EXPECTATION OF A STOPPING TIME FOR RANDOM SUMS WITH NONZERO MEAN

by

Clarence Allan Foy, Jr.

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Doctoral Committee:

Associate Professor Michael B. Woodroofe, Chairman
Professor Paul G. Federbush
Assistant Professor Arthur O. Pittenger
Associate Professor Chung-Tuo Shih
Associate Professor Norman Starr

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CHAPTER 1

INTRODUCTION

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed (i. i. d.) random variables having a finite mean μ and a finite positive variance σ^2. For any positive constant c, the stopping time $N(c)$, defined by

$$N(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

is finite with probability one by the law of the iterated logarithm.

An optional stopping time of this form played a central role in a controversy over the evaluation of ESP experiments in the late 1930's. Card-guessing experiments were conducted and were assumed to be independent Bernoulli trials with probability 1/5 of a correct guess. Experimenters felt that a large "critical ratio"--the deviation between the observed number of correct guesses and the expected number of correct guesses divided by the standard deviation--would indicate the presence of ESP.

However, it was noted that the performance of a subject tended to decline after a long period and sometimes experimenters, rather than specifying in advance the length of a run, would terminate it when conditions seemed to warrant such a move. That the effect of optional stopping had to be considered in the calculation of various probabili-

ties was pointed out by several critics. In particular, Feller [8] demonstrated that the normal approximation for the probability of occurrence of a given critical ratio could not be used. Further, citing the law of the iterated logarithm, he remarked that with probability one any given critical ratio could be attained infinitely often and, hence, stopping at critical points could certainly give biased results.[1]

More generally, for any sequence $\{X_n\}$ of i. i. d. random variables with mean zero and variance σ^2, $\sum_1^n X_i/\sigma\sqrt{n}$ may be regarded as a "critical ratio." In this case $N(a)$ defined above is just the number of trials that must be conducted to achieve a critical ratio that is greater than a or less than $-a$. A natural question then is just how long must one wait to attain a desired critical ratio.

Several people have studied the expected value of the stopping time $N(a)$ for sequences of random variables with zero mean. One of the earliest papers was by Blackwell and Freedman [2] in which they studied coin-tossing random variables. Later, Chow, Robbins, and Teicher [4] proved that for all i. i. d. sequences with zero mean, $EN(a) = +\infty$ for all $a \geq 1$.

To study the case of sequences with nonzero mean, μ will be regarded as a translation parameter. In other words, if $\{X_n\}$ is a fixed sequence with common mean equal to zero, the behavior of the stopping time $N(a)$ defined with respect to the sequence $\{X_n(\mu)\}$,

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[1]Feller [8, p. 288] gives a hint of his future important work on the law of the iterated logarithm when he states that "we are here concerned with a trivial special case of deep and interesting problems to which a considerable part of the modern theory of probability is devoted."

$X_n(\mu) = X_n + \mu$, is analyzed. In particular, for $\sigma \geq 1$, I am able to obtain bounds for $EN(\sigma)$ in terms of the mean μ and thus obtain information on the way in which $EN(\sigma)$ diverges to infinity as $\mu \to 0$. This problem has been studied by Woodroofe [16]. For sequences with $E|(X_1-\mu)\sigma^{-1}|^{4\theta} < \infty$ for some $\theta > 1$, he proved the existence of positive constants b_1, b_2, γ_1, γ_2 with $0 < \gamma_1 < \gamma_2 < 1$ such that

$$b_1|\mu|^{-(1+\gamma_1)} \leq EN(\sigma) \leq b_2|\mu|^{-(1+\gamma_2)}$$

for all sufficiently small values of μ. Here, the constants γ_1 and γ_2 depend on the common distribution of the sequence $\{X_n\}$. In this paper, the exponents of $|\mu|$ above are replaced by expressions involving a single constant which is independent of the distribution of the given sequence; however, a finite moment generating function must be assumed.

