil (fig. 1) the cross is still formed.

* A photograph of the figure was obtained by placing the plate on a varnished black background, illuminating it by an arc lamp, so that the beam of light fell upon it at an angle of about 30 degrees, and the camera was placed directly in front of the plate. Process plates were used, and the exposure was from two to two and a-half minutes.

If the plate be triangular in form, then three rays are formed, again each starting from an angle (fig. 2), and if the plate be an octagon, then a star with eight rays is produced (fig. 3). An angle in a plate always tends to give rise to a ray. This is often very fine at the point, and thickens considerably afterwards. If a flat circular plate is used, then no deposit takes place, but if it is concave, a uniform deposit over

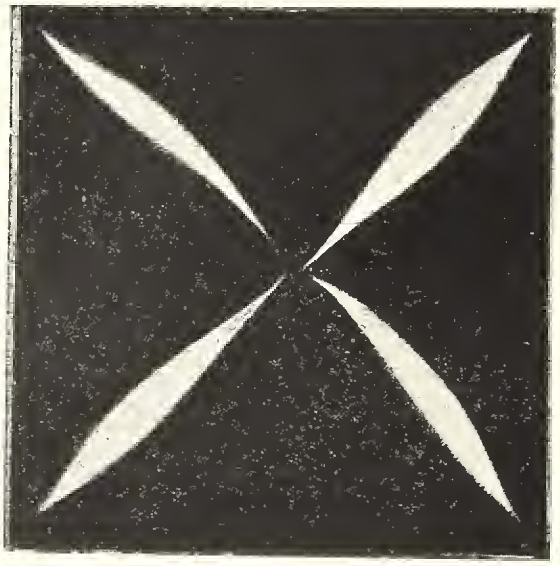

Fig. 1.

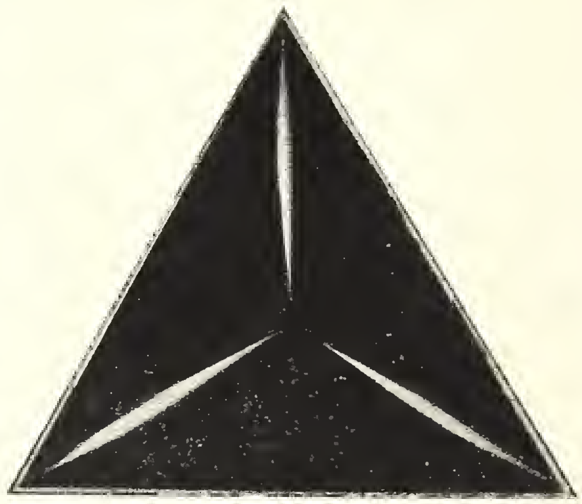

Fig. 2.

the whole of it occurs, and if it be convex, then little or no deposit is formed, if any, it is in the form of a star. When an oblong rectangular plate is used, then the rays similar to those formed on a square plate are produced, but they do not meet, but

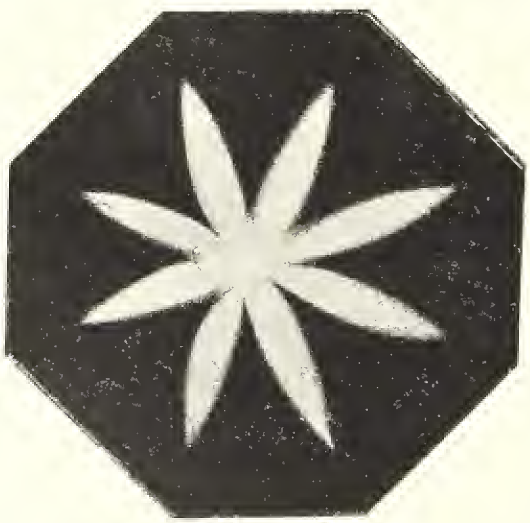

Fig. 3.

Fig. 4.

are, as it were, drawn asunder, and remain at each end of the plates, being however often connected by a thin straight line (fig. 4).

In all cases, the angles of the plate determine the figure formed on it. With regard to other general points connected with the formation of these figures. The nature of the dust used is not a matter of importance, it may be composed of organic or inorganic matter; the spores of a fungus, or magnesia, or the dust from ashes or fumes of ammonium chloride. In fact, the necessary condition is that the dust be very fine, then always the same figure is formed. The product formed by burning magnesium is, however, the best form of dust to use. It is easily obtained, and has a silvery whiteness in appearance, which gives distinctness to the figures. With regard to the plate on which the figure is to form, its composition, like that of the dust, is of no importance; the shape determines the figure, not its constitution. Glass, for several reasons, is the best material for the plate, but copper, zinc, silver, antimony or other metal may be used, or ebonite, celluloid, black india-rubber, cardboard, &c., in fact, the receiving surface is not necessarily a solid substance; mercury in a square vessel will have deposited on it a figure similar to that on a piece of glass of the same size and shape, and, still more, the surface of the glass plate may be coated with oil, gum, copal-varnish, &c., and the cross will form as if they were not present. Obviously, with regard to the visibility of the figures formed, the nature of the plate is of considerable importance; on some substances the figures are more easily seen than on others. In the following experiments glass plates have been used, except when mention is made to the contrary.

Passing from the materials used to the active agent in producing the figures, namely heat, it should be stated that there are many different ways of applying it, and different results are produced. The simplest way is to pass the plate two or three times over the flame of a small Bunsen or spirit lamp. If it be a glass plate, a good indication of sufficient heating is when the condensed moisture disappears—it is of little importance whether the heated side or the other one is uppermost—then the plate is enveloped in the dust atmosphere by placing the receiver, filled with dust, over it, and leaving it there for the six or seven minutes. To obtain a figure in its simplest form and as dense and clear as possible, it is necessary that the plate be equally warmed all over; a convenient way of doing this is to lay the plate on one of copper, heated to about 20° C., for about half a minute, or an ordinary air or water bath will answer the same purpose. As long as the plate and the surrounding dust atmosphere have approximately the same temperature, the deposit formed is nearly uniform; there is only a slight appearance of any figure, but as soon as any rise of temperature occurs, then a figure begins to appear. At first the indications are very slight, and occur only round the edge of the plate; but as the temperature is raised, the figure spreads over the whole of it. A figure may also be developed by having the plate at a lower temperature than that of the surrounding atmosphere, provided that the plate is not below 17° C., but the figures produced in this way are slight and imperfect and disappear altogether when the plate is 6° below that of the atmosphere. In order to determine roughly the temperatures of the plate and its surrounding atmosphere, a receiver, of the same shape and size as the glass one, was made of asbestos cloth and covered with cardboard; in the top of it a hole was made, and a delicate thermometer introduced. A few of the results obtained will show the nature of the alterations produced by differences of temperature between plate and surrounding atmosphere. When filling the receiver with the magnesia smoke, it should be turned round over the burning magnesium, so as to render the whole of the inside of as uniform a temperature as possible, and before placing the receiver over the plate, after the combustion is over, it should be allowed to stand for about a minute, so that any coarse particles may settle. First, with regard to cases in which

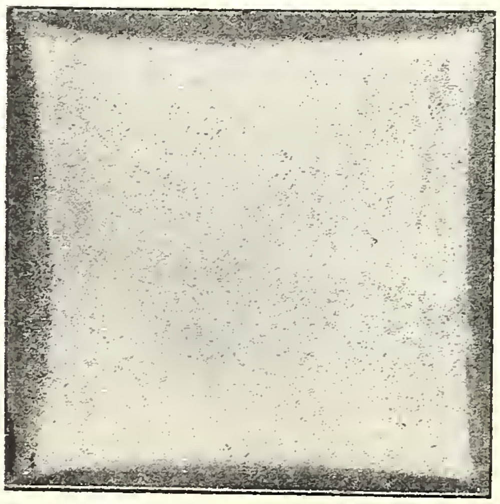

Fig. 5.

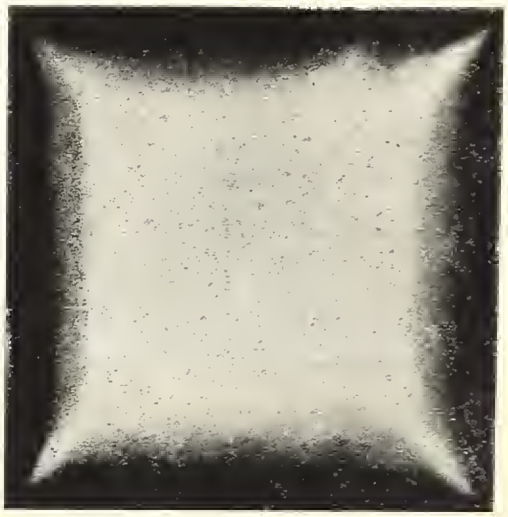

Fig. 6.

the plate is lower in temperature than the surrounding atmosphere. If the plate be at a temperature of 19° C., and the maximum temperature of the dust atmosphere be 24°, then a nearly uniform deposit is produced, but at the corners of the plate there is a short line of deposit, and along the sides there is somewhat less deposit (fig. 5). If the plate be warmer, 20·6°, and the atmosphere 24°, then the above characters are

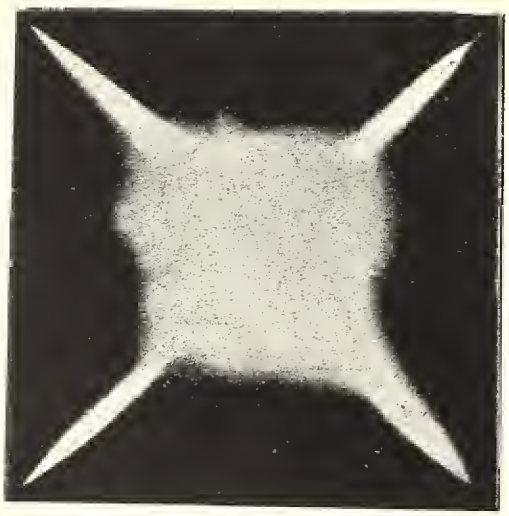

Fig. 7.

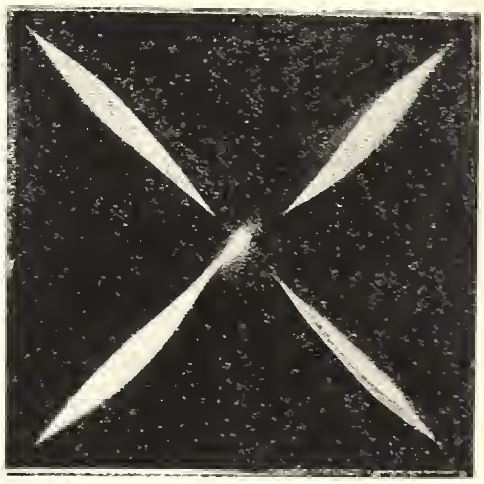

Fig. 8.

still further developed and a bag-shaped deposit is formed (fig. 6), and this is characteristic of what takes place when the plate is below the temperature of the atmosphere, but sufficiently warm to act. When the plate is slightly warmer than the dust atmosphere, 1·8° for instance, then again a figure is produced similar in character to the last one, but a further development of it (fig. 7). If the plate be made warmer and warmer, and the surrounding atmosphere kept nearly at the same temperature, then the figure gradually alters and becomes more perfect. If the difference of temperature between plate and atmosphere be about 5°, there is only a small amount of deposit on the central part of the plate, and the four rays are well developed. When the difference of temperature is about 12°, then a good clear cross is formed, its only imperfection being a slight fuzz in the centre (fig. 8). At a difference of temperature of 100° or 120°, the same figure, a cross, is formed, but the amount of dust deposited is less than at lower temperatures. Hence, whether the difference of temperature between plates and atmosphere be very considerable or very slight, the same effect is produced. A thick piece of glass held in the hand for 30 seconds and then placed in the dust atmosphere will have a figure deposited upon it, but the amount of deposit will be small and the figure faint. The figures form best between certain limits of temperature, and when there is a marked difference between the

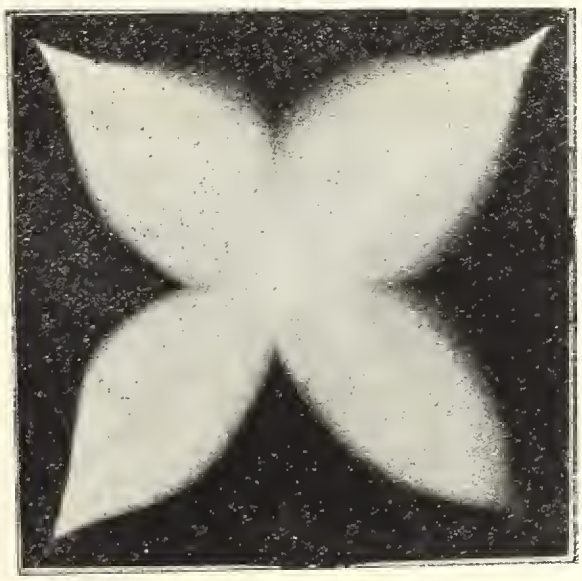

Fig. 9.

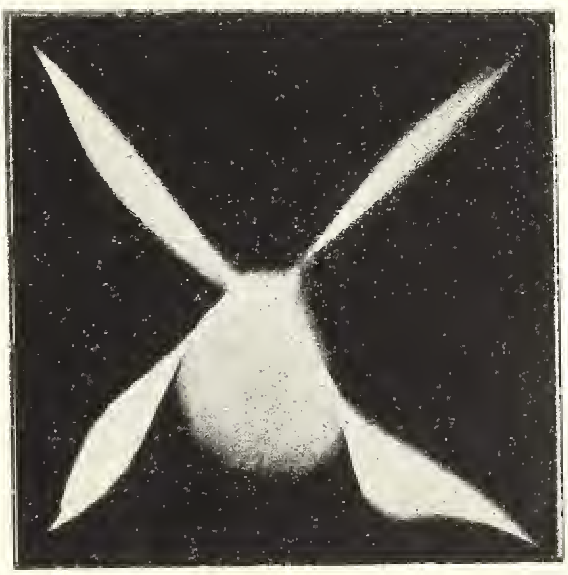

Fig. 10.

plate and the surrounding atmosphere. They are very sensitive to change of temperature; in fact, to get a perfect cross or other figure, both plate and atmosphere must each be uniformly heated. If, in addition to uniformly heating a plate, a warmed body be placed below it and kept there during the time that the dust is depositing, there is a considerable increase in the amount of deposit and a modification of the figure formed; for instance, if a copper cylinder, 12 millims. in diameter and 14 millims. high, heated to 55°, be placed 30 millims. below the centre of a square plate, then the figure shown in fig. 9 is produced. If a piece of glass be only warmed by holding it in the hand, and is then placed immediately below the plate, but not touching it, a marked and peculiar effect on the cross is produced, as seen in fig. 10. If this heating below the plate be increased, either by raising the temperature of the small copper cylinder, or by using taller cylinders, so as to bring the source of heat nearer to the plate, the amount of deposit is increased, and ultimately the figure of the cross disappears, and there is uniform deposit over the whole of the plate. On still further increasing the heat below the plate the reverse action sets in, and the amount of deposit decreases. These changes will be described in detail later on. Some experiments made with a Bunsen lamp show how these figures are affected by radiant heat, and the singular effects which it produces. The flame of an ordinary Bunsen burner was placed on a level with the plate and allowed to burn while the deposit was being formed. When the flame was at a distance of 12 inches from the centre of the plate, the cross was distorted, as shown in fig. 11, the heat having travelled not only the 12 inches to the plate, but also passed through the glass of the receiver containing the fumes. In the next experiment the lamp was removed to a distance of 16 inches, then less distortion took place. At a distance of 21 inches the effect produced was still visible (fig. 12), and even with the lamp at 26 inches the

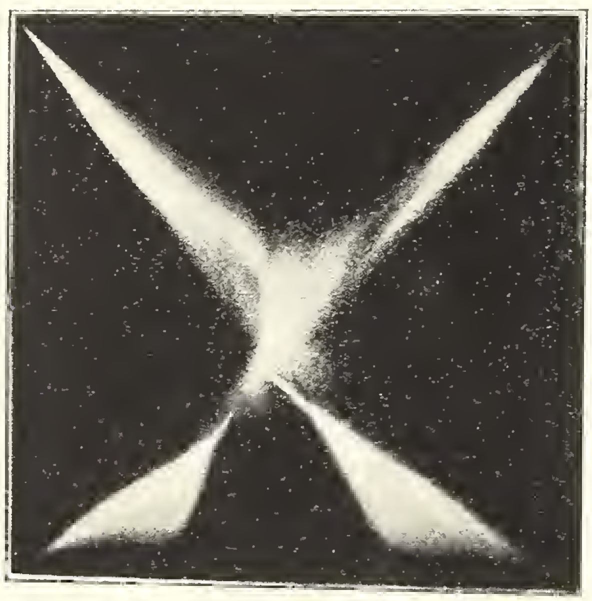

Fig. 11.

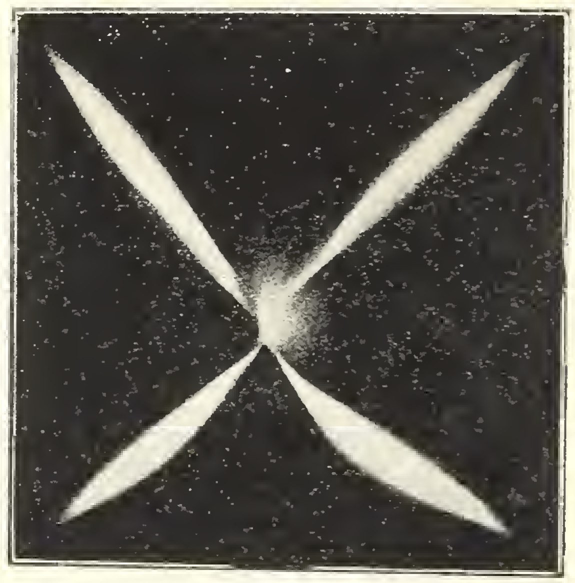

Fig. 12.

two rays nearest to it are slightly thickened and distorted (fig. 13), but at 30 inches no effect was produced. Another experiment of a little more definite character was tried. A small copper cylinder, 95 millims. in diameter and 100 millims. high, was filled with boiling water and placed at a distance of 12 inches from the centre of the plate outside the fume vessel; the cross was affected as before, the nearest rays were shortened and bulged out. A small candle burning at a distance of 8 inches from the plate is also sufficient to distort the figure which is being produced upon it.

It has already been shown that by increasing the heat below a plate the amount of deposit is increased; but if this heating be carried on to still higher temperatures, the phenomena are reversed, and less and less deposit occurs. If the copper cylinder used in the former experiment be heated to 200°, and be placed below the centre of the plate, no deposit forms immediately above it. The same effect is more readily produced, and at a lower temperature, if the plate is in absolute contact with the warmed copper cylinder, and it may be mentioned here, that the only way of obtaining the cross on the square glass entirely free from all fuzz at the centre, is by using as a source of heat a metal plate, and placing on it another thin piece of metal about 1 inch in diameter, and allowing the centre of the glass plate to rest upon it. If the copper cylinder under the plate be heated to about 150° C., and the plate

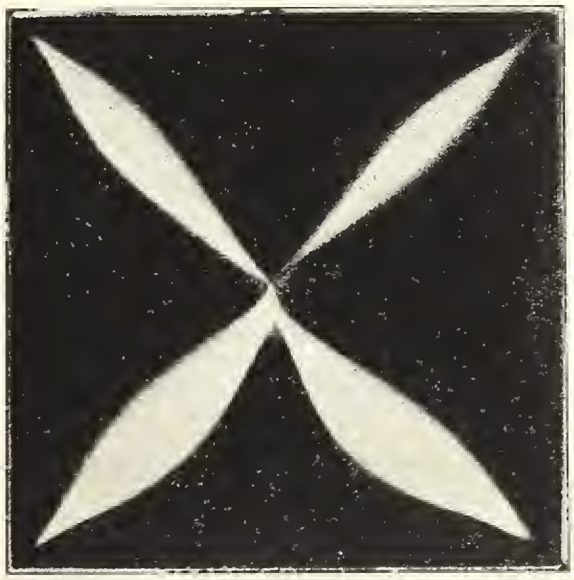

Fig. 13.

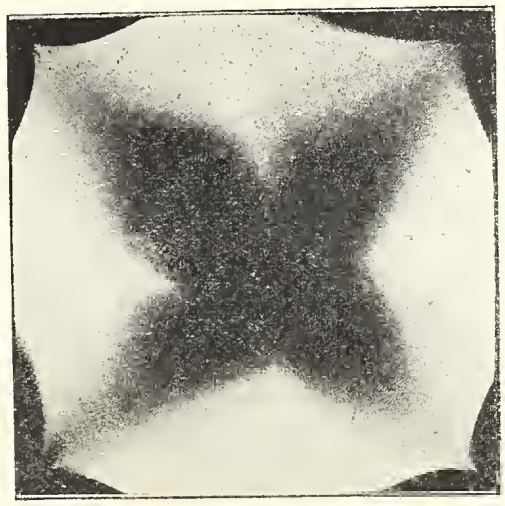

Fig. 14.

rests upon it, no deposit occurs immediately above it, and this open space assumes the form of a cross (fig. 14). That dust does not deposit on a sufficiently heated surface has long been known; but it is interesting that in this case the portion

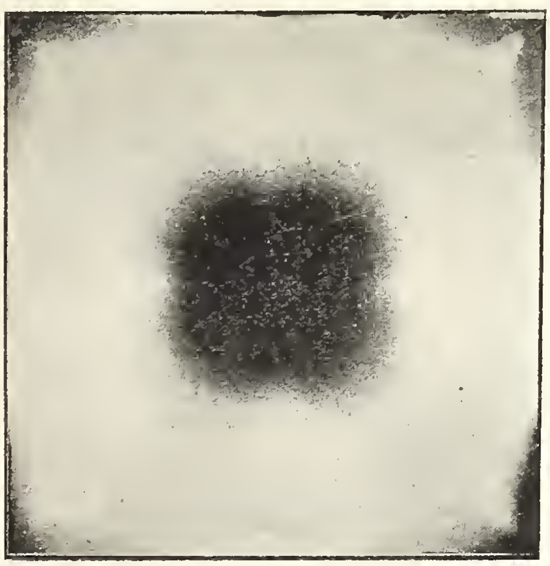

Fig. 15.

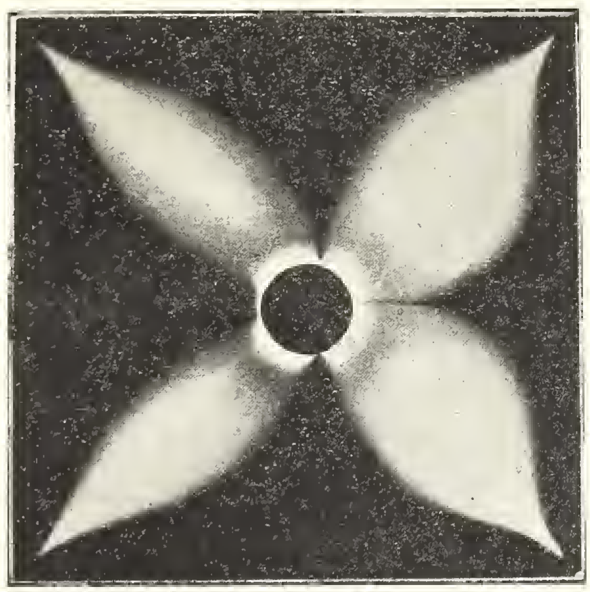

Fig. 16.

where there is no deposit should be in the shape of a cross. If the copper cylinder be heated as before and the plate not heated, but placed on it at an ordinary temperature, then there is an open space, square in general form—possibly the former cross filled up—formed, and in the centre there always appears a very small white cross, and over the rest of the plate, except at the corners, there is an even deposit of dust (fig. 15).

Other curious results are produced if the copper cylinder, heated to about 130°, be placed on the upper side of the heated plate, instead of the under one. Then a cross is formed, but it is very much broadened out, and a deposit of dust has formed round the base of the cylinder (fig. 16). If the plate be not heated, but the hot cylinder put upon it, then a modified effect, shown in fig. 17, is produced, and lastly, again reversing the heating, putting a cold cylinder on a heated plate, the cross is well formed, and a curious deposit, square in shape, is found round the base of the cylinder (fig. 18). All these forms are readily and constantly produced when the centre of the plate is heated or cooled as above described. It will now be obvious why three wires form the best kind of support for the plates on which a symmetrical figure is to be formed. If a large solid support be required, a cork is probably better

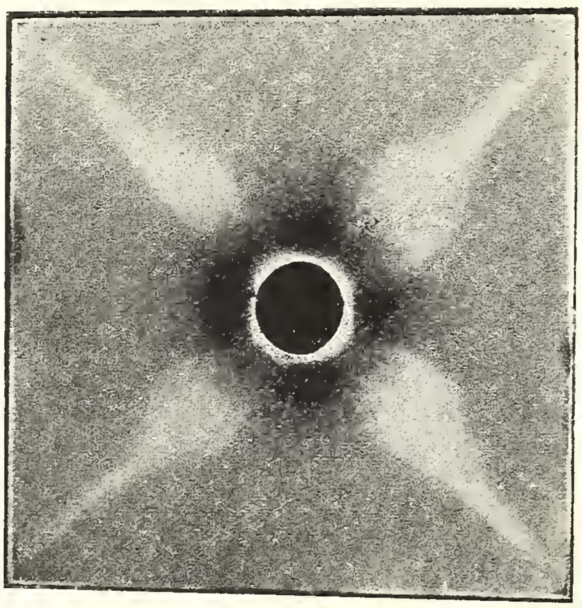

Fig. 17.

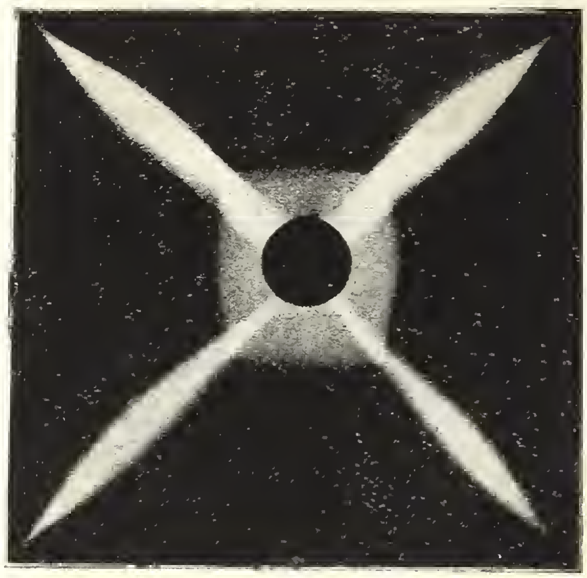

Fig. 18.

than anything else, but a cork heated to 100° C. caused, when supporting a square plate, a uniform deposit to take place over very nearly the whole surface.

There is still another condition which affects the formation of these figures, and that to a very considerable extent : it is whether the plate on which the deposit is forming be horizontal or not. If not horizontal, the figure always has a tendency, as it were, to slide down the plate. The smoothness of the glass is not essential to this effect, for if a copper plate be painted over with lampblack and a little shellac in alcohol, which gives it a rough surface, identical figures are formed. Fig. 19 shows the deposit formed if the plate is placed on a slope of only 2 degrees, but if the slope be increased to 5 degrees, then the deposit assumes the form shown in fig. 20, and if the slope be 15 degrees, then the deposit has the form shown in fig. 21. These three figures show in an interesting way the great effect which the slope of the plate produces. There is another way by which the formation of these figures may be controlled and altered to a remarkable extent, and which should throw light on the mode of their formation. It is by the proximity to the plate of other bodies. For instance, if a piece of glass or metal as long as the plate and 10 millims. wide be fixed against it so as to project above it, then an even deposit forms under its shadow. If holes are

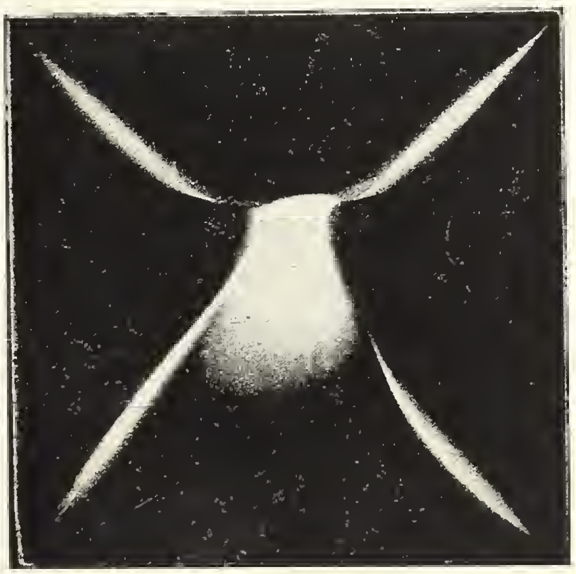

Fig. 19.

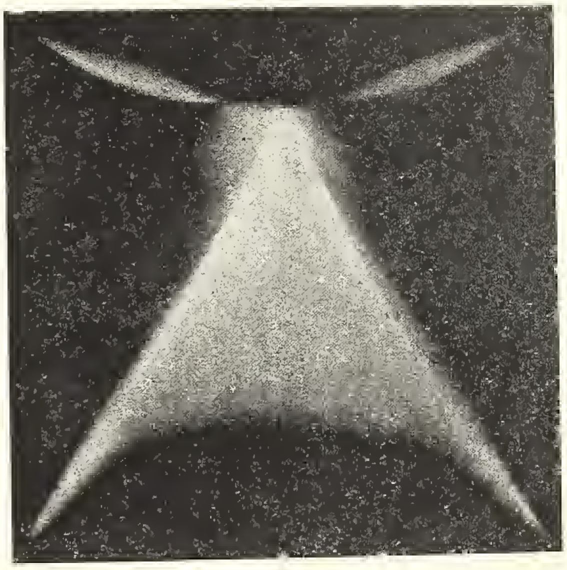

Fig. 20.

cut in this screen, no deposit takes place on the glass in front of the holes. Fig. 22 shows what happened when a square glass had a piece of metal with holes cut in it pressed against it. In front of each opening in the screen there is a clear space on the plate. Another curious, but very complicated effect is produced by cutting a

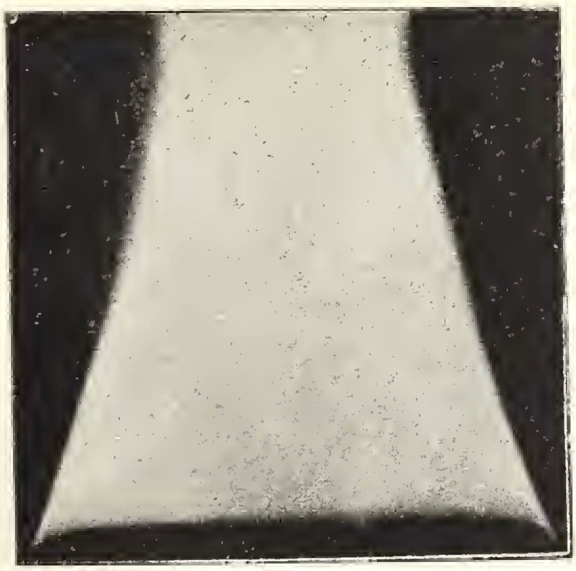

Fig. 21.

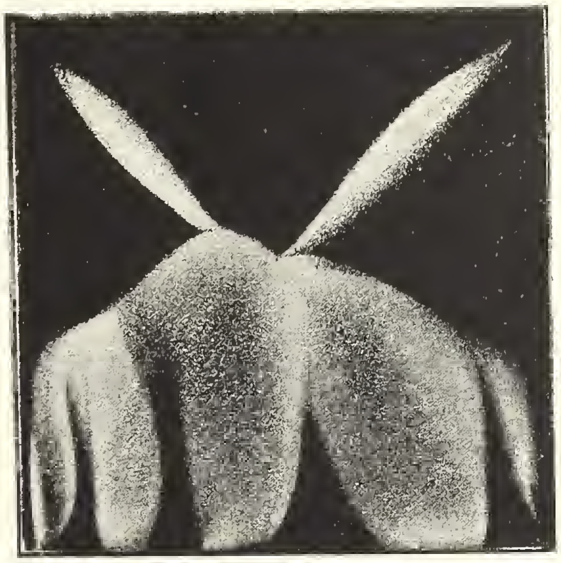

Fig. 22.

re-entering angle out of a square of copper. It is difficult to follow how the deposit can form in the way shown in fig. 22A.

Bearing on this same point is the fact that if a warmed plate be placed on the floor of the vessel in which it is exposed to the dust, instead of being raised above it, no figure, only an even deposit is formed. Now, if in place of a screen extending the whole length of the plate a small one be set up, a piece of glass 5 millims. wide, for instance, the same kind of action occurs, the plate immediately behind the screen is protected, and there a deposit of dust forms, of a curious rounded shape (fig. 23).

Fig. 22A.

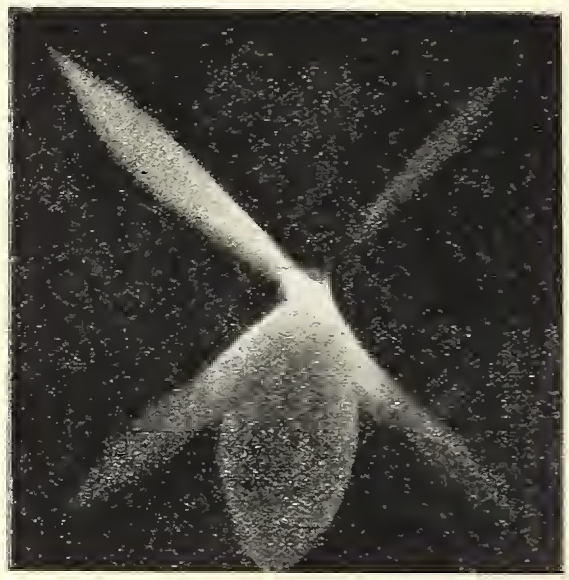

Fig. 23.

A still narrower obstruction may be used. The effect produced by a pin fixed against the plate is shown in fig. 24, and fig. 25 shows the effect of a fine human hair. In neither of these cases does the deposit commence at the obstruction, but a little way

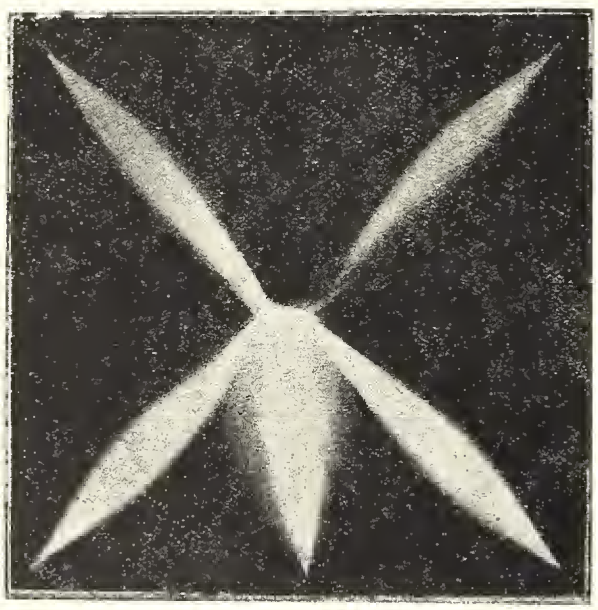

Fig. 24.

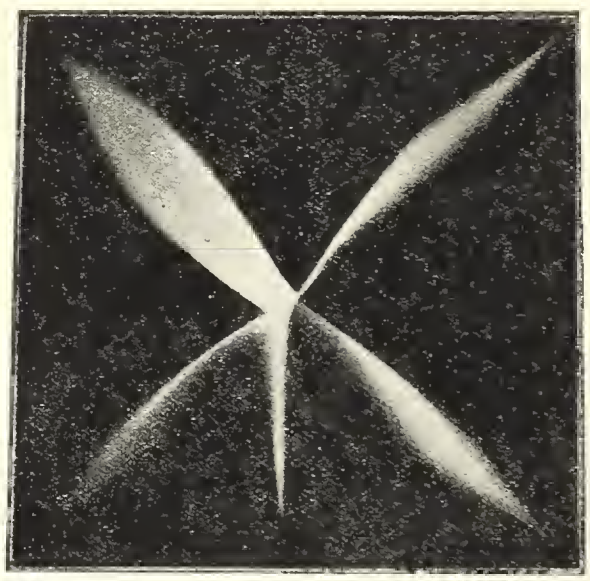

Fig. 25.

from it. A piece of thin wire acts exactly in the same way as a hair. Experiments were then made to ascertain what effects altering the position of the pin would have on the figure produced, and it was found, that as long as the pin is in contact with the plate, its height above it does not affect the deposit formed. In all these cases, a considerable amount of deposit was formed, commencing at an appreciable distance from the pin. The pin was then lowered, so that the edge was immediately below the plate; when it was 2 millims. below, it produced a considerable amount

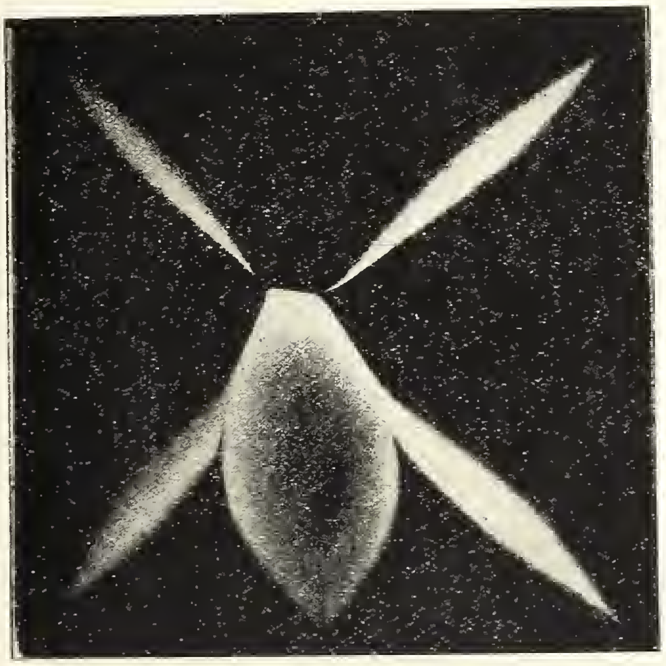

Fig. 26.

Fig. 27.

of deposit; when 4 millims. below, the amount was much diminished, and when 8 millims., only a trace of deposit was formed. It was found that as the pin

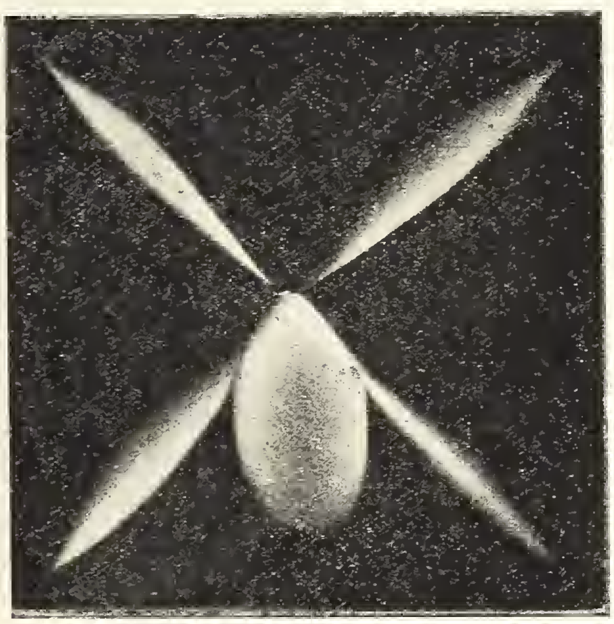

Fig. 28.

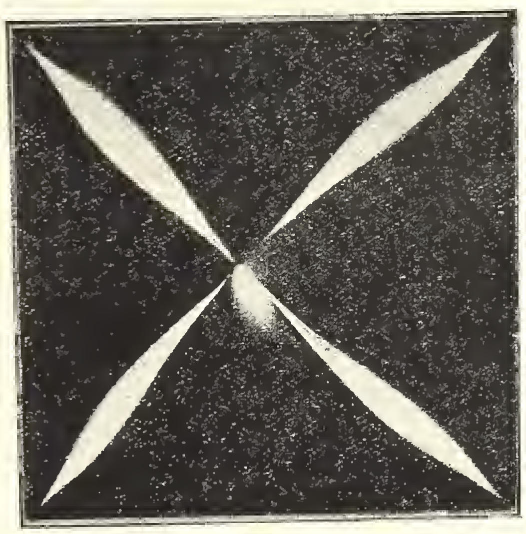

Fig. 29.

receded, so did the deposit recede from the edge of the plate, becoming at the same time smaller in amount.

Fig. 27 shows the deposit formed when the pin was 3 millims. away, and still at the level of the plate. Fig. 28 is the figure formed when the pin was 6 millims. from the plate, and fig. 29 is the effect produced when 8 millims. away; but when the pin was 10 millims. away, no effect was produced. If the pin does not touch the plate, as it did in the former case, its height again does not affect the deposit formed. If the pin be placed at a lower level than the plate, and at different distances from it, it is still able to produce a deposit on the plate, as was proved by trying it at a constant distance of $1\frac{1}{2}$ millim. below the level of the plate, and at distances of 2, 4, 6, 8, and 10 millims. from the plate. At 2 millims. a considerable amount of deposit was formed, and the amount gradually diminishes and recedes from the edge of the plate as the distance increases. At 2 millims. from the plate, the deposit is nearly up to the edge. At 4 millims. it commences at 18 millims. from the edge, at 6 millims. at 27 millims., at 8 millims. at 30 millims., and at 10 millims. there is no deposit formed.

If the pin be placed at a still greater depth below the level of the plate, it is still able to produce a deposit on the plate, the deposit, of course, becoming less as the

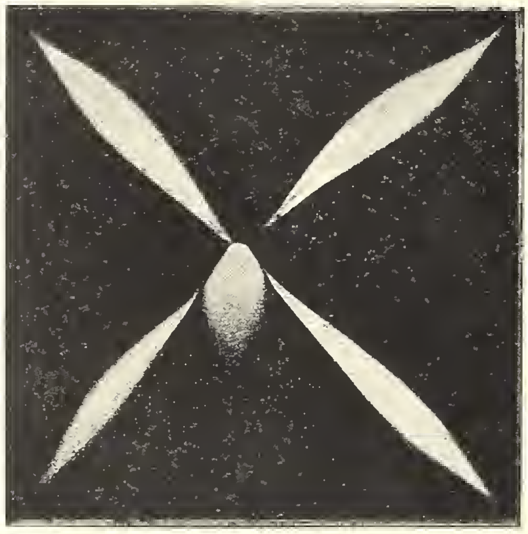

Fig. 29A.

Fig. 30.

depth increases. At 4 millims. below the level of the plate, and 2 millims. away from it, a small deposit is formed (fig. 29A), and even when it is 6 millims. below, a visible deposit is formed at the centre of the plate. The amount of deposit produced by a pin on the same level as that of the plate, may be equalled, but apparently is never exceeded. It has already been shown that no deposit takes place on warming and exposing to dust a circular plate, but if a pin be placed at different distances from it, and either above or below it, deposits are produced similar to those formed on any other shaped plate.

It is certainly remarkable that a pin so far from the plate and so much below it should be able in so definite a way to affect what is taking place upon it.

There still remained another way in which the pin could be presented to the plate, namely, by holding it above the plate. If a pin 50 millims. long be held 6 millims. above and 3 millims. beyond a plate it produces no effect on the figure, but if the pin be simply lowered, so that it is only 4 millims. above the plate, then a slight deposit at the centre is formed, and when the pin was only 2 millims. above the plate and still 3 millims. from it, increase in the deposit occurred. In all these cases with the pin supported from above much less deposit was formed than when the pin was pointing upwards.

If the pin be bent at a right angle, it produces on the plate a deposit similar in form and amount to that produced by a vertical pin at the same distance from the plate.

This action of any neighbouring body on the dust deposit is shown by any rough edge which the plate itself may have. If, for instance, a glass plate be used and it has been cut in the usual way, in addition to the figure which is dependent on the shape, there will be certain lines of deposit darting out in different directions; these are produced by small splinters of glass attached to the edge. Fig. 30 shows this on an oblong glass and fig. 31 on a circular glass. If the edges of the plate be carefully

Fig. 31.

Fig. 32.

ground, then these lines of deposit cease to be formed. Fig. 32 shows a square glass, two of whose edges were left rough and the other two were ground.

There are many curious alterations in the forms of the figures produced by placing on the plate obstructions to the flow of these lines of dust. For instance, taking again a square plate, if a strip of glass 1 millim. high and 1 millim. wide be placed across one corner of the plate and then the cross be developed, it has no effect, the cross forms as if no obstruction were there, but if the strip be 7 millims. high, then a marked effect is produced. In front of the strip the ray retains its usual form, but on the other side and round the centre there is a great widening-out of the ray and a slight banking-up of the dust against the sides of the glass strip. This effect of the obstruction strip is shown in fig. 33. If the strip be even 20 millims. high it acts in the same kind of way. If a strip 5 millims. high and 30 millims. long be placed parallel with the edge of the plate, and nearly at the centre, the cross is altered in a remarkable way, shown in fig. 34.

The following figures show the effect which other forms of obstruction have on these dust figures. A glass ring 4 millims. thick and 0·75 millim. high was placed at the centre of a square plate, and produced no alteration of the cross (fig. 35). Then a ring 1·5 millim. high was used, and it produced but little effect (fig. 36);

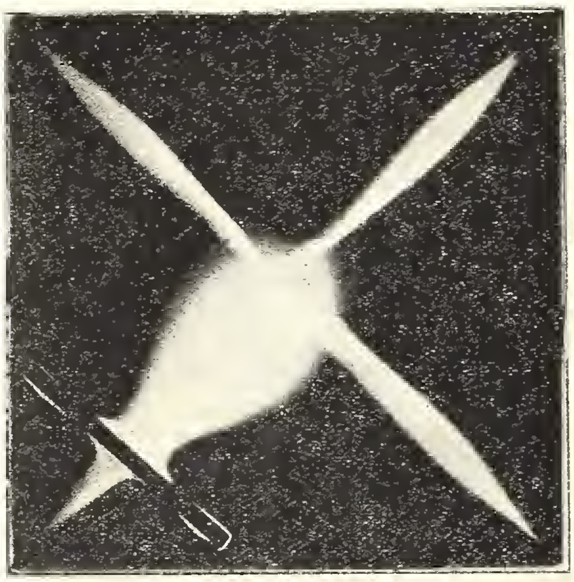

Fig. 33.

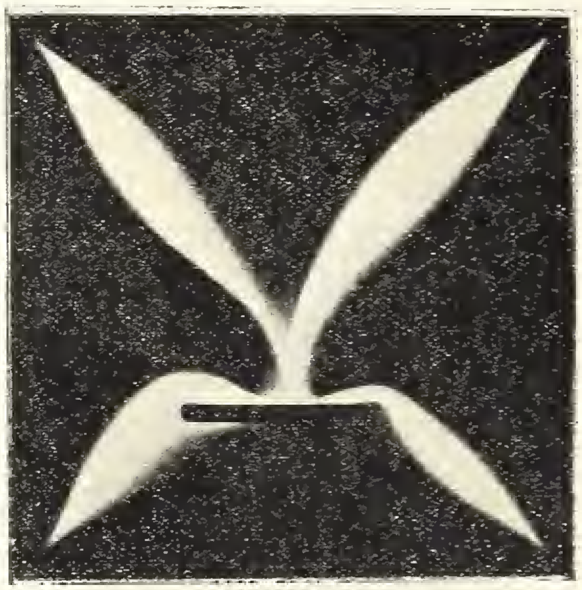

Fig. 34.

but when a ring 3 millims. high was used, then the central part within the ring became to a considerable extent thickened, and much deposit was formed (fig. 37), and when the ring was 5 millims. high an even deposit was formed inside the ring, but the rays of the cross outside were not affected (fig. 38). The effect of offering

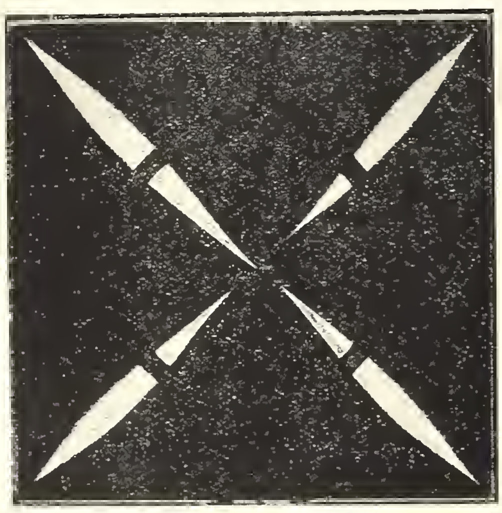

Fig. 35.

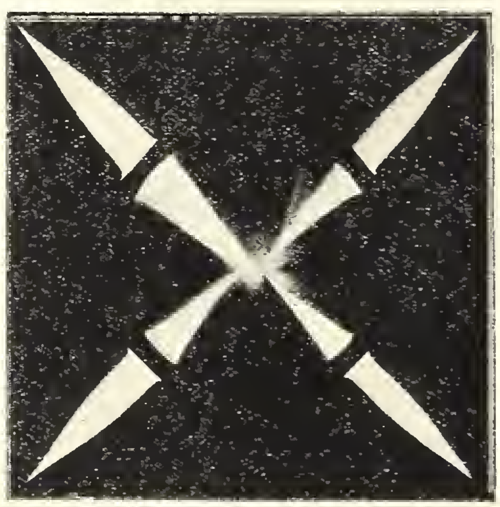

Fig. 36.

obstructions of different kinds to the flow of these dust currents was further tested by supporting from above, instead of from below, a strip of glass longer than the square plate on which the deposit was to be formed. When this is hung against the side of the plate, a dense deposit takes place all along this edge, but when the screen extends about 10 millims. on both sides beyond the plate, the deposit stops at

4 millims. from the edge of the plate at both ends, and is conical in form. If the strip had been of the same length as the plate, the deposit would have reached, as shown in former experiments, very nearly the whole length of the plate. If the hanging screen be raised 2 millims. above the plate, then in place of a long line of

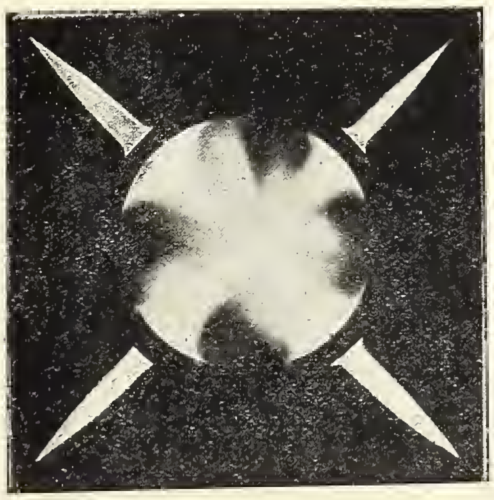

Fig. 37.

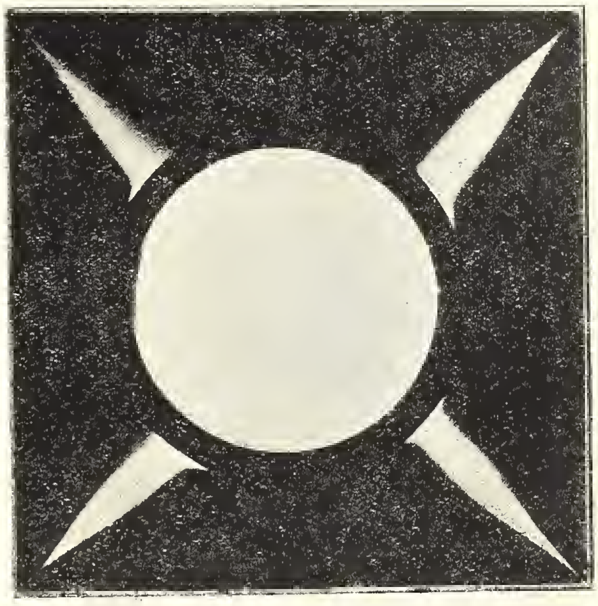

Fig. 38.

deposit there is a line of clear space some 7 millims. wide, and at the ends are delicate curved lines extending to the corners of the plate, and beyond this open space there is the conical deposit as in the former case (fig. 39). In raising the screen so that it was 4 millims. above the plate, the depth of the clear space increased and was now

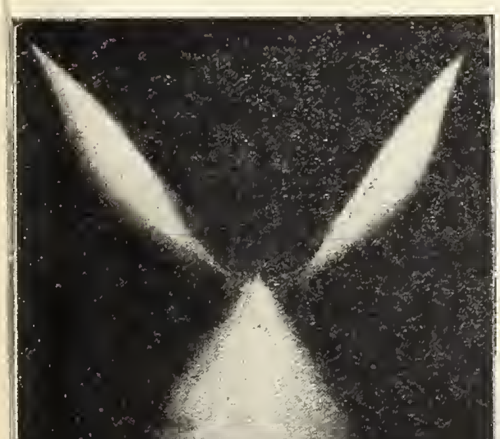

Fig. 39.

Fig. 40.

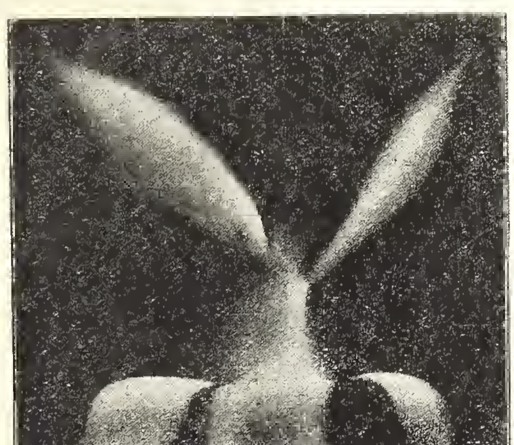

Fig. 41.

12 millims. from the edge. When the screen was 6 millims. above the plate, the clear space became strongly curved, and at the top of the curve was 20 millims. from the edge of the plate, and when 10 millims. above the plate the normal rays of the cross were well developed, and a well defined but slightly distorted cross was formed,

so that the hanging screen at this height exercises but little influence on the figure formed below it. The same hanging screen was now allowed to rest on the plate, but at a distance of 15 millims. from the edge, and a pin was placed against the edge of the plate. Fig. 40 shows well the different actions which came into play, that of the pin, of the corners of the plate, of the screen, and of the broken corner producing a forked ray. The hanging screen was now raised to 3 millims. above the plate, the current now passed under it, and gave the curious picture (fig. 41) with distinct side wings.

Another way of offering obstruction to these dust currents was to place above the plate on which the deposit is to take place, other plates at different heights, and of varying sizes, some larger and some smaller than the plate which is to receive the figure. First taking the case of placing a plate larger than the one on which the deposit is to form above it, the large one was $7\frac{1}{2}$ inches by $4\frac{3}{4}$ inches and the smaller was 3 inches square. The upper part was supported on vulcanite pillars, so far from

Fig. 42.

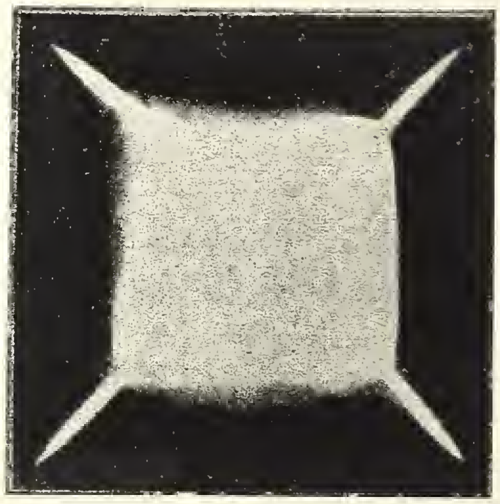

Fig. 43.

the plate as not to influence the figures formed. It was found that when the distance between these plates was 1 millim. no deposit took place; but when this distance was 5 millims. a deposit did take place, covering most of the curved outline, and passing into each of the corners, but less deposit occurred in the middle of the figure (fig. 42), and was apparently an early stage of the cross. If the distance between the two plates be 10 millims., the amount of even deposit is less, and when the distance is increased to 15 millims. a considerable change has occurred, and fig. 43 is formed, the cross still further developed, and when the distance between the plates is 20 millims., then a perfect cross forms.

If the upper plate, in place of being larger, is of the same size as the lower one. different results take place. Plates $3\frac{1}{4}$ inches square were used, and the upper one was suspended above the lower one. When the distance between the plates was 1 millim., again no dust entered, but when it was 2 millims. there was a small amount of deposit at each of the corners, and when 3 millims. a considerable increase of deposit occurred, but it was still limited to the corners (fig. 44). When the distance between the plates was 4 millims., a further inroad of dust took place, and when 5 millims., the centre is the only part without deposit, but from the entrance of the dust being principally at the corners, a rough cross, formed by absence of dust, and pointing to the centre, is distinguishable (fig. 44A), and at 7 millims. there is an even deposit.

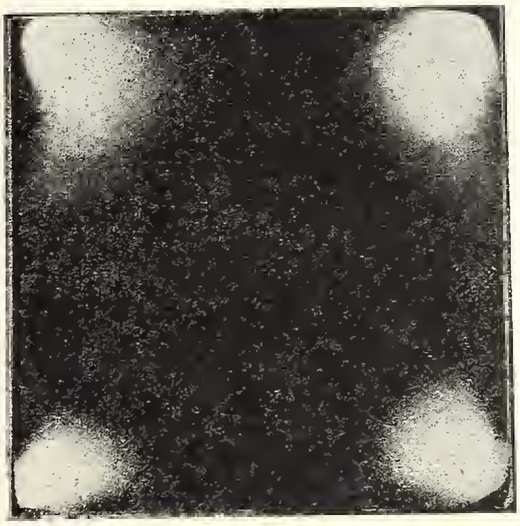

Fig. 44.

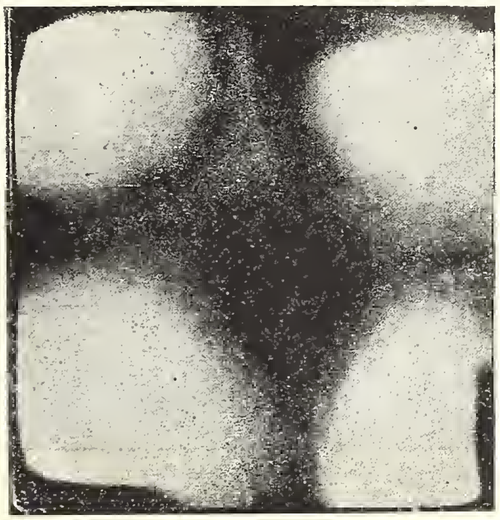

Fig. 44A.

In the next set of experiments the covering glass, in place of being as large as the lower glass, was only a strip 14 millims. wide and 190 millims. long, and $1\frac{1}{2}$ millim. thick. It was supported on vulcanite pillars which did not influence the depositions

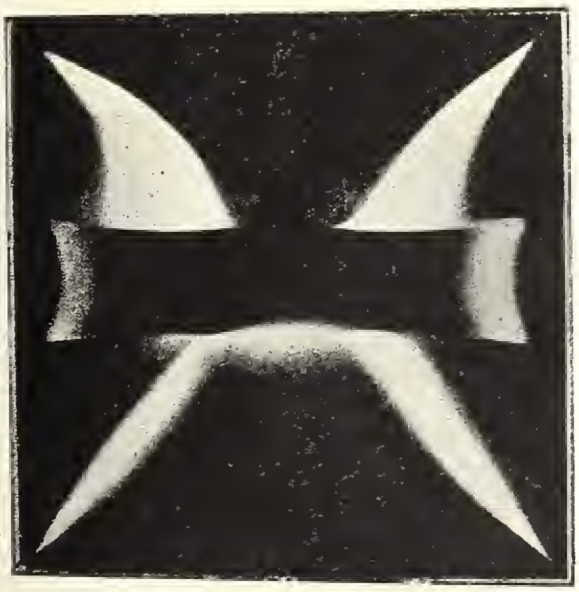

Fig. 45

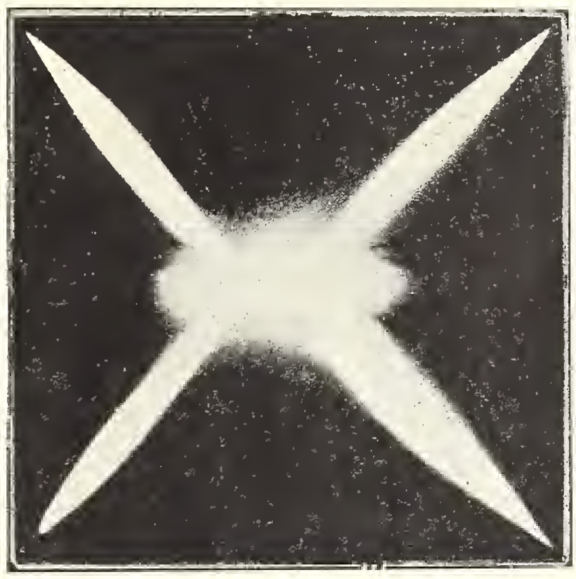

Fig. 46.

of the dust. When this strip was 1 millim. above the plate no deposit took place; when 2 millims. above the plate a small amount occurred, and this was at a distance of 7 millims. from the edge of the plate, and of a curved form, of course, under the strip (fig. 45). The strip was now raised to a height of 3 millims., and the amount of deposit not only increased, but receded further from the edge of the plate, and was now 12 millims. from it. On still further increasing the distance between the strip and the plate, the amount of deposit goes on increasing and travels nearer the centre. When raised to 4 millims. above the plate the deposits have met in the centre, and when the height between the plates is 7 millims., then the deposit is 15 millims. from the edge, and when 10 millims. above the lower plate the deposit is 18 millims. from the edge, and is central to the large cross (fig. 46). At a distance of 15 millims. the strip no longer produces any effect, the ordinary large cross forms.

In order to ascertain whether any figure could be formed by a dusty atmosphere when in motion, magnesium was burnt in an asbestos tube, while a current of air was being drawn through it. The asbestos tube was attached to a glass tube, 32 millims. in diameter, and in this tube pieces of glass of different lengths were introduced for the figures to form on. It was found that a peculiar and characteristic figure was always produced. It consists of a multitude of dust streams which unite into a single stream, as shown in fig. 47. If the tube be wider, the same picture is formed by increasing the amount of air drawn through the tube. It may also be stated that if the dust atmosphere be violently disturbed by means of a stirrer, while the dust is settling on the plate, it produces no alteration of the figure which is forming without the stirrer comes very close to the plate.

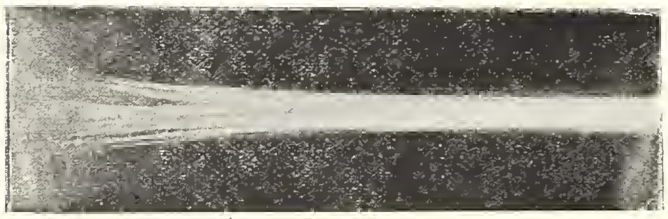

Fig. 47.

This figure, formed in the tube, is probably of a somewhat different character from the previous ones, for it forms quite as readily when the plate is not warmed as it does when it is warmed.

When the dust is obtained by burning magnesium, the magnesia formed undergoes some curious changes. The figure when first formed lies loosely on the plate, the slightest friction will remove it. If, however, it be left exposed to the air, it loses its silvery whiteness and becomes more and more attached to the glass, so that after about a week or fortnight the figure may be lightly rubbed without its being removed. Again, the magnesia itself undergoes a change of form immediately after its production. If the dust be collected at once, that is, while the magnesium is still burning, and be examined under a microscope, it will be seen that it is made up of small separate irregular-shaped particles about 0·005 millim. long (fig. 48); but if the dust be collected after the combustion is over, and it has stood for one or two minutes, then its form is different, for it now

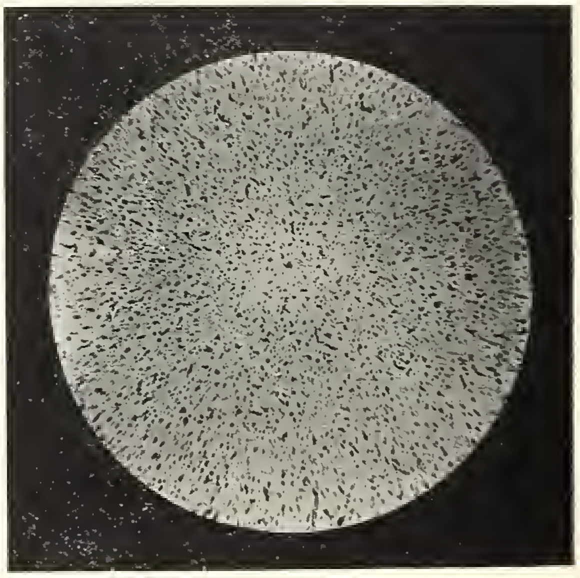

Fig. 48.

consists of particles strung together and having a distinctly fibrous structure (fig. 49). It is in this form that the dust exists when forming pictures. It has already been stated that magnesia dust, if allowed to deposit on mercury, forms the ordinary cross; on the contrary, if it be allowed to deposit on water at about 17°, or on a

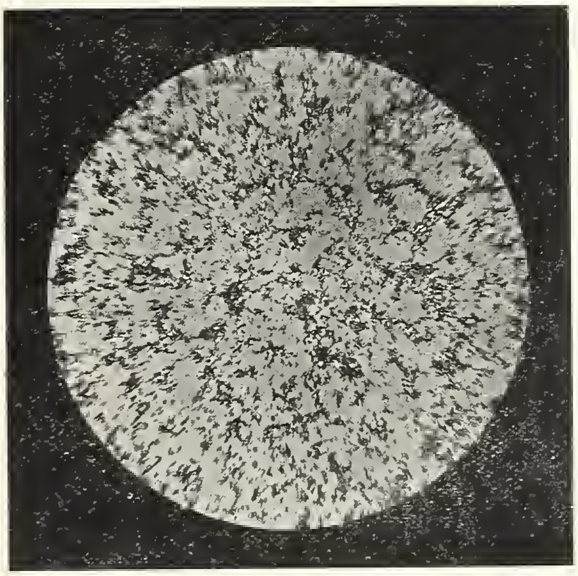

Fig. 49.

Fig. 49A.

mixture of water with a little alcohol or glycerine, then the deposit which forms on the surface breaks up, as the dust sinks, into a figure having a cellular form (fig. 49A).

As before stated, other powders than magnesia act in the same way. For instance, a figure, corresponding exactly with those described as produced by the action of a

Fig. 50.

Fig. 51.

pin, and magnesia, is also produced with fine fungus spores, dust from ashes, or ammonium chloride.

It is interesting to note that if the warmed glass be rubbed with a piece of flannel, and then exposed to the dust, in place of a fine even deposit a very strongly fibrous one forms (fig. 50). Even small specks of dust in this fibrous form act very strongly in the same way as the pin or rough edge of a glass in inducing deposits to take place. If the plate be charged with negative electricity, then a deposit much finer in character is produced (fig. 51).

It is remarkable that these figures deposited by a dust-laden atmosphere, should be so sharp in outline and definite in form. They originate, no doubt, in the currents set up by the warming of the plate, but that these feeble currents should so completely and persistently prevent the deposition of dust at certain places, and determine its precipitation at others, was hardly to be anticipated. Especially may reference be made to the singular action of the pin both near and at a distance from the plate, and the apparently complicated way in which obstructions act in altering the form of the deposits. The formation of the figures taking place as readily on copper or other metals, as on glass or ebonite, indicates that the phenomena are not purely electrical.

It is hoped that by the foregoing records and descriptions of these singular figures, physicists may be enabled to explain their formation.

I wish to record that this investigation was carried out in the Davy-Faraday Laboratory at the Royal Institution, also that my best thanks are due to my assistant, Mr. Olaf Block, for the important aid which he has given me.

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PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

Series A, VOL. 201, pp. 205–222.

[Plate 1.]

THE SPECTRUM OF γ CYGNI

BY

Sir NORMAN LOCKYER, K.C.B., F.R.S.,

AND

F. E. BAXANDALL, A.R.C.Sc.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 337. 19.6.03

Price Two Shillings and Sixpence.

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VII. *The Spectrum of γ Cygni.*

By Sir NORMAN LOCKYER, *K.C.B., F.R.S., and* F. E. BAXANDALL, *A.R.C.Sc.*

Received December 3,—Read December 11, 1902.

[PLATE 1.]

IN a paper on "The Chemical Classification of the Stars," communicated to the Royal Society on May 4, 1899,* one of us indicated that it was then possible to classify the stars according to their chemistry. In the case of type stars of some of the groups lists have been given† of the wave-lengths and probable origins of the lines on which the classification is based. The type stars thus dealt with represent the groups of higher temperature, viz., α Cygni (Cygnian), Rigel (Rigelian), ζ Tauri (Taurian), Bellatrix (Crucian), ε Orionis (Alnitamian), and Sirius (Sirian).

The spectrum of stars of the Polarian type—representing a temperature stage next lower than that of α Cygni,—is, so far as the relative intensities of the metallic lines are concerned, closely allied to that of the chromosphere. It is also interesting as the connecting link between the spectrum of the Aldebarian stars, in which the arc lines of the metallic elements predominate, and that of α Cygni, chiefly composed of the enhanced lines of some of the metals. It has hence been thought important to make a careful reduction of the spectrum of a star of this group. Of the existing photographs of Polarian type spectra at Kensington, that of γ Cygni is the best for the purpose of reduction, and for this reason has been selected.

Method of Reduction.

The wave-lengths have been determined by measuring the relative positions of the lines on the plate with a micrometer, and subsequent use of HARTMANN'S interpolation formula. In selecting the lines to be used as bases for the reduction, only sharply-defined lines with well-authenticated origins, and of the simple nature of which there

* 'Roy Soc. Proc.,' vol. 65, p. 186.

† 'Catalogue of 470 Brighter Stars,' published by the Solar Physics Committee.

is little doubt, were taken; lines which were suspected, however slightly, of having a double or complex origin were rejected. A list of the lines used is here given:—

λ.	Origin.	λ.	Origin.
3900·68	*p* Ti	4501·45	*p* Ti
4012·54	*p* Ti	4584·02	*p* Fe
4215·70	*p* Sr	4657·38	*p* Ti
4415·29	Fe	4780·20	*p* Ti

The result of a previous reduction of the spectrum of α Cygni, already published, serves as a valuable check on the accuracy of the reduced wave-lengths, as there are many lines common to the two spectra, and there can be no doubt as to the identity of most of the stronger α Cygni lines with enhanced lines of some of the metals, as has been shown in a previous paper.*

In the table at the end of the paper the γ Cygni lines are compared with those reduced at Kensington from the spectrum of α Cygni and that of the chromosphere, and also with those recorded by PICKERING† in the spectrum of δ Canis Majoris. The latter star is selected by PICKERING as typical of Group XIIIc. in his classification, in which group he also includes γ Cygni. In the case of the chromosphere, in order to keep the table within moderate limits, only those lines which agree with γ Cygni lines have been inserted, but of the chromospheric lines omitted none are prominent except those of helium.

Comparison of γ Cygni and Chromosphere.

Reference to the table will show that the metallic and protometallic lines have, speaking broadly, about the same relative intensities in the spectra of γ Cygni and the chromosphere. It would thus appear that the temperature and electrical conditions prevailing in the chromospheric vapours which furnish the metallic lines are nearly identical with those appertaining to the absorbing atmosphere of γ Cygni. To arrive at any conclusion as to which of the two light sources in question represents the higher temperature, it is necessary to study in detail the comparative intensities of well-known lines. For this purpose, two sets of lines have been considered: (1) the strongest unenhanced lines of the metals represented; (2) the most marked enhanced lines of the metals. In the following table a comparison is given of the intensities of the strongest lines of iron, manganese, chromium, cobalt, barium, calcium, aluminium, and titanium, as they occur in γ Cygni and the chromosphere.

* 'Roy. Soc. Proc.,' vol. 64, p. 321.

† 'Annals Harv. Coll. Obs.,' vol. 28, Part I., p. 79.

Comparative Intensities of the Strongest Metallic Lines in γ Cygni and the Chromosphere.

Strongest arc lines. λ.	Origin.	Intensity. γ Cygni. Max. = 10.	Intensity. Chromosphere. Max. = 10.	Strongest arc lines. λ.	Origin.	Intensity. γ Cygni. Max. = 10.	Intensity. Chromosphere. Max. = 10.
⎧ 4045·98	Fe	8	7	4528·80	Fe	4	3
⎨ 4063·76	Fe	8	6–7	4030·92	Mn	5	5
⎩ 4071·91	Fe	5	6	4033·22	Mn	4	3–4
4132·24	Fe	5–6	3	3995·46	Co	5	3–4
4144·04	Fe	8	5–6	4226·90	Ca	8	7
4202·20	Fe	5	3	3989·91	Ti	4	2–3
4260·64	Fe	6	4	3998·79	Ti	5	4
4271·33	Fe	6 ⎫	4–5	3944·16	Al	3–4	5
4271·93	Fe	6 ⎭		3961·67	Al	5–6	6
⎧ 4383·72	Fe	4–5	5	4554·21	Ba	5–6	7–8
⎨ 4404·93	Fe	3–4	4	4254·51	Cr	4	6
⎩ 4415·29	Fe	5–6	4	4274·96	Cr	4	5

These intensities cannot be accepted as absolute, but as the same limits (1 to 10) are used in the two spectra, it may be conceded that the intensities are roughly comparable. It will be noticed that in the majority of cases the lines appear to be somewhat weaker in the chromosphere than in γ Cygni. Notable exceptions, however, to this are the lines of aluminium, chromium, and barium.

In the next table, the intensities of the more prominent enhanced lines of iron, magnesium, chromium, titanium, and strontium are similarly compared.

Comparative Intensities of Enhanced Lines in γ Cygni and the Chromosphere.

Enhanced lines. λ.	Origin.	Intensity. γ Cygni. Max. = 10.	Intensity. Chromosphere. Max. = 10.	Enhanced lines. λ.	Origin.	Intensity. γ Cygni. Max. = 10.	Intensity. Chromosphere. Max. = 10.
4233·33	*p* Fe	7–8	6–7	4399·94	*p* Ti	5–6	5–6
⎧ 4508·46	*p* Fe	4	5	4443·98	*p* Ti	9	7
⎪ 4515·51	*p* Fe	4	4	4450·65	*p* Ti	4	5
⎨ 4520·40	*p* Fe	3	3	4468·66	*p* Ti	6	6
⎩ 4522·69	*p* Fe	4	4	4501·45	*p* Ti	6	7
4549·64	*p* Fe	8	7–8	4534·14	*p* Ti	6	7–8
4584·02	*p* Fe	8	7	4549·81	*p* Ti	8	7–8
3900·68	*p* Ti	4–5	4	4563·94	*p* Ti	4–5	7–8
3913·61	*p* Ti	4	6	4572·16	*p* Ti	6–7	7
4012·54	*p* Ti	5	5–6	4558·83	*p* Cr	3	3–4
4161·68	*p* Ti	6–7	3	4588·38	*p* Cr	3	3
4163·82	*p* Ti	5–6	4	4077·89	*p* Sr	8	10
4300·21	*p* Ti	6	5	4215·70	*p* Sr	9	10
4321·20	*p* Ti	8	5	4481·30	*p* Mg	5–6	—
4338·08	*p* Ti	9	5				

Here we find that of the 29 lines included 12 have a greater intensity in γ Cygni, 11 in the chromosphere, while 6 have been estimated as having equal intensities in the two spectra, thus showing a very evenly-balanced state of affairs.

Taking the two comparisons together, it would appear that the evidence points to the unenhanced lines being, upon the whole, somewhat weakened in the chromosphere at the expense of the enhanced lines. This result tends to show that if any distinction is to be made between the temperature conditions of the two light sources in question, the chromosphere must be placed on a slightly higher level.

The most marked difference between the spectrum of γ Cygni and that of the chromosphere occurs in the case of the helium lines. There is no evidence of their presence in the former spectrum, while in the latter the stronger helium lines are quite conspicuous. We do not, however, know much about the relative positions of the helium vapour and the metallic vapours in the chromosphere, and it is quite possible that the temperature conditions of the two are vastly different. Another notable difference between the two spectra is in regard to the well-known enhanced line of magnesium, λ 4481·3. This is fairly prominent in γ Cygni, but appears to be entirely lacking in the chromospheric spectrum. As the enhanced lines of other elements are well developed in the chromospheric spectrum, this is a very curious result, and difficult to account for, especially as the line in question is well marked in both γ Cygni and α Cygni, between which the chromosphere must apparently be placed from temperature considerations.

In the transition from stars resembling the Sun, through γ Cygni (Polarian), the chromosphere, to α Cygni (Cygnian), the gradual strengthening or weakening of well-known groupings of metallic lines can be traced. There cannot be any doubt about the authenticity in the spectra of γ Cygni and the chromosphere of such groups and pairs of metallic lines as the aluminium pair ($\lambda\lambda$ 3944·16, 3961·67), manganese triplet ($\lambda\lambda$ 4030·88, 4033·22, 4034·64), iron triplets ($\lambda\lambda$ 4045·98, 4063·76, 4071·91) and ($\lambda\lambda$ 4383·72, 4404·93, 4415·29), chromium triplet ($\lambda\lambda$ 4254·51, 4274·96, 4289·89), and the enhanced iron quartette ($\lambda\lambda$ 4508·46, 4515·51, 4520·40, 4522·69).

Moreover, reference to the Kensington publications of eclipse results,* in addition to those of FROST,† EVERSHED,‡ MITCHELL,§ and HUMPHREYS‖ will show that there is a general consensus of opinion that the chromospheric lines have, upon the whole, metallic origins. This is entirely at variance with the conclusion arrived at by Professor DEWAR, and embodied in his Presidential Address to the British Association, 1902, that the chromospheric lines are to be accounted for by the lines of krypton, xenon, and those of the most volatile atmospheric gases. In this connection,

* 'Phil. Trans.,' A, vol. 197, p. 208.

† 'Astrophysical Journal,' vol. 12, p. 307.

‡ 'Phil. Trans.,' A, vol. 197, p. 381.

§ 'Astrophysical Journal,' vol. 15, p. 97.

‖ 'Astrophysical Journal,' vol. 15, p. 313.

he says,* "In the 'Astrophysical Journal' for June last is a list of 339 lines in the spectrum of the corona, photographed by HUMPHREYS. . . . . Of these, no fewer than 209 do not differ from lines we have measured in the most volatile gases of the atmosphere, or of krypton, or xenon, by more than one unit of wave-length on ÅNGSTRÖM's scale, a quantity within the limit of probable error."

It may be here pointed out that HUMPHREYS' list of 339 lines referred to the spectrum of the solar chromosphere, and not to that of the corona. The latitude allowed (one tenth-metre) in comparing the wave-lengths of the lines in the solar and terrestrial spectra is far greater than can be accepted in modern exact work, and as the average error of HUMPHREYS' wave-lengths is probably less than 0·2 tenth-metre, it is obvious that, until Professor DEWAR can give the wave-lengths of his lines to a greater accuracy than that of the nearest tenth-metre, little weight can be attached to the results of his comparison. His conclusion, moreover, appears to have been based merely on apparent similarity of wave-lengths, without taking into account the relative intensities of the lines in the spectra compared, or of the correspondence of conspicuous groupings of lines, which would certainly tend to clear matters.

The extreme limits of HUMPHREYS' 339 eclipse lines are, roughly speaking, 2000 tenth-metres apart, which gives an average interval of 6 tenth-metres. In Professor DEWAR's three lists of gaseous lines there occur between the same limits 564 lines, with an average interval of 4 tenth-metres. If we assume, then, that the lines of each set are evenly distributed over the region involved, there will be certain to be a large number of lines in the two sets which agree in position within the limits of error allowed (one tenth-metre).

Many lines have gaseous origins assigned to them which have been hitherto universally acknowledged by the various workers in the subject to be representatives of well-known metallic lines, and groups of lines previously given as due to some particular metal are split up by Professor DEWAR's analysis, some members being ascribed to krypton, others to xenon, &c., while other members remain clear of his gaseous lines. The following table contains several groups of chromospheric lines, which are all included in both HUMPHREYS' list† and that given in the Kensington eclipse publication,‡ and which have been ascribed to the same metals in the two records. In the comparison column, LIVEING and DEWAR's gaseous lines are given which agree within one tenth-metre (this being the difference accepted by Professor DEWAR in his analytical comparison) with the chromospheric lines.

* 'Nature,' vol. 66, p. 475.

† 'Astrophysical Journal,' vol. 15, p. 318.

‡ 'Phil. Trans.,' A, vol. 197, p. 208.

COMPARISON of Groups of Chromospheric Lines belonging to Various Metals with LIVEING and DEWAR'S Gaseous Lines.

Chromosphere (HUMPHREYS). λ.	Origin.		Atmospheric Gases (LIVEING and DEWAR).		
	HUMPHREYS.	Kensington.	Most volatile.	Xenon.	Krypton.
3829·5	Mg	Mg	3830	3829	—
3832·5	Mg	Mg	—	—	—
3838·4	Mg	Mg	—	—	—
3944·0	Al	Al	—	3944·0	—
3961·6	Al	Al	—	—	—
4046·0	Fe	Fe	4047	—	4045
4063·7	Fe	Fe	—	—	—
4071·9	Fe	Fe	—	—	—
4077·9	Sr	*p* Sr	—	—	—
4215·7	Sr	*p* Sr	—	4215	—
4254·5	Cr	Cr	—	—	—
4274·9	Cr	Cr	—	—	—
4289·9	Cr	Cr	4290	—	—
4383·6	Fe	Fe	—	—	—
4404·9	Fe	Fe	—	—	—
4415·2	Fe	Fe	4415	—	—
4508·5	Fe ?	*p* Fe	4508	—	—
4515·5	—	*p* Fe	—	—	—
4520·7	Fe ?	*p* Fe	—	—	—
4522·9	Ti	*p* Fe	4523	—	—

From this comparison it would appear that Professor DEWAR claims for xenon, one member of the magnesium triplet (λλ 3829·5–3838·4), one component of the aluminium double (λλ 3944·0, 3961·6) and one member of the strontium pair (λλ 4077·9, 4215·7); for krypton one member of the iron triplet (λλ 4046·0–4071·9); and for the most volatile gases, one member of the magnesium triplet, one of each of two iron triplets, one of a chromium triplet, and two members of the enhanced iron quartette (λλ 4508·5–4522·9). It is, of course, quite possible that some of these gaseous lines may account for the coronal lines; but that the chromospheric lines are, in the main, produced by metallic vapours, there can be no doubt.

Comparison of γ Cygni and α Cygni.

It will be seen that there is a much greater number of lines in the spectrum of γ Cygni than in that of α Cygni. The lines occurring solely in γ Cygni which have been traced to any terrestrial origin are found to be attributable to the ordinary metallic arc lines, as distinguished from the enhanced lines. These, which occur so prominently in α Cygni, are, with certain exceptions, present also in γ Cygni, so that the latter spectrum practically consists of the α Cygni spectrum (with modifications of the intensities of the enhanced lines of various metals) with the ordinary arc lines added, and the two sets are of about equal importance. This is a condition of affairs intermediate to that of the Aldebarian stars—in which the ordinary lines are well-developed and the proto-metallic lines weak or missing—and α Cygni, where the enhanced lines are very prominent, to the nearly total exclusion of the metallic arc lines.

The only line of any prominence which occurs solely in α Cygni is the silicium line λ 4131·1 This is one component of the silicium double which is so conspicuous in the spectra of α Cygni, Rigel, Sirius, &c. There is certainly a line in γ Cygni apparently coincident with the other component λ 4128·1, but in the absence of its companion it must be concluded that the γ Cygni line in question has probably an origin entirely distinct from silicium. The silicium double mentioned is also absent from the chromospheric spectrum, which closely resembles that of γ Cygni.

In a paper "On the Order of Appearance of Chemical Substances at different Stellar Temperatures,"* it was pointed out that the enhanced lines of the various metals attained a maximum intensity at varying levels of the stellar temperature sequence. The present detailed investigation of the γ Cygni spectrum confirms this result, the enhanced lines of strontium, scandium, and titanium being at their strongest in γ Cygni and much stronger than in α Cygni, while in the latter spectrum the enhanced lines of iron, chromium, and magnesium, attain their maximum intensity, being more prominent than in γ Cygni.

Of the better known arc lines of some of the metals which are prominent in γ Cygni, but very weak or lacking in α Cygni, the following may be mentioned: the iron triplets ($\lambda\lambda$ 4045·98, 4063·76, 4071·91) and ($\lambda\lambda$ 4383·72, 4404·93, 4415·29); the manganese quartette ($\lambda\lambda$ 4030·92, 4033·22, 4034·64, 4035·80); the chromium triplet ($\lambda\lambda$ 4254·51, 4274·96, 4289·89); the aluminium pair ($\lambda\lambda$ 3944·16, 3961·67); the calcium line, λ 4226·90; and the barium line, λ 4554·21.

General Conclusions.

The investigation of the photographic spectrum of γ Cygni in its relation to other spectra has led to the following conclusions:—

1. The majority of the lines are due to metallic vapours, the enhanced lines and the arc lines being of about equal prominence.

2. The temperature conditions are thus intermediate between those of Aldebaran

* 'Roy. Soc. Proc.,' vol. 64, p. 396.

(arc lines prominent, enhanced lines weak or absent) and those of α Cygni (enhanced lines prominent, arc lines weak or absent).

3. The enhanced lines of scandium, strontium, and titanium are better developed than in α Cygni, but those of iron, chromium, and magnesium are less conspicuous than in α Cygni.

4. The relative intensities of the metallic and proto-metallic lines are about the same as in the spectrum of the solar chromosphere, which, if anything, represents a slightly higher temperature.

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni.

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
3872·9	5	Fe	3872·64	3872·7	7	3872·6	4	3872·4	3	
76·0	3	Fe	76·19	—	—	76·1	1–2	—	—	
78·8	7	Fe	78·72	78·5	5	78·7	3	78·7	4	
80·6	1–2	—	—	—	—	80·8	2	80·5	1–2	
82·4	3	—	—	—	—	82·5	2	82·2	2	
83·5	1	? C	83·55	83·2	3 ?	83·4	4	—	—	
85·1	2	Fe	{ 84·81 85·29	— —	— —	— —	— —	84·5 —	1–2 —	 ? double.
86·9	3–4	Fe	87·20	—	—	87·2	2–3	86·3	2	
89·1	5	H	89·15	89·1	11 ?	89·1	8	89·1	10	Hζ.
91·1	1	Fe	90·99	—	—	91·4	2	—	—	
91·9	2–3	Fe	92·07	—	—	92·2	2	—	—	
93·6	1–2	Fe	93·54	—	—	94·0	2	—	—	? double.
96·1	4–5	Fe	95·80	—	—	95·7	3	95·8	2	? fine double.
98·0	3–4	{ Fe Y	98·03 98·15 }	—	—	98·0	2	98·1	1–2	
99·4	3	*p* V	99·30	99·9	2	99·2	2	—	—	
3900·7	4–5	*p* Ti	3900·68	3900·7	7	3900·7	4	3900·7	5–6	
03·2	4–5	Fe	03·09	03·1	3	03·1	2–3	03·4	2	
03·8	2	Fe	04·05	—	—	—	—	—	—	
05·6	4	Si *p* Cr	05·66	—	—	05·3	2	05·7	4	
06·7	4	Fe	{ 06·63 06·89 }	06·6	4	06·8	2	06·7	2	
08·7	2–3	Cr	08·90	—	—	08·4	1	09·0	1	
09·8	2–3	Fe	{ 09·80 09·98 }	—	—	09·6	<1	—	—	
11·0	3	Fe V	10·98	—	—	—	—	11·5	1	
12·4	2	? Ni	12·45	—	—	—	—	—	—	
13·6	4	*p* Ti	13·61	13·6	3	13·6	6	13·6	4–5	
14·5	4	{ Fe Ti	{ 14·43 14·48 }	—	—	—	—	14·5	3	
16·1	4	Cr	15·95	—	—	16·2	3	16·4	<1	
16·7	4	Fe	16·88	—	—	—	—	—	—	
18·7	4–5	Fe	{ 18·46 18·56 18·79 }	—	—	18·6	3	18·8	<1	? double.
20·7	4	Fe	20·44	—	—	20·4	3	20·4	1	? double.
22·9	3–4	Fe	23·05	—	—	23·1	3	23·1	<1	
26·2	3	Fe	26·09	—	—	25·9	2	26·2	1	Broad line.
—	—	—	—	—	—	—	—	{ 28·2 30·4 32·1	1 2–3 1	} Probably masked in γ Cygni by broad K line.

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
3933 ·8	10	Ca	3933 ·83	3933 ·8	220	3933 ·8	10	3933 ·8	10	
—	—	—	—	—	—	—	—	36 ·0 37 ·3 38 ·6 39 ·3	3 1 3 3	Possibly masked in γ Cygni by broad K line.
41 ·0	2–3	Fe	41 ·03	—	—	—	—	41 ·8	<1	
42 ·5	3	Fe	42 ·59	—	—	41 ·9	1–2	42 ·6	<1	
44 ·1	3–4	Al	44 ·16	44 ·1	3 (44 ·1 to 45 ·2)	44 ·2	5	44 ·2	<1	
45 ·2	6	*p* Y Fe Fe	44 ·94 45 ·03 45 ·26	— — 45 ·2		45 ·2	2	45 ·2	3	
—	—	—	—	—	—	—	—	47 ·2	1	
47 ·9	1	? Ti	47 ·92	—	—	—	—	—	—	
48 ·9	2–3	Ti Fe	48 ·82 48 ·93	49 ·0	3	48 ·6	2–3	—	—	
50 ·1	4	Fe	50 ·10	—	—	50 ·3	2–3	—	—	
51 ·3	4	Fe	51 ·31	—	—	51 ·8	1	—	—	
52 ·7 to 53 ·3	6	Fe Fe Mn Co Fe Cr	52 ·75 52 ·85 53 ·04 53 ·12 53 ·30	53 ·0	4	52 ·3	3–4	52 ·1	1–2	Broad hazy line, probably including all the solar lines given.
55 ·5	1	Fe	55 ·48	—	—	—	—	54 ·4	<1	
56 ·6	6	Co Ti Fe Fe	56 ·48 56 ·60 56 ·82	56 ·6	2	56 ·6	4	56 ·6	1	Probably compound line.
58 ·4	4–5	Ti	58 ·36	58 ·4	2	58 ·2	4	59 ·0	1	
60 ·0	1	—	—	—	—	—	—	—	—	
61 ·7	5–6	Al	61 ·67	61 ·6	3	61 ·7	6	61 ·6	2	
63 ·3	2	Fe	63 ·25	—	—	63 ·3	1	—	—	
—	—	—	—	—	—	—	—	64 ·9	1	Possibly masked in γ Cygni by broad H line of calcium.
—	—	—	—	—	—	—	—	66 ·4	1	
68 ·6 70 ·2	10	Ca H	68 ·63 70 ·18	68 ·6 70 ·2	180	68 ·6 70 ·2	10 10	68 ·6 70 ·2	10 10	
—	—	—	—	—	—	—	—	71 ·4	1	Possibly masked in γ Cygni by broad H line of calcium.
73 ·8	4	Ni Zr Fe Ca *p* Y	73 ·70 73 ·80 73 ·86	—	—	73 ·5	2	74 ·0	3–4	
74 ·9	4	Co Fe	74 ·90	—	—	—	—	—	—	
76 ·8	3–4	Fe Cr	76 ·77 76 ·84	76 ·8	2	76 ·7	1	—	—	
77 ·9	2–3	Fe	77 ·89	—	—	77 ·8	2	77 ·3	1	
78 ·6	2	—	—	—	—	78 ·1	Tr	—	—	
79 ·7	3	*p* Cr Co Fe	79 ·66 79 ·78	—	—	79 ·3	1	79 ·6	3	
81 ·9	7	Ti	81 ·92	82 ·0	6	82 ·0	6	82 ·0	2–3	Very broad line.
84 ·0	3–4	Cr Fe	84 ·06 84 ·11	—	—	83 ·8	1	—	—	
86 ·2	3	Fe	86 ·32	—	—	86 ·3	1–2	—	—	
87 ·3	3	? Mn	87 ·24	87 ·0	3	—	—	—	—	
88 ·6	3	—	—	—	—	88 ·3	1	—	—	
89 ·9	4	Ti Fe	89 ·91 90 ·01	—	—	89 ·9	2–3	—	—	
91 ·1	3–4	Cr Zr	91 ·33	91 ·6	2	91 ·3	3	—	—	
92 ·1	1–2	—	—	—	—	—	—	—	—	

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γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
3993·7	2	—	—	—	—	—	—	3993·7	<1	
95·5	5	Co	3995·46	3995·5	3	3995·2	3–4	95·7	<1	
97·3	5	Fe	97·11 97·55	97·6	6	97·7	3	97·3	1	
98·9	5	Ti	98·79	98·8		98·8	4	—	—	
4000·1	3	—	—	—	—	4000·4	1–2	4000·0	1	
01·8	3–4	Fe	4001·81	—	—	—	—	—	—	
03·0	3	—	—	4003·0	3	03·3	1	02·7	3–4	
04·0	2	Ce Fe Ti	03·91	—	—	—	—	—	—	
—	—	—	—	—	—	—	—	04·9	<1	
05·4	8	Fe	05·41	05·3	4	05·4	5	05·5	2–3	
06·8	1	—	—	—	—	06·8	1–2	—	—	
09·9	2–3	Fe	09·86	09·4	1	09·5	2–3	09·4	1	
12·5	5	*p* Ti	12·54	12·6	3	12·5	5–6	12·5	4	
14·4	2	Fe	14·42	—	—					
14·8	3–4	Fe	14·68	14·7	3	14·5	3	14·1	<1	
15·8	2	—	—	—	—	—	—	15·7	2–3	
17·2	3	Un Fe	17·24 17·31	—	—	17·5	2	17·2	<1	
18·1	3–4	Mn	18·23 18·27	18·4	2	18·5	1	—	—	
20·6	1–2	Sc Fe	20·55 20·64	—	—	20·6	<1	—	—	
22·0	3–4	Fe	22·02	22·0	2	21·6	3	—	—	
23·2	3–4	—	—	—	—	23·1	1	23·6	1–2	
24·8	7	Ti Fe	24·73 24·88	24·8	5	24·7	3	24·6	3	
25·7	1	—	—	—	—	—	—	25·2	1	
28·5	4	*p* Ti	28·50	28·5	2	28·5	2–3	28·5	3	
29·8	2	Fe	29·80	—	—	—	—	—	—	
30·8	5	Mn	30·92	30·8	5	30·9	5	30·9	1	
33·2	4	Mn	33·22	33·2	2	33·2	3–4	33·2	2–3	
34·6	3	Mn	34·64	34·6	1	34·6	3–4	34·6	<1	
35·9	2	Mn *p* V	35·80 35·80	35·8	1	35·9	1	35·8	2	
37·3	1	—	—	37·2	1	37·7	1	—	—	
—	—	—	—	—	—	—	—	38·3	1–2	
40·8	2–3	Fe	40·79	40·8	2	40·8	4	40·4	<1	Close double, components merging into each other.
41·5	5	Mn	41·53	—	—	—	—	41·9	<1	
43·0	1–2	? La	43·05	—	—	43·4	1	—	—	
44·4	1	Fe Fe	44·06 44·77	—	—	44·4	1	44·4	1–2	Probably close double.
46·0	8	Fe	45·98	45·9	4	45·9	7	46·0	3–4	
47·5	1	Fe	47·46	—	—	—	—	—	—	
48·9	4	*p* Fe Mn Cr	48·82 48·91	48·9	1	49·0	3	48·9	3	
50·6	2	—	—	50·8	1	51·0	1	—	—	
52·5	2	Fe	52·45 52·65	52·6	1	—	—	52·3	2	
54·0	4–5	*p* Ti *p* V	53·98	53·8	2	53·8	3	53·9	3	
55·2	2	*p* Ti *p* Fe	55·19 55·63	—	—	55·6	2	—	—	
56·2	2	—	—	—	—	—	—	—	—	
57·6	2	Fe	57·50	—	—	57·4	1–2	57·6	1	? Double.
to		Co Fe	58·37	—	—	58·2	1–2	—	—	
58·8		Fe Cr	58·92	59·0	1	59·2	1–2	—	—	
61·3	2	? Ni	61·24	—	—	61·2	1–2	—	—	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
4063·8	8	Fe	4063·76	4063·7	7	4063·7	6–7	4063·8	2	
65·2	1–2	Mn Ti Fe	65·24 65·54	—	—	—	—	—	—	
67·3	8	Fe *p* Ni Fe	67·14 67·30 67·43	67·0	5	67·3	3	67·2	4	
70·9	1	Fe	70·93	—	—	—	—	70·0	1	
71·9	5	Fe	71·91	71·9	4	71·8	6	71·9	2	
73·7	3–4	? Fe	73·92	—	—	73·9	1	—	—	
75·6	2	—	—	75·0	1	75·3	1	75·6	1	
77·9	8	*p* Sr	77·89	77·9	8	77·9	10	77·9	3	
79·4	2	Fe Mn	79·34 79·39	—	—	—	—	—	—	
80·4	1–2	Fe	80·37	—	—	80·3	1–2	80·0	<1	
82·8	1–2	Sc Fe Ti Mn	82·59 83·10	—	—	83·1	2	82·0	<1	
83·7	4–5	Fe Mn Y	83·72 83·78	83·8	2	84·0	1	—	—	
85·3	4–5	Fe	85·16 85·47	85·4	1	85·0	1–2	86·2	<1	Probably close double.
86·9	4-5	? La	86·86	87·2	3	86·7	3			
89·0		Fe	88·73 89·37	—	—	—	—	—	—	
90·5	3	—	—	90·2	1	—	—	—	—	
92·6	4	Fe Co Mn V	92·43 92·55 92·82	92·7	1	92·5	2	92·5	1	
94·6	1	—	—	—	—	—	—	94·5	<1	
96·2	4	Fe Fe	96·13 96·26	96·2	2	96·2	1–2	—	—	Rather broad, possibly double.
98·5	4	Fe	98·34	98·5	1	98·2	1	—	—	
4100·6	2	? Fe	4100·90	—	—	—	—	4100·3	1	Merging into Hδ.
02·0	8	H	02·00	4101·8	11	4101·9	10	01·8	10	Hδ.
04·1	2	Fe	04·29	—	—	—	—	—	—	
05·2	1	? V	05·32	—	—	—	—	—	—	
06·6	3	Fe Fe	06·42 06·58	06·6	1	—	—	—	—	
07·8	3	Fe	07·65	—	—	07·6	2	—	—	
09·9	5	V Fe	09·91 09·95	10·4	4	09·9	3	—	—	
11·5	4	—	—	11·0	?	11·9	1	11·2	2	
13·5	3	—	—	13·1	1	—	—	13·3	<1	
15·4	3–4	V	15·33	14·7	1	—	—	—	—	
17·5	1	—	—	—	—	—	—	—	—	
18·9	5	Fe	18·71	18·9	4	18·7	3–4	19·7	<1	
20·3	3	Fe	20·37	—	—	—	—	21·0	1	
22·0	1–2	Fe	21·96	—	—	—	—			
23·0	4	—	—	22·8	3	23·0	3	22·9	3–4	
23·8	4	Fe	23·91	—	—	—	—	—	—	
25·0	2	—	—	—	—	—	—	24·7	1–2	
26·2	2	Fe	26·04 26·34	—	—	—	—	25·8	1	
28·0	5	Fe Fe V	27·77 27·97 28·25	28·1 28·5	5	28·0	3	28·1 28·6	5-6 2-3	α Cygni line 4128·1 undoubtedly due to silicium.
29·2	4	—	—	—	—	29·6	2–3	—	—	
30·6	2	? Ba	30·80	—	—	—	—	—	—	
—	—	—	—	—	—	—	—	31·1	5–6	
32·2	5–6	Fe	32·23	32·2	3	32·4	3	32·4	1	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. = 10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. = 220.	λ.	Intensity. Max. = 10.	λ.	Intensity. Max. = 10.	
4134·8	3-4	Fe	4134·84	4134·8	3	4134·8	3	—	—	Hazy, probably double.
36·9	5	Fe Fe	36·68 37·16	— —	— —	— —	— —	— —	— —	
				37·4	3	37·5	3	—	—	
37·9	5	Fe	37·81	—	—	—	—	4138·4	1	
40·1	3	Fe	40·09	—	—	40·1	1	—	—	
42·3	3	Fe Cr	42·03 42·33	—	—	42·3	1	—	—	
43·8	8	Fe Fe	43·57 44·04	43·9	5	43·8	5–6	43·9	1–2	
46·0	2–3	Fe	46·23	—	—	46·0	1	46·0	2	
47·7	3	Mn Fe	47·65 47·84	—	—	47·5	1	—	—	
49·4	5–6	Fe	49·53	49·5	2	49·4	3-4	49·7	<1	? double.
50·4	1	—	—	—	—	—	—	—	—	
52·3	3	Fe	52·34	—	—	52·1	2	—	—	
54·5	5	Fe	54·07 54·67 54·98	54·9	2	54·8	2-3	—	—	Broad, probably compounded of the three solar Fe lines.
56·7	6	—	—	56·7	2	56·5	3	—	—	Probably identical with unknown solar line 4156·47.
57·9	1	Fe	57·95	—	—	57·8	1	—	—	
59·2	3	Fe Un	58·96 59·35	—	—	58·9	1	—	—	Un = strong solar line, to which ROWLAND assigns no origin
60·5	3	—	—	—	—	—	—	—	—	
61·7	6–7	*p* Ti	61·68	61·7	3	61·7	3	61·7	1-2	
—	—	—	—	—	—	—	—	63·0	<1	
63·8	5-6	*p* Ti	63·82	63·9	2	63·8	4	63·8	3-4	
65·5	3-4	Fe	65·55	—	—	—	—	—	—	
67·6	3-4	—	—	67·5	1	67·5	2-3	67·6	1	Probably identical with strong solar line 4167·44, to which ROWLAND assigns no origin.
—	—	—	—	—	—	—	—	69·8	<1	
71·2	1-2	Fe *p* Ti	71·07 71·21	—	—	—	—	—	—	
72·1	3-4	*p* Ti	72·07	—	—	72·1	3-4	72·0	2-3	
73·6	4-5	*p* Fe *p* Ti	73·52 73·70	72·9 73·6	13	73·5	4-5	73·5	6-7	Probably close double.
75·4	1	—	—	—	—	—	—	—	—	
77·7	4	*p* Y	77·75	77·8	3	77·8	5	77·8	2-3	
79·0	5	*p* Fe	78·95	79·5	4	79·0	4-5	79·0	6-7	
81·9	4-5	Fe	81·92	82·0	1	81·9	3	81·8	<1	
84·5	5	*p* Ti	84·40	85·0	2	84·6	2-3	85·0	<1	? double.
87·2 87·8	6	Fe Fe	87·20 87·94	87·6	4	87·6	4-5	88·0	2	Close double.
90·7	1	—	—	—	—	—	—	—	—	
91·7	5	Fe Fe	91·59 91·84	91·8	3	91·7	3-4	92·0	1	
93·4	1	—	—	—	—	—	—	—	—	
95·5	4	Fe	95·49	—	—	95·5	1	—	—	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
4196·4	4	Fe	4196·37	4196·8	3	4196·4	2–3	—	—	
98·9	7	Fe Fe Fe	98·49 98·80 99·27	98·5	4	98·8	4	4198·5	1	
4201·1	1	Fe	4201·09	—	—	—	—	—	—	
02·2	5	Fe	02·20	4202·2	3	4202·2	3	4202·3	1	
03·9	1	? Fe ? La	04·10 04·16	— —	— —	— —	— —	— —	— —	
05·0	5–6	*p* Y *p* V	04·89 05·24	— 05·3	— 3	— 05·1	— 3	— 05·2	— 1	
06·9	3	Fe	06·86 07·29	06·9	2	07·1	1	—	—	
09·2	4–5	? Zr	09·14	08·8	2	09·6	2–3	—	—	
10·5	3	Fe	10·49	10·5	2	10·9	1	10·8	<1	
12·0	3	? Zr	12·05	12·1	1	12·4	1	—	—	
13·7	2	Fe	13·81	—	—	—	—	—	—	
15·7	9	*p* Sr	15·70	15·7	5	15·7	10	15·7	2	
17·2	3	—	—	17·6	1	17·0	<1	—	—	
19·5	3–4	Fe	19·52	19·6	1	19·4	2	—	—	
20·4	3	Fe	20·51	—	—	—	—	—	—	
22·4	5–6	Fe	22·38	22·4	1	22·4	3	22·2	<1	
24·2	3	Fe	24·34	24·7	1	—	—	—	—	
25·2	3	—	—			—	—	24·9	1	
26·9	8	Ca	26·90	27·0	5	26·9	7	27·2	1	
29·8	3	Fe Fe	29·68 29·93	—	—	29·4	<1	—	—	
—	—	—	—	—	—	—	—	30·7	1	
32·2	1–2	—	—	—	—	—	—	32·1	<1	
33·3	7–8	*p* Fe	33·33	33·6	3	33·3	6–7	33·3	8	
36·1	5–6	Fe	36·11	36·0	2	35·9	4	35·7	1	
—	—	—	—	—	—	—	—	37·6	<1	
39·0	5	Fe	38·97	—	—	38·0	1–2	39·2	<1	
40·1	3	Mn Fe	39·89 40·04	40·0	3	40·3	1	40·6	<1	
42·6	5	*p* Cr	42·62	42·5	1	42·8	2–3	42·6	3–4	
45·5	2	Fe	45·42	—	—	45·0	1–2	45·0	1	
47·0	7	Sc	47·00	47·3	4	47·0	7	47·2	3	
50·3	4	Fe	50·29	—	—	50·4	4–5	—	—	
50·9	4	Fe	50·95	51·0	2			51·0	1	
52·5	2–3	? Co	52·47	53·0	?	—	—	53·1	2	Possibly double.
54·5	4	Cr	54·51	54·5	2	54·5	6	54·5	1–2	
56·2	3	—	—	—	—	55·6	1–2	—	—	
58·4	6	Fe	58·48	58·7	2	58·2	2	58·6	3	
60·6	6	Fe	60·64	60·5	2	60·6	4	60·7	<1	
62·1	3	*p* Cr	—	62·2	1	61·6	1–2	62·2	3	
64·2	1	Fe	64·37	—	—	64·6	1–2	64·4	<1	
65·1	1	Fe	64·90	—	—	65·5	<1	—	—	
67·3	2	—	—	—	—	67·7	2–3	67·5	<1	
69·5	2–3	—	—	70·0	1	69·8	1	69·8	1–2	
71·2	6	Fe	71·33	71·7	4	71·6	4–5	71·7	1	
71·9	6	Fe	71·93							
73·5	3	Fe Zr	73·48 73·64	73·8	1	73·8	1	73·6	3	
75·1	4	Cr	74·96	75·0	3	75·0	5	75·0	<1	
75·6	4	—	—	75·0	3	—	—	75·8	2	
—	—	—	—	—	—	—	—	76·3	1	
77·6	2	—	—	—	—	—	—	—	—	Faint close double.
78·4	2	Fe	78·39	78·4	1	—	—	78·4	2	
80·4	2	—	—	80·5	1	80·2	1–2	—	—	Faint close double.
81·0	2	—	—	—	—	—	—	—	—	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
4282·8	5	{ Fe Ca	4282·57 83·17 }	4282·9	1	4283·0	2–3	4282·8	1	
84·4	2	—	—	84·4	1	—	—	84·4	2	
—	—	—	—	—	—	—	—	86·8	<1	
88·0	3–4	? Ti	88·04	88·1	2	87·6	1	88·3	2	
90·1	9	{ Cr p Ti	89·89 90·38	89·8	4	90·2	6–7	90·4	4	
92·2	1–2	—	—	—	—	—	—	92·4	1	
94·2	6	{ p Ti Fe	94·20 94·30 }	94·3	2	94·2	5	94·2	4	
96·7	5	p Fe	96·74	97·1	2	96·7	2–3	96·7	4	
99·4	4	Ti Fe	99·41	—	—	—	—	—	—	
4300·2	6	p Ti	4300·21	4300·2	5	4300·2	5	4300·2	5	
—	—	—	—	—	—	—	—	02·1	2	
03·3	6	p Fe	03·34	02·6	5	03·0	4	03·3	5	
05·8	5–6	—	—	05·8	1	05·8	1–2	06·0	1	
—	—	—	—	—	—	—	—	07·6	1	
08·1	5	{ Ca Fe p Ti	07·91 08·08 08·10 }	08·0	2	08·1	5	08·1	4	
09·6	5	Fe	09·54	09·5	2	—	—	09·7	1	
11·3	1	—	—	—	—	—	—	10·9	1	
13·0	5	p Ti	13·03	—	—	13·0	2	13·1	2–3	
14·3	7–8	Sc	14·25	14·3	2	14·0	2	—	—	
15·1	7–8	{ p Ti Fe	15·13 15·26 }	15·2	5	15·1	4–5	15·1	4	
17·0	3–4	p Ti	16·96	17·0	1	—	—	17·2	1	
—	—	—	—	17·6	1	—	—	—	—	
18·8	3–4	Ca	18·82	—	—	18·3	2	—	—	
—	—	—	—	—	—	—	—	19·9	1	
21·2	8	{ Sc p Ti	20·91 21·20 }	21·0	3	21·2	5	21·2	2–3	
23·4	1	—	—	—	—	—	—	—	—	
26·0	9	Fe	25·94	26·0	5	25·8	6	26·0	3–4	{ Apparently close double.
27·3	2	Fe	27·27	—	—	—	—	—	—	
30·6	7	p Ti	{ 30·50 30·87	30·9	2	30·6	2–3	30·7	2	
34·0	4–5	? La	33·93	—	—	33·9	3	—	—	
38·1	9	p Ti	38·08	37·6	10	38·1	5	38·1	4	
40·6	10	H	40·63	40·7	11	40·7	10	40·7	10	H_γ.
44·3	4–5	{ p Mn p Ti	44·19 44·45 }	44·7	1	44·3	3	44·3	2	
46·8	1	Fe	46·72	—	—	} 47·4	<1	—	—	
48·0	1	Fe	48·00	—	—			—	—	
—	—	—	—	—	—	—	—	49·1	1–2	
51·9	7	{ p Fe Cr Mg	51·93 52·08 }	52·0	5	51·9	6	51·9	7	
55·2	3–4	Ca	55·26	55·3	1	55·0	1–2	54·9	1	
58·7	3	Fe	58·67	—	—	} 59·2	3–4	{ 57·8	2	
—	—	p Y	58·88	—	—			—	—	
59·8	3	Cr	59·78	59·9	3			60·0	1	
62·3	1–2	p Ni	62·40	—	—	62·0	1	62·4	1–2	
64·6	1	—	—	—	—	64·1	1	64·0	1	
—	—	—	—	—	—	—	—	65·4	<1	
67·8	4	{ Fe p Ti	67·75 67·84 }	67·9	1	67·8	2–3	67·9	1–2	
—	—	—	—	—	—	68·7	?	—	—	
69·9	3	Fe	69·94	70·0	1	70·2	2–3	69·9	2	
71·7	2–3	—	—	71·5	1	—	—	71·7	1	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
4374·7	8	Sc p Ti	4374·63 74·90	— 4374·7	— 9	— 4374·9	— 7	— 4374·9	— 2	
76·1	2	Fe	76·11	76·1		—	—	—	—	
79·4	3	V	79·40	79·4	1	79·7	2	78·9	<1	
—	—	—	—	—	—	—	—	80·4	1	
83·7	4–5	Fe	83·72	83·7	8	83·7	5	83·7	2	
85·3	5–6	p Fe	85·55	85·2	4	85·5	3	85·5	5–6	
87·0	1	? Ti	87·01	—	—	—	—	—	—	
88·6	1	Fe	88·57	—	—	88·1	1–2	88·1	1	
91·2	4	Fe p Ti	91·12 91·19	90·5	2	91·2	2–3	91·0	2–3	
94·1	2	? Ti	94·22	—	—	—	—	93·6	1	
95·2	6	p Ti V	95·20 95·41	95·3	7	95·2	7	95·2	5	
98·2	3	? Yt	98·18	—	—	—	—	98·0	1	
4400·2	5–6	p Ti Sc	99·94 4400·55	4400·2	7	99·9	5–6	99·9	3	Probably close double.
01·0	2	—	—	—	—	—	—	—	—	
03·3	1	—	—	—	—	—	—	4402·8	1	
04·9	3-4	Fe	04·93	05·0	2	4404·9	4	04·9	1-2	
08·4	3	V Fe V	08·31 08·58 08·68	08·5	2	08·1	3	—	—	
09·3	4	? Fe	09·29	—	—	—	—	—	—	
11·3	3	p Ti	11·20	11·5	2	11·2	1–2	11·2	1	
13·6	1	—	—	—	—	—	—	13·5	1	
15·3	5-6	Fe	15·29	15·3	4	15·3	4	15·3	<1	
—	—	—	—	—	—	—	—	17·0	5	
17·9	6–7	p Ti	17·88	17·9	6	17·9	4–5	17·9	2–3	
—	—	—	—	—	—	—	—	19·5	<1	
20·7	1–2	—	—	—	—	—	—	—	—	
22·7	3-4	Fe Y	22·74	22·8	3	22·7	3	22·0	1	
25·6	1	Ca	25·61	—	—	25·6	1	—	—	
27·4	3–4	Ti Fe	27·27 27·48	27·4	2	27·4	3	—	—	
—	—	—	—	—	—	—	—	28·7	<1	
30·6	3	Fe	30·78	30·8	2	30·1	2–3	—	—	
33·4	1–2	Fe	33·39	—	—	—	—	—	—	
—	—	—	—	—	—	—	—	34·4	1	
35·5	4–5	Ca	35·13 35·85	35·2	2	35·5	4–5	—	—	Probably close double.
38·5	1	Fe	38·51	—	—	—	—	—	—	
41·8	1–2	V	41·88	—	—	41·8	1–2	41·8	1	
42·5	4–5	Fe	42·51	42·5	9	—	—	—	—	
44·0	9	p Ti	43·98	44·0		44·0	7	44·0	4–5	
47·5	3	Fe	47·30 47·80	47·7	1	47·0	1	47·8	1	
50·7	4	p Ti	50·65	50·6	3	50·6	5	50·6	2–3	
55·0	4–5	Ca p Fe	54·95 55·30	55·0	2	55·0	5	55·3	2	
59·3	2–3	Fe	59·30	60·0	1	59·9	1–2	—	—	
62·0	5	Fe	61·82 62·17	62·0	2	62·3	3	61·8	1–2	
64·8	3	p Ti	64·62	64·8	1	64·6	2–3	64·5	1	
66·7	2	Fe	66·73	—	—	66·5	<1	—	—	
68·7	6	p Ti	68·66	69·5	3	68·7	6	68·7	4	
70·7	3	? Ni Zr	70·65	71·0	2	—	—	—	—	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

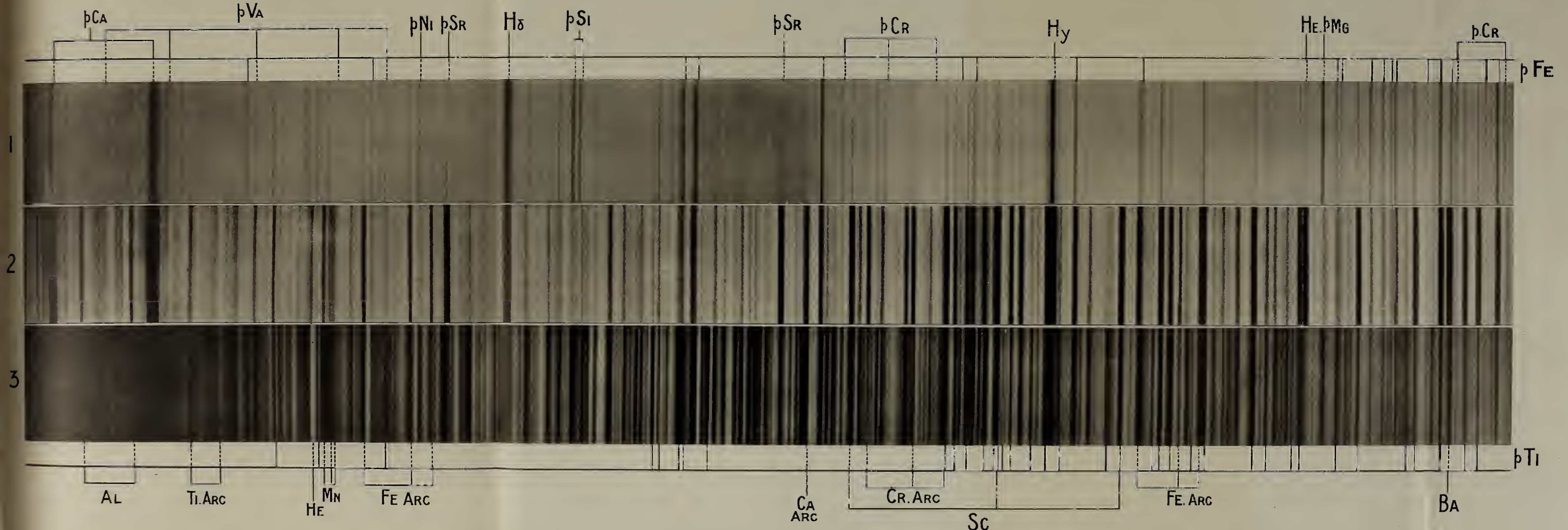
γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
—	—	—	—	—	—	—	—	4471 ·6	1–2	Probably He 4471 ·65.
4473 ·0	2–3	Fe	4472 ·88	4473 ·0	1	—	—	73 ·1	1–2	
76 ·2	3	Fe	76 ·19	76 ·2	1	4476 ·2	3	—	—	
80 ·3	2	Fe	80 ·31	—	—	80 ·6	1–2	—	—	
81 ·3	5–6	*p* Mg	81 ·30	82 ·0	4	—	—	81 ·3	8	
82 ·3	1–2	Fe	—	—	—	82 ·3	4	—	—	
—	—	—	—	—	—	—	—	84 ·0	<1	
85 ·2	1	—	—	—	—	—	—	86 ·6	<1	
88 ·6	4	*p* Ti	88 ·49	—	—	} 89 ·3	3			
89 ·4	4	*p* Fe	89 ·35	89 ·6	3			89 ·0	3	
91 ·6	3	*p* Fe	91 ·57	91 ·6	3	91 ·6	2–3	91 ·6	3–4	
94 ·7	3–4	Fe	94 ·74	94 ·8	2	94 ·3	2	—	—	
97 ·6	3–4	—	—	97 ·0	2	96 ·8	—	—	—	
99 ·6	1	—	—	—	—	—	—	—	—	
4501 ·5	6	*p* Ti	4501 ·45	4501 ·5	4	4501 ·5	7	4501 ·5	3–4	
07 ·0	1	—	—	—	—	—	—	—	—	
08 ·5	4	*p* Fe	08 ·46	08 ·5	3	08 ·5	5	08 ·5	5	
12 ·9	1	Ti	12 ·91	—	—	12 ·3	1	12 ·3	<1	
15 ·5	4	*p* Fe	15 ·51	15 ·4	2	15 ·5	4	15 ·5	5	
18 ·2	2	Ti	18 ·20	—	—	18 ·3	1	18 ·2	1	
20 ·4	3	*p* Fe	20 ·40	23 ·0	1	20 ·4	3	20 ·4	4	
22 ·8	4	*p* Fe	22 ·69	22 ·9	1	22 ·7	4	22 ·7	5	
25 ·3	2	Ti	27 ·49	—	—	—	—	25 ·5	<1	
28 ·8	4	Fe	28 ·80	28 ·8	2	28 ·8	3	—	—	
29 ·5	2	? Fe	29 ·85	—	—	—	—	29 ·6	1	
31 ·2	2	{ Co Fe	31 ·12 31 ·33 }	31 ·2	1	31 ·0	1	—	—	
—	—	—	—	—	—	—	—	32 ·2	<1	
34 ·1	6	*p* Ti	34 ·14	34 ·2	4	34 ·1	7–8	34 ·1	5	
36 ·0	2–3	Ti	{ 35 ·74 36 ·09 36 ·22 }	—	—	35 ·9	2	—	—	
—	—	—	—	—	—	—	—	38 ·8	<1	
40 ·1	3	—	—	—	—	40 ·0	1–2	—	—	
41 ·4	4	*p* Fe	41 ·40	41 ·6	2	41 ·7	3	41 ·4	3	
44 ·8	4	{ Cr Ti	44 ·79 44 ·86 }	44 ·9	2	44 ·8	3	45 ·0	<1	
48 ·0	1–2	Fe	48 ·02	—	—	—	—	47 ·2	<1	
49 ·7	8	{ *p* Fe *p* Ti	49 ·64 49 ·81 }	49 ·7	3	49 ·7	7–8	49 ·8	7	
52 ·6	2	? Ti	52 ·63	—	—	—	—	52 ·8	<1	
54 ·2	5–6	Ba	54 ·21	54 ·2 }	5	{ 54 ·2	7–8	—	—	
						—	—	55 ·3	2	
56 ·2	5–6	{ *p* Fe Fe	56 ·06 56 ·31 }	56 ·0 }		56 ·1	3–4	56 ·1	5	
58 ·8	3	*p* Cr	58 ·83	58 ·9	1	58 ·8	3–4	58 ·8	5	
61 ·6	1	—	—	—	—	61 ·3	<1	61 ·6	1	
63 ·9	4–5	*p* Ti	63 ·94	64 ·0	4	63 ·9	7–8	63 ·9	3	
65 ·8	3	{ Cr Co Fe	65 ·69 65 ·84 }	—	—	66 ·3	1	66 ·0	<1	
68 ·9	3	Fe	68 ·94	—	—	—	—	68 ·0	<1	
—	—	—	—	—	—	—	—	70 ·6	<1	
72 ·2	6–7	*p* Ti	72 ·16	72 ·2	3	72 ·2	7	72 ·2	4	
74 ·9	1	Fe	74 ·90	—	—	—	—	74 ·9	<1	
76 ·5	3	*p* Fe	76 ·51	76 ·5	1	76 ·5	3	76 ·5	3–4	
—	—	—	—	—	—	—	—	77 ·2	<1	
80 ·6	3–4	{ V Fe Ni	80 ·59 80 ·76 }	80 ·0	1	80 ·0	2	80 ·3	2	
—	—	—	—	—	—	—	—	83 ·0	2	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
4584·0	8	*p* Fe	4584·02	4584·0	5	4584·0	7	4584·0	7	
—	—	—	—	86·1	1	—	—	86·0	<1	
88·4	3	*p* Cr	88·38	—	—	88·4	3	88·4	4	
90·2	2–3	*p* Ti	90·13	—	—	90·1	3	90·2	1–2	
92·5	3–4	{ *p* Cr Fe	92·25 92·84 }	92·8	1	92·5	3	92·5	2–3	
94·1	1	? V	94·30	—	—	—	—	—	—	
95·6	1	Fe	95·54	95·9	2	95·1	2	—	—	
—	—	—	—	—	—	—	—	96·6	1-2	
98·1	1	? Fe	98·30	—	—	—	—	—	—	
4600·7	4	—	—	—	—	4600·8	3	—	—	
03·2	2	Fe	4603·13	—	—	03·0	2	—	—	
05·2	1	Ni	05·17	—	—	05·5	2	—	—	
13·9	2-3	—	—	4613·5	1	13·3	2	—	—	
16·9	3–4	*p* Cr	16·80	16·9	1	—	—	4616·8	2	? double.
19·2	4	{ Fe *p* Cr Fe	18·97 19·47 }	19·2	4	19·0	3–4	19·1	3	
20·2	3–4	—	—	—	—	—	—	21·1	2	
—	—	—	—	—	—	—	—	23·5	<1	
—	—	—	—	—	—	—	—	24·9	1	
26·2	2	Cr	26·36	25·8	1	—	—	26·6	1	
29·5	6	*p* Fe Ti Co	29·52	29·9	4	29·5	5–6	29·6	5–6	
32·8	1–2	—	—	—	—	32·8	1–2	32·6	1	
34·2	2–3	*p* Cr	34·25	34·8	1	34·3	1–2	34·3	3	
—	—	—	—	—	—	—	—	35·6	2	
—	—	—	—	—	—	—	—	38·9	1	
42·6	1	—	—	—	—	42·8	1	—	—	
46·3 to 49·0	} 4	{ Cr Fe Ni	46·35 47·62 48·84	46·3 — 48·9	1 — 1	46·3 — 48·4	4–5 — 2	— — —	— — —	Band probably consisting of the three well-marked solar lines whose λλ are given.
52·5	3	Cr	52·34	—	—	51·8	3–4	—	—	
55·3	2	—	—	—	—	55·4	2	—	—	
57·4	4–5	*p* Ti	57·38	57·0	4	57·4	3–4	57·4	2	
60·6	1	—	—	—	—	—	—	60·8	<1	
63·6	2-3	—	—	63·7	2	64·5	2	63·8	2	
—	—	—	—	—	—	—	—	66·5	<1	
67·4	8	{ ? Fe ? Ti	67·63 67·77 }	68·0	3	67·6	3–4	67·2	2–3	
70·4	7	Sc	70·4*	70·0	3	70·8	3–4	70·5	2	*THALÉN'S spark λ corrected to ROWLAND.
73·5	1	Fe	73·35	—	—	74·0	1	73·5	<1	
75·6	1–2	? Ti	75·29	—	—	76·0	1	—	—	
78·1	3	? Cd	78·35	79·0	1	—	—	—	—	
82·5	3–4	*p* Y	82·60	82·2	1	82·5	2–3	—	—	
86·0	1	—	—	—	—	—	—	—	—	
89·2	1	—	—	—	—	—	—	—	—	
91·6	1–2	{ Ti Fe	91·52 91·61	— 91·6	— 1	— 91·6	— 3	—	—	
99·6	4–5	—	—	98·8	2	99·5	2–3	—	—	
4703·4	4–5	Mg	4703·18	4703·1	2	4703·2	3–4	—	—	
08·6	5	—	—	09·0	3	{ 07·8 09·6	2–3 2–3 }	—	—	Broad and hazy.
15·0	3	—	—	14·5	1	—	—	—	—	
17·8	1–2	—	—	—	—	18·5	1	—	—	
19·6	3	—	—	20·5	1	—	—	—	—	
28·3	3	—	—	27·6	1	27·7	2	—	—	

Wave-lengths, Intensities, and Probable Origins of γ Cygni Lines, compared with those of δ Canis Majoris, the Chromosphere, and α Cygni—*continued.*

γ Cygni (Kensington).				δ Canis majoris (Harvard).		Chromosphere (Kensington).		α Cygni (Kensington).		Remarks.
λ.	Intensity. Max. =10.	Probable origin.	λ of probable origin.	λ.	Intensity. Max. =220.	λ.	Intensity. Max. =10.	λ.	Intensity. Max. =10.	
4731·3	4	—	—	4731·7	1	4731·4	3–4	4731·7	3·4	
34·1	2–3	? Fe	4733·78	—	—	33·8	1	—	—	
37·6	4	—	—	37·0	1	37·0	3	—	—	
40·9	1	—	—	—	—	40·5	2	—	—	
44·9	1	—	—	—	—	45·5	1	—	—	
48·6	1–2	—	—	—	—	48·0	1	—	—	
52·2	2	—	—	—	—	—	—	—	—	
—	—	—	—	54·2	1	—	—	—	—	
55·3	3	—	—	—	—	—	—	—	—	Broad, probably double.
64·2	7	Ti Ni	64·11	64·1	8	—	—	—	—	
67·8	1	—	—	—	—	67·0	2	—	—	
71·2	2	—	—	71·8	1	—	—	—	—	
80·2	3–4	*p* Ti	80·20	80·1	1	79·9	3–4	80·1	2	
83·1	2–3	—	—	—	—	83·1	2	—	—	
86·7	2–3	—	—	86·8	1	86·7	2–3	—	—	
98·7	2–3	—	—	98·7	2	98·7	2	—	—	
4805·2	3	*p* Ti	4805·25	4805·2	3	4805·2	5	4805·2	2	
11·0	3	—	—	—	—	11·0	3	—	—	
24·3	7	Fe *p* Cr	24·33	24·0	5	24·3	6	24·3	4	
40·4	2	—	—	—	—	40·4	2–3	—	—	
48·4	3–4	*p* Cr	48·44	48·4	3	48·5	3	48·4	3–4	
55·4	5	—	—	55·7	3	55·0	2–3	—	—	
61·5	8	H	61·49	—	—	61·5	10	61·5	10	H_β.



1. α CYGNI 2. CHROMOSPHERE 3. γ CYGNI

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OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 223–327.

QUATERNIONS AND PROJECTIVE GEOMETRY

BY

PROFESSOR CHARLES JASPER JOLY,

ROYAL ASTRONOMER OF IRELAND.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 338. 20.6.03

Price Five Shillings.

INDEX SLIP.

VIII. *Quaternions and Projective Geometry.*

By Professor Charles Jasper Joly, *Royal Astronomer of Ireland.*

Communicated by Sir Robert Ball, *F.R.S.*

Received November 27,—Read December 11, 1902.

Introduction.

A quaternion q adequately represents a point Q to which a determinate weight is attributed, and, conversely, when the point and its weight are given, the quaternion is defined without ambiguity. This is evident from the identity

$$q = \left(1 + \frac{Vq}{Sq}\right) Sq \qquad \text{(A)},$$

in which Sq is regarded as a weight placed at the extremity of the vector

$$OQ = \frac{Vq}{Sq} \qquad \text{(B)},$$

drawn from any assumed origin O. It is sometimes convenient to employ capitals Q concurrently with italics q to represent the same point, it being understood that

$$Q = \frac{q}{Sq} = 1 + OQ \qquad \text{(C)}.$$

Thus Q represents the point Q affected with a unit weight. The point O may be called the *scalar* point, for we have

$$O = 1 \qquad \text{(D)}.$$

In order to develop the method, it becomes necessary to employ certain special symbols. With one exception these are found in Art. 365 of 'Hamilton's Elements of Quaternions,' though in quite a different connection. We write

$$(a, b) = bSa - aSb, \quad [a, b] = V . VaVb \qquad \text{(E)};$$

and in particular for points of unit weight, these become

$$(A, B) = B - A, \quad [A, B] = V.VAVB = V.VA.(B - A) \qquad \text{(F)}.$$

Thus (ab) is the product of the weights $SaSb$ into the vector connecting the points, and $[ab]$ is the product of the weights into the moment of the vector connecting the points with respect to the scalar point. The two functions (ab) and $[ab]$ completely define the line ab.

Again HAMILTON writes

$$[a,b,c]=(a,b,c)-[b,c]Sa-[c,a]Sb-[a,b]Sc;\ (a,b,c)=S[a,b,c]=SVaVbVc\,. \quad (G);$$

or if we replace a, b, c by $(1+\alpha)Sa$, $(1+\beta)Sb$, $(1+\gamma)Sc$, where α, β and γ are the vectors from the scalar point to three points a, b and c, we have

$$[\text{A}, \text{B}, \text{C}] = S\alpha\beta\gamma - V(\beta\gamma + \gamma\alpha + \alpha\beta);\quad (\text{A}, \text{B}, \text{C}) = S\alpha\beta\gamma \quad . \quad . \quad . \quad (H).$$

Hence it appears that $[a, b, c]$ is the symbol of the plane a, b, c; for $-V[a, b, c](a, b, c)^{-1}$ is the reciprocal of the vector perpendicular from the scalar point on that plane. Also (A, B, C) is the sextupled volume of the tetrahedron OABC.

Again, HAMILTON writes for four quaternions

$$(abcd) = S\,.\,a[bcd] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I);$$

and in terms of the vectors this is seen to be the products of the weights into the sextupled volume of the pyramid (ABCD).

Other notations may of course be employed for these five combinatorial functions of two, three, or four quaternions or points, but HAMILTON's use of the brackets seems to be quite satisfactory.

In the same article HAMILTON gives two most useful identities connecting any five quaternions. These are

$$a(bcde) + b(cdea) + c(deab) + d(eabc) + e(abcd) = 0\,. \quad . \quad . \quad . \quad (J),$$

and

$$e(abcd) = [bcd]Sae - [acd]Sbe + [abd]Sce - [abc]Sde \quad . \quad . \quad . \quad (K),$$

which enable us to express any point in terms of any four given points, or in terms of any four given planes.

The equation of a plane may be written in the form

$$Slq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (L);$$

and thus l, any quaternion whatever, may be regarded as the symbol of a plane as well as of a point.

On the whole, it seems most convenient to take as the auxiliary quadric the sphere of unit radius

$$S\,.\,q^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (M),$$

whose centre is the scalar point. With this convention the plane $Slq = 0$ is the polar of the point l with respect to the auxiliary quadric; or the plane is the reciprocal of the point l. Thus the principle of duality occupies a prominent position.

The formulæ of reciprocation

$$([abc];[abd]) = [ab](abcd);\ [[abc];[abd]] = -(ab)(abcd)\,. \quad . \quad . \quad (N)$$

connecting any four quaternions are worthy of notice, and are easily proved by replacing the quaternions by $1+\alpha$, $1+\beta$, $1+\gamma$, and $1+\delta$ respectively. In complicated relations it may be safer to separate the quaternions as in these formulæ by semi-colons, but generally the commas or semi-colons may be omitted without causing any ambiguity.

These new interpretations are not in the least inconsistent with any principle of the calculus of quaternions. We are still at liberty to regard a quaternion as the separable sum of a vector and a scalar, or as the ratio or product of two vectors, or as an operator, as well as a symbol of a point or of a plane.

In particular, in addition to HAMILTON's definition of a vector as a right line of given direction and of given magnitude, and in addition to his subsequent interpretations of a vector as the ratio or product of two mutually rectangular vectors, or as a versor, we may now consider a vector as denoting the point at infinity in its direction, or the plane through the centre of reciprocation. For the vector OQ of equation (B) becomes infinitely long if $Sq = 0$, and the plane $Slq = 0$ passes through the scalar point if $Sl = 0$. We may also observe that the difference of two unit points A − B is the vector from one point B to the other A, and this again is in agreement with the opening sections of the "Lectures."

Additional illustrations and examples may be found in a paper on "The Interpretation of a Quaternion as a Point-symbol," 'Trans. Roy. Irish Acad.,' vol. 32, pp. 1–16.

The only other symbols peculiar to this method are the symbols for quaternion arrays. The five functions (ab), $[ab]$, $[abc]$, (abc), and $(abcd)$ are particular cases of arrays, being, in fact, arrays of one row. In general the array of m rows and n columns

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & \ldots & a_n \\ b_1 & b_2 & b_3 & \ldots & b_n \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ p_1 & p_2 & p_3 & \ldots & p_n \end{matrix}\right\} \quad \ldots \ldots \ldots \quad (O)$$

may be defined as a function of mn quaternion constituents, which vanishes if, and only if, the groups of the constituents composing the rows were connected by linear relations with the same set of scalar multipliers. In other words, the array vanishes if scalars $t_1, t_2 \ldots t_n$ can be found to satisfy the m equations

$$\begin{aligned} t_1a_1 + t_2a_2 + \ldots + t_na_n &= 0, \\ t_1b_1 + t_2b_2 + \ldots + t_nb_n &= 0, \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ t_1p_1 + t_2p_2 + \ldots + t_np_n &= 0. \end{aligned}$$

The expansion of arrays is considered in a paper on "Quaternion Arrays," 'Trans. Roy. Irish Acad.,' vol. 32, pp. 17–30.

SECTION I.

Fundamental Geometrical Properties of a Linear Quaternion Function.

1. The quaternion equation

$$f(p+q)=fp+fq \quad \ldots \ldots \quad (1),$$

may be regarded as a definition of the nature of a linear quaternion function f, the quaternions p and q being perfectly arbitrary. As a corollary, if x is any scalar,

$$f(xp)=xfp \quad \ldots \ldots \quad (2),$$

and on resolving fq in terms of any four arbitrary quaternions a_1, a_2, a_3, a_4, we must have an expression of the form

$$fq = a_1Sb_1q + a_2Sb_2q + a_3Sb_3q + a_4Sb_4q \quad \ldots \ldots \quad (3),$$

because the coefficients of the four quaternions a must be scalar and distributive functions of q. Sixteen constants enter into the composition of the function f, being four for each of the quaternions b.

2. When a quaternion is regarded as the symbol of a point, the operation of the function f produces a linear transformation of the most general kind.

The equations

$$f(xa+yb)=xfa+yfb\,; \quad f(xa+yb+zc)=xfa+yfb+zfc \quad \ldots \quad (4),$$

show that the right line a, b is converted into the right line fa, fb, and the plane containing three points a, b, c into the plane containing their correspondents, fa, fb and fc.

The homographic character of the transformation is also clearly exhibited.

3. In order to specify a function of this kind it is necessary to know the quaternions a', b', c', d' into which any set of four unconnected quaternions, a, b, c, d, are converted. Thus, from the identical relation

$$q(abcd) + a(bcdq) + b(cdqa) + c(dqab) + d(qabc) = 0 \quad . \quad . \quad . \quad (5),$$

connecting one arbitrary quaternion with the four given quaternions, is deduced the equation

$$fq(abcd) + a'(bcdq) + b'(cdqa) + c'(dqab) + d'(qabc) = 0 \quad . \quad . \quad . \quad (6),$$

which determines the result of operating by f on q.

When we are merely concerned with the geometrical transformation of points, the absolute magnitudes* of the representative quaternions cease to be of importance, and the function

$$fq = x\text{A}'(\text{BCD}q) + y\text{B}'(\text{CD}q\text{A}) + z\text{C}'(\text{D}q\text{AB}) + w\text{D}'(q\text{ABC}) \quad . \quad . \quad . \quad . \quad (7),$$

which involves four arbitrary scalars, converts the four points A, B, C, D into four others, A$'$, B$'$, C$'$, $'$D. Given a fifth point E and its correspondent E$'$, the four scalars are determinate to a common factor, and subject to a scalar multiplier, the function which produces the transformation is

$$fq = \text{A}'(\text{BCD}q) \cdot \frac{(\text{B}'\text{C}'\text{D}'\text{E}')}{(\text{BCDE})} + \text{B}'(\text{CD}q\text{A}) \cdot \frac{(\text{C}'\text{D}'\text{E}'\text{A}')}{(\text{CDEA})} + \text{C}'(\text{D}q\text{AB}) \cdot \frac{(\text{D}'\text{E}'\text{A}'\text{B}')}{(\text{DEAB})}$$
$$+ \text{D}'(q\text{ABC}) \cdot \frac{(\text{E}'\text{A}'\text{B}'\text{C}')}{(\text{EABC})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

It is only necessary to replace q by E in order to verify this result.

4. *A linear quaternion function, f, being regarded as effecting a transformation of points, the inverse of its conjugate f'^{-1} produces the corresponding tangential transformation.*

For any two quaternions, p and q,

$$\text{S}pq = \text{S}pf^{-1}q' = \text{S}f'^{-1}pq' = \text{S}p_{,}q' \text{ if } q' = fq, \; p_{,} = f'^{-1}p \quad . \quad . \quad . \quad (9).$$

Hence any plane $\text{S}pq = 0$, in which the quaternion q represents the current point, transforms into the plane $\text{S}p_{,}q' = 0$, and the proposition is proved.

Thus, when symbols of points (q) are transformed by the operation of f, symbols of planes (p), or of points reciprocal to the planes, are transformed by the operation of f'^{-1}.

5. HAMILTON's beautiful method of inversion of a linear quaternion function receives a geometrical interpretation from the results of the last article.

* In accordance with the notation proposed ('Trans. Roy. Irish Acad.,' vol. 32, p. 2), capital letters are used in this article concurrently with small letters to denote the same points, but the weights for the capital symbols are unity; thus $q = \text{Q}\text{S}q = (1 + \text{OQ})\,\text{S}q$.

The symbol of the plane containing three points a, b, c, may be written in the form

$$p = [a, b, c] \quad \ldots\ldots\ldots\ldots \quad (10);$$

and on transformation this becomes

$$np_{\prime} = [fa, fb, fc] = F'[abc] = F'p = F'f'p_{\prime} \quad \ldots\ldots \quad (11),$$

where n is a certain scalar and where F' is an auxiliary function.

In fact, the first equation sums up the last article; in the second a new function F' is introduced, and in the fourth equation (9) is utilized.

Since $p_{\prime}$ is quite arbitrary (11) may be replaced by the symbolical equations

$$n = F'f'\,;\; F' = nf'^{-1}\,;\; f' = n^{-1}F'^{-1}\,;\; n = f'F' \quad \ldots\ldots \quad (12),$$

an arbitrary quaternion being understood as the subject of the operations.

Moreover, because

$$n\mathrm{S}pq = \mathrm{S}pF'f'q = \mathrm{S}Fpf'q = \mathrm{S}qfFp \quad \ldots\ldots \quad (13),$$

where p and q are arbitrary quaternions and where F is the conjugate of F', it appears that

$$n = fF\,;\; F = nf^{-1}\,;\; f = n^{-1}F^{-1}\,;\; n = Ff \quad \ldots\ldots \quad (14).$$

And for any three arbitrary quaternions

$$F[abc] = [f'af'bf'c] \quad \ldots\ldots\ldots\ldots \quad (15)$$

as appears from symmetry, or, anew geometrically, by considering a point as the intersection of three planes.

Operating on the last equation by $\mathrm{S}f'd$ we find, since $n = fF = f'F'$,

$$n\,(abcd) = (f'af'bf'cf'd) = (fafbfcfd) \quad \ldots\ldots \quad (16).$$

The fact that $(abcd)$ is a combinatorial function of a, b, c and d proves that n is an invariant, or that it is quite independent of any particular set of quaternions, a, b, c, d. This invariance is, however, established by the form of the equations (12) and (14).

6. Replacing f by $f_t = f - t$, where t is an arbitrary scalar, HAMILTON denotes by F_t and n_t the auxiliary function and the invariant which bear the same relations to f_t that F and n bear to f.

By (15) and (16), F_t and n_t are of the forms

$$F_t = F - tG + t^2H - t^3 \quad \ldots\ldots\ldots\ldots \quad (17),$$

where G and H are new auxiliary functions; and

$$n_t = n - tn' + t^2n'' - t^3n''' + t^4 \quad \ldots\ldots\ldots \quad (18),$$

where n', n'' and n''' are new invariants.

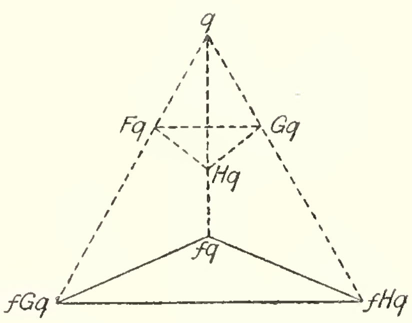
He then equates the coefficients of the arbitrary scalar t in the symbolical equation

$$n_t = f_t F_t = F_t f_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

and obtains the symbolical equations

$$n = Ff,\ n' = F + Gf,\ n'' = G + Hf,\ n''' = H + f \quad . \quad . \quad . \quad (20),$$

which will be found to be of great importance in the geometrical theory.



In virtue of (19), all these functions are commutative, in order of operation.

These equations establish certain collineations which are illustrated in the annexed figure.

From the relations (20) HAMILTON deduces

$$H = n''' - f;\ G = n'' - n'''f + f^2;\ F = n' - n''f + n'''f^2 - f^3 \quad . \quad . \quad (21),$$

and the symbolic quartic satisfied by f

$$f^4 - n'''f^3 + n''f^2 - n'f + n = 0 \text{ or } (f - t_1)(f - t_2)(f - t_3)(f - t_4) = 0 \quad . \quad (22),$$

if t_1, t_2, t_3, t_4 are the roots of the quartic

$$t^4 - n'''t^3 + n''t^2 - n't + n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

or the *latent* roots of the function f.

It appears by (12) and (14) that exactly similar equations are valid for the conjugate function f', it being only necessary to replace F, G and H by their conjugates F', G' and H', as the invariants n, n', n'' and n''' are the same in both cases.

7. The united points of the transformation are represented by the quaternions q_1, q_2, q_3 and q_4, which satisfy the equations

$$fq_1 = t_1q_1\,;\ fq_2 = t_2q_2\,;\ fq_3 = t_3q_3\,;\ fq_4 = t_4q_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24);$$

and they are determined by operating on an arbitrary quaternion by the function obtained by omitting one factor of the second form of (22). In like manner by omitting two or three factors of the same quartic, the equations of the lines joining two, and of the planes through three, of the united points are obtained by operating on a variable quaternion. Thus

$$q = (f - t_3)(f - t_4)r \quad \text{and} \quad q = (f - t_4)r \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

are respectively the equation of the line through the points q_1 q_2 and of the plane through the points q_1, q_2, q_3. These results are obvious when the arbitrarily variable point is referred to the united points as points of reference, or when we write

$$r = x_1q_1 + x_2q_2 + x_3q_3 + x_4q_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

8. *The united points of a function and of its conjugate form reciprocal tetrahedra with respect to the unit sphere* $Sq^2 = 0$.

For when the roots are all unequal

$$t_1 Sq'_2 q_1 = Sq'_2 f q_1 = Sq_1 f' q'_2 = t_2 Sq_1 q'_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (27),$$

if q'_1 q'_2, q'_3 and q'_4 are the united points of the conjugate. Thus the points q_1 and q'_2 are conjugate with respect to the sphere.

Since the plane $Sq'_1 q = 0$ contains the points q_2, q_3, q_4, the weights may be chosen so that

$$q'_1 = \frac{[q_2 q_3 q_4]}{(q_1 q_2 q_3 q_4)},\ q'_2 = \frac{[q_3 q_4 q_1]}{(q_2 q_3 q_4 q_1)},\ q'_3 = \frac{[q_4 q_1 q_2]}{(q_3 q_4 q_1 q_2)},\ q'_4 = \frac{[q_1 q_2 q_3]}{(q_4 q_1 q_2 q_3)} \quad . \quad . \quad (28);$$

and these relations imply*

$$Sq_1 q'_1 = Sq_2 q'_2 = Sq_3 q'_3 = Sq_4 q'_4 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (29);$$

and from symmetry

$$q_1 = \frac{[q'_2 q'_3 q'_4]}{(q'_1 q'_2 q'_3 q'_4)},\ q_2 = \frac{[q'_3 q'_4 q'_1]}{(q'_2 q'_3 q'_4 q'_1)},\ q_3 = \frac{[q'_4 q'_1 q'_2]}{(q'_3 q'_4 q'_1 q'_2)},\ q_4 = \frac{[q'_1 q'_2 q'_3]}{(q'_4 q'_1 q'_2 q'_3)}. \quad (30).$$

To these relations may be added the quaternion identities

$$q_1 q'_1 + q_2 q'_2 + q_3 q'_3 + q_4 q'_4 = 4 = q'_1 q_1 + q'_2 q_2 + q'_3 q_3 + q'_4 q_4 \quad . \quad . \quad . \quad . \quad (31),$$

$$q_1 Sq'_1 + q_2 Sq'_2 + q_3 Sq'_3 + q_4 Sq'_4 = 1 = q'_1 Sq_1 + q'_2 Sq_2 + q'_3 Sq_3 + q'_4 Sq_4 \quad . \quad (32),$$

which are probably more elegant than important. The second shows that the centre of the sphere is the centre of mass of the weights $Sq_1Sq'_1$, $Sq_2Sq'_2$, $Sq_3Sq'_3$, $Sq_4Sq'_4$ placed at the vertices of either of the tetrahedra, and that the sum of their weights is unity.

From these identities we deduce the vector equations

$$(q_1 q'_1) + (q_2 q'_2) + (q_3 q'_3) + (q_4 q'_4) = 0 = [q_1 q'_1] + [q_2 q'_2] + [q_3 q'_3] + [q_4 q'_4] \quad (33),$$

which express that equilibrating forces can be placed along the lines joining corresponding vertices, or that any line which meets three of these lines meets the fourth, or that the lines are generators of a quadric.†

* Writing $q_1 = w_1 (1 + \alpha_1)$, $q'_1 = w'_1 (1 + \alpha'_1)$, equations (29) give $w_1 w'_1 (1 + S\alpha_1 \alpha'_1) = 1$. Hence the product of the weights $w_1 w'_1$ is the reciprocal of the product of the perpendiculars from the centre of the sphere and from the point q_1 (or q'_1) on the opposite face of the tetrahedron $q_1 q_2 q_3 q_4$ (or $q'_1 q'_2 q'_3 q'_4$). Observe that only the products $w_1 w'_1$ have been assigned, not w_1 and w'_1 separately.

† In the notation of the last note (33) becomes $\Sigma w_1 w'_1 (\alpha'_1 - \alpha_1) = \Sigma w_1 w'_1 V\alpha_1 \alpha'_1 = 0$. The equilibrating forces are proportional to the distances between the vertices divided by the products of perpendiculars mentioned in the note cited.

It is also possible to obtain relations connecting pairs of the points ((N), p. 224),

$$[q_1q_2] = +\frac{(q'_3q'_4)}{(q'_1q'_2q'_3q'_4)};\ (q_1q_2) = -\frac{[q'_3q'_4]}{(q'_1q'_2q'_3q'_4)};\ [q'_1q'_2] = +\frac{(q_3q_4)}{(q_1q_2q_3q_4)};$$

$$(q'_1q'_2) = -\frac{[q_3q_4]}{(q_1q_2q_3q_4)} \quad \ldots \quad (34),$$

from which we learn that

$$-1 = (q_1q_2q_3q_4)(q'_1q'_2q'_3q'_4) \quad \ldots \quad (35);$$

and we are at liberty to write separately on further selection of the weights (for the products of the weights $Sq_1Sq'_1$ alone have been assigned),

$$(q_1q_2q_3q_4) = (q'_1q'_2q'_3q'_4) = \sqrt{-1} \quad \ldots \quad (36),$$

with corresponding simplifications in the formulæ.

When the function is self-conjugate, the tetrahedron of united points is self-reciprocal to the unit sphere.

9. Introducing two new linear functions defined by the equations

$$f = f_0 + f_i,\ f' = f_0 - f_i, \text{ or } 2f_0 = f + f',\ 2f_i = f - f' \quad \ldots \quad (37),$$

it is obvious that for any two quaternions, p and q,

$$Spf_0q = Sqf_0p;\ Spf_iq = -Sqf_ip \quad \ldots \quad (38),$$

or symbolically

$$f_0 = f_0',\quad f_i = -f_i' \quad \ldots \quad (39),$$

and f_0 is self-conjugate, and f_i is the negative of its conjugate.

10. The equation

$$Sqf_0q = 0 \quad \ldots \quad (40)$$

is the general equation of a quadric surface, and

$$Sqf_ip = 0 \quad \ldots \quad (41)$$

is that of a linear complex, p and q being both variable points.

In fact (40) is the most general scalar quadratic function homogeneous in q, and the surface represented meets the arbitrary line $q = a + tb$ in the points determined by the roots of the quadratic

$$Saf_0a + 2tSaf_0b + t^2Sbf_0b = 0 \quad \ldots \quad (42).$$

In like manner (41) is the most general scalar function linear in two quaternions and combinatorial with respect to both, for by (38)

$$Sqf_iq = 0 \quad \ldots \quad (43)$$

whatever quaternion q may be. It is therefore immaterial if we replace q and p in (41) by any other points on their line, provided the two points are not coincident, and the equation therefore imposes a single linear restriction on the line pq, and represents a linear complex.

In terms of vectors, putting $q = 1 + \rho$, $p = 1 + \varpi$, and using the expression given in the 'Elements' (Art. 364, XII.) for a linear quaternion function, we have

$$fq = e + \epsilon + S\epsilon'\rho + \phi\rho,\ f'q = e + \epsilon' + S\epsilon\rho + \phi'\rho;$$
$$f_0q = e_0 + \epsilon_0 + S\epsilon_0\rho + \phi_0\rho,\ f_,q = \epsilon_, - S\epsilon_,\rho + V\eta\rho \quad . \quad . \quad . \quad (44);$$

where

$$e_0 = e,\ 2\epsilon_0 = \epsilon + \epsilon',\ 2\epsilon_, = \epsilon - \epsilon';\ \phi = \phi_0 + V\eta,\ \phi' = \phi_0 - V\eta,$$

and the equations of the quadric and linear complex assume well-known forms

$$e_0 + 2S\epsilon_0\rho + S\rho\phi_0\rho = 0,\ S\epsilon_,(\varpi - \rho) + S\eta V\rho\varpi = 0 \quad . \quad . \quad . \quad (45).$$

11. The equation of the polar plane of a point a with respect to the quadric (compare (42)) is

$$Sqf_0a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

and f_0a is the pole of this plane with respect to the unit sphere.

Thus f_0a is the symbol of the polar plane of the point a.

With respect to the quadric the pole of the plane

$$Sqb = 0 \text{ is } p = f_0^{-1}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

and the reciprocal of the quadric has for its equation

$$Sqf_0^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

The lines of the complex through a given point a lie in the plane

$$Sqf_,a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49),$$

while the point of concourse of the lines in the plane

$$Sqb = 0 \text{ is } p = f_,^{-1}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

and

$$Spf_,^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

is the equation of the reciprocal of the complex.

12. The nature of the united points of the function $f_,$ is easily ascertained.

Since the function is the negative of its conjugate, its symbolic quartic (22) must be of the form

$$f_,^4 + n_,''f_,^2 + n_, = 0, \text{ or } (f_,^2 - s_1^2)(f_,^2 - s_2^2) = 0 \quad . \quad . \quad . \quad . \quad (52).$$

And if

$$f_,p_1 = s_1p_1,\ f_,p'_1 = -s_1p'_1,\ f_,p_2 = s_2p_2,\ f_,p'_2 = -s_2p'_2 \quad . \quad . \quad . \quad (53),$$

it follows in the first place (43) that the united points all lie on the unit sphere, and in the second by (27) that

$$Sp_1p_2 = Sp_1p'_2 = Sp'_1p_2 = Sp'_1p'_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54).$$

Hence in this order $p_1p_2p'_1p'_2$ is a quadrilateral situated on the unit sphere.

These results may be verified for the vector form (44). Actually solving

$$f_{,}(1 + \varpi) = s(1 + \varpi) = \epsilon_{,} - S\epsilon_{,}\varpi + V\eta\varpi,$$

we see that $s = - S\epsilon_{,}\varpi$, $sS\eta\varpi = S\eta\epsilon_{,}$, and therefore

$$(s - \eta)\varpi = \epsilon_{,} - s^{-1}S\eta\epsilon_{,}, \text{ or } (s^2 - \eta^2)\varpi = (s + \eta)(\epsilon_{,} - s^{-1}S\eta\epsilon_{,}),$$

so that operating by $S\epsilon_{,}$ the result is the quartic in s

$$s^4 + s^2(\epsilon_{,}^2 - \eta^2) - (S\eta\epsilon_{,})^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55);$$

and for a real function two roots of this quartic are always real and two are imaginary. Two of the united points are consequently real (Art. 7) and two are imaginary.

SECTION II.

The Classification of Linear Quaternion Functions.

13. Linear quaternion functions may be classified according to the nature of the united points :—

I. The first class consists of those functions which have no line or plane locus of united points, and it is divisible into sub-class :—

I_1, the four united points distinct.

I_2, two united points coincident.

I_3, three united points coincident.

I_4, all four coincident.

I_5, two distinct pairs of coincident united points.

II. The second class consists of functions having a line locus of united points, with the following sub-classes :—

II_1, the two remaining united points distinct.

II_2, the two remaining united points coincident.

II_3, one of the remaining united points on the line locus.

II_4, the two remaining points coincident and on the line locus.

III. The third class consists of functions having a plane locus of united points, and there are two sub-classes :—

III_1, the remaining united point is not in the plane.

III_2, the remaining united point is in the plane.

IV. The functions of the fourth class have two line loci of united points.

It is to be noticed that any peculiarity in a function is exactly reproduced in its conjugate. This will appear clearly from the following discussion, but the proposition is virtually proved in the concluding remarks of Art. 6.

To assist in the examination of the different cases, it is convenient to repeat HAMILTON'S relations (20) and (21), and in addition to obtain the symbolic quartics for the function H, G, and F. These quartics are deducible from the relations (20) or (21) without much trouble. The group of formulæ is thus :—

$$Ff = n, \quad F + Gf = n', \quad G + Hf = n'', \quad H + f = n''' ;$$

$$H = n''' - f, \quad G = n'' - n'''f + f^2, \quad F = n' - n''f + n'''f^2 - f^3 ;$$

$$f^4 - n'''f^3 + n''f^2 - n'f + n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

$$H^4 - 3n'''H^3 + (n'' + 3n'''^2) H^2 + (n' - 2n''n''' - n'''^3) H + n - n'n''' + n''n'''^2 = 0.$$

$$G^4 - 2n''G^3 + (2n + n'n''' + n''^2) G^2 - (2nn'' - nn'''^2 + n'^2 + n'n''n''') G + n^2 - nn'n''' + n'^2n'' = 0.$$

$$F^4 - n'F^3 + nn''F^2 - n^2n'''F + n^3 = 0.$$

14. For the sake of brevity in discussing the various classes, one root of the scalar quartic is supposed to be reduced to zero by replacing the function by one of the four functions $f - t_1$, $f - t_2$, $f - t_3$, $f - t_4$ of Art. 6 ; and whenever there is a multiple root, it is the multiple root which is reduced to zero.

I. *One quaternion, "a," is reduced to zero by the operation of the function.*

Remembering that the conjugate also reduces a quaternion a' to zero, it follows if

$$fa = 0, f'a' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

that the locus of the transformed points, $p = fq$, is a fixed plane,

$$Spa' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58),$$

because $Sqf'a' = 0$. Every plane through the point a is reduced to a line; every line through the point becomes a point; the scalar n is zero; the function F reduces every point to a and destroys every point in the fixed plane (58).

The quadrinomial (3) must reduce to a trinomial, for f cannot destroy a quaternion unless there is a relation between a_1, a_2, a_3, a_4, or else between b_1, b_2, b_3, b_4. The type of functions of this kind is

$$fq = a_1 Sa'_1 q + a_2 Sa'_2 q + a_3 Sa'_3 q\,;\ a = [a'_1 a'_2 a'_3],\ a' = [a_1 a_2 a_3] \quad . \quad . \ (59).$$

II. *The function destroys two distinct points.*

If

$$fa = 0, fb = 0\,;\ f'a' = 0, f'b' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

the line a, b is destroyed. The locus of the transformed points is the line of intersection of the planes

$$Spa' = 0,\ Spb' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

Every plane and every line through the line a, b is reduced to a point. The function is reducible to the binomial type

$$fq = a_1 Sa'_1 q + a_2 Sa'_2 q\,;\ a + tb = [a'_1 a'_2 r'],\ a' + t'b' = [a_1 a_2 r] \quad . \quad . \quad . \quad . \quad (62),$$

when r and r' are quite arbitrary, and it is evident (15) that the function F vanishes identically.

III. *The function destroys three non-collinear points.*

$$fa = 0, fb = 0, fc = 0\,;\ f'a' = 0, f'b' = 0, f'c' = 0 \quad . \quad . \quad . \quad . \quad (63)\,;$$

and every point is reduced to a fixed point, the intersection of the planes

$$Spa' = 0,\ Spb' = 0,\ Spc' = 0,\ \text{or}\ p = [a'b'c'] \quad . \quad . \quad . \quad . \quad (64).$$

Hence the function is a monomial,

$$fq = [a'b'c']\, S\, [abc] q = a_1 Sa'_1 q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65),$$

and the function G vanishes identically.

IV. *The function destroys two distinct points, a and b, and alters the weights of two others, c and d, in the same ratio, but otherwise leaves these points unchanged.*

The type is

$$fq \,.\, (abcd) = t_0 c(abqd) + t_0 d(abcq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

15. In order to illustrate the nature of the solution of the equation

$$fq = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

in the different cases, we employ HAMILTON'S relations (56), which give the solution on substitution.

I_1. *One latent root is zero.* In this case

$$n = 0, \; Fp = 0 \; ; \quad n'q = Gp + Fq = Gp + xa \quad . \quad . \quad . \quad . \quad . \quad (68),$$

because F reduces every quaternion to the fixed quaternion a multiplied by a scalar x. Here x is arbitrary, provided the condition $Fp = 0$ is satisfied; the point Gp lies in the fixed plane (58); and q may be any point on the line joining this point to a, or in other words, this line is the solution of the equation (67).

If the condition $Fp = 0$ is not satisfied, the scalar x must be infinite, so that in the limit $f(Gp + xa)$ may have a component at the point a, which escapes destruction by F. The solution is simply the point a affected with an infinite weight.

When $n = 0$, it appears from HAMILTON's relations that F satisfies the depressed equation

$$F(F - n') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69),$$

and the interpretation is, F reduces an arbitrary quaternion to a; $F - n'$ destroys a.

I_2. *Two latent roots are zero.* Here

$$n = n' = 0, \quad Fp = 0 \; ; \quad Gp + Fq = 0 \; ; \quad n''q = Hp + Gq \quad . \quad . \quad (70),$$

and q must be allowed the full extent of arbitrariness consistent with the conditions.

Observing that the relations (56) now give

$$f^2G = 0, \quad Gf^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

it appears that the double operation of f destroys the result of operating on any quaternion by G, and that G destroys f^2q. Hence,

$$Gq = xa' + ya, \quad \text{where} \quad fa' = a, f^2a' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (72).$$

The scalar x is determinate for

$$fGq = Gp = xa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

but y is arbitrary, and the solution is any point on the line, y variable,

$$n''q = Hp + xa' + ya \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

As before, if Fp is not zero, the solution is a multiplied by an infinite scalar.

The character of the function G has now completely changed. It now destroys a line (f^2q), and because $Gf^2 = 0$, or $Gf^2H^2 = 0$, and also $n' = 0$, the symbolic equations of G and F are both degraded, and are

$$G(G - n'')^2 = 0, \quad F^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

I_3. The solution in this case is

$$n = n' = n'' = 0 \; ; \quad Fp = 0, \quad Gp + Fq = 0, \quad Hp + Gq = 0 \; ; \quad n'''q = p + Hq \quad . \quad (76).$$

The symbolic equations now give

$$F^2 = 0, \quad G^3 = 0, \quad Hf^3 = 0, \quad G = -Hf, \quad F = Hf^2 \quad . \quad . \quad . \quad (77).$$

and

$$Hq = xa'' + ya' + za \quad \text{where} \quad fa'' = a', \quad fa' = a, \quad fa = 0 \quad . \quad . \quad (78).$$

The solution thus takes the more explicit form,

$$n'''q = p + xa'' + ya' + za\,; \quad Hp = xa' + ya\,; \quad Gp = -xa, \quad Fp = 0\,. \quad (79),$$

and z alone is arbitrary.

If the last condition is not fulfilled, z is infinite.

I_4. Again, where $n''' = 0$, the solution is any point on the line, w variable,

$$q = xa''' + ya'' + za' + wa\,; \; p = xa'' + ya' + za\,; \; fp = ya' + ya\,; \; f^2p = xa\,; \; f^3p = 0\,. \quad (80).$$

The symbolical equations satisfied by F, G, H and f are now

$$F^2 = 0, \quad G^2 = 0, \quad H^4 = 0, \quad f^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

Although the forms of the equations for F and G are identical, the nature of these functions are widely different; G reduces an arbitrary point to the line $xa' + ya$, which is destroyed by a further application of the same function; F reduces an arbitrary point at once to the point wa, which is destroyed by a successive operation.

The type of a function of this class I_4 is

$$fq\,(aa'a''a''') = a\,(aqa''a''') + a'\,(aa'qa''') + a''\,(aa'a''q) \quad . \quad . \quad . \quad (82),$$

in which a, a', a'' and a''' are arbitrary quaternions.

The function,

$$f(q)\,.\,(aa'bb') = a\,(aqbb') + t_0b\,(aa'qb') + (b + t_0b')\,(aa'bq)\,. \quad . \quad . \quad (83)$$

belongs to the sub-class I_5.

16. II_1. A function of the second class destroys two points, a and b, and in virtue of the distributive property it destroys the line a, b.

Since the locus of fq is a line (61), the function F vanishes identically (15), and likewise the invariant n' as well as n.

HAMILTON's relations become,

$$n = n' = 0\,; \quad F = 0, \quad Gf = 0\,; \quad Hf + G = n'', \quad H + f = n''' \quad . \quad . \quad (84)$$

and the symbolic equations for f and G degrade into

$$F = f^3 - n'''f^2 + n''f = 0\,; \quad G\,(G - n'') = 0 \quad . \quad . \quad . \quad . \quad (85).$$

The function $G - n''$ destroys the line a, b, which is consequently the locus of Gq.

For the solution of the equation $fq = p$, the relations (84) give

$$n''q = Hp + Gq\,; \quad Gp = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86);$$

and since Gq may be any point on the line a, b, the locus of q is the plane $[Hp, a, b]$.

If $Gp = 0$ is not satisfied, the solution is an arbitrary point on the line a, b affected with an infinite weight.

II_3. If $n'' = 0$, the solution is

$$n'''q = p + Hq\,; \quad Hp + Gq = 0, \quad Gp = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (87),$$

and

$$G^2 = 0, \quad Hf^2 = 0, \quad Hf = -G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88),$$

whence

$$Hq = xa' + ya + zb, \quad Hp = xa \quad \text{if} \quad a' = fa \quad . \quad . \quad . \quad . \quad (89).$$

II_4. If further, $n''' = 0$, the solution is

$$q = xa'' + ya' + za + wb, \quad p = xa' + ya, \quad fp = xa, \quad f^2p = 0 \quad . \quad . \quad (90);$$

and the general function of this type is

$$fq\,(aba'a'') = a\,(abqa'') + a'\,(aba'q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91),$$

and the function G of I_4 is of this sub-class.

17. III_1. The third class is that in which f destroys three points a, b, c, which are not situated on a common line

Here

$$n = n' = n'' = 0\,; \quad F = G = Hf = 0\,; \quad n''' = f + H, \quad f^2 - n'''f = 0 \quad . \quad (92),$$

and the solution is

$$n'''q = p + xa + yb + zc \quad \text{where} \quad Hp = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (93).$$

III_2. If $n''' = 0$,

$$q = xa' + ya + zb + wc \quad \text{where} \quad p = xa, \quad fa' = a \quad . \quad . \quad . \quad (94).$$

The type of the function is

$$f(q)\,.\,(abca') = a\,(abcq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

to which the function F of I_4 belongs.

18. IV. The fourth class is that in which two lines ab and cd are destroyed.

$$n = n' = 0, \quad F = 0, \quad Gf = 0, \quad Hf + G = n'' = \tfrac{1}{4}n'''^2, \quad H + f = n''' \quad . \quad (96)$$

and the symbolic equations are

$$f\,(f - \tfrac{1}{2}n''') = 0\,; \quad G\,(G - n'') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

The function

$$f(q)\,.\,(abcd) = t_0c(abqd) + t_0d\,(abcq) \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

is of this type. f destroys the line a, b and reduces an arbitrary point to c, d; $f - t_0$ destroys c, d and reduces an arbitrary point to ab.

19. As the theory of the self-conjugate linear vector function differs in various details from that of the self-conjugate quaternion function, it is necessary to devote a few remarks to the latter.

The four united points of a self-conjugate function form a tetrahedron self-conjugate to the unit sphere, for in this case the two tetrahedra of Art. 8 coincide. If two united points coincide, they must coincide with a point on the sphere, and the scalar quartic has a pair of equal roots. But in the case of a *real* self-conjugate vector function when two latent roots are equal, the function has an infinite number of axes in a certain plane, and not a single axis resulting from the coalescence of a pair; and the reason is simply that a real vector cannot be perpendicular to itself, while each axis of a self-conjugate vector function must be perpendicular to two others. For a quaternion function, on the other hand, a real point may be its own conjugate with respect to the unit sphere, and there may be in this case coincidence of united points without a locus of united points and consequent degradation of the symbolic quartic.

Again, the roots and axes of a self-conjugate vector function must be real, because two conjugate imaginary vectors, $\alpha + \sqrt{-1}\,\beta$, $\alpha - \sqrt{-1}\,\beta$, cannot be at right angles to one another, since the condition is $\alpha^2 + \beta^2 = 0$, while $\alpha^2 + \beta^2$ is essentially negative. But two united points of a real self-conjugate quaternion function may be conjugate imaginaries, the condition

$$S\,(a + \sqrt{-1}b)\,(a - \sqrt{-1}b) = Sa^2 + Sb^2 = 0 \quad . \quad . \quad . \quad (99),$$

merely showing that the real points a and b are situated one inside and one outside the unit sphere.

20. On account of the importance of the self-conjugate function, it may not be superfluous to illustrate cases of coalesced united points.

Writing for the general self-conjugate function,

$$f(1 + \rho) = e + \epsilon + S\epsilon\rho + \Phi\rho\,; \quad S\,(1 + \rho)\,f(1 + \rho) = e + 2S\epsilon\rho + S\rho\Phi\rho \quad . \,(100),$$

the latent quartic is

$$t^4 - t^3\,(e + m'') + t^2\,(em'' + m' - \epsilon^2) - t\,(em' + m + S\epsilon\,(\Phi - m'')\,\epsilon)$$
$$+ m\,(e - S\epsilon\Phi^{-1}\,\epsilon) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101).$$

The quadric surface $Sqfq = 0$ has its centre at the extremity of the vector $-\Phi^{-1}\epsilon$, or say at the point c.

One root is zero if

$$e - S\epsilon\Phi^{-1}\,\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102),$$

and the quadric is a cone with its vertex at the point c. A second root is zero if

$$m = -\,S\epsilon\,(\Phi - m'' + m'\Phi^{-1})\,\epsilon = -\,mS\epsilon\Phi^{-2}\epsilon, \text{ or if } T\Phi^{-1}\epsilon = 1 \quad . \quad . \,(103)\,;$$

that is, if the vertex is on the unit sphere.

A third root is zero if

$$m' = S\epsilon\,(1 - m''\Phi^{-1})\,\epsilon, \text{ or if } S\epsilon\,(1 - m''\Phi^{-1} + m'\Phi^{-2})\,\epsilon = mS\epsilon\Phi^{-3}\epsilon = 0 \quad . \quad (104),$$

and this simply requires $\Phi^{-2}\epsilon$ to be parallel to a generator of the cone. and perpendicular to the vector to its vertex. This generator touches the sphere.

The condition that the fourth root may vanish reduces to

$$mT\Phi^{-2}\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105),$$

and requires $m = 0$ for a real function, and in this case the cone breaks into a pair of planes, and the symbolic quartic degrades.

Admitting that $T\Phi^{-2}\epsilon = 0$ (for an imaginary function), it appears that the generator $-\Phi^{-1}\epsilon + x\Phi^{-2}\epsilon$ is common to the quadric and the sphere when four roots are zero.

The preceding analysis establishes the fact that a real self-conjugate function may belong to the classes, I_1, I_2, I_3, II_4 but not to I_4.

A real self-conjugate function cannot belong to I_5 if its two united points are real, for certain of the conditions of self-conjugation of the tetrahedron in the limit require $Sa^2 = Sab = Sb^2 = 0$, or the line a, b must be a generator of the sphere; and matters are not changed when we assume a and b to be conjugate imaginaries. We conclude therefore that no self-conjugate function belongs to I_5.

Since self-conjugate functions of the type II_4 exist, *à fortiori* they will exist for the less restricted types II_1, II_2, II_3.

Self-conjugate functions may belong to the types III_1, III_2. and to type IV, the lines being now conjugate with respect to the sphere (compare the following Article).

21. *If a function converts any tetrahedron into its reciprocal, it is self-conjugate.*

Here if

$$fa = x\,[bcd], \quad fb = y\,[acd], \quad fc = z\,[abd], \quad fd = w\,[abc] \quad . \quad . \quad (106),$$

the function producing the transformation is

$$fq\,(abcd) = x\,[bcd]\,(qbcd) - y\,[acd]\,(qacd) + z\,[abd]\,(qabd) - w\,[abc]\,(qabc) \quad . \quad . \quad . \quad (107),$$

which is manifestly self-conjugate.

This includes as a particular case the deduction from Art. 8.

The following theorems may be stated here :—

If a function has a scalar for a principal solution, its conjugate has three vector principal solutions.

If a function has a line or a plane locus of united points, it has a vector or a linear system of vector principal solutions.

The nature of the function $f_{\prime}$, which is the negative of its conjugate, has been sufficiently considered in Art. 12.

22. It may be as well to show the geometrical meaning of changing from a function f to another $f - t_0$, as in Art. 14.

Writing

$$p' = (f - t_0)\, q = p - t_0 q, \quad p = fq \quad . \quad . \quad . \quad . \quad . \quad . \quad (108);$$

it is obvious that p' is some point on the line pq. To determine the point, let P′, P and Q be the points p', p and q with unit weights, then

$$\mathrm{P}' = \frac{p - t_0 q}{\mathrm{S}p - t_0 \mathrm{S}q} = \frac{f\mathrm{Q} - t_0\mathrm{Q}}{\mathrm{S}f\mathrm{Q} - t_0} = \frac{\mathrm{PS}f\mathrm{Q} - t_0\mathrm{Q}}{\mathrm{S}f\mathrm{Q} - t_0} \quad . \quad . \quad . \quad . \quad (109);$$

and we have the ratio of segments

$$\frac{\mathrm{P'Q}}{\mathrm{P'P}} = \frac{\mathrm{Q} - \mathrm{P}'}{\mathrm{P} - \mathrm{P}'} = \frac{\mathrm{S}f\mathrm{Q}}{t_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110),$$

or its ratio is directly proportional to the perpendicular from the point Q on the plane $\mathrm{S}fq = 0$, which is projected to infinity by the transformation.*

Hence it is easy to form a geometrical conception of the nature of a transformation by reducing it to some simpler type, as in Art. 14; the point P for instance may always be supposed to lie in a fixed plane, while in the case of functions of the classes II and III it may be supposed to lie on a fixed line or to be a fixed point.

SECTION III.

Scalar Invariants.

23. From the results of Arts. 5 and 6, it appears that

$$((f - t)\, a, (f - t)\, b, (f - t)\, c, (f - t)\, d) = (abcd)\,(n - n't + n''t^2 - n'''t^3 + t^4) \quad (111)$$

is identically true, no matter what the value of t may be or what quaternions a, b, c, and d may represent. In this sense the four scalars n, n', n'', and n''' are invariants, and every relation connecting them implies some peculiarity in the geometrical transformation produced by f.

* In vectors, if $\mathrm{Q} = 1 + \rho$, the ratio is $t_0^{-1}\,(e + \mathrm{S}\epsilon'\rho). = t_0^{-1} x \mathrm{T}\epsilon'$ if x is the length of the perpendicular.

But there is a wider sense in which these four scalars are invariants. If n_1 and n_2 are the fourth invariants of any two functions f_1 and f_2, the relation

$$((f_1ff_2 - tf_1f_2)\,a, (f_1ff_2 - tf_1f_2)\,b, (f_1ff_2 - tf_1f_2)\,c, (f_1ff_2 - tf_1f_2)\,d) \\ = (abcd)\,n_1n_2\,(n - n't + n''t^2 - n'''t^3 + t^4) \quad (112)$$

is evidently true or may be verified at once by repeated application of (16). Thus any relation implying a peculiarity of the function f and depending on its four invariants, implies also a corresponding peculiarity in the mutual relations of the functions f_1ff_2 and f_1f_2, that is, of any two functions F_1 and F_2 decomposible in the manner indicated. In particular, if in (112) f_2 is replaced by f_1^{-1}, it is evident that the invariants of $f_1ff_1^{-1}$ are identical with those of f. And, moreover, the functions may be replaced by their conjugates without altering the invariants.

We now propose to examine the meaning of a few invariants, bearing in mind the remarks of this article, and remembering also that the invariants are more general than those of quadrics, for the function f is not supposed to be self-conjugate.

24. For brevity, replacing fa by a', we have

$$n'''\,(abcd) = (a'bcd) + (ab'cd) + (abc'd) + (abcd') \quad . \quad . \quad . \quad . \quad (113).$$

If n''' vanishes, it is possible to determine an infinite number of tetrahedra a, b, c, d, so that the corners of a derived tetrahedron shall lie on the faces of the original.

For taking any three points a, b, c, and their deriveds a', b', c', three planes are found

$$(a'bcd) = 0, \quad (ab'cd) = 0, \quad (abc'd) = 0 \quad . \quad . \quad . \quad . \quad . \quad (114),$$

whose common point d enjoys the property of having its derived in the plane of a, b, and c if, and only if, $n''' = 0$.

Conversely, if this is true for any tetrahedron and its derived, the invariant n''' vanishes, and the property is true for an infinite number of tetrahedra.

Interchanging the words *corner* and *face*, we have the corresponding interpretation of the vanishing of n'.

More generally, when n''' vanishes, an infinite number of tetrahedra exists, so that the pairs derived from them by the operations of the functions f_1ff_2 and f_1f_2 are related in the manner described.

Analogous extensions will be understood in the sequel.

25. Again, suppose that the sum of the squares of the roots of $n_t = 0$ is zero, or that

$$n'''^2 - 2n'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

In this case, tetrahedra may be found related to their correspondents in such a manner that the deriveds of these correspondents have their corners on the faces of the originals.

Of greater interest, however, is the case in which the sum of the square roots of the roots of $n_t = 0$ is zero, or when

$$(n'''^2 - 4n'')^2 = 64n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

Here the n''' invariant of one of the square roots of the function (compare Art. 36) vanishes, so that by the operation of this square root $f^{\frac{1}{2}}$, it is possible, from a suitably selected tetrahedron (one of an infinite number), to derive a second, and from that again a third, so that the second has its corners on the faces of the first, while its faces contain the corners of the third. But directly by the operation of $f\,(=f^{\frac{1}{2}}.f^{\frac{1}{2}})$ the third tetrahedron is transformed from the first, and these are so related that it is possible to inscribe to the first a tetrahedron circumscribed to the third.

Similarly, we can interpret invariants arising from relations such as

$$t_1^{\ m} + t_2^{\ m} + t_3^{\ m} + t_4^{\ m} = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117),$$

where m is the ratio of two integers, and where t_1, t_2, t_3, and t_4 are the latent roots of f.

26. Before passing on to invariants of a rather different type, we shall consider the relation connecting two quadric surfaces when an *infinite* number of tetrahedra can be inscribed to one and circumscribed to another.

Let the equations of the quadrics be

$$SqF_1q = 0, \quad SqF_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)\,;$$

let the tetrahedron $(abcd)$ be inscribed to the first, and let its faces touch the second at the points a', b', c', d'; let the function f derive the tetrad of points of contact from the corresponding vertices. Then there are four equations of inscription to the first quadric

$$SaF_1a = 0, \quad SbF_1b = 0, \quad ScF_1c = 0, \quad SdF_1d = 0 \quad . \quad . \quad . \quad (119)\,;$$

twelve equations of conjugation of the points a', b, &c., to the second quadric

$$Sa'F_2b = Sb'F_2a = 0 \quad \text{or} \quad Saf'F_2b = SaF_2fb = 0 \quad . \quad . \quad . \quad (120)\,;$$

and four equations of contact such as

$$Sa'F_2a' = 0 \quad \text{or} \quad Saf'F_2fa = 0\,.$$

The equations of conjugation require the function F_2f to be self-conjugate, so that

$$f'F_2 = F_2f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121),$$

and the conditions of contact may therefore be replaced by four equations such as

$$SaF_2f^2a = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122).$$

An infinite number of tetrahedra may consequently be respectively inscribed and circumscribed to the quadrics

$$SqF_2f^2q = 0, \quad SqF_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123),$$

when the condition (121) is satisfied and when the n''' of f vanishes; and if this is likewise possible for the given quadrics, we must have

$$F_2f^2 = F_1, \quad \text{or} \quad f^2 = F_2^{-1}F_1, \quad \text{or} \quad f = (F_2^{-1}F_1)^{\frac{1}{2}} \quad . \quad . \quad . \quad (124).$$

The sum of the square roots of the latent roots of the function $F_2^{-1}F_1$ must consequently vanish, or the invariant* (116) of this function is zero.

It has been proved incidentally in this article if a tetrahedron circumscribed to $SqF_2q = 0$ is self-conjugate to $SqF_3q = 0$, that the invariant n''' of the function $F_2^{-1}F_3$ is zero; and if the tetrahedron is self-conjugate to $SqF_3q = 0$ and inscribed to $SqF_1q = 0$, that the same invariant of the function $F_3^{-1}F_1$ is zero. Here $F_3 = F_2f$.

It must be carefully observed that in dealing with quadrics the extent of the invariance (Art. 23) is limited. If F_1 and F_2 are self-conjugate, the functions $f_1F_1f_2$ and $f'_1F_2f_2$ must be self-conjugate before theorems can be extended from the quadrics determined by the simpler to those determined by the more complex functions.

27. The invariant n'' vanishes if

$$(a'b'cd) + (a'bc'd) + (a'bcd') + (ab'c'd) + (ab'cd') + (abc'd') = 0 \quad . \quad (125).$$

To save verbiage in the interpretation, the edges ab and $c'd'$ may be called the *opposite* edges of a tetrahedron and its derived. If each edge of $(abcd)$ intersects the opposite edge of $(a'b'c'd')$, the invariant will manifestly vanish, for every term will be zero.

To display the nature of the conditions requisite for determining a tetrahedron possessing this property, when $n'' = 0$, let a and b be assumed fixed, and then five of the terms may be written in forms

$$Scf_nd = 0 \qquad (n = 1, 2, 3, 4 \text{ or } 5) \quad . \quad . \quad . \quad . \quad . \quad (126),$$

where f_n is one of five linear quaternion functions. Three equations give

$$c = [f_1d, f_2d, f_3d] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127),$$

and substitution in the fourth and fifth require the point d to be on the curve of the quartic surfaces

$$(f_1d, f_2d, f_3d, f_4d) = 0, \quad (f_1d, f_2d, f_3d, f_5d) = 0 \quad . \quad . \quad . \quad . \quad (128),$$

* This condition appears to answer in every particular the condition (compare 'Elements of Quaternions,' New Ed., vol. ii., p. 377) that a triangle can be inscribed to one conic and circumscribed to another (*see*, however, SALMON'S 'Three Dimensions,' Note to Art. 207).

which is complementary to the sextic curve (compare Art. 64),

$$[f_1 d, f_2 d, f_3 d] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129).$$

Selecting any point d on this complementary curve of the tenth order, c is determined by (127), and the sixth condition must be satisfied.

Hence it appears that any two vertics may be assumed at random, and a plane locus for the third. Ten points d lie in this plane, and ten tetrahedra satisfy the conditions.

Generally, also, if the sum of the products of the square roots of the latent roots of the function vanishes, an infinite number of tetrahedra may be found related to their correspondents, so that corresponding edges a, b ; a', b', are intersected by opposite edges of intermediate tetrahedra. (Compare Art. 25.)

28. The case in which the two invariants n' and n''' vanish simultaneously is of considerable importance in the theory of the linear function. These conditions are always satisfied for the functions $2f_{\prime} = f - f'$; and also for functions of a more general type ; in fact, for functions whose squares satisfy a depressed equation

$$(f^2)^2 + n''f^2 + n = 0, \text{ or } (f^2 - s^2)(f^2 - s'^2) = 0 \quad . \quad . \quad . \quad (130).$$

It appears from Art. 24 that two systems of tetrahedra exist, one set having their correspondents inscribed to them, the other set being inscribed to their correspondents. We shall prove that *one system of tetrahedra exists which are at once inscribed and circumscribed to their correspondents.*

Let q_1 and q_2 be the united points of f for the roots $\pm s$, and q'_1 and q'_2 for the roots $\pm s'$. Take any line whatever

$$q = x(q_1 + uq_2) + y(q'_1 + vq'_2) \qquad (x, y \text{ variable}) \quad . \quad . \quad . \quad . \quad (131),$$

intersecting the lines $q_1 q_2$ and $q'_1 q'_2$. The function f converts this line into the line

$$p = xs(q_1 - uq_2) + ys'(q'_1 - vq'_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132),$$

which intersects the connectors of the united points in the harmonic conjugates of the points of intersection of the original line. Repeating the operation, the line p is restored to q.

In other words, when n' and n'' vanish, *the transformation interchanges lines which cut harmonically the connectors of the united points;* or it transforms a certain congenency of lines into itself.

Take any tetrahedron having opposite edges, ab and cd, on two conjugate lines of this congenency ; the corresponding tetrahedron has the two edges $c'd'$ and $a'b'$ respectively on those two lines, and either tetrahedron may be said to be at one and the same time inscribed and circumscribed to the other.

If the line a, b intersects the connectors in the points Q_1 and Q'_1, and if a', b' intersects them in Q_2, Q'_2 (compare (131), (132)), we may write

$$a = Q_1 + t_1 Q'_1; \quad b = Q_1 + t_2 Q'_1; \quad c' = sQ_1 + s't_3 Q'_1; \quad d' = sQ_1 + s't_4 Q'_1;$$
$$a' = sQ_2 + s't_1 Q'_2; \quad b' = sQ_2 + s't_2 Q'_2; \quad c = Q_2 + t_3 Q'_2; \quad d = Q_2 + t_4 Q'_2;$$

and the anharmonics of the ranges $abc'd'$ and $a'b'cd$ are

$$\frac{(ab)(c'd')}{(bc')(d'a)} = \frac{ss'(t_1 - t_2)(t_3 - t_4)}{(st_2 - s't_3)(s't_4 - st_1)}; \quad \frac{(a'b')(cd)}{(b'c)(da')} = \frac{ss'(t_1 - t_2)(t_3 - t_4)}{(s't_2 - st_3)(st_4 - s't_1)}.$$

For a pair of quadrics (118) a quadrilateral on one determines a self-conjugate tetrahedron with respect to the other if n' and n''' of the function $F_1^{-1}F_2$ vanish. Moreover, in this case the quadrics

$$SqF_1q = 0, \quad SqF_2F_1^{-1}F_2q = 0$$

intersect in a common quadrilateral.

29. It may be worth while drawing attention to a simple rule for obtaining in a convenient form certain scalar invariants of linear functions. These invariants are the coefficients of powers and products of x_1, x_2, &c., in the latent quartic of the function

$$x_1 f_1 + x_2 f_2 + \ldots + x_n f_n$$

and the rule is to distinguish by accents or suffixes the symbols in $(abcd)$ just as if this expression had been differentiated. For instance, there is the twelve-term invariant

$$n_{12}(abcd) = \Sigma(a_1 b_2 cd)$$

where a_1 stands for $f_1 a$, and a_2 for $f_2 a$.

It would appear that when a twelve-term invariant vanishes, every term will vanish provided the tetrahedron $(abcd)$ is suitably inscribed to a definite curve.

Suppose eleven terms vanish. Let three be solved for a, and substitution in the remaining eight leaves eight equations in b, c and d. From three of these find b, and five are left in c and d; and on elimination of c, two equations in d remain, which represent a definite curve. From symmetry the remaining three vertices trace out a curve or curves. These curves are covariant with the functions.

SECTION IV.

The Relations of a Pair of Quadrics, $SqF_1q = 0$, $SqF_2q = 0$, which depend on the Nature of the Function $F_2^{-1}F_1$.

30. We shall briefly consider the relations of a pair of quadrics which depend on the peculiarities of the function $F_2^{-1}F_1$, where

$$SqF_1q = 0, \quad SqF_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

are the equations of the two quadrics.

If the polar plane of the point a is the same with respect to the two quadrics,

$$F_1a = t_1F_2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134),$$

where t_1 is a scalar, because (Art. 11) the symbols of the polar planes are F_1a and F_2a. Here t_1 is a latent root of the function $F_2^{-1}F_1$ and a is a united point.

If b is a second united point answering to the latent root t_2, we have, on account of the self-conjugate character of the functions F_1 and F_2,

$$t_1SbF_2a = SbF_1a = SaF_1b = t_2SaF_2b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (135),$$

provided the latent roots are distinct. Thus the polar plane of a contains the points b, c, and d; and the tetrahedron is self-conjugate to both quadrics. The function $F_1^{-1}F_2$ belongs to the general type I_1, in which all the united points are distinct (Art. 13).

31. Let two united points a and b approach coincidence. The relation (135) remains true up to the limit, and ultimately

$$SaF_1a = 0, \quad SaF_2a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136);$$

and the coalesced point is situated on the curve of intersection of the surface. By (134) the symbols of the tangent planes to the two surfaces are identical, and the two surfaces touch.

If then $F_2^{-1}F_1$ belongs to the type I_2, the surfaces touch, and conversely; and if the quadrics touch in two distinct points the type of the function is I_5, and the intersection is a line and a cubic.

Let c and d be the remaining united points. By (135) the line c, d lies in the common tangent plane; so in order to determine the generators of the two quadrics in the plane, it is only necessary to determine the points in which the quadrics meet the line c, d. For the first and second quadrics, the equations determining the points $c + xd$ are respectively (134)

$$t_3 \mathrm{S}c\mathrm{F}_2 c + x^2 t_4 \mathrm{S}d\mathrm{F}_2 d = 0 \quad \mathrm{S}c\mathrm{F}_2 c + x^2 \mathrm{S}d\mathrm{F}_2 d = 0 ; \quad . \quad . \quad . \quad (137).$$

The quadrics consequently have distinct generators unless $t_3 = t_4$, and unless the points c and d are distinct.

For quadrics having a pair of common co-planar generators, $\mathrm{F}_2^{-1}\mathrm{F}_1$ is of the type II_2, and conversely.

32. In the next place, let three roots t_1 be equal, so that a is the union of three united points of $f = \mathrm{F}_2^{-1}\mathrm{F}_1$. The point a' of Art. 15 (78) is now in the common tangent plane, because it has been derived by the operation of $f - t_1$ from another point a''. In fact we have

$$(\mathrm{F}_1 - t_1 \mathrm{F}_2)\, a'' = \mathrm{F}_2 a', \quad (\mathrm{F}_1 - t_1 \mathrm{F}_2)\, a' = \mathrm{F}_2 a \quad . \quad . \quad . \quad . \quad (138);$$

and from the first of these it is obvious that $\mathrm{S}a\mathrm{F}_2 a' = 0\, (= t_1^{-1} \mathrm{S}a\mathrm{F}_1 a')$, while the second may be written in the form

$$\mathrm{F}_1 (a + xa') = (t_1 + x)\, \mathrm{F}_2 \left(a + \frac{xt_1}{t_1 + x} a' \right) \quad . \quad . \quad . \quad . \quad . \quad (139).$$

This equation shows that the polar plane of the point $a + xa'$ with respect to the first quadric is identical with the polar plane of $a + \dfrac{xt_1}{t_1 + x} a'$ with respect to the second; and because a' lies in the tangent plane, in the limit where x becomes infinitesimally small, the two points become identical to the first order of x, and the common polar plane becomes a consecutive tangent plane to both quadrics. The quadrics have, therefore, stationary contact, and their function $\mathrm{F}_2^{-1}\mathrm{F}_1$ is of the class I_3.

The generators in the tangent plane are now found by expressing that $xa' + d$ is on one of the quadrics; the equations may be written in the form

$$x^2 t_1 \mathrm{S}a'\mathrm{F}_2 a' + 2xt_1 \mathrm{S}a'\mathrm{F}_2 d + t_4 \mathrm{S}d\mathrm{F}_2 d = 0 ; \; x^2 \mathrm{S}a'\mathrm{F}_2 a' + 2x \mathrm{S}a'\mathrm{F}_2 d + \mathrm{S}d\mathrm{F}_2 d = 0 \quad . \quad (140),$$

where the equation for the first quadric has been reduced by the aid of (138), in order that it may be compared with that for the second quadric. The generators are common if, and only if, $t_1 = t_4$, and the function is then of the type II_4.

33. When the four united points coincide, the point a'' as well as a' lies in the common tangent plane, a'' having been derived, as a' was in the last article, from a third point a'''. From the three equations

$$(F_1 - t_1F_2)\,a''' = F_2a'' ; \quad (F_1 - t_1F_2)\,a'' = F_2a' ; \quad (F_1 - t_1F_2)\,a' = F_2a \quad . \quad (141),$$

we see that, in addition to the conditions that the points should lie in the tangent plane, we have

$$Sa''(F_1 - t_1F_2)\,a' = 0 ; \quad Sa'F_2a' = 0, \quad \text{and} \quad Sa'F_1a' = 0 \quad . \quad . \quad (142),$$

as appears from operating on the third by Sa'' and using this result in operating on the second by Sa', and finally operating on the third by Sa'. The line $a + xa'$ is consequently a generator of both quadrics, and the function belongs to the class I_4.

The remaining generators, determined by the point in which $a' + ya''$ meets the surfaces again, are deducible from the equations

$$t_1Sa'F_2a'' + yt_1Sa''F_2a'' + ySa''F_2a' = 0 ; \quad Sa'F_2a'' + ySa''F_2a'' = 0 \quad . \quad (143).$$

If these remaining generators are common to both quadrics we must have $Sa''F_2a' = 0$, and then they coincide of necessity with the other generator, and the quadrics become a pair of cones touching along a generator.

34. Suppose the function to have a line locus of united points, so that

$$F_1a = t_1F_2a ; \quad F_1b = t_1F_2b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144) ;$$

it immediately follows that one quadric meets the line a, b in two points common to the other, and the quadrics touch at these two points. Substituting in the equations of the quadrics

$$q = xa + yb + z\,(c + ud) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145),$$

the equations become,

$$t_1S\,(xa + yb)\,F_2\,(xa + yb) + z^2\,(t_3ScF_2c + u^2t_4SdF_2d) = 0$$

$$S\,(xa + yb)\,F_2\,(xu + yb) + z^2\,(ScF_2c + u^2SdF_2d) = 0 \quad . \quad . \quad . \quad . \quad (146),$$

and for a constant value of u these represent the sections by an arbitrary plane through the line a, b. These sections are identical if

$$(t_3 - t_1)\,ScF_2c + u^2\,(t_4 - t_1)\,SdF_2d = 0 \quad . \quad . \quad . \quad . \quad . \quad (147),$$

and as this is a quadratic in u, the quadrics have two plane sections common. The function f is of the type II. The case of coincidence of the points c, d has occurred in Art. 31, one of the conics breaking up (type II_2).

If $t_3 = t_1$, while c is not situated on ab, the quadrics have two coincident plane sections, or ring-contact. The type of the function is III_1.

If $t_3 = t_4$, but c not coincident with d, the function is of the class IV., and the quadrics intersect in common points on the line c, d. Let ab meet the quadrics in a', b' and cd in $c'd'$, then it is very easy to see that a', c', b', d' is a quadrilateral common to both surfaces.

When c coincides with a point a on the line, let a' be the point for which (Art. 15)

$$(F_1 - t_1F_2)\,a' = F_2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148),$$

then $SaF_2a = 0$, and $SbF_2a = 0$, and the line ab touches the two quadrics at a. The conics in the common plane sections touch (type II_3).

If, further, d coincides with the point a (type II_4), the point F_2a' is derived by the operation of $F_1 - t_1F_2$ from some other point a'' (Art. 32), and therefore

$$SaF_2a' = 0\,;\ SbF_2a' = 0\,;\ \text{and}\ SaF_1a' = 0\,;\ SbF_1a' = 0 \quad . \quad . \quad (149).$$

Hence it appears that the line $a' + xb$ meets the two quadrics in the same two points, and the lines from a to these points are common generators. The intersection of the quadrics consists, therefore, of a pair of lines and a conic passing through their common point (type II_4).

Finally, it remains to notice the case of a plane locus of united points with the fourth point in the plane (III_2). It may be proved that in this case the coincident plane sections consist of a pair of lines along which the quadrics touch.

35. Summing up, the intersection of two quadrics according to the types of the function $F_2^{-1}F_1$, is

I_1, a twisted quartic with two apparent double points ;

I_2, a twisted quartic with three apparent double points ;

I_3, a twisted quartic with two apparent and one real double point ;

I_4, a right line and a cubic touching it ;

I_5, a right line and a cubic ;

II_1, two conics ;

II_2, a pair of lines and a conic ;

II_3, two conics in contact ;

II_4, a pair of lines and a conic through their intersection ;

III_1, the surfaces touch along a conic ;

III_2, the surfaces touch along two generators ;

IV, the intersection is a quadrilateral.

SECTION V.

The Square Root of a Linear Quaternion Function.

36. When the same effect is produced by the twice-repeated operation of one linear quaternion function and by the single operation of another, the former may be said to be a square root of the latter.

TAIT first extracted the square root of a linear vector function, and pointed out the great utility of the conception. We now proceed to examine some of the properties of a square root of a quaternion function, and to illustrate their bearing on certain geometrical investigations.

The united points of a square root are also united points of the primitive function.

If

$$f^{\frac{1}{2}}a = t_1^{\frac{1}{2}}a, \quad \text{then} \quad fa = t_1 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150).$$

The converse does not hold, for it may happen that loci of united points exist for the primitive and not for the square root. For example, if

$$f^{\frac{1}{2}}a = b, \quad f^{\frac{1}{2}}b = t_1 a \, ; \quad \text{then} \quad fa = t_1 a, \quad fb = t_1 b \quad . \quad . \quad . \quad . \quad (151),$$

and though every point on the line $a\,b$ is a united point for the primitive, this is not generally true for its square roots. (Compare Art. 13.)

When there is no locus of united points, the square roots have the same four united points as the primitive, and their latent roots are sets of the square roots

$$\pm t_1^{\frac{1}{2}}, \quad \pm t_2^{\frac{1}{2}}, \quad \pm t_3^{\frac{1}{2}}, \quad \pm t_4^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

of the latent roots of the primitive. Thus in the general case a function has sixteen square roots.

37. *When the primitive has a line locus of united points* (a, b), *any two points on the line may be assumed as united points of the square root.*

By the last article it may be seen that the square root must have united points on the line. Assume these to be $a + xb$, $a + yb$, then

$$f^{\frac{1}{2}}(a + xb) = \pm t_1^{\frac{1}{2}}(a + xb) \, ; \quad f^{\frac{1}{2}}(a + yb) = \pm t_1^{\frac{1}{2}}(a + yb) \quad . \quad . \quad (153),$$

and the square root satisfies the condition that its twice repeated operation is equivalent to the operation of f. If the signs are alike and x and y distinct, the square root has a locus of united points; otherwise it has not.

If a square root has coalesced points, so has the primitive.

If

$$f^{\frac{1}{2}}a' = t^{\frac{1}{2}}a' + a \, ; \quad f^{\frac{1}{2}}a' = t^{\frac{1}{2}}a \, ; \quad \text{then} \; fa' = ta' + 2t^{\frac{1}{2}}a \, ; \quad fa = ta. \quad . \quad (154),$$

and therefore the repeated operation of $f - t$ is required to destroy a'; and the primitive has a coalesced united point.

The square root of a function having a plane locus of united points possesses at least a line locus of united points.

The only escape is the assumption that the square root has a united point coalesced from three points, and this has just been shown to involve a coalesced point for the primitive, contrary to hypothesis.

When the primitive has coalesced points but no loci of united points, the number of square roots is limited.

This follows from (154).

38. *Except in the case in which the primitive has loci of united points, the square roots are all commutative with one another and with the primitive,* for they possess a common system of united points.*

Moreover, for a definite square root,

$$(f+x)^{\frac{1}{2}}(f+y)^{\frac{1}{2}} = ((f+x)(f+y))^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (155),$$

with liberty to change the order of the factors. This follows most easily by operating on the united points.

In general also, for any two functions f_1 and f_2, and a definite square root,

$$f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} = (f_1^{\frac{1}{2}} f_2^2 f_1^{-\frac{1}{2}})^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156),$$

because

$$(f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}})^2 = f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} \cdot f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} = f_1^{\frac{1}{2}} f_2^2 f_1^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (157);$$

and in particular a relation which is occasionally useful is

$$(f_1^{-1} f_2 + t)^{\frac{1}{2}} = f_1^{-\frac{1}{2}} (f_1^{-\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} + t)^{\frac{1}{2}} f_1^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (158).$$

39. It is evident from the foregoing that the square roots of a function and of its conjugate are conjugate when they have the same latent roots.

Thus we may write

$$(f^{\frac{1}{2}})' = f'^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159),$$

to signify that the conjugate of a square root is the corresponding square root of the conjugate function.

In particular, taking the conjugate of (158),

$$(f_2' f_1'^{-1} + t)^{\frac{1}{2}} = f_1'^{\frac{1}{2}} (f_1'^{-\frac{1}{2}} f_2' f_1'^{-\frac{1}{2}} + t)^{\frac{1}{2}} f_1'^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (160).$$

SECTION VI.

THE SQUARE ROOT OF A FUNCTION IN RELATION TO THE GEOMETRY OF QUADRICS.

40. The transformation

$$p = f^{\frac{1}{2}} q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

converts the quadric $Sqfq = 0$ into the unit sphere $Sp^2 = 0$, f being a self-conjugate function.

* Compare 'Elements of Quaternions,' New Ed., vol. ii., Appendix, p. 364.

This suggests a quaternion equation such as

$$q = (f+x)^{\frac{1}{2}}(f+y)^{\frac{1}{2}}(f+z)^{\frac{1}{2}}e = \sqrt{\{(f+x)(f+y)(f+z)\}}\,e \quad . \quad (162),$$

where e is some constant quaternion, as equivalent to the equation of a system of generalized confocals

$$Sq(f+x)^{-1}q = 0 \quad . \quad . \quad . \quad (163).$$

On substitution in the scalar from the quaternion equation the result is

$$Se(f+y)(f+z)e = 0 \quad . \quad . \quad . \quad (164),$$

and y and z disappear, provided e is chosen to be one of the eight points satisfying

$$Se^2 = Sefe = Sef^2e = 0 \quad . \quad . \quad . \quad (165).$$

Thus e is one of the intersections of three known quadrics.

It is not necessary to dwell on HAMILTON's theory of the umbilicar generatrices, as the subject will be resumed in an extended form.* Accordingly it is sufficient to mark that the equation of such a generator is

$$q = (f+y)(f+x)^{\frac{1}{2}}e = (f+x)^{\frac{3}{2}}e + \tfrac{2}{3}(y-x)\frac{d}{dx}(f+x)^{\frac{3}{2}}e \quad . \quad . \quad (166),$$

where y is variable; and the form of this equation shows that when x varies the generator sweeps out the developable of which the cuspidal edge is the curve

$$q = (f+x)^{\frac{3}{2}}e \quad . \quad . \quad . \quad (167).$$

41. More generally, starting from any two quadrics,

$$Sqf_1q = 0, \quad Sqf_2q = 0 \quad . \quad . \quad . \quad (168);$$

the equation of the system of quadrics inscribed to their common circumscribing developable (compare Art. 11) is

$$Sq(f_1^{-1} + xf_2^{-1})^{-1}q = 0 \quad . \quad . \quad . \quad (169).$$

This by the principles of Art. 38 may be replaced by

$$Sqf_2^{\frac{1}{2}}(f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + x)^{-1}f_2^{\frac{1}{2}}q = 0 \quad . \quad . \quad . \quad (170);$$

and on comparison with (163) and (162) it is manifestly equivalent to the quaternion equation

$$f_2^{\frac{1}{2}}q = (f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + x)^{\frac{1}{2}}(f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + y)^{\frac{1}{2}}(f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + z)^{\frac{1}{2}}e' \quad . \quad . \quad . \quad (171);$$

or, by an application of (158), to

$$q = (f_1^{-1}f_2 + x)^{\frac{1}{2}}(f_1^{-1}f_2 + y)^{\frac{1}{2}}(f_1^{-1}f_2 + z)^{\frac{1}{2}}e \quad . \quad . \quad . \quad . \quad (172),$$

* Compare Arts. 41 and 71.

where $e = f_2^{-\frac{1}{2}}e'$. By (165) it is seen that the quaternion e of this formula satisfies the three equations

$$Sef_2e = 0, \quad Sef_2f_1^{-1}f_2e = 0, \quad Sef_2f_1^{-1}f_2f_1^{-1}f_2e = 0 \quad . \quad . \quad . \quad (173),$$

and is therefore one of the intersections of three quadrics.

42. In particular the equation of the curve of intersection of the original quadrics (168) is

$$q = (f_1^{-1}f_2 + x)^{\frac{1}{2}}a, \text{ where } Saf_1a = Saf_2a = Saf_1f_1^{-1}f_2a = 0 \, . \quad . \quad (174),$$

as may be proved by direct transformation from the general result (172), or perhaps more shortly by assuming the form $q = (f + x)^{\frac{1}{2}}a$ and determining f, or by verification. remembering (158).

Hence the equation

$$Sa\,(f_2f_1^{-1} + x)^{\frac{1}{2}}f_3(f_1^{-1}f_2 + x)^{\frac{1}{2}}a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

determines the eight points of intersection of the three quadrics

$$Sqf_1q = 0, \quad Sqf_2q = 0, \quad Sqf_3q = 0.$$

SECTION VII.

The Family of Curves $q = (f + t)^m a$ and their Developables.

43. Instead of writing down and discussing the equations of the circumscribing developable and of its cuspidal edge of the quadrics (169), which are in fact of the same form as (166) and (167), except that $f = f_1^{-1}f_2$ is not self-conjugate, we shall devote a few remarks to the family of curves

$$q = (f + t)^m a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

and their developables, m being a scalar, a a constant quaternion, t a scalar variable, and f an arbitrary linear quaternion function. This family includes the right line, the conic, the twisted cubic, the quartic intersection of two quadrics, the quartic which is not the intersection of two quadrics, and the cuspidal edge of the developable circumscribed to two quadrics; the corresponding values of m being $m = 1, 2, -1$ or 3, 4 and $\frac{3}{2}$.

44. The equation of a tangent to the curve (176) is

$$q = (f + s)(f + t)^{m-1}a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177),$$

when the scalar parameter s alone varies. When s and t both vary the equation is that of the developable of the tangent lines.

If for suitable weights of the united points q_1, q_2, q_3, q_4, we write

$$a = q_1 + q_2 + q_3 + q_4 \quad \text{. . . . .} \quad (178),$$

the equation of the developable becomes

$$q = \sum_1^4 (t_1 + s)(t_1 + t)^{m-1} q_1 \quad \text{. . . . .} \quad (179).$$

When $m - 1$ is positive, the result of putting $t = -t_1$ is

$$q = (t_2 + s)(t_2 - t_1)^{m-1} q_2 + (t_3 + s)(t_3 - t_1)^{m-1} q_3 + (t_4 + s)(t_4 - t_1)^{m-1} q_4 \quad \text{. .} \quad (180);$$

and this represents a certain number of right lines in the united plane $[q_2, q_3, q_4]$, the number being determined by the nature of m, being as we know 4 when the developable is circumscribed to a pair of quadrics, or when $m = \frac{3}{2}$.

The remaining part of the intersection in the united plane is obtained by putting s equal to $-t_1$, and its equation is

$$q = (t_2 - t_1)(t_2 + t)^{m-1} q_2 + (t_3 - t_1)(t_3 + t)^{m-1} q_3 + (t_4 - t_1)(t_4 + t)^{m-1} q_4 \quad \text{. .} \quad (181);$$

or more simply

$$q = (f + t)^{m-1} a_1, \quad \text{where} \quad a_1 = (t_2 - t_1) q_2 + (t_3 - t_1) q_3 + (t_4 - t_1) q_4 \quad \text{. .} \quad (182).$$

The plane curve is likewise included in the family (176), and for $m = \frac{3}{2}$ it is a quartic (174), or rather a conic counted twice,

$$x_2^2 \frac{t_3 - t_4}{(t_2 - t_1)^2} + x_3^2 \frac{t_4 - t_2}{(t_3 - t_1)^2} + x_4^2 \frac{t_2 - t_3}{(t_4 - t_1)^2} = 0 \quad \text{. . . .} \quad (183),$$

as we see from (181) on putting $q = x_2 q_2 + x_3 q_3 + x_4 q_4$.

In case $m - 1$ is negative it is necessary first to multiply (179) by the product $\Pi (t_1 + t)^{1-m}$ before putting $-t$ equal to a latent root. Then, on making $t = -t_2$, we find only the point q_2, which shows there are no right lines in the plane $[q_2 q_3 q_4]$, and which indicates multiplicity of the curve at the united points.

45. Just as the equation of the tangent line was obtained in the last article from that of the curve, the equation of the osculating plane may be written in the form

$$q = (f + u)(f + s)(f + t)^{m-2} a \quad \text{. . . . . . .} \quad (184);$$

where t is supposed to remain constant, while s and u vary together. It is easy to verify that this plane contains two consecutive tangents to the curve.

The reciprocal of the plane is the point (compare Art. 5)

$$p = (f' + t)^{2-m} a', \qquad a' = [a, fa, f^2 a] \quad \text{. . . . . .} \quad (185);$$

and consequently the cuspidal edges of the reciprocals of curves of the family (176) belong to a similar family obtained by altering a into a' and f into its conjugate. Also the sum of the exponents m for a curve and the cuspidal edge of its reciprocal is equal to 2.

The developable formed by the tangents to the new cuspidal edge is

$$p = (f' + s')(f' + t)^{1-m}a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186);$$

and it may be worth while to verify directly that lines of this reciprocal developable are reciprocal to the corresponding lines of (179). Also lines in a united plane reciprocate into lines through a united point of the conjugate function; so that we can assert that the number of lines of the developable of a curve whose exponent is m which lie in a united plane is the number of lines of the developable of a curve whose exponent is $2 - m$ which pass through a united point.

46. The points (s) in which an osculating plane (184) at (t) cuts the curve again are found by combining this equation with (176) and putting

$$Sqp = 0 = S(f+s)^m a(f'+t)^{2-m}a' = Sa'(f+t)^{2-m}(f+s)^m a \quad . \quad . \quad (187).$$

In this, when we use the expression (178) for a and when we observe (185) that

$$a' = [afaf^2a] = \Sigma[q_2q_3q_4](t_2 - t_3)(t_3 - t_4)(t_4 - t_2) \quad . \quad . \quad . \quad . \quad (188),$$

equation (187) becomes

$$\Sigma \frac{(t_1 + s)^m}{(t_1 + t)^m}(t_1 + t)^2(t_2 - t_3)(t_3 - t_4)(t_4 - t_2) = 0 \quad . \quad . \quad . \quad . \quad (189).$$

The points at which the plane meets the curve four times are determined by

$$\Sigma (1)^m (t_1 + t)^2(t_2 - t_3)(t_3 - t_4)(t_4 - t_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (190).$$

SECTION VIII.

The Dissection of a Linear Function.

47. In addition to the decomposition of a function into its self-conjugate and non-conjugate parts by addition and subtraction, there is another very useful resolution by multiplication and division analogous to Tait's resolution of a linear

vector function into a function representing a pure strain following or followed by a rotation.

Multiply any function into its conjugate, and write

$$ff' = F^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191),$$

where F is the self-conjugate function whose double operation is equivalent to the operation of the self-conjugate function ff' (Art. 36).

Introducing a new linear function g and its conjugate g' defined by the relations

$$f = Fg, \quad f' = g'F \quad \text{or} \quad g = F^{-1}f, \; g' = f'F^{-1} \quad . \quad . \quad . \quad . \quad (192),$$

it appears that this function is the inverse of its conjugate, for

$$g'g = 1 = gg' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

is a consequence of the equations of definition.

The geometrical property of this new function is, that *points conjugate to the unit sphere remain conjugate after transformation.*

For if

$$Spq = 0, \quad \text{then} \quad Sgpgq = Spg'gq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (194).$$

In particular the unit sphere is converted into itself by the transformation.

This transformation is orthogonal, points and planes being transformed by the same function (Art. 4).

48. On counting the constants, it appears that an arbitrary function f cannot be reduced to the product of a self-conjugate function and a conical rotator

$$R = r(\;)r^{-1}, \quad R' = R^{-1} = r^{-1}(\;)r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (195),$$

there being sixteen constants in f, ten in F, and three in R.

In order to determine the conditions, observe that by the last article

$$F^2 = ff' \quad \text{if} \quad f = FR, \quad \text{and} \quad RR' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (196).$$

Now I say that *if a scalar remains a scalar after the operation of* R, *the function is a conical rotator.* For then

$$SR'\rho = S\rho R(1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (197),$$

and therefore $R'\rho$ or $R\rho$ remains a vector whatever vector ρ may be; and, moreover, the angle between any two vectors is unaltered by the transformation.*

Thus the condition required is simply

$$f(1) = F(1), \quad \text{where} \quad F^2 = ff' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (198);$$

and when the reduction is possible it is generally determinate.

* Compare the Appendix to the New Edition of HAMILTON'S 'Elements,' vol. ii., p. 366.

49. *A function which is the inverse of its conjugate is in general reducible in an infinite variety of ways to the product of a self-conjugate function and a rotator.*

Because $gg' = 1$ in the notation of Art. 47, the conditions (198) that g should be reducible are

$$g(1) = G(1), \quad \text{where} \quad G^2 = 1, G = G' \quad . \quad . \quad . \quad . \quad . \quad (199),$$

for simplicity writing

$$1 + g(1) = a, \quad 1 - g(1) = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200);$$

it is evident from the last equation that

$$Ga = a, \quad Gb = -b, \quad Sab = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201);$$

so a and b are united points of G, and conjugate with respect to the unit sphere.

Take any point c in the polar plane of b, and any point d in the polar line of ac; and assume

$$Gc = c, \quad Gd = -d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202);$$

then the function determined by the four relations (201) and (202) is self-conjugate, and its symbolic equation is $G^2 - 1 = 0$. By the construction it follows that

$$Sab = Sbc = Sad = Scd = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203),$$

and the function is consequently self-conjugate.

We have now determined a self-conjugate function, one of an infinite number, which satisfies (199), and the proposition is proved.

The rotator corresponding to G is of course

$$R = G^{-1}g = Gg \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204).$$

50. The results of recent articles establish the possibility of reducing an arbitrary function to the form

$$f = FGR \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205);$$

where F, G, and R satisfy the equations

$$F^2 = ff', \quad F^{-1}f(1) = G(1), \quad G^2 = 1, \quad R = GF^{-1}f \quad . \quad . \quad . \quad (206);$$

and by analogous processes the function may also be reduced to other forms such as $G_1F_1R_1$, but on these we need not delay.

51. *An arbitrary function may be reduced to a quotient or product of two self-conjugate functions.*

Assuming

$$f = F_2^{-1}F_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207),$$

it appears that the united points of f (compare Art. 30) satisfy the equations

$$F_1a = t_1F_2a; \quad F_1b = t_2F_2b; \quad F_1c = t_3F_2c; \quad F_1d = t_4F_2d \quad . \quad . \quad (208);$$

but on the supposition that F_1 and F_2 are self-conjugate, it follows (135) that these united points form a tetrahedron self-conjugate to the two quadrics $SqF_1q = 0$, $SqF_2q = 0$. Take therefore any quadric to which this tetrahedron is self-conjugate; F_1 is determined and F_2 follows from (207).

Otherwise the function

$$F_2(q) \, . \, (abcd) = xa'(qbcd) + yb'(aqcd) + zc'(abqd) + wd'(abcq) \quad . \quad (209)$$

is self-conjugate (Art. 21) when $(a'b'c'd')$ is the tetrahedron reciprocal to $(abcd)$; and on comparison with (208) the function F_1 may be written down. The four scalars x, y, z, w are arbitrary, as might have been expected, since each self-conjugate function involves ten constants, while f involves sixteen.

If two functions can be simultaneously reduced to the forms

$$f_1 = F^{-1}F_1, \quad f_2 = F^{-1}F_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210),$$

the united points of f_1 and f_2 must form tetrahedra self-conjugate to a common quadric, or

$$Fa_1 = x_1[b_1c_1d_1], \text{ \&c.} \quad Fa_2 = x_2[b_2c_2d_2], \text{ \&c.} \quad . \quad . \quad . \quad . \quad (211).$$

In this case the eight united points are so related that any quadric

$$SqF_3q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

which passes through seven, passes also through the eighth.

The condition that the point a_1 should be on the quadric may be written (211)

$$Sa_1F_3F^{-1}[b_1c_1d_1] = 0, \quad \text{or} \quad (F^{-1}F_3a_1, b_1, c_1, d_1) = 0 \quad . \quad . \quad . \quad (213),$$

and if b_1, c_1, and d_1 are likewise on the quadric, it follows (Art. 24) that the first invariant of the function $F^{-1}F_3$ vanishes. Hence if the points a_2, b_2, c_2 are also on the quadric, the remaining point d_2 must lie on the quadric too.* Thus one of the united points is fixed with respect to the others, and the functions f_1 and f_2 must satisfy three conditions, which reduce the number of their constants to 29, and this is precisely the number involved in the two quotients $F^{-1}F_1$, $F^{-1}F_2$.

* Compare Appendix to the New Edition of the 'Elements of Quaternions,' vol. ii., p. 364.

SECTION IX.

The Determination of Linear Transformations which satisfy Certain Conditions.

52. The results of Art. 47 afford a simple solution of such problems as to find a *transformation which shall convert one quadric into another.*

Symbolically this problem amounts to solving the equation

$$F_1 = f' F_2 f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

which connects two known self-conjugate functions F_1 and F_2 with an unknown function f and its conjugate.

The first quadric is reduced to the unit sphere by the transformation

$$q_1 = F_1^{\frac{1}{2}} q, \quad \text{so that } SqF_1q = Sq_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215).$$

The unit sphere is converted into itself by the transformation (Art. 47)

$$q_2 = gq_1, \quad \text{so that } Sq_1^2 = Sq_2^2 \text{ if } gg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (216);$$

and finally the sphere is converted into the second quadric by the transformation

$$q_3 = F_2^{-\frac{1}{2}} q_2, \quad \text{so that } Sq_2^2 = Sq_3F_2q_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217).$$

Thus the transformation

$$f = F_2^{-\frac{1}{2}} g F_1^{\frac{1}{2}}, \quad gg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

converts the first quadric into the second; and evidently this is the most general transformation fulfilling the conditions.

53. *To convert an arbitrary function "f" into itself,* observe that the transformation must belong to the group (compare (218)),

$$F = f_0^{-\frac{1}{2}} g f_0^{\frac{1}{2}}; \quad \text{where } f = f_0 + f_{\prime}, \ gg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (219),$$

which converts the self-conjugate part f_0 of the function into itself.

The problem therefore reduces to the determination of g from the equation (compare (214))

$$f_{\prime} = f_0^{\frac{1}{2}} g' f_0^{-\frac{1}{2}} \cdot f_{\prime} \cdot f_0^{-\frac{1}{2}} g f_0^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220).$$

The form of this equation suggests the new function

$$f_{\prime\prime} = f_0^{-\frac{1}{2}} f_{\prime} f_0^{\frac{1}{2}}, \quad f_{\prime\prime} + f'_{\prime\prime} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221);$$

and the equation (220) reduces to

$$f_{\prime\prime} = g' f_{\prime\prime} g \text{ or } g f_{\prime\prime} = f_{\prime\prime} g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222);$$

and the problem reduces to the determination of a function g commutative with the known function $f_{\prime\prime}$.

The function g must possess the same* united points as $f_{\prime\prime}$; or g must be of the form (compare (221))

$$g = x + y f_{\prime\prime} + z f_{\prime\prime}^2 + w f_{\prime\prime}^3; \quad g' = x - y f_{\prime\prime} + z f_{\prime\prime}^2 - w f_{\prime\prime}^3 \quad . \quad . \quad (223).$$

Actually multiplying these expressions we find (219)

$$g g' = 1 = (x + z f_{\prime\prime}^2)^2 - (y f_{\prime\prime} + w f_{\prime\prime}^3)^2 \quad . \quad . \quad . \quad . \quad . \quad (224);$$

and as this equation must be equivalent to the latent quartic of the function $f_{\prime\prime}$, it must vanish when for $f_{\prime\prime}$ are substituted its latent roots. Now (Art. 23) the latent roots of $f_{\prime\prime}$ are identical with those of $f_{\prime}$, and the latent roots of the latter function (Art. 12) are of the form $\pm\sqrt{s}$, $\pm\sqrt{-s'}$. Substituting and reducing, we find in terms of the two invariants $n_{\prime}''$ and $n_{\prime}$ of $f_{\prime}$, two equations

$$\begin{aligned} 1 &= x^2 + n_{\prime}(2yw - z^2 - n_{\prime}'' w^2), \\ 0 &= 2xz - y^2 + n_{\prime}''(2yw - z^2 - n_{\prime}'' w^2) + n_{\prime} w^2 \quad . \quad . \quad . \quad . \quad (225) \end{aligned}$$

connecting the four scalars x, y, z and w. Hence, reverting to the original functions, the transformation

$$F = x + y f_0^{-1} f_{\prime} f_0 + z f_0^{-1} f_{\prime}^2 f_0 + w f_0^{-1} f_{\prime}^3 f_0 \quad . \quad . \quad . \quad . \quad . \quad (226)$$

converts the function f into itself; in other words, it converts the quadric and the linear complex

$$S q f_0 q = 0, \quad S' p f_{\prime} q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

into themselves.

54. Passing on to the general case, let us consider the relations which must be satisfied when one function f can be converted into another F; or the conditions that a quadric and a complex can be simultaneously converted into another given quadric and another given complex.

* Compare Art. 38, and the Appendix to HAMILTON'S 'Elements,' vol. ii., p. 364.

The first quadric is converted into the second by the transformation (218) $F_0^{-\frac{1}{2}}gf_0^{\frac{1}{2}}$, and this converts the first complex into

$$Sp\,.\,F_0^{\frac{1}{2}}gf_0^{-\frac{1}{2}}.f_{\prime}.f_0^{-\frac{1}{2}}g'F_0^{\frac{1}{2}}q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228);$$

and, on comparison with the second complex, it appear that we must have

$$F_0^{-\frac{1}{2}}F_{\prime}F_0^{-\frac{1}{2}} = gf_0^{-\frac{1}{2}}f_{\prime}f_0^{-\frac{1}{2}}g', \text{ or } F_{\prime\prime} = gf_{\prime\prime}g^{-1} \quad . \quad . \quad . \quad . \quad . \quad (229),$$

where we have introduced two new symbols for greater convenience.

Equation (229) requires the functions $F_{\prime\prime}$ and $f_{\prime\prime}$ to have the same latent roots (Art. 23); or again, $F_0^{-1}F_{\prime}$ and $f_0^{-1}f_{\prime}$ must have the same latent roots, and this is the sufficient condition, for it appears, on substituting a united point of $F_{\prime\prime}$ in (229), that the function g' must convert the united points of $F_{\prime\prime}$ into those of $f_{\prime\prime}$; and it is always possible to find a function g' capable of doing this, because (Art. 12) the united points of the two functions are quadrilaterals upon the unit sphere, and a function g' always converts this sphere into itself.

Thus, given two quadrics and two linear complexes, *it is possible to transform simultaneously one quadric into the other and one complex into the other whenever the latent roots of the functions* $f_0^{-}f_{\prime}$ and $F_0^{-1}F_{\prime}$ *are proportional.*

55. *To find a transformation which shall convert one conic into another.*

The essentials of the problem are contained in the equation

$$f(at^2 + 2bt + c) = w\,(a's^2 + 2b's + c') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230).$$

In order that the right-hand number may be a quadratic function of t, it is necessary to have

$$w = (vt + v')^2, \quad s = \frac{ut + u'}{vt + v'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231);$$

so that on equating powers of t we obtain, in the usual notation for binary quantics,

$$fa = (a'b'c'\between uv)^2\,;\; fb = (a'b'c'\between uv\between u'v')\,;\; fc = (a'b'c'\between u'v')^2 \quad . \quad . \quad . \quad . \quad (232).$$

These relations are not sufficient to determine the function; we may arbitrarily assume two quaternions d and d' and write $fd = d'$ (Art. 3). The function thus determined involves eleven arbitrary constants, the four u, u', v, v' which regulate the correspondence of point to point on the conics, and the seven (eight less one) involved in the two quaternions d and d', for multiplying these by a common factor is without effect.

56. *In order to transform simultaneously two given conics into two other conics, a single relation must exist connecting the conics.*

Affixing numeral suffixes, 1, 2, to the various symbols in (232), we obtain the system of six equations which the function f must satisfy. Any six quaternions are connected by two relations, and the equations

$$(s_1x_{11}+s_2x_{12})\,a_1+2\,(s_1y_{11}+s_2y_{12})\,b_1+(s_1z_{11}+s_2z_{12})\,c_1$$
$$+\,(s_1x_{21}+s_2x_{22})a_2+2\,(s_1y_{21}+s_2y_{22})\,b_2+(s_1z_{21}+s_2z_{22})\,c_2=0 \quad . \quad (233)$$

$$(s'_1x'_{11}+s'_2\,x'_{12})\,a'_1+2\,(s'_1y'_{11}+s'_2y'_{12})\,b'_1+(s'_1z'_{11}+s'_2z'_{12})c'_1$$
$$+\,(s'_1x'_{21}+s'_2\,x'_{22})a'_2+2\,(s'_1y'_{21}+s'_2y'_{22})\,b'_2+(s'_1z'_{21}+s'_1z'_{22})\,c'_2=0\,;$$

in which s_1, s_2, s'_1, s'_2 are arbitrary, but the other scalars given may be taken as determining the two pairs of relations connecting the two sets of six quaternions.

When the left-hand members of the equations analogous to (232) are multiplied by $s_1x_{11}+s_2x_{12}$, &c., and added, the sum is zero; and the sum of the right-hand members is (with an obvious abbreviation)

$$\{(s_1x_{11}+s_2x_{12},\ s_1y_{11}+s_2y_{12},\ s_1z_{11}+s_2z_{12}\between u_1v_1)^2a'_1+2\between u_1v_1\between u'_1v'_1)b'_1+\between u'_1v'_1)^2c'_1\}$$
$$+\,\{(s_1x_{21}+s_2x_{22},\ s_1y_{21}+s_2y_{22},\ s_1z_{21}+s_2z_{22}\between u_2v_2)^2a'_2+2\between u_2v_2\between u'_2v'_2)\,b'_2+$$
$$\between u'_2v'_2)^2c'_2\}=0 \quad . \quad . \quad . \quad . \quad (234),$$

or, for simplicity,

$$(s_1X_{11}+s_2X_{12})\,a'_1+2\,(s_1Y_{11}+s_2Y_{12})\,b'_1+(s_1Z_{11}+s'_2Z_{12})\,c'_1$$
$$+\,(s_1X_{21}+s_2X_{22})\,a'_2+2\,(s_1Y_{21}+s_2Y_{22})\,b'_2+(s_1\,Z_{21}+s_2Z_{22})\,c'_2=0\,. \quad . \quad (235)$$

where X_{11} is a quadratic in $u_1\ v_1$, and $Y_{11}\ Z_{11}$ its successive polars to $u'_1v'_1$. This relation connecting the six quaternions must be equivalent to the second equation (233), so we may equate corresponding coefficients of quaternions, when we shall obtain six equations linear in s_1, s_2, s'_1, s'_2. Let s'_1 and s'_2 be eliminated from them. The result is the system of determinants

$$\left\|\begin{matrix} X_{11}s_1+X_{12}s_2 & Y_{11}s_1+Y_{12}s_2 & Z_{11}s_1+Z_{12}s_2 & X_{21}s_1+X_{22}s_2 & Y_{21}s_1+Y_{22}s_2 & Z_{21}s_1+Z_{22}s_2 \\ x'_{11} & y'_{11} & z'_{11} & x'_{21} & y'_{21} & z'_{21} \\ x'_{12} & y'_{12} & z'_{12} & x'_{22} & y'_{22} & z'_{22} \end{matrix}\right\|=0. \quad (236),$$

which is equivalent to four equations. But s_1 and s_2 are arbitrary; consequently this system of determinants breaks up into two independent systems, equivalent to eight equations among the eight scalars u, v. The eight scalars enter homogeneously into the equations, and may be eliminated, leaving a single condition connecting the four conics, in order that it may be possible to find a transformation which shall convert two of them into the remaining two.

57. *A twisted cubic may be transformed into another twisted cubic with arbitrary correspondence of the points.*

The equation of transformation of one arbitrary twisted cubic into another is (compare Art. 43)

$$f(abcd\between t,\ 1)^3=(a'b'c'd'\between ut+u',\ vt+v')^3 \quad . \quad . \quad . \quad . \quad . \quad (237).$$

Hence equating coefficients of t, four equations are obtained which serve to determine f for arbitrary values of u, v, u', v' (Art. 3). These four scalars may be selected in any way we please.

58. *A single condition connects two quartics of the second class* when it is possible to transform one into the other.*

The equation of transformation is

$$f(abcde \between t, 1)^4 = (a'b'c'd'e' \between ut + u', vt + v')^4 \quad . \quad . \quad . \quad . \quad . \quad (238),$$

and five equations of condition may be written down analogous to (232).

Let the relations connecting the sets of five quaternions be

$$x_0 a + 4x_1 b + 6x_2 c + 4x_3 d + x_4 e = 0,\ y_0 a' + 4y_1 b' + 6y_2 c' + 4y_3 d' + y_4 e' = 0 \ . \ (239);$$

then, as in Art. 56 (234), we obtain the equation

$$X_0 a' + 4X_1 b' + 6X_2 c' + 4X_3 d' + X_4 e' = 0 \quad . \quad . \quad . \quad . \quad . \quad (240),$$

where

$$X_0 = (x_0 x_1 x_2 x_3 x_4 \between uv)^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241),$$

and where X_1, X_2, X_3, and X_4 are its successive polars to $u'v'$.

On comparison of (240) and (239) the equality of ratios

$$\frac{X_0}{y_0} = \frac{X_1}{y_1} = \frac{X_2}{y_2} = \frac{X_3}{y_3} = \frac{X_4}{y_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (242)$$

is seen to be necessary. This is equivalent to four quartic equations in the homogeneous variables u, v, u', v', and the resultant of these four equations equated to zero is the single condition in question.

SECTION X.

Covariance of Functions.

59. The subject of covariance naturally arises in connection with the various transformations lately considered, but as the principles laid down in the note on Invariants of Linear Vector Functions printed in the Appendix to the new edition of 'Hamilton's Elements' apply with but slight modification to the more general case of quaternion functions, it does not seem desirable to go into any great detail.

* A quartic of the second class is the partial intersection of a cubic and quadric surface, and only one quadric can be drawn through it.

We propose to obtain functions from given functions f_1, f_2, f_3, &c., which fall into certain classes connected by invariantal relations. We denote two arbitrary functions by the Roman capitals X, Y, and we consider the transformations effected by multiplying a given function by X and into Y.

This transformation changes the series of functions

$$f_1, f_2, f_3, \ldots f_1 f_2^{-1} f_3, \ldots f_1 f_2^{-1} f_3 f_4^{-1} f_5 \ldots \quad (243)$$

into the series

$$\mathrm{X}f_1\mathrm{Y}, \mathrm{X}f_2\mathrm{Y}, \mathrm{X}f_3\mathrm{Y}, \ldots \mathrm{X}f_1 f_2^{-1} f_3\mathrm{Y}, \ldots \mathrm{X}f_1 f_2^{-1} f_3 f_4^{-1} f_5\mathrm{Y} \quad (244);$$

and we shall speak of this as the (XY) class.

The series

$$f_1^{-1}, f_2^{-1}, f_3^{-1}, \ldots f_1^{-1} f_2 f_3^{-1}, \ldots f_1^{-1} f_2 f_3^{-1} f_4 f_5^{-1} \ldots \quad (245)$$

becomes

$$\mathrm{Y}^{-1}f_1^{-1}\mathrm{X}^{-1}, \mathrm{Y}^{-1}f_2^{-1}\mathrm{X}^{-1}, \mathrm{Y}^{-1}f_3^{-1}\mathrm{X}^{-1}, \ldots \mathrm{Y}^{-1}f_1^{-1}f_2f_3^{-1}\mathrm{X}^{-1}, \ldots$$
$$\mathrm{Y}^{-1}f_1^{-1}f_2f_3^{-1}f_4f_5^{-1}\mathrm{X}^{-1} \ldots \quad (246),$$

and this is the (Y^{-1}, X^{-1}) class.

The series

$$f_1 f_2^{-1}, f_2 f_3^{-1}, \ldots f_1 f_2^{-1} f_3 f_4^{-1} \quad \ldots \quad (247)$$

is the (XX^{-1}) class, transforming into the series

$$\mathrm{X}f_1 f_2^{-1}\mathrm{X}^{-1}, \mathrm{X}f_2 f_3^{-1}\mathrm{X}^{-1}, \ldots \mathrm{X}f_1 f_2^{-1} f_3 f_4^{-1}\mathrm{X}^{-1} \ldots \quad (248);$$

and finally the series

$$f_1^{-1}f_2, \quad f_2^{-1}f_3, \ldots \quad f_1^{-1}f_2f_3^{-1}f_4, \quad \ldots \quad (249)$$

forms the ($\mathrm{Y}^{-1}\mathrm{Y}$) class, as it transforms into

$$\mathrm{Y}^{-1}f_1^{-1}f_2\mathrm{Y}, \quad \mathrm{Y}^{-1}f_2^{-1}f_3\mathrm{Y}, \ldots \mathrm{Y}^{-1}f_1^{-1}f_2f_3^{-1}f_4\mathrm{Y} \quad \ldots \quad (250).$$

Inverse functions of the (XY) class belong to the ($\mathrm{Y}^{-1}\mathrm{X}^{-1}$) class, and conversely; inverse functions of the classes (XX^{-1}) or ($\mathrm{Y}^{-1}\mathrm{Y}$) belong to their own class, and so also do products and quotients of functions of these classes. The product of an (XY) function into a ($\mathrm{Y}^{-1}\mathrm{X}^{-1}$) function is an ($\mathrm{XX}^{-1}$) function, and so on.

In like manner there are four classes for the conjugate functions, as appears on taking the conjugates of a typical function. The annexed scheme exhibits the eight classes, the conjugates being printed under their correspondents:—

$$\begin{array}{cccc} (\mathrm{XY}), & (\mathrm{Y}^{-1}\mathrm{X}^{-1}) & (\mathrm{XX}^{-1}), & (\mathrm{Y}^{-1}\mathrm{Y}) \\ (\mathrm{Y'X'}) & (\mathrm{X'}^{-1}\mathrm{Y'}^{-1}) & (\mathrm{X'}^{-1}\mathrm{X'}) & (\mathrm{Y'Y'}^{-1}) \end{array} \quad \ldots \quad (251).$$

60. When we deal with quadrics or complexes, or when the condition is imposed that self-conjugate functions remain self-conjugate, the classes of the conjugate type

coincide with those originally found, but in a different order. In this case Y is the conjugate of X, and the scheme (251) becomes

$$\begin{array}{cccc} (XX') & (X'^{-1}X^{-1}) & (XX^{-1}) & (X'^{-1}X') \\ (XX') & (X'^{-1}X^{-1}) & (X'^{-1}X') & (X\,X^{-1}) \end{array} \quad . \quad . \quad . \quad . \quad (252).$$

In this case the conjugate of a transformed function is the transformed function of the conjugate.

Again, in the general case, when $Y = X^{-1}$, the types of the upper row (251) merge in the single type (XX^{-1}), and the conjugates in the type $(X'^{-1}X')$.

Finally, all types unite in the single class (XX') when X is the inverse of its conjugate (Art. 47).

61. Covariant functions may be derived by the following general process, as well as by multiplication and division. For arbitrary scalars, t_1, t_2, t_3, &c.,

$$n_t(\Sigma tf)^{-1}[abc] = [\Sigma tf'a,\ \Sigma tf'b,\ \Sigma tf'c] = \Sigma t_1t_2t_3\ F_{123}[abc] \quad . \quad . \quad . \quad (253),$$

where n_t is the fourth invariant of Σtf, and where

$$F_{123}[abc] = \Sigma[f'_1a, f'_2b, f'_3c] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (254),$$

the summation in this last equation referring to permutation of the suffixes.

These functions belong to the $(Y^{-1}X^{-1})$ class, because

$$\Sigma[Y'f'_1X'a,\ Y'f'_2X'b,\ Y'f'_3X'c] = n_YY^{-1}F_{123}n_XX^{-1}[abc] \quad . \quad . \quad . \quad (255),$$

n_X and n_Y being the fourth invariants of X and Y.

In like manner functions of the (XY) class are obtainable in the form

$$f_{123}[abc] = \Sigma[F_1'a, F_2'b, F_3'c]\,;\ F_1'[abc] = [f_1a, f_1b, f_1c] \quad . \quad . \quad . \quad (256).$$

62. Although rather foreign to the subject of this paper, it may be as well to indicate the nature of the Hamiltonian quaternion invariants of a system of functions. It was stated in a paper on Quaternion Arrays* that these invariants are included in the quotient

$$\left\{ \begin{array}{cccc} f_1a & f_1b & f_1c & f_1d \\ f_2a & f_2b & f_2c & f_2d \\ . & . & . & . \\ . & . & . & . \\ f_na & f_nb & f_nc & f_nd \\ a & b & c & d \end{array} \right\} \div (abcd) \quad . \quad . \quad . \quad . \quad . \quad (257),$$

formed by dividing a four-column array by $(abcd)$, each row of the array consisting of the results of operation by a single function on four arbitrary quaternions. Briefly,

* 'Trans. Roy. Irish Acad.,' vol. 32, p. 30.

a quaternion array may be defined as a function which vanishes if, and only if, the constituents of every row can be linearly connected by the same set of scalar multipliers. It is multiplied by a scalar if every constituent in a column is multiplied by that scalar; and the sign of the array is changed if contiguous columns are transposed.

These laws are precisely the laws which govern the function $(abcd)$, which is in fact a one-row array, so that if in (257) we replace any one of the four quaternions by any quaternion $xa + yb + zc + wd$, the quotient remains unchanged. The quotient is therefore an invariant in the Hamiltonian sense; it remains unchanged when the four quaternions a, b, c, d are operated on by the function Y.

If we regard the lowest row as consisting of the results of operating by the special linear function *unity* on a, b, c and d, and if we replace $f_1, f_2 \ldots f_n$ by $\mathrm{X}f_1\mathrm{Y}$, $\mathrm{X}f_2\mathrm{Y}, \ldots \mathrm{X}f_n\mathrm{Y}$ and unity in the last row by XY; to a factor, $n_\mathrm{Y}n_\mathrm{X}$, the quotient becomes the corresponding quotient for the system of functions

$$\mathrm{X}f_1\mathrm{X}_1^{-1},\ \mathrm{X}f_2\mathrm{X}_2^{-1}, \ldots \mathrm{X}f_n\mathrm{X}^{-1}.$$

SECTION XI.

The Numerical Characteristics of Certain Curves and Assemblages of Points.

63. In order to facilitate future investigations, we shall determine the numerical characteristics of certain curves and systems of points which frequently occur.

Using the symbol Q_n to denote a homogeneous quaternion function of q of the order M_n, it appears from Salmon's chapter on the "Order of Restricted Systems of Equations" in his 'Modern Higher Algebra,' that

$$\{Q_1Q_2\} = 0, \quad \text{or} \quad t_1Q_1 + t_2Q_2 = 0 \quad \ldots \ldots \quad (258)$$

represents a system of points whose number is

$$M_1^3 + M_1^2M_2 + M_1M_2^2 + M_2^3 \quad \ldots \ldots \quad (259).$$

64. In like manner the chapter cited enables us to write down the order of the curve represented by

$$[Q_1Q_2Q_3] = 0, \quad \text{or} \quad t_1Q_1 + t_2Q_2 + t_3Q_3 = 0 \quad \ldots \quad (260);$$

but as it is desirable to determine also its rank and the number of its apparent double points, we shall adopt a different method.

The quaternions a and b being arbitrary, the identity

$$Q_1(Q_2Q_3ab) + Q_2(Q_3abQ_1) + Q_3(abQ_1Q_2) + a(bQ_1Q_2Q_3) + b(Q_1Q_2Q_3a) = 0 \quad . \; (261),$$

shows that the two surfaces

$$(aQ_1Q_2Q_3) = 0, \quad (bQ_1Q_2Q_3) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; (262)$$

intersect in the curve (260), and also in a complementary curve common to the three surfaces

$$(abQ_2Q_3) = 0, \quad (abQ_3Q_1) = 0, \quad (abQ_1Q_2) = 0 \quad . \; . \; . \; . \; . \; (263);$$

for when (262) is satisfied, the identity shows that either (260) or (263) must be satisfied.

Let m denote the order of the curve (260); then the order of the complementary is

$$(M_1 + M_2 + M_3)^2 - m = m' \quad . \; . \; . \; . \; . \; . \; . \; . \; (264),$$

the orders of the two surfaces (262) being $M_1 + M_2 + M_3$.

Again, considering the intersection of the second and third surfaces (263), it follows from the identity that they intersect in the complementary curve and in the new curve

$$[Q_1ab] = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (265);$$

and because the orders of the surfaces are $M_1 + M_3$ and $M_1 + M_2$, the order $m_{,}$ of this new curve is connected with m' by the relation

$$(M_1 + M_2)(M_1 + M_3) - m' = m_{,} \quad . \; . \; . \; . \; . \; . \; . \; (266).$$

Again, writing down the identity

$$a(bcqQ_1) + b(cqQ_1a) + c(qQ_1ab) + q(Q_1abc) + Q_1(abcq) = 0 \quad . \; (267),$$

in which q is the variable quaternion, while a, b and c are constants, it appears exactly as before that the surfaces

$$(abqQ_1) = 0, \quad (abcQ_1) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; (268),$$

of orders $M_1 + 1$ and M_1, intersect in the curve (265) and in a complementary curve which is obviously the complete intersection of the surfaces

$$(abcq) = 0, \quad (abcQ_1) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; (269);$$

that is, a plane and a surface of order M_1.

Now the relations*

* SALMON'S 'Geometry of Three Dimensions,' Arts. 345, 346.

$$2(h - h') = (m - m')(\mu - 1)(\nu - 1), \quad r - r' = (m - m')(\mu + \nu - 2) \quad . \quad (270)$$

connect the number of apparent double points (h) and the rank (r) of a curve with those of its complementary in the intersection of two surfaces of orders μ and ν. But we know the characteristics of the plane curve (269) to be

$$m_{\prime}' = M_1, \quad r_{\prime}' = M_1(M_1 - 1), \quad h_{\prime}' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (271);$$

and hence we find the characteristics of its complementary (265),

$$m_{\prime} = M_1^2, \quad r_{\prime} = 2M_1^2(M_1 - 1), \quad 2h_{\prime} = M_1^2(M_1 - 1)^2 \quad . \quad . \quad (272);$$

and these in turn give the characteristics for the curve m',

$$m' = \Sigma_2; \quad r' = \Sigma_2(\Sigma_1 - 2) + \Sigma_3; \quad 2h' = \Sigma_2(\Sigma_2 - \Sigma_1 + 1) - \Sigma_3 \quad . \quad (273),$$

and, finally, for the original curve (260) we have

$$m = \Sigma_1^2 - \Sigma_2; \quad r = 2\Sigma_1^3 - 3\Sigma_1\Sigma_2 + \Sigma_3 - 2(\Sigma_1^2 - \Sigma_2);$$
$$2h = (\Sigma_1^2 - \Sigma_2)^2 - (2\Sigma_1^3 - 3\Sigma_1\Sigma_2 + \Sigma_3) + (\Sigma_1^2 - \Sigma_2) \quad . \quad (274),$$

where Σ_1, Σ_2 and Σ_3 are the sum, the sum of the products in pairs, and the product of the three quantities M_1, M_2 and M_3.

As examples, for the twisted cubic

$$[f_1qf_2qa] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (275),$$

$M_1 = M_2 = 1$, $M_3 = 0$, and $\Sigma_1 = 2$, $\Sigma_2 = 1$, $\Sigma_3 = 0$, so that $m = 3$, $r = 4$, $h = 1$. For the curve

$$[f_1qf_2qf_3q] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (276),$$

$\Sigma_1 = 3$, $\Sigma_2 = 3$, $\Sigma_3 = 1$; and $m = 6$, $r = 16$, $h = 7$.

These numbers admit of course of simple verification.*

65. In like manner proceeding one step further we calculate the characteristics of the curve common to the five surfaces obtained by equating to zero the coefficients in the identity

$$Q_1(Q_2Q_3Q_4Q\ ) + Q_2(Q_3Q_4Q_5Q_1) + Q_3(Q_4Q_5Q_1Q_2) + Q_4(Q_5Q_2Q_2Q_3) + Q_5(Q_1Q_2Q_3Q_4)$$
$$= 0 \quad . \quad . \quad (277)$$

to be

$$m = \Sigma M_1M_2, \quad r = \Sigma M_1\Sigma M_1M_2 + \Sigma M_1M_2M_3 - 2\Sigma M_1M_2 \quad . \quad . \quad . \quad (278);$$

this curve being the complementary of (260) for the fourth and fifth surfaces.

The curve common to the five surfaces may be conveniently designated by the equation in double brackets

* The expression for the rank of a curve, 'Modern Higher Algebra,' Art. 284, seems to require modification.

$$((Q_1Q_2Q_3Q_4Q_5)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (279)$$

which is intended to denote that every set of four of the included quaternions is linearly connected.

66. For the number of points common to the surfaces whose equations are obtained by deleting two of the quaternions included in triple brackets

$$(((Q_1Q_2Q_3Q_4Q_5Q_6))) = 0 \quad . \quad . \quad . \quad . \quad . \quad (280),$$

SALMON'S formula ('Modern Higher Algebra') gives

$$N = \Sigma M_1M_2M_3 \quad . \quad . \quad . \quad . \quad . \quad (281).$$

67. To complete the scheme, we may regard the equation

$$[[Q_1Q_2Q_3Q_4]] = 0 \quad . \quad . \quad . \quad . \quad . \quad (282),$$

as requiring the four quaternions Q_1, Q_2, Q_3, Q_4 to be collinear; or the four curves (260), obtained by omitting one quaternion, to have common points. If these points exist they satisfy the equation (compare (279))

$$((aQ_1Q_2Q_3Q_4)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (283),$$

or lie on the complementary common to the five surfaces.

A curve meets its complementary ('Geometry of Three Dimensions,' Art. 346) in

$$t = m(\mu + \nu - 2) - r \quad . \quad . \quad . \quad . \quad . \quad (284)$$

points, and in particular for the curve $[Q_1Q_2Q_3]$ and the two surfaces $(aQ_1Q_2Q_3) = 0$, $(Q_1Q_2Q_3Q_4) = 0$, we find the number to be (compare (274))

$$t_4 = \Sigma_1\Sigma_2 - \Sigma_3 + M_4(\Sigma_1^2 - \Sigma_2) \quad . \quad . \quad . \quad . \quad . \quad (285).$$

These points are generally variable with the arbitrary quaternion a.

Again, the surface

$$(aQ_1Q_2Q_3)(bQ_1Q_2Q_4) + u(aQ_1Q_2Q_4)(bQ_1Q_2Q_3) = 0 \quad . \quad . \quad . \quad . \quad (286)$$

intersects $(Q_1Q_2Q_3Q_4) = 0$ in $[Q_1Q_2Q_3] = 0$, $[Q_1Q_2Q_4] = 0$, and in the complementary corresponding to b. When we seek the intersection of the curve $[Q_1Q_2Q_3] = 0$ with its complex complementary on this surface, the number of points is found to be $2t_4 + M_1^3 + M_1^2M_2 + M_1M_2^2 + M_2^3$, and these can all be accounted for by (285) and (259).

We can also in this manner determine the points common to the two complementaries (283) answering to a and b to be $\Sigma M_1M_2M_3$, employing the characteristics (278), and putting $M_5 = 0$.

SECTION XII.

On the Geometrical Relations depending on Two Functions and on the Four Functions f, f', f_0 and $f_{\prime}$.

68. We devote this section to the study of the geometrical relations connecting a function f with its conjugate f', its self-conjugate part f_0 and its non-conjugate part $f_{\prime}$ (Art. 9), and to the relations connecting a pair of arbitrary functions f_1 and f_2.

The quadric

$$Sqfq = Sqf'q = Sqf_0q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (287)$$

is the locus of a point which is conjugate with respect to the unit sphere to its correspondent in each of the transformations due to f, f' and f_0.

The linear complex

$$Spf_{\prime}q = 0, \text{ or } Spfq = Sqfp, \text{ or } Spf'q = Sqf'p \quad . \quad . \quad . \quad . \quad (288),$$

may be written in the form (compare p. 223).

$$S\text{PQ}'Sf\text{Q} = S\text{QP}'Sf\text{P}, \quad (\text{P}Sp = p,\ \text{P}'Sf\text{P} = f\text{P}) \quad . \quad . \quad . \quad . \quad . \quad (289),$$

which expresses that the product of the perpendiculars from Q′, the derived of one point Q, and from the centre of reciprocation on the polar plane of another point P with respect to the unit sphere, multiplied by the perpendicular (SfQ) from Q on the plane which is projected to infinity by the transformation, is equal to the corresponding product of three perpendiculars found by interchanging P and Q. This property is also true when f is replaced by its conjugate f'.

The equation of the complex may also be regarded as representing the assemblage of lines converted by f_i into conjugate lines with respect to the unit sphere.

69. In order to determine the four lines common to the quadric and the linear complex, observe that the point of contact $(f_0^{-1}h)$ of a plane $Shq = 0$ with the quadric must also be the point of concourse $(f_i^{-1}h)$ of the lines of the complex in that plane, in order that the plane may contain lines common to the two assemblages. Therefore the points e in which the pairs of common lines intersect satisfy the equations

$$e = f_i^{-1}h = u^{-1}f_0^{-1}h, \text{ or } h = f_i e = uf_0 e \quad . \quad . \quad . \quad . \quad . \quad (290).$$

Thus four points e are determined, the united points of the function $f_0^{-1}f_i$.

It appears, as in Art. 12, that the latent roots of this function are equal and opposite, and that the united points form a quadrilateral on the quadric.

Otherwise, the invariants of $f_0^{-1}f_i$ and of $f_i f_0^{-1}$ are identical (Art. 23), and these functions satisfy the same symbolic quartic; and because their conjugates, $-f_i f_0^{-1}$ and $-f_0^{-1}f_i$ likewise satisfy the same quartic, it must be of the form

$$(f_0^{-1}f_i)^4 + N''(f_0^{-1}f_i)^2 + N = 0, \text{ or } ((f_0^{-1}f_i)^2 - u_1^2)((f_0^{-1}f_i)^2 - u_2^2) = 0 \quad . \quad (291).$$

Hence the lines in question are determined on solution of a quadratic equation.

When these four points e_1, e'_1, e_2, e'_2 are taken as points of reference,* so that

$$q = \frac{xe_1 + ye'_1}{\sqrt{Se_1 f_0 e'_1}} + \frac{ze_2 + we'_2}{\sqrt{Se_2 f_0 e'_2}},\ p = \frac{x'e_1 + y'e'_1}{\sqrt{Se_1 f_0 e'_1}} + \frac{z'e_2 + w'e'_2}{\sqrt{Se_2 f_0 e'_2}} \quad . \quad . \quad (292)$$

the equations of the quadric and complex may by the aid of (290) (compare again Art. 12) be reduced to the forms

$$xy + zw = 0 \qquad u_1(xy' - x'y) + u_2(zw' - z'w) = 0 \, . \quad . \quad . \quad . \quad (293).$$

70. The locus of points whose correspondents are in perspective with a fixed point a is the twisted cubic

$$fq + tq = a \quad \text{or} \quad [fq, q, a] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (294),$$

and the locus of lines which pass through a fixed point a and connect a point and its correspondent is the cone

$$fq + tq = xfa + ya \quad \text{or} \quad (fqqfaa) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (295).$$

* Observe that these four points e are the only points for which

$$fq \doteq f'q \doteq f_0 q \doteq f_i q,$$

the signs $\doteq$ being used to denote equality when the quaternions are multiplied by a suitable factor. For vector functions

$$\phi\rho = \phi'\rho = \phi_0\rho$$

only when $\rho = \epsilon$, where ϵ is the spin-vector.

The complex of lines connecting points and their correspondents has for its equation*

$$(fppfqq) = 0 \quad \ldots\ldots\ldots\ldots \quad (296);$$

and the locus of points whose connectors to correspondents intersect a fixed line ab is the quadric surface

$$(fqqab) = 0 \quad \ldots\ldots\ldots\ldots \quad (297).$$

The reciprocal of the complex (296) is the complex of the conjugate

$$(f'ppf'qq) = 0 \quad \ldots\ldots\ldots\ldots \quad (298);$$

for the line $p'q'$ is reciprocal to the line p, fp if $Spp' = Spq' = Spf'p' = Spf'q' = 0$, which requires p', q', $f'p'$, $f'q'$ to be coplanar.

The formulæ of this article comprise many theorems with respect to the normals of confocal quadrics. It may also be observed that the complex (296) is unchanged when f is replaced by $(f+x)(f+y)^{-1}$.

71. *An arbitrary quadric has eight generators which connect a point and its correspondent in an arbitrary transformation.* This is the extension of HAMILTON'S celebrated theory of the umbilical generators. (Compare Art. 40.)

The conditions that the line $q = fa + sa$ should be a generator of the arbitrary quadric surface

$$SqFq = 0 \quad \ldots\ldots\ldots\ldots \quad (299)$$

are

$$SaFa = 0, \quad Sa(f'F + Ff')a = 0, \quad Saf'Ffa = 0 \quad \ldots\ldots \quad (300);$$

so that we can determine eight points a as the intersections of three known quadrics, and the lines joining these points to their correspondents are the common generators of the complex and the quadric.

Four of these lines are generators of one system of the quadric and four of the other system.

Four of the lines must belong to one system of generators. Let these be determined by the points a_1, a_2, a_3, a_4. The condition that the line pq should meet the line $a_1 fa_1$ is

$$(pqa_1fa_1) = 0 \quad \text{or} \quad S(pq)[a_1fa_1] + S[pq](a_1fa_1) = 0 \quad \ldots \quad (301);$$

and because any line which meets three of these four lines likewise meets the fourth, we must have for proper selection of the weights

$$(a_1fa_1)+(a_2fa_2)+(a_3fa_3)+(a_4fa_4) = 0, \; [a_1fa_1]+[a_2fa_2]+[a_3fa_3]+[a_4fa_4] = 0 \quad (302).$$

* When we refer p and q to the united points of f, the equation of this complex takes the forms

$$\Sigma(t_2t_3 + t_1t_4)(yz' - y'z)(xw' - x'w) = 0, \; \Sigma(t_2 - t_3)(t_1 - t_4)(yzx'w' + y'z'xw) = 0,$$

where

$$p = xa + yb + zc + wd, \quad q = x'a + y'b + z'c + w'd.$$

A vector equation may also be employed, for if we put $p = 1 + \alpha$, $q = p + \rho$, the equation of the complex may be replaced by

$$(f+t)\rho = u(f+s)(1+\alpha), \quad \text{or} \quad \rho = v(V(f+t)^{-1}(1+\alpha) - \alpha S(f+t)^{-1}(1+\alpha)),$$

when we eliminate s by separating the scalar and vector parts after inversion of $f+t$.

Hence the eight points common to three of the quadrics

$$(qfqa_nfa_n) = 0 \qquad (n = 1, 2, 3, \text{ or } 4) \quad . \quad . \quad . \quad . \quad . \quad (303)$$

are likewise common to the fourth. But four of these points are the united points of the function f, while the remaining four determine (297) four lines of the complex (296) which meet the four generators. These four lines are common to the quadric and the complex, and make up with the other four the complete system of eight lines.

In accordance with (302) we may write for the two sets of four lines*

$$\{a_1fa_1\} + \{a_2fa_2\} + \{a_3fa_3\} + \{a_4fa_4\} = 0,$$
$$\{a'_1fa'_1\} + \{a'_2fa'_2\} + \{a'_3fa'_3\} + \{a'_4fa'_4\} = 0 \quad . \quad . \quad . \quad . \quad (304),$$

and it may be remarked that a direct interpretation of (302) is that four equilibrating forces can be placed along the lines of either set, for the first equation (302) expressed that the resultant of four forces vanishes, and the second requires their moment with respect to the centre of reciprocation to be zero† (see (33), p. 230).

72. The locus of the united points of all functions of the system

$$(x'f + y'f' + z')^{-1}(xf + yf' + z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (305)$$

is the curve

$$[fqf'qq] = 0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (306);$$

and this curve (276) is a sextic whose rank is 16, and the number of whose apparent double points is 7.

If q is a united point of a function (305) and t the corresponding latent root, we obviously have

$$(x - tx')fq + (y - ty')f'q + (z - tz')q = 0 \quad . \quad . \quad . \quad . \quad (307),$$

whence (306) follows immediately.

The sextic curve is evidently the locus of united points of the conjugates $(xf' + yf + z)$ of functions $xf + yf' + z$, but it is not the locus of united points of conjugates of functions of the general type (305).

In the following articles we shall consider some part of the theory of two arbitrary functions f_1 and f_2, as it is partially applicable to the subject under discussion.

73. The loci of the united points of all functions of the two systems

$$(x'f_1 + y'f_2 + z')^{-1}(xf_1 + yf_2 + z) \text{ and } (x'f_1' + y'f_2' + z')^{-1}(xf_1' + yf_2' + z) \quad (308)$$

are respectively the sextic curves

$$[f_1qf_2qq] = 0, \qquad [f_1'qf_2'qq] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (309).$$

These two curves unite in the special case of $f_2 = f_1'$. The first is the locus of the united points of the system $xf_1 + yf_2 + z$, and the second is the corresponding locus for the conjugate system.

* In the notation of arrays $\Sigma\{p_nq_n\} = 0$ implies $\Sigma(p_nq_n) = \Sigma[p_nq_n] = 0$.

† If $a_n = A_nSa_n$, $fa_n = B_nSfa_n$, $A_n = 1 + \alpha_n$, $B_n = 1 + \beta_n$, the equations (302) become

$$\Sigma(\beta_n - \alpha_n)Sa_nSfa_n = 0\,; \; \Sigma V\alpha_n\beta_nSa_nSfa_n = 0.$$

The locus of the united planes of the system $xf_1 + yf_2 + z$ *is the reciprocal of the conjugate sextic.*

By the conjugate sextic we mean the second curve (309), and the proposition is obvious when we reflect that a united plane of a function is the reciprocal of the corresponding united point of its conjugate (Art. 8).

The united plane of a function of the system $xf_1 + yf_2 + z$ *cuts the sextic in three united points and in three other collinear points.*

The equation of a united plane of the function $xf_1 + yf_2 + z$ is $Sa'q = 0$, where a' is a united point of the conjugate. Writing the equation of the sextic in the form

$$x'f_1q + y'f_2q + z'q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (310),$$

and expressing that q lies in the plane, the result is

$$Sq\,(x'f_1'a' + y'f_2'a') = 0, \text{ or } Sq\,((x' - sx)\,f_1'a'_1 + (y' - sy)f_2'a') = 0 \quad (311),$$

where s is arbitrary, because $xf_1'a' + yf_2'a' + za' = ta'$.

Hence either $x' = x$, $y' = y$, and q is a united point of the function, or else

$$Sqa' = Sqf_1'a' = Sqf_2'a' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (312);$$

and the three remaining points are collinear.

In particular for the functions f, f', f_0, f_r, the polar plane with respect to the quadric and the plane of rays of the complex, corresponding to the reciprocal of a united plane of the function f, intersect in that united plane; and their common line is a three-point chord of the sextic (306).

74. Knowing the rank and number of apparent double points of the sextic, its characteristics are

$$r = 16,\ m = 6,\ n = 30,\ \alpha = 48,\ \beta = 0,\ x = 96,\ y = 72,\ g = 355,\ h = 7 \quad (313),$$

as may be verified by the formulæ printed in Arts. 326–7 of SALMON's 'Geometry of Three Dimensions.' Also the deficiency of the curve is $D = 3$.

These numbers apply reciprocally to the developable of the last article generated by the united planes. Thus the order of its cuspidal curve is 30, and six united planes pass through an arbitrary point, while sixteen pass through a line.

Through a united point the six united planes consist of the three planes which are united planes of the function possessing the united point, and three other planes intersecting in a common line (compare (312)) which is the reciprocal of a three-point chord of the second sextic.

75. *The triple chords of the sextic generate a surface of the eighth order.*

The three-point chords of a curve generate a surface of order ('Three Dimensions,' Art. 471)

$$\tfrac{1}{6}\,(m - 2)\,(6h + m - m^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (314),$$

and this reduces to 8 in the present case.

The characteristics of the cone, whose vertex is a point on the sextic and which contains the sextic, are deducible from the data of Art. 330 of the 'Geometry of Three

Dimensions.' The cone has $h - m + 2$ double edges, and consequently three triple chords pass through an arbitrary point on the sextic. The sextic is thus a triple curve on the regulus of triple chords, and the surface has no other multiple line.

76. There is yet another quadratic complex of importance in the study of a pair of functions. A point a is transformed into $xf_1a + yf_2a$ by the operation of $xf_1 + yf_2$, and as x and y vary, the locus of the transformed point is a line which we shall call the *satellite* of a.

The satellites generate the complex

$$(f_1^{-1}pf_2^{-1}pf_1^{-1}qf_2^{-1}q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (315),$$

and the form of this equation should be compared with (296) and (298). There is also the complex of conjugate satellites obtained by replacing f_1 and f_2 by their conjugates, but when the functions are self-conjugate, or when one is the conjugate of the other, the two complexes combine into one. For the functions f and f' this is

$$(f^{-1}pf'^{-1}pf^{-1}qf'^{-1}q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316).$$

The four points $fa, f'a, f_0a, f_ia$ form a harmonic range on the satellite of the point a. There are also harmonic properties connecting pencils of planes $Sqfa = 0$, $Sqf'a = 0$, $Sqf_0a = 0$, $Sqf_ia = 0$; and it may be verified that these four planes intersect in a satellite for the inverse functions. This we shall prove for the general case.

The reciprocal of the complex of satellites is the complex of the conjugate satellites for the inverse functions.

If p and q are any two points on the reciprocal of the satellite of a,

$$Spf_1a = Spf_2a = 0, \quad Sqf_1a = Sqf_2a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (317),$$

and on taking conjugates we see that the four points $f_1'p$, $f_2'p$, $f_1'q$, $f_2'q$ are co-planar, so that

$$(f_1'pf_2'pf_1'qf_2'q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (318).$$

The locus of points whose satellites meet the line ab is the quadric surface (compare (297))

$$(f_1qf_2qab) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (319).$$

77. The satellite of a point which describes a line $q = a + tb$ constructs one system of generators of the quadric

$$q = (f_1 + sf_2)(a + tb) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (320),$$

but the regulus degrades into a system of lines enveloping a conic whenever

$$(f_1af_2af_1bf_2b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (321),$$

that is, whenever the line belongs to the reciprocal of the complex of conjugate satellites (318).

The conic is co-planar with the line when the further conditions

$$(abf_1af_1b) = 0, \quad (abf_2af_2b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (322),$$

are satisfied (compare (296), (298)).

But when we are given, as here, a series of tangents to a conic homographic with a series of points on a line in its plane, in three cases a tangent passes through its corresponding point; and evidently when a point lies on its satellite, it also lies on the sextic $[f_1qf_2qq] = 0$; so the line under discussion is a triple chord of the sextic.

It seems worth while noticing (compare Art. 66) the remarkable equation

$$(((a, b, f_1a, f_1b, f_2a, f_2b))) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (323)$$

of the *assemblage of triple chords of the sextic*, for this equation is equivalent to (321) and (322).

78. Again, in an arbitrary plane $Slq=0$, it is generally possible to find one point p whose satellite lies in the plane. The conditions are

$$Slp = 0, \quad Slf_1p = 0, \quad Slf_2p = 0, \quad \text{so} \quad p = [l, f_1'l, \ f_2'l] \quad . \quad . \quad . \quad (324);$$

and the point is determinate unless the reciprocal of the plane lies on the conjugate sextic (Art. 73), or, in other words, unless the plane is a united plane for some function of the system. In this case (compare (312)) there exists a line locus for points p whose satellites lie in the plane.

This is precisely the case of the last article, so when the envelope of satellites is a conic co-planar with the line, the plane is a united plane.

79. For an arbitrary plane, the locus of points whose deriveds by $f_1 + xf_2$ remain in the plane is the line of intersection of $Sl(f_1 + xf_2)q = 0$ or $Sq(f_1' + xf_2')l=0$ with the given plane $Slq = 0$. All these lines pass through the point p, which may be called the *focus* of the plane.

Assuming an arbitrary point p to be a focus, the plane of which it is the focus is (compare (324)) the reciprocal of the point

$$l = [pf_1pf_2p] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (325).$$

The relation between a focus and the reciprocal of the plane is of the same nature as the correspondence discussed in Section XIX. (compare (526) with (324)).

The points whose satellites pass through a given point a lie on a twisted cubic

$$[af_1qf_2q] = 0,$$

and the locus of points whose satellites lie in a plane is a right line. The satellite of a point q and the plane $Slq = 0$ pierces the plane in the point

$$q_{\prime} = f_1qSlf_2q - f_2qSlf_1q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (326),$$

and from this quadratic transformation connecting the points q and $q_{\prime}$, it follows that q (or $q_{\prime}$) describes a conic when $q_{\prime}$ (or q) describes a right line. In the former case the conics pass through the focus of the plane. Thus again an arbitrary line qq' meets the satellites of two points on the line (compare (320)).

It would take too long to explain the various geometrical relations in the plane

$Slq = 0$, but subjects such as that just mentioned may be readily investigated by writing

$$q = xa + yb + zc, \quad q' = x'a + y'b + z'c,$$

where a, b and c are any three points in the plane. Then the array

$$\{qq'\} = \lambda\{bc\} + \mu\{ca\} + \nu\{ab\} \quad \text{if} \quad \lambda = yz' - y'z, \ \mu = zx' - z'x, \ \nu = xy' - x'y \quad . \quad (327)$$

and

$$\{f_tq, f_tq'\} = \lambda\{f_tb, f_tc\} + \mu\{f_tc, f_ta\} + \nu\{f_ta, f_tb\} \quad . \quad . \quad . \quad (328),$$

if

$$f_t = f_1 + tf_2.$$

Hence (compare (301) and (296)) the line qq' joins a point to its correspondent in the transformation produced by f_t if

$$\Sigma\lambda^2(bcf_tbf_tc) + \Sigma\mu\nu\{(caf_taf_tb) + (abf_tcf_ta)\} = 0 \quad . \quad . \quad . \quad . \quad (329).$$

This equation may be regarded as the tangential equation of a conic involving a parameter t quadratically. For six values of t the equation represents a pair of points—one point of each pair being one of the six points in which the plane meets the critical sextic, and the second point being the intersection of the plane with the *line* into which the plane is transformed by the function $(f_t - s)$ which destroys the aforesaid point (compare Art. 14, I).

In a united plane, the theory is simpler. Let a, b, c be the united points in the plane, united points of f_1. Then (327) and (328) become

$$\{qq'\} = \lambda\{bc\} + \mu\{ca\} + \nu\{ab\},$$

$$\{(f_1+tf_2)\,q, (f_1+tf_2)\,q'\} = \lambda\{t_2b+tf_2b, t_3c+tf_2c\} + \mu\{t_3c+tf_2c, t_1a+tf_2a\}$$
$$+ \nu\{t_1a + tf_2a, t_2b + tf_2b\} \quad . \quad (330);$$

and we get the conics

$$t^2\{\Sigma\lambda^2(bcf_2bf_2c) + \Sigma\mu\nu[(caf_2af_2b) + (abf_2cf_2a)]\} - t\Sigma(t_2 - t_3)\mu\nu(abcf_2a) = 0 \quad . \quad (331).$$

In this case the system of conics is inscribed to a common quadrilateral.

The conic enveloped by the satellites is

$$\Sigma\lambda^2t_1(bcf_2^{-1}bf_2^{-1}c) + \Sigma\mu\nu[t_2(caf_2^{-1}af_2^{-1}b) + t_3(abf_2^{-1}cf_2^{-1}a)] = 0,$$

or

$$\Sigma\lambda^2t_1(bcf_2bf_2c) + \Sigma\mu\nu[t_2(abf_2cf_2a) + t_3(caf_2af_2b)] = 0 \quad . \quad . \quad . \quad . \quad . \quad (332).$$

80. More particularly for the functions $ff'f_0f_r$ in a united plane of f, the united points a, b, c form a triangle (I) in perspective with the triangle (II) of the traces of the united planes of the conjugate; for these planes are

$$Sqa = 0, \quad Sqb = 0, \quad Sqc = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (333);$$

and the centre of perspective is given by

$$SqaSbc = SqbSca = SqcSab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (334);$$

while corresponding sides intersect in the points

$$b\mathrm{S}ca - c\mathrm{S}ab, \quad c\mathrm{S}ab - a\mathrm{S}bc, \quad a\mathrm{S}bc - b\mathrm{S}ca \quad . \quad . \quad . \quad . \quad (335).$$

The point of concourse of lines of the linear complex in the plane is $f_{\prime}^{-1}[abc]$, or,

$$(t_2 - t_3)a\mathrm{S}bc + (t_3 - t_1)b\mathrm{S}ca + (t_1 - t_2)c\mathrm{S}ab \quad . \quad . \quad . \quad . \quad (336),$$

since this point is the intersection of the planes

$$\mathrm{S}qf_{\prime}a = 0, \quad \mathrm{S}qf_{\prime}b = 0, \quad \mathrm{S}qf_{\prime}c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (337).$$

for which the united points are points of concourse. This point lies on the axis of perspective (335), and the equation of that axis may be written in the form

$$q = (f - t)f_{\prime}^{-1}[abc] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (338).$$

The three lines of the complex which pass through the united points intersect the sides of the triangle (I) in a triangle (III) in perspective with (I), and through the vertices of this third triangle pass the polars of the united points with respect to the quadric $\mathrm{S}qf_0q = 0$, and the traces of these planes form a triangle (IV) likewise in perspective with (I).

SECTION XIII.

The System of Quadrics $\mathrm{S}q\dfrac{f+s}{f+t}q = 0$, and some Questions relating to Poles and Polars.

81. In this section we shall notice some properties of the system of quadrics

$$\mathrm{S}q\frac{f+s}{f+t}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (339).$$

The self-conjugate function f in this homographic system may be supposed reduced to the type noticed in Art. 28, for by a linear transformation the symbolic quartic may be reduced in three ways to the form

$$f^4 + \mathrm{N}''f^2 + \mathrm{N} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (340).$$

The system (339) is its own reciprocal, and it includes confocal and concyclic systems. If a is the pole of the plane $Sbq = 0$ with respect to one of the quadrics, a and b are connected by the equation

$$b = \frac{f+s}{f+t}a, \text{ or } (f+t)b = (f+s)a, \text{ or } a = \frac{f+t}{f+s}b \quad . \quad . \quad . \quad (341).$$

Given b, the locus of a is a twisted cubic if s alone varies, a right line if s is constant, and a quadric

$$(afa\ bfb) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (342)$$

when s and t are both variable. (Compare Art. 70.)

The points of contact of the plane with quadrics of the system are found by adding the condition $Sab = 0$, when we find three points, one point or a conic locus.

A generalized normal joins a point to the reciprocal of its tangent plane, thus for u variable,

$$q = \frac{f+u}{f+t}a, \qquad \text{when} \qquad Sa\frac{f+s}{f+t}a = 0 \quad . \quad . \quad . \quad . \quad (343)$$

is the generalized normal at the point a; or deleting the condition and allowing t and u to vary, we have the equation of the assemblage of normals through the point a, and when a itself varies, we see that (342) represents the complex of normals to the system.

82. In general, two quadrics $s_1\ t_1$ and $s_2\ t_2$ intersect in a curve through which no third quadric of the system can pass, but when $t_1 = t_2$, an infinite number of the quadrics intersect in the curve. This follows from the consideration that

$$Sq \,.\, \frac{x(f+s_1)(f+t_2) + y(f+s_2)(f+t_1)}{(f+t_1)(f+t_2)}q = 0 \quad . \quad . \quad . \quad . \quad (344)$$

is the general equation of a quadric through the curve; and a factor will not cancel unless $t_1 = t_2$.

If q is any point on the curve of intersection, the poles of the tangent planes at that point with respect to some third quadric of the system will be conjugate to that quadric if

$$Sq\frac{(f+s_1)(f+s_2)(f+t_3)}{(f+t_1)(f+t_2)(f+s_3)}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (345).$$

In order that this may be the case for every point on the curve, the factor $f + s_3$ must cancel. Thus we must have s_3 equal s_1, s_2 or t_3. But further, on comparison with (344), it appears that the third quadric must coincide with one of the others, or else that $t_3 = s_3$ and $s_1 = s_2$.

This theory embraces the laws of confocals, their orthogonal section, and the property that the pole of the tangent plane to one, at a point of intersection with a second, taken with respect to the second, lies in its tangent plane at the point.

83. More generally, given any three quadrics

$$Sqf_1q = 0,\ Sqf_2q = 0,\ Sqf_3q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (346);$$

take the polar planes of a point q with respect to the first and second, and the poles of these planes with respect to the reciprocal of the third; these poles are conjugate to that reciprocal provided the point lies upon the quadric

$$Sqf_2f_3f_1q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (347).$$

If the quadrics have a common self-conjugate tetrahedron with the quadric of reciprocation, the three functions have the same united points, and are consequently commutative; and the three surfaces (347) obtainable for different selections of the quadrics (346) are identical.

84. Before leaving this subject, it may be of interest to show how the invariant condition that three quadrics should be polar quadrics of a cubic presents itself.

We have, if the quadrics are polars of the cubic $F(qqq) = 0$,

$$Sqf_1q = F(aqq),\ Sqf_2q = F(bqq),\ Sqf_3q = F(cqq) \quad . \quad . \quad . \quad (348),$$

if a, b, c are the poles. Hence

$$Sqf_1b = Sqf_2a;\ Sqf_2c = Sqf_3b;\ Sqf_3a = Sqf_1c \quad . \quad . \quad . \quad (349);$$

and on identifying the planes

$$f_1b = f_2a;\ f_2c = f_3b;\ f_3a = f_1c \quad . \quad . \quad . \quad . \quad . \quad . \quad (350);$$

so that

$$a = f_2^{-1}f_1f_3^{-1}f_2f_1^{-1}f_3a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (351);$$

and the function $f_2^{-1}f_1f_3^{-1}f_2f_1^{-1}f_3$ must have one latent root equal to unity.

SECTION XIV.

Properties of the General Surface.

85. If Q is a homogeneous and scalar function of a variable quaternion q of order m, the equation

$$Q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (352)$$

represents a surface. We shall write generally for any differential

$$dQ = mSp\, dq. \quad \ldots \quad (353),$$

where p is a homogeneous quaternion function of q and of the order $m-1$. Since p is a determinate function of q, q may be regarded as a function of p; and using EULER's theorem for homogeneous functions we have

$$Q = Spq = P \quad \ldots \quad (354)$$

where P is the function of p into which Q transforms.

86. Again, we shall write generally for the differential of the quaternion p regarded as a function of q,

$$dp = (m-1) f\, dq \quad \ldots \quad (355)$$

where $f\,dq$ is a linear function of dq, involving q homogeneously in the order $m-2$.

This function is self-conjugate, for taking two successive and independent differentials of Q,

$$d'\,dQ = mSp\, d'\,dq + m(m-1)\, S\,.\, f d'q\,.\, dq$$
$$= dd'Q = mSp\, dd'q + m(m-1)\, S\,.\, f dq\,.\, d'q \quad \ldots \quad (356);$$

and because the differentials are independent,

$$d'dq = dd'q, \text{ and therefore } Sdq\, f\, d'q = Sd'q\, f dq \quad \ldots \quad (357),$$

consequently the function f is self-conjugate, for dq and $d'q$ are quite arbitrary.

87. Differentiating (354) we find on comparison with (353)

$$dP = nSq\, dp, \qquad \text{where } (n-1)(m-1) = 1 \quad \ldots \quad (358),$$

and it is easy to verify that n is the order in which p is involved in P. Also introducing a new linear function g, we write

$$dq = (n-1) g\, dp \quad \ldots \quad (359),$$

and, as in the last article, g is self-conjugate and involves p in the order $n-2$ in its constitution.

Thus for any differential by (355) and (359)

$$dp = (m-1) f dq = (m-1)(n-1) fg\, dp = fg\, dp \quad \ldots \quad (360);$$

or symbolically

$$1 = fg = gf \quad \ldots \quad (361),$$

and one function produces on an arbitrary quaternion the same effect as the inverse of the other. In particular, employing EULER's theorem in (355) and (359) we have

$$p = fq = g^{-1}q\,; \quad q = gp = f^{-1}p \quad \ldots \quad (362).$$

88. When dq, instead of being perfectly arbitrary, satisfies

$$dQ = 0 \quad \text{or} \quad Sp\, dq = 0, \quad \text{where} \quad Q = 0 \quad \ldots \ldots \quad (363),$$

dq represents some point in the tangent plane at q to the surface $Q = 0$. The point p is the reciprocal of the tangent plane with respect to the unit sphere $Sq^2 = 0$; and the surface $P = 0$ is the reciprocal of the given surface. The relations of reciprocity are clearly exhibited by the equations (compare (354))

$$Sp\, dq = 0, \quad Sq\, dp = 0, \quad dP = 0 \text{ if } Q = 0, \quad dQ = 0 \quad \ldots \quad (364);$$

$$-S\, dpdq = Sp\, d^2q = Sq\, d^2p, \quad d^2P = 0 \text{ if also } d^2Q = 0 \quad \ldots \quad (365).$$

89. For the asymptotic lines, in addition to (364) and (365), the new relation

$$0 = S\, dp\, dq = Sp\, d^2q = Sq\, d^2p \quad \ldots \ldots \quad (366);$$

and thus for arbitrary scalars x and y

$$S\,(p + x\, dp)\,(q + y\, dq) = 0 \quad \ldots \ldots \quad (367),$$

or the reciprocal of an asymptotic tangent is the asymptotic tangent to the reciprocal surface at the corresponding point. Hence also, if corresponding tangents are reciprocal they touch asymptotic lines.

The tangents to the asymptotic lines of the original surface are also represented by the equations

$$Srfr = 0, \quad Spr = 0 \quad \ldots \ldots \quad (368);$$

and those of the reciprocal surface by

$$Srgr = 0, \quad Sqr = 0 \quad \ldots \ldots \quad (369);$$

r being allowed to vary arbitrarily, but p and q being kept constant. These lines are, in fact, the generators of the reciprocal quadrics

$$Srfr = 0 \quad \text{or} \quad Srg^{-1}r = 0, \quad \text{and} \quad Srgr = 0 \quad \text{or} \quad Srf^{-1}r = 0 \quad . \quad (370),$$

(compare (362)) which lie in the corresponding tangent planes.

90. The generalized normal to a surface at any point is the line joining that point to the pole of the tangent plane with respect to the quadric of reciprocation. But as there is practically no additional labour involved in the following discussion when the auxiliary quadric is arbitrarily selected, we assume it to be

$$Sqhq = 0 \quad \ldots \ldots \quad (371);$$

and then the equation of the normal at q to the surface $Q = 0$ is

$$r = q + th^{-1}p \quad \ldots \ldots \quad (372).$$

If c is a centre of generalized curvature, or a point at which consecutive normals intersect, we have for intersecting normals

$$c = q + th^{-1}p, \quad \mathrm{d}c = \mathrm{d}q + th^{-1}\,\mathrm{d}p + h^{-1}p\,\mathrm{d}t = c\,\mathrm{d}u \quad . \quad . \quad . \quad (373),$$

where $\mathrm{d}u$ is some small scalar, and $\mathrm{d}c = c\mathrm{d}u$, because on the hypothesis that consecutive normals intersect in c, c and $\mathrm{d}c$ represent the same point and differ only in weight. On elimination of c, (373) becomes

$$\mathrm{d}q + th^{-1}\,\mathrm{d}p + v\,(q + th^{-1}p) + wq = 0, \quad (tv = \mathrm{d}t - t\,\mathrm{d}u,\ v + w = -\,\mathrm{d}u) \quad . \quad (374);$$

and as this may be written

$$(1 + th^{-1}f)\,(\mathrm{d}q + vq) + wq = 0, \quad \text{or} \quad \mathrm{d}q + vq + w\,(1 + th^{-1}f)^{-1}q = 0 \quad . \quad (375),$$

we find, on operating by $\mathrm{S}p$ or $\mathrm{S}fq$, the equation

$$\mathrm{S}qf(1 + th^{-1}f)^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (376).$$

On inversion of the function this becomes a quadratic in t whose roots determine the two centres of curvature.

91. This equation may be thrown into the more suggestive form*

$$\mathrm{S}q\,(f^{-1} + th^{-1})^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (377),$$

which shows that the roots t are the parameters of two of the quadrics of the singly infinite system $\mathrm{S}r\,(f^{-1} + th^{-1})^{-1}\,r = 0$, which pass through the point q. The third quadric of the system through that point is of course $\mathrm{S}rfr = 0$, which corresponds to $t = 0$. The quadric $t = \infty$ is the auxiliary (371).

The two centres of curvature (373) are (t_1 and t_2 being the roots of (377))

$$c_1 = (f^{-1} + t_1h^{-1})p, \quad c_2 = (f^{-1} + t_2h^{-1})p \quad . \quad . \quad . \quad . \quad (378);$$

and the form of these equations shows that the points are the poles of the tangent plane $\mathrm{S}rp = 0$ with respect to the two quadrics t_1 and t_2.

The equation of the tangent to a line of curvature, $r = q + x\,\mathrm{d}q$ may by (375) be thrown into the form

$$r = q + yf^{-1}\,(f^{-1} + th^{-1})^{-1}q = q\,(1 + y) - yth^{-1}\,(f^{-1} + th^{-1})^{-1}q \quad . \quad (379),$$

where $t = t_1$ or t_2, and the form of this equation shows that the tangents are the generalized normals to the quadrics t_1 and t_2.

The first form of (379) shows that the tangent t_1 touches the quadric t_2, for

$$\mathrm{S}q\,(f^{-1} + t_2h^{-1})^{-1}f^{-1}\,(f^{-1} + t_1h^{-1})^{-1}q = 0 \quad . \quad . \quad . \quad . \quad (380),$$

as appears on replacing the middle function by

* Because $(1 + th^{-1}f)^{-1} = ((f^{-1} + th^{-1})f)^{-1} = f^{-1}\,(f^{-1} + th^{-1})^{-1}$.

$$(t_2 - t_1)f^{-1} = t_2(f^{-1} + t_1h^{-1}) - t_1(f^{-1} + t_2h^{-1}) \quad . \quad . \quad . \quad . \quad (381);$$

and, moreover, the lines of curvature form a conjugate *réseau* on the surface, for (380) gives

$$Sr_1fr_2 = 0 \text{ if } r_1 = f^{-1}(f^{-1} + t_1h^{-1})^{-1}q, \; r_2 = f^{-1}(f^{-1} + t_2h^{-1})^{-1}q \quad . \quad (382),$$

(compare (379)).

The other usual properties analogous to those for confocals may be easily obtained, but it must suffice to state that the centres of curvature for the quadric t_1 are

$$c' = f^{-1}(f^{-1} + t_1h^{-1})^{-1}q, \quad c_2' = (f^{-1} + t_2h^{-1})(f^{-1} + t_1h^{-1})^{-1}q \quad . \quad (383).$$

92. To reduce the equation (377) to a quadratic, let the symbolic quartic of $h^{-1}f$ be

$$(h^{-1}f)^4 - N'''(h^{-1}f)^3 + N''(h^{-1}f)^2 - N'(h^{-1}f) + N = 0 \quad . \quad . \quad (384);$$

then on multiplying by t^4 and dividing by $1 + th^{-1}f$, the result is

$$t^3\{(h^{-1}f)^3 - N'''(h^{-1}f)^2 + N''(h^{-1}f) - N'\} - t^2\{(h^{-1}f)^2 - N'''(h^{-1}f) + N''\}$$
$$+ t\{(h^{-1}f) - N'''\} - 1 = -N_t(1 + th^{-1}f)^{-1} \quad . \quad . \quad . \quad . \quad (385).$$

Observing that the coefficient of t^3 on the left is $-N(h^{-1}f)^{-1}$ or $-Nf^{-1}h$, the equation (376) becomes

$$t^3NSqhq + t^2Sqf\{(h^{-1}f)^2 - N'''(h^{-1}f) + N''\}q$$
$$- tSfq\{h^{-1}f - N'''\}q + Sqfq = 0 \quad . \quad . \quad . \quad . \quad (386);$$

and this immediately reduces to

$$t^2NSqhq + tSp(h^{-1}fh^{-1} - N'''h^{-1})p + Sph^{-1}p = 0 \quad . \quad . \quad . \quad (387),$$

when we replace fq by p, and discard the extraneous factor t.

If n and n_1 are the fourth invariants of f and h, $N = nn_1^{-1}$; and it is easy to see that n is the result of substituting q in the equation of the Hessian of the surface if Q is an integral as well as a homogeneous function of q. Thus one root is infinite in either of two cases, if the point is on the Hessian, and if it is on the auxiliary quadric; in either case the centre of curvature is the pole of the tangent plane with respect to the auxiliary. A root is zero if $Sph^{-1}p = 0$, and in this case the tangent plane touches the auxiliary, and a centre of curvature is the point q itself. These special cases depend on two distinct conditions, the relation of the auxiliary quadric to the surface, and the relation of the Hessian to the surface.

93. A curve is a generalized geodesic when consecutive tangents are coplanar with the pole of the tangent plane with respect to the auxiliary quadric; or, symbolically,

$$(q, \mathrm{d}q, \mathrm{d}^2q, h^{-1}p) = 0, \quad \text{or} \quad xq + y\,\mathrm{d}q + z\,\mathrm{d}^2q + wh^{-1}p = 0 \quad . \quad . \quad (388)$$

is the equation of a geodesic.

Operate with $\mathrm{S}p$, $\mathrm{Sd}p$, $\mathrm{S}hq$, $\mathrm{S}h\mathrm{d}q$ and by (364), (365),

$$z\mathrm{S}p\mathrm{d}^2q + w\mathrm{S}ph^{-1}p = 0\,; \quad y\mathrm{Sd}p\,\mathrm{d}q + z\mathrm{Sd}p\,\mathrm{d}^2q + w\mathrm{Sd}ph^{-1}p = 0\,;$$
$$x\mathrm{S}qhq + y\mathrm{S}qh\,\mathrm{d}q + z\mathrm{S}qh\,\mathrm{d}^2q = 0\,; \quad x\mathrm{S}qh\,\mathrm{d}q + y\mathrm{Sd}qh\,\mathrm{d}q + z\mathrm{Sd}qh\,\mathrm{d}^2q = 0 \quad . \quad (389).$$

Introducing the function f and eliminating the scalars $xyzw$, we find

$$\frac{\mathrm{Sd}ph^{-1}p}{\mathrm{S}ph^{-1}p} = -\frac{y\mathrm{Sd}qf\,\mathrm{d}q + z\mathrm{Sd}qf\,\mathrm{d}^2q}{z\mathrm{Sd}qf\,\mathrm{d}q}$$
$$= -\frac{\mathrm{Sd}qf\,\mathrm{d}^2q}{\mathrm{Sd}qf\,\mathrm{d}q} + \frac{\mathrm{S}qhq\mathrm{Sd}qh\,\mathrm{d}^2q - \mathrm{S}qh\,\mathrm{d}q\mathrm{S}qh\,\mathrm{d}^2q}{\mathrm{S}qhq\mathrm{Sd}qh\,\mathrm{d}q - (\mathrm{S}qh\,\mathrm{d}q)^2} \quad . \quad . \quad . \quad (390)\,;$$

and this, when the surface is a quadric so that f is constant, immediately integrates, and gives

$$\mathrm{S}ph^{-1}p\mathrm{Sd}qf\,\mathrm{d}q = u\,(\mathrm{S}qhq\mathrm{Sd}qh\,\mathrm{d}q - (\mathrm{S}qh\,\mathrm{d}q)^2) \quad . \quad . \quad . \quad . \quad (391),$$

where u is the constant of integration.

SECTION XV.

The Analogue of Hamilton's Operator ∇.

94. In applications of quaternions to projective geometry an operator analogous to Hamilton's ∇ is occasionally useful. I define it by the equation (compare Art. 85)

$$\mathrm{D}Q = p \quad \text{when} \quad \mathrm{d}Q = \mathrm{S}p\mathrm{d}q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (392).$$

To render this operator available for use, take any four independent differentials of q and write down the identity

$$p\,(\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q) = [\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q]\,\mathrm{S}p\,\mathrm{d}q - [\mathrm{d}q\,\mathrm{d}''q\,\mathrm{d}'''q]\,\mathrm{S}p\,\mathrm{d}'q$$
$$+ [\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}'''q]\,\mathrm{S}p\,\mathrm{d}''q - [\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}''q]\,\mathrm{S}p\,\mathrm{d}'''q \quad . \quad (393),$$

which suggests the symbolical equation

$$\mathrm{D} = \Sigma\,\frac{\pm\,[\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q]\,\mathrm{d}}{(\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (394),$$

where the summation refers to the four symbols d.

95. Otherwise, if the quaternion variable q is a function of four parameters, x, y, z, w, we may replace the arbitrary differentials in terms of the deriveds of q with respect to these parameters, and then (394) becomes

$$D = \Sigma \frac{\pm [q_y q_z q_w]}{(q_x q_y q_z q_w)} \frac{\partial}{\partial x} \quad \ldots \ldots \quad (395),$$

where

$$q_x = \frac{\partial q}{\partial x}, \quad q_y = \frac{\partial q}{\partial y}, \quad q_z = \frac{\partial q}{\partial z}, \quad q_w = \frac{\partial q}{\partial w} \quad \ldots \ldots \quad (396).$$

In particular, if these four deriveds satisfy the six equations

$$Sq_x q_y = Sq_y q_z = Sq_z q_x = Sq_x q_w = Sq_y q_w = Sq_z q_w = 0 \quad \ldots \quad (397),$$

it easily appears that the symbolic equation (395) reduces to

$$D = \frac{q_x}{Sq_x^2} \frac{\partial}{\partial x} + \frac{q_y}{Sq_y^2} \frac{\partial}{\partial y} + \frac{q_z}{Sq_z^2} \frac{\partial}{\partial z} + \frac{q_w}{Sq_w^2} \frac{\partial}{\partial w} \quad \ldots \ldots \quad (398).$$

More particularly if q is referred to the vertices of a tetrahedron self-conjugate to the unit sphere, so that

$$q = ax + by + cz + dw, \quad \text{and if} \quad Sa^2 = Sb^2 = Sc^2 = Sd^2 = 1 \quad . \quad (399)$$

for suitable selection of the weights of these four points, the operator takes its simplest form

$$D = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} + d \frac{\partial}{\partial w} \quad \ldots \ldots \quad (400),$$

while

$$SD^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 + \left(\frac{\partial}{\partial w}\right)^2 \quad \ldots \ldots \quad (401).$$

If, on the other hand, $$q = t + ix + jy + kz \quad \ldots \ldots \quad (402),$$

the operator reduces to $$D = \frac{\partial}{\partial t} - \nabla \quad \ldots \ldots \quad (403).$$

96. It may be useful to collect a few formulæ which may serve as examples of the application of the operator. We therefore give the following:

$$Dq = 4; \; DKq = -2 = KDq; \; DSq = 1 = SDq; \; DVq = 3 = VDq;$$
$$DSaq = a; \; DS.q^2 = 2q; \; DTq^2 = 2Kq; \; Dq^2 = 4(q + Sq); \; D(Vq)^2 = 2Vq;$$
$$DT(q + a) = KU(q + a); \; DSqfq = (f + f')q.$$

To these we may add

$$D^2T(q + a)^2 = -4 = TD^2S(q + a)^2; \; TD^2T(q + a)^2 = 8 = D^2S(q + a)^2;$$
$$TD^2.Tq^n = nKD.KqTq^{n-2} = n(4Tq^{n-2} + (n-2)qKqTq^{n-4}) = n(n+2)Tq^{n-2}.$$

And again

$$D^2(S.q^2)^n = 2nD.q(S.q^2)^{n-1} = 8n(S.q^2)^{n-1} + 4n(n-1)q^2(S.q^2)^{n-2};$$

and on taking the scalar of both sides

$$S.D^2.(S.q^2)^n = 4n(n+1)(S.q^2)^{n-1}.$$

From these results follow certain analogues of LAPLACE'S equation

$$TD^2Tq^{-2} = 0, \quad TD^2.f(D).T(q+a)^{-2} = 0 \quad . \quad . \quad . \quad . \quad (404);$$

and

$$S.D^2(S.q^2)^{-1} = 0, \quad S.D^2.f(D).(S.(q+a)^2)^{-1} = 0 \quad . \quad . \quad . \quad (405).$$

Moreover, the general expression for the operator in terms of arbitrary differentials a, b, c, d of q enables us to write down a number of invariants and identities. For instance, operating on fq, we find

$$D.fq.(abcd) = [bcd]fa - [acd]fb + [abd]fc - [abc]fd \quad . \quad . \quad (406).$$

Other examples relating to integration will be found in a paper in 'Proc. Roy. Irish Acad.,' vol. 24, Sect. A, pp. 6–20.

97. So far as projective geometry is concerned, the use we make of the operator D is to form successive polars of a point with respect to a surface and to show that it leads directly to ARONHOLD'S notation.

The n^{th} polar of a point r with respect to a surface $Q = 0$ of order m is

$$(SrD)^n.Q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (407).$$

If $n = m$, the operator simply multiplies Q by a numerical factor and changes the quaternion involved from q to r. Thus we may write the equation of the surface in the form

$$(SrD)^mQ = 0, \quad \text{or} \quad (Sra)^m = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (408),$$

where a is a symbolic quaternion devoid of meaning unless it enters into a term homogeneous in a to the order m. This is equivalent to ARONHOLD'S method.

SECTION XVI.

THE BILINEAR QUATERNION FUNCTION.

98. We shall now explain a method which promises to be of considerable value in the application of quaternions to projective geometry.

A bi-linear quaternion function $f(pq)$ is a function of two quaternions (p and q) linear and distributive with respect to both. It may be reduced to the form

$$f(pq) = a_1 Spf_1 q + a_2 Spf_2 q + a_3 Spf_3 q + a_4 Spf_4 q \quad . \quad . \quad . \quad . \quad (409),$$

where a_1, a_2, a_3, a_4 are any four quaternions and where f_1, f_2, f_3, and f_4 are four linear quaternion functions. The bilinear function involves sixty-four constants, sixteen for each of the four functions.

99. Writing generally for all quaternions p and q

$$f(pq) = f_{\prime}(qp) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (410),$$

we may call the new bilinear function $f_{\prime}$ the *permutate* of the function f. When a function is unaltered by transposition of the quaternions, it may be called a *permutable* function. Thus

$$P(pq) = \tfrac{1}{2} f(pq) + \tfrac{1}{2} f_{\prime}(pq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (411)$$

is a permutable function, the permutable part of f or $f_{\prime}$. A permutable function involves forty constants, the functions f_1, f_2, f_3, f_4 of (409) being then self-conjugate.

100. When a bilinear function changes sign with transposition of its quaternions, it may be called a *combinatorial* function. Thus

$$C(pq) = \tfrac{1}{2} f(pq) - \tfrac{1}{2} f_{\prime}(pq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (412)$$

is combinatorial. It vanishes for $p = q$, and, regarded geometrically, it relates not to a pair of points, but to the line joining the points.

A bilinear function is thus reducible to the form

$$f(pq) = P(pq) + C(pq)\,; \quad f_{\prime}(pq) = P(pq) - C(pq) \quad . \quad . \quad (413)\,;$$

and is uniquely resoluble into its permutable and combinatorial parts.

101. Writing generally for any three quaternions, p, q, and r,

$$Srf(pq) = Spf'(rq) = Sqf''(pr) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (414),$$

we shall call the new functions $f'(pq)$, $f''(pq)$ the *first and second conjugates* of $f(pq)$. In fact $f'(pq)$ is the conjugate when the first quaternion p alone varies, and $f''(pq)$ is the conjugate when the second varies.

102. As the accents employed to denote the permutate and the first and second conjugates are not commutative in order of application, it is safer to use brackets in the rare cases in which double accents are necessary. Thus

$$f(pq) = (f')'(pq) = (f'')''(pq) = (f_{\prime})_{\prime}(pq) \quad . \quad . \quad . \quad . \quad (415).$$

because the first conjugate of the first conjugate of $f(pq)$ is simply the function $f(pq)$ itself.

When the successive accents are different, the laws connecting the various functions are deducible from the relations (compare (414))

$$\begin{aligned} Srf(pq) &= Spf'(rq) = Sq(f')''(rp) = Sp(f')_{,}(qr) \\ &= Sqf''(pr) = Sq(f'')_{,}(rp) = Sp(f'')'(qr) \\ &= Srf_{,}(qp) = Sq(f_{,})'(rp) = Sp(f_{,})''(qr) \quad . \quad . \quad . \quad (416), \end{aligned}$$

in which p, q and r are perfectly arbitrary.

These relations show that

$$\begin{aligned} (f')''(pq) = (f'')_{,}(pq) = (f_{,})'(pq) = f''(qp), \\ (f')_{,}(pq) = (f'')'(pq) = (f_{,})''(pq) = f'(qp) \quad . \quad . \quad . \quad . \quad (417); \end{aligned}$$

and thus any multiply accented function may be reduced to one or other of six fundamental functions, the function and its two conjugates and the permutates of these three functions.

103. Exactly as in Arts. 5 and 6, the equations

$$\begin{aligned} &(f(aq) - ta\,;\, f(bq) - tb\,;\, f(cq) - tc\,;\, f(dq) - td) \\ &\qquad = (f'(aq) - ta\,;\, f'(bq) - tb\,;\, f'(cq) - tc\,;\, f'(dq) - td) \quad (418), \end{aligned}$$

$$\begin{aligned} &(f(pa) - ta\,;\, f(pb) - tb\,;\, f(pc) - tc\,;\, f(pd) - td) \\ &\qquad = (f''(pa) - ta\,;\, f''(pb) - tb\,;\, f''(pc) - tc\,;\, f''(pd) - td) \quad (419) \end{aligned}$$

are identities for all quaternions p, q, a, b, c and d, and for every value of the scalar t. The first is obtained on the supposition that $f(pq)$ is a function of p, and the second on the supposition that it is a function of q. Dividing each member of the identities by $(abcd)$, we obtain the biquadratics

$$\begin{aligned} J(q) - tJ'(q) + t^2J''(q) - tJ'''(q) + t^4, \\ I(p) - tI'(p) + t^2I''(p) - tI'''(p) + t^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (420); \end{aligned}$$

and $J(q)$, $J'(q)$, $J''(q)$, $J'''(q)$, of the fourth, third, second and first order respectively in q, are the invariants of $f(pq)$ considered as a function of p. Equating these biquadratics to zero, we obtain the equations whose roots are the latent roots of $f(pq)$ as a function of p and as a function of q.

It is evident from (418) and (419) that these relations are equivalent when the function is permutable, and then $I(q) = J(q)$, &c.

104. The quartic surfaces

$$I(p) = 0, \quad J(q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (421)$$

we shall call respectively the first and second Jacobians. Whenever a pair of quaternions satisfies the equation

$$f(pq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (422),$$

the point p must lie on the surface $I(p) = 0$ and q must lie on $J(q) = 0$; for $f(pr)$, a linear function of r, has then one zero latent root, and $f(rq)$ has also a zero latent root.

On reference to (409), it appears that (422) is equivalent to

$$Spf_1q = Spf_2q = Spf_3q = Spf_4q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (423);$$

and in the particular case when the function is permutable, the four linear functions are self-conjugate, and the equations assert that the polar planes of one point (p) intersect in the other (q). In this case the surfaces (421) coincide with one another and with the Jacobian of the four quadrics; and although it does not appear that in general the surfaces are the Jacobians of four quadrics, we have retained the name as being convenient and suggestive.

Two points related as in (422) will be called *Jacobian correspondents*, or more particularly IJ Jacobian correspondents.

105. When a function has a zero latent root, so has its conjugate. Consequently, whenever p and q are Jacobian correspondents, or whenever (422) is satisfied, it must be possible to find two other points r' and r'', so that

$$f'(r'q) = 0, \quad f''(pr'') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (424).$$

There are thus two new types of Jacobian correspondence; and the argument of Art. 102 shows that there can be no more, for the conditions (422) and (424) may be re-written in the form

$$f_{\prime}(qp) = 0, \quad (f')_{\prime}(qr') = 0, \quad (f'')_{\prime}(r''p) = 0 \quad . \quad . \quad . \quad . \quad (425),$$

without altering the signification of the equations, and we have now exhausted the six fundamental functions of the article cited.

106. *The points "r'" and "r''" of the second and third Jacobian correspondences lie upon the third Jacobian $K(r)$.*

A latent root of $f'(r'q)$ considered as a function of q (424) is zero, and therefore r' satisfies the equation

$$(f'(r'a), f'(r'b), f'(r'c), f'(r'd)) = ((f')''(r'a), (f')''(r'b), (f')''(r'c), (f')''(r'd)) = 0 \;. (426),$$

in the second number of which the function of q has been replaced by its conjugate. But (417) the second number is equivalent to

$$(f''(ar), f''(br), f''(cr), f''(dr)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (427),$$

and consequently r'', which satisfies (427), satisfies also (426), or r' and r'' lie upon the same quartic surface.

As in Art. 103. we deduce the identity

$$(f'(ra) - ta\,;\; f'(rb) - tb\,;\; f'(rc) - tc\,;\; f'(rd) - td)$$
$$= (f''(ar) - ta\,;\; f''(br) - tb\,;\; f''(cr) - tc\,;\; f''(dr) - td) = 0 \quad . \quad (428);$$

and the result of dividing by $(abcd)$ may be written in the form

$$K(r) - tK'(r) + t^2K''(r) - t^3K'''(r) + t^4 \quad . \quad . \quad . \quad . \quad . \quad (429),$$

and the latent quartic of $f'(rq)$ or $f''(qr)$ (functions of q) is obtained by equating this to zero.

The scheme of the Jacobians is now complete. the six fundamental functions of Art. 102 having been employed.

The points $r'q$ of (424) may be said to be JK Jacobian correspondents, and p and r'' are IK correspondents.

When $f(pq)$ is permutative, the JK and IK types unite and I coincides with J; when $f(pq)$ is self-conjugate with respect to p, K coincides with I, and the JK and IJ correspondences coalesce.

It readily appears from (416) that when the function is doubly self-conjugate it is also permutable, and when it is permutable and self-conjugate to one element it is likewise self-conjugate to the other. In this case the three Jacobians coincide with the Hessian of the cubic surface

$$Sqf(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (430).$$

SECTION XVII.

The Four-System of Linear Functions.

An Example of the Use of the Bilinear Function.

107. When one of the quaternions in a bilinear function is regarded as a quaternion parameter, the function represents a triply-infinite system of linear quaternion functions, or a *four-system* of linear functions, to borrow a convenient phrase from Sir ROBERT BALL's 'Theory of Screws.'

Thus

$$f(pq) = x_1 f(p_1 q) + x_2 f(p_2 q) + x_3 f(p_3 q) + x_4 f(p_4 q),$$
$$\text{where} \quad p = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 \quad . \quad . \quad (431)$$

is a linear combination of four given linear functions $f(p_{\prime\prime} q)$, the quaternions $p_{\prime\prime}$ being supposed given while the scalars $x_{\prime\prime}$ are variable.

It is frequently of advantage to use the notation

$$f(pq) = f_p(q) = f_q(p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (432),$$

when the bilinear function is regarded as a function of q or as a function of p.

108. *An arbitrary point is a united point of a definite function of the four-system, provided it does not lie on a critical curve of the tenth order.*

If q is assumed to be a united point of a function determined by p,

$$\{f(pq), q\} = 0, \quad \text{or} \quad f(pq) = tq, \quad \text{or} \quad f_q(p) = tq \quad . \quad . \quad . \quad (433);$$

and the solution of the equation in its third form is

$$pJ(q) = tF_q(q), \quad \text{or} \quad p = F_q(q), \; t = J(q) \quad . \quad . \quad . \quad . \quad (434),$$

where F_q is HAMILTON's auxiliary function corresponding to f_q and where $J(q)$ is the fourth invariant of f_q (Art. (103)).

This solution is definite (Art. 15), provided q does not lie upon the critical curve

$$F_q(q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (435).$$

To exhibit the nature of this curve, observe that

$$0 = \mathrm{S} \,.\, F_q(q)\,[p_1p_2p_3] = \mathrm{S}qF_q'\,[p_1p_2p_3] = [\,q,\ f_qp_1,\ f_qp_2,\ f_qp_3] \quad . \quad . \quad (436)$$

for all quaternions $[p_1p_2p_3]$, $[p_1p_2p_4]$, &c.; or, in the notation of Art. 65.

$$((q,\ f(p_1q),\ f(p_2q),\ f(p_3q),\ f(p_4q))) = 0 \quad . \quad . \quad . \quad . \quad . \quad (437),$$

whenever (435) is satisfied. But we have seen that (437) represents a curve of order $m = 10$ and rank $r = 40$ (278), which is common to all the quartic surfaces obtained by deleting one quaternion within the double brackets (436).

The solution may be expressed in a more explicit form by means of the identity

$$q(f(p_1q), f(p_2q), f(p_3q), f(p_4q)) = \Sigma \pm f(p_1q)\,(q, f(p_2q), f(p_3q), f(p_4q)) \quad . \quad (438),$$

so that we may write (434) in the form

$$p\,(p_1p_2p_3p_4) = \Sigma \pm p_1\,(q,\ f(p_2q),\ f(p_3q),\ f(p_4q))\,; \quad t = J(q) \quad . \quad (439).$$

109. *When the point lies on the critical curve it is generally a united point of every function of a determinate two-system.*

In this case the solution of (433) is (Art. 15)

$$pJ'(q) = tG_q(q) + F_q(p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (440)\,;$$

or

$$p = G_q(q) + xp_0, \quad f(p_0q) = 0, \quad t = J'(q) \quad . \quad . \quad . \quad . \quad (441).$$

Thus p may be any point on the line joining the point $G_q(q)$ to p_0 — the Jacobian correspondent of q; and consequently a determinate two-system exists, every function of which has q for a united point (compare Art. 123).

110. Similarly for the conjugate four-system $f''(pr)$, a point r is a united point of a definite function, unless it happens to lie upon the conjugate critical curve

$$F_r^{\backprime}(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (442),$$

where $F_r^{\backprime}$ is the auxiliary function of $f_r^{\backprime}(p) = f''(pr)$, but we must observe that $f_q^{\backprime}$ is not the conjugate of f_q.

Now the reciprocal of a united point of $f''(pr)$ (the conjugate to r of $f(pr)$) is a united plane of the original four-system. And thus an arbitrary plane is the united plane of some definite function, but if the plane belongs to the *developable surface* (442) it is a common united plane of a definite two-system of functions determined by

$$p = G_r^{\backprime}(r) + xp_0^{\backprime}, \quad f''(p_0^{\backprime}r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (443).$$

Ten of these singular planes pass through an arbitrary point; the order of the developable surface is $r = 40$; and the order of the cuspidal curve* is $n = 3\,(r - m) + \beta = 90$.

* 'Three Dimensions,' Art. 327.

111. It is obvious from this theory that the united points of functions of this system compose definite tetrads, so that one point of a tetrad being given the remaining three are generally determinate.

In fact (434) is a quartic transformation connecting united points q with the auxiliary points p, so that one point p corresponds to one point q, while four points q correspond to one point p. For a given point p, these four points are by (434) the intersections of the quartic surfaces, for arbitrary quaternions l,

$$\frac{Sl_1F_q(q)}{Sl_1p} = \frac{Sl_2F_q(q)}{Sl_2p} = \frac{Sl_3F_q(q)}{Sl_3p} = \frac{Sl_4F_q(q)}{Sl_4p} \quad . \quad . \quad . \quad . \quad . \quad (444).$$

But these surfaces have a common curve (435); and three surfaces having a common curve intersect in

$$\mu\nu\rho - m(\mu + \nu + \rho - 2) + r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (445)$$

points not on the common curve, and this number is 4 when $\mu = \nu = \rho = 4$, $m = 10$, $r = 40$, as in the present case.

112. *The locus of points "p" determining functions, each of which has a united point on a given line, is a unicursal twisted quartic.*

When we replace q by $q + xq'$ in the second form of (434), we may write

$$p = (p_0p_1p_2p_3p_4 \between x, 1)^4 = p_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (446),$$

and the form of the equation establishes the proposition.

In like manner we have

$$t = (t_0t_1t_2t_3t_4 \between x, 1)^4 = t_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (447).$$

113. *For every intersection of the line with the critical curve, the quartic breaks up.*

If x' is the value of the scalar x for a point on the critical curve, $p_{x'}$ and $t_{x'}$ both vanish, or

$$0 = (p_0p_1p_2p_3p_4 \between x', 1)^4, \qquad 0 = (t_0t_1t_2t_3t_4 \between x', 1)^4 \quad . \quad . \quad . \quad . \quad (448).$$

We may employ these equations to eliminate p_4 and t_4 from (446) and (447); and discarding the factor $x - x'$, we find

$$p = (p'_0p'_1p'_2p'_3 \between x1)^3, \qquad t = (t'_0t'_1t'_2t'_3 \between x1)^3 \quad . \quad . \quad . \quad . \quad (449).$$

The locus of p is now a twisted cubic, and the discarded factor corresponds to a line of the nature of those of Art. 109.

When the line qq' meets the critical curve twice, the locus is a conic and a pair of lines. If the line is a triple chord, the locus is one line of a new type and three lines of the type already mentioned. Finally, for a quadruple chord, the quartic reduces to a point and four lines, as we shall see immediately.

But first we notice that the arguments of Art. 110 apply, so that we may write down the equation of the quartic curve whose points determine functions, each of which has a united plane through a given line. If the line lies in one or more of the planes of the developable (442), the quartic degrades in the manner explained.

114. Otherwise we may say that (446) and (447) determine a system of functions $f(p_x q) - t_x q$ which destroys the line $q + xq'$ point by point. Or counting unity as one function, it may be said that a five-system is required to destroy a line point by point. However, when the line intersects the critical curve once, twice, or thrice, it can be destroyed seriatim by a four-, three-, or two-system of functions. For example, in the case of triple intersection we may write

$$p_x = p_0 x + p_1,\ t_x = t_0 x + t_1;\ f(p_0 x + p_1,\ q'x + q) - (t_0 x + t_1)(q'x + q) = 0 \quad (450);$$

and, going one step further, in the case of a quadruple chord

$$f(p_0,\ q'x + q) = t_0 (q'x + q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (451).$$

Thus a quadruple chord of the critical curve is a line locus of united points of a determinate function. And because the number of quadruple chords of a curve is ('Three Dimensions,' Art. 274)

$$\tfrac{1}{24}(- m^4 + 18m^3 - 71m^2 + 78m - 48mh + 132h + 12h^2) \quad . \quad . \quad (452),$$

or 20 for $m = 10$, $h = 25$, we learn that *twenty functions of the four-system have line loci of united points—quadruple chords of the critical curve.*

The formula (314) gives 80 as the order of the surface of triple chords.

115. *The locus of a point which determines a function having a united point in a given plane is a sextic surface.*

The functions H_p, G_p and F_p being HAMILTON's auxiliary functions for $f_p(q) = f(pq)$, the relations

$$H_p(q) = t'q; \quad G_p(q) = t''q; \quad F_p(q) = t'''q \quad . \quad . \quad . \quad . \quad . \quad . \quad (453)$$

are satisfied, provided q is a united point of $f(pq)$, t', t'' and t''' being suitable scalars.

If q lies in a given plane, these equations, with that of the given plane, afford the relations

$$Sql = 0, \quad SqH_p'(l) = 0, \quad SqG_p'(l) = 0, \quad SqF_p'(l) = 0 \quad . \quad . \quad (454).$$

linear in q and of orders 0, 1, 2 and 3 in p. Expressing that q is a common point, we have the equation of the sextic surface

$$(l, \quad H_p'(l), \quad G_p'(l), \quad F_p'(l)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (455).$$

116. *The sextic surface has a double curve of the seventh order answering to pairs of united points in the plane.*

If the first, second and third of equations (454) regarded as planes in q intersect in a common line, the fourth plane will also pass through that line. The condition for a common line is

$$ul + vH_p'(l) + wG_p'(l) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (456),$$

where u, v and w are certain scalars. Operating on this by f_p', we have by Art. 6

$$u(I'''(p) - H_p')l + v(I''(p) - G_p')l + w(I'(p) - F_p')l = 0 \quad . \quad . \quad (457),$$

remembering (Art. 103) that $I'''(p)$, $I''(p)$, $I'(p)$ and $I(p)$ are the invariants of f_p'. But this relation gives $F_p'(l)$ linearly in terms of l, $H_p'(l)$, $G_p'(l)$, and therefore, as asserted, the fourth plane will also pass through the common line.

Hence it appears that (456), or its equivalent

$$[l, H_p'(l), G_p'(l)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (458),$$

represents a double curve on the sextic (455); for if p is any point on this curve, not only will (455) be satisfied, but the equation of the tangent plane at that point will also vanish, since every set of three quaternions included in the brackets of (455) is then linearly connected. The order of this curve is 7, by Art. 64.

Moreover, (456) expresses that a united line of the function f_p' passes through the point l, or, reciprocally, that a united line of the function f_p lies in the plane $Slq = 0$.

117. *The point determining the function for which the plane is a united plane is a triple point on the sextic.*

If p_0 is this point, and if t_1, t_2, t_3 are the roots of the function $f(p_0q)$ answering to the united points in the plane, it follows from the fundamental properties of the auxiliary functions that

$$H_{p'_0}(l) = \Sigma t_1 \,.\, l, \quad G_{p'_0}(l) = \Sigma t_2t_3 \,.\, l, \quad F_{p'_0}(l) = t_1t_2t_3 \,.\, l \quad . \quad . \quad (459);$$

and consequently the tangent plane and the polar quadric of the point p_0 to the surface (455) vanish identically. The point is therefore a triple point.

118. It may be noticed that in terms of a, b, c, any three points in the plane, the triple point is

$$p_0 = [f'(la), \ f'(lb), \ f'(lc)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (460);$$

also in terms of these three points, if $l = [abc]$,

$$H_p'(l) = \Sigma\,[f(pa), b, c], \quad G_p'(l) = \Sigma\,[a, f(pb), f(pc)],$$
$$F_p'(l) = [f(pa), f(pb), f(pc)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (461).$$

Consequently if $q = xa + yb + zc$, we may replace the system of equations (454) by

$$xX + yY + zZ = 0, \quad xX_2 + yY_2 + zZ_2 = 0, \quad xX_3 + yY_3 + zZ_3 = 0 \,. \quad (462),$$

where

$$X = SaH_p'(l) = (a, f(pa), b, c);$$
$$X_2 = SaG_p'(l) = (a, f(pa), f(pb), c) + (a, f(pa), b, f(pc));$$
$$X_3 = SaF_p'(l) = (a, f(pa), f(pb), f(pc)) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (463);$$

and Y, Y_2, Y_3 and Z, Z_2, Z_3 may be written down from symmetry.

Moreover, when we specially select the points a, b, c as the united points of the function $f(p_0q)$, and when we form successive polars of p_0 with respect to X, X_2 and X_3, we find (Art. 97) in terms of the latent roots t_1, t_2, t_3 corresponding to a, b and c,

$$Sp_0D \,.\, X = 0, \quad Sp_0D \,.\, X_2 = (t_2 + t_3)\, X, \quad (Sp_0D)^2 X_3 = 2t_2tX_3 \quad . \quad (464),$$

because

$$Sp_0D \,.\, Y_3 = (a, f(p_0a), f(pb), f(pc)) + (a, f(pa), f(p_0b), f(pc)) + (a, f(pa), f(pb), f(p_0c))$$
$$= t_2\,(a, f(pa), b, f(pc)) + t_3\,(a, f(pa), f(pb), c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (465),$$

and similarly in the other cases.

Thus the equation of the sextic may be written in the form

$$\begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (466),$$

and the third polar of the point p_0 is

$$(t_2 - t_3)\,(t_3 - t_1)\,(t_1 - t_2)\, XYZ = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (467).$$

Thus the tangent cone at the triple point breaks up into three planes.

In the same notation the double curve is represented by

$$\begin{Vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \end{Vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (468);$$

and forming the polars, the point p_0 is seen to be triple and

$$\begin{Vmatrix} X & Y & Z \\ (t_2 + t_3)X, & (t_3 + t_1)Y, & (t_1 + t_2)Z \end{Vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (469)$$

represents the system of tangents at the triple points—the lines of intersection of the planes X, Y and Z.

We may add that the equation of the cone, vertex p_0, standing on the curve is

$$(t_2 - t_3)\, XYZ_2 + (t_3 - t_1)\, XYZ_2 + (t_1 - t_2)\, XYZ_2 = 0 \quad . \quad . \quad . \quad (470).$$

119. This surface resembles a STEINER's quartic in many particulars, but it is a degraded case of the general surface

$$p = (xyz)^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (471),$$

where $(xyz)^4$ is the general quaternion function of three homogeneous scalar parameters x, y, z. The general surface is of the 16th order. The STEINER quartic may be written $p = (xyz)^2$, a general quaternion quadratic function of x, y, z. Surfaces of this type arise from the general transformation

$$p = f(q, q, \ldots q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (472)$$

of the nth order, being the transformations of planes.

The twisted quartics of Art. 112 correspond to the conics on the STEINER quartic. The sextic surface contains ten lines corresponding to the ten points in which the plane intersects the critical curve of the tenth order, for to every point on that curve corresponds a two-system of functions or a line in the space p (Art. 109). Again, the sextic contains an infinite number of twisted cubics corresponding to the lines in the plane which pass through one of these ten points (Art. 113); and it likewise contains 45 conics answering to the connectors of these points. More generally (Art. 113) a conic through five of these points transforms into a twisted cubic, and similarly for other cases.

120. When we express that the twisted cubic (449) is plane, the condition

$$(p'_0, p'_1, p'_2, p'_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (473)$$

is of the tenth order in q' and of the sixth in q, which latter point we may suppose to be on the critical curve. This condition will then represent a cone of the tenth order of the lines through the point q which transform into plane curves in the p space. But this cone must consist in part of the cone of the ninth order containing the critical curve. The remaining part is a plane, and every line in this plane through q transforms into a plane cubic.

In particular, an arbitrary plane cuts the critical curve in ten points and intersects ten planes of the type just mentioned in lines which transform into plane cubics on the sextic surface. Here again is a point of similarity with the STEINER quartic, for the plane containing one of these cubics cuts the sextic again in another cubic.

121. Corresponding to a plane $[p_1p_2p_3]$ in the p space there is a Jacobian quartic

$$(q, f(p_1q), \ f(p_2q), \ f(p_3q)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (474)$$

in the q space, the locus of united points of functions of the three-system determined by points in the plane. All these quartics intersect in the critical curve (437).

In like manner to a line in the p space corresponds the twisted sextic curve

$$[q, f(p_1q), \ f(p_2q)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (475),$$

the locus of united points of a two-system.

The locus of Jacobian correspondents of points in the plane is the sextic curve

$$[f(p_1q), \ f(p_2q), \ f(p_3q)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (476).$$

Now any one of these sextics is the residual of the critical curve in the intersection of a pair of Jacobian quartics, and a curve meets its residual in t points, where

$$r + t = m(\mu + \nu - 2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (477).$$

In particular for $r = 40$, $m = 10$, $\mu = \nu = 4$, we have $t = 20$; and so there are twenty intersections, but I propose to show that these in reality correspond to ten contacts.

Take, for example, the curve (476), and let q_1 be a point of intersection and take p_1 to be the Jacobian correspondent of q_1, so that $f(p_1, q_1) = 0$. Then the tangent to the curve at q_1 is

$$[f(p_1 q),\ f(p_2 q_1),\ f(p_3 q_1)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (478).$$

But this tangent lies in the tangent planes at the same point to the system of quartics $(f(p_1 q), f(p_2 q), f(p_3 q), f(p_4 q) + uq) = 0$, where u is arbitrary, and as these quartics contain the critical curve, the sextics touch this curve where they meet it.

122. Hence, *the locus of the Jacobian correspondents of points on the critical curve is a curve of the tenth degree*; for in the plane $[p_1 p_2 p_3]$ there are ten points which are Jacobian correspondents of points on the critical curve.

The Jacobian quartic of the plane $[p_1 p_2 p_3]$ *contains ten lines.*

The tangent plane to the Jacobian quartic at a point on the critical curve, corresponding to one of the ten points just mentioned, intersects the plane of Art. 120 in a line which transforms into a plane cubic on the sextic surface into which the tangent plane to the quartic transforms. But the quartic transforms into a tangent plane to this sextic, and therefore contains the cubic, consequently the quartic contains the line.

123. We shall now consider the orders of the surfaces and curves into which given surfaces and curves are transformed by the relation connecting p and q (434).

With an arbitrary surface $Q = 0$ in the q space is associated a complementary $Q' = 0$, so that the points of the two surfaces compose tetrads of united points of functions of the four-system. These two surfaces, of orders m and m' respectively, transform into a common surface of order n.

An arbitrary line in the p space cuts the surface (n) in n points, and to these correspond $4n$ points in the q space situated on a sextic curve (475). This curve cuts the surface Q in $6m$ points, and these are generally united points of $6m$ distinct functions, because the surface Q is arbitrary. Hence $n = 6m$.

Again, the sextic cuts the surface Q' in $6m'$ points, but these fall into triads of united points complementary to the $6m$ points. Hence $n = \frac{1}{3}\, 6m'$; and we have the complete formula

$$n = 6m = 2m' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (479).$$

More generally, if the surface Q is wholly composed of sets of ν united points,

$$n = \frac{6m}{\nu} = \frac{6m'}{4 - \nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (480).$$

There is a case of exception for a Jacobian quartic $(q, f(p_1 q), f(p_2 q), f(p_3 q)) = 0$ which transforms into a plane and not a surface of the sixth degree as (480) would give for $\nu = m = 4$. But here the sextic curve cuts the quartic in 4 points and touches it in 10 points on the critical curve (Art. 121), and the four points correspond to the intersection of the line with the plane in the p space, while to the ten points correspond lines of the type mentioned in Art. 109. We learn, therefore, that an arbitrary right line in the p space intersects ten of these lines, and that they compose a critical surface of the tenth order. This is otherwise justified from the consideration that an arbitrary quartic transformation converts a plane into a surface of the sixteenth order; and the fact that a plane transforms into a sextic shows that a critical surface of the tenth order has been discarded.

The equation of the complementary of the Jacobian $J(q) = 0$ will be found in Art. 127.

124. In like manner, taking an arbitrary curve in the q space of order M, let its complementary be of order M′, and let both transform into a curve of order N. The curve, being arbitrary, will not intersect the critical curve, and the 4M points in which it cuts the quartic, transformed from an arbitrary plane in the p space, will correspond point for point to the N points in which the transformed curve cuts the plane. Thus N = 4M.

Consider further the intersections of the curve and its complementary with an arbitrary surface (m) and its complementary (m'). The curve meets the complementary of the surface in Mm' points, and the complementary of the curve meets the surface in $M'm$ points. In general, each point of one set corresponds to one point of the other set, and the two sets compose pairs of united points. Thus $Mm' = M'm$, or $M' = 3M$ by (479); and accordingly we have the complete formula

$$N = 4M = \frac{4M'}{3} \qquad (481).$$

The whole set of points of intersection of the curve and surface and their complementaries is arranged as follows:—The Mm points unite with $3Mm$ of the $M'm'$ points in Mm tetrads. The Mm' points and the $M'm$ unite with $2(Mm' + M'm)$ of the $M'm'$ points to form tetrads, and thus by (481) and (479) all the $M'm'$ points are exhausted; and there are but $4Mm$ $(= Mm + Mm' + M'm)$ tetrads. But the curve (N) intersects the surface (n) in $Nn = 4M \times 6m$ points, and consequently there remain over 20 Mm points, which are critical points on the transformed curve and surface. These points evidently must lie on the critical surface of Art. 123.

When a curve is wholly composed of pairs of united points, the order of the transformed curve is N = 2M, and from symmetry the order of the complementary is M′ = M.

An arbitrary surface and its complementary do not intersect in a curve wholly composed of pairs of united points, though of course the curve of intersection will contain all the pairs of united points which lie on the surface. It does not seem to be easy to assign any general relation connecting the order of a curve of this nature with that of its transformed curve. Thus 7 is the order of the curve transformed from the cubic intersection of a plane with its complementary (Art. 116).

125. We may account for the curve of intersection of the pair of sextics derived from two arbitrary planes in the following manner.

Call the two planes P and P′, and their complementary cubics C and C′. The complementary of the line (PP′) forms part of the intersection of the cubics C and C′, and this curve is a cubic (481). There remains, therefore, a sextic as part of the intersection of C and C′. The complementary of the cubic curve (PC′) is a curve of the ninth order, part being the cubic (P′C), and the remaining part the residual sextic on C and C′. This sextic is wholly composed of pairs of united points. The line and its complementary cubic transform into a common quartic. The cubic (PC′), the cubic (P′C) and the residual sextic transform into a common curve of order $3 \times 4 = 2 \times 6 = 12$ (compare the last article). Thus we can only account for a curve of order 16 $(= 4 + 12)$, and the sextics consequently intersect in a singular curve of order 20.

126. *The complex of lines joining pairs of united points is of the fourth order.*

If a and b are any two points on a line joining united points,

$$f(p, a) = xa + yb, \quad f(p, b) = za + wb \quad . \quad . \quad . \quad . \quad . \quad (482),$$

where p determines the function. The theory of quaternion arrays allows us to write the condition that these two equations should be simultaneously satisfied in the form*

$$\left\{ \begin{matrix} f(e_1a) & f(e_2a) & f(e_3a) & f(e_4a) & a & b & 0 & 0 \\ f(e_1b) & f(e_2b) & f(e_3b) & f(e_4b) & 0 & 0 & a & b \end{matrix} \right\} = 0 \quad . \quad .. \quad (483)$$

where e_1, e_2, e_3, e_4 are arbitrary quaternions; and by the rules of expansion of arrays, this equation is equivalent to

$$\Sigma \pm (f(e_1a), f(e_2a), a, b) \quad (f(e_3b), f(e_4b), a, b) = 0 \quad . \quad . \quad . \quad (484),$$

where the signs follow the rules of determinants. As this is of the fourth order in a and b, and also combinatorial with respect to both, it represents a complex of the fourth order.

127. By (433) and (434) we have

$$f(pq) = qJ(q), \quad p = F_q(q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (485);$$

and throughout this article we shall suppose p expressed as a quartic function of q.

One root of the latent quartic of $f(pq)$ is thus equal to $J(q)$, so that when we substitute in the equation of that quartic (Art. 103 (420)), we have identically

$$J(q)^4 - J(q)^3 I'''(p) + J(q)^2 . I''(p) - J(q) I'(p) + I(p) = 0 \quad . \quad (486).$$

* The equations of the various assemblages of chords of Art. 113 may also be discussed by the aid of arrays.

The direct interpretation of this identity is that the transformation converts the Jacobian $I(p) = 0$ into two surfaces, one being the Jacobian $J(q) = 0$ and the other the surface of the twelfth order

$$J(q)^3 - J(q)^2 I'''(p) + J(q) I''(p) - I'(p) = 0 \quad . \quad . \quad . \quad (487).$$

This surface is the locus of three of the united points of functions which have a zero latent root, the fourth united point lying on the Jacobian $J(q) = 0$.

The critical curve is triple upon this surface, and the surface meets the Jacobian again in a residual curve of the eighteenth order, which is the locus of *united points corresponding to a double zero root.*

128. Making the substitution $s = t - J(q)$ in the latent quartic of the function $f(p, q)$ the equation reduces to

$$s^4 + s^3(4J(q) - I'''(p)) + s^2(6J(q)^2 - 3I'''(p)J(q) + I''(p))$$
$$+ s(4J(q)^3 - 3I'''(p)J(q)^2 + 2I''(p)J(q) - I'(p)) = 0 \ . \ (488).$$

A second root of the original quartic is equal to $J(q)$ if

$$4J(q)^3 - 3I'''(p)J(q)^2 + 2I''(p)J(q) - I'(p) = 0 \quad . \quad . \quad . \quad (489),$$

and this is the *locus of united points which correspond to double latent roots.* This surface is of the twelfth order, the critical curve is a triple curve upon it, and it meets the Jacobian in the same curves as (487).

The locus of united points corresponding to triple latent roots is the curve of intersection of this surface with the surface of the eighth order

$$6J(q)^2 - 3I'''(p)J(q) + I''(p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (490).$$

But the critical curve is double on this surface, and accordingly it counts six times in the intersection, so that the *locus of triple united points* is a curve of order $36 (= 8 \times 12 - 6 \times 10)$.

129. Further, quadruple united points are the points common to the surfaces (489), (490), and

$$4J(q) - I'''(p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (491),$$

which do not lie upon the common critical curve.

In order to calculate the number of these quadruple points it is necessary to find the number of points common to the critical curve and the curve locus of triple points. Now $24 (= 4 \times 3 \times 2)$ functions have triple zero roots, this being the number of points common to the surfaces $I(p) = 0$, $I'(p) = 0$, $I''(p) = 0$ in the p space; and the curve locus of triple points being of the 36th order meets $J(q) = 0$ in 144 points. Subtracting 24, there remain 120 points on the critical curve.

The triple curve therefore intersects (491) in 24 quadruple united points, and in

120 points on the critical curve; and thus *twenty-four functions of the system have four equal latent roots* and four coalesced united points.

130. Again, suppose that two roots of (488) are zero and that the remaining two are equal. In this case

$$8\mathrm{J}(q)^2 - 4\mathrm{J}(q)\,\mathrm{I}'''(p) + 4\mathrm{I}''(p) - \mathrm{I}'''(p)^2 = 0 \quad . \quad . \quad . \quad (492);$$

and this equation, combined with (489), gives a curve locus of order 36 ($= 8 \times 12 - 2 \times 3 \times 10$), which is the *locus of united points of functions whose roots are equal in pairs.*

We have now outlined the general theory of the four-system, but in a later section some supplementary remarks will be made on this subject.

SECTION XVIII.

The Quadratic Transformation of Points in Space.

The Second Example of the Use of the Bilinear Function.

131. The general quadratic transformation in space is represented by the equation

$$p = f(qq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (493),$$

in which it is obviously permissible to regard the bilinear function as *permutable*, or the four linear functions (409) as self-conjugate. The transformation involves 40 constants.

To a plane in the p space corresponds a quadric, or

$$Slp = 0, \quad Slf(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (494)$$

transform one into the other; and thus to one point p correspond eight points q—the intersections of three quadrics—and to one point q corresponds in general one point p.

We use the word *octad* to denote the group of eight points corresponding to p.

132. The right line $q = a + tb$ transforms into the conic

$$p = f(aa) + 2tf(ab) + t^2f(bb) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (495),$$

and $f(aa)$ and $f(bb)$ are two points on the conic, while $f(ab)$ is the pole of their chord.

The condition for the collinearity of these three points is

$$[f(aa), \quad f(ab), \quad f(bb)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (496);$$

and this equation may be replaced by

$$f(aa) + (x + y)f(ab) + xyf(bb) = 0, \quad \text{or} \quad f(a + xb, a + yb) = 0 \quad . \quad (497);$$

and this expresses that the original line joins Jacobian correspondents. *Thus lines joining Jacobian correspondents transform into lines.*

In this case (Art. 104) of the permutable function, if

$$f(rr') = 0 = f(r'r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (498),$$

the points r and r' are conjugate to every quadric of the system (494).

We may replace (498) by

$$f(r \pm tr', r \pm tr') = f(rr) + t^2f(r'r') \quad . \quad . \quad . \quad . \quad . \quad . \quad (499),$$

or *points harmonically conjugate to a pair of Jacobian correspondents transform into a single point.*

Thus we may speak of the rays of the assemblage of lines represented by (496) as *connectors*, (1) of a pair of Jacobian correspondents, (2) of a pair of points of an octad, (3) of an infinite number of pairs of points of octads.

It is evident that when two points of an octad coincide, they unite on the Jacobian; and that every point on the Jacobian is the union of a pair of points of an octad.

133. *The Jacobian correspondents transform into limiting points, separating the points derived from real from those derived from imaginary points.*

The points on the transformed connector

$$p = f(rr) + sf(r'r') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (500)$$

are transformed from the points $r \pm \sqrt{s}\, r'$; these latter are real if s is positive; otherwise they are imaginary.

To discriminate between the *outer* and the *inner* region on the line (500), observe that the vectors from the centre of reciprocation to the limiting points are

$$\rho = \frac{Vf(rr)}{Sf(rr)}, \quad \rho' = \frac{Vf(r'r')}{Sf(r'r')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (501);$$

and that the vector to the point p is

$$\text{OP} = \frac{Vf(rr) + sVf(r'r')}{Sf(rr) + sSf(r'r')} = \frac{\rho Sf(rr) + s\rho' Sf(r'r')}{Sf(rr) + sSf(r'r')} \quad . \quad . \quad . \quad (502).$$

The point P lies on the inner region if $Sf(rr)$ and $sSf(r'r')$ are of like sign; and the inner region corresponds to real points if the points r and r' are either both inside or both outside the quadric

$$Sf(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (503).$$

This quadric is the locus of points projected to infinity; it may of course be imaginary, so that $Sf(rr)$ and $Sf(r'r')$ are essentially one-signed if r and r' are real. In this case the region is always inner. If the quadric is real, the points r and r' (if real) cannot both lie inside, for they are conjugate to it. The nature of the intersection of a line with this quadric controls the nature of the conic into which it is transformed.

134. The locus of the Jacobian correspondents of points in a plane is a sextic curve, and for the permutable function this sextic cuts an arbitrary plane in points which correspond in pairs. There are therefore three connectors in a plane.

The vertices of the triangle of connectors belong to the same octad; for if q_1 is one vertex and q_2 and q_3 the points, one on each of the connectors through q_1, which (Art. 132) belong to the same octad as q_1, then q_2 and q_3 belong to a common octad, and their line is a connector—the third connector in the plane.

We may suppose the weights of the points q_1, q_2 and q_3 chosen so that the Jacobian correspondents are

$$q_2 \pm q_3, \quad q_3 \pm q_1, \quad q_1 \pm q_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (504),$$

the vertices of the triangle being (Art. 132) harmonically conjugate to these points in pairs.

135. Let the eight quaternions which represent points of an octad have their weights chosen so that*

$$p_0 = f(q_1 q_1) = f(q_2 q_2) = \&\text{c.} = f(q_8 q_8) \quad . \quad . \quad . \quad . \quad . \quad (505),$$

* It follows from Art. 132, that this convention is the same as that made at the end of the last article.

and let the twenty-eight points $f(q_1q_2)$ be denoted by

$$p_{12} = f(q_1q_2), \text{ \&c. } p_{78} = f(q_7q_8) \quad . \quad . \quad . \quad . \quad . \quad (506).$$

It may be remarked that these relations lead to

$$\pm 2\sqrt{-1}p_{12} = f(q_1 \pm \sqrt{-1}q_2, \quad q_1 \pm \sqrt{-1}q_2) \quad . \quad . \quad . \quad (507);$$

so that the points (506), although real, if the points of the octad are real, have been transformed from imaginary points, and consequently do not lie in the same region (Art. 133) as the point p_0.

The Jacobian correspondents transform into $p_0 \pm p_{12}$, &c.

136. *A plane transforms into a* STEINER'S *quartic.*

In the notation of the last article, the plane

$$q = t_1q_1 + t_2q_2 + t_3q_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (508)$$

transforms into the surface

$$p = p_0(t_1^2 + t_2^2 + t_3^2) + 2p_{23}t_2t_3 + 2p_{31}t_3t_1 + 2p_{12}t_1t_2 \quad . \quad . \quad . \quad (509);$$

and if we write the identity connecting the five quaternions in the form

$$p = p_0w + p_{23}x + p_{31}y + p_{12}z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (510),$$

comparison with (509) gives

$$2xyzw = y^2z^2 + z^2x^2 + x^2y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (511)$$

on elimination of the parameters t. This is the scalar equation of the surface (509), and the existence of the three intersecting double lines (y, z; z, x; and x, y), which characterize a STEINER'S quartic, is manifest.

Evidently the three connectors transform into the double lines; and the points $p_0 \pm p_{23}$, $p_0 \pm p_{31}$, $p_0 \pm p_{12}$ separate (Art. 133) the lines into regions intersected by a pair of real and a pair of imaginary sheets of the surface.*

137. The nature of the surface into which a plane transforms may be established from purely geometrical considerations. A tangent plane to the surface transforms back into a quadric touching the plane, that is, cutting it in a pair of lines. These lines transform back into conics in the tangent plane and on the surface. One point of intersection of these conics corresponds to the point of intersection of the lines. The other three points must result from the union of pairs of points of octads, and therefore the lines must cut the sides of the triangle in points harmonically conjugate to the Jacobian correspondents. The conics consequently intersect the lines into which the three connectors transform, and these three lines must be double. In terms

* It is easy to verify this by determining the greatest and least value of $t_2t_3(t_2^2 + t_3^2)^{-1}$ for real values of t_2 and t_3. Compare (509).

of the parameters, the equations of a pair of lines transforming into conics in a common plane must be

$$u_1t_1 + u_2t_2 + u_3t_3 = 0, \quad \frac{t_1}{u_1} + \frac{t_2}{u_2} + \frac{t_3}{u_3} = 0 \quad . \quad . \quad . \quad . \quad . \quad (512);$$

this is a consequence of the harmonic section. Two lines thus related may be said to be conjugate, and there exist four self-conjugate lines

$$t_1 \pm t_2 \pm t_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (513),$$

any one of which transforms into a conic having ring-contact with the quartic. The planes of these four conics transform back into cones, touching the plane along the self-conjugate lines. The self-conjugate lines join triads of non-corresponding Jacobian points, such as $q_1 + q_2$, $q_2 + q_3$, $q_3 - q_1$.

It is easy to see that the four conics are inscribed to the faces of a tetrahedron, and that each touches the other three. Consider, for example, the conics transformed from the sides of the triangle, $q_2 + q_3$, $q_3 + q_1$, $q_1 + q_2$. The equation of one conic is

$$p = f(q_2 + q_3 + t(q_3 + q_1), \quad q_2 + q_3 + t(q_3 + q_1))$$
$$= 2(p_0 + p_{23}) + 2t(p_0 + p_{23} + p_{31} + p_{12}) + 2t^2(p_0 + p_{31}) \quad . \quad . \quad (514);$$

and this shows that the conic passes through a limiting point on each of two of the double lines; and as the pole of the chord is symmetrical with respect to the suffixes, it is likewise the pole of corresponding chords for the conics into which the other sides of the triangle transform.

It is not difficult to prove that every line in the plane through one of the six Jacobian points transforms into a conic having a fixed tangent. The tangent for the point $q_1 + q_2$ is

$$p = p_0 + p_{12} + t(p_{23} + p_{31}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (515).$$

138. Let a connector meet the Jacobian in the points a, a', b and c, a and a' being correspondents so that $f(aa') = 0$; let b' and c' be the correspondents of b and c; and consider the points of an octad in the plane $[b'aa']$. The two connectors aa' and bb' in this plane intersect in the point b, and as b is its own harmonic conjugate with respect to b and b', two sides of the triangle of Art. 134 unite in the line aa'. Let $b_{\prime}$ be the harmonic conjugate of b with respect to a and a', then $b_{\prime}$ is a vertex of the infinitely slender triangle, the remaining two being the point b counted twice. (Compare Arts. 132 and 134.)

The point $b_{\prime}$, being the intersection of the connector aa' with a consecutive connector, is a *focal point* on the ray aa' of the congruency (496); and similarly $c_{\prime}$, the harmonic conjugate of c to a and a', is the second focal point; and by HAMILTON'S theory the ray touches the focal surface at these two points.*

* This theorem of the construction of the focal points is an extension of Mr. RUSSELL'S theorem for the congruency of lines joining corresponding points on the Hessian of a cubic surface. R. RUSSELL, "Geometry of Surfaces derived from Cubics," 'Proc. Roy. Irish Acad.,' vol. 5, p. 464.

In this case the plane transforms into the surface

$$p = f(t_1b + t_2b_, + t_3b', t_1b + t_2b_, + t_3b')$$
$$= t_1^2 f(bb) + t_2^2 f(b_,b_,) + t_3^2 f(b'b') + 2t_2t_3 f(b_,b') + 2t_1t_2 f(bb_,) \quad . \quad (516),$$

and if we take (as we may) $f(bb) = f(b_,b_,)$, the scalar equation of the surface takes the form

$$4xy^2w = 4x^2z^2 + y^4, \text{ where } w = t_1^2 + t_2^2,\ x = t_3^2,\ y = 2t_2t_3,\ z = 2t_1t_2 \quad (517).$$

On comparison with (511) we see that two of the lines of the STEINER's quartic have united; for $x = 0$ we have the line x, y counted four times.

139. By a process similar to that of Arts. 123 and 124, but much simpler, we can determine the order (m') of the complementary of a surface of order m, and the order (n') of the surface into which both transform. The formula is

$$\frac{4m}{\nu} = \frac{4m'}{8 - \nu} = n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (518),$$

where ν is the number of points of octads of which the surface is wholly composed.* And this formula is proved without trouble, remembering that a line in the p space transforms into a twisted quartic—the intersection of two quadric surfaces.

In like manner† for a curve (M), its complementary (M′) and its transformed (N),

$$\frac{2\mathrm{M}}{\nu} = \frac{2\mathrm{M}'}{8 - \nu} = \mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (519).$$

Thus the complementary of a connector is a twisted cubic;‡ the complementary of a plane is a surface of the seventh order, which cuts the plane in the triangle of connectors and in a quartic—probably the four lines of Art. 137.

The formulæ of this article are not directly applicable to the Jacobian, which is a critical surface of the transformation. The twisted quartic into which a line in the p space transforms, cuts the Jacobian in 16 points and does not in general touch it. For if it did the twisted quartic would have a double point. Consequently, the Jacobian transforms into a surface of the sixteenth order. Every point on the

* For the general transformation of order μ, the relation is

$$\frac{\mu^2 m}{\nu} = \frac{\mu^2 m'}{\mu^3 - \nu} = n.$$

† For a transformation of order μ,

$$\frac{\mu \mathrm{M}}{\nu} = \frac{\mu \mathrm{M}'}{\mu^3 - \nu} = \mathrm{N}.$$

‡ For example,

$$q = \sum_1^4 \frac{q_n}{x_n - ty_n}, \text{ where } q_5 = \sum_1^4 \frac{q_n}{x_n},\ q_6 = \sum_1^4 \frac{q_n}{y_n},$$

is the equation of the twisted cubic through six points $q_1, q_2 \ldots q_6$, and it is not difficult to verify that this curve and the line $q = q_7 + tq_8$ transform into a common line $p = p_0 + tp_{78}$ if the eight points form an octad.

Jacobian is the union of a pair of points of an octad (Art. 132), and therefore the complementary surface is composed of hexads of points of octads, and its order is consequently 24, or six times that of the Jacobian, because the quartic cuts it in a hexad for every point of intersection with the Jacobian.

140. *The complementary of the Jacobian is the focal surface of the congruency of connectors.**

When two points of a set transforming into a common point approach coincidence, they close in on the Jacobian, and simultaneously the remaining points of the set reach the complementary surface. Through any one of these remaining points two consecutive connectors pass; and therefore, by HAMILTON's beautiful theory, the remaining points are *focal points* on the rays connecting them to the coincident points.

Every ray touches the focal surface in two points—the two focal points on the ray; and for a quadratic transformation it cuts that surface in twenty other points. *These twenty points are harmonically conjugate in pairs to the Jacobian correspondents.* For (Art. 132) the harmonic conjugate of any one of the points belongs to the same octad as that point; but the focal surface is complementary and is wholly composed of hexads of points of octads, and therefore the harmonic conjugate is also on the focal surface.

141. *The focal surface of the transformed connectors is the transformed Jacobian.*

On transformation the harmonic conjugates on a connector unite. In the notation of Art. 138, the point b and the focal point $b_{\prime}$ unite in a focal point of the transformed connector, for through $b_{\prime}$ pass two consecutive connectors which transform into consecutive connectors through $f(b_{\prime}b_{\prime})$. Similarly the points c and $c_{\prime}$ transform into the second focal point and the transformed Jacobian is consequently the focal surface. The twenty points of the last article transform into ten points. The Jacobian correspondents a and a' transform into limiting points (Art. 133). Thus we have accounted for the sixteen points in which the transformed connector meets its focal surface.

The class of the transformed Jacobian is $n' = 4$. In the p space draw a plane through an arbitrary line to touch the surface. This plane contains a pair of consecutive transformed connectors, and on passing back to the q space it becomes a quadric containing consecutive intersecting connectors. This quadric is therefore a cone. The system of planes through the arbitrary line transforms into a system of quadrics through a twisted quartic, and four of these quadrics are cones. To these four cones correspond four tangent planes to the focal surface through the arbitrary line. Hence we may write down the equation of the reciprocal of the transformed Jacobian. The condition that the quadric $Slf(qq) = 0$ should be a cone† is

* This theorem is true for the connectors of a set of points to a coincident pair of the set for all transformations.

† If $f(qq) = \Sigma a_1 Sqf_1q$, then $f'(lq) = \Sigma f_1 q Sla_1$.

$$f'(lq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (520),$$

where r is the vertex, for the tangent plane $Slf(qr) = Sqf'(lr) = 0$ must vanish identically. Hence the fourth invariant of $f'(lr)$ must vanish, or

$$(f'(la),\ f'(lb),\ f'(lc),\ f'(ld)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (521),$$

and this is the equation of the reciprocal of the surface.

Thus the transformed Jacobian is the reciprocal of a Jacobian surface, but one of less generality than those previously considered. We may replace (520) by four equations

$$Slf(ra) = 0,\quad Slf(rb) = 0,\quad Slf(rc) = 0,\quad Slf(rd) = 0 \quad . \quad . \quad (522);$$

and because f is a permutable function, on replacing r by $xa + yb + zc + wd$ and eliminating x, y, z and w, we obtain the symmetrical determinant

$$\begin{vmatrix} Slf(aa), & Slf(ab), & Slf(ac), & Slf(ad) \\ Slf(ab), & Slf(bb), & Slf(bc), & Slf(bd) \\ Slf(ac), & Slf(bc), & Slf(cc), & Slf(cd) \\ Slf(ad), & Slf(bd), & Slf(cd), & Slf(dd) \end{vmatrix} = 0 \quad . \quad . \quad . \quad (523).$$

But ('Three Dimensions,' Art. 528) a surface, whose equation is a symmetrical determinant with constituents linear in the variables, has ten double points. This accounts for the class of the surface being 16 instead of 36 $(= 4(4-1)^2)$.

In the case in which the function is self-conjugate as well as permutable, that is when p, q and r may be transposed in $Spf(qr)$ in any manner, we have the theory of the corresponding points on the Hessian of the general cubic surface

$$Sqf(qq) = 0$$

and Mr. RUSSELL's paper may be referred to for various examples.

142. The characteristics of the two congruencies are found thus. The order of the congruency of connectors is obviously $\mu = 7$, as seven connectors can be drawn from an arbitrary point to the remaining points of the octad to which the point belongs. The class is $\nu = 3$, for three connectors lie in a plane. The order of the focal surface (Art. 139) is $M = 24$. Its class is $N = 16$. This follows from the relation ('Three Dimensions,' Art. 510)

$$M - N = 2(\mu - \nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (524);$$

or independently by Mr. RUSSELL's elegant method* which is applicable in this more general case.

For the transformed congruency, the order is $\mu' = 28$ (Art. 135), the order of the focal surface is $M' = 16$, and its class is $N' = 4$ (Arts. 139, 141); and therefore (524) the class of the congruency is $\nu' = 22$.

* 'Proc. Roy. Irish Acad.,' vol. 5, p. 473.

Consequently twenty-two connectors are generators of a quadric $Slf(qq) = 0$; and in particular the polar quadric of a point with respect to a cubic surface contains 22 generators joining corresponding points on the Hessian.

SECTION XIX.

Homography of Points in Space.

The Third Example of the Use of the Bilinear Function.

143. Writing generally

$$f(pq) = r \quad \text{or} \quad \{f(pq), r\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (525),$$

and regarding r as a constant quaternion, a one-to-one relation is established between the points p and q, so that one may be said to be the homograph of the other.

This is equivalent to three relations of the form

$$Spf_1q = 0, \quad Spf_2q = 0, \quad Spf_3q = 0 \quad . \quad . \quad . \quad . \quad . \quad (526);$$

and accordingly the bilinear function is not utilized to its full extent, but it seems to be the most convenient instrument for investigating the subject.

144. We have generally in the notation of Arts. 107, 108,

$$qI(p) = F_p(r), \quad pJ(q) = F_q(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad (527),$$

and thus the critical curves of the transformation are

$$F_p(r) = 0 \quad \text{and} \quad F_q(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (528)$$

respectively; or (compare (437))

$$((r, f(pa), f(pb), f(pc), f(pd))) = 0 \text{ and } ((r, f(aq), f(bq), f(cq), f(dq))) = 0 \quad . \quad (529).$$

These curves are sextics, and because (528) may be replaced by

$$[f''(pr_1), f''(pr_2), f''(pr_3)] = 0, \quad [f'(r_1q), f'(r_2q), f'(r_3q)] = 0 \quad . \quad (530),$$

where $[r_1r_2r_3] = 0$, they may be described as the locus of Jacobian correspondents of points in the plane reciprocal to the point r (424).

As in Art. 109, when a point (q) is on the critical curve, its homograph is a line

$$pJ'(q) = tG_q(r) + F_q(p), \quad F_q(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad (531),$$

and not a point; and as in Art. 112 the homograph of a line $q + xq'$ is a twisted cubic

$$p = (p_0 p_1 p_2 p_3 \between x 1)^3 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (532);$$

and a line of the type (531) breaks off the cubic for every intersection with the critical curve.

Thus, when the line is a chord of the critical curve, its homograph is also a line, so that

$$\{f(p + xp', \ q + xq'), r\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (533).$$

Symmetry shows that $p + xp'$ must be a chord of the second critical curve.

If the homograph of a line is plane, it is at most a conic. For the condition of planarity (compare Art. 120)

$$(p_0 p_1 p_2 p_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (534)$$

is of the sixth order in q and in q', and this equation represents a complex of the sixth order. But this complex can include nothing except intersectors of the critical sextic, for the cone of intersectors from the arbitrary point q is of the sixth order.

The ruled surface of triple chords has been noticed in Art. 75.

145. The homograph of a plane

$$Slq = 0 \quad \text{is} \quad SlF_p(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (535),$$

a general cubic surface through the critical curve.

This cubic surface also passes through the sextic

$$F_p'(l) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (536),$$

and it intersects the Jacobian $I(p) = 0$ in this sextic and in the critical curve. The equation of the Jacobian may be written in the forms

$$Sf_p'(l)\, F_p(r) = Sf_p(r)\, F_p'(l) = I(p)\, Slr = 0 \quad . \quad . \quad . \quad . \quad . \quad (537),$$

and for l and r, both variable, the curves $F_p(r) = 0$, $F_p'(l) = 0$ generate the Jacobian in a manner analogous to the double generation of a quadric. Since the rank of the sextic is $r = 16$ (Art. 64), the two curves intersect in 14 points (477).

146. It may be of interest to show how we can fully account for the lines on the cubic surface (535). Let the six points in which the critical curve $F_q(r) = 0$ cuts the plane $Slq = 0$ be denoted by the symbols 1, 2, 3, 4, 5, 6 ; and let (12), (23), &c., denote the fifteen connectors of these points. Further let [1], [2], . . . [6] denote the six conics that can be drawn through all but one of the six points.

The curves and points represented by these 27 symbols transform into the lines on the cubic. By (531) and (533) we account for the lines and the points. In general a unicursal curve transforms into a curve of thrice the order, but for every intersection with the critical curve a line breaks off. Thus the six conics likewise transform into lines.

Any pair of these loci, which intersect in a point which is not critical, continue to intersect after transformation, and this consideration enables us to write down the full scheme of double-sixes on the cubic surface. These fall into three types :—

$$\text{I.}\quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ [1] & [2] & [3] & [4] & [5] & [6] \end{pmatrix}.$$

$$\text{II.}\quad \begin{pmatrix} 1 & 2 & 3 & (56) & (64) & (45) \\ (23) & (31) & (12) & [4] & [5] & [6] \end{pmatrix}.$$

$$\text{III.}\quad \begin{pmatrix} 1 & [1] & (23) & (24) & (25) & (26) \\ 2 & [2] & (13) & (14) & (15) & (16) \end{pmatrix}.$$

In these schemes, every line represented by a symbol in one row intersects every line in the other row, except that denoted by the symbol in the same column. There are thus 36 double-sixes ; one of the first type, twenty of the second, fifteen of the third.

The schemes are easily obtained by taking two non-intersecting lines, say 1 and [1], when we have

1 intersects (12), (13), (14), (15), (16), [2], [3], [4], [5], [6],

[1] „ (12), (13), (14), (15), (16), 2, 3, 4, 5, 6,

and, discarding the common lines, the double-six is found. In like manner the 45 triple tangent planes belong to one or other of the types

(1, [2], (12)) or ((12), (34), (56)).

147. One or two relations respecting a point on a critical curve and its line homograph may be mentioned. Since the line (531) has a point for its homograph, it must be a triple chord of the sextic $F_p(r) = 0$. It meets this sextic in three points, p_1, p_2, p_3, and intersects the Jacobian in a fourth point p_0 or $F_q(p)$. To the three points p_1, p_2, p_3 correspond the three triple chords of the q sextic which pass through q ; and the homograph of every plane through the line p_1, p_2, p_3 is a cubic

having q as a double point and containing the three triple chords which pass through q.

The cubic homograph of any plane contains the critical sextic which counts thrice in its intersection with the octic surface of triple chords, and the remainder of the intersection consists of the six line-homographs of the critical points in the plane.

The homograph of the surface of chords of the p sextic, which meet the line $p_1p_2p_3$, is the cone whose vertex is q and which contains the q sextic.

The homograph of one sextic is the surface of triple chords of the other.

One chord can be drawn to meet two non-intersecting triple chords in points not on the sextic. Its homograph is the line joining the homographs of these chords.

The locus of the points $F_q(p)$, the Jacobian correspondents of points on the critical curve, is a curve of the fourteenth order. For the octic surface intersects the Jacobian in the second critical curve counted thrice, and in a residual curve of order 14.

148. Connectors of points with their homographs compose the complex of the sixth order

$$(f(pp), f(pq), f(qp), r)\,(f(pq), f(qp), f(qq), r)$$
$$= (f(pp), f(pq), f(qq), r)\,(f(pp), f(qp), f(qq), r) \quad . \quad (538),$$

as appears on elimination of x, y, z and w from

$$f(xp + yq,\ zp + wq) = r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (539).$$

Or in other words, this is the assemblage of lines which meet their twisted cubic homographs.

The condition that two pairs of homographs should be on the same line is

$$((f(pp), f(pq), f(qp), f(qq), r)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (540),$$

for if two sets of values of x, y, z, w satisfy (539), the five quaternions included in (540) must be co-planar. Now (540) imposes two conditions on the line pq, and therefore represents a congruency of lines; and from the conditions implied in (540) we can select but two combinatorial functions with respect to p and q. These are

$$(f(pp), f(pq), f(qp), f(qq)) = 0,\ (f(pp), f(pq) + f(qp), f(qq), r) = 0 \quad . \quad (541);$$

and the congruency is therefore common to two complexes of the fourth and third orders respectively. But these complexes contain the congruency

$$[f(pp), f(pq) + f(qp), f(qq)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (542),$$

and this is foreign to the question, being, in fact, the congruency (496) of Art. 132 of connectors for the permutable function $f(pq) + f(qp)$. When this is rejected, there remains the congruency of connectors of two pairs of homographs, and its order and class are $\mu = 5\,(= 4 \times 3 - 7)$, $\nu = 9\,(= 4 \times 3 - 3)$, for the congruency (542) has been shown to be of the seventh order and third class.

Equations (541) being supposed satisfied, they are equivalent to

$$u_1 f(pp) + u_2 f(pq) + u'_2 f(qp) + u_3 f(qq) = 0,$$

$$v_1 f(pp) + v_2\,(f(pq) + f(qp)) + v_3 f(qq) = r \quad . \quad . \quad . \quad . \quad (543);$$

and multiplying the first by t and adding it to the second, we find that t must satisfy the quadratic

$$(v_2 + tu_2)(v_2 + tu'_2) = (v_1 + tu_1)(v_3 + tu_3) \quad . \quad . \quad . \quad . \quad (544),$$

if the sum can be reduced to the form (539). The roots of this equation lead to the determination of the two pairs of homographs.

The bi-connectors of homographs which pass through a point are double edges of the cone of connectors of homographs, and those which lie in a plane are bi-tangents to the curve enveloped by the connectors. This appears from the forms of the equations (538) and (540).

149. The congruency of connectors of Jacobian correspondents is intimately connected with the theory of the last article.

We have already considered the case in which the function is permutable, but matters now are much more complicated.

The congruency may be expressed by

$$f(pp) + uf(pq) + vf(qp) + uvf(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad (545),$$

and it is obvious that it is included in the quartic complex, the first of (541), and it is easy to verify that it is also included in the sextic complex (538) and that *no matter what quaternion "r" may be.* Replacing uv by w in (545) and substituting in the equations of these two complexes we find that either $w = uv$, or else the lines must belong to the congruency (540). In other words, the congruency of this article is complementary to the congruency of the last as regards the two complexes. But the rays of the former congruency count double as edges of cones or as tangents in planes. Hence the order and class of the congruency under discussion are $\mu = 14\,(= 4 \times 6 - 2 \times 5)$, $\nu = 6\,(= 4 \times 6 - 2 \times 9)$.

These numbers are exactly double the corresponding numbers for the permutable function, and as regards the class there is no difficulty in seeing how this arises. In general there are two sextic loci of Jacobian correspondents of the points in a plane (528), and the connectors in the plane join the six points of one to the corresponding six points of the other. For the permutable function the two loci coalesce, and the number of connectors is halved.

Again, we may say that the lines of this new congruency through a point are *fixed* edges of the cone (538), and the lines in a plane fixed tangents to a sextic curve, because they are independent of r; the lines of the former congruency are double edges and double tangents.

We proceed to determine the class of the focal surface. The equations

$$(pqab) = 0, \quad f(pq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (546)$$

require a ray to intersect the fixed line a, b. Eliminating p, the equation of the locus of q is

$$f(qq) + xf(aq) + yf(bq) = 0, \quad \text{or} \quad [f(qq), f(aq), f(bq)] \quad . \quad . \quad (547);$$

and this (274) is a curve of order $m = 11$ and rank $r = 48$. But this curve is a complex curve consisting of the line ab and a residual which intersects it in four points on the Jacobian. The order and rank of the residual are $m = 10$, $r = 40$, the rank being diminished by twice the number of intersections. The number (r) of tangent planes through ab to this curve minus twice the number of intersections gives the number of planes through ab containing consecutive rays. Thus the class of the focal surface is N = 32, and its order (524) is M = 48. Every one of these numbers is double the corresponding number obtained in Art. 142 for the permutable function.

For the sake of completeness we wish to show the nature of the assemblage of lines common to the complex (538) and the second complex (541), as we have already completely considered the lines common to the remaining two pairs. Evidently the congruency of bi-connectors belongs to these two complexes and is counted twice among their common lines. There remains an assemblage of lines of order $\mu = 3 \times 6 - 2 \times 5 = 8$, and of class $\nu = 3 \times 6 - 2 \times 9 = 0$. It is easy to prove by the method of this article that these lines join an arbitrary point to the eight correspondents of r in the quadratic transformation $f(pp) = r$.

SECTION XX.

The Method of Arrays.

Applications to n-Systems of Linear Functions.

150. We shall illustrate the method of quaternion arrays* by a few examples on systems of linear functions. These functions may be supposed to be of the most general kind, functions of a point in space of μ dimensions, but we pay particular attention to the case of three dimensions.

An array of n rows and m columns vanishes if, and only if, the constituents in the rows are connected by the same set of scalar coefficients $x_1, x_2 \ldots x_m$. Thus

$$\left\{ \begin{matrix} a_1 & a_2 & a_3 & \ldots & a_m \\ b_1 & b_2 & b_3 & \ldots & b_m \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ l_1 & l_2 & l_3 & \ldots & l_m \\ p_1 & p_2 & p_3 & \ldots & p_m \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ r_1 & r_2 & r_3 & \ldots & r_m \end{matrix} \right\} = 0 \quad \ldots \ldots \quad (548),$$

when

$$\Sigma x_s a_s = 0, \; \Sigma x_s b_s = 0, \; \ldots \; \Sigma x_s r_s = 0 \quad \ldots \ldots \quad (549).$$

It is proved in the memoir that the expansion of the array is of the form†

$$\Sigma \pm (a_1 a_2 a_3 a_4)(b_5 b_6 b_7 b_8) \ldots (l_{4n'-3}, \; l_{4n'-2}, \; l_{4n'-1}, \; l_{4n'})$$
$$\times \left\{ \begin{matrix} p_{4n'+1} & p_{4n'+2} & \ldots & p_m \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_{4n'+1} & r_{4n'+2} & \ldots & r_m \end{matrix} \right\} \quad \ldots \ldots \quad (550);$$

and we take definitely $m = 4n' + n''$, where $n'' = 0$, 1, 2 or 3. The number of equivalent scalar conditions is $4m - n + 1$ for the vanishing of a quaternion array, and $(\mu + 1)m - n + 1$ for an array of points in μ dimensions.

The scalars x_1, x_2, &c., are determined when (548) is satisfied by the system of arrays of $m - 1$ columns and n rows, of which this array

$$\left\{ \begin{matrix} x_1 a_1 + x_2 a_2, & a_3, & \ldots & a_m \\ x_1 b_1 + x_2 b_2, & b_3 & & b_m \\ \cdot & & & \\ \cdot & & & \\ x_1 r_1 + x_2 r_2, & r_3 & \ldots & r_m \end{matrix} \right\} = 0 \quad \ldots \ldots \quad (551)$$

is a type.

* 'Trans. Roy. Irish Acad.,' vol. 32, pp. 17–30.

† Every row must be represented in the expansion, and it may be gathered from the Memoir how to expand if one row involves only four constituents. In this case the general method fails.

If all the minor arrays formed by omitting one column of (548) vanish, we take any two of these minors, and forming second minors corresponding to (551) we obtain two sets of relations (549), and so on in general.

151. In order to find the conditions that a linear function of an n-system should convert m given *weighted* points $a_1, \ldots a_m$ into m others, $b_1, \ldots b_m$, we write down the array in m rows and $n+1$ columns,

$$\left\{\begin{matrix} f_1a_1 & f_2a_1 & \cdot & \cdot & f_na_1 & b_1 \\ f_1a_2 & f_2a_2 & \cdot & \cdot & f_na_2 & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1a_m & f_2a_m & \cdot & \cdot & f_na_m & b_m \end{matrix}\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (552),$$

whose vanishing requires

$$\Sigma x_s f_s a_t = b_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (553).$$

The vanishing of this array requires $4m - n$ scalar equations to be satisfied. If then $n = 4m$, the array vanishes without restriction, and a single condition must be satisfied for the vanishing of the arrays, such as (551),

$$\left\{\begin{matrix} x_1f_1a_1 + b_1 & f_2a_1 & \cdot & \cdot & f_na_1 \\ x_1f_1a_2 + b_2 & f_2a_2 & \cdot & \cdot & f_na_2 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ x_1f_1a_m + b_m & f_2a_m & \cdot & \cdot & f_na_m \end{matrix}\right\} = 0, \text{ \&c.} \quad . \quad . \quad . \quad . \quad (554),$$

and these determine the coefficients x without ambiguity.

Thus from a given $4m$-system can be found one function which shall convert m given weighted points into other given weighted points. (Compare Art. 3.)

152. When the weights are disregarded, the equations of condition are

$$\Sigma x_s f_s a_1 = y_1 b_1, \quad \Sigma x_s f_s a_2 = y_2 b_2, \ \ldots \ \Sigma x_s f_s a_m = y_m b_m \quad . \quad . \quad . \quad (555);$$

and these furnish the array

$$\left\{\begin{matrix} f_1a_1 & f_2a_1 & \cdot & \cdot & \cdot & f_na_1 & b_1 & 0 & 0 & \cdot & \cdot & 0 \\ f_1a_2 & f_2a_2 & \cdot & \cdot & \cdot & f_na_2 & 0 & b_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ f_1a_m & f_2a_m & \cdot & \cdot & \cdot & f_na_m & 0 & 0 & 0 & \cdot & \cdot & b_m \end{matrix}\right\} = 0 \quad . \quad . \quad . \quad . \quad (556),$$

of $m + n$ columns and m rows. Its vanishing requires $3m - n + 1$ conditions to be satisfied, and the vanishing of the minor arrays such as (551) requires a single condition if $n = 3m + 1$, and these definitely determine the function. Thus from a $(3m + 1)$-system can be found one function which converts m points to m others when the weights are neglected. In particular, a linear transformation can be found (out of the whole sixteen-system) to convert five points into five others (Art. 3).

153. When lines are to be converted into lines, the conditions are

$$\Sigma x_s f_s a_t = y_t b_t + y'_t b'_t ; \quad \Sigma x_s f_s a'_t = z_t b_t + z'_t b'_t \quad . \quad . \quad . \quad . \quad (557),$$

and the array

$$\left\{ \begin{array}{cccccccccccccc} f_1 a_1 & f_2 a_1 & . \; . & f_n a_1 & b_1 & b'_1 & 0 & 0 & . \; . \; . & 0 & 0 & 0 & 0 \\ f_1 a'_1 & f_2 a'_1 & . \; . & f_n a'_1 & 0 & 0 & b_1 & b'_1 & . \; . \; . & 0 & 0 & 0 & 0 \\ . & . & & . & . & . & . & . & & . & . & . & . \\ f_1 a_m & f_2 a_m & . \; . & f_n a_m & 0 & 0 & 0 & 0 & . \; . \; . & b_m & b'_m & 0 & 0 \\ f_1 a'_m & f_2 a'_m & . \; . & f_n a'_m & 0 & 0 & 0 & 0 & . \; . \; . & 0 & 0 & b_m & b'_m \end{array} \right\} \quad . \quad . \quad (558)$$

of $n + 2m$ columns and of $2m$ rows must vanish. The number of conditions is $6m - n + 1$. Thus a function of a seven-system and of a thirteen-system respectively converts one and two lines into one and two others.

In like manner, when planes are to be converted into planes, the array is of $n + 3m$ columns and of $3m$ rows, and requires $9m - n + 1$ conditions for its vanishing.

In general for space of μ dimensions a function of an n-system is completely defined if

$$n = \mu (m_1 + 2m_2 + 3m_3 + \&c.) + 1 = \mu M + 1 \quad . \quad . \quad . \quad . \quad (559),$$

which converts m_1 given points, m_2 lines, m_3 planes, &c., into other given points, lines and planes, &c.

154. We shall now suppose that the array (556) does not vanish without conditions restricting the generality of the points. Let all the points except a_m and b_m be given. It is sufficient to consider the cases in which the number of conditions does not exceed three.

By the expansion (550) we have, if $3m - n + 1 = \nu$, so that ν conditions must be satisfied, or if $n = 3m - \nu + 1 = 3(m - 1) + (4 - \nu)$,

$$\begin{aligned} \Sigma \pm (f_1(a_1), f_2(a_1), f_3(a_1), b_1)\,(f_4(a_2), f_5(a_2), f_6(a_2), b_2) \; . \; . \; . \\ (f_{3n-5}(a_{m-1}), f_{3m-4}(a_{m-1}), f_{3m-3}(a_{m-1})\, b_{m-1}) \\ \times \{f_{3m-2}(a_m), f_{3m-1}(a_m) \; . \; . \; . \; f_{3n-\nu+1}(a_m), b_m\} = 0 \quad . \quad . \quad . \quad (560). \end{aligned}$$

For it is obviously no use retaining any term $(f_1(a_1), f_2(a_1), f_3(a_1), f_4(a_1))$, in which a b does not enter, as the minor array of this term has a column of zeros and vanishes.

We thus have three types of conditions for $\nu = 1$, 2 or 3, and these are of the forms, the functions F being linear,

$$\begin{aligned} &\text{I.} \;\; (F_1 a_m, F_2 a_m, F_3 a_m, b_m) = 0 ; \\ &\text{II.} \;\; [F_1 a_m, F_2 a_m, b_m] = 0 ; \\ &\text{III.} \;\; \{F_1 a_m, b_m\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (561). \end{aligned}$$

In type I, if a_m is given, b_m lies in a plane; and a_m lies on a general cubic surface if b_m is given.

In type II, if a_m is given, b_m may be any point on a line; and if b_m is given, a_m may be any point on a twisted cubic.

In the third case, $F_1 a_m = t b_m$, and either point is determined if the other is given.

There is no difficulty in applying this method to the case of Art. 153. We must, however, include the case of four conditions being requisite. The last line must belong to a complex, a congruency, a ruled surface, or be one of a definite number of lines.

155. We shall now consider the critical cases when every first minor of (552) vanishes.

The minor obtained by omitting the last column expands into

$$\Sigma \pm (f_1(a_1), f_2(a_1), f_3(a_1), f_4(a_1)) \ldots \{f_{4m-3}(a_m), f_{4m-2}(a_m) \ldots f_n(a_m)\} \quad . \quad (562).$$

Here, as in the last article, we have the types

$$\text{I. } (F_1 a_m, F_2 a_m, F_3 a_m, F_4 a_m) = 0;$$
$$\text{II. } [F_1 a_m, F_2 a_m, F_3 a_m] = 0;$$
$$\text{III. } \{F_1 a_m, F_2 a_m\} = 0;$$
$$\text{IV. } F a_m = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (563)$$

corresponding to $n = 4m$, $n = 4m + 1$, $n = 4m + 2$, and $n = 4m + 3$.

Now, from the nature of arrays, though it does not appear directly from the form of the expansion, these conditions are all combinatorial functions of the m points a.

I. In the first place, for the type I we have for $m = 1$ the Jacobian of a four-system. Next, for $n = 8$, $m = 2$ we have a one-conditioned assemblage of lines of the fourth order, or a complex of the fourth order. These are the lines which can be destroyed by single functions of the system. For $n = 12$, $m = 3$, (562) represents a one-conditioned assemblage of planes, and these planes envelope a surface of the fourth class, and each can be destroyed by a corresponding definite function of the system.

For $n = 16$, $m = 4$, the same equation represents a constant multiplied by the volume of the tetrahedron $(a_1 a_2 a_3 a_4)$ to the fourth power.

II. Again, for $n = 4m - 1$, and more particularly for $m = 1$, we have the critical sextic

$$[f_1 a f_2 a f_3 a] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (564),$$

of three functions; and for seven functions a congruency of lines common to a set of quartic complexes; while for eleven functions we have a two-conditioned assemblage of planes, or a developable of planes enveloping certain surfaces of the fourth class.

III. For $n = 4m - 2$ there is first the system of united points of $f_1^{-1} f_2$ for a pair of functions, or $\{f_1 a, f_2 a\} = 0$. Secondly, a ruled surface of lines destroyed by functions of a six-system; and thirdly, a determinate number of planes destroyed by functions of a ten-system. For a fourteen-system it requires an invariant relation to vanish.

IV. This case requires a single function to destroy a point; it gives the lines destroyed by functions of a five-system (of these there are 20, compare Art. 114); and it imposes a condition on a nine-system of functions, so that some function of the system may be capable of destroying a plane. For a thirteen-system an invariant relation must vanish if a critical case arises for non-coplanar points.

I calculate the order of the Kummer surface of the quartic complex for the eight-system to be 72, and the order and class of the congruency of the double lines to be 24. The lines of this congruency would seem to be capable of being destroyed by two-systems of functions selected from the eight-system.

156. More particularly, if the line ab can be destroyed by a single function of an n-system,

$$\Sigma x_1 f_1 a = 0, \quad \Sigma x_1 f_1 b = 0 \quad \text{. . . . . (565);}$$

and the array

$$\begin{Bmatrix} f_1 a & f_2 a & \ldots f_n a \\ f_1 b & f_2 b & \ldots f_n b \end{Bmatrix} = 0 \quad \text{. . . . . (566)}$$

must vanish. The number of conditions is now $9 - n$, so that from a nine-system one function can be found to destroy an arbitrary line. For $n = 8$, we have the complex

$$\Sigma \pm (f_1 a f_2 a f_3 a f_4 a)(f_5 b f_6 b f_7 b f_8 b) = 0 \quad \text{. . . . . (567).}$$

If the plane a, b, c can be destroyed by a single function

$$\begin{Bmatrix} f_1 a & f_2 a \ldots f_n a \\ f_1 b & f_2 b \ldots f_n b \\ f_1 c & f_2 c \ldots f_n c \end{Bmatrix} = 0 \quad \text{. . . . . (568),}$$

and this requires $13 - n$ conditions. For $n = 12$ we have the surface enveloped by the plane (compare the last article)

$$\Sigma \pm (f_1 a f_2 a f_3 a f_4 a)(f_5 b f_6 b f_7 b f_8 b)(f_9 c f_{10} c f_{11} c f_{12} c) = 0 \quad \text{. . (569).}$$

157. When a line can be destroyed point by point by functions of a two-system selected from an n-system,

$$\Sigma (x_1 + t y_1) f_1 (a + t b) = 0, \text{ or } \Sigma x_1 f_1 a = 0, \ \Sigma x_1 f_1 b + \Sigma y_1 f_1 a = 0, \ \Sigma y_1 f_1 b = 0 \quad \text{(570);}$$

and the array

$$\begin{Bmatrix} f_1 a & f_2 a \ldots f_n a & 0 & 0 \ldots 0 \\ f_1 b & f_2 b \ldots f_n b & f_1 a & f_2 a \ldots f_n a \\ 0 & 0 \ldots 0 & f_1 b & f_2 b \ldots f_n b \end{Bmatrix} = 0 \quad \text{. . . . (571)}$$

must vanish, or $13 - 2n$ conditions must be satisfied when the line is arbitrary. The *functions* must satisfy $9 - 2n$ conditions, as the line may be made to satisfy four. For a four-system one condition must be satisfied for the existence of a line of this nature, but for a five-system (compare Art. 114) a ruled surface of such lines exists, triple chords of a curve of the tenth order.

If the line can be destroyed by functions of a three-system we have (compare Art. 114)

$$\Sigma (x_1 + ty_1 + t^2z_1) f_1 (a + tb) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (572),$$

and the resulting array is of 4 rows and $3n$ columns, and vanishes if $13 - 3n$ conditions are satisfied. Finally, if the line is destroyed seriatim by functions of an included four-system, $21 - 4n$ conditions must be satisfied.

We may state that the number of conditions required to determine an N-system included in an n-system is

$$\mathrm{N}\,(n - \mathrm{N}) = \mathrm{N}'\,(n - \mathrm{N}'), \quad (\mathrm{N} + \mathrm{N}' = n) \quad . \quad . \quad . \quad . \quad . \quad (573).$$

158. As regards the destruction of planes, a plane may be destroyed *en bloc*, as in (568), or line by line, or point by point. In the second case,

$$\Sigma (x_1 + sy_1) f_1 (a + tb + sc + std) = 0,$$

$$\text{or} \quad \Sigma (x_1 + sy_1) f_1 (a + sc) = 0, \; \Sigma (x_1 + sy_1) f_1 (b + sd) = 0 \quad . \quad . \quad (574),$$

with the condition $(abcd) = 0$.

Thus the array is

$$\left\{ \begin{array}{cccccccc} f_1a & f_2a & .. & f_na & 0 & ... & 0 \\ f_1c & f_2c & .. & f_nc & f_1a & ... & f_na \\ 0 & 0 & .. & 0 & f_1c & ... & f_nc \\ f_1b & f_2b & .. & f_nb & 0 & ... & 0 \\ f_1d & f_2d & .. & f_nd & f_1b & ... & f_nb \\ 0 & 0 & .. & 0 & f_1d & ... & f_nd \end{array} \right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (575)$$

of 6 rows and $2n$ columns, requiring $25-2n$ conditions when we disregard $(abcd)=0$. This is the case in which *a function can destroy a hyperboloid* generator by generator*. The same number of conditions must be satisfied even when the four points are supposed co-planar.

Finally, the case in which the points are destroyed seriatim gives an array of $3n$ columns and 6 rows, requiring $25 - 3n$ conditions for its vanishing.

From these articles we can clearly trace the way in which a Jacobian of four functions may degrade, one of the most interesting being where it breaks up into a pair of quadrics, one of which is destroyed generator by generator by a two-system.

* In the paper on the interpretation of a quaternion as a point symbol, the equation $q = a + tb + sc + std$ is considered. It represents a ruled quadric and exhibits the dual generation.

SECTION XXI.

The Extension of the Method to Hyper-Space.

159. Exactly as in quaternions we may regard the sum of a scalar and a line vector in space of n dimensions as the symbol of a weighted point.

If

$$q = \mathrm{S}q + \mathrm{V}q = \left(1 + \frac{\mathrm{V}q}{\mathrm{S}q}\right) \mathrm{S}q = (1 + \mathrm{OQ})\, \mathrm{S}q\, ; \; \mathrm{OQ} = \frac{\mathrm{V}q}{\mathrm{S}q} \quad . \quad . \quad (576);$$

q is the symbol of the point Q to which a weight Sq is attributed.

The point represented by a sum of point symbols is the centre of mass of the weighted points, and the weight attributable to that point is the sum of the weights.

The equation

$$q = a + tb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (577),$$

in which t is a variable scalar, is the equation of the line ab.

The most general homographic divisions on two lines ab and cd are represented by

$$q = a + tb, \quad q = c + td \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (578),$$

in which the weights Sa, Sb, Sc, Sd have been suitably selected.

The equation

$$q = t_1 a_1 + t_2 a_2 + t_3 a_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (579)$$

represents the plane of the points a_1, a_2, a_3; and more generally

$$q = t_1 a_1 + t_2 a_2 + \&\text{c.} \ldots + t_m a_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (580)$$

is the equation of the $(m-1)$-flat containing the m points $a_1, a_2 \ldots a_m$.

I believe it is more convenient to call generally a plane space of m dimensions an m-flat, and to retain the name *plane* for its ordinary signification—a two-flat.

160. In accordance with Hamilton's notation ('Elements,' Art. 365) we propose to write

$$[a_1 a_2 a_3 \ldots a_m] = \mathrm{V}_m . \mathrm{V}a_1 \mathrm{V}a_2 \ldots \mathrm{V}a_m - \Sigma \pm \mathrm{V}_{m-1} . \mathrm{V}a_2 \mathrm{V}a_3 \ldots \mathrm{V}a_m \mathrm{S}a_1 \quad . \quad (581);$$

or briefly

$$[a]_m = \mathrm{V}_m [a]_m + \mathrm{V}_{m-1} [a]_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (582);$$

as the symbol of the $(m-1)$-flat containing the m points $a_1, a_2 \ldots a_m$.

In order to justify this proposal, we observe that the array $[a_1a_2 \ldots a_m]$ changes sign whenever two contiguous elements are transposed. It consequently vanishes whenever one element is a scalar multiple of another, or whenever any group of elements is linearly connected by scalar coefficients; and it does not vanish under any other conditions. It is equivalent to the most general one-row array that can be formed from the m symbols a, because, according to the principles laid down on the subject of quaternion arrays, the general one-row array must be of the form

$$\{a_1a_2 \ldots a_m\} = x\mathrm{V}_m.\mathrm{V}a_1\mathrm{V}a_2 \ldots \mathrm{V}a_m + y\Sigma \pm \mathrm{V}_{m-1}.\mathrm{V}a_2\mathrm{V}a_3 \ldots \mathrm{V}a_m\mathrm{S}a_1 \quad . \quad (583);$$

and the separable parts V_m and V_{m-1} of $[a]_m$ afford all the information contained in the general array with indeterminate scalars x and y.

The equation of the flat containing m points a may be written in the form

$$[qa_1a_2a_3 \ldots a_m] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (584),$$

as this implies (580)

$$q = t_1a_1 + t_2a_2 + t_3a_3 + \ldots + t_ma_m,$$

in which $t_1, t_2 \ldots t_m$ are variable scalars.

161. Returning to the relation (582)

$$[a]_m = \mathrm{V}_m[a]_m + \mathrm{V}_{m-1}[a]_m,$$

it is evident that $\mathrm{V}_{m-1}[a]_m$ is equal to the product of a scalar and a set of $m-1$ mutually rectangular unit vectors $i_2, i_3 \ldots i_m$, in the $(m-1)$-flat containing the m points $a_1, a_2 \ldots a_m$. It is also apparent that $\mathrm{V}_m[a]_m$ is the product of a set of m mutually rectangular unit vectors in the m-flat containing the origin and the points a multiplied by a scalar. We may take this product of m vector units to be $i_1i_2i_3 \ldots i_m$. Thus we have

$$[a]_m = (yi_1 - x)\, i_2i_3 \ldots i_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (585),$$

where $i_si_t + i_ti_s = 0$, $i_s^2 = -1$, and where x and y are certain scalars. (Compare CLIFFORD'S 'Mathematical Papers,' p. 398.)

From this we find the symbol of a definite point

$$\mathrm{A}_m = 1 + \frac{\mathrm{V}_{m-1}[a]_m}{\mathrm{V}_m[a]_m} = 1 - \frac{xi_2i_3 \ldots i_m}{yi_1i_2i_3 \ldots i_m} = 1 - \frac{x}{yi_1} = 1 + \frac{x}{y}i_1 \quad . \quad (586);$$

and we verify at once that this point is a conjugate of all the points a with respect to the quadric

$$\mathrm{S}.q^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (587),$$

because for any one of these points we have

$$\mathrm{S}a_t\mathrm{A}_m = \mathrm{S}a_t\left(1 + \frac{\mathrm{V}_{m-1}[a]_m}{\mathrm{V}_m[a]_m}\right) = \mathrm{S}a_t + \frac{\mathrm{V}_m a_t \mathrm{V}_{m-1}[a]_m}{\mathrm{V}_m[a]_m} = \mathrm{S}a_t - \mathrm{S}a_t = 0 \quad . \quad (588),$$

as appears on reference to the equation (581).

In other words, A_m is the reciprocal in the m-flat which contains the origin and the points a of the $(m-1)$-flat which contains the points a.

For example, in three dimensions,

$$A_2 = 1 + \frac{Va_2Sa_1 - Va_1Sa_2}{VVa_1Va_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (589)$$

is the point in the plane oa_1a_2 which is reciprocal to the line a_1a_2.

162. A comparison of the equations (581) and (585) shows that the m points

$$x + yi_1, i_2, i_3 \ldots i_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (590)$$

(of which $i_2, i_3 \ldots i_m$ are at infinity) may be taken as defining the $(m-1)$-flat containing the points a.

Hence, conversely, if $[a]_m$ is any function satisfying the equations of condition

$$[a]_m = V_m[a]_m + V_{m-1}[a]_m\,; \quad \frac{V_{m-1}[a]_m}{V_m[a]_m} = V\,.\,\frac{V_{m-1}[a]_m}{V_m[a]_m} \quad . \quad . \quad . \quad (591),$$

it is the symbol of an $(m-1)$-flat. In fact, we can reduce this function to the form (585) and the proposition is evident by (590).

163. *The symbol of the flat reciprocal to* $[a]_m$ *with respect to the auxiliary quadric* (587), $S\,.\,q^2 = 0$, *in an n-space is*

$$[a]_m\,\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (592),$$

where Ω *is the product of "n" mutually rectangular vector units in the n-space, or*

$$\Omega = i_1i_2i_3 \ldots i_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (593).$$

In fact, from (585) we obtain

$$[a]_m\,\Omega = (-)^{m-1}(yi_1 - x)\,i_1i_{m+1}i_{m+2}\ldots i_n\,(i_2i_3i_4\ldots i_m)^2$$
$$= (-)^{\frac{1}{2}m(m+1)}(y + xi_1)\,i_{m+1}i_{m+2}\ldots i_n = [a]_{n+1-m} \quad . \quad . \quad . \quad (594)\,;$$

and $n+1-m$ defining points of this new $(n-m)$-flat are (590)

$$y + xi_1, i_{m+1}, i_{m+2} \ldots i_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (595).$$

But all these points are conjugates, with respect to the auxiliary quadric, of the m points (590); and therefore the flat $[a]_m\Omega$ is the reciprocal of the flat $[a]_m$.

More symbolically, we have the relations

$$V_m[a]_m\,.\,\Omega = V_{n-m}\,.\,[a]_m\,\Omega\,; \quad V_{m-1}[a]_m\,.\,\Omega = -V_{n-m+1}\,.\,[a]_m\,\Omega \quad . \quad (596),$$

and in particular for three dimensions we deduce the relations

$$[ab] = -(a'b')\,; \quad (ab) = [a'b'] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (597),$$

connecting a line and its reciprocal (compare p. 224).

For odd spaces, if

$$n - m + 1 = m \quad \text{or} \quad m = \tfrac{1}{2}(n + 1)\,,$$

the flat and its reciprocal, $[a]_m$ and $[a]_m\Omega$, are of the same order. This is the case for a line in three dimensions, and we recover from the general formulæ

$$[ab] = -(a'b')\,; \quad (ab) = [a'b']\,,$$

relations which I have elsewhere given connecting the symbols of reciprocal lines.

We are now prepared with all the necessary machinery for the geometry of flats and of their reciprocals.

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PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 329–402.

THE DIFFERENTIAL INVARIANTS OF A SURFACE, AND THEIR GEOMETRIC SIGNIFICANCE

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LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 339.

24.6.03

Price Three Shillings and Sixpence.

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IX. *The Differential Invariants of a Surface, and their Geometric Significance.*

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Received February 14,—Read March 5, 1903.

The present memoir is devoted to the consideration of the differential invariants of a surface; and these are defined as the functions of the fundamental magnitudes of the surface and of quantities connected with curves upon the surface which remain unchanged in value through all changes of the variables of position on the surface. The idea of differential parameters for relations of space appears to have been introduced by Lamé; it is to Beltrami* that the earliest investigations of the corresponding quantities in the theory of surfaces are due, as well as many detailed results.†

It is natural to expect that these differential invariants would belong to the general class of differential invariants which constitute Lie's important generalisation of the original theory of invariants and covariants of homogeneous forms. This association has been effected‡ for some classes of differential invariants by Professor Žorawski, and he has obtained the explicit expression of several of the individual functions.

Professor Žorawski's method is used in the present memoir. In applying it, a considerable simplification proves to be possible; for it appears that, at a certain stage in the solution of the partial differential equations characteristic of the invariance, the equations which then remain unsolved can be transformed so that they become the partial differential equations of the system of concomitants of a set of simultaneous binary forms. The known results of the latter theory can therefore be used to complete the solution of the partial differential equations, and the result gives the algebraic aggregate of the differential invariants.

This memoir consists of two parts. In the first, the investigation just indicated is carried out; and the explicit expressions of the members of an aggregate, algebraically

* In his memoir, "Sulla teorica generale dei parametri differenziali," 'Mem. Acc. Bologna,' 2nd Series, vol. 8 (1869), pp. 549–590, Beltrami gives a sketch of the early history of the subject.

† An account of the theory, developed on the basis of Beltrami's researches, is given by Darboux, 'Théorie générale des surfaces,' vol. 3, pp. 193–217; he also gives references to Bonnet and Laguerre.

‡ In a memoir hereafter quoted (§ 1).

complete up to a certain order, are obtained. In the second part, the geometric significance of the different invariants is the goal; in attaining it, some modifications are made in the aggregate, but they leave it algebraically complete.

The investigation reveals new relations among the intrinsic geometric properties of a curve upon a surface. To the order considered, four such relations exist; and their explicit expressions have been constructed.

PART I.

Construction of the Invariants.

1. In an interesting memoir* published in the 'Acta Mathematica,' Professor Žorawski has developed a method, outlined by Lie,† and has applied it to the determination of certain properties of functions which appertain to a surface and are invariantive, alike under any transformation of the two independent variables and under any deformation of the surface that involves neither tearing nor stretching. In particular, he obtains the number of these functions of any order which are algebraically independent of one another; he also obtains expressions for several functions of the lowest orders belonging to recognised types.

The method, and much of Professor Žorawski's analysis, can be applied to obtain the more extensive class of all the differential functions which, appertaining to a surface and to any set of curves upon the surface, are invariantive under any transformation of the two independent variables. The process, which involves the solution of complete Jacobian systems of the first order and the first degree, only gives the invariantive functions which are algebraically independent of one another; it is not adapted to the construction of the asyzygetic aggregate. Moreover, only some of these functions are invariantive when the surface is deformed without tearing or stretching; they can be selected by inspection, on using the fundamental theorem connected with the theory of the deformation of surfaces.

As far as possible, the notation adopted by Professor Žorawski is used. The analysis, preliminary to the construction of the differential equations which are characteristic of the invariance, is set out briefly; it is needed to make the process intelligible. There is some difference from Professor Žorawski's analysis, mainly (but not entirely) because a beginning is made from the consideration of relative invariants and not of absolute invariants.

2. The independent variables of position on the surface are taken to be x and y. A function f of these variables and of the derivatives of any number of functions

* "Ueber Biegungsinvarianten: eine Anwendung der Lie'schen Gruppentheorie," 'Acta Math.,' vol. 16 (1892–93), pp. 1–64.

† 'Math. Ann.,' vol. 24 (1884), pp. 574, 575.

which involve the invariables is said to be a relative invariant when, if the same function F of new independent variables X and Y and of corresponding new derivatives of the transformed functions be constructed, the relation

$$f = \Omega^{\mu} F$$

is satisfied, where

$$\Omega = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}.$$

The invariants actually considered are rational, so that μ is an integer. The invariant is said to be absolute where $\mu = 0$.

Now it is known, by LIE's theory, that the property of invariance will be established if it is possessed for the most general infinitesimal transformation of x and y; accordingly, we shall take

$$X = x + \xi(x, y)\, dt, \quad Y = y + \eta(x, y)\, dt,$$

where ξ and η are arbitrary integral functions of x and y. Derivatives with regard to x and y are required; we write

$$u_{mn} = \frac{\partial^{m+n} u}{\partial x^m \partial y^n},$$

for all values of m and n. Thus, as only the first power of dt is retained, we have

$$\Omega = 1 + (\xi_{10} + \eta_{01})\, dt.$$

The possible Arguments in the Invariants.

3. Next, we have to consider the possible arguments of a differential invariant of a surface. Broadly speaking, these may belong to one or other of three classes :—

(i) the fundamental magnitudes associated with the surface, and their derivatives of any order with respect to x and y;

(ii) functions $\phi(x, y)$, $\psi(x, y)$, . . . and their derivatives of any order with respect to x and y;

(iii) the variables x and y, and the derivatives of y of any order with regard to x.

We consider them briefly in turn.

4. Firstly, as regards the fundamental magnitudes: by a known theorem, a surface is defined uniquely (save only as to position and orientation) by the three magnitudes of the first order, usually denoted by E, F, G, and the three magnitudes of the second order, denoted by L, M, N. (If only E, F, G be given, the surface is defined as above, subject also to any deformation that does not involve tearing or stretching.)

These six quantities can occur in the invariantive function required, as well as their derivatives of any order with respect to x and y.

But there is a difficulty as regards the derivatives of L, M, N; for there are two relations, commonly known as the MAINARDI-CODAZZI equations, which express

$$\frac{\partial \mathrm{L}}{\partial y}-\frac{\partial \mathrm{M}}{\partial x}, \quad \frac{\partial \mathrm{M}}{\partial y}-\frac{\partial \mathrm{N}}{\partial x}$$

in terms of L, M, N, E, F, G, and the first derivatives of E, F, G. To avoid this difficulty, it is convenient to introduce the four fundamental magnitudes of the third order, denoted by P, Q, R, S; the six first derivatives of L, M, N can be expressed in terms of P, Q, R, S linearly, together with additive combinations of L, M, N and of the first derivatives of E, F, G.

The second derivatives of L, M, N will thus be expressible in terms of the first derivatives of P, Q, R, S, together with the appropriate additive combinations free from those derivatives. But again there is a difficulty as regards these; for there are three relations, which express

$$\frac{\partial \mathrm{Q}}{\partial x}-\frac{\partial \mathrm{P}}{\partial y}, \quad \frac{\partial \mathrm{R}}{\partial x}-\frac{\partial \mathrm{Q}}{\partial y}, \quad \frac{\partial \mathrm{S}}{\partial x}-\frac{\partial \mathrm{R}}{\partial y}$$

in terms of P, Q, R, S, L, M, N, E, F, G, and the first derivatives of E, F, G. To avoid this new difficulty, it is convenient to introduce the five fundamental magnitudes of the fourth order, denoted by α, β, γ, δ, ϵ; the first derivatives of P, Q, R, S (and therefore the second derivatives of L, M, N) can be expressed linearly in terms of α, β, γ, δ, ϵ, together with additive combinations of P, Q, R, S, L, M, N, E, F, G, and the first derivatives of E, F, G.

And so on, for the derivatives of successive orders of L, M, N; we avoid the difficulty of linear relations among them by the introduction of the successive fundamental magnitudes. The analytical definition* of these magnitudes can be taken in the form

$$\begin{aligned}
ds^2 &= \mathrm{E}\,dx^2 + 2\mathrm{F}\,dx\,dy + \mathrm{G}\,dy^2, \\
\frac{1}{\rho} &= \mathrm{L}\left(\frac{dx}{ds}\right)^2 + 2\mathrm{M}\frac{dx}{ds}\frac{dy}{ds} + \mathrm{N}\left(\frac{dy}{ds}\right)^2, \\
&= \left(\mathrm{L}, \mathrm{M}, \mathrm{N} \between \frac{dx}{ds}, \frac{dy}{ds}\right)^2, \\
\frac{d}{ds}\left(\frac{1}{\rho}\right) &= \left(\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S} \between \frac{dx}{ds}, \frac{dy}{ds}\right)^3, \\
\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) &= \left(\alpha, \beta, \gamma, \delta, \epsilon \between \frac{dx}{ds}, \frac{dy}{ds}\right)^4,
\end{aligned}$$

where ρ is the radius of curvature of the normal section of the surface through the

* See a paper by the author, 'Messenger of Mathematics,' vol. 32 (1903), pp. 68 *et seq.*; see also § 31, *post.*

tangent-line defined by $dx : dy$, and the arc derivatives are effected along the geodesic tangent.*

Accordingly, the quantities of the class under consideration that may occur are E, F, G and their derivatives up to any order, together with the fundamental magnitudes of any order above the first, but without any derivatives of these fundamental magnitudes.†

5. Secondly, as regards functions $\phi(x, y)$, $\psi(x, y)$, . . . and their derivatives: we do not retain the functions themselves, but only their derivatives, for the following reason. The invariantive property is usually some intrinsic geometric property connected with a curve on the surface represented by ϕ = constant or zero, ψ = constant or zero, and the like. Accordingly, we retain only derivatives of these functions up to any order; the equations of transformation will show the connection of the order of these derivatives with the order of the derivatives of E, F, G retained.

6. Thirdly, as regards x, y, and the derivatives of y with respect to x up to any order: it is clear that x and y will not occur explicitly, for their presence cannot contribute any element to the factor Ω; it is also clear that they will not occur explicitly, for the further reason that their increments involve ξ and η but not derivatives of ξ or η, whereas all other increments involve derivatives of ξ or η, but neither ξ nor η themselves. Further, after the retention of quantities of the second class, we shall not retain y'. For let the value of y' belong to a curve $\psi = 0$ on the surface, so that

$$\psi_{10} + y' \psi_{01} = 0.$$

We know that

$$\frac{E\psi_{01}^2 - 2F\psi_{01}\psi_{10} + G\psi_{10}^2}{EG - F^2} = I,$$

where I is an absolute invariant; if then we have a differential invariant involving y', we turn it into one involving ψ_{10} and ψ_{01}, by writing

$$y' = -\frac{\psi_{10}}{\psi_{01}};$$

while if we have one involving ψ_{10} and ψ_{01}, we turn it into one involving y', by writing

$$\frac{\psi_{01}}{I} = \frac{\psi_{10}}{y'} = \left\{I \frac{EG - F^2}{E + 2Fy' + Gy'^2}\right\}^{\frac{1}{2}}.$$

It would therefore be unnecessary to retain y', when we retain first derivatives of any number of functions in an earlier class.

Similarly, it can be shown to be unnecessary to retain y'', when we retain second derivatives of any number of functions in an earlier class; and so for other derivatives of y with respect to x.

* See § 31, *post.*

† It will appear that the introduction of these magnitudes not merely avoids the difficulty as regards the derivatives of L, M, N, but also secures a substantial simplification of the expressions of the differential invariants.

Hence we retain none of the third class of possible magnitudes. But after the reasons adduced, we should only be justified in dropping y' from the set of magnitudes when it was otherwise required, if we associated the first derivatives of the appropriate function ψ with the functions already retained; or in dropping y'', if we associated the second derivatives of ψ with the functions already retained; and so for the other derivatives of y. (An example occurs later in § 24.)

NOTE.—In calculations subsidiary to the determination of the geometric significance, it is found necessary to use the relations involving the derivatives of L, M, N, P, Q, R, S; it may therefore be convenient to give their explicit expressions.* They are :—

$$\left.\begin{aligned}
P &= L_{10} - 2(L\Gamma + M\Delta) \\
Q &= L_{01} - 2(L\Gamma' + M\Delta') \\
&= M_{10} - (L\Gamma' + M\Delta') - (M\Gamma + N\Delta) \\
R &= M_{01} - (L\Gamma'' + M\Delta'') - (M\Gamma' + N\Delta') \\
&= N_{10} - 2(M\Gamma' + N\Delta') \\
S &= N_{01} - 2(M\Gamma'' + N\Delta'')
\end{aligned}\right\},$$

where

$$\left.\begin{aligned}
2V^2\Gamma &= GE_{10} - F(2F_{10} - E_{01}) \\
2V^2\Gamma' &= GE_{01} - FG_{10} \\
2V^2\Gamma'' &= G(2F_{01} - G_{10}) - FG_{01}
\end{aligned}\right\}, \qquad
\left.\begin{aligned}
2V^2\Delta &= E(2F_{10} - E_{01}) - FE_{10} \\
2V^2\Delta' &= EG_{10} - FE_{01} \\
2V^2\Delta'' &= EG_{01} - F(2F_{01} - G_{10})
\end{aligned}\right\};$$

and

$$\left.\begin{aligned}
\alpha &= P_{10} - 3(P\Gamma + Q\Delta) \\
\beta &= P_{01} - 3(P\Gamma' + Q\Delta') - \tfrac{3}{2}\frac{T^2}{V^2}(FL - EM) \\
&= Q_{10} - (P\Gamma' + Q\Delta') - 2(Q\Gamma + R\Delta) + \tfrac{1}{2}\frac{T^2}{V^2}(FL - EM) \\
\gamma &= Q_{01} - (P\Gamma'' + Q\Delta'') - 2(Q\Gamma' + R\Delta') - \tfrac{1}{2}\frac{T^2}{V^2}(GL - EN) \\
&= R_{10} - 2(Q\Gamma' + R\Delta') - (R\Gamma + S\Delta) + \tfrac{1}{2}\frac{T^2}{V^2}(GL - EN) \\
\delta &= R_{01} - 2(Q\Gamma'' + R\Delta'') - (R\Gamma' + S\Delta') - \tfrac{1}{2}\frac{T^2}{V^2}(GM - FN) \\
&= S_{10} - 3(R\Gamma' + S\Delta') + \tfrac{3}{2}\frac{T^2}{V^2}(GM - FN) \\
\epsilon &= S_{01} - 3(R\Gamma'' + S\Delta'')
\end{aligned}\right\},$$

where $T^2 = LN - M^2$.

* They are quoted from the author's paper, mentioned in § 4.

Increments of the Arguments.

7. We now require the increments of the various arguments, corresponding to the increments of x and y. We denote by E′, F′, . . . the same functions of X and Y as E, F, . . . are of x and y; thus, if dE be the increment of E, we have

$$\mathrm{E}' = \mathrm{E} + d\mathrm{E};$$

and so for the other magnitudes.

Since the relation

$$\mathrm{E}\, dx^2 + 2\mathrm{F}\, dx\, dy + \mathrm{G}\, dy^2 = \mathrm{E}'\, d\mathrm{X}^2 + 2\mathrm{F}'\, d\mathrm{X}\, d\mathrm{Y} + \mathrm{G}'\, d\mathrm{Y}^2$$

holds for all values of dx and dy, we have

$$\begin{aligned}
\mathrm{E} &= \mathrm{E}'\left(\frac{\partial \mathrm{X}}{\partial x}\right)^2 + 2\mathrm{F}'\frac{\partial \mathrm{X}}{\partial x}\frac{\partial \mathrm{Y}}{\partial x} + \mathrm{G}'\left(\frac{\partial \mathrm{Y}}{\partial x}\right)^2 \\
&= \mathrm{E}'(1 + 2\xi_{10}\, dt) + 2\mathrm{F}'\eta_{10}\, dt, \\
\mathrm{F} &= \mathrm{E}'\frac{\partial \mathrm{X}}{\partial x}\frac{\partial \mathrm{X}}{\partial y} + \mathrm{F}'\left(\frac{\partial \mathrm{X}}{\partial x}\frac{\partial \mathrm{Y}}{\partial y} + \frac{\partial \mathrm{X}}{\partial y}\frac{\partial \mathrm{Y}}{\partial x}\right) + \mathrm{G}\frac{\partial \mathrm{Y}}{\partial x}\frac{\partial \mathrm{Y}}{\partial y} \\
&= \mathrm{E}'\xi_{01}\, dt + \mathrm{F}'(1 + \xi_{10}\, dt + \eta_{01}\, dt) + \mathrm{G}'\eta_{10}\, dt, \\
\mathrm{G} &= \mathrm{E}'\left(\frac{\partial \mathrm{X}}{\partial y}\right)^2 + 2\mathrm{F}'\frac{\partial \mathrm{X}}{\partial y}\frac{\partial \mathrm{Y}}{\partial y} + \mathrm{G}'\left(\frac{\partial \mathrm{Y}}{\partial y}\right)^2 \\
&= 2\mathrm{F}'\xi_{01}\, dt + \mathrm{G}'(1 + 2\eta_{01}\, dt).
\end{aligned}$$

We thus have

$$-\,d\mathrm{E} = (2\mathrm{E}'\xi_{10} + 2\mathrm{F}'\eta_{10})\, dt.$$

Now, the differences between E and E′, F and F′, are small quantities of the order dt; hence, when we are retaining only small quantities of the order dt on the right hand side, we can replace E′, F′, G′ by E, F, G respectively; and we find

$$\left.\begin{aligned}
-\frac{d\mathrm{E}}{dt} &= 2\mathrm{E}\xi_{10} + 2\mathrm{F}\eta_{10} \\
-\frac{d\mathrm{F}}{dt} &= \mathrm{E}\xi_{01} + \mathrm{F}\xi_{10} + \mathrm{F}\eta_{01} + \mathrm{G}\eta_{10} \\
-\frac{d\mathrm{G}}{dt} &= 2\mathrm{F}\xi_{01} + 2\mathrm{G}\eta_{01}
\end{aligned}\right\}.$$

Similarly, the relation

$$\mathrm{L}\, dx^2 + 2\mathrm{M}\, dx\, dy + \mathrm{N}\, dy^2 = \frac{ds^2}{\rho} = \mathrm{L}'\, d\mathrm{X}^2 + 2\mathrm{M}'\, d\mathrm{X}\, d\mathrm{Y} + \mathrm{N}'\, d\mathrm{Y}^2$$

holds for all values of dx and dy; so that the laws of transformation for L, M, N are the same as for E, F, G. Hence

$$\left.\begin{aligned}-\frac{dL}{dt} &= 2L\xi_{10} + 2M\eta_{10}\\ -\frac{dM}{dt} &= L\xi_{01} + M\xi_{10} + M\eta_{01} + N\eta_{10}\\ -\frac{dN}{dt} &= 2M\xi_{01} + 2N\eta_{01}\end{aligned}\right\}.$$

Using the relation

$$(P, Q, R, S\between dx, dy)^3 = ds^3 \cdot \frac{d}{ds}\left(\frac{1}{\rho}\right) = (P', Q', R', S'\between dX, dY)^3$$

in the same way, we find

$$\left.\begin{aligned}-\frac{dP}{dt} &= 3P\xi_{10} + 3Q\eta_{10}\\ -\frac{dQ}{dt} &= P\xi_{01} + 2Q\xi_{10} + Q\eta_{01} + 2R\eta_{10}\\ -\frac{dR}{dt} &= 2Q\xi_{01} + R\xi_{10} + 2R\eta_{01} + S\eta_{10}\\ -\frac{dS}{dt} &= 3R\xi_{01} + 3S\eta_{01}\end{aligned}\right\}.$$

Using the relation

$$(\alpha, \beta, \gamma, \delta, \epsilon\between dx, dy)^4 = ds^4 \cdot \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = (\alpha', \beta', \gamma', \delta', \epsilon'\between dX, dY)^4$$

similarly, we find

$$\left.\begin{aligned}-\frac{d\alpha}{dt} &= + 4\alpha\xi_{10} + 4\beta\eta_{10}\\ -\frac{d\beta}{dt} &= \alpha\xi_{01} + 3\beta\xi_{10} + \beta\eta_{01} + 3\gamma\eta_{10}\\ -\frac{d\gamma}{dt} &= 2\beta\xi_{01} + 2\gamma\xi_{10} + 2\gamma\eta_{01} + 2\delta\eta_{10}\\ -\frac{d\delta}{dt} &= 3\gamma\xi_{01} + \delta\xi_{10} + 3\delta\eta_{01} + \epsilon\eta_{10}\\ -\frac{d\epsilon}{dt} &= 4\delta\xi_{01} + 4\epsilon\eta_{01}\end{aligned}\right\}.$$

And so for the increments of the other fundamental magnitudes.

8. The increments of the derivatives of E, F, G are required; they can be obtained by the following method, differing from that which is adopted by Professor Žorawski. Let x and y become $x + h$ and $y + k$ respectively, and let the consequent new values of X and Y be X + H, Y + K; then

$$\mathrm{H} = (\mathrm{X} + \mathrm{H}) - \mathrm{H} = h + \{\xi(x+h, y+k) - \xi(x, y)\}\, dt = h + \mathrm{A}\, dt,$$

where

$$\mathrm{A} = \sum_{r=0}\sum_{s=0}{}' \xi_{rs} \frac{h^r k^s}{r\,!\, s\,!},$$

and Σ' implies that r and s may not be zero together.

Similarly

$$\mathrm{K} = k + \mathrm{B}\, dt,$$

where

$$\mathrm{B} = \sum_{r=0}\sum_{s=0}{}' \eta_{rs} \frac{h^r k^s}{r\,!\, s\,!},$$

with the same signification for Σ' as before; and thus, for all values of p and q, we have

$$\mathrm{H}^p \mathrm{K}^q = h^p k^q + (p h^{p-1} k^q \mathrm{A} + q h^p k^{q-1} \mathrm{B})\, dt.$$

Now, as the relation

$$\mathrm{E} = \mathrm{E}'(1 + 2\xi_{10}\, dt) + 2\mathrm{F}'\eta_{10}\, dt$$

holds for all values of x and y, it follows that

$$\begin{aligned}\mathrm{E}(x+h, y+k) = \mathrm{E}(\mathrm{X}+\mathrm{H}, \mathrm{Y}+\mathrm{K})\{1 + 2\xi_{10}(x+h, y+k)\, dt\} \\ + 2\mathrm{F}(\mathrm{X}+\mathrm{H}, \mathrm{Y}+\mathrm{K})\,\eta_{10}(x+h, y+k)\, dt.\end{aligned}$$

Let both sides be expanded in powers of h and k; then $\dfrac{\mathrm{E}_{mn}}{m\,!\, n\,!}$ = coefficient of $h^m k^n$ in the expansion of

$$\left[\sum_{p=0}\sum_{q=0} \frac{\mathrm{E}'_{pq}}{p\,!\, q\,!}\{h^p k^q + (p h^{p-1} k^q \mathrm{A} + q h^p k^{q-1}\mathrm{B})\, dt\}\right]\left[1 + 2\sum_{r=0}\sum_{s=0} \xi_{r+1,s} \frac{h^r k^s}{r\,!\, s\,!}\, dt\right]$$

$$+ 2\left[\sum_{p=0}\sum_{q=0} \frac{\mathrm{F}'_{pq}}{p\,!\, q\,!}\{h^p k^q + (p h^{p-1} k^q \mathrm{A} + q h^p k^{q-1}\mathrm{B})\, dt\}\right]\left[\sum_{r=0}\sum_{s=0} \eta_{r+1,s} \frac{h^r k^s}{r\,!\, s\,!}\, dt\right].$$

Remembering that the first power of dt alone is to be retained, we find this coefficient to be

$$\begin{aligned}&\frac{\mathrm{E}'_{mn}}{m\,!\, n\,!} \\ &\quad + \Sigma\Sigma' \frac{1}{(m-r+1)\,!\,(n-s)\,!\, r\,!\, s\,!}(m-r+1)\,\xi_{rs}\mathrm{E}'_{m-r+1,\, n-s}\, dt \\ &\quad + 2\Sigma\Sigma \frac{1}{(m-r)\,!\,(n-s)\,!\, r\,!\, s\,!}\xi_{r+1,s}\mathrm{E}'_{m-r,\, n-s}\, dt \\ &\quad + \Sigma\Sigma' \frac{1}{(m-r)\,!\,(n-s+1)\,!\, r\,!\, s\,!}(n-s+1)\,\eta_{rs}\mathrm{E}'_{m-r,\, n-s+1}\, dt \\ &\quad + 2\Sigma\Sigma \frac{1}{(m-r)\,!\,(n-s)\,!\, r\,!\, s\,!}\eta_{r+1,s}\mathrm{F}'_{m-r,\, n-s}\, dt;\end{aligned}$$

the first summation $\Sigma\Sigma'$ does not occur if $r = m + 1$, the second summation $\Sigma\Sigma'$ does not occur if $s = n + 1$, and in neither of them may r and s vanish together. Writing

$$\binom{m}{r} = \frac{m\,!}{(m-r)\,!\,r\,!}, \qquad \binom{n}{s} = \frac{n\,!}{(n-s)\,!\,s\,!},$$

$$E'_{mn} = E_{mn} + dE_{mn},$$

we have

$$\begin{aligned} -\frac{dE_{mn}}{dt} = &\sum_{r=0}^{m}\sideset{}{'}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{rs}E'_{m-r+1,\, n-s} \\ &+ 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{r+1,\, s}E'_{m-r,\, n-s} \\ &+ \sum_{r=0}^{m}\sideset{}{'}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{rs}E'_{m-r,\, n-s+1} \\ &+ 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{r+1,\, s}F'_{m-r,\, n-s}. \end{aligned}$$

Proceeding similarly from the expressions for F and G, we find

$$\begin{aligned} -\frac{dF_{mn}}{dt} = &\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{r,\, s+1}E'_{m-r,\, n-s} \\ &+ \sum_{r=0}^{m}\sideset{}{'}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{rs}F'_{m-r+1,\, n-s} \\ &+ \sum_{r=0}^{m}\sideset{}{'}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{rs}F'_{m-r,\, n-s+1} \\ &+ \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}(\xi_{r+1,\, s} + \eta_{r,\, s+1})\,F'_{m-r,\, n-s} \\ &+ \sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{r+1,\, s}G'_{m-r,\, n-s}, \end{aligned}$$

and

$$\begin{aligned} -\frac{dG_{mn}}{dt} = &\,2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{r,\, s+1}F'_{m-r,\, n-s} \\ &+ 2\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{r,\, s+1}G'_{m-r,\, n-s} \\ &+ \sum_{r=0}^{m}\sideset{}{'}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\xi_{rs}G'_{m-r+1,\, n-s} \\ &+ \sum_{r=0}^{m}\sideset{}{'}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\eta_{rs}G'_{m-r,\, n-s+1}. \end{aligned}$$

NOTE.—As we now have the first increment of the quantities E_{mn}, F_{mn}, G_{mn}, and as the second increments are not required, the quantities E', F', G' on the right-hand sides can be replaced by E, F, G, without affecting the values of the first increments.

9. In particular, we have

$$\left.\begin{aligned}
-\frac{dE_{10}}{dt} &= 3\xi_{10}E_{10} + 2\xi_{20}E + \eta_{10}(E_{01} + 2F_{10}) + 2\eta_{20}F \\
-\frac{dE_{01}}{dt} &= 2\xi_{10}E_{01} + \xi_{01}E_{10} + 2\xi_{11}E + \eta_{01}E_{01} + 2\eta_{10}F_{01} + 2\eta_{11}F \\
-\frac{dE_{20}}{dt} &= 4\xi_{10}E_{20} + 5\xi_{20}E_{10} + 2\xi_{30}E \\
&\quad + 2\eta_{10}(E_{11} + F_{20}) + \eta_{20}(E_{01} + 4F_{10}) + 2F\eta_{30} \\
-\frac{dE_{11}}{dt} &= 3\xi_{10}E_{11} + \xi_{01}E_{20} + 2\xi_{20}E_{01} + 3\xi_{11}E_{10} + 2\xi_{21}E \\
&\quad + \eta_{10}(E_{02} + 2F_{11}) + \eta_{01}E_{11} + 2\eta_{20}F_{01} + \eta_{11}(E_{01} + 2F_{10}) + 2\eta_{21}F \\
-\frac{dE_{02}}{dt} &= 2\xi_{10}E_{02} + 2\xi_{01}E_{11} + 4\xi_{11}E_{01} + \xi_{02}E_{10} + 2\xi_{12}E \\
&\quad + 2\eta_{01}E_{02} + 2\eta_{10}F_{02} + 4\eta_{11}F_{01} + \eta_{02}E_{01} + 2\eta_{12}F
\end{aligned}\right\} ;$$

$$\left.\begin{aligned}
-\frac{dF_{10}}{dt} &= 2\xi_{10}F_{10} + \xi_{01}E_{10} + \xi_{20}F + \xi_{11}E \\
&\quad + \eta_{10}(F_{01} + G_{10}) + \eta_{01}F_{10} + \eta_{20}G + \eta_{11}F \\
-\frac{dF_{01}}{dt} &= \xi_{10}F_{01} + \xi_{01}(E_{01} + F_{10}) + \xi_{11}F + \xi_{02}E \\
&\quad + \eta_{10}G_{01} + 2\eta_{01}F_{01} + \eta_{11}G + \eta_{02}F \\
-\frac{dF_{20}}{dt} &= 3\xi_{10}F_{20} + \xi_{01}E_{20} + 3\xi_{20}F_{10} + 2\xi_{11}E_{10} + \xi_{21}E + \xi_{30}F \\
&\quad + \eta_{01}F_{20} + 2\eta_{10}F_{11} + \eta_{10}G_{20} + \eta_{20}(2G_{10} + F_{01}) + 2\eta_{11}F_{10} + \eta_{30}G + \eta_{21}F \\
-\frac{dF_{11}}{dt} &= 2\xi_{10}F_{11} + \xi_{01}(E_{11} + F_{20}) + \xi_{20}F_{01} + \xi_{11}(E_{01} + 2F_{10}) + \xi_{02}E_{10} + \xi_{21}F + \xi_{12}E \\
&\quad + 2\eta_{01}F_{11} + \eta_{10}(F_{02} + G_{11}) + \eta_{20}G_{01} + \eta_{11}(2F_{01} + G_{10}) + \eta_{02}F_{10} + \eta_{21}G + \eta_{12}F \\
-\frac{dF_{02}}{dt} &= \xi_{10}F_{02} + \xi_{01}(E_{02} + 2F_{11}) + 2\xi_{11}F_{01} + \xi_{02}(2E_{01} + F_{10}) + \xi_{12}F + \xi_{03}E \\
&\quad + 3\eta_{01}F_{02} + \eta_{10}G_{02} + 2\eta_{11}G_{01} + 3\eta_{02}F_{01} + \eta_{12}G + \eta_{03}F
\end{aligned}\right\} ;$$

$$\left.\begin{aligned}
-\frac{dG_{10}}{dt} &= \xi_{10}G_{10} + 2\xi_{01}F_{10} + 2\xi_{11}F + \eta_{10}G_{01} + 2\eta_{01}G_{10} + 2\eta_{11}G \\
-\frac{dG_{01}}{dt} &= \xi_{01}(G_{10} + 2F_{01}) + 2\xi_{02}F + 3\eta_{01}G_{01} + 2\eta_{02}G \\
-\frac{dG_{20}}{dt} &= 2\xi_{10}G_{20} + 2\xi_{01}F_{20} + \xi_{20}G_{10} + 4\xi_{11}F_{10} + 2\xi_{21}F \\
&\quad + 2\eta_{01}G_{20} + 2\eta_{10}G_{11} + \eta_{20}G_{01} + 4\eta_{11}G_{10} + 2\eta_{21}G \\
-\frac{dG_{11}}{dt} &= \xi_{10}G_{11} + \xi_{01}(G_{20} + 2F_{11}) + \xi_{11}(2F_{01} + G_{10}) + 2\xi_{02}F_{10} + 2\xi_{12}F \\
&\quad + 3\eta_{01}G_{11} + \eta_{10}G_{02} + 3\eta_{11}G_{01} + 2\eta_{02}G_{10} + 2\eta_{12}G \\
-\frac{dG_{02}}{dt} &= 2\xi_{01}(F_{02} + G_{11}) + \xi_{02}(G_{10} + 4F_{01}) + 2\xi_{03}F \\
&\quad + 4\eta_{01}G_{02} + 5\eta_{02}G_{01} + 2\eta_{03}G
\end{aligned}\right\} .$$

10. We require expressions for the increments of the derivatives of functions such as $\phi(x, y)$, $\psi(x, y)$, . . . ; for this purpose, we proceed as before. We have

$$\phi(x+h, y+k) = \phi'(X+H, Y+K);$$

and therefore

$$\frac{\phi_{mn}}{m!\,n!} = \text{coefficient of } h^m k^n \text{ in expansion of } \phi(X+H, Y+K)$$

$$= \ldots\ldots\ldots \sum_p \sum_q \frac{\phi'_{pq}}{p!\,q!} H^p K^q$$

$$= \ldots\ldots\ldots \sum_p \sum_q \frac{\phi'_{pq}}{p!\,q!} \{h^p k^q + (ph^{p-1}k^q A + qh^p k^{q-1} B)\, dt\}$$

$$= \frac{\phi'_{mn}}{m!\,n!} + U\, dt,$$

where U is the coefficient of $h^m k^n$ in

$$\sum_p \sum_q \frac{\phi'_{pq}}{p!\,q!}(ph^{p-1}k^q A + qh^p k^{q-1} B),$$

that is, in

$$\sum_p \sum_q \sum_r \sum_s \frac{\phi'_{pq}}{p!\,q!\,r!\,s!}(ph^{r+p-1}k^{q+s}\xi_{rs} + qh^{r+p}k^{q+s-1}\eta_{rs}),$$

where in the summation r and s do not vanish together and, if either p or q be zero, the corresponding term ceases to occur.

Writing

$$\phi'_{mn} = \phi_{mn} + d\phi_{mn},$$

we have

$$-\frac{d\phi_{mn}}{dt} = \sum_{r=0}^{m} \sum_{s=0}^{n}{}' \binom{m}{r}\binom{n}{s}\{\phi'_{m+1-r,\, n-s}\xi_{rs} + \phi'_{m-r,\, n+1-s}\eta_{rs}\},$$

which gives the required increments for derivatives of a function ϕ. Similarly of course for the increments of the derivatives of all functions similar to ϕ.

NOTE.—Just as in the expressions for the increments of the various derivatives of E, F, G, we can replace, in the expressions for the increments of the various derivatives of a function ϕ, the various quantities $\phi'_{\mu\nu}$ on the right-hand sides by $\phi_{\mu\nu}$ without affecting the values of the first increments. As before, second increments are not needed for our purpose.

11. In particular, we have

$$\left.\begin{aligned} -\frac{d\phi_{10}}{dt} &= \phi_{10}\xi_{10} + \phi_{01}\eta_{10} \\ -\frac{d\phi_{01}}{dt} &= \phi_{10}\xi_{01} + \phi_{01}\eta_{01} \end{aligned}\right\} ;$$

$$\left.\begin{aligned} -\frac{d\phi_{20}}{dt} &= 2\phi_{20}\xi_{10} + \phi_{10}\xi_{20} + 2\phi_{11}\eta_{10} + \phi_{01}\eta_{20} \\ -\frac{d\phi_{11}}{dt} &= \phi_{11}\xi_{10} + \phi_{20}\xi_{01} + \phi_{10}\xi_{11} + \phi_{02}\eta_{10} + \phi_{11}\eta_{01} + \phi_{01}\eta_{11} \\ -\frac{d\phi_{02}}{dt} &= 2\phi_{11}\xi_{01} + \phi_{10}\xi_{02} + 2\phi_{02}\eta_{01} + \phi_{01}\eta_{02} \end{aligned}\right\} ;$$

$$\left.\begin{aligned} -\frac{d\phi_{30}}{dt} &= 3\phi_{30}\xi_{10} + 3\phi_{20}\xi_{20} + \phi_{10}\xi_{30} + 3\phi_{21}\eta_{10} + 3\phi_{11}\eta_{20} + \phi_{01}\eta_{30} \\ -\frac{d\phi_{21}}{dt} &= 2\phi_{21}\xi_{10} + \phi_{30}\xi_{01} + \phi_{11}\xi_{20} + 2\phi_{20}\xi_{11} + \phi_{10}\xi_{21} \\ &\quad + 2\phi_{12}\eta_{10} + \phi_{21}\eta_{01} + \phi_{02}\eta_{20} + 2\phi_{11}\eta_{11} + \phi_{01}\eta_{21} \\ -\frac{d\phi_{12}}{dt} &= \phi_{12}\xi_{10} + 2\phi_{21}\xi_{01} + 2\phi_{11}\xi_{11} + \phi_{20}\xi_{02} + \phi_{10}\xi_{12} \\ &\quad + \phi_{03}\eta_{10} + 2\phi_{12}\eta_{01} + 2\phi_{02}\eta_{11} + \phi_{11}\eta_{02} + \phi_{01}\eta_{12} \\ -\frac{d\phi_{03}}{dt} &= 3\phi_{12}\xi_{01} + 3\phi_{11}\xi_{02} + \phi_{10}\xi_{03} + 3\phi_{03}\eta_{01} + 3\phi_{02}\eta_{02} + \phi_{01}\eta_{03} \end{aligned}\right\} .$$

12. A comparison of the expressions of the increments of the derivatives of E, F, G on the one hand, and those of the derivatives of a typical function ϕ on the other, leads to one immediate inference as to the arguments that enter into the composition of a differential invariant. Suppose that such an invariant is required to involve derivatives of a function ϕ up to order M in x and y combined; the increments of these derivatives involve (among others) the quantities

$$\xi_{M0}, \quad \xi_{M1}, \ldots, \quad \xi_{0M}; \quad \eta_{M0}, \quad \eta_{M1}, \ldots, \quad \eta_{0M}.$$

The invariantive property requires that the terms involving these quantities must (if they do not balance one another) be balanced by other terms involving these same quantities; and therefore derivatives of E, F, G up to order M − 1 in x and y combined must occur. And conversely.

In particular, if derivatives of ϕ of the third order occur in an invariantive function, it must contain derivatives of E, F, G of the second order.

The Differential Equations Defining the Invariants.

13. The invariantive property is used, exactly as in Professor Żorawski's application of Lie's method, to obtain partial differential equations of the first order satisfied by any invariantive function. We proceed from an equation such as

$$f' = \Omega^{-\mu} f;$$

we substitute, in each of the arguments such as u', where

$$u' = u + dt \cdot \frac{du}{dt}$$

the proper value of $\frac{du}{dt}$ obtained above for the various arguments; we also write

$$\Omega = 1 + (\xi_{10} + \eta_{01})\, dt\,;$$

and then, according to LIE's theory, we equate the coefficient of dt on the two sides. The functions ξ and η are arbitrary; and therefore, in this new equation, the coefficients of the various derivatives of ξ and η on the two sides are equal. We thus obtain a number of partial differential equations of the first order satisfied by f. The construction of the form of f depends upon the manipulation of the equations.

14. The whole process will be sufficiently illustrated in its details if we construct the algebraically independent aggregate of differential invariants which involve derivatives of two* functions ϕ and ψ up to the third order inclusive. In order to take full account of the increments of such derivatives, it is desirable and necessary to retain derivatives of E, F, G up to the second order and, in place of the derivatives of L, M, N of that order, to retain the fundamental magnitudes of the second, the third, and the fourth orders. Thus the invariantive function involves some or all of the quantities

$$\begin{array}{l} E,\ E_{10},\ E_{01},\ E_{20},\ E_{11},\ E_{02}\,; \\ F,\ F_{10},\ F_{01},\ F_{20},\ F_{11},\ F_{02}\,; \\ G,\ G_{10},\ G_{01},\ G_{20},\ G_{11},\ G_{02}\,; \\ L,\ M,\ N\,; \\ P,\ Q,\ R,\ S\,; \\ \alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon\,; \\ \phi_{10},\ \phi_{01},\ \phi_{20},\ \phi_{11},\ \phi_{02},\ \phi_{30},\ \phi_{21},\ \phi_{12},\ \phi_{03}\,; \\ \psi_{10},\ \psi_{01},\ \psi_{20},\ \psi_{11},\ \psi_{02},\ \psi_{30},\ \psi_{21},\ \psi_{12},\ \psi_{03}. \end{array}$$

Denoting any one of these arguments by u, the invariantive property gives

$$f(\ldots, u', \ldots) = \Omega^{-\mu} f(\ldots, u, \ldots),$$

that is,

$$f\left(\ldots, u + \frac{du}{dt}\, dt, \ldots\right) = \{1 + (\xi_{10} + \eta_{10})\, dt\}^{-\mu} f(\ldots, u, \ldots),$$

and therefore

$$\sum_u \frac{\partial f}{\partial u} \frac{du}{dt} = -\mu\, (\xi_{10} + \eta_{10}) f.$$

* The form of the results indicates the form of the results when more than two functions occur. Moreover, if more than two functions of the type of ϕ and ψ be considered, they are connected by an identical relation.

Substituting for $\frac{du}{dt}$ the respective values for the respective arguments, and equating the coefficients of the various derivatives of ξ and η, we have the requisite partial differential equations. They are :—

$$\mu f = 2E\frac{\partial f}{\partial E} + F\frac{\partial f}{\partial F} + 2L\frac{\partial f}{\partial L} + M\frac{\partial f}{\partial M} + 3P\frac{\partial f}{\partial P} + 2Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R}$$

$$+ 4\alpha\frac{\partial f}{\partial \alpha} + 3\beta\frac{\partial f}{\partial \beta} + 2\gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta}$$

$$+ 3E_{10}\frac{\partial f}{\partial E_{10}} + 2E_{01}\frac{\partial f}{\partial E_{01}} + 4E_{20}\frac{\partial f}{\partial E_{20}} + 3E_{11}\frac{\partial f}{\partial E_{11}} + 2E_{02}\frac{\partial f}{\partial E_{02}}$$

$$+ 2F_{10}\frac{\partial f}{\partial F_{10}} + F_{01}\frac{\partial f}{\partial F_{01}} + 3F_{20}\frac{\partial f}{\partial F_{20}} + 2F_{11}\frac{\partial f}{\partial F_{11}} + F_{02}\frac{\partial f}{\partial F_{02}}$$

$$+ G_{10}\frac{\partial f}{\partial G_{10}} + 2G_{20}\frac{\partial f}{\partial G_{20}} + G_{11}\frac{\partial f}{\partial G_{11}}$$

$$+ \phi_{10}\frac{\partial f}{\partial \phi_{10}} + 2\phi_{20}\frac{\partial f}{\partial \phi_{20}} + \phi_{11}\frac{\partial f}{\partial \phi_{11}} + 3\phi_{30}\frac{\partial f}{\partial \phi_{30}} + 2\phi_{21}\frac{\partial f}{\partial \phi_{21}} + \phi_{12}\frac{\partial f}{\partial \phi_{12}}$$

$$+ \psi_{10}\frac{\partial f}{\partial \psi_{10}} + 2\psi_{20}\frac{\partial f}{\partial \psi_{20}} + \psi_{11}\frac{\partial f}{\partial \psi_{11}} + 3\psi_{30}\frac{\partial f}{\partial \psi_{30}} + 2\psi_{21}\frac{\partial f}{\partial \psi_{21}} + \psi_{12}\frac{\partial f}{\partial \psi_{12}} \quad . \quad . \quad . \quad (I_1),$$

$$\mu f = F\frac{\partial f}{\partial F} + 2G\frac{\partial f}{\partial G} + M\frac{\partial f}{\partial M} + 2N\frac{\partial f}{\partial N} + Q\frac{\partial f}{\partial Q} + 2R\frac{\partial f}{\partial R} + 3S\frac{\partial f}{\partial S}$$

$$+ \beta\frac{\partial f}{\partial \beta} + 2\gamma\frac{\partial f}{\partial \gamma} + 3\delta\frac{\partial f}{\partial \delta} + 4\epsilon\frac{\partial f}{\partial \epsilon}$$

$$+ E_{01}\frac{\partial f}{\partial E_{01}} + E_{11}\frac{\partial f}{\partial E_{11}} + 2E_{02}\frac{\partial f}{\partial E_{02}}$$

$$+ F_{10}\frac{\partial f}{\partial F_{10}} + 2F_{01}\frac{\partial f}{\partial F_{01}} + F_{20}\frac{\partial f}{\partial F_{20}} + 2F_{11}\frac{\partial f}{\partial F_{11}} + 3F_{02}\frac{\partial f}{\partial F_{02}}$$

$$+ 2G_{10}\frac{\partial f}{\partial G_{10}} + 3G_{01}\frac{\partial f}{\partial G_{01}} + 2G_{20}\frac{\partial f}{\partial G_{20}} + 3G_{11}\frac{\partial f}{\partial G_{11}} + 4G_{02}\frac{\partial f}{\partial G_{02}}$$

$$+ \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \phi_{11}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{02}\frac{\partial f}{\partial \phi_{02}} + \phi_{21}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{12}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{03}\frac{\partial f}{\partial \phi_{03}}$$

$$+ \psi_{01}\frac{\partial f}{\partial \psi_{01}} + \psi_{11}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{02}\frac{\partial f}{\partial \psi_{02}} + \psi_{21}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{12}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{03}\frac{\partial f}{\partial \psi_{03}} \quad . \quad . \quad (I_2),$$

which come from the coefficients of ξ_{10}, η_{01} respectively ;

$$\begin{aligned}
&\mathrm{E}\frac{\partial f}{\partial \mathrm{F}} + 2\mathrm{F}\frac{\partial f}{\partial \mathrm{G}} + \mathrm{L}\frac{\partial f}{\partial \mathrm{M}} + 2\mathrm{M}\frac{\partial f}{\partial \mathrm{N}} + \mathrm{P}\frac{\partial f}{\partial \mathrm{Q}} + 2\mathrm{Q}\frac{\partial f}{\partial \mathrm{R}} + 3\mathrm{R}\frac{\partial f}{\partial \mathrm{S}} \\
&\quad + \alpha\frac{\partial f}{\partial \beta} + 2\beta\frac{\partial f}{\partial \gamma} + 3\gamma\frac{\partial f}{\partial \delta} + 4\delta\frac{\partial f}{\partial \epsilon} \\
&\quad + \mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{E}_{01}} + \mathrm{E}_{20}\frac{\partial f}{\partial \mathrm{E}_{11}} + 2\mathrm{E}_{11}\frac{\partial f}{\partial \mathrm{E}_{02}} \\
&\quad + \mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{F}_{10}} + (\mathrm{E}_{01} + \mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{F}_{01}} + \mathrm{E}_{20}\frac{\partial f}{\partial \mathrm{F}_{20}} + (\mathrm{E}_{11} + \mathrm{F}_{20})\frac{\partial f}{\partial \mathrm{F}_{11}} + (\mathrm{E}_{02} + 2\mathrm{F}_{11})\frac{\partial f}{\partial \mathrm{F}_{02}} \\
&\quad + 2\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{G}_{10}} + (\mathrm{G}_{10} + 2\mathrm{F}_{01})\frac{\partial f}{\partial \mathrm{G}_{01}} + 2\mathrm{F}_{20}\frac{\partial f}{\partial \mathrm{G}_{20}} + (\mathrm{G}_{20} + 2\mathrm{F}_{11})\frac{\partial f}{\partial \mathrm{G}_{11}} + (2\mathrm{F}_{02} + 2\mathrm{G}_{11})\frac{\partial f}{\partial \mathrm{G}_{02}} \\
&\quad + \phi_{10}\frac{\partial f}{\partial \phi_{01}} + \phi_{20}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{02}} + \phi_{30}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{21}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{12}\frac{\partial f}{\partial \phi_{03}} \\
&\quad + \psi_{10}\frac{\partial f}{\partial \psi_{01}} + \psi_{20}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{02}} + \psi_{30}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{21}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{12}\frac{\partial f}{\partial \psi_{03}} = 0 \quad . \quad . \quad (\mathrm{I}_3),
\end{aligned}$$

$$\begin{aligned}
&2\mathrm{F}\frac{\partial f}{\partial \mathrm{E}} + \mathrm{G}\frac{\partial f}{\partial \mathrm{F}} + 2\mathrm{M}\frac{\partial f}{\partial \mathrm{L}} + \mathrm{N}\frac{\partial f}{\partial \mathrm{M}} + 3\mathrm{Q}\frac{\partial f}{\partial \mathrm{P}} + 2\mathrm{R}\frac{\partial f}{\partial \mathrm{Q}} + \mathrm{S}\frac{\partial f}{\partial \mathrm{R}} \\
&\quad + 4\beta\frac{\partial f}{\partial \alpha} + 3\gamma\frac{\partial f}{\partial \beta} + 2\delta\frac{\partial f}{\partial \gamma} + \epsilon\frac{\partial f}{\partial \delta} \\
&\quad + (\mathrm{E}_{01} + 2\mathrm{F}_{10})\frac{\partial f}{\partial \mathrm{E}_{10}} + 2\mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{E}_{01}} + (2\mathrm{E}_{11} + 2\mathrm{F}_{20})\frac{\partial f}{\partial \mathrm{E}_{20}} + (\mathrm{E}_{02} + 2\mathrm{F}_{11})\frac{\partial f}{\partial \mathrm{E}_{11}} + 2\mathrm{F}_{02}\frac{\partial f}{\partial \mathrm{E}_{02}} \\
&\quad + (\mathrm{F}_{01} + \mathrm{G}_{10})\frac{\partial f}{\partial \mathrm{F}_{10}} + \mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{F}_{01}} + (2\mathrm{F}_{11} + \mathrm{G}_{20})\frac{\partial f}{\partial \mathrm{F}_{20}} + (\mathrm{F}_{02} + \mathrm{G}_{11})\frac{\partial f}{\partial \mathrm{F}_{11}} + \mathrm{G}_{02}\frac{\partial f}{\mathrm{F}_{02}} \\
&\quad + \mathrm{G}_{01}\frac{\partial f}{\partial \mathrm{G}_{10}} + 2\mathrm{G}_{11}\frac{\partial f}{\partial \mathrm{G}_{20}} + \mathrm{G}_{02}\frac{\partial f}{\partial \mathrm{G}_{11}} \\
&\quad + \phi_{01}\frac{\partial f}{\partial \phi_{10}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{20}} + \phi_{02}\frac{\partial f}{\partial \phi_{11}} + 3\phi_{21}\frac{\partial f}{\partial \phi_{30}} + 2\phi_{12}\frac{\partial f}{\partial \phi_{21}} + \phi_{03}\frac{\partial f}{\partial \phi_{12}} \\
&\quad + \psi_{01}\frac{\partial f}{\partial \psi_{10}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{20}} + \psi_{02}\frac{\partial f}{\partial \psi_{11}} + 3\psi_{21}\frac{\partial f}{\partial \psi_{30}} + \psi_{12}\frac{\partial f}{\partial \psi_{21}} + \psi_{03}\frac{\partial f}{\partial \psi_{12}} = 0 \quad . \quad (\mathrm{I}_4),
\end{aligned}$$

which come from the coefficients of ξ_{01}, η_{10} respectively ;

$$\begin{aligned}
&2\mathrm{E}\frac{\partial f}{\partial \mathrm{E}_{10}} + 5\mathrm{E}_{10}\frac{\partial f}{\partial \mathrm{E}_{20}} + 2\mathrm{E}_{01}\frac{\partial f}{\partial \mathrm{E}_{11}} \\
&+ \mathrm{F}\frac{\partial f}{\partial \mathrm{F}_{10}} + 3\mathrm{F}_{10}\frac{\partial f}{\partial \mathrm{F}_{20}} + \mathrm{F}_{01}\frac{\partial f}{\partial \mathrm{F}_{11}} \\
&\qquad\qquad + \mathrm{G}_{10}\frac{\partial f}{\partial \mathrm{G}_{20}} \\
&+ \phi_{10}\frac{\partial f}{\partial \phi_{20}} + 3\phi_{20}\frac{\partial f}{\partial \phi_{30}} + \phi_{11}\frac{\partial f}{\partial \phi_{21}} + \psi_{10}\frac{\partial f}{\partial \psi_{20}} + 3\psi_{20}\frac{\partial f}{\partial \psi_{30}} + \psi_{11}\frac{\partial f}{\partial \psi_{21}} = 0 \quad . \quad . \quad (\mathrm{II}_1),
\end{aligned}$$

$$2E\frac{\partial f}{\partial E_{01}} + 3E_{10}\frac{\partial f}{\partial E_{11}} + 4E_{01}\frac{\partial f}{\partial E_{02}}$$

$$+ E\frac{\partial f}{\partial F_{10}} + F\frac{\partial f}{\partial F_{01}} + 2E_{10}\frac{\partial f}{\partial F_{20}} + (E_{01} + 2F_{10})\frac{\partial f}{\partial F_{11}} + 2F_{01}\frac{\partial f}{\partial F_{02}}$$

$$+ 2F\frac{\partial f}{\partial G_{10}} + 4F_{10}\frac{\partial f}{\partial G_{20}} + (2F_{01} + G_{10})\frac{\partial f}{\partial G_{11}}$$

$$+ \phi_{10}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{20}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{12}} + \psi_{10}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{20}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{12}} = 0 \quad . \quad (\mathrm{II}_2),$$

$$E_{10}\frac{\partial f}{\partial E_{02}}$$

$$+ E\frac{\partial f}{\partial F_{01}} + E_{10}\frac{\partial f}{\partial F_{11}} + (2E_{01} + F_{10})\frac{\partial f}{\partial F_{02}}$$

$$+ 2F\frac{\partial f}{\partial G_{01}} + 2F_{10}\frac{\partial f}{\partial G_{11}} + (G_{10} + 4F_{01})\frac{\partial f}{\partial G_{02}}$$

$$+ \phi_{10}\frac{\partial f}{\partial \phi_{02}} + \phi_{20}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{11}\frac{\partial f}{\partial \phi_{03}} + \psi_{10}\frac{\partial f}{\partial \psi_{02}} + \psi_{20}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{11}\frac{\partial f}{\partial \psi_{03}} = 0 \quad . \quad . \quad (\mathrm{II}_3),$$

$$2F\frac{\partial f}{\partial E_{10}} + (E_{01} + 4F_{10})\frac{\partial f}{\partial E_{20}} + 2F_{01}\frac{\partial f}{\partial E_{11}}$$

$$+ G\frac{\partial f}{\partial F_{10}} + (2G_{10} + F_{01})\frac{\partial f}{\partial F_{20}} + G_{01}\frac{\partial f}{\partial F_{11}}$$

$$+ G_{01}\frac{\partial f}{\partial G_{20}}$$

$$+ \phi_{01}\frac{\partial f}{\partial \phi_{20}} + 3\phi_{11}\frac{\partial f}{\partial \phi_{30}} + \phi_{02}\frac{\partial f}{\partial \phi_{21}} + \psi_{01}\frac{\partial f}{\partial \psi_{20}} + 3\psi_{11}\frac{\partial f}{\partial \psi_{30}} + \psi_{02}\frac{\partial f}{\partial \psi_{21}} = 0 \quad . \quad . \quad (\mathrm{II}_4),$$

$$2F\frac{\partial f}{\partial E_{01}} + (E_{01} + 2F_{10})\frac{\partial f}{\partial E_{11}} + 4F_{01}\frac{\partial f}{\partial E_{02}}$$

$$+ F\frac{\partial f}{\partial F_{10}} + G\frac{\partial f}{\partial F_{01}} + 2F_{10}\frac{\partial f}{\partial F_{20}} + (2F_{01} + G_{10})\frac{\partial f}{\partial F_{11}} + 2G_{01}\frac{\partial f}{\partial F_{02}}$$

$$+ 2G\frac{\partial f}{\partial G_{10}} + 4G_{10}\frac{\partial f}{\partial G_{20}} + 3G_{01}\frac{\partial f}{\partial G_{11}}$$

$$+ \phi_{01}\frac{\partial f}{\partial \phi_{11}} + 2\phi_{11}\frac{\partial f}{\partial \phi_{21}} + 2\phi_{02}\frac{\partial f}{\partial \phi_{12}} + \psi_{01}\frac{\partial f}{\partial \psi_{11}} + 2\psi_{11}\frac{\partial f}{\partial \psi_{21}} + 2\psi_{02}\frac{\partial f}{\partial \psi_{12}} = 0 \quad . \quad (\mathrm{II}_5),$$

$$E_{01}\frac{\partial f}{\partial E_{02}}$$

$$+ F\frac{\partial f}{\partial F_{01}} + F_{10}\frac{\partial f}{\partial F_{11}} + 3F_{01}\frac{\partial f}{\partial F_{02}}$$

$$+ 2G\frac{\partial f}{\partial G_{01}} + 2G_{10}\frac{\partial f}{\partial G_{11}} + 5G_{01}\frac{\partial f}{\partial G_{02}}$$

$$+ \phi_{01}\frac{\partial f}{\partial \phi_{02}} + \phi_{11}\frac{\partial f}{\partial \phi_{12}} + 3\phi_{02}\frac{\partial f}{\partial \phi_{03}} + \psi_{01}\frac{\partial f}{\partial \psi_{02}} + \psi_{11}\frac{\partial f}{\partial \psi_{12}} + 3\psi_{02}\frac{\partial f}{\partial \psi_{03}} = 0 \quad . \quad (\mathrm{II}_6),$$

which come from the coefficients of ξ_{20}, ξ_{11}, ξ_{02}, η_{20}, η_{11}, η_{02} respectively; and

$$2E\frac{\partial f}{\partial E_{20}} + F\frac{\partial f}{\partial F_{20}} + \phi_{10}\frac{\partial f}{\partial \phi_{30}} + \psi_{10}\frac{\partial f}{\partial \psi_{30}} = 0 \quad \text{(III}_1\text{)},$$

$$2E\frac{\partial f}{\partial E_{11}} + E\frac{\partial f}{\partial F_{20}} + F\frac{\partial f}{\partial F_{11}} + 2F\frac{\partial f}{\partial G_{20}} + \phi_{10}\frac{\partial f}{\partial \phi_{21}} + \psi_{10}\frac{\partial f}{\partial \psi_{21}} = 0 \quad \text{(III}_2\text{)},$$

$$2E\frac{\partial f}{\partial E_{02}} + E\frac{\partial f}{\partial F_{11}} + F\frac{\partial f}{\partial F_{02}} + 2F\frac{\partial f}{\partial G_{11}} + \phi_{10}\frac{\partial f}{\partial \phi_{12}} + \psi_{10}\frac{\partial f}{\partial \psi_{12}} = 0 \quad \text{(III}_3\text{)},$$

$$E\frac{\partial f}{\partial F_{02}} + 2F\frac{\partial f}{\partial G_{02}} + \phi_{10}\frac{\partial f}{\partial \phi_{03}} + \psi_{10}\frac{\partial f}{\partial \psi_{03}} = 0 \quad \text{(III}_4\text{)},$$

$$2F\frac{\partial f}{\partial E_{20}} + G\frac{\partial f}{\partial F_{20}} + \phi_{10}\frac{\partial f}{\partial \phi_{30}} + \psi_{01}\frac{\partial f}{\partial \psi_{30}} = 0 \quad \text{(III}_5\text{)},$$

$$2F\frac{\partial f}{\partial E_{11}} + F\frac{\partial f}{\partial F_{20}} + G\frac{\partial f}{\partial F_{11}} + 2G\frac{\partial f}{\partial G_{20}} + \phi_{01}\frac{\partial f}{\partial \phi_{21}} + \psi_{01}\frac{\partial f}{\partial \psi_{21}} = 0 \quad \text{(III}_6\text{)},$$

$$2F\frac{\partial f}{\partial E_{02}} + F\frac{\partial f}{\partial F_{11}} + G\frac{\partial f}{\partial F_{02}} + 2G\frac{\partial f}{\partial G_{11}} + \phi_{01}\frac{\partial f}{\partial \phi_{12}} + \psi_{10}\frac{\partial f}{\partial \psi_{12}} = 0 \quad \text{(III}_7\text{)},$$

$$F\frac{\partial f}{\partial F_{02}} + 2G\frac{\partial f}{\partial G_{02}} + \phi_{01}\frac{\partial f}{\partial \phi_{03}} + \psi_{01}\frac{\partial f}{\partial \psi_{03}} = 0 \quad \text{(III}_8\text{)},$$

which come from the coefficients of ξ_{30}, ξ_{21}, ξ_{12}, ξ_{03}, η_{30}, η_{21}, η_{12}, η_{03} respectively.

15. Consider the set of equations (III_1) to (III_8); all the POISSON-JACOBI conditions of coexistence are satisfied so that, in so far as the third derivatives of ϕ and ψ and the second derivatives of E, F, G are concerned, the set may be regarded as a complete JACOBIAN system. The total number of variables occurring in the derivatives of f is

4, for the derivatives of ϕ of the third order,

+ 4, ... ψ . . . .

+ 9, ... E, F, G of the second order,

= 17 in all; hence as the total number of equations is 8, there will be *nine* algebraically independent solutions involving these 17 quantities. When we integrate the set of equations in the usual manner, we find a set of nine solutions, apparently in their simplest form when given by

$$\left.\begin{aligned} u_1 &= 2V^2\phi_{30} - (2F_{20} - E_{11})r - E_{20}s \\ u_2 &= 2V^2\phi_{21} - G_{20}r - E_{11}s \\ u_3 &= 2V^2\phi_{12} - G_{11}r - E_{02}s \\ u_4 &= 2V^2\phi_{03} - G_{02}r - (2F_{02} - G_{11})s \\ v_1 &= 2V^2\psi_{30} - (2F_{20} - E_{11})\rho - E_{20}\sigma \\ v_2 &= 2V^2\psi_{21} - G_{20}\rho - E_{11}\sigma \\ v_3 &= 2V^2\psi_{12} - G_{11}\rho - E_{02}\sigma \\ v_4 &= 2V^2\psi_{03} - G_{02}\rho - (2F_{02} - G_{11})\sigma \\ \theta &= E_{02} - 2F_{11} + G_{20} \end{aligned}\right\},$$

where $V^2 = EG - F^2$, and

$$\left.\begin{aligned} E\phi_{01} - F\phi_{10} &= r \\ G\phi_{10} - F\phi_{01} &= s \end{aligned}\right\}, \qquad \left.\begin{aligned} E\psi_{01} - F\psi_{10} &= \rho \\ G\psi_{10} - F\psi_{01} &= \sigma \end{aligned}\right\}.$$

Any functional combination of these nine quantities will satisfy the set of eight equations which have been considered, as will also any functional combination of the derivatives of ϕ and ψ of orders lower than 3, of the derivatives of E, F, G of orders lower than 2, and of L, M, N, P, Q, R, S, α, β, γ, δ, ϵ. We therefore have to find the functional combinations which will satisfy the remaining equations.

16. For this purpose, we make $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, \theta$; E, E_{10}, E_{01}; F, F_{10}, F_{01}; G, G_{10}, G_{01}; $\phi_{10}, \phi_{01}, \phi_{20}, \phi_{11}, \phi_{02}$; $\psi_{10}, \psi_{01}, \psi_{20}, \psi_{11}, \psi_{02}$, the variables; and we transform the set of equations $(\mathrm{II}_1) \ldots (\mathrm{II}_6)$, so that the derivatives of f are taken with regard to these variables. Denoting f with the new variable by $\bar{f}$ for a moment, we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial \bar{f}}{\partial \xi} + \Sigma \frac{\partial \bar{f}}{\partial u_n}\frac{\partial u_n}{\partial \xi} + \Sigma \frac{\partial \bar{f}}{\partial v_n}\frac{\partial v_n}{\partial \xi} + \frac{\partial \bar{f}}{\partial \theta}\frac{\partial \theta}{\partial \xi},$$

for all the quantities ξ in the original equations; the magnitude $\partial\bar{f}/\partial\xi$ is zero if ξ be not one of the new variables.

The result of the transformation is to replace the set of equations $(\mathrm{II}_1) \ldots (\mathrm{II}_6)$ by the set

$$2E\frac{\partial f}{\partial E_{10}} + F\frac{\partial f}{\partial F_{10}} + \phi_{10}\frac{\partial f}{\partial \phi_{20}} + \psi_{10}\frac{\partial f}{\partial \psi_{20}} + (G_{10} - 2F_{10})\frac{\partial f}{\partial \theta}$$

$$+ \frac{\partial f}{\partial u_1}[6V^2\phi_{20} + (2E_{01} - 6F_{10})r - 5E_{10}s] + \frac{\partial f}{\partial u_2}[2V^2\phi_{11} - G_{10}r - 2E_{01}s]$$

$$+ \frac{\partial f}{\partial v_1}[6V^2\psi_{20} + (2E_{01} - 6F_{10})\rho - 5E_{10}\sigma] + \frac{\partial f}{\partial v_2}[2V^2\psi_{11} - G_{10}\rho - 2E_{01}\sigma] = 0 \quad . \quad (\mathrm{II}_1)',$$

$$2F\frac{\partial f}{\partial E_{10}} + G\frac{\partial f}{\partial F_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{20}} + \psi_{01}\frac{\partial f}{\partial \psi_{20}} - G_{01}\frac{\partial f}{\partial \theta}$$
$$+ \frac{\partial f}{\partial u_1}[6V^2\phi_{11} - 4G_{10}r - (E_{01} + 4F_{10})s] + \frac{\partial f}{\partial u_2}[2V^2\phi_{02} - G_{01}r - 2F_{01}s]$$
$$+ \frac{\partial f}{\partial v_1}[6V^2\psi_{11} - 4G_{10}\rho - (E_{01} + 4F_{10})\sigma] + \frac{\partial f}{\partial v_2}[2V^2\psi_{02} - G_{01}\rho - 2F_{01}\sigma] = 0 \quad . \quad (\text{II}_4)',$$

$$2E\frac{\partial f}{\partial E_{01}} + E\frac{\partial f}{\partial F_{10}} + F\frac{\partial f}{\partial F_{01}} + 2F\frac{\partial f}{\partial G_{10}} + \phi_{10}\frac{\partial f}{\partial \phi_{11}} + \psi_{10}\frac{\partial f}{\partial \psi_{11}} + 2E_{01}\frac{\partial f}{\partial \theta}$$
$$- E_{10}r\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_2}[4V^2\phi_{20} - 4F_{10}r - 3E_{10}s] + \frac{\partial f}{\partial u_3}[4V^2\phi_{11} - (G_{10} + 2F_{01})r - 4E_{01}s]$$
$$+ (G_{10} - 2F_{01})s\frac{\partial f}{\partial u_4}$$
$$- E_{10}\rho\frac{\partial f}{\partial v_1} + \frac{\partial f}{\partial v_2}[4V^2\psi_{20} - 4F_{10}\rho - 3E_{10}\sigma] + \frac{\partial f}{\partial v_3}[4V^2\psi_{11} - (G_{10} + 2F_{01})\rho - 4E_{01}\sigma]$$
$$+ (G_{10} - 2F_{01})\sigma\frac{\partial f}{\partial v_4} = 0 \quad . \quad (\text{II}_2)',$$

$$2F\frac{\partial f}{\partial E_{01}} + F\frac{\partial f}{\partial F_{10}} + G\frac{\partial f}{\partial F_{01}} + 2G\frac{\partial f}{\partial G_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{11}} + \psi_{01}\frac{\partial f}{\partial \psi_{11}} + 2G_{10}\frac{\partial f}{\partial \theta}$$
$$+ (E_{01} - 2F_{10})r\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_2}[4V^2\phi_{11} - 4G_{10}r - (E_{01} + 2F_{10})s]$$
$$+ \frac{\partial f}{\partial u_3}[4V^2\phi_{02} - 3G_{01}r - 4F_{01}s] - G_{01}s\frac{\partial f}{\partial u_4}$$
$$+ (E_{01} - 2F_{10})\rho\frac{\partial f}{\partial v_1} + \frac{\partial f}{\partial v_2}[4V^2\psi_{11} - 4G_{10}\rho - (E_{01} + 2F_{10})\sigma]$$
$$+ \frac{\partial f}{\partial v_3}[4V^2\psi_{02} - 3G_{01}\rho - 4F_{01}\sigma] - G_{01}\sigma\frac{\partial f}{\partial v_4} = 0 \quad . \quad (\text{II}_5)',$$

$$E\frac{\partial f}{\partial F_{01}} + 2F\frac{\partial f}{\partial G_{01}} + \phi_{10}\frac{\partial f}{\partial \phi_{02}} + \psi_{10}\frac{\partial f}{\partial \psi_{02}} - E_{10}\frac{\partial f}{\partial \theta}$$
$$+ \frac{\partial f}{\partial u_3}[2V^2\phi_{20} - 2F_{10}r - E_{10}s] + \frac{\partial f}{\partial u_4}[6V^2\phi_{11} - (G_{10} + 4F_{01})r - 4E_{01}s]$$
$$+ \frac{\partial f}{\partial v_3}[2V^2\psi_{20} - 2F_{10}\rho - E_{10}\sigma] + \frac{\partial f}{\partial v_4}[6V^2\phi_{11} - (G_{10} + 4F_{01})\rho - 4E_{01}\sigma] = 0 \quad . \quad (\text{II}_3)',$$

$$F\frac{\partial f}{\partial F_{01}} + 2G\frac{\partial f}{\partial G_{01}} + \phi_{01}\frac{\partial f}{\partial \phi_{02}} + \psi_{01}\frac{\partial f}{\partial \psi_{02}} + (E_{01} - 2F_{10})\frac{\partial f}{\partial \theta}$$
$$+ \frac{\partial f}{\partial u_3}[2V^2\phi_{11} - 2G_{10}r - E_{01}s] + \frac{\partial f}{\partial u_4}[6V^2\phi_{02} - 5G_{01}r + (2G_{10} - 6F_{01})s]$$
$$+ \frac{\partial f}{\partial v_3}[2V^2\psi_{11} - 2G_{10}\rho - E_{01}\sigma] + \frac{\partial f}{\partial v_4}[6V^2\psi_{02} - 5G_{01}\rho + (2G_{10} - 6F_{01})\sigma] = 0 \quad . \quad (\text{II}_6)'.$$

17. A special case of these six equations is discussed* by Professor Żorawski in his memoir already quoted, viz., that in which there occurs a single function ϕ with its derivatives up to the second order inclusive, and there are no derivatives of E, F, G of order higher than the first; and he obtains three independent solutions. These are

$$\left.\begin{aligned} a &= 2V^2\phi_{20} + (E_{01} - 2F_{10})\,r - E_{10}s \\ b &= 2V^2\phi_{11} - G_{10}r - E_{01}s \\ c &= 2V^2\phi_{02} - G_{01}r + (G_{10} - 2F_{01})\,s \end{aligned}\right\}.$$

Manifestly, a, b, c are independent solutions of the equations in the present case: also, other three independent solutions are given by

$$\left.\begin{aligned} a' &= 2V^2\psi_{20} + (E_{01} - 2F_{10})\,\rho - E_{10}\sigma \\ b' &= 2V^2\psi_{11} - G_{10}\rho - E_{01}\sigma \\ c' &= 2V^2\psi_{02} - G_{01}\rho + (G_{10} - 2F_{01})\,\sigma \end{aligned}\right\}.$$

All these six solutions are independent of θ; u_1, u_2, u_3, u_4; v_1, v_2, v_3, v_4.

The Jacobi-Poisson conditions of coexistence of the six equations are satisfied either identically or in virtue of the eight equations (III_1) to (III_8), which are definitely satisfied; so that, taking account of the variables that occur in the derivatives of f, the set of six equations is a complete system. The number of these variables is

6, from the first derivatives of E, F, G,
+ 6, . . . second . . . . ϕ, ψ,
+ 1, being θ,
+ 8, being u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4,

= 21 in all; hence the total number of algebraically independent solutions of the complete system of six equations is 15. Of these, we already possess six in a, b, c, a', b', c', so that other nine are required.

The form of the equations suggests that there will be four solutions of the type

$$u_n + aP_n + bQ_n + cR_n + S_n,$$

four of the type

$$v_n + a'P'_n + b'Q'_n + c'R'_n + S'_n,$$

and one of the type

$$\theta + T_n;$$

which, when obtained, will be the necessary nine.

18. One mode of obtaining these solutions is as follows:—We use the values of a, b, c, a', b', c' to eliminate from f the second derivatives of ϕ and of ψ; the effect is

* *Loc. cit.*, § 26.

to modify the form of the equations $(\mathrm{II}_1)' \ldots (\mathrm{II}_6)'$ by removing from them all the terms that involve those derivatives. The substituted derivatives with respect to a, b, c, a', b', c' do not occur—a result only to be expected, because these quantities are simultaneous solutions of all the six equations. Consequently, in any differential operations, the quantities a, b, c, a', b', c' behave like constants.

In order that

$$f = u_1 + a\mathrm{P}_1 + b\mathrm{Q}_1 + c\mathrm{R}_1 + \mathrm{S}_1,$$

where $\mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1, \mathrm{S}_1$ are functions of ϕ_{10}, ϕ_{01}, and of the first derivatives of E, F, G, but are independent of the quantities u and v, θ, a, b, c, a', b', c', may satisfy $(\mathrm{II}_1)'$, we must have

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{P}_1 = -3,$$

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{Q}_1 = 0,$$

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{R}_1 = 0,$$

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{S}_1 = \mathrm{E}_{01}r + 2\mathrm{E}_{10}s.$$

In order that the same quantity may satisfy $(\mathrm{II}_4)'$, we must have

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{P}_1 = 0,$$

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{Q}_1 = -3,$$

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{R}_1 = 0,$$

$$\left(2\mathrm{F}\frac{\partial}{\partial \mathrm{E}_{10}} + \mathrm{G}\frac{\partial}{\partial \mathrm{F}_{10}}\right)\mathrm{S}_1 = \mathrm{G}_{10}r - 2(\mathrm{E}_{01} - 2\mathrm{F}_{10})s.$$

Similarly, the equation $(\mathrm{II}_3)'$ requires

$$\left(\mathrm{E}\frac{\partial}{\partial \mathrm{F}_{01}} + 2\mathrm{F}\frac{\partial}{\partial \mathrm{G}_{01}}\right)\Theta = 0,$$

and the equation $(\mathrm{II}_6)'$ requires

$$\left(\mathrm{F}\frac{\partial}{\partial \mathrm{F}_{01}} + 2\mathrm{G}\frac{\partial}{\partial \mathrm{G}_{01}}\right)\Theta = 0,$$

for $\Theta = \mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1, \mathrm{S}_1$. The equation $(\mathrm{II}_2)'$ requires

$$\left(2\mathrm{E}\frac{\partial}{\partial \mathrm{E}_{01}} + \mathrm{E}\frac{\partial}{\partial \mathrm{F}_{10}} + \mathrm{F}\frac{\partial}{\partial \mathrm{F}_{01}} + 2\mathrm{F}\frac{\partial}{\partial \mathrm{G}_{10}}\right)\Phi = 0$$

for $\Phi = \mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1$, and

$$\left(2E\frac{\partial}{\partial E_{01}} + E\frac{\partial}{\partial F_{10}} + F\frac{\partial}{\partial F_{01}} + 2F\frac{\partial}{\partial G_{10}}\right)S_1 = E_{10}r;$$

and the equation $(II_5)'$ requires

$$\left(2F\frac{\partial}{\partial E_{01}} + F\frac{\partial}{\partial F_{10}} + G\frac{\partial}{\partial F_{01}} + 2G\frac{\partial}{\partial G_{10}}\right)\Phi = 0,$$

for $\Phi = P_1, Q_1, R_1$, and

$$\left(2F\frac{\partial}{\partial E_{01}} + F\frac{\partial}{\partial F_{10}} + G\frac{\partial}{\partial F_{01}} + 2G\frac{\partial}{\partial G_{10}}\right)S_1 = -(E_{01} - 2F_{10})r.$$

We thus have 24 equations giving the derivatives of the four quantities P_1, Q_1, R_1, S_1 with respect to $E_{10}, E_{01}, F_{10}, F_{01}, G_{10}, G_{01}$. Each of the four quantities is then given by effecting the quadrature

$$\Theta = \int\left(\frac{\partial\Theta}{\partial E_{10}}dE_{10} + \ . \ \ . \ \ . + \frac{\partial\Theta}{\partial G_{01}}dG_{01}\right).$$

The results are

$$\begin{aligned} 2V^2P_1 &= 3(-E_{10}G - E_{01}F + 2F_{10}F), \\ 2V^2Q_1 &= 3(E_{10}F + E_{01}E - 2F_{10}E), \\ 2V^2R_1 &= 0, \\ 2V^2S_1 &= \{EG_{10}(2F_{10} - E_{01}) + F(E_{01}^2 - 2E_{01}F_{10} - E_{10}G_{10}) + GE_{10}E_{01}\}r \\ &\quad + \{E(E_{01}^2 - 4E_{01}F_{10} + 4F_{10}^2) + 2FE_{10}(E_{01} - 2F_{10}) + GE_{10}^2\}s. \end{aligned}$$

The solution in question remains a solution when it is multiplied by $2V^2$; denoting this product by κ', we have

$$\kappa' = 2V^2u_1 + ap_1 + bq_1 + cr_1 + re_1 + sf_1.$$

Similarly we obtain

$$\left.\begin{aligned} \lambda' &= 2V^2u_2 + ap_2 + bq_2 + cr_2 + re_2 + sf_2 \\ \mu' &= 2V^2u_3 + ap_3 + bq_3 + cr_3 + re_3 + sf_3 \\ \nu' &= 2V^2u_4 + ap_4 + bq_4 + cr_4 + re_4 + sf_4 \end{aligned}\right\},$$

where

$$\left.\begin{aligned} p_1 &= 3\{-F(E_{01} - 2F_{10}) - GE_{10}\} \\ q_1 &= 3\{E(E_{01} - 2F_{10}) + FE_{10}\} \\ r_1 &= 0 \\ e_1 &= -EG_{10}(E_{01} - 2F_{10}) + F(E_{01}^2 - 2E_{01}F_{10} - E_{10}G_{10}) + GE_{10}E_{01} \\ f_1 &= E(E_{01} - 2F_{10})^2 + 2FE_{10}(E_{01} - 2F_{10}) + GE_{10}^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} p_2 &= 2FG_{10} - 2GE_{01} \\ q_2 &= -2EG_{10} + F(E_{01} + 2F_{10}) - GE_{10} \\ r_2 &= E(E_{01} - 2F_{10}) + FE_{10} \\ e_2 &= EG_{10}^2 - 2FE_{01}G_{10} + GE_{01}^2 \\ f_2 &= e_1, \text{ above} \end{aligned}\right\},$$

$$\left.\begin{aligned}
p_3 &= FG_{01} - G(2F_{01} - G_{10}) \\
q_3 &= -EG_{01} + F(2F_{01} + G_{10}) - 2GE_{01} \\
r_3 &= -2EG_{10} + 2FE_{01} \\
e_3 &= EG_{10}G_{01} + F(-E_{01}G_{01} - 2F_{01}G_{10} + G_{10}^2) + GE_{01}(2F_{01} - G_{10}) \\
f_3 &= e_2, \text{ above}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
p_4 &= 0 \\
q_4 &= 3\{FG_{01} - G(2F_{01} - G_{10})\} \\
r_4 &= 3\{-EG_{01} + F(2F_{01} - G_{10})\} \\
e_4 &= EG_{01}^2 - 2FG_{01}(2F_{01} - G_{10}) + G(2F_{01} - G_{10})^2 \\
f_4 &= e_3, \text{ above}
\end{aligned}\right\}.$$

Other four solutions are given by

$$\left.\begin{aligned}
\kappa'' &= 2V^2v_1 + a'p_1 + b'q_1 + c'r_1 + \rho e_1 + \sigma f_1 \\
\lambda'' &= 2V^2v_2 + a'p_2 + b'q_2 + c'r_2 + \rho e_2 + \sigma f_2 \\
\mu'' &= 2V^2v_3 + a'p_3 + b'q_3 + c'r_3 + \rho e_3 + \sigma f_3 \\
\nu'' &= 2V^2v_4 + a'p_4 + b'q_4 + c'r_4 + \rho e_4 + \sigma f_4
\end{aligned}\right\};$$

and there is a last solution given by

$$\begin{aligned}
\nabla = E\,&\{(E_{01} - 2F_{10})G_{01} + G_{10}^2\} \\
&+ F\{E_{10}G_{01} - E_{01}(2F_{01} + G_{10}) + 2F_{10}(2F_{01} - G_{10})\} \\
&+ G\{E_{01}^2 - E_{10}(2F_{01} - G_{10})\} \\
&- 2V^2(E_{02} - 2F_{11} + G_{20}).
\end{aligned}$$

Consequently, it follows that every simultaneous solution of the fourteen equations made up of the eight $(III_1) \ldots (III_8)$ and of the six $(II_1) \ldots (II_6)$, is a functional combination of the fifteen quantities

$$a,\ b,\ c,\ a',\ b',\ c',$$

$$\kappa',\ \lambda',\ \mu',\ \nu',\ \kappa'',\ \lambda'',\ \mu'',\ \nu'',$$

$$\nabla,$$

and of the quantities derivatives with regard to which have not occurred in those fourteen equations, viz., E, F, G, L, M, N, P, Q, R, S, α, β, γ, δ, ϵ, ϕ_{10}, ϕ_{01}, ψ_{10}, ψ_{01}, making 34 arguments in all What now is required is the algebraically independent aggregate of the functional combinations of these 34 arguments satisfying the remaining four differential equations $(I_1) \ldots (I_4)$.

19. As regards these equations, we replace (I_1) and (I_2) by two equations composed of their sum and their difference. The former is

$$\begin{aligned}
2\mu f = {} & 2\left(E\frac{\partial f}{\partial E} + F\frac{\partial f}{\partial F} + G\frac{\partial f}{\partial G}\right) + 2\left(L\frac{\partial f}{\partial L} + M\frac{\partial f}{\partial M} + N\frac{\partial f}{\partial N}\right) \\
& + 3\left(P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R} + S\frac{\partial f}{\partial S}\right) \\
& + 4\left(\alpha\frac{\partial f}{\partial \alpha} + \beta\frac{\partial f}{\partial \beta} + \gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} + \epsilon\frac{\partial f}{\partial \epsilon}\right) \\
& + 3\left(E_{10}\frac{\partial f}{\partial E_{10}} + E_{01}\frac{\partial f}{\partial E_{01}} + F_{10}\frac{\partial f}{\partial F_{10}} + F_{01}\frac{\partial f}{\partial F_{01}} + G_{10}\frac{\partial f}{\partial G_{10}} + G_{01}\frac{\partial f}{\partial G_{01}}\right) \\
& + 4\left(E_{20}\frac{\partial f}{\partial E_{20}} + E_{11}\frac{\partial f}{\partial E_{11}} + E_{02}\frac{\partial f}{\partial E_{02}} + F_{20}\frac{\partial f}{\partial F_{20}} + F_{11}\frac{\partial f}{\partial F_{11}} + F_{02}\frac{\partial f}{\partial F_{02}}\right. \\
& \qquad \left. + G_{20}\frac{\partial f}{\partial G_{20}} + G_{11}\frac{\partial f}{\partial G_{11}} + G_{02}\frac{\partial f}{\partial G_{02}}\right) \\
& + \phi_{10}\frac{\partial f}{\partial \phi_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} + \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
& + 2\left(\phi_{20}\frac{\partial f}{\partial \phi_{20}} + \phi_{11}\frac{\partial f}{\partial \phi_{11}} + \phi_{02}\frac{\partial f}{\partial \phi_{02}} + \psi_{20}\frac{\partial f}{\partial \psi_{20}} + \psi_{11}\frac{\partial f}{\partial \psi_{11}} + \psi_{02}\frac{\partial f}{\partial \psi_{02}}\right) \\
& + 3\left(\phi_{30}\frac{\partial f}{\partial \phi_{30}} + \phi_{21}\frac{\partial f}{\partial \phi_{21}} + \phi_{12}\frac{\partial f}{\partial \phi_{12}} + \phi_{03}\frac{\partial f}{\partial \phi_{03}}\right. \\
& \qquad \left. + \psi_{30}\frac{\partial f}{\partial \psi_{30}} + \psi_{21}\frac{\partial f}{\partial \psi_{21}} + \psi_{12}\frac{\partial f}{\partial \psi_{12}} + \psi_{03}\frac{\partial f}{\partial \psi_{03}}\right) \quad . \quad . \quad (I_1).
\end{aligned}$$

The latter is

$$\begin{aligned}
0 = {} & 2\left(E\frac{\partial f}{\partial E} - G\frac{\partial f}{\partial G} + L\frac{\partial f}{\partial L} - N\frac{\partial f}{\partial N}\right) + 3P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} - R\frac{\partial f}{\partial R} - 3S\frac{\partial f}{\partial S} \\
& + 4\alpha\frac{\partial f}{\partial \alpha} + 2\beta\frac{\partial f}{\partial \beta} - 2\delta\frac{\partial f}{\partial \delta} - 4\epsilon\frac{\partial f}{\partial \epsilon} \\
& + 3E_{10}\frac{\partial f}{\partial E_{10}} + E_{01}\frac{\partial f}{\partial E_{01}} + F_{10}\frac{\partial f}{\partial F_{10}} - F_{01}\frac{\partial f}{\partial F_{01}} - G_{10}\frac{\partial f}{\partial G_{10}} - 3G_{01}\frac{\partial f}{\partial G_{01}} \\
& + 4E_{20}\frac{\partial f}{\partial E_{20}} + 2E_{11}\frac{\partial f}{\partial E_{11}} + 2F_{20}\frac{\partial f}{\partial F_{20}} - 2F_{02}\frac{\partial f}{\partial F_{02}} - 2G_{11}\frac{\partial f}{\partial G_{11}} - 4G_{02}\frac{\partial f}{\partial G_{02}} \\
& + \phi_{10}\frac{\partial f}{\partial \phi_{10}} - \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} - \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
& + 2\left(\phi_{20}\frac{\partial f}{\partial \phi_{20}} - \phi_{02}\frac{\partial f}{\partial \phi_{02}} + \psi_{20}\frac{\partial f}{\partial \psi_{20}} - \psi_{02}\frac{\partial f}{\partial \psi_{02}}\right) \\
& + 3\phi_{30}\frac{\partial f}{\partial \phi_{30}} + \phi_{21}\frac{\partial f}{\partial \phi_{21}} - \phi_{12}\frac{\partial f}{\partial \phi_{12}} - 3\phi_{03}\frac{\partial f}{\partial \phi_{03}} \\
& + 3\psi_{30}\frac{\partial f}{\partial \psi_{30}} + \psi_{21}\frac{\partial f}{\partial \psi_{21}} - \psi_{12}\frac{\partial f}{\partial \psi_{12}} - 3\psi_{03}\frac{\partial f}{\partial \psi_{03}} \quad . \quad . \quad . \quad . \quad (I_2).
\end{aligned}$$

Of these four equations $(\mathrm{I}_1)'$, $(\mathrm{I}_2)'$, (I_3), (I_4), the first will be found to be satisfied for the various forms of f that satisfy the other three, by the appropriate determination of the constant μ to be associated with each such form. Also, $(\mathrm{I}_2)'$ is the condition to be satisfied in order that (I_3) and (I_4) may possess common solutions. To obtain these common solutions, we proceed as follows.

Let the equations (I_3) and (I_4) be written

$$\nabla_1 f = 0, \quad \nabla_2 f = 0.$$

Then by actual substitution we obtain the results

$$\left.\begin{aligned} \nabla_1 a &= 0, & \nabla_2 a &= 2b \\ \nabla_1 b &= a, & \nabla_2 b &= c \\ \nabla_1 c &= 2b, & \nabla_2 c &= 0 \end{aligned}\right\} ;$$

$$\left.\begin{aligned} \nabla_1 a' &= 0, & \nabla_2 a' &= 2b' \\ \nabla_1 b' &= a', & \nabla_2 b' &= c' \\ \nabla_1 c' &= 2b', & \nabla_2 c' &= 0 \end{aligned}\right\} ;$$

$$\nabla_1 \nabla = 0, \qquad \nabla_2 \nabla = 0.$$

Also

$$\begin{aligned} \nabla_1 \kappa' &= 0, & \nabla_2 \kappa' &= 3\lambda' - \nabla r, \\ \nabla_1 \lambda' &= \kappa', & \nabla_2 \lambda' &= 2\mu' - \nabla s, \\ \nabla_1 \mu' &= 2\lambda' - \nabla r, & \nabla_2 \mu' &= \nu', \\ \nabla_1 \nu' &= 3\mu' - \nabla s, & \nabla_2 \nu' &= 0, \\ \nabla_1 r &= 0, & \nabla_2 r &= -s, \\ \nabla_1 s &= -r, & \nabla_2 s &= 0 ; \end{aligned}$$

and therefore

$$\begin{aligned} \nabla_1 \kappa' &= 0, & \nabla_2 \kappa' &= 3(\lambda' - \tfrac{1}{3}\nabla r), \\ \nabla_1 (\lambda' - \tfrac{1}{3}\nabla r) &= \kappa', & \nabla_2 (\lambda' - \tfrac{1}{3}\nabla r) &= 2(\mu' - \tfrac{1}{3}\nabla s), \\ \nabla_1 (\mu' - \tfrac{1}{3}\nabla s) &= 2(\lambda' - \tfrac{1}{3}\nabla r), & \nabla_2 (\mu' - \tfrac{1}{3}\nabla s) &= \nu', \\ \nabla_1 \nu' &= 3(\mu' - \tfrac{1}{3}\nabla s), & \nabla_2 \nu' &= 0. \end{aligned}$$

We write

$$\kappa' = k, \quad \lambda' - \tfrac{1}{3}\nabla r = l, \quad \mu' - \tfrac{1}{3}\nabla s = m, \quad \nu' = n ;$$

and then these equations give

$$\left.\begin{aligned} \nabla_1 k &= 0, & \nabla_2 k &= 3l \\ \nabla_1 l &= k, & \nabla_2 l &= 2m \\ \nabla_1 m &= 2l, & \nabla_2 m &= n \\ \nabla_1 l &= 3m, & \nabla_2 n &= 0 \end{aligned}\right\} .$$

Similarly, we write

$$\kappa'' = k', \quad \lambda'' - \tfrac{1}{3}\nabla\rho = l', \quad \mu'' - \tfrac{1}{3}\nabla\sigma = m', \quad \nu'' = n' ;$$

and we find

$$\left.\begin{aligned} \nabla_1 k' &= 0, & \nabla_2 k' &= 3l' \\ \nabla_1 l' &= k', & \nabla_2 l' &= 2m' \\ \nabla_1 m' &= 2l', & \nabla_2 m' &= n' \\ \nabla_1 n' &= 3m', & \nabla_2 n' &= 0 \end{aligned}\right\}.$$

These quantities $k, l, m, n, k', l', m', n'$ replace $\kappa', \lambda', \mu', \nu', \kappa'', \lambda'', \mu'', \nu''$; moreover, ∇ is a simultaneous solution of the equations. What we require are the functional combinations of the quantities

$$a, b, c, \; a', b', c',$$

$$k, l, m, n, \; k', l', m', n',$$

$$\text{E, F, G, L, M, N, P, Q, R, S}, \; \alpha, \beta, \gamma, \delta, \epsilon, \phi_{10}, \phi_{01}, \psi_{10}, \psi_{01},$$

making 33 arguments in all.

For this purpose, we transform the equations, so that these 33 arguments may become the independent variables. The process would be laborious but not intrinsically difficult, were it not that the effect of the operators ∇_1 and ∇_2 upon the various arguments has already been obtained; and the results are

$$\begin{aligned} 2\text{F}\frac{\partial f}{\partial \text{G}} &+ \text{E}\frac{\partial f}{\partial \text{F}} \\ + 2\text{M}\frac{\partial f}{\partial \text{N}} &+ \text{L}\frac{\partial f}{\partial \text{M}} \\ + 3\text{R}\frac{\partial f}{\partial \text{S}} + 2\text{Q}\frac{\partial f}{\partial \text{R}} &+ \text{P}\frac{\partial f}{\partial \text{Q}} \\ + 4\delta\frac{\partial f}{\partial \epsilon} + 3\gamma\frac{\partial f}{\partial \delta} + 2\beta\frac{\partial f}{\partial \gamma} &+ \alpha\frac{\partial f}{\partial \beta} \\ + 2b\frac{\partial f}{\partial c} &+ a\frac{\partial f}{\partial b} \\ + 2b'\frac{\partial f}{\partial c'} &+ a'\frac{\partial f}{\partial b'} \\ + 3m\frac{\partial f}{\partial n} + 2l\frac{\partial f}{\partial m} &+ k\frac{\partial f}{\partial l} \\ + 3m'\frac{\partial f}{\partial n'} + 2l'\frac{\partial f}{\partial m'} &+ k'\frac{\partial f}{\partial l'} \\ &+ \phi_{10}\frac{\partial f}{\partial \phi_{01}} \\ &+ \psi_{10}\frac{\partial f}{\partial \psi_{01}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (\text{I}_3)', \end{aligned}$$

$$
\begin{aligned}
&2F\frac{\partial f}{\partial E} + G\frac{\partial f}{\partial F} \\
&+ 2M\frac{\partial f}{\partial L} + N\frac{\partial f}{\partial M} \\
&+ 3Q\frac{\partial f}{\partial P} + 2R\frac{\partial f}{\partial Q} + S\frac{\partial f}{\partial R} \\
&+ 4\beta\frac{\partial f}{\partial \alpha} + 3\gamma\frac{\partial f}{\partial \beta} + 2\delta\frac{\partial f}{\partial \gamma} + \epsilon\frac{\partial f}{\partial \delta} \\
&+ 2b\frac{\partial f}{\partial a} + c\frac{\partial f}{\partial b} \\
&+ 3l\frac{\partial f}{\partial k} + 2m\frac{\partial f}{\partial l} + n\frac{\partial f}{\partial m} \\
&+ 2b'\frac{\partial f}{\partial a'} + c'\frac{\partial f}{\partial b'} \\
&+ 3l'\frac{\partial f}{\partial k'} + 2m'\frac{\partial f}{\partial l'} + n'\frac{\partial f}{\partial m'} \\
&+ \phi_{01}\frac{\partial f}{\partial \phi_{10}} \\
&+ \psi_{01}\frac{\partial f}{\partial \psi_{10}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}_1)',
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{10}\frac{\partial f}{\partial \phi_{10}} - \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} - \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
&+ 2\left(E\frac{\partial f}{\partial E} - G\frac{\partial f}{\partial G} + L\frac{\partial f}{\partial L} - N\frac{\partial f}{\partial N} + a\frac{\partial f}{\partial a} - c\frac{\partial f}{\partial c} + a'\frac{\partial f}{\partial a'} - c'\frac{\partial f}{\partial c'}\right) \\
&+ 3P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} - R\frac{\partial f}{\partial R} - 3S\frac{\partial f}{\partial S} \\
&+ 3k\frac{\partial f}{\partial k} + l\frac{\partial f}{\partial l} - m\frac{\partial f}{\partial m} - 3n\frac{\partial f}{\partial n} \\
&+ 3k'\frac{\partial f}{\partial k'} + l'\frac{\partial f}{\partial l'} - m'\frac{\partial f}{\partial m'} - 3n'\frac{\partial f}{\partial n'} \\
&+ 4\alpha\frac{\partial f}{\partial \alpha} + 2\beta\frac{\partial f}{\partial \beta} - 2\delta\frac{\partial f}{\partial \delta} - 4\epsilon\frac{\partial f}{\partial \epsilon} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I}_2)'',
\end{aligned}
$$

$$
\begin{aligned}
2\mu f = {} & \phi_{10}\frac{\partial f}{\partial \phi_{10}} + \phi_{01}\frac{\partial f}{\partial \phi_{01}} + \psi_{10}\frac{\partial f}{\partial \psi_{10}} + \psi_{01}\frac{\partial f}{\partial \psi_{01}} \\
& + 2\left(\mathrm{E}\frac{\partial f}{\partial \mathrm{E}} + \mathrm{F}\frac{\partial f}{\partial \mathrm{F}} + \mathrm{G}\frac{\partial f}{\partial \mathrm{G}} + \mathrm{L}\frac{\partial f}{\partial \mathrm{L}} + \mathrm{M}\frac{\partial f}{\partial \mathrm{M}} + \mathrm{N}\frac{\partial f}{\partial \mathrm{N}}\right) \\
& + 3\left(\mathrm{P}\frac{\partial f}{\partial \mathrm{P}} + \mathrm{Q}\frac{\partial f}{\partial \mathrm{Q}} + \mathrm{R}\frac{\partial f}{\partial \mathrm{R}} + \mathrm{S}\frac{\partial f}{\partial \mathrm{S}}\right) \\
& + 4\left(\alpha\frac{\partial f}{\partial \alpha} + \beta\frac{\partial f}{\partial \beta} + \gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} + \epsilon\frac{\partial f}{\partial \epsilon}\right) \\
& + 6\left(a\frac{\partial f}{\partial a} + b\frac{\partial f}{\partial b} + c\frac{\partial f}{\partial c} + a'\frac{\partial f}{\partial a'} + b'\frac{\partial f}{\partial b'} + c'\frac{\partial f}{\partial c'}\right) \\
& + 8\nabla\frac{\partial f}{\partial \nabla} \\
& + 11\left(k\frac{\partial f}{\partial k} + l\frac{\partial f}{\partial l} + m\frac{\partial f}{\partial m} + n\frac{\partial f}{\partial n}\right) \\
& + 11\left(k'\frac{\partial f}{\partial k'} + l'\frac{\partial f}{\partial l'} + m'\frac{\partial f}{\partial m'} + n'\frac{\partial f}{\partial n'}\right) \quad \ldots\ldots\ldots \quad (\mathrm{I}_1)''.
\end{aligned}
$$

Association with Binariants.

20. The expression of these equations at once associates the solution with known results in the theory of the concomitants of a system of simultaneous binary forms. The equations $(\mathrm{I}_3)'$, $(\mathrm{I}_4)'$, $(\mathrm{I}_2)''$ are the differential equations of the invariants of the system of binary forms

$$
\begin{aligned}
& (\phi_{10}, \phi_{01} \between *)^1, \quad (\psi_{10}, \psi_{01} \between *)^1, \\
& (\mathrm{E}, \mathrm{F}, \mathrm{G} \between *)^2, \quad (\mathrm{L}, \mathrm{M}, \mathrm{N} \between *)^2, \quad (a, b, c \between *)^2, \quad (a', b', c' \between *)^2, \\
& (\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S} \between *)^3, \quad (k, l, m, n \between *)^3, \quad (k', l', m', n' \between *)^3, \\
& (\alpha, \beta, \gamma, \delta, \epsilon \between *)^4,
\end{aligned}
$$

or, what is the same thing, they are the differential equations of the invariants and covariants of the system of binary forms

$$
\begin{aligned}
w_1 &= (\psi_{10}, \psi_{01} \between \phi_{01}, -\phi_{10}), \\
w_2 &= (\mathrm{E}, \mathrm{F}, \mathrm{G} \between \phi_{01}, -\phi_{10})^2, \\
w'_2 &= (\mathrm{L}, \mathrm{M}, \mathrm{N} \between \phi_{01}, -\phi_{10})^2, \\
w''_2 &= (a, b, c \between \phi_{01}, -\phi_{10})^2, \\
w'''_2 &= (a', b', c' \between \phi_{01}, -\phi_{10})^2, \\
w_3 &= (\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S} \between \phi_{01}, -\phi_{10})^3, \\
w'_3 &= (k, l, m, n \between \phi_{01}, -\phi_{10})^3, \\
w''_3 &= (k', l', m', n' \between \phi_{01}, -\phi_{10})^3, \\
w_4 &= (\alpha, \beta, \gamma, \delta, \epsilon \between \phi_{01}, -\phi_{10})^4.
\end{aligned}
$$

We therefore require an algebraically complete aggregate of this set of invariants and covariants.

It is to be noticed that the argument ∇ does not appear in the equations $(I_2)''$, $(I_3)'$, $(I_4)'$; so that it is a solution of the equations, and it must be associated with the required algebraically complete aggregate of concomitants of the binary forms.

The three equations $(I_2)''$, $(I_3)'$, $(I_4)'$ constitute a complete Jacobian system, and the number of arguments which occur is 33; hence the algebraically complete aggregate of solutions contains 30 solutions, which thus give the algebraically complete aggregate of concomitants of the system of binary forms.

This aggregate is known* to be (or to be equivalent to) the following:—

the linear quantic, w_1;

the quadratic w_2, and its Hessian (discriminant) $EG - F^2$;

. . . . . w'_2, ... $LN - M^2$;

. . . . . w''_2, ... $ac - b^2$;

. . . . . w'''_2, ... $a'c' - b'^2$;

the cubic w_3, its Hessian, and either its discriminant or its cubicovariant;

the cubic w'_3, its Hessian, and either its discriminant or its cubicovariant;

the cubic w''_3, its Hessian, and either its discriminant or its cubicovariant;

the quartic w_4, its Hessian, its quadrinvariant, and its cubinvariant;

together with the Jacobians of any one of the forms w, say w_2, with all the rest of the forms. This makes up the requisite total of 30.

The asyzygetic aggregate is, of course, vastly more extensive; but for the present purpose it is only an algebraically independent aggregate that is wanted. Many modifications in the latter are possible: what is necessary to secure is that any modification does not interfere with the algebraical completeness of the aggregate. For instance, consider the set composed of

$$w_2, \quad EG - F^2, \quad w''_2, \quad ac - b^2, \quad J(w_2, w''_2),$$

where

$$4J(w_2, w''_2) = \frac{\partial w_2}{\partial \phi_{10}} \frac{\partial w''_2}{\partial \phi_{01}} - \frac{\partial w_2}{\partial \phi_{01}} \frac{\partial w''_2}{\partial \phi_{10}};$$

in the asyzygetic system, there is an intermediate invariant $Ec - 2Fb + Ga$; we have

$$J^2 = w_2 w''_2 (Ec - 2Fb + Ga) - w_2^2 (ac - b^2) - w''^2_2 (EG - F^2),$$

and therefore, in the algebraical aggregate, $Ec - 2Fb + Ga$ can be included when any other (such as $ac - b^2$) is excluded. Such a change would be desirable if differential invariants, linear in the quantities a, b, c, were required.

* See a memoir by the author, 'American Journal of Mathematics,' vol. 12 (1890), §§ 17, 22, 30.

21. Accordingly, we can take as an algebraically complete aggregate, containing the 31 necessary members, the set which follows :—

(i.) ∇,

(ii.) w_1,

(iii.) $J(w_1, w_2) = (E\psi_{01} - F\psi_{10})\phi_{01} - (F\psi_{01} - G\psi_{10})\phi_{10}$,

(iv.) w_2,

(v.) $H(w_2) = EG - F^2 = V^2$,

(vi.) w'_2,

(vii.) $J(w_2, w'_2) = (EM - FL)\phi_{01}^2 - (EN - GL)\phi_{01}\phi_{10} + (FN - GM)\phi_{10}^2$,

(viii.) $H(w'_2) = LN - M^2$, or $I(w_2, w'_2) = EN - 2FM + GL$,

(ix.) w''_2,

(x.) $J(w_2, w''_2) = (Eb - Fa)\phi_{01}^2 + \ldots$,

(xi.) $H(w''_2) = ac - b^2$, or $I(w_2, w''_2) = Ec - 2Fb + Ga$,

(xii.) w'''_2,

(xiii.) $J(w_2, w'''_2) = (Eb' - Fa')\phi_{01}^2 + \ldots$,

(xiv.) $H(w'''_2) = a'c' - b'^2$, or $I(w_2, w'''_2) = Ec' - 2Fb' + Ga'$,

(xv.) w_3,

(xvi.) $H(w_3) = (PR - Q^2)\phi_{01}^2 + \ldots$,

(xvii.) $\Phi(w_3) = (P^2S - 3PQR + 2Q^3)\phi_{01}^3 + \ldots$, or $\Delta(w_3)$,

(xviii.) $J(w_2, w_3) = (EQ - FP)\phi_{01}^3 + \ldots$,

(xix.) w'_3,

(xx.) $H(w'_3) = (km - l^2)\phi_{01}^2 + \ldots$,

(xxi.) $\Phi(w'_3) = (k^2n - 3klm + 2l^3)\phi_{01}^3 + \ldots$, or $\Delta(w'_3)$,

(xxii.) $J(w_2, w'_3) = (El - Fk)\phi_{01}^3 + \ldots$,

(xxiii.) w''_3,

(xxiv.) $H(w''_3) = (k'm' - l'^2)\phi_{01}^2 + \ldots$,

(xxv.) $\Phi(w''_3) = (k'^2n' - 3k'l'm' + 2l'^3)\phi_{01}^3 + \ldots$, or $\Delta(w''_3)$,

(xxvi.) $I(w_2, w''_3) = (El' - Fk')\phi_{01}^3 + \ldots$,

(xxvii.) w_4,

(xxviii.) $H(w_2) = (\alpha\gamma - \beta^2)\phi_{01}^2 + \ldots$,

(xxix.) $I(w_2) = \alpha\epsilon - 4\beta\delta + 3\gamma^2$,

(xxx.) $J(w_2) = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \beta^2\epsilon - \gamma^3$,

(xxxi.) $J(w_2, w_4) = (E\beta - F\alpha)\phi_{01}^4 + \ldots$

22. It is still necessary to satisfy equation $(I_1)''$. This will be effected as follows :

when f involves ϕ_{10} and ϕ_{01}, it must be homogeneous in them, say of degree m_1; when f involves ψ_{10} and ψ_{01}, it must be homogeneous in them, say of degree m_2; when f involves E, F, G, it must be homogeneous in them, say of degree m_3; likewise for L, M, N, say of degree m_4; likewise for P, Q, R, S, say of degree m_5: for α, β, γ, δ, ϵ, say of degree m_6; for a, b, c, say of degree m_7; for a', b', c', say of degree m_8; for k, l, m, n, say of degree m_9; for k', l', m', n', say of degree m_{10}; and for ∇, of degree m_{11}; provided the value of μ, the index of the invariant, is given by

$$2\mu = m_1 + m_2 + 2(m_3 + m_4) + 3m_5 + 4m_6$$
$$+ 6m_7 + 6m_8 + 11(m_9 + m_{10}) + 8m_{11}.$$

This relation determines the indices of the whole system, as follows:—

Index = 1, w_1;

Index = 2, $J(w_1, w_2)$, w_2, $H(w_2)$, w'_2, $H(w'_2)$ and $I(w_2, w'_2)$;

Index = 3, $J(w_2, w'_2)$, w_3;

Index = 4, ∇, w''_2, $I(w_2, w''_2)$, w'''_2, $I(w_2, w'''_2)$, $H(w_3)$, $J(w_2, w_3)$, w_4, $I(w_4)$;

Index = 5, $J(w_2, w''_2)$, $J(w_2, w'''_2)$, $J(w_2, w_4)$;

Index = 6, $H(w''_2)$, $H(w'''_2)$, $\Phi(w_3)$ and $\Delta(w_3)$, $H(w_4)$, $J(w_4)$;

Index = 7, w'_3, w''_3;

Index = 8, $J(w_2, w'_3)$, $J(w_2, w''_3)$;

Index = 12, $H(w'_3)$, $H(w''_3)$;

Index = 18, $\Phi(w'_3)$, $\Phi(w''_3)$;

Index = 22, $\Delta(w'_3)$, $\Delta(w''_3)$.

23. All these are relative invariants, that is to say, when the same function F of new variables is formed as the function f is of the old variables, then

$$\Omega^\mu F = f,$$

where μ is the index of f, and $\Omega = \dfrac{\partial(X, Y)}{\partial(x, y)}$. In order to have the absolute invariants, it is sufficient to divide each of them by a proper power of any one of them. For this purpose, we choose

$$V^2 = H(w_2) = EG - F^2;$$

we can regard V as of index unity, and therefore it will be sufficient to divide the relative invariants by a power of V equal to its index. We therefore have the set of 30 absolute invariants, given by

$$\frac{w_1}{V}, \frac{J(w_1, w_2)}{V^2}, \frac{w_2}{V^2}, \frac{w'_2}{V^2}, \frac{H(w'_2) \text{ and } I(w_2, w'_2)}{V^2}, \frac{w''_2}{V^4},$$

$$\frac{H(w''_2)}{V^6} \text{ and } \frac{I(w_2, w''_2)}{V^4}, \frac{w'''_2}{V^4}, \frac{H(w'''_2)}{V^6} \text{ and } \frac{I(w_2, w'''_2)}{V^4}, \frac{J(w_2, w'_2)}{V^3},$$

$$\frac{J(w_2, w''_2)}{V^5}, \frac{J(w_2, w'''_2)}{V^5}, \frac{w_3}{V^3}, \frac{w'_3}{V^7}, \frac{w''_3}{V^7}, \frac{\nabla}{V^4}, \frac{H(w_3)}{V^4}, \frac{J(w_2, w_3)}{V^4},$$

$$\frac{H(w'_3)}{V^{12}}, \frac{J(w_2, w'_3)}{V^8}, \frac{H(w''_3)}{V^{12}}, \frac{J(w_2, w''_3)}{V^8}, \frac{w_4}{V^4}, \frac{I(w_4)}{V^4}, \frac{J(w_2, w_4)}{V^5},$$

$$\frac{\Phi(w_3) \text{ and } \Delta(w_3)}{V^6}, \frac{\Phi(w'_3)}{V^{18}} \text{ and } \frac{\Delta(w'_3)}{V^{22}}, \frac{\Phi(w''_3)}{V^{18}} \text{ and } \frac{V(w''_3)}{V^{22}}, \frac{H(w_4)}{V^6}, \frac{J(w_4)}{V^6}.$$

These thirty differential invariants constitute *the algebraically complete aggregate in terms of which all invariants*, involving (i.) some or all of the derivatives of the fundamental magnitudes E, F, G, L, M, N, up to the second order inclusive, as well as the magnitudes themselves, (ii.) the magnitudes P, Q, R, S, α, β, γ, δ, ϵ, (iii.) and the derivatives of two functions ϕ and ψ up to the third order inclusive, *can be expressed algebraically.* But it is to be noted that this inference is concerned solely with the partial differential equations, and it assumes that the various quantities E, F, G, L, M, N, and their derivatives are independent of one another; if any relations should subsist, owing to the intrinsic nature of the magnitudes, then the number of invariants in the above complete aggregate will be diminished by the number of relations.

Now one such relation is known; it is the relation commonly associated with GAUSS's name, and it expresses $LN - M^2$ in terms of E, F, G, and their derivatives up to the second order inclusive. But $LN - M^2$ is $H(w'_2)$ in the foregoing set; and, as will be seen later (§ 35) in the course of the geometrical interpretation, we have

$$\nabla = 4H(w_2)\,H(w'_2),$$

so that the number must be diminished by unity. Accordingly, *the algebraically complete aggregate of differential invariants, involving the magnitudes up to the specified order of derivation, contains* 29 *members; in terms of these members, every other invariant, involving the same magnitudes up to the specified order of derivation, can be expressed algebraically.*

24. As an illustration of the remark in § 6, we can obtain MINDING's expression for the geodesic curvature, quoted* by Professor ŽORAWSKI as an invariant. Let $\phi = 0$ be the equation of the curve, then

$$\phi_{10} + \phi_{01}y' = 0,$$

$$\phi_{20} + 2\phi_{11}y' + \phi_{02}y'^2 + \phi_{01}y'' = 0,$$

so that

$$-\phi_{01}^3 y'' = \phi_{20}\phi_{01}^2 - 2\phi_{11}\phi_{10}\phi_{01} + \phi_{02}\phi_{10}^2.$$

* *Loc. cit.*, p. 63.

Now $w''_2 V^{-4}$ is an invariant, as also is $w_2 V^{-2}$; hence

$$\frac{w''_2}{V w_2^{\frac{3}{2}}}$$

is an invariant, say U, so that

$$\begin{aligned} U &= \frac{1}{V} \frac{a\phi_{01}^2 - 2b\phi_{01}\phi_{10} + c\phi_{10}^2}{(E\phi_{01}^2 - 2F\phi_{01}\phi_{10} + G\phi_{10}^2)^{\frac{3}{2}}} \\ &= \frac{1}{V(E\phi_{01}^2 - 2F\phi_{01}\phi_{10} + G\phi_{10}^2)^{\frac{3}{2}}} [\{2V^2\phi_{20} + (E_{01} - 2F_{10})r - E_{10}s\}\phi_{01}^2 \\ &\qquad - 2\{2V^2\phi_{11} - G_{10}r - E_{01}s\}\phi_{01}\phi_{10} \\ &\qquad + \{2V^2\phi_{02} - G_{01}r + (G_{10} - 2F_{01})s\}\phi_{10}^2] \\ &= \frac{1}{V(E + 2Fy' + Gy'^2)^{\frac{3}{2}}} [-2V^2 y'' + (2GF_{01} - GG_{10} - FG_{01})y'^3 \\ &\qquad + (2GE_{01} + 2FF_{01} - 3FG_{10} - EG_{01})y'^2 \\ &\qquad - (2EG_{10} + 2FF_{10} - 3FE_{01} - GE_{10})y' \\ &\qquad - (2EF_{10} - EE_{01} - FE_{10})] \\ &= -\frac{2}{\rho''}, \end{aligned}$$

according to MINDING's expression for the geodesic curvature; or the geodesic curvature of the curve $\phi = 0$ is the invariant

$$-\tfrac{1}{2}\frac{w''_2}{V w_2^{\frac{3}{2}}}.$$

25. It is possible to make further inferences from the results. Thus we can settle the algebraically complete aggregate of invariants up to the order of derivatives retained, when those invariants are required which involve derivatives of E, F, G, and only one function, say ϕ. They manifestly constitute the aggregate, complete up to the order specified, of all the functions that remain invariant when the surface is deformed in any way without tearing or stretching, account being taken of a particular curve $\phi = 0$, and the invariance persisting through all changes of the independent variables of the surface. This aggregate, algebraically complete up to the order specified, consists of the nine members

$$\frac{w_2}{V^2}, \quad \frac{\nabla}{V^4},$$

$$\frac{w''_2}{V^4}, \quad \frac{H(w''_2)}{V^6} \text{ and } \frac{I(w_2, w''_2)}{V^4}, \quad \frac{J(w_2, w''_2)}{V^5},$$

$$\frac{w'_3}{V^7}, \quad \frac{H(w'_3)}{V^{12}}, \quad \frac{J(w_2, w'_3)}{V^8}, \quad \frac{\Phi(w'_3)}{V^{18}} \text{ and } \frac{\Delta(w'_3)}{V^{22}},$$

the first five of which were given by Professor ŽORAWSKI, who considered the specific aggregate only up to one order lower.

26. If we require the aggregate of invariants of this class involving derivatives of E, F, G up to order $n-1$ and derivatives of ϕ up to order n, the number of members in that algebraically complete aggregate can be obtained. *The total number of members is*

$$n^2;$$

it is composed of $\frac{1}{2}(n-1)(n-2)$ quantities which do not involve the derivatives of ϕ, these quantities being called Gaussian invariants of deformation, and their number having been determined* by Žorawski; and of $\frac{1}{2}n(n+3)-1$ quantities, each of which involves derivatives of ϕ. To make up the latter aggregate of $\frac{1}{2}n(n+3)-1$ quantities, we need (in addition to the binary forms already used) other binary forms of orders 4, 5, . . . , n; among these, the binary form of order m (for all values of m) has ϕ_{01} and $-\phi_{10}$ for its variables, and its coefficients are linear in the derivatives of ϕ up to order m inclusive; and the members, that would occur in the simplest expression of the aggregate through the existence of the binary form of order m, would be the quotients (by proper powers of V) of the binary form itself, of the $m-1$ (Hermite's) associated covariants, and of the Jacobian of w_2 and the binary form, making $m+1$ in all. Thus the total number† up to order n is

$$\begin{aligned} &1+3+4+\ldots+n \\ &=\tfrac{1}{2}n(n+3)-1, \end{aligned}$$

the number in question.

27. If we require the aggregate of differential invariants, which involve derivatives of E, F, G, L, M, N up to order $n-1$ and derivatives of a single function ϕ up to order n, the number in that algebraically complete aggregate can be obtained as follows. We can replace the derivatives of L, M, N of the specified orders by the introduction of the fundamental magnitudes of orders 3, 4, . . . , $n+1$ defined as the coefficients in the various powers of $\frac{dx}{ds}$ and $\frac{dy}{ds}$ in the complete expression of the quantities

$$\frac{d}{ds}\left(\frac{1}{\rho}\right),\quad \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),\ \ldots\ ,\ \frac{d^{n-1}}{ds^{n-1}}\left(\frac{1}{\rho}\right),$$

where ρ is the radius of curvature of the normal section through the tangent defined by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and the arc-differentiation of $\frac{dx}{ds}$, $\frac{dy}{ds}$ is taken along the geodesic tangent‡.

When $n=2$, the system of binariants is composed of three quadratic forms with their three discriminants, a cubic form with its set of two associated covariants, and

* In his memoir, § 13.

† It will be noted that ∇V^{-4} in the aggregate in § 22 is a Gaussian invariant of deformation, and so is included among the $\frac{1}{2}(n-1)(n-2)$ quantities which do not involve ϕ.

‡ For the significance of this remark, see § 31, *post*.

the Jacobian of one of the quadratic forms with the other two quadratic forms and with the cubic form, being 12 in all. To include the next higher order given by $n = 3$, we need a cubic form with its set of two associated covariants, a quartic form with its set of three associated covariants, and the Jacobian of each of the forms with the originally selected quadratic form, being 9 in all. And so on in succession: the total number of binariants is

$$12 + \{(9 + 11 + 13 + \ldots + (2n + 3)\}$$
$$= n^2 + 4n.$$

With these must be associated the $\frac{1}{2}(n - 1)(n - 2)$ quantities that do not involve the derivatives of ϕ, these being the Gaussian invariants of deformation; hence the total number is

$$\tfrac{3}{2}n^2 + \tfrac{5}{2}n + 1.$$

But these are relative invariants; each of them must be divided by the appropriate power of V so that, as one of them is V^2 and the quotient is unity, thus making the function no longer an invariant of the surface, the number of absolute invariants is

$$\tfrac{3}{2}n^2 + \tfrac{5}{2}n$$
$$= \tfrac{1}{2}n(3n + 5).$$

28. Lastly, if we require the aggregate of differential invariants which involve derivatives of E, F, G, L, M, N up to order $n - 1$, and derivatives of two functions ϕ, ψ up to order n, the number can be obtained in a similar manner. As in § 27, we replace the derivatives of L, M, N of the specified orders by the fundamental magnitudes of orders $3, 4, \ldots, n + 1$. The algebraically complete aggregate of relative invariants of the surface up to the orders specified is composed of two portions. The first includes the $\frac{1}{2}(n - 1)(n - 2)$ quantities which do not involve the derivatives of ϕ and ψ, these being the Gaussian invariants of deformation, as before. The second is the algebraically complete aggregate of the system of concomitants of a set of binary forms, each divided by a proper power of V in order to give rise to an absolute invariant of the surface. This set of binary forms contains

1 quantic of order 1,
4 quantics . . 2,
3 3,
3 4,
. . . .
3 . . . . . . n,
1 quantic . . . $n + 1$,

being $3n$ in all. With them must be coupled (a) their (HERMITE'S) associated covariants, the number of which is

$$1 \cdot 0 + 4 \cdot 1 + 3\{2 + 3 + \ldots + (n - 1)\} + 1 \cdot n$$

$= \frac{3}{2}n^2 - \frac{1}{2}n + 1$; and ($b$) the Jacobian of any one of the quantics with each of the rest, being $3n - 1$ in all. Thus the tale of the concomitants of the binary forms

$$= \tfrac{3}{2}n + (\tfrac{3}{2}n^2 - \tfrac{1}{2}n + 1) + 3n - 1$$
$$= \tfrac{3}{2}n^2 + \tfrac{11}{2}n.$$

But these are relative invariants; each of them must be divided by the appropriate power of V, so that, as one of them is V^2, and the quotient is unity, thus making the function no longer an invariant of the surface, the number of absolute invariants from this source is $\frac{3}{2}n^2 + \frac{11}{2}n - 1$. Thus the required aggregate of invariants of the kind specified up to order n is, in all, equal to

$$\tfrac{1}{2}(n-1)(n-2) + \tfrac{3}{2}n^2 + \tfrac{11}{2}n - 1$$
$$= 2n^2 + 4n.$$

29. But all these numbers are subject to diminution by as many units as there are algebraically independent relations among the invariants, which do not occur merely through algebraical forms, but arise through intrinsic relations associated with the general theory of surfaces. One such relation, being GAUSS's equation, has already (§ 23) been mentioned; so that the number $2n^2 + 4n$ would certainly be diminished by unity. It might happen that certain other combinations of the fundamental magnitudes of the various orders could be expressed in terms of E, F, G and their derivatives, the combinations being invariants of the set of binary forms, and the expressions in terms of E, F, G, and their derivatives being invariants of deformation. Each such relation would diminish the number $2n^2 + 4n$ by a single unit.

So far as I am aware, GAUSS's equation is the only relation of the type indicated which has already been established; but there is reason (§ 56) for surmising that other relations of that type do actually subsist.

PART II.

GEOMETRIC SIGNIFICANCE OF THE INVARIANTS.

30. The algebraically complete aggregate of the invariants of a given surface and of any two curves drawn upon it has been proved to be determinable by the development of LIE's method, as used by Professor ŽORAWSKI for the invariants of deformation. The actual determination of the members of those aggregates, which belong to the lowest orders, has been made. Each such invariant has a geometric significance,

and the significance of some of them is known; we proceed to consider this aspect of the invariants.

In dealing with binariants, several methods are possible. There is the symbolical method. There is the method dependent upon the use of canonical forms for the various functions; the complete expression of each binariant must be used through each operation; in the present instance, the canonical form would arise by taking ϕ and ψ as the independent variables on the surface.* There is the method that depends upon the characteristic property of binariants, by which the leading term alone, being sufficient to determine the binariant uniquely, is used to replace the binariant. The last of these methods will be used.

31. We denote by s an arc of the curve $\phi = 0$, so that d/ds implies differentiation along the curve; and we denote by d/dn differentiation in a direction on the surface perpendicular to the curve. Where no confusion will arise, we shall use x', x'', . . . in place of $\frac{dx}{ds}$, $\frac{d^2x}{ds^2}$, . . . ; and so with quantities other than x.

In constructing the fundamental quantities of order higher than the second, a normal section through the tangent to ϕ is drawn; successive derivatives of the curvature of this section at the point are constructed, and the values of the second derivatives of x and y are those connected with the geodesic property at the point.† Accordingly, it is effectively the geodesic tangent to ϕ that is drawn; we shall denote by t an arc of this geodesic, so that d/dt implies differentiation along the geodesic. As the curve and the geodesic touch one another, we have

$$\frac{du}{ds} = \frac{du}{dt},$$

when the quantities relate to tangential properties only; but

$$\frac{du}{ds} - \frac{du}{dt}$$

is not zero when the quantities relate to contact of higher orders. Thus

$$\frac{dx}{ds} = \frac{dx}{dt}, \qquad \frac{dy}{ds} = \frac{dy}{dt};$$

but

$$\frac{d}{ds}\left(\frac{1}{\rho'}\right) - \frac{d}{dt}\left(\frac{1}{\rho'}\right),$$

where $\frac{1}{\rho'}$ is the circular curvature of the geodesic tangent, is not zero.

* This method is used by DARBOUX, 'Théorie générale des Surfaces,' vol. 3, p. 203.

† See the paper quoted in § 4.

The Independent Magnitudes connected with the Curve.

32. Various magnitudes connected with the curve $\phi = 0$ are required; we take

$\frac{1}{\rho}$ = its circular curvature,

$\frac{1}{\tau}$ = its curvature of torsion,

$\left.\begin{array}{l}\frac{1}{\rho'} = \text{the circular curvature} \\ \frac{1}{\tau'} = \text{the curvature of torsion}\end{array}\right\}$ of the geodesic tangent,

$\frac{1}{\rho''}$ = its geodesic curvature,

R = the radius of the osculating sphere,

ϖ = the angle between the normal to the surface and the principal normal of $\phi = 0$, and

$$B = \frac{d\phi}{dn},$$

where dn is the normal distance at the point of $\phi = 0$ from the curve $\phi + d\phi = 0$. Further, we write

$$A = Lx'^2 + 2Mx'y' + Ny'^2,$$

$$W = \frac{1}{V}\begin{vmatrix} Ex' + Fy', & Fx' + Gy' \\ Lx' + My', & Mx' + Ny' \end{vmatrix},$$

$$N = \frac{1}{V}\begin{vmatrix} Ex' + Fy', & mx'^2 + 2m'x'y' + m''y'^2 + Ex'' + Fy'' \\ Fx' + Gy', & nx'^2 + 2n'x'y' + n''y'^2 + Fx'' + Gy'' \end{vmatrix},$$

with the customary notation for m, m', m'', n, n', n''; then $A = 0$ gives the asymptotic lines, $W = 0$ gives the lines of curvature, $N = 0$ gives geodesic lines. Moreover,

$$W^2 = AH - A^2 - K,$$

where H and K are the mean curvature and the specific curvature of the surface at the point, viz.,

$$H = \frac{1}{\rho_1} + \frac{1}{\rho_2}, \qquad K = \frac{1}{\rho_1\rho_2},$$

ρ_1 and ρ_2 being the principal radii of curvature. We have the relations*

* See STAHL und KOMMERELL, 'Die Grundformeln der allgemeinen Flächentheorie,' § 14, for some of them.

$$\frac{1}{\rho'} = \frac{\cos\varpi}{\rho} = \mathrm{A},$$

$$\frac{1}{\rho''} = \frac{\sin\varpi}{\rho} = \mathrm{D},$$

$$\frac{1}{\tau} = \frac{d\varpi}{ds} - \mathrm{W},$$

$$\frac{1}{\tau'} = -\mathrm{W},$$

$$\mathrm{R}^2 = \rho^2 + \tau^2\left(\frac{d\rho}{ds}\right)^2,$$

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) = (\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}\between x', y')^3,$$

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) = (\alpha, \beta, \gamma, \delta, \epsilon\between x', y')^4;$$

in the last two equations x' and y' are used in place of dx/dt and dy/dt, to which they are equal respectively. The relation

$$\mathrm{W}^2 = \mathrm{AH} - \mathrm{A}^2 - \mathrm{K}$$

at once gives

$$\frac{1}{\tau'^2} = \left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\rho_2}\right);$$

and we also have

$$\frac{1}{\tau} - \frac{1}{\tau'} = \frac{1}{\rho'^2 + \rho''^2}\left(\rho''\frac{d\rho'}{ds} - \rho'\frac{d\rho''}{ds}\right),$$

$$\frac{1}{\rho^2} = \frac{1}{\rho'^2} + \frac{1}{\rho''^2},$$

so that τ, ρ, and R are expressible in terms of ρ', ρ'', τ' and of their derivatives.

The Values of $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dx}{dn}, \frac{dy}{dn}$.

33. As regards $\frac{dx}{ds}(=x')$ and $\frac{dy}{ds}(=y')$, we have

$$\phi_{10}x' + \phi_{01}y' = 0,$$

$$\mathrm{E}x'^2 + 2\mathrm{F}x'y' + \mathrm{G}y'^2 = 1;$$

and therefore

$$x' = \frac{\phi_{01}}{\sqrt{w_2}}, \quad y' = -\frac{\phi_{10}}{\sqrt{w_2}}.$$

Next, differentiating along a direction in the surface that is perpendicular to the tangent to ϕ, we take the direction determined by dx/dn and dy/dn as being perpendicular to the direction determined by x' and y'; hence

$$Ex'\frac{dx}{dn} + F\left(y'\frac{dx}{dn} + x'\frac{dy}{dn}\right) + Gy'\frac{dy}{dn} = 0.$$

Moreover

$$E\left(\frac{dx}{dn}\right)^2 + 2F\frac{dx}{dn}\frac{dy}{dn} + G\left(\frac{dy}{dn}\right)^2 = 1;$$

and therefore

$$\frac{dx}{dn} = -\frac{1}{V}(Fx' + Gy') = \frac{1}{V\sqrt{w_2}}(-F\phi_{01} + G\phi_{10}),$$

$$\frac{dy}{dn} = \frac{1}{V}(Ex' + Fy') = \frac{1}{V\sqrt{w_2}}(E\phi_{01} - F\phi_{10}),$$

the quantities in the brackets in the last expressions being the quantities r and s of § 15.

Identification of the Simplest Invariants.

34. Using these results, we can at once obtain the interpretation of several of the invariants. We have

$$\begin{aligned} B &= \frac{d\phi}{dn} \\ &= \phi_{10}\frac{dx}{dn} + \phi_{01}\frac{dy}{dn} \\ &= \frac{\sqrt{w_2}}{V}, \end{aligned}$$

and therefore

$$\frac{w_2}{V^2} = B^2.$$

Again,

$$\begin{aligned} A &= (L, M, N\between x', y')^2 \\ &= \frac{1}{w_2}(L, M, N\between \phi_{01}, -\phi_{10})^2 \\ &= \frac{w'_2}{w_2}; \end{aligned}$$

and therefore by the relations in § 32, we have

$$\frac{w'_2}{V^2} = \frac{B^2}{\rho'}.$$

Also

$$\begin{aligned} W &= \frac{1}{V}\{(EM - FL)x'^2 + \ldots\} \\ &= \frac{1}{Vw_2}\{(EM - FL)\phi_{01}^2 + \ldots\} \\ &= \frac{J(w_2, w'_2)}{Vw_2}; \end{aligned}$$

and therefore, by the relations in § 32, we have

$$\frac{J(w_2, w'_2)}{V^3} = -\frac{B^2}{\tau'}.$$

The result of § 24 gives

$$D = -\tfrac{1}{2}\frac{w''_2}{Vw_2^{\frac{3}{2}}},$$

and therefore, also by the relations in § 32, we have

$$\frac{w''_2}{V^4} = -2\frac{B^3}{\rho''}.$$

Also

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) = (P, Q, R, S\between x', y')^3$$

$$= \frac{w_3}{w_2^{\frac{3}{2}}},$$

so that

$$\frac{w_3}{V^3} = B^3\frac{d}{dt}\left(\frac{1}{\rho'}\right);$$

and

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) = (\alpha, \beta, \gamma, \delta, \epsilon\between x', y')^4$$

$$= \frac{w_4}{w_2^{2}},$$

so that

$$\frac{w_4}{V^4} = B^4\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right).$$

35. Certain invariants occur as belonging to the surface, independent of all curves such as $\phi = 0$. Of these, the most important is ∇V^{-4}; its value is given by

$$\frac{\nabla}{V^4} = \frac{4}{\rho_1\rho_2} = 4K.$$

But, as is well known, we also have

$$\frac{LN - M^2}{EG - F^2} = \frac{1}{\rho_1\rho_2} = K,$$

so that we have

$$\nabla = 4V^2 H(w'_2).$$

This is a relation among the differential invariants, and it is due to the intrinsic nature of the quantities E, F, G, L, M, N; accordingly, the number of algebraically independent invariants up to the present order must (§ 23) be diminished by unity, on account of the preceding relation.

It was noted, in § 21, that $H(w'_2)$ and $I(w_2, w'_2)$ are alternatives in a complete system, when w_2, $H(w_2)$, w'_2, $J(w_2, w'_2)$ are retained; as a matter of fact, the relation

$$J^2(w_2, w'_2) = I(w_2, w'_2)\, w_2 w'_2 - H(w_2)\, w'^2_2 - H(w'_2) w_2^2$$

subsists. Now the significance of $I(w_2, w'_2)$ is known: we have

$$\frac{I(w_2, w'_2)}{V^2} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Accordingly, we substitute the values that have been obtained, and we find

$$\frac{1}{\tau'^2} = \frac{1}{\rho'}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \frac{1}{\rho'^2} - \frac{1}{\rho_1\rho_2}$$

$$= \left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\rho_2}\right),$$

again the well-known relation giving the torsion of a geodesic at any point. This torsion vanishes when the geodesic is a tangent to a line of curvature.

Interpretation of the Remaining Invariants Associated with w_2, w'_2, w''_2, w_3.

36. We require the derivatives of w_2, w'_2, w''_2 with respect to the arc; for this purpose we shall use the property already quoted (§ 30)—that a binariant is uniquely determined by its leading term which, in the present instance, is the term involving the highest power of ϕ_{01}. Writing generally

$$u = f\phi^2_{01} - 2g\phi_{01}\phi_{10} + h\phi^2_{10},$$

we have

$$\frac{du}{ds} = 2f\phi_{01}(\phi_{11}x' + \phi_{02}y') - 2g\phi_{01}(\phi_{20}x' + \phi_{11}y') + \ldots$$

$$+ \phi_{01}^2(f_{10}x' + f_{01}y') + \ldots,$$

so that

$$\sqrt{w_2}\frac{du}{ds} = \phi_{01}^2(2f\phi_{11} - 2g\phi_{20}) + \ldots$$

$$+ \phi_{01}^3 f_{10} + \ldots$$

$$= \frac{1}{V^2}\{(fb - ga)\phi_{01}^2 + \ldots\}$$

$$+ \frac{1}{V^2}[\{f(EG_{10} - FE_{01}) + g(EE_{01} - 2EF_{10} + FE_{10}) + V^2 f_{10}\}\phi_{01}^3 + \ldots].$$

Firstly, let $f, g, h = E, F, G$, so that u becomes w_2; then, on reduction, we find

$$\sqrt{w_2}\frac{dw_2}{ds} = \frac{J(w_2, w''_2)}{V^2} + \frac{w_2}{V^2}\{(EG_{10} - 2FF_{10} + GE_{10})\phi_{01} + \ldots\},$$

and consequently

$$\sqrt{w_2}\frac{d}{ds}\left(\frac{w_2}{V^2}\right) = \frac{J(w_2, w''_2)}{V^4}.$$

Secondly, let $f, g, h = L, M, N$, so that u becomes w'_2; then, on reduction, we find

$$\sqrt{w_2}\frac{dw'_2}{ds} = w_3 + \frac{1}{V^2 w_2}\{w'_2 J(w_2, w''_2) - w''_2 J(w_2, w'_2)\}$$
$$+ \frac{w'_2}{V^2}\{(EG_{10} - 2FF_{10} + GE_{10})\phi_{01} + \ldots\},$$

and consequently

$$\sqrt{w_2}\frac{d}{ds}\left(\frac{w'_2}{V^2}\right) = \frac{w_3}{V^2} + \frac{1}{V^2 w_2}\{w'_2 J(w_2, w''_2) - w''_2 J(w_2, w'_2)\}.$$

Thirdly, let $f, g, h = a, b, c$, so that u becomes w''_2; then, on reduction, we find

$$\sqrt{w_2}\frac{dw''_2}{ds} = \frac{1}{2}\frac{w'_3}{V^2} + \frac{2w''_2}{V^2}\{(EG_{10} - 2FF_{10} + GE_{10})\phi_{01} + \ldots)\},$$

and consequently

$$\sqrt{w_2}\frac{d}{ds}\left(\frac{w''_2}{V^4}\right) = \frac{1}{2}\frac{w'_3}{V^6}.$$

The first of these gives

$$\frac{J(w_2, w''_2)}{V^5} = 2B^2\frac{dB}{ds},$$

and the third of them, taking account of the value of w''_2 which has already been obtained, gives

$$\frac{w'_3}{V^7} = -4B\frac{d}{ds}\left(\frac{B^3}{\rho''}\right).$$

The second of them can also be used to identify $J(w_2, w''_2)$, because all the other quantities occurring in the relation have been identified; the value is

$$\frac{J(w_2, w''_2)}{V^5} = B\rho'\frac{d}{ds}\left(\frac{B^2}{\rho'}\right) - B^3\rho'\left\{\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{2}{\rho''\tau'}\right\}.$$

Substituting the earlier value on the left-hand side, we have (after a slight reduction)

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{d}{ds}\left(\frac{1}{\rho'}\right) = \frac{2}{\rho''\tau'},$$

being an illustration of the remark in § 31, and showing that in general the rate of change of the curvature of a normal section is not the same along the curve $\phi = 0$ and the geodesic, both of which touch that section. The result can also be written in the form

$$\frac{dA}{ds} = \Upsilon + 2DW,\ \frac{dA}{dt} = \Upsilon,$$

with the earlier significance for A, D, W, and Υ is given by

$$\Upsilon = (P, Q, R, S \between x', y')^3;$$

and another form is

$$\frac{d}{ds}\left(\frac{w'_2}{w_2}\right) = \frac{w_3}{w_2^{\frac{3}{2}}} - \frac{w''_2 J(w_2, w'_2)}{V^2 w_2^{\frac{3}{2}}}.$$

37. We require derivatives of some of the binary quadratics with respect to an arc in the surface normal to the curve $\phi = 0$; for this purpose, we proceed as in § 36. We take

$$u = f\phi_{01}^2 - 2g\phi_{01}\phi_{10} + h\phi_{10}^2,$$

and we have

$$\begin{aligned} V\sqrt{w_2}\frac{du}{dn} = {} & 2f\phi_{01}\{\phi_{11}(-F\phi_{01} + \ldots) + \phi_{02}(E\phi_{01} + \ldots)\} \\ & - 2g\phi_{01}\{\phi_{20}(-F\phi_{01} + \ldots) + \phi_{11}(E\phi_{01} + \ldots)\} \\ & + \phi_{01}^2\{f_{10}(-F\phi_{01} + \ldots) + f_{01}(E\phi_{01} + \ldots)\} + \ldots\ldots; \end{aligned}$$

and so, after some transformation and reduction, we find

$$\begin{aligned} V^3\sqrt{w_2}\frac{du}{dn} = {} & \phi_{01}^2\{f(Ec - Fb) - g(Eb - Fa)\} + \ldots \\ & + \phi_{01}^3\{fE(EG_{01} + FG_{10} - 2FF_{01}) - (fF + gE)(EG_{10} - FE_{01}) \\ & + gF(-EE_{01} + 2EF_{10} - FE_{10}) + V^2(-Ff_{10} + Ef_{01})\} + \ldots\ldots \end{aligned}$$

Firstly, let $f, g, h = E, F, G$, so that u becomes w_2. The coefficient of the first term in the earlier aggregate is

$$= E^2c - 2EFb + F^2a$$

$$= E(Ec - 2Fb + Ga) - V^2a,$$

and therefore that aggregate is

$$= w_2 I(w_2, w''_2) - V^2 w''_2.$$

The coefficient of the first term in the later aggregate is

$$E\{-F(EG_{10} - 2FF_{10} + GE_{10}) + E(EG_{01} - 2FF_{01} + GE_{01})\},$$

and therefore that aggregate is

$$= w_2 V\sqrt{w_2}\frac{dV^2}{dn}.$$

Consequently

$$V^3\sqrt{w_2}\frac{dw_2}{dn} = w_2 I(w_2, w''_2) - V^2 w''_2 + w_2 V\sqrt{w_2}\frac{dV^2}{dn},$$

and therefore

$$V^5\sqrt{w_2}\frac{d}{dn}\left(\frac{w_2}{V^2}\right) = w_2 I(w_2, w''_2) - V^2 w''_2.$$

Inserting the values of the invariants that are already known, we have

$$\mathrm{B}\,.\,2\mathrm{B}\frac{d\mathrm{B}}{dn} = \mathrm{B}^2\frac{\mathrm{I}(w_2, w''_2)}{\mathrm{V}^4} + 2\frac{\mathrm{B}^3}{\rho''},$$

and therefore

$$\frac{\mathrm{I}(w_2, w''_2)}{\mathrm{V}^4} = 2\left(\frac{d\mathrm{B}}{dn} - \frac{\mathrm{B}}{\rho''}\right).$$

Secondly, let $f, g, h = \mathrm{L, M, N}$, so that u becomes w'_2. Proceeding in the same way, we find

$$\mathrm{V}^5\sqrt{w_2}\frac{d}{dn}\left(\frac{w'_2}{\mathrm{V}^2}\right) = w'_2\mathrm{I}(w_2, w''_2) + \mathrm{V}^2\mathrm{J}(w_2, w_3) - \frac{1}{w_2}\{\mathrm{V}^2w'_2w''_2 + \mathrm{J}(w_2, w'_2)\,\mathrm{J}(w_2, w''_2)\}.$$

Inserting the values of those invariants which have already been obtained, we have (after a little reduction)

$$\frac{\mathrm{J}(w_2, w_3)}{\mathrm{V}^4} = \mathrm{B}^3\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2\mathrm{B}^2}{\tau'}\frac{d\mathrm{B}}{ds}.$$

Thirdly, let $f, g, h = a, b, c$, so that u becomes w''_2. Proceeding in the same way as for w_2, we find

$$\mathrm{V}^7\sqrt{w_2}\frac{d}{dn}\left(\frac{w''_2}{\mathrm{V}^4}\right) = w_2\mathrm{H}(w''_2) - \tfrac{2}{3}w_2^2\nabla + \tfrac{1}{2}\mathrm{J}(w_2, w'_3).$$

Now we have retained $\mathrm{I}(w_2, w''_2)$ in our aggregate, in place of $\mathrm{H}(w''_2)$, so that the latter must be removed from the foregoing expression: as the relation

$$\mathrm{J}^2(w_2, w''_2) = w_2w''_2\mathrm{I}(w_2, w''_2) - w_2^2\mathrm{H}(w''_2) - w''^2_2\mathrm{V}^2$$

holds, we have

$$\mathrm{V}^7w_2^{\frac{3}{2}}\frac{d}{dn}\left(\frac{w''_2}{\mathrm{V}^4}\right) = \tfrac{1}{2}w_2\mathrm{J}(w_2, w'_3) - \tfrac{2}{3}w_2^3\nabla + w_2w''_2\mathrm{I}(w_2, w''_2) - w''^2_2\mathrm{V}^2 - \mathrm{J}^2(w_2, w''_2).$$

Inserting the values of those invariants which have already been obtained, and reducing the equation, we ultimately have

$$\frac{\mathrm{J}(w_2, w'_3)}{\mathrm{V}^8} = -4\mathrm{B}^3\frac{d}{dn}\left(\frac{\mathrm{B}}{\rho''}\right) + 8\mathrm{B}^2\left(\frac{d\mathrm{B}}{ds}\right)^2 + \tfrac{16}{3}\mathrm{B}^4\mathrm{K}.$$

It may be noted, in passing, that the above equation, which gives the relation between $\mathrm{I}(w_2, w''_2)$ and $\mathrm{H}(w''_2)$, leads to the expression for $\mathrm{H}(w''_2)$ in the form

$$\frac{\mathrm{H}(w''_2)}{\mathrm{V}^6} = -4\frac{\mathrm{B}}{\rho''}\frac{d\mathrm{B}}{dn} - 4\left(\frac{d\mathrm{B}}{ds}\right)^2.$$

38. Again, it is known that

$$\left.\begin{aligned} \mathrm{V}^2\frac{\partial\mathrm{H}}{\partial x} &= \mathrm{GP} - 2\mathrm{FQ} + \mathrm{ER}, & \mathrm{V}^2\frac{\partial\mathrm{K}}{\partial x} &= \mathrm{NP} - 2\mathrm{MQ} + \mathrm{LR} \\ \mathrm{V}^2\frac{\partial\mathrm{H}}{\partial y} &= \mathrm{GQ} - 2\mathrm{FR} + \mathrm{ES}, & \mathrm{V}^2\frac{\partial\mathrm{K}}{\partial y} &= \mathrm{NQ} - 2\mathrm{MR} + \mathrm{LS} \end{aligned}\right\};$$

and therefore

$$\sqrt{w_2}\,V^2\frac{dH}{ds} = (GP - 2FQ + ER)\,\phi_{01} + (GQ - 2FR + ES)\,(-\phi_{10})$$
$$= \mathfrak{a}_1,$$

say, where $\mathfrak{a}_1$ is a covariant of the system with index easily seen to be equal to 3. Now it is easy to verify that

$$(EQ - FP)^2 = (GP - 2FQ + ER)\,EP - (EG - F^2)\,P^2 - (PR - Q^2)\,E^2,$$

and therefore that

$$J^2(w_2, w_3) = w_2 w_3 \mathfrak{a}_1 - V^2 w_3^2 - w_2^2 H(w_3).$$

Consequently $\mathfrak{a}_1$ is expressible in terms of the members of the system; when the expression is substituted above, the result enables us to obtain the value of $H(w_3)\,V^{-4}$. But it is simpler to modify the original system of concomitants in § 21: we can replace $H(w_3)$ in that aggregate by $\mathfrak{a}_1$, and the modified aggregate still is complete. For the significance of $\mathfrak{a}_1$, we have

$$\frac{\mathfrak{a}_1}{V^3} = B\frac{dH}{ds}.$$

Further, we have

$$\sqrt{w_2}\,V^3\frac{dH}{dn} = (GP - 2FQ + ER)\,(-F\phi_{01} + G\phi_{10})$$
$$+ (GQ - 2FR + ES)\,(E\phi_{01} - F\phi_{10})$$
$$= \mathfrak{a}_2,$$

say, where $\mathfrak{a}_2$ is a covariant of the system with index easily seen to be 4. It is easy to verify that

$$E^3(P^2S - 3PQR + 2Q^3) - EP^2\{E^2S - 3EFR + (EG + 2F^2)\,Q - FGP\}$$
$$= -3EP\,(EQ - FP)\,(GP - 2FQ + ER) + 2\,(EQ - FP)^3 + 2V^2P^2\,(EQ - FP),$$

and therefore

$$w_2^3\Phi(w_3) - w_2 w_3^2 \mathfrak{a}_2 + 3w_2 w_3 \mathfrak{a}_1 J(w_2, w_3) - 2J^3(w_2, w_3) - 2V^2 w_3^2 J(w_2, w_3) = 0.$$

Consequently $\mathfrak{a}_2$ is expressible in terms of members of the system; when the expression is substituted above, the result enables us to obtain the value of $\Phi(w_3)\,V^{-6}$. But, as in the last case, it is simpler to modify the original system of concomitants in § 21: we can replace $\Phi(w_3)$ in that aggregate by $\mathfrak{a}_2$, and the modified aggregate still is complete. For the significance of $\mathfrak{a}_2$, we have

$$\frac{\mathfrak{a}_2}{V^4} = B\frac{dH}{dn}.$$

An Aggregate for the Lowest Orders of Derivatives.

39. It may be remarked (and it is easy to verify the statement) that, if we desire an algebraically complete aggregate of invariants, involving derivatives of ϕ alone up to order 2 at the utmost, and derivatives of E, F, G up to order 1, and the fundamental magnitudes of the first three orders and no other quantities, such an aggregate is composed of

$$\frac{w_2}{V^2},\ \frac{w'_2}{V^2},\ \frac{H(w'_2)}{V^2} \text{ or } \frac{I(w_2, w'_2)}{V^2};\ \frac{w''_2}{V^4};\ \frac{H(w''_2)}{V^6} \text{ or } \frac{I(w_2, w''_2)}{V^4},$$

$$\frac{w_3}{V^3},\ \frac{H(w_3)}{V^4} \text{ or } \frac{\mathfrak{a}_1}{V^3},\ \frac{\Phi(w_3)}{V^6} \text{ or } \frac{\Delta(w_3)}{V^6} \text{ or } \frac{\mathfrak{a}_2}{V^4},\ \frac{J(w_2, w'_2)}{V^3},$$

$$\frac{J(w_2, w''_2)}{V^5}, \text{ and } \frac{J(w_2, w_3)}{V^4}.$$

Every other invariant of the surface involving only the same quantities that occur in these invariants can be expressed algebraically in terms of the members of this aggregate. The geometrical significance of each of the members has been obtained; if, therefore, the geometrical significance of any additional invariant is known, the algebraic equation expressing the invariant in terms of the retained aggregate will express a property of the surface and the curve. Such additional invariants are provided by $\frac{dK}{ds}$ and $\frac{dK}{dn}$; they should accordingly lead to properties of the surface and the curve.

Two New Relations.

40. We have

$$\sqrt{w_2}\, V^2 \frac{dK}{ds} = (NP - 2MQ + LR)\, \phi_{01} + (NQ - 2MR + LS)\,(-\phi_{10}).$$

But

$$(LQ - MP)^2 = (NP - 2MQ + LR)\, LP - (LN - M^2)\, P^2 - L^2 (PR - Q^2),$$

and

$$LQ - MP = \frac{1}{E}\{L(EQ - FP) - P(EM - FL)\};$$

hence, taking account of the relation among the leading terms of the various concomitants, we have

$$\frac{\{w'_2 J(w_2, w_3) - w_3 J(w_2, w'_2)\}^2}{w_2^2} = \sqrt{w_2}\, w'_2 w_3 V^2 \frac{dK}{ds} - w_3^2 H(w'_2) - w'^2_2 H(w_3).$$

Consequently

$$\sqrt{w_2}\,V^2\frac{dK}{ds} = \frac{w'_2}{w_2^2 w_3}\{J^2(w_2, w_3) + w_2^2 H(w_3)\} - \frac{2}{w_2^2} J(w_2, w_3) J(w_2, w'_2)$$
$$+ \frac{w_3}{w_2^2 w'_2}\{J^2(w_2, w'_2) + w_2^2 H(w'_2)\}$$
$$= \frac{w'_2}{w_2^2 w_3}(w_2 w_3 \mathfrak{a}_1 - V^2 w_3^2) - \frac{2}{w_2^2} J(w_2, w_3) J(w_2, w'_2)$$
$$+ \frac{w_3}{w_2^2 w'_2}\{w_2 w'_2 I(w_2, w'_2) - V^2 w'^2_2\},$$

so that $\frac{dK}{ds}$ is expressed in terms of the members of the retained aggregate. Substituting the values of the invariants in the equation and dividing out by V^3B after substitution, we find

$$\frac{dK}{ds} - \frac{1}{\rho'}\frac{dH}{ds} = \left(H - \frac{2}{\rho'}\right)\frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{2}{\tau'}\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{4}{B\tau'^2}\frac{dB}{ds},$$

a property which can be changed also into other forms, *e.g.*, by using the relation in § 36, which expresses $\frac{d}{dt}\left(\frac{1}{\rho'}\right)$ in terms of $\frac{d}{ds}\left(\frac{1}{\rho'}\right)$ and other magnitudes. Effecting this change, and substituting for $\frac{dK}{ds}$, $\frac{dH}{ds}$, $\frac{d}{ds}\left(\frac{1}{\rho'}\right)$, their values in terms of derivatives of ρ_1, ρ_2, ρ', we can express the relation in the form

$$-\frac{d}{ds}\left[\left(\frac{1}{\rho_1} - \frac{1}{\rho'}\right)\left(\frac{1}{\rho'} - \frac{1}{\rho_2}\right)\right] = \frac{2}{\tau'}\left\{\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{2}{\rho'}\right)\frac{1}{\rho''} + \frac{d}{dn}\left(\frac{1}{\rho'}\right)\right\} - \frac{4}{B\tau'^2}\frac{dB}{ds},$$

and therefore

$$\frac{1}{\tau'^2}\frac{d\tau'}{ds} = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{2}{\rho'}\right)\frac{1}{\rho''} + \frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{B\tau'}\frac{dB}{ds},$$

or, what is the same thing,

$$-\frac{d}{ds}\left(\frac{1}{\tau'}\right) = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{2}{\rho'}\right)\frac{1}{\rho''} + \frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{B\tau'}\frac{dB}{ds}$$
$$= \frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(H - \frac{2}{\rho'}\right)\frac{1}{\rho''} - \frac{2}{\tau' B}\frac{dB}{ds}.$$

Proceeding to construct the other invariant that was suggested in § 39, we have

$$\sqrt{w_2}\,V^3\frac{dK}{dn} = (NP - 2MQ + LR)(-F\phi_{01} + G\phi_{10})$$
$$+ (NQ - 2MR + LS)(E\phi_{01} - F\phi_{10})$$

Let u denote the leading coefficient on the right-hand side, so that

$$u = SEL - R(2EM + FL) + Q(EN + 2FM) - PFN;$$

and let α_1, α_2 denote the leading coefficients of $\mathfrak{a}_1$, $\mathfrak{a}_2$ respectively, so that

$$\alpha_1 = \mathrm{GP} - 2\mathrm{FQ} + \mathrm{ER},$$
$$\alpha_2 = \mathrm{E^2S} - 3\mathrm{EFR} + (\mathrm{EG} + 2\mathrm{F^2})\,\mathrm{Q} - \mathrm{FGP}.$$

Then it is not difficult to establish the identity

$$\mathrm{E}u = \mathrm{L}\alpha_2 - 2\,(\mathrm{EM} - \mathrm{FL})\,\alpha_1 + (\mathrm{EN} - 2\mathrm{FM} + \mathrm{GL})\,(\mathrm{EQ} - \mathrm{FP})$$
$$- \frac{2}{\mathrm{E}}\,(\mathrm{EG} - \mathrm{F^2})\,\{\mathrm{L}\,(\mathrm{EQ} - \mathrm{FP}) - \mathrm{P}\,(\mathrm{EM} - \mathrm{FL})\}.$$

Noting that all the quantities on the right-hand side are leading coefficients of covariants, we change the identity into a relation among covariants; and the result, on division throughout by w_2, is

$$\sqrt{w_2}\,\mathrm{V}^3\frac{d\mathrm{K}}{dn} = \frac{w'_2}{w_2}\,\mathfrak{a}_2 - \frac{2}{w_2}\,\mathrm{J}\,(w_2 w'_2)\,\mathfrak{a}_1 + \frac{1}{w_2}\,\mathrm{I}\,(w_2 w'_2)\,\mathrm{J}\,(w_2 w_3)$$
$$- 2\mathrm{V}^2\frac{w'_2}{w_2^{\,2}}\,\mathrm{J}\,(w_2,\, w_3) + 2\mathrm{V}^2\frac{w_3}{w_2^{\,2}}\,\mathrm{J}\,(w_2,\, w'_2),$$

so that $d\mathrm{K}/dn$ is expressed in terms of the members of the retained aggregate. Substituting the values of the invariants in the equation and dividing out by $\mathrm{V}^4\mathrm{B}$ after substitution, we find

$$\frac{d\mathrm{K}}{dn} - \frac{1}{\rho'}\frac{d\mathrm{H}}{dn} = \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{\tau'}\frac{d}{dt}\left(\frac{1}{\rho'}\right) + \frac{4}{\mathrm{B}\rho'\tau'}\frac{d\mathrm{B}}{ds} + \frac{2}{\tau'}\frac{d\mathrm{H}}{ds} - \frac{2\mathrm{H}}{\mathrm{B}\tau'}\frac{d\mathrm{B}}{ds}.$$

Effecting the same transformation as before, by taking

$$\frac{d}{dn}\left(\mathrm{K} - \frac{\mathrm{H}}{\rho'} + \frac{1}{\rho'^2}\right) = \frac{d}{dn}\left(-\frac{1}{\tau'^2}\right),$$

we find

$$\frac{d}{dn}\left(\frac{1}{\tau'}\right) = \frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{2}{\rho'\mathrm{B}}\frac{d\mathrm{B}}{ds} - \frac{d\mathrm{H}}{ds} + \frac{\mathrm{H}}{\mathrm{B}}\frac{d\mathrm{B}}{ds}$$
$$= \frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\rho''\tau'} + \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds} - \frac{d\mathrm{H}}{ds}.$$

Identification of the remaining Invariants obtained in § 23, with some Modifications of the System.

41. We proceed now to the identification of the invariants of the next higher order of derivatives; these involve derivatives of ϕ of the third order, derivatives of ψ of the third order, and the fundamental magnitudes of the fourth order. The method used is similar to that adopted in the preceding sections; we form derivatives, with

regard to s and to n, of the invariants already interpreted, identify the new forms by means of some member of the complete aggregate, and thus we obtain the interpretation of that member. Accordingly, we shall usually state the results without the calculations, the laborious character of which is greatly lightened by using the leading terms of the covariants only.

We have

$$\sqrt{w_2}\frac{d}{ds}\left(\frac{w_3}{V^3}\right) = \frac{w_4}{V^3} + \frac{3}{2V^5 w_2}\{w_3 J(w_2, w''_2) - w''_2 J(w_2, w_3)\}.$$

Inserting the values of the invariants which occur in this equation, and using the relation

$$\frac{d}{dt}\left(\frac{1}{\rho'}\right) = \frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\rho''\tau},$$

obtained in § 36, we have

$$\mathrm{B}\frac{d}{ds}\left\{\mathrm{B}^3\frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2\mathrm{B}^3}{\rho''\tau'}\right\}$$

$$= \mathrm{B}^4\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) + 3\mathrm{B}^3\frac{d\mathrm{B}}{ds}\frac{d}{ds}\left(\frac{1}{\rho'}\right) + 3\frac{\mathrm{B}^4}{\rho''}\frac{d}{dn}\left(\frac{1}{\rho'}\right),$$

and this easily leads to the relation

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) - \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right) = \frac{2}{\mathrm{B}^3}\frac{d}{ds}\left(\frac{\mathrm{B}^3}{\rho''\tau'}\right) - \frac{3}{\rho''}\frac{d}{dn}\left(\frac{1}{\rho'}\right)$$

$$= 2\frac{d}{ds}\left(\frac{1}{\rho''\tau'}\right) + \frac{3}{\rho''}\frac{d}{ds}\left(\frac{1}{\tau'}\right) + \frac{3}{\rho''^2}\left(\mathrm{H} - \frac{2}{\rho'}\right),$$

on using the expression for $\frac{d}{ds}\left(\frac{1}{\tau'}\right)$ obtained in § 40. The fact that the value of

$$\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right) - \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right)$$

is different from zero is another illustration of the remark in § 36.

We also have

$$V^4\sqrt{w_2}\frac{d}{dn}\left(\frac{w_3}{V^3}\right) = J(w_2, w_4) - \tfrac{3}{2}\frac{\mathrm{H}(w'_2)}{V^2}w_2 J(w_2, w'_2)$$

$$+ \frac{3}{2V^2}\left\{w_3 I(w_2, w''_2) - \frac{1}{w_2}J(w_2, w''_2)\,J(w_2, w_3) - V^2\frac{w''_2\,w_3}{w_2}\right\};$$

when we substitute for the respective invariants and reduce, we obtain an expression for $J(w_2, w_4)$ in the form

$$\frac{J(w_2, w_4)}{\mathrm{B}^3 V^5} = \frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right) + 2\frac{d}{dn}\left(\frac{1}{\rho''\tau'}\right) - \tfrac{3}{2}\frac{\mathrm{K}}{\tau'} + \frac{3}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\left\{\frac{d}{dn}\left(\frac{1}{\rho'}\right) - \frac{2}{\mathrm{B}\tau'}\frac{d\mathrm{B}}{ds}\right\},$$

and the expression can be further modified by substituting the value of $\frac{1}{B}\frac{dB}{ds}$ given in § 40.

42. As the quantity H, the measure of the mean curvature of the surface at the point, has occurred in the invariants $\mathfrak{a}_1$ and $\mathfrak{a}_2$, and as the quantities $\frac{d^2H}{ds\,dn}$ and $\frac{d^2H}{dn\,ds}$ are not equal to one another, we construct the quantities

$$\frac{d}{ds}\left(\frac{\mathfrak{a}_1}{V^3}\right),\quad \frac{d}{dn}\left(\frac{\mathfrak{a}_1}{V^3}\right),\quad \frac{d}{ds}\left(\frac{\mathfrak{a}_2}{V^4}\right),\quad \frac{d}{dn}\left(\frac{\mathfrak{a}_2}{V^4}\right).$$

It will appear that, by means of the second and the third of these, we can obtain the value of $\frac{d^2H}{ds\,dn} - \frac{d^2H}{dn\,ds}$.

We have

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{\mathfrak{a}_1}{V^3}\right) = \frac{1}{V^3}\{(E\gamma - 2F\beta + G\alpha)\,\phi_{01}{}^2 + \ldots\}$$

$$+ \tfrac{1}{2}\frac{H(w'_2)}{V^5}\{I(w_2, w'_2)\,w_2 - 2V^2 w'_2\}$$

$$+ \frac{1}{2V^5 w^2}\{\mathfrak{a}_1 J(w_2, w''_2) - \mathfrak{a}_2 w''_2\}.$$

Let

$$\mathfrak{h}_1 = (E\gamma - 2F\beta + G\alpha)\,\phi_{01}{}^2 + \ldots;$$

then as

$$(E\beta - F\alpha)^2 = E\alpha(E\gamma - 2F\beta + G\alpha) - (\alpha\gamma - \beta^2)E^2 - (EG - F^2)\alpha^2,$$

we have

$$J^2(w_2, w_4) = w_2 w_4 \mathfrak{h}_1 - w_2{}^2 H(w_4) - V^2 w_4{}^2.$$

Hence the invariant $\mathfrak{h}_1$ is expressible in terms of the members of the system; when the expression is substituted above, the result enables us to obtain the value of $H(w_4)V^{-6}$. But it is simpler to modify the original system of concomitants in § 21; we can replace $H(w_4)$ in the aggregate by $\mathfrak{h}_1$, and the modified aggregate is still complete. The index of $\mathfrak{h}_1$ is manifestly 4.

When the various values are substituted, we find

$$\frac{\mathfrak{h}_1}{V^4} = B^2\left\{\frac{d^2H}{ds^2} - \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right) - \frac{1}{\rho''}\frac{dH}{dn}\right\}.$$

43. We have

$$V\sqrt{w_2}\,\frac{d}{dn}\left(\frac{\mathfrak{a}_1}{V^3}\right) = \frac{1}{V^3}[\{E(G\beta - 2F\gamma + E\delta) - F(G\alpha - 2F\beta + E\gamma)\}\,\phi_{01}{}^2 + \ldots]$$

$$+ \frac{1}{2V^5 w_2}[\{w_2 I(w_2, w''_2) - V^2 w''_2\}\,\mathfrak{a}_1 - J(w_2, w''_2)\,\mathfrak{a}_2]$$

$$- \frac{H(w'_2)}{V^3}J(w_2, w'_2).$$

Let

$$\mathfrak{h}_2 = \{E(G\beta - 2F\gamma + E\delta) - F(G\alpha - 2F\beta + E\gamma)\}\,\phi_{01}^2 + \ldots ;$$

then as

$$E^3(\alpha^2\delta - 3\alpha\beta\gamma + 2\beta^3) - E\alpha^2\{E^2\delta - 3EF\gamma + (EG + 2F^2)\beta - FG\alpha\}$$
$$= -3E\alpha(E\beta - F\alpha)(G\alpha - 2F\beta + E\gamma) + 2(E\beta - F\alpha)^3 + 2V^2\alpha^2(E\beta - F\alpha),$$

we have

$$w_2^3\Phi(w_4) - w_2 w_4^2\mathfrak{h}_2 + 3w_2 w_4\mathfrak{h}_1 J(w_2, w_4) - 2J^3(w_2, w_4) - 2V^2 w_4^2 J(w_2 w_4) = 0.$$

Consequently $\mathfrak{h}_2$ is expressible in terms of members of the system and of $\Phi(w_4)$; and $\Phi(w_4)$ is expressible in terms of w_4, $H(w_4)$, $I(w_4)$, $J(w_4)$. When the various expressions are substituted, we can modify the system of concomitants in § 21; we replace any one of them, say $I(w_4)$, by $\mathfrak{h}_2$, and the modified aggregate is still complete. The expression for $\frac{d}{dn}\left(\frac{\mathfrak{a}_1}{V^3}\right)$ then gives the significance of $\mathfrak{h}_2$, the index of which is 5; when the values of the invariants already interpreted are substituted, we find

$$\frac{\mathfrak{h}_2}{V^5} = B^2\left\{\frac{d^2H}{dn\,ds} - \frac{K}{\tau'} + \frac{1}{B}\frac{dB}{ds}\frac{dH}{dn}\right\}.$$

Similarly, we have

$$\sqrt{w_2}\,\frac{d}{ds}\left(\frac{\mathfrak{a}_2}{V^4}\right) = \frac{\mathfrak{h}_2}{V^4} - \frac{H(w'_2)}{V^4}J(w_2, w'_2) + \frac{1}{2V^2 w_2}\{J(w_2, w''_2)\,\mathfrak{a}_2 + V^2 w''_2\mathfrak{a}_1\};$$

and thus we obtain another expression for $\mathfrak{h}_2 V^{-5}$ in the form

$$\frac{\mathfrak{h}_2}{V^5} = B^2\left\{\frac{d^2H}{ds\,dn} - \frac{K}{\tau'} + \frac{1}{\rho''}\frac{dH}{ds}\right\}.$$

Comparing the two values thus obtained for $\mathfrak{h}_2$, we at once have an expression for $\frac{d^2H}{ds\,dn} - \frac{d^2H}{dn\,ds}$, given by

$$\frac{d^2H}{ds\,dn} - \frac{d^2H}{dn\,ds} = -\frac{1}{\rho''}\frac{dH}{ds} + \frac{1}{B}\frac{dB}{ds}\frac{dH}{dn}.$$

44. We proceed in the same way with $\frac{d}{dn}\left(\frac{\mathfrak{a}_2}{V^4}\right)$. To simplify the system we introduce a covariant $\mathfrak{h}_3$ of index 6, defined as

$$\mathfrak{h}_3 = \{E^2(G\gamma - 2F\delta + E\epsilon) - 2EF(G\beta - 2F\gamma + E\delta) + F^2(G\alpha - 2F\beta + E\gamma)\}\,\phi_{01}^2 + \ldots ;$$

this covariant is expressible in terms of w_4, $H(w_4)$, $I(w_4)$, $J(w_4)$, $J(w_2, w_4)$ and, as

$H(w_4)$ and $I(w_4)$ have already been replaced by $\mathfrak{h}_1$ and $\mathfrak{h}_2$, we replace $J(w_4)$ by $\mathfrak{h}_3$, leaving the modified aggregate still complete. Then we have

$$V\sqrt{w_2}\frac{d}{dn}\left(\frac{\mathfrak{a}_2}{V^4}\right) = \frac{\mathfrak{h}_3}{V^4} + Kw'_2 - \tfrac{1}{2}HKw_2 + \frac{\mathfrak{a}_2}{2V^6}I(w_2 w''_2)$$
$$+ \frac{1}{2V^4 w_2}\{\mathfrak{a}_1 J(w_2, w''_2) - w''_2\mathfrak{a}_2\},$$

which, after substitution of the known invariants and some reduction, leads to an expression for $\mathfrak{h}_3$ in the form

$$\frac{\mathfrak{h}_3}{V^6} = B^2\left\{\frac{d^2H}{dn^2} + \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right) - \frac{1}{B}\frac{dB}{ds}\frac{dH}{ds}\right\},$$

giving also the value of $\dfrac{d^2H}{dn^2}$ as an invariant.

45. The expressions for $\dfrac{d^2H}{ds^2}$, $\dfrac{d^2H}{ds\,dn}$, $\dfrac{d^2H}{dn\,ds}$, $\dfrac{d^2H}{dn^2}$ can be obtained in another way; it will be sufficiently illustrated by constructing the first of them. From the expressions for $V^2\dfrac{\partial H}{\partial x}$, $V^2\dfrac{\partial H}{\partial y}$ in § 38, we find the following by differentiation:

$$V^2H_{20} - \tfrac{1}{2}\frac{LN - M^2}{V^2}\{E(EN - 2FM + GL) - 2V^2L\}$$
$$= G\alpha - 2F\beta + E\gamma + PG\Gamma + Q(G\Delta - 2F\Gamma) + R(E\Gamma - 2F\Delta) + ES\Delta.$$

$$V^2H_{11} - \tfrac{1}{2}\frac{LN - M^2}{V^2}\{F(EN - 2FM + GL) - 2V^2M\}$$
$$= G\beta - 2F\gamma + E\delta + PG\Gamma' + Q(G\Delta' - 2F\Gamma') + R(E\Gamma' - 2F\Delta') + ES\Delta'.$$

$$V^2H_{02} - \tfrac{1}{2}\frac{LN - M^2}{V^2}\{G(EN - 2FM + GL) - 2V^2N\}$$
$$= G\gamma - 2F\delta + E\epsilon + PG\Gamma'' + Q(G\Delta'' - 2F\Gamma'') + R(E\Gamma'' - 2F\Delta'') + ES\Delta'',$$

where (§ 6)

$$\left.\begin{aligned} 2V^2\Gamma &= GE_{10} - F(2F_{10} - E_{01}) \\ 2V^2\Gamma' &= GE_{01} - FG_{10} \\ 2V^2\Gamma'' &= G(2F_{01} - G_{10}) - FG_{01} \end{aligned}\right\}, \qquad \left.\begin{aligned} 2V^2\Delta &= E(2F_{10} - E_{01}) - FE_{10} \\ 2V^2\Delta' &= EG_{10} - FE_{01} \\ 2V^2\Delta'' &= EG_{01} - F(2F_{01} - G_{10}) \end{aligned}\right\}.$$

Knowing the values of x' and y', we form $\dfrac{dx'}{dx}$, $\dfrac{dx'}{dy}$, $\dfrac{dy'}{dx}$, $\dfrac{dy'}{dy}$, and then we have

$$\frac{d^2x}{ds^2} = x'\frac{dx'}{dx} + y'\frac{dx'}{dy},$$

$$\frac{d^2y}{ds^2} = x'\frac{dy'}{dx} + y'\frac{dy'}{dy}.$$

The actual values are found to be

$$\frac{d^2x}{ds^2} = \frac{s}{w_2^2}(-\phi_{01}^2\phi_{20} + 2\phi_{01}\phi_{10}\phi_{11} - \phi_{10}^2\phi_{02})$$

$$-\tfrac{1}{2}\frac{\phi_{01}^2}{w_2^2}(E_{10}\phi_{01}^2 - 2F_{10}\phi_{01}\phi_{10} + G_{10}\phi_{10}^2)$$

$$+\tfrac{1}{2}\frac{\phi_{01}\phi_{10}}{w_2^2}(E_{01}\phi_{01}^2 - 2F_{01}\phi_{01}\phi_{10} + G_{01}\phi_{10}^2),$$

$$\frac{d^2y}{ds^2} = \frac{r}{w_2^2}(-\phi_{01}^2\phi_{20} + 2\phi_{01}\phi_{10}\phi_{11} - \phi_{10}^2\phi_{02})$$

$$+\tfrac{1}{2}\frac{\phi_{01}\phi_{10}}{w_2^2}(E_{10}\phi_{01}^2 - 2F_{10}\phi_{01}\phi_{10} + G_{10}\phi_{10}^2)$$

$$-\tfrac{1}{2}\frac{\phi_{10}^2}{w_2^2}(E_{01}\phi_{01}^2 - 2F_{01}\phi_{01}\phi_{10} + G_{01}\phi_{10}^2),$$

where r and s on the right-hand sides have the values given in § 15. Now

$$V^2\frac{d^2H}{ds^2} = V^2H_{20}x'^2 + 2V^2H_{11}x'y' + V^2H_{02}y'^2 + V^2H_{10}x'' + V^2H_{01}y'';$$

when we substitute the values of the various quantities and reduce, we have

$$\frac{d^2H}{ds^2} = \frac{1}{B^2}\frac{\mathfrak{h}_1}{V^4} + \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right) + \frac{1}{\rho''}\frac{dH}{dn},$$

the same value as before (§ 42).

Again, we know* that

$$\frac{dx}{dt} = x', \qquad \frac{dy}{dt} = y',$$

$$\frac{d^2x}{dt^2} = \Gamma x'^2 + 2\Gamma'x'y' + \Gamma''y'^2,$$

$$\frac{d^2y}{dt^2} = \Delta x'^2 + 2\Delta'x'y' + \Delta''y'^2.$$

Hence, as

$$\frac{d^2H}{dt^2} = H_{20}x'^2 + 2H_{11}x'y' + H_{02}y'^2 + H_{10}\frac{d^2x}{dt^2} + H_{01}\frac{d^2y}{dt^2},$$

we find, after substitution,

$$\frac{d^2H}{dt^2} = \frac{1}{B^2}\frac{\mathfrak{h}_1}{V^4} + \tfrac{1}{2}K\left(H - \frac{2}{\rho'}\right).$$

Consequently we have

$$\frac{d^2H}{ds^2} - \frac{d^2H}{dt^2} = \frac{1}{\rho''}\frac{dH}{dn},$$

* See the paper by the author, quoted in § 4.

another illustration of the remark in § 36. It may similarly be proved that

$$\frac{d^2K}{ds^2} - \frac{d^2K}{dt^2} = \frac{1}{\rho''}\frac{dK}{dn}.$$

These are particular cases of the theorem, which can similarly be established: *If Ω denote any quantity, which is connected with any point on the surface and the expression of which is independent of the curve $\phi = 0$ through the point, then*

$$\frac{d^2\Omega}{ds^2} - \frac{d^2\Omega}{dt^2} = \frac{1}{\rho''}\frac{d\Omega}{dn}.$$

46. Proceeding to the identification of the two invariants $H(w'_3)$ and $\Phi(w'_3)$, which involve the coefficients of w'_3, we construct $\frac{d}{ds}I(w_2, w''_2)$ and $\frac{d}{dn}I(w_2, w''_2)$, and we find

$$\sqrt{w_2}\frac{d}{ds}\left\{\frac{I(w_2, w''_2)}{V^4}\right\} = \frac{1}{2V^6}\{(Em - 2Fl + Gk)\phi_{01} + \ldots\},$$

$$V\sqrt{w_2}\frac{d}{dn}\left\{\frac{I(w_2, w''_2)}{V^4}\right\} = \frac{1}{2V^6}[\{E(En - 2Fm + Gl) - F(Em - 2Fl + Gk)\}\phi_{01} + \ldots] - \tfrac{1}{3}w''_2\frac{\nabla}{V^4}.$$

Let these covariants be denoted by $\mathfrak{c}_1$, $\mathfrak{c}_2$, respectively, so that

$$\mathfrak{c}_1 = (Em - 2Fl + Gk)\phi_{01} + \ldots$$

$$\mathfrak{c}_2 = \{E(En - 2Fm + Gl) - F(Em - 2Fl + Gk)\}\phi_{01} + \ldots$$

Then $\mathfrak{c}_1$ can be used to replace $H(w'_3)$ in the aggregate as $\mathfrak{a}_1$ replaced $H(w_3)$, and $\mathfrak{c}_2$ can be used to replace $\Phi(w'_3)$ in the aggregate as $\mathfrak{a}_2$ replaced $\Phi(w_3)$, in each case without affecting the completeness of the aggregate. The index of $\mathfrak{c}_1$ is 7; that of $\mathfrak{c}_2$ is 8. Their significance is given by

$$\frac{\mathfrak{c}_1}{V^7} = 4B\frac{d^2B}{ds\,dn} - 4B\frac{d}{ds}\left(\frac{B}{\rho''}\right),$$

and

$$\frac{\mathfrak{c}_2}{V^8} = 4B\frac{d^2B}{dn^2} - 4B\frac{d}{dn}\left(\frac{B}{\rho''}\right) + \tfrac{8}{3}B^2K.$$

47. We have

$$\sqrt{w_2}\frac{d}{ds}\left\{\frac{J(w_2, w'_2)}{V^3}\right\} - \frac{J(w_2, w_3)}{V^3}$$

$$= \frac{1}{V^3 w_2}J(w_2, w'_2)\,J(w_2, w''_2) - \frac{1}{2V^3}w''_2\,I(w_2, w'_2) + \frac{w'_2 w''_2}{Vw_2},$$

and

$$V\sqrt{w_2}\frac{d}{dn}\left\{\frac{J(w_2, w'_2)}{V^3}\right\} - \frac{w_2\mathfrak{a}_1 - w_3V^2}{V^3}$$

$$= \frac{1}{V^5}J(w_2, w'_2)\,I(w_2, w''_2) - \frac{1}{2V^5}J(w_2, w''_2)\,I(w_2, w'_2)$$

$$+ \frac{1}{V^3 w_2}\{w'_2\,J(w_2, w''_2) - w''_2\,J(w_2, w'_2)\}.$$

When substitution is made for the various invariants, and the reduction is effected, we find

$$-\frac{d}{ds}\left(\frac{1}{\tau'}\right) = \frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(H - \frac{2}{\rho'}\right)\frac{1}{\rho''} - \frac{2}{B}\frac{dB}{ds}\frac{1}{\tau'},$$

$$-\frac{d}{dn}\left(\frac{1}{\tau'}\right) = -\frac{d}{ds}\left(\frac{1}{\rho'}\right) - \frac{2}{\rho''\tau'} - \left(H - \frac{2}{\rho'}\right)\frac{1}{B}\frac{dB}{ds} + \frac{dH}{ds}$$

$$= -\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \left(H - \frac{2}{\rho}\right)\frac{1}{B}\frac{dB}{ds} + \frac{dH}{ds},$$

which are relations obtained earlier (§ 40). They show that $\frac{d}{ds}\left(\frac{1}{\tau'}\right)$ and $\frac{d}{dn}\left(\frac{1}{\tau'}\right)$ can be expressed in terms of the other magnitudes.

We also have

$$V^3\sqrt{w_2}\frac{d}{dn}\left\{\frac{J(w_2, w''_2)}{V^5}\right\} = \frac{1}{2V^5}J(w_2, w''_2)\,I(w_2, w''_2) + \tfrac{1}{2}\frac{w_2\mathfrak{c}_1}{V^5} - \tfrac{1}{2}\frac{w'_3}{V^3}.$$

All the covariants that occur in this relation are known; when we substitute their values and reduce the resulting expression, we find

$$\frac{d^2B}{ds\,dn} - \frac{d^2B}{dn\,ds} = \frac{1}{B}\frac{dB}{ds}\frac{dB}{dn} - \frac{1}{\rho''}\frac{dB}{ds}.$$

This result, and the corresponding result obtained for H in § 43, are special cases of the theorem, which can be established by using the invariantive forms: *If Ω denote any quantity, which is connected with any point on the surface and the expression of which is independent* of the curve $\phi = 0$ through the point, then*

$$\frac{d^2\Omega}{ds\,dn} - \frac{d^2\Omega}{dn\,ds} = \frac{1}{B}\frac{dB}{ds}\frac{d\Omega}{dn} - \frac{1}{\rho''}\frac{d\Omega}{ds}.$$

48. Similarly

$$\sqrt{w_2}\frac{d}{ds}\left\{\frac{J(w_2, w''_2)}{V^5}\right\} = \frac{1}{V^7}\{\tfrac{1}{2}w''_2\,I(w_2, w''_2) - w_2H(w''_2)\} + \frac{J(w_2, w'_3)}{2V^7} + \frac{\nabla}{3V^7}w^2_2;$$

* The *value* of B is not independent of the curve; but B is one of the fundamental quantities for the expression of properties of the curve, and its *expression* is an irresoluble variable.

after substitution and reduction, we have

$$\frac{1}{B}\frac{d^2B}{ds^2} = 2K - \frac{d}{dn}\left(\frac{1}{\rho''}\right) + \frac{1}{\rho''^2} + 2\left(\frac{1}{B}\frac{dB}{ds}\right)^2.$$

Also, we have

$$\sqrt{w_2}\,\frac{d}{ds}\left\{\frac{J(w_2, w_3)}{V^4}\right\} - \frac{J(w_2, w_4)}{V^4} - \tfrac{1}{2}\frac{Kw_2J(w_2, w'_2)}{V^4}$$

$$= \frac{1}{2V^6}\left\{-2w''_2\mathfrak{a}_1 + \frac{3}{w_2}J(w_2, w''_2)\,J(w_2, w_3) + V^2\frac{w''_2w_3}{w_2}\right\};$$

thence a value of $J(w_2, w_4)$ is obtained in the form

$$\frac{J(w_2, w_4)}{B^3V^5} = \frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right) + \tfrac{1}{2}\frac{K}{\tau'} - \frac{2}{\rho''}\frac{dH}{ds} + \frac{1}{\rho''}\frac{d}{dt}\left(\frac{1}{\rho'}\right) - \frac{2}{\tau'}\left(\frac{1}{B}\frac{dB}{ds}\right)^2$$

$$+ \frac{2}{B}\frac{dB}{ds}\left\{\frac{d}{dn}\left(\frac{1}{\rho'}\right) + \left(H - \frac{2}{\rho'}\right)\frac{1}{\rho''}\right\}.$$

Comparing this value of $J(w_2, w_4)$ with the value that was obtained in § 41, we find

$$\frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right) - \frac{d^2}{dn\,ds}\left(\frac{1}{\rho'}\right)$$

$$= \frac{2}{\tau'}\frac{d}{dn}\left(\frac{1}{\rho''}\right) + \frac{1}{B}\frac{dB}{ds}\frac{d}{dn}\left(\frac{1}{\rho'}\right) + \frac{1}{\rho''}\frac{d}{dt}\left(\frac{1}{\rho'}\right) - 2\frac{K}{\tau'} - \frac{4}{\tau'}\left(\frac{1}{B}\frac{dB}{ds}\right)^2.$$

Lastly, we have

$$V\sqrt{w_2}\,\frac{d}{dn}\left\{\frac{J(w_2, w_3)}{V^4}\right\} - \frac{w_2\mathfrak{b}_1}{V^4} + \frac{w_4}{V^4} - \tfrac{1}{2}\frac{K}{V^4}\left\{2w_2w'_2V^2 - I(w_2, w'_2)\,w_2^2\right\}$$

$$= -\frac{J(w_2, w''_2)}{V^6}\mathfrak{a}_1 + \frac{3}{2V^6}J(w_2, w_3)\,I(w_2, w''_2)$$

$$+ \frac{3}{2w_2V^4}\left\{w_3J(w_2, w''_2) - w''_2J(w_2, w_3)\right\}.$$

When we substitute the values of the invariants in this expression and reduce the result, we find

$$\frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right) = \frac{d^2H}{ds^2} - K\left(H - \frac{2}{\rho''}\right) - \frac{1}{\rho''}\frac{dH}{dn} - \frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right)$$

$$+ \frac{1}{B}\left[\frac{2}{\tau'}\frac{d^2B}{dn\,ds} - 4\frac{dB}{ds}\frac{dH}{ds} + 5\frac{dB}{ds}\frac{d}{dt}\left(\frac{1}{\rho'}\right)\right]$$

$$+ \frac{2}{B^2}\frac{dB}{ds}\left\{\left(H - \frac{2}{\rho'}\right)\frac{dB}{ds} - \frac{1}{\tau'}\frac{dB}{dn}\right\}.$$

It is to be noted, from the results obtained in this section, that $\dfrac{d^2B}{ds^2}$ and $\dfrac{d^2}{dn^2}\left(\dfrac{1}{\rho'}\right)$ are expressed in terms of the other magnitudes retained; or, if we choose, we can regard the last relation as determining $\dfrac{d^2B}{dn\,ds}$ in terms of the other magnitudes retained.

Invariants involving Derivatives of Two Functions.

49. Among the aggregate of invariants set out in § 23, there still remain nine as yet uninterpreted; but their expressions involve derivatives of the function ψ. Five out of these nine involve derivatives of no higher order than those which occur in the invariants interpreted in §§ 34–38. In order to obtain their interpretation, it is convenient to associate with them invariants which depend upon ψ alone and bear the same relation to ψ alone as some of those already interpreted bear to ϕ alone; and then the complete aggregate can be simplified by replacing some of the original forms by some of the associated forms.

For this purpose, let

$$W'''_2 = (a', b', c' \between \psi_{01}, -\psi_{10})^2,$$

$$W_2 = (E, F, G \between \psi_{01}, -\psi_{10})^2,$$

$$J(W_2, W'''_2) = (Eb' - Fa')\psi_{01}^2 - (Ec' - Ga')\psi_{01}\psi_{10} + (Fc' - Gb')\psi_{10}^2,$$

$$\Delta_1 = a'\phi_{01}\psi_{01} - b'(\phi_{01}\psi_{10} + \phi_{10}\psi_{10}) + c'\phi_{10}\psi_{10},$$

$$\Delta_2 = 2(Eb' - Fa')\phi_{01}\psi_{01} - (Ec' - Ga')(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + 2(Fc' - Gb')\phi_{10}\psi_{10};$$

then we establish (and it is easy to verify) the equations

$$J^2(w_1, w_2) - w_2 W_2 + V^2 w_1^2 = 0,$$

$$w_2\Delta_1 - w'''_2 J(w_1, w_2) + w_1 J(w_2, w'''_2) = 0,$$

$$\Delta_1^2 - w'''_2 W'''_2 + w_1^2 H(w'''_2) = 0,$$

$$w_2\Delta_2 - 2J(w_2, w'''_2) J(w_1, w_2) + w_1 w_2 I(w_2, w'''_2) = 2V^2 w_1 w'''_2,$$

$$\Delta_2^2 - 4J(w_2, w'''_2) J(W_2, W'''_2) = w_1^2\{I^2(w_2, w'''_2) - 4V^2 H(w'''_2)\}.$$

The first of these equations shows that W_2 can be regarded as known; it is not an independent invariant but, if we wished, we could replace w_2 by W_2 in the complete aggregate without affecting the completeness. This change will not be made; we shall retain W_2 as a quantity convenient for other purposes and alternative to w_2 in the aggregate.

The second and the third equations, combined so as to eliminate Δ, show that W'''_2 can be regarded as known; it is not an independent invariant but, if we wished, we could replace w'''_2 by W'''_2 in the complete aggregate without affecting the completeness. This change will be made.

The fourth and the fifth equations, combined so as to eliminate Δ_2, show that $J(W_2, W'''_2)$ can be regarded as known; it is not an independent invariant but, if we wished, we could replace $J(w_2, w'''_2)$ by $J(W_2, W'''_2)$ in the complete aggregate without affecting the completeness. This change also will be made.

The five invariants that remained for interpretation were

$$\frac{w_1}{V}, \quad \frac{J(w_1, w_2)}{V^2}, \quad \frac{w'''_2}{V^4}, \quad \frac{J(w_2, w'''_2)}{V^5}, \quad \frac{I(w_2, w'''_2)}{V^4};$$

after the changes that have been made, the five are

$$\frac{w_1}{V}, \quad \frac{J(w_1, w_2)}{V^2}, \quad \frac{W'''_2}{V^4}, \quad \frac{J(W_2, W'''_2)}{V^5}, \quad \frac{I(w_2, w'''_2)}{V^4},$$

of which the last may also be written $I(W_2, W'''_2)V^{-4}$. The interpretation of the first two of these is easily obtained; for the interpretation of the remaining three, which involve derivatives of ψ but not of ϕ, the results of earlier interpretations can be used.

50. For the purpose of the interpretation, we need certain geometric properties of the curve $\psi = 0$. Let ds' denote an elementary arc along the curve, and dn' an element along the normal to the curve; and let

$$A = \frac{d\psi}{dn'}.$$

Further, let $\frac{1}{\rho'_\psi}$ denote the circular curvature of the geodesic tangent to $\psi = 0$, and $\frac{1}{\tau'_\psi}$ the curvature of torsion of that tangent; also, let $\frac{1}{\rho''_\psi}$ denote the geodesic curvature of $\psi = 0$. Then W_2, W'''_2, $I(W_2, W'''_2)$, $J(W_2, W'''_2)$, stand to $\psi = 0$ in precisely the same relation as w_2, w''_2, $I(w_2, w''_2)$, $J(w_2, w''_2)$ to $\phi = 0$; and therefore

$$\frac{W_2}{V^2} = A^2,$$

$$\frac{W'''_2}{V^4} = -\frac{2A^3}{\rho''_\psi},$$

$$\frac{I(W_2, W'''_2)}{V^4} = 2\left(\frac{dA}{dn'} - \frac{A}{\rho''_\psi}\right),$$

$$\frac{J(W_2, W'''_2)}{V^5} = 2A^2\frac{dA}{ds'}.$$

Moreover, we have

$$\frac{dx}{ds'} = \frac{\psi_{01}}{\sqrt{W_2}}, \quad \frac{dy}{ds'} = \frac{-\psi_{10}}{\sqrt{W_2}};$$

so that, if λ be the angle at which $\phi = 0$ and $\psi = 0$ intersect, we have

$$\cos\lambda = E\frac{dx}{ds}\frac{dx}{ds'} + F\left(\frac{dx}{ds}\frac{dy}{ds'} + \frac{dy}{ds}\frac{dx}{ds'}\right) + G\frac{dy}{ds}\frac{dy}{ds'}$$

$$= \frac{J(w_1, w_2)}{\sqrt{w_2 W_2}},$$

and therefore

$$\frac{\mathrm{J}(w_1, w_2)}{\mathrm{V}^2} = \mathrm{AB}\cos\lambda.$$

Also

$$\sin\lambda = \mathrm{V}\left(\frac{dx}{ds'}\frac{dy}{ds} - \frac{dy}{ds'}\frac{dx}{ds}\right)$$

$$= \frac{\mathrm{V}w_1}{\sqrt{w_2 \mathrm{W}_2}},$$

and therefore

$$\frac{w_1}{\mathrm{V}} = \mathrm{AB}\sin\lambda.$$

We can regard the quotient of the last two invariants as giving the angle λ; and we can regard the sum of their squares as defining the magnitude A. Clearly

$$\mathrm{J}^2(w_1, w_2) + \mathrm{V}^2 w^2_1 = \mathrm{V}^4\mathrm{A}^2\mathrm{B}^2$$

$$= w_2 \mathrm{W}_2,$$

a relation already used; it may be further used to replace $\mathrm{J}(w_1, w_2)$ by W_2.

51. The general theory shows that all other invariants, which involve derivatives of ϕ and ψ up to the second order inclusive, derivatives of E, F, G of the first order, and the fundamental magnitudes of the first three orders, can be expressed in terms of the aggregate already retained, composed of the eleven invariants selected in § 39 and the five just identified, viz.:—

$$\frac{w_1}{\mathrm{V}}, \ \frac{\mathrm{J}(w_1, w_2)}{\mathrm{V}^2} \ \text{or} \ \frac{\mathrm{W}_2}{\mathrm{V}^2}, \ \frac{\mathrm{W}'''_2}{\mathrm{V}^4}, \ \frac{\mathrm{J}(\mathrm{W}_2, \mathrm{W}'''_2)}{\mathrm{V}^3}, \ \frac{\mathrm{I}(\mathrm{W}_2, \mathrm{W}'''_2)}{\mathrm{V}^4}.$$

It is not without interest to illustrate the property by one or two simple examples.

Consider the circular curvature of the geodesic tangent to $\psi = 0$; after the result in § 34, it manifestly will be given by

$$\frac{\mathrm{W}'_2}{\mathrm{V}^2} = \frac{\mathrm{A}^2}{\rho'_\psi};$$

according to the theory, it ought to be expressible in terms of the invariants retained. Take

$$\nabla_1 = \mathrm{L}\phi_{01}\psi_{01} - \mathrm{M}(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + \mathrm{N}\phi_{10}\psi_{10};$$

then we have the equations

$$\nabla_1^2 = w'_2\mathrm{W}'_2 - w_1^2\mathrm{H}(w'_2),$$

$$w_2\nabla_1 = w'_2\mathrm{J}(w_1, w_2) - w_1\mathrm{J}(w_2, w'_2);$$

and therefore

$$w_2^2\{w'_2\mathrm{W}'_2 - w_1^2\mathrm{H}(w'_2)\} = \{w'_2\mathrm{J}(w_1, w_2) - w_1\mathrm{J}(w_2, w'_2)\}^2.$$

When the geometric values of all the invariants are substituted, the preceding relation (after mere simplification and division throughout by $A^2B^6V^8$) becomes

$$\frac{1}{\rho'\rho'_\psi} = \frac{\sin^2\lambda}{\rho_1\rho_2} + \left(\frac{\cos\lambda}{\rho'} + \frac{\sin\lambda}{\tau'}\right)^2,$$

a relation which can be verified independently by means of EULER'S theorem on the curvature of a normal section and of the expression in § 35 for the torsion of the geodesic tangent.

Similarly for the curvature of torsion of the geodesic tangent to $\psi = 0$; after the result in § 34, it manifestly will be given by

$$\frac{J(W_2, W'_2)}{V^3} = -\frac{A^2}{\tau'_\psi}.$$

According to the theory, it also ought to be expressible in terms of the invariants retained. Take

$$\Phi = 2(EM - FL)\phi_{01}\psi_{01} - (EN - GL)(\phi_{01}\psi_{10} + \phi_{10}\psi_{01}) + 2(FN - GM)\phi_{10}\psi_{10};$$

then we have the equations

$$\Phi^2 = 4J(w_2, w'_2)\,J(W_2, W'_2) + w_1^2V^4\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2,$$

$$w_2\Phi = 2J(w_1, w_2)\,J(w_2, w'_2) - w_1w_2I(w_2, w'_2) + 2V^2w_1w'_2,$$

and therefore

$$w_2^2\left\{4J(w_2, w'_2)\,J(W_2, W'_2) + w_1^2V^4\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\right\}$$
$$= \{2J(w_1, w_2)\,J(w_2, w'_2) - w_1w_2I(w_2, w'_2) + 2V^2w_1w'_2\}^2,$$

which gives an expression in terms of the invariants. When we substitute the values of all the invariants and divide out by $A^2B^6V^{10}$, we find

$$\frac{4}{\tau'\tau'_\psi} = -\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\sin^2\lambda + \left\{\frac{2\cos\lambda}{\tau'} - \frac{2\sin\lambda}{\rho'} + \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\sin\lambda\right\}^2.$$

That some results of this kind, connecting ρ' and ρ'_ψ, should exist, can easily be seen. When ρ_1 and ρ_2 are given, ρ' is determined by the inclination of $\phi = 0$ to a line of curvature; λ being given, we then know the inclination of $\psi = 0$ to that line of curvature, and so ρ'_ψ is known. Similarly for some result connecting τ' and τ'_ψ.

As a last illustration of this kind, consider the invariantive expressions for $\frac{dH}{ds'}$ and $\frac{dH}{dn'}$. Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be the invariants corresponding to $\mathfrak{a}_1$ and $\mathfrak{a}_2$, so that

$$\mathfrak{b}_1 = (GP - 2FQ + ER)\psi_{01} + (GQ - 2FR + ES)(-\psi_{10}),$$

$$\mathfrak{b}_2 = (GP - 2FQ + ER)(-F\psi_{01} + G\psi_{10}) + (GQ - 2FR + ES)(E\psi_{01} - F\psi_{10});$$

then

$$\frac{\mathfrak{b}_1}{V^3} = A\frac{dH}{ds'}, \quad \frac{\mathfrak{b}_2}{V^4} = A\frac{dH}{dn'}.$$

Now we have the equations

$$\left.\begin{aligned} w_2\mathfrak{b}_1 &= J(w_1, w_2)\,\mathfrak{a}_1 - w_1\mathfrak{a}_2 \\ w_2\mathfrak{b}_2 &= V^2 w_1\mathfrak{a}_1 + J(w_1, w_2)\,\mathfrak{a}_2 \end{aligned}\right\},$$

which are easily established; substituting in them the values of the invariants that occur, we find (on removing a factor AB^2V^5), the relations

$$\left.\begin{aligned} \frac{dH}{ds'} &= \frac{dH}{ds}\cos\lambda - \frac{dH}{dn}\sin\lambda \\ \frac{dH}{dn'} &= \frac{dH}{ds}\sin\lambda + \frac{dH}{dn}\cos\lambda \end{aligned}\right\},$$

which are the ordinary differential relations for transference from directions* ds and dn to ds' and dn', when the subject of operation is a function of position only and involves no properties of tangency and no properties of contact of order higher than the first. But for a function of position (and, *a fortiori*, for a function which involves properties of contact of the first order or of higher orders), the operators $\frac{d}{ds}$ and $\frac{d}{dn}$ are not interchangeable. Thus, in particular, $\frac{d^2H}{ds\,dn}$ and $\frac{d^2H}{dn\,ds}$ are not equal to one another, except for special curves; an expression for their difference has already been obtained.

52. It still remains to identify the four invariants w''_3, $H(w''_3)$, $\Phi(w''_3)$, $J(w_2, w''_3)$, which involve the derivatives of both ϕ and ψ. Instead of proceeding to obtain their values, we use the method adopted in § 49; we replace them by four equivalent invariants involving derivatives of ψ only, and the change does not affect the completeness of the aggregate. These four invariants are

$$W''_3 = (k', l', m', n' \between \psi_{01}, -\psi_{10})^3,$$

$$H(W''_3) = (k'm' - l'^2)\psi_{01}{}^2 + \dots$$

$$\Phi(W''_3) = (k'^2n' - 3k'l'm' + 2l'^3)\psi_{01}{}^3 + \dots,$$

$$J(W_2, W''_3) = (El' - Fk')\psi_{01}{}^3 + \dots$$

* The value of $\sin\lambda$ shows that the direction of dn' falls within the angle between ds and dn.

We then modify this set of four, and replace $H(W''_3)$ and $\Phi(W''_3)$ by $\mathfrak{E}_1$ and $\mathfrak{E}_2$ where

$$\mathfrak{E}_1 = (Em' - 2Fl' + Gk')\psi_{01} + \ldots \quad ,$$

$$\mathfrak{E}_2 = \{E(En' - 2Fm' + Gl') - F(Em - 2Fl + Gk)\}\psi_{01} + \ldots \quad ;$$

and the set W''_3, $J(W_2, W''_3)$, $\mathfrak{E}_1$, $\mathfrak{E}_2$ replace w''_3, $H(w''_3)$, $\Phi(w''_3)$, $J(w_2, w''_3)$ in the aggregate, which remains complete after the change. The set of equations, which exhibit the equivalence of the four inserted forms to the four ejected forms, is similar to the corresponding set in § 42; it is more complicated because the ground-forms w_3'', W''_3 are of the third order.

The geometric significance of the four inserted forms can be obtained from the consideration that they stand related to the curve $\psi = 0$ exactly as w'_3, $J(w_2, w'_3)$, $\mathfrak{e}_1$, $\mathfrak{e}_2$ to the curve $\phi = 0$. Adopting the notation of § 51, we thus have

$$\frac{W''_3}{V^7} = -4A\frac{d}{ds'}\left(\frac{A^3}{\rho''_\psi}\right),$$

$$\frac{J(W_2, W''_3)}{V^8} = -4A^3\frac{d}{dn'}\left(\frac{A}{\rho''_\psi}\right) + 8A^2\left(\frac{dA}{ds'}\right)^2 + \tfrac{16}{3}A^4K,$$

$$\frac{\mathfrak{E}_1}{V^7} = 4A\frac{d^2A}{ds'dn'} - 4A\frac{d}{ds}\left(\frac{A}{\rho''_\psi}\right),$$

$$\frac{\mathfrak{E}_2}{V^8} = 4A\frac{d^2A}{dn'^2} - 4A\frac{d}{dn'}\left(\frac{A}{\rho''_\psi}\right) + \tfrac{8}{3}A^2K.$$

All other properties of the curve $\psi = 0$ up to the order retained can be expressed in terms of the invariants of the aggregate; the examples given in § 51 will be a sufficient illustration of the remark.

The Aggregate for a Single Curve $\phi = 0$ up to the Order Retained.

53. The 29 invariants in the preceding set have a closer affinity to the curve $\phi = 0$ than to the curve $\psi = 0$, the chief reason being that the first derivatives of ϕ were made the variables for the binary forms. By taking the first derivatives of ψ for these variables an equivalent set of 29 invariants could be obtained, having a closer affinity to the curve $\psi = 0$ than to the curve $\phi = 0$. And it would be possible to modify each of these two sets, so as to construct a new equivalent set of 29, symmetrically related to the two curves. All that is necessary in each modification is to secure that the retained aggregate remains algebraically complete.

Out of the set of 29 invariants retained, there are 20 which are not affected by the curve $\psi = 0$ in their expression; and therefore we infer that all the differential invariants of a surface and a curve $\phi = 0$ upon the surface, involving derivatives of ϕ up to the third order inclusive, involving the magnitudes E, F, G and their derivatives up to the second order inclusive, involving also the fundamental magnitudes of the second, the third, and the fourth orders, can be expressed algebraically in terms of an algebraically complete aggregate of 20 members.

This aggregate is composed of 20 quantities, each divided by an appropriate power of V; the sections quoted give the significance of the respective invariants. These quantities are as follows:—

[§ 34] $$w_2 = (\mathrm{E},\ \mathrm{F},\ \mathrm{G} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 34] $$w'_2 = (\mathrm{L},\ \mathrm{M},\ \mathrm{N} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 34] $$\mathrm{J}(w_2, w'_2) = \begin{array}{|c|c|c|}\hline \mathrm{EM} & \mathrm{EN} & \mathrm{FN} \\ -\mathrm{FL} & -\mathrm{GL} & -\mathrm{GM} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 35] $$\mathrm{I}(w_2, w'_2) = \mathrm{EN} - 2\mathrm{FM} + \mathrm{GL}.$$

[§ 34] $$w''_2 = (a,\ b,\ c \between \phi_{01},\ -\phi_{10})^2,$$

[§ 36] $$\mathrm{J}(w_2, w''_2) = \begin{array}{|c|c|c|}\hline \mathrm{E}b & \mathrm{E}c & \mathrm{F}c \\ -\mathrm{F}a & -\mathrm{G}a & -\mathrm{G}b \\ \hline \end{array} \between \phi_{01},\ -\phi_{10})^2,$$

[§ 37] $$\mathrm{I}(w_2, w''_2) = \mathrm{E}c - 2\mathrm{F}b + \mathrm{G}a.$$

[§ 34] $$w_3 = (\mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S} \between \phi_{01},\ -\phi_{10})^3,$$

[§ 37] $$\mathrm{J}(w_2, w_3) = \begin{array}{|c|c|c|c|}\hline \mathrm{EQ} & 2\mathrm{ER} & \mathrm{ES} & \\ -\mathrm{FP} & -\mathrm{FQ} & +\mathrm{FR} & \mathrm{FS} \\ & -\mathrm{GP} & -2\mathrm{GQ} & -\mathrm{GR} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10})^3,$$

[§ 38] $$\alpha_1 = \begin{array}{|c|c|}\hline \mathrm{ER} & \mathrm{ES} \\ -2\mathrm{FQ} & -2\mathrm{FR} \\ +\mathrm{GP} & +\mathrm{GQ} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10}),$$

[§ 38] $$\alpha_2 = \begin{array}{|c|c|}\hline \mathrm{E}^2\mathrm{S} & \mathrm{EFS} \\ -3\mathrm{EFR} & -(\mathrm{EG} + 2\mathrm{F}^2)\,\mathrm{R} \\ +(\mathrm{EG} + 2\mathrm{F}^2)\,\mathrm{Q} & +3\mathrm{FGQ} \\ -\mathrm{FGP} & -\mathrm{G}^2\mathrm{P} \\ \hline \end{array} \between \phi_{01},\ -\phi_{10}),$$

[§ 36] $$w'_3 = (k,\ l,\ m,\ n \between \phi_{01},\ -\phi_{10})^3,$$

[§ 37]
$$J(w_2, w'_3) = \left|\begin{array}{c|c|c|c} El & 2Em & En & \\ -Fk & -Fl & +Fm & Fn \\ & -Gk & -2Gl & -Gm \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big)^3,$$

[§ 46]
$$\mathfrak{e}_1 = \left|\begin{array}{c|c} Em & En \\ -2Fl & -2Fm \\ +Gk & +Gl \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big),$$

[§ 46]
$$\mathfrak{e}_2 = \left|\begin{array}{c|c} E^2 n & EFn \\ -3EFm & -(EG + 2F^2)\,m \\ +(EG + 2F^2)\,l & +3FGl \\ -FGk & -G^2 k \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big),$$

[§ 34]
$$w_4 = (\alpha, \beta, \gamma, \delta, \epsilon\,\Big)\!\Big(\phi_{01},\, -\phi_{10})^4,$$

[§ 41]
$$J(w_2, w_4) = \left|\begin{array}{c|c|c|c|c} E\beta & 3E\gamma & 3E\delta & E\epsilon & \\ -F\alpha & -2F\beta & & +2F\delta & F\epsilon \\ & -G\alpha & -3G\beta & -3G\gamma & -G\delta \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big)^4,$$

[§ 42]
$$\mathfrak{h}_1 = \left|\begin{array}{c|c|c} E\gamma & E\delta & E\epsilon \\ -2F\beta & -2F\gamma & -2F\delta \\ +G\alpha & +G\beta & +G\gamma \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big)^2.$$

[§ 43]
$$\mathfrak{h}_2 = \left|\begin{array}{c|c|c} E^2\delta & E^2\epsilon & EF\epsilon \\ -3EF\gamma & -2EF\delta & -(EG + 2F^2)\,\delta \\ +(EG + 2F^2)\,\beta & -2FG\beta & +3FG\gamma \\ -FG\alpha & -G^2\alpha & -G^2\beta \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big)^2,$$

[§ 44]
$$\mathfrak{h}_3 = \left|\begin{array}{c|c|c} E^3\epsilon & E^2F\epsilon & EF^2\epsilon \\ -4E^2F\,\delta & -4EF^2\,\delta & -(2F^3 + 2EFG)\,\delta \\ +(E^2G + 5EF^2)\,\gamma & +(2EFG + 4F^3)\,\gamma & +(5F^2G + EG^2)\,\gamma \\ -(2EFG + 2F^3)\,\beta & -4F^2G\beta & -4FG^2\beta \\ +F^2G\alpha & +FG^2\alpha & +G^3\alpha \end{array}\right| \Big)\!\Big(\phi_{01},\, -\phi_{10}\Big)^2.$$

The various indices of these quantities, being the powers of V by which they must be divided to become absolute invariants, are :—

Index 2, w_2, w'_2, $\mathrm{I}(w_2, w'_2)$;

Index 3, $\mathrm{J}(w_2, w'_2)$, w_3, $\mathfrak{a}_1$;

Index 4, w''_2, $\mathrm{I}(w_2, w''_2)$, $\mathrm{J}(w_2, w_3)$, $\mathfrak{a}_2$, w_4, $\mathfrak{h}_1$;

Index 5, $\mathrm{J}(w_2, w''_2)$, $\mathrm{J}(w_2, w_4)$, $\mathfrak{h}_2$;

Index 6, $\mathfrak{h}_3$;

Index 7, w'_3, $\mathfrak{e}_1$;

Index 8, $\mathrm{J}(w_2, w'_3)$, $\mathfrak{e}_2$.

54. It will be seen from these forms that all the invariants retained are linear in all the quantities L, M, N, P, Q, R, S, α, β, γ, δ, ϵ, a, b, c, k, l, m, n which occur in them. This property facilitates the expression of any other invariant in terms of the various members; thus

$$\frac{\mathrm{LN} - \mathrm{M}^2}{\mathrm{V}^2} = \frac{w_2 w'_2 \mathrm{I}(w_2, w'_2) - \mathrm{J}^2(w_2, w'_2)}{\mathrm{V}^2 w_2^{\,2}},$$

$$\frac{\alpha\epsilon - 4\beta\delta + 3\gamma^2}{\mathrm{V}^4} = \frac{w_4 \mathfrak{h}_3 - 4\mathrm{J}(w_2, w_4)\,\mathfrak{h}_2 + 3w_2 \mathfrak{h}_1^{\,2}}{w_2^{\,3}},$$

and so for others. Moreover, in the invariants which contain a, b, c linearly, the effect is that the derivatives of ϕ of the second order (being the highest that occurs) are contained linearly; and in those invariants which contain k, l, m, n linearly, the effect is that the derivatives of ϕ of the third order (being the highest order that occurs) are contained linearly, as well as those of the second order.

Moreover, the forms can be used to obtain the value of any given invariant; all that is necessary for this purpose is to obtain the expression of the invariant in terms of the members of the selected aggregate, and to substitute the values of the members that occur. Thus, consider the simultaneous invariant

$$\begin{vmatrix} a, & b, & c \\ \mathrm{L}, & \mathrm{M}, & \mathrm{N} \\ \mathrm{E}, & \mathrm{F}, & \mathrm{G} \end{vmatrix};$$

when expressed in terms of the members of the aggregate, it is equal to

$$\frac{1}{w_2}\{\mathrm{J}(w_2, w''_2)\,\mathrm{I}(w_2, w'_2) - \mathrm{J}(w_2, w'_2)\,\mathrm{I}(w_2, w''_2)\}$$
$$+ \frac{1}{w_2^{\,2}}\{w''_2\,\mathrm{J}(w_2, w'_2) - w'_2\,\mathrm{J}(w_2, w''_2)\},$$

and the value of the latter expression is

$$2V^5 \left\{ \left(H - \frac{1}{\rho'} \right) \frac{dB}{ds} + \frac{1}{\tau'} \frac{dB}{dn} \right\}.$$

In this way the actual values of a large number of the invariants belonging to the asyzygetic aggregate can be obtained. The asyzygetic aggregate of two cubics is known. The asyzygetic aggregate, arising when a quadratic is associated with a system asyzygetically complete in itself, is also known; so that the asyzygetic aggregate belonging to w_2, w'_2, w''_2, w_3, w'_3 can be obtained by the application of known theorems.

Further, the asyzygetic aggregate of a cubic and a quartic is known, so that the asyzygetic aggregate could be obtained for w_2, w'_2, w''_2, w_3, w_4, and also for w_2, w'_2, w''_2, w'_3, w_4. But, so far as I am aware, the asyzygetic aggregate of either two cubics and one quartic, or a cubic and any system asyzygetically complete in itself, is not known; as soon as either is known, the results could be applied to obtain the asyzygetic aggregate for w_2, w'_2, w''_2, w_3, w'_3, w_4, that is, the complete system of concomitants in terms of which any rational integral invariant can be expressed as a rational integral function.

The Geometrical Magnitudes which are Independent.

55. As regards the quantities, which have served to assign the geometrical significance of the several invariants, some inferences can be drawn from the results obtained. Denoting by χ the angle between the curve and the line of curvature connected with ρ_1, we have

$$\frac{1}{\rho'} = \frac{\cos^2 \chi}{\rho_1} + \frac{\sin^2 \chi}{\rho_2},$$

$$\frac{1}{\tau'} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \cos \chi \sin \chi,$$

$$H = \frac{1}{\rho_1} + \frac{1}{\rho_2}, \quad K = \frac{1}{\rho_1 \rho_2},$$

so that not more than three of the quantities $\frac{1}{\rho'}$, $\frac{1}{\tau'}$, H, K are independent. For purposes of expression, we have retained $\frac{1}{\rho'}$, $\frac{1}{\tau'}$, H. There are also the quantities B and $\frac{1}{\rho''}$.

To the order of derivatives which occur in the invariants that have been constructed, the geometric magnitudes, which might be expected to occur in the values of the invariants, are as follows:—

$$\frac{1}{\rho'},\ \frac{d}{ds}\left(\frac{1}{\rho'}\right),\ \frac{d}{dn}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{dsdn}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{dnds}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right),$$

$$\mathrm{H},\ \frac{d\mathrm{H}}{ds},\ \frac{d\mathrm{H}}{dn},\ \frac{d^2\mathrm{H}}{ds^2},\ \frac{d^2\mathrm{H}}{dsdn},\ \frac{d^2\mathrm{H}}{dnds},\ \frac{d^2\mathrm{H}}{dn^2},$$

$$\mathrm{B},\ \frac{d\mathrm{B}}{ds},\ \frac{d\mathrm{B}}{dn},\ \frac{d^2\mathrm{B}}{ds^2},\ \frac{d^2\mathrm{B}}{dsdn},\ \frac{d^2\mathrm{B}}{dnds},\ \frac{d^2\mathrm{B}}{dn^2},$$

$$\frac{1}{\rho''},\ \frac{d}{ds}\left(\frac{1}{\rho''}\right),\ \frac{d}{dn}\left(\frac{1}{\rho''}\right),$$

$$\frac{1}{\tau'},$$

and the derivatives of $\frac{1}{\tau'}$. But not all of these can be retained as independent magnitudes. In § 40 it was proved that

$$\frac{d}{ds}\left(\frac{1}{\tau'}\right) = \frac{2}{\tau'\mathrm{B}}\frac{d\mathrm{B}}{ds} - \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\rho''} - \frac{d}{dn}\left(\frac{1}{\rho'}\right),$$

$$\frac{d}{dn}\left(\frac{1}{\tau'}\right) = \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds} + \frac{d}{ds}\left(\frac{1}{\rho'}\right) + \frac{2}{\rho''\tau'} - \frac{d\mathrm{H}}{ds};$$

so that the first derivatives of $\frac{1}{\tau'}$, and consequently also the second (and higher) derivatives, are expressible in terms of the derivatives of the other quantities retained.* Again, in §§ 41, 43, 47 it has been shown that the quantities

$$\frac{d^2}{dsdn}\left(\frac{1}{\rho'}\right) - \frac{d^2}{dnds}\left(\frac{1}{\rho'}\right),$$

$$\frac{d^2\mathrm{H}}{dsdn} - \frac{d^2\mathrm{H}}{dnds},$$

$$\frac{d^2\mathrm{B}}{dsdn} - \frac{d^2\mathrm{B}}{dnds},$$

are expressible in terms of the derivatives of the first order; so that it is sufficient to retain $\frac{d^2}{dsdn}\left(\frac{1}{\rho'}\right)$, $\frac{d^2\mathrm{H}}{dsdn}$, $\frac{d^2\mathrm{B}}{dsdn}$, and reject $\frac{d^2}{dnds}\left(\frac{1}{\rho'}\right)$, $\frac{d^2\mathrm{H}}{dnds}$, $\frac{d^2\mathrm{B}}{dnds}$. Further, in § 41, it was proved that

* It is proved in DARBOUX's 'Théorie générale des Surfaces,' vol. 2, p. 360, that the quantity

$$\frac{d}{ds}\left(\frac{1}{\tau'}\right) + \left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{1}{\rho''},$$

which occurs in the first of the two equations, is the same for two curves that have the same tangent.

$$\frac{1}{\mathrm{B}}\frac{d^2\mathrm{B}}{ds^2} = 2\mathrm{K} - \frac{d}{dn}\left(\frac{1}{\rho''}\right) + \frac{1}{\rho''^2} + 2\left(\frac{1}{\mathrm{B}}\frac{d\mathrm{B}}{ds}\right)^2,$$

$$\frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right) = \frac{d^2\mathrm{H}}{ds^2} - \mathrm{K}\left(\mathrm{H} - \frac{2}{\rho''}\right) - \frac{1}{\rho''}\frac{d\mathrm{H}}{dn} - \frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right)$$
$$+ \frac{1}{\mathrm{B}}\left\{\frac{2}{\tau'}\frac{d^2\mathrm{B}}{dn\,ds} - 4\frac{d\mathrm{B}}{ds}\frac{d\mathrm{H}}{ds} + 5\frac{d\mathrm{B}}{ds}\frac{d}{dt}\left(\frac{1}{\rho'}\right)\right\}$$
$$+ \frac{2}{\mathrm{B}^2}\frac{d\mathrm{B}}{ds}\left\{\left(\mathrm{H} - \frac{2}{\rho'}\right)\frac{d\mathrm{B}}{ds} - \frac{1}{\tau'}\frac{d\mathrm{B}}{dn}\right\},$$

and the values of $\frac{d}{dt}\left(\frac{1}{\rho'}\right)$ and $\frac{d^2}{dt^2}\left(\frac{1}{\rho'}\right)$ have been given in §§ 36, 41; hence it is unnecessary to retain $\frac{d^2\mathrm{B}}{ds^2}$, $\frac{d^2}{dn^2}\left(\frac{1}{\rho'}\right)$.

We therefore retain the quantities

$$\frac{1}{\rho'},\ \frac{d}{ds}\left(\frac{1}{\rho'}\right),\ \frac{d}{dn}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds^2}\left(\frac{1}{\rho'}\right),\ \frac{d^2}{ds\,dn}\left(\frac{1}{\rho'}\right),$$

$$\mathrm{H},\ \frac{d\mathrm{H}}{ds},\ \frac{d\mathrm{H}}{dn},\ \frac{d^2\mathrm{H}}{ds^2},\ \frac{d^2\mathrm{H}}{ds\,dn},\ \frac{d^2\mathrm{H}}{dn^2},$$

$$\mathrm{B},\ \frac{d\mathrm{B}}{ds},\ \frac{d\mathrm{B}}{dn},\ \frac{d^2\mathrm{B}}{ds\,dn},\ \frac{d^2\mathrm{B}}{dn^2},$$

$$\frac{1}{\rho''},\ \frac{d}{ds}\left(\frac{1}{\rho''}\right),\ \frac{d}{dn}\left(\frac{1}{\rho''}\right),$$

$$\frac{1}{\tau'},$$

being 20 in all; their association with the 20 algebraically independent differential invariants set out in § 53 has already been made.

56. These results would seem to have an important bearing when we proceed to the next higher order of derivatives. As $\frac{d^2\mathrm{B}}{ds^2}$ is rejected from the aggregate of quantities, the quantities $\frac{d^3\mathrm{B}}{ds^3}$ and $\frac{d^3\mathrm{B}}{dn\,ds^2}$ will also be rejected; also, as $\frac{d^2\mathrm{B}}{dn\,ds} - \frac{d^2\mathrm{B}}{ds\,dn}$ is expressible by quantities of lower order, the quantities $\frac{d^3\mathrm{B}}{ds^2\,dn}$ and $\frac{d^3\mathrm{B}}{dn\,ds\,dn}$ will be rejected; thus, in this order, the only derivatives of B to be retained are

$$\frac{d^3\mathrm{B}}{ds\,dn\,ds},\ \frac{d^3\mathrm{B}}{dn^2\,ds},\ \frac{d^3\mathrm{B}}{ds\,dn^2},\ \frac{d^3\mathrm{B}}{dn^3},$$

four in number. Moreover, these four may reduce to two; for the first may be equivalent to the rejected $\frac{d^3B}{dn\,ds^2}$, while the second and the third may be equivalent to one another. Similarly, the only derivatives of $\frac{1}{\rho'}$ to be retained are

$$\frac{d^3}{ds^3}\left(\frac{1}{\rho'}\right),\quad \frac{d^3}{ds^2\,dn}\left(\frac{1}{\rho'}\right),\quad \frac{d^3}{dn\,ds\,dn}\left(\frac{1}{\rho'}\right),\quad \frac{d^3}{dn\,ds^2}\left(\frac{1}{\rho'}\right),$$

four in number; and these may reduce to two. There are six derivatives of H, viz.:

$$\frac{d^3H}{ds^3},\quad \frac{d^3H}{ds^2\,dn},\quad \frac{d^3H}{dn\,ds^2},\quad \frac{d^3H}{dn^2\,ds},\quad \frac{d^3H}{ds\,dn^2},\quad \frac{d^3H}{dn^3},$$

which may reduce to four; and there are four derivatives of $\frac{1}{\rho''}$, viz.:

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho''}\right),\quad \frac{d^2}{ds\,dn}\left(\frac{1}{\rho''}\right),\quad \frac{d^2}{dn\,ds}\left(\frac{1}{\rho''}\right),\quad \frac{d^2}{dn^2}\left(\frac{1}{\rho''}\right),$$

which may reduce to three. Hence there are, in all, eighteen new geometrical quantities arising through the inclusion of derivatives of the next higher order; and these eighteen quantities may reduce to eleven.

Now the number of differential invariants, which involve derivatives of ϕ up to order n and the corresponding quantities of proper order, is $\frac{1}{2}n(3n+5)$ by § 27; and this number is certainly subject to diminution by 1 unit, as explained at the beginning of § 29, so that it is $\frac{1}{2}n(3n+5)-1$. When $n=4$, this is 33; and we know that there are 20 invariants for $n=3$; so that 13 new invariants are introduced by the differential equations for the new order. It has been indicated that there may be only 11 new geometrical quantities available for their expression; if so, the inference would be that there are two algebraic relations among these 13. These relations are outside the differential equations; and the only cause from which they could arise would be owing to the intrinsic significance of the magnitudes. As there actually is one* differential invariant of deformation of this order (that is, a function involving E, F, G and their derivatives up to the third order, and no other quantities), the obvious suggestion is that it would behave like the invariant of the lower order, due to GAUSS, and would be expressible in terms of invariants in the binariant system composed of the fundamental magnitudes; but this inference is only a suggestion, and cannot be regarded as an established result.

[NOTE: *added*, 12 *May*, 1903.

After the manuscript of this memoir had been sent to the Royal Society but before the memoir itself had been read, I succeeded in definitely establishing the inference suggested at the end of § 56. The necessary calculations are long and are of the

* ŻORAWSKI, *l.c.*, p. 31.

same general character as those in §§ 14–22; their aim is to obtain the one solution, other than ∇ and V^2, of the twenty-eight partial differential equations satisfied by differential invariants of deformation, which are of order not higher than three. The mode of dealing with such a system of equations has been amply illustrated in Part I. of the memoir; accordingly, only the results of the analysis will be given.

We denote by Γ, Γ', Γ'', Δ, Δ', Δ'', quantities defined in § 6; and we write

$$u = E_{12} - 2F_{21} + G_{30} + \Gamma''E_{20} - (2\Gamma' + \Delta'')\,E_{11} + \Gamma E_{02} + 2\Delta''F_{20} - 2\Delta'G_{20} + \Delta G_{11},$$

$$v = E_{03} - 2F_{12} + G_{21} + \Gamma''E_{11} - 2\Gamma'E_{02} + 2\Gamma F_{02}, + \Delta''G_{20} - (\Gamma + 2\Delta')\,G_{11} + \Delta G_{02},$$

$$\theta = E_{02} - 2F_{11} + G_{20},$$

these being simultaneous solutions of the eighteen equations, which correspond to the vanishing of the derivatives of ξ and η of order 4 and of order 3 in the various arguments (§ 13). Further, we write

$$\begin{aligned} p = {} & E^2(-4E_{01}G_{10}G_{01} + 8F_{10}G_{10}G_{01} - 4G_{10}^3) \\ & + EF(-4E_{10}G_{10}G_{01} + 8E_{01}F_{10}G_{01} - 16F_{10}F_{01}G_{10} - 16F_{10}^2G_{01} + 16F_{10}G_{10}^2 \\ & \qquad + 4E_{01}G_{10}^2 + 8E_{01}F_{01}G_{10}) \\ & + EG(-2E_{10}E_{01}G_{01} + 6E_{10}F_{10}G_{01} + 4E_{10}F_{01}G_{10} - 4E_{10}G_{10}^2 - 4E_{01}F_{10}F_{01} \\ & \qquad - 2E_{01}F_{10}G_{10} + 8F_{01}F_{10}^2 - 4F_{10}^2G_{10} - 2E_{01}^2G_{10}) \\ & + F^2(-2E_{10}E_{01}G_{01} + 4E_{10}F_{01}G_{10} + 10E_{10}F_{10}G_{01} - 4E_{10}G_{10}^2 - 12F_{10}^2G_{10} \\ & \qquad + 24F_{10}^2F_{01} - 6E_{01}F_{10}G_{10} - 12E_{01}F_{10}F_{01} - 2E_{01}^2G_{10}) \\ & + FG(8E_{10}E_{01}F_{01} - 4E_{10}^2G_{01} + 4E_{10}E_{01}G_{10} - 32E_{10}F_{10}F_{01} + 16E_{10}F_{10}G_{10} + 8E_{01}^2F_{10}) \\ & + G^2(8E_{10}^2F_{01} - 4E_{10}^2G_{10} - 4E_{01}^2E_{10}), \end{aligned}$$

$$\begin{aligned} q = {} & E^2(8F_{10}G_{01}^2 - 4E_{01}G_{01}^2 - 4G_{01}G_{10}^2) \\ & + EF(-4E_{10}G_{01}^2 - 32F_{10}F_{01}G_{01} + 8F_{10}G_{10}G_{01} + 4E_{01}G_{10}G_{01} + 16E_{01}F_{01}G_{01} \\ & \qquad + 8F_{01}G_{10}^2) \\ & + EG(6E_{10}F_{01}G_{01} - 2E_{10}G_{10}G_{01} + 4E_{01}F_{10}G_{01} + 8F_{10}F_{01}^2 - 4F_{10}F_{01}G_{10} \\ & \qquad - 4E_{01}^2G_{01} - 2E_{01}F_{01}G_{10} - 4E_{01}F_{01}^2 - 2E_{01}G_{10}^2) \\ & + F^2(10E_{01}F_{01}G_{01} - 2E_{10}G_{10}G_{01} + 4E_{01}F_{10}G_{01} + 24F_{01}^2F_{10} - 12F_{10}F_{01}G_{10} \\ & \qquad - 2E_{01}G_{10}^2 - 6E_{01}F_{01}G_{10} - 4E_{01}^2G_{01} - 12E_{01}F_{01}^2) \\ & + FG(-4E_{10}E_{01}G_{01} - 16E_{10}F_{01}^2 + 8E_{10}F_{01}G_{10} - 16E_{01}F_{10}F_{01} + 8E_{01}F_{10}G_{10} \\ & \qquad + 16E_{01}^2F_{01} + 4E_{01}^2G_{10}) \\ & + G^2(8E_{10}E_{01}F_{01} - 4E_{10}E_{01}G_{10} - 4E_{01}^3). \end{aligned}$$

Also, we write

$$\lambda_1 = 4V^4u - 4V^4\theta\,(2\Delta' + 3\Gamma) \;- p,$$

$$\lambda_2 = 4V^4v - 4V^4\theta\,(2\Gamma' + 3\Delta'') - q.$$

Then *a first expression for the differential invariant of deformation of the third order* is found to be

$$(E\lambda_2{}^2 - 2F\lambda_1\lambda_2 + G\lambda_1{}^2)\,V^{-14}.$$

This expression can be modified by means of the relation (§ 35)

$$4V^2(LN - M^2) = \nabla$$

$$= -2V^2\theta + E\,\{(E_{01} - 2F_{10})\,G_{01} + G_{10}{}^2\} + G\,\{E_{01}{}^2 - E_{10}\,(2F_{01} - G_{10})\}$$

$$+ F\,\{E_{10}G_{01} - E_{01}\,(2F_{01} + G_{10}) + 2F_{10}\,(2F_{01} - G_{10})\}.$$

Dividing both sides by V^2 and taking the derivative with regard to x, we find (on using the relations in § 6, and after reduction) that we have

$$\lambda_1 = -8V^4(NP - 2MQ + LR), = -8V^4\mathfrak{a},$$

say. Proceeding similarly from the derivative with regard to y, we have

$$\lambda_2 = -8V^4(NQ - 2MR + LS), = -8V^4\mathfrak{b},$$

say. It thus appears that *the two combinations of* E, F, G *and their derivatives up to the third order, represented by* λ_1 *and* λ_2, *are expressible in terms of the fundamental magnitudes of the second and the third order.* Moreover, dropping the numerical factor 64, we have *an expression for the differential invariant of deformation of the third order* (say I) in the form

$$IV^6 = E\mathfrak{b}^2 - 2F\mathfrak{a}\mathfrak{b} + G\mathfrak{a}^2.$$

By the theory in the preceding memoir, this invariant (which now involves only fundamental magnitudes of the first three orders and none of their derivatives) ought to be expressible in terms of the members of the system set out in § 53. Writing

$$w'_1 = (\mathfrak{a}, \mathfrak{b}\between\phi_{01}, -\phi_{10}),$$

$$w''_1 = (E\mathfrak{b} - F\mathfrak{a},\; F\mathfrak{b} - G\mathfrak{a}\between\phi_{01}, -\phi_{10}),$$

we find

$$w_2V^6I = V^2w'_1{}^2 + w''_1{}^2,$$

$$w_2{}^2w'_1 = w_2w_3I\,(w_2, w'_2) + w_2w'_2\mathfrak{a}_1 - 2J\,(w_2, w'_2)\,J\,(w_2, w_3) - 2V^2w'_2w_3,$$

$$w_2{}^2w''_1 = w_2I\,(w_2, w'_2)\,J\,(w_2, w_3) + 2V^2w_3J\,(w_2, w'_2) - 2V^2w'_2J\,(w_2, w_3)$$

$$- 2w_2\mathfrak{a}_1J\,(w_2, w'_2) + w_2w'_2\mathfrak{a}_2.$$

When the geometric values of the several invariants are substituted, we find

$$w'_1 = BV^3 \frac{dK}{ds},$$

$$w''_1 = BV^4 \frac{dK}{dn};$$

and therefore

$$I = \left(\frac{dK}{ds}\right)^2 + \left(\frac{dK}{dn}\right)^2,$$

which is *the geometric significance of the differential invariant of deformation of the third order.* Its expression appears to involve association with the curve $\phi = 0$; but the relations in § 51 shew that the association is the same for all curves, so that the quantity is a function solely of position on the surface, being the sum of the squares of the first derivatives of K along any two perpendicular directions along the surface.]

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OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 403–433.

ON THE LAWS GOVERNING ELECTRIC DISCHARGES IN GASES AT LOW PRESSURES

BY

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LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 340. 8.7.03

Price Two Shillings.

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X. *On the Laws Governing Electric Discharges in Gases at Low Pressures.*

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Communicated by Professor J. J. THOMSON, *F.R.S.*

Received, February 11,—Read, March 5, 1903.

I. *Introduction.*

THE researches of recent years have conclusively settled the general connection between the spark potential and the pressure of a gas. It is now well known that as the pressure of a gas diminishes the difference of potential necessary to produce a discharge between electrodes in the gas, a fixed distance apart, also diminishes, until, at a critical pressure, the spark potential reaches a minimum value. It is further established that below the critical pressure the potential difference required to produce discharge rapidly increases as the pressure is lowered.

This connection between the spark potential and the corresponding pressure of a gas has been well illustrated in a series of curves drawn by PEACE,* who investigated the sparking potentials between a pair of parallel plates at pressures ranging from one-half an atmosphere down to a little below the critical pressure.

Among others, STRUTT† and BOUTY‡ have carried on the investigation at pressures considerably below the critical point, and their results show that, once the critical pressure has been passed, the rise in potential difference necessary to produce discharge is exceedingly rapid.

The effect of varying the distance between the electrodes was first determined by PASCHEN,§ who observed the existence of a simple law connecting the pressure at which discharge took place with the corresponding spark potential and the distance between the electrodes.

PASCHEN'S results showed that when a given potential difference was applied to two spherical electrodes, whose distance apart could be varied, the maximum pressure at which discharge occurred varied inversely with the distance between the spheres.

The range of pressures over which he found the law to apply, while considerable, did not extend below 2 centims. of mercury, and his results do not in any case indicate that the critical pressure had been reached. It is evident, then, that PASCHEN'S conclusions are confined to pressures higher than the critical pressures.

* PEACE, 'Roy. Soc. Proc.,' vol. 52, p. 99.

† STRUTT, 'Phil. Trans.,' A, vol. 193, p. 377.

‡ BOUTY, 'Compt. Rend.,' vol. 131 (2), p. 443.

§ PASCHEN, 'Ann. d. Phys.,' vol. 37, p. 69.

Since the statement of this law by PASCHEN, PEACE* alone seems to have published results which could throw any additional light on the conditions holding for discharge in a gas at very low pressures. PEACE experimented in air, with parallel plates as electrodes, at various distances apart, and found that the value of the critical pressure increased greatly as the distance between the electrodes was lessened, but his results at points below the critical pressure give no evidence of the existence of any such law as had been enunciated by PASCHEN.

This can be readily seen from the numbers recorded in his paper, a few of which, selected from readings taken below the critical pressure, are given in the following table. These results admit of easy comparison, since the potential differences in the cases chosen are very nearly the same. The product of pressure and spark length should be a constant quantity if PASCHEN'S law held.

TABLE of PEACE'S Results.

Applied potential difference in volts.	Pressures in millims. of mercury.	Distance between electrodes in inches.	Product of pressure and spark length.
649	2·5	·082	·205
660	6	·005	·030
670	5	·021	·105
731	2·5	·030	·075

If we compare the first and second of these results where the difference in spark potentials is only 11 volts, we find the product in the first case nearly seven times that in the second. Again, the product corresponding to the spark potential 660 volts is less than one-third that corresponding to 670 volts, a large difference in the opposite direction. The same irregularity is exhibited by the product corresponding to the spark potential 731 volts, and it seems difficult to understand how experimental errors could be made to explain such a wide divergence of results.

At the critical pressure PEACE'S results point to the existence of the law, but, as stated above, it would appear that as soon as lower pressures were approached the indications were uniformly against the existence of the relation which PASCHEN found to hold at high pressures.

Owing to the special precautions taken by PEACE to obtain accurate values for the spark potentials, it is possible to arrive at but one of two conclusions regarding the departure from PASCHEN'S law indicated by PEACE'S numbers. Judging by the results, either the law ceases to hold when the critical pressure is passed, or else the apparatus used by him in his experiments did not admit of an accurate measurement of the actual spark lengths corresponding to different spark potentials.

* PEACE, 'Roy. Soc. Proc.,' vol. 52, p. 99.

A short discussion of the apparatus will reveal one considerable defect. The object of the investigations of both PASCHEN and PEACE was to determine the electromotive intensity requisite to cause discharge in a gas. Throughout the range of pressures investigated by PASCHEN the discharge always took place along the shortest distance between the spherical electrodes, and the electromotive intensity requisite to break down the gas was therefore directly proportional to the spark potentials obtained by him. At points below the critical pressure, as PEACE's results indicate, discharge occurs more easily over a longer distance than over a shorter one, and if the values of the electromotive intensities necessary to break down a gas at different pressures are to be compared, it is necessary to know in each case not only the potential difference applied to the electrodes, but also the path between the electrodes along which the initial discharge occurs.

To insure passage of the discharge over the same length of path PEACE used plane parallel plates of very large diameter as electrodes, but while in this way he obtained a uniform field of considerable extent, and so was able to obtain an accurate measure of the electromotive intensity between the electrodes, he failed to make certain that the path along which the gas initially broke down was always confined to the uniform part of the field. As mentioned in his paper, there was considerable tendency, at low pressures, to a brush discharge from the edges of the plates, and this indicated a defect in his apparatus, which apparently he did not completely eliminate.

In the present paper an account is given of an investigation on the potentials necessary to produce discharge in a gas, with a form of apparatus which insured the passage of the discharge in a uniform electric field.

With this apparatus the discharge potentials have been determined, for different distances between the electrodes, over a range extending considerably above and below the critical pressure. The results of the investigation not only confirm the truth of the law enunciated by PASCHEN for discharges at high pressures, but also demonstrate, beyond doubt, the applicability of the same law to the critical pressure and to all pressures below it.

The existence of the same relation has been sought in each of the gases air, hydrogen, and carbon dioxide, and the result of the investigation has been the establishment with equal certainty of the same general law for all pressures, viz., that with a given potential difference, the field being uniform, the product of the pressure at which discharge occurs and the distance between the electrodes is constant.

II. *Description of Apparatus.*

The form of the discharge chamber is shown in fig. 1.

The electrodes consisted of two plane brass plates a, a, 3·6 centims. in diameter, embedded in ebonite, as shown in the figure, the outer faces of the electrodes being flush with the surface of the ebonite. These pieces of ebonite which carried the electrodes served also to close the glass tube T, T, which thus constituted a discharge chamber. In order to confine the gas in this chamber to the region where the electric field was uniform, a ring of ebonite C, C, which projected over the edges of the brass plates, was inserted. In the construction of the apparatus special precautions were taken to insure that the plugs B, B pressed tightly against the ebonite ring. As a result of this device, that portion of the electric field which was not uniform was entirely confined to the space occupied by ebonite, so that in this way it was rendered impossible for a discharge to occur through the gas in any but a uniform field. The thickness of the ebonite ring, which could be made accurate to $\frac{1}{1000}$ millim., determined the distance between the electrodes and consequently the length of the discharge. The length of the discharge could be varied at will, therefore, by inserting rings of different thicknesses.

The gas was admitted and removed from the chamber by glass tubes sealed into the ebonite plugs, and these tubes were connected with the air-space by two very fine channels leading through the ebonite ring.

Before closing the discharge tube, which was made air-tight with ordinary commercial soft wax, the inner surface of the ebonite ring was carefully rubbed with glass paper to remove any conducting material from its surface.

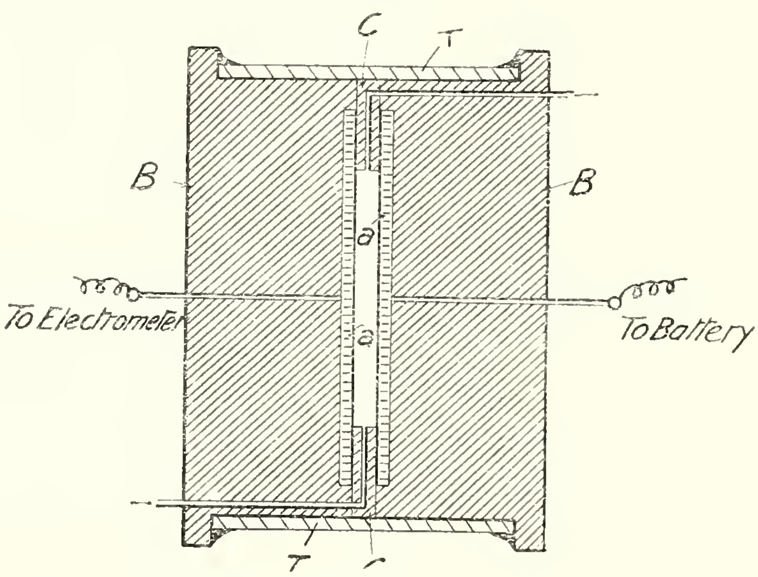


Fig. 1.

The potential differences used in these experiments were obtained from a series of small storage cells, similar to those used in the Reichsanstalt, Berlin. As these cells have a large capacity, their voltage remained constant over long intervals of time, and as a consequence it was possible to make the readings with the greatest accuracy. The potential differences were measured by a Weston voltmeter, which was carefully calibrated by means of a potentiometer furnished with a standard Weston cadmium element.

Throughout the investigation the discharge chamber was connected in series with a drying tube containing phosphoric pentoxide, a glass reservoir about 2 litres in volume, a McLeod pressure gauge giving readings accurate to $\frac{1}{1000}$ of a millimetre, and a mercury pump of small capacity. By using this reservoir and the pump of small capacity it was possible to diminish the pressure in the discharge tube by such exceedingly small amounts that it was easy to obtain a series of discharge potentials over the whole range of pressures investigated without the necessity of admitting fresh gas to the chamber.

In making measurements, one terminal of the battery was joined to earth and the other terminal was connected through a resistance of xylol to one of the electrodes of the discharge tube. The other electrode was permanently joined to one pair of quadrants of a quadrant electrometer, the second pair of which was kept to earth. In determining the potential difference necessary to produce discharge at a given pressure, the electrometer electrode was first earthed, a given potential applied to the battery electrode, and the earth connection of the electrometer electrode then removed.

If after waiting some minutes no discharge passed, the operation was repeated with a slightly higher potential applied to the battery electrode. This procedure was followed until a potential sufficiently high was reached to break down the gas and cause a discharge. The passage of the discharge could be readily noted, as it was accompanied by a violent deflection of the electrometer needle.

The well-known phenomenon of delay in the passing of the discharge, which has been investigated at length by WARBURG,* was observed throughout the experiments. It was especially marked in the neighbourhood of the critical pressure, discharge being frequently obtained ten or even fifteen minutes after the requisite voltage had been applied.

In every case, therefore, as the minimum sparking potential for any pressure was approached, a considerable time was allowed to elapse, with a given applied potential difference, before any increase was made.

III. *Experiments in Air.*

In the experiments on atmospheric air the whole discharge apparatus was first exhausted to a very low pressure and then re-filled by fresh air, which bubbled in very slowly, first through a wash-bottle of sulphuric acid and then through a tube tightly packed with phosphoric pentoxide. The discharge chamber was then exhausted to about 20 millims. of mercury and allowed to stand at this pressure for a period of from eight to twelve hours.

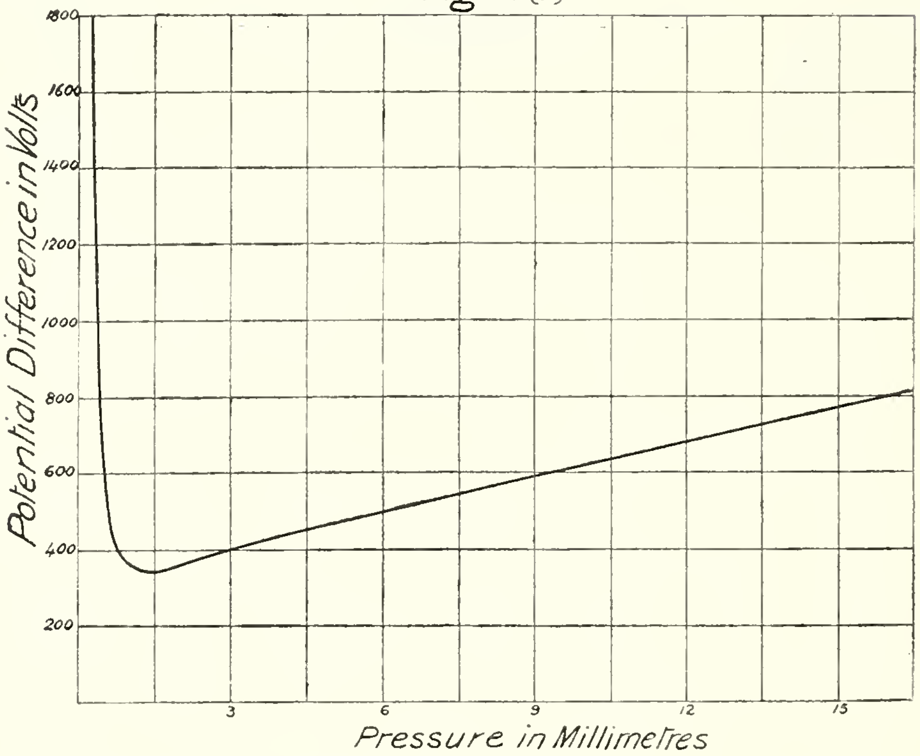
During this time the air was always in contact with phosphoric pentoxide in the drying tube, and was therefore entirely free from moisture when the measurements were taken.

The first measurements were made with the electrodes 3 millims. apart, and the spark potentials were determined over a range of pressures extending from 51 millims. down to ·35 millim. of mercury. The spark potentials corresponding to

* WARBURG, 'Ann. d. Phys.,' vol. 62, p. 385.

the various pressures are recorded in Columns V. and VI. of Table I., and the results are represented graphically in fig. 2A.

Fig. II (a)



In making these determinations, the precaution was always taken of allowing eight or ten minutes to intervene between consecutive readings, in order to make certain that the air was in its normal condition when the discharge occurred. As can be seen from the figure, the curve is quite regular and exhibits all the peculiarities already noted by PEACE,* STRUTT,† and BOUTY.‡ The curve, however, is carried much higher than those drawn by any of these experimenters, discharges corresponding to potential differences of over 1800 volts being recorded.

The distance between the electrodes was then varied and five different sets of readings were taken, in air, with the electrodes 1, 2, 3, 5, and 10 millims. apart. respectively. The complete set of numbers for these different spark lengths is given in Table I., and curves showing the readings taken over that portion of the range of pressure below 5 millims. of mercury are exhibited in fig. 2B.

It is apparent from the relative positions of these curves in the figure, that at points at and below the critical pressures, with a given potential difference applied to the electrodes, the pressures at which discharges occurred regularly decreased as the distance between the electrodes was increased. But a critical examination of the curves and also a reference to the numbers which they represent show that PASCHEN's law is rigidly applicable over the whole series of discharge potentials recorded.

* PEACE, 'Roy. Soc. Proc.,' vol. 52, p. 111.

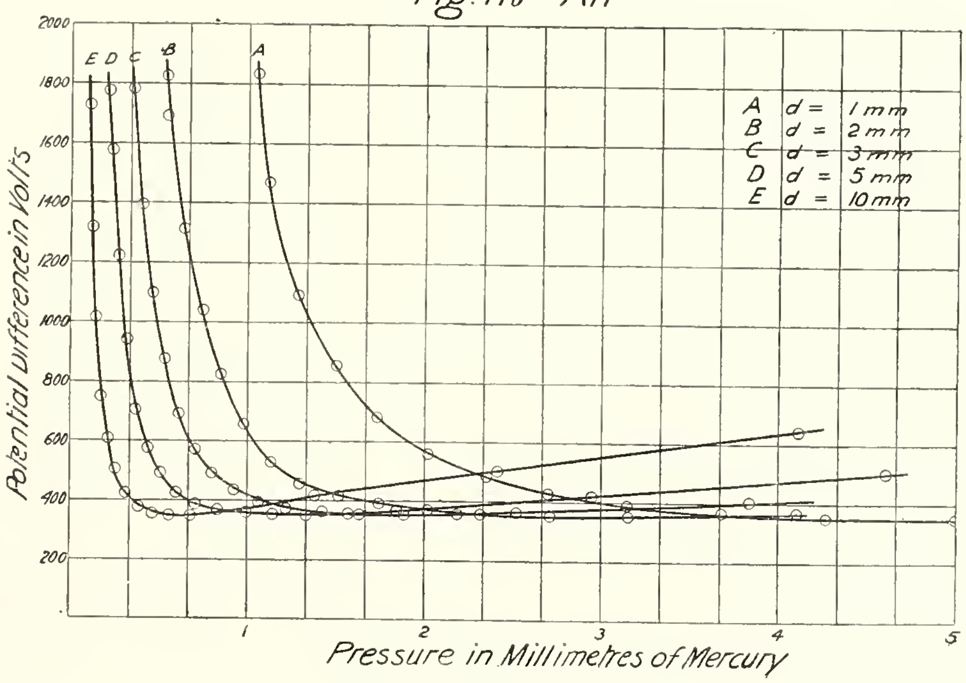
† STRUTT, 'Phil. Trans.,' A, vol. 193, p. 384.

‡ BOUTY, 'Compt. Rend.,' vol. 131 (2), p. 446.

TABLE I.—Air.

Spark length = 1 millim.		Spark length = 2 millims.		Spark length = 3 millims.		Spark length = 5 millims.		Spark length = 10 millims.	
Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.
150	1510	20	620	51	1480	7·34	600	7·09	831
120	1265	13·2	527	41·5	1275	4·61	504	4·12	645
90	1025	8·73	455	31·5	1015	2·95	418	2·39	504
61	784	5·52	400	21·4	790	1·85	368	1·39	420
40·8	634	4·11	373	14·1	630	1·57	356	·982	372
21·6	489	3·16	355	9·31	526	1·34	349	·805	355
19·4	477	2·71	351	5·99	452	1·14	352	·679	348
12·4	417	2·32	357	3·84	405	·982	359	·562	351
7·77	367	2·02	371	2·51	371	·839	370	·466	359
6·66	357	1·75	389	2·18	361	·714	388	·384	377
5·80	352	1·52	419	1·89	356	·607	427	·312	425
4·98	349	1·30	460	1·64	358	·517	484	·259	504
4·27	355	1·13	534	1·42	364	·440	575	·219	605
3·67	368	·982	654	1·22	375	·375	705	·180	757
3·15	392	·857	826	1·06	397	·321	935	·152	1020
2·70	429	·750	1042	·928	441	·276	1223	·125	1315
2·35	481	·643	1312	·804	494	·232	1585	·105	1730
2·02	558	·549	1695	·710	576	·216	1774	—	—
1·74	681	·536	1829	·616	691	—	—	—	—
1·51	855	—	—	·536	863	—	—	—	—
1·29	1090	—	—	·465	1092	—	—	—	—
1·12	1463	—	—	·411	1395	—	—	—	—
1·05	1826	—	—	·357	1786	—	—	—	—

Fig. IIb– Air



For example, the pressures at which discharge took place with an applied potential of 1800 volts were, for the different distances between the electrodes, approximately:

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	1·05
2	·536
3	·351
5	·216
10	·105

and it will be seen that the numbers in Column II. are almost exactly in inverse proportion to the numbers in Column I.

Again, with an applied potential of 500 volts (say), the approximate pressures at which discharge occurred were:

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	2·35
2	1·30
3	·804
5	·517
10	·259

where the pressures are in the ratio 1·00 : ·55 : ·34 : ·22 : ·11, numbers which are again very nearly inversely proportional to the distance between the electrodes.

Further, we notice that the spark potential corresponding to the critical pressure in all cases was practically the same, 350 volts, and the values of the critical pressures for the different spark lengths were, from Table I.:

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	4·98
2	2·71
3	1·89
5	1·34
10	·679

and these numbers, while not exactly in the ratio 10 : 5 : 3 : 2 : 1, are still very close to it.

In finding the values for portions of the curves around the critical pressures the results given in Table I. show that a small variation in potential difference was associated with a relatively very large change in the pressures, so that a very small error in reading the potential difference would result in a large error in the pressure readings. It is interesting to note, however, that even under these unfavourable conditions a striking agreement is presented between the results obtained at critical pressures and the results demanded by PASCHEN's law.

In order to make the agreement between the numbers demanded by PASCHEN's law and those obtained in these experiments still more evident, the results recorded in Table I. are again given in a slightly different form in Table II., where each potential difference is associated with the product of the pressure at which discharge took place and the corresponding spark length. PASCHEN* found that at high pressures these products were constant for different distances between the electrodes, as long as the applied potential difference was the same.

The numbers recorded in Table II. show that the same law is rigidly applicable to all pressures, both high and low.

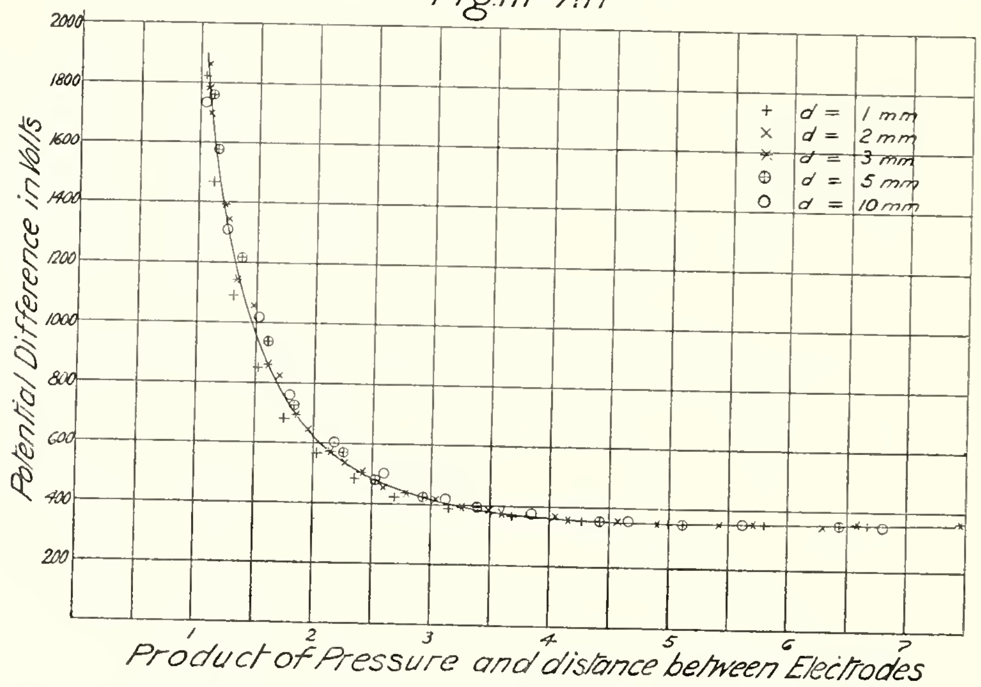
TABLE II.—Air.

Spark length = 1 millim.		Spark length = 2 millims.		Spark length = 3 millims.		Spark length = 5 millims.		Spark length = 10 millims.	
Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.
150	1510	40	620	153	1480	36·7	600	70·9	831
120	1265	26·4	527	124·5	1275	23·0	504	41·2	645
90	1025	17·4	455	94·5	1015	14·7	418	23·9	504
61	784	11·0	400	64·2	790	9·25	368	13·9	420
40·8	634	8·22	373	42·3	630	7·85	356	9·82	372
21·6	489	6·32	355	27·9	526	6·70	349	8·05	355
19·4	477	5·42	351	17·9	452	5·70	352	6·79	348
12·4	417	4·64	357	11·5	405	4·91	359	5·62	351
7·77	367	4·04	371	7·53	371	4·19	370	4·66	359
6·66	357	3·50	389	6·54	361	3·57	388	3·84	377
5·80	352	3·04	419	5·67	356	3·03	427	3·12	425
4·98	349	2·60	460	4·92	358	2·58	484	2·59	504
4·27	355	2·26	534	4·26	364	2·20	575	2·19	605
3·67	368	1·96	654	3·66	375	1·87	705	1·80	757
3·15	392	1·71	826	3·18	397	1·60	935	1·52	1020
2·70	429	1·50	1042	2·78	441	1·38	1223	1·25	1315
2·35	481	1·28	1312	2·41	494	1·16	1585	1·05	1730
2·02	558	1·09	1695	2·13	576	1·08	1774	—	—
1·74	681	1·07	1829	1·84	691	—	—	—	—
1·51	855	—	—	1·60	863	—	—	—	—
1·29	1090	—	—	1·39	1092	—	—	—	—
1·12	1463	—	—	1·23	1395	—	—	—	—
1·05	1826	—	—	1·07	1786	—	—	—	—

* PASCHEN, 'Ann. d. Phys.,' vol. 37, p. 69.

A like conclusion must be drawn from the curve shown in fig. 3, which graphically represents the numbers in Table II. In plotting this curve the products of spark lengths and discharge pressures were taken as abscissæ and the sparking potentials

Fig. III—Air



as ordinates. The regularity of the curve which represents the products for the five different electrode distances shows clearly that there can be no doubt regarding the applicability of Paschen's law to electric discharges, in air, at pressures at and below the critical point as well as at pressures above it.

IV. *Experiments in Hydrogen.*

In order to demonstrate, if possible, the generality of the law which has just been proven to hold for discharges in air, a series of measurements were made on the spark potentials in the gases hydrogen and carbon dioxide.

In these experiments exactly the same apparatus was used as in the previous experiments in air.

Preparatory to making the measurements in hydrogen the apparatus was first exhausted of air to a pressure of 1 millim. of mercury, or less, and then filled with hydrogen to atmospheric pressure. It was then exhausted and refilled with hydrogen several times to make certain that all air was removed.

The hydrogen was prepared from zinc and sulphuric acid in a Kipp apparatus, and, in order to ensure purity and freedom from moisture, was passed through wash-bottles containing potassium permanganate and caustic potash, and through a tube tightly packed with phosphoric pentoxide, before being led into the discharge chamber.

Also, just as in the experiments in air, the gas was always allowed to stand for several hours, at a pressure of about 20 millims. of mercury, in the presence of phosphoric pentoxide before any readings were recorded.

Table III.—Hydrogen.

Spark length = 1 millim.		Spark length = 2 millims.		Spark length = 3 millims.		Spark length = 5 millims.		Spark length = 10 millims.	
Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.
21·7	328	23	435	13·6	415	13·6	469	7·58	526
16·2	300	14·8	360	8·54	356	9·35	415	4·37	427
11·9	281	11·0	323	5·40	301	6·02	350	2·55	335
10·3	278	8·08	299	4·66	286	3·80	300	1·77	299
8·94	287	6·95	285	4·02	278	3·28	287	1·46	283
7·74	306	5·93	279	3·44	282	2·80	281	1·22	287
6·52	335	5·04	284	2·93	292	2·41	282	1·01	295
5·57	374	4·30	293	2·52	310	2·05	285	·846	313
4·73	487	3·72	305	2·15	356	1·76	293	·700	343
4·11	649	3·23	333	1·85	440	1·51	305	·575	426
3·54	905	2·77	399	1·59	564	1·26	345	·470	595
3·04	1275	2·36	523	1·35	780	1·09	410	·390	850
2·60	1781	2·03	727	1·16	1054	·928	539	·330	1142
—	—	1·73	1010	1·00	1382	·808	706	·276	1477
—	—	1·48	1380	·861	1789	·700	975	·264	1710
—	—	1·33	1746	—	—	·600	1373	—	—
—	—	—	—	—	—	·516	1775	—	—

In the experiments with this gas, readings were taken for the same electrode distances 1, 2, 3, 5 and 10 millims., and the values of the spark potentials and their corresponding pressures are given in Table III. These numbers are also graphically set forth in fig. 4.

We see from this table that the readings corresponding to the spark potential 1800 volts are :

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	2·60
2	1·33
3	·861
5	·516
10	·264

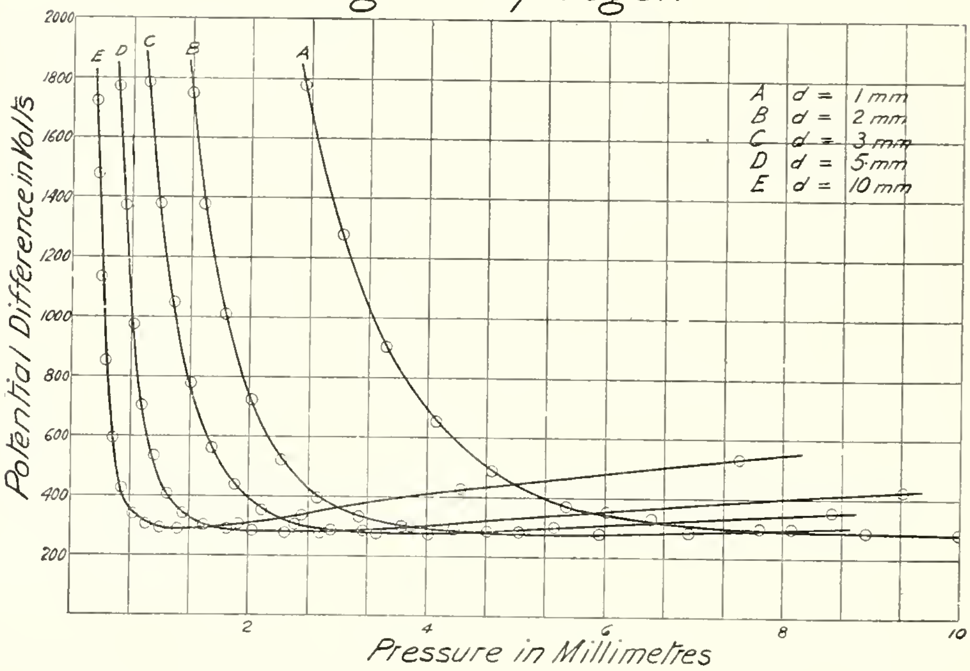
which pressures are in the ratio 9·9 : 5·0 : 3·2 : 2·0 : 1.

Again, with a spark potential of 500 volts the readings give :

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	4·7
2	2·4
3	1·7
5	·94
10	·51

the pressures being in the ratio 9·3 : 4·8 : 3·3 : 1·9 : 1.

Fig. IV—Hydrogen



The minimum spark potential in hydrogen was about 280 volts, and the critical pressures corresponding to the different spark lengths were :

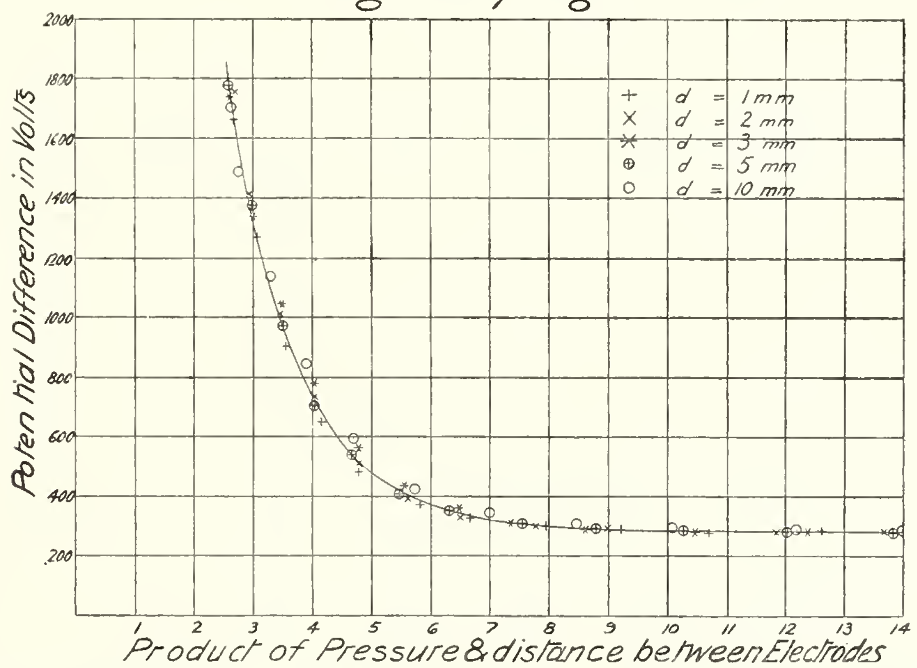
Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	10·3
2	5·93
3	4·02
5	2·80
10	1·46

where the various discharge pressures are once more nearly inversely proportional to the distance between the electrodes.

TABLE IV.—Hydrogen.

Spark length = 1 millim.		Spark length = 2 millims.		Spark length = 3 millims.		Spark length = 5 millims.		Spark length = 10 millims.	
Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.
21·7	328	46	435	40·8	415	68	469	75·3	526
16·2	300	29·6	360	25·6	356	46·7	415	43·7	427
11·9	281	22·0	323	16·2	301	30·1	350	25·5	335
10·3	278	16·1	299	13·9	286	19·0	300	17·7	299
8·94	287	13·9	285	12·0	278	16·4	287	14·6	283
7·74	306	11·8	279	10·3	282	14·0	281	12·2	287
6·52	335	10·0	284	8·79	292	12·0	282	10·1	295
5·57	374	8·60	293	7·56	310	10·2	285	8·46	313
4·73	487	7·44	305	6·45	356	8·80	293	7·00	343
4·11	649	6·46	333	5·55	440	7·55	305	5·75	426
3·54	905	5·54	399	4·77	564	6·30	345	4·70	595
3·04	1275	4·72	523	4·05	780	5·45	410	3·90	850
2·60	1781	4·06	727	3·48	1054	4·64	539	3·30	1142
—	—	3·46	1010	3·00	1382	4·04	706	2·76	1477
—	—	2·96	1380	2·58	1789	3·50	975	2·64	1710
—	—	2·66	1746	—	—	3·00	1373	—	—
—	—	—	—	—	—	2·58	1775	—	—

Fig. V – Hydrogen



To indicate further that the law is applicable at all points, a table of products, similar to that recorded for air, was calculated, and is given in Table IV. A single curve, fig. 5, represents these five sets of readings, and again the close grouping of the different results about this common curve shows that the law is equally applicable above and below the critical pressure to all spark potentials.

It is evident, then, that with hydrogen, just as with air, PASCHEN'S law is rigidly applicable over the whole range of pressures.

V. *Experiments in Carbon Dioxide.*

These further experiments were made with a view to corroborate the results already obtained in air and hydrogen. The same apparatus as had been used with these two gases again served for the experiments in carbon dioxide, and the distance between the electrodes was varied as before, so that readings were obtained at the five different distances 1, 2, 3, 5, and 10 millims. The carbon dioxide was prepared by treating marble with hydrochloric acid, and was purified and dried by being bubbled through a wash-bottle of water and passed through a tube tightly packed with phosphoric pentoxide before reaching the discharge apparatus. In each case the operation of exhausting the whole discharge apparatus to 1 millim., or less, of mercury, and then refilling with carbon dioxide was repeated five or six times, and finally the gas was allowed to stand as in both previous cases, in the presence of a bulb of phosphoric pentoxide for several hours.

The complete set of results is given in Table V., and the corresponding curves set forth in fig. 6, and if we again compare the discharge pressures and spark lengths corresponding to any value of the applied potential, the same law is seen to hold here also with even greater rigidity than in the other cases.

TABLE V.—Carbon Dioxide.

Spark length = 1 millim.		Spark length = 2 millims.		Spark length = 3 millims.		Spark length = 5 millims.		Spark length = 10 millims.	
Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.
19·8	516	21·3	802	8·75	674	9·10	790	7·27	993
12·6	480	13·8	645	5·57	563	5·77	674	4·26	790
9·41	443	8·76	519	3·55	477	3·64	579	2·43	656
6·83	425	5·41	464	2·25	427	2·33	498	1·44	553
5·86	421	4·02	439	1·91	420	1·45	438	·860	473
5·02	419	3·46	426	1·63	419	1·25	423	·612	428
4·31	420	2·95	421	1·41	425	1·07	421	·510	423
3·73	427	2·52	419	1·20	432	·919	428	·409	440
3·18	443	2·15	420	1·02	449	·786	441	·340	470
2·73	475	1·84	427	·875	487	·678	464	·280	506
2·34	503	1·58	443	·758	542	·572	495	·239	563
2·00	559	1·34	473	·651	599	·492	533	·196	639
1·72	636	1·16	525	·558	699	·419	599	·162	761
1·47	763	·980	605	·482	815	·360	704	·134	973
1·26	916	·848	702	·420	971	·310	820	·111	1219
1·08	1127	·728	847	·362	1162	·266	969	·094	1550
·946	1432	·625	1026	·314	1445	·232	1159	·089	1730
·817	1801	·536	1258	·274	1756	·196	1373	—	—
—	—	·455	1574	—	—	·169	1662	—	—
—	—	·421	1762	—	—	·164	1770	—	—

For 1800 volts the figures are approximately :

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	·817
2	·421
3	·274
5	.164
10	·0892

where the pressures are almost in the required ratio, being 9·2 : 4·8 : 3·0 : 1·9 : 1.

For 500 volts the numbers are :

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	2·34
2	1·23
3	·84
5	·57
10	·28

where the pressures are as 8·4 : 4·4 : 3 : 2 : 1.

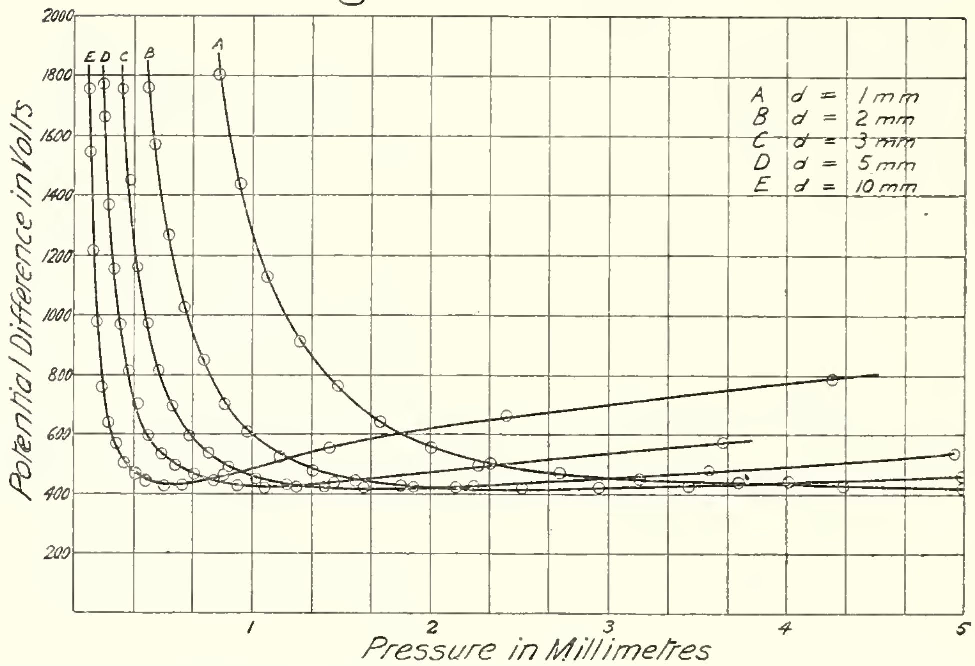
And at the minimum discharge potentials, which are again constant, 420 volts, the readings given are :

Distance between electrodes in millims.	Discharge pressures in millims. of mercury.
1	5·02
2	2·52
3	1·63
5	1·07
10	·510

Special attention is directed to these latter results, inasmuch as the exactness of the ratio indicated by the pressures is very remarkable. The ratios of the pressures are practically 10 : 5 : 3·1 : 2 : 1, the nearest approximation to the numbers demanded by PASCHEN's law which has been shown by any of the comparisons, and this result is all the more convincing in that these figures were obtained at the critical points, where, in the other two gases, the results obtained indicated the law in a somewhat less marked degree.

Though it would appear that further evidence was unnecessary, the table of products was again calculated and is given in Table VI. Also the corresponding curve is shown in fig. 7.

Fig. VI–Carbon Dioxide

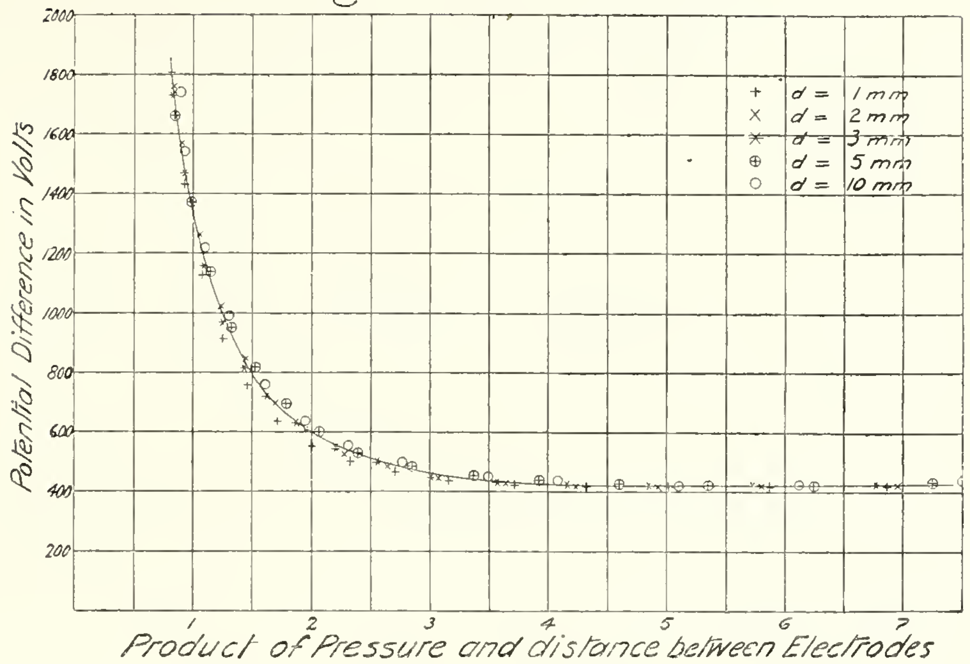


Once more the regularity of the curve shows that, as in air and hydrogen, so in carbon dioxide, Paschen's law is rigidly applicable to all spark potentials both above and below the critical pressure.

Table VI.—Carbon Dioxide.

Spark length = 1 millim.		Spark length = 2 millims.		Spark length = 3 millims.		Spark length = 5 millims.		Spark length = 10 millims.	
Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.	Product of pressure and spark length.	Spark potential in volts.
19·8	516	42·6	802	26·2	674	45·5	790	72·7	993
12·6	480	27·6	645	16·7	563	28·8	674	42·6	790
9·41	443	17·5	519	10·6	477	18·2	579	24·3	656
6·83	425	10·8	464	6·75	427	11·6	498	14·4	553
5·86	421	8·04	439	5·73	420	7·25	438	8·60	473
5·02	419	6·92	426	4·89	419	6·25	423	6·12	428
4·31	420	5·90	421	4·23	425	5·35	421	5·10	423
3·73	427	5·04	419	3·60	432	4·59	428	4·09	440
3·18	443	4·30	420	3·06	449	3·93	441	3·40	470
2·73	475	3·68	427	2·62	487	3·39	464	2·80	506
2·34	503	3·16	443	2·27	542	2·86	495	2·39	563
2·00	559	2·68	473	1·95	599	2·46	533	1·96	639
1·72	636	2·32	525	1·67	699	2·09	599	1·62	761
1·47	763	1·96	605	1·44	815	1·80	704	1·34	973
1·26	916	1·69	702	1·26	971	1·55	820	1·11	1219
1·08	1127	1·45	847	1·08	1162	1·33	969	·946	1550
·946	1432	1·25	1026	·942	1445	1·16	1159	·892	1730
·817	1801	1·07	1258	·822	1756	·98	1373	—	—
—	—	·910	1574	—	—	·845	1662	—	—
—	—	·842	1762	—	—	·820	1770	—	—

Fig. VII Carbon Dioxide



VI. *Spark Potentials with different Electrodes.*

It has now been shown, using brass electrodes of constant size, that, for discharges in a uniform field, in any gas, the values of the spark potentials are determined solely by the product of the pressure of the gas and the distance between the electrodes. From this result it appeared that if the size or material of the electrodes did not affect the results, the spark potentials were dependent only upon the quantity of the gas per unit cross section between the electrodes.

In order to determine this point, the brass electrodes which had been used up to this time were replaced in turn by electrodes of iron, zinc and aluminium, of exactly the same size. The results of the experiments showed that there was no variation in the different sets of readings, and it was evident that there was not the slightest effect produced in any case by a change in the material of which the electrodes were made.

In order to see if the size of the electrodes affected the values of the spark potentials for the different pressures, provided the discharge took place in a uniform field, a reduction was made in the surface of the electrodes exposed to the gas. This was done by replacing the ebonite rings C, C, fig. 1, which had an inner diameter of 3 centims., by others whose inner diameter was but 1 centim. By this device the areas of the electrodes exposed to the gas were reduced to about $\frac{1}{10}$ of their value in the early experiments, and the condition that the discharge could only take place in a uniform field still held. Using this apparatus with air, no difference could be observed in the values of the discharge potentials corresponding to the different pressures, and it was therefore certain that the value of the spark potential was in no way influenced by the size of the electrodes.

It is therefore clearly established that the only factors affecting the spark potentials are pressure and the distance between the electrodes, and hence PASCHEN's law is most accurately expressed by saying, "that, with a given applied potential difference, discharge in a uniform field, in any gas, is dependent solely on the constancy of the quantity of matter per unit cross section between the electrodes."

VII. *Minimum Spark Potentials.*

An interesting result in connection with these experiments is the almost constant value obtained for the minimum spark potential with the different electrode distances in each of the gases.

PEACE,* in the paper already referred to, was able to point to the probable existence of such a condition, but his results were not sufficiently regular to allow him to speak with certainty from the evidence at that time in his possession. This is seen from the following table of results taken from his paper, which appear to be the readings upon which he based his conclusions :—

PEACE'S Table of Minimum Spark Potentials.

Spark length in millims.	Minimum discharge potential in volts.
·01	326
·025	330
·05	333
·1	354
·2	370
·3	390
·5	400
·7	428
1	458
2	475

While these results are of the same order, it will be noticed that the spark potential rapidly increases with the distance between the electrodes, and that the smallest value differs from the greatest by nearly 150 volts.

In the results recorded in the present experiments, however, it cannot be said that there is any indication of an increase in spark potential for an increasing spark length.

The minimum spark potentials observed in these experiments, for the three different gases, are given in the following table :—

* PEACE, 'Roy. Soc. Proc.,' vol. 52, pp. 107, 112.

OBSERVED Minimum Spark Potential in Volts.

Spark length in millims.	Air.	Hydrogen.	Carbon dioxide.
1	349	278	419
2	351	279	419
3	356	278	419
5	349	281	421
10	348	283	423

where it will be seen that the values of the minimum spark potentials, for air, over this large range of spark lengths vary by only 7 volts. The values for hydrogen, over the same large range of spark lengths, vary by only 5 volts, and those for carbon dioxide by only 4 volts.

These results, then, seem to establish the fact that the least spark potential required to break down a gas is entirely independent of the spark length.

It is evident, too, from figs. 3, 5 and 7, that the constancy of the minimum spark potential is a necessary condition to PASCHEN'S law holding for discharges at different electrode distances.

R. J. STRUTT,* in his paper "On the Least Potential Differences required to produce Discharge through Gases," has drawn the conclusion that the minimum spark potential for discharges in any selected gas is probably equal to the cathode fall, in the same gas, measured over the whole extent of the negative glow in the vacuum tube. Since the cathode fall in any gas has been shown by WARBURG† to be a constant, over a very large range of pressures, the constancy of the values obtained in these experiments for the least spark potential gives strong support to STRUTT'S conclusion.

Moreover, the value of the least spark potential found by STRUTT in air, using a spark length of $\frac{3}{4}$ millim., was 341 volts. This agrees very well with the numbers given above, the difference being only about 8 or 9 volts. For hydrogen, however, the agreement between the results is not so good, his values for the least spark potential, 302–308 volts, being somewhat higher than those found for hydrogen in these experiments.

In this connection it may be pointed out that special precautions were taken in the neighbourhood of the critical pressure to make certain that the gas was in its normal condition when the discharge occurred, and so make sure that the spark potential obtained was not too small. The procedure followed was to apply a low voltage to the electrodes, and then gradually increase it until the discharge passed. By this procedure there could be no doubt that the gas was always in its

* STRUTT, 'Phil. Trans.,' A, vol. 193, p. 393.

† WARBURG, 'Ann. d. Phys.,' vol. 31, p. 545.

normal state, and that therefore no discharge could occur until the correct potential difference was reached.

After discharge did occur the gas was allowed to stand for a considerable time before the operation was repeated.

On account of the "delay" in the discharge, already referred to, special care was taken at the critical pressure to see that no voltage applied to the electrodes was replaced by a higher one, until a sufficient time had elapsed to make sure that discharge would not occur with the lower voltage.

VIII. *Connection between Spark Lengths and Spark Potentials.*

In the preceding experiments the spark potentials and corresponding pressures have been found for spark lengths ranging from 1 to 10 millims. It is evident from PASCHEN'S law, which has been shown to govern these discharges, that the different curves in figs. 2, 4 or 6 are interdependent, and that if one were given in each figure all the others could be deduced. It is clear, too, providing PASCHEN'S law applies, that curves can be deduced for spark lengths not included within these limits. This has been done in fig. 8, where curves corresponding to a number of spark lengths, in air, ranging from 1 millim. down to 5 micra* have been plotted.

The numbers corresponding to these curves were calculated by PASCHEN'S law from the experimental results obtained with a spark length of 1 millim. The values for spark lengths shorter than 5 micra have not been calculated, as there is evidence to show that PASCHEN'S law does not apply beyond this point. It can be seen that as the spark length is gradually decreased a length will be reached finally when the gas between the electrodes will consist of but two surface layers. It will then be subject to special molecular forces and, in all probability, a departure from the laws governing electric discharges in a gas under normal conditions will appear when this limiting spark length is reached.

This point has been well brought out by EARHART† in a paper on spark potentials for very short distances. He has shown, for a series of pressures, that a direct proportionality exists between spark potential and spark length, down to a spark length of about 5 micra. For shorter lengths than this he has shown that, while a law of proportionality still holds between spark potentials and spark lengths, the spark potentials diminish more rapidly for the same change in the spark length than they do in the range of longer distances.

It is true this critical spark length of 5 micra is of a higher order than most of the values found by a number of experimenters for the distance over which molecular forces act. The value of this distance given by QUINCKE,‡ deduced from the results

* 1 micron = ·001 millim.; EARHART, 'Phil. Mag.,' January, 1901.

† EARHART, 'Phil. Mag.,' January, 1901.

‡ QUINCKE, 'Pogg. Ann.,' 1869, vol. 137, p. 402.

of experiments on capillary phenomena, is about ·05 micron. REINOLD and RÜCKER* found that the range of unstable thickness of a film began somewhere between ·096

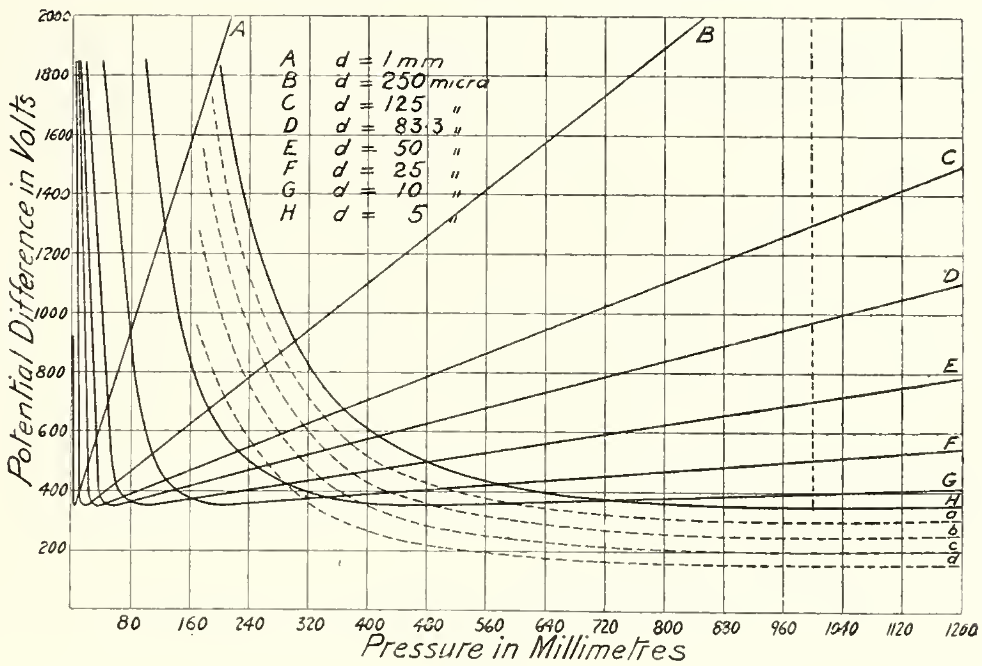


Fig. 8.

and ·045 micron. A value of the same order is given by PLATEAU,† who fixes the superior limit of the radius of the sphere of molecular action at ·118 micron.

Results of a higher order, however, were obtained by MÜLLER-ERZBACH‡ and KAYSER.§ The former of these made experiments on the thickness of water and carbon bisulphide films, and finally concluded that the radius of the sphere of molecular action is at least 1·5 micron. KAYSER, experimenting on condensation of gases on glass threads, fixed the range of molecular action at from 2 to 3 micra.

Now the distance between the electrodes when the air film is reduced to the two surface layers is equal to the diameter of the sphere of molecular action, and there is thus strong experimental evidence from the data given above to support our adopting EARHART'S value of 5 micra for the smallest length to which we can legitimately apply PASCHEN'S law.

The experiments described in this paper have been made with a view to finding the relation between spark potentials and corresponding pressures for a constant spark length in air and other gases, but, as all the results for different spark lengths are connected by PASCHEN'S law, it is easy to deduce curves, for any gas, expressing the relation between potential differences and corresponding spark lengths at selected pressures. Such curves, for air, deduced from those exhibited in figs. 2 and 8, have been plotted for a series of different pressures and are shown in fig. 9.

It will be seen that these curves present a number of points of special interest.

* REINOLD and RÜCKER, 'Phil. Trans.,' vol. 177, Part II., p. 684, 1886.

† PLATEAU, 'Statique des Liquides,' 1873, vol. 1, p. 210.

‡ MÜLLER-ERZBACH, 'Wied. Ann.,' vol. 28, p. 696, 1886.

§ KAYSER, 'Wied. Ann.,' vol. 14, p. 468, 1881.

The curve B, corresponding to a pressure of 1000 millims., which is the critical pressure for the spark length of 5 micra, fig. 8, is a straight line and shows that the spark lengths are directly proportional to the spark potentials for the whole range of spark lengths. It will be noticed, too, that at this pressure the minimum spark potential, 350 volts, to which special attention has been drawn in this paper, is that

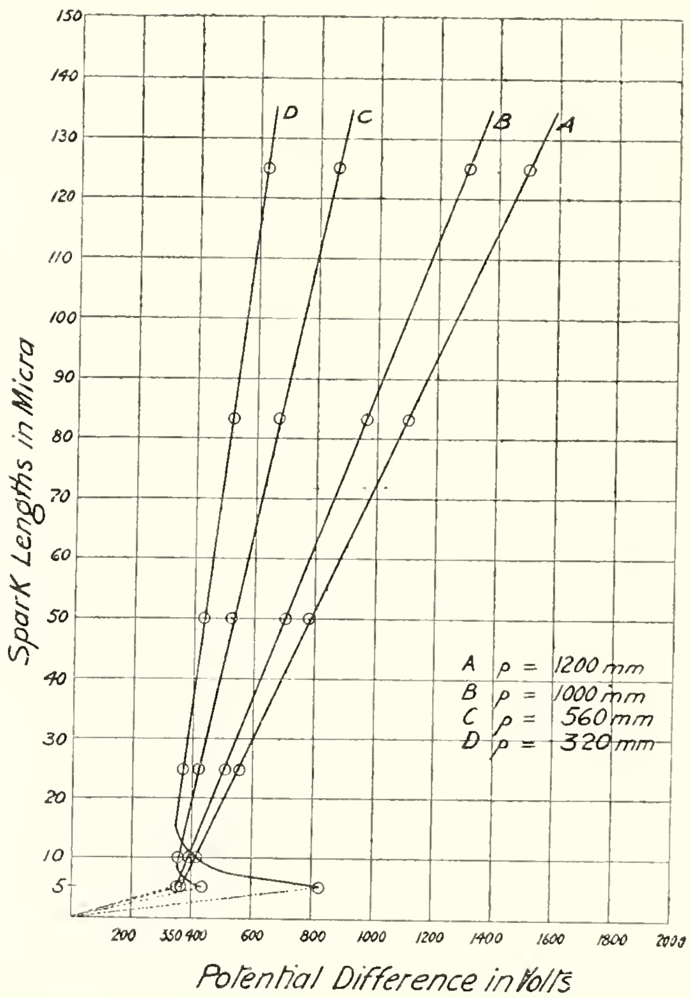


Fig. 9.

potential necessary to break down the gas for the shortest spark length to which we have considered PASCHEN's law is applicable.

Again, the curve D, which is typical of all the curves for pressures below 1000 millims. of mercury, expresses the relation between spark potentials and spark lengths for a pressure of 320 millims.

It shows that the ratio of spark potential to sparking distance is constant for all spark lengths greater than 15 or 16 micra. For shorter spark lengths the spark potential increases with decreasing spark lengths until finally the 5-micra line is reached at a potential of about 820 volts.

The curve A, which is a type of those which can be drawn for pressures above 1000 millims. of mercury, differs from B in but one feature. The law of proportionality again holds throughout for this pressure down to the shortest spark length, but it will be seen that a potential difference of about 365 volts is necessary to produce discharge when the spark length of 5 micra is reached.

While the three types of curves which have been described all present different characteristics, it will be seen that all are confined to spark lengths above 5 micra, and to spark potentials greater than 350 volts.

EARHART has shown that for spark lengths below 5 micra the spark potentials again decrease as the spark lengths are shortened, until finally the two electrodes come together and direct electrical contact is established. Throughout this lower range of spark lengths his results also show, for a series of pressures, that the spark potentials vary directly with the spark lengths.

These experimental results of EARHART give an indication of the forms the curves in fig. 9 would have taken had the experiments with the apparatus used in this investigation been extended to the shorter set of spark lengths. Had this been done,

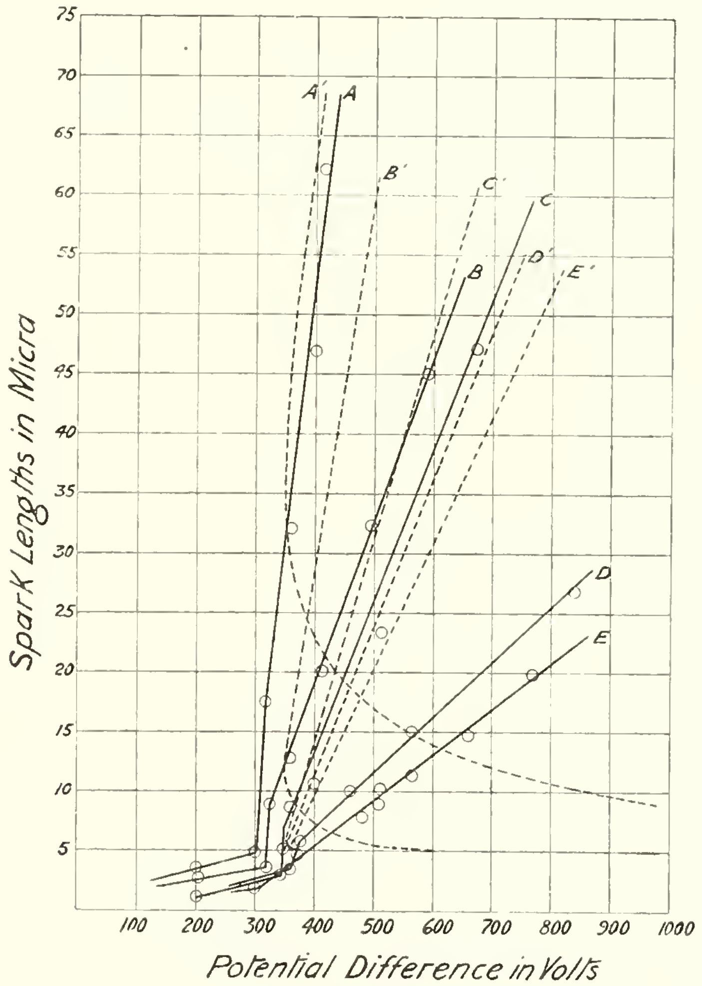


Fig. 10.

it is highly probable that the curves in fig. 9, on reaching the 5-micra line, would have followed courses such as are indicated by the dotted straight lines in the figure.

On this view it is of interest to examine the character of the pressure-spark potential curves that can be drawn in fig. 8 for spark lengths shorter than 5 micra.

It is clear that all these curves will lie below the 5-micra curve for every pressure. and since a law of proportionality applies between spark length and spark potential, at all pressures, it is easy to show that they fall off regularly down to the zero potential line. The dotted lines *a*, *b*, *c*, *d*, shown in fig. 8, indicate the relative positions of these curves.

In order to make a direct comparison between EARHART's curves and those which we have deduced, in fig. 9, by PASCHEN's law, a series of each is reproduced in fig. 10. In this figure, A, B, C, D, and E are drawn from the numbers given in EARHART's paper and correspond to pressures of 15 centims., 40 centims., 1. 2, and 3 atmospheres. respectively, while the dotted curves A′, B′, C′, D′ and E′ are deduced from fig. 8 for the pressures 15 centims., 40 centims., 1 atmosphere, 1000 millims. and 1200 millims.. respectively.

For the higher range of spark lengths it will be seen that EARHART's values are invariably larger than those deduced for the same pressures in this investigation. This difference is especially noticeable in connection with the curves B and B′, which correspond to a pressure of 40 centims. With a spark length of 50 micra, for example, EARHART's spark potential for this pressure is 625 volts, while that indicated by the curve B′ is but 470 volts, a difference of about 25 per cent. This difference, however, is exceptionally great, and extends over a very limited range of spark lengths. For distances greater than 100 micra, the values of the spark potentials do not appear to differ by more than 8 or 10 per cent. It is evident, too, from EARHART's diagram, that an irregularity exists in regard to his curve for this pressure, as it does not take up the position one should expect from his curves for higher and lower pressures.

A comparison of the curves corresponding to pressures of 15 centims. and 1 atmosphere also shows that the average difference between the spark potentials for each of these curves, over the higher range of spark lengths, does not exceed 8 per cent. This constant difference in the two sets of results in all probability is due, at least in part, to the difference in the form of the electrodes used in the two investigations, as both BAILLE* and PASCHEN† give results which show that, for spark lengths of this order, the spark potentials obtained with spherical electrodes are in every case considerably higher than those obtained when the electrodes are parallel plates.

When spark lengths slightly greater than 5 micra are reached, EARHART's curves A, B, C, and D become more nearly vertical, and indicate that over a considerable range of spark lengths the spark potentials remain approximately constant. It will be seen, too, that the vertical portion of the curves becomes shorter and shorter with increasing pressures, until finally, at 3 atmospheres, curve E, it disappears altogether.

The deduced curves A′, B′, C′ also present some characteristic features over the same range of spark lengths. They each exhibit a minimum spark potential which is reached in each case at approximately the spark length where the constancy of spark

* BAILLE, 'Annales de Chimie,' (5), vol. 25, p. 531. 1882.

† PASCHEN, 'Wied. Ann.,' vol. 37, p. 79, 1889.

potentials first appears in the corresponding curves of EARHART. These deduced curves then indicate rapidly increasing spark potentials down to the 5-micra line. It will be seen, too, that this feature of the curves extends over a range of spark lengths which diminishes with increasing pressures and finally disappears, as the curves D′ and E′ show, when a pressure of 1000 millims. is reached.

It is evident also from fig. 8 that the potential-spark length curves for all pressures greater than 1000 millims. (which is the critical pressure for the pressure-potential curve corresponding to 5 micra) will be similar in form to D′ and E′.

It thus appears that the two sets of curves, though differing widely in form for the lower range of pressures, yet present a resemblance as higher pressures are selected which becomes more and more marked. This can be seen very clearly from fig. 10, where each of the curves D′ and E′ has practically the same form as the curve E down to the 5-micra line and shows no indication of not following a course similar to E for spark lengths below 5 micra.

The explanation of the vertical portion of EARHART'S curves seems evident. The results are in reality precisely what one should expect to obtain for low pressures when electrodes other than parallel plates were used. Take, for example, a pressure of 500 millims., fig. 8, which is the critical pressure for a spark length of 10 micra. With parallel plates as electrodes, it is clear that the spark potential-spark length curve would consist of a straight line down to a spark length of 10 micra, at which distance the spark potential is 350 volts, the minimum spark potential for a gas under normal conditions. If the distance between the electrodes is still further reduced, the resistance offered by the gas increases and a potential difference higher than 350 volts will be necessary in order to obtain discharge. At a pressure of 500 millims., therefore, a spark length of 10 micra is the one which offers least resistance. With spherical electrodes, for all spark lengths above 10 micra, the shortest distance between the electrodes is that of least resistance, and the discharge will take place along this line. But when the shortest distance between the spherical surfaces is less than 10, but greater than 5 micra, this distance is no longer the one which offers least resistance to the passage of the discharge, and under these circumstances a longer, but less difficult path will be followed. The path which offers least resistance is clearly the one which corresponds to the minimum spark potential. It follows, then, that while the shortest distance between the electrodes is decreased from 10 to 5 micra and the gas is kept at a pressure of 500 millims. discharge will always occur with a constant spark potential of 350 volts and will follow the path which corresponds to this difference of potential. As EARHART'S experiments were performed with electrodes one of which was spherical and the other plane, the explanation will, in all probability, account for the ranges of constant spark potentials, which his results for different pressures indicate.

The explanation which has just been given evidently requires that the constant spark potential corresponding to the vertical portions of EARHART'S curves should be the same, 350 volts, for all pressures. But it will be seen that he obtained, for pressures up to two atmospheres, values varying from 325 to 370 volts. This discrepancy, however, though marked, is not large and possibly is within the range of experimental error.

The results which EARHART obtained for spark lengths shorter than 5 micra cannot in any way affect the validity of this explanation, for he has shown without doubt that the discharges in this range are governed by a law which does not apply to the gas under ordinary circumstances.

IX. *Spark Potentials in Different Gases.*

In a paper on the cathode fall of potential in gases, by CAPSTICK,* an attempt has been made to show that the cathode fall in a compound gas is related to the cathode falls in the elementary gases of which it is composed by a simple additive law. Experiments were made with hydrogen, oxygen, nitrogen, ammonia gas, and water vapour. The results, though not conclusive, were yet of sufficient weight to lead the author to the observation that the cathode fall is approximately an additive quantity and is probably a property of the atom rather than the molecule of a gas.

Owing to the difficulties experienced by CAPSTICK and others in overcoming the intermittence of the current in the case of compound gases, the effort to extend his investigations to compound gases other than those mentioned was abandoned, and the question up to the present time has remained unsettled.

As already pointed out in this paper, experimental evidence has been brought forward by STRUTT to show that the minimum spark potential should be equal to the cathode fall measured in the same gas. In view of this conclusion it seemed desirable to extend the experiments described in the first part of this paper to include a larger number of compound gases, in order to throw light, if possible, on the question raised by CAPSTICK. Measurements were therefore carried out with the gases oxygen, nitrous oxide, hydrogen sulphide, sulphur dioxide and acetylene. A constant spark length of 3 millims. was used throughout in order that a direct comparison could be made between the results obtained with these gases and those already recorded for hydrogen and carbon dioxide. The results obtained with all the gases, using this spark length, are recorded in Table VII., and curves corresponding to the readings at the critical and lower pressures are shown in fig. 11.

All the curves present the general characteristic of a minimum spark potential, followed, at lower pressures, by rapidly increasing spark potentials. It will be seen, also, that the critical pressures and the minimum spark potentials vary with the different gases. The curves, too, cut each other in regular order, and at the lowest pressures their relative arrangement with regard to the ordinate axis is practically the inverse of that assumed by them with reference to the abscissa axis above the critical pressures.

* CAPSTICK, 'Roy. Soc. Proc.,' vol. 63, p. 356.

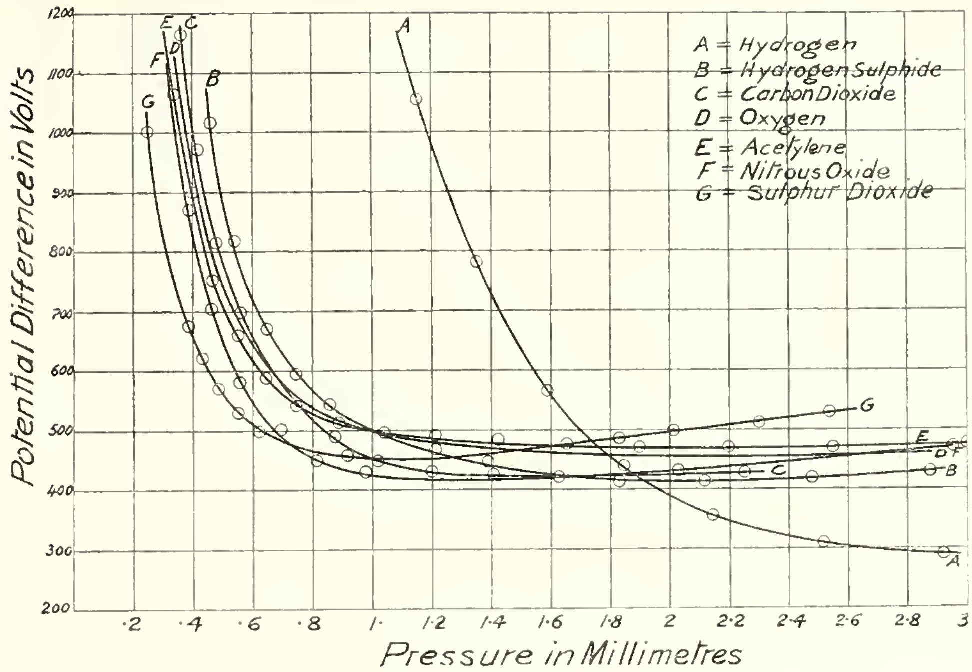


Fig. 11.

The values of the minimum spark potentials obtained in these experiments for the different gases are given in the following table :—

Gas.	Minimum spark potential in volts.
H_2	278
O_2	455
H_2S	414
CO_2	419
N_2O	418
SO_2	457
C_2H_2	468

Owing to the special precautions taken by STRUTT* to obtain an accurate value for the minimum spark potential in nitrogen, measurements were not taken with this gas. Adopting STRUTT's value of 251 volts for nitrogen, it will be seen that, with the exception of oxygen, all the minimum spark potentials given above obey an additive law; that is, if H′, N′, O′, &c., represent the spark potential constant corresponding to an atom of the gases H_2, N_2, O_2, &c., respectively, the minimum spark potential for any compound gas whose formula is $H_x . N_y . O_z$, &c., will be equal to $xH' + yN' + zO' +$ &c.

If we assume the truth of this law and calculate H′, N′, O′, &c., from the minimum spark potentials for H_2, N_2, H_2S, SO_2 and CO_2 we find :

$$H' = 139, \quad N' = 126, \quad C' = 98, \quad S' = 136, \quad O' = 161,$$

* STRUTT, 'Phil. Trans.,' A, vol. 193, p. 385.

TABLE VII.—Spark Length = 3 millims.

Hydrogen.		Carbon dioxide.		Nitrous oxide.		Acetylene.		Hydrogen sulphide.		Sulphur dioxide.		Oxygen by electrolysis.		Oxygen from potassium permanganate.		Oxygen from potassium chlorate.	
Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.	Pressures in millims. of mercury.	Spark potential in volts.
13·6	415	8·75	674	8·64	716	14	765	9·73	617	13·5	1145	8·42	576	8·07	556	8·75	596
8·54	356	5·57	563	5·00	560	8·7	633	5·29	467	7·7	842	4·85	523	4·71	494	4·92	524
5·40	301	3·55	477	2·95	470	7·4	583	3·90	442	4·5	651	3·42	483	3·25	466	3·44	477
4·66	286	2·25	427	2·03	430	6·4	555	2·88	428	2·54	531	2·82	468	2·72	456	2·86	464
4·02	278	1·91	420	1·70	420	5·5	530	2·48	418	2·30	511	2·34	461	2·27	453	2·37	439
3·44	282	1·63	419	1·41	418	4·7	509	2·12	414	2·01	501	1·94	458	1·88	462	1·97	457
2·93	292	1·41	425	1·18	422	4·1	490	1·83	414	1·83	486	1·60	455	1·57	474	1·62	457
2·52	310	1·20	432	·982	430	3·5	480	1·57	432	1·61	471	1·33	456	1·30	486	1·35	467
2·15	356	1·02	449	·816	450	3·0	474	1·39	457	1·40	466	1·10	463	1·08	501	1·12	484
1·85	440	·875	487	·670	500	2·55	468	1·20	472	1·23	461	·910	493	·901	516	·941	503
1·59	564	·758	542	·560	581	2·2	468	·86	543	1·20	459	·759	533	·750	536	·782	544
1·35	780	·651	599	·466	706	1·9	471	·73	592	1·04	457	·625	631	·616	566	·646	621
1·16	1054	·558	699	·389	866	1·66	476	·63	668	·92	459	·517	766	·501	645	·535	711
1·00	1382	·482	815	·326	1061	1·42	483	·54	817	·80	465	·428	1001	·428	763	·446	832
·861	1789	·420	971	·277	1370	1·21	490	·46	1013	·70	481	·354	1250	·357	987	·366	1003
—	—	·362	1162	·251	1830	1·04	498	·39	1286	·62	498	·294	1578	·303	1289	·305	1283
—	—	·314	1445	—	—	·89	513	·33	1650	·55	531	·268	1802	·260	1726	·255	1710
—	—	·274	1756	—	—	·65	588	—	—	·43	621	—	—	—	—	—	—
—	—	—	—	—	—	·47	754	—	—	·27	1000	—	—	—	—	—	—
—	—	—	—	—	—	·34	1064	—	—	·23	1590	—	—	—	—	—	—
—	—	—	—	—	—	·28	1302	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	·24	1650	—	—	—	—	—	—	—	—	—	—

and if we use these values to calculate the minimum spark potentials in the remaining gases, we obtain :

Gas.	Value found by experiment.	Calculated value.
C_2H_2	468	474
N_2O	418	412
O_2	455	321

The agreement between the observed and calculated values for each of the gases N_2O and C_2H_2 is very marked, and is a strong evidence that the additive law holds. The only case in which there is any serious difference between the observed and the calculated values is that of oxygen. Judging that this discrepancy might be due to impurities, three specimens of this gas were prepared by independent methods. It was prepared in turn by electrolysis, by heating potassium permanganate, and by heating a mixture of potassium chlorate and manganese dioxide. In every case the gas was purified by being passed through a mixed concentrated solution of potassium iodide and caustic potash and through concentrated sulphuric acid. It was also carefully dried in the usual way. It will be seen from Table VII. that the three sets of readings practically coincide at every pressure, and, since it is not possible that the same impurity could be present in each of these specimens to the same degree, it does not seem reasonable that the irregularity in oxygen could be traced to impurities arising from any lack of precaution in the preparation of the gases.

It is well known, however, that when an electric discharge is passed through oxygen, a considerable quantity of ozone is produced. It is in fact by this method that ozone in its purest form can be obtained. It is highly probable, then, that after the first discharge had passed between the electrodes, in the experiments on oxygen, a considerable percentage of ozone was present in the gas, and it may be that the discrepancy noted above is due to this cause. The experimental value of 455 volts found for oxygen seems to bear out this conclusion, for, ascribing the value of 161 volts to the atom of oxygen, we get by addition 483 volts as the calculated value of the minimum spark potential for ozone. The difference between the two values is but 28 volts, and assuming that the discharge occurs initially through the dissociation of ozone rather than of oxygen, the result is not in opposition to the additive law which has been shown to hold for the other gases. This large influence of a small amount of a denser gas when mixed with one less dense is in accord with the results obtained by previous experimenters, for WARBURG* and CAPSTICK† in their experiments on the cathode fall, and STRUTT‡ in his experiments on the

* WARBURG, 'Wied. Ann.,' vol. 31, p. 545.

† CAPSTICK, 'Roy. Soc. Proc.,' vol. 63, p. 360.

‡ STRUTT, 'Phil. Trans.,' A, vol. 193, p. 385.

minimum spark potential, found that a very small percentage of oxygen increased their values for nitrogen by an amount out of all proportion to the quantity of the denser gas present.

The calculated values for the minimum spark potential in water vapour and in ammonia are 439 volts and 543 volts respectively, and the values found by CAPSTICK for the cathode fall in these gases are respectively 469 and 582 volts. When we consider that the values in the one case are calculated from the measurements made on one effect, while the values in the second case are the direct experimental results on an entirely different effect, this comparatively close agreement not only forms a corroboration of STRUTT's conclusions, but also lends support to the view that the minimum spark potential has to do with the atoms rather than the molecules of a gas, and is determined, in any special case, by the application of a simple additive law.

In this connection it may be mentioned that the value found by STRUTT* for the cathode fall in the monatomic gas argon, 167 volts, corresponds very closely with the constants which we have ascribed to the atoms of the various gases mentioned above. His value, 226 volts, for the monatomic gas helium, however, is considerably larger than any of the atomic constants we have deduced.

In performing these experiments, all ordinary precautions were taken to ensure the purity of the gases. The nitrous oxide was prepared by heating ammonium nitrate in a flask, and the gas was collected over water, but was well dried with phosphorous pentoxide before being passed into the discharge tube. The sulphur dioxide was prepared from copper and sulphuric acid. In order to purify it the better, the gas was dried and then liquefied. It was further dried by being passed through a phosphoric pentoxide tube before reaching the discharge apparatus.

Acetylene was obtained in the usual way by the action of water on calcium carbide, and was carefully dried with sulphuric acid and phosphoric pentoxide. Hydrogen sulphide was prepared in a Kipp apparatus from ferric sulphide and sulphuric acid. It was slowly bubbled through wash-bottles of water and then carefully dried in the usual way.

In every case, as in the early part of the experiments, the gas remained in the discharge chamber in the presence of phosphoric pentoxide for several hours before any readings were taken.

X. *Summary of Results.*

1. The law governing electric discharges between parallel plates, in a uniform field, in any gas, for pressures at and below the critical pressures, is that which PASCHEN found to hold with spherical electrodes for high pressures, viz., that, with a given spark potential, the pressure at which discharge occurs is inversely proportional to the distance between the electrodes.

* STRUTT, 'Phil. Mag.,' March, 1900.

2. The values of the spark potentials are not influenced at any pressure by the size of the electrodes, provided the discharge takes place in a uniform field.

3. Plates of iron, zinc, aluminium, and brass were in turn used as electrodes, but the material out of which the electrodes were made was not found to affect the values of the spark potentials at any pressure.

4. When the discharge was compelled to pass in a uniform field between parallel plates the minimum spark potential in any gas was found to be a physical constant for that gas, being independent of the pressure and of the distance between the electrodes.

5. Evidence has been adduced which indicates that PASCHEN'S law is applicable to discharges in a uniform field between parallel plates as long as the distance between the electrodes is greater than the diameter of the sphere of molecular action.

6. The minimum spark potential has been shown to vary with different gases. The results obtained with a large number of elementary and compound gases show that the minimum spark potential is a property of the atom rather than the molecule, and that for any selected gas it may be calculated by the application of a simple additive law.

In conclusion, I desire to thank President LOUDON for the kindly interest he has always shown in my work by placing at my disposal every facility the laboratory afforded.

To Dr. J. C. McLENNAN, also, under whose immediate supervision these experiments were carried out, I am deeply indebted for valued assistance. I cannot adequately express how much I owe to his encouragement and advice.

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OF THE

ROYAL SOCIETY OF LONDON

Series A, VOL. 201, pp. 435–455.

ON THE DEPENDENCE OF THE REFRACTIVE INDEX OF GASES ON TEMPERATURE

BY

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LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS FRIEDLÄNDER AND SON, BERLIN.

1903.

A 341. 14.7.03

Price One Shilling and Sixpence.

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Phil. Trans., A., vol. 201, 1903, pp. 435–455.

XI. *On the Dependence of the Refractive Index of Gases on Temperature.*

By George W. Walker, *M.A.*, *A.R.C.Sc.*, *Fellow of Trinity College, Cambridge.*

Communicated by Professor J. J. Thomson, *F.R.S.*

Received February 26,—Read March 26, 1903.

The importance of this question was first impressed on me in the course of some theoretical investigations on refraction in gases and the closely related property of electric susceptibility.

A comparison of the actual temperature effect on a property of a body, with a theoretical formula professing to explain the property, is a very severe test, and one which has proved fatal to many theories.

According to Gladstone and Dale's law, of which most theories of refraction are particular cases, the refractive power of a gas is proportional to its density; or, as a formula,

$$\mu - 1 = \kappa\rho,$$

where μ is the refractive index,
ρ is the density,

and κ a constant depending on the gas, but independent of temperature. If, then, the gas closely obeys Boyle's and Charles' laws, we must have

$$\frac{(\mu - 1)(1 + \alpha t)}{p} = \frac{(\mu_0 - 1)(1 + \alpha t_0)}{p_0},$$

where p is the pressure, t is the temperature, and α the coefficient of expansion of the gas at constant pressure.

If the pressure is kept constant, we must have

$$\mu - 1 \propto \frac{1}{1 + \alpha t} \quad \ldots \ldots \ldots \quad (1).$$

Several observers have attempted to test this point.

Mascart,* Lorenz,† Benoît,‡ Von Lange§ made observations on the refractive

* 'Annales de l'École Normale Supérieure,' Series 2, vol. 6, 1877, p. 9.

† Wiedemann, 'Annalen der Physik,' vol. 11, 1880.

‡ 'Travaux et Mémoires du Bureau International des Poids et Mesures,' vol. 6 1888.

§ Poggendorff, 'Annalen der Physik,' vol. 153, p. 488.

index of air at different temperatures and made use of the formula (1) to calculate α. Their results are briefly as follows: LORENZ and BENOÎT obtained a value of α equal to the ordinary coefficient of expansion for air, VON LANGE obtained a value considerably less, while MASCART obtained a value considerably greater. LORENZ does not indicate what degree of accuracy he obtained, while BENOÎT and VON LANGE do not appear to have obtained as great accuracy as MASCART.

MASCART experimented on a number of gases, and in almost every case obtained a value of α appreciably greater than the corresponding coefficient of expansion of the gas. This range of temperature was from about 5° C. to 40° C.

The disagreement between the results of the above-mentioned experimenters in the case of air, and the somewhat limited range of temperature used by MASCART, led me to think that a repetition of the experiments on a few gases would be of value. I set myself the task of obtaining an accuracy of 1 in 500 over a range of temperature from 10° C. to 90° C., and I think the results show that this accuracy has been attained and in some cases surpassed.

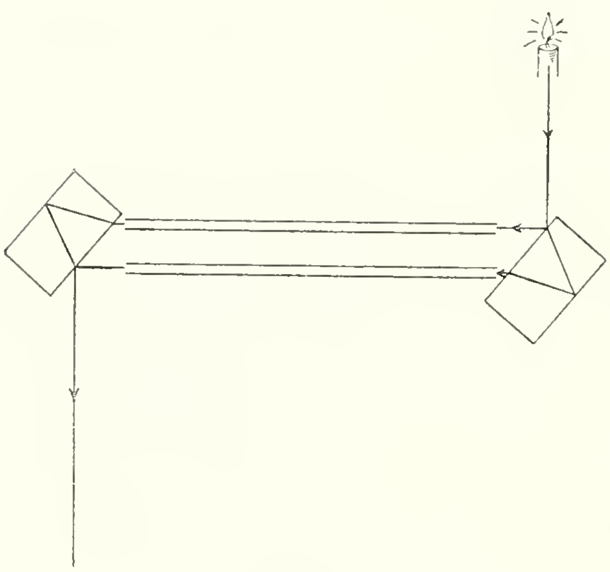

Fig. 1.

The method used was JAMIN's interference method, which I shall briefly describe, although it is well known (see fig. 1). The rays of light from a monochromatic flame fall on a thick glass block, whose faces are optically plane and parallel, the back face being silvered. Two parallel beams of light are thus produced and proceed through the two tubes filled with the gas and reach a second block of glass identical with the first. The two beams unite on emerging from the second glass block and produce interference bands, which may be observed through a telescope. When the pressure in one of the tubes is altered, the bands move across the field of view. As will be proved later, the number of bands displaced for a given difference of pressure enables us to calculate the refractive index of the gas.

The glass blocks which I used were made by REINFELDER UND HERTEL in Munich. The dimensions were 6 × 4 × 3 centims., the faces, 6 × 3, being optically plane and parallel. One of the blocks was placed on an adjustable screw stand, so that the necessary adjustments might be made. The other block was placed on a heavy block of hard wood.

The tubes for holding the gas were made of brass, and were about 100 centims. long and 1 centim. diameter, and had soldered to them at each end a stuffing box B (see fig. 2). The tubes were soldered to an outer jacket E, which was also made of brass, and was tightly wound on the outside with a thick layer of cotton wool. The vertical tubes *F* at each end of the jacket admitted the introduction of a thermometer fitted through a rubber cork. Steam or water entered at *G* and was pumped out at the corresponding hole at the other end of the jacket. *C* is an optically plane and parallel plate of glass, 17 millims. diameter, 1·5 millims. thick. The four plates were all cut from the same plate of worked glass by REINFELDER UND HERTEL. *D* is a piece of hollow cork to reduce eddies of cold air.

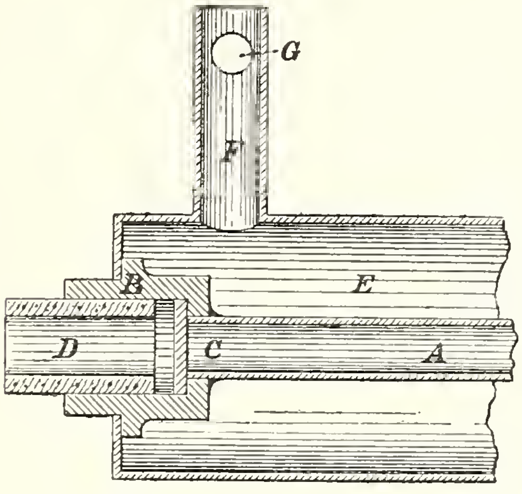

Fig. 2.

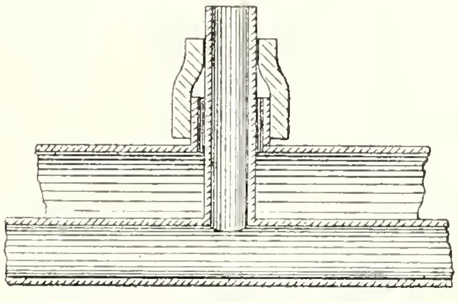

Fig. 3.

I experienced very great difficulty in making the joints between the glass plates and the brass tubes absolutely air-tight under the varying conditions of temperature and pressure. I succeeded finally by using a rubber washer $\frac{1}{5}$ millim. thick between the glass and the brass, and then painting bicycle enamel round the junction. This material dries rapidly and hardens, but still with sufficient elasticity to avoid straining of the glass. It is not porous, nor does it melt or even soften at 100° C. It is, moreover, soluble in ether, so that the glass plate can be recovered unimpaired.

Small brass tubes (see fig. 3) passed through the side of the jacket and were screwed and soldered, one to each of the long brass tubes. These served to connect the tubes with the manometer for recording the pressure.

These small brass tubes passed through short brass tubes of slightly larger diameter, soldered to the jacket, thus allowing play during alteration of temperature. The joint was made by a short piece of thick rubber tube, wired, and painted over with black enamel.

The steady temperatures required were obtained as follows:—Tap water was drawn through the jacket by means of a water pump. This gave temperatures about 10° C. Higher temperatures, such as 20° C., or 30° C., were obtained by drawing the water through lead spirals of different sizes, immersed in a saucepan of water kept boiling. Temperatures from 50° C. to 100° C. were obtained by boiling water under reduced pressure in an old mercury bottle, and drawing the steam through the jacket by the water pump.

The arrangements for altering the pressure in the tubes of gas and measuring the differences are shown in fig. 4. The glass tube D connected the brass tube M with

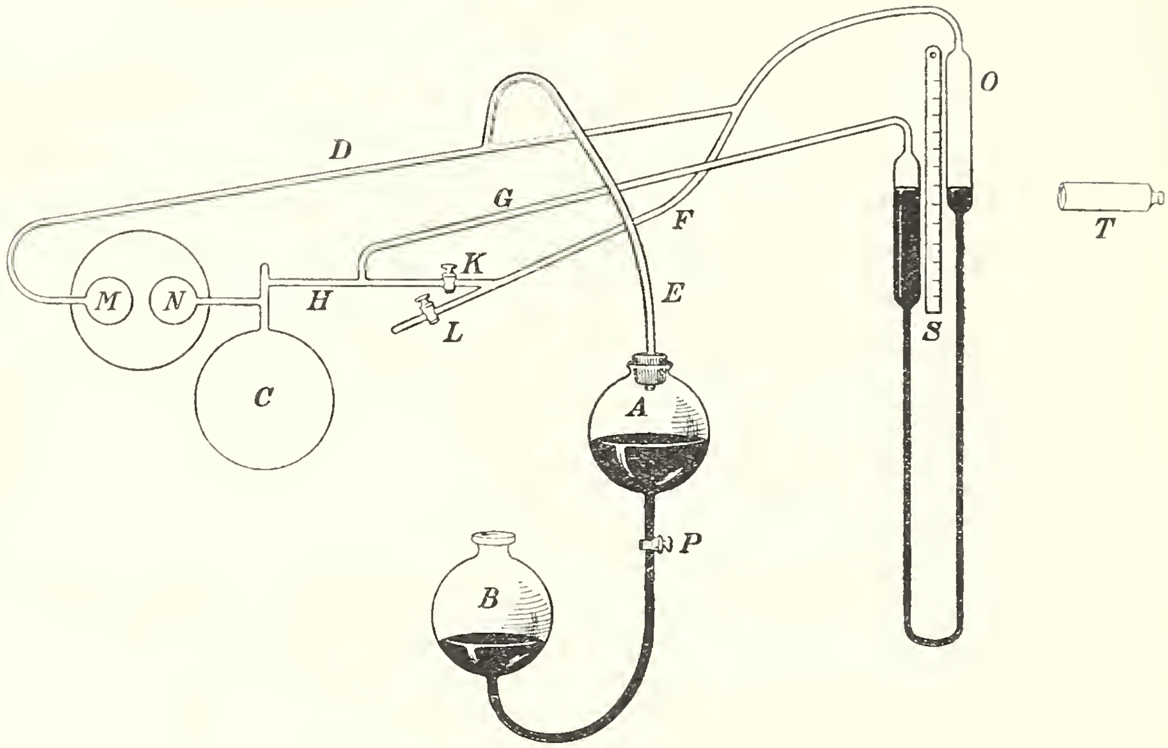

Fig. 4.

one limb of the manometer. The branch F led to a tap L, through which different gases could be passed into the apparatus.

The branch H led to the second brass tube N, and from H the branch G led to the second limb of the manometer. The branch E was connected to a glass reservoir A of about $\frac{3}{4}$ litre capacity, the tube being drawn to a fine capillary just before entering the reservoir. A was connected with a second reservoir B by a rubber tube. On lowering B, mercury ran from A to B, and thus the pressure could be altered. The tap K being shut, a difference of pressure in the two brass tubes was produced, which could be measured on the manometer. C is a glass bulb introduced to keep the pressure in N more nearly constant than it would otherwise be. The manometer was over an inch in diameter in the wide portions, and thus capillary error was avoided. A steel scale S was hung between the two limbs and the level of the mercury, read by means of a telescope of a cathetometer placed in front of the manometer and about 150 centims. from it.

Wherever it was possible, the glass joints were made by means of a blow-pipe, and the only other joints were at the reservoir A and at the small tubes connected with M and N. These were made with thick rubber tube, wired, and painted with black enamel. The glass taps K and L were fitted with mercury seals. Very great care was taken in testing to see that the whole apparatus was absolutely air-tight.

As a source of light, I used an ordinary Bunsen burner, placed about 150 centims. from the first glass block, and a small piece of bicarbonate of soda was held in the flame. This gave a brilliant yellow flame for a long time. The position of the interference bands was observed in a telescope with a micrometer scale in the eye-piece.

Theory of the Measurement.

MASCART and others have established that for pressures in the vicinity of atmospheric pressure the refractive power is proportional to the pressure, or

$$\mu = 1 + \kappa p,$$

where κ is a function of the temperature.

This is only true in cases where BOYLE'S law practically holds. In the case of such gases as ammonia, where the deviation from BOYLE'S law is appreciable, a correction is required.

Let d be the length of either tube,
λ the wave-length of Na light,
p_0 the initial pressure in the tubes,
p_1 the final pressure in first tube,
p_2 the final pressure in second tube, and
n the number of bands displaced, then

$$\kappa = \frac{n \times \lambda}{(p_1 - p_2)\, d}.$$

The measurements were made as follows:—Steam or water was allowed to run through the jacket for over an hour until the temperature was steady, and no drift of the bands was observed when the tap K was open. The bands were then adjusted so that a band was on the cross wire in the telescope. The two limbs of the manometer were read, and also the two thermometers in the jacket, and the thermometer hung beside the manometer. The tap K was then shut, the reservoir B lowered, and the tap P opened. When about 100 bands had passed, the tap P was shut and the position of the band observed, the manometer read, and also the three thermometers. B was then raised, the tap P opened, and the mercury allowed to flow back to A. The tap P was shut when the original position was attained and the readings again made. This method provides a test of any possible drift of the bands in one direction due to creeping changes of temperature. The proper temperature to take is readily seen to be the temperature as recorded when the pressure has been reduced, and not the mean of the initial and final temperatures.

With regard to the accuracy I consider that I was able to estimate $\frac{1}{10}$ millim. on the manometer scale. An interference band being a fuzzy thing and not sharp, I found it impossible to estimate more than $\frac{1}{10}$th of a band. The bands appeared about

5 millims. apart in the eye-piece, and the breadth of a band about 1 millim., or $\frac{1}{5}$th of the distance between two bands. It is therefore useless to have a screw micrometer reading to $\frac{1}{100}$th of a band when the eye cannot judge more than $\frac{1}{10}$th. In the case of air at 10° C., 100 bands corresponds to a difference in level in the limbs of the manometer of about 16 centims. We may therefore consider the quantity $\frac{\text{bands}}{\text{pressure}}$ to be accurate to 1 part in 500.

With regard to the thermometers, they were made by R. MITTELBACH, in Göttingen, and divided in half degrees from 0° C. to 100° C. They could easily be read to $\frac{1}{10}$th of a degree. but this accuracy is not necessary. I had one of the thermometers standardised at Kew and compared the others with it under the same conditions as in the experiments. The thermometers used in the experiments were placed in a bath at constant temperature, the same amount of stem being exposed as in the actual experiments, while the standardised thermometer was completely immersed.

Atmospheric Air.

The air in the laboratory was used, and dried by means of phosphoric pentoxide. The tap L was kept shut so that the same air was in the apparatus throughout the experiments.

It would serve little purpose to give all the readings taken; and I shall confine myself to a few specimens. Throughout the initial pressure was as nearly as possible atmospheric pressure.

The thermometers in the jacket were marked 7 and 9, and the thermometer placed under the scale of the manometer marked 6.

15th November, 1901.

	Readings of thermometers in degrees Centigrade.			Readings of manometer in centimetres.		Number of bands.	Differences.		Ratio, $\frac{\text{bands}}{\text{pressure}}$.
	9	7	6	Right limb.	Left limb.		Pressure.	Bands.	
	10·3	10	13·4	15·60	15·60	—	—	—	—
(1) . .	10·3	10	13·5	23·49	7·58	100	15·91	100	6·285
(2) . .	10·2	9·9	13·5	15·65	15·57	99·4	15·83	99·4	6·279
	10·2	9·9	13·5	15·60	15·60	—	—	—	—
(3) . .	10·3	10	13·5	23·48	7·58	100	15·90	100	6·289
(4) . .	10·4	10·1	13·5	15·60	15·60	99·8	15·90	99·8	6·276
	10·4	10·1	13·5	15·60	15·60	—	—	—	—
(5) . .	10·4	10·1	13·7	21·88	9·2	79·6	12·68	79·6	6·277
(6) . .	10·4	10·1	13·7	15·81	15·37	76·6	12·24	76·6	6·258

The correction for each of the thermometers at this temperature was + ·4° C.

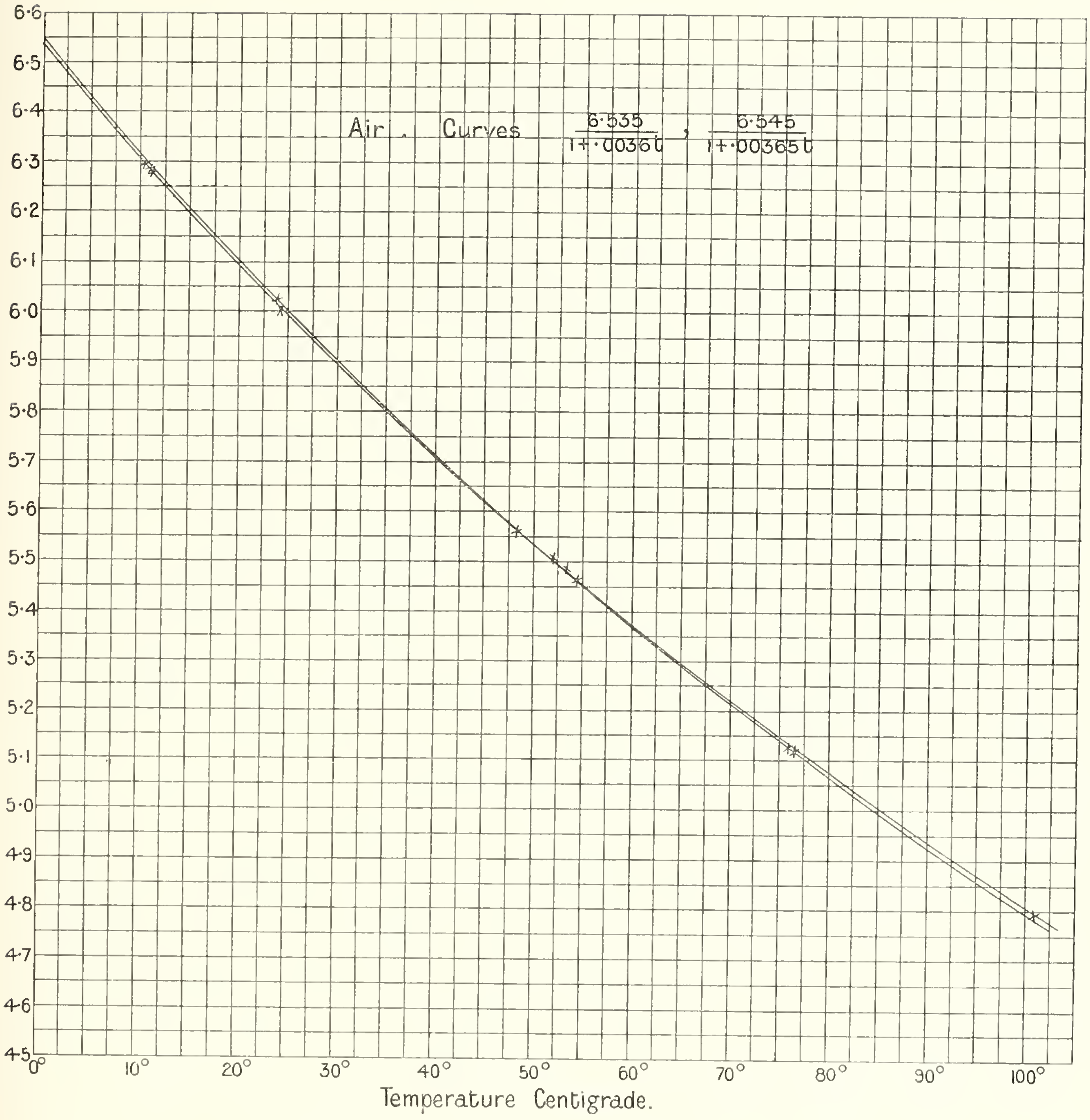

Diagram I.

Taking means we get

	Ratio.	Mean temperature of jacket, ° C.	Temperature of manometer, ° C.
From (1) and (2) . . .	6·282	10·55	13·9
,, (3) ,, (4) . . .	6·282	10·55	13·9
,, (5) ,, (6) . . .	6·267	10·65	14·1

19th November, 1901.

	Readings of thermometers in degrees Centigrade.			Readings of manometer in centimetres.		Number of bands.	Differences.		Ratio, bands / pressure
	9	7	6	Right limb.	Left limb.		Pressure.	Bands.	
	51·6	52·2	16·5	15·61	15·61	—	—	—	—
(1) . .	51·4	52·1	16·5	22·39	8·70	75·2	13·69	75·2	5·493
(2) . .	51·4	52	16·4	15·61	15·58	75·2	13·66	75·2	5·505
	51·4	52·1	16·4	15·61	15·61	—	—	—	—
(3) . .	51·6	52·2	16·3	22·37	8·71	75	13·66	75	5·490
(4) . .	51·9	52·6	16·3	15·61	15·61	75·2	13·66	75·2	5·505

Making the necessary corrections on thermometer readings we get

	Ratio.	Mean temperature of jacket, ° C.	Temperature of manometer, ° C.
From (1) and (2) . . .	5·499	52·0	16·9
,, (3) ,, (4) . . .	5·497	52·1	16·7

25th November, 1901.

	Readings of thermometers in degrees Centigrade.			Readings of manometer in centimetres.		Number of bands.	Differences.		Ratio, bands / pressure.
	9	7	6	Right limb.	Left limb.		Pressure.	Bands.	
	100	100	12·2	15·60	15·60	—	—	—	—
(1) . .	100	100	12·4	23·90	7·13	80·4	16·77	80·4	4·794
(2) . .	100	100	12·6	15·61	15·57	80	16·73	80	4·781
	100	100	12·6	15·60	15·60	—	—	—	—
(3) . .	100·1	100	13	23·35	7·69	75·2	15·66	75·2	4·803
(4) . .	100·2	100·2	13	15·54	15·62	75·2	15·74	75·2	4·777

	Ratio.	Mean temperature of jacket, ° C.	Temperature of manometer, ° C.
From (1) and (2) . . .	4·787	100·9	12·8
,, (3) ,, (4) . . .	4·790	100·95	13·4
Mean . .	4·788	100·9	13·1

The next table gives a complete statement of the values of the ratio $\frac{\text{bands}}{\text{pressure}}$ at different temperatures.

It is convenient to reduce these values to what they would be if the mercury in the manometer was at 0° C. and the tubes of the length at 0° C.

The coefficient of expansion of brass was taken as ·000019.

The values of the corrected ratio are given in the fifth column. The next columns are the values obtained by multiplying the corrected ratio by the factors $(1+{\cdot}00355t)$, $(1+{\cdot}00360t)$ and $(1+{\cdot}00365t)$ respectively, t being the temperature Centigrade.

The values are calculated to the nearest 5 in the third decimal place.

DRY Atmospheric Air.

Date.	Temperature of tubes.	Temperature of manometer.	Ratio.	Corrected ratio.	Multiplied by		
					$1+{\cdot}00355t$.	$1+{\cdot}00360t$.	$1+{\cdot}00365t$.
	° C.	° C.					
15th Nov.	10·55	13·9	6·282	6·295	6·530	6·535	6·540
15th ,,	10·65	14·1	6·267	6·280	6·515	6·520	6·525
22nd ,,	11·0	15·8	6·267	6·285	6·530	6·535	6·540
22nd ,,	11·05	16·4	6·262	6·280	6·525	6·530	6·535
22nd ,,	11·05	16·9	6·270	6·290	6·535	6·540	6·545
26th ,,	23·9	13·6	6·011	6·025	6·540	6·545	6·550
26th ,,	24·3	14·0	5·989	6·000	6·520	6·525	6·530
19th ,,	48·4	15·9	5·551	5·560	6·515	6·530	6·540
19th ,,	52·0	16·9	5·499	5·510	6·515	6·530	6·545
19th ,,	52·1	16·7	5·497	5·510	6·515	6·530	6·545
18th ,,	54·3	14·4	5·450	5·460	6·510	6·525	6·540
18th ,,	53·2	14·9	5·482	5·490	6·525	6·540	6·555
18th ,,	53·0	14·9	5·489	5·495	6·530	6·545	6·560
21st ,,	76·5	18·4	5·109	5·120	6·510	6·530	6·550
21st ,,	76·1	18·1	5·115	5·125	6·510	6·530	6·550
25th ,,	100·9	13·1	4·788	4·790	6·505	6·530	6·555
Mean					6·520	6·535	6·545
Greatest variation . . .					+ ·020 − ·015	+ ·010 − ·015	+ ·015 − ·020

The superiority of the coefficient ·00360 is clear from the numbers. The results are also shown in the diagram on page 441. Moreover, the agreement between the results on the 15th and 22nd shows that no measurable alteration in the gas had taken place.

I think the numbers justify one in taking the ratio as

$$\frac{6{\cdot}535 \pm {\cdot}005}{1 + t({\cdot}00360 \pm {\cdot}00003)}.$$

The length of each tube was 99·9 centims. between the inner surfaces of the glass plates.

The wave-length for Na light may be taken as 5890×10^{-10} metre.

The standard atmosphere as 76 centims. of mercury at 0° C.

Hence we get for the refractive index of dry atmospheric air

$$\mu = 1 + \frac{\cdot 0002928 \pm \cdot 0000003}{\{1 + t(\cdot 00360 \pm \cdot 00003)\}} \frac{p}{76}.$$

Carbon Dioxide.

The gas was made by warming a bulb containing sodium bicarbonate and drying by means of phosphoric pentoxide. The whole apparatus was exhausted to under 1 centim. pressure by means of an oil-pump, and then the bulb containing the bicarbonate gently warmed until atmospheric pressure was attained. This process of exhausting and refilling was repeated about six times, so that the apparatus might be considered filled with practically pure CO_2.

Observing for nearly a week at about 10° C., I noticed a gradual diminution in the value of the refractive index, which was comparatively rapid at first and became very slow in a few days. The whole change was about 1 per cent. I consider that this was due to gas, probably air, coming off the walls slowly, and later results seem to support this view.

I refilled with pure CO_2, and now there appeared no change. The results are given in the following table. Throughout the initial pressure was maintained as nearly as possible at atmospheric pressure by adjusting the reservoir B.

Carbon Dioxide, put in 28th February, 1902.

Date.	Temperature of tubes.	Temperature of manometer.	Ratio.	Corrected ratio.	Multiplied by		
					$1 + \cdot 00375t$.	$1 + \cdot 00380t$.	$1 + \cdot 00385t$.
	° C.	° C.					
5th March	9·7	15·8	9·687	9·715	10·070	10·075	10·080
5th "	9·65	16·1	9·698	9·725	10·075	10·080	10·085
5th "	9·6	16·4	9·698	9·725	10·075	10·080	10·085
5th "	9·65	17·1	9·698	9·725	10·075	10·080	10·085
6th "	60·7	18·9	8·159	8·175	10·035	10·060	10·085
7th "	61·3	18·1	8·165	8·180	10·060	10·085	10·110
12th "	84·1	17·3	7·619	7·630	10·040	10·070	10·100
12th "	84·5	17·4	7·609	7·620	10·040	10·070	10·100
12th "	84·6	17·6	7·613	7·625	10·045	10·075	10·105
14th "	77·1	17·7	7·773	7·785	10·035	10·065	10·095
14th "	76·9	17·8	7·740	7·755	9·990	10·020	10·050
17th "	18·5	16·8	9·310	9·335	9·980	9·990	10·000
17th "	18·5	17·1	9·307	9·330	9·975	9·985	9·995
17th "	18·6	17·3	9·280	9·305	9·955	9·965	9·975
Mean, excluding 14th and 17th					10·055	10·075	10·090
Greatest variation					+ ·020 − ·020	+ ·010 − ·015	+ ·02 − ·01

The results on 14th and 17th March are quite anomalous and beyond ordinary error of observation, and my inference is that more impure gas had come off at the highest temperature. I therefore refilled with fresh CO_2, keeping the tubes at about 80° C. while filling.

The following table gives the results obtained on the new gas.

New Carbon Dioxide, put in 18th March, 1902.

Date.	Temperature of tubes.	Temperature of manometer.	Ratio.	Corrected ratio.	Multiplied by		
					$1+\cdot00375t$.	$1+\cdot00380t$.	$1+\cdot00385t$.
	° C.	° C.					
19th March	10·5	17·3	9·673	9·700	10·080	10·085	10·090
19th ,,	10·5	17·4	9·659	9·685	10·075	10·070	10·075
20th ,,	74·8	17·3	7·832	7·845	10·045	10·075	10·105
20th ,,	74·9	17·4	7·833	7·845	10·045	10·075	10·105
21st ,,	21·7	16·7	9·259	9·285	10·040	10·050	10·060
21st ,,	21·7	17·0	9·276	9·300	10·055	10·065	10·075
24th ,,	31·55	14·4	8·955	8·975	10·035	10·050	10·065
24th ,,	31·45	14·4	8·984	9·000	10·060	10·075	10·090
				Mean	10·055	10·070	10·085
				Greatest variation . . .	+ ·025 − ·020	+ ·015 − ·020	+ ·020 − ·025

No alteration appears to have taken place in the gas during the experiments. The results are also in very close agreement with the results on the former gas. Both sets are shown on the curve for CO_2.

We may take the ratio as

$$\frac{10\cdot070 \pm \cdot01}{1 + t\,(\cdot00380 \pm \cdot00003)}$$

and the refractive index for CO_2 is

$$\mu = 1 + \frac{\cdot0004510 \pm \cdot0000005}{\{1 + t\,(\cdot00380 \pm \cdot00003)\}}\,\frac{p}{76}.$$

Hydrogen.

The gas was prepared from zinc and moderately diluted hydrochloric acid. The apparatus was exhausted and filled about seven or eight times at the temperature of the room. The gas was dried by phosphoric pentoxide.

Observations for a week at about 10° C. indicated a gradual increase in the refractive index, which I attribute to carbon dioxide coming off the walls. When this effect had ceased, the apparatus was exhausted and kept exhausted for a few hours, while the temperature of the tubes was maintained at about 70°. I hoped in this way to remove all impure gas from the walls, but it will be seen from the results that

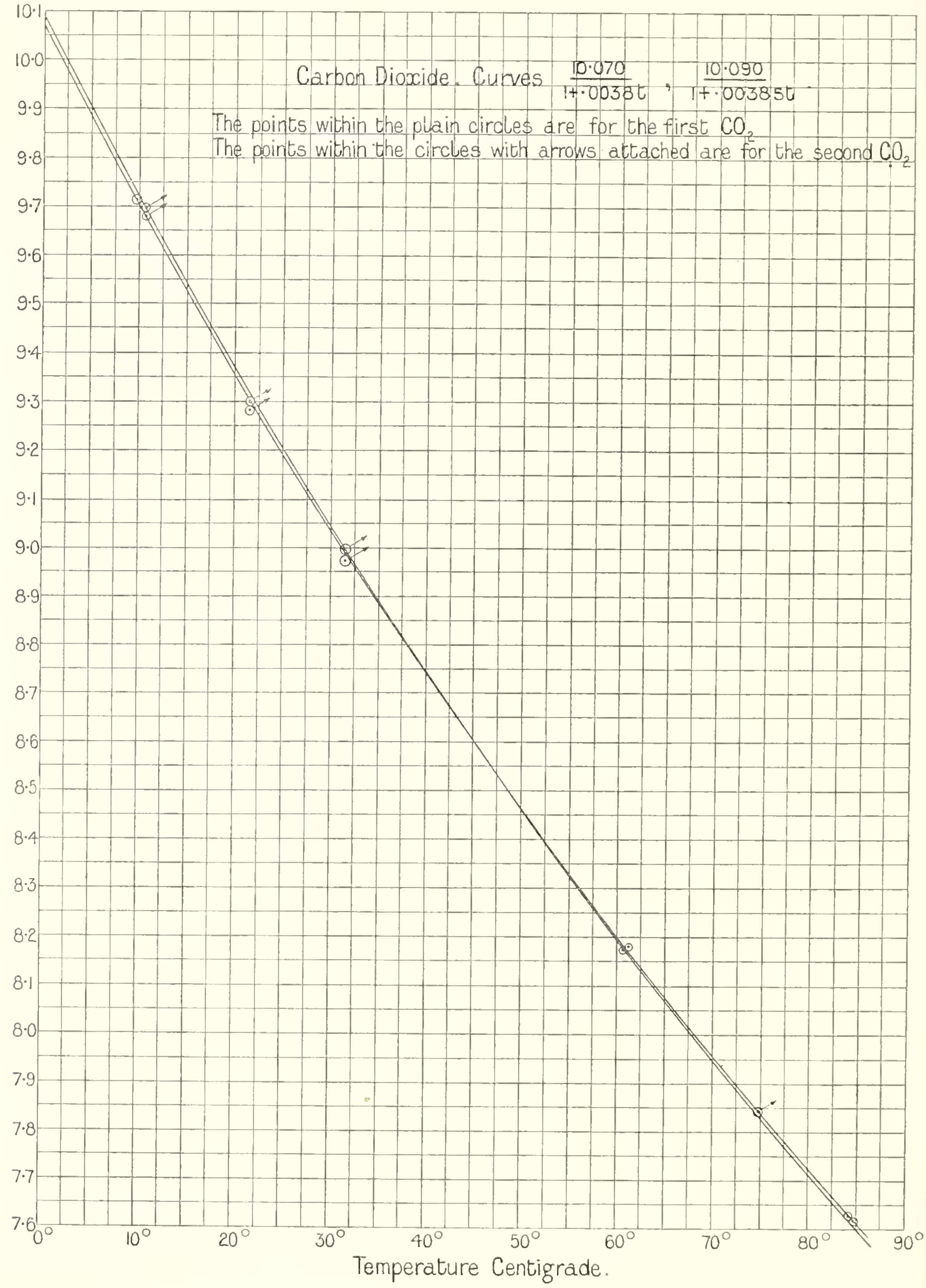

Diagram II.

I was not quite successful. The exhaustion and re-filling with new hydrogen was repeated about four or five times.

The results are given in the following table :—

HYDROGEN put in 21st April, 1902.

Date.	Temperature of tubes.	Temperature of manometer.	Ratio.	Corrected ratio.	Multiplied by		
					$1 + \cdot 00345t$.	$1 + \cdot 00350t$.	$1 + \cdot 00355t$.
	° C.	° C.					
22nd April	11·7	17·4	3·010	3·019	3·142	3·143	3·144
22nd ,,	11·7	17·6	3·007	3·016	3·138	3·139	3·141
23rd ,,	59·7	18·2	2·583	2·588	3·121	3·129	3·137
24th ,,	71·0	19·0	2·509	2·515	3·131	3·140	3·149
25th ,,	83·5	19·8	2·442	2·447	3·156	3·164	3·172
26th ,,	89·7	17·1	2·406	2·409	3·154	3·165	3·176
28th ,,	23·3	16·3	2·913	2·920	3·155	3·158	3·161
28th ,,	22·65	16·3	2·916	2·923	3·152	3·155	3·158
29th ,,	33·0	14·8	2·821	2·827	3·148	3·153	3·158
29th ,,	32·4	15·4	2·822	2·828	3·145	3·149	3·153
29th ,,	32·3	15·4	2·826	2·832	3·147	3·152	3·157
30th ,,	65·8	15·2	2·556	2·560	3·140	3·149	3·158
30th ,,	65·5	15·2	2·561	2·565	3·145	3·153	3·161
1st May	10·8	14·5	3·031	3·038	3·152	3·153	3·154
1st ,,	10·9	14·6	3·033	3·040	3·156	3·157	3·158
2nd ,,	77·5	14·4	2·476	2·479	3·141	3·151	3·161
3rd ,,	84·1	15·3	2·438	2·441	3·148	3·159	3·170
3rd ,,	83·2	15·6	2·442	2·445	3·147	3·157	3·167
5th ,,	81·7	13·2	2·447	2·449	3·139	3·149	3·159
5th ,,	10·65	13·2	3·042	3·048	3·159	3·161	3·163
		Mean			3.148	3·154	3·160
		Greatest variation . . .			+ ·011 − ·008	+ ·007 − ·005	+ ·010 − ·007

The results on the 22nd, 23rd, and 24th April are fairly consistent, but on raising the temperature to over 80° the values are distinctly higher. From the 28th onwards the agreement is quite satisfactory. I conclude that some impurity (CO_2) had come off at the higher temperature, and after that the composition remained constant. The results from the 28th onwards give a ratio

$$\frac{3{\cdot}154 \pm {\cdot}003}{1 + t\,({\cdot}00350 \pm {\cdot}00003)}.$$

The value of the ratio at 0° for the original hydrogen may be taken as the mean of the first two observations on 22nd April. This gives 3·141. The amount of impurity is thus 13 in 3141, or 1 part in 240. This would not alter the value of the temperature coefficient to the present order of accuracy. I did not think it worth while to put in new hydrogen, as the absolute purity could hardly be relied on by this method of preparation.

We may take the ratio for the original hydrogen as

$$\frac{3{\cdot}141 \pm {\cdot}003}{1 + t\,({\cdot}00350 \pm {\cdot}00003)},$$

which gives for the refractive index

$$\mu = 1 + \frac{{\cdot}0001407 \pm {\cdot}00000015}{\{1 + t\,({\cdot}00350 \pm {\cdot}00003)\}}\,\frac{p}{76}.$$

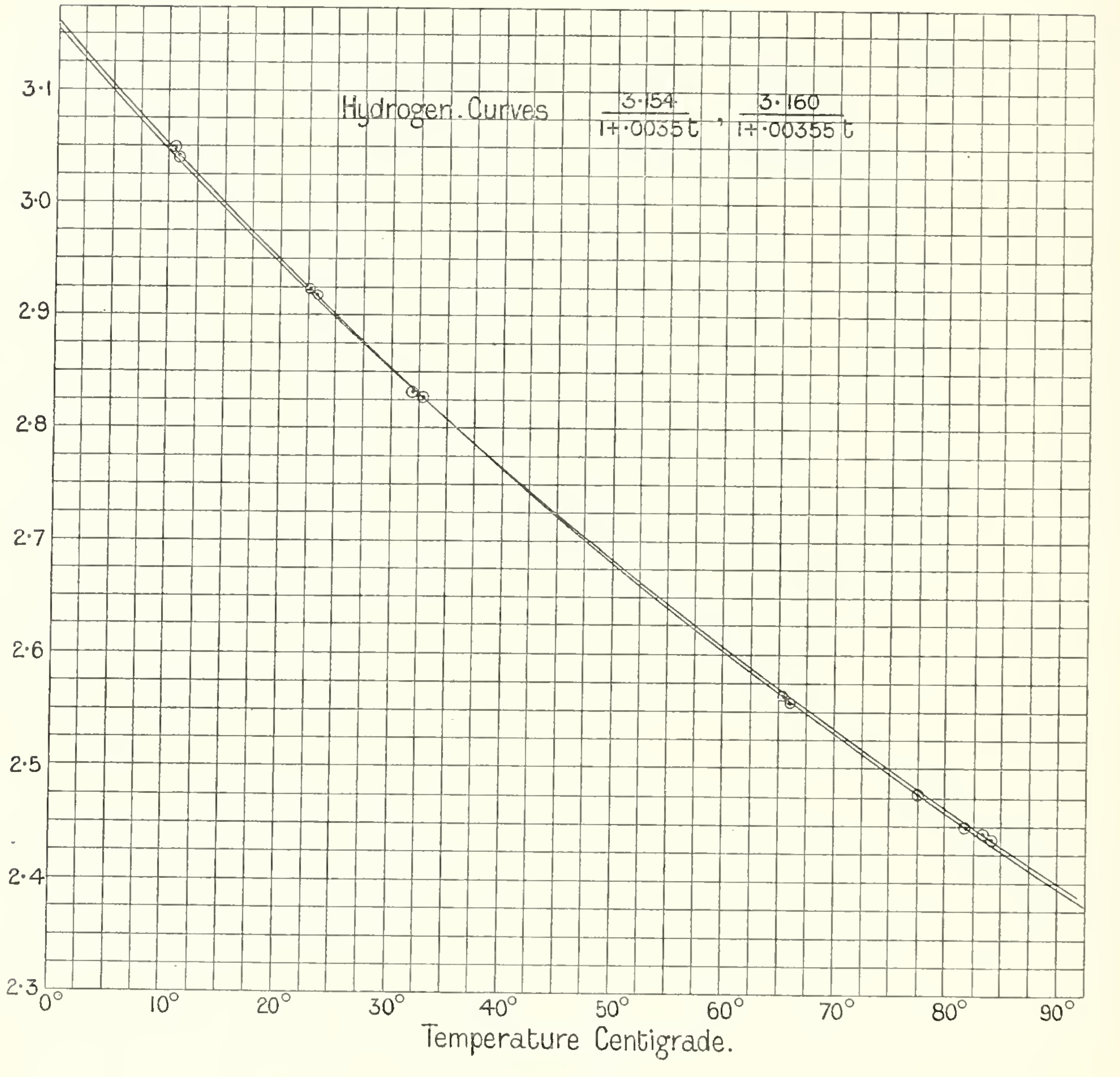

Diagram III.

Ammonia.

The gas was prepared dry by the following method, for which I am indebted to Dr. SCOTT, of the Davy-Faraday Laboratory.

A strong solution of ammonia, in a glass flask, was gently warmed, and the gas passed first through a tube containing dry caustic potash and next through a tube containing dry calcium chloride. The calcium chloride absorbs large quantities of the gas. One end of the tube was then sealed up and the other attached to the apparatus, and the whole exhausted with the oil-pump. On gently warming the calcium chloride tube, the ammonia gas was liberated. The former method, of exhausting and filling several times while the tubes were kept at about 90° C., was adopted; and after the gas had been in the apparatus for a few days, the process was repeated.

More care must be taken in the case of ammonia, since the gas does not strictly follow the ordinary gaseous law.

If p be the pressure and t the temperature, the refractive index may be written

$$\mu = 1 + \frac{\kappa p (1 + \lambda p)}{(1 + \alpha t)}.$$

Hence if p_0 be the initial pressure in the tubes,
p_1 ,, final ,, ,, one tube,
p_2 ,, ,, ,, ,, second tube,

the number of bands displaced

$$\propto \frac{\kappa (p_1 - p_2) \{1 + \lambda (p_1 + p_2)\}}{(1 + \alpha t)}.$$

According to MASCART* λ for ammonia = ·000178 per centimetre of mercury.

We must, therefore, take care that the value of $p_1 + p_2$ does not vary to any extent throughout the series of measurements. This point was carefully attended to, and the value of $(p_1 + p_2)$ was equal to 120 centims. throughout, the variation not exceeding 2 centims.

My main object being the temperature coefficient, and not so much the absolute value of μ, I did not make any measurements with another value of $p_1 + p_2$. This omission I now regret; but MASCART's value may be used, as he made experiments specially on this point, and had much greater ranges of pressure than my apparatus was arranged for. In his papers I cannot find that he measured the temperature coefficient.

* 'Comptes Rendus,' vol. 86, 1878, p. 321.

AMMONIA Gas, put in 13th June, 1902.

Date.	Temperature of tubes.	Temperature of manometer.	Ratio.	Corrected ratio.	Multiplied by		
					$1 + \cdot 00385t$.	$1 + \cdot 00390t$.	$1 + \cdot 00395t$.
	° C.	° C.					
17th June	11·8	15·5	8·139	8·160	8·530	8·535	8·540
17th ,,	11·8	15·6	8·140	8·161	8·530	8·535	8·540
18th ,,	56·25	18·2	6·997	7·012	8·530	8·550	8·570
18th ,,	56·0	18·2	7·003	7·018	8·530	8·550	8·570
19th ,,	60·6	20·2	6·887	6·904	8·515	8·535	8·555
19th ,,	60·3	20·4	6·904	6·921	8·530	8·550	8·570
20th ,,	75·5	19·4	6·597	6·610	8·530	8·555	8·580
20th ,,	76·4	19·4	6·572	6·585	8·525	8·550	8·575
20th ,,	76·8	19·8	6·562	6·575	8·520	8·545	8·570
23rd ,,	90·5	22·3	6·291	6·305	8·500	8·530	8·560
23rd ,,	90·5	22·7	6·283	6·298	8·490	8·520	8·550
23rd ,,	90·4	22·8	6·287	6·302	8·495	8·525	8·555
24th ,,	32·0	23·6	7·580	7·607	8·545	8·555	8·565
24th ,,	31·95	23·9	7·574	7·602	8·540	8·550	8·560
24th ,,	31·75	24·2	7·575	7·603	8·530	8·545	8·555
25th ,,	23·75	23·9	7·780	7·810	8·525	8·535	8·545
25th ,,	23·65	23·8	7·777	7·807	8·515	8·525	8·535
25th ,,	23·75	23·8	7·781	7·811	8·525	8·535	8·545
				Mean	8·523	8·540	8·557
				Greatest variation . . .	+ ·022 − ·033	+ ·015 − ·020	+ ·023 − ·022

I was unable to make a final observation at 12° C., owing to the fact that one of the glass joints had cracked during the night of the 25th June. However, the results of 17th and 25th June are a fairly good test that no change in the gas had taken place.

We may take the ratio as

$$\frac{8\cdot 540 \pm \cdot 010}{1 + t(\cdot 00390 + \cdot 00003)}.$$

Throughout $p_1 + p_2 = 120$ centims. of mercury

$$\lambda = \cdot 000178 \text{ per centimetre of mercury (MASCART).}$$

We thus obtain for the refractive index

$$\mu = 1 + \frac{(\cdot 0003743 \pm \cdot 0000005)(1 + \cdot 000178p)}{1 + t(\cdot 00390 \pm \cdot 00003)} \frac{p}{76},$$

p being expressed in centimetres of mercury at 0° C.

At 0° C. and 76 centims. pressure

$$\mu_0 = 1 + \cdot 0003793 \pm \cdot 0000005.$$

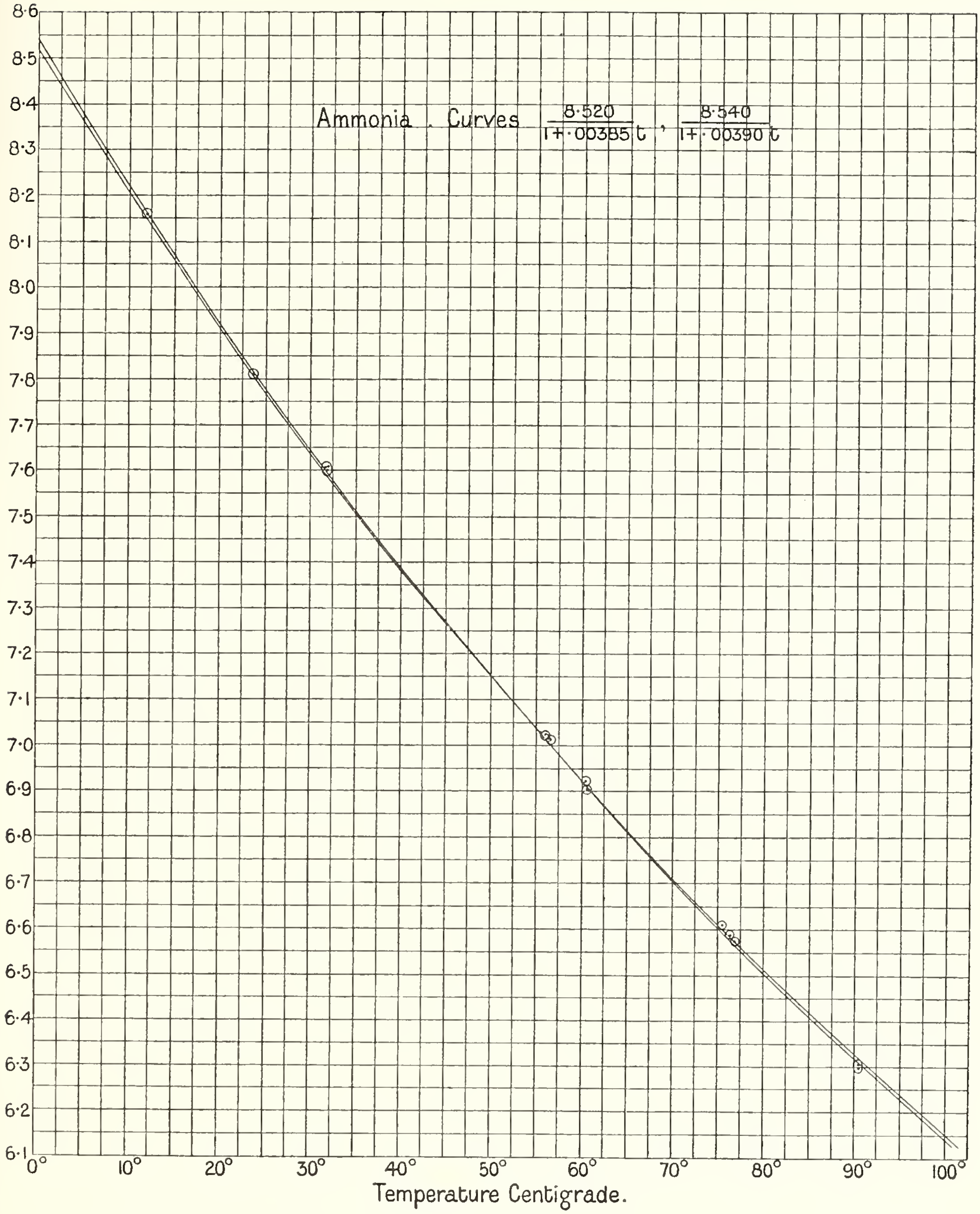

Diagram IV.

Sulphur Dioxide.

The gas was obtained from a syphon of the liquefied gas and dried by means of a phosphoric pentoxide bulb inserted in the apparatus. The former process of exhausting and refilling at a temperature of about 90° C. was adopted.

As in the case of ammonia, the pressure conditions must remain the same throughout.

My first experiments gave a temperature coefficient about ·00415; but, after having been at a temperature about 90° C., there seemed to have been a considerable absorption of gas, so that I could not obtain the former pressure conditions at lower temperatures. I therefore put in new gas and kept a record of the pressures. There was still a gradual absorption of the gas, although not so great as before. Whether this was due solely to the walls of the apparatus or to the phosphoric pentoxide I am not in a position to say.

New Sulphur Dioxide, put in 16th August, 1902.

Date.	Temperature of tubes.	Temperature of manometer.	Ratio.	Corrected ratio.	p_1+p_2.	$\frac{\kappa}{1+\alpha t}$.	Multiplied by $1+ ·00415t$.	Multiplied by $1+ ·00416t$.	Multiplied by $1+ ·00417t$.
	° C.	° C.							
18th Aug.	80·7	21·2	11·524	11·550	137	10·955	14·625	14·635	14·645
18th „	81·1	21·3	11·507	11·535	137	10·940	14·620	14·630	14·640
18th „	81·2	21·4	11·495	11·520	137	10·925	14·605	14·615	14·625
19th „	37·4	22·2	13·278	13·320	130	12·665	14·630	14·635	14·640
19th „	37·3	22·4	13·250	13·295	130	12·640	14·595	14·600	14·605
19th „	37·5	22·5	13·248	13·295	130	12·640	14·605	14·610	14·615
19th „	15·0	22·7	14·409	14·465	126	13·775	14·630	14·635	14·635
19th „	14·9	22·7	14·404	14·460	126	13·770	14·620	14·625	14·625
20th „	14·15	20·0	14·437	14·485	121	13·820	14·630	14·635	14·635
20th „	14·15	20·2	14·442	14·490	121	13·825	14·635	14·640	14·640
20th „	14·20	20·4	14·221	14·270	81	13·825	14·635	14·640	14·640
Mean							14·620	14·625	14·630
Greatest variation . . .							+ ·015 − ·025	+ ·015 − ·025	+ ·015 − ·025

The last two observations were made in order to obtain the coefficient of increase with pressure.

We have

$$\frac{\kappa}{1+\alpha t}(1+\lambda\, 121) = 14·490,$$

$$\frac{\kappa}{1+\alpha t}(1+\lambda\, 81) = 14·270.$$

Hence

$$\frac{\kappa}{1+\alpha t} = 13·825, \qquad \lambda = ·000398.$$

The accuracy attained seems much greater than in the case of the former gases; but to be safe we take the ratio as

$$\frac{(14 \cdot 625 \pm \cdot 01)(1 + \cdot 000398p)}{1 + t(\cdot 00416 \pm \cdot 00002)}.$$

This gives for the refractive index

$$\mu = 1 + \frac{(\cdot 0006553 \pm \cdot 0000005)(1 + \cdot 000398p)}{1 + t(\cdot 00416 \pm \cdot 00002)} \frac{p}{76}.$$

At 76 centims. pressure and 0° C.

$$\mu_0 = 1 + \cdot 0006758 \pm \cdot 0000005.$$

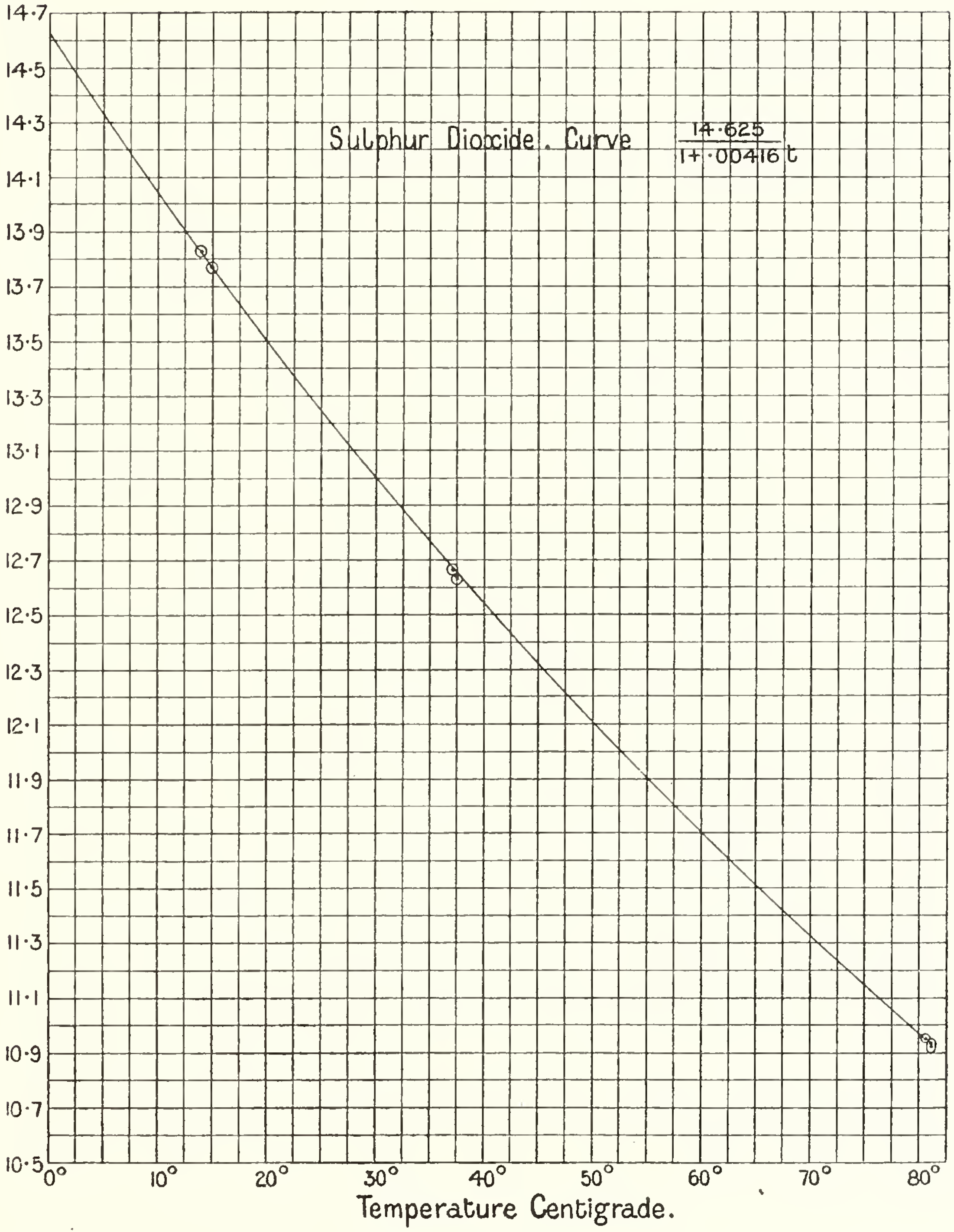

Diagram V.

The following table gives a comparison of the results with those of former observers* :—

REFRACTIVE Index for Na Light at 76 centims. Pressure and 0° C.

Observer.	Air.	Hydrogen.	Carbon dioxide.	Ammonia.	Sulphur dioxide.
Present	1·0002928 ± 3	1·0001407 ± 15	1·0004510 ± 5	1·0003793 ± 5	1·0006758 ± 5
MASCART	1·0002927	1·000139	1·000454	1·000379	1·0007038
LORENZ	—	1·000139	—	1·000373	—
KETTELER	—	1·000143	1·000449	—	1·000686
DULONG	1·000294	1·000138	1·000449	1·000385	1·000685

The following table gives a comparison of MASCART's temperature coefficients with those obtained in this paper :—

	Air.	Hydrogen.	Carbon dioxide.	Ammonia.	Sulphur dioxide.
Coefficient of expansion	·00367	·00366	·00371	—	·00390
MASCART, refractive index coefficient	·00382	·00378	·00406	—	·00460
Present	·00360 ± 3	·00350 ± 3	·00380 ± 3	·00390 ± 3	·00416 ± 2

The values of the temperature coefficient of refractive index obtained are, in every case, less than those obtained by MASCART. It is somewhat futile to attempt to explain the difference; but perhaps the following points are worthy of attention. In my apparatus the tubes were about 1 metre long and the two rubber washers together about $\frac{2}{5}$ millim. thick, while MASCART used tubes about 25 centims. long and his rubber washers were probably 1 millim. thick each. He does not mention the thickness, but LORENZ, who appears to have used an almost identical apparatus, used washers $1\frac{1}{2}$ millims. thick each. The somewhat irregular behaviour of rubber under varying conditions of temperature and pressure may have produced errors in MASCART's observations, from which I consider that mine are entirely free.

I have already referred to the apparent escape of impurities from the walls of the apparatus. MASCART makes no reference to this point, and gives no indication of how he tested the constancy of composition of the gas during the experiments. It is true he analysed the gas chemically after the experiments, but this is hardly accurate enough for the point in view.

* A very useful table of the results of different observers is given by BRÜHL, 'Zeitschrift für Physikalische Chemie,' vol. 7, 1891.

The difference between the temperature coefficient of refraction and the coefficient of expansion has naturally attracted my attention; but I do not propose to discuss the matter theoretically in this paper, mainly because I am now taking up experiments on the temperature coefficients of the dielectric constants, which I hope will give me a more complete basis for generalization.

In conclusion, I wish to express my great obligation to Professor THOMSON for having placed the appliances of the laboratory at my disposal, and for his kind interest in what has necessarily been a very tedious work.

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PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 457–496.

[PLATES 2–3.]

SOLAR ECLIPSE OF 1900, MAY 28.

GENERAL DISCUSSION OF SPECTROSCOPIC RESULTS

BY

J. EVERSHED, F.R.A.S.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 342.

Price Three Shillings.

15.7.03

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XII. *Solar Eclipse of* 1900, *May* 28.—*General Discussion of Spectroscopic Results.*

By J. Evershed, *F.R.A.S.*

Communicated by the Joint Permanent Eclipse Committee.

Received December 17, 1902,—Read January 22, 1903.

[Plates 2, 3.]

In the preliminary report of an expedition to the south limit of totality, in Algeria, I described in detail the methods adopted and the instruments employed in obtaining photographs of the "flash" spectrum in high solar latitudes.*

The present paper deals with the results obtained from a detailed study and measurement of four of the best negatives of the series of sixteen which were secured with the principal instrument, a reflecting prismatic camera.

This instrument was an ordinary reflecting telescope of 188 centims. focus, fitted with two prisms of light flint glass at the upper end of the tube near the position usually occupied by the small mirror of the Newtonian reflector. The prisms had an effective aperture of 8 centims. and angles of 60° and 45° respectively; they were set approximately at minimum deviation for K, and gave a linear dispersion at the focus of the large mirror equal to 93 millims. between F and K.

Description of the Photographs.

The plates were exposed near the time of greatest phase of the eclipse, which was not quite total at my station. The first plate was exposed at 45 seconds before, and the last at 32 seconds after, the computed time of mid-eclipse. Owing to the position of my station, near the extreme limit of the zone of total-eclipse, and just outside that limit, there appears in all the photographs a considerable amount of continuous spectrum due to the uneclipsed photosphere. Notwithstanding this, all the exposures which were made within 15 seconds of mid-eclipse yielded good images of the flash spectrum, and the sky illumination was sufficiently reduced to allow of the fainter

* 'Roy. Soc. Proc.,' vol. 67, p. 370.

spectrum arcs being impressed during quite half a minute at the time of greatest obscuration (see Plate 2).

In all the images the continuous spectrum extends from λ 3500 to λ 5100, and throughout this long range the focus appears to be almost perfect, a striking testimony to the good qualities of the reflector as compared with a lens.

Some of the stronger arcs show a diffuseness on the violet side, a defect which has been traced to a want of homogeneity in the glass at the base of the 60° prism. In the ultra-violet region this shading becomes scarcely noticeable, and the definition here is very fine; this is no doubt owing to the almost complete absorption of the ultra-violet rays in traversing the thickest part of the prisms.

The four negatives selected for special study, and which are reproduced in Plate 2, are from the exposures numbered 9, 10, 11, and 13.

No. 9 was exposed for 2 seconds, beginning 15 seconds before mid-eclipse. The flash spectrum is impressed in a rather narrow rift in the continuous spectrum, extending from position angle 140° to 148°, and including a region between 70° and 77° south latitude. The bright arcs crossing the rift are exceedingly narrow thread-like lines, well defined throughout the spectrum, and are therefore well adapted for accurate wave-length determinations. Although the arcs are inclined about 30° from the normal to the direction of dispersion, this was found in making the measures to detract but very little from the accuracy of a setting.

In the ultra-violet the Fraunhofer lines are particularly well-defined in this image right up to the end of the plate on the continuous spectrum, but between Hζ and Hβ they are obliterated by over-exposure.* The stronger dark lines, and many of the weaker ones, are continuous with and run into bright lines in the rift, and in several instances the density of the silver deposit is the same in the bright line as it is in the dark line, giving the impression that the change from dark to bright is entirely one of contrast resulting from the withdrawal of the bright background of continuous spectrum. Some of the more intense lines, such as those of titanium at λλ 3685, 3759, and 3761, do not become dark lines on the continuous spectrum, but, being more intense than the latter, appear bright even upon the bright background.

This may be accounted for by the great altitude to which the titanium vapour extends, not to its being intrinsically brighter than the photosphere at the limb. For the continuous spectrum is produced by what is virtually a slit of extreme fineness, defined by the limbs of the sun and moon, and subtending an angle of less than 1″, whilst the titanium spectrum comes from a stratum, or virtual slit, of 7″ or 8″ in angular width.

No. 10 was exposed also for 2 seconds at about 10 seconds before mid-eclipse. The limb of the moon, advancing eastwards, has covered up the lowest strata of the flash in the position where the lines are so well developed in No. 9. There is, however, a

* In the reproductions the Fraunhofer lines are almost invisible except at the extreme ultra-violet end in No. 13.

good but narrow image of the flash at about position angle 137°, or latitude − 63° to − 66° E.; and on the west side also there is a fine thread of faint continuous spectrum in latitude − 56° W., upon which the flash lines appear as minute dots, like beads on a string. About twelve of these dots may be counted between H and K.

No. 11 spectrum was exposed during 10 seconds near the time of mid-eclipse. Judging by the symmetrical distribution of the bands of continuous spectrum on each side of the central line of the image, the middle of the exposure must have been timed almost at the moment of greatest phase, which appears to have coincided with the computed time of mid-eclipse.

The continuous spectrum in this negative is reduced to nine or ten narrow bands, due to indentations in the moon's limb, and the flash spectrum appears in the form of long arcs crossing the bands and extending over the whole of the south-polar region of the sun. Most of the arcs cover 80° degrees of the limb, extending from latitude − 75° on the east side to latitude − 28° on the west. The sharpest definition is along a band at position angle 212° in latitude − 41° on the west side, and this portion of the image was selected for measures of wave-length and estimates of intensity.

The bright lines on this negative are more strongly impressed, and can be traced further towards the more refrangible end of the spectrum than in any of the other images. Some of the Fraunhofer dark lines can still be traced near the end of the image in the ultra-violet, crossing the narrow strips of continuous spectrum. These lines therefore do not wholly disappear before the last remnants of continuous spectrum vanish, but they become exceedingly faint, and are easily obliterated by over-exposure.

No. 13 spectrum, exposed for 2 seconds about 14 seconds after mid-phase, shows a considerable arc of the photosphere uncovered over the south-west limb, and the negative is somewhat fogged from the increasing sky illumination. There is, however, a good image of the flash spectrum near the middle line of the image in the south-east quadrant. The lines are here very short, the flash layer being exposed in a narrow depression of the moon's limb, but they are well adapted for measurement.

Methods of Measurement.

The photographs numbered 9, 11, and 13 were measured with a micrometer microscope, lent to me for this purpose by Major Hills, R.E. This instrument has a screw of 1 millim. pitch and about 200 millims. in length. The head of the screw being divided to 100 parts, readings can be made to ·01 millim., and by estimation to ·001 millim.

In practice it was found that ·01 millim. was about the limit of accuracy attainable with the best defined lines. A preliminary set of measures of the sharpest lines over the whole length of spectrum photographed was made, to test the accuracy of the screw over long runs, a duplicate series of measures being made over the same portion of the screw, but with the negative reversed end for end. A comparison of the direct and reversed measures revealed systematic differences amounting to as much as ·02 millim. in a run of 100 millims.

Although this error of run would have very little effect on the resulting wave-lengths, which depend ultimately on short measured distances from known lines, it was considered more satisfactory to measure the photographs in three sections of about 70 millims. each, selecting a portion of the screw which gave consistent results over this range. In order to reduce the accidental errors of setting, and to detect blunders, each section was measured twice, one set of measures with the red end to the right, and the other with the red end to the left.

This method involved some extra labour in combining the measures, and in joining up the three sections into one consistent whole by means of the lines which overlapped between the sections. However, the definition of some of the images is so good that any amount of trouble taken in getting satisfactory measures seemed to be justified.

The relation between wave-length and measured distances at all points in the spectrum was determined approximately by graphical methods, using 42 well-known lines, including lines of hydrogen, calcium, titanium and iron, &c. A large number of the finer lines were then identified with certainty, and in the final reduction the broad over-exposed hydrogen and calcium lines were rejected as standards, and about 65 more suitable lines were selected which are well distributed throughout the spectrum, using in the ultra-violet region lines which I considered thoroughly well identified in the spectra obtained in 1898.*

From the standard lines the position in millimetres of each 50 tenth-metre of wave-length was computed, taking the mean value given by four or five of the nearest standards in each case. A table of differences was then made giving the intermediate values by interpolation and the value in millimetres of one tenth-metre at every 25 units.

The wave-lengths of all the lines, including the standards, were computed from this table, using second differences.

Each of the three spectra measured was reduced independently, using the same standard lines, but computing a separate table for each. A direct comparison of the three sets of measures showed that they were very nearly identical, and one table might have served for all. But there appear small systematic differences, due in part the fact that the measures were made at different distances from the centre of the arcs, and probably in part also to slight irregular contraction of the photographic films in drying.

It was therefore considered more satisfactory to treat each spectrum entirely independently, combining in the end the wave-length values obtained to arrive at the most probable values measured on all three spectra.

* 'Phil. Trans.,' A, vol. 197.

From the accordance between the two sets of measures for each spectrum the accidental errors of the mean positions may be estimated at about ·01 millim. The mean error is less than this for the best defined lines, but greater for the broad or diffuse lines. This error corresponds to an error in wave-length of ·16 tenth-metre at 5000, decreasing to ·07 at 4000, and ·04 at the end of the spectra at 3500.

It does not, of course, follow that the wave-lengths in the tables can be relied on within these limits, except for isolated lines of which the measures are unaffected by any disturbing causes, such as faint companion lines or shadings. But this degree of accuracy seems actually to have been attained in a large proportion of the iron and titanium and other well-identified lines (see Table I., p. 478).

The mean values for the hydrogen lines, given separately in Table II., agree very closely indeed with the computed values. Thus in nineteen lines, in a total of twenty-eight, the differences do not exceed ·04 tenth-metre, and in four lines only the differences reach ·1 tenth-metre, three of these being the lines γ, δ and ϵ, which are difficult to bisect on account of their great width. This result will, perhaps, best indicate the general accuracy of the wave-length work.

No corrections of any kind have been applied to the results, and it may be well to emphasise here the fact that no corrections are needed for apparent displacements due to the different altitudes to which the various gases ascend in the chromosphere. In making the measures, the settings were made at the position of maximum density in the case of the broad over-exposed hydrogen lines, the finer lines being simply bisected without reference to the apparent edge of the moon's limb.

It is probable that the positions of maximum density of the images of the stronger lines correspond to radiations coming from a region within 2″ of the photosphere, whilst in the fainter ultra-violet hydrogen series the radiations are almost confined to the flash spectrum layer, the emission from the upper chromosphere being almost insensible for these lines.

Assuming that all the settings were made on arcs radiated from a region within 2″ of the photosphere, this being the approximate limit to which the reversing layer extends, no appreciable error will be made by bisecting the images; for a difference of 1″ of arc in the positions of the various gases above the moon's limb would make an apparent shift on the plate at 188 centims. focus of ·0091 millim., a quantity about equal to the accidental errors of measurement.

On the other hand, if the settings were made on the inner edge of the arcs, that is, at the apparent limb of the moon, a considerable error would be introduced, depending on the intensities of lines, which spread inwards as well as outwards as a result of irradiation.

It is particularly noticeable that a large number of the arcs of the flash spectrum in these prismatic camera photographs are narrow lines sharply defined on both sides, there is no diffuseness on the outer side, as might be expected were the arcs *true images* of the strata producing them. They are in reality, as I have previously pointed out,* diffraction images more or less enlarged by photographic diffusion, and they appear to be as well adapted for bisection and wave-length determination as are the lines given by a slit-spectroscope.

Exception may, perhaps, be taken in the case of the helium lines and the somewhat remarkable line at 4685·7. These do not increase in intensity towards the photosphere, and it is possible that they are very weak and even absent from the flash layer. A bisection of these arcs may, therefore, represent a point in the chromosphere higher than the flash layer.

No allowance has, however, been made for this, yet the values obtained indicate only a small displacement towards the red end, averaging ·16 tenth-metre for the three lines 4713, 4471, and 4026, when compared with the principal components of the double lines as determined by RUNGE and PASCHEN, and which they are assumed to represent.

But it is noticeable that in these spectra the helium lines become broad and faint in the flash layer, although narrow strong lines outside; the measures are, therefore, somewhat uncertain, and it is possible that they may be partly affected by the less refrangible components of the double lines. In any case they serve to show what a small correction is needed, even for the lines of a substance like helium, which is characteristic of the upper chromosphere rather than the flash layer.

Estimates of Intensity.

On account of the great range of intensity between the weakest and the strongest lines, a scale was adopted ranging from 0 to 100. This is practically equivalent to adopting two orders of intensity, 0 to 10 representing the weak lines, 10 to 100 the strong lines, the latter progressing by fives.

The intensities of all the lines, with the exception of those of hydrogen and H and K, were estimated while making the measures, two independent estimates of each line being obtained from the two sets of measures of each spectrum. The mean of the two estimates is set down for each spectrum in Table I.

The hydrogen and calcium lines were estimated separately. From $H\zeta$ to $H\rho$ they form a nearly uniformly diminishing series, giving a convenient scale of reference with which to compare the strong lines of the flash spectrum.

At the ends of the spectra, where the density of the image falls off considerably, the estimates are of course very rough and uncertain, and throughout the middle portions of the spectra the intensities are not perhaps strictly comparable, except over a limited range of wave-length.

* 'Phil. Trans.,' A, vol. 197, p. 394.

General Discussion of Results.

The identification of the bright lines in these spectra with the dark lines of the Fraunhofer spectrum presents very little difficulty in the case of the strong, or well-defined flash lines, and it appears to be generally true that the more reliable the values of wave-length obtained in photographs of the flash, the more closely do they correspond with Rowland's values of the dark lines. Thus many of the lines measured on small-scale photographs obtained in 1898 show apparent displacements considerably greater than the accuracy of the measures seemed to warrant, and which rendered many of the identifications doubtful. This is particularly the case in the region between 3700 and 3900, where the iron lines especially seemed to be systematically of smaller wave-length than the corresponding dark lines, whilst the hydrogen lines in the same region agreed very closely indeed with their theoretical positions. In the present measures, however, in which the scale of the plates is nearly four times greater, these displacements are not confirmed, and the same lines are found to agree with Rowland's values within ·04 tenth-metre.

As regards the fainter ill-defined lines and groups there is, of course, considerable uncertainty in assigning the particular dark lines of which they are supposed to be the reversals, or which lines in a group of dark lines are reversed in the flash.

It is, however, abundantly clear, from an examination of Table I., that every well-defined bright line of the flash (excluding hydrogen and helium lines and the line at 4685·7) can be assigned to a dark line of Rowland's table of an intensity exceeding 2 of his scale. There are no bright lines of even medium strength which occur in blank spaces of the solar spectrum where the lines are weaker than 0, and only a few of the very weakest lines in the table coincide with solar lines with an intensity less than 2.

As a corollary to this, it may be stated that in general the greater the intensity of a dark line in the solar spectrum, the more probable is its presence as a bright line in any given image of the flash, and in the long range of spectrum covered by the spectra under discussion, λ 3500 to λ 5000, the dark lines of intensities exceeding 7 are all present as bright lines, except in two or three instances where they are obviously obscured by strong hydrogen or calcium lines.

In the tables of flash-spectrum lines published by Frost and by Mitchell, the same general fact is apparent in the large number of identifications made with prominent Fraunhofer lines. Professor Frost concludes that "at least 60 per cent. (and probably many more) of the stronger dark lines of the solar spectrum are found bright in a stratum not exceeding 1″ in height above the photosphere."*

It will probably be generally admitted, therefore, that the flash spectrum as photographed hitherto is a reversal of the more prominent of the Fraunhofer lines,

* 'Astrophysical Journal,' vol. XII., p. 345.

and does not include lines (other than those of He and H) which are not present in the dark line spectrum.

The most important point remaining open for discussion is the relation of the intensities of the bright lines to those of their dark line equivalents, for on this point turns the question whether the flash spectrum layer is in truth the stratum which by its absorption gives rise to the Fraunhofer spectrum.

In discussing the results of the flash spectra obtained in India in 1898,* I stated certain conclusions leading to the belief that the flash spectrum does, in fact, represent the upper more diffused portion of an absorbing stratum which, taken as a whole, produces the Fraunhofer lines. The conclusions relating to the relative intensities of the lines I now recapitulate in the following three paragraphs :—

(1) The relative intensities of the lines of any one element in the flash spectrum are practically the same as those of the same element in the solar spectrum.

(2) The relative intensities between groups of lines belonging to different elements are widely different in the flash and in the solar spectrum.

(3) The apparent intensity of the radiation from an element in the lower chromosphere is determined by the extent to which that element is diffused above the photosphere, and the real relative intensities between the different elements cannot be judged in photographs of the flash spectrum.

The statements in the second and third paragraphs will now probably be generally admitted, and do not need further discussion. It remains to determine how far the statement given in the first paragraph is borne out by the present results, which cover a somewhat different range of the spectrum, give more accurate values of the wave-lengths, and which give very much more complete and reliable values of the intensities of the lines.

Probably this is the most important conclusion deduced from my former results, and as it is one which is most open to criticism, I propose to deal with it in some detail, and with especial reference to the results obtained by FOWLER and BAXANDALL under Sir NORMAN LOCKYER. These investigators have found that the relative intensities of the lines of an element in the flash approximate to those in the spark spectrum, whilst the intensities of the dark lines closely resemble those in the arc spectrum ; whence they conclude that the flash spectrum layer is not the seat of the Fraunhofer absorption lines.†

In making comparisons of intensity in the bright line and dark line spectra of an element, a serious difficulty is encountered in the probably compound nature of many of the apparently single lines of the flash spectrum. In such cases it is, of course, impossible to assign the true value of intensity to the components; even when the unequal components of an obviously double line are easily distinguished, it is difficult

* 'Phil. Trans.,' A, vol. 197.

† See FOWLER on the Flash Spectrum, 'Observatory,' April, 1902. Also Sir N. LOCKYER and BAXANDALL, 'Monthly Notices, R.A.S.,' vol. LXI., Appendix.

to estimate the intensities correctly, the weaker component being liable to be considerably under-estimated.

Another difficulty occurs when single Fraunhofer lines have a compound origin assigned, such as Fe–Ti, &c., the proportion of intensity of each element in the "make up" of the dark line being unknown. In such cases the relative proportions of intensity in the corresponding flash line may be quite different or even reversed, the predominating element being in general the one which ascends to the greatest elevation in the chromosphere, not necessarily the one which predominates in the dark line.

In these circumstances it is impossible to make anything like a complete or final comparison of intensities. The best that can be done is to select for each element isolated lines which are least open to the suspicion of being made up of more than one line in the flash spectrum, and also lines of supposed single origin as given in ROWLAND's tables.

Unfortunately, there are only three elements which have a sufficient number of lines in their spectra to be treated satisfactorily in this way; they are iron, titanium, and chromium. In the following tables I give the results for these elements, selecting 219 Fe lines of ROWLAND's intensity 3 and upwards, 124 Ti lines of intensity 1 and upwards, and 157 Cr lines of intensity 0 and over.

These are represented in the flash spectra by 93 Fe lines, 39 Ti lines, and 25 Cr lines respectively. The selection of suitable lines was made entirely from ROWLAND's table, and without reference to the flash spectra, so as to avoid bias in the selection.

ROWLAND's intensities of the solar lines are given in the first column of each table, and the number of lines selected between λλ 3500 and 5000 in the second column, the third and fourth columns give respectively the numbers and percentages of the lines which are found as bright lines in the flash spectrum, the fifth column giving the average intensity of these lines.

A glance at the first and last column of each table will show the general relation between the flash intensity and the dark-line intensities for the three elements considered. The numbers indicating intensities for the bright and dark lines are not, of course, directly comparable, since they depend on methods of judging intensity which may differ widely in the two cases. It is a mere coincidence in the case of iron that the numbers representing the stronger lines practically correspond in the first and last columns.

From the columns of percentages the general rule is obvious, that the stronger the dark line of an element, the more probable is its occurrence as a bright line in any given image of the flash spectrum. Thus we find that of the Ti lines none are present in the spectra under discussion corresponding to ROWLAND's intensity 1, and the percentage of dark lines exceeding intensity 1 which are present as bright lines increases with each increase of dark-line intensity up to intensity 4; of the 12 dark lines exceeding intensity 4, all are present in the flash. Similarly with iron, all of the 32 lines exceeding ROWLAND's intensity 8 are present in the flash and none under his intensity 3.

This general law of correspondence of intensity between bright lines and dark lines is, however, far from being exact in detail even with the selected lines used in these comparisons, and the average intensities of the bright lines are in some instances made up of rather widely diverging units.

This is more particularly the case with the weaker dark lines of each element, which are often of abnormal intensity in the flash. In the case of chromium most of the flash lines corresponding with solar lines of intensity 2 and 3 may be considered abnormally strong, for the average intensities for these lines are greater than the average of the lines corresponding with the solar lines of intensity 4.

The percentage columns show also that many dark lines of medium intensity may be absent in the flash, whilst other weaker lines are present.

It must be remembered that estimates of intensity in the flash spectrum, however carefully made, are liable to considerable errors for several reasons. The great weakening of the spectrum near the ends of the plate materially affects the percentages of the weaker lines as given above, and the low dispersion of the plates compared with those on which ROWLAND's estimates were based introduces other sources of discrepancy. Moreover, ROWLAND's table itself is admittedly a "preliminary" table, in which some of the assignments of origin may be erroneous or incomplete, lines having a single origin assigned being really made up of two or more elements.

IRON Lines in Sun and Flash.

Including all isolated lines in ROWLAND's table assigned to Fe only between λ 3500 and λ 5000, excepting those which are obscured in the flash by strong hydrogen and calcium lines.*

ROWLAND's intensity in ⊙.	Number of lines in ⊙.	Number of lines in flash.	Percentage of lines in flash.	Average intensity in flash.
Under 3	Very large number	0	0	—
3 and 4	94	12	13	5
5 ,, 6	66	26	40	3
7 ,, 8	27	23	85	6
9 to 14	13	13	100	8
15 ,, 20	14	14	100	17
25 and over	5	5	100	24

* Including three lines ascribed to Fe only by LOCKYER at λλ 4179, 4233, and 4515.

Abnormal Fe Lines.

In sun.		In flash.		If enhanced.	Remarks.
Wave-length.	Intensity.	Wave-length.	Intensity.		
Lines strong in flash.					
3558·67	8	3558·9	20	?	
3570·27	20	3570·33	30	?	
3634·47	3	3634·48	5	?	
3647·99	12	3647·98	15	?	
3856·52	8	3856·47	15	No	
4179·03	3	4179·1	8	Yes	
4233·33	4	4233·3	10	Yes	
4325·94	8	4325·8	15	No	
4404·93	10	4404·8	20	No	
4515·51	3	4515·6	8	Yes	
4584·02	4	4583·9	25	Yes	
4924·11	5	4924·1	25	Yes	
5018·63	4	5018·5	20	Yes	
Lines weak or absent in flash and exceeding intensity 6 in ⊙.					
3536·71	7	Absent	—	?	Spectrum very weak here.
3651·61	7	3651·85	0	?	
3680·07	9	3680·4 ±	0	?	Hτ interferes at 3679·4.
3684·26	7	3684·29	0	?	Ti line at 3685·3 interferes.
3701·23	8	3701·28	3	?	
3705·71	9	3705·67	5 ?	No	Flash line confused with strong line at 3706·09.
3850·12	10	3850·26	2	No	
3878·15	8	Absent ?	—	No	This line seems to be present on some images not measured.
4528·80	8	4529·0	1	No	

Titanium Lines in Sun and Flash.

Including all lines in Rowland's table assigned to Ti only between λ 3500 and λ 5000, excepting those which are obscured in the flash by strong hydrogen lines.

Rowland's intensity in ⊙.	Number of lines in ⊙.	Number of lines in flash.	Percentage of lines in flash.	Average intensity in flash.
1	38	0	0	—
2	28	5	18	10*
3	27	9	33	8
4	19	13	68	16
5	7	7	100	21
6 and 7	2	2	100	37
Over 7	3	3	100	55

* The average intensity of the five flash lines is increased by two very abnormal lines at λλ 3505·06 and 3520·40, omitting these the average would be 4.

Abnormal Ti Lines.

In sun.		In flash.		If enhanced.	Remarks.
Wave-length.	Intensity.	Wave-length.	Intensity.		
		Lines strong in flash.			
3505·06	2	3505·1	20	?	
3510·99	5	3511·1	30	?	
3520·40	2	3520·4	20	?	
3535·55	4	3535·75	50	?	Perhaps compounded of Ti and a line at 3535·87, intensity 3 ? origin.
3641·47	4	3641·48	40	?	
4294·20	2	4294·35	12	Yes	
4395·20	3	4395·15	30	Yes	
4417·88	3	4417·7	20	Yes	
4501·45	5	4501·5	30	Yes	
		Lines weak or absent in flash and exceeding intensity 3 in ⊙.			
3653·64	5	3653·67	5	?	
3753·00	4	3652·72	0	No	
3924·67	4	3924·8	0	No	
3948·82	4	3949·11	4	No	
3981·92	4	3981·3 } 3982·2 }	5	No	
3989·91	4	3990·12	2	No	
4171·21	4	Absent	—	No	
4291·11 } ·28 }	3 2	Absent	—	No	
4306·08	4	4306·0	1	No	
4533·42	4	Absent	—	No	} Perhaps obscured by strong Ti line at 4534·14.
4534·95	4	Absent	—	No	}
4981·91	4	Absent	—	No	Spectrum very weak here.

Chromium Lines in Sun and Flash.

Including all isolated lines in Rowland's table assigned to Cr only.

Rowland's intensity in ⊙.	Number of lines in ⊙.	Number of lines in flash.	Percentage of lines in flash.	Average intensity in flash.
0 and 1	109	3	3	0
2	22	4	18	1
3	15	8	46	2
4	3	3	100	1
5	3	2	67	3
6 to 8	3	3	100	13
9 and 10	2	2	100	25

ABNORMAL Cr LINES.

In sun.		In flash.		If enhanced.	Remarks.
Wave-length.	Intensity.	Wave-length.	Intensity.		
		Lines strong in flash.			
3593·64	9	3593·65	30	?	
4242·54	2	4242·6	1	Yes	
4359·78	3	4358·9 to 4360·2	5 ?	No	The Cr line is confused with other lines in flash.
4539·95	0	4539·8	0	No	
4541·69	2	4541·6	1	No	
4558·82	3	4558·8	8	Yes	
4588·38	3	4588·0	2	Yes	
4666·39 } ·66 }	0 1	4666·5	5	No	
4708·20	2	4708·1	1	No	
		Lines weak or absent in flash intensity exceeding 3 in ⊙.			
4626·36	5	Absent	—	No	Present in FROST'S and MITCHELL'S lists.
4651·46	4	4651·3	2 ?	No	} Spectrum very weak here.
4652·34	5	Absent	—	No	

These sources of error would all tend to produce discordances in the relative intensities between the dark lines of an element and their bright reversals in the flash, and the question arises whether the apparent anomalies which are indicated above are to be ascribed to such imperfections in our knowledge of the spectra, or to fundamental differences such as might be expected were the emission and absorption spectra produced in separate and distinct layers of the sun's atmosphere, and under different conditions of temperature and pressure.

Under the heading "Abnormal lines" I give with each table a list of the lines with intensities in the flash considerably above the average, corresponding with the dark line intensity, and a list also of the exceptionally weak or absent lines.

In these lists the wave-lengths and intensities of the solar lines, from ROWLAND, are entered in the first two columns, followed by the wave-lengths and intensities taken from Table I in columns 3 and 4. The fifth column indicates whether the line is an "enhanced" line or not, *i.e.*, a line which is relatively brighter in the spark than in the arc spectrum of the element as determined by LOCKYER.

It is at once apparent that many of the abnormally bright flash lines are enhanced lines, whilst none of the abnormally weak lines are enhanced lines. The lists of enhanced lines published by Sir NORMAN LOCKYER do not include the ultra-violet region beyond λ 3800, it is uncertain, therefore, whether the flash lines in this region are enhanced lines or not. If these are omitted, all the titanium lines abnormally strong in the flash, and all the iron lines excepting the three at λλ 3856·5, 4325·9, and 4404·8, are enhanced lines.

If all the enhanced lines in the above-mentioned lists are considered, it is found that all the more strongly enhanced lines of iron and titanium coincide with strong lines in the flash (11 Fe lines and 21 Ti lines). But since many of these lines are of compound origin in the flash, it is not possible to say whether they are all of *abnormal* intensity, *e.g.*, 4351·9, 4549·6, 4556·1, 4629·6, and others. The quartette of enhanced iron lines at 4508·5, 4515·5, 4520·4 and 4522·7 are all abnormally strong in the flash considered as Fe lines only, but according to ROWLAND three of these are of compound origin, one including Ti. However, it seems probable that the abnormal intensity of this group is chiefly due to the fact that the lines are enhanced lines.

There can be little doubt from this inquiry that the enhanced lines *do* play a significant part in the flash spectrum, and the abnormal intensities of these lines are not due to errors in the assignment of origin in ROWLAND's tables or to over-estimates of intensity in the flash.

Of the abnormally weak lines a considerable number are probably the result of under-estimates due to the close proximity of very strong lines of other elements. There remain a few, however, which cannot be thus explained; among these particular attention may be called to the titanium lines at λλ 3753·00, 3924·67, 4171·21, and 4306·08, all of intensity 4 in the solar spectrum, and the chromium line at λ 4626·36. No satisfactory reason can at present be given for the weakness or absence of these lines in the flash spectrum.

Notwithstanding these instances of disagreement between the intensities of the Fraunhofer lines of an element and their flash spectrum equivalents, the general agreement between the two spectra is so striking that it can scarcely be maintained that there is a fundamental difference in the conditions under which they are produced. The abnormally strong lines in the flash, which in so many cases are also lines which are enhanced in the spark, would, it is true, indicate that some of the radiating gas at all events must be in a condition differing from that in the absorbing layer, and this, it must be acknowledged, is of great interest and importance, particularly in view of the fact pointed out by FOWLER, that under some stellar conditions, *e.g.*, in α Cygni, these particular lines constitute a separate and much simpler spectrum quite free from admixture with the ordinary arc lines.*

But, as I hope to show in what follows, the prominence of these enhanced lines in the flash can be simply explained without abandoning the view that the flash region is really identical with the absorbing layer, and in the great majority of cases the flash lines are true reversals of the dark lines.

In all photographs hitherto obtained at stations near the central line of eclipse, the flash spectrum must represent the more elevated region of the radiating gases, since

* 'Observatory,' June, 1902.

this portion of the layer remains uncovered by the moon for an appreciable time after the sky glare is withdrawn at totality, whilst the lower dense strata immediately in contact with the photosphere are instantaneously occulted.

It might reasonably be assumed, therefore, that the intensities of the bright lines in the lowest strata differ to some extent from those in the spectra photographed, and even more closely approximate to the intensities in the Fraunhofer spectrum.

But the photographs under discussion portray a *grazing* contact, in which the motion of the moon was not across but parallel to the flash layer. These spectra, therefore, should more truly represent the radiation from the entire depth of the layer, at any rate at points near the apex of the bright arcs, and where the layer is sufficiently uncovered, because at such points the very lowest strata would remain visible throughout the time the plate was exposed.*

A careful comparison between the intensities of the lines at points near to and far from the apex, or centre line of the spectra, shows, however, that there are no appreciable differences.

Moreover, the intensities given in Table I., which were estimated at points not far from the apex, and where the continuous spectrum of the photosphere was just beginning to appear, will be found to be in substantial agreement with the results of LOCKYER (1898), FROST (1900), and MITCHELL (1901), all of which were obtained near the central line.

It seems, therefore, that there can be no very striking differences between the spectra of the higher and lower regions of the flash layer as regards the intensities of the lines, unless absorption by the upper regions through which the line of sight passes should neutralise such differences. In particular it may be noted that the enhanced lines seem to predominate throughout the entire region.

If it is assumed that the differences between spark and arc spectra are conditioned by temperature, the spark being the hotter, it would seem at first sight that the flash region must have a higher temperature, and must consequently be distinct from the absorbing layer, since in the latter the intensities of the lines closely approximate to those in the arc. I think it can be shown, however, that the spark and arc conditions may *co-exist* at the same altitude above the photosphere.

It is well known that the outer limit of the chromosphere, as seen in the line of hydrogen, presents a structure of small filaments like blades of grass covering the entire surface, and very unlike the diffused, indefinite limit which a true atmospheric envelope might be expected to present.

According to SECCHI, "at the base of the chromosphere the hydrogen has the shape of small, close filaments which seem to correspond with the granulations of the photosphere." †

* The terms *layer* and *strata* are here used for convenience, but it is not intended to imply that the gases of the chromosphere are in reality stratified.

† 'Popular Astronomy,' S. NEWCOMB, p. 275.

This structure suggests that the chromosphere is in reality a region of innumerable small eruptions of the same nature as the jets of highly luminous gas which are constantly to be seen with the spectroscope in all regions of the sun's limb. It is probable, indeed, that these jets, and the larger eruptive prominences, are in reality only the more pronounced manifestations of a phenomenon occurring on a smaller scale everywhere over the solar surface.

The highly-heated gases composing these eruptions, which may be supposed to originate below the photospheric level, would lose heat as they ascended by adiabatic expansion and by radiation, and at a certain elevation would precipitate the more refractory substances as highly luminous clouds, forming, in fact, the photospheric granules and the columnar filaments observed in sunspots. But the gaseous streams, deprived of their condensable materials, would continue to ascend above the photosphere, finally becoming diffused in the region of the chromosphere. The expanded gases, subsequently subsiding in a relatively cooled condition, would form a strongly absorbing atmosphere settling down uniformly and slowly upon the photosphere and through which the ascending streams would be forced.

If this really represents roughly the actual state of things, it is clear that the temperature conditions represented by the electric spark and by the arc may both exist at the same altitude above the photosphere, the spark condition in the highly-heated ascending gases and the arc condition in the cooler descending gases.

Seen at the sun's limb, as under the conditions of a total eclipse, the more intense spectrum of the ascending gases would be neutralised to a considerable extent by the absorption of the cooler gases in which the jets would be immersed, and through which for immense distances the line of sight must pass. But just those particular rays which are characteristic of the high temperature spectrum would not suffer absorption to nearly the same extent, consequently these rays (the enhanced spark lines) would stand out conspicuously in a spectrum which in its main features would be the emission spectrum of the cooler descending gases, *i.e.*, the reversed Fraunhofer spectrum.

The relatively cool gases would obviously determine the character of the absorption spectrum of the disk, and the only effect of the hotter eruptions, supposing them to be too small to be individually distinguishable in the spectroscope, would be to produce a faint emission line of about the same intensity as the background of continuous spectrum, and tending to diminish the intensity and width of all the dark lines, particularly the enhanced spark lines.

In this way, by assuming the presence of innumerable eruptions of hot gas and cooler but quietly descending absorbing gases, the abnormal intensity of the enhanced lines in the flash can be simply explained without abandoning the view that the flash spectrum is really the reversed Fraunhofer spectrum, and that the entire depth of the flash region, and, indeed, of the chromosphere itself, is effective in producing the absorption lines.

That there really exists a circulation of the solar gases in a radial direction is strikingly shown in the detailed structure of some of the Fraunhofer lines themselves. DESLANDRES has called attention to certain peculiarities in the structure of the lines H and K in the general light of the sun and in particular regions of the solar surface.*

These lines consist of three distinct portions—a broad diffuse absorption shading, a bright rather wide emission line near the centre of the shading, and a narrow absorption line which obliterates all but the edges of the underlying bright line.

DESLANDRES finds that over undisturbed regions of the disk, and at some distance from the limb, the central absorption line is always displaced towards the red with respect to the underlying emission line, producing a dissymmetry in the edges of the latter. This he attributes to a vertical circulation of the calcium vapour, the ascending gas producing the emission line slightly displaced to the violet, whilst the cooler descending gas gives rise to the central absorption line displaced to the red.

According to JEWELL, all the strongly shaded lines exhibit an emission line, which is very nearly obscured by a central strong absorption line usually unsymmetrically placed. Traces of an emission line are also visible at the sides of some of the narrow unshaded lines. The effect of motion of the hot gas he considers, however, to be masked to a certain extent by pressure shift, the displacement of the emission line to the violet by reason of the ascending motion being partly neutralised by an opposite displacement due to pressure.†

Some sort of circulation of the solar gases in a radial direction and all over the surface, such as is demanded by the theory of "convective equilibrium," would seem, therefore, to be established, the ascending gases rising with sufficient velocity to appreciably displace the emission lines when observed on the sun's disk, whilst the more diffused absorbing gases descending with a more uniform motion produce the well-defined dark lines very slightly displaced to the red compared with the same lines from a terrestrial source. Obviously such motion of the gases being in a radial direction will not affect the position or definition of the bright lines of the flash spectrum as seen at the limb during an eclipse.

A difficulty has to be faced, however, when we try to account for the apparent sorting out of the different elements in the chromosphere, which seems to depend in a general way on atomic weight, the lighter elements ascending to greater elevations than the heavier.‡

But an eruption in the ordinary sense due to an explosion would give equal

* 'Comptes Rendus,' August, 1894.

† 'Astrophysical Journal,' vol. III., p. 100, *et seq.*

‡ The exceptional altitudes reached by the elements Ca and Ti do not materially affect this general law, which asserts itself by the absence in the chromosphere of nearly all the elements having atomic weights exceeding that of Zr (91), Ba and La being, perhaps, the only elements with a higher atomic weight that have been identified with tolerable certainty in the flash spectrum.

velocities to the whole mass of mixed gases, and it is difficult to see why these should not be projected to equal altitudes in the chromosphere, yet most of the metals with atomic weights between 20 and 100 stop short at from 1″ to 2″ elevation, whilst the elements H, He, Ca, and Ti ascend to 8″ or 10.″

The same lagging behind of the elements of the lower chromosphere occurs, however, in the so-called "metallic" and great eruptive prominences.* In these the higher parts usually consist solely of H, He, Ca and probably Ti, the other elements only appearing at the base or stem of the prominence, or frequently only in the surrounding chromosphere. In the more violent eruptions, too, the distortions due to motion in the line of sight affect chiefly the hydrogen and calcium lines, the lines of other elements present in such outbursts being usually undisturbed, or but slightly affected, showing that these elements, although apparently mixed up with the hydrogen, do not share in the motion.

Although it may be difficult at present to understand the nature of these great eruptions, it would seem reasonable to suppose that the entire chromosphere consists of miniature eruptive prominences of the same nature as the greater outbursts, the base of the eruptions giving the metallic lines of the flash spectrum and the higher parts the lines of H, He, Ca and Ti only.

This conclusion is strengthened when it is remembered that the strongly enhanced lines of iron at 5317, 5269, 5018, and 4924 so prominent in the flash, are always the first to appear as bright lines in the metallic eruptions, other iron lines, although stronger than the above in the Fraunhofer spectrum, being seldom or never seen reversed. This is doubtless owing to the relatively high temperature of the gases in these eruptions compared with the absorbing gases, and in the lower chromosphere the enhanced lines indicate a similar state of things, the highly-heated ascending jets giving a high temperature emission spectrum more nearly resembling that of the spark than of the arc.

The Flash Spectrum in High Latitudes.

It is of interest to compare the images at different points on the limb to determine whether the flash spectrum is the same in all latitudes. The limited distribution of the metallic prominences, which, in the writer's experience, are only to be found in the latitudes of spot formation, would perhaps lead one to anticipate some modification of the spectrum in high latitudes.

At the date of the eclipse (May 28th, 1900) the sun's south pole was at position angle 164° and very nearly coincident with the limb. Unfortunately, this point, and the region within 10° of it on either side, is occupied by the continuous spectrum

* No spectroscopic distinction can be made between the metallic eruptions and the more quiescent forms of prominence, for the latter, when photographed at an eclipse, exhibit the same metallic lines at their bases as the former.

in all the images obtained before mid-eclipse, and in those obtained after that phase only the stronger lines are impressed, the moon's limb having occulted the stratum very rapidly, notwithstanding that the motion was nearly parallel to it. This would indicate an extreme shallowness of the layer near the pole.

In the mid-eclipse photograph, No. 11, the continuous spectrum being broken up into narrow bands, the flash spectrum arcs can be traced right across the polar region near the more refrangible end of the plate. In the portion of spectrum between F and K the bands coalesce from over-exposure and obscure the bright arcs entirely.

The highest latitudes in which really good images of the flash spectrum occur are $-70°$ to $-77°$ on the east side in No. 9, and $-76°$ on the west side in No. 13; and the lowest latitude is in $-36°$ to $-41°$ on the west side in No. 11. Intermediate between these there are the excellent images in latitude $-56°$ west and $-64°$ east in No. 10. From this material comparisons can be made between the spectra at fairly high latitudes and those at mid-latitudes, and as a check on the results the east and west limbs at about the same latitudes can be compared.

All these images are indicated on Plate 2 by arrows at the ends of the spectra, and the position of the south pole is similarly shown for each spectrum. In Plate 3 a limited portion of the spectrum is shown for the three images which were measured. These are on a scale equal to 4·3 times that of the original negatives, and the curved arcs have been converted into linear spectra by means of a cylindrical lens during the process of enlargement. Great care was taken to avoid the production of spurious lines due to defects in the negatives.

Comparing the two high-latitude spectra shown in the upper and lower figures of Plate 3 with the mid-latitude spectrum placed between them, it is not easy to detect differences which can fairly be ascribed to latitude. It may be noticed that the titanium line at about λ 3900 and the aluminium line at λ 3944 are both relatively weak in the upper spectrum (latitude $-74°$ East) compared with the middle spectrum (latitude $-41°$ West). But in the lower spectrum, from an equally high latitude on the opposite side of the pole, these lines are as strong as in No. 11 spectrum.

There are many other minor differences in relative intensities between the three spectra, as will be apparent on comparing the three columns of intensities given in Table I., but these seem to bear no relation to difference of latitude.

A special effort was made to discover any modification of intensity in the enhanced lines near the pole, and the average intensity of all the more prominent enhanced lines of iron and titanium in Nos. 9 and 13 spectra was compared with the average of these lines in No. 11 spectrum, making due allowance for the greater intensity of No. 11 spectrum, as a whole, compared with the others.

The result is shown in the following table :—

AVERAGE Intensity of the Principal Enhanced Lines of Fe and Ti.

	Ti (23 lines).	Fe (14 lines).
In latitude $-74°/75°$ (from No. 9 and 13 spectra)*. . .	16·3	18·6
,, $-41°$ (from No. 11 spectrum)	18·4	14·6

It will be seen from this that while the Ti lines appear to be slightly weaker in high latitudes than in mid-latitudes, the Fe lines give just the opposite result; so that by taking both elements together there is found to be practically no difference at all. The small differences indicated for the lines of each element alone, may safely be put down to the uncertainties of the original estimates of intensity.

It is to be inferred, therefore, that the enhanced lines are of the same intensity in all latitudes, and that the general character of the flash spectrum remains unaltered in passing from the equator to the poles. This does not, however, preclude the possibility that the flash stratum is shallower in the polar regions than near the equator.

An interesting subject for future inquiry would be as to whether the flash spectrum undergoes any modifications such as an increase or decrease of depth of the layer or changes in the intensity of the enhanced lines at different epochs in the sun-spot period. If the chromosphere is really of an eruptive character one might expect, at times of maximūm spot activity, when also the metallic eruptions are most frequent, that the flash spectrum region would extend to greater altitudes in the chromosphere and that the enhanced lines would be relatively brighter than during the quiet periods of minimum.

The evidence so far obtained is, it is true, against any marked changes occurring. I have compared the spectrum obtained by SHACKLETON in 1896 with those herein discussed and with others obtained in 1898 and 1901, but without detecting any certain changes in the intensities of the lines. The flash appears, in fact, to be of as constant and unchanging a character as is the Fraunhofer spectrum, which is only what would naturally be expected, seeing that it appears to be in the main the reversal of that spectrum.

Summary of Conclusions.

In a general way the conclusions arrived at from the discussion of the spectra obtained in 1898 are amply confirmed and extended by the present results.

It is now shown that every strong dark line of the solar spectrum exceeding ROWLAND'S intensity 7 is found in these spectra as a bright line; and the great

* The observed intensities of the high latitude spectra have been multiplied by the factor 1·81, this number representing the ratio of intensities between these two spectra taken together and No. 11 spectrum, when all the lines between λ 39 and λ 50 are compared.

majority of the bright lines of the flash spectrum, excluding hydrogen and helium lines, coincide with dark lines of intensity not less than 3.

Most of the bright arcs of the flash spectrum are well-defined narrow lines admitting of considerable accuracy in the measures, and the present determinations of wave-length indicate that the coincidence of the bright lines with the dark lines is exact within ·05 tenth-metre for all the well-defined lines.

As regards the relative intensities of the lines of any one element in the flash and Fraunhofer spectra, my previous results require modification and extension as follows :—

The relative intensities of isolated lines of an element in the flash spectrum are in general, but not exact, agreement with those of the same element in the solar spectrum, and those lines which are exceptionally strong in the flash are in most cases lines which are enhanced in the spark spectrum of the element.

All of the more prominent enhanced lines of iron and titanium are found to coincide with strong lines in the flash, but owing to the compound nature of some of the lines, it is not certain that all of these have abnormal intensities in the flash.

There is no evidence of differences in the relative intensities of the lines of an element in the higher or lower regions of the flash layer, and the enhanced lines appear to predominate throughout the entire depth of the radiating stratum.

The enhanced lines are equally prominent in the polar regions and in low latitudes, and the flash spectrum generally is now found to be the same in all latitudes and shows no essential change after an interval of five years.

An explanation of the abnormal intensities of the enhanced lines in the flash spectrum is now offered which depends on the assumption of a continuous circulation of the solar gases in a radial direction ; the highly heated ascending gases giving the predominant features to the flash spectrum, whilst the cooler more diffused gases, slowly subsiding, determine the character of the absorption spectrum.

The final conclusion is that the flash spectrum represents the emission of both ascending and descending gases, whilst the Fraunhofer spectrum represents the absorption of the descending gases only.

Tables of Wave-length and Intensity.

In Table I. the wave-lengths and intensities of the bright lines in the three measured spectra (Nos. 9, 11, and 13) are entered in the first six columns ; the 8th and 9th columns give the "adopted" wave-lengths and intensities, *i.e.*, the most probable values deduced from all the measures, and these are compared in the two following columns with ROWLAND's values of the nearest absorption lines.* The origins, mostly from ROWLAND, are given in the last column. A vertical line

* The wave-lengths of the helium lines from RUNGE and PASCHEN are also given in the tenth column. They are placed within brackets to distinguish them from the absorption lines.

connecting two or more wave-lengths means that a shading extends between them, and probably indicates an unresolved group of lines. In many instances the whole group presents the appearance of a single wide line or band, and in these cases the intensity of the group as a whole is given.

The intensities of the flash lines are, of course, entirely relative, and the higher absolute values obtained for No. 11 spectrum are simply due to the greater intensity of this spectrum as a whole compared with No. 9 or No. 13. But since these higher values are probably more reliable than the others, the "adopted" values are mostly taken from No. 11 spectrum.

In Table II. the wave-lengths of the hydrogen lines in each spectrum are compared in the 2nd, 3rd, and 4th columns. The mean values for the three sets of measures are entered in the 5th column, and these are compared with the theoretical values computed from BALMER'S formula, given in the 6th column. The last column gives the differences, observation — calculation.

In the formula $\lambda = \frac{s^2}{a(s^2 - 4)}$ the constant $a (= 27418{\cdot}75)$ is derived from ROWLAND'S measures of the lines α, β, and γ reduced to a vacuum, and the resulting wave-lengths are corrected to air in accordance with a table of RUNGE ('Astronomy and Astrophysics,' vol. 12, No. 5).

TABLE I.—Eclipse Spectra, May 28, 1900.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
—	3488·90	—	—	10	—	. . . .	3488·9	10	3488·817	4	Mn
—	91·31	—	—	20	—	. . . .	91·3	20	91·195	5	Ti
—	93·69	—	—	5	—	Very indistinct (probable group) best on continuous spectrum	93·7	5	93·430	1	Fe
									93·834	1	
—	—	—	—	—	—		—	—	94·308	2	
—	95·02	—	—	5	—		95·0	5	94·815	2	
—	96·39	—	—	30	—	. . . .	96·4	30	96·224	0	Co V
									96·348	2	Zr
—	97·54	—	—	10	—	Best on continuous spectrum .	97·5	10	97·668	3	Mn
									97·982	8	Fe
—	3505·12	—	—	20	—	. . . .	3505·1	20	3505·056	—	Ti
—	11·10	—	—	30	—	Well defined	11·1	30	10·985	5	Ti
—	12·40	—	—	0	—	. . . .	12·4 ±	0	—	—	—
—	12·98	—	—	0	—	. . . .	13·0 ±	0	—	—	—
—	14·22	—	—	5	—	. . . .	14·2	5	13·965	7	Fe
									14·138	3	Ni
—	15·41	—	—	5	—	. . . .	15·4	5	15·206	12	Ni
—	17·49	—	—	10	—	. . . .	17·5	10	17·446	3	V
—	20·40	—	—	20	—	. . . .	20·4	20	20·397	2	Ti
—	3524·75	—	—	10	—	Dark line on continuous spectrum, bright line outside	3524·7	10	3524·677	20	Ni

TABLE I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
—	3526·19	—	—	5	—	. . . .	3526·2	5	3525·986	4	Fe
									26·183	6	Fe
—	31·01	—	—	10	—	Equal pair, best on continuous spectrum	31·0	10	30·919	3	?
									31·982	3	Mn
—	32·00	—	—	10	—		32·0	10	32·143	4	Mn
									32·262	3	Mn
—	33·93	—	—	2	—	Very vague	33·9	2	—	—	—
3535·65	35·85	—	–	50	—	. . . .	35·75	50	35·554	4	Ti
—	41·69	—	—	0	—	. . . .	41·7 ±	0	—	—	—
—	42·76	—	—	2	—	. . . .	42·8 ±	2	—	—	—
—	45·39	—	—	20	—	. . . .	45·4	20	45·336	4	–, V
—	48·23	—	—	2	—	. . . .	48·2	2	48·175	3	Mn Fe
									48·332	5	Mn Ni
—	49·36	—	—	10	—	. . . .	49·4	10	49·151	2	Y ?
52·35	52·19	—	—	10	—	. . . .	52·27	10	52·098	1	Zr
									52·253	2	Fe
—	53·83	—	—	0	—	Visible at centre only . . .	53·5 ±	0	53·624	3	Ni
—	53·72	—	—	0	—		53·7 ±	0	53·887	5	Fe
56·92	56·93	—	—	40	—	. . . .	56·92	40	56·738	2	Zr
									56·830	2	Fe
									56·944	4	Fe
									57·036	3	?
—	58·87	—	—	20	—	. . . .	58·9	20	58·672	8	Fe
—	61·04	—	—	3	—	On continuous spectrum only .	61·0	3	61·037	4	Co
—	62·07	—	—	8	—	. . . .	62·1	8	61·898	3	Ni
									62·043	1	Fe, Ti
65·80	65·53	3565·62	—	15	—	. . . .	65·65	15	65·535	12	Fe
—	66·44	66·70	—	15	—	. . . .	66·57	15	66·522	10	Ni
67·98	67·91	67·81	—	20	—	. . . .	67·90	20	67·835	4	?
70·40	70·29	70·31	—	30	—	Strong absorption line coincides	70·33	30	70·183	4	Mn
									70·273	20	Fe
									70·415	4	?
—	72·18	—	—	0	—	. . . .	72·2 ?	0	72·014	6	Ni
									72·155	5	Fe
72·77	72·76	72·48	5	40	—	Perhaps two lines in 13 . . .	72·71	40	72·712	6	Sc, –
74·11	74·08	74·02	—	8	—	. . . .	74·07	8	—	—	—
76·66	76·59	76·59	10	50	10	Shaded line in 11; no absorption line in 9	76·61	50	76·527	3	–, Sc ?
—	78·88	78·92	—	20	5	. . . .	78·90	20	78·832	10	Cr
3581·28	81·27	81·30	—	35	10	Strong absorption line in 9 with weak emission line	81·28	35	81·349	30	Fe
—	84·97	84·78	—	8	5	. . . .	84·88	8	84·800	6	Fe
									84·940	5	Co
—	3585·70	3585·48	—	20	5	Ill defined and shaded; wide in 13	3585·60	20	85·310	5	Co
									85·479	7	Fe
									3585·658	2	?

TABLE I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
									3587·130	8	Fe
—	3587·27	—	—	7	—	. . . .	3587·27	7	87·286	2	Ti
						Group of lines			87·370	7	Co
									87·899	5	Fe
—	88·04	—	—	2	—	. . . .	88·04	2	88·084	6	Ni
3589·71	89·91	3589·97	5	25	5	. . . .	89·86	25	89·773	5	?
									89·908	5	?
90·74	90·70	90·75		20	5	. . . .	90·73	20	90·609	2	?
									90·651	2	?
—	91·72	—	—	0	—	On continuous spectrum only in 11	91·72	0	—	—	—
92·19	92·22	92·16	—	12	0	. . . .	92·19	12	92·169	2	Y ?
93·61	93·66	93·69	5	30	5	. . . .	93·65	30	93·636	9	Cr
—	94·94	—	—	5	—	. . . .	94·94	5	94·784	6	Fe
									95·017	3	Co
96·22	96·19	96·24	10	25	5	. . . .	96·22	25	96·195	4	Ti
—	98·00	97·86	—	1	0	Narrow line on inner continuous band in 11	97·93	1	97·854	8	Ni
3600·92	3600·88	3600·87	15	30	10	. . . .	3600·89	30	3600·880	3	Y
02·04	02·10	02·11	5	25	10	. . . .	02·09	25	02·060	1	Y
—	03·38	—	—	2	—	Wide line	03·38	2	03·354	5	Fe
03·92	03·90	03·70	5	8	0	. . . .	03·84	8	03·832	2	?
									03·922	3	Ti
05·49	05·49	—	4	15	—	. . . .	05·49	15	05·479	7	Cr
—	06·83	—	—	5	—	. . . .	06·83	5	06·838	6	Fe
08·73	08·99	08·93	—	20	2	. . . .	08·88	20	09·008	20	Fe
09·56	—	—	0	—	—	. . . .	09·56	0	09·467	5d ?	Ni
—	10·56	—	—	0	—	. . . .	10·56	0	—	—	—
11·27	11·16	11·20	5	15	2	. . . .	11·21	2	11·189	2	Y, Mg ?
14·11	14·00	13·96	10	60	10	Shaded on V side in 11; very long line	14·02	60	13·947	4	–, Sc
									14·019	3	?
14·99	14·87	14·80	1	10	1	Narrow line stronger nearer centre in 11	14·89	10	14·922	2	?
18·59	18·92	18·93	—	25	5	. . . .	18·81	25	18·919	20	Fe
	—	—	5			. . . .					
19·64	19·57	19·51	—	10	1	. . . .	19·57	10	19·539	8	Ni
21·11	21·42	—	0	8	—	. . . .	21·26	8	21·244	3	?
		21·70	—	—	0	Band in 11, ill defined in 13 .	—	—	21·612	6	Fe
22·00	22·38	—	3	3	—	. . . .	22·03	3	22·147	6	Fe
24·95	24·99	25·07	10	40	5	Good isolated line in 11 . . .	25·00	40	24·979	5	Ti, Fe
—	27·9	—	—	0	—	. . . .	27·9	0	27·953	4	Co
28·85	28·94	28·75	2	7	0	Stronger at centre in 11 and absorption line on R side	28·85	7	28·847	2	Y, Mg ?
									30·876	4	?
30·96	30·91	30·91	20	45	10	. . . .	30·93	45	30·918	3	?
									31·124	2	Ca
3631·64	3631·59	3631·54	5	25	5	. . . .	3631·59	25	3631·605	15	Fe

TABLE I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Inten-sity.	Wave-length in sun (ROWLAND).	Inten-sity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
3633·20	3633·22	3633·23	4	20	4	. . . .	3633·22	20	3633·277	2	Y
34·42	34·55	—	2	5	—	Diffuse in 9	34·48	5	(3634·393)	—	He
—	35·57	—	—	1	—	. . . .	35·57	1	35·608	4	Ti, Fe
—	36·73	—	—	7	—	. . . .	36·73	7	36·608	4	Cr ?
—	38 50	—	—	2	—	. . . .	38·50	2	38·442	3	Fe
40·56	—	—	1	—	—	. . . .	40 56	1	40·535	6	Cr–Fe
41·50	41·51	41·42	20	40	10	Well defined in 13 . . . . (41·50 and 42·90 lines)	41·48	40	41·473	4	Ti
42·90	42·96	42·93	20	45	10		42·93	45	42·820 42·912 42·965	7 2 3	Ti Sc ?
—	43·9	—	—	0	—	Exceedingly fine line 11 . . .	43·9	0	—	—	—
45·45	45·46	45·36	5	15	5	. . . .	45·43	15	45·429 45·475	3 3	? Sc ?
47·95	47·98	48·02	4	15	5	Diffuse in 9	47·98	15	47·988	12	Fe
49·46	49·54	49·51	—	5	0	. . . .	49·50	5	49·476 49·654	3 5	Co Fe, La
			3								
50·51	50·41	—	—	2	—	. . . .	50·46	2	50·178 50·423	4 5	Fe Fe
51·87	51·88	51·81	10	18	8	Wide in 13.	51·85	18	51·614 51·940	7 4	Fe Sc
53·77	53·62	53·61	1	5	0	Very diffuse in 11	53·67	5	53·637	5	Ti
55·59	55·87	—	1	5	—	Ditto	55·73	5	55·609 55·801	3 3	Fe ?
57·94	57·97	—	0	0	—	. . . .	57·96	0	58·044	3	Co, Fe, Mn
59·96	59·85	59·71	5	12	5	. . . .	59·84	12	59·901	5	Fe–Ti
—	61·31	—	—	1	—	. . . .	61·31	1	—	—	H
62·34	62·34	62·39	8	15	5	. . . .	62·35	15	62·378	5	H, Ti
63·49	63·58	63·58	0	4	0	. . . .	63·55	4	63·541 63·596	3 3	? Fe, H
64·72	64·73	64·76	3	10	4	. . . .	64·74	10	64·760	2	H, Y
66·31	66·11	66·33	1	6	2	. . . .	66·25	6	—	—	H
67·90	67·77	67·81	1	8	2	. . . .	67·83	8	—	—	Hω
69·52	69·58	69·55	3	10	3	. . . .	69·55	10	69·666	4	H–Fe
71·50	71·48	71·61	3	12	3	. . . .	71·53	12	—	—	Hχ
73·96	73·82	73·90	3	15	4	. . . .	73·89	15	—	—	Hϕ
74·85	74·85	74·87	1	5	1	. . . .	74·85	5	74·865	1	Zr
76·56	76·42	76·55	5	18	5	. . . .	76·51	18	—	—	Hυ
77·85	77·75	77·95	5	15	3	. . . .	77·85	15	77·831 77·991	3 3	? ?
79·59	79·45	79·52	8	30	8	Very broad in 11; probably two lines	79·52	30	—	—	Hτ
80·84	—	80·00	0	—	1	. . . .	80·43 ?	0	80·069	9	Fe
82·98	82·88	82·95	5	20	5	Wide and ill-defined in 9. . .	82·94	20	—	—	Hσ
84·29	—	—	0	—	—	. . . .	84·29	0	84·258	7	Fe
3685·34	3685·31	3685·42	40	60	30	. . . .	3685·36	60	3685·339	10	Ti

Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (Rowland).	Intensity.	Element.
3686·97	3686·89	3687·01	8	20	7	Much narrower than σ and τ .	3686·96	20	—	—	Hρ
87·64	—	87·66	0	—	0	. . . .	87·65	0	3687·610	6	Fe
88·47	—	88·62	0	—	0	. . . .	88·54	0	88·558	4	Ni
—	89·58	89·65	—	2	0	. . . .	89·61	2	89·614	6	Fe
—	90·51	—	—	0	—	. . . .	90·51	0	—	—	—
—	—	91·08	—	—	0	. . . .	91·08	0	—	—	—
91·75	91·62	91·74	10	25	8	. . . .	91·70	25	—	—	Hπ
94·24	94·24	94·22	4	10	3	On continuous spectrum only .	94·24	10	94·164	4	Fe
									94·344	3	Eb ?
94·96	95·08	95·16	1	3	0	. . . .	95·07	3	95·194	5	Fe
—	96·01	—	—	0	—	. . . .	96·01	0	96·006	1	Fe, V
97·34	97·20	97·30	15	30	10	. . . .	97·28	30	—	—	Ho
98·29	98·30	98·27	0	4	0	Strong on 4th band of continuous spectrum, and at centre in 11	98·29	4	98·303	2	Ti, Zr
—	3700·15	—	—	0	—	. . . .	3700·15	0	—	—	—
3701·26	01·29	3701·28	0	3	0	Stronger at centre	01·28	3	3701·234	8	Fe
02·15	—	02·43	0	—	0	. . . .	02·29	0	02·170	4	Fe
									02·382	2	Co
04·03	04·01	03·98	20	35	12	. . . .	04·01	35	—	—	Hξ
—	05·67	—	—	5	—	. . . .	05·67	5	05·708	9	Fe
06·14	06·03	06·10	15	20	5	Diffuse on V side in 9, rather wide in 13	06·09	20	06·175	6	Ca, Mn
07·28	08·02	07·97	1	5	1	. . . .	07·28	0	07·186	5	Fe
08·03							08·01	5	08·068	5	Fe
09·19	09·43	09·41	1	8	1	Diffuse in 9	09·34	8	09·389	8	Fe
10·46	10·44	10·51	10	20	3	Narrow line in 11	10·47	20	10·431	3	Y
12·12	12·16	12·12	25	40	15	. . . .	12·13	40	—	—	Hν
12·96	12·99	13·01	3	8	0	Very fine line in 11	12·99	8	13·037	2	—
									13·087	3	Cr
14·75	15·07	15·02	—	—	0	. . . .	15·04	—	—	—	—
			3	15		. . . .		15	—	—	—
15·71	15·68	15·57	—	—	1	. . . .	15·62	—	15·615	4	Mn ?
16·39	16·59	16·49	1	4	0	. . . .	16·49	4	16·591	7	Fe
20·08	20·13	20·13	12	25	8	. . . .	20·11	25	20·084	20	Fe
21·92	22·03	22·00	30	45	20	. . . .	21·98	45	—	—	Hμ
24·19	24·65	24·55	2	5	0	Very diffuse and broad in 11 .	24·61	5	24·526	6	Fe
25·06									24·716	1	Ti
26·80	27·13	27·47	—	—	1		27·30	—	—	—	—
			5	10		Absorption band coincides in 9		10	—	—	—
28·08	28·01	—	—	—	—		28·04	—	—	—	—
29·68	30·01	29·71	1	1	0	. . . .	29·80	1	29·952	3	Ti
31·53	31·51	31·46	1	1	—	. . . .	31·50	1	31·523	3	Fe
32·67	—	32·70	2	—	1	. . . .	32·69	2	32·545	6	Co, Fe
33·54	33·26	—	2	5	—	In shade of Hλ	33·40	5	3733·469	7d ?	Fe
3734·48	3734·58	3734·52	30	45	20	Trace of line on less refrangible side of hydrogen line, coinciding with Fe absorption	3734·53	45	—	—	Hλ

Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9	Intensities. No 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity	Wave-length in sun (Rowland).	Intensity.	Element.
3737·08	3737·13	3737·05	15	35	10	Absorption line coincides in 9; probably 2 lines in 13	3737·09	35	3737·059	5	Mn, Ca
									37·281	30	Fe
38·39	38·41	—	0	3	—	. . . .	38·40	3	38·466	6	?
39·33	39·51	—	1	3	—	. . . .	39·42	3	39·370	3	Ni
—	40·59	—	—	1	—	. . . .	40·59	1	—	—	—
41·76	41·81	41·69	15	25	5	Shaded on V-side in 9. . . .	41·75	25	41·791	4	Ti
43·64	43·53	43·45	5	8	1	Diffuse in 9	43·54	8	43·508	6	Fe-Ti
									43·626	2	?
45·80	45·83	45·76	15	30	5	Absorption line coincides in 9 .	45·80	30	45·717	8	Fe
									46·058	6	Fe
48·24	48·42	48·26	5	10	5	Diffuse	48·31	10	48·408	10	Fe
—	—	49·47	—	—	1	. . . .	49·47	1	49·631	20	Fe
50·25	50·29	50·26	35	50	25	. . . .	50·27	50	—	—	Hκ
51·66	51·71	—	1	3	—	Very fine line in 9 and 11 . .	51·68	3	—	—	—
52·80	52·64	—	0	0	—	. . . .	52·72	0	—	—	—
53·69	53·86	—	1	2	—	Very fine line in 9	53·77	2	53·732	6	Fe-Ti
54·74	—	—	0	—	—	. . . .	54·74	0	—	—	—
55·56	55·72	—	0	0	—	. . . .	55·64	0	—	—	—
57·72	57·84	57·81	5	10	1	Poor definition in 11; diffuse on V-side in 9	57·79	10	57·824	4	Cr-Ti
59·41	59·40	59·38	40	50	20	. . . .	59·40	50	58·375	15	Fe
									59·447	12	Ti
61·44	61·38	61·49	40	55	20	. . . .	61·44	55	61·464	7	Ti
63·89	63·96	63·91	3	12	3	Diffuse in 9	63·93	12	63·945	10	Fe
—	65·71	65·59	—	2	0	. . . .	65·65	2	65·689	6	Fe
66·98	67·29	67·34	2	10	3	Very diffuse in 9	67·31	10	67·341	8	Fe
68·39	68·38	68·34	2	1	0	Diffuse; on continuous spectrum only in 11	68·37	1	68·385	2	C-Cr-Fe-C
69·63	—	—	1	—	—	. . . .	69·63	1	—	—	—
70·70	70·78	70·70	45	55	30	. . . .	70·73	55	—	—	Hι
74·46	74·47	—	10	15	—	. . . .	74·46	15	74·473	3	Y
75·71	75·59	75·54	—	—	3	Uniform band or wide line in 11	75·61	—	75·717	7	Ni
—	—	—	2	8	—		—	8	76·198	2	Ti
76·52	76·77	76·11	—	—	0		76·64	—	76·600	3	Fe
									76·698	1	Mn
77·40	77·69	—	—	—	—	. . . .	77·54	—	77·593	3	Fe
—	—	—	1	1	—	. . . .	—	1	78·203	2	Ni
78·49	78·73	—	—	—	—	. . . .	78·61	—	78·463	3	Fe
									78·652	2	Fe
79·64	79·58	—	1	3	—	Interrupted line in 11. . . .	79·61	3	79·569	4	Fe
									79·657	2	Fe
80·52	—	—	—	—	—	. . . .	80·52	—	—	—	—
								0			
82·43	—	—	0	—	—	. . . .	82·43	—	—	—	—
83·40	3783·57	3783·46	3	8	1	. . . .	83·48	8	3783·674	6	Ni
3784·03	—	—	0	—	—	. . . .	3784·03	0	—	—	—

Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (Rowland).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
3785·27	3785·79	—	—	—	—	. . . .	3785·53	—	—	—	—
			1	5		. . . .		5			
86·45	86·47	3786·25			0	Wide in 13.	86·39	—	—	—	—
—	88·05	—	—	5	—	Absorption line at 3787·8 in No. 9; strong on continuous spectrum in 13	88·05	5	3788·046	9	Fe
88·78	88·85	88·79	10	12	3	. . . .	88·81	12	88·839	2	Y
90·11	90·18	—	1	1	—	. . . .	90·14	1	90·238	5	Fe
90·82	90·93	—	1	1	—	. . . .	90·87	1	—	—	—
92·44	92·46	—	1	0	—	Broad in 9.	92·45	0	92·294 92·482	3 1	Fe Cr-C Ni
93·72	93·94	—	0	1	—	. . . .	93·83	1	93·745	4	Ni
94·89	95·13	95·06	1	5	1	. . . .	95·03	5	95·147	8	Fe
98·02	98·04	97·95	50	60	35	. . . .	98·01	60	—	—	Hθ
99·88	99·61	99·60	1	5	1	Probably 2 lines in 9	99·70	5	99·693	7	Fe
3801·51	3801·66	3801·66	3	4	0	On continuous spectrum only .	3801·61	4	3801·679	0N	-C
03·00	03·16	03·32	2	2	0	Ditto, at centre in 11	03·16	2	03·141	0	C
—	04·91	04·85	—	—	0	. . . .	04·9	0	04·934	2	Fe-Cr-C
05·45	05·87	—	1	1	—	. . . .	05·5	1	05·486	6	Fe
06·44	06·35	—	1	1	—	Group of perhaps 4 lines in 13	06·39	—	06·357 06·865	2 8	Fe-C Mn-Fe
07·49	07·96	07·57	1	—	0	. . . .	07·66	1	07·293 07·681	6 6	Ni V-Fe
09·48	09·38	—	1	0	—	Wide in 9	09·43	0	—	—	—
—	10·00	—	—	0	—	. . . .	10·00	0	09·894 10·061	0 00	C C
10·69	10·80	10·81	1	0	0	. . . .	10·77	0	10·681 10·761	0 0	C C
12·10	12·23	—	1	0	—	. . . .	12·16	0	12·126 12·205	0 0	C C
12·80	—	—	0	—	—	. . . .	12·80	0	13·100 13·219	5 2	Fe Fe
13·42	13·30	13·18	3	10	1	Enhanced Ti at 3813·54 (Lockyer)	13·30	10	—	—	—
14·69	14·81	14·70	2	7	1	. . . .	14·73	7	14·698	8	Fe-C
15·99	15·92	15·90	2	10	2	. . . .	15·94	10	15·987	15	Fe
—	—	17·76	—	—	0	. . . .	17·76	0	—	—	—
19·56	19·88	—	0	4	—	Diffuse in 11	19·72	4	(3819·75)	—	He
20·63	20·62	26·58	3	10	3	. . . .	20·61	10	20·586	25	Fe-C
21·75	22·02	—	1	1	—	. . . .	21·89	1	21·866 21·981	1N 4	C Fe
—	—	23·88	—	—	0	. . . .	23·88	0	—	—	—
24·55	24·61	24·63	1	10	4	. . . .	24·60	10	24·591	6	Fe
26·07	26·05	25·96	1	10	4	. . . .	26·03	10	26·027	20	Fe
3827·97	3827·91	3827·92	2		3	. . . .	3827·93	6	3827·980	8	Fe

Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Inten-sity.	Wave length in sun (Rowland).	Inten-sity.	Element.
3829·51	3829·53	3829·59	5	15	6	. . . .	3829·54	15	3829·501	10	Mg
30·71	30·93	30·85	1	1	0	. . . .	30·83	1	—	—	—
32·51	32·42	32·45	12	25	8	. . . .	32·46	25	32·450	15	Mg
—	—	—	—	—	–	A line is distinctly visible, but was not measured, on V side of Hη	—	—	34·364	10	Fe
35·45	35·51	35·56	50	65	40	. . . .	35·51	65	—	—	Hη
36·87	36·99	—	0	1	—	. . . .	36·93	1	36·905	1	Zr ?, Ti
38·47	38·39	38·44	20	30	10	. . . .	38·43	30	38·435	25	Mg
40·88	40·57	40·90	1	4	1	. . . .	40·78	4	40·580	8	Fe
—	41·11	—	—	4	—	. . . .	41·11	4	41·195	10	Fe-Mn
42·18	42·11	—	0	0	—	. . . .	42·14	0	42·191	3	Co
									43·127	3	C
43·25	43·28	43·14	1	3	0	. . . .	43·20	3	43·195	2	Fe-C
									43·404	4	Fe
—	45·51	45·56	—	1	0	. . . .	45·53	1	45·606	8	C-Co
46·80	46·97	—	0	1	—	. . . .	46·88	1	46·943	5	Fe
47·70	48·16	—	0	0	—	. . . .	47·93	0	—	—	—
49·07	48·92	—	0	0	—	. . . .	49·00	0	49·140	3	La-C
49·84	50·15	—	0	2	—	Diffuse but well defined in outer band in 11	50·15	2	50·118	10	Fe-Cr
51·93	—	—	0	—	—	. . . .	51·9	0	—	—	—
52·77	—	—	0	—	—	. . . .	52·8	0	52·714	4	Fe CCCC, &c.
54·68	54·71	—	0	1	—	. . . .	54·70	1	54·707	2N	C
56·45	56·55	56·41	2	15	5	. . . .	56·47	15	56·524	8	Fe
—	58·56	58·51	—	1	0	Ill defined in 9	58·53	1	58·442	7	Ni
59·99	60·04	59·96	10	25	6	Strong on outer band in 11 . .	60·00	25	60·055	20	Fe-C
61·61	61·68	61·72	2	2	0	Only visible at centre in 11; ill defined in 9	61·67	2	61·681	1	C
62·70	62·91	—	0	0	—	. . . .	62·80	0	—	—	—
63·50	—	—	0	—	—	. . . .	63·50	0	63·533	3	C
—	65·64	—	—	2	—	. . . .	65·64	2	65·674	7	Fe-C
71·75	71·88	71·58	1	2	1	. . . .	71·74	2	—	—	—
—	72·71	—	—	2	—	. . . .	72·71	2	72·639	6	Fe
74·25	74·14	74·34	—	—	—	Head of second cyanogen band	74·24	1	—	—	–
78·74	78·80	78·65	6	15	5	A double line on 2 images not measured, apparently single on others	78·73	15	78·720	7N	Fe
79·70	—	—	0	—	—	. . . .	79·70	0	79·716	1	C
83·61	83·65	83·68	—	—	—	Head of cyanogen band, scarcely visible on outer band in 11	83·65	—	83·568	—	Edge of C band
85·64	—	—	0	—	—	. . . .	85·64	0	85·657	4	Fe
86·47	86·49	86·39	5	10	4	. . . .	86·45	10	3886·434	15	Fe
89·24	89·09	89·12	60	75	50	. . . .	89·15	75	—	—	Hζ
3891·17	90·95	3891·08	1	2	0	On continuous spectrum only in 11	91·07	2	—	—	—
—	3892·30	—	—	0	—	. . . .	3892·3	0	—	—	—

TABLE I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
3893·31	3893·17	3893·59	0	0	0		3893·35	0	—	—	—
94·24	94·49	—	—	—	—		94·26	0	3894·165	3	Fe, Cr, V ?
									94·211	8	Cr, CO
95·14	—	—	1	—	—		—	—	95·119	3	Co
	95·81	95·91	—	10	3	In No. 9 perhaps 2 lines, 3895·1 and 96·5	95·85	10	95·803	7	Fe
96·52	—	—	1	—	—		—	—	—	—	—
98·15	98·31	98·09	0	1	0		98·18	1	98·151	5	V
99·89	99·88	99·98	1	3	0	Narrow line in 11	99·92	3	99·850	8	Fe
3900·75	3900·65	3900·63	10	25	8	Poor definition in 13	3900·68	25	3900·681	5	*Ti*-Fe
03·17	03·19	03·13	1	7	1		03·16	7	03·090	10	Cr, –Fe, Mo
05·62	05·69	05·74	1	5	1		05·68	5	05·660	12	Si
—	06·45	06·65	—	3	0		06·55	3	06·628	10	Fe
07·24	07·31	—	1	0	—		07·27	0	—	—	—
08·48	08·71	—	0	1	—		08·60	1	—	—	—
09·95	09·98	—	0	0	—		09·97	0	09·802	4	Fe
									09·976	5	Fe, V
—	11·22	—	—	0	—		11·22	0	—	—	—
—	12·37	—	—	1	—		12·37	1	—	—	—
13·57	13·55	13·55	10	25	6		13·56	25	13·609	5	Ti-
14·52	14·52	14·67	0	1	1	On continuous spectrum only in 11; diffuse in all	14·57	1	14·566	1	?
16·07	16·15	16·17	10	3	2		16·10	3	16·079	1	Zr
									16·207	0	Zr, La
18·53	18·53	18·51	1	1	4	Wide in 13	18·53	1	18·464	4	Fe
									18·563	4	Fe
20·28	20·31	20·41	0	8	5	Wide in 13	20·33	8	20·410	10	Fe
—	—	21·90	—	—	0	Narrow lines in 13	21·9	0	21·855	4	Ce, Mn-Zr
—	—	22·68	—	—	1		22·7	1	22·560	1N	V
23·13	23·13	23·18	1	8	3		23·14	8	23·054	12	Fe
—	—	24·82	—	—	0		24·8	0	24·673	4	Ti
26·01	—	26·28	1	—	0		26·14	0	26·123	7	Fe-
28·19	28·14	28·08	1	7	5		28·14	7	28·075	8	Fe
30·86	30·49	30·41	3	5	5		30·58	5	30·450	8	Fe
33·98	33·79	33·94	100	100	100	K	33·90	100	33·825	1000	Ca
38·39	38·26	38·49	2	5	2		38·38	5	38·552	4	?
40·29	—	—	0	—	—		40·3	0	—	—	—
44·10	44·14	44·14	2	20	5		44·13	20	44·160	15	Al
45·22	45·29	45·29	0	0	1	On continuous spectrum only in 11	45·27	0	45·260	3	Fe
47·65	47·81	47·66	0	0	0	Perhaps 2 lines in 13	47·70	0	47·675	4	Fe
									47·918	2	Ti
									48·82	4	Ti
49·01	49·14	49·17	1	4	1		49·11	4	49·039	1	Ca
									49·199	1	La
3950·38	3950·38	3950·48	1		2		3950·41	6	3950·497	2	Y

Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	Intensities. No. 9.	No. 11.	No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (Rowland).	Intensity.	Element.
3951·39	3951·31	—	0	—	—		3951·35	0	3951·311	5	Fe
52·35	—	3952·42	1	3	—		52·38	3	—	—	—
—	53·31	—	—	—	1		53·3	0	53·303	3	Fe-Cr
55·06	—	—	0	—	—		55·1	0	—	—	—
									56·476	4	Ce, Co-Ti
56·35	56·82	56·57	1	5	1	Very diffuse in 9	56·58	5	56·603	4	Fe
									56·819	6	Fe
58·19	58·34	58·28	1	5	2		58·27	5	58·355	5	Zr, Ti, Ce
61·75	61·73	61·58	4	20	8		61·69	20	61·674	20	Al
65·05	—	—	0	—	—		65·0	0	—	—	—
66·65	—	—	1	—	—	In shade of H	66·6 ?	0	—	—	—
68·79	68·38	68·56	80	90	90	H	68·58	90	68·625	700	Ca
70·33	70·31	70·34	50	70	50		70·33	70	70·177	5N	He
—	72·06	—	—	0	—		72·1	0	—	—	—
73·47	73·77	—	0	2	—		73·6	2	73·702	3	Ni, Zr
76·73	76·68	76·44	0	1	0		76·62	1	—	—	—
—	77·95	—	—	0	—		77·9	0	77·891	6	Fe
—	81·32	—	—	—	—		81·3	—	81·917	4	Ti
82·19	—	82·26	1	7	2		82·2	7	82·630	2	Ti
82·68	82·74	82·92	1	—	2	Very wide and diffuse on V side in 11	82·8	—	82·790	3	Mn-Y
—	84·08	—	—	1	—		84·1	1	84·091	7	Cr-Fe
88·76	88·60	—	—	1	—		88·68	1	88·659	0	La
90·15	90·07	90 15	—	2	—		90·12	2	89·912	4	Ti
91·23	91·33	91·28	1	3	1		91·28	3	91·333	3	Cr, Zr
93·97	94·40	94·91	Shading	Shading	Shading		94·4	0	—	—	—
95·89	95·87	95·73	1	3	—		95·83	3	95·899	1N	La
96·66	97·62	97·66	Shading	4	1		97·65	4	97·64	2	?
99·22	99·10	99·14	3	6	2	Apparent absorption line at 3999·9 in 9	99·15	6	98·790	4	Ti
4000·44	4000·46	4000·55	0	1	0	Strong on bands of continuous spectrum in 11	4000·48	1	—	—	—
—	01·73	—	—	1	—		01·7	1	4001·814	3	Fe
05·42	05·45	05·44	1	7	4	Very diffuse line in 9	05·44	7	05·408	7	Fe
—	09·55	09·27	—	1	1		09·41	1	—	—	—
12·52	12·52	12·47	6	12	5		12·50	12	12·541	4	Ti, Ce
—	13·76	—	—	2	—		13·7	2	13·798	3	Ti-Fe
									13·964	5	Fe
14·54	14·41	14·62	0	2	1		14·52	2	14·677	5	Fe
—	17·77	—	—	2	—	Wide and ill-defined	17·8	2	—	—	—
—	18 56	18·3	—	—	0		18·4	0	—	—	—
—	20·28	—	—	3	—		20·3	3	—	—	—
22·01	21·90	21·83	1	—	1		21·91	1	—	—	—
23·57	23·49	23·64	0	1	1		23·57	1	23·533	3	V, Co
4025·05	4025·03	4025·09	1	4	1	Perhaps a double line in 11	4025·06	4	24·73	3	T
									4025·29	3	Ti

TABLE I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
4026·78	4026·49	4026·61	0	20	1	Very strong outside, weaker at centre in 11	4026·63	20	(4026·342)	—	He
28·45	28·48	29·55	1	3	1	. . . .	28·49	3	4028·497	4	Ti–Ce
29·91	—	—	0	—	—	. . . .	29·9	0	29·796	5	Fe–Zr
30·82	30·88	30·95	1	10	4	Clearest line in complex group of ill-defined lines in 11	30·88	10	30·918	10	Mn
31·86	31·92	—	0	0	—	. . . .	31·89	0	31·865	2	La, –
33·09	33·17	33·26	0	9	3	. . . .	33·17	9	33·224	8	Mn
34·55	34·56	34·83	0	8	3	. . . .	34·64	8	34·644	6	Mn
35·79	35·79	36·0	0	1	0	. . . .	35·86	1	35·883	4	Mn
40·87	41·00	41·0	2	3	1	Diffuse in 9 and 11.	40·96	3	40·937	1	Ce–Nd–Co
42·95	42·92	—	1	2	—	. . . .	42·93	2	42·743 43·054	0 0	Cr–Nd La
—	44·50	—	—	1	—	Narrow line in 11	44·5	1	—	—	—
45·83	45·94	45·97	8	15	5	Poor definition in 9	45·91	15	45·975	30	Fe
48·68	48·71	48·90	1	3	1	. . . .	48·76	3	48·883	6	Zr–Mg–Cr
—	53·79	53·9	0	3	0	. . . .	53·9	3	53·981	3	Cr–Fe–Ti
54·05	—	—	0	—	—	Centre of faint group in 9 . .	—	—	—	—	—
—	55·21	—	—	0	—	. . . .	55·2	0	—	—	—
57·40	57·64	57·7	0	1	0	. . . .	57·58	1	57·499 57·668	3 7	Fe ?
58·97	58·88	—	0	0	—	. . . .	58·9	0	58·915 59·081	3 3	Fe, Cr Mn
63·72	63·73	63·75	2	13	4	. . . .	63·73	13	63·759	20	Fe
67·13	67·07	67·32	1	4	1	. . . .	67·17	4	67·139	5	Cr–Fe
72·01	71·91	71·95	2	10	4	Long, well-defined in 11 . . .	71·96	10	71·908	15	Fe
73·49	73·59	—	0	1	—	. . . .	73·5	1	73·637	0	Ce
—	74·73	—	—	0	—	. . . .	74·7	0	—	—	—
77·86	77·95	77·94	45	50	30	. . . .	77·92	50	77·885	8	Sr
—	80·13	80·09	—	0	0	. . . .	80·1	0	80·368	3	Fe, Nd, Cr
83·13	83·32	83·8	0	1	0	. . . .	83·2	1	83·095	4	V–Mn
85·28	85·36	85·1	0	1	0	. . . .	85·2	1	85·161 85·467	4 4	Fe Fe
86·67	86·85	87·0	1	5	0	. . . .	86·8	5	86·861	1	La
—	88·90	—	—	0	—	. . . .	88·9	0	—	—	—
90·50	90·75	—	0	0	—	. . . .	90·6	0	—	—	—
92·59	92·60	92·9	1	3	0	. . . .	92·7	3	92·821	3	V, Ca
4101·96	4102·02	4102·02	70	80	55	. . . .	4102·00	80	4102·000	40N	Hδ
07·80	07·80	07·96	0	0	0	. . . .	07·9	0	07·649	5	Ce–Fe
09·53	09·50	09·93	2	5	1	Diffused on R side in 9 . . .	09·6	5	09·609 09·905	1 2	Nd ? V
—	14·4	—	—	0	—	. . . .	14·4	0	14·606	4	Fe
18·79	18·70	18·98	3	5	2	Poor definition	18·82	5	18·708	5	Fe
—	—	—	—	—	—	. . . .	—	—	18·934	4	Co
21·41	21·21	21·44	2	3	0	Very long and narrow in 11. .	21·35	3	21·477	6	Cr–Co
4123·81	4123·24	4123·20	3	5	1	Diffuse and wide in 11; ditto in 13	4123·4	5	4123·384	1	La

TABLE I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
—	4126·17	—	—	0	—	. . . .	4126·2	0	4126·344	4	V–Fe
4127·48	27·96	4128·08	2	2	1	. . . .	27·8	2	{ 27·767	4	Fe
									27·957	4	Ce–Fe
29·72	29·73	30·08	2	2	1	. . . .	29·8	2	—	—	—
32·21	32·3	32·53	1	5	2	. . . .	32·3	5	32·235	10	Fe–Co
									{ 34·492	3	Fe ?
34·54	34·53	34·70	0	1	1	Very poor in 9; diffuse and wide in 11	34·5	1	34·589	3	Fe ?
									34·846	5	Fe
37·40	37·19	37·77	2	3	1	Diffuse in 9	37·4	3	37·156	6	Fe
—	40·21	—	—	0	—	. . . .	40·2	0	40·089	6	Fe
41·79	—	—	—	—	—	. . . .	41·8	2	—	—	—
	43·57	43·91	6	15	3	. . . .	43·9	15	{ (4143·919)	—	He
44·06	—	—	—	—	—	. . . .	—	—	44·038	15	Fe
46·02	46·32	—	0	0	—	. . . .	46·2	0	46·225	3	Fe
—	47·80	—	—	0	—	. . . .	47·8	0	47·836	4	Fe
49·45	49·30	49·36	3	7	2	. . . .	49·4	7	{ 49·360	2	C, Zr
									49·533	4	Fe
52·16	52·31	51·9	1	1	1	. . . .	52·1	1	52·108	2	Fe, La, Ce
54·46	54·53	54·57	1	1	1	. . . .	54·5	1	54·667	4	Fe
56·27	56·43	56·6	4	5	1	. . . .	56·4	5	56·391	1	Zr
61·30	61·39	61·5	3	4	1	. . . .	61·4	4	61·369	2	Zr
63·71	63·81	63·8	3	10	1	. . . .	63·8	10	63·818	4	Cr–Ti,-
65·12	65·81	—	0	1	—	. . . .	65·4	1	—	—	—
67·20	67·39	67·5	2	2	0	Wide line (over 1 tenth-metre) in 9	67·4	2	67·438	8	?
69·70	—	—	0	—	—	. . . .	69·7	0	.	—	—
71·98	72·04	72·10	2	10	2	. . . .	72·0	10	72·07	2	Ti Fe
73·46	73·59	73·52	3	10	2	. . . .	73·5	10	—	—	—
75·55	75·77	—	0	0	—	. . . .	75·7	0	75·806	5	Fe
77·57	77·54	77·85	6	10	3	. . . .	77·65	10	{ 77·698	3	?
									77·772	3	Fe
78·83	79·15	79·26	4	8	3	Enhanced Fe (LOCKYER) . . .	79·1	8	79·025	3	?
80·51	81·09	—	0	0	—	. . . .	81·3	0	—	—	—
81·70	81·94	81·9	1	1	0	. . . .	81·8	1	81·919	5	Fe
84·58	—	84·7	0	—	1	. . . .	84·6	1	84·47	2	?
86·65	—	—	—	—	—	. . . .	86·6	0	{ 87·204	6	Fe
87·58	87·30	87·7	4	8	1	Very wide in 11 and 13 . . .	87·5	8	87·409	00	La, C
91·51	91·40	91·44	1	5	1	. . . .	91·45	5	91·595	6	Fe
94·77	95·17	—	2				95·0	—	95·492	5	Fe
		95·98	—	} 1	1	Very diffuse in 13	—	1	—	—	—
96·31	96·72	—	3				96·5	—	96·372	4	Fe
98·38	—	—	—	—	—	. . . .	98·4	—	98·494	4	Fe
98·98	98·96	—	4	5	—	2 lines in 11	98·9	5	98·800	3	Fe
4200·70	—	4201·08	0	—	2	. . . .	4200·9	1	4200·946	1	Ti

TABLE I.—Eclipse Spectra. May 28, 1900—continued.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
4202·18	4202·15	4202·17	1	6	1	. . . .	4202·17	6	4202·198	8	Fe
04·86	04·92	05·0	5	4	0	. . . .	04·93	4	04·916	2	?
06·59	06·92	07·3	1	1	0	. . . .	06·9	1	06·862	3	Fe
08·68	08·92	09·1	1	3	0	. . . .	08·9	3	09·14	1	Zr
—	10·40	—	—	—	—	. . . .	10·4	1	10·494	4	Fe
11·97	—	—	0	1	—	. . . .	12·0	0	12·048	2	Zr
13·63	—	—	0	—	—	. . . .	13·6	0	13·812	3	Fe
15·68	15·79	15·70	25	40	20	Long line in 13	15·72	40	15·703	5	Sr
17·53	17·64	—	0	0	—	Lumps on continuous spectrum in 11	17·6	0	17·720	5	La, Fe-Cr
19·52	19·71	—	0	0	—	. . . .	19·6	0	19·516 19·580	4 3	Fe ?
20·41	—	—	0	—	—	. . . .	20·4	0	20·509	3	Fe
22·61	22·43	22·65	4	1	0	Diffuse in 11	22·56	1	22·382	5	Fe
27·13	26·96	26·96	15	30	10	Poor definition in 9	27·0	30	26·904	20	Ca
29·79	—	—	0	—	—	. . . .	29·8	0	—	—	—
33·27	33·25	33·4	10	10	5	Enhanced Fe (LOCKYER) . . .	33·3	10	33·328	4	Fe
35·95	35·99	36·1	3	4	1	. . . .	36·01	4	36·112	8	Fe
38·26	38·5	—	0	0	—	. . . .	38·4	0	—	—	—
39·88	40·44	—	0	0	—	. . . .	40·1	0	39·890	3	Fe, Mn
42·67	42·60	—	1	1	—	. . . .	42·6	1	—	—	—
47·05	47·03	47·1	20	25	8	. . . .	47·06	25	46·996	5	Sc*
50·32	50·50	50·6	1	5	1	Diffuse in 9	50·5	5	50·287 50·945	8 8	Fe Fe
—	52·36	—	—	—	—	. . . .	52·4	1	52·468	0	Co
54·45	54·55	54·4	5	12	5	. . . .	54·47	12	54·505	8	Cr
58·29	58·33	58·5	2	1	0	On continuous spectrum only in 11	58·37	1	—	—	—
60·31	—	60·7	3	—	2	. . . .	60·5	5	60·640	10	Fe
—	61·61	—	—	5	—	. . . .	61·6	5	61·679	2	–, Cr
—	62·03	62·2	—	0	0	Very diffuse in 9	62·1	0	—	—	—
64·50	—	—	0	—	–	Ditto	64·5	0	—	—	—
68·00	67·65	67·7	1	0	0	. . . .	67·8	0	—	—	—
71·89	71·81	71·8	2	8	4	. . . .	71·83	8	71·934	15	Fe
73·55	73·38	—	0	0	—	. . . .	73·5	0	73·482 73·643	3 2	Fe Zr
75·08	74·95	75·0	8	12	4	. . . .	75·01	12	74·958	7	Cr
—	—	78·1	—	—	0	. . . .	78·1	1	—	—	—
80·25	80·53	80·1	2	1	0	. . . .	80·3	1	—	—	—
82·76	82·82	82·7	3	1	1	. . . .	82·76	1	82·565	5	Fe
84·42	84·46	—	0	0	—	Shading ends in 9	84·44	0	—	—	—
88·22	88·11	87·9	0	1	0	. . . .	88·1	1	88·04	2	Ti
90·11	90·04	90·2	12	25	5	. . . .	90·12	25	89·885	5	Cr
94·22	94·33	94·5	5	12	4	Narrow line in 9	94·35	12	94·204 94·301	2 5	Ti Fe
4296·54	4296·93	4296·9	1	1	1	Diffuse in 9	4296·8	1	4296·74	3	?

* One of the strongest lines of Scandium.

Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (Rowland).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11.	No. 13.						
4299·1	—	—	1	—	—	. . . .	4299·1	1	4299·149	3	Ca
4300·1	4300·38	4300·1	7	15	5	. . . .	4300·2	15	4300·211	3	Ti
02·2	02·04	02·8	2	2	3	Wide in 13	02·3	2	02·353	2	Fe
						Well defined group of two or three lines in 9		—	02·460	2	?
								—	02·692	4	Ca
03·5	03·65	—	2	2	—	. . . .	03·6	2	02·913	2N	?
05·6	—	—	0	—	—	. . . .	05·6	0	—	—	—
—	06·03	06·0	—	1	1	. . . .	06·0	1	06·078	4	Ti
08·06	08·07	08·2	5	15	4	Well defined in 9 . . . . (G)	08·1	15	07·907	3	Ca
									08·081	6	Fe
09·6	09·6	09·9	1	0	1	Very fine line in 11.	09·7	0	—	—	—
10·5	—	—	4	—	—	. . . .	10·5	4	—	—	—
12·9	13·1	12·8	5	10	2	. . . .	12·9	10	13·034	3	Ti
14·3	14·5	14·4	5	6	2	. . . .	14·4	6	14·479	1	Ti
								—	14·964	1	Ti
								—	15·138	3	Ti
15·4	15·2	15·4	5	6	2	. . . .	15·3	6	15·262	4	Fe
17·3	17·2	—	0	1	—	. . . .	17·2	1	—	—	—
18·9	19·0	—	1	4	—	. . . .	18·95	4	18·817	4	Ca, Mn?
20·9	20·7	20·7	10	12	4	Narrow line in 13	20·8	12	20·907	3	Sc
23·4	—	—	1	—	—	. . . .	23·4	1	23·386	2N	?
25·8	25·9	25·6	8	15	5	Perhaps two lines in 9; wide on V side in 11	25·8	15	25·939	8	Fe
30·7	31·0	30·8	1	1	0	Diffuse in 9	30·8	1	30·866	2	Ti, Ni
33·86	34·1	33·7	2	1	0	Fine narrow line in 9	33·9	1	33·925	1N	La
38·0	—	—	10	—	—	Difficult to measure in shade of Hγ	38·0	10 ?	38·084	4	Ti
40·7	40·2	40·7	80	90	60	. . . .	40·5	90	40·634	20N	Hγ
44·2	44·6	44·4	5	8	1	Diffuse in 9	44·4	8	44·451	2	Ti
									44·670	4	Cr
46·36	—	—	0	—	—	. . . .	46·4	0	—	—	—
—	—	47·8	—	—	0	. . . .	47·8	0	—	—	—
51·76	51·9	51·9	8	20	4	Diffuse in 9. Enhanced Fe (Lockyer)	51·9	20	51·930	5	Cr
									52·083	5N	Mg
54·8	54·9	—	0	1	—	. . . .	54·8	1	—	—	—
58·9	58·6	59·1	3	—	1	Wide in 13.	58·9	5	58·879	0	Y-Zr
				8	—	. . . .	—	—	59·784	3	Cr
60·7	59·8	—	1	—	—	. . . .	60·2	3	59·907	0	Zr
62·9	—	—	0	—	—	. . . .	62·9	0	—	—	—
—	63·9	64·1	—	0	0	. . . .	64·0	0	—	—	—
67·55	67·7	67·7	2	1	1	Very ill defined lines in 13 and very diffuse in 11	67·7	1	67·749	5	Fe
									67·839	2	Ti
69·7	70·0	4370·1	0	0	1	. . . .	69·9	0	69·873	0	Ti
									69·941	4	Fe
4371·0	4371·5	—	1	0	—	. . . .	4371·2	0	4371·144	1	Zr

TABLE I.—Eclipse Spectra, May 28, 1900—*continued*.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun. (ROWLAND).	Intensity.	Element.
4374·9	4374·9	4374·9	20	25	12	Best defined outside in 11; wide in 13. Enhanced Ti (LOCKYER)	4374·9	25	4374·981 75·103	0 2	Zr V, Mn
79·65	79·6	80·0	0	1	2	Diffuse in 11	79·7	1	79·927	0	Zr
83·6	83·7	83·7	5	25	8	Very diffuse in 9	83·7	25	83·720	15	Fe
85·16	85·3	85·5	2	1	1		85·3	1	85·406	1	La
—	88·2	—	—	1	—	No. 11 only, visible outside to W, not at line of measures, or on continuous spectrum	88·2	1	—	—	—
99·9	—	91·4	1	—	0	. . . .	91·1	1	91·123 91·192	2 1	Fe Ti
95·25	95·1	95·1	10	30	15	. . . .	95·15	30	95·201 95·413	3 2	Ti V, Zr
98·1	98·0	98·1	1	5	1	. . . .	98·1	5	98·178	1	(Zircon not Zr)
4400·4	4400·1	4400·2	5	20	5	Wide in No. 9	4400·2	20	4400·343 00·555	0 3	Zr Sc
04·75	04·9	04·8	3	20	5	. . . .	04·8	20	04·927	10	Fe
—	07·7	—	—	—	—	. . . .	07·7	0	08·364	2	V
08·1	—	08·6	2	3	2	. . . .	08·1	2	08·582	3	Fe
—	09·3	—	—	—	—	. . . .	09·3	1	08·683	2	V
—	11·5	11·2	—	0	0	. . . .	11·4	0	—	—	—
15·5	15·5	15·5	5	20	5	. . . .	15·5	20	15·293	8	Fe
—	—	—	—	—	—	. . . .	—	—	15·72	3	?
17·7	17·6	17·8	5	20	5	. . . .	17·7	20	17·884	3	Ti
—	20·5	—	—	0	—	. . . .	20·5	0	20·686	00	Zr
22·5	22·7	22·2	—	1	1	Diffuse in 13	22·5	1	22·741	3	Fe, Y
24·1	—	—	0	—	—	. . . .	24·1	0	24·006	2	Fe?
—	25·2	—	—	0	—	. . . .	25·2	0	25·608	4	Ca
27·5	27·4	27·5	2	4	3	Narrow line in 11	27·5	4	27·482	5	Fe
29·9	30·2	30·1	1	3	2	Diffuse in 11	30·1	3	30·070 30·221	00 00	La La
33·8	—	—	0	—	—	. . . .	33·8	0	—	—	—
35·4	35·4	35·1	2	5	5	Diffuse in 11, perhaps another line on V side. Diffuse in 13	35·3	5	35·129	5	Ca
—	41·6	—	—	0	—	. . . .	41·6	0	41·881	3N	V –
44·2	44·0	43·9	15	25	12	Diffuse on V side in 9, probably a line on V side	44·0	25	43·976	5	Ti
50·5	50·6	50·6	3	10	1	Diffuse in 9	50·6	10	50·482 50·654	1 2	Zr–Fe Ti?
54·8	55·1	55·1	2	10	1	. . . .	55·0	10	54·953	5	Ca, Zr
59·1	59·4	59·3	0	0	0	. . . .	59·3	0	59·199 59·301	2 3	Ni Fe
61·7	61·8	61·6	1	2	5	. . . .	61·7	2	61·818	4	Fe
64·6	64·5	64·6	1	1	0	Narrow line in 11	64·6	1	64·617 64·844	2 2	Ti? Mn
—	66·5	66·5	—	1	0	. . . .	66·5	1	66·727	5	Fe
4468·6	4468·7	4468·7	12	15	25	. . . .	4468·7	15	4468·663	5	Ti

TABLE I.—Eclipse Spectra, May 28, 1900—*continued*.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (ROWLAND).	Intensity.	Element.
4471·6	4471·65	4471·9	5	40	12	Faint and wide in middle, strong and narrow outside in 9	4471·7	40	(4471·646)	—	He
75·9	76·0	76·3	3	2	5	. . . .	76·1	2	76·185	4	Fe
79·3	79·4	—	—	—	—	Probable group	79·4	1	—	—	—
82·5	82·1	82·4	2	3	5	Diffuse in 13	82·4	3	{ 82·338	5	Fe, –
									{ 82·438	3	Fe
89·2	89·6	89·3	2	3	10	} Diffuse band or group of lines .	89·4	3	89·351	2	?
—	—	—	—	—	—		—	—	89·911	4	Fe
—	—	—	—	—	—		—	—	90·253	3N	Mn-Fe
91·3	91·7	91·5	1	1	10		91·5	1	91·570	2	?
93·9	94·3	94·7	1	0	2	Very diffuse in 9	94·3	0	94·738	6	Fe
96·7	96·9	97·3	1	1	5	Ditto.	96·9	1	97·023	3	Cr
4501·4	4501·5	4501·6	10	30	25	. . . .	4501·5	30	4501·445	5	Ti, –
08·4	08·4	08·7	3	10	10	Very narrow line in 9 . . . *	08·5	10	08·455	4	Fe ? –
15·4	15·6	15·7	3	8	10	Poor definition in 9. . . . *	15·6	8	15·508	3	?
18·3	18·3	—	0	2	—	. . . .	18·3	2	18·189	3	Ti
20·6	20·5	20·6	2	5	5	. . . . *	20·6	5	20·397	3	Fe ? –
22·9	23·0	23·0	5	10	10	. . . . *	23·0	10	{ 22·802	3	?
									{ 22·974	2	Ti
28·8	28·9	29·3	0	1	0	. . . .	29·0	1	28·798	8	Fe
31·4	—	—	0	—	—	. . . .	31·4	0	31·327	5	Fe
34·3	34·3	34·2	12	30	20	. . . .	34·3	30	{ 34·139	6	Ti-Co
									{ 34·95	4	Ti
36·4	36·6	—	2	1	—	Continuous spectrum only No. 11	36·3	1	—	—	—
39·6	40·0	—	1	0	—	. . . .	39·8	0	39·946	0	Cr
41·1	42·1	—	1	1	—	. . . .	41·6	1	41·690	2	Cr
—	44·2	—	—	0	—	} Very diffuse and wide in 9; probably a group in 11	—	—	—	—	—
45·0	—	—	1	—	—		45·0	0	44·864	3	Ti
—	45·8	—	—	0	—		—	—	—	—	—
49·8	50·0	49·7	15	30	33	Enhanced Fe at 4549·64 . . .	49·8	30	49·808	6	Ti-Co
54·4	54·2	54·2	8	30	30	. . . .	54·3	30	54·211	8	Ba
56·2	56·3	56·1	4	8	8	Double Fe line; V line enhanced, R line arc (LOCKYER)	56·2	8	56·306	4	Fe-Cr
58·7	58·9	58·9	2	8	5	Enhanced Cr (LOCKYER) . . .	58·8	8	58·827	3	Cr
60·5	—	—	0	—	—	. . . .	60·5	0	—	—	—
62·1	—	—	0	—	—	. . . .	62·1	0	—	—	—
64·1	64·1	63·9	10	25	25	. . . .	64·0	25	63·939	4	Ti
72·2	72·3	72·0	12	30	25	. . . .	72·2	30	72·156	6	Ti-
76·7	76·4	76·5	2	0	5	On continuous spectrum only in 11. Enhanced Fe (LOCKYER)	76·5	0	76·512	2	?
80·4	80·1	79·9	0	0	5	. . . .	80·1	0	80·228	3	Cr
84·1	84·0	83·6	10	25	25	Enhanced Fe	83·9	25	84·018	4	Fe–
88·0	88·0	—	0	2	—	} Long line in 11	88·0	2	88·381	3	?
89·8	90·1	4589·3	0	2	3	Ditto ? double in 13	89·7	2	4590·126	3	?
92·2	4592·6	—	0	0	—	} Narrow lines in No. 9, but too faint for good measures	92·3	0	—	—	—
4593·9	—	—	0	—	—		4593·9	0	—	—	—

* Enhanced Fe (LOCKYER).

Table I.—Eclipse Spectra, May 28, 1900—*continued*.

Wave-lengths. No. 9. Latitude −74° E.	Wave-lengths. No. 11. Latitude −41° W.	Wave-lengths. No. 13. Latitude −75° W.	Intensities. No. 9.	Intensities. No. 11.	Intensities. No. 13.	Remarks.	Adopted wave-length.	Intensity.	Wave-length in sun (Rowland).	Intensity.	Element.
4600·4	4600·8	4600·5	1	1	5	. . . .	4600·6	1	4600·541	2	Ni
									00·932	3	Cr
02·6	—	02·8	0	—	0	. . . .	02·7	0	03·126	6	Fe
13·4	13·2	—	1	0	—	. . . .	13·3	0	13·386	3	Fe
									13·544	3	Cr, La
15·8	16·1	16·2	1	1	0	. . . .	16·0	1	16·305	4	Cr
19·0	19·2	19·0	1	2	0	Enhanced Cr at 4618·97 (Lockyer)	19·1	2	18·971	4	Fe
22·1	—	—	0	—	—	. . . .	22·1	0	22·065	0	Cr
									22·128	1	Cr
25·5	—	—	1	—	—	. . . .	25·5	1	25·227	5	Fe
29·2	29·3	28·5	8	15	25	Line seems displaced to V in 13	29·1	15	29·521	6	Ti-Co
32·1	—	—	—	—	—	. . . . Diffuse band in 9	32·1	0	—	—	—
33·6	33·7	—	1	1	—	Diffuse in 11	33·6	1	—	—	—
36·7	—	—	1	—	—	. . . .	36·7	1	—	—	—
—	37·7	—	—	0	—	. . . .	37·7	0	—	—	—
39·2	—	—	—	—	—	. . . .	39·2	1	—	—	—
46·1	45·7	44·8	3	5	5	Diffuse band in 9	45·5	5	46·347	5	Cr
—	47·8	—	—	1	—	. . . .	47·8	1	—	—	—
50·9	51·8	51·1	1	2	5	. . . .	51·3	2	51·461	4	Cr
55·1	54·8	—	2	2	—	Perhaps 2 lines in No. 9	55·3	2	—	—	—
—	—	56·0	—	1	5	. . . .	56·0	5	56·644	3	Ti
—	57·0	—	—	2	—	. . . .	57·0	2	57·380	2	Ti ?
59·8	—	—	0	—	—	. . . .	59·8 ?	0	—	—	—
—	61·4	62·2 (61·4 and 63·1)	—	0	3	. . . .	61·4	0	—	—	—
63·1	63·1		1	1		. . . .	63·1	1	—	—	—
66·6	66·6	66·3	2	5	7	. . . .	66·5	5	66·387	0	Cr
									66·655	1	Cr
69·3	70·1	69·2	2	5	7	. . . .	69·5	5	69·504	1	Cr
77·3	78·4	79·0	—	0	2	On continuous spectrum only in No. 11	78·2	0	78·347	3N	Cd
			1	—	—			—	79·027	6	Fe
81·3	81·9	81·3	—	0	1		81·5	0	—	—	—
—	85·7	—	—	5	—	Corona ? or upper chromosphere, visible outside only on West side	85·7	5	—	—	—
97·9	98·6	—	—	1	—	. . . .	98·2	1	—	—	—
4701·9	4702·4	4702·5	0	1	7	. . . .	4702·2	1	—	—	—
08·6	07·8	07·6	0	1	5	. . . .	08·0	1	4708·196	2	Cr
13·6	13·2	13·4	4	15	7	Helium line very long and narrow in all three spectra	13·4	15	(4713·252)	—	He
—	22·0	—	—	0	—	On continuous spectrum only	22·0	0	—	—	—
27·0	27·3	—	1	0	—		27·0	0	—	—	—
30·4	31·0	30·9	1	0	3		30·8	0	4730·897	1	Cr
4736·1	4736·7	4735·5	1	3	0	. . . .	4736·1	3	—	—	—

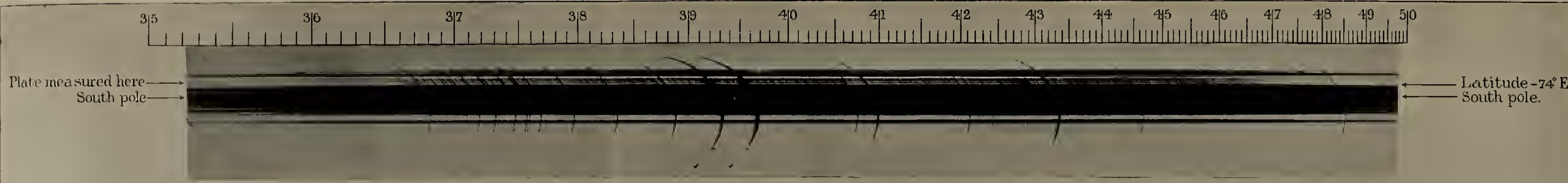
Table I.—Eclipse Spectra, May 28, 1900—continued.

Wave-lengths.			Intensities.			Remarks.	Adopted wave-length.	Intensity.	Wave length in sun (Rowland).	Intensity.	Element.
No. 9. Latitude −74° E.	No. 11. Latitude −41° W.	No. 13. Latitude −75° W.	No. 9.	No. 11	No. 13.						
—	—	—	—	—	—	. . . .	—	—	4761·718	3	Mn
4761·9	4762·8	4761·5	5	—	—	. . . .	4762·0	—	62·567	5	Mn
				5	10	Probably a group of lines . .		5	64·108	4	Ti–Ni
—	66·9	65·9	—	—	—	. . . .	66·4	—	66·050	3	Mn
—	—	—	—	—	—	. . . .	—	—	66·621	4	Mn
—	—	71·4	—	—	0	. . . .	71·4	0	—	—	—
78·8	79·9	79·0	1	1	0	. . . .	79·2	1	—	—	—
82·0	83·0	—	1	1	—	. . . .	82·5	1	—	—	—
85·3	86·2	—	2	1	—	. . . .	85·8	1	—	—	—
88·3	89·2	—	0	1	—	Poor definition in 9	88·7	1	—	—	—
—	91·8	—	—	1	—	. . . .	91·8	1	—	—	—
97·3	98·6	98·6	2	2	0	Diffuse in 9 and 11	98·2	2	—	—	—
4804·5	4805·1	—	2	5	—	. . . .	4804·6	5	—	—	—
09·6	—	—	1	—	—	. . . .	09·6	1	—	—	—
23·2	23·6	4823·5	5	5	7	. . . .	23·4	5	4823·697	5	Mn
48·8	48·5	—	1	0	—	. . . .	48·6	0	—	—	—
61·5	61·5	61·7	60	75	70	F	61·6	75	61·527	30	Hβ
71·5	71·3	71·8	1	5	7	Diffuse in 9	71·5	5	71·512	5	Fe
—	—	77·8	—	—	0	. . . .	77·8	0	—	—	—
82·1	83·3	83·6	2	5	5	Wide and diffuse in all spectra (width = 2·3 tenth-metres in 11)	83·0	5	—	—	—
91·0	91·0	91·0	1	5	7	Wide line	91·0	7	90·948 91·683	6 8	Fe Fe
4900·1	4900·4	4900·2	3	5	5	. . . .	4900·2	5	4900·095 00·301	2 2	Ti La Y ?
03·5	04·6	—	2	1	—	. . . .	04·0	1	—	—	—
09·8	11·2	—	5	2	—	Diffuse in 9, very wide in 11 .	10·5	2	—	—	—
19·2	18·7	20·3	Shading	Shading	5	Diffuse shading in 9 and 11, ill-defined group in 13	19·4	?	19·174	6	Fe
24·2	24·1	23·9	15	30	25	Enhanced Fe	24·1	25	24·107	5	Fe
34·1	34·2	34·2	10	20	22	. . . .	34·2	20	34·214 34·277	6	Ba–Fe ?
57·7	57·8	57·9	?	15	7	. . . .	57·8	10	57·785	8	Fe
—	71·2	—	—	1	—	. . . .	71·2	1	—	—	—
—	83·8	82·9	—	10	0	Very wide in 11	83·3	5	82·682 83·433	4 3	Fe Fe
—	91·8	—	—	2	—	. . . .	91·8	2	—	—	—
—	99·8	—	—	2	—	Line, or blue edge of group .	99·8	2	—	—	—
—	5006·0	—	—	5	—	. . . .	5006·0	5	—	—	—
5018·2	18·5	5018·6	15	30	20	Enhanced Fe	18·5	20	5018·629	4	Fe
—	31·3	—	—	5	—	. . . .	31·3	5	5031·199	3	?
—	5040·8	—	—	15	—	. . . .	5040·8	15	—	—	—

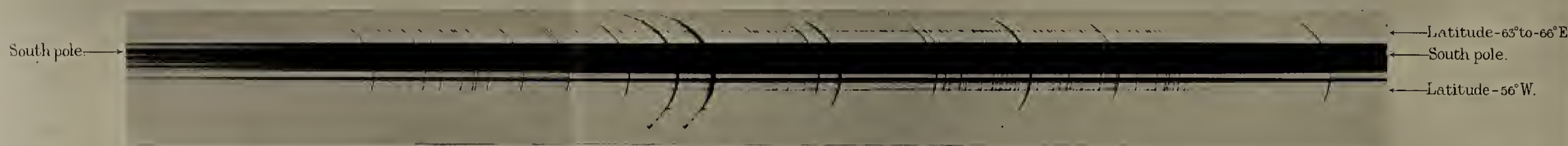
TABLE II.—Hydrogen Lines.

Designation.	No. 9.	No. 11.	No. 13.	Mean.	Computed.	O – C.
β	4861·5	4861·5	4861·7	4861·57	4861·52	+ ·05
γ	4340·7	4340·2	4340·7	4340·53	4340·63	– ·10
δ	4101·96	4102·02	4102·02	4102·00	4101·90*	+ ·10
ε	3970·33	3970·31	3970·34	3970·33	3970·22	+ ·11
ζ	3889·24	3889·09	3889·12	3889·15	3889·20	– ·05
η	3835·45	3835·51	3835·56	3835·51	3835·53	– ·02
θ	3798·02	3798·04	3797·95	3798·00	3798·04	– ·04
ι	3770·70	3770·78	3770·70	3770·73	3770·77	– ·04
κ	3750·25	3750·29	3750·26	3750·27	3750·30	– ·03
λ	3734·48	3734·58	3734·52	3734·53	3734·51	+ ·02
μ	3721·92	3722·03	3722·00	3721·98	3722·08	– ·10
ν	3712·12	3712·16	3712·12	3712·13	3712·11	+ ·02
ξ	3704·03	3704·01	3703·98	3704·01	3704·00	+ ·01
ο	3697·34	3697·20	3697·30	3697·28	3697·29	– ·01
π	3691·75	3691·62	3691·74	3691·70	3691·70	± ·00
ρ	3686·97	3686·89	3687·01	3686·96	3686·97	– ·01
σ	3682·98	3682·88	3682·95	3682·94	3682·95	– ·01
τ	3679·59	3679·45	3679·52	3679·52	3679·49	+ ·03
υ	3676·56	3676·42	3676·55	3676·51	3676·50	+ ·01
φ	3673·96	3673·82	3673·90	3673·87	3673·90	– ·03
χ	3671·50	3671·48	3671·61	3671·53	3671·48	+ ·05
ψ	3669·52	3669·58	3669·55	3669·55	3669·60	– ·05
ω	3667·90	3667·77	3667·81	3667·83	3667·82	+ ·01
Series No. 27	3666·31	3666·11	3666·33	3666·25	3666·24	+ ·01
,, ,, 28	3664·72	3664·73	3664·76	3664·74	3664·82	– ·08
,, ,, 29	3663·49	3663·58	3663·58	3663·55	3663·54	+ ·01
,, ,, 30	3662·34	3662·34	3662·39	3662·36	3662·40	– ·04
,, ,, 31	—	3661·31	—	3661·31	3661·35	– ·04
,, ,, ∞	—	—	Theoretical	limit . . .	3646·13	—

* The solar absorption line is at 4102·00 according to JEWELL.



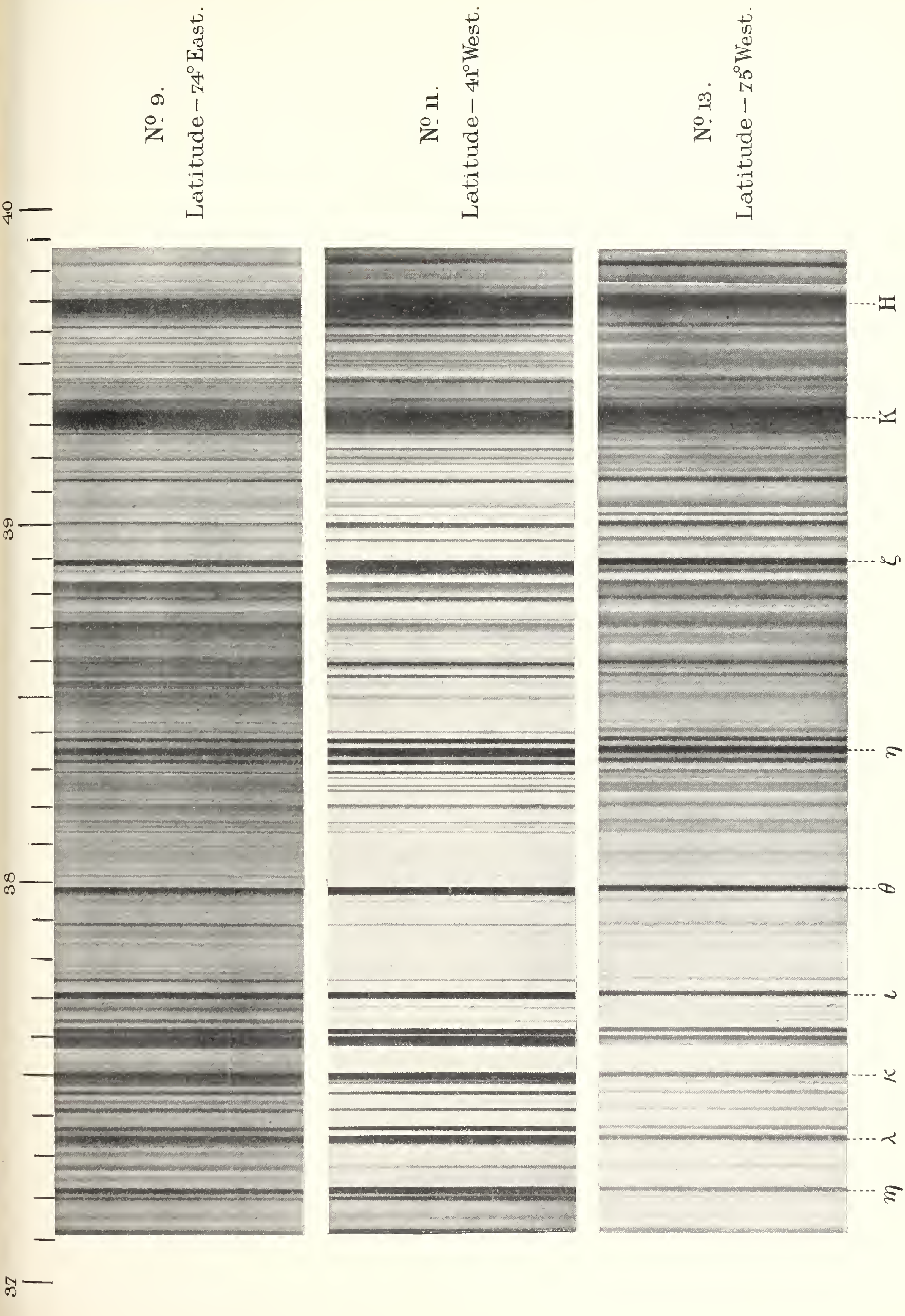
SPECTRUM N° 9. Exposed for 2 seconds, beginning 15 seconds before mid-eclipse.



SPECTRUM N° 10. Exposed for 2 seconds, beginning 10 seconds before mid-eclipse.

SPECTRUM N° 11. Exposed for 10 seconds. Middle of exposure at mid-eclipse.

SPECTRUM N° 13. Exposed for 2 seconds, beginning 14 seconds after mid-eclipse.



No. 9. Latitude – 74° East.
No. 11. Latitude – 41° West.
No. 13. Latitude – 75° West.
40
39
38
37
H
K
ζ
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θ
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μ

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PHILOSOPHICAL TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A, VOL. 201, pp. 497–549.

THE ELECTRICAL CONDUCTIVITY IMPARTED TO A VACUUM BY HOT CONDUCTORS

BY

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LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS FRIEDLÄNDER AND SON, BERLIN

1903.

A 343.

20.7.03

Price Three Shillings and Sixpence.

INDEX SLIP.

XIII. *The Electrical Conductivity Imparted to a Vacuum by Hot Conductors.*

By O. W. Richardson, *B.A.*, *B.Sc.*, *Fellow of Trinity College, Cambridge.*

Communicated by Professor J. J. Thomson, *F.R.S.*

Received February 28,—Read March 26, 1903.

Introduction.

The experimental part of the present paper is an investigation of the electrical conductivity of the space surrounding hot surfaces of platinum, carbon, and sodium at low pressures. A preliminary account of some of the experiments on platinum was read before the Cambridge Philosophical Society on November 25th, 1901.*

The conductivity produced by hot metals has been the subject of a great number of researches by different authors. The phenomena are, however, very complicated; for the quantity and sign of the ionisation is found to vary in the most remarkable manner with the nature, temperature, and previous history of the metal, with the nature and pressure of the surrounding gas, and with small changes in the state of the metal surface. The present investigation was undertaken with the idea that in the negative ionisation at high temperatures the conductivity produced by metals took its simplest form. This idea is supported by the observation of Professor McClelland,† that the negative current is to a great extent independent of the nature of the gas, and is independent of its pressure over a range from ·04 to ·004 millim.

The chief problem which is here attacked experimentally is the way in which the saturation current from the hot metal surface to a neighbouring electrode varies with the temperature of the metal. The value of the saturation current corresponds to the total number of ions which are produced by the surface per second. Incidentally it was found necessary to examine, in addition to the above, the relation between the current and the electromotive force for the conductivity produced by the three abovementioned conductors at various pressures.

The theory, by which it is proposed to explain the phenomena, is based on the

* 'Proc. Cambr. Phil. Soc.,' vol. 11, p. 286.

† 'Proc. Cambr. Phil. Soc.,' vol. 10, p. 241, and vol. 11, p. 296.

hypothesis of conduction in metals by corpuscles which has been elaborated by Professors DRUDE* and J. J. THOMSON.† According to that theory a metal is to be regarded as a sponge-like structure of molecules and comparatively large fixed positive ions, with small negative ions or corpuscles moving freely with great velocity throughout the mass. Since the corpuscles do not all leave the metal when they strike the surface, it is evident that there must be a surface discontinuity of potential which prevents their escape. If now we raise the temperature of the metal we increase the average velocity of the corpuscles, and, provided the energy required to take an ion through the surface does not increase with the temperature, many more of the ions which strike the surface will pass through than before. In this way we can calculate the way in which the number of corpuscles shot off from unit area of a metal surface varies with the temperature. The formula so obtained involves two new constants, viz., the number of ions in unit volume of the metal and the work done by an ion in passing through the surface.

It may be permissible to state in anticipation that almost the whole of the experimental results are in striking agreement with the theory. In particular the theoretical formula makes the saturation current increase enormously rapidly with the temperature of which it is an exponential function. The experiments show that this is actually the case, and the saturation current has been followed over the following ranges of values for the three conductors examined :

For platinum from 10^{-10} to 10^{-3} ampère per sq. centim.
,, carbon ,, 10^{-8} ,, 2 ,, ,, ,,
,, sodium ,, 10^{-11} ,, 2×10^{-2} ,, total current.

The corresponding ranges of temperature for platinum and sodium are roughly from 1000° C. to 1600° C., and from 100° C. to 450° C. respectively.

Perhaps the most surprising result of the investigation is the relatively enormous currents which have been obtained. The biggest leak measured was ·4 ampère from a carbon filament to an electrode placed near it ; this corresponded to a current of 2 ampères per sq. centim. of the carbon surface, the potential on the wire being -60 volts. In this case the gas pressure was only $\frac{1}{600}$ millim. of mercury, so that the ionisation produced by collisions was negligible. In all cases care was taken that the field which was put on the filaments was insufficient either to start a discharge or to maintain one when started.

The smaller currents with sodium were measured by means of a quadrant electrometer : the largest (·04 ampère) was registered on a Weston ammeter.

It is evident from these experiments that a metal if placed in a vacuum and heated to a sufficiently high temperature makes the space around it an extremely

* 'DRUDE's Annalen,' vol. 1, p. 566.

† 'Rapports présentés au Congrès International de Physique,' Paris, 1900.

good conductor of electricity. The results show that in the case of an incandescent lamp, heated to the highest temperature it will stand, the specific conductivity of the surrounding space is comparable with that of the filament.

In the case of a hot conductor the current across the intervening space to the electrode will, of course, only go in one direction. The current when the hot metal is charged positively, and the electrode put to earth, is vanishingly small in comparison with the current when the wire is charged negatively.

The remainder of the present paper is divided up as follows :—

A.—*Theoretical Investigation.*

I. Calculation of the saturation current.

II. Equilibrium of corpuscles near a hot plane of infinite area.

B.—*Experimental Investigation.*

I. Experiments with platinum.

II. ,, ,, carbon.

III. ,, ,, sodium.

C.—*Conclusion.*

A.—Theoretical Investigation.

I. *Calculation of the Saturation Current*

§ 1. The application of the kinetic theory of gases to the equilibrium of the free negative electrons or corpuscles inside a metal scarcely needs justification here, since it has already been made use of by Professor Drude.* It may, however, be permissible to point out some results which show that the similarity between a corpuscle in a metal and a molecule in a gas under ordinary conditions is very close indeed. Professor Thomson† has shown, from the change of resistance of bismuth in a magnetic field, that the mean free path of a corpuscle in that metal has the value 10^{-4} centim.; while a series of experiments by Mr. Patterson‡ indicate that for platinum, gold, tin, silver, copper, zinc, cadmium, mercury, and carbon the mean free path has values which lie between $5{\cdot}9 \times 10^{-7}$ and $4{\cdot}1 \times 10^{-6}$ centim. The mean free path for a nitrogen molecule in air under standard conditions is 10^{-5} centim.; so that the mean free path of a corpuscle in bismuth is the same as that of a molecule in air at $\frac{1}{10}$th of an atmosphere pressure, whereas for other metals the mean free path is the same as that in air at about 10 atmospheres pressure. The free time is,

* 'Drude's Annalen,' vol. 1, p. 572, &c.

† 'Rapports présentés au Congrès International de Physique,' Paris, 1900, vol. 3, p. 138.

‡ 'Phil. Mag.' (6), vol. 3, p. 655.

of course, only about one-hundredth that of an air molecule possessing the same free path, owing to the great velocity of agitation of the corpuscles. Nevertheless we are quite justified in assuming that the time during which the corpuscles are moving freely is great compared with that during which they are colliding. In fact this assumption follows at once if we are to attach any definite meaning to the ideas of free paths and collisions.

If, in addition to neglecting the number ot corpuscles which are colliding at any moment in comparison with those which are not, we assume that the atoms of the metal and the positive ions oscillate about fixed centres and are subject to forces of restitution which are functions of their displacements only, we obtain at once, by the application of the ordinary analysis of the kinetic theory, the distribution of velocity among the corpuscles. This is found to be the same as that for an equal number of similarly constituted gas molecules. Thus the number of corpuscles ($N_u N_v N_w$) which have velocity components in three mutually perpendicular directions between u and $u + du$, v and $v + dv$, and w and $w + dw$ respectively are given by

$$\left.\begin{aligned} N_u &= N\left(\frac{km}{\pi}\right)^{\frac{1}{2}} e^{-km(u-\alpha)^2}\,du \\ N_v &= N\left(\frac{km}{\pi}\right)^{\frac{1}{2}} e^{-km(v-\beta)^2}\,dv \\ \text{and}\qquad N_w &= N\left(\frac{km}{\pi}\right)^{\frac{1}{2}} e^{-km(w-\gamma)^2}\,dw \end{aligned}\right\} \quad \text{. . . . . (1),}$$

where N is the total number of corpuscles considered, m is the mass of a corpuscle, α, β, and γ are the impressed velocity components of the corpuscles in the direction of u, v, and w respectively, whilst $\frac{3}{4}k$ is the average energy of translation of a corpuscle, and is equal to that of a gas molecule at the same temperature as the metal considered. The velocities α, β, γ are connected with the components q, r, s of the current density according to the relation

$$(q, r, s) = n(\alpha, \beta, \gamma)\,\epsilon,$$

where ϵ is the charge on a corpuscle, and n is the number per cub. centim.

§ 2. If we suppose the impressed velocities to be nil or to be negligible compared with the velocities of agitation, the number of molecules in unit volume having velocity components between u and $u + du$, v and $v + dv$, and w and $w + dw$ becomes

$$n\left(\frac{km}{\pi}\right)^{\frac{3}{2}} e^{-km(u^2+v^2+w^2)}\,du\,dv\,dw \quad \text{. . . . . (2),}$$

whilst the number with these velocity components which hit unit area perpendicular to u per second is

$$n\left(\frac{km}{\pi}\right)^{\frac{3}{2}} ue^{-km(u^2+v^2+w^2)}\,du\,dv\,dw \quad \text{. . . . . (3).}$$

If we suppose the surface of the hot conductor to be perpendicular to the axis of u, then the total number of corpuscles which hit unit area of the surface per second is

$$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty n \left(\frac{km}{\pi}\right)^{\frac{3}{2}} u e^{-km(u^2+v^2+w^2)}\, du\, dv\, dw.$$

We now suppose that there is a discontinuity in the electrostatic potential at the surface of the metal which is great enough to prevent the escape of the corpuscles at low temperatures. If the work done by an ion in passing through the surface layer is Φ, then the discontinuity in the potential is Φ/ϵ, where ϵ is the charge on an ion. We have further, by symmetry,

$$\frac{1}{\epsilon}\frac{\partial \Phi}{\partial y} = \frac{1}{\epsilon}\frac{\partial \Phi}{\partial z} = 0,$$

the surface being perpendicular to the axis of x.

Moreover $$m\dot{u} = -\frac{\partial \Phi}{\partial x}, \quad \text{whence} \quad u_0 = \sqrt{u^2 - \frac{2}{m}\Phi} \quad \ldots \ldots \quad (4),$$

where u_0 is the normal component of the velocity of the corpuscle after it has escaped from the metal.

It is evident from this that not all the corpuscles which strike the surface of the metal escape from it, but only those which have a normal velocity component which is $\geqq \sqrt{\frac{2}{m}\Phi}$. Hence, to get the total number which pass through the surface layer, we have to integrate expression (3) with respect to du not from 0 to ∞, but from $\sqrt{\frac{2}{m}\Phi}$ to ∞. Thus the total number which escape per second from unit area is given by

$$\mathrm{N} = \int_{\sqrt{\frac{2}{m}\Phi}}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty n \left(\frac{km}{\pi}\right)^{\frac{3}{2}} u e^{-km(u^2+v^2+w^2)}\, du\, dv\, dw \quad \ldots \ldots \quad (5)$$

$$= \frac{n}{2}(km\pi)^{-\frac{1}{2}} e^{-2k\Phi} = n\sqrt{\frac{\mathrm{R}\theta}{2m\pi}}\, e^{-\Phi/\mathrm{R}\theta} \quad \ldots \ldots \ldots \quad (6),$$

since k is connected with θ, the absolute temperature, by the relation $k = (2\mathrm{R}\theta)^{-1}$, R being the gas constant for a single corpuscle. The saturation current being equal to the quantity of electricity carried by the ions which are shot off from the surface in one second, is given by

$$\mathrm{C} = \mathrm{N}\epsilon\mathrm{S} = n\epsilon\mathrm{S}\sqrt{\frac{\mathrm{R}\theta}{2m\pi}}\, e^{-\Phi/\mathrm{R}\theta} \quad \ldots \ldots \ldots \quad (7),$$

where S is the area of the metal surface and ϵ, as before, the charge on an ion.

§ 3. When the ions are removed by an external electric field as quickly as they are set free at the surface of the metal, as in the case of the experiments to be described later, the metal must be continually losing energy owing to the emission of the corpuscles. This energy is composed of two parts : the first being represented by the work done by the corpuscles in passing through the surface layer, while the second is equal to the energy of translation which they possess when they have reached the outside of the metal. The sum of the two is easily calculated, since it is equal to the energy of translation which the corpuscles that have passed through the surface layer possessed while they were inside the metal. We have therefore merely to multiply the number of corpuscles which strike the surface by the energy each possesses and integrate between limits which embrace all values that pass through the surface layer. The total loss of energy per second is therefore

$$T = \frac{n}{2}\left(\frac{km}{\pi}\right)^{\frac{3}{2}} \int_{\sqrt{\frac{2}{m}\Phi}}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m\,(u^2 + v^2 + w^2)\, u e^{-km(u^2+v^2+w^2)}\, du\, dv\, dw \quad . \quad . \quad (8).$$

$$= \frac{n}{2}\,\frac{(1 + k\Phi)\, e^{-2k\Phi}}{\pi^{\frac{1}{2}} m^{\frac{1}{2}} k^{\frac{3}{2}}} = n\left(1 + \frac{\Phi}{2R\theta}\right)\sqrt{\frac{2R^3\theta^3}{\pi m}}\; e^{-\Phi/R\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Now the work done in a second by the corpuscles passing through the surface layer is obviously $= N\Phi$, so that the part of the energy lost by the hot metal per second which appears in the form of the translational energy of the corpuscles, is given by

$$T - N\Phi = \frac{n}{2}\,\frac{e^{-2k\Phi}}{\pi^{\frac{1}{2}} m^{\frac{1}{2}} k^{\frac{3}{2}}} = n\left(\frac{2R^3\theta^3}{m\pi}\right)^{\frac{1}{2}} e^{-\Phi/R\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

This calculation of the rate of emission of energy only applies, of course, to the case where the ions are removed by an external field as fast as they are formed. If there is no external field and the ions are allowed to remain, we soon arrive at a steady state when as many corpuscles possessing a given amount of energy enter the surface of the metal in a given time as leave it ; so that in this case there is no loss of energy due to this cause.

The following proof of formula (6), which is due to Professor J. J. THOMSON, is interesting, since it does not involve the methods of the kinetic theory of gases. Suppose we have a closed space which is bounded by a surface of hot metal, then the corpuscles will be given off from the metal until a steady state is reached. In this steady state as many corpuscles will pass through the bounding surface from the vacuum to the metal as from the metal to the vacuum, but the pressure will not be the same on both sides of the surface owing to the forces which tend to retain the corpuscles in the metal. There will thus be a discontinuity in the pressure at the surface of separation, and Φ being the work done on an ion when taken through the surface, we have

$$\int_1^2 p\, dv = \Phi,$$

where 1 refers to the metal and 2 to the neighbouring space, p being the pressure and v the volume occupied by a corpuscle at any point. Substituting for p its value $R\theta/v$ from the gas equation, we get

$$\log v_2 - \log v_1 = \Phi/R\theta;$$

whence, if n_2 be the number of corpuscles per unit volume outside and n_1 the number per unit volume inside the metal, we have

$$n_2 = n_1 e^{-\Phi/R\theta}.$$

Now the number of corpuscles shot off from the surface per second is not equal to the number per unit volume of the space, but is equal to this multiplied by the average velocity perpendicular to the surface. So that, in the steady state, $N = n_1 u$, where

$$u = \left(\frac{km}{\pi}\right)^{\frac{1}{2}} \int_0^\infty u e^{-kmu^2}\, du = \sqrt{\frac{R\theta}{2m\pi}}; \quad \text{whence} \quad N = n\sqrt{\frac{R\theta}{2m\pi}}\, e^{-\Phi/R\theta},$$

which is the same formula as has been deduced above without postulating the existence of a steady state.

By following up the analogy between the emission of corpuscles and evaporation, the preceding formulæ, connecting the corpuscular pressure with the temperature, can be obtained thermodynamically in a manner involving still fewer assumptions.

II. *The Equilibrium of Corpuscles near a Plane Surface of Hot Metal of Infinite Extent.*

§ 4. Both this problem and the corresponding problem in spheres are of considerable importance, not only in connection with experiments in vacuum tubes, but also with regard to the behaviour of hot celestial bodies in space. For instance, the aurora borealis and allied phenomena indicate that large quantities of ions continually reach the earth from some extraneous source, while certain variations of the earth's magnetic field and other meteorological phenomena seem to be intimately connected with events which take place at the surface of the sun. The present paper does not attempt to solve these questions, but the above facts indicate that the subject of the ionisation produced by hot bodies is not without importance in regard to meteorology.

The problem under consideration may be specified in the following terms:—Given an infinite quantity of hot metal bounded on one side by a plane surface of infinite extent which is maintained at a given potential, find the charge on unit area of the metal surface and the potential at any point in the space outside the metal when the steady state has been attained.

Let us take the surface of separation perpendicular to the axis of x, and let the suffix 1 refer to points inside the metal, the suffix 2 referring to points in the neighbouring empty space. There is, as we have seen already, a discontinuity in the pressure of the corpuscles at the surface of the metal, and by the conservation of energy

$$\int_1^2 p\,dv = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where w is the work done in taking unit-mass of the corpuscles through the surface layer, p is the pressure, and v the volume of unit mass of the corpuscles at any point. Similarly, in order to obtain the equations satisfied by the corpuscles outside the metal when the equilibrium stage has been reached, we use the principle that the work along any path extending from a point a to a point b due to expansion is equal to the work done by the electric forces. This gives

$$\int_b^a p\,dv + \int_b^a nve_0 \frac{dV}{dx}\,dx = 0.$$

V being the electrostatic potential, e_0 the charge on a corpuscle, and n the number of corpuscles in unit volume, since everything is independent of y and z. Now $nv = N_0$ the number of corpuscles in unit mass, whence

$$\frac{R\theta}{v}\frac{dv}{dx} + N_0 e_0 \frac{dV}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

In addition to this the electrostatic potential has to satisfy POISSON'S equation, which takes the form

$$\frac{d^2V}{dx^2} = -4\pi\zeta = \frac{4\pi N_0 e_0}{v};$$

e_0 being the numerical value of the negative charge.

The equation to be satisfied is therefore

$$R\theta \frac{d^2(\log v)}{dx^2} + 4\pi \frac{N_0^2 e_0^2}{v} = 0$$

$$\text{or} \quad \frac{d^2v}{dx^2} - \frac{1}{v}\left(\frac{dv}{dx}\right)^2 + C = 0, \quad \text{where} \quad C = \frac{4\pi N_0^2 e_0^2}{R\theta} \quad . \quad . \quad (13).$$

A first integral of this equation is

$$\frac{d(\log v)}{dx} = \left(B + \frac{2c}{v}\right)^{\frac{1}{2}}$$

B being an integration constant.

Now when v is infinite $\frac{d \log v}{dx} = 0$, and therefore $B = 0$, so that

$$\sqrt{2c}\,dx = v^{-\frac{1}{2}}\,dv,$$

whence

$$v = \tfrac{1}{4}(\sqrt{2c}\,x - A)^2,$$

A being a second integration constant.

We have the further conditions:

$$\left.\begin{array}{lll} v = \infty & \text{when} & x = \infty \\ v = v_1 e^{w/R\theta} & \text{when} & x = 0 \end{array}\right\}$$

which are satisfied if $A^2 = 4v_1 e^{w/R\theta}$.

Taking the positive root for v_1 (since negative values are inadmissible) and putting $v_1 = N_0/n_1$ we obtain

$$v^{\frac{1}{2}} = \left(\frac{2\pi}{R\theta}\right)^{\frac{1}{2}} N_0 e_0 x + \left(\frac{N}{n_1}\right)^{\frac{1}{2}} e^{\frac{1}{2}w/R\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

This equation gives the concentration (v^{-1}) of the corpuscles at any distance x from the plane when the temperature is maintained at $\theta°$ absolute.

Returning to equation (12) we see that integration and substitution for v yield the electrostatic potential V in the form

$$V = -2\frac{R\theta}{N_0 e_0} \log\left\{\left(\frac{2\pi}{R\theta}\right)^{\frac{1}{2}} N_0 e_0 x + \left(\frac{N_0}{n_1}\right)^{\frac{1}{2}} e^{\frac{1}{2}w/R\theta}\right\} + \gamma.$$

If $V = V_0$ for $x = 0$, the integration constant γ is determined as

$$\gamma = V_0 + 2\frac{R\theta}{N_0 e_0} \log\left\{\left(\frac{N_0}{n_1}\right)^{\frac{1}{2}} e^{\frac{1}{2}w/R\theta}\right\},$$

so that V is finally to be obtained from

$$V = V_0 - 2\frac{R\theta}{N_0 e_0} \log\left\{1 + \left(\frac{2\pi n_1 N_0}{R\theta}\right)^{\frac{1}{2}} e_0 e^{-\frac{1}{2}w/R\theta} x\right\} \quad . \quad . \quad . \quad . \quad (15).$$

The electric intensity at any point x is given by

$$-\frac{dV}{dx} = \frac{2\left(2\pi R\theta \frac{n_1}{N_0}\right)^{\frac{1}{2}}}{1 + \left(\frac{2\pi n_1 N_0}{R\theta}\right)^{\frac{1}{2}} e_0 e^{-\frac{1}{2}w/R\theta} x} e^{-\frac{1}{2}w/R\theta} \quad . \quad . \quad . \quad . \quad . \quad (16),$$

and the charge on unit area of the radiating plane by

$$\sigma = -\frac{1}{4\pi}\left(\frac{dV}{dX}\right)_{X=0} = \left(\frac{n_1 R\theta}{2\pi N_0}\right)^{\frac{1}{2}} e^{-\frac{1}{2}w/R\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

the volume density ζ at any point x being

$$\zeta = -\frac{1}{4\pi}\frac{d^2V}{dx^2} = -\frac{n_1 e_0 e^{-w/R\theta}}{\left\{1 + \left(\frac{2\pi n_1 N_0}{R\theta}\right)^{\frac{1}{2}} e_0 e^{-\frac{1}{2}w/R\theta} x\right\}^2} \quad . \quad . \quad . \quad . \quad (18).$$

It is evident that $\int_0^\infty \zeta\, dx = \sigma$, since $dV/dx = 0$ for $x = \infty$. Thus, as we should expect, the charge on the surface is equal and opposite to the total charge in the space outside the metal.

As a numerical illustration we may calculate the potential at a point distant 10 centims. from a plane surface of platinum which is put to earth and maintained at a temperature of, say, 1500° absolute. Taking the number of molecules in a cubic centimetre at 0° and 760 millims. as 2×10^{19}, the charge on an ion as $6{\cdot}5 \times 10^{-10}$ and the value of w/R, which has been determined experimentally, to be $4{\cdot}93 \times 10^4$, we find the potential at a point 10 centims. from the surface to be 1·5 volts, while at a point 1 centim. distant it would be about 1·2 volts.

The experiments in the sequel were not intended to test this part of the theory, but they show, as we should expect, that practically the whole of the current is stopped by a fall of potential of the order of one volt when it tends to drive the corpuscles back to the hot metal.

It will be seen by inspection of formula (15) that even at the highest temperatures we can attain the potential differences at small distances from the hot surface never become very great. For instance, at the temperature of the sun (6000° C.) the difference of potential between the surface and a point 1 centim. distant from it would be only about sixteen times its value at 1300° C. On the other hand, the surface density increases very quickly with the temperature, as will be seen from formula (17). In the case of carbon at 6000° C., taking 10^{24} as a probable maximum value of n and $7{\cdot}8 \times 10^4$ as the value of w/R, we find that σ has the enormous value of 300 electrostatic units, whereas at 1300° C. σ would have been less than this in the ratio of 1 to 3×10^8.

These numbers are to be taken as purely illustrative. It is not supposed that any conductor could possibly exist in a vacuum at 6000° C.

It will be noticed that the preceding theory of the equilibrium of corpuscles near a surface where they are being emitted is quite independent of any hypothesis as to the nature of the mechanism by which they are set free. The results are therefore of interest even if the hypothesis, that the negative ions from hot conductors are the same as those which carry the current inside the metal, is ultimately found to be untrue.

B.—Experimental Investigation.

I. *Experiments with Platinum.*

§ 1. *Description of the Apparatus.*

The ultimate object of the experiments was to determine the way in which the saturation current from a hot platinum wire to a surrounding electrode, both placed in a vacuum, varied with the temperature of the wire. For this purpose the type of bulb shown in fig. 1 was found to be most convenient.

The wire to be heated was in the form of a spiral, with its axis passing centrally along the length of the tube; the current through the wire was supplied by means of the two thick leads AA_1 and BB_1. The electrode to which the current was measured was an aluminium cylinder which surrounded the hot part of the wire. The cylinder was supported by a stout aluminium wire E, sealed through the side tube D by means of platinum. The end E was connected to the electrometer or galvanometer which served to measure the current. The side tube F connected the bulb with the pump and McLeod gauge.

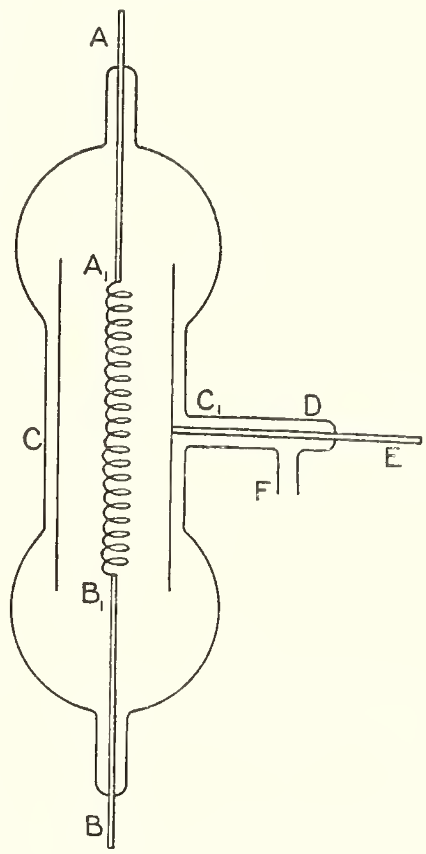

Fig. 1.

In the earlier experiments, trouble was experienced owing to loose contacts appearing at A_1 and B_1 when the platinum wire had been heated. In the final form of the tube this was obviated by making the leads AA_1 and BB_1 of platinum wire 1 millim. thick, to which the ends of the platinum spiral A_1B_1 were welded electrically. This made the platinum quite continuous through the tube. The support E of the electrode CC_1 was insulated outside the tube at D by means of sealing wax. Inside the tube there was only glass insulation, which, however, is very good at low pressures.

The temperature of the platinum wire was obtained by measuring its resistance. The arrangement of apparatus which was used to do this and to measure the current from the surface of the wire is indicated in fig. 2. The whole of the apparatus below AFK_2 is the part which was used to determine the resistance and was insulated on paraffin blocks. It could be charged to any desired potential up to 400 volts by means of the battery B_1 through the key K_1 and water resistance A. The potential was measured by the volt-meter W. In this way any desired potential could be maintained on the hot wire F. The cylindrical electrode C was put to earth through the galvanometer G_1, which thus served to measure the current. In some of the experiments a Thomson galvanometer, of 4058 ohms resistance, which gave a deflection of 1 scale division for 7×10^{-10} ampère, was used, but this was replaced later by a less sensitive D'Arsonval galvanometer. The sensitiveness of the galvanometer deflections could be lowered at will by means of the shunt resistances, R_5 and R_6.

The resistance of the hot wire was determined by placing it in one arm of a Wheatstone's bridge, the other three arms being the resistances R_2, R_3, and R_4. The battery B_2, which worked the bridge, also supplied the current necessary to heat F. As there was a current of up to 1·5 ampères continually running through the arm F_1, the corresponding arm R_2 had to be constructed so as to carry this current

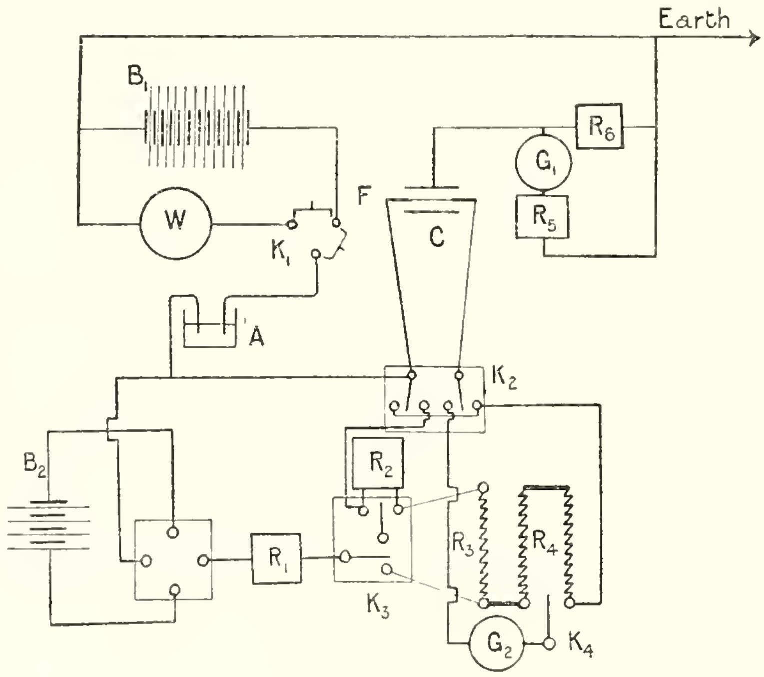

Fig. 2.

without heating. A German silver rheostat of very thick wire mounted on an iron frame and having a resistance of 1·7 ohms was found to fulfil the required conditions. The arm R_3 had a resistance of 1000 ohms, and plugs were taken out of the box R_4 till a balance was obtained in the galvanometer G_2. The maximum current which flowed through the arms R_3 and R_4 was thus about $\frac{1}{700}$th of that flowing through R_2 and F. There was therefore no danger of R_3 and R_4 being heated by the currents used. The adjustable resistance R_1 served to regulate the heating current. By means of the keys, K_2, K_3, K_4, the battery could be put on between the two resistances R_3 and R_4 at the one end, and R_2 and F at the other. In this way the currents flowing through the various arms could be made small, F would not become heated, and, by making the proper connections for the galvanometer G_2, the resistance of the arm F at the ordinary temperature could be determined. In the earlier experiments an arrangement was used which enabled the same galvanometer to be employed both for the resistance and leakage circuits; but it was found to be more convenient to use two galvanometers.

§ 2. *Variability of the Current.*

As some unsteadiness had been observed in the galvanometer readings for the leak during the earlier observations, a series of experiments was made in order to examine if the current from the wire varied when the conditions were kept as steady as possible. When the current was passed through the wire the tube became hot and gas was given off from the walls and from the hot wire, so that it was impossible to keep the pressure absolutely constant. However, by continuously pumping out the gas the pressure was kept practically constant, the limits of variation being very small. A constant current was run through the wire so that its temperature and resistance were invariable except in so far as they depended on the pressure of the gas in the apparatus.

Since the rate of escape of heat from the wire is determined largely by the gas pressure, the temperature of the wire is a function of the pressure. In fact, the galvanometer spot was a far more sensitive indicator of the pressure than the McLeod gauge. By carefully watching the galvanometer and pumping accordingly, the variations of both pressure and temperature were kept very small indeed.

Under these conditions it was hoped that the rate of leak from the wire with a constant voltage would remain approximately constant. It was found, however, that it varied in the most haphazard manner, oscillating irregularly between the limits of 10^{-6} and 10^{-4} ampère. The current did not become any steadier with continuous heating. Readings taken every three or four minutes for the space of three hours showed the same continuous irregular periodicity. The irregularities were quite independent of the potential that was or had been applied to the wire, and also seemed to have no relation to the rate at which gas was given off. There was no measurable falling off with time.

It ought, perhaps, to be mentioned that the tube used for this experiment seemed far more variable than those used for the temperature experiments, though they were never examined systematically. The platinum wire used for this tube was the purest obtainable.

These results are taken to indicate that the negative ionisation depends to a great extent on small changes in the condition of the surface of the hot wire. We should expect this to be the case on the view that the phenomena are due to the escape of corpuscles from the metal, since an alteration of 14 per cent. in the work done by an ion in going through the surface would multiply the current by 100.

Further experiments showed that fairly steady readings were obtained if the heating current was stopped and the tube allowed, as it were, to recover itself between each observation. The initial value of the current was almost constant, it then began to decrease slowly and afterwards varied in the irregular manner described above. The following readings taken with constant voltage, temperature, and pressure at the times stated, indicate the sort of agreement which is observed :—

Time of observation.	Current observed (in scale divisions).
12.18 P.M.	183
12.34 „	191
12.43 „	201
12.56 „	218
1.15 „	205
3.16 „	208
3.48 „	193
11.5 A.M. (next day)	193

This mode of observation is practically that followed in § 5, where the current is observed immediately after the temperature of the wire has been increased by a given amount.

§ 3. *Experiments with Alternating Currents.*

A mode of observation which is especially well calculated to show the relation between the positive and negative ionisation produced by hot platinum in a vacuum is to heat the wire by putting it on a 200-volt alternating circuit and to observe the current to the cylindrical electrode. The ions of both signs are alternately driven away from and attracted to the hot wire owing to the alternating field between the wire and the cylinder. The cylinder is connected to one quadrant of an electrometer, the other quadrant being put to earth. The direction of the current, which is indicated by the direction in which the spot of the electrometer moves, is determined by the sign of the ions which reach the cylinder in greatest quantity under the alternating electromotive force. At low temperatures all the ions produced by the wire are positive, so that the current is necessarily in the positive direction. At higher temperatures negative ions are also produced in gradually increasing quantity, so that at one temperature the same number of positive and negative ions reach the cylinder in a given time. At this temperature, which may be called the transition temperature, there is no current from the hot wire to the surrounding electrode under the given alternating field. At still higher temperatures, owing to the rapid rate at which the negative ionisation increases with the temperature, the current is always negative.

In these experiments the temperature of the wire was not determined, but a rough idea of it can be obtained from the resistance. This was determined with the apparatus indicated in fig. 2, except that the galvanometer G was replaced by a telephone. The following table gives corresponding values of the leak and the resistance of the wire :—

Resistance of wire.	Current from wire to cylinder in ampères.
ohms	
3·20	$+1·38 \times 10^{-12}$
3·30	$+3·7 \times 10^{-12}$
3·48	$+3·7 \times 10^{-12}$
3·78	$+1·6 \times 10^{-12}$
4·00	$-7·2 \times 10^{-12}$
4·22	$-7·5 \times 10^{-11}$
4·42	$-6·8 \times 10^{-10}$
4·56	$-2·5 \times 10^{-9}$

It will be seen from these experiments and those to be described later that the negative ionisation increases very rapidly with the temperature, and becomes enormous compared with the positive. The transition temperature for platinum at low pressures is about 900° C.

§ 4. *The Relation between the Current and the Applied Electromotive Force.*

As the ultimate object of these experiments was to measure the saturation current from the wire, it was thought advisable to investigate the relation between the current and the potential applied. A large number of current-E.M.F. curves for hot platinum have been given by Professor McClelland.* As, however, my apparatus, though similar, was not quite the same as, and the currents employed were much greater than, in the case investigated by Professor McClelland, it was considered necessary to make new experiments on the subject.

As the absolute value of the current was continually varying in the way previously described, the current was continually referred to its value with a given potential on the wire. This "standard" potential was − 41 volts. The current with 41 volts on the wire was measured both before and after taking a reading with any assigned potential: the ratio of this reading to the mean of the readings with 41 volts was taken to be what the ratio of the current under the given voltage to the current at 41 volts would have been if the state of the tube had remained constant. In this way the variability of the hot wire could be satisfactorily eliminated.

In these experiments the value of the saturation current was about 3×10^{-6} ampère, and was probably about ten thousand times as big as the current used by Professor McClelland. In making the observations readings of the current were taken for every 6 or 7 volts up to 80, and afterwards at intervals of 40 volts up to 400. The numbers so obtained are plotted in the following curve (fig. 3), the value of the current with − 41 volts on the wire is fixed arbitrarily as unity. The voltages refer to the positive end of the wire.

The current rises to about one-third its final value with ten volts on the wire but

* 'Cambr. Phil. Proc.,' vol. 11, p. 296.

does not become saturated till about 160 volts. It will be seen that this curve is very similar to the one given by Professor McClelland for the same pressure (·1 millim.). The similarity of the two leads to the conclusion that the form of the current E.M.F. curve is largely independent of the amount of ionisation produced by

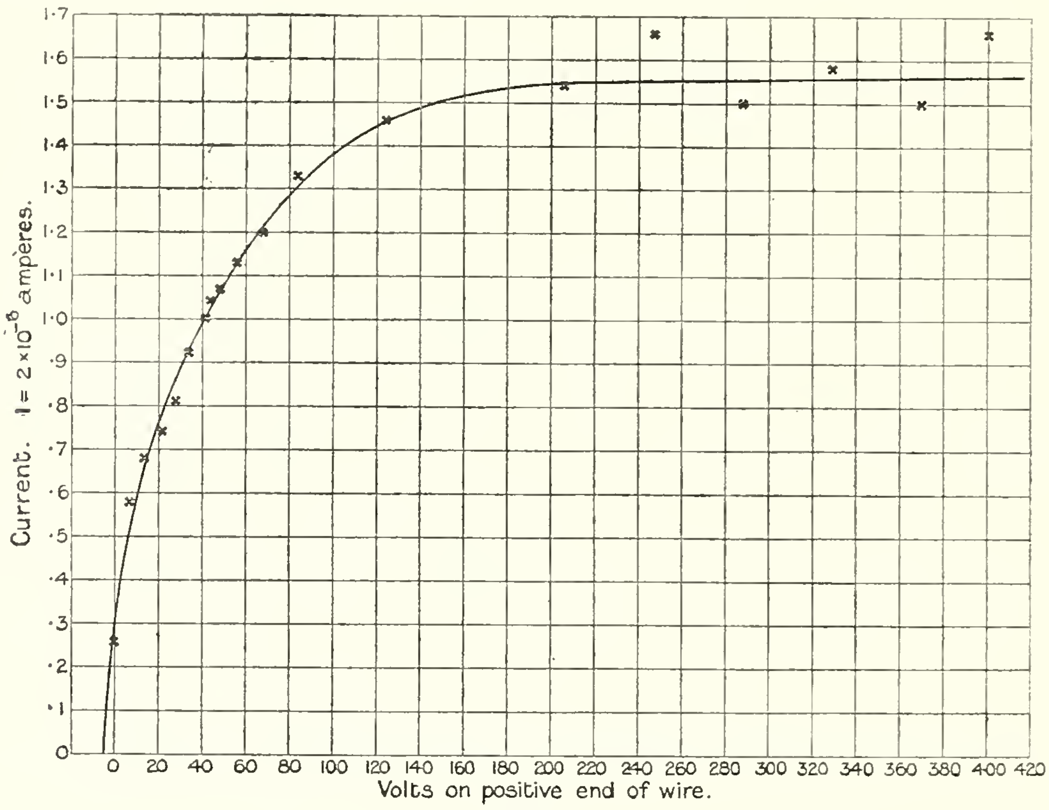

Fig. 3.

the wire. In many of the experiments on the variation of the current with the temperature the pressure was considerably less than ·1 millim., but we should expect that a voltage which would saturate the current at a given pressure would saturate it at any lower pressure. At any rate, the experiments to be described later show that this is true for the negative ionisation produced by hot carbon. Another set of experiments on platinum showed that at ·008 millim. the current was saturated by less than 80 volts.

§ 5. *The Relation between the Saturation Current and the Temperature of the Wire.*

The temperature of the wire was obtained from its resistance, and in order to determine this the apparatus indicated in fig. 2 was employed in the manner already described. During each observation it was found that the temperature of the wire, which was run at constant voltage, fell slightly, owing to the gas given off from the walls of the tube and elsewhere. A reading for the resistance was therefore taken immediately before and after the reading for the current, and the mean of the two resistances was taken to be that which corresponded to the current reading. The wire was heated for a long time and the tube constantly pumped out previous to making the observations, in order to reduce the evolution of gas as far as possible. When the tubes were heated at first large quantities of gas came off, but after a time further heating and pumping did not seem to effect much reduction in the rate of evolution of gas. Despite constant pumping the pressure always rose slightly in the McLeod gauge; the increase was, of course, more marked the higher the temperature of the wire.

To obtain the temperature from the resistance measurements use was made of the determinations of the melting points of potassium and sodium sulphates of Messrs. HEYCOCK and NEVILLE.* The wire was set up in air and its resistance determined, first at the ordinary temperature and afterwards when the smallest possible grain of potassium sulphate placed on it just melted. In this way the resistance for two temperatures differing by about 1000° was obtained, and the temperature corresponding to any other resistance reading could be got by interpolation from the curves given by Professor CALLENDAR.† To test the method the melting point of sodium sulphate was determined and no determination was more than 20° from the true value (883° C.). This agreement was held to be quite good enough for the purpose. The temperature as found from the resistance in this way is the average temperature of the wire whereas what is required is the temperature at the surface. A calculation showed, however, that the temperature at the centre of the hot wire only differed from that at the circumference by 4° C., a quantity which is negligible compared with the experimental error.

In the experiments on platinum a potential of 120 volts was maintained on the wire; this was more than enough to saturate the current at practically all the pressures which occurred. It was found afterwards that at pressures greater than ·09 millim. the current was not saturated by this potential but the deviation from the saturation value due to this cause is smaller for the observations taken than the error due to unavoidable irregularities.

The values of the pressures are given in the tables for comparison. Two numbers are inserted in each case in the resistance column, these are the resistances as determined immediately before and after the value of the saturation current was read. The difference between the two numbers is a measure of the rate at which the temperature of the wire was changing and, therefore, of the rate at which the pressure of the gas in the tube was increasing.

The following table gives the results which were obtained at temperatures below 1450° C.

* 'Jour. Chem. Soc.,' vol. 67, p. 160.

† 'Phil. Mag.,' vol. 48, p. 519.

Pressure of gas in millims. of mercury.	Resistance of hot wire in ohms.	Current from wire to cylinder, $1 = \text{ampère} \times 10^{-9}$.	Temperature of wire in ° C.
·023	8·338 8·335	2·52	1031
·025	8·438 8·430	8·28	1058
·021	8·642 8·625	30·6	1105
·025	8·795 8·782	100·5	1146
·024	8·894 8·875	188	1170
·028	8·969 8·950	300	1190
·028	9·106 9·088	728	1224
·032	9·163 9·131	858	1243
·032	9·263 9·230	1,414	1269
·037	9·381 9·350	2,600	1298
·044	9·472 9·445	4,025	1323
·063	9·603 9·574	11,320*	1354
·063	9·925 9·883	11,740*	1445

The next table gives another series of observations extending over a higher range of temperature. Owing to the greater unsteadiness of the tube at the higher temperatures the points do not fall quite so accurately on the curve. The current at 1600° was the biggest measured and corresponded to $1{\cdot}03 \times 10^{-3}$ ampère per sq. centim. of platinum surface.

* The current became rather unsteady here.

Pressure in millims. of mercury.	Resistance of wire in ohms.	Saturation current, 1 = ampère × 10^{-7}.	Temperature in ° C.
·024	9·725 9·718	1·04	1194
·044	10·16 10·14	13·62	1298
·091	10·63 10·61	116	1419
·106	10·79 10·75	578	1449
·152	10·95 10·91	1370	1490
·180	11·13 11·07	1730	1533
·162	11·35 11·355	4180	1599

The relation between the saturation current and the temperature is shown graphically in fig. 4. The ordinates give the value of the saturation current, the

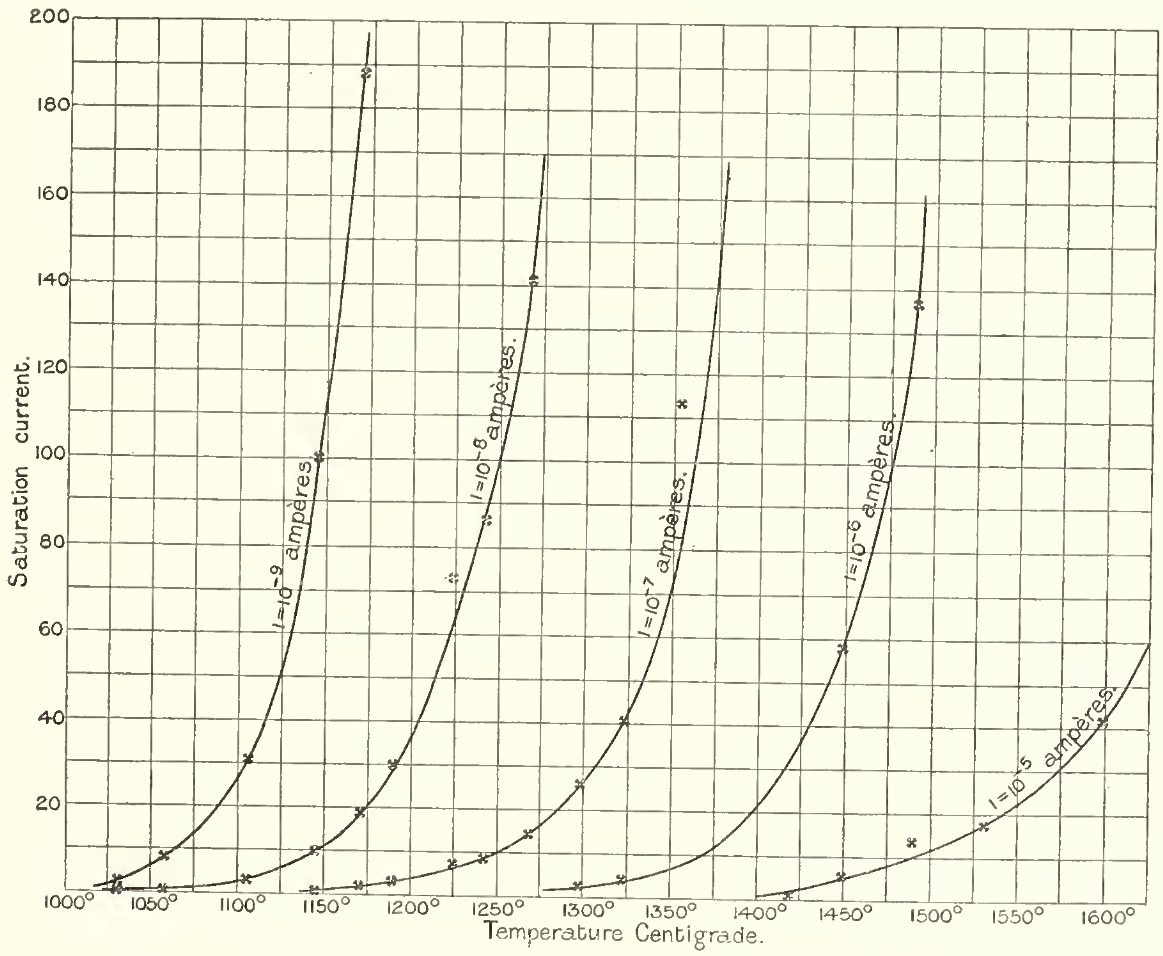

Fig. 4.

abscissæ being temperatures in °C. In the first curve, starting from the left of the diagram, each unit of the ordinate represents 10^{-9} ampère; in each succeeding curve as we pass to the right the value of the ordinate is successively multiplied by ten, so that in the last curve each unit is equal to 10^{-5} ampère. To obtain the saturation current per unit-area the values on the curve have to be multiplied by 2·5.

The curves show that the negative ionisation increases very rapidly with the temperature of the wire (in fact the saturation current varies roughly as the 70th power of the absolute temperature). It will be seen that the current never vanishes absolutely, but only in an asymptotic manner, so that it should be observable at any temperature provided sensitive enough instruments are employed. As a matter of fact at low temperatures it would of course be masked by other effects, which become large by comparison. The curves seem also to tend continuously to an infinite value of the saturation current; but the theory indicates that at higher temperatures the current would increase much more slowly with the temperature. This falling off of the rate of increase has not yet been observed with any of the conductors which have been examined.

We are now in a position to apply formula (7) to the reduction of the experimental results. For the sake of convenience we may write for the number of corpuscles shot off from unit area of the metal per second

$$N = (C/\epsilon S) = A\theta^{\frac{1}{2}}e^{-b/\theta},$$

where $A = n\,(R/2m\pi)^{\frac{1}{2}}$ and $b = \Phi/R$. The saturation current C is here to be measured in electrostatic units. In order to test the formula we may write the above equation in the form:

$$\log_{10} C - \log_{10} \epsilon S = \log_{10} A + \tfrac{1}{2}\log_{10}\theta - \frac{b}{2{\cdot}303\theta}.$$

If we put, for convenience, $\log_{10} C - \frac{1}{2}\log_{10}\theta - \log_{10} 3 + 1{\cdot}5 = y$ and $\theta^{-1} = x$, we may write our equation

$$y = a - b_0 x_0,$$

so that plotting the values of y against those of θ^{-1} should give a straight line. In the accompanying graph the ordinates are the values of $\log_{10} C - \frac{1}{2}\log_{10}\theta$, the abscissæ being $\theta^{-1} \times 10^5$. The curve got is very approximately indeed a straight line; though any variation from strict rectilinearity might be explained by the variation with temperature of the coefficient A, that is, if our theory is correct, of n the number of corpuscles per cubic centimetre of platinum. We may therefore say with certainty that the main features of the phenomenon are to be represented by a formula of the type $A\theta^{\frac{1}{2}}e^{-b/\theta}$.

Interesting conclusions are also to be drawn from the actual values of the constants themselves. From the constant A we obtain the number n of free corpuscles in a cubic centimetre of solid platinum, since we have the relation $n = \left(\frac{2m\pi}{R}\right)^{\frac{1}{2}} A$. A is obtained by putting corresponding values of θ and C in the equation

$$\log_{10} A = \log_{10} C - \tfrac{1}{2} \log_{10} \theta - \log_{10} {\cdot}788 + 9{\cdot}523 + (2{\cdot}24 \times 10^4)\,\theta^{-1}.$$

At θ (absolute) $= 1542°$ this gives $A = 1{\cdot}51 \times 10^{26}$. The various constants in the logarithmic equation come from the area of the wire, which was ·394 sq. centim., and the value of the charge on an ion, which was taken to be 6×10^{-10} electrostatic unit. The value of m/R (m being the mass of, and R the gas-constant for, one corpuscle) was found to be $= 1{\cdot}204 \times 10^{-11}$. Putting this in the expression for n, we find $1{\cdot}3 \times 10^{21}$ free negative ions in a cubic centimetre of platinum at 1542° absolute. An independent value of n has been obtained by Mr. PATTERSON* from experiments on the change of resistance of platinum in a magnetic field. This when calculated by the method given by Professor THOMSON† yields $n = 1{\cdot}37 \times 10^{22}$. The agreement of the value found above with this is really very good, when one considers the numerous sources of error to which the measurements are liable, and that an error of 7 per cent. in the absolute temperature, among other things, would multiply the value of n by ten.

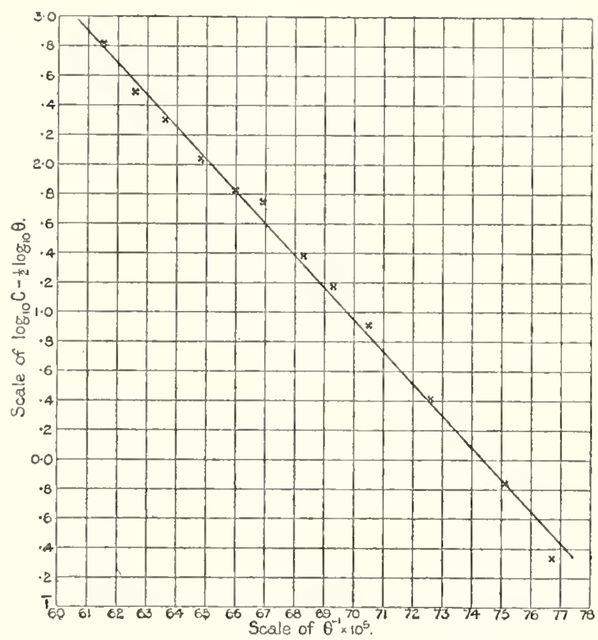

Fig. 5.

It was thought that possibly some regular change in the value of n with the temperature might be found if values were calculated for different temperatures. It was found, however, that n oscillated in an irregular manner between $\cdot 43 \times 10^{21}$ and $2{\cdot}0 \times 10^{21}$, so that the experiments yielded no evidence of any detectable variation of n. This method of obtaining n is extremely inaccurate, so that the agreement between the above numbers is really better than would be expected.

The signification of the constant $b = \Phi/R$ which occurs in the exponential factor is equally important, since Φ is the work done by an ion in passing through the surface layer. We obtain b from the equation

$$b = \frac{\log_e C/C' - \frac{1}{2}\log_e \theta/\theta'}{\theta'^{-1} - \theta^{-1}},$$

where C, C′ and θ, θ' are corresponding currents and absolute temperatures.

* 'Phil. Mag.' (6), vol. 3, p. 643.

† J. J. THOMSON, 'Rapports présentés au Congrès International de Physique,' vol. 3, p. 138, Paris, 1900.

Substituting the values of C and C′ for $\theta = 1571$, $\theta' = 1378$ respectively, we get the average value of b from 1378° to 1571° absolute as $4 \cdot 93 \times 10^4$. If we assume that all the work done by the corpuscles in passing through the surface is electrical, we can calculate from this the discontinuity in the potential. Since R is equal to $\left(\frac{2}{1 \cdot 204}\right) \times 10^{-16}$, we have $\Phi = 4 \cdot 93 \times \frac{2}{1 \cdot 204} \times 10^{-12} = \epsilon\delta\phi$, where ϵ is the charge on an ion and $\delta\phi$ is the discontinuity in the potential at the surface of the metal. From this we obtain

$$\delta\phi = 1 \cdot 365 \times 10^{-2} \text{ electrostatic unit} = 4 \cdot 1 \text{ volts.}$$

The further discussion of these results will be postponed until the experiments on sodium and carbon have been considered. (See additional Note at end of this paper.)

II. *Experiments with Carbon.*

§ 1. *Description of Apparatus.*

In order to detect and examine the negative leak from carbon, the hot wire previously employed was replaced by a filament from an ordinary incandescent lamp. The thick filaments from small 8 or 12-volt lamps were found to be most suitable. In the form of apparatus which was used to investigate the relation between the negative leak and the resistance of the carbon (see fig. 6) the filament was allowed to remain inside the lamp. The lamps were opened up by snipping off the glass point at the top with a pair of pliers. The wide tube A was then fixed on by drawing it out at the end which was to be joined and blowing the junction out until it was wide enough to allow the aluminium electrode E to be introduced. This process required some care, as the lamps are liable to crack when hot. It was found that air leaks due to small cracks in the part of the lamp which is covered with plaster of Paris could be effectually stopped by embedding the whole lamp in melted paraffin wax. The tube L, into which the electrode was fixed with sealing-wax, was joined to a bulb C, which was somewhat wider than A, into which it was inserted, the joints being made air-tight by means of sealing-wax. The side tube D led to the pump and McLeod gauge. The filament F could be charged either positively or negatively, and the leak from it to the electrode E was measured in exactly the same way as has already been described in the case of platinum.

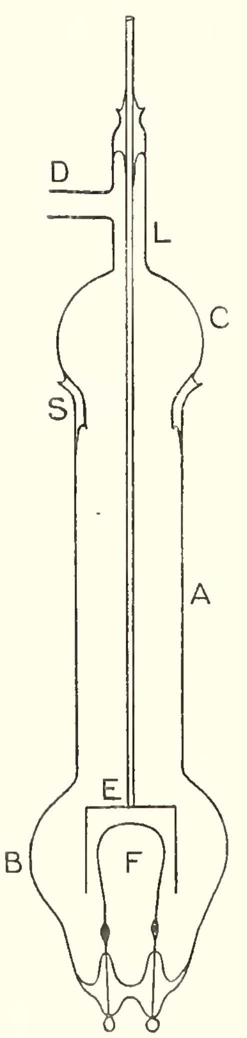

Fig. 6.

This form of apparatus was found to be quite satisfactory for investigating the connection between the current from the carbon on the one hand and the electromotive force, the resistance of the filament, and the current required to heat it, respectively, on the other. The variation of the saturation current with the resistance gives an approximation to its variation with the temperature of the filament. It was thought that a better estimate might be obtained if the temperature of the incandescent filament were determined by means of a thermocouple. With this object a second form of tube was set up, which we shall proceed to describe.

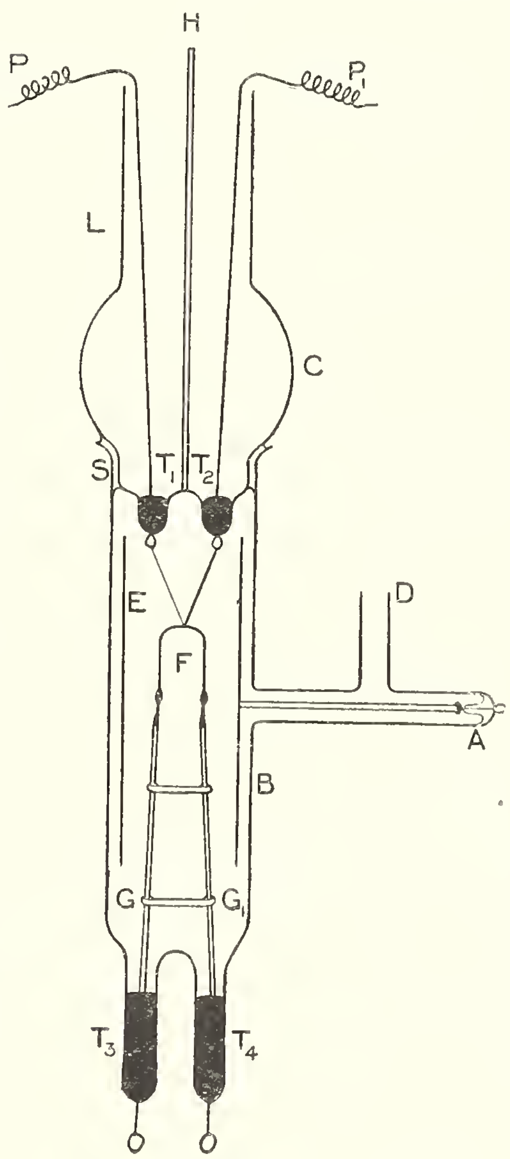

Fig. 7.

In this case the filament F, together with the platinum wires which support it (see fig. 7), was cut out of the original lamp and fastened to two stout copper wires G and G_1. This was done by placing the platinum and copper wires alongside, wrapping them round closely with fine copper wire and then soldering the whole. The filament was thus supported on two long copper legs; the rigidity of the structure was ensured by melting on two cross-pieces of blue glass in the positions shown in the figure. The ends of the copper terminals rested in the small tubes, T_3 and T_4, which contained mercury, and which were fused in to the end of the large tube B. The current which heated the filament entered by platinum wires, which were melted into the tubes T_3 and T_4. The electrode, to which the current was measured, was a long narrow aluminium cylinder E, which practically surrounded the hot filament. The cylinder was supported by a stout wire let in through the side-tube A. The tube D was connected with the pump and McLeod gauge.

The thermocouple was of platinum and iridio-platinum, the wires being the finest obtainable. The pure platinum wire was ·0025 centim. in diameter, whilst the 10 per cent. iridium alloy had a diameter of ·0035 centim. The wires were tied together on to the filament by means of a slip-knot, so as to make good contact but not to increase the diameter of the filament materially. They were then suspended from platinum wires let in to the tubes T_1 and T_2, which were inserted in the tube LC in exactly the same way as T_3 and T_4 were fixed in to B. The wires P and P_1, which were prevented from touching by the cardboard partition H, connected the mercury cups T_1 and T_2 with the rest of the thermocouple circuit. The tubes B and C were connected by a sealing-wax joint S just as in the former apparatus.

In the first experiments with carbon the apparatus shown in fig. 2 was used, just as for platinum, except that the tube with the incandescent platinum wire (fig. 1) was replaced by the tube shown in fig. 6. In commencing an experiment the apparatus was exhausted to ·001 millim., so that the gas which was afterwards in the tube was all given off from the hot filament and the walls of the tube. The observations were generally taken so as to keep the tube as cool as possible, but by letting it get hot enough pressures up to a millimetre could be registered on the McLeod gauge, even with constant pumping. The first experiments were made to determine the way in which the leak varied with the applied electromotive force, other conditions being, so far as possible, kept constant.

§ 2. *Relation between the Current and the Applied E.M.F.*

In all cases there was no current which would show a deflection in the galvanometers used when the filament was charged positively. The positive leak from hot wires in a vacuum, though large when measured by an electrometer, is always negligible compared with the currents measured in these experiments. Some of the subjoined current E.M.F. curves were obtained by using the apparatus in fig. 6, others by using that in fig. 7. As we should expect the curves to vary considerably with the shape and position of the electrodes, the apparatus from which the curves were obtained will be definitely specified in each case. As an abbreviation for the "apparatus shown in fig. 6" we shall write "apparatus 6," and so on.

The relation between the current and the electromotive force depends largely on the pressure of the gas in the apparatus. It may also depend on the value of the maximum current which can be obtained, *i.e.*, on the temperature of the wire. At very low pressures (below, say, ·02 millim.) the current rises very rapidly with the E.M.F. till it reaches a certain value, after which it becomes practically independent of the E.M.F. This "saturation current" generally increased slightly with the electromotive force, the increase being attributable to the extra ions produced by collisions with the gas molecules. The following curve, given by apparatus 7, shows the phenomenon of saturation very clearly. The flow of the heating current was accompanied by a P.D. of 3·8 volts between the two ends of the filament, so that it is important to state which end of the filament the voltages refer to. There was found to be no current when the negative end of the filament was earthed, the whole of the filament being then positive to the surrounding earthed electrode. The values are the means of a considerable number of observations. This curve (fig. 8) is for a pressure of ·003 millim.; the voltage given is that of the negative end of the filament.

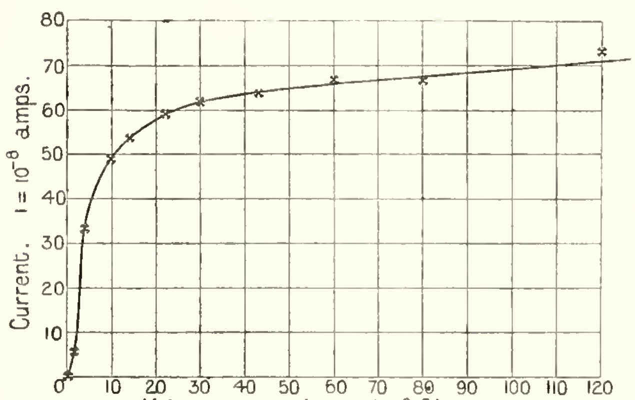

Fig. 8.

From the preceding curve it will be seen that the current was practically saturated by a potential of about 15 volts. With higher pressures of gas in the apparatus, the saturating potential might become much greater, as is shown by the following curve, taken, also with apparatus 7, at a pressure of ·02 millim. In this case the current is not saturated till a potential of about 280 volts is reached. During this experiment the temperature of the filament, as indicated by the deflection produced by the thermo-couple, was kept constant.

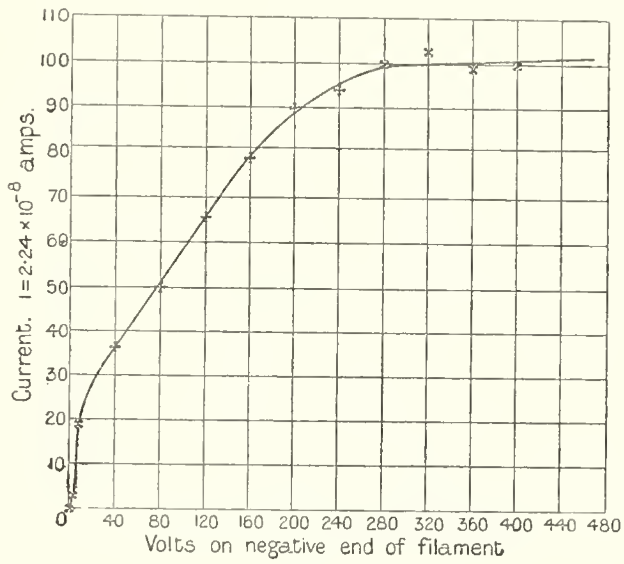

Fig. 9.

The bend in the curve at about 20 volts seems to indicate that a sort of saturation occurs here. The subsequent increase of current would then be explained by the ions produced by collisions as the electromotive force was increased. On this supposition, when we again reach the flat part of the curve at 280 volts we must suppose that every collision possible at this pressure produces ions. Similar considerations explain the gradual slope of the curve in fig. 8 after saturation. Owing to the peculiar shape of the electrodes it was not possible to calculate the magnitude of the effects.

With this curve it is interesting to compare one obtained at a slightly lower pressure (·013 millim.) with the other form of apparatus. In this case the current used to heat the filament was kept constant while its resistance decreased in the ratio of 1·025 to 1 during the observations. This does not imply that the temperature altered, since heating a carbon filament steadily decreases its resistance when cooled and measured again at the original temperature. The absolute value of the current is also some twenty times as great as that in the preceding curve. It will be seen that in this case saturation was reached with about 160 volts.

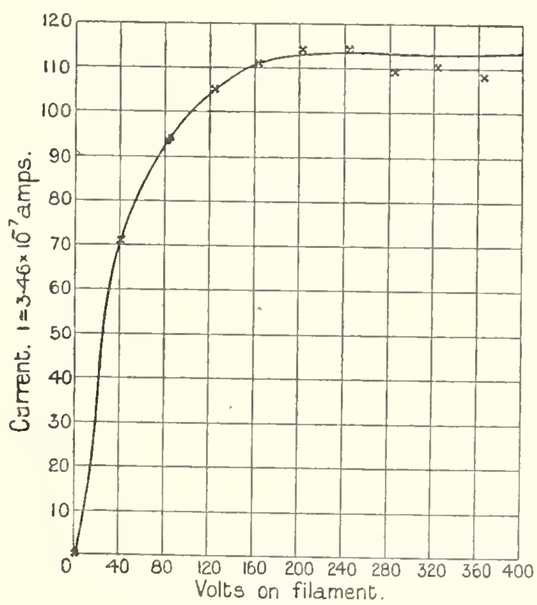

Fig. 10.

It is evident that in this case the bend at 20 volts does not appear. This may be due to the greater magnitude of the current and smaller pressure, which makes the Townsend effect less by comparison. With still higher pressures the current for low voltages is small and increases more rapidly as the voltage is raised than it would if it were proportional to the potential difference. The curve then passes through a singular point, the current increasing less and less rapidly with the voltage until the saturation value is reached. The current afterwards remains stationary for some time until it begins to increase again as the voltage is raised. This increase is doubtless due to new ions produced by the collisions of the negative ions, since it is in all respects similar to the effects described by Professor TOWNSEND.* All these characteristics are shown by the upper curve in fig. 11, which was obtained with apparatus 6. In this case the pressure increased from ·65 to ·85 millim. during the experiment; the heating current was maintained constant and equal to 1·07 ampères.

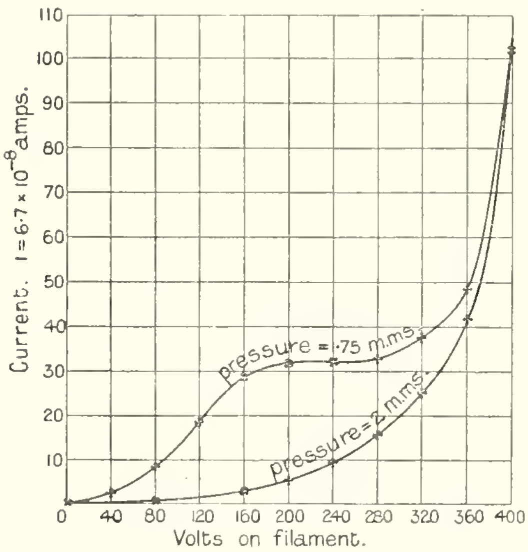

Fig. 11.

When the pressure of the gas is raised further, the current at low voltages becomes still smaller than before, the current E.M.F. curve being always concave to the axis of current. In this case the current never becomes saturated, owing to the collision effect coming in before the saturating potential is reached. Under these circumstances the current increases more and more rapidly with the potential as the latter is raised. These characteristics are very well shown by the lower curve in fig. 11, which was taken with apparatus 6 at a pressure of 2 millims. The filament was heated by a constant current.

It will be noticed that several of these curves are very similar to those obtained with hot platinum wires by Professor MCCLELLAND.†

In all these cases it was found that in retracing the observations backwards, the curves never quite coincided with those obtained first. These effects, which were of the nature of hysteresis, were partly attributable to change in the conditions while the observations were being made. Such changes were, for example, increase of pressure due to gas given off from the walls, change in the temperature of the carbon heated by a constant current owing to the permanent alteration of the resistance of a carbon filament produced by heating, &c. Even when such disturbances were eliminated as far as possible, the curve could never be made to return on its outward path. The form of the curve was always the same, but the value of the current on the return curve was invariably smaller than on the outward one; in a particular case, when the pressure increased from 2·4 to 2·6 millims., the current with about 200 volts was reduced to one-third its value on the return journey.

After having investigated in some detail the connection between the current and the electromotive force when the filament was maintained at a constant temperature, the connection between the saturation current and the other conditions was next examined. The experiments to be described are therefore concerned with the relation between the saturation current and

* TOWNSEND, 'Phil. Mag.,' Feb., 1902.

† 'Camb. Phil. Proc.,' vol. 11, p. 296.

(1) The resistance of the filaments ;
(2) The currents used to heat the filaments ; and
(3) The temperature of the carbon surface, respectively.

In what follows a section will be devoted to each of the above headings.

§ 3. *The Relation between the Saturation Current and the Resistance of the Filament.* This was investigated in the same manner and with the same apparatus (fig. 2) as in the case of platinum. In all cases the apparatus which has already been described and is shown in fig. 6 was employed. The thick German silver resistance R (fig. 2), which served as an intermediate standard, had now a resistance of 1·62 ohms. The smaller currents were measured with the sensitive Thomson galvanometer. For the larger currents a D'Arsonval galvanometer, which gave a deflection of 1 millim. for a current of $3{\cdot}46 \times 10^{-7}$ ampère, was employed, owing to its greater convenience. As the resistance of the filament decreased slightly during the observations, a reading was taken both before and after each observation of the leak, the mean of the two readings being taken as the value of the resistance which corresponded to the reading for the current. Resistance readings were taken over a range of saturation current extending from $2{\cdot}8 \times 10^{-7}$ to 6×10^{-3} ampère per sq. centimetre of surface. The corresponding range of the value of the ratio of the resistance of the filament to its resistance at 11° C. at the commencement of the experiment was from ·610 to ·567. In other words, while the resistance of the filament only alters in the ratio of 610 to 567, the negative leak has become twenty thousand times as big as it was at first. It is evident, therefore, that, as in the case of platinum, the number of negative ions produced at the surface increases with enormous rapidity as the temperature rises. It will be shown later that, over a much greater range of temperature than this, there is no perceptible falling off in the rate at which the current increases.

The corresponding values which were obtained for the saturation current and the resistance of the carbon filament are given in the following table. The resistances are expressed as fractions of the resistance which the filament possessed at 10° C. before it was heated.

Saturation current in ampères.	Resistance as a fraction of initial resistance.
$3{\cdot}9 \times 10^{-8}$	·609
$9{\cdot}43 \times 10^{-8}$	·604
$3{\cdot}25 \times 10^{-7}$	·600
$8{\cdot}55 \times 10^{-7}$	·594
$19{\cdot}04 \times 10^{-7}$	·588
$4{\cdot}32 \times 10^{-6}$	·581
$13{\cdot}3 \times 10^{-6}$	·571
$3{\cdot}53 \times 10^{-5}$	·560
$7{\cdot}80 \times 10^{-5}$	·547
$2{\cdot}47 \times 10^{-4}$	·528
$3{\cdot}95 \times 10^{-4}$	·509
$9{\cdot}05 \times 10^{-4}$	·48

These numbers when plotted against one another on squared paper yield curves very like the current temperature curves for platinum (fig. 4). They have not, however, been inserted, since they are much the same as the current-temperature curves for carbon (fig. 13, p. 526), which have been plotted from the same observations.

Just as in the former experiments, the current was never found to be a function of the electromotive force alone, so also here the cooling curves never exactly coincided with those obtained as the temperature of the filament was raised. This was partly due to the permanent change in the resistance of carbon produced by heating.* It was attempted to correct for this by taking, instead of the ratio of the resistance at moment to the original resistance at 11° C. before commencing the experiment, the ratio to the resistance which the filament would possess if at a temperature of 11° C. at that moment. The permanent change in the resistance was assumed to be proportional to the rate at which the resistance changed during an experiment, the conditions being kept, as far as possible, constant. In this way it was possible to obtain by extrapolation the resistance which the filament would possess if allowed to cool down to 11° C. at any stage during the experiments. That this process brings the two curves more nearly into coincidence will be seen at once on comparing the numbers in columns I., VI., and VII. of the following table. In this case the potential on the filament was − 204 volts, and the heating current was run at constant voltage. The results of these corrections are shown in the accompanying table :—

I.	II.	III.	IV.	V.	VI.	VII.	VIII.
Saturation current, $1 = 10^{-7}$ ampère.	Heating current, ampères.	Initial resistance proportional to	Amount resistance decreased during experiment.	Corrected zero resistance 11° C., proportional to	Ratio of resistance to corrected zero resistance.	Ratio of resistance to original zero resistance.	Pressure, millim.
1·7	·59	1755	0	2910	·604	·604	—
5·2	·64	1726	0	2910	·594	·594	·003
20·7	·69	1698	0	2907	·584	·584	—
46·5	·735	1682	1	2901	·580	·578	·008
118	·78	1664	2	2889	·576	·572	—
294	·83	1645	2	2877	·573	·566	·02
735	·89	1624	2	2865	·567	·559	—
310	·87	1637	2	2853	·574	·563	·035
145	·84	1646	3	2835	·581	·566	—
72	·81	1654	2	2823	·586	·568	·04
34·6	·79	1665	1	2817	·592	·572	—
17·3	·76	1675	1	2811	·596	·576	·035
8·3	·72	1687	1	2805	·602	·580	·03

* LE CHÂTELIER, 'Journal de Phys.,' ser. 3, vol. 1, p. 185.

In order to illustrate the magnitude of these changes, the numbers in columns I. and III. have been plotted in the accompanying curve (fig. 12).

The chief objects of these experiments on the relation between the negative current from and the resistance of, a carbon filament, was to determine the dependence of the former in the temperature. Le Châtelier,* using an optical method, has given numbers connecting the temperature of a carbon filament with its resistance. If we plot a curve from these numbers between the temperature of the filament and the ratio of its resistance at that temperature to its resistance at 15° C., we can use this to obtain the temperature of any other filament from its resistance. In this case we have again to face the uncertainty caused by the permanent change in the resistance of the filaments when heated. I have attempted to correct for this in the same manner as has been described above. Le Châtelier states that in his experiments there was a permanent lowering of the resistance of the filament amounting to about 10 per cent. This change has been distributed among the observations in such a way that the observations at the highest temperatures are responsible for the greater part of the alteration. The corrected curve thus coincides with the original one up to about 1000° C., after which it branches off, the divergence between the two becoming gradually greater, until finally at about 2000° it ends 10 per cent. higher than the one plotted from Le Châtelier's numbers.

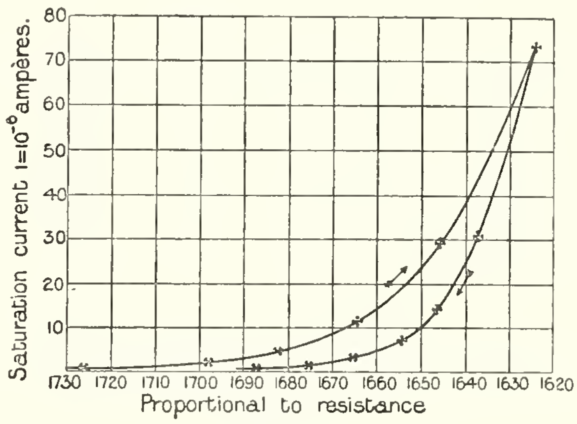

Fig. 12.

The numbers in the table on p. 523, when treated in this manner, yield the following :—

I.	II.	III.
·39	·610	1250
·943	·606	1265
3·25	·602	1285
8·55	·599	1305
19·04	·595	1325
43·2	·592	1345
133·4	·589	1365
353	·586	1380
780	·582	1400
2475	·577	1430
3950	·572	1460
9050	·567	1490

I. = Saturation current, unit being 10^{-7} ampère.

II. = Ratio of resistance to corrected resistance at 11° C.

III. = Temperature in degrees Centigrade.

* 'Journal de Phys.,' *loc. cit.*

The numbers in columns I. and III., when plotted against one another, yield the following curves (fig. 13) for the variation of the negative leak from carbon with the temperature. The various curves represent successively greater units of current as in the case of the curves for platinum (fig. 4).

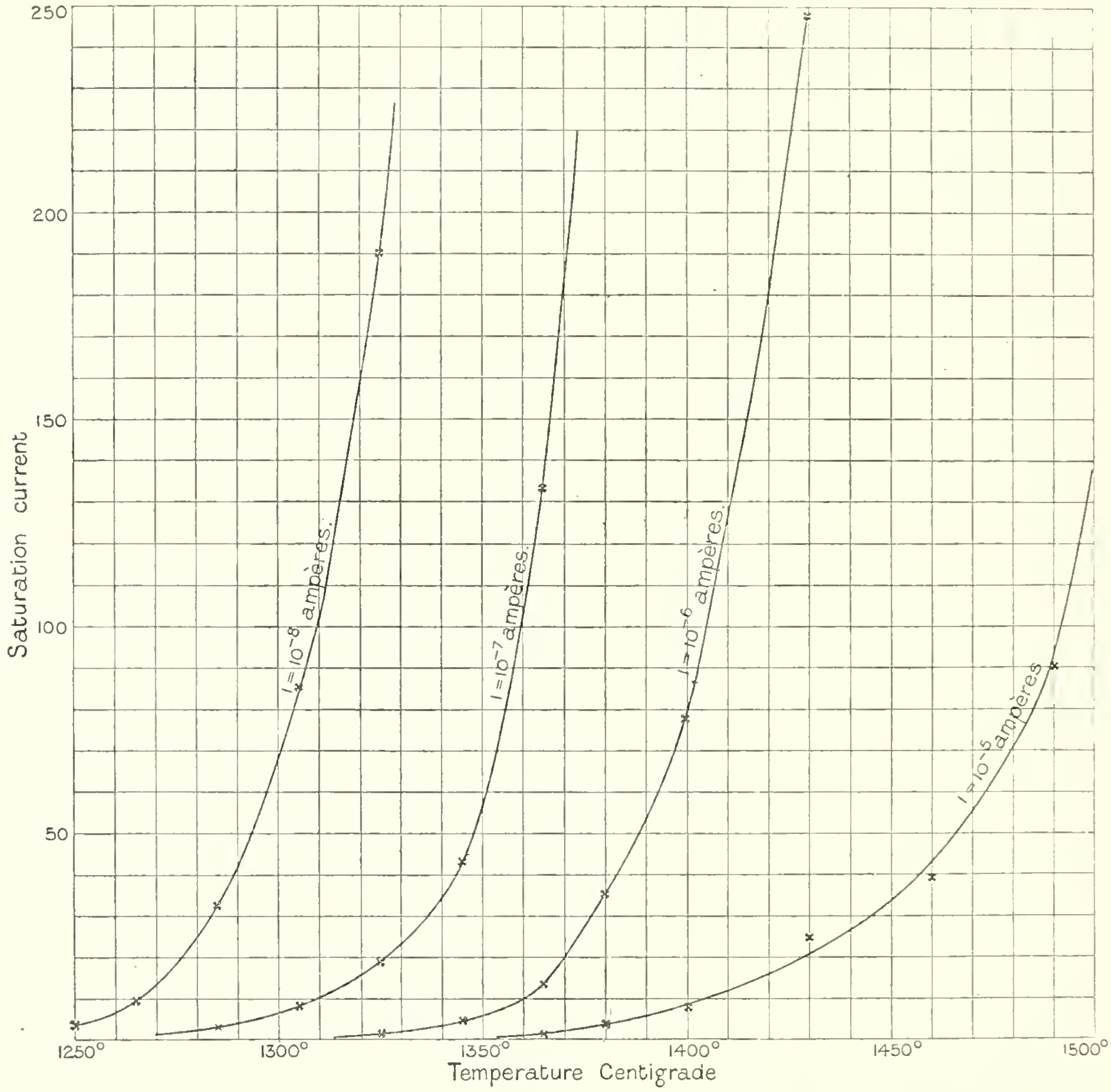

Fig. 13.

The further consideration of these results on the temperature variation of the negative leak from carbon will be postponed till § 5.

§ 4. *The Relation between the Negative Leak and the Current used to Heat the Filaments.*

A series of experiments was made in which the saturation current and the corresponding current required to heat the filament were measured; since it was thought that these measurements would be of especial interest at temperatures so high that they could not be determined, at any rate by the methods used in the present paper. In these experiments the portion of the apparatus used to measure the leak was unchanged, while the whole of the arrangement used to measure the resistance was removed. The only part of the heating circuit which remained was the battery of twelve storage cells and the adjustable resistance used to regulate the current. The magnitude of the latter was determined by means of a small vertical ammeter, reading up to 4 ampères, which was inserted in the circuit.

The first experiments were made with the apparatus shown in fig. 6, and were pushed to very high temperatures. In fact, THE MAXIMUM CURRENT FROM THE FILAMENT TO THE ALUMINIUM ELECTRODE REACHED THE ENORMOUS VALUE OF 1·5 AMPÈRE PER SQUARE CENTIMETRE OF CARBON SURFACE. These experiments were made with a lamp which possessed a small air leak that had been stopped by embedding in paraffin in the manner already described. When the greatest currents were put on the lamp became hot so that the paraffin melted and the pressure inside the apparatus rose to 1 millim. During the course of the experiments the pressure was therefore not constant, but increased gradually from ·006 millim. to 1 millim. The potential on the filament was − 250 volts, and was sufficient to saturate the current at all the pressures concerned. The results of these observations are shown graphically in the accompanying diagram (fig. 14). The values of the ordinates are successively multiplied by ten as we move to the left from one curve to the next.

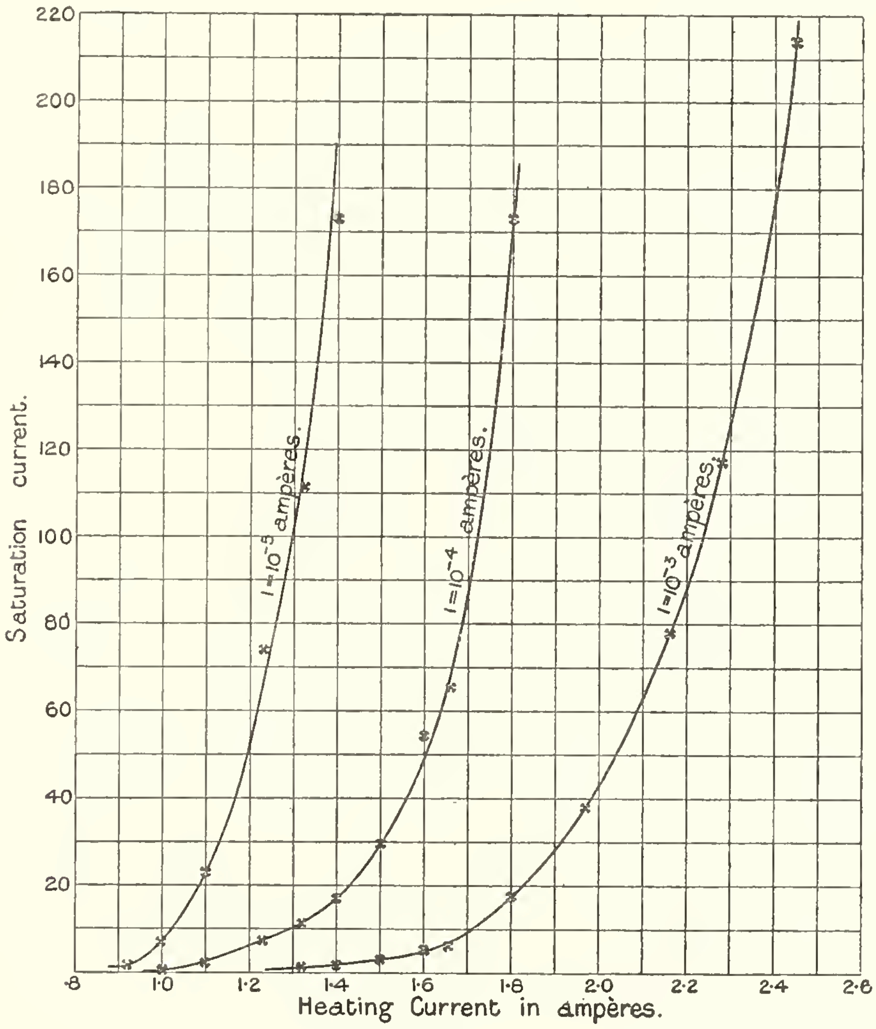

Fig. 14.

It will be noticed that there is a great similarity between fig. 14 and the curves connecting the saturation current with the resistance and temperature respectively. This is merely due to the rate of variation of the saturation current being so rapid that the differences in the alteration of resistance, temperature, and heating current becomes insignificant in comparison.

Since in the last experiments the big currents were always accompanied by a high pressure of gas in the apparatus, it might be thought that part of the increased current was due to the gas present. To investigate this point a series of experiments was made with the apparatus shown in fig. 7, so that the gas pressure could be kept down to a very low value. By heating the filament for a short time only and taking the observations very quickly, it was found that the temperature of the bulb and electrode could be prevented from rising perceptibly. Under these conditions it was found that the amount of gas given off was greatly diminished, the highest pressure recorded during the observations being ·006 millim. A reading of the McLeod gauge was taken between each reading of the galvanometer deflection for the saturation current. The potential on the filament was − 80 volts, this being more than enough to saturate the current (*cf.* fig. 8). The numbers which were obtained are given in the accompanying table :

Heating current, ampères.	Saturation current, $1 = \text{ampère} \times 10^{-8}$.	Pressure, millims. of Hg.
1·49	2·1	·002
1·59	10·5	·002
1·70	35	·0025
1·84	143	·0025
2·0	540	·003
2·26	$2·24 \times 10^3$	·003
2·43	$10·1 \times 10^3$	·005
2·68	42×10^3	·005
2·92	122×10^3	·006
3·45	760×10^3	·006
3·65	1640×10^3	·005

The greatest observed value of the saturation current is not given in the above table, since the corresponding reading of the ammeter was not taken. This enabled the reading to be taken much more quickly, so that the pressure only changed from ·0022 to ·0025 millim. The corresponding saturation current was ·04 ampère; in other words, a square centimetre of surface would have given a current of ·28 ampère across a vacuum at $\frac{1}{400}$th millim. pressure.

In the preceding series of experiments the highest possible value of the temperature had not been reached, so that further experiments were instituted to determine the maximum current which could be obtained from a square centimetre of a carbon filament when the temperature was pushed to the highest limit, *i.e.*, just before the filament melted. The arrangement of the apparatus was altered somewhat, the leak being measured by a Weston ammeter instead of a galvanometer as before; in other respects the arrangement was unchanged. The pressure was kept very low and a potential of about − 60 volts was maintained on the filament, the surrounding cylinder being earthed.

With this apparatus it was found possible to maintain an actual current of ·4 ampère (corresponding to 2 AMPÈRES per square centimetre of filament surface) at a pressure of less than $\frac{1}{600}$ millim. The current could not be made to surpass this value since the filament melted on raising it to a slightly higher temperature. The fact that such large currents can be produced at such low pressures has an important bearing on the theory of the mechanism by which the corpuscles are produced, which will be considered later.

§ 5. *The Relation between the Saturation Current and the Temperature.*

From the experiments on the variation of the saturation current with the resistance we have been able to give numbers which indicate, roughly at any rate, the way in which the former depends on the temperature. It was thought that a more reliable estimate might be obtained if the temperature of the filament were determined by means of a thermal junction of platinum and iridio-platinum. With this object the following experiments were made:—

The tube employed was that shown in fig. 7, and already described. The filament in this tube was in the form of a simple U and had the following linear dimensions: length = 1·2 centim., diameter = ·0376 centim., and total area of surface = ·142 sq. centim. For these experiments the apparatus shown in fig. 2 had to be altered, the portions below AFK_2 being entirely reconstituted. The apparatus used for measuring the saturation current was unchanged, the only alterations being made in the portion used to measure the temperature. The thermocouple circuit was completed by taking the lead P_1 (fig. 7) to the cold junction, which was placed in a test-tube immersed in water at 12° C.; the other wire from the cold junction passed through a resistance box to a D'Arsonval galvanometer, and thence through P (fig. 7) to the hot junction. The adjustable resistance R_1 (fig. 2) still served to regulate the current which was used to heat the filament.

In order to standardise the thermocouple the melting-point of potassium sulphate was again taken as the fixed point. A junction of the same wire as that used during the experiments was fixed on to a stout platinum wire, which was clamped horizontally in the hottest part of a Bunsen burner. The Bunsen was arranged to burn vigorously with a bright green inner cone and was carefully protected from draughts. Very small portions of the salt were then placed on the stout wire on the side of the flame opposite to the thermocouple, and matters were so arranged that when the salt just melted it was exactly the same distance from the edge of the flame on the one side as the thermocouple on the other. The reading of the galvanometer was then taken to correspond to the melting-point of the salt. The method was then tested by placing a small portion of salt on the thermocouple itself and observing when it began to melt. This was found to agree with the previous observations. The greatest difference between the observations taken was less than 3 per cent. A further test was supplied by determining the melting-point of sodium sulphate; the value found was within 20° of that given by Messrs. HEYCOCK and NEVILLE.* This agreement was considered to be quite good enough for the purpose in hand. The thermocouple was finally found to give an electromotive force of 17·7 millivolts when its junctions were at 1067° C. and 12° C. respectively.

The platinum temperatures given by the galvanometer readings have been corrected to the air thermometer scale by means of the curves given by Professor CALLENDAR.†

The thermocouple method possesses one great advantage over the resistance method of determining the temperature of hot wires, in that the observations can be taken much more quickly, and so the wire has to be heated for a much shorter time. In this way the apparatus need never get hot, and far less gas is given off, so that the readings generally are much steadier.

The accompanying table represents a series of observations with this apparatus. The pressure was always less than $\frac{1}{500}$th of a millimetre of mercury, while the potential on the wire was − 44 volts, this being more than enough to saturate the current. The platinum temperatures are given under the column headed Pt, the numbers under t are the temperatures (degrees Centigrade) reduced to the air thermometer scale.

Scale-divisions of thermocouple.	Pt.	t.	Leak, $1 = 10^{-8}$ ampère.	Pressure.
				millims.
108·5	1122	1110	3·7	·001
110·8	1145	1129	8·2	—
112·8	1165	1145	25	—
114·8	1186	1162	39	·001
117	1209	1180	78	—
119	1229	1197	167	·0015
120·6	1245	1209	295	—
121·5	1254	1216	662	—
119·2	1231	1199	266	·0015
117·3	1212	1183	110	—
116·3	1202	1173	79	—
113	1168	1148	37	—
110·7	1144	1128	16·5	—
109	1127	1115	7·5	—
107	1107	1097	3·7	—
104·1	1077	1075	1·5	·0016

It will be noticed here again that the current for a given temperature is smaller as the temperature is being increased than when it is falling.

* 'Chem. Soc. Journal,' vol. 67, p. 160.

† 'Phil. Mag.,' vol. 48, p. 519.

The following table represents a series of observations taken with the thermocouple apparatus at somewhat higher temperatures. The potential on the filament was here $= -87$ volts.

Reading for thermocouple.	Pt.	t.	Saturation current $1 = 10^{-8}$ ampère.	Pressure in millims.
114·6	1184	1160	9	·003
117	1209	1180	60	·005
123	1270	1227	346	·007
131·3	1354	1290	2700	·007
138·5	1428	1359	22000	·007

The temperatures given by the thermocouple method are on the average about 120° lower than those obtained from the resistance for the same current. The temperature registered by the couple would be lower than that of the more remote parts of the filament for several reasons, the chief one being the conduction of heat away locally by the leads of the thermocouple itself. It is difficult to say whether we should expect this difference to amount to 120° C.

We are now in a position to test whether the experimental results are in agreement with the theoretical formula for the saturation current, viz. :—

$$C/\epsilon S = n\sqrt{\frac{R}{2m\pi}}\,\theta^{\frac{1}{2}}e^{-\Phi/R\theta} = A\theta^{\frac{1}{2}}e^{-b/\theta},$$

using the notation employed before. If we take, as in the case of platinum, $y = \log_{10}C - \frac{1}{2}\log_{10}\theta$ and $x_0 = \theta^{-1}$, the above equation reduces, as before, to the straight line

$$y = a - b_0 x_0.$$

The following curve (fig. 15) has been plotted in this manner from the numbers

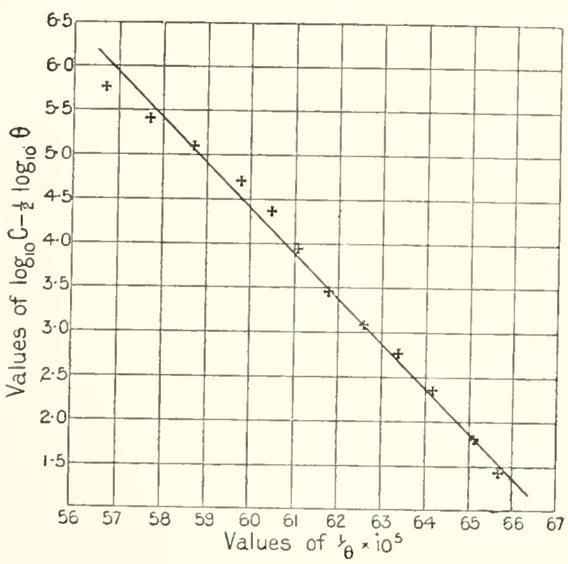

Fig. 15.

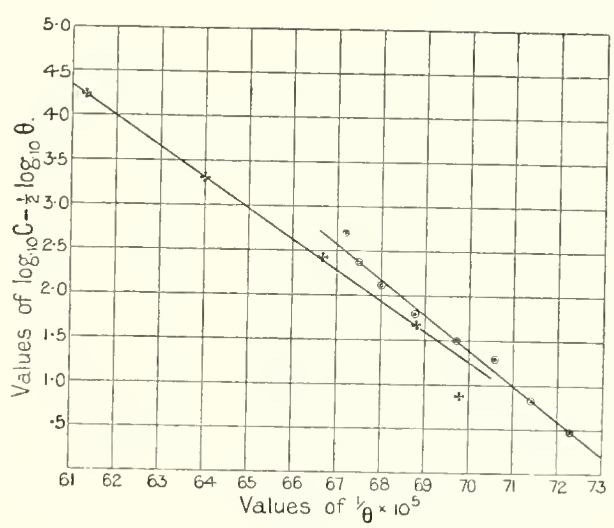

Fig. 16.

given in the table on p. 525. The ordinates are values of $\log_{10}C - \frac{1}{2}\log_{10}\theta$, while the abscissæ are values of $\theta^{-1} \times 10^5$.

The numbers on pp. 530 and 531 yield the curves in fig. 16. The straight line on the right is drawn from the observations on p. 530, and corresponds to lower temperatures than the other. The experimental points for this curve are denoted by ⊙. The other curve from the observations on p. 531 refers to somewhat higher temperatures. The experimental points for this curve are indicated thus : ×.

All these three curves are fairly close approximations to a straight line ; it is therefore quite evident that the observations are represented very closely by assigning to the saturation current a formula of the type $C/\epsilon S = A\theta^{\frac{1}{2}} e^{-b/\theta}$.

When we come to the actual values of the constants in the above formula, the agreement with the simple theory is not so good as in the case of platinum, though possibly this is partly due to the greater difficulty of the experiments. The curves in figs. 15 and 16 give for the value of b

$$11{\cdot}9 \times 10^4,\ 9{\cdot}7 \times 10^4, \text{ and } 7{\cdot}8 \times 10^4 \text{ respectively.}$$

In order that the differences of the values of b should be proportional to the contact E.M.F. between carbon, platinum, and other metals, b for carbon should be $5{\cdot}2 \times 10^4$, since its value for platinum is $4{\cdot}93 \times 10^4$. The difference between this and the above numbers does not appear to be very great, but the effect of a small error in b is enormous when we come to calculate from it the value of n, the number of corpuscles in a cub. centim. of carbon.

If we take $7{\cdot}8 \times 10^4$ as the best value of b, and $C = 2180$ at 1515° absolute as being the mean of the two series of temperature measurements, we find A is of the order 10^{34} and n is of the order 10^{29}. Now, Mr. PATTERSON* finds that at ordinary temperatures $n = 10^{19}$. The effect of temperature on the resistance of carbon indicates that the concentration of the corpuscles would be at least ten times as great at 1000° as at 0° C., so that we should expect to find n of the order 10^{20}. As a matter of fact, if we take $b = 5{\cdot}2 \times 10^4$ instead of $7{\cdot}8 \times 10^4$, we find $n = 5 \times 10^{21}$ instead of 10^{29}.

Reasons which might make this method of determining n give values which are too large, will be considered at some length after the experiments on sodium have been described.

III. *Experiments with Sodium.*

§ 1. *Nature of Problem.*

Sodium was selected as the next metal to be investigated on account of its strong electropositive character. Since this implies a great attraction for positive electricity, we should expect its power of retaining the negative corpuscles to be much smaller than that of the conductors hitherto examined. If the foregoing theory is correct the corpuscles ought to escape from the alkali metals at a much lower temperature than

* 'Phil. Mag.,' 6, III., 655.

from metals which are low down in the volta series. In fact, assuming (1) that the difference in the discontinuity of potential at a platinum vacuum and sodium vacuum surface is equal to the contact difference of potential for sodium and platinum (taken roughly to be equal to two volts), (2) that the value of the discontinuity (4·1 volts) previously obtained for platinum is correct, (3) that the concentration of the corpuscles for sodium is of the same order of magnitude as for copper, and (4) the correctness of the present theory, a preliminary calculation showed that currents of the order of some 10^{-6} ampère per square centimetre ought to be obtained at as low a temperature as 500° C.

The problem we have to face in the case of sodium is not quite the same as in the case of non-volatile substances such as carbon and platinum. For in this case the metal has an appreciable vapour pressure at the temperature at which the experiments are carried out, and part of the conductivity present is doubtless due to the spontaneous ionisation of the metal vapour. A second inconvenience, which is more of a practical nature, is caused by the distillation of the metal from the hotter to the colder parts of the tube, causing the state of the latter to continually vary. For the same reason some of the sodium condenses on the electrode which is supposed to be free from it, so that both electrodes emit negative ions.

We have seen that in the case of platinum and carbon no current was obtained when the hot conductor was positively charged; in other words, the conductivity was perfectly unipolar. In the case of sodium, owing to the spontaneous ionisation of the vapour and the condensation of the metal on the inserted electrode, we should expect to get a current in both directions. In the following experiments the first effect must have been small, owing to the low vapour-pressure at the temperatures employed, while the effect of the second was made small by using an electrode with a very small superficial area (a thin platinum wire). It will be seen that in every case the current when the sodium surface was negative was more than twenty times its value when the surface was positive.

The apparatus which was used to detect and measure the negative leak from sodium will now be described.

§ 2. *Description of Apparatus.*

After a great number of trials of various forms of glass apparatus, all of which came to an untimely end owing to the joints not being able to stand the continued heating or otherwise, the metal apparatus shown diagrammatically in fig. 17 was set up. The weldless steel tube ABDC was 76 centims. long and 3·2 centims. in diameter, and was kept at zero potential by means of the earth wire shown. The straight platinum wire A_1B_1 was insulated with sealing-wax at each end and could be charged positively or negatively to any desired potential. It formed the electrode mentioned above. The whole tube was placed in a small combustion furnace, by means of which the central portions could be heated to any desired temperature. The temperature was determined by means of a copper-nickel thermocouple, C_1D_1, attached to a hollow semicircular cylinder of brass, E, placed at the middle point of the line CD. The brass piece E was cut from a tube which before the operation fitted easily

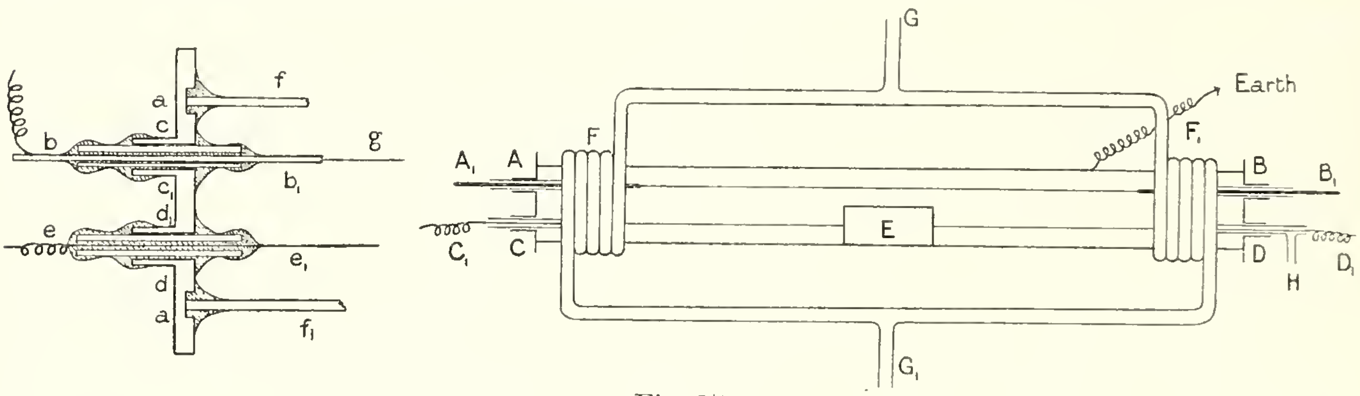

Fig. 17.

into ABCD, so that when the copper and nickel wires were tied round it, it fitted quite tightly. The spirals FF_1 were made of composition tubing wound tight round ABDC; they served to keep the ends of the tube cool and thus prevent the sealing-wax joints from softening. They were fed with cold water at G_1 and emptied at G. The side-tube H from DD_1 led to the pump and McLeod gauge.

The manner in which the wires A_1B_1 and C_1D_1 were fixed in at the ends is shown more clearly in the enlarged diagram on the left of the figure. The shaded parts represent the distribution of the sealing-wax which was used to make the joints. The ends ff_1 of the tube ABDC fitted into an annular depression on the brass plate aa. The platinum wire A_1B_1 was soldered at each end on to a stout copper wire, bb_1, which fitted fairly tight in a glass tube passing through the brass tube cc_1. The whole was fixed in air-tight by means of sealing-wax. One of the leads from the thermocouple was fixed in exactly the same way into the brass tube dd_1.

The sodium was originally placed in the form of small cubes on and around E, and it was considered that after heating for a short time in a vacuum a fairly uniform distribution of sodium over the central portions of the steel tube would be obtained. This was certainly what happened in the case of the glass apparatus which had been tried previously and in which the effect could be observed. The leak from the hot sodium to the platinum wire electrode A_1B_1 was then measured, according to its magnitude, either by an electrometer or by a galvanometer. At the lower temperatures where the quadrant electrometer was employed, one of the quadrants was connected to the case of the instrument which was insulated. The other quadrant was connected with a standard condenser and, by means of a wire passing axially on sealing-wax supports along a brass tube, with the electrode A_1B_1. The outside of the brass shielding-cylinder was connected with the case of the electrometer. In making an experiment the whole of the electrometer system was charged to a given potential, and the time required for the spot to move over a given number of scale divisions was noted. This measured the current from the insulated electrode

A_1B_1 to the surrounding earthed tube ABDC. By altering the capacity of the condenser a suitable rate of movement of the spot of the electrometer could be obtained each time.

In using the galvanometer the arrangement was practically the same as that employed before. One end of the battery was put to earth while the other was connected through the galvanometer to A_1B_1. The battery was capable of supplying any number of volts up to 420, the potential being measured by the Weston voltmeter used previously. A D'Arsonval galvanometer giving 1 millim. deflection for 2×10^{-8} ampère, and having a resistance of 500 ohms, was used.

With the exception of the change in the materials of the couple the thermoelectric circuit was exactly the same as that employed in the experiments on carbon. To reduce the galvanometer readings to temperatures use was made of the recent observations of Mr. E. P. HARRISON.* Only one fixed point was determined, viz., that of the boiling-point of sulphur. The electromotive force at that temperature was found to correspond to 22·7 microvolts per degree over the whole range, a result which agrees very accurately with that given by Mr. HARRISON. The relation between electromotive force and temperature was not assumed to be linear, but corresponding values for intermediate points were calculated from Mr. HARRISON'S curves. From these figures a curve was plotted which gave temperatures in terms of galvanometer readings directly. The galvanometer employed gave 1 millim. deflection for $1·39 \times 10^{-7}$ ampère, and the total resistance of galvanometer and thermoelectric circuit with no resistance out of the box was 19·7 ohms.

In the various forms of glass apparatus previously tried it was found that considerable currents were obtained at ordinary temperatures when the sodium was charged negatively. This was ultimately found to be due to the photoelectric effect produced by the light present in the room, since it disappeared when the experiments were made in the dark. The steel tube finally used in the experiments had the great advantage that it could easily be made absolutely light-tight. In order to make sure that no light reached the sodium, the glass tubes through which the wires were let in at each end of the steel tube were painted over with black enamel. The leak was then tested and found to be small and the same in both directions; so that it was all due to imperfections in the insulation.

§ 3. *The Relation between the Current and the Electromotive Force.*

After testing the insulation and pumping down the apparatus experiments were first made to see how the current varied with the direction of the electromotive force. The first measurements showed that, at a temperature of about 300° C., the current when the wire was at a potential of + 40 volts was 3500 times its value when the wire was charged to − 40 volts. The value of the current when the wire was positive was $1·5 \times 10^{-6}$ ampère. Later experiments showed, however, that the

* 'Phil. Mag.,' (6), vol. 3, p. 177.

positive current was invariably about 30 times as big as the negative. This was probably due to sodium having condensed on the wire electrode. For, although the wire had a surface per unit-length of less than one-hundredth that of the steel tube, the surface ionisation would be far easier to saturate; so that we should expect the currents in the two directions under a given voltage to have a ratio considerably less than 100 to 1. The above high value of the ratio obtained initially would correspond to the stage when no sodium had condensed on the wire.

In making these experiments the apparatus was first pumped down to a pressure of about ·1 millim., but it was found that on heating the steel tube a considerable amount of gas was given off. At first the amount of gas evolved was so great that the pressure in the apparatus, the volume of which was very considerable, rose to several centimetres of mercury. This evolution of gas was noticed in every case when sodium was heated, but by continued heating it usually became very small. In this particular instance, even after heating for several days, on pumping the apparatus out and heating again it was found that the pressure rapidly rose to about 5 millims. It was thought that the gases from the furnace might perhaps diffuse through the steel tube. To prevent this the latter was covered with a layer of soluble glass, which was carefully dried on; this seemed to have the desired effect, for it was found that afterwards there was no difficulty in keeping the pressure below a millimetre even when the tube was heated to 450° C.

Experiments were next made to investigate the way in which the current varied with the potential when the wire A_1B_1 was charged positively. It was found that the current E.M.F. curves were markedly different from those previously obtained with carbon and platinum. The current was small at first and increased much more rapidly with the voltage than if the two were proportional. In fact, the general shape of the current E.M.F. curves was much like that of the curves for current and temperature obtained with carbon and platinum. There was no indication of saturation at any potential.

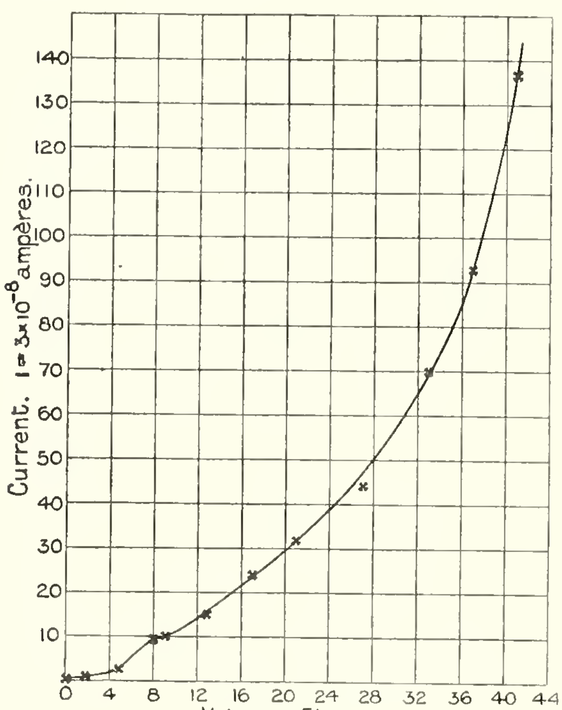

Fig. 18.

These differences are to be attributed to the difference in the experimental conditions and especially in the shape of the electrodes. In the case of sodium we have a large ionisation produced at the inner surface of a wide tube, and it is a well known fact that it is difficult to saturate the current to a wire inside the tube in such a case, owing to the weakness of the electric field near the surface.

The accompanying curve (fig. 18) gives the relation between the current and the electromotive force for voltages in the wire electrode between 0 and + 40. The pressure was about 5 millims. The sudden increase in the current between 4 and 8 volts was obtained every time and did not seem to be due to experimental error.

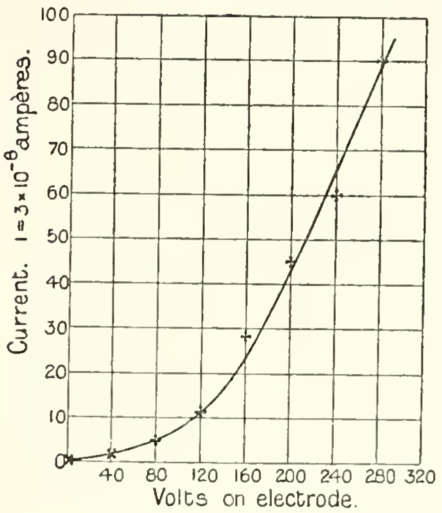

Fig. 19.

The current E.M.F. curve from 40 to 240 volts is very similar to that in fig. 18, except that it approximates very closely to a straight line between 160 and 240 volts. It is given in fig. 19.

When the voltage was increased above 240 it was found that the current rose rapidly to several thousand times its previous value. The increased current was quite steady at 320 volts, but at 280 volts it seemed to be in a very unstable state, since all kinds of intermediate readings could be obtained. Above 320 volts the current increased in a linear manner with the voltage. The experimental numbers are given below.

Volts on wire	200	240	280	280	320	360	400
Currents. Ampères × 10^{-7} . . .	3·6	5·7	93·6	155·4	7750	14350	20250

These numbers seem to indicate that with potentials greater than 240 volts an ordinary vacuum discharge took place at some point or points in the tube; in the following experiments care was therefore taken never to use potentials greater than 80 volts

§ 4. *Relation between the Current under a given Voltage and the Temperature.*

In the case of sodium, owing to the fact that the current could not be saturated, its value *under a given electromotive force* was measured at different temperatures.

This comes to practically the same thing as measuring the saturation current, since we should expect, *ceteris paribus*, the current with a given electromotive force always to be proportional to the number of ions liberated at the metal surface. In order to be sure of not getting a discharge, a potential of about 80 volts between the wire A_1B_1, and the cylinder was always employed. The following table represents a series of observations of current and temperature ranging from 217° C. to 427° C. It will be seen that the corresponding range of current is from 10^{-9} to 10^{-2} ampère; in other words, raising the temperature of the metal from 217° to 427° increases the current to ten million times its original value. The currents below 10^{-7} ampère were measured with the electrometer. In this series of experiments very low values of the currents were not measured; in a later series the current was taken nearly down to 10^{-12} ampère. In that case it was found that the leak increased less rapidly with the temperature below than above about 180°, so that presumably the ionisation present below 180° is not due to the emission of corpuscles from the metal surface. In the series of experiments which gave the numbers in the following table the pressure of the gas was about 1·5 millims.

Reading of thermocouple.	Temperature, centigrade.	Current, ampères.	Reading of thermocouple.	Temperature, centigrade.	Current, ampères.
170	427°	$1 \cdot 39 \times 10^{-2}$	139	334°	$5 \cdot 14 \times 10^{-6}$
165	410	$8 \cdot 34 \times 10^{-3}$	136	327	$3 \cdot 06 \times 10^{-6}$
160	393	$8 \cdot 34 \times 10^{-4}$	132	317	$1 \cdot 53 \times 10^{-6}$
158	387	$3 \cdot 61 \times 10^{-4}$	129	310	$5 \cdot 56 \times 10^{-7}$
156	381	$1 \cdot 65 \times 10^{-4}$	123	296	$1 \cdot 39 \times 10^{-7}$
152	370	$6 \cdot 34 \times 10^{-5}$	122	284	$6 \cdot 25 \times 10^{-8}$
145	350	$1 \cdot 39 \times 10^{-5}$	102	248	$9 \cdot 72 \times 10^{-9}$
143	345	$1 \cdot 11 \times 10^{-5}$	89	217	$1 \cdot 8 \times 10^{-9}$
140	337	$7 \cdot 0 \times 10^{-6}$	—	—	—

Corresponding values of current and temperature have been plotted on the accompanying curve (fig. 20) in order to facilitate comparison with the results obtained for carbon and platinum. It will be seen that the general appearance of the curves is much the same as before. The unit of current is successively multiplied by ten as we pass to the right from one curve to the next.

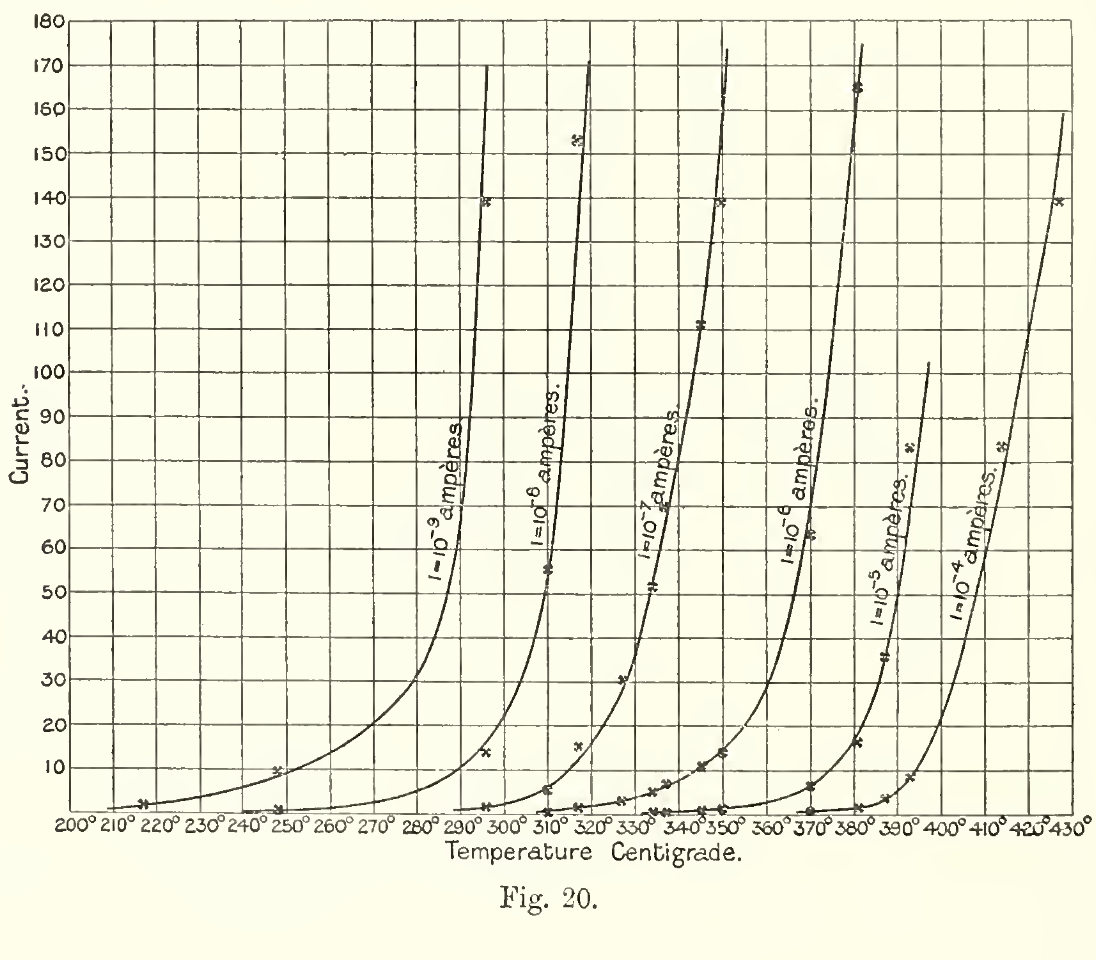

Fig. 20.

We now come to the application of the theoretical formula

$$C/\epsilon S = n\sqrt{\frac{R}{2m\pi}}\,\theta^{\frac{1}{2}}\,e^{-\Phi/R\theta}$$

to the reduction of these results. This equation may be written as before

$$y = \log_{10}C - \tfrac{1}{2}\log_{10}\theta = a + b_0x_0,$$

where $x_0 = \theta_1^{-1}$, θ being the absolute temperature. To test the theory, values of y have been plotted against values of $\theta^{-1} \times 10^4$ in the following curve (fig. 21).

It will be seen that all the points fall very nearly on a straight line except the first two. They are all, however, fairly accurately represented by the dotted curve shown. As the two lowest points correspond to a low temperature, it is possible that some other effect is coming in here which would account for their deviation from rectilinearity.

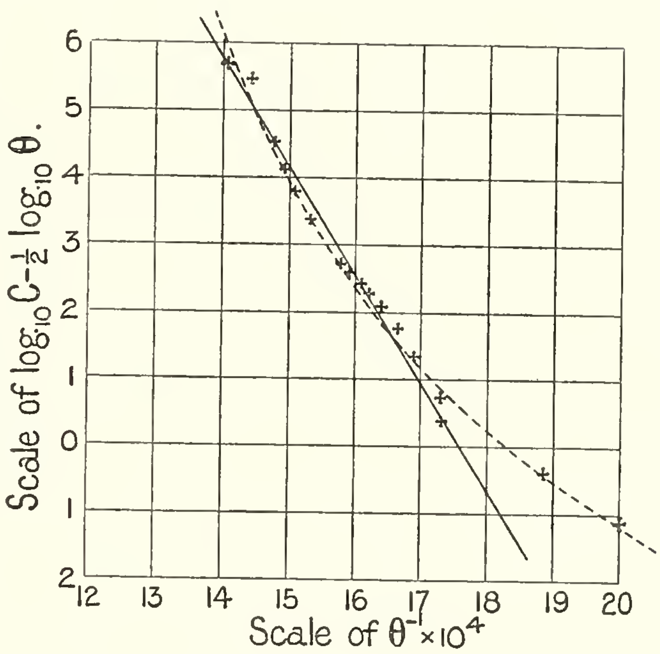

Fig. 21.

In calculating the value of b $(= \Phi/R)$ we may either confine our attention to the straight part of curve 4, and neglect the two first observations, or we may take the average over the whole range of the experiments. The two values differ by about 24 per cent.; if we take the mean we find $b = 3{\cdot}16 \times 10^4$. This gives, for the discontinuity of potential at the surface, the value 2·63 volts, and would therefore give 1·47 volt as the difference of its values for sodium and platinum.

The above value of the difference, which is approximately equal to the contact electromotive force for sodium and platinum, forms a strong confirmation of the theory; but when we come to calculate n, the number of free corpuscles in a cubic centimetre of sodium, the agreement is not so good. In fact, we find from the experimental results that for $\theta = 628$, $\log_{10} C - \frac{1}{2}\log_{10}\theta = 2{\cdot}615$; whence, putting in an estimated value of the area of the sodium surface, we get $n = 10^{26}$ about. The value of n has not been determined for sodium by any other method, but we should expect it to be not greatly different from that for copper, which is given by Mr. Patterson as 3×10^{22}. The value given by this method is thus far too great, for it is hard to imagine that the corpuscles can have a pressure of ten million atmospheres. I believe the discrepancy here is greater than can be explained by errors of experiment, although that was possible in the case of the high values found for carbon.

A further series of current measurements was made, using the quadrant electrometer, in order to investigate the leak at somewhat lower temperatures. It was also thought desirable to measure the currents under a given voltage in each of the two possible directions and to see if there was any relation between them. The experiments also served to test whether the current temperature curve, obtained when the tube was cooling, followed the same path as that obtained with rising temperature. It was scarcely to be expected that the two curves would coincide, even approximately, owing to the continual distillation of the sodium from the hotter parts of the tube to the cooler.

The results of the experiments are given in the accompanying table. In making the observations readings were generally taken with the wire electrode alternately positive and negative, a potential of 84 volts being used. In taking each reading the capacity was adjusted so as to give a convenient rate of movement of the electrometer spot. It was attempted to take corresponding positive and negative readings at as near the same temperature as possible, the gas furnace being adjusted after each pair of readings had been taken. The pressure in the apparatus varied from ·25 to ·4 millim.

Volts on wire = +84.		Volts on wire = −84.		Volts on wire = +84.		Volts on wire = −84.	
Temperature Centigrade.	Current 1 = ampère $\times 10^{-12}$.	Temperature Centigrade.	Current 1 = ampère $\times 10^{-12}$.	Temperature Centigrade.	Current 1 = ampère $\times 10^{-12}$.	Temperature Centigrade.	Current 1 = ampère $\times 10^{-12}$.
°		°		°		°	
10	2·4	10	2·2	306	$3·24 \times 10^4$	—	—
92·5	23·4	92	6·65	325	$2·12 \times 10^5$	323	890
104	31·6	97	6·3	340	$3·66 \times 10^5$	338	$1·01 \times 10^4$
131	247	—	—	340	$11·4 \times 10^5$	340	$3·26 \times 10^4$
141	318	—	—	311	$1·96 \times 10^5$	314	$7·1 \times 10^3$
146	352	145	16·4	289	$3·66 \times 10^4$	290	228
183	1210	182	49·6	241	$2·7 \times 10^3$	242	14·6
202	$2·26 \times 10^3$	204	76	226	372	228	20·2
235	$5·12 \times 10^3$	237	161	195	27	197·5	4·4
270	$8·3 \times 10^3$	272	373	123·5	8·7	—	—
296	$3·3 \times 10^4$	296	733				

The meaning of these numbers is best expressed graphically. In fig. 22 the logarithm of the current has been plotted against the temperature. The unit of current is 10^{-12} ampère. Curves 1 and 2 were taken with the wire charged positively, 3 and 4 with the wire charged negatively. The observations for curves 1 and 3 were made simultaneously with the temperature of the tube rising, whereas the curves 2 and 4 correspond to the second set of observations with the temperature falling.

The various marks refer to observational points for the different curves as follows :—○ to No. 1, ∗ to No. 2, ⊙ to No. 3, and · to No. 4.

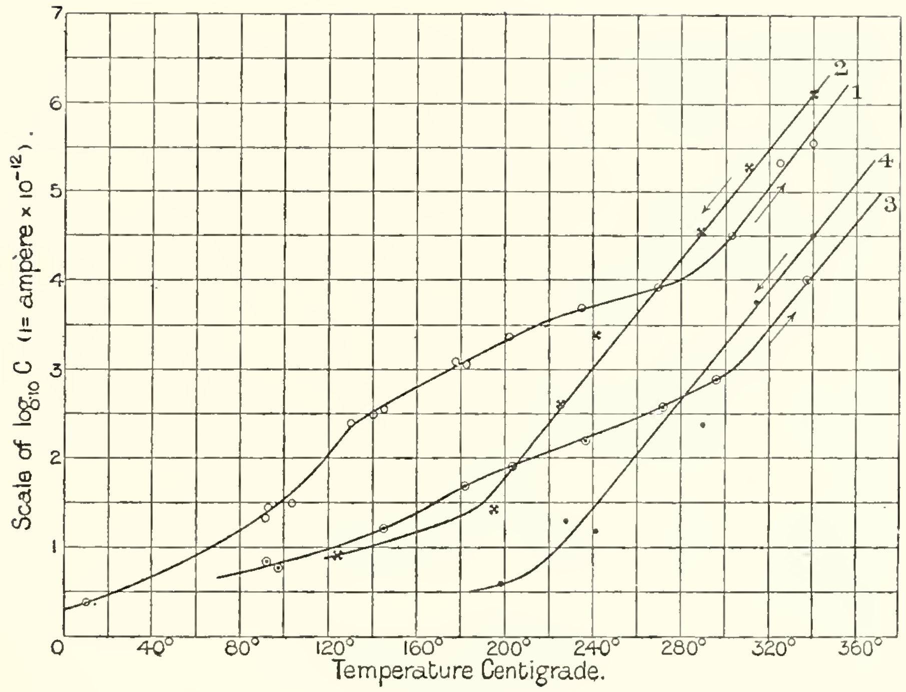

Fig. 22.

It will be noticed that the two curves belonging to any one set of observations become parallel above about 180°. The constant distance apart of the curves, which is approximately the same for the two sets of observations when measured along the vertical ordinate, has an average value of about 1·6. This shows that the ratio of the currents in opposite directions remains constant and independent of the temperature (above 180°) and is equal to about 40 to 1. The point of inflexion at about 280° on the up curves probably indicates an accidental change in the sodium surface, since it was not repeated on cooling.

C.—Conclusion.

§ 1. *The Determination of the Number of Ions in a Cubic Centimetre of Metal.*

The preceding results show that the number of negative ions produced by one square centimetre of surface of platinum, carbon, and sodium at temperature θ can be represented with fair accuracy by the formula $N = A\theta^{\frac{1}{2}}e^{-b/\theta} = A_1 n\theta^{\frac{1}{2}}e^{-b/\theta}$, where A and b are assumed to be definite constants for each metal. As an empirical result we find A and b have the values given in the following table :—

Conductor.	A.	b.
Platinum	10^{26}	$4 \cdot 93 \times 10^4$
Carbon	10^{34}	$7 \cdot 8 \times 10^4$
		$9 \cdot 7 \times 10^4$
		$11 \cdot 9 \times 10^4$
Sodium	10^{31}	$3 \cdot 16 \times 10^4$

The value of A is determined from that of b and so depends very largely on the value selected for b. An error of 10 per cent. in b multiplies A by about 100, whilst if b were determined wrongly to one part in three, A would be multiplied by 3×10^7. For this reason only the order of magnitude of A has been given in the table.

If we can assume that A and b are constants independent of the temperature, we obtain the value of n, the number of free corpuscles per cubic centimetre of the conductor, at once from the theory by dividing A by 10^5. Treating the values of A in this way, we find that the value of n for platinum agrees satisfactorily with that obtained by Mr. PATTERSON. On the other hand, the values (10^{29}) for carbon and (10^{26}) for sodium are greater than the maximum possible value. Moreover, the error in each case seems greater than can be accounted for by experimental uncertainties.

This error is probably due in part to the assumption that A and b are constants, whereas it is evident that they must both be functions of the temperature. It is possible on the preceding theory to say something about the forms of these functions which indicate that they both vary with the temperature.

With regard to the number n of corpuscles per cubic centimetre of metal, we suppose they are formed by decomposition of the neutral atoms in much the same way as in any case of chemical dissociation. If C_1 be the number of positive and C_2 of negative ions per cubic centimetre ($C_1 = C_2 = C$ as a rule), C_m being the number of undissociated atoms per cubic centimetre, and $C_0 (= C_m + C)$ being the value which C_m would possess if there were no dissociation, then :

$$C^2 \equiv C_1 C_2 = k C_m \equiv k (C_0 - C),$$

since the number of re-combinations per second is proportional to C_1C_2, whilst the number of dissociations is proportional to C_m, and these two must be equal in the steady state.

Now VAN 'T HOFF* has shown that for all re-actions of this type the quantity k varies with the temperature according to the equation

$$\frac{d}{d\theta} (\log k) = q/2\theta^2,$$

q being the heat evolved when two ions re-combine, whence

$$k = Ae^{-q/2\theta}.$$

A must be very large, for when $\theta = \infty$, $k = (A)$ must be great compared with C_0. We may write C in the form

$$C = \sqrt{(\tfrac{1}{4}k^2 + kC_0)} - \tfrac{1}{2}k,$$

whence we see that when $\theta = 0$, $k = 0$, and $C = 0$; when $\theta = \infty$, k is large compared with C_0 and $C = C_0$.

We see from the nature of the above function that the value of k would decrease

* 'Lectures on Phys. Chem.,' vol. 1, p. 141.

with enormous rapidity in the neighbourhood of the absolute zero; so that, although the resistance of metals decreases steadily with decreasing temperature down to the lowest temperatures yet reached, it is quite possible that it becomes infinite again at the absolute zero. The fact that the resistance of pure metals is proportional to the absolute temperature over a wide range, together with the high values of n which prevail at ordinary temperatures, seems to indicate that for most metals k has practically reached its maximum value, where it varies only slightly with θ.

For this reason we are led to the conclusion that the discrepancies of n are not due so much to disturbances produced by its temperature-variation (except, perhaps, in the case of carbon) as to the fact that the exponential coefficient b is a function of θ. We have seen that $b = \Phi/R$, where Φ is the work done by a corpuscle in escaping from the metal, and R is the gas constant for a single corpuscle. Now $R = \frac{5}{273} \times 10^{-14}$ roughly, and $b = 5 \times 10^{-4}$ for platinum, so that Φ is approximately equal to 10^{-11}.

A second approximation to the value of Φ is obtained when we consider the nature of the forces which retain the corpuscles inside the metal. These are a sort of integrated effect of the attractions of the positive and negative ions scattered about in the metal near the corpuscle. The field would thus be much the same as if the corpuscles were surrounded by a perfect spherical conductor of molecular dimensions. The quantity Φ is therefore of the same order as the energy required to remove a corpuscle from inside such a charged sphere, which is $\frac{1}{2}e^2/\zeta$, where e is the charge on an ion and ζ is the radius of an atom. Taking $\zeta = 2 \times 10^{-8}$ centim., this gives $\Phi = 9 \times 10^{-12}$.

If this view is correct—it hardly seems likely that the above numerical agreement is entirely a coincidence—we should expect the value of b to decrease as the temperature is raised owing to the greater distance of the atoms apart. We should therefore expect b to decrease in much the same way as the linear dimensions of the metal increase with the temperature. It is probable, therefore, that b can be represented with sufficient accuracy as a function of the temperature of the form $b = a_1 - a_2\theta$. Writing the equation at the beginning of this section in the form

$$\log C = \log A_1 + \tfrac{1}{2} \log \theta + \log n - b/\theta,$$

we see that the first three terms (with the possible exception of $\log n$, which we are not considering) vary extremely slowly with θ, if at all, so that we may use as an approximation

$$\log C = a_3 - b/\theta,$$

where $a_3 = \log A_1 + \frac{1}{2} \log \theta + \log n$. If now we put $b = a_1 - a_2\theta$, we see that $\log C = a_3 + a_2 - a_1/\theta$. So that, as we found in the experiments, $\log C$ is a linear function of $1/\theta$, but the constant A from which n is determined is much larger than it ought to be, owing to part of b having become added to it.

As a numerical example, we may give to b the value $5 \times 10^4 - 7\theta$, which corresponds to a temperature coefficient of ·00014 per degree, and would change the value of b by 20 per cent. in a range of 1400°. On calculating out we find that this small temperature coefficient would leave b practically unaltered, but would make the apparent value of n one thousand times its true value, whilst doubling the coefficient would square the error in n, and so on. It is therefore evident that the temperature variation of b is quite adequate to explain the large values of n which have been found. Moreover, owing to the peculiar nature of the functions, it is impossible to arrive at the true values of n by this method.

The value of A found in these experiments are therefore not irreconcileable with the values of n given by Mr. PATTERSON, but the two values of n can be made identical by assigning to b a small temperature coefficient. The coefficients necessary have been calculated, and, together with corresponding orders of magnitude of A and of n, are given in the following table.

Conductor.	Order of A.	Order of n.	Value of b with temperature coefficient to give value of n in last column.
Platinum	10^{26}	10^{21}	$4 \cdot 93 \times 10^4$
Carbon	10^{34}	10^{20}	$7 \cdot 8 \times 10^4 (1 - \cdot 00027\theta)$*
Sodium	10^{31}	10^{23}	$3 \cdot 16 \times 10^4 (1 - \cdot 00022\theta)$†

From the values of b we can calculate the work done by an ion in passing through the surface, and hence the discontinuity of potential between the metal and the surrounding space. For the case of platinum this has already been done, the value obtained being 4·1 volts. For carbon and sodium, taking into account the temperature coefficients given above, we find for the discontinuity at 15° C. the values 6·1 and 2·45 volts respectively. It will be noticed that these numbers follow the same order as the Volta series, though their differences (at any rate for carbon and platinum) are not equal to the corresponding contact electromotive force.

§ 2. *The work done by a Corpuscle in passing through the Surface Layer.*

It has been shown on p. 543 that the value of Φ is of the same order of magnitude as $\frac{1}{2}e^2/\zeta$, where e is the charge on an ion and ζ is the radius of a molecule; it is therefore also of the same order of magnitude as the energy set free when two ions ot opposite sign re-combine, and as the work required to produce an ion by collision. Theoretical considerations, in conjunction with the experimental results, render it probable that Φ may be represented very approximately as a linear function of the

* θ is the absolute temperature.

† It is noteworthy that this number ·00022 is practically equal to the coefficient of cubical expansion of sodium (·000204).

temperature, whilst the numbers given on p. 544 show that the temperature coefficient of Φ is of the same order of magnitude as the coefficient of linear expansion of the corresponding solid conductor.

These facts render it probable that Φ is a function of the size and distance apart of the molecules of which the conductor consists. If we consider ζ in the formula $\Phi = \frac{1}{2} e^2/\zeta$ as the distance apart of the centres of the molecules in the solid state, it will be proportional to the cube root of the atomic volume. We should therefore expect the work done by a corpuscle in passing through the surface layer of different metals to be approximately equal to a constant divided by the cube root of the atomic volume. Up to the present Φ has been determined only for sodium, platinum and carbon, but fortunately these three elements furnish a considerable range of atomic volume. As a matter of fact, carbon has the smallest atomic volume of all elements, whilst that of sodium is only exceeded by the alkali metals of greater atomic weight.

In the accompanying table values of the atomic volume and the inverse of its cube root are given in the first two columns. The third contains the surface discontinuity in the potential $\delta\phi$, which is proportional to Φ; whilst the numbers in the last column are the ratios of those in the second and third. In the case of carbon there is some doubt as to what the value of the atomic volume should be, since the density has different values for the different allotropic forms. Thus for charcoal the density is 1·9, for graphite 2·2, and for diamond 3·5.

Element.	At. vol.	(At. vol.)$^{-\frac{1}{3}}$.	$\delta\phi$ [volts].	(At. vol.)$^{-\frac{1}{3}}/\delta\phi$.
Sodium	23	·35	2·45	·14
Platinum	9·3	·476	4·1	·12
Carbon charcoal	6·3	·55	6·1	·09
„ diamond	3·46	·66	—	·11

It will be seen that the numbers in the last column are not quite constant; but they only change by about 40 per cent., whilst the atomic volume changes in the ratio of 6 to 1, and the atomic weight varies from 12 to 195. It seems therefore fair to conclude that the work done by a corpuscle in passing through the surface layer is, to a first approximation, inversely proportional to the cube root of the atomic volume of the element.

§ 3. *The Effect of Gas on the Current.*

The negative leak from a hot platinum wire surrounded by air at atmospheric pressure is always much smaller under a given voltage than at low pressures, when the wire is maintained at the same temperature in both cases. In one case, when the wire was giving a current of 3×10^{-6} ampère at a pressure of ·05 millim., air was let into the apparatus and the current again measured under atmospheric pressure at the same temperature. It was found that there was no detectable leak till a temperature of 60° higher was reached, when one division of the galvanometer scale (3×10^{-8} ampère) was obtained with the wire at a potential of − 200 volts. The small value of the currents at atmospheric pressure is probably due to the difficulty of saturating them.

It was thought conceivable that the ionisation at low pressures might be due to the gas molecules hitting the hot wire and becoming ionised thereby. If we assume that the maximum current would correspond to each molecule producing one ion, we can calculate its value in any given case. If we take the number of molecules in a cubic centimetre of gas at 0° C. and 760 millims. to be 2×10^{19}, then the number which hit unit area of the wire per second is $\frac{1}{6}u \times 2 \times 10^{19}$ approximately, where u (the square root of the mean velocity square) may be taken as 5×10^4 centims. per second for air. The number which strike unit area of the wire per second at 1 millim. pressure is therefore $2{\cdot}2 \times 10^{20}$, which gives a saturation current of $14{\cdot}3 \times 10^{10}$ electrostatic units, or 47·3 ampères per square centimetre. At a pressure ·0016 this current would become ·08 ampère per square centimetre. As a matter of fact, during the experiments, a current of 2·0 ampères per square centimetre was obtained at ·0016 millim. pressure. This is twenty-five times the maximum value obtained by supposing each molecule to produce one ion; so that it is highly improbable that any considerable part of the conductivity investigated is due to ions produced in this way.

Another way of considering this question is to calculate the number of times each molecule of air inside the cylindrical electrode must collide with the filament per second to produce the observed current, assuming that each collision sets free one corpuscle. In the experiments in question the cylinder had a volume of about 1 cub. centim., so that each molecule present would have to pass backwards and forwards between the filament and the cylinder some 10^5 times each second. This seems to be an impossible feat for an uncharged molecule.

Both these points of view lead to the conclusion that the corpuscles are not produced by a dynamical action between the molecules of the surrounding gas and the surface of the metal. In fact, all the experimental results seem to point to the view that the corpuscles are produced from the metal by a process similar to evaporation. The effect of the surrounding gas, of impurities in the wire, and of its previous history are to be regarded as due to alterations in the property of the metal which corresponds to latent heat in the theory of evaporation.

§ 4. *The Edison Effect.*

It will readily be seen that the results which have been obtained furnish a complete explanation of the phenomenon known as the Edison effect. EDISON first discovered this effect by connecting an insulated electrode, which was symmetrically placed between the ends of the filament of an incandescent lamp, through a galvanometer to the positive end of the filament. A current was then observed which amounted in some cases to several milliampères, although there was no current when the electrode was joined to the negative terminal. Evidently the current was carried by corpuscles passing from the negative portions of the hot carbon to the relatively positive electrode; and, on this view, we should expect the current to vanish by comparison when the electrode was negative with respect to the filament.

This observation was confirmed and extended by Professor FLEMING,* who showed, by using cylindrical electrodes which he placed round various parts of the filament, that the current only came from the negative end. He also found, in agreement with the results of the present paper, that a platinum filament likewise gave an effect. This was in the same direction as, but greater in magnitude than, that given by carbon. Finally, the Edison effect was found to increase rapidly with the temperature of the filament, which confirms its identification with the phenomena here investigated.

§ 5. *The Energy Emitted.*

It is of interest to compare the energy lost by a hot body owing to the emission of corpuscles with the energy given off in the form of electro-magnetic radiation. The recent measurements of F. KURLBAUM† show that the energy radiated in 1 second from 1 sq. centim. of the surface of an absolutely black body at 1° absolute is

$$S = 2{\cdot}12 \times 10^{-4} \frac{\text{erg}}{\text{centim. sec. deg.}^4},$$

whilst we have seen that the total rate of loss of energy of a conductor owing to the emission of corpuscles at temperature θ absolute is

$$E_\theta = n\,\{1 + \Phi/2R\theta\}\sqrt{\frac{2R^3\theta^3}{\pi m}}\,e^{-\Phi/R\theta}.$$

Since the quantities Φ and n in this formula have now been determined for carbon and platinum, we can calculate E at any temperature for these substances. The first term in brackets represents the part of the energy due to the motion of translation of the emitted corpuscles, and is less than 5 per cent. of the second term at all temperatures at which experiments have been carried out. We may therefore leave it out to a first approximation and calculate only the second term, which is equal to the work done by the corpuscles in passing through the surface layer. This is obviously equal to $N\Phi$

$$= C_0\delta\Phi,$$

* 'Phil. Mag.' [5], vol. 42, p. 52.

† 'Wied. Ann.,' vol. 65, p. 759.

where $\delta\Phi$ is the discontinuity in the potential, and C_0 is the saturation current per unit area.

Calculating in this way we find that for platinum at $\theta = 1900$ $E = 8 \times 10^4$ ergs/centim. sec. The value of S_{1900} is a much larger quantity, viz., $2 \cdot 75 \times 10^9$ ergs/centim. sec. The largest experimental values of E were obtained with carbon. Since the greatest value of the saturation current attained was $C_0 = 1 \cdot 5$ ampère $= 4 \cdot 5 \times 10^9$ electrostatic units per square centimetre and $\delta\Phi = 6$ volts, we have the rate at which energy is lost by the wire $= 9 \times 10^7$ ergs/centim. sec. The temperature corresponding to this current was not measured, but was certainly greater than 2000° absolute. The energy radiated from an absolutely black body at 2000° absolute would have been $3 \cdot 36 \times 10^9$ ergs/centim. sec.

We see then that at all the temperatures at which experiments were made the loss of energy due to the escape of the corpuscles is much less than that due to the emission of ordinary electromagnetic radiation ; on the other hand, it increases much more rapidly with the temperature, so that, in the case of carbon at any rate, it would become first equal to, and finally great compared with, the electromagnetic radiation, at temperatures not much above 2000° C. It must not be forgotten that for this calculation the hot conductor is supposed to be placed in a vacuum and surrounded by an electric field which removes the ions ; otherwise all the ions diffuse back to the metal and there is no loss of energy due to this cause.

In all these experiments we are a long way from the region where an appreciable fraction of the total number of ions which strike the surface of the conductor pass through. This is easily seen if we calculate the value of the saturation current per unit area on the supposition that every corpuscle which hits the surface escapes. Let us take $n = 10^{22}$ as a probable maximum for the number of corpuscles in a cubic centimetre of, say, carbon ; then, at 2730° absolute $\frac{1}{6}nu = 3 \times 10^{28}$, so that the saturation current would be 18×10^{18} electrostatic units or 6×10^9 ampères per square centimetre. As the largest current which has been yet obtained is $2 \cdot 0$ ampères per square centimetre, it is evident that we are still a long way from the limit. This calculation seems to indicate that the region on the current temperature diagram when the current begins to be proportional to the square root of the absolute temperature is much higher than any temperature which can be reached in the ordinary way.

The magnitude of the currents which have been obtained with low voltages indicate that a vacuum bounded by a hot conductor is, at any rate under certain circumstances, an extremely good conductor of electricity. In fact, it seems probable that such a vacuum is capable of becoming the best conductor that can possibly be obtained. The conductivity of metals is limited by the shortness of the mean free path of the ions, whereas the mean free path of a corpuscle in an atmosphere of corpuscles is probably very large. All that is necessary, therefore, to produce a big current is to supply the ions quickly enough at the hot surface, that is, to raise the temperature of the hot conductor to a sufficient extent. The experiments also seem to show that as far as electrical conductivity is concerned, the boundary of a hot conductor is an indefinite term; since so many of the corpuscles pass freely to the outside of the metal it is evident that at high enough temperatures quite an appreciable fraction of the current along a wire must be carried by the ions in the surrounding space.

In conclusion, I wish to thank Professor THOMSON for his never-failing advice and encouragement during the course of these experiments, which were carried out in the Cavendish Laboratory.

[*Note, added June* 30, 1903.—Since the present paper was written Mr. H. A. WILSON has made some experiments on the conductivity produced by hot platinum at low pressures, in which he finds that by carefully treating the wire the current can be reduced to about one two hundred thousandth of the value found by the author at the same temperature. Mr. WILSON also shows that the current is greatly increased by admitting hydrogen into the apparatus, and concludes that the high values found in this paper are due to hydrogen absorbed by the wire, which is only given off very slowly, if at all, by mere heating.

These results are not, however, inconsistent with the view that the effects are due to electrons shot out of the metal. To obtain the observed facts we have only to suppose that the occlusion of hydrogen diminishes the work which a corpuscle has to do in escaping from the surface. Mr. WILSON's own results are in agreement with this theory, for he finds that raising the pressure of hydrogen from 0 to 133 millims. reduces the value of the work in question in the ratio of 155 to 36. It might be thought that on this view the constant A which determines the number of ions per cub. centim. of platinum should be independent of the pressure of the hydrogen outside. The numbers found by Mr. WILSON do not support this supposition, but the numerous practical and theoretical difficulties demand that little weight should be attached to the difference.

It is possible that Mr. WILSON's process of removing hydrogen from a wire by oxidation may, as it were, overshoot the mark by leaving an electrical double layer with negatively charged oxygen on the outside. Such a double layer would increase the work for the corpuscles to get out and so would reduce the leak in the manner observed.]

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SERIES A, VOL. 201, pp. 551–558.

ON THE FORMATION OF DEFINITE FIGURES BY THE DEPOSITION OF DUST

BY

JOHN AITKEN, F.R.S.

LONDON:

PUBLISHED FOR THE ROYAL SOCIETY

BY DULAU AND CO., 37, SOHO SQUARE, W.

CONTINENTAL AGENTS, MESSRS. FRIEDLÄNDER AND SON, BERLIN.

1903.

A 344. 25.8.03

Price One Shilling.

INDEX SLIP.

XIV. *On the Formation of Definite Figures by the Deposition of Dust.*

By JOHN AITKEN, *F.R.S.*

Received July 13, 1903.

OWING to the kindness of Dr. W. J. RUSSELL, F.R.S., I received in June an advance copy of his paper on the above subject.* After reading this paper it appeared to me that all the figures illustrated in it could be explained on well-known principles. I shall therefore do what I can to fulfil the hope expressed by Dr. RUSSELL at the end of his paper that physicists from his descriptions may be enabled to explain their formation.

The formation of these dust figures appears to be due principally to three causes: (1) the convection currents set up by the hot plate; (2) to gravitation; and (3) to the repelling action of the hot surface. It seems trivial to remind the reader that gravitation plays a part in the formation of these figures, but it is to be feared that it is from not keeping the effects of gravitation fully in view that difficulty has been experienced in explaining them. It is principally owing to gravitation, or rather to an after-effect of gravitation, that no dust is deposited on certain parts of the plate. Gravitation acts on the dust under the plate as well as on the dust over it, thus causing the film of air flowing along the under surface of the plate to be dust-free, all the dust having fallen out of it. This dust-free film of air, after flowing along the under surface of the plate, turns round the edges and flows over the top surface, presenting its dustless side to the plate, and the air has to travel some distance over the top surface before the dust falls through the dustless film. That is, it takes some time for the upper current to undo the work of the under current, and the result is no dust falls on the plate till the current has flowed some distance from the edge. As stated, the third influence at work in the formation of these dust figures is the repelling action of the hot surface. It is well known that a hot surface tends to keep itself free from dust while surrounded by dusty air. The hot surface may in a manner be said to repel the dust, the action being probably due to the air next the hot body being warmer than the air at a slight distance from it, and the dust particles, being more strongly bombarded by the hotter air molecules on the one side than by the colder ones on the other, are driven away from the hot surface. The energy of this action will probably be the greater the quicker the temperature gradient in the air in a direction at right angles to the hot surface.

* 'Phil. Trans.,' series A, vol. 201, pp. 185–204.

The above principles seemed to offer the explanation of the dust figures, but as reasoning on physical phenomena should always be put to the test of observation when possible, I prepared apparatus to repeat Dr. RUSSELL'S experiments, and made arrangements for seeing the directions of the air currents and the condition of the air at the under and upper surfaces of the plate. For these experiments it was found most convenient to use small plates on which to deposit the dust figures. Metal plates were used, as they were easily prepared, and they were coated with black varnish to show the figures. To prevent any obstruction that might result from the use of three wire supports to rest the plate on, only one wire was used fixed in the centre of the plate. The plates were about 2·5 centims. square and 2 millims. thick. A thick plate is best, as it keeps its heat longest and gives time for observations to be made under fairly constant conditions.

The object of using small plates was that the observations could be made with a lens of greater magnifying power than was possible with large plates. An ordinary glass shade 12 centims. in diameter, made of thin glass, was used for confining the dusty air. A thin glass receiver has the advantage of being better made, the glass being of more even thickness than the thick glass ones, so enabling a more perfect image to be obtained by the lens. For illumination a narrow strip of incandescent gas mantle was hung over a Bunsen burner, exposing two thicknesses of the mantle to the flame. The Bunsen burner was enclosed in a lantern, the glass condenser of which was removed and its place filled with a globular flask of water. This had to be adopted, as the heat coming from, and through, the glass lens interfered with the formation of the figures. The gas mantle and flask were mounted at the same level as the plate, and a small hand lens was used to further concentrate the light, and by means of it the light could also be directed to any part of the plate where illumination was desired. The air was examined by means of a hand lens of as high a magnifying power as possible.

When making observations on the air currents, it was found in some cases to be an advantage not to use much dust, only as thick as might be called a haze, because when the dust is dense the beam of light illuminates by reflected light the whole interior of the receiver and makes observation difficult, whereas with few dust particles the illumination is confined to the part under investigation. In some cases it was found best to reduce the dust to such an amount that the individual particles of magnesia could be seen in the narrow illuminated area.

Turning now to the results of the observations made with this apparatus, the following points may be noted :—Bringing the light to bear on the under surface of the hot plate it is seen that there is a dust-free space below the plate, the dust in the film of air next the plate having fallen out and been repelled by the hot air above. This dustless film is seen to flow horizontally along the under surface, turn sharply round the edges of the plate, and flow horizontally over the top surface, no dusty air being in contact with the plate at any point when the plate is first put in and fairly warm

Fig. 1 represents the appearance of the plate at this stage. As the temperature of the plate falls, the rate of flow slackens and the repulsive action of the heat grows less, and at last a certain stage is reached when the current has become so slow that the under side of the dusty air comes close to the plate over the area where the

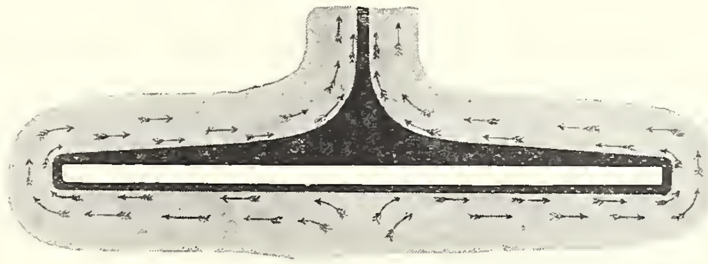

Fig. 1

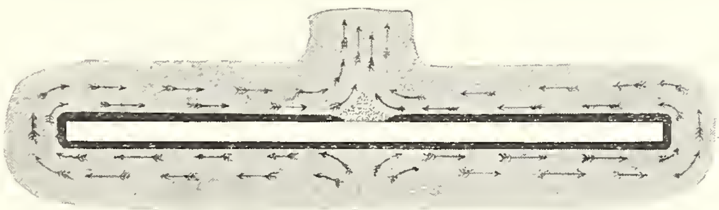

Fig. 2

currents meet, and as these currents here turn sharply upwards, there is a small space under their meeting point where the air is still and into which the dust collects and is seen showering down on the plate, as shown in fig. 2.

As these air currents rising from beneath all flow round the edges of the plate and move horizontally in a direction at right angles to the edges, they thus meet over the diagonals of the square; hence the deposition of the dust in Dr. RUSSELL's figures on the diagonal lines in his fig. 1 and on the lines bisecting the angles of the triangle in fig. 2 and the octagon fig. 3.* In fig. 4, for evident reasons, the currents here meet about the same angle as in fig. 1, and deposits take place on the lines bisecting the angles, but the stronger currents provided by the longer sides of the oblong plate prevent much deposit taking place where they meet in the centre of the plate. Further, in all these four cases, the currents meeting over the diagonals do not turn directly upwards, but flow also towards the centre of the plate, taking more or less of a horizontal movement along the diagonals, so tending to give time for the dust to fall. Further, when the currents have met and their direction has become partly vertical and partly horizontal, the repelling effect of the hot surface nearly ceases, as the temperature gradient perpendicular to the hot surface is in the rising current practically *nil*. The reason for the narrowing of the deposits as they approach the centre of the plate would appear to be due to the greater velocity of the currents where they meet over the centre of the plates caused by the union of all the currents from the different sides.

* Reference must be made to Dr. RUSSELL's paper for these dust figures, as they are not reproduced here.

The importance of the dustless film from the under side of the plate is evidenced by the fact observed by Dr. RUSSELL that no figure is obtained unless the plate be supported above the bottom of the receiver. When we examine by means of the beam of light the surface of a plate laid on the bottom of the receiver, we still find the dust-free film of air over the plate when the plate is pretty hot. This dustless film is very thin at the edges and thickens towards the centre, and a rising current can be seen flowing over it towards the centre, the rising current having a dustless core. This dustless film soon disappears as the temperature of the plate falls, and long before the plate is cold dust falls all over it, but no definite figures are formed.

Turning again to Dr. RUSSELL's figures, the effect of the velocity of the current is well shown in figs. 7 and 8. Fig. 7 was obtained with only a slight heating of the plate, and fig. 8 by a higher temperature. In the former figure the slow currents produced by the slight heating only kept the outside edges free from dust and allowed a large deposit to take place over the centre of the plate; while in the latter the higher temperature gave a current strong enough to prevent almost any deposit at the centre.

The cause of the extension in the breadth of the arms of the cross in Dr. RUSSELL's fig. 9, which was obtained by placing a hot cylinder some distance below the plate, is not so evident, but the probable explanation seems to be the following: When the currents are due to the hot plate alone the circulation is mostly horizontal from centre to edge below and from edge to centre above the plate, and the area where the currents meet is narrow, but when there is a hot body under the plate there will be an upward current all round it of hot air. This upward current will prevent the horizontal movements above described being so markedly horizontal, and will cause them to turn upwards at an easier curve, so broadening the dead dust-depositing area under the up-curving air.

In fig. 10 the extra deposit is probably due to some interference with the under air current produced by the piece of glass held under the plate. This subject will be referred to later.

Turning now to the effect on the figures of flames, &c., placed at a distance from the apparatus, as shown in Dr. RUSSELL's figs. 11, 12, and 13. These alterations in the figures appear to be due not to any direct effect of the flames, &c., on the dust or on the plate, but to the heat radiated by them heating the receiver and so giving rise to convection currents at the side of the plate. These currents entirely change the symmetrical flow of the air over the plate and cause the centre of the current rising over it to move to one side, as shown in figs. 11 and 12. When making observations with the apparatus described in this paper, it was not possible to get any of the figures quite regular; even the slight amount of heat given off by the small incandescent strip of mantle after passing through water interfered with the results, and air currents could be seen rising in the receiver on the side next the light. In support of this convection explanation, it may be further stated that the same deformations as shown in figs. 11, 12, and 13 can be equally well produced by heating the receiver by means of the hand held on it.

Turning now to the results obtained by Dr. RUSSELL when the plate was rested on a hot cylinder of metal, as shown in fig. 14. Here everything is reversed; black centre and black diagonal arms where in the other figures it was white, and white to the edges of the plate where before it was black. How, it may be asked, stands the explanation now; where is the protecting effect of the dustless film from under the plate? For explanation let us turn to experiment. When the beam of light is turned on to this new condition of matters we find the air circulation is all changed. The dustless film from under the plate no longer turns round the edge and flows horizontally over the upper surface, but the large amount of hot air coming from the hot cylinder below the plate causes the dustless film to rise straight up from the edge, and an induced current of air is seen flowing over the plate from the centre to the edge, depositing its dust as it goes. It is only at the corners of the plate, where the mutual influences of the neighbouring currents and the amount of hot air is less, and where the currents approach and bring the dustless film over the plate, that there is any protection.

Both of Dr. RUSSELL'S figures, shown in figs. 9 and 14, were produced by somewhat similar conditions. In both cases a hot body was placed beneath the plate, but in the case shown in fig. 9 the hot cylinder was placed some distance below the plate and only heated to 55° C.; whereas, in the other case, the hot cylinder was at a temperature of 150° C., and the plate rested on it. Referring to fig. 9, Dr. RUSSELL points out that as the temperature of the body underneath the plate is increased the amount of deposit also increases, and ultimately the figure of the cross disappears; but, as will be seen from fig. 14, it reappears in a reversed form when the temperature is high enough and the plate rests on the hot body, all of which is easily understood by what has been said above.

Figs. 15 and 16 do not call for any special observation. Fig. 17 is interesting as showing the effect when the plate is cold and the currents are produced by an influence above the plate. In this case the currents flow over the cold plate towards the hot cylinder placed at the centre. As these currents do not come from the under side of the plate, they do not have a dustless film. So the plate has dust deposited all over it, but the figure of the cross can still be seen and is produced by the currents from the different sides flowing towards the centre, meeting over the diagonals, and causing the calm depositing areas as in the previous cases, only more feebly. In fig. 18 the white deposit round the cold cylinder is caused by the cold air flowing down the cylinder and forming a calm dust-depositing area round it.

The effect of placing the plate in a sloping position is shown in figs. 19, 20 and 21. These alterations in the forms of the deposited dust are evidently due to the slope of the plate interfering with the flow of the dustless film from underneath the plate and to the change produced by the slope on the currents over the upper surface.

Take, for instance, fig. 20. Here most of the air from beneath the plate flows to the higher edge and but little curves round the lower one, while the side streams keep about the usual strength. The current, however, round the higher edge being warmer and stronger than usual, does not flow to the centre of the plate, which is in a downward direction, but rises in an easy curve, with the result that over a large area of the plate the air is nearly motionless and the dust is free to deposit itself on the plate. In fig. 21 no dustless film seems to have come from the lower edge owing to the high angle of the plate, and all the hot air from the under side has flowed to the higher edge; where the rising current has been so strong it has curved in but little, and as the side currents are weak, as most of the hot air has flowed to the upper edge, the greater part of the plate is therefore exposed to the dusty air flowing over it from the lower edge.

The figure shown in fig. 22 seems to be due to the obstruction placed on the plate interfering with the regular flow and causing eddies and deposition of dust, while the dustless film enters the holes in the obstruction and, as usual, protects the surface in front of them over which they flow.

The curious effect of cutting a re-entering angle out of the plate, as shown in fig. 22A, is very interesting, and shows that the cutting out of that angular piece has in some way introduced new conditions which have interfered with the protecting action of the dustless film. Referring this to experimental observation, it is seen at once how this peculiar deposit is produced. The beam of light shows that the air streaming up through the angular opening does not turn over and flow over the plate but rises straight up, owing to the large quantity of hot air drawn to the one point. This upward-moving current induces another current over the plate moving towards it from the centre, and as this current flows slowly and is composed of dusty air without a dustless film, the particles settle out of it and cause the peculiar marking extending from the centre to the angular opening.

The next series of figures, from fig. 23 to 29A, produced by the action of a piece of glass, a pin, a hair, or other obstruction touching, or even near the edge of, the plate, are most curious and unexpected. On putting these conditions to the test of observation, it was seen that all these obstructions cause deposits to form by the interference they offer to the stream-lines of air moving over the surface of the plate. Where the obstruction cuts the stream the current is slackened, and more or less eddying probably takes place, enabling the dust to settle. What is seen when the air is examined with the lens while illuminated by a narrow beam of light is as follows:—While the beam of light is moved about on either side of the obstruction the air is seen to flow in well-defined stream-lines, the lower surface of the dusty air being distinct and clearly defined, but when the light shines on the air that has passed the obstruction, the upper limit of the dustless air has lost its definition. And further, if the beam of light were moved backwards and forwards, from one side to the other of the obstruction, it was observed that not only the upper limit of the dustless air on each side of the obstruction was well defined, but there was always a greater thickness of dustless air on each side than opposite the obstruction. As the light travelled backwards and forwards the lower limit of the dust always seemed to dip just when the light was opposite the obstruction. As the plate cools, dust begins to fall behind the obstruction, while as yet there is a space of dustless air on each side of it. While it may be strange that so small an obstruction as a hair should produce these deposits, yet it is known that when stream-lines are interfered with, unexpected results frequently happen.

Figs. 30, 31 and 32 call for no special remarks, as these cases are explained in the previous paragraph, the deposits being due to the rough edges of the plates interfering with the stream-lines.

Fig. 33 is the same as fig. 23, already explained, only in the former the obstruction is placed on one of the diagonals, and not on the side of the square, as in fig. 23.

It seems unnecessary to consider in detail the other figures in Dr. RUSSELL's paper, which show the effects of different kinds of obstructions placed on or above the hot surface. The manner in which these figures are formed can be easily understood with the aid of what has been said. Only a few remarks may be made as to figs. 44 and 44A. When we examine by means of a beam of light the conditions in these two cases, it is seen that the dust-free film from underneath the plate rises and flows upwards past the edge of the top plate. In doing so, it seems to draw away with it some of the air from between the plates, with the result that a very slight negative pressure is established in the space between them. The beam of light shows that the upward current does not move in a straight line but curves inwards, drawn in by the lower pressure between the plates. The amount of this in-curving is least at the middle of the sides of the plate and greatest at the corners, the reason for this being that at the middle of the sides the air currents are stronger, as they contain more hot air than the currents at the corners, with the result that the weak currents at the corners are drawn more out of their course than the stronger currents at the sides. As the temperature falls the currents weaken, and at last the currents at the corners yield and the air is drawn in there and brings its dust with it, but its velocity being small, only the edges get any protection from the dust-free film and the dust settles on the plate before it travels far, giving the patches of dust shown in the corners of the plates in figs. 44 and 44A.

What perhaps surprises one most in the formation of these dust figures is the important part the dustless film from the under side of the plate plays in protecting the upper surface from deposits of dust; and we have seen that whatever tends to destroy this dustless film tends to bring about conditions favourable for the dust settling on the plate.

[*Note by* Dr. RUSSELL, *July* 23, 1903.—The interesting observations made by Mr. AITKEN by means of his exploring beam of light have contributed substantially to the elucidation of the dust-figures. It was, of course, clear from the first that the agency was the currents of air creeping over the edge of the plate, the dust being deposited where the drift was slowest; but the sharpness of the patterns and the cleanness of the rest of the plate are made more intelligible by the existence of Mr. AITKEN's clear layer of drifting air through which the dust from above has to fall before reaching the plate. It will be observed that the explanation requires that the dust has had time to fall completely out of the layer when it was travelling underneath the plate, but has not had time to fall through it to any extent from above when the layer was above the plate. The thinning out of the clear layer in the wake of a pin or hair, owing to eddies or broken motion, as described by Mr. AITKEN, throws light on the features of the deposit thus produced by revealing that it is denser near the centre of the plate, because the dust has had more time to fall through this thinned-out layer of clean drifting air; yet the persistence of the effect when the obstacle is far removed from the edge of the plate remains very remarkable.]

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HARRISON AND SONS, PRINTERS IN ORDINARY TO HIS MAJESTY, ST. MARTIN'S LANE, LONDON, W.C.