KEY AND SUPPLEMENT

TO

ELEMENTARY MECHANICS.

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NOTICE.

This work contains not only solutions of the examples and answers to the Exercises of the Elementary Mechanics of the author; but also additions to the text, and other new matter intended to interest the general student of mechanics, and be of service to teachers of the science in connection with any other text book. Valuable assistance has been rendered in correcting proofs by Professor H. A. Howe, of the University of Denver.

DeVolson Wood.

KEY AND SUPPLEMENT

TO

ELEMENTARY MECHANICS.

PAGE 1, ARTICLE 1.—Motion is a change of position. Motion is determined by the relative position of bodies at different times. If bodies retain the same relative position during successive times, they are said to have no motion in reference to each other; in other words, they are said to be at rest in reference to each other. All bodies of which we have any knowledge are in motion; hence all motion is, so far as we know, relative. Absolute motion implies reference to a point absolutely at rest, but as no such point is known, such motion has only an ideal existence.

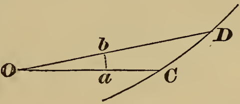
PAGE 2, ART. 6.—No definition of space will give a better idea than that obtained by experience. Metaphysicians have indulged in speculations in regard to its nature, but they are able to assert with certainty only that it has the property of extension. Descartes taught that the properties of extension, known as length, breadth, and thickness, were solely properties of matter, and hence when a body was removed no space remained in the place formerly occupied by it. So far as we know, no space exists which is perfectly de-void of matter. Perfectly void space is an ideality; still modern philosophy distinguishes between the thing contained, and that which contains it—between matter and the place occupied by matter. It abstracts (so to speak) space from matter, and, in a measure, matter from space. It seems impossible to conceive of matter not occupying space, but it is not difficult to conceive of a given quantity of matter as occupying a very small or a very large space. We are able to consider matter in the abstract without considering the dimensions of the space it occupies; and also we may consider space in the abstract as not including matter. The latter is called *absolute space;* it is conceived as remaining always similar to itself and immovable.

Time is duration. We gain a knowledge of it by the order of events. Every event has its place in time and space, and by means of memory we gain a knowledge of the order in which events occur. Without memory we would gain *by experience* no knowledge of time. Sir Isaac Newton considered mathematical time as *flowing* at a uniform rate, unaffected by the motions of material things. This idea induced him to call his new calculus *fluxions.*

Rate refers to some unit as a standard. Thus, to illustrate, rate of interest is a certain amount of money paid for the use of *one* dollar; passenger rates refer to the amount paid for *one* passenger; rate of shipping per ton is the amount paid for carrying *one* ton; *rate of motion* is the space passed over in *one* second, *one* minute, *one* hour, *one* day, or *one* of any other unit. The term *velocity* is simply the equivalent of *the rate of motion.* Angular velocity is *rate* of angular motion

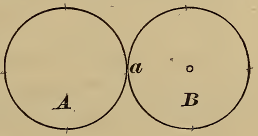
(p. 4, Art. 12). Acceleration is the *rate* of change of velocity, being the amount of *change* in the velocity for *one* second, or *one* of any other unit (p. 10, Art. 22). Mechanical power is the *rate* of doing work (p. 55, Art. 99). Rates may involve two units. Thus, rate per ton-mile implies a certain amount paid for *one* ton for *one* mile; passenger rates are often an amount for *one* person for *one* mile; mechanical power, or rate of doing work, is the amount of work done in *one* foot for *one* second, or by *one* pound for *one* second, etc. *Rate* is a thing used for measuring quantities, as a yard-stick is used for measuring cloth, a chain for measuring land, the pound for weighing groceries, ton for measuring merchandise, etc.

PAGE 4, ART. 12.—The definition here given for rotary motion is applicable to the case where the motion is in a curved path not circular, as *CD*. But the analysis given in the text is not applicable to this case.

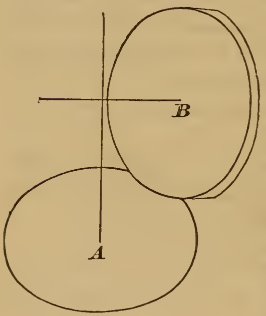


PAGE 6, ART. 14.—Just after Fig. 5, for *If two velocities*, etc., read *If two concurrent simultaneous velocities*, etc.

PAGE 8, ART. 20.—Speaking of the rotation of the moon, suggests an interesting question in practical mechanics. If the wheel *B* rolls around on the circumference of an equal wheel *A*, will the former turn once or twice on its own axis?



Mark the point *a* which, initially, is in contact with the wheel A, and roll B half around A; it will be observed that the marked point will then be at the left of the centre of B, as it was at the start. Continuing the rolling, the point a will again be at the left of the centre when B has gone completely around A; hence it is sometimes asserted that the wheel B has turned twice on its axis. But we wish to show that it has turned but once on its own axis, and the whole wheel has been rotated once about the axis of the wheel A. Let the axis of the wheels be at right angles with each other, then it will be evident, from mere inspection, that when B has turned once on its axis it will have gone once around A. Thus B will have gone once around the axis of A, and once about its own axis. Next, incline the axis of B upward, so as to approach a parallelism to A, and the same result will be seen from mere inspection,



and it will continue to remain evident as it becomes nearer and nearer parallel, and when they become actually parallel, the same condition will hold true. Hence, in the former figure, the wheel B will turn but once on its own axis in rolling once around A. The same result may be shown in another way. Let a block be placed at a facing a mark on the axis of B, and conceive this axis to be rigidly connected with the axis of A while the wheel B is free to turn on its own axis. In this way the axis of B will be carried bodily about A. When B has rolled half around A, it will be found that the block will face the same direction in space—say towards the east—but that it will not face the mark on the axis, for the mark will be on the opposite side of the axis. Continuing the rotation, it will be found that the block will face the mark only once at each revolution about A.

Similarly, the moon turns but once on its *own* axis in one revolution about the earth, but the rotation about the two centres are not exactly coincident; for it is found by observation, that in some parts of the orbit more of the surface of the moon is seen on the eastern (or western) side than in other parts of the orbit; thus showing that the rotation about the earth is sometimes faster, and at other times slower than the rotation of the moon about its own axis. This phenomenon is called *Libration.*

EXERCISES.

PAGE 9.

1. $4\frac{1}{6}$ miles.
2. The former.
3. 66 feet per second.
4. 17 feet; $17n$ feet.
5. $400 \div \dfrac{40 \times 5280}{60 \times 60} = 6\frac{9}{11}$ seconds.
6. $\dfrac{200 \times 2\pi}{60} = 6\frac{2}{3}\pi$ in arc; or $\dfrac{200 \times 360}{60} = 1200$ degrees.
7. $\sqrt{3^2 + 2^2} = \sqrt{13} = 3{\cdot}605 +$ miles per hour.

PAGE 10.

8. $\sqrt{15^2 + \left(\frac{44}{3}\right)^2} = \frac{1}{3}\sqrt{3961} = 20{\cdot}98$ ft. per second.

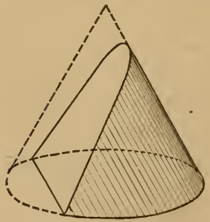
9. $0{\cdot}0009\frac{1}{9}$.

10. See Article 14. $v = \sqrt{5^2 + 10^2 + 100 \cos 60^\circ} = \sqrt{125 + 50} = \sqrt{175} = 13{\cdot}22 +$ feet; hence the distance between them in two seconds will be $2 \times 13{\cdot}22 + = 26{\cdot}44 +$ feet.

PAGE 10, ART. 22.—Observe that *acceleration* is not the rate of change of motion, but the rate of change of the rate of motion. It is the rate of change of a rate. *The rate of change* is usually measured in the same units as the *rate of motion*. If one is in feet per second, the other is also. It is possible to conceive of mixed units. Thus in the case of falling bodies, the velocity at the end of the first second is $16\frac{1}{12}$ feet, and the acceleration is $643\frac{1}{3}$ yards per minute.

Strive to get a clear conception of the meaning of acceleration and of its measure. It is one of the elements of the absolute measure of force.

PAGE 13, ART. 26.—The expression "The locus of these points will be a parabola," means that if any number of points in the path be determined in the same manner, they will all be in the arc of a parabola.



A parabola is a curve which may be cut from a right cone by a plane parallel to one of its elements. (See Author's Coördinate Geometry.)

EXAMPLES.

1. Space 250 feet; velocity 50 feet per second.

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2. $f = 25$ feet; $s = 12\frac{1}{2}$ feet.
3. During 4 seconds the space will be $s = \frac{1}{2}$ of 32×4^2 (Art. 14), and during 3 seconds the space will be $\frac{1}{2}$ of 32×3^2; hence during the 4th second the space will be $\frac{1}{2} \times 32\ (4^2 - 3^2) = 112$ feet.
4. By means of equation (1), Art. 25, find $6\frac{1}{4}$ feet per second.
5. $t = \sqrt{12 \cdot 5} = 3 \cdot 533 +$ seconds.
6. $f = (20 \div 120)\ 3 \cdot 28 = 0 \cdot 54\frac{2}{3}$ feet per second.
7. $32\frac{1}{6} \div 3 \cdot 28 = 9 \cdot 8$ metres per second.

PAGE 15.—*Matter* and *force* are two grand realities of the external world, and of these we know nothing directly. Our knowledge of the former is confined to its *properties*, and of the other to its *laws* of action. But we have no reason to believe that one exists independently of the other. In our earlier experiences matter is conceived to be hard, gross, and unyielding; but later we find that it is yielding, and that many solids, as iron and lead, may be changed to liquids by heat, and that liquids may be changed to gases—so that matter is proved to be more or less viscid or attenuated. Solids are porous. Changes of form are effected by forces, so that some metaphysicians have reasoned that there may possibly be no gross matter, but, instead thereof, those things which we consider as bodies are only aggregations of forces. On the other hand, all investigations in mechanics proceed on the hypothesis that matter is in no sense a force, or an aggregation of forces, but that it is something distinct from force, *something* upon which force acts. In the case of attraction, the matter in two bodies may remain constant, while the force exerted by each upon the other will depend upon the distance between them. It is true, however, that we gain a knowledge of matter only through the action of forces. Every avenue to the mind through the senses is an agent for transmitting the result of the action of certain forces, and the very act of transmission brings into play certain forces.

PAGE 16, ART. 28.—Mathematics applied to the laws of physical science enables us to determine magnitudes which far transcend the powers of accurate measurement or even of conception.

Sir Wm. Thompson gives four methods for ascertaining the mean distance between molecules.

Optical dynamics.

Contact electricity of metals.

Capillary attraction.

Kinetic theory of gases.

"Optical dynamics leaves no alternative but to admit that the diameter of a molecule, or the distance from the centre of a molecule to the centre of a contiguous molecule in glass, water, or any other of our transparent liquids and solids, exceeds one ten-thousandth of the wave length of light, or a two-hundred-millionth of a centimetre" ($\frac{1}{20000000000}$ of a metre).

"However difficult it may be even to imagine what kind of thing the molecule is, we may regard it as an established truth of science that a gas consists of moving molecules disturbed from rectilineal paths and constant velocities by collisions or mutual influences, so rare that the mean length of proximately rectilineal portions of the path of each molecule is many times greater than the average distance from the centre of each molecule to the centre of the molecule nearest it at any time. If for a moment we suppose the molecules to be hard elastic globes all of one size, influencing one another only through actual contact, we have for each molecule simply a zigzag path composed of rectilineal portions, with abrupt changes of direction."

"If the particles were hard elastic globes, the average time from collision to collision would be inversely as the average velocity of the particle. But Maxwell's experiments on the variation of the viscosities of gases with change of temperature prove that the mean time from collision to collision is independent of the velocity, if we give the name collision to those mutual actions only which produce something more than a certain specified degree of deflection of the line of motion. This law could be fulfilled by soft elastic particles (globular or not globular), but not by hard elastic globes." "By Joule, Maxwell, and Clausius we know that the average velocity of the molecules of oxygen, or nitrogen, or common air, at ordinary atmospheric temperature and pressure, is about 50,000 centimetres per second (500 metres per second, or about 1,600 feet per second), and the average time from collision to collision a five-thousand-millionth of a second ($\frac{1}{5000000000}$). Hence the average length of path of each molecule between collisions is about $\frac{1}{100000}$ of a centimetre" ($\frac{1}{10000000}$ of a metre).

"The experiments of Cagniard de la Tour, Faraday, Regnault, and Andrews on the condensation of gases do not allow us to believe that any of the ordinary gases could be made forty thousand times denser than at ordinary atmospheric pressure and temperature without reducing the whole volume to something less than the sum of the volumes of the gaseous molecules, as now defined.

"Hence, according to Clausius, the average length of path from collision to collision cannot be more than five thousand times the diameter of the gaseous molecule; and the number of molecules in unit of volume cannot exceed 25,000,000 divided by the volume of a globe whose radius is that average length of path. Taking now the estimated $\frac{1}{100000}$ of a centimetre for the average length of path from collision to collision we conclude that the diameter of the gaseous molecules cannot be less than $\frac{1}{5000000000}$ of a centimetre ($\frac{1}{500000000000}$ of a metre); nor the number of molecules in a cubic centimetre of the gas (at ordinary density) greater than 6,000,000,000,000,000,000,000."

"The densities of known liquids and solids are from five hundred to sixteen thousand times that of atmospheric air at ordinary pressure and temperature: and, therefore, the number of molecules in a cubic centimetre may be from 3×10^{24} to 10^{26} (that is, from three million million million million, to a hundred million million million million). From this the distance from center to nearest center in solids and liquids may be estimated at from $\frac{1}{1400000000}$ to $\frac{1}{460000000}$ of a centimetre ($\frac{1}{140000000000}$ to

$\frac{1}{460000000000}$ of a metre). The four lines of argument lead all to substantially the same estimate of the dimensions of molecular structure. Jointly they establish with what we cannot but regard as a very high degree of probability the conclusion that, in any ordinary liquid, transparent, solid, or seemingly opaque solid, the mean distance between the centers of contiguous molecules is less than the hundred-millionth, and greater than the two thousand-millionth of a centimetre. To form some conception of the coarse-grainedness indicated by this conclusion, imagine a rain drop, or a globe of glass as large as a pea, to be magnified up to the size of the earth, each constituent molecule being magnified in the same proportion. The magnified structure would be coarser-grained than a heap of small shot, but probably less coarse-grained than a heap of cricket-balls." (Extracts from a paper by Prof. Sir Wm. Thomson on the size of Atoms, *Am. Jour. of Science and Art.* 1870, vol. ii., pp. 38–45.)

Mr. N. D. C. Hodges, in an article on the size of molecules (*Phil. Mag.* and *Jour.* of *Science*, 1879, vol. ii., p. 74), says: "If we consider unit mass of water, the expenditure on it of an amount of energy equivalent to 636.7 units of heat will convert it from water at zero into steam at 100°. I am going to consider this conversion into steam as a breaking-up of the water into a large number of small parts, the total surface of which will be much greater than that of the water originally. To increase the surface of a quantity of water by one square centimetre requires the use of .000825 metre gramme of work. The total superficial area of all the parts, supposing them spherical, will be $4 \pi r^2 N$, the number of parts being N. The work done in dividing the water will be $4 \pi r^2 N \times .000825$. For the volume of all the parts we have $\frac{4}{3} \pi r^3 N$. This volume is, in accordance with the requirements of the kinetic theory of gases, about 3,000 of the total volume of the water. The volume of the steam is 1,752 times the original unit volume of water. Hence—

$$\frac{4}{3} \pi r^3 N 3000 = 1752$$
$$4 \pi r^2 N .000825 = 636.7423.$$

One unit of heat equals 423 units of work (in French units); solving these equations for r and N, we get $r = .000000005$ centimetre (or diameter $= .00000001$ centimetre $= \frac{1}{10000000000}$ metre), a quantity closely corresponding with the previous results of Sir Wm. Thomson, Maxwell, and others; and N equals 9000 (million)^3, or for the number in one cubic centimetre 5 to 6 (million)^3."

The extreme tenuity of a gas is further shown by the following extract taken from the Beiblätter *zu den Annalen der Physik und Chemie*, 1879, No. 2, p. 59. "At 0°C. and 760*mm* pressure a cubic centimetre (.061 cubic inch) holds nearly one hundred trillions of gas molecules. Under these conditions the molecules themselves fill nearly the $\frac{3}{1000}$ of the space occupied by the gas. The absolute weight of a hydrogen molecule is represented by $\frac{15}{10^{23}} g$, (g in metres)."

Mr. G. J. Stoney, in an article on Polarization on

Stress in Gases (*Phil. Mag. and Jour. of Science*, 1878, vol. ii., p. 407), says that " the number of molecules in a cubic millimetre of atmospheric air is about 10^{18} (= 1,000,000,000,000,000,000),* and that the average distance between them is about $\frac{1}{1000000000}$ of a metre. The average striking distance (*i. e.* the average length of path between encounters) of the molecules is about $\frac{1}{15000000}$ of a metre. The average velocity at ordinary temperature may be taken as 500 metres per second (1,600 feet per second), and the molecules meet with so many encounters, that the direction of the path of each is changed somewhere about 10,000,000,000 times every second." In one movement the particle travels $\frac{1}{20000000}$ of a metre, or $\frac{1}{20000}$ of a millimetre ; † and it makes this movement in $\frac{1}{10000000000}$ of a second. Now the wave length of a chemical ray is about $\frac{1}{5000}$ of a millimetre, hence we find that the molecule of air travels through a distance which is one-fourth as long as the length of this particular wave in this fraction of a second. ‡

According to Pouillet, the mechanical energy of a cubic mile of sun light at the earth equals 12,050 ft. lbs.

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* Clausius's limit is 6,000 times this amount.

† Clausius's estimate is one-half this value.

‡ Mr. E. H. Cook, in an article on the Existence of the Luminiferous Ether (*Phil. Mag. and Jour. of Science*, 1879, vol. I, p. 235), after quoting the above figures from Mr. Stoney's paper, adds : "Then in one moment the particle travels $\frac{1}{2000000}$ of a metre in $\frac{1}{10000000000}$ of a second;" also in his deductions he adds : "hence we find that the molecule of air travels through a distance which is more than twice as long as the length of this particular wave in this fraction of a second." The reader can easily see that this deduction is erroneous.

(*Phil. Mag.*, 1855, vol. ix., p. 39); accepting which Sir Wm. Thomson calculated the weight in pounds of a *cubic foot* of the ether of space (that which is instrumental in transmitting light, and called ether) by the formula:

$$W = \frac{83g}{V^3 n^2},$$

where $g = 32\frac{1}{6}$ (acceleration per second of gravity), $V =$ the velocity of light per second, being about 192,000 miles per second, and $n = \frac{1}{50}$, being the ratio of the greatest velocity of a rotating particle to the velocity of light. Herr Glan asserts that n is not constant, that he found $n = \frac{1}{27}$ in one case, and $\frac{1}{4306}$ in another. (*Am. Jour. of Arts and Sciences*, 1879, vol. xviii., 404. *Annalen der Physik und Chemie*, No. 8, 1879, p. 584.) Assuming $n = \frac{1}{50}$, we find that a *cubic foot* would weigh about

$$W = \frac{83 \times 32\frac{1}{6} \times 50^2}{(5280 \times 192000)^3}$$
$$= \frac{1{\cdot}5}{10^{20}} \text{ lb. nearly};$$

and for the weight of a cubic mile $\frac{1}{10^9}$ pound. The weight of a volume of the size of the earth would be about 240 pounds. Admitting this result, it follows that a sphere equal in diameter to the diameter of the earth's orbit (or say 190,000,000 miles) will contain an amount of ethereal matter nearly equal to $\frac{1}{1000}$ of that of the mass of the earth.

Probable tension of the ether of space. The above results combined with one or two other plausible assumptions enable us to find a probable value of the tension of the light ether. As stated above, the velocity of the particles of air producing a pressure of 15 lbs. per square inch, is about 1,600 feet per second, which is about 50 per cent. more than the velocity of sound in air. Assuming now that the normal velocity of the ether particles is somewhat more than the velocity of light—or, more definitely, that it is 195,000 miles per second, and that the tension is directly as the masses, and also as the square of the velocities of the particles; also that the weight of a cubic mile of the ether is $\frac{1}{10^9}$ lb., as found above, and observing that 100 cubic inches of ordinary air weigh 31 grains, and 7,000 grains make a pound, we have for the pressure P in pounds of the ether upon a square inch

$$P = 15 \times \left(\frac{195000 \times 5280}{1600}\right)^2 \times \frac{\frac{1}{10^9}}{\frac{31}{100} \times \frac{1728}{7000} \times (5280)^3}$$

$$= 0{\cdot}00000185 = \frac{1{\cdot}85}{10^6} \text{ lb. per inch of section.}$$

A Mr. Preston, an English writer, in his work on the *Physics of Ether*, estimates, or rather assumes, 500 tons per square inch as a probable inferior limit of the pressure (p. 18), and, with this as a basis, he finds the weight of a cubic mile of ether to be about 220 lbs. (p. 120). But as he has used 56·5 grains for the weight of a cubic foot of air when it is nearly ten times this amount, he should have found for the weight of a cubic mile of ether *nearly one ton.* His assumption, however, in regard to pressure is quite arbitrary, and does not seem to be well founded. It seems improbable that there should be so much mass in the ether of space. Even 240 lbs. for a volume equal to that of the earth *seems* a high value when we consider the amount that must be displaced by the planets while moving about their orbits.

Temperature of the Sun. The following may be interesting, although it does not fall under the article above referred to. "The effective temperature of the sun may be defined as that temperature which an incandescent body of the same size placed at the same distance ought to have in order to produce the same thermal effect if it had the maximum emissive power. If we consider the surrounding temperature during the observation to have been about 240° we obtain . . . for the effective temperature in degrees centigrade 9,965.4°. . . I think, then, that I may fairly conclude that the temperature of the sun is not very different from its effective temperature, and that it is not less than 10,000°, nor much more than 20,000° centigrade." *Phil. Mag.*, 1879, vol. ii., pp. 548–550. See also *Am. Jour. Sc. and Arts*, 1870, vol. ii., p. 68.

Sir Wm. Thomson calculates the mechanical energy of the solar rays falling annually on a square foot of land in latitude 50° to equal 530,000,000 foot pounds, or 396 H. P. per yard per day. He finds that the heat alone hourly given out by each square yard of the solar surface is equivalent to 63,000 horse power, and would require then the hourly combustion of 13,500 lbs. of coal.
Appleton's Cyclopædia, 1868, vol. ix. p. 23.

PAGE 16, ART. 29.—A better definition is—*Force is an action between bodies*—for this form of the statement recognizes the existence of at least two bodies in every action. A force never acts upon one body without producing an equal opposite action upon another body. We speak of the action of a force upon a body because in most cases the second body is so large, relatively, that the force produces little or no perceptible effect upon it. But according to the law of Universal Gravitation each particle in the universe is attracted by every other particle with a force which depends upon their masses and the distances between them; hence, in a highly refined sense, it may be said that every force producing motion involves every body in the universe. The entire universe of matter is bound together by a something—an action—which we call *Force*. *Every phenomenon which we witness in the physical world is the result of force*, acting through space, or during a certain time.

Force alone is *stress*. In other words *stress* is force abstracted from time and space. The science of Stress is the science of Statics. Stress is always measured in pounds or its equivalent. If a force produces motion, that part of the phenomenon which is abstracted from time and space is stress, so that the attractive force between the earth and moon, or between the earth and sun, measured in pounds, is *stress*.

When force is compounded with time or space the result is work, or energy, or momentum, as will hereafter be shown.

The following are some definitions of force as given by different authors:

La Place says: "The nature of that singular modification, by means of which a body is transported from one place to another, is now, and always will be, unknown; it is denoted by the name of *Force*.

We can only ascertain its effects, and the laws of its action." (*Mécanique Céleste*, p. 1).

"Force is an action between bodies, causing or tending to cause change in their relative rest or motion" (*Rankine's Applied Mechanics*, p. 15).

"Force is that principle of which, considered simply as a mechanical agent, we know but little more than that when it is *imparted*, that is, *put into*, a body, it produces either motion alone; or strain, with or without motion." "What is called overcoming inertia, is simply *putting in force*." (Trautwine's *Engineer's Pocket Book*, p. 445 and p. 447). We consider this as a misuse of terms. We cannot safely say that force is put into a body. When force acts upon a body free to move, *energy* is put into the body, when force and space are involved in the result, and time is abstracted (see text, p. 66); or *momentum* when the elements involved are force and time, space being abstracted (see Chap. V). Although it is too early in the text for a full discussion, we add a remark for the benefit of those who have some knowledge of the subject. When a constant force, F, acts through a space, s, it does the work represented by the product of F and s, or Fs. If the body upon which it acts is wholly free, the entire *work* will be stored in the body, and is then called energy, the measure of which is $\frac{1}{2} Mv^2$; hence $Fs = \frac{1}{2} Mv^2$. Eliminating s by means of Eq. (2), p. 12, of the text, gives $Ft = Mv$, which is the measure of the effect of force combined with time, and the second member is the measure of the momentum (text p. 78). It would be better to say that force and time, or force and space, are put into a body than that force alone is put into it. "Nothing but force can resist force." "Matter, in itself, cannot resist force" (*ib.* p. 445). By *resistance* is here understood to be such a condition of things as that motion will not result, and, in this sense, these statements are correct.

"Force put into a body" is, properly speaking, putting it under stress, and the body is said to be strained, but no amount of internal stress will produce motion of the body.

"Whatever changes the state of a body or the elements of a body, *with respect to rest and motion*, is called *force*." (Bartlett's *Analyt. Mech.*, p. 17).

"Force is defined as that which changes or tends to change a body's state of rest, or motion, and any given force may be measured by the acceleration it imparts to a gramme." (Cumming's *Theory of Electricity*, p. 5).

"Force is whatever changes or tends to change the motion of a body by altering either its dimension or its magnitude; and a force acting on a body is measured by the momentum it produces in its own direction in a unit of time." (Maxwell's *Theory of Heat*, p. 83).

It will appear that *the momentum produced* in a unit of time is the same as *the acceleration*, and hence the two last definitions are equivalent. We, however, deem it advisable to avoid the expression *momentum produced* because it is liable to be confounded with *the actual momentum* of a body, although it is not intended to even imply the latter. *Acceleration* is specific and correct.

"Force is matter in motion, nothing more, nothing less; the abstract idea of force without matter is a nonentity." (Nystrom *On the Force of Falling Bodies and Dynamics of Matter*, p. 20).

The preceding remarks will show that this definition contains a misapplication of terms. Matter in motion is either Energy or Momentum, according as time or space is abstracted in considering the elements which enter into the combination.

"*Force* is a mere name, but the *product of a force into the displacement of its point of application* has an objective existence."

"*Force* is the rate at which an agent does work per unit of length."

"The mere rate of transference of energy per unit of length of that motion is, in the present state of science, very conveniently called force." (Lecture by Prof. G. P. Tait, *Nature*, 1876, vol. xiv. p. 462).

In regard to these views, we observe that the name applied to anything is a mere name. In a certain sense it is an *ideality*, but generally the name stands for a reality. In this case, if *force* is a mere name, what is the sense of the remainder of the sentence? How can a force have a *point of application* if the force is a mere name? And granting that it may have, how can the product have an objective existence?

In regard to the second definition, it is analytically correct, but we consider it rather as a deduction than as a fundamental definition. It is shown on pages 52 and 67 that

$$Fs = \tfrac{1}{2}Mv^2 = K:$$

or, in terms of the calculus,

$$Fds = d'(\tfrac{1}{2}Mv^2) = \tfrac{1}{2}Md(v^2) = dK.$$

From the former we have

$$F = \frac{\tfrac{1}{2}Mv^2}{s},$$

and from the latter

$$F = \frac{dK}{ds};$$

hence, generally, force is the rate of doing work per unit of length. But is force merely a rate? What shall be said of force as a stress, where no transference of energy takes place? That this definition is not elementary, but a mere deduction, is not only evident, but may be more forcibly shown by means of other deductions. Thus, from the former equation we have

$$s = \frac{\tfrac{1}{2}Mv^2}{F},$$

hence, *space* is the rate of doing work per unit of force (per pound).

Or again,

$$v = \sqrt{2\frac{Fs}{M}},$$

hence velocity is the square root of twice the rate of doing work per unit of mass!

Or, again, in regard to momentum, page 78,

$$Ft = Mv;$$

$$\therefore F = \frac{Mv}{t};$$

hence, force is the rate of producing momentum per unit of time.

Or, again,

$$t = \frac{Mv}{F};$$

hence, time is the rate of producing momentum per unit of force!

Now all of these are correct deductions; but the fundamental equations are established on the hypothesis that all, except velocity, are substantial quantities. The value of F is fundamentally measured in *pounds*. Indirectly it may be measured in a variety of ways as shown above, and as will be still further shown in Article 86 of the text.

Page 17, Arts. 31 and 32.—*Terrestrial Gravitation* as a force causes, or tends to cause, bodies to move towards the earth, and when a spring balance or other weighing machine is interposed to prevent any movement, the intensity of the *impelling* force may be determined in pounds or an equivalent.

Page 19, Art. 35.—The fundamental idea of a point of application of a force is that of a definite attachment, like the attachment of a rope, or chain, or rod of iron, to a body; but it is certain, in regard to the forces of nature—as gravity, chemical forces, etc.—that the conception is erroneous, for there is no attachment.

Still it may be conceived that the force acts upon a particle which may still be considered as the point of application, and thus the old term, with its gross associations, is useful in the most refined sense.

Page 20, Art. 36.—*Inertia* is a name merely to express the fact that matter has not of itself power to put itself in motion, or being in motion to bring itself to rest, or even to change its rate of motion. Yet some writers call *Inertia* a force, and others, with little if any more propriety, speak of the *force* of inertia. M. Morin, a French physicist, attempted to prove that *inertia* is a force. He took a prism standing on its base, and by a sudden pull or push applied to its base caused the prism to fall backwards. He argued that the falling over of the prism indicated the action of a force. A force might have been applied at the top of the prism which would have overturned it directly, and Morin argued, that when it overturned by a sudden action at the base, there must have been a force equivalent to one at the top, and this he called the force of inertia. The fact is, any force applied to a body, not acting in a line through its centre, tends to rotate the body; and in all cases where the body is free, will cause it to actually rotate. If the prism referred to, standing on its base, be acted upon by a force applied at any point above the base, it will not be overturned unless the *moment* of the force exceeds the moment of the weight in reference to the point about which it tends to turn. If the force applied at the base be so intense as to produce a rotary moment exceeding the moment of the weight above referred to, it will cause the prism to fall by rotating backwards; but if the force be less intense, it will simply cause the body to slide on the plane.

Again, inertia does not fulfill any of the conditions of a force. It is not an action between bodies. It is not an action in any sense. It cannot be measured by pounds. It is a *negation*—an entire lack of something—a lack of force. We repeat, it is not a force.

EXERCISES.

PAGE 22.

1. The least force. One object of some of these exercises is to enable the student to get a correct idea of the relation between force and the resultant motions of bodies by the inductive method. The student who has not correct notions of these relations will doubtless insist that it requires more force to move a large body than a small one; and he may go so far as to say that it will take 10 pounds of pull or push to move a body weighing 10 pounds. But the fact must be perceived that the smallest force (10 lbs. of push, for instance) will just as certainly move 100 lbs. of matter free to move in the direction of the push, as it will one pound. It will not move the former as rapidly as the latter—or, in other words, it will not move it as far in the same time. In this way we get an idea of the fact that the *visible effect* of a force depends conjointly upon the mass moved, and the space through which it is moved in a given time. If necessary make an experiment by suspending different weights with equally long strings, and pull them sidewise by a string attached to the body passing over a pulley—or edge of the

table—and holding at the suspended end a small weight. It will be found that any weight which is sufficient to overcome the friction of the string on the pulley—or edge of the table—will pull the heaviest weight sidewise. Observe that the experiment is simply to show *actual* movement, and not the amount of movement.

2. The least force. Also the least force would deflect it from its course. This shows that a force will have the same effect upon a moving body as upon one at rest.

PAGE 23.—3. Because it is opposed by an equal opposite force.

4. 100 pounds. A man once pulled a spring balance so as to indicate, say, 100 pounds. Another gentleman asserted that he could pull two balances attached end to end so that each would indicate 100 pounds. This he did to the astonishment of the observers, but their astonishment ceased when they found that it was no more difficult to pull two in that way than one.

5. Yes. The fact that the boat is in motion, does not affect the result. Hence we have the principle that "action and reaction are equal and opposite."

6. No. To show it (should there be doubt) assume that a string passes from each sled to the hand, then will the tension on *each* be less than 10 pounds, but on both strings it will be just 10 lbs. Conceive that the strings become *one* back to the first sled—the tension on the one will be 10 lbs.—but on the part back of the first, it will be the same as before—which was evidently less than 10 lbs. If the two sleds are

of equal weight the tension on the connecting cord will be 5 lbs. I have heard students assert that no tension could be produced unless there were a resisting force—showing that they had not yet a correct conception of the relation between forces and masses. It requires force to move a free body. In such cases I have asked them to conceive that a cord were attached to the moon, and that they pulled upon it; when they will severally admit that a pull of ten or more pounds may be easily exerted. If the sleds were not very heavy, it would not be possible for the boy to run sufficiently fast to maintain a constant pull of 10 pounds for a long distance; but if it can be done for a few feet only, it will answer the purpose of the illustration.

7. No tension.

Page 23, Art. 48.—The so-called Three Laws of Motion did not spring suddenly into philosophy. There was a long period of darkness succeeded by twilight and dawn before the truth shone out clearly. The *principles* of the three laws were known, and to a considerable extent realized, before Newton's time, but perhaps they had not been so clearly and sharply defined by any preceding *writer*, and much less had they been made the foundation of mechanical science; hence there is a certain propriety in calling them Newton's Laws. Correct notions in regard to these principles date from the time of Galileo. Prior to his day, the leading philosophy was Aristotelian. Aristotle flourished between 300 and 400 years before Christ. In his philosophy there was no distinct idea of *force* as a cause, much less any idea of a relation between the cause of motion and the momentum produced. He taught that a heavy body would fall faster than a lighter one—that when a body is thrown by the hand it ought to cease to move as soon as it left the hand were there no surrounding impulses, but that it continued to move because the hand sets in motion the air about the body, and that the air acted afterwards in impelling the body. He divided motions into Natural and Violent; the former of which is illustrated by a falling body, in which the motion is constantly increasing, and the latter by a body moving on the ground, where the motion is constantly decreasing. It must not, however, be inferred that philosophers had no idea of cause and effect. Some *general notions* of this kind have always been entertained.

Between the period of Aristotle and Galileo, many important principles were established. Archimedes (born 287 B. C.) developed some important properties of the Centre of Gravity, established the principle of the Lever, and some of the principles of Hydrostatics. History seems to show that the advanced position secured by this eminent philosopher was not maintained, and that little or no advance was made until the time of Stevin—or Stevainus, as commonly written (1548–1620). His determination of the conditions of equilibrium of the inclined plane is so ingenious, it is worth repeating. Consider two inclined planes having a common vertex and horizontal base. Conceive a uniform long chain to be placed on them, and joined underneath so as to hang freely. He showed that it would hang at rest without friction, because any motion would only bring it into the same condition in which it was at first. The part hanging below would evidently be in equilibrium by itself, hence if that part be cut off the remaining part will be at rest; hence the condition of equilibrium is—*the weights on each part must be exactly proportional to the lengths of the planes.* If one side becomes vertical the same proportion holds true.

Galileo forms the grand connecting link between the philosophers of the ancient and modern physical sciences. He was born at Pisa, February 18th, 1564, 39 years before the death of Michael Angelo, and 21 years after the death of Copernicus, and died on the 8th of January, 1642, the year in which Sir Isaac Newton was born. The *science of motion* began with him. He taught that motion was due to force—that all bodies in a vacuum would fall with equal velocities—that inertia of matter implies persistence of condition—he gave a satisfactory definition to momentum—also stated with approximate precision the principle that "action and reaction are equal"—also established the principle of "virtual velocities," which was made by Lagrange to include all of mechanical science in one expression. He made a mathematical analysis of the strength of beams, of projectiles, of the pendulum, of floating bodies, and of the inclined plane. For his investigations in other fields of science—see some biographical sketch.

PAGE 23, ART. 49.—*First Law of Motion.* Says Whewell, in his History of the Inductive Sciences: "It may be difficult to point out who first announced this Law in a general form." We have already seen that the facts involved in it were recognized by Galileo.

It is equivalent to saying that every change is due to a cause, and yet, to cover the entire ground, this statement needs modification. Motion is due to a cause, but when the cause ceases, the motion of a free body does not cease, it simply becomes uniform. *Change of position* is not then necessarily due to a coexisting cause, but may be due to a cause remote in time. *Change of condition*, however, requires a present acting cause; and the latter will produce either a change in the rate of motion, or of the direction of motion or of both.

The law cannot be proved by direct experiment, for it is practically impossible to remove from the body all acting forces, and hence uniform motion under the action of no forces is not realized by experiment. It may, however, be observed that the less the resistance the more nearly uniform will be the motion, and hence we are led to *infer* that if all resistance could be removed, the motion would be strictly uniform. Similarly, it is observed that a body projected on a very smooth plane moves so nearly in a straight line that we are led to *infer* that if there were no deflecting causes the path would be exactly straight. The law is the result of induction rather than of proof, but it appears so perfectly reasonable that we assent to it as soon as it is properly illustrated. The strongest proof of its correctness lies in the fact that deductions founded on this hypothesis agree with the results of observation.

The process of induction consists in conceiving clearly the law, and in perceiving the subordination of facts to it.

Page 23, Art. 50.—Second Law. *Change of motion is in proportion to the acting force.* If all bodies were of equal size it would only be necessary to consider their relative velocities in determining the effect of forces. But a larger body requires more force than a smaller one of the same substance to produce the same velocity under the same circumstances; in other words —mass and velocity are both involved—and the term *motion* here means *momentum.* This also agrees with Newton's explanation of the term. It is better, therefore, to word the law thus: *Change of momentum is in proportion to the acting force.* It must be particularly observed that the law does not assert that *momentum varies as the force;* but that it is a *change* of the momentum that varies as the force.

This law was clearly perceived by Galileo, and by means of it and the first law, he determined that the path of a projectile in a vacuum was a parabola. The law, however, was not considered fully established until the theory in regard to the motion of the earth, involving both this law and the law of universal gravitation, was realized. The triumph of both laws was complete at the same time.

Page 24, Art. 51.—Third Law. *Action and Reaction are equal and opposite.* The first and second laws refer to one body only; this involves two bodies. It asserts that an action between two bodies is of the same intensity upon each, but that the direction of action upon one is directly opposed to that upon the other. Every action implies an equal opposite action —one being called a reaction in reference to the other. No force, acting in one direction only, exists;

it is always accompanied by an equal but opposite action. No force is known to exist without the presence of matter. Force does not act in curved lines; the action and reaction between two particles is in the right line adjoining them.

Newton gave three examples illustrating this law:

If any one presses a stone with his finger, his finger is also pressed by the stone.

If a horse draws a stone fastened to a rope, the horse is drawn backward, so to speak, equally towards the stone.

If one body impinges on another, changing the motion of the other body, its own motion experiences an equal change in the opposite direction.

It does not seem rational that the stone will push the finger in the same sense that the finger presses the stone. The finger appears to be an active agent while the stone is inert. In the strictest sense we should say that, in the attempt to press a stone *a force is developed* between the finger and the stone, which force acts equally in opposite directions; in one direction against the finger, in the other against the stone.

Similarly, in regard to the horse and stone. In the former example the condition is statical, but in this the horse is supposed to move the stone. The horse, evidently, is not *actually* pulled backward, although there is an actual backward pull upon the horse by the rope. The fact is, that, in the effort to draw the stone, a force is developed which produces tension in the rope, which tension acts to pull the stone one way, and the horse the opposite way. As the horse is able to take a footing on the earth, he is able to exert a force on the rope equal to, or exceeding, that necessary to overcome the friction of the stone, and thus move the stone. The pressure between the horse and the earth also acts in opposite directions.

In the third illustration another idea is presented. *Motion* is used in the sense of *momentum*, as before stated, and it should read, *the change of momentum in two impinging bodies is the same in both bodies but in opposite directions.* This is a necessary result of the second and third laws. The forces being measured by the change in the momentum, and the pressures being equal between the impinging bodies, it follows that the changes in their momenta must be equal. Hence if one loses momentum the other must gain, and the loss in one case must equal the gain in the other.

Thus if the body whose mass is M_1 impinges upon another whose mass is M_2, v_1 and v_2, their respective velocities before impact, and v'_1 and v'_2 their respective velocities at any instant after the first contact, and assuming, as we will in establishing the formula, that the velocities are in the same direction, in which case it will only be necessary to change the sign of one of the velocities if they move in opposite directions; then will the momentum of M_1 before impact be

$$M_1v_1,$$

and after impact

$$M_1v'_1;$$

and as M_1 is supposed to be the impinging body it will lose velocity, and we have for the momentum lost by M_1

$$M_1v_1 - M_1v'_1.$$

Similarly, the momentum gained by M_2 will be

$$M_2v'_2 - M_2v_2,$$

which, according to the third law, must equal that lost by the former body; hence for all stages of the motion after the first contact, not only during compression, but also at and after separation, we have

$$M_1(v_1 - v'_1) = M_2(v'_2 - v_2).$$

If M_2 be the impinging body, we have

$$M_2(v_2 - v'_2) = M_1(v'_1 - v_1),$$

which easily reduces to the former equation. If they move in opposite directions before impact make v_1 or v_2 negative, the impact is here supposed to be direct and central, for which case the equations are true whether the bodies be elastic or non-elastic. Several cases are discussed on pp. 85–90 of the text.

(It appears that Whewell, in his History of the Inductive Sciences, has not drawn a sufficiently clear distinction between the second and third laws. He appears to hold that the third law gives a measure of the pressure or force, whereas the second law is the only one of the three that gives it).

Page 24, Art. 52.—We are not informed who first gave the laws for the composition and resolution of forces; but Galileo was one of the first to make use of them in explaining curvilinear motions. The method was systematized by Descartes by the aid of the systems of rectilinear axes.

EXERCISES.

PAGE 26.

1. Because the force of gravitation draws it from the rectilineal path in which it was projected.
2. The least force.
3. 500 pounds.

PAGE 27.

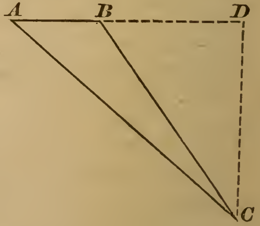
4. Midway between their initial positions.
5. He must aim to walk southwesterly. To find the direction, draw a line, AB, of any length to represent the one mile due east, and a line, AC, perpendicular to AB, and of such length that the hypothenuse, BC, will be three times as long as AB; then will the angle ABC represent the required direction.

We have

$$\cos B = \frac{1}{3};$$

$$\therefore ABC = 40° \ 32°.$$

6. Make $AB = 3$, the angle $DBC = 45°$, and BC equal to 8; join AC, then will AC represent the resultant direction. A numerical solution gives $DAC = 33° \ 10'$ and the course will be S. 56° 50′ E.



$BD = DC = 8 \ \sin 45° = 8 \times \frac{1}{2}\sqrt{2} = 4\sqrt{2}$

$AD = 3 + 4\sqrt{2}$

$\tan. DAC = \frac{4\sqrt{2}}{3 + 4\sqrt{2}} = \frac{32 - 12\sqrt{2}}{23} =$

$0{\cdot}6535$; $\therefore DAC = 33° \ 10'$.

The velocity will be 10·34 miles per hour.

7. It will be $10 \times \cos 45° = \frac{1}{2}\sqrt{2} \times 10 = 5 \times 1.4142 + = 7.0710 +$.

8. Yes, and will move towards the rear end of the car (First Law). Because it tends to preserve the velocity which it had just before the collision.

9. In the first edition F_2 was 30 lbs. It was intended to be 20 lbs. so as to show that the resultant of two velocities may be the same as those producing it. In this case the triangle of velocities will be equilateral, and hence each of the angles will be 60°.

If 20 and 30 pounds be used, we have for the diagonal of the parallelogram

$$R = \sqrt{900 + 400 + 2 \times 20 \times 30 \times 0{\cdot}5}$$
$$= 43{\cdot}59 \text{ lbs.};$$

and for the angles 36° 35′ 12″, and 23° 24′ 48″ respectively.

PAGE 27, ARTS. 59, 60.—The intensity of a force is known only by its results.

PAGE 29.

ART. 63.—The case of a force acting normal to the path of a body was a difficult one with the philosophers living about Galileo's time, and is not the easiest to explain by elementary processes at the present time. It will be considered in Chap. xvi., p. 220 of the text.

Art. 64.—We say that gravity tends to draw bodies towards each other; but we only know that it causes them to move towards each other when free. It is quite as proper to speak of its *pushing* as of *drawing* them towards each other.

Some attempts have been made to explain the essential nature—or the cause—of gravitation. One of the most celebrated of these theories was given by one Le Sage. According to his theory, lines of force acted in all conceivable directions through space, and as two bodies intercept those lines which would pass through both bodies, there would be more force to drive them towards than from each other. (See *Theories of Gravitation* by Wm. B. Taylor, of the Smithsonian Institute, Washington, D. C.) But no theory thus far suggested is considered sufficient to account for the fact of gravitation.

Page 29, Art. 65.—The law of universal gravitation is one of the discoveries which aided in immortalizing the name of Sir Isaac Newton. He first conceived the nature of the law, and then proceeded to prove it. He assumed that if the law was correct it ought to explain the circular motion of the moon about the earth—in other words, that the pulling force (so to speak) of gravity at the moon would be just sufficient to draw it the required amount, from a tangent to the orbit.

If g be the acceleration of a pulling body at the surface of the earth, at the moon it will be $g \div D^2$—where D is the number of diameters of the earth between the center of the earth and center of the moon, and is about 60·3612; and as $g = 32·246$ ft., we would have at the moon, the acceleration $32·246 \div (60·3612)^2 = 0·0088 +$. The force at the moon which would produce this acceleration equals the centrifugal force, and is given by the last equation of Art. 315 of the text, and is

$$Force = m\,\frac{4\pi^2 r}{T^2},$$

where r is the radius of the orbit, T the time of a complete revolution, and m the mass of the body. But the *force* divided by the *mass* equals the *acceleration*—as is shown by the last of the equations of Article 86;

$$\therefore acceleration = \frac{Force}{m} = \frac{4\pi^2 r}{T^2},$$

which applied to the moon, and reduced, gives 0·0089 + (see Art. 319). The two results should agree if the law and data are both correct. It will be seen that the value of the radius of the earth enters into the computation, the correct value of which was not known to Newton at the time of his first investigations. Some fifteen years after he began the investigation, while attending a lecture in London, he obtained the correct value of the radius, with which he proved the truth of his proposition.

The analysis by which the above result is reached is anticipated, but it will enable the reader to understand why an error in the true value of the radius of the earth vitiated the first result.

It is questionable whether the story—that Newton conceived the law of gravitation by seeing an apple fall from a tree—so often taught to juveniles, is not purely fictitious. It is certain that he did not consider the law established for fifteen years after he first conceived it, during which time, it is said, he reviewed his work many times.

Page 30, Art. 67.—A history of pendulum experiments would furnish material for a book. The *mathematical pendulum* is an ideality, but a very useful one in discussing the subject. Compound pendulums are necessarily used in making experiments. The most practical method of determining the acceleration due to gravity is by means of a pendulum; and some of the results thus found are given on p. 244 of the text. The length of the seconds' pendulum has also been used for determining the standard of linear measure. Thus, the English law requires that the length of the yard shall be to the length of the simple pendulum vibrating seconds at the Tower of London reduced to the level of the sea as 36 to 39·13908. (See Art. 328.)

Page 31, Art. 68.—The formula

$$g = 32{\cdot}1726 - 0{\cdot}08238 \cos. 2L,$$

is given in *The Mécanique Céleste*, Tome iii. v., § 42, [2,049].

EXAMPLES.

PAGE 35.

1. $h = \frac{v^2}{2g} \therefore v = \sqrt{2gh} = \sqrt{2 \times 32\frac{1}{6} \times 100} =$ 80·20 feet.

2. $t = \frac{v}{g} = \frac{300}{32\frac{1}{6}} = 9{\cdot}3$ seconds.

3. $t = \frac{v_\circ}{g} = 3{\cdot}109$ seconds.

$h = v_\circ t - \frac{1}{2}gt^2 = 100 \times 3{\cdot}109 - \frac{1}{2} \times 32\frac{1}{6} \times (3{\cdot}109)^2 =$ 155·4 feet.

PAGE 36.

4. Acceleration $= 32\frac{1}{6}$ ft. per sec.
$= 32\frac{1}{6} \times 60 = 1{,}930$ ft. per minute.

5. Acceleration per second $= 32{\cdot}16666$ feet, $= 32{\cdot}16666 \div 3{\cdot}28 = 9{\cdot}807 +$ metres. For 4 seconds, $9{\cdot}807 \times 4 = 39{\cdot}228$ metres.

6. $h = \frac{1}{2}gt^2 + v_\circ t$, (Eq. (6) p. 34.);

$$\therefore 120 = \frac{1}{2} \times \frac{193}{6} t^2 + 25t.$$

Solving for t we find

$$t = 2{\cdot}0627 + \text{sec.}$$

Also (Eq. (11) p. 13.)

$$v = v_\circ + gt$$

$$= 25 + \frac{193}{6} \times 2{\cdot}0627$$

$$= 91{\cdot}35 \text{ feet.}$$

7. $$150 = \tfrac{1}{2} \times \frac{193}{6} t^2 + 25t\,;$$

$$\therefore t = -\frac{150}{193} + \sqrt{\frac{1800 \times 193 + 22500}{193^2}}$$

$$= \frac{-150 + 608{\cdot}19}{193}$$

$$= 2{\cdot}37 \text{ seconds.}$$

8. For the falling body (Eq. (2) p. 34)

$$h = \tfrac{1}{2}gt^2\,;$$

and for the body projected (Eq. (8) p. 34)

$$h = vt - \tfrac{1}{2}gt^2.$$

Eliminating h gives

$$v = gt\,;$$

but

$$t = \frac{2h}{v}\,;$$

hence eliminating t gives

$$v^2 = 2gh\,;$$

$$\therefore v = \sqrt{2h{\cdot}g}$$

$$= \sqrt{AB{\cdot}g}.$$

9. The sound will be $\frac{h}{1130}$ seconds in returning; hence the time of falling will be

$$4 - \frac{h}{1130} = t$$

$$= \sqrt{\frac{2h}{g}}, \text{ (Eq. (4) p. 34)};$$

$$\therefore \left(4 - \frac{h}{1130}\right)^2 = \frac{2h}{g},$$

or

$$h^2 - (1130)^2\left(\frac{8}{1130} + \frac{1}{16\frac{1}{12}}\right) h = -16 \times (1130)^2;$$

$$\therefore h = 231{\cdot}6 + \text{feet}.$$

10. Let $x =$ the distance upward from the lower point to the point of meeting; then will the point of meeting be $a - x$ from the upper point. If t be the time of meeting, we have for the falling body

$$a - x = vt + \tfrac{1}{2}gt^2,$$

and for the body projected upward

$$x = Vt - \tfrac{1}{2}gt^2.$$

Adding we have

$$a = (V + v)\, t;$$

$$\therefore t = \frac{a}{V + v};$$

which in the second equation gives

$$x = \frac{a}{V + v}\left(V - \tfrac{1}{2}g\frac{a}{V + v}\right).$$

The distance below the highest point will be

$$a - x = \frac{a}{V + v}\left(V + \tfrac{1}{2}g\,\frac{a}{V + v}\right).$$

EXERCISES.

PAGE 36.—1. He will. To draw it at a uniform velocity he must overcome the friction only, but at an increasing velocity he not only overcomes the friction but also exerts additional force to overcome the inertia of the mass (see Second Law).

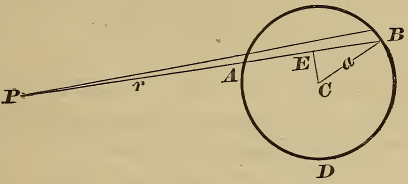
2. Attraction is more.
3. Because the air resists less.
4. $\frac{1}{180}$ ounce.

PAGE 37.

5. 10 inches.
6. More, for the force of gravity is less at the equator than at the poles.
7. It will neither gain nor lose in weight if weighed with the same beam scales. It will lose in weight if weighed with the same spring balance, for the resistance of the spring will remain constant, while the force of gravity will be less.
8. 9·807 + metres per second.

PAGE 38, ART. 78.—In the last two lines of the page it is *assumed* that the attraction of a *sphere* upon an external particle varies inversely as the square of the distance from the center of the sphere. We here submit a proof of the truth of the statement. Newton, in his Principia, proved the proposition of the statement geometrically; we now use the calculus.

Let ABD be a spherical shell, center C, radius a, P the position of an external particle. Conceive two consecutive radial lines drawn from P, cutting the shell in the points A and B.



Proof. Let ds be an element of length of the circle at A, $PA = r$, $PC = c$, $CB = a$, $k =$ thickness of shell, $\delta =$ density. Conceive a line joining P and C, $\theta = APC$, and $y =$ perpendicular from A upon PC.

Then

$2\,\pi ykds =$ volume of shell generated by the revolution of A about PC,

$2\,\pi y\delta kds =$ mass thus generated.

The attraction upon the particle will be

$$-\frac{2\,\pi\delta kyds}{r^2},$$

which resolved along the axis PC gives

$$-\frac{2\,\pi\delta kyds}{r^2}\cos\theta.$$

Let $p = CE =$ the perpendicular from C upon PB, then

$$p = c\sin\theta\,;\ \therefore\ dp = c\cos\theta\,d\,\theta\,;$$

also $$r^2 - 2rc\cos\theta + c^2 = a^2\,;$$

$$\therefore\ \frac{dr}{d\theta} = -\frac{rc\sin\theta}{r - c\cos\theta};$$

and $$\frac{ds}{d\theta} = \frac{ar}{r - c\cos\theta},$$

(for $$ds^2 = dr^2 + r^2d\theta^2).$$

Hence

$$\frac{2\pi\delta k y ds}{r^2}\cos\theta = \frac{2\pi\delta k y \cos\theta}{r^2}\cdot\frac{ard\theta}{r - c\cos\theta}$$

$$= \frac{2\pi\delta k y a dp}{cr(r - c\cos\theta)}$$

$$= \frac{2\pi\delta k a}{c^2}\times\frac{pdp}{\sqrt{a^2 - p^2}}$$

which gives the attraction for a circular element of the shell; hence for the entire shell we have

$$-\frac{4\pi\delta k a}{c^2}\int_o^a \frac{pdp}{\sqrt{a^2 - p^2}}$$

$$= \frac{4\pi\delta k a^2}{c^2};$$

hence for a constant radius a, the attraction varies inversely as the square of the distance of the particle from the center of the shell, which was to be proved.

If $c = a$, we have

$$4\pi\delta k,$$

hence the attraction of a spherical shell upon a particle in its surface is independent of the dimensions of the shell.

To find the attraction of a homogeneous sphere upon an external particle, make $k = da$ and integrate, and we have

$$\frac{4\pi\delta}{c^2}\int_o^a a^2 da = \frac{4}{3}\frac{\pi\delta a^3}{c^2}$$

$$= \text{volume} \times \frac{\delta}{c^2}.$$

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ART. 79.—*Weight*, to an uneducated person, is conceived to be essential to matter. Such an one would doubtless assert that a body falls to the earth because it is heavy; but to the student of mechanics, weight is nothing but the measure of a force, the magnitude of which depends upon the quantity of matter constituting the body.

PAGE 41.

ART. 83.—The *unit of mass* might be the piece of platinum which is used as the standard pound (see Art. 33), but as we have occasion to compare the force of gravity at different places, and as the force of gravity at London is assumed to be $32\frac{1}{6}$ feet, we have chosen to consider the unit of mass as about $\frac{1}{32\frac{1}{6}}$ of the weight at that place.

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ART. 85.—*Density* is sometimes used in the sense of specific gravity, and if the density of water be taken as unity, the specific gravity of any substance (compared with water) will equal its density. We prefer to make the definition conform to the sense used in mechanics. The following are the definitions given by several authors:

By the density of a body is meant its mass or quantity of matter compared with the mass or quantity of matter of an equal volume of some standard body arbitrarily chosen.—TOWNE'S *Elementary Chem.*, p. 29.

Density is a term employed to denote the degree of proximity of the atoms of a body. Its measure is the ratio arising from dividing the number of atoms the body contains by the number contained in an equal volume of some standard substance whose density is assumed as unity. The standard substance usually taken is distilled water at the temperature of 38°.75 F.—BARTLETT'S *Elements of Analytical Mechanics*, p. 29.

Enfin si l'on représente par D la masse, sous l'unité de volume du corps que l'on considère, D sera ce qu'on nomme la Densité de ce corps. On prend communément pour unité de densité celle de l'eau destillée à cette dernière température (4° du thermometre centigrade).—POISSON, *Traité de Mécanique*, page 108.

The quantity of matter in a body does not depend on the size of the body only, but also on the closeness with which the particles are packed. This difference is defined as a difference of density. Thus there is more matter in a cubic inch of lead than in a cubic inch of oak, and this is expressed by saying that the *density* of lead is greater than the density of oak.—MAGNUS, *Lessons in Elementary Mechanics*, p. 60.

Experiment shows that the weight of a certain volume of one substance is not necessarily the same as the weight of an equal volume of another substance. Thus seven cubic inches of iron weigh about as much as five cubic inches of lead.

We say then that lead is denser than iron, and we adopt the following definitions. When the weight of any portion of a body is proportional to the volume of that portion, the body is said to be of uniform density. And the densities of two bodies of uniform density are proportional to the weights of equal volumes of the bodies.—TODHUNTER, *Mechanics for Beginners*, p. 7.

The density of a body is the mass comprised under a unit of volume.—SILLIMAN, *First Principles of Philosophy*, p. 67.

Density is the quantity of matter contained in a unit volume; the absolute density or the closeness with which the particles are packed being uniform throughout that unit volume. This definition is directly applicable if a body is homogeneous; but if it is heterogeneous, and the density varies from point to point, the density at any point is the quantity of matter contained in a unit volume throughout which the density is the same as that at the point. Density is usually measured by means of comparison with some substance the density of which is assumed to be the unit-density.—PRICE, *Infinitesimal Calculus*, p. 164.

The quantity of matter in a body, or as we now call it, the mass of a body, is proportional, according to Newton, to the volume and the density conjointly. In reality the definition gives us the meaning of *density* rather than of mass, for it shows us that if twice the original quantity of matter, air for example, be forced into a vessel of given capacity, the density will be doubled and so on. But it also shows us that, of matter of uniform density, the mass or quantity is proportional to the volume or space it occupies.—THOMSON AND TAIT, *Treatise on Natural Philosophy*, p. 162.

Heaviness (Fr., densité, Ger., dichtigkeit) is the intensity with which matter fills space. The heavier a body is, the more matter is contained in the space it occupies. The natural measure of heaviness is that quantity of matter (the mass) which fills the unity of volume; but since matter can only be measured by weight, the weight of a unit volume, *e. g.* of a cubic meter or of a cubic foot of another matter, must be employed as a measure of its *heaviness.*

The product of the volume and the heaviness is the weight.

The heaviness of a body is uniform or variable according as equal portions of the volume have equal or different weights.—WEISBACH, *Mechanics of Engineering*, p. 160.

The *density* of a body is the degree of closeness between its particles. The term depends upon the hypothesis that the ultimate particles of matter have weight, and therefore mass proportional to their bulk. It coincides with specific gravity.—PROF. NICHOL, *Cyclopædia of the Physical Sciences*, page 177.

On sait que le poids d'un corps varie avec l'intensité de la pesanteur mais que sa masse ne varie pas. Sous l'influence de la même pesanteur, par exemple en un même lieu du globe, le poids est évidemment proportionnel à la masse et le rapport des poids de deux substances sous le même volume sera précisément celui de leurs masses sous le même volume ; de lá, la synonymie, qui existe entre les mots poids spécifique et densité, qui exprime ses rapports.—WURTZ, *Dictionnaire de Chimie.* Sous Densité.

The *density* of a body is the ratio of its mass to its volume.—SMITH, *Elementary Treatise of Mechanics*, p. 44.

EXERCISES.

PAGE 43.

1. The matter outside of one-half the radius would produce no effect; and that within would attract as if it were all at the centre. The sphere of one-half the radius will contain one-eighth the matter, and the inverse square of the distance will be 4; hence the weight will be $\frac{1}{8} \times 4 \times 10 = 5$ lbs. We get this result more directly by saying, as in Art. 78, that the attraction will be directly as the distance from the centre.
2. $10 \div 2^2 = 2\frac{1}{2}$ lbs.
3. Nothing.
4. At a distance from the outside somewhat less than half the thickness of the shell. The exact distance cannot be found unless the thickness of the shell be given. (If R be the outside radius, r_1 the inside, and r the required distance from the centre; then

$$\frac{\frac{4}{3}\pi\,(r^3 - r_1^3)}{r^2} : \frac{\frac{4}{3}\pi\,(R^3 - r_1^3)}{R^2} :: 5 : 10$$

or

$$2R^2\,(r^3 - r_1^3) = r^2\,(R^3 - r_1^3);$$

which is a cubic equation, and may be solved by Cardan's method.)

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5. *Yes* for the first answer; *No* for the second.
6. On the opposite side. He could not stop at the

centre by a mere effort of the will. Some entertain the idea that the will of a person, causing a movement of parts of the body, could, in a measure, control the movement of the body as a whole; but, as a fact, the matter of the body is subject to the action of forces, like any other matter. No part of the body can be moved except in accordance with the three laws of motion. If an arm is moved in one direction, some other part of the body will be moved in an opposite direction (the body being free). While passing across the hollow referred to in the Exercise, the person might throw his arms about, or kick, or perform the evolutions of swimming, or rowing; but as there is supposed to be no matter, or no body, for him to act against, he could not change his rate nor direction of movement. In drawing his feet forward he would necessarily pull some part of the body backward. It is shown, by higher analysis, that the centre of gravity of a moving system is unaffected by the mutual actions of internal forces—so, in this case, the motion of the centre of gravity of the person would be entirely unaffected by any contortions the person might make.

7. At the centre of the sphere. At the same point. Uniform.

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8. The ball. If the ball were very small compared with the person, the movement of the person might be neglected compared with that of the ball; just as the motion of the sun is neglected compared with that of the planets. In the case of the planets, the pulling force is dependent upon their masses and dis-

tances, but in the case of the person this force is dependent upon his muscular exertion.

9. He could not. In the effort to throw the ball away, a force is developed between the hand and ball, which acts equally between the ball and hand, but in opposite directions in accordance with the *third law;* and hence the ball would move one way and the person the opposite way; and both would move in straight lines in accordance with the *first law,* and their relative velocities will be inversely as their masses in accordance with the *second law.*

If the person were placed at rest in any position in the hollow and unable to reach anything, he could not turn over, nor change ends; that is, if his head were towards the north and his feet towards the south, he could not so change as to have his head towards the south and his feet towards the north. Should he attempt to so turn as to bring his head towards the south, he would cause his feet to approach his head, and they would meet about half way. He might succeed in kicking his own head, or, if he were very flexible, the head and feet might pass each other; but *the body could not turn over so as to change ends. Neither could he roll over.* If one end of the body turns one way, the other end necessarily turns the opposite way.

If a cat be held with her back down two feet or so from the floor, and dropped, she will strike on her feet; how does she do it? According to the principles of mechanics, if there were nothing for her to act against it would be impossible for her to turn, and she would necessarily strike on her back.

While experimenting, I was surprised to find how near the floor the cat might be held, and often apparently perfectly unwatchful when dropped, and yet alight on her feet. Now the movement of the body should be accounted for on mechanical principles. Instinct operates quicker than reason, and it appears to be certain that the cat, instinctively, initiates a rotation of her body at the instant she is dropped. While it is difficult to see how muscular action can be quick enough to produce this result, yet I see no other way of accounting for the rotation. Suspend the cat with a string at each foot, then suddenly cut the strings, and she will rotate herself. It is also an interesting fact that she will strike on her feet if let fall several feet, say six or eight feet. Now if she had the same initial rotation when falling six feet, as when falling two feet, why would she not turn too far in the former case and strike on her side? To explain this, we here state a principle not yet proved in this course. *If a rotating body self-contracts, it will rotate more rapidly, but if it self-expands it will rotate more slowly.* The cat has the ability, within a limited range, of expanding or contracting the transverse dimensions of her body, and to that extent of regulating the amount of her rotation.

Consider still further the relations of the man and ball in the ninth exercise. He puts his hand in close contact with the ball, but without grasping it, and they move away from each other, until both strike the walls of the hollow. Then suppose that the man springs for the ball and seizes it, and then springs towards the centre of the sphere, but before reaching the opposite side throws the ball in anger. If the ball goes in a direction perpendicular to the line of his body (or, generally, in any line not passing through the centre of his body), the man will be thrown into a rotary motion as well as a motion of translation, and he will inevitably perform somersaults while backing away to the opposite side of the hollow.

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10. $100 \div 32\frac{1}{6} = 3\frac{21}{193} = 3{\cdot}109.$

11. $Density = \dfrac{Mass}{Volume} = \dfrac{200 \times 6}{193 \times 2} = 3{\cdot}109.$

12. $Mass = \dfrac{2{\cdot}2 \text{ lb} \times 5 \times 6}{193} = {\cdot}342.$

13. $Density = \dfrac{3 \times 2{\cdot}2 \times 6}{193 \times 35{\cdot}3} = {\cdot}00581.$

14. If the resistance of the air be considered it would. In the second case, it would not stop, but would go from surface to surface with the regularity of a pendulum. (See text, p. 249.) In the third case the velocity would be greatest at the centre, if the hole be a vacuum; but if it be filled with air, the greatest velocity would be passed before the ball reached the centre.

ART. 86. The value of $F = Mf$ is sometimes called the *absolute measure of force*, but nothing is gained by the term, except that it distinguishes it from the mere stress which it would produce if no motion resulted. Some English writers call the value of F when thus expressed *The Poundal*, but this term has not come largely into use.

Observing that acceleration is the rate of increase of velocity (Key, p. 6), it follows that this value of F is the same as that given by Newton's *second law*, that the force is proportional to the change of momentum produced. But we are confident that the constant use of the term *acceleration* for *rate of change of velocity* possesses great advantage, since *rate of change* is liable to be considered the same as actual velocity. Indeed some text books assert that the momentum of a body is a measure of the force acting upon it—and call it the *second law*. Now a body may have momentum when no force is acting upon it. It is the *rate of change of momentum* that measures the force producing the change; in other and better words, *the mass into the acceleration.*

Observe that the establishment of this equation contains a very important principle. There is, strictly speaking, no relation between pounds and feet; but the ratio of two weights may be the same as the ratio of two linear measures. A ratio is an abstract number, and often serves to connect concrete quantities, forming an equation. Thus, in this case,

$$\frac{F}{W} = \text{a ratio, a mere number,} = \frac{f}{g}.$$

The equation being established, it is operated upon algebraically. This use of ratio has many applications in physics.

ANSWERS TO EXAMPLES.

PAGE 47.

1. *Velocity* $= \sqrt[2]{\left(\frac{2 \times 2000 \times 193 \times 1}{6 \times 100}\right)} =$

$\sqrt[2]{1286 \cdot 66} = 35 \cdot 87$ feet.

2. *Space* $= \frac{25 \times 193 \times 100}{6 \times 2 \times 500} = \frac{965}{12} =$

$80 \cdot 42$ or $80\frac{5}{12}$ feet.

3. *Velocity* $= \sqrt[2]{\left(\frac{2 \times (25 - 10) \times 193 \times 100}{6 \times 500}\right)} =$

$\sqrt[2]{193} = 13 \cdot 88$ feet.

4. *Force* $= \frac{100 \times 100}{32 \times 10} = 31 \cdot 25$ lbs.

SOLUTIONS OF PROBLEMS.

PAGE 50.

4. The tension equals the weight, P, plus the force which will produce the acceleration. $\frac{W - P}{W + P} g$ is the acceleration when P is raised vertically. The mass multiplied by the acceleration is the moving force, or $\frac{P}{g} \cdot \frac{W - P}{W + P} g$; hence the tension is $P + \frac{W - P}{W + P} P$ $= \frac{2WP}{W + P}$. Similarly, it equals W minus the accel-

erating force, or $W - \frac{W - P}{W + P} W = \frac{2WP}{W+P}$.

5. The effective moving force is $W - T$, hence from Prob. 2, $W - T = \frac{W}{g} f.$

Substitute $f = \frac{1}{4}g$, and $W - T = \frac{1}{4} W$;

$\therefore T = \frac{3}{4} W.$

If ascending, $T - W = \frac{W}{g} f$, or $T - W = \frac{1}{4} W$;

$\therefore T = 1\frac{1}{4} W.$

$T = W +$ the force which will produce the acceleration $= W + \frac{W}{g} \frac{1}{4}g = \frac{5}{4} W.$

EXAMPLES.

PAGE 50.

1. $s = \frac{1}{2} \cdot \frac{P}{2P} gt^2 = \frac{193 \times 25}{4 \times 6} = 201\frac{1}{24}$ ft.

2. $s = \frac{1}{2} \cdot \frac{2P - P}{2P + P} gt^2 = \frac{1}{6} \times \frac{193}{6} \times 25 = 134\frac{1}{36}$ ft.

3. 1° $s = \frac{1}{2} \cdot \frac{P - W}{P + W} gt^2$, or $P = \frac{gt^2 + 2s}{gt^2 - 2s} W$

$= \frac{\frac{193}{6} \times 25 + 2 \times 10}{\frac{193}{6} \times 25 - 2 \times 10} 50 = 52{\cdot}55$ lbs.

2° $s = \frac{1}{2} \frac{W - P}{W + P} gt^2$, or $P = 47{\cdot}57$ lbs.

4. $s = \frac{1}{2} \frac{P - W}{P + W} gt^2$, or $g = \frac{2s(P + W)}{(P - W)t^2}$

$$= \frac{2 \times 6{\cdot}8 \times 9\frac{1}{2}}{\frac{1}{2} \times 4} = 64{\cdot}6 \text{ ft.}$$

5. $$1 = \frac{2 \times 10 \times P}{P + 10}; \therefore P = \tfrac{10}{19} \text{ lb.}$$

$$5 = \frac{2 \times 10 \times P}{P + 10}; \therefore P = 3\tfrac{1}{3} \text{ lbs.}$$

$$10 = \frac{2 \times 10 \times P}{P + 10}; \therefore P = 10 \text{ lbs.}$$

$$20 = \frac{2 \times 10 \times P}{P + 10}; \therefore P = \infty.$$

If the tension exceeds 20 lbs. P will be negative.

6. $$s = \tfrac{1}{2} \frac{P}{P + W} gt^2; \therefore t = \sqrt{\frac{2s(P + W)}{Pg}}.$$

$$= \sqrt{\frac{2 \times 10 \times 22}{2 \times \frac{193}{6}}} = 2{\cdot}615 \text{ seconds.}$$

ANSWERS TO EXERCISES.

PAGE 51.

1. The balance will indicate no tension. This question was given, because the author sometimes found that, in a case like Fig. 21, some students would assert that the tension of the string ought to be P; but by taking an extreme case, like the one in the exercise, the fallacy would be apparent. One of the best conditions of mind for searching after truth is to be convinced of one's error. When one admits that

his position is erroneous, he is generally in a condition to admit the truth.

2. If the acceleration is increasing, the tension will exceed the weight; if uniform it will equal the weight; if decreasing it will be less than the weight.

3. It will be less than his weight while descending, and greater if ascending. In both cases, if the motion be uniform, it will equal his weight.

4. The tension will be less than his weight.

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ART. 91.—It has already been stated that, in the relations between force and motion, we have four fundamental elements, force, space, time, and mass. Force and matter are so intimately connected that it is impossible to completely divorce them; but force may be abstracted from space and time—not in the sense that it can exist without them, but in the sense that it may be considered independently of them, and when so considered it is called *stress.* But a force acting upon a body may move it through space, and the space may be considered independently of time. The product of force and space both considered in the same line, *and abstracted from time, is work.*

The term originated from the grosser ideas of labor, but the definition given in the text is applicable to the most refined actions in mechanics. All known forces in nature are constantly working. Thus, rivers wear the beds of their streams; wind drives the sail, uproots trees, produces drifts of sand, etc.; heat expands bodies, and may overcome their cohesion, etc.

According to the definition, a man merely supporting a weight cannot be said to work, and yet he soon becomes conscious of fatigue. But a more critical examination of his case shows that the weight is not strictly at rest. The natural action of his organism, especially the beating of his heart, causes slight elevations and depressions of his load, so that he is, in the strictest sense, constantly laboring.

The following are some examples, of the average work accomplished by a man under various conditions.

WORK OF MAN AGAINST KNOWN RESISTANCES.

	lbs.	hrs per day.	ft. lbs. per day.
1. Raising his own weight up stairs or ladder	145	8	2,088,000
2. Hauling up weights with rope, and lowering the rope unloaded	40	6	648,000
3. Lifting weights by hand	44	6	522,720
4. Carrying weights up stairs returning unloaded	143	6	399,600
5. Shoveling up earth to a height of 5 ft. 3 in.	6	10	280,800
6. Wheeling earth in barrow up slope of 1 in 12, one-half horiz. veloc. 0.9 ft. per sec., and returning unloaded.	132	10	356,400
7. Pushing or pulling horizontally (capstan or oar)	26.5	8	1,526,400
8. Turning a crank or winch	18	8	1,296,000
9. Working pump	13.2	10	1,188,000
10. Hammering	15	8 ?	480,000

PERFORMANCE OF A MAN TRANSPORTING LOADS HORIZONTALLY.

	lbs.	hrs. per day.	lbs. con'd 1 ft.
11. Walking unloaded, transport of own weight	140	10	25,200,000

	lbs.	hrs. per day.	lbs. con'd 1 ft.
12. Wheeling load in 2-wheel barrows, return unloaded	224	10	13,428,000
13. Wheeling load in 1-wheel barrow, return unloaded	132	10	7,920,000
14. Traveling with burden	90	7	5,670,000
15. Carrying burden, returning unloaded.	140	6	5,032,800

See Rankine's *Steam Engine*, pp. 84–85, where the rate of doing the above works is also given.

Morin and Weisbach give 2,387 ft. lbs. per minute as the work which a man is capable of doing when working eight hours consecutively. This equals $\frac{2387}{33000}=0.07+H.P.$ nearly.

At an experiment made at Dresden in 1880, men working only 2 minutes at a time on a hand fire-engine did 0·277 *H.P.*—or nearly four times that given above.

Page 54, Art. 98.—Nystrom asserts in his writings, that work is not independent of time, for it requires time to move a body over space; also that if one horse drew twice the load over the same space as another, he did twice the work and was twice as efficient. But we hold, and trust we have clearly shown, that time is properly abstracted in the idea of work—that efficiency is very different from work. We have also shown that if time is considered even *implicitly*, the velocity must also be considered.

Art. 99.—The simple definition—Power is *rate* of doing work—is coming more generally into use.

Page 56, Art. 104.—Friction is a *force;* its value can be measured in pounds. It does not directly produce a positive acceleration, but a negative one. Its direct office is to destroy motion; not to produce it. But indirectly it may produce motion by producing heat. Heat produces motion, and the work of friction has its equivalent in heat, and this heat if collected would produce the same motion as that which it has destroyed. But in practice it is so quickly dissipated that, in most cases, it is apparently lost.

A *smooth* surface is one from which the idea of *roughness is abstracted.*

PAGE 58, ART. 107.—The laws of Morin are only approximately correct. In machinery, the character of the surfaces in contact, the mechanical execution of the fitting up, and of the lubricants, are each and all important elements. See practical treatises and articles upon the subject. Rankine, *Steam Engine*, pp. 14–18, Thurston, *Friction and Lubrication.*

PAGE 59, ART. 109.—The frictional resistance of railroad trains is principally rolling friction under good working conditions. A railroad train in good order, and on a good road, will not be safe against starting under the action of gravity alone, unless the gradient is less than eighteen or twenty feet to the mile; once started it will continue in motion on gradients as low as thirteen feet to the mile. The coefficient of rolling friction for trains in good order is $\frac{13}{5280} = 0{\cdot}0025$, or less than six pounds per ton. The resistance at starting is $\frac{20}{5280} = 0{\cdot}0038$ or $8\frac{1}{2}$ pounds per ton.

The resistance of a locomotive is about 12 pounds per ton.

The resistances on railroads, under average conditions, and including all forms of resistances, are given by Clarke.

When the permanent way is straight, rails dry and clean, he gives for trains only

$$R = 6 + \frac{v^2}{246};$$

for engine and train

$$R = 8 + \frac{v^2}{171};$$

where R is in pounds per ton gross, and v the velocity of the train in miles per hour. (*Manual of Rules, etc.*, p. 965.) A Mr. Hughes found on an English "tramway" a resistance of twenty-six pounds per ton.

On railroads, frictional resistances are sometimes greatly increased by the resistance of the air, called "head resistance," and amounts, in pounds per square foot of front exposed, to 0·005 of the square of the velocity in miles per hour with which the air meets the head of the train. Side winds often increase the flange resistance seriously.

The value of the coefficient of friction on ordinary railroads is 0·003, on well laid railroad tracks 0·002, on best possible railroad 0·001.

Mr. S. Whinery (Trans. Am. Soc. C. E., April, 1878) gives a formula for the total resistances of a train running on curves,

$$R = D\frac{(g + t)}{25} + a\,n,$$

where R = total resistance, D = degrees of curvature, g = gauge of track, t = length of rigid wheel base, a

and n are quantities expressing resistances due to accidental and irregular conditions. These resistances are inversely as the radius of curvature, directly as the load, and nearly independent of the velocity.

Mr. O. Chanute (Trans. Am. Soc. C. E., April, 1878) analyzes this increase of resistance as follows:

Due to twist of wheel...........	0·001
" slip "	0·1713
" flange friction............	0·2450
" loss of couplings.........	0·0213
Total................	0·4386

Loose wheels reduce this loss 20 or 25 per cent. The rigid form of wheel-base of European cars and locomotives doubles the increase due to curves as well as increases the resistance on the straight line. According to Mr. Chanute the "coning" of wheels increases the resistance from 0·125 to 0·25 pounds per degree of curve per ton. (Thurston, *Friction and Lubrication*, pp. 13–18.)

Recent experiments on the New York and Erie R. R. show that on a track of steel rails in first-class condition, the friction of a train at low velocities may be reduced to 3½ or 4 pounds per ton on a horizontal road; and that the rolling resistance on such a track in the summer may be safely taken at 5 pounds per ton. (*R. R. Gazette*, March 24, 1882, or Haswell's *Pocket-Book for Engineers*, edition of 1882.)

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Example 3.—If $\mu = 0$, the expression $\mu b W + h W$ becomes $h W$, in which case the work equals that necessary to raise the weight through the height of the plane.

SOLUTIONS OF EXAMPLES.

PAGE 64.

1. It will raise $50 \times 33{,}000 \times 60$ ft. lbs. in 1 hour, which divided by $500 \times 62{\cdot}5$ will give the cubic ft. $= 3{,}168$.
2. The average height to which the material is raised will be 10 ft. Hence the work $= 140 (\frac{1}{4} \pi \times 3^2 \times 20) \times 10 = 197{,}920{\cdot}8$. lbs.
3. $$s = \frac{F + F'}{2F} a = \frac{1000 + 200}{2000} \times 12 = 7{\cdot}2 \text{ in.}$$
4. $39{\cdot}37$ inches $= 3{\cdot}2808$ ft. $\times 2{\cdot}2$ lbs. $= 7{\cdot}217 +$ ft. lbs.
5. $43{,}333 \times 7{\cdot}217 + \div (32{,}808)^2 = 29{,}057$ ft. lbs.
6. Substituting in the answer to Prob. 2, p. 61, we have H. P. $= 0{\cdot}9114 \times 1 \times 12^{\frac{3}{2}} = 37{\cdot}89$.
7. Work of the fall $= 2{,}000 \times 8 = 16{,}000$ ft. lbs. Let $x =$ the distance driven, then $10{,}000x = 16{,}000$; $\therefore x = 1{\cdot}6$ ft.
8. Find the velocity in feet per minute. We have

$v = \frac{2 \times 5280}{60}$. The horse-power $= Fv \div$ 33,000, or $\frac{200 \times 2 \times 5280}{60 \times 33000} = 1\frac{1}{15}$.

9. $\mu = \frac{gt^2P - 2(P + W)s}{gt^2 W} = \frac{\frac{193}{6} \times 9 \times 8 - 2(8+40)4}{\frac{193}{6} \times 9 \times 40}$ $= 0{\cdot}1668.$

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10. $s = \frac{1}{2} \cdot \frac{P - \mu W}{P + W} gt^2 = \frac{1}{2} \times \frac{5 - 25 \times 0{\cdot}15}{5 + 25}$ $\times \frac{193}{6} \times 25 = 1206{\cdot}25 \div 72 = 16{\cdot}75$ ft.

ANSWERS TO EXERCISES.

1. One pound raised one foot. Work is a compound quantity, compounded of stress and space.
2. See preceding remarks, text, p. 56.
3. It is dependent upon time *only implicitly*, and in the sense that motion requires time. But, strictly, time should be abstracted.
4. Tang. 15° = 0·268.
5. 5 lbs.
6. Mechanical power is not work, but *rate* of doing work—just as velocity is not space, but *rate* at which space may be passed. Mechanical power involves a *unit* of time, but work does not.

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ART. 41.—The doctrine of energy is the grand, general principle of modern physics. All the changing phenomena of nature are but manifestations of the transmutation of energy. Its principles are not deduced by any system of mathematics, but by a long series of inductions. We *accept* its general principles without attempting a general demonstration.

Even *work* in a higher sense is but a means of transmitting energy. Thus, a horse works by drawing a load, but it is simply a means of transmitting the energy possessed by the horse, first into energy stored in the mass of the load, and second into heat by means of the friction overcome. Still, *work* is not only a convenient, but a useful term. *Work done* is one idea, *energy produced* as its equivalent is another —force and space are the elements of the former, mass and velocity of the latter.

DEFINITIONS OF WORK AND ENERGY.

"Work is the overcoming of resistance continually recurring along some path."—BARTLETT'S *Elements of Analytical Mechanics*, p. 26.

"Work consists in moving against resistance. The work is said to be *performed*, and the resistance overcome."—RANKINE'S *Applied Mechanics*, p. 477.

"Work is the effect of strain and motion combined. —TRAUTWINE'S *Engineers' Pocket-Book*, p. 445.

Remarks on above from p. 446 same book: "Gravity acting in a body falling freely in a vacuum, and consequently unresisted, exerts no effort upon it, it neither goes before, and pulls it along, nor behind, and pushes it; for there can be no pull or push except when there is some *force* to pull or push against. But it simply, as it were, animates the body, or endows it with the power of locomotion. As the body falls, the force of gravity which gives it motion all remains unimpaired, and stored up in it, ready to exert an effort *against* any other *force* which it may chance to meet with. Therefore a body falling unresistedly has no weight; for gravity, which gives it *weight* alone while at rest, now gives it *motion* alone."

(The word *strain* in the above definition is improperly used for *stress.*—AUTHOR.)

"A force is said to do work if its place of application has a positive component motion in its direction; and the work done by it is measured by the product of its amount into this component motion."—THOMSON AND TAIT, *Nat. Philos.*, p. 176.

"Work done on a body by a force is always shown by a corresponding increase of *vis viva*, or kinetic energy, if no other forces act on the body which can do work or have work done against them. If work be done against any force, the increase of kinetic energy is less than in the former case by the amount of work so done. In virtue of this, however, the body possesses an equivalent in the form of potential energy, if its physical conditions are such that these forces will act equally, and in the same directions, if the motion of the system is reversed."—*Ib.*, p. 177.

"An agent is said to do work when it causes the point of application of the force it exerts to move through a certain space. Motion is essential to work."—Twisden, *Elementary Introduction to Practical Mechanics*, p. 18.

"Mechanical effect, or work done, is that effect which a force accomplishes in overcoming a resistance. It depends not only on the force, but also on the space during which it is in action, or during which it overcomes a resistance."—Weisbach's *Mechanics of Engineering*, p. 168.

"Work is the production of motion against resistance."—Todhunter, *Mechanics for Beginners*, p. 337.

"Work, same as before. According to this definition, a man who merely supports a load does not work; for here there is resistance without motion. Also while a free body moves uniformly no work is performed; for here there is motion without resistance."—Todhunter, *Nat. Philos. for Beginners*, p. 255.

"Whenever a body moves through any space in a direction opposite to that in which a force is acting on it, work is said to be performed. It is evident that the application of force is necessary to overcome resistance, and it is very often found convenient to measure the work done by the amount of force expended, and the distance in the direction of the force through which it has been employed."—Magnus's *Elementary Mechanics*, p. 102.

"Thus the increase of *vis viva*, which is also the work done by the acting forces on the body."—Price's *Infinitesimal Calculus*, vol. iii., p. 636.

"Work is done when resistance is overcome, and the quantity of work done is measured by the product of the resisting force and the distance through which that force is overcome."—MAXWELL'S *Theory of Heat*, p. 87.

"Work is the overcoming of mechanical resistance of any kind."—NYSTROM'S *Pamphlet on Force of Falling Bodies, etc.*, p. 25.

DEFINITIONS OF ENERGY AND VIS VIVA.

"Energy expresses power to do work, or force stored and ready for use."—MCCULLOCH'S *Treatise on Mechanical Theory of Heat*, p. 40.

"*Vis viva* (energy) is a quantity which varies as the product of the mass of a particle and the square of its velocity."—PRICE'S *Infinitesimal Calculus*, vol. 3, p. 360.—2d Ed. Oxford.

"Living force, or *vis viva* (or energy), is nothing more than an expression referring to the quantity of work (motion and strain combined) which the force in a body at any given instant could perform, if left to itself, without afterwards receiving any additional force."—TRAUTWINE, *Civil Eng. Pocket Book*, p. 446, Ed. 1872.

"Energy measures the quantity of working power of a moving body."—BARTLETT, *Elements of Analyt. Mech.*, 9th Ed., p. 116.

"The product of the mass of a body by the square of its velocity is called its living force or *vis viva*."—*Mécanique Céleste*, p. 99.

"Energy means capacity for performing work."—RANKINE, *Applied Mechanics*, p. 477.

"*Vis viva*, or living force (or energy), is the power of a moving body to overcome resistance, or the measure of work which can be performed before the body is brought to a state of rest."—SILLIMAN, *Principles of Physics*, 2d Ed., p. 78.

"Energy is the capacity a body has, when in a given condition, for performing a certain measurable quantity of work."—TODHUNTER, *Natural Philosophy for Beginners*, p. 264.

"Energy of a body is power of doing work."—MAGNUS, *Lessons in Elementary Mechanics*, p. 110.

"Energy of a body is the capacity which it has of doing work, and is measured by the quantity of work which it can do. The kinetic energy of a body is the energy which it has in virtue of being in motion."—CUMMING, *Theory of Electricity*, p. 5.

"Kinetic energy or *vis viva* is defined as half the product of the mass into the square of the velocity."—*Ibd.*, p. 13.

"Energy is defined to be capacity for doing work." "It is of two kinds—kinetic or actual when the body is in actual motion. Potential or latent when the body, in virtue of work done upon it occupies a position of advantage, so that the work can be at any time recovered by the return of the body to its old position."—*Ibd.*, p. 14.

"Energy is the capacity of a body to perform work. Energy is said to be stored when this capacity is increased, and to be restored when it is diminished. The units of work and of energy are the same."—WEISBACH, *Mech. and Eng.* Translator's note, bottom of page 168.

"If we adopt the same units of mass and velocity as before there is particular advantage in defining kinetic energy as half the product of the mass into the square of the velocity."—ROUTH's *Rigid Dynamics.*

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ART. 112.—If the body had an initial velocity v_o, the work done upon it in passing from a velocity v_o to v, or the work which would be given out by it in passing from the velocity v to v_o, will be

$$\frac{1}{2}Mv^2 - \frac{1}{2}Mv_\circ^2 = \frac{1}{2}M(v^2 - v_\circ^2).$$

This may be deduced in another way, if we anticipate the equations for momentum.
Thus, Eq. (2), p. 78, of the text is

$$Ft = M(v - v_\circ).$$

The space over which F acts will be equivalent to the average velocity into the time, or

$$s = \frac{1}{2}(v + v_\circ)t.$$

Multiplying these equations, member by member, and canceling t, gives

$$Fs = \frac{1}{2}M(v^2 - v_\circ^2),$$

as before. But we do not consider this method as good as the first, for when the velocity changes irregularly, it is not so evident that the space equals the average of the extreme velocities into the time.

Art. 113.—Potential energy is relative. Thus if a body whose weight is 10 lbs. is 40 feet above the earth, its potential energy in reference to the earth is $10 \times 40 = 400$ ft. lbs.; but if it be over a well 20 feet deep, the potential energy in reference to the bottom of the well, will be $10 \times 60 = 600$ ft. lbs.; and in reference to its own position it is nothing. It is the work which the body may do in reference to some point—or condition—arbitrarily chosen.

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Art. 115.—For the mathematical theory of heat, see Poisson *Traité des Cheleur*, Fourier *Theorie de la Cheleur*, Rankine on *The Steam Engine*, Clausius on *The Theory of Heat*, Hirn's investigations, Maxwell's *Theory of Heat*, McCulloch's *Theory of Heat*, Tait's *History of Thermodynamics, etc., etc.*

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Art. 117.—This is one of the most important physical constants which has been determined in recent times. For a comprehensive and able review of the methods by which it has been determined, as well as for a description of that author's methods and results, see *Mechanical Equivalent of Heat*, by Professor H. A. Rowland. (Proceedings of the American Academy of Arts and Sciences, 1879, p. 45.) Prof. Rowland's results differ by only a small per cent. from those given by Joule. He states,

p. 44, that the value found by Joule at 14°, agrees with his results at 18° *C.* The value 772 foot pounds still stands as a practically correct one.

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ART. 118.—In regard to energy generally, it appears that all of it—or at least very nearly all—originates with the sun. It was a beautiful remark of John Stephenson—as he saw a railroad train winding its way through the country—"that train is drawn by the heat of the sun." The heat and light of the sun caused the growth of vegetation; that vegetation in time was gathered into great masses, which in time became coal in the mine. This coal was brought forth and used as fuel in the locomotive; so that it originated in the action of the sun. All energy on the earth is due to the light and heat of the sun. Activity in the commercial world is directly dependent upon it; for if the sun, on account of spots upon it, or from other causes, does not dispense with its usual heat, short crops will result, and thus affect all the business of a country, if not of the world. In this way the sun may be charged with causing—more or less directly—depressions in trade, or activity in commerce, as the case may be; and hence, in some cases at least, of producing sadness or cheerfulness in the home circle. Even the religious world is affected by the action of the sun. In earlier times, on account of the superstitions of the people, religious leaders appealed more or less effectually to the fears of their followers when the sun's rays were cut off by an eclipse; and in modern times the feeling of dependence upon the Creator is too often modified by the prosperity or depression caused by the circumstances surrounding them, the conditions of which were dependent upon the elements of nature, and these caused, more or less directly, by the action of the sun.

The sun appears to be dispensing its energy to his family of planets, and in this way wasting itself away. Sir Isaac Newton realized this condition of things and saw—or thought he saw—the necessity of the sun's being replenished in order to maintain its stock of energy; and he conceived that this might be done by one comet after another falling into the sun. As the comets come from remote regions of space, they would possess a large amount of energy when they struck the sun, and by falling into it would produce intense heat. No comet, however, has within historical times been known to fall into the sun; but, on the other hand, the most critical examination of their orbits shows that their paths are nearly as well defined—and nearly as fixed in position—as any of the planets; and no cause is *known* to exist that will cause them to fall into the sun. It was formerly supposed that the ether of space caused a resistance to the motion of all bodies in space; and if it did, not only would the comets, but also the planets, ultimately fall into the sun. However, nothing—absolutely nothing—is *known* in regard to the effect of this ether upon the motion of bodies in it. Comets and planets have moved in their orbits for untold ages, and, for aught we know, have maintained their relative positions. If the comets were destined to fall into the sun, they must have been doing so for ages and ages, and hence must originally have been comparatively very numerous. It would seem that sufficient time had elapsed since the existence of the solar system to have exhausted this stock of energy—still, at the present time, comets are numerous. Similarly, in regard to the planets, the most refined observations, combined with the most refined analysis, have failed to detect any modification of motion due to the ether of space.

More recently, the late Professor Benj. Pierce, of Cambridge, Mass., put forth the theory that the energy of the sun was supplied by meteorolites falling from an immense distance directly into the sun. The meteors are dark bodies, and it is assumed that there may be multitudes of them scattered through space. That there are many, is shown by the fact that they frequently fall upon the earth. Admitting the truth of this hypothesis, it seems inevitable that, in the course of time, the supply will be exhausted—and then the question, What will follow? becomes a serious one to science.

Space, which, in our younger days, we conceived to be void, is really *filled* with something, and there may be vastly more inert matter scattered through it than we have imagined.

A mere contraction of the volume of the sun, caused by the mutual attraction of its own particles, will produce heat. The author has shown that if the earth contracted from twice its present diameter to its present size, and all the energy thus produced be changed into heat, and uniformly disseminated throughout the mass, the temperature would be raised 44,655 degrees F., if it had the specific heat of water, or 357,240 degrees F., if it had the specific heat of iron. (See *Analytical Mechanics*, p. 229, or *Mathematical Visitor*, 1880, p. 134.)

We believe that the solar system is stable, that it is not made to run down, that it has the elements of self-preservation; but we cannot prove it. Neither can those who entertain the opposite view prove their position. This problem is, at present, beyond the reach of science.

SOLUTIONS OF EXAMPLES.

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1. $\frac{25}{32\frac{1}{6}}$ = the mass, and as $32\frac{1}{6}$ is the acceleration in *feet* per *second*, the velocity should be in the same units; hence $v = \frac{100}{60} = \frac{5}{3}$ feet per second, and the work will be

$$\tfrac{1}{2} \cdot \frac{25}{32\frac{1}{6}} \cdot \left(\frac{5}{3}\right)^2 = 1{\cdot}07 + \text{ft. lbs.}$$

2. The work will vary as the depth, and the energy as the square of the velocity; hence,

$$depth = 2\left(\tfrac{15}{5}\right)^2 = 18 \text{ feet.}$$

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3. Reducing the tons to pounds, we have

$$60 \text{ tons} = 60 \times 2000 \text{ lbs.} = 120{,}000 \text{ lbs.}$$

Similarly,

$$40 \text{ miles per hour} = \frac{40 \times 5280}{60 \times 60} \text{ ft. per second}$$

The friction $= 60 \times 8 = 480$ lbs.

Let $x =$ the required distance in miles $= 5280x$ feet, then the work will be $480 \times 5280x$; hence we have

$$480 \times 5280x = \tfrac{1}{2} \cdot \frac{120000}{32\frac{1}{6}} \left(\frac{40 \times 5280}{60 \times 60}\right)^2;$$

which solved gives

$$x = \tfrac{1}{2} \times \frac{100}{32\frac{1}{6} \times 4 \times 528} \left(\frac{176}{3}\right)^2 = 2{\cdot}53 \text{ miles.}$$

4. Let $x =$ the required number of pounds of water; then the energy put into water in raising it from 32° F. to 212° will be

$$772\,(212 - 32)x = 772 \times 180x.$$

The kinetic energy stored in the train will be

$$\tfrac{1}{2} \cdot \frac{200000}{32\frac{1}{6}} \left(\frac{20 \times 5280}{60 \times 60}\right)^2;$$

which, by the conditions of the problem, equals the heat energy to be put into the water; hence

$$772 \times 180x = \tfrac{1}{2} \times \frac{200000}{32\frac{1}{6}} \left(\frac{88}{3}\right)^2;$$

$$\therefore x = 19{\cdot}2 \text{ lbs.}$$

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5. $$f = \frac{P - \mu W}{P + W} g = \frac{4 - 0{\cdot}2 \times 10}{14} \times \frac{193}{6} = \frac{193}{42}$$

$$= 4{\cdot}6;$$

also for the velocity

$$v^2 = 2fs = 2 \times 4{\cdot}6 \times 5 = 46.$$

The work will be

$$10 \times \mu \times \textit{distance},$$

and the energy will be $\frac{1}{2}Mv^2$;

$$\therefore 0{\cdot}2 \times 10 \times \textit{distance} = \tfrac{1}{2} \times \frac{10 \times 6}{193} \times 46;$$

$$\therefore \textit{distance} = 3\tfrac{4}{7} \text{ feet.}$$

6. The friction $= 200 \times 0{\cdot}2 = 40$ lbs. Let the velocity per minute be x, then the work per minute will be $40x$, and for 3 minutes it will be $120x$. The energy of one pound of water raised one degree F. is 772 ft. lbs., and of 5 lbs. it will be 5×772, and for 50 degrees it will $50 \times 5 \times 772$.
Hence

$$120x = 50 \times 5 \times 772;$$

$$\therefore x = 1{,}608 \text{ ft. per minute.}$$

ANSWERS TO EXERCISES.

PAGE 76.

1. It is not; force is only one of the elements producing energy.

2. Ability to do work. Work has, however, been done.

3. It produces action of the stomach, thus in-

volving energy; promotes action of the heart; causes the growth of the bone, muscle, and flesh, and these enable the animal to move about—to do work—to swallow more food—to lie down—to get up, etc., etc.

4. Because, in the first place it is not as concentrated, and, in the second place, the heat is more quickly conducted away.

5. It will. It is due to this cause that meteors are visible. The meteorolites which fall upon the earth have the appearance of having been partly melted, and hence must have been subjected to great heat. This is due to the compression of the air in front of the meteor and of the friction of the air against its sides as it passes swiftly through the air, and the heat thus produced is so great that the meteor is heated to redness, and thus appears like a shooting star, as it really is. It is probable that the smaller meteorolites become so nearly consumed by the great heat that they could scarcely be found after they had fallen upon the earth, but larger ones have been found which struck the earth with such violence that they nearly or quite buried themselves. It has been suggested that the great iron deposit in the upper peninsula of Michigan, a short distance west of Marquette, was probably a large meteor, and a similar suggestion has been made in regard to the iron mountain in the State of Missouri.

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6. The friction of the water produced by moving against the banks and bed of the stream produces heat, which, escaping into the surrounding air, modifies its temperature.

7. First by the heat due to the friction, and second, by inducing a quicker circulation of the blood more heat is supplied to the parts.

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8. $2.50. This exercise is intended to draw attention to the fact that the value of wood for fuel depends upon its capacity for producing heat; or, in other words, of its inherent heat energy. If the heating power of a given weight of hickory be 100, it has been found that the heating power of the same quantity of oak will be about 80, and of maple about 50. The practical value, however, of fuel may be governed largely by other circumstances. Thus, when a fire is wanted for only a short time, as a kitchen fire in midsummer, or where steam must be raised quickly, etc., the cheaper fuel may be quite as valuable as the more costly.

Pages 78, 79.

Arts. 122–125.—*Momentum*, according to Newton's definition, *is* strictly *quantity of motion*. He says (*Principia* B. 1, Def. II.) "The whole motion is the sum of all its parts, and therefore in a body, double in quantity, with equal velocity, the motion is double; with twice the velocity it is quadruple." It is said in the text that *quantity of motion* does not fully express the desired meaning, but this is due simply to the fact that *quantity* had not been defined. Including Newton's definition, it does express the meaning correctly and fully.

If a force of constant intensity acts upon a free body moving from rest, the product of the force and time equals the momentum produced. Space is here entirely *abstracted from force and time.* Although the body cannot move without involving space, yet all considerations of space must be discarded. It is immaterial whether the space over which the body must move in acquiring the velocity v be great or small; and hence, so far as the momentum is concerned, the space may vanish.

From the equation $Ft = Mv$, we have $F = M\frac{v}{t}$, where $\frac{v}{t}$ $\left(\text{or for a variable force, we have in the notation of the calculus } \frac{dv}{dt}\right)$ is the rate of change of the velocity, and hence, $M\frac{v}{t}$ is the rate of change of momentum, which is, according to the *second* law, the measure of the force F. We thus reproduce the expression for that law.

The expression Ft is not the momentum, but simply its equal under the restrictions given above. It has been proposed to give a special name to this product, just as Fs is called the measure of work, while its equal $\frac{1}{2}Mv^2$, in case of a free body, is called energy. Maxwell called Ft *an impulse* (*Matter and Motion*, p. 44), to which we do not object, since the effect will be the same whether it be produced in an imperceptibly short time, or in a longer time. But the product has no meaning except where the force moves a body. If F be a mere stress, like the pressure of a stone upon the earth where no motion is involved, then Ft produces nothing. The time effect of a stress, is its effect in moving a body—producing velocity.

Much has been written in regard to *force, vis viva* (now called energy) and *momentum.* For many years, in the earlier history of the science of mechanics, there were long and sharp discussions as to whether *work* or *momentum* was the proper measure of *force;* but, as we have shown in what precedes, neither is the proper measure, and hence theirs was merely a war of words. One factor in the product Fs is the force; and one factor in the product Ft is also force. The former placed equal to $\frac{1}{2}Mv^2$, and solved gives $F = \frac{\frac{1}{2}Mv^2}{s}$; hence the measure of force is the *rate of change of energy per unit of space;* and we have already shown that it is also the *rate of change of momentum* per unit of time.

Efforts are sometimes made to determine a relation between momentum and energy; but no physical relation exists, and hence none can be found. In order that there shall be a ratio between them, they must have a common unit. Since one is compounded of force and time in which space is excluded, and the other of force and space in which time is excluded, they have not a common unit.

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We here note the three following statements from different authors.

First, In Van Nostrand's *Engineering Magazine* for 1877 and 1878 is a series of articles on momentum, and *Vis Viva* by Prof. J. J. Skinner. On pp. 129, 130, 131, it is stated that momentum which equals MV represents the number of pounds pressure which the mass M with the velocity V is capable of exerting under the conditions that the pressure is constant and capable of bringing the body to rest in one second. This is numerically correct as a deduction, but in the articles referred to there is apparently a labored effort to show that momentum is *pressure only*, and not quantity of motion (see also p. 137 of the *Eng. Mag.*). Also on page 132 it is stated that, "The unit of momentum, then, is a force of pressure equal to one pound" (see also p. 240). In this definition, the element of time does not appear, but it is not proper to drop it simply because it is *one* second. The above-named writer corrected himself in a later article, page 501 of the same *Magazine.*

To measure anything requires a unit of the same kind considered as a standard. Strictly speaking, the unit of momentum is the momentum of a unit of mass moving with a unit velocity. *Momentum cannot be measured by pounds only.* (See also article by the author, *Eng. Mag.*, vol. xviii., 1878, p. 33.)

Second, It is stated that Beaufoy determined that a body of one pound weight, with a velocity of one foot in a second, strikes with a pressure equal to 0·5003 lb.; and hence to find the pressure produced by the impact of any projectile, we have the general formula, $pressure = 0{\cdot}5003 Wv^2$ (Silliman's *Physics*). Now we assert that the formula is false. Admitting, for the sake of the argument, that he did find such a result when the projectile struck a hard body, like a piece of iron, it would have been very much less had the body struck been more yielding, like a gas-bag, or a sack of loose feathers, and so a great range of values might be found depending upon the character of the bodies.

Third, Professor Tait, in an interesting lecture upon force, delivered before the British Association, 1876, says (see *Nature*, 1876, p. 462), "With a moderate exertion you can raise a hundred weight a few feet, and *in its descent it might be employed to drive machinery*, or *to do some other species of work.* But tug as you please at a ton, you will not be able to lift it; and, therefore, after all your exertion, it will not be capable of doing any work in descending again.

"Thus, it appears, that *force* is a mere name, and that the *product of the force into the displacement of its point of application* has an objective existence."

. . .

"*Force is the rate at which an agent does work per unit of length.*" . . .

These definitions have already been referred to on p. 20 of the Key, and the remarks there made will be more readily understood in this place, after having passed over energy and momentum. Force, fundamentally, is a *quantity* instead of *rate*; just as interest is a *quantity*—an amount of money—and not *rate.* It is true that the amount of money paid for the use of a hundred dollars is identical with the rate per cent., but every one readily distinguishes between rate per cent. and interest. Professor Tait made use of *interest* in an illustration of this principle, but, we think, used it improperly.

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ART. 131.—If $\lambda = l$, and $k = 1$, we have $E = F$, hence—as a deduction—the coefficient of elasticity may be defined as the force (stress) necessary to elongate a prismatic bar whose section is unity to double its length, provided the original conditions remain constant except that of length; that is, the elasticity and the cross section must both remain constant. But these conditions are never realized, hence this definition is highly ideal. In fact, the coefficient of elasticity is constant for any material for an elongation of only a very small fraction of the length; but even with this limitation it is of untold importance in certain physical sciences.

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ART. 138.—Substituting from equations (1) and (2), page 89, we have

$$Mv_1^2 + M'v_1'^2 =$$

$$M\left[\frac{Mv + M'v'}{M + M'} - \frac{eM'}{M + M'}(v - v')\right]^2$$

$$+ M'\left[\frac{Mv + M'v'}{M + M'} + \frac{eM}{M + M'}(v - v')\right]^2$$

$$Mv_1^2 = \frac{M}{(M + M')^2}\Big[M^2v^2 + 2MM'vv' + M'^2v'^2$$

$$- 2eMM'v^2 - 2eM'^2vv' + 2eMM'vv' + 2eM'^2$$

$$v'^2 + e^2M'^2v^2 - 2e^2M'^2vv' + e^2M'^2v'^2\Big];$$

or

$$= \frac{1}{(M + M')^2}\Big[M^3v^2 + 2M^2M'vv' + MM'^2v'^2$$

$- 2eM^2M'v^2 - 2eMM'^2vv' + 2eM^2M'vv' + 2eMM'^2v'^2 + e^2MM'^2v^2 - 2e^2MM'^2vv' + e^2M M'^2v'^2\Big].$ (2)

Similarly,

$$M'v_1'^2 = \frac{M'}{(M + M')^2}\Big[M^2v^2 + 2MM'vv' + M'^2v'^2 + 2eM^2v^2 + 2eMM'vv' - 2eM^2vv' - 2eMM' v'^2 + e^2M^2v^2 - 2e^2M^2vv' + e^2M^2v^2\Big]$$

$$= \frac{1}{(M + M')^2}\Big[M^2M'v^2 + 2MM'^2vv' + M'^3 v'^2\ 2eM^2M'v^2 + 2eMM'^2vv' - 2eM^2M'vv' - 2eMM'^2v'^2 + e^2M^2M'v^2 - 2e^2M^2M'vv' + e^2M^2 M'v'^2\Big].$$ (3)

Adding equations (2) and (3) we have

$$Mv_1^2 + M'v_1'^2 = \frac{1}{(M + M')^2}\Big[M^3v^2 - 2eM^2M'v^2 + e^2MM'^2v^2 + M^2M'v^2 + 2eM^2M'v^2 + e^2M^2 M'v^2 + 2M^2M'vv' - 2eMM'^2vv' + 2eM^2M' vv' - 2e^2MM'^2vv' + 2MM'^2vv' + 2eMM'^2vv' - 2eM^2M'vv' - 2e^2M^2M'vv' + MM'^2v'^2 + 2eMM'^2v'^2 + e^2MM'^2v'^2 + M'^3v'^2 - 2eMM'^2v'^2 + e^2M^2M'v'^2\Big]$$

$$= \frac{1}{(M + M')^2}\Big[(M + M')M^2v^2 +$$
$$(M + M')e^2MM'v^2 + (M + M')\,2MM'vv' -$$
$$(M + M')\,2e^2MM'vv' + (M + M')M'^2v'^2 +$$
$$(M + M')e^2MM'v'^2\Big]$$

$$= \frac{1}{M + M'}\Big[M^2v^2 + e^2MM'v^2 +$$
$$2MM'vv' - 2e^2MM'vv' + M'^2v'^2 + e^2MM'$$
$$v'^2\Big]$$

$$= \frac{1}{M + M'}\Big[M^2v^2 + M'^2v'^2 + MM'$$
$$(e^2v^2 + 2vv' - 2e^2vv' + e^2v'^2)\Big].$$

Adding and subtracting $MM'(v^2 + v'^2)$ we have :

$$Mv_1^2 + M'v_1'^2 = \frac{1}{M + M'}\Big[(M + M')Mv^2 + (M$$
$$+ M')M'v'^2 + MM'(e^2v^2 - v^2 + 2vv'(1 - e^2) +$$
$$e^2v'^2 - v'^2)\Big]$$

$$= Mv^2 + M'v'^2 + \frac{MM'}{M + M'}\Big[(1 -$$
$$e^2)(2vv' - v^2 - v'^2)\Big]$$

$$= Mv^2 + M'v'^2 - \frac{(1 - e^2)MM'}{M + M'}$$
$$(v - v')^2. \qquad (4)$$

SOLUTIONS OF EXAMPLES.

PAGE 91.

1. $5 \times 5 = 25$ lbs. sec.

2. $\lambda = \dfrac{Fl}{EK} = \dfrac{9000 \times 10 \times 12}{\frac{3}{4} \times 26000000}$ 0·05538 + inches.

3. $E = \dfrac{Fl}{\lambda K} = \dfrac{2500 \times 2 \times 12}{\frac{1}{4}\pi(\frac{1}{2})^2 \times \frac{1}{10}} = 24,446,200$ lbs. very nearly.

4. $V = \dfrac{Wv - W'v'}{W + W'} = \dfrac{120 - 120}{18} = 0.$

5. Eqs. (1) and (2), p. 89 of text, give

$$v_1 = \frac{Wv + W'v'}{W + W'} - \frac{eW'}{W + W'}(v - v')$$

$$= \frac{20 \times 100 + 50 \times 40}{70} - \frac{\frac{1}{2} \times 50}{70}(60) = 35\tfrac{5}{7}$$

feet;

$$v_1' = \frac{20 \times 100 + 50 \times 40}{70} + \frac{\frac{1}{2} \times 20}{70} \times 60 =$$

$65\frac{5}{7}$ feet.

PAGE 92.

6. Here $v' = 0$, and we have

$$v_1 = \frac{2000}{70} - \frac{25}{70} \times 100 = -7\tfrac{1}{7} \text{ feet,}$$

$$v_1' = \frac{2000}{70} + \frac{10}{70} \times 100 = 42\tfrac{6}{7} \text{ feet.}$$

7. Here $v' = -40$ feet, hence

$$v_1 = \frac{2000 - 2000}{70} - \frac{\frac{1}{2} \times 50 \times 140}{70} = -50$$

feet per second.

$$v_1' = \frac{2000 - 2000}{70} + \frac{\frac{1}{2} \times 20 \times 40}{70} = 20 \text{ ft.}$$

per second.

8. By Art. 132, p. 84, we have

$$e^2 = \frac{h'}{h},$$

where h is the height of fall, and h' the height of rebound. The sum of an infinite decreasing progression is

$$s = \frac{\textit{first term}}{1 - \textit{ratio}}$$

$$= \frac{h + h'}{1 - e^2}$$

$$= \frac{h + he^2}{1 - e^2}$$

$$= \frac{1 + e^2}{1 - e^2} h.$$

9. If $e = 1$, $s = \infty$.

$e = \frac{1}{2}$, $s = \frac{5}{3}h$

$e = \frac{1}{4}$, $s = \frac{17}{15}h$

$e = 0$, $s = h$.

10. Let M be the mass of one of the bodies, and v the velocity of the impinging one; its kinetic energy will be

$$\tfrac{1}{2}Mv^2.$$

The kinetic energy of the two bodies after impact will be least when both are non-elastic; in which case the common velocity after impact will be (Eq. (1) p. 86),

$$V = \tfrac{1}{2}v;$$

and the kinetic energy will be

$$\tfrac{1}{2}(2M(\tfrac{1}{2}v)^2) = \tfrac{1}{4}Mv^2,$$

and the kinetic energy before impact will be just twice that after, and cannot exceed that ratio.

ANSWERS TO EXERCISES.

1. Yes.
2. No.
3. No.
4. By equation of work = energy.
5. They will, at the instant of impact.
6. Yes.
7. No.
8. No.

9. By preventing so great a loss of energy. When the wheels of a car strike "dead" against a rail, it batters the rail, thereby doing an amount of work which is lost to the train; but the springs, when in action, prevent such a dead blow in the first place, and then by their reaction restore a portion of the energy to the train. In short, if all the parts were perfectly elastic there would be no permanent battering of the parts, and no energy would be lost by the impact.

Page 94.

Statics is a limiting case of dynamics, in which the applied forces mutually destroy each other, and leave the body, so far as its condition in regard to rest or motion is concerned, the same as if no forces were acting. As such, its principles may be established independently of motion. Many writers hold that the conditions of equilibrium should be determined independently of all considerations of motion, and we have accordingly given the usual proofs, although the Newtonian method, given in Art. 52 of the text, is, to us, quite satisfactory.

We consider that the confirmation of the results, flowing from the parallelogram of forces, is a stronger confirmation of its truth than any formal demonstration ever made. Nearly all the formulas of mechanics are founded upon it. The principles of mechanism, and the places of planets and comets all involve it. Such a proposition might be *assumed* without proof, and its truthfulness be confirmed by its leading to results confirmed by daily experience.

Formerly a proof was considered so important, that many different methods were devised, and one work gave forty-five different proofs of the parallelogram of forces.

SOLUTIONS OF EXAMPLES.

PAGE 99.

1. Let P be one stress and F the other, then we have (Eq. on p. 97 of text),

$$R = \sqrt{P^2 + F^2 + 2PF \cos 90^\circ} = \sqrt{P^2 + F^2}.$$

If $\theta = 0$, then

$$R = \sqrt{P^2 + F^2 + 2PF} = P + F.$$

If $\theta = 180^\circ$, then

$$R = \sqrt{P^2 + F^2 - 2PF} = P - F.$$

2. Here the force 5 reversed will be the resultant of the other two; then

$$5^2 = 3^2 + 4^2 + 2 \cdot 3 \cdot 4 \cos \theta;$$

$$\therefore \cos \theta = \frac{25 - 9 - 16}{24} = 0;$$

$$\therefore \theta = 90^\circ.$$

3. We have

$$R^2 = R^2 + R^2 + 2RR \cos \theta;$$

$$\therefore \cos \theta = -\tfrac{1}{2};$$

$$\therefore \theta = 120^\circ.$$

4. We have

$$R^2 = 100^2 + 100^2 + 2 \times 100 \times 100 \cos 60°.$$

But $\cos 60° = \frac{1}{2}$, hence

$$R^2 = 3 \times 100^2;$$

$$\therefore R = 100\sqrt{3}.$$

5 We have

$$(P + F)^2 = P^2 + F^2 + 2PF \cos\theta;$$

or,

$$P^2 + 2PF + F^2 = P^2 + F^2 + 2PF \cos\theta;$$

$$\therefore \cos\theta = 1,$$

and

$$\theta = 0°.$$

6. We have

$$R^2 = (P - F)^2 = P^2 + F^2 + 2PF \cos\theta,$$

or,

$$P^2 - 2PF + F^2 = P^2 + F^2 + 2PF \cos\theta;$$

$$\therefore \cos\theta = -1;$$

$$\therefore \theta = 180°.$$

7. From the proportion on p. 98 of text we have

$$50 : F :: \sin(F,R) : \sin 115°,$$

$$50 : R :: \sin(F,R) : \sin 35°,$$

$F : R :: \sin 115° : \sin 35°,$

$P : R :: \sin 30° : \sin 35°.$

From these we have

$$F = \frac{\sin 115°}{\sin 35°} R = 1{\cdot}58\ R.$$

$$R = \frac{\sin 35°}{\sin 30°} \times 50 = 57{\cdot}35 \text{ lbs.}$$

$$\therefore F = 90{\cdot}63 \text{ lbs.}$$

$$\sin (F,R) = \frac{50 \times \sin 35}{57{\cdot}35} = 0{\cdot}5000;$$

$$\therefore \text{angle } (F,R) = 30°.$$

8. The string will make a right angle at the point where the weight is applied, and the sides of the triangle representing the forces will be as 3 to 4 to 5; hence we have

$$5 : 4 :: 20 : x = 16;$$

$$5 : 3 :: 20 : x = 12.$$

PAGE 100.

9. The parallelogram representing the forces will be a rectangle, of which the diagonal will be a diameter of the circle.

ANSWERS TO EXERCISES.

1. It will be the resultant of two forces acting away from C, one of which will equal CA, the other $CD = AB$.

2. A line through A equal and parallel to a line joining E and B.
3 They would not.
4. When acting upon the same particle, in opposite directions, and equal in magnitude.
5. With 4, 5, and 9 they can, if 4 and 5 act opposite to 9. But forces 3, 4, and 8 cannot, since two of them, 3 and 4 together, do not equal the third.
6. It will not. The resultant takes the place of the other two.

Page 102.

Art. 158. It will be observed that, in Fig. 45, β is the complement of α; hence $\cos \beta = \cos (90 - \alpha) = \sin \alpha$, hence the equations for X and Y become

$$X = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \text{etc.} = 0,$$
$$Y = F_1 \sin \alpha_2 + F_2 \sin \alpha_2 + \text{etc.} = 0;$$

from which we see that the equation for Y may be deduced directly from that of X by writing *sin* in place of *cos* in the first equation.

SOLUTIONS OF EXAMPLES.

Page 103.

1. We have

$$X = 20 \cos 30° + 30 \cos 90° + 40 \cos 150°$$
$$50 \cos 180° = R \cos \alpha,$$

$$Y = 20 \sin 30^\circ + 30 \sin 90^\circ + 40 \sin 150^\circ + 50 \sin 180^\circ = R \sin \alpha;$$

in which all the terms are written as positive, and their essential signs made to depend upon the trigonometrical functions.

Reducing gives

$$20 \times \tfrac{1}{2}\sqrt{3} + 0 - 40 \times \tfrac{1}{2}\sqrt{3} - 50 = R \cos \alpha,$$

$$10 + 30 + 20 + 0 = R \sin \alpha;$$

or

$$-67{\cdot}32 \;+\; = R \cos \alpha,$$

$$60 = R \sin \alpha.$$

Dividing gives

$$\frac{R \cos \alpha}{R \sin \alpha} = -\frac{67{\cdot}32}{60},$$

or

$$\cot \alpha = -1{\cdot}122;$$

$$\therefore \alpha = 138^\circ\ 17'.$$

Squaring and adding gives

$$R^2(\cos^2 \alpha + \sin^2 \alpha) = (67{\cdot}32)^2 + (60)^2.$$

But $\cos^2 \alpha + \sin^2 \alpha = 1$;

$$\therefore R = \sqrt{8131{\cdot}98}$$

$$= 90{\cdot}18 \text{ lbs.}$$

2. $R \cos \alpha = 20 \cos 180° + 10 \cos 270°,$

$R \sin \alpha = 20 \sin 180° + 10 \sin 270°;$

hence

$$R \cos \alpha = -20,$$
$$R \sin \alpha = -10.$$

Squaring and adding gives

$$R = \sqrt{500} = 22{\cdot}36.$$

3. We have

$$R \cos \alpha = P \cos 0° + P \cos 90° + P \cos 225° + P \cos 270°,$$

$$R \sin \alpha = P \sin 0° + P \sin 90° + P \sin 225° + P \sin 270°;$$

hence

$$R \cos \alpha = P(1 - \tfrac{1}{2}\sqrt{2}),$$
$$R \sin \alpha = P(1 - \tfrac{1}{2}\sqrt{2} - 1)$$
$$= -\tfrac{1}{2}\sqrt{2}P.$$

Dividing the second by the first gives

$$\tan \alpha = -\frac{\tfrac{1}{2}\sqrt{2}}{1 - \tfrac{1}{2}\sqrt{2}} = -(1 + \sqrt{2}) = -2{\cdot}4162 +$$

$\therefore \alpha = 292°\ 30'.$

Squaring and adding gives

$$R^2(\sin^2 \alpha + \cos^2 \alpha) = P^2(1 - \tfrac{1}{2}\sqrt{2})^2 + P^2(-\tfrac{1}{2}\sqrt{2})^2;$$

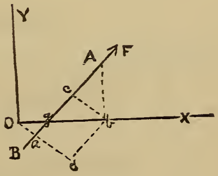
or

$$R^2 = P^2(2 - \sqrt{2})$$
$$= 0.58579P^2;$$
$$\therefore R = 0.765P.$$

Page 104, Art. 162.—A single force whose line of action does not pass through the centre of a free body, produces rotation as well as translation. The measure of the effect of a force in producing rotation is proportional to the moment of the force; as is shown from the fact that the moment is proportional to the *work done by the force*.

The theory of moments is here discussed without reference to the bodies upon which the forces act.

Page 110, Art. 176.—The cut, Fig. 55, should be as here given; that is, in the typical figure the force should be positive away from the origin, and so placed that it would produce positive rotation about the origin, O, and the angles α and β be acute, as shown in Fig. 42 of the text; so that the signs of the terms in the analytical expression for the moment will flow directly from the trigonometrical functions. This being done the



typical form of the expression for the arm of the force will be

$$Oa = x \cos \beta - y \cos \alpha.$$

In the old Fig. 55, the angle between the axis of x and the direction line of the force was, as shown in Art. 157,

$$\alpha = 180° + dOb;$$

and similarly,

$$\beta = 180° + YOd;$$

$$\therefore \cos \alpha = -\cos dOb,$$

$$\cos \beta = -\cos YOd,$$

which values give for the arm

$$Oa = y \cos \alpha - x \cos \beta,$$

as given in the text. We call the former value the *typical* one, and the latter a *deduced* one. The former should always be used in connection with the true value of the angles a and β.

SOLUTIONS OF EXAMPLES.

PAGE 115.

1. We have

$$t = \frac{AC}{AE} W,$$

in which

$$W = 20 \text{ lbs.},$$

$AC = 24$ inches, $AB = 6$ inches, $AD = 4$ inches.

To find AE; in the right-angled triangle DAB we have

$$\text{tang } B = \frac{AD}{AB} = \tfrac{4}{6} = 0 \cdot 666 + ;$$

$$\therefore B = 33^\circ \, 41'.$$

Then

$$AE = AB \sin B$$
$$= 3 \cdot 327 + \text{ inches.}$$

Substituting above gives

$$t = \frac{24 \times 20}{3 \cdot 327} = 144 \cdot 2 + \text{lbs.}$$

2. Taking the origin of moments at D we have

$$P = \frac{AC}{AD} \; W$$

$$= \frac{20 \times 24}{4} = 120 \text{ lbs.}$$

3. Taking the origin of moments at B we have

$$W. \, BC = F.AB,$$

or

$$F = \frac{20 \times 18}{6} = 60 \text{ lbs.}$$

Page 116.

4. If $t = W$, we have

$$W = \frac{AC}{AE} W,$$

or

$$AE = AC.$$

From the right-angled triangle AEB we have

$$AB = \frac{AE}{\sin 45^\circ}$$

substituting,

$$= \frac{AC}{\sqrt{\frac{1}{2}}}$$

$$= \sqrt{2}\, AC.$$

5. Taking the origin of moments at A we have

$$t \times AC = BE \times W.$$

But from the example

$DB = 2AB,\ \theta = 45^\circ,\ W = 50$ lbs.

From the figure

$$BE = AB \sin 45^\circ$$

$$= \tfrac{1}{2}\sqrt{2}\, AB = 0{\cdot}7071 AB.$$

$$\sin\varphi = \frac{BE}{DB} = \frac{AB \sin 45^\circ}{2AB}$$

$$= \tfrac{1}{4}\sqrt{2};$$

$$\therefore \varphi = 20^\circ\ 42'\ 17''.$$

To find AC we have from the figure

$$AC = AB \sin ABC$$

$$= AB \sin(\theta - \varphi)$$

$$= AB \sin 24° \, 17' \, 43''$$
$$= 0{\cdot}4115 \, AB.$$

Substituting in the first equation above gives

$$t = \frac{0{\cdot}7071 \times 50}{0{\cdot}4115}$$
$$= 85{\cdot}97 + \text{lbs.}$$

To get the compression on the bar, take the origin of moments at D, in which case the moment of the tension will be zero. The perpendicular from D upon AB prolonged will be $AD \sin \theta$, and from D perpendicular upon the vertical through B will equal $BE = AB \sin \theta$; hence we have—calling c the compression—

$$c . \, AD \sin \theta = W . \, AB \sin \theta;$$

$$\therefore c = \frac{AB}{AD} W.$$

To find AD we have

$$\sin ABD : \sin \varphi :: AD : AB;$$

$$\therefore AD = \frac{AB \times 0{\cdot}4115}{\frac{1}{4}\sqrt{2}}$$

$$= \frac{AB}{0{\cdot}8592 +};$$

which substituted above gives

$$c = 0{\cdot}8592 \, W$$
$$= 42{\cdot}96 \text{ lbs.}$$

PAGE 116.

6. Let fall a perpendicular p from B upon AC, then

$$p = AB \sin A.$$

Let D be directly under W, then taking the origin of moments at B we have

$$t.\, p = W.\, BD;$$

$$\therefore t = \frac{BC \cos CBD}{AB \sin A} W.$$

7. If $t = W$, then $AB \sin BAC = BD$, or BC will bisect the angle ACD.

8. Take the origin of moments at A. Let fall a perpendicular from A upon CB produced; its length will be

$$p = AB \sin CBD$$

$$= 6 \times \frac{8}{BC}$$

$$= 6 \times \frac{8}{\sqrt{4^2 + 8^2}}$$

$$= \frac{48}{8 \cdot 9442} \text{ feet.}$$

Let c be the compression on BC, then the equation of moments becomes,

$$c.\, p = W.\, AD;$$

$$\therefore c = \frac{AD}{p} W$$

$$= \frac{10 \times 8 \cdot 9442}{48} 500$$

$$= 931 \cdot 7 \text{ lbs.}$$

9. For equilibrium we have

$$Ob = \frac{P_1}{P_2 - P_1} bc$$
$$= \frac{2P_1}{0}$$
$$= \infty;$$

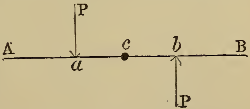
or there can be no equilibrium.

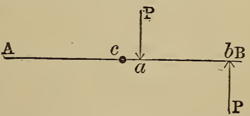
10. We will have

$$bc = \frac{P_2 - P_1}{P_1} Ob$$
$$= \frac{2 \times 0}{P_1}$$
$$= 0;$$

or the forces must act at the same point.

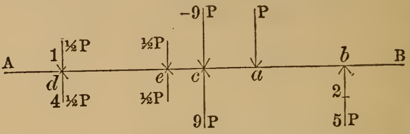
PAGE 118, ART. 187.—It is well to illustrate this article still further. If the forces constituting the couple be equidistant from the centre c of the body, it is sufficiently evident, without a thorough demonstration, that it will produce rotation only. But is it equally evident that, if the same couple act upon the same body in such a way that the points of application are both on one side of the centre, it will produce rotation only, and the same amount of rotation as in the preceding case? According to





the proposition it will, and we will prove it by a special solution.

At a point d, such that $dc = cb$, introduce two equal and opposite forces, each equal to $\frac{1}{2}P$, and call the upper $\frac{1}{2}P$, 1, and the lower half 4.



Since these are in equilibrium the problem will be the same as before. Separate the force at b into two equal parts, and for the sake of convenience call one part 2 and the other 5. Now the resultant of 4 and 5 will be a force equal to their sum, or P, applied at the centre c—which force call 9. Combining 1 with 2 we have a couple whose arm is db, and the moment will be $\frac{1}{2}P.db = P.cb$. Similarly at e, at a distance $ec = ca$, introduce two equal and opposite forces, each equal to $\frac{1}{2}P$; and separate the force at a into two equal parts. Combining $\frac{1}{2}P$ at a with the $\frac{1}{2}P$ above e gives a resultant equal to P applied at c acting down, and marked $-$ 9, which will equilibrate + 9, and there will be no motion of translation. There will remain the couple $\frac{1}{2}P.ea$, which will produce rotation only about the centre c. Finally, the two couples $\frac{1}{2}P.db$, and $\frac{1}{2}P.ea$, each producing rotation about the centre c, but in opposite senses, are equivalent to the single couple.

$$P(\tfrac{1}{2}db - \tfrac{1}{2}ea) = P(cb - ca) = P.ab;$$

hence, the body will rotate about its centre of gravity, and the rate of rotation will be independent of the points of application of the forces, and dependent only upon the moment of the couple.

PAGE 120, ART. 190.—This article contains all the principles of the simple lever. In some works levers are divided into three classes, but, mechanically, there is no distinction between them. It is only necessary for equilibrium that the sum of the moments be zero —or that the moments of the forces which turn the lever one way equals the moments of those which tend to turn it the opposite way.

PAGE 121.

ART. 192.—The force F' and D produces equilibrium in the system; hence a single force equal and opposite to F' at D will produce the same effect as the three forces F, P, and P.

ART. 197.—Assume that any number of forces act upon a body in any manner; they may produce both translation and rotation. The measure of their effort to produce rotation will be the sum of their moments, wherever be the origin, and the sum of these moments will be equivalent to a single couple, Arts. 184 and 186. Hence, if there be no effort at rotation, the sum of the moments will vanish for any and every point assumed for the origin of moments. The only other tendency to motion is that of translation; in which case there will be a single resultant passing through the centre of the body. If there be a resultant, *the moment* will be zero when the origin of moments is on the line of the resultant, Art. 189; and will have a finite value when the origin is not on the line of the resultant; and if there be *no* resultant the latter moment will also vanish.

SOLUTIONS OF EXAMPLES.

PAGE 125.

1. For, according to the triangle of forces, the resultant of two of them will equal in magnitude that represented by the third side, but its direction of action will be opposite to that represented by the third side, and at a distance from it equal to the altitude of the triangle. See Fig. 71 of the text, only in this case the third force will act *from* A towards B.

2. Inscribe a circle in the triangle; then will the radius r be the common arm of the three forces in reference to the centre of the circle, and we will have the equation of moments

$$R.r = P.r + F.r;$$
$$\therefore R = P + F.$$

3. The point of application of the resultant of two of them will be at the middle point of the side of the triangle between them; and the point of application of this resultant and the third weight will be at two-thirds the distance from the third weight on the line joining them; and this will be the required point. It is at the intersection of the medians of the triangle (Art. 222).

4. Let P and F be the respective amounts; then taking the origin of moments at the weight we have the relative moments $2.P$ and $1.F$; hence

$$2P = F.$$

Also, since the sum of P and F equals the entire weight,

$$P + F = 175.$$

Substituting,

$$3P = 175 \text{ pounds};$$
$$\therefore P = 58\tfrac{1}{3} \text{ pounds},$$

and

$$F = 116\tfrac{2}{3} \text{ pounds}.$$

5. Let $W = 500$, the arm of which in reference to the point B will be one-half of AB, or 1 foot; and the arm of F will be $DB = 3$ feet; hence the equation of moments will be

$$3F = 1\,W$$
$$= 500 \text{ lbs.};$$
$$\therefore F = 166\tfrac{2}{3} \text{ lbs.}$$

6. Let $x =$ the required distance, then will the lever arm of the weight be x, and of the man $8 - x$; hence we have, taking the origin of moments at the fulcrum,

$$175\,(8 - x) = 4000x,$$

or

$$(4000 + 175)x = 8 \times 175;$$
$$\therefore x = \tfrac{1400}{4175} \text{ feet}$$
$$= 4\tfrac{4}{167} \text{ inches}.$$

ANSWERS TO EXERCISES.

PAGE 126.

1. It is. Foot-pounds of rotary effort.
2. The resistance in pounds which is overcome through a certain number of linear feet.
3. It is the momentum of a given number of

pounds of mass moving at a given rate in feet per second.

4. This question is defective, because velocity involves a unit of time, which may be *one* second, *one* minute, or any other unit. Assuming that the velocity is feet per second, the unit will be 1 pound of mass × 1 foot per second × 1 foot for the arm.

5. It can, and will always do so in a free body if the line of action of the force does not pass through the centre of the mass.

6. They cannot. Since neither couple acting separately can produce translation, they cannot produce it when acting together. The resultant of two couples is a single couple, Arts. 185 and 186, and for this reason can produce rotation only.

7. It will be 100 lbs. more. Pulling down with his hands will add nothing to the pressure of his feet, for that effort is resisted by an equal upward push of his shoulder. A man by pulling upward on the straps of his boots, does not, thereby, diminish the pressure of the boot upon the floor, although it increases the pressure between his foot and the boot. No "perpetual motion man" has yet thrown himself over a fence by pulling on the straps of his boots.

8. If the cutting is uniform it is; but if the timber at one side of the hole is harder than at the

other, there will be a side push, tending to force the auger out of line.

9. It will.

SOLUTIONS OF EXAMPLES.

PAGE 133.

1. Let R be the resultant, and x the distance of its point of application from the force 6. Take the origin moments at 6; then we have the equation of moments

$$Rx = 11 \times 5,$$

and of forces

$$R = 6 + 11 = 17;$$
$$\therefore x = \tfrac{55}{17} = 3\tfrac{4}{17} \text{ feet.}$$

2. We have, using the same notation as before,

$$Rx = 11 \times 5 = 55,$$
$$R = 11 - 6 = 5;$$
$$\therefore x = \tfrac{55}{5} = 11 \text{ feet.}$$

3. Take the origin of moments at the extremity of the line near the weight 2, and retaining the same notation as above, we have

$$Rx = 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 = 54$$
$$R = 2 + 3 + 4 + 5 = 14 \text{ lbs.};$$
$$\therefore x = 54 \div 14 = 3\tfrac{6}{7} \text{ feet.}$$

4. Take the origin of moments at A, then we have

$$Rx = 3 \times 0 + 4 \times 3 + 5 \times 7 + 6 \times 5 = 77;$$
$$R = 3 + 4 + 5 + 6 = 18 \text{ lbs.};$$
$$\therefore x = 77 \div 18 = 4 \text{ feet } 3\tfrac{1}{3} \text{ inches.}$$

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5. Let x be distance from A where P is applied, to the point of application of the resultant; take the origin of moments at the point of application of the resultant, then will the arm of P be $x \sin \varphi$, and of F, $(x + AB) \sin \varphi$, assuming $P > F$; hence

$$Px \sin \varphi = F(x + AB) \sin \varphi;$$
$$\therefore x = \frac{F}{P - F} AB;$$

which substituted in the preceding equation gives

$$\frac{PF}{P - F} AB \sin \varphi = \frac{PF}{P - F} AB \sin \varphi.$$

ANSWERS TO EXERCISES.

PAGE 134.

1. It has not. We may say that it has one at infinity, which is equivalent to saying that it has none.
2. No. The sum of the forces must also be zero. If the sum of the moments in reference to three arbitrary points is zero, they will be in equilibrium (see Art. 195 of the text).
3. When they form a couple.
4. Because the resultant is zero.
5. It will.
6. It will.

7. It may vary directly as the distance from the centre; or inversely as the distance; or as any power of the distance; or any root of the distance; or as any power for a part of the distance, and any root for the remaining distance; or in any other way, provided the concentric shells comprising the sphere shall be of uniform density.

PAGE 135.—Many of the properties of the centre of gravity were developed as long ago as in the days of Archimedes. The properties are not only of great importance in statics; but when the principles of dynamics were developed—after Galileo's time—they were found to be no less important in that science. We mention only one—A free rotating body rotates about an axis through its centre of gravity; or, more strictly, through the centre of the mass.

ANSWERS TO EXERCISES.

PAGE 138.

1. The vertical through the centre of gravity of the carriage and load must intersect the ground *between* the wheels. In order that it shall overturn, the vertical must fall outside the base of the wheels. A carriage in motion may overturn when it would not if standing, on account of the inertia of the mass. A sudden side "lurch" may induce a rotary movement sufficient to overturn it.

2. He can stand so long as the vertical through the centre of gravity of his body falls within the base occupied by his feet.

3. Some parts of his body must move backward. The space occupied by his feet being of finite size, he may move his head, or other parts of his body, to some extent without endangering his stability.

4. Because the base being so very narrow, with only two legs, a small displacement will cause the vertical through the centre of gravity to fall without the support.

5. Because in Fig. 77 the centre of gravity must be moved further than in Fig. 78, and also must be raised through a greater height. Some writers consider the height through which the centre of gravity must be raised in order to overturn a body, a measure of its stability.

6. Because the line through the point of support and the centre of gravity of the book will be inclined to the edges, and as the former will be vertical, the latter must be inclined.

7. It may. Such will be the case with a cylinder resting on its convex surface. It will be in indifferent equilibrium in reference to rolling, but stable in reference to a longitudinal motion.

8. The centre of gravity is below the top of the post.

9. In both these exercises the balls are so much heavier than the bodies to which they are attached, that the centre of gravity of the whole device is below the support.

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$\bar{x}$ is read "*x dash*."

SOLUTIONS OF EXAMPLES.

PAGE 140.

1. Let l be the length of the line, and x the distance of the centre of gravity to the smaller weight; then will $l - x$ be the distance to the other, and the equation of moments gives

$$1.x = n(l - x);$$

$$\therefore x = \frac{n}{1+n} l.$$

If $n = 2$, then $x = \frac{2}{3}l$.

If $n = 3$, then $x = \frac{3}{4}l$.

2. The middle of one side will be the centre of gravity of two of them, and the centre of gravity of the three will be in the line joining this point with the vertex, and, according to the preceding example, it will be at two-thirds the distance from the apex. Hence the centre of gravity will be at two-thirds the distance from any apex, on a line drawn to the middle of the opposite side.

3. The centre of gravity of the weights 1 and 2 will be in the line joining them, and at two-thirds the distance from 1; and of the three weights it will be in the line joining the former point, and the weight 3, and at its

middle point. The solution is the same whether the triangle be equilateral or scalene.

4. The centre of gravity of the three weights at the base will be at the centre of the base, and of the four weights in a line joining the apex with the centre of the base, and, according to the first example, at three-fourths the distance from the apex.

ART. 217.—A line, in mechanics, is a body from which all dimensions are abstracted except that of length.

PAGE 142, ART. 220.—This article is introduced here—in advance of its proof—partly to classify it with lines, and partly to furnish exercises.

SOLUTIONS OF EXAMPLES.

1. Join the centre of gravity of one edge with that of another, and the centre of this line with the centre of another edge, and so on.

2. Half the diagonal of the base (a side being 1) will be $\frac{1}{2}\sqrt{2}$, and the length of one of the lateral edges will be $\sqrt{1 + (\frac{1}{2}\sqrt{2})^2} = \sqrt{\frac{3}{2}} = \frac{1}{2}\sqrt{6}$. There are four edges, and hence the entire lengths will be $2\sqrt{6}$. The lengths of the sides of the base will be 4. The centre of gravity of the four lateral edges will be at one-half the altitude, and of the edges of the base it will be at the centre of the base. Taking the origin of moments at the apex, and $\bar{x}$ the distance of the centre of gravity from the apex, we have

$$(2\sqrt{6} + 4)\,\bar{x} = 2\sqrt{6} \times \tfrac{1}{2} + 4 \times 1,$$
$$\therefore \bar{x} = 0{\cdot}724 +.$$

3. In the equation in Article 220, make $BO = r$, $AC = 2r$; then arc $ABC = \pi r$, and we have

$$Oc = \frac{r.2r}{\pi r} = \frac{2r}{\pi}.$$

PAGE 143.

4. We will have $BO = r$, $AC =$ the chord of $60° = r$, the radius, and $AB = 2\pi r \frac{60}{360} = \frac{1}{3}\pi r$; hence

$$Oc = \frac{r.r}{\frac{1}{3}\pi r} = \frac{3r}{\pi}.$$

5. We will have $BO = r$, $AC =$ the side of an inscribed square $= r\sqrt{2}$, and $ABC =$ the arc of a quadrant $= \frac{1}{2}\pi r$; hence

$$Oc = \frac{r.r\sqrt{2}}{\frac{1}{2}\pi r} = 2\sqrt{2}\frac{r}{\pi}.$$

ART. 221.—A surface is a body from which all dimensions are abstracted except length and breadth.

SOLUTIONS OF EXAMPLES.

PAGE 145.

1. It will not, for the moment of the part next to the apex will be less than the part next to the base.

2. When the sides adjacent to the vertical angle are equal, the bisecting line will pass through

the centre of gravity of the triangle. In other cases it will not.

3. The altitudes will be the same, and the centre of gravity will be at one-third the altitude from the base.

4. The area of the larger circle will be πR^2; of the smaller πr^2; and of the remaining part $\pi(R^2 - r^2)$. Taking the point A for the origin of moments, we have

$$\pi(R^2 - r^2)Ac = \pi R^2.R - \pi r^2.r;$$

$$\therefore Ac = \frac{R^3 - r^3}{R^2 - r^2}.$$

$$= \frac{R^2 + Rr + r^2}{R + r}$$

If $r = \frac{1}{2}R$, then

$$Ac = \frac{7}{6}R.$$

If $r = R$, $Ac = \frac{3}{2}R$.

5. Let $AF = AD = a$; then the diagonal $AE = a\sqrt{2}$, $CE = \frac{1}{2}a\sqrt{2}$, and the equation of moments will be

$$\tfrac{3}{4}a^2.AB = a^2.\tfrac{1}{2}a\sqrt{2} - \tfrac{1}{4}a^2(a\sqrt{2} - \tfrac{1}{4}a\sqrt{2});$$

$$\therefore AB = \tfrac{5}{12}a\sqrt{2} = \tfrac{5}{6}\cdot\tfrac{1}{2}a\sqrt{2} = \tfrac{5}{6}AC.$$

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6. The centre of gravity of the triangle ABC will be at $\frac{1}{3}$ of CF from F; of DCE, $\frac{1}{3}$ of CG from G; or from F it will be $FG + \frac{1}{3}CG$. The triangle ABC will be to triangle DCE as $(FC)^2$

is to $(GC)^2$; and $ACB : ADEB = (FC)^2 : (FC)^2 - (GC)^2$.

Taking the origin of moments at F we have,

$$ADEB \times Fg = ACB \times \tfrac{1}{3}FC - DCE \times (FG + \tfrac{1}{3}CG).$$

From the proportions, we have,

$$ADEB = ACB \times \frac{(FC)^2 - (GC)^2}{(FC)^2},$$

$$DCE = ACB \times \frac{(CG)^2}{(FC)^2}.$$

Substituting,

$$((FC)^2 - (GC)^2)Fg = \tfrac{1}{3}(FC)^3 - (CG)^2(FG + \tfrac{1}{3}CG).$$

From the figure we have

$$\frac{CG}{FC} = \frac{DE}{AB},$$

and

$$FC = CG + FG;$$

$$\therefore \frac{CG}{CG + FG} = \frac{DE}{AB};$$

$$\therefore CG = \frac{DE}{AB - DE} \times FG;$$

hence we find

$$FC = \frac{AB}{AB - DE} FG.$$

Substituting above gives

$$\left[\left(\frac{AB}{AB - DE}\right)^2 - \left(\frac{DE}{AB - DE}\right)^2\right](FG)^2 \times Fg =$$

$$\tfrac{1}{3}\left(\frac{AB}{AB-DE}\right)^3 (FG)^3 - \left(\frac{DE}{AB-DE}\right)^2$$
$$\left(1+\tfrac{1}{3}\frac{DE}{AB-DE}\right) FG^3.$$

Reducing gives

$$\frac{(AB)^2-(DE)^2}{(AB-DE)^2} Fg = \tfrac{1}{3}FG\left[\frac{(AB)^3}{(AB-DE)^3} - \right.$$
$$\left.\frac{(DE)^2(3AB-2DE)}{(AB-DE)^3}\right]$$
$$= \tfrac{1}{3}FG\left[\frac{(AB)^3-3AB.(DE)^2+2(DE)^3}{(AB-DE)^3}\right];$$

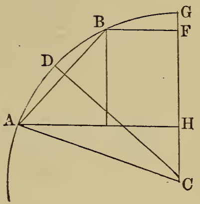
$$\therefore Fg = \tfrac{1}{3}FG$$
$$\left[\frac{(AB)^3-3AB.(DE)^2+2(DE)^3}{(AB-DE)(AB-DE)(AB+DE)}\right]$$
$$= \tfrac{1}{3}Fg\left[\frac{(AB)^3-3AB.(DE)^2+2(DE)^3}{(AB^2-2AB.DE+DE^2)(AB+DE)}\right]$$
$$= \tfrac{1}{3}Fg\,\frac{AB+2DE}{AB+DE}.$$

7. The slant height will be $\sqrt{r^2+h^2}$, and the lateral area, $2\pi r.\frac{1}{2}\sqrt{r^2+h^2}$; and the centre of gravity of the lateral area will be in the axis at $\frac{2}{3}h$ from the apex. Let $\bar{x}$ be the required distance, then

$$(\pi r^2+\pi r\sqrt{r^2+h^2})\,\bar{x} = \pi r^2.h + \pi r\sqrt{r^2+h^2}.\tfrac{2}{3}h;$$

$$\therefore \bar{x} = \frac{\frac{2}{3}\sqrt{r^2+h^2}+r}{\sqrt{r^2+h^2}+r}\,h.$$

PAGE 149, ART. 230.—PROBLEM. *To find the centre of gravity of a segment of a sphere.*



A segment of one base, GAH, is considered in the text.

To find the volume of the segment AGH, we have from geometry

$$\tfrac{1}{6}\pi(GH)^3 + \tfrac{1}{2}GH \times \pi(AH)^2$$
$$= \pi GH(\tfrac{1}{6}GH^2 + \tfrac{1}{2}(AH)^2).$$

We put this under another form—thus, in the right-angled triangle AHC, we have

$$(AH)^2 = (AC)^2 - (CH)^2$$
$$= (CG)^2 - (CG - GH)^2$$
$$= 2CG.GH - (GH)^2$$
$$= GH(2CG - GH),$$

which, substituted in the preceding expression, gives for the volume of the segment,

$$\pi(GH)^2(CG - \tfrac{1}{3}GH),$$

which is the value used in the text. If r be the radius of the sphere, and h the altitude of the segment, the expression becomes

$$\pi h^2(r - \tfrac{1}{3}h).$$

For the volume of the spherical sector ACG, we have, from geometry,

$$2\pi(CG)^2 \times \frac{GH}{CG} \times \tfrac{1}{3}CG$$
$$= \tfrac{2}{3}\pi(CG)^2 \times GH$$
$$= \tfrac{2}{3}\pi r^2 h,$$

which is also the value used in the text.

The volume of the cone the radius of whose base is AH, and altitude HC, is

$$\pi(AH)^2 \times \tfrac{1}{3}CH$$
$$= \pi(r^2 - (r-h)^2)\cdot\tfrac{1}{3}(r-h)$$
$$= \tfrac{1}{3}\pi(2rh - h^2)(r-h)$$
$$= \tfrac{1}{3}\pi h(2r-h)(r-h).$$

These values in the first equation of the article gives

$$Cg' = \frac{8(CG)^2.GH.Cg - 3(AH)^2.(CH)^2}{12(GH)^2(CG - \tfrac{1}{3}GH)}$$

$$= \frac{8(r^2.h.\tfrac{3}{8}(2r-h) - 3(r^2-(r-h)^2)(r-h)^2}{12h^2.(r-\tfrac{1}{3}h)}$$

$$= \tfrac{1}{4}\,\frac{(2r-h)^2}{r-\tfrac{1}{3}h}.$$

To put this under another form, let the angle $ACG = \theta$, then

$$h = r - CH = r - r\cos\theta = r(1-\cos\theta);$$

hence

$$2r - h = 2r - (r - r\cos\theta)$$
$$= 2r - r + r\cos\theta$$
$$= r\,(1 + \cos\theta).$$

From trigonometry

$$2\cos^2\tfrac{1}{2}\theta = 1 + \cos\theta;$$
$$\therefore (2r - h)^2 = 4r^2\cos^4\tfrac{1}{2}\theta.$$

Also we find

$$\begin{aligned} r - \tfrac{1}{3}h &= r - \tfrac{1}{3}(CG - CH) \\ &= r - \tfrac{1}{3}(r - r\cos\theta) \\ &= \tfrac{1}{3}r(2 + \cos\theta); \end{aligned}$$

and these substituted above, give

$$Cg' = \frac{3\cos^4\tfrac{1}{2}\theta}{2 + \cos\theta}\,r.$$

If $h = 0$, we have $\theta = 0$, and the segment will vanish, and we have

$$Cg' = r,$$

as it should.

If $h = 2r$, $\theta = 180°$, and we have

$$Cg' = 0,$$

hence the centre of gravity of the sphere will be at its geometrical centre, as it should.

For the hemisphere, $\theta = 90°$, and we have

$$\begin{aligned} Cg' &= \tfrac{3}{2}(\tfrac{1}{2}\sqrt{2})^4 r \\ &= \tfrac{3}{8}r. \end{aligned}$$

If the segment has two bases, let θ be the angle subtended by the radius of the upper base, and φ the angle subtended by the radius of the lower base, then by taking the difference of the moments of both segments, and the segment on the upper base, we find

$$Cy' = 3\frac{(1-\cos\varphi)^2.\cos^4\tfrac{1}{2}\varphi - (1-\cos\theta)^2.\cos^4\tfrac{1}{2}\theta}{(1-\cos\varphi)^2(2+\cos\varphi) - (1-\cos\theta)^2(2+\cos\theta)}\,r.$$

SOLUTIONS OF EXAMPLES.

PAGE 150.

1. In Article 229, CG is the radius of a circle, and by the conditions of the problem GH will also be a radius; hence we have

$$\begin{aligned} &\tfrac{3}{8}(2CG - GH) \\ &= \tfrac{3}{8}(2r - r) \\ &= \tfrac{3}{8}r. \end{aligned}$$

2. Let $\bar{x}$ be the required distance to the centre from the common tangent point; then, according to Article 202, the moment of the difference of the spheres will equal the difference of their moments, and we have

$$(\tfrac{4}{3}\pi R^3 - \tfrac{4}{3}\pi r^3)\bar{x} = \tfrac{4}{3}\pi R^3.R - \tfrac{4}{3}\pi r.^3r;$$

$$\therefore \bar{x} = \frac{R^4 - r^4}{R^3 - r^3}$$

$$= \frac{R^3 + R^2r + Rr^2 + r^3}{R^2 + Rr + r^2}.$$

If $r = 0$, we have

$$\bar{x} = R,$$

and the centre of gravity will be at the centre of the sphere.

If $r = R$, we have

$$\bar{x} = \tfrac{4}{3}R,$$

which gives the centre of gravity when the thickness of the solid, opposite the tangent point, is infinitesimal.

3. To find the versed-sine GH, Fig. 94, we have in this case $CB = r$, $HB = \frac{1}{4}r$;

$$\therefore CH = \sqrt{r^2 - \tfrac{1}{16}r^2} = \tfrac{1}{4}r\sqrt{15};$$

$$\therefore GH = (1 - \tfrac{1}{4}\sqrt{15})r.$$

In the equation of Article 230,

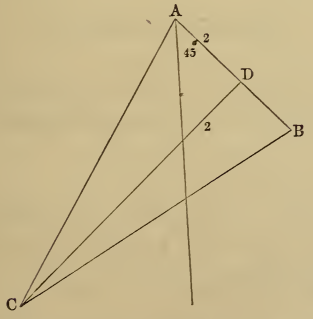
$$Cg = \tfrac{3}{8}[2r - (1 - \tfrac{1}{4}\sqrt{15})r] = \tfrac{3}{8}(1 + \tfrac{1}{4}\sqrt{15})r,$$

and we have

$$Cg' = \frac{8r^2(1-\frac{1}{4}\sqrt{15})r . \frac{3}{8}(1+\frac{1}{4}\sqrt{15})r - 3(\frac{1}{4}r)^2 . (\frac{1}{4}r\sqrt{15})^2}{12(1 - \frac{1}{4}\sqrt{15})^2 \times r^2 \times [(r - \frac{1}{3})(1 - \frac{1}{4}\sqrt{15})r]}$$

$$= \frac{\frac{3}{16} - \frac{45}{256}}{12(\frac{31}{16} - \frac{1}{2}\sqrt{15})(\frac{1}{3} + \frac{1}{12}\sqrt{15})} r$$

$$= 0{\cdot}977\, r \text{ nearly.}$$



4. The centre of gravity will be $\frac{1}{4}$ of 8 inches, or 2 inches from the base, hence the line joining

the point of suspension with the centre of gravity forms a triangle of which the two sides are each 2, and as the altitude is perpendicular to the base, the oblique angles will each be 45 degrees, which will equal the required inclination.

5. In this case the angle between the perpendicular and the radius of the base will be 30°, and we have

$$\tan 30° = \tfrac{1}{4}Alt. \div radius\ of\ base;$$

$$\therefore \frac{Alt.\ of\ cone}{radius\ of\ base} = 4 \tan 30°$$

$$= 4\frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$= \tfrac{4}{3}\sqrt{3}$$

$$= 2{\cdot}30940 +.$$

SOLUTIONS OF EXAMPLES.

Page 154.

1. In this case $\theta = \frac{1}{2}\pi$ in the equation of Art. 235, and $\sin\theta = 1$; hence, by substitution, we have

$$Cg = \tfrac{4}{3}\frac{r}{\pi}.$$

2. The area of the segment will equal the area of the sector $ACBG$, minus the area of the triangle ACB. Let θ be the angle ACG, then $ACB = 2\theta$, arc $AGB = r.2\theta = 2r\theta$, and the area of the sector $= \frac{1}{2}r.2r\theta = r^2\theta$; $HB = r \sin\theta$, $CH = r\cos\theta$, and area of the triangle $ACB = r^2 \sin\theta\cos\theta$.

Hence

$$r^2(\theta - \sin\theta\cos\theta)\,\bar{x} = r^2\theta.\tfrac{2}{3}\frac{r\sin\theta}{\theta} - r^2\sin\theta$$

$$\cos\theta.\tfrac{2}{3}r\cos\theta;$$

$$\therefore\ \bar{x} = \tfrac{2}{3}\cdot\frac{\sin\theta - \sin\theta\cos^2\theta}{\theta - \sin\theta\cos\theta}\,r$$

$$= \tfrac{2}{3}\cdot\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\,r.$$

3. The sphere may be generated by the revolution of a semicircle about a diameter. Area of the semicircle $= \frac{1}{2}\pi r^2$. The circumference described by the centre of gravity of the semicircle, will be (Ex. 1), $\frac{4}{3}\frac{r}{\pi}\cdot 2\pi = \frac{8}{3}r$. Hence, according to Art. 233, we have

$$volume = \tfrac{1}{2}\pi r^2\cdot\tfrac{8}{3}r = \tfrac{4}{3}\pi r^3.$$

4. Referring to the solution of Example 2 above, we find,

$$area\ of\ segment = (\theta - \sin\theta\cos\theta)r^2,$$

which multiplied by $2\pi\bar{x}$, where $\bar{x}$ is the answer to the 2d Example, gives the required result; hence the required volume is

$$\tfrac{4}{3}\pi r^3\sin^3\theta$$
$$= volume\ of\ the\ sphere \times \sin^3\theta.$$

5. Following the method of Art. 233, and using the values already found, we have

$$\begin{aligned} volume &= r^2\theta \times 2\pi Cg \\ &= 2\pi r^2\theta \times \tfrac{2}{3}\frac{r \sin\theta}{\theta} \\ &= \tfrac{4}{3}\pi r^3 \sin\theta \\ &= volume\ of\ the\ sphere \times \sin\theta \\ &= \tfrac{4}{3}\pi r^2.AH \\ &= area\ of\ great\ circle \times \tfrac{4}{3}AH. \end{aligned}$$

If $\theta = 90°$ we have $\frac{4}{3}\pi r^3$, which is the volume of a sphere, as it should be.

The volume generated as in Example 4 = vol. in Ex. 5 $\times \sin^2\theta$.

PAGE 158, ART. 241.—In the first edition of this work, the solution of this problem is erroneous. The line EF will not generally be tangent to the curve MN, and hence the analysis founded on that supposition is erroneous. They will be in equilibrium when the point g is vertically under C.

Let $EgC = \varphi$, then the equation of moments will be

$$W_1.Eg.\sin\varphi = W_2.gF.\sin\varphi;$$
$$\therefore W_1.Eg. = W_2.gF.$$

From the figure we have

$$Eg + gF = EF = r_1 + r_2,$$

which combined with the preceding equation gives

$$Eg = \frac{W_2}{W_1 + W_2}(r_1 + r_2);$$

also

$$CF = R - r_2,$$
$$CE = R - r_1,$$

and as the three sides of the triangle ECF thus become known, the angles E and F may be found; for we have from trigonometry

$$\sin \tfrac{1}{2}E = \frac{\sqrt{(s-a)(s-b)}}{ab},$$

where $s = \frac{1}{2}(a + b + c)$, a, b, c, being the sides of the triangle, and E the angle opposite the side c, or CF. Similarly,

$$\sin \tfrac{1}{2}F = \frac{\sqrt{(s-b)(s-c)}}{bc},$$

where F is opposite a, or EC.

In the triangle ECg the two sides CE and Eg, and the angle included by them becoming known, the angle $ECg - CgE$ may be found from the proportion

$$EC + Eg : EC - Eg : : \tan \tfrac{1}{2}(ECg + CgE) : \tan \tfrac{1}{2}(ECg - CgE),$$

and finally

$$ECg = \tfrac{1}{2}(ECg + CgE) + \tfrac{1}{2}(ECg - CgE).$$

SOLUTIONS OF EXAMPLES.

PAGE 169.

1. P will move down. Solving for s in Equation (3), p. 166 of text, we have

$$s = \frac{P \sin B - W \sin A}{2(P + W)} gt^2$$

$$= \frac{5347 \cdot 71}{110} = 48 \cdot 6 \text{ feet}.$$

2. If the weights be in equilibrium, $v = 0$ and Eq. (1), p. 165, becomes

$$0 = \frac{P \sin B - W \sin A}{P + W} 2gs,$$

or

$$P \sin B = W \sin A.$$

3. In this example the acceleration $= \frac{1}{5}g$;

$$\therefore s = \tfrac{1}{2} \cdot \tfrac{1}{5}gt^2 = \tfrac{1}{10}gt^2,$$

which in Eq. (3), p. 166 of the text, gives

$$1 = \frac{\frac{1}{5}(P + W)}{P \sin 45^\circ - W \sin 30^\circ},$$

or

$$P \times 0{\cdot}7071 - \tfrac{1}{2} W = \tfrac{1}{5}(P + W);$$

$$\therefore P = 1{\cdot}38\, W +.$$

4. From the conditions of the problem, and the figure, we have

$$CA = 3 \text{ ft.},\quad AE = 1 \text{ ft.},\quad FB = \tfrac{1}{2} \text{ ft.};$$

$$\therefore CE = 2 \text{ ft.},\quad CF = 2\tfrac{1}{2} \text{ ft.}, EF = 1\tfrac{1}{2} \text{ ft.}$$

The cylinders being of the same material, their weights will be as the squares of their radii; hence

$$W_1 = 4\, W_2.$$

Take the origin of moments at g; or since gC is vertical, the moment will be the same if the origin be anywhere on that line. The arm of W_1 will be the perpendicular from E to the line Cg, which is

$$Eg \,.\, \sin EgC = Eg \,.\, \sin \varphi.$$

Similarly, the arm of W_2 will be

$$gF. \sin \varphi.$$

Hence the equation of moments will be

$$W_1 \,.\, Eg \sin \varphi = W_2 \,.\, gF \sin \varphi,$$

or, substituting the value of W_1 and cancelling $\sin \varphi$,

$$4 W_2 . Eg = W_2 . gF,$$

$$\therefore 4 Eg = gF.$$

But

$$Eg + gF = 1\tfrac{1}{2} \text{ ft.};$$

substituting,

$$5 Eg = 1\tfrac{1}{2} \text{ ft.};$$

$$\therefore Eg = \tfrac{3}{10} \text{ ft.}$$

and

$$gF = \tfrac{6}{5} \text{ ft.}$$

From trigonometry we have

$$\cos CEF = \frac{(CE)^2 + (EF)^2 - (CF)^2}{2 CE . EF}$$

$$= \frac{4 + 2\frac{1}{4} - 6\frac{1}{4}}{6} = 0;$$

$$\therefore CEF = 90°,$$

and we have

$$\tan ECg = \frac{Eg}{EC} = \frac{\frac{3}{10}}{2} = 0{\cdot}15;$$

$$\therefore ECg = 8° \; 31' \; 50'' +.$$

5. The mass moved remains the same, but the

effective moving force is reduced by the amount of the friction; hence we have at once

$$v = \sqrt{\left[\frac{P - W \sin A - \mu W \cos A}{P + W} 2gs\right]}$$

$$= \frac{P - W \sin A - \mu W \cos A}{P + W} gt$$

$$= \frac{P - W(\sin A + \mu \cos A)}{P + W} gt.$$

Similarly,

$$t = \sqrt{\left[\frac{P + W}{P - W(\sin A + \mu \cos A)} \cdot \frac{2s}{g}\right]}.$$

6. Let $R \doteq AC$, $r = AE = FB$;

$\therefore EC = R - r = CF$, $EF = 2r$.

EF is bisected by a perpendicular from C;

$$\therefore \sin CEF = \frac{r}{R - r}.$$

By moments, as in Example 4, we find

$$Eg = \tfrac{1}{3} EF = \tfrac{2}{3} r.$$

Since Cg is not perpendicular to EF, the triangle is oblique, and we have

$$EC + Eg : EC - Eg :: \tan \tfrac{1}{2}(ECg + EgC) : \tan \tfrac{1}{2}(ECg - EgC),$$

or

$$R - \tfrac{1}{3} r : R - \tfrac{5}{3} r :: \tan \tfrac{1}{2}(180 - E) : \tan \tfrac{1}{2}(ECg - EgC);$$

from which the angle ECg may be found. The angle may be found by means of right-angled triangles. Thus, let fall a perpendicular from C upon EF, and let the foot be represented by G, which the reader can supply in the figure. Then

$$CG = \sqrt{(R-r)^2 - r^2},$$

and

$$\tan ECG = \frac{r}{\sqrt{(R-r)^2 - r^2}};$$

$$\therefore ECG = \tan^{-1}\frac{r}{\sqrt{R^2 - 2Rr}}.$$

Also

$$\tan gCG = \frac{gG}{CG}$$

$$= \frac{\frac{1}{3}r}{\sqrt{(R-r)^2 - r^2}};$$

$$\therefore gCG = \tan^{-1}\frac{r}{3\sqrt{R^2 - 2Rr}};$$

and, finally,

$$ECg = ECG - gCG.$$

7. The velocity of discharge will be

$$v = \sqrt{2gh},$$

and the quantity discharged in one second will be

$$k\sqrt{2gh},$$

and in the time t the quantity of discharge will be

$$kt\sqrt{2gh};$$

hence

$$kt\sqrt{2gh} = q;$$

$$\therefore t = \frac{q}{k\sqrt{2gh}}.$$

SOLUTIONS OF EXAMPLES.

PAGE 179.

1. The formula, p. 171 of text, becomes

$$F = W\frac{\sin A}{\cos \varphi}$$

$$= 50\frac{\sin 30°}{\cos (60° - 30°)}$$

$$= 50\frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}}$$

$$= 28{\cdot}8 \text{ lbs.}$$

2. We will have

$$F = 60\frac{\sin 45°}{\cos(-45°)} = 60 \text{ lbs.}$$

3. We find

$$F = W\frac{\sin A}{\cos 0°} = W \sin A \text{ lbs.}$$

4. We will have

$$F = W\frac{\sin A}{\cos(-A)} = W \tan A \text{ lbs.}$$

5. From Article 254 we have, in reference to motion down the plane,

$$F = \frac{\frac{1}{2} - 0{\cdot}2 \cos(30^\circ)}{\cos(-5^\circ) - 0{\cdot}2 \sin(-5^\circ)} 50 = 16{\cdot}1 \text{ lbs.};$$

up the plane

$$F = \frac{\frac{1}{2} + 0{\cdot}2 \cos 30^\circ}{\cos 5^\circ - 0{\cdot}2 \sin 5^\circ} 50 = 34{\cdot}4 \text{ lbs.}$$

Page 180.

6. We will have, from the first value of F, p. 172 of the text,

$$F = \frac{\frac{1}{2}\sqrt{2} - 0{\cdot}15 \times \frac{1}{2}\sqrt{2}}{1} 100 = 60 \text{ lbs.}$$

7. From Article 256 we have

$$\sin A = \frac{cg}{R}.$$

But from Example 4, p. 146 of the text, where $r = \frac{1}{2}R$, we find (see solution in this Key)

$$cg = \tfrac{7}{6}R - R = \tfrac{1}{6}R;$$

$$\therefore \sin A = \tfrac{1}{6};$$

$$\therefore A = 9^\circ\ 35'\ 40.$$

8. From the formula of Article 259 we have

$$Cc = \tfrac{15}{20} \times 4 = 3 \text{ feet},$$

and, therefore, the three sides of the triangle OCc become known; hence, from trigonometry, we have

$$\cos \tfrac{1}{2}C = \sqrt{\frac{\tfrac{1}{2}s(s-c)}{ab}}$$

$$= \sqrt{\frac{4\tfrac{1}{2}(4\tfrac{1}{2}-2)}{12}}$$

$$= \sqrt{\tfrac{15}{16}} = 0{\cdot}96824;$$

$$\therefore C = 28^\circ\ 57'\ 20.''$$

9. The formula of Article 259 gives

$$P = W.$$

The triangle will be equilateral, hence each of the angles will be 60°.

10. The equation on page 179 gives

$$F = CD = \frac{3}{2\sqrt{1-\tfrac{4}{16}}}$$

$$= \sqrt{3} \text{ feet.}$$

ANSWERS TO EXERCISES.

PAGE 180.

1. It will.
2. It cannot without a force to hold it.
3. When the centre of gravity is highest the potential energy will be greatest, and least when lowest. In Fig. 113 it is greatest.
4. In Fig. 114 the potential energy is a maximum. In Fig. 115 the potential energy is least.
5. In Fig. 117 it is indifferent, for, since the weights are in equilibrium at all points on the curve,

if the weight W be moved from one position to another on the curve, the weight P will be raised or lowered so that their common centre of gravity will remain at the same height.

The same general relations hold in Figs. 118 and 119; hence the potential energy is indifferent in these also.

PAGE 181.

6. In this case they are in equilibrium in only one position. If $P > W$, we will have $e > 1$, and the value of y will be imaginary; hence W necessarily exceeds P.
7. They cannot, for the eccentricity would be less than unity.
8. If the weight W be at the upper extremity of the axis, at A, there will be equilibrium.
9. The length will equal twice the diameter of the bowl, and it will rest on the edge F.
10. Horizontal.
11. It would rest on the horizontal plane.

PAGE 186.

The expression for the moment in the 4th line of page 186 should be

$$F_1(x_1 \cos \beta_1 - y_1 \cos \alpha_1),$$

for reasons given in Article 176 of this Key. This is the same as

$$F_1 (x_1 \sin \alpha_1 - y_1 \cos \alpha_1).$$

The signs of all the other expressions on that page should also be changed from + to − and − to +.

SOLUTIONS OF EXAMPLES.

PAGE 192.

1. $$t = \frac{1.2}{2 \times 0.6633} 100 = 90.5 \text{ lbs.}$$

2. Let
$$AB = x, \; BC = y,$$
$$AD = u, \; DC = v,$$
$$t = \text{the tension of } AB,$$
$$t_1 = \text{the tension of } BC;$$

then
$$x + y = 10, \quad (1)$$
$$u + v = 5, \quad (2)$$
$$t = 2t_1; \quad (3)$$

and the last formula of Article 269 of the text gives
$$\sec BAD = 2 \sec BCD,$$

or
$$\frac{x}{u} = 2\frac{y}{v}. \quad (4)$$

From the figure,
$$x^2 - u^2 = y^2 - v^2 = BD^2. \quad (5)$$

Eliminating between equations (1), (2), and (4) gives
$$u = \frac{5x}{20 - x}. \quad (6)$$

Equations (1), (2), and (5) give

$$x^2 - u^2 = (10 - x)^2 - (5 - u)^2,$$

or

$$4x - 2u = 15. \qquad (7)$$

Combining (6) and (7) gives

$$x = 4{\cdot}47 + \text{ft.},$$
$$u = 1{\cdot}44 + \text{ft.};$$
$$\therefore y = 5{\cdot}53 + \text{ft.},$$
$$v = 3{\cdot}56 + \text{ft.}$$

PAGE 193.

3. These conditions require that the points A and C shall not be in the same horizontal; and in the values of t and t_1, page 188 of the text, $BAD = 0$, and we have

$$t = W \cot BCD,$$
$$t_1 = W \operatorname{cosec} BCD$$
$$= \frac{W}{\sin BCD}.$$

4. The angle DAC will be 45°, and the last formulas on page 191 of the text give

$$c \cos 45° = t$$
$$c \sin 45° = 250 \text{ lbs.};$$
$$\therefore c = \frac{250}{0{\cdot}7071} = 353{\cdot}5 \text{ lbs.}$$
$$t = \frac{250}{0{\cdot}7071} \times 0{\cdot}7071$$
$$= 250 \text{ lbs.}$$

5. The last equation on page 191 of the text gives

$$W \sin CAD = \tfrac{1}{2} W;$$

$$\therefore \sin CAD = \tfrac{1}{2};$$

hence

$$CAD = 30^\circ,$$

and the depth CD will be one-half the length of the rafter.

6. The equation

$$t = \tfrac{1}{2} W \cot CAD,$$

page 191 of the text, gives

$$W = \tfrac{1}{2} W \cot CAD;$$

$$\therefore \cot CAD = 2;$$

$$\therefore CAD = 26^\circ\ 33'\ 54''.$$

ANSWERS TO EXERCISES.

PAGE 193.

1. They will not. The two added forces R and $-R$ will form a couple.

2. They will not, for the resultant of F and F_1 combined with $-R$ will constitute a couple. If the force equal to R were to act in a direction opposite to the resultant, then, in Exercise 1, there will be a resultant equal and opposite to the fourth force, and in the 2d Exercise, after the 3d force be removed, there will be a resultant equal to $2R$.

3. No modification is needed, for the moment of

the components will equal the moment of the single force.

4. It cannot—to do so would require an infinite tension.

5. If the weight at B is free to adjust itself, the tension on each will be equal, and each equal to $\frac{1}{2}W$.

6. The tension will remain the same. The tension is dependent only upon the weight and slope of the parts, as shown by the equations on p. 188 of the text.

7. An ellipse for the sum of the distances from the fixed points A and C will constantly equal the length of the string.

8. Decreased—and the thrust at the lower ends will be diminished. See the last equations of Art. 271 in the text.

9. The thrust and stresses on the braces will both be increased.

10. If the strut supports the weight, there will be no stress on the rafters.

Page 197.

Galileo was the first writer, of whom we have any knowledge, who established formulas for the strength of beams. His work was published at Bologna in 1656. Although the hypotheses upon which the formulas were founded were false, yet the law of variation of strength which he deduced for rectangular beams was correct. This law is—the strength *varies* directly as the first power of the breadth and the square of the depth jointly, and inversely as the first power of the length of the beam. But the factor used by him for determining the *value* of the strength was three times too large.

SOLUTIONS OF EXAMPLES.

PAGE 201.

1. Let $w =$ the load per unit of length of the beam, $l =$ the length of the beam, then will

$$W = wl.$$

We have from the last equation on page 200 of the text,

$$b = \tfrac{3}{4}\frac{Wl}{Rd^2}$$

$$= \tfrac{3}{4}\frac{500 \times 8 \times 8 \times 12}{1400 \times 8 \times 8}$$

$$= 3\tfrac{3}{14} \text{ inches.}$$

REMARK.—It is best to reduce all the dimensions to inches, since the tabular value of R is given per square inch.

2. From equation (2) of Art. 282, we have

$$bd^2 = \frac{6Pl}{R}$$

But

$$d = 4b;$$

$$\therefore b^3 = \tfrac{3}{8}\frac{Pl}{R}$$

$$= \tfrac{3}{8}\frac{1000 \times 8 \times 12}{1200}$$

$$= 30;$$

$$\therefore b = 3{\cdot}10 + \text{ inches};$$

and

$$d = 12{\cdot}43 + \text{ inches}.$$

3. From Problem 3, p. 200, we have

$$d^2 = \tfrac{3}{2}\frac{Pl}{Rb}$$

$$= \tfrac{3}{2} \cdot \frac{8000 \times 12 \times 12}{2 \times 10000}$$

$$= 86{\cdot}4;$$

$$\therefore d = 9{\cdot}29 \text{ inches}.$$

4. From the same equation as the preceding, we have

$$R = \tfrac{3}{2}\frac{Pl}{bd^2}$$

$$= \tfrac{3}{2}\frac{20000 \times 10 \times 12}{4 \times 9 \times 9}$$

$$= 11111{\cdot}1 + \text{ lbs}.$$

5. The required stress will be the value of R found from the equation above,

$$\therefore l = \tfrac{2}{3}\frac{Rbd^2}{P}$$

$$= \tfrac{2}{3}\frac{20000 \times 1\tfrac{1}{2} \times (3\tfrac{1}{2})^2}{10000}$$

$$= 24{\cdot}5 \text{ inches.}$$

6. Problem 4, p. 200 of the text, gives

$$W = \tfrac{4}{3}\frac{Rbd^2}{l}$$

$$= \tfrac{4}{3}\frac{12000 \times 6 \times 144}{15 \times 12}$$

$$= 76{,}800 \text{ lbs.}$$

7. The load will be uniform, and will equal the weight of the beam. We have

$$W = 2 \times 2 \times l \times \tfrac{1}{4}$$

$$= l,$$

and the formula of problem 2, p. 199 of the text, gives

$$l = \tfrac{1}{3}\frac{Rbd^2}{W}$$

$$= \tfrac{1}{3}\frac{30000 \times 2 \times 4}{l};$$

$$\therefore l^2 = 80{,}000;$$

and

$$l = 282{\cdot}8 \text{ inches}$$

$$= 23 \text{ feet } 6{\cdot}8 \text{ inches.}$$

PAGE 205.

The *straight* line of quickest descent is not *the line* of quickest descent. Curves of quickest descent are called *Brachistochrones.* Their form depends upon the conditions assumed. The forces may be assumed to vary according to the inverse squares, or directly as the distance, or inversely as the distance, or according to some other law, and they may be assumed to act in parallel lines or radiate from a point. If they are constant and parallel, as in the case of terrestrial gravitation, the curve will be a cycloid. It was problems of this character that gave rise to the Calculus of Variations—a very high order of analysis.

SOLUTIONS OF EXAMPLES.

PAGE 208.

1. From the 3d equation, p. 203 of the text, we have

$$\sin\varphi = \frac{2s}{gt^2}$$

$$= \frac{200}{32\frac{1}{6}\times 25}$$

$$= \cdot 2487;$$

$$\therefore \varphi = 14^\circ\ 24'.$$

PAGE 209.

2. From the 2d equation on p. 204 of the text we have

$$s = v_0 t - \tfrac{1}{2}gt^2\sin\varphi;$$

$$\therefore s = 50t - 16\tfrac{1}{12}\times\tfrac{1}{2}\sqrt{2}t^2$$

$$= 50t - 8\tfrac{1}{24}\sqrt{2}t^2.$$

Hence, in 3 seconds

$$s = 150 - 72\tfrac{3}{8}\sqrt{2} = 47{\cdot}65 \text{ feet.}$$

At the end of 5 seconds

$$s = 250 - 201\tfrac{1}{24}\sqrt{2} = -\ 34{\cdot}3 \text{ ft.},$$

that is, it will have ceased to ascend the plane, and returning, will, at the end of 5 seconds, be 34·3 feet below the starting-point.
At the end of 10 seconds

$s = 500 - 804\tfrac{1}{6}\sqrt{2} = -\ 637{\cdot}3$ feet below the starting-point.

3. The required velocity will be the same as that acquired by the body in sliding down half the length of the plane; hence the required velocity will be

$$\begin{aligned} v &= \sqrt{gs \sin \varphi} \\ &= \sqrt{32\tfrac{1}{6} \times 100 \times \tfrac{20}{100}} \\ &= 25{\cdot}36 \text{ feet.} \end{aligned}$$

If a body starts from the middle of the plane at the same time as the one at the upper end, it will reach the foot in the same time that the upper one reaches the middle; hence, if it be projected upward with the velocity acquired, at the instant the body starts from the upper end, they will meet at the middle of the plane.

4. The point must be higher than the lower extremity of the diameter—otherwise the solution is not possible. The required line will be

the distance from the point to where the line cuts the circle when drawn to the extremity of the diameter.

5. From equation (2), p. 208 of the text,

$$\begin{aligned} v &= \tfrac{1}{9}\sqrt{(h - 17\cdot6)s} \\ &= \tfrac{1}{9}\sqrt{22\cdot4 \times 5280} \\ &= 38\cdot2 + \text{ feet per second} \\ &= 26\cdot05 + \text{ miles per hour.} \end{aligned}$$

(To reduce feet per second to miles per hour, multiply the former by $\frac{60\times60}{5280} = \frac{15}{22}$.)

6. From the 5th equation we have

$$s = 4\cdot6v^2;$$

and from the 2d (writing s' for s so as to distinguish it from the preceding s)

$$\begin{aligned} v^2 &= \tfrac{1}{18}(h - 17\cdot6)s' \\ &= \tfrac{1}{18} \times 32\cdot4 \times 2640; \\ \therefore s &= \frac{4\cdot6 \times 32\cdot4 \times 2640}{81} \\ &= 4857\cdot6 \text{ feet.} \end{aligned}$$

7. The time down the plane may be deduced from the 3d equation of the text. We have

$$\begin{aligned} t &= \sqrt{\frac{328s}{h - 17\cdot6}} \\ &= \sqrt{\frac{328 \times 2640}{32\cdot4}} \end{aligned}$$

PAGE 209.

$$= 163{\cdot}4 \text{ sec.}$$
$$= 2{\cdot}72 \text{ minutes.}$$

For this time on the horizontal, we have from the 4th and 5th equations

$$t = 9\tfrac{1}{3}\sqrt{\frac{s}{4{\cdot}6}}$$
$$= 9\tfrac{1}{3}\sqrt{\frac{4858}{4{\cdot}6}}$$
$$= 303{\cdot}3 \text{ seconds,}$$
$$= 5{\cdot}05 + \text{ minutes.}$$

8. Equation (5) will give the velocity which it must acquire in moving down the plane. We have

$$v^2 = \frac{s}{4{\cdot}6}$$
$$= \frac{1000}{4{\cdot}6}$$
$$= 217{\cdot}3 \text{ feet per second.}$$

The required height will be given by equation (2); we have

$$h = \frac{81v^2 + 17{\cdot}6s}{s}$$
$$= \frac{81 \times 217{\cdot}3 + 17{\cdot}6 \times 1200}{1200}$$
$$= 32{\cdot}27 \text{ feet.}$$

9. Equation (5), page 208 of the text, gives

$$v = \sqrt{\frac{s}{4 \cdot 6}}$$

$$= \sqrt{\frac{800}{4 \cdot 6}}$$

$$= 13 \cdot 19 \text{ ft. per sec.}$$

Equation (2) gives

$$s = \frac{81 v^2}{h - 17 \cdot 6}$$

$$= \frac{81 \times 173 \cdot 91}{7 \cdot 4}$$

$$= 1903 \cdot 64 \text{ ft.}$$

The author once had occasion to use the principles of the last example in constructing the approach to an ore dock at Marquette, Mich.

SOLUTIONS OF EXAMPLES.

PAGE 217.

According to Article 299, the range will be

$$2h \sin 2\alpha,$$

where

$$h = \frac{v^2}{2g};$$

hence the range will be

$$\frac{v^2}{g} \sin 2\alpha$$

$$= \frac{25g^2}{g} \sin 90°$$

$$= 25g$$

$$= 804\tfrac{1}{6} \text{ feet.}$$

The greatest height will be, (Art. 301)

$$h \sin^2 \alpha$$

$$= \frac{v^2}{2g} \sin^2 45°$$

$$= \frac{25g^2}{2g} \times \tfrac{1}{2}$$

$$= 6\tfrac{1}{4} \times 32\tfrac{1}{6}$$

$$= 201{\cdot}04 + \text{ feet.}$$

PAGE 218.

2. $$BC = 2\sqrt{h \cdot y},$$

$$= 2\sqrt{15 \times 12}$$

$$= 26{\cdot}8 + \text{ feet.}$$

3. From Article 299, we have

$$2h = \frac{x}{\sin 2\alpha} = \frac{v^2}{g},$$

which in Article 300 gives

$$t^2 = \frac{4x}{2g \sin \alpha \cos \alpha} \sin^2 \alpha,$$

or solving for tan α gives

$$\tan \alpha = \frac{gt^2}{2x}$$

$$= \frac{32\frac{1}{6} \times 225}{2 \times 1000}$$

$$= 3{\cdot}6187$$

$$\therefore \alpha = 74^\circ\ 33'\ 9''.$$

PAGE 218.

From Article 300, we have

$$v = \frac{gt}{2 \sin \alpha}$$

$$= \frac{32\frac{1}{6} \times 15}{2 \times {\cdot}96387}$$

$$= 250{\cdot}29 \text{ feet.}$$

From Article 301, we have

$$CD = h \sin^2 \alpha$$

$$= \frac{(250{\cdot}29)^2}{64\frac{1}{3}} (0{\cdot}96387)^2$$

$$= 904{\cdot}69 \text{ feet.}$$

4. From Article 299, we have

$$h = \frac{x}{2 \sin 2\alpha} = \frac{v^2}{2g};$$

$$\therefore v = \sqrt{\frac{2 \times 32\frac{1}{6} \times 25000}{2 \times 1}}$$

$$= 896{\cdot}8 + \text{feet.}$$

From Art. 301, we have

$$A = h \sin^2 \alpha$$

$$= \frac{(897)^2}{64\frac{1}{3}} \times \tfrac{1}{2}$$

$$= 6{,}250 \text{ feet.}$$

From Article 300, we have

$$t = \frac{2v \sin \alpha}{g}$$

$$= \frac{2 \times 897 \times \frac{1}{2}\sqrt{2}}{32\frac{1}{6}}$$

$$= 39 + \text{seconds.}$$

5. We have from Articles 299 and 301,

$$AB = 4CD,$$

or

$$4h \sin \alpha \cos \alpha = 4h \sin^2 \alpha,$$

dividing by $\sin \alpha \cos \alpha$,

$$\tan \alpha = 1;$$

$$\therefore \alpha = 45°.$$

6. In equation (3) page 214, make $y = -150$, which is negative, because in Fig. 139 y is positive upwards, and the point where it will strike the plane, in this example, is below the point of starting, $\tan \alpha = 1$, and we have

$$-150 = x - \frac{32\frac{1}{6}x^2}{2(75)^2 \times \frac{1}{2}},$$

which solved for x gives

$$x = 271{\cdot}5 \text{ feet.}$$

7. In this example

$$\alpha = -30°,\ y = -25 \text{ feet.}$$

The velocity with which the body will leave the eaves will equal that of a body falling through the vertical height of the ridge above the eaves, and this value will be considered as the velocity of projection. We then have

$$v^2 = 2g \times 7 = 14g,$$

and these substituted in equation (3), page 214, give

$$-25 = -0{\cdot}57735x - \frac{x^2}{28 \times 0{\cdot}86603},$$

which solved gives

$$x = 17{\cdot}6 + \text{feet.}$$

8. Substituting the values given in the example for α and y in equation (3) page 214 gives

$$60 = 400 \tan \alpha - \frac{32\frac{1}{6}(400)^2}{2v^2 \cos^2 \alpha};$$

$$50 = 600 \tan \alpha - \frac{32\frac{1}{6}(600)^2}{2v^2 \cos^2 \alpha}.$$

Multiplying the first by 9 and the second by 4, and subtracting the latter from the former, gives

$$340 = 1200 \tan \alpha;$$

$$\therefore \tan \alpha = \tfrac{17}{60},$$

and

$$\alpha = 15^\circ\ 49'\ 9''.$$

Substituting this value in the first of the preceding equations gives

$$60 = \tfrac{17}{60} \times 400 - \frac{32\frac{1}{6}(400)^2}{2 \times (0{\cdot}96213)^2 \times v^2};$$

$$\therefore v = \sqrt{\frac{96\frac{1}{2}}{320(0{\cdot}96213)^2}} \times 400$$

$$= 228{\cdot}2 \text{ feet per second.}$$

EXERCISES.

Page 219.

1. Zero.
2. The velocity of projection being the same, they will strike the sea at the same time, and their range from the point where the ship will be at that time will be the same; but not the same if reckoned from the point of projection.
3. 15 miles per hour $= \frac{15 \times 5280}{60 \times 60}$ feet per second $= 22$ feet per second; hence the actual velocity will be 11 feet per second in the direction of motion of the ship in reference to the point from which the projection is made.
4. In reference to the point on the earth, it will be the same; but not in reference to the point in space from which the projection is made.
5. It will reach it in the same time. A horizontal motion does not affect the time of descent due to gravity. The projectile falls from the

highest point of its path (in a vacuum) in the same time that it would fall vertical downwards.

6. They will; for according to Article 304 of the text, we have for equal ranges the angle α and $90° - \alpha$. Let $\alpha = 45° - \delta$; then will the angles be $45° - \delta$ and $90 - (45° - \delta) = 45° + \delta$; but $45° + \delta$ is the complement of $45° - \delta$.

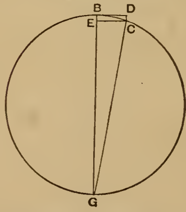
7. The lines will be the sides of an angle, and since the velocities are uniform, they will be divided into equal parts in equal times, by the motion of the bodies; hence, by geometry, the lines passing these equal divisions will be parallel.

Page 223.

The relation between centripetal and centrifugal forces has been the subject of much discussion. In an article which appeared some time since in "*Nature*," it was asserted that the term *centrifugal force* had done much harm in mechanical science, and ought not to be used. The basis of the trouble with such writers is, they consider that centrifugal force is to be applied to the same body as the centripetal; but, as stated in the text, such is not the case. Centripetal force is generally conceived to be the action of the ruling, or larger body, upon the smaller one, while centrifugal force is the equal opposite action upon the other body. Thus, if a nail holds one end of a string, while a body attached to the other end is made to rotate rapidly about it, the nail represents the ruling body, and the centripetal force is the pull of the string upon the rotating body, and the centrifugal force the pull of the string upon the nail. Similarly the attractive force between the earth and sun is centripetal if applied to the earth, and centrifugal if applied to the sun. These are illustrations of the third Law of Motion—that action and reaction are equal but opposite.

A constant centripetal force, caused by uniform motion in a circle, does not produce an acceleration, for it does not act along the same line. The action being constantly normal to the path, its effect is constantly expended in deflecting the body from a rectilineal path. Should it cease to act as soon as the deflection is made, the body would move in a right line, in accordance with the First Law of Motion.

The analysis for determining the value of the centrifugal force in the text is lengthy but strictly logical. The following solution may be more acceptable. Assume that the body is moving around the circle at a uniform rate; then will the centrifugal force be constant, and at any point, as B, the direction of motion will be that of the tangent BD. The centripetal force must be such as to draw the body from the tangent BD a distance equal to DC in the same time that it would move over BD. Draw EC and CG, and a chord BC. If now the point C be *indefinitely* near B, the limit of the sine EC (or its equal BD), and of the chord BC will be the arc BC.



Let BD be the space moved over in time t, and v the constant velocity, then

$$t = \frac{BD}{v},$$

since the motion in the arc is uniform; and since the centrifugal force is constant, the space BE will be given by equation (2), p. 12 of the text, or

$$BE = \tfrac{1}{2}v't,$$

where v' is the velocity which would be produced by the centripetal force in passing over the space BE in time t, if the force acted along the line BE. But since the times are equal, we eliminate t, between these equations, and find

$$v' = \frac{2BE}{EC} v.$$

From the figure we have, since EC is a mean proportional between BE and EG,

$$(EC)^2 = BE(BG - BE),$$

which ultimately becomes

$$(EC)^2 = BE \,.\, BG;$$

$$\therefore BE = \frac{(EC)^2}{BG}.$$

Substituting this value above gives

$$v' = \frac{2EC}{BG} v$$

$$= \frac{EC}{r} v,$$

since $BG = 2r$. But *ultimately* the velocity along BD or EC is that along the arc BC, and ultimately

$$EC \doteq vt,$$

where v is the velocity along the arc. This substituted, gives

$$\frac{v'}{t} = \frac{v^2}{r},$$

or, multiplying by m,

$$\frac{mv'}{t} = m\frac{v^2}{r},$$

But the left member is, according to Article 122, the value of a constant force, hence

$$f = m\frac{v^2}{r},$$

the required result.

Page 229, Articles 319, 320.—Sir Isaac Newton conceived the fact that if the attraction of gravitation varied as the inverse square of the distance from the centre of the force, it ought to account for the motion of the moon; that is, the force of gravity exerted by the earth should just equal that necessary to cause the proper deviation of the moon from a tangent to its orbit. His first efforts to prove this law failed, due to the fact that an erroneous value of R, the radius of the earth, was used. Instead, however, of abandoning the idea, and attempting to account for the motion according to any other hypothesis, he returned to his calculation from time to time, but with no better results. Finally, while attending a lecture in London, he obtained a corrected value of the radius, which, when substituted in the equation he had so often reviewed, established his theory. He was so overcome by the grandeur of the problem as the final proof was becoming apparent, that he was unable to complete the numerical reduction, and called a friend to do it for him.

It will be seen, in Article 319, that the radius of the earth enters the formula, in determining the distance of the moon, in the expression $60{\cdot}36R$. The value of R which he at first used was too small by $\frac{1}{16}$ to $\frac{1}{17}$ of its true value. See also remarks on pp. 36 and 37 of this Key.

The law of gravitation was not, at once, universally accepted. Several times, especially in the history of astronomy, certain phenomena appeared to conflict with this law, when it was called in question, and its truth assailed. But all opposition to it disappeared after Laplace, by his truly wonderful analysis, explained all those paradoxes, and accounted for all the motions of the solar system, on the simple law of Universal Gravitation. It is now believed to be true, not only for the solar system, but for every particle of matter in the universe. Newton believed that the ether of space, whatever it might be, was more dense in the vicinity of the planets, than in remote space; that, indeed, it might be only air extremely rarefied.

To find the stress due to the attraction between the earth and moon.

It equals the centrifugal force, the value of which is

$$m\omega^2 r.$$

The mass m of the moon is about $3\frac{1}{2}$ times that of a mass of water of equal volume, and as a cubic foot of water weighs $62\frac{1}{2}$ lbs. when $g = 32\frac{1}{6}$ feet, and the diameter of the moon is 2,160 miles, we have

$$m = 3\tfrac{1}{2} \times \tfrac{1}{6}\pi(2160 \times 5280)^3 \times \frac{62\frac{1}{2}}{32\frac{1}{6}} \text{ lbs.}$$

The time of the revolution of the moon about the earth is about $27\frac{1}{3}$ days; hence the angular velocity per second is

$$\omega = \frac{2\pi}{27\frac{1}{3} \times 24 \times 3600}.$$

The mean distance between the centres of the moon and earth is about 240,000 miles; hence

$$r = 240{,}000 \times 5280;$$

all the magnitudes being in feet, and all the times reduced to seconds. Hence we have

$$stress = \frac{3\frac{1}{2} \times \frac{4}{6}\pi^3 \times (2160)^3 \times (5280)^4 \times 240000 \times 62\frac{1}{2}}{32\frac{1}{6} \times (27\frac{1}{3} \times 24 \times 3600)^2}$$

which reduced gives, approximately,

$$44{,}000{,}000{,}000{,}000{,}000{,}000 \text{ lbs.}$$
$$= 44 \times 10^{18}.$$

A steel rod one square inch of section will sustain a pull of 120,000 lbs.; hence it would require (approximately)

$$370,000,000,000,000$$
$$= 37 \times 10^{13} \text{ square inches}$$

of steel to hold the moon in her orbit if substituted for the attraction between the earth and moon.

In one square mile are

$$4,014,489,600 \text{ sq. inches};$$

which, divided into the above, gives, for the equivalent section in miles,

$$90,000 \text{ sq. miles nearly.}$$

Since the radius of the moon is 1,080 miles, the area of a great circle will be

$$3,660,000 \text{ sq. miles nearly},$$

which, divided by the solid section of the steel rod, gives 40 + ; hence, if the rods were each one square inch in section, and the great circle of the moon be divided into inch-square spaces, the rods would cover one space in 40.

The square of the diameter of the earth is nearly 15 times the square of the diameter of the moon, hence such a steel rod would cover about $\frac{1}{600}$ of the meridian circle of the earth.

If the material be iron instead of steel, and if 10,000 lbs. be taken to represent the tenacity, a value quite commonly used in engineering structures, the rod—or rods—would cover more than one-fourth the cross-section of the moon, and about $\frac{1}{50}$ of a great circle of the earth.

The same problem applied to the attraction between the sun and earth gives

$$\frac{5\frac{1}{2} \times \frac{2}{3}\pi^3(20500000)^3 \times 5280 \times 92500000 \times 62\frac{1}{2}}{32\frac{1}{6} \times (365 \times 24 \times 3600)^2} \text{ lbs.},$$

where it is assumed that the mean density of the earth is $5\frac{1}{2}$ times that of water, the distance between the centre of the sun and the earth 92,500,000 miles, the radius of the earth 20,500,000 feet, and the time of the revolution in the orbit 365 days. This reduced gives, approximately,

$$912 \times 10^{18} \text{ lbs.},$$

or

912,000,000,000,000,000,000 lbs.,

or more than 20 times that between the earth and moon. According to this result it would require a solid steel rod of a cross-section equal nearly to one-half the great circle of the moon, the tenacity being 120,000 lbs. per square inch; or if the rod be of iron, and 10,000 lbs. be used for its tenacity, the section of the rod will be about $\frac{2}{5}$ of the area of a great circle of the earth. These examples show the immense stress of gravitation when large masses are involved.

The following examples will show that the same force, under certain circumstances, is comparatively weak.

Required the time that it will take two spheres of the same material as the earth, each one foot in diameter, placed $12\frac{1}{4}$ *inches from centre to centre, to come together by their mutual attractions, in void space.*

According to one of Newton's laws of attraction, the force varies as the mass. If the diameter of the earth be 41,700,000 feet (p. 33 of the text), then will the mass of the sphere 1 foot in diameter be

$$\frac{1}{(41700000)^3} \text{ of the earth,}$$

and therefore the acceleration at the surface of the earth due to the attraction of such a sphere placed at the centre of the earth would be

$$\frac{g}{(41700000)^3},$$

an inappreciable quantity.

According to another of Newton's laws, combined with the analysis on pp. 33 and 34 of this Key, the force varies inversely as the square of the distance from the centre of the sphere; hence at the distance of one foot from the centre of the small sphere we have

$$1^2 : (20850000)^2 :: \frac{g}{8(20850000)^3} : f;$$

$$\therefore f = \frac{g}{8 \times 20850000},$$

for tne acceleration of a particle one foot from the centre of the sphere. This will also be the acceleration of any uniform sphere whose centre is one foot from the first sphere, if the diameter of the second sphere does not exceed one foot; or should it exceed that diameter, it will still be true for the *mass* of the sphere if reduced to a sphere whose diameter is less

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than one foot, and hence will be the acceleration produced upon an equal sphere. If the first sphere were fixed in space, the second sphere would move the $\frac{1}{4}$ of an inch; but as both are free each will move one-half the distance between them, or $\frac{1}{8}$ of an inch.

In order to simplify the problem, we will assume that the acceleration is uniform, while the spheres are moving the $\frac{1}{8}$ of an inch, and is that due to the attraction at a distance of 12 inches from their centres; then will equation (4), page 12 of the text, be applicable, and we have

$$t = \sqrt{\frac{2s}{f}}$$

$$= \sqrt{\frac{2 \times \frac{1}{8} \times \frac{1}{12}}{\frac{g}{8 \times 20850000}}}$$

$$= \sqrt{108031}$$

$$= 328{\cdot}7 \text{ seconds, nearly,}$$

or less than $5\frac{1}{2}$ minutes.

This problem is in "The System of the World," by Sir Isaac Newton, p. 527 of our copy of the *Principia*. It is there stated that "the attraction of homogeneous spheres near their surfaces are (Prop. lxxii.) as their diameters. Whence a sphere of one foot in diameter, and of a like nature to the earth, would attract a small body placed near its surface with a force 20,000,000 times less than the earth would do if placed near its surface, but so small a force could produce no sensible effect. If two such spheres were distant but $\frac{1}{4}$ of an inch, they would not, even in spaces void of resistance come together by the force of their mutual attraction in less than a month's time."

We have sought for the source of the error in the *Principia*, by determining the conditions necessary for giving his result, but have not satisfied ourselves. We observe that he made an error in saying that the force of attraction on the surface of the small sphere is 20,000,000 times less than on the earth; for, according to his proposition—considering the *radius* of the earth as 20,000,000 feet—it should be 40,000,000 times less; and according to the inverse squares, at the distance of one foot from the centre of the small sphere, it would be 160,000,000 times less. If now we assume that the particle is moved $\frac{1}{8}$ of a *foot*, under the action of this force, we would have

$$t = \sqrt{\frac{\frac{2 \times \frac{1}{8}}{32}}{160000000}}$$

$$= \sqrt{2,500,000 \text{ (nearly)}} \text{ seconds,}$$

where the quantity under the radical is nearly the number of seconds in one month. Whether this gives any clue to the source of the error, we are unable to say.*

* The author presented the above result to the Physical Section of the Am. Asso. for Ad. Sc., at Montreal, 1882. The genuineness of "The System of the World, by Sir Isaac Newton," was there called in question. The work was originally issued in a separate volume, but in the author's copy it is bound with the *Principia*, and paged with it. But it evidently forms no part of the *Principia* proper.

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The assumptions made in order to simplify the problem, not being strictly accurate, we now propose the following problem :

Assume that two equal spheres of the same material as the earth, each one foot in diameter, are reduced in size to a mere point at their centres, and placed one foot from each other ; required the time it would take for them to come together by their mutual attraction, they being uninfluenced by any external force.

We first make a general solution.

Let E = the mass of the earth,

m = the mass of one of the spheres,

m' = the mass of the other sphere,

R = the radius of the earth,

r = the radius of one of the spheres,

r' = the radius of the other sphere,

g = the acceleration due to gravity on the earth,

μ = the acceleration due to the attraction of a sphere of mass unity upon another sphere of mass unity, the distance between their centres being unity,

a = the original distance between the centres of m and m', and

x = the distance between their centres at the end of time t.

The values of units are determined by measurements on the surface of the earth, and the unit mass will be so taken as to correspond with the units in use. The acceleration produced by a mass E, conceived to be reduced to a point, upon one of the units of mass at a distance unity, will be E times as great as that produced by the other unit, or

$$\mu E,$$

and at a distance R it will be

$$\mu \frac{E}{R^2},$$

which, according to the notation, will be the acceleration on the surface of the earth due to gravity; hence

$$g = \mu \frac{E}{R^2}, \tag{1}$$

and

$$\mu = \frac{R^2}{E} g, \tag{2}$$

by means of which the numerical value of the unit of acceleration may be determined.

Again, the acceleration produced by the attraction of the mass m upon one of the units of mass at distance, unity will be

$$\mu m,$$

and at a distance, x, the acceleration will be

$$\frac{\mu m}{x^2}, \tag{3}$$

which will also be the acceleration produced upon m, by the attraction between m and m' at the distance x between them, since the result will be the same as if both masses were concentrated at their centres of gravity; for all of m will exert the same force upon each particle of m' as upon each particle of the unit. But the pull in *pounds* will equal the mass into the acceleration (p. 44, Art. 86 of the text), or

$$\mu \frac{mm'}{x^2}. \tag{4}$$

Similarly, considering the attraction of m' upon m, the acceleration produced upon m' will be

$$\frac{\mu m'}{x^2}, \tag{5}$$

and the pull in pounds will be m times this amount, or

$$\mu \frac{m'm}{x^2}, \tag{6}$$

which is the same as (4), as it should be. Substituting μ from (2) in (4) or (6) gives for the pull in pounds (or their equivalent), of any two masses m and m',

$$\frac{mm'R^2g}{E} \cdot \frac{1}{x^2}. \tag{7}$$

The origin of the axis of x being at the centre of one of the masses and moving with it, and the total mass moved by the stress being $m + m'$, we have

$$(m + m')\frac{d^2x}{dt^2} = -\frac{mm'R^2g}{E} \cdot \frac{1}{x^2}, \quad (8)$$

which integrated (*Analyt. Mech.*, pp. 33, 34), observing that for $t = 0$, $x = 0$, and $v = 0$, and that μ in the reference equals $\frac{mm'R^2g}{(m + m')E}$ in this case, gives

$$t = \left[\frac{(m + m')Ea}{2mm'R^2g}\right]^{\frac{1}{2}} \times \left[(ax - x^2) + x\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right]_a^o, \quad (9)$$

which for the limits gives

$$t = \tfrac{1}{2}\pi a\left[\frac{(m + m')Ea}{2mm'R^2g}\right]^{\frac{1}{2}}.$$

If both spheres are of the same density, their masses will be as the cubes of their radii; or

$$\left.\begin{aligned} m &= \frac{r^3}{R^3}E, \\ m' &= \frac{r'^3}{R^3}E; \end{aligned}\right\} \quad (10)$$

and we have

$$t = \tfrac{1}{2}\pi a\left(\frac{r^3 + r'^3}{2r^3r'^3g}Ra\right)^{\frac{1}{2}};$$

and if the spheres are equal, as in the problem, we have

$$t = \tfrac{1}{2}\pi a\left(\frac{Ra}{r^3g}\right)^{\frac{1}{2}}.$$

If $a = 1$ foot, $R = 20{,}850{,}000$, $r = \frac{1}{2}$ foot, $g = 32\frac{1}{6}$, we have

$$= 1{\cdot}57079\left(\frac{166800000}{32{\cdot}166}\right)^{\frac{1}{2}}$$

$$= 3577 \text{ seconds, nearly}$$

$$= 59{\cdot}6 \text{ minutes, nearly.}$$

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An exact solution of the former problem may be made by means of equation (9), by substituting in it $a = 12\frac{1}{4}$ inches $= 1\frac{1}{48}$ feet, and making $x = a$ for one limit and 1 foot for the other. We would thus have

$$t = \left(\frac{1\frac{1}{48} \times 20850000}{\frac{1}{8} \times 32\frac{1}{6}}\right)^{\frac{1}{2}} \left[(1\tfrac{1}{48} - 1) + 1\tfrac{1}{48}\cos^{-1}\left(\frac{12}{12\frac{1}{4}}\right)^{\frac{1}{2}} - 0\right]$$

$$= 384 \text{ seconds nearly,}$$

or less than $6\frac{1}{2}$ minutes.

The following is the reduction of the preceding expression. To find the value of $\cos^{-1}\left(\frac{12}{12\frac{1}{4}}\right)^{\frac{1}{2}}$, we have

$$\log. 12 = 1{\cdot}079181$$
$$\log. 12{\cdot}25 = 1{\cdot}088136$$

$$\text{Dif.} = \bar{1}{\cdot}991045$$

Dividing by 2, $\bar{1}{\cdot}995523$,

and $\log. 0{\cdot}98974 = \bar{1}{\cdot}995522$,

or $\log. \cos 8° 12' 46'' = 9{\cdot}995523$.

The length of arc will be

$$\frac{8° 12' 46''}{180° \times 60 \times 60} 3{\cdot}1416,$$

or

$$\frac{29566}{648000}\,3{\cdot}1416\,;$$

which may be reduced as follows:

$$\begin{aligned} \text{log. } 29566 &= 4{\cdot}470892 \\ \text{log. } 3{\cdot}1416 &= 0{\cdot}497150 \\ \text{ar. co. log. } 648000 &= 4{\cdot}188425 \\ \hline \text{subtracting 10, log. } 0{\cdot}14337 &= \bar{1}{\cdot}156467 \\ \text{adding log. } 1\tfrac{1}{48}\text{, log. } 1{\cdot}02083 &= 0{\cdot}008951 \\ \hline \text{gives} \quad \text{log. } 0{\cdot}14636 &= \bar{1}{\cdot}165418\,; \end{aligned}$$

hence,

$$1\tfrac{1}{48}\cos^{-1}\left(\frac{12}{12\frac{1}{4}}\right)^{\frac{1}{2}} = 0{\cdot}14636.$$

We also have $(1\frac{1}{48} - 1)\,1 = \frac{1}{48} = 0{\cdot}02083 +$ which added to the preceding result gives $0{\cdot}16719 +$ for the value in the brackets.

For the first parenthesis, we have

$$\begin{aligned} \text{log. } 1\tfrac{1}{48} &= 0{\cdot}008951 \\ \text{log. } 20850000 &= 7{\cdot}319106 \\ \text{log. } 8 &= 0{\cdot}903090 \\ \text{log. } 32\tfrac{1}{6}\text{, ar. co.} &= 8{\cdot}492594 \\ \hline \text{subtracting 10 and dividing by 2,} \quad & \ 6{\cdot}723741 \\ \hline \text{log. } 2300{\cdot}7 &= 3{\cdot}361870. \\ \text{Adding} \quad \text{log. } 0{\cdot}16719 &= \bar{1}{\cdot}223209 \\ \hline \text{gives} \quad \text{log. } 384{\cdot}6 &= 2{\cdot}585079\,; \end{aligned}$$

that is, the time will be 385 seconds nearly, or less than $6\frac{1}{2}$ minutes.

To find the stress in pounds which would be exerted by the mutual action of two such spheres at a distance of one foot between their centres, we have from equations (7) and (10), since $m = m'$, and $x = 1$,

$$\frac{E}{R^4 g}.$$

The mass of the earth is $5\frac{1}{2}$ times an equal mass of water. The weight of a cubic foot of water is $62\frac{1}{2}$ lbs. at the place where $g = 32\frac{1}{6}$, and the volume of the earth is $\frac{4}{3}\pi R^3$; hence

$$E = 5\tfrac{1}{2} \times 62\tfrac{1}{2} \times \tfrac{4}{3}\pi R^3,$$

which substituted above gives for the stress

$$\frac{\frac{4}{3}\pi \times 5\frac{1}{2} \times 62\frac{1}{2}}{20850000 \times 32\frac{1}{6}} \text{ lbs.,}$$
$$= 0{\cdot}00000215 + \text{lbs.,}$$

or less than $\frac{22}{10000000}$ of a pound, a quantity inappreciably small.

A would-be inventor once proposed to weigh the varying force of gravity by means of very delicate scales, and by using them on board a steamer, thus determine whether the water underneath were deep or shallow. Since the density of the solid earth exceeds that of water, the force of gravity at the surface of deep water will be less than on shallow water, but it is evident that the rocking and heaving of the vessel would probably produce more effect upon such delicate mechanism than that due to the variations of the force of gravity.

It is also stated that mariners have observed that two ships at rest in a quiet sea tend to approach each other, but it will be found that the gravitating stress due to their mutual attractions is so small that it might be more than neutralized by a very slight breeze, or by the beating of very small waves.

To give some idea of the magnitude of this stress, assume that the vessels are of equal mass, and each equivalent to a sphere of the average mass of the earth, each 30 feet in diameter, and 200 feet between their centres.

Conceive that the masses are reduced to their centres, then since the mutual attraction of unit-spheres at distance unity between them is $\frac{22}{100000000}$ of a pound, the attraction of the masses of the ships, reduced to their centres, at the same distance (one foot) will be

$$\tfrac{22}{100000000} \times 30^3 \times 30^3 = 1471 \text{ pounds};$$

and at the distance of 200 feet, it will be

$$\frac{1471}{200^2} = 0.04 \text{ pound nearly.}$$

If the distance between them be 500 feet, the stress would be

$$\frac{1471}{500^2} = \tfrac{1}{170} \text{ of a pound, nearly.}$$

If all external forces, such as the wind and action of the sea were neutralized, this slight stress would be sufficient to cause the ships in question to collide in a short time.

SOLUTIONS OF EXAMPLES.

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1. We have from equation (5), page 226,

$$f = m\frac{v^2}{r} = \frac{W}{g}\cdot\frac{v^2}{r}.$$

But f must equal $2W$;

$$\therefore 2W = \frac{W}{g}\cdot\frac{v^2}{r};$$

$$\therefore v = \sqrt{2gr};$$

or the velocity must be that acquired by a body falling freely through a distance equal to the radius of the circle (see eq. (3), p. 36 of the text).

2. If the tension is $3W$, we have

$$3W = \frac{W}{g}\cdot\frac{v^2}{r};$$

$$\therefore v = \sqrt{3gr}.$$

Let n be the number of révolutions, then the velocity will be the space ($2\pi rn$) divided by the time, or 60 seconds. r must be reduced to feet and the time to seconds, for g is given in feet per second. Hence we have

$$v = \frac{2\pi rn}{60} = \frac{\pi rn}{30};$$

$$\therefore \frac{\pi rn}{30} = \sqrt{3gr};$$

$$\therefore n = \frac{30}{\pi}\sqrt{\frac{3g}{r}}$$

$$= \frac{30}{3{\cdot}1416}\sqrt{16\tfrac{1}{12}}$$

$$= 38{\cdot}3.$$

3. The centrifugal force will equal the weight; hence

$$W = m\frac{v^2}{r} = \frac{W}{g}\cdot\frac{v^2}{r};$$

$$\therefore v = \sqrt{gr} = \frac{\pi rn}{30} \text{ (see preceding example)};$$

$$\therefore n = \frac{30}{3{\cdot}1416}\sqrt{\frac{g}{r}}$$

$$= 38{\cdot}3,$$

as in the preceding example.

4. When the body is at the lowest position its weight will be added to the centrifugal force,

but the tension due to the centrifugal force equals the weight; hence the tension will equal $2W$.

5. The friction will be μ times the pressure due to the centrifugal force, and must equal the weight. Let n be the number of revolutions per minute; then will the angular velocity per second be

$$\frac{2\pi r . n}{60r} = \frac{2\pi n}{60};$$

and the centrifugal force will be (Art. 314)

$$f = \frac{W}{g}\left(\frac{2\pi n}{60}\right)^2 R,$$

and the friction will be

$$\mu f = \frac{\mu \pi^2 n^2 WR}{(30)^2 g} = W;$$

$$\therefore n = \frac{30}{\pi}\sqrt{\frac{g}{\mu R}}.$$

6. The body is assumed to be in a radial groove, and the string slightly elastic so as to allow the body to move slightly along the groove, and thus give the friction a chance to act.
The angular velocity per minute will be

$$2\pi \times 250:$$

and the centrifugal force will be

$$f = \frac{W}{g}\left(\frac{500\pi}{60}\right)^2 \cdot \frac{30}{12}$$

$$= \tfrac{15}{193} \times \tfrac{625}{9} (3{\cdot}1416)^2 W.$$

The friction will be 0·15 of this amount, and the tension of the string 0·85 of the same; hence the tension will be

$$T = 0{\cdot}85 \times \tfrac{15}{193} \times \tfrac{625}{9} (3{\cdot}1416)^2 W$$

$$= 45{\cdot}27 + \text{pounds.}$$

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7. The friction will be

$$\tfrac{1}{10} W.$$

The centrifugal force will be (Art. 313),

$$\frac{W}{g}\frac{v^2}{R};$$

hence

$$\frac{Wv^2}{gR} = \tfrac{1}{10} W,$$

or

$$v^2 = \frac{gR}{10};$$

$$\therefore v = \sqrt{\frac{32\frac{1}{6} - 2500}{10}}$$

$$= 89{\cdot}675 \text{ feet per second}$$

$$= 61{\cdot}14 \text{ miles per hour.}$$

8. According to Article 322, we have

$$h = \frac{v^2}{g} \cdot \frac{b}{r}$$

$$= \frac{\left(\frac{30 \times 5280}{60 \times 60}\right)^2 \times 4\frac{2}{3}}{32\frac{1}{6} \times 3000}$$

$$= 0{\cdot}09\tfrac{1}{3} \text{ feet}$$

$$= 1{\cdot}12 \text{ inches.}$$

9. The weight will be to the centrifugal force as the length of the string is to the lateral movement of the body; or

$$W : f :: 6 : x;$$

$$\therefore x = 6\frac{f}{W}.$$

To find f we have equation (5), Art. 313,

$$f = \frac{W}{g} \frac{\left(\frac{40 \times 5280}{60 \times 60}\right)^2}{4000};$$

which substituted above, gives

$$x = \frac{6}{32\frac{1}{6} \times 4000} \left(\frac{40 \times 5280}{3600}\right)^2;$$

$$= 0{\cdot}160 \text{ ft.}$$

$$= 1{\cdot}92 \text{ inches.}$$

The value is independent of the weight.

10. To find the time of making one revolution, we have

$$T = 60 \div 100 = \tfrac{3}{5}.$$

Then from p. 232 of the text, we have

$$\cos\varphi = \frac{gT^2}{4\pi^2 \,.\, AB}$$

$$= \frac{32\tfrac{1}{6} \times (\tfrac{3}{5})^2}{5 \times (3{\cdot}1416)^2}$$

$$= 0{\cdot}23466\,;$$

$$\therefore \varphi = 76^\circ\ 25'\ 43''.$$

Or by logarithms

$$\log.\ 32\tfrac{1}{6} = 1.507406$$
$$\log.\ (0.6)^2 = \bar{1}.556303$$

adding, 1 063709.

$$\log.\ 5 = 0.698970$$
$$\log.\ (3.1416)^2 = 0.994300$$

adding, 1.693270 which subtracted from the above gives

$$\log.\ \cos\varphi = \bar{1}.370439;$$
$$\therefore \varphi = 76^\circ\ 25'\ 43''.$$

If $\cos\varphi = 1$, or $\varphi = 0$, a relation will be established between AB and T, or we will have

$$AB = \frac{g}{4\pi^2}T^2\,;$$

and if AB is assumed to be less than the value found by this formula, $\cos\varphi$ will exceed unity, and hence φ will be imaginary, or, in other words, the conditions will be impossible. We also have

$$\begin{aligned} AC &= AB\cos\varphi \\ &= 15\cos 76^\circ\ 25'\ 43'' \\ &= 15 \times 0{\cdot}23466 \\ &= 3{\cdot}52 \text{ inches}; \end{aligned}$$

and

$$\begin{aligned} BC &= AB\sin\varphi \\ &= 15 \times \sin 76^\circ\ 25'\ 43'' \\ &= 14{\cdot}58 \text{ inches}. \end{aligned}$$

11. Let $n =$ the number of revolutions per minute, $= n \div 60$ per second. The distance BC will be

$$BC = 14 \sin 10^\circ \text{ inches},$$

and the velocity in feet per second will be

$$\begin{aligned} v &= \frac{2\pi n \times \frac{14}{12}\sin 10^\circ}{60}. \\ &= \tfrac{7}{180}\pi n \sin 10^\circ. \end{aligned}$$

Employing the equation at the bottom of page 232 of the text, making $F = 4$ lbs., and substituting the value of v given above, we have

$$(\tfrac{7}{180}\pi n \sin 10^\circ)^2 = 32\tfrac{1}{6} \times \tfrac{14}{12} \times \sin 10^\circ \tan 10^\circ \times \frac{4 \times 32\frac{1}{6} \times 14 \sin 10^\circ}{5 \times 12};$$

$$\therefore n = \frac{180}{7\pi}\sqrt{\frac{193}{6}\times\frac{14}{3}\left[\frac{1}{4\cos 10^\circ}+\frac{1}{5\sin 10^\circ}\right]}$$

$$= 118{\cdot}86.$$

The reduction is as follows :

$$n = \frac{180}{7\times 3{\cdot}1416}\sqrt{\frac{1351}{9}\left(\frac{1}{4\times 0{\cdot}98481}+\frac{1}{5\times 0{\cdot}17365}\right)}$$

$$= \frac{180}{21{\cdot}9912}\sqrt{\frac{1351}{9}(0{\cdot}25383+1{\cdot}15174)}$$

$$= \frac{180}{21{\cdot}9912}\sqrt{\frac{1351}{9}\times 1{\cdot}40557.}$$

log. 1·40557	0·147852
log. 1351	3·130655
log. 9 ar. co.	9·045757
Dividing by 2,	2·324264
	1·162132
log. 180	2·255273
log. 21·9912 a. c.	8·657750
log. 118·86	2·075055

12. The tenacity to be overcome will be that in a section through the axis of the stone, and hence will be

$$4\times 12\times 4\times 600 = 115200 \text{ lbs.}$$

The centrifugal force producing rupture will be that due to either half of the stone into which it is divided by the plane section; and this will equal the mass of one-half multiplied by the distance of its centre of gravity from the axis of rotation.

The centre of gravity of a semicircle is $\frac{4}{3}\frac{r}{\pi}$ from the centre of the circle (Ex. 1, p. 154 of the text).

If δ be the weight of a cubic foot of the stone, the weight of one-half will be

$$\begin{aligned} &\tfrac{1}{2}\pi r^2 \times \text{thickness} \times \delta \\ &= \tfrac{1}{2}\pi \times 4 \times \tfrac{1}{3} \times \delta \\ &= \tfrac{2}{3}\pi\delta; \end{aligned}$$

and mass

$$= \tfrac{2}{3}\frac{\pi\delta}{g}.$$

If n be the number of revolutions per minute, then the angular velocity per second will be

$$2\pi\frac{n}{60} = \frac{n\pi}{30};$$

and the centrifugal force will be

$$\tfrac{2}{3}\frac{\pi\delta}{g}\left(\frac{n\pi}{30}\right)^2 \times \tfrac{4}{3}\frac{r}{\pi} = 115200;$$

$$\therefore n = \sqrt{\frac{115200 \times 9 \times 30^2 \times 193}{8 \times 2 \times (3{\cdot}1416)^2 \times 6 \times \delta}}$$

$$= \frac{13786{\cdot}7}{\sqrt{\delta}}.$$

EXERCISES.

Page 235.

1. It will not. The deviating force will be the normal component of the force at the centre.

2. Because the centrifugal force causes a pressure against the side of the vessel, to balance which requires an increase of height of the water.

   The same *tendency* exists with other substances, but in order that any substance shall actually be elevated at the outer surface, the centrifugal force must overcome the friction between its particles.

3. The diameter of the earth at the equator is about 26 miles more than at the poles. If the earth were changed to a sphere, the polar diameter would be increased about 13 miles, and the equatorial decreased about the same amount. The exact change in the dimensions involves a knowledge of the volume of ellipsoids.

4. It does. The angular velocity of the earth being constant, the centrifugal force varies as the distance from the centre.

5. It would come to rest on the surface of the hollow.

6. According to Article 318, if the earth revolved in 84 m. $42\frac{6}{17}$ sec., bodies on the equator would weigh nothing; hence if it revolved in 84 minutes they would fly off if free, but if held

by cohesion or otherwise, the holding force must be overcome before they could fly off.

7. It would be nearer.

8. Because the velocity in their orbits is necessarily greater in order to balance the attractive force of the sun, and the circumference of the orbit is less than those more remote. Kepler's law is—the squares of the times of the revolutions are as the cubes of the mean distances from the sun.

9. The centrifugal force is not destroyed—it is only equilibrated.

10. Tangentially.

11. It is proper to say that the centrifugal force—and hence the centripetal—is due to the velocity of the stone in the sling. Strictly speaking, the centripetal force has nothing to do with the velocity; but indirectly it has, for the velocity could not be produced without the centripetal force—and it is only in this sense that it has something to do with it.

12. The centrifugal force causes the clothes to *press* against the vertical inside surface of the vessel.

13. They are not affected by the rotation; but if the earth should cease to rotate, the surface at the poles would be elevated; and hence cause bodies there to weigh less by being at a greater distance from the centre of the earth.

14. There is. If the moment of the centrifugal force in reference to the outer rail as an origin exceeds the moment of the weight of the car in reference to the same point, they will overturn.

Page 236.—The analysis of this chapter very properly belongs to higher analysis; but we have succeeded, by means of curves and special artifices, in bringing it within geometrical and algebraic analysis.

In the language of the calculus, if m be the mass and η the force at a unit's distance, we would have

$$m\frac{d^2x}{dt^2} = -\eta x.$$

Multiplying by dx gives

$$m\frac{dx\, d^2x}{dt^2} = -\eta x\, dx,$$

and integrating,

$$m\frac{dx^2}{2dt^2} = -\tfrac{1}{2}\eta x^2 + \tfrac{1}{2}C,$$

or

$$m\frac{dx^2}{dt^2} = -\eta x^2 + C.$$

Assuming for the initial conditions that the velocity is zero, and $x = a$, we have

$$0 = -\eta a^2 + C,$$

$$\therefore C = \eta a^2,$$

and we have

$$\frac{dx^2}{dt^2} = \frac{\eta}{m}(a^2 - x^2);$$

which is the square of the velocity, and is the same as equation (2), page 238 of the text. From the last equation we have

$$dt = \sqrt{\frac{m}{\eta}}\,\frac{dx}{\sqrt{a^2 - x^2}},$$

which integrated gives

$$t = \sqrt{\frac{m}{\eta}} \sin^{-1} \frac{x}{a} + C'.$$

Assuming that $t = 0$ for $x = a$, we have

$$0 = \sqrt{\frac{m}{\eta}} \tfrac{1}{2}\pi + C' ;$$

$$\therefore C' = -\tfrac{1}{2}\pi \sqrt{\frac{m}{\eta}} ;$$

$$\therefore t = \sqrt{\frac{m}{\eta}} (\sin^{-1} \frac{x}{a} - \tfrac{1}{2}\pi)$$

If $x = 0$, we have

$$t = -\tfrac{1}{2} \sqrt{\frac{m}{\eta}} \pi.$$

But we also have $\sin^{-1} 0 = \pi$, for which value, we have

$$t = \tfrac{1}{2}\pi \sqrt{\frac{m}{\eta}},$$

which is the same as Equation (4), page 241 of the text.

Page 243, Art. 328.—Captain Kater used the principle of the convertibility of the centres of suspension and oscillation for determining the length of a simple seconds pendulum, and hence the acceleration due to gravity.—*Phil. Trans.*, 1818.

Let a body, furnished with a movable weight, be provided with a point of suspension A, and another point on which it may vibrate, fixed as nearly as can be estimated in the centre of oscillation B, and in a line with the point of suspension and the centre of gravity. The oscillations of the body must now be observed when suspended from A, and also when suspended from B. If the vibrations in each position should not be equal in equal times, they may readily be made so by shifting the movable weight. When this is done, the distance between the two points A and B is the length of the simple equivalent pendulum. Thus the length L and the corresponding time T of vibration will be found uninfluenced by any irregularity of density or figure. In these experiments, after numerous trials of spheres, etc., knife edges were preferred as a means of support. At the centres of suspension and oscillation there were two triangular apertures to admit the knife edges on which the body rested while making its oscillations.

Having thus the means of measuring the length L with accuracy, it remains to determine the time T. This is effected by comparing the vibrations of the body with those of a clock. The time of a single vibration or of any small arbitrary number of vibrations cannot be observed directly, because this would require the fraction of a second of time, as shown by the clock, to be estimated either by the eye or ear. The vibrations of the body may be counted, and here there is no fraction to be estimated, but these vibrations will not probably fit in with the oscillations of the clock pendulum, and the differences must be estimated. This defect is overcome by "the method of coincidences." Supposing the time of vibration of the clock to be a little less than that of the body, the pendulum of the clock will gain on the body, and at length at a certain vibration the two will for an instant coincide. The two pendulums will now be seen to separate, and after a time will again approach each other, when the same phenomenon will take place. If the two pendulums continue to vibrate with perfect uniformity, the number of oscillations of the pendulum of the clock in this interval will be an integer, and the number of oscillations of the body in the same interval will be less by one complete oscillation than that of the pendulum of the clock. Hence by a simple proportion the time of a complete oscillation may be found.

The coincidences were determined in the following manner: Certain marks made on the two pendulums were observed by a telescope at the lowest point of their arcs of vibration. The field of view was limited by a diaphragm to a narrow aperture across which the marks were seen to pass. At each succeeding vibration the clock pendulum follows the other more closely, and at last the clock-mark completely covers the other during their passage across the field of view of the telescope. After a few vibrations it appears again preceding the other. The time of disappearance was generally considered as the time of coincidence of the vibrations, though in strictness the mean of the times of disappearance and reappearance ought to have been taken, but the error thus produced is very small. (*Encyc. Met.*, Figure of the Earth.) In the experiments made in Hartan coal-pit in 1854, the Astronomer Royal used Kater's method of observing the pendulum. (*Phil. Trans.*, 1856.)

The value of T thus found will require several corrections. These are called "Reductions." If the centre of oscillation does not describe a cycloid, allowance must be made for the alteration of time depending on the arc described. This is called "the reduction to infinitely small arcs." If the point of support be not absolutely fixed, another correction is required (*Phil. Trans.*, 1831). The effect of the buoyancy and the resistance of the air must also be allowed for. This is the "reduction to a vacuum." The length L must also be corrected for changes of temperature.

The time of an oscillation thus corrected enables us to find the value of gravity at the place of observation. A correction is now required to reduce this result to what it would have been at the level of the sea. The attraction of the intervening land must be allowed for by Dr. Young's rule (*Phil. Trans.*, 1819). We thus obtain the force of gravity at the level of the sea, supposing all the land above this level were cut off and the sea constrained to keep its present level. As the sea would tend in such a case to change its level, further corrections are still necessary if we wish to reduce the result to the surface of that spheroid which most nearly represents the earth. (See *Camb. Phil. Trans.*, vol. x.)

There is another use to which the experimental determination of the length of a simple equivalent pendulum may be applied. It has been adopted as a standard of length on account of being invariable and capable at any time of recovery. An Act of Parliament (5 Geo. IV.) defines the yard to contain thirty-six such parts, of which parts there are 39.1393 in the length of the pendulum vibrating seconds of mean time in the latitude of London, in vacuo, at the level of the sea, at temperature 62° F. The commissioners, however, appointed to consider the mode of restoring the standards of weight and measure which were lost by fire in 1834, report that several elements of reduction of pendulum experiments are yet doubtful or erroneous, so that the results of a convertible pendulum are not so trustworthy as to serve for supplying a standard for length; and they recommend a material standard, the distance, namely, between two marks on a certain bar of metal under given circumstances, in preference to any standard derived from measuring phenomena in nature. (*Report*, 1841.)

All nations, practically, use this simple mode of determining the length of the standard of measure, that of placing two marks on a bar, and by a legal enactment declaring it to be a certain length.

For length of seconds pendulum see *Mech. Céleste*, T. II., pp. 327, 343, 479.

SOLUTIONS OF EXAMPLES.

PAGE 248.

1. From the equation on page 243 of the text we have

$$t = \pi\sqrt{\frac{l}{g}},$$

but in the example $t = \frac{1}{2}$; hence

$$\tfrac{1}{2} = \pi\sqrt{\frac{l}{g}};$$

$$\therefore l = \frac{g}{4\pi^2} = \frac{32\frac{1}{6}}{4 \times (3{\cdot}1416)^2}$$

$$= 0{\cdot}827 \text{ feet.}$$

$$= 9{\cdot}77 \text{ inches.}$$

2. For this example we have

$$2 = \pi\sqrt{\frac{l}{g}};$$

$$\therefore l = \frac{4g}{\pi^2} = \frac{4 \times 32\frac{1}{6}}{(3{\cdot}1416)^2}$$

$$= 13{\cdot}036 \text{ feet.}$$

3. First find the time, in seconds, of one vibration. In one day there are $24 \times 60 \times 60 = 86{,}400$ seconds; hence the time of one vibration will be

$$\tfrac{86400}{86420} = 0{\cdot}99976 \text{ seconds.}$$

Let x be the required length, then from the equation on page 243 (t' being the time for length x) we have

$$t^2 : t'^2 :: x : l,$$

or

$$1 : (0{\cdot}99976)^2 :: x : 39{\cdot}1;$$

$$\therefore x = \frac{39{\cdot}1}{(0{\cdot}99976)^2} = 39{\cdot}1181 + \text{ inches};$$

hence it must be elongated

$$39{\cdot}1181 - 39{\cdot}1 = 0{\cdot}0181 + \text{ inches.}$$

PAGE 249.

4. From the equation on page 246 of the text we have

$$h = \frac{45{\cdot}5}{86354{\cdot}5} r$$

$$= \frac{45{\cdot}5}{86354{\cdot}5} \text{ 20,923,161 feet,}$$

$$= 11024{\cdot}3 \text{ feet.}$$

5. From the equation

$$5t = 5\pi\sqrt{\frac{l}{g}},$$

we have

$$5t = 15{\cdot}7080\sqrt{\frac{2}{32\frac{1}{6}}}$$
$$= 3{\cdot}9168 \text{ seconds.}$$

6. From Eq. (2), page 246, we have

$$v_1 = \sqrt{gr}$$
$$= \sqrt{32{\cdot}0902 \times 20{,}923{,}161}$$
$$= 25911{\cdot}93 \text{ feet per second}$$
$$= 4{\cdot}9 + \text{ miles per second.}$$

7. We will have from Eq. (1), page 246 of the text,

$$t = \pi\sqrt{\frac{r}{g}}$$
$$= 3{\cdot}1416\sqrt{\frac{20{,}923{,}161}{32{\cdot}0902}}$$
$$= 42 \text{ m. } 17 \text{ sec.}$$

8. From Equations (1) and (4), pages 247 and 248, we have

$$v_1 = \frac{Fl}{Ek}\sqrt{\frac{Ekg}{Pl}}$$
$$= F\sqrt{\frac{gl}{PEk}}$$

$$= 3000 \sqrt{\frac{32\frac{1}{6} \times 5}{1000 \times 28000000 \times \frac{1}{4}}}$$

$$= 0{\cdot}453 \text{ ft. per sec.}$$

9. From Eq. (6) we have

$$t = \pi \sqrt{\frac{\lambda}{g}}$$

$$= \pi \sqrt{\frac{Pl}{gEk}}$$

$$= 3{\cdot}1416 \sqrt{\frac{1000 \times 5}{32\frac{1}{6} \times 28000000 \times \frac{1}{4}}}$$

$$= 0{\cdot}0148 \text{ of a second.}$$

SOLUTIONS OF EXAMPLES.

PAGE 258.

1. In this example the weight of the fluid is abstracted, and we have only to consider the effect of the pressure of the piston. The area of the piston is $\frac{1}{4}\pi d^2 = 0{\cdot}7854$ inches; hence the pressure p per square inch will be

$$p = 20 \div 0{\cdot}7854 = 25{\cdot}46 + \text{lbs.}$$

The area of the bottom of the box $= 2 \times 3 \times 144 = 864$ sq. inches. The area of the sides $= (2 \times 1 \times 2 + 2 \times 1 \times 3)\ 144 = 1440$ sq. inches. Hence the entire area of the bottom and sides will be 2,304 sq. inches; hence the pressure will be

$$P = 2304 \times 25{\cdot}46$$

$$= 58659{\cdot}8 \text{ lbs.}$$

2. It will equal the weight of the water; hence

$$P = \frac{6 \times 8}{1728} \times 62\tfrac{1}{2} \text{ lbs.}$$
$$= 1{\cdot}736 \text{ lbs.}$$

3. The area of the base will be 3·1416 × 16. The pressure due to the liquid will equal the weight of a cylinder of water whose base is 8 inches in diameter and height 10 inches, or

$$\frac{3{\cdot}1416 \times 16 \times 10}{1728} \times 62\tfrac{1}{2} = 18{\cdot}18 \text{ lbs.}$$

The pressure upon the base due to the external pressure of 100 lbs. will be

$$\frac{8^2}{6^2} \times 100 = 177{\cdot}777 \text{ lbs.};$$

hence the total pressure will be

$$P = 18{\cdot}208 + 177{\cdot}777$$
$$= 195{\cdot}96 \text{ lbs.}$$

4. The upward pressure equals the weight of water displaced—or 63 lbs.; hence the pull on the string will be the difference of the upward pressure and the weight of the block, or

$$63 - 35 = 28 \text{ lbs.}$$

5. We have from Article 86, p. 44 of the text,

$$F = Mf;$$

$$\therefore f = \frac{F}{M} = \frac{28}{35} g = \frac{28 \times 32\frac{1}{6}}{35}.$$

And, from Eq. (3), p. 12 of the text,

$$v = \sqrt{2fs}$$

$$= \sqrt{\frac{28 \times 64\frac{1}{3} \times 50}{35}}$$

$$= 50{\cdot}73 \text{ ft. per sec.}$$

PAGE 259.

6. We have, for the volume of the block $(\frac{3}{2})^3 = \frac{27}{8}$ feet. Hence the weight $= \frac{27}{8} \times 180 = 607{\cdot}5$ lbs. The upward pressure of the water will be $\frac{27}{8} \times 62\frac{1}{2}$ lbs. $= 210{\cdot}937$ lbs., which, subtracted from the weight, will equal the tension, or 396·563 lbs.

7. Let a equal half the length of the bar, and x the distance of the point of attachment from the middle of the bar, which point will be in the iron part. Let s be the ratio of the weight of a given volume of wood to that of an equal volume of water. If the weight of the water displaced per unit of length of the bar be called *unity*, then will the weight of the wooden part of the bar be represented by sa, and of the iron part by $8sa$. The upward pressure of the water will be $2a$, since it equals the volume displaced. The resultant upward stress on the wooden part will be $a - sa = (1 - s)a$; and its moment will be $(1 - s)a \times (a + x)$. The resultant downward force of the

bar between the middle of the beam and point of attachment will be $(8s - 1)x$, and the moment will be $\frac{1}{2}(8s - 1)x^2$. The moment of the remaining part will be $\frac{1}{2}(8s - 1)(a - x)^2$. Hence we have the equation

$$\tfrac{1}{2}(8s - 1)(a - x)^2 = a(1 - s)(a + x) - \tfrac{1}{2}(8s - 1)x^2$$

$$\therefore x = \frac{7s \pm \sqrt{-111s^2 + 68s - 6}}{16s - 2}\, a$$

8. If one end is depressed 3 inches, the other will be raised the same amount, hence the difference of level will be 6 inches, and the column of water necessary to produce this difference of level will be

$$6 \times 13\tfrac{1}{2} = 81 \text{ inches.}$$

ANSWERS TO EXERCISES.

1. It would not disperse, but would remain in the same form.

2. The gas would fill the hollow space.

3. If the sides of the pail were vertical, it probably would ; but even in this case, if it were poured in and run down the side, it would require a short time to overcome the adhesion on the side and permit the entire weight to be exerted on the bottom of the vessel.

4. As much less as the weight of an equal volume of air.

5. Of the same density of air—or the weight must be the same as that of an equal volume of air.

SOLUTIONS OF EXAMPLES.

PAGE 270.

1. $w_1 = \frac{(s_2 - s)s_1}{(s_2 - s_1)s} w.$

$$s_1 = \frac{w_1 s s_2}{w_1 s + w s_2 - w s}$$

substituting the value of $s = 1{\cdot}3077$, we have

$$s = \frac{12 \times 1{\cdot}3077 \times 11}{34 \times 11 + (12 - 34)1{\cdot}3077} \doteq \tfrac{1}{2}.$$

2. $s = \frac{w}{w - w_1} = \frac{32}{32 - 25} = \frac{32}{7} = 4\tfrac{4}{7}.$

3. $s = \frac{W + w_1}{W + w}$. In this case $w_1 = 0$; ∴ solving for W we have

$$W = \frac{ws}{1 - s} = \frac{{\cdot}8 \times 60}{1 - {\cdot}8} = \frac{48}{.2} = 240 \text{ grains.}$$

4. A stone 5 ft. on each edge = 125 cu. ft. and will displace 125 × 62·5 = 7812·5 lbs. water, × 2·3 = 17968·75 lbs. = weight of stone.

5. $s_1 = \tfrac{3}{4}.$

6. $s = \frac{w - w_2}{w - w_1} = \frac{40 - 32}{40 - 35} = \frac{8}{5} = 1{\cdot}6.$

7. $s = \frac{vs + v_1 s_1}{v + v_1} = \frac{35 + 18}{5 + 2} = \frac{53}{7} = 7{\cdot}572.$

8. $w_1 = \frac{(s_2 - s)s_1}{(s_2 - s_1)s} w = \frac{(10{\cdot}5 - 14)19{\cdot}3 \times 10}{(10{\cdot}5 - 19{\cdot}3)14} =$

$$\frac{-675{\cdot}5}{-123{\cdot}2} = 5{,}483.$$

$$w_2 = \frac{(s_1 - s)s_2}{(s_1 - s_2)s}\,w = \frac{(19{\cdot}3 - 14)10{\cdot}5 \times 10}{(19{\cdot}3 - 10{\cdot}5)14} =$$

$$\frac{556{\cdot}5}{123{\cdot}2} = 4{\cdot}518.$$

9. $$\frac{1}{n} = 1 - \frac{vs + v_1 s_1}{(v + v_1)s_2} = 1 - \frac{27 \times 1 + 39{\cdot}4915 \times 1{\cdot}8485}{(27 + 39{\cdot}4915)1{\cdot}6321}$$

$$= 1 - \frac{100}{108{\cdot}52} = 1 - {\cdot}9215 = 0{\cdot}0785.$$

10. $$s = \frac{b_1 + 0{\cdot}0013\,(b_2 - c)}{b_1 + b_2 - c} = \frac{14 + 0{\cdot}0013(10 - 7)}{14 + 10 - 7}.$$

$$= \frac{14{\cdot}0039}{17} = 0{\cdot}8237.$$

ANSWERS TO EXERCISES.

Page 271.

1. It will not; for water being more compressible than the solid, will be relatively more dense in the air than in a vacuum, and hence when in air will force the body upward more than when in a vacuum.

2. Because the smoke is lighter than the surrounding air; but if it be heavier it will fall in the air.

3. See answer to Exercise 1.

4. This assumes that the weight of the bag—or some other cause—causes the bag to sink, and if it sinks at all it will go to the bottom of the vessel. If now a pressure be exerted upon the surface of the liquid, it will cause the bag (or

gas) to condense more than the liquid, and hence it will not rise. Toys have been made involving this principle.

5. Water is more compressible than iron for the same pressure, hence it seems possible, theoretically, for water to be subjected to such a pressure as to be as dense as iron at the same pressure.

6. If both are incompressible, their relative densities will be unchanged by pressure, and hence the heavier body will sink indefinitely. If the body be compressible, it will become relatively more dense, and hence there will be no limit.

7. If the brine be sufficiently "strong"—according to popular language—it will float the egg. In order that it may float between the top and bottom, the brine must be more dense near the bottom than at the top.

8. It will. It is related of Benjamin Franklin, that he asked a company of savants why a pail of water containing a fish would weigh no more than without the fish. Several reasons were given, and finally he was appealed to for the reason. He thus replied: "Are you sure it will weigh no more?" They had been trying to explain a false assumption.

SOLUTIONS OF EXAMPLES.

Page 276.

1. The equation on page 273 gives us $\frac{f}{g} = \frac{F}{W} = \tan \varphi$;

$$\tan \varphi = \tfrac{15}{50} = 0{\cdot}3\,; \ \therefore \varphi = 16° \, 41' \, 57''.$$

2. $f = g \tan \varphi$. The section of the liquid will be a triangle with a base of 3 ft. and a height of 2 ft., hence $\tan \varphi = \frac{2}{3}$;

$$\therefore f = \tfrac{2}{3} \times 32\tfrac{1}{6} = 21\tfrac{4}{9} \text{ feet per second.}$$

3. $f = g \tan \varphi$. In this case the slope of the free surface will be 45°, and $\tan 45° = 1$; $\therefore f = g$.

4. Let A be the edge of the vessel. From A conceive a horizontal line drawn to XE, and mark the foot with the letter Z. The volume generated by the revolution of the semi-parabola is one-half the product of the base and altitude (See *Mensuration*, or works on the Integral *Calculus*). And as the volume of this paraboloid is the unoccupied portion of the cylinder, the altitude ZE will be $2 \times 3 = 6$ inches. AZ is 12 inches. From a property of the parabola, we have

$$EZ : AZ :: AZ : \textit{the parameter},$$
$$(\text{or } x : y :: y : 2p)\,;$$
$$\therefore \text{the parameter} = \frac{12^2}{6} = 24 \text{ inches.}$$

This is known to be twice the subnormal DC; hence $DC = 12$ inches, which, as shown on p. 275 of the text, is $g \div \omega^2$, therefore

$$\omega^2 = \frac{32\frac{1}{6}}{12}.$$

Let n be the number of turns per minute sought. Then $2\pi \frac{n}{60} = \frac{n\pi}{30}$ will be the angular velocity per second, and we have

$$\left(\frac{n\pi}{30}\right)^2 = \frac{32\frac{1}{6}}{12};$$

$$\therefore n = 11{\cdot}77 \text{ turns.}$$

5. The angular velocity will be $\frac{30}{60}2\pi = \pi$, which is the value of ω in the value of DC, p. 275 of the text; hence $DC = \frac{g}{\pi^2}$. This is one-half the parameter in the equation $y^2 = 2px$; hence the equation becomes $y^2 = 2\frac{g}{\pi^2}x$.

SOLUTIONS OF EXAMPLES.

PAGE 282.

1. To fulfill this condition we must have

$$\tfrac{1}{2}\delta b h^2 = \tfrac{1}{2}\delta b(9 - h^2)$$
$$h^2 = 4{\cdot}5$$
$$h = 2{\cdot}121 \text{ feet.}$$

2. The area of each triangle will be $\frac{1}{2} \times 1{\cdot}4 \times 2{\cdot}6 = 1{\cdot}82$. The centre of gravity of the triangle whose base will be in the surface will be $\frac{1}{3} \times 2{\cdot}6 \times \sin 56° \, 35'$ below the free surface; hence the pressure on it will be

$$62\tfrac{1}{2} \times 1{\cdot}82 \times \tfrac{1}{3} \times 2{\cdot}6 \times \sin 56° \, 35' = 82{\cdot}28 \text{ lbs.};$$

and of the other triangle,

$62\frac{1}{2} \times 1{\cdot}82 \times \frac{2}{3} \times 2{\cdot}6 \times \sin 56° 35' = 164{\cdot}57$ lbs.

3. Pressure on concave surface $= \frac{1}{2}\delta b h^2$, but $b = 2\pi r$;

$\therefore p = \frac{1}{2} \times 62{\cdot}5 \times 2 \times 3{\cdot}1416 \times 9 = 1767{\cdot}15$ lbs.

Weight of liquid $= \pi r^2 \times h \times \delta = 3{\cdot}1416 \times 3 \times 62{\cdot}5 = 589{\cdot}05$ lbs.

Pressure on base = weight of liquid = $589{\cdot}05$ lbs.

4. The weight of the liquid $= \frac{4}{3}\delta\pi r^3 = \frac{4}{3} \times 62{\cdot}5 \times 3{\cdot}1416 \times 125 = 32725{\cdot}0$ lbs.

Normal pressure $= 4\delta\pi r^3 = 4 \times 62{\cdot}5 \times 3{\cdot}1416 \times 125 = 98175$ lbs.

5. Pressure on flood-gate $= \frac{1}{2}\delta b(h_2{}^2 - h_1{}^2) = \frac{1}{2} \times 62{\cdot}5 \times 2 \times (\overline{13}^2 - \overline{10}^2) = 62{\cdot}5 \times 69 = 4312{\cdot}5$ lbs.

6. Pressure on opposite side $= \frac{1}{2}\delta b(h'^2 - h''^2) = \frac{1}{2} \times 2 \times 62{\cdot}5(7^2 - 4^2) = 62{\cdot}5 \times 33 = 2062{\cdot}5$, and

$4312{\cdot}5 - 2062{\cdot}5 = 2250$ lbs.

SOLUTIONS OF EXAMPLES.

PAGE 287.

1. By Art. 372, the centre of pressures of rectangle is at a distance from the top equal to $\frac{2}{3}$ the height, $\therefore \frac{2}{3}$ of $3 = 2$ feet.

2. By Art. 373, we have for the required depth,

$$\frac{35}{6} - \frac{1}{3}\cdot\frac{5}{6}\left(\frac{\frac{35}{6} + 10}{\frac{35}{6} + 5}\right) = \frac{35}{6} - \frac{5}{18}\cdot\frac{19}{13} = \frac{1365}{234} - \frac{95}{234}$$

$= \frac{1270}{234} = 5{\cdot}427$ ft.; or $\cdot 427$ ft. $= 5{\cdot}19$ inches below the top of the flood-gate.

3. To resist overturning we have

$\delta b^2 h = \frac{125}{6} h_1^3$ substituting the values given, and we have

$$b^2 = \frac{125 \times 512}{6 \times 180 \times 8} = \frac{64000}{8640} = 7{\cdot}4074;$$

$$\therefore b = \sqrt{7{\cdot}4074} = 2{\cdot}72 \text{ ft.} = 2 \text{ feet } 8\tfrac{2}{3} \text{ inches.}$$

4. In this case we have, p. 279 of the text, $31\frac{1}{4} b h^2$ lbs. for the pressure of the liquid. Its moment in reference to the edge of the wall will be $\frac{1}{3} \times 31\frac{1}{4} b h^3$, and the wall must be capable of resisting twice this amount. The moment of the wall will be

$$\tfrac{1}{2} \times 4 \times 8 \times 120 \times \tfrac{2}{3} \text{ of } 4 = 5120 \text{ lbs.};$$

$$\therefore \tfrac{2}{3} \times 31\tfrac{1}{4} b h^3 = 5120;$$

$$\therefore h = 6{\cdot}2 \text{ feet.}$$

5. The pressures are proportional to the areas, and the areas are as $(15)^2 \div (1{\cdot}5)^2 = 100$ to 1;
$\therefore$ total pressure $= 500 \times 100 = 50{,}000$ lbs.

SOLUTIONS OF EXAMPLES.

Page 310.

1. $$t = \frac{2Q}{ms\sqrt{2gh}} = \frac{2\pi r^2 h}{0{\cdot}62 \times \pi(\frac{1}{48})^2 \sqrt{64\frac{1}{3} \times 3}}$$

$$= \frac{2 \times \frac{1}{4} \times 3 \times 2304}{0.62\sqrt{193}} = 401.2 \text{ sec.} = 6\text{m. } 41.2 \text{ sec.}$$

2. $Q = \frac{2}{3} mbt\sqrt{2gh^3} = \frac{2}{3} \times 06.2 \times 2(45 \times 60)$
$\sqrt{64\frac{1}{3} \times (\frac{5}{6})^3}$
$= 13618.8$ cubic feet.

3. The equation on p. 309 of the text is $x = \frac{h}{r^2} y^4$. But $h = 24$ inches, and $r = 3$ inches, hence we have $x = \frac{24}{81} y^4 = \frac{8}{27} y^4$.

To find the area of the orifice, we have on p. 309 of the text,

$$ms = \frac{\pi r^2 c}{\sqrt{2gh}}, \text{ when } c = \frac{h}{T} = \frac{2}{600} = \frac{1}{300};$$

$r = \frac{1}{4}$ ft., $h = 2$ ft.;

$$\therefore s = \frac{3.1416 \times \frac{1}{16} \times \frac{1}{300}}{0.62\sqrt{64\frac{1}{3} \times 2}} = 0.0000931 \text{ sq. ft.}$$

$$= 0.01341 \text{ sq. in.}$$

4. From the equation on p. 303 of the text we have

$$25 - 0.025187 \frac{Q^2}{(\frac{1}{6})^4} = 0.0006769 \frac{1500}{(\frac{1}{6})^2}$$
$$(Q^2 + 0.141724Q(\tfrac{1}{6})^2)$$

or

$$25 - 32.64235Q^2 = 7895.36(Q^2 + 0.003936Q);$$

$$\therefore Q^2 + \frac{31.076}{7928} Q = \frac{25}{7928};$$

$$\therefore Q = 0.0542 \text{ cu. ft. per sec.}$$

$$= 195.1 + \text{ cu. ft. per hour.}$$

ANSWERS TO EXERCISES.

PAGE 310.

1. The time will be the same for each.

2. It will not. The time will be less, for the head producing the velocity will be equivalent to what it would be if for the *weight* of the water an equal weight of mercury be substituted.

3. It will not. The flow of the water in this case will exceed that of the mercury in the preceding.

4. It will. It may be observed that when the pressure producing the flow of a liquid is the weight of the same liquid, the head equals the height of the liquid above the orifice.

5. It will not, but the depth of submergence will gradually increase. The block receives an initial velocity downward which is being gradually lessened as the surface of the liquid descends.

6. It will be lowest near the orifice.

7. It will be greater; for the acceleration upward of the vessel will have the same effect as an increased pressure on the surface.

8. It will be greater; for the head over the orifice will be greater.

9. It is not; a part of the pressure is engaged in producing motion of the mass.

PAGES 326–329.—The expression for the pressure (or rather the tension) of the air at any height, x, above the earth, p. 326, reduces to

$$p = p_\circ e^{-\frac{x}{H}} = \frac{p_\circ}{e^{\frac{x}{H}}};$$

where $p_\circ = 15$ lbs., $e = 2{\cdot}71828 +$, and $H = 26214$ feet, according to Rankine;

$$\therefore p = \frac{15}{(2{\cdot}71828)^{\frac{x}{26214}}}.$$

If $x = 20{,}000{,}000$ feet, about the radius of the earth, we have

$$p = \frac{15}{(2{\cdot}71828)^{763}} = \frac{15}{2 \times 10^{330}} \text{ lbs. per sq. inch,}$$

nearly.

But this expression gives too rapid a diminution of the tension, since the effect of gravity is discarded. Newton, in the *Principia*, gives the following:

PROPOSITION xxii., B. II.—*Let the density of any fluid be proportional to the compression, and its parts attracted downwards by a gravitation reciprocally proportional to the squares of the distance from the centre. I say, that if the distances be taken in harmonic progression, the densities of the fluid at those distances will be in geometrical progression.* In accordance with this proposition the following table was computed in *The System of the World* by Sir Isaac Newton:

HEIGHT IN MILFS.	COMPRESSION OF THE AIR. INITIAL PRESSURE EQUAL A COLUMN OF WATER 33 FEET HIGH.	EXPANSION OF THE AIR, THE INITIAL VOLUME BEING UNITY.
0	33	1
5	17.1815	1.8486
10	9.6717	3.4151
20	2.852	11.571
40	0.2525	136.83
400	$0.(10^{17})1224$	26956×10^{15}
4,000	$0.(10^{105})465$	73907×10^{102}
40,000	$0.(10^{192})1628$	20263×10^{189}
400,000	$0.(10^{210})7895$	41798×10^{207}
4,000,000	$0.(10^{212})9878$	33414×10^{209}
Infinite.	$0.(10^{212})6041$	54622×10^{209}

where $(10^{17})1224$, implies that there are 17 cyphers before 1224 in the decimal, thus 0·000000000000000001224, and similarly for the others. The following inference is drawn: "But from this table it appears that the air, in proceeding upwards, is rarefied in such a manner, that a sphere of that air which is nearest to the earth, of but one inch in diameter, if dilated with that rarefaction which it would have at the height of one semi-diameter of the earth, would fill all the planetary regions as far as the sphere of Saturn, and a great way beyond; and at the height of ten semi-diameters of the earth would fill up more space than is contained in the whole heavens on this side the fixed stars, according to the preceding computation of their distance. And though, by reason of the far greater thickness of the atmospheres of comets, and the great quantity of the circum-solar centripetal force, it may happen that the air in celestial spaces, and in the tails of comets, is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapor is abundantly sufficient to produce all the appearances of the tails of comets, for that they are indeed of a very notable rarity appears from the shining of the stars through them." Similar remarks are made in the *Principia* under Prop. xli.

SOLUTIONS OF EXAMPLES.

Page 331.

1. In the first formula on p. 317 of the text, making $V_1 = 2V_\circ$ and $\alpha = 0{\cdot}002\dot{0}39$, we find

$$t_1 - t_\circ = \frac{1}{0{\cdot}002039} = 490. +$$

Page 332.

2. At the surface of the earth the pressure of the air will balance a column of water 34 feet high. The pressures will be inversely as the amount of compression; hence

$$1 : 30 :: 34 : x = 1020 \text{ feet of water.}$$

But the first 34 is due to atmospheric pressure; hence the depth will be $1020 - 34 = 986$ feet.

3. From the equation in Prob. 3, p. 323 of the text we have

$$2y = 2\sqrt[3]{\frac{34(\frac{1}{2})^3}{34+1000}} = 2\sqrt[3]{\frac{34}{8272}} = 2\sqrt[3]{0{\cdot}0041102}$$
$= 0{\cdot}322$ of an inch.

4. Assuming that a cubic foot of air weighs 0·08072 of a pound, 10,000 cubic feet will weigh 807·2

lbs., and the weight of the gas will be $807{\cdot}2 \times 0{\cdot}069 = 55{\cdot}696$ lbs. This subtracted from the weight of an equal volume of air will give the lifting capacity, or $807{\cdot}2 - 55{\cdot}696 = 741{\cdot}504$ lbs.

5. From the last equation of Article 419, p. 317 of the text, we have ($t_\circ$ being 32°)

$$p_2 = \frac{V_1}{V_2} \times \frac{1 + \alpha(t_2 - t_\circ)}{1 + \alpha(t_1 - t_\circ)} p_1$$

$$= \frac{5}{5{\cdot}5} \times \frac{1 + 0{\cdot}002039(400 - 32)}{1 + 0{\cdot}002039(32 - 32)} 15$$

$$= 23{\cdot}868 \text{ lbs.}$$

Why will a wheel with nearly all the matter concentrated in the axle roll down a plane in less time than if it be nearly all concentrated in the rim?

Because, in descending the plane the entire work is done by gravity, and the measure of that work is the weight into the vertical descent of the centre of the wheel; and this work is changed into kinetic energy in the wheel. When the matter is concentrated in the axle, the energy will be $\frac{1}{2}mv^2$, where m is the mass, and v the final velocity of the centre; but if it be all concentrated in the rim its energy will be $\frac{1}{2}mv_1^2 + \frac{1}{2}m(r\omega)^2$, where m is the same mass as before, v_1 the final velocity of the centre, ω the final angular velocity of the rim, and r the radius of the rim. Since the total kinetic energy must be the same in both cases, it is evident that v must exceed v_1, and hence the time of descent in the latter case will exceed that in the former. It is shown in *Analytical Mechanics*, p. 215, that if a given mass be entirely concentrated in the axle, or uniformly distributed as a disc, or entirely concentrated in the rim, all having the same radius, the times will be as $\sqrt{2} : \sqrt{3} : \sqrt{4}$.

THE END.