Since $N(\sigma) < \infty$ almost surely, $EN(\sigma) = \sum_0^\infty P[N(\sigma) > n]$ and, so, in order to obtain bounds for $EN(\sigma)$ one is led to consider the behavior of the probabilities $P[N(\sigma) > n]$ for large n. Breiman [3] determined the asymptotic behavior of these tail probabilities for a slightly different stopping time $N^*(\sigma)$ defined by

$$N^*(\sigma) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| \geq \sigma\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

For ease of reference, his conclusions are presented here as

Theorem 1.1. Let $\{X_n\}_{n=1}^\infty$ be a sequence of i. i. d. random variables with common mean zero, finite, positive variance σ^2, and $E|X_1|^3 < \infty$. If, for any positive σ, $N^*(\sigma)$ is the stopping time defined above, then either there exists an integer n such that $P[N^*(\sigma) > n] = 0$ or there

exist positive numbers $\beta(c)$ and $\alpha = \alpha(c)$ such that $P[N^*(c) > n] \sim \alpha n^{-\beta(c)}$ as $n \to \infty$.[2]

It is important to note that although α depends on the common distribution of the sequence, $\beta(c)$ is a constant that is the same for all sequences satisfying the stated conditions. In fact, $-\beta(c)$ is the largest zero of the confluent hypergeometric function $M(\lambda, \frac{1}{2}, c^2/2)$ regarded as a function of the real variable λ. The definition of $M(a, b, x)$ and some properties of β as a function of c that will be useful are included in the Appendix.

The bounds obtained in this paper for $EN(c)$ are dependent upon the parameter $\beta(c)$. Specifically, we show for certain sequences $\{X_n\}$ that for $\varepsilon > 0$ and $c > 1$, there exist positive constants A and B, independent of μ, such that

$$A|\mu|^{-2[1-\beta(c)]+\varepsilon} \leq EN(c) \leq B|\mu|^{-2[1-\beta(c)]-\varepsilon}$$

for all sufficiently small values of μ. The upper bound remains valid for $c = 1$.

Breiman determined the constant $\beta(c)$ first by considering a stopping time similar to $N^*(c)$ but defined on a normalized Brownian motion process. If $\{W_\mu(t), t \geq 0\}$ is a Brownian motion process with drift μ, then the stopping time $T_\mu(c)$ is defined by

$$T_\mu(c) = \inf \left\{t: |W_\mu(t)| > c\sqrt{t+1}\right\}.$$

Shepp [13] proved that $ET_0(c) = +\infty$ if $c \geq 1$ and $ET_0(c) < \infty$ if $c < 1$. Applying Breiman's results, I am able to show that there exist positive

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[2]Breiman states his results for a one-sided stopping time but, in fact, his proof is for the two-sided time $N^*(c)$ defined here.

constants C and D such that as $\mu \to 0$,

$$ET_\mu(\sigma) \sim C\mu^{-2[1-\beta(\sigma)]} \qquad \sigma > 1,$$

$$ET_\mu(1) \sim D \log \mu^{-2} \qquad \sigma = 1,$$

$$ET_\mu(\sigma) \to ET_0(\sigma) \qquad \sigma < 1.$$

We remark that these relations hold even for $\mu < 0$; in the first case, we simply regard $\mu^{-2[1-\beta(\sigma)]}$ as $\left(\mu^{-2}\right)^{1-\beta(\sigma)}$.

The above results for Brownian motion are not hard to obtain and are presented first in Chapter 2. In considering sequences of random variables, the general methods used are based on the case for which the sequence has a common normal distribution. For this reason and, also, since slightly sharper bounds may be obtained (i.e., bounds hold for $\varepsilon = 0$), this case is considered separately in Chapter 3.

In order to obtain the desired bounds for $EN(\sigma) = \sum_0^\infty P[N(c) > n]$, it is necessary to obtain bounds on the tail probabilities which hold uniformly for sufficiently small μ. In other words, the dependence of the constant α in Theorem 1.1 on the distribution of the sequence must be eliminated. This is done in Chapter 4, where a correction to the proof of Breiman's theorem is given and the appropriate generalization is made.

Chapter 5 concerns sequences of random variables which have a continuous distribution function and for which the moment generating function exists. The desired bounds for $EN(\sigma)$ are obtained. Finally, in Chapter 6, extensions to more general cases are presented.

CHAPTER 2

BROWNIAN MOTION

Let us suppose that $\{W(t), t \geq 0\}$ is a Brownian motion process defined on the probability space $(\Omega, \mathcal{F}, P)$. When necessary, we write $W(t, \omega)$ for $W(t)$. We shall assume that this process is normalized; i.e., $EW(t) = 0$, $VarW(t) = t$, and $W(0) \equiv 0$. In addition, we shall have occasion to use the notation P^x and E^x when dealing with probabilities and the associated expectations for a Brownian motion process that begins at the point x [i.e., $W(0) \equiv x$]. We note that $P \equiv P^0$ and that $P^x[W(t) \in B] = P[W(t) \in B-x]$ for any Borel set $B \subset R^1$.

Now, for $\mu \neq 0$, we wish to consider the process $\{W(t)+\mu t, t \geq 0\}$, a Brownian motion with drift μ. For convenience, we write $W(t) + \mu t = W_\mu(t)$ for all $t \geq 0$. For any positive number c, we define the stopping time $T = T_\mu(c)$ for the Brownian motion with drift μ by

$$T = T_\mu(c) = \inf \left\{t: |W_\mu(t)| \geq c\sqrt{t+1}\right\}.$$

The same stopping time defined for normalized Brownian motion (i.e., $\mu = 0$) will be denoted by $T_0 = T_0(c)$. In this chapter we determine the asymptotic behavior of $ET_\mu(c)$ as $\mu \to 0$.

Lemma 2.1. The distribution of the stopping time T_0 is absolutely continuous.

Proof. To show that $F(t) = P(T_0 \leq t)$ is absolutely continuous, it suffices to show that for some constant K, $P(t < T_0 \leq t+h) \leq Kh$ for all

$t \geq 0$ and $h > 0$. By the symmetry of Brownian motion, it is clear that

$$(1) \qquad P(t < T_0 \leq t+h) = 2P[t < T_0 \leq t+h,\ W(T_0) > 0].$$

Now, let us denote the first hitting time of the level a by S_a; i.e., $S_a = \inf\{s\colon W(s) = a\}$. It is well known (e.g., [9, p. 174]) that S_1 has density g where

$$g(s) = \frac{1}{\sqrt{2\pi s^3}} \exp\left(-\frac{1}{2s}\right),\ s > 0.$$

A change of scale shows that S_a has the same distribution as $a^2 S_1$. Thus, in particular, for $a = c\sqrt{t+1}$, it follows from (1) that

$$\begin{aligned} P(t < T_0 \leq t+h) &\leq 2P(t < S_{c\sqrt{t+1}} \leq t+h) \\ &= 2\int_{t[c^2(t+1)]^{-1}}^{(t+h)[c^2(t+1)]^{-1}} \frac{1}{\sqrt{2\pi s^3}} \exp\left(-\frac{1}{2s}\right) ds \\ &\leq \frac{2M}{c^2} h \end{aligned}$$

where M is a bound for g. |||

It follows then that the random variable T_0 has a density which we will denote by f_0. We now show that the stopping time T also has a density and we obtain a relation between its density and f_0.

Lemma 2.2. For any $t > 0$ and $h > 0$

$$P(t < T \leq t+h) = \exp\left(-\frac{\mu^2(t+h)}{2}\right)\int_{\{t < T_0 \leq t+h\}} e^{\mu W(t+h)}\, dP\ .$$

Proof. We make use of the function space representation. For fixed $t > 0$ and $h > 0$, we consider the space $C[0,\ t+h]$ of continuous functions on the interval $[0,\ t+h]$. P_0 will denote the Wiener measure on

$C[0, t+h]$; i.e., P_0 is the unique measure such that

$$P_0[x(\cdot): x(t_1) \in B_1, \ldots, x(t_n) \in B_n]$$
$$= P[W(t_1) \in B_1, \ldots, W(t_n) \in B_n]$$

for $0 \leq t_1 < t_2 < \ldots < t_n \leq t+h$ and for Borel sets $B_i \subset R^1$, $i = 1,\ldots,n$, $n \geq 1$. Similarly, we denote by P_μ the measure induced on $C[0, t+h]$ by $\{W_\mu(s), 0 \leq s \leq t+h\}$. Shepp [12, p. 348] has shown that P_μ is absolutely continuous with respect to P_0 with Radon-Nikodym derivative $dP_\mu/dP_0 = \exp\left(-\frac{\mu^2(t+h)}{2} + \mu x(t+h)\right)$; i.e., for any $A \subset C[0, t+h]$,

$$P_\mu(A) = \exp\left(-\frac{\mu^2(t+h)}{2}\right)\int_A e^{\mu x(t+h)}\, dP_0(x) \ . \tag{2}$$

Now, in particular, we let A be the set in $C[0, t+h]$ defined by

$$\{x(\cdot): |x(s)| < c\sqrt{s+1},\ 0 \leq s \leq t;$$
$$|x(s)| \geq c\sqrt{s+1} \text{ for some } s \in (t, t+h]\}.$$

Then it is clear that $P_\mu(A) = P(t < T \leq t+h)$ and

$$\int_{\{t < T_0 \leq t+h\}} e^{\mu W(t+h)}\, dP = \int_A e^{\mu x(t+h)}\, dP_0(x) \ .$$

These equalities together with (2) yield the desired result. |||

Lemma 2.3. The stopping time T has a density function f_μ. Further, if f_0 is the density function of T_0, then for almost every (a. e.) $t > 0$

$$f_\mu(t) = \exp\left(-\frac{\mu^2 t}{2}\right)\cosh\left(\mu c\sqrt{t+1}\right) f_0(t). \tag{3}$$

Proof. As in Lemma 2.1, for $t > 0$ and $h > 0$ we consider $P(t < T \leq t+h)$. Writing $D = \{t < T_0 < t+h\}$, we see by the previous lemma that

$$(4)\qquad P(t < T \le t+h) = \exp\left(-\frac{\mu^2(t+h)}{2}\right)\int_D e^{\mu W(t+h)}\, dP \,.$$

We write the integral above as $E\left(I_D\, e^{\mu W(t+h)}\right)$ where I_D denotes the indicator function of the set D.

Next, we define the random variable $S = t + h - T_0$ and note that on the set D, $0 \le S < h$. Clearly, S is $\mathcal{F}(T_0)$-measurable where $\mathcal{F}(T_0)$ is the σ-algebra consisting of the sets Λ such that $\Lambda \cap \{T_0 \le t\} \in \sigma[W(s),\ s \le t]$ for all $t \ge 0$. It follows from the strong Markov property that

$$E\left(I_D\, e^{\mu W(S+T_0)} \,|\mathcal{F}(T_0)\right)(\omega) = I_D(\omega) E^{W[T_0(\omega),\omega]}\left(e^{\mu W[S(\omega),\omega']}\right).$$

Using the properties of Brownian motion, we obtain

$$\begin{aligned} E^{W(T_0)}\left(e^{\mu W(S,\omega')}\right) &= E\{\exp\left(\mu[W(S,\omega') + W(T_0)]\right)\} \\ &= \exp\left(\mu W(T_0) + \frac{\mu^2 S}{2}\right). \end{aligned}$$

Thus,

$$\begin{aligned} E\left(I_D\, e^{\mu W(t+h)}\right) &= E\left(E[I_D\, e^{\mu W(S+T_0)}\,|\mathcal{F}(T_0)]\right) \\ &= E\left(I_D \exp\,[\mu W(T_0) + \frac{\mu^2 S}{2}]\right). \end{aligned}$$

By the symmetry of Brownian motion, this last expectation is equal to

$$\int_t^{t+h} \cosh\,(\mu\sigma\sqrt{s+1})\, \exp\left(\frac{\mu^2(t+h-s)}{2}\right) f_0(s)\, ds.$$

Combining this and (4), we find that

$$(5)\qquad P(t < T \le t+h) = \int_t^{t+h} \cosh\,(\mu\sigma\sqrt{s+1})\, \exp\left(-\frac{\mu^2 s}{2}\right) f_0(s)\, ds.$$

Since this is true for all $t > 0$ and $h > 0$, it follows immediately that

the distribution function of T is absolutely continuous and that the density f_μ is given by (3) for a. e. $t > 0$. |||

Before proceeding to the main theorem, it is necessary to make one final remark. Breiman [3] studied the stopping time

$$T^* = T^*(c) = \inf\{t \geq 1: |W(t)| \geq c\sqrt{t}\}$$

and proved that $P[T^* > t \mid W(1) = 0] \sim \alpha t^{-\beta(c)}$ as $t \to \infty$. Using the definition of T^* and the Markov property, we find that

$$\begin{aligned} P[T^* > t \mid W(1)] &= P[|W(s+1)| < c\sqrt{s+1},\ 0 \leq s \leq t-1 \mid W(1)] \\ &= P^{W(1)}[|W(s)| < c\sqrt{s+1},\ 0 \leq s \leq t-1] \end{aligned}$$

Thus, $P[T^* > t \mid W(1) = 0] = P[|W(s)| < c\sqrt{s+1},\ 0 \leq s \leq t-1] =$ $P(T_0 > t-1)$ by the definition of T_0; consequently, $P(T_0 > t) \sim \alpha t^{-\beta(c)}$ as $t \to \infty$.

We show in the Appendix that as a function of c, β is strictly decreasing and continuous on $[1, +\infty)$ and that $\beta(1) = 1$. Hence, for all $c > 1$, we see that $\beta(c) < 1$. Also, we remark that $\beta(c) > 1$ for $c < 1$.

Theorem 2.1. If $\{W_\mu(t),\ t \geq 0\}$ is a Brownian motion process with drift μ and if we define for any positive c the stopping time $T(c)$ by

$$T(c) = \inf\{t: |W_\mu(t)| \geq c\sqrt{t+1}\},$$

then for each $c > 1$ there exists a positive constant C, independent of μ, such that $ET(c) \sim C\mu^{-2[1-\beta(c)]}$ as $\mu \to 0$. For $c = 1$, there exists a positive constant D, independent of μ, such that $ET(1) \sim D \log \mu^{-2}$. For $c < 1$, $\lim_{\mu \to 0} ET(c) = ET_0(c)$.

Proof. We define the stopping time $T_0(c)$ on the normalized Brownian

motion process $\{W(t),\ t \geq 0\}$ as before. Again, f_μ and f_0 will denote the densities of $T(c)$ and $T_0(c)$ respectively. First, we fix $c > 1$ and we write T, T_0, and β for $T(c)$, $T_0(c)$, and $\beta(c)$ respectively.

It follows from Lemma 2.3 that

$$ET = \int_0^\infty t \exp\left(-\frac{\mu^2 t}{2}\right) \cosh\left(\mu c\sqrt{t+1}\right) f_0(t)\,dt.$$

We integrate by parts and perform the change of variable $s = \mu^2 t$ to obtain

$$(6) \qquad ET = \mu^{-2(1-\beta)} \int_0^\infty P(T_0 > s\mu^{-2})(s\mu^{-2})^\beta \exp\left(-\frac{s}{2}\right) s^{-\beta} v_\mu(s)\,ds$$

where $v_\mu(s) = \cosh\left(c\sqrt{s+\mu^2}\right) - \frac{s}{2}\cosh\left(c\sqrt{s+\mu^2}\right) + \frac{cs}{2\sqrt{s+\mu^2}}\sinh\left(c\sqrt{s+\mu^2}\right)$.

Hence, it remains to show that the limit as $\mu \to 0$ of the integral in (6) is a finite positive constant.

As noted previously, $P(T_0 > t) \sim \alpha t^{-\beta}$ as $t \to \infty$; it follows that the above integrand converges for all $s > 0$ to $\alpha \exp\left(-\frac{s}{2}\right) s^{-\beta} v_0(s)$ as $\mu \to 0$. Further, there is a $t_1 > 0$ such that for all $t \geq t_1$, $t^\beta P(T_0 > t) < \alpha+1$. Hence, for all $\mu \neq 0$ and all $s > 0$,

$$(s\mu^{-2})^\beta P(T_0 > s\mu^{-2}) \leq \begin{cases} \alpha+1 & s\mu^{-2} \geq t_1 \\ t_1^{\ \beta} & s\mu^{-2} < t_1 \end{cases}.$$

Thus, for all $\mu \neq 0$,

$$\left|(s\mu^{-2})^\beta P(T_0 > s\mu^{-2}) \exp\left(-\frac{s}{2}\right) s^{-\beta} v_\mu(s)\right| \leq (\alpha+1+t_1^{\ \beta}) \exp\left(-\frac{s}{2}\right) s^{-\beta} |v_\mu(s)|.$$

We wish to apply the dominated convergence theorem as $\mu \to 0$ and, so, it suffices to bound $|v_\mu(s)|$ in a neighborhood of zero, say for $|\mu| < 1$. For such μ,

$$|v_\mu(s)| \leq \cosh\left(c\sqrt{s+1}\right) + \frac{s}{2}\cosh\left(c\sqrt{s+1}\right) + c\sqrt{s}\,\sinh\left(c\sqrt{s+1}\right).$$

Letting the right side above equal $w(s)$, it follows from the fact that $\beta < 1$ that $\int_0^\infty \exp\left(-\frac{s}{2}\right) s^{-\beta} w(s)\,ds < \infty$. Thus, by the dominated convergence theorem,

$$\lim_{\mu \to 0} \mu^{2(1-\beta)} ET = \alpha \int_0^\infty \exp\left(-\frac{s}{2}\right) s^{-\beta} v_0(s)\,ds. \tag{7}$$

The latter integral is finite; in fact, letting C denote the limit in (7), we find from known integral formulas [10, p. 365, #3.562.1 and #3.562.2] that

$$\begin{aligned} C = \alpha \exp\left(\frac{c^2}{4}\right) \Big(& \Gamma(2-2\beta)[D_{-2+2\beta}(-c) + D_{-2+2\beta}(c)] \\ & - \tfrac{1}{2}\Gamma(4-2\beta)[D_{-4+2\beta}(-c) + D_{-4+2\beta}(c)] \\ & + \tfrac{c}{2}\Gamma(3-2\beta)[D_{-3+2\beta}(-c) - D_{-3+2\beta}(c)]\Big) \end{aligned}$$

where $D_\lambda(z)$ denotes the parabolic cylinder function. Therefore, from (7), $ET \sim C\mu^{-2(1-\beta)}$ as $\mu \to 0$.

Now, we consider $c = 1$. Again, we write T and T_0 for $T(1)$ and $T_0(1)$; since $\beta(1) = 1$, (6) becomes

$$ET = \int_0^\infty P(T_0 > s\mu^{-2})(s\mu^{-2}) \exp\left(-\frac{s}{2}\right) s^{-1} v_\mu(s)\,ds$$

where $v_\mu(s)$ is as above with $c = 1$. If we define the functions w_1 and w_2 by $w_1(s) = \frac{s}{2}\cosh\sqrt{s}$ and $w_2(s) = \frac{\sqrt{s}}{2}\sinh\sqrt{s}$, then it follows as before that

$$\lim_{\mu \to 0} \int_0^\infty P(T_0 > s\mu^{-2})(s\mu^{-2}) \exp\left(-\frac{s}{2}\right) s^{-1}\left(-\frac{s}{2}\cosh\sqrt{s+\mu^2} + \frac{s}{2\sqrt{s+\mu^2}}\sinh\sqrt{s+\mu^2}\right) ds$$

$$= \alpha \int_0^\infty \exp\left(-\frac{s}{2}\right) s^{-1}[-w_1(s) + w_2(s)]\,ds < \infty.$$

Thus,

$$\lim_{\mu\to 0} \frac{ET}{\log \mu^{-2}} = \lim_{\mu\to 0} \frac{\int_0^\infty P(T_0 > s\mu^{-2})(s\mu^{-2}) \exp\left(-\frac{s}{2}\right) s^{-1} \cosh \sqrt{s+\mu^2}\, ds}{\log \mu^{-2}} .$$

We denote the integrand in the numerator above by $w_3(s)$.

Let $\varepsilon > 0$ be given. There exists a $t_0 > 0$ such that for $t \geq t$, $|tP(T_0 > t) - \alpha| < \varepsilon$. Also, there exists a $\delta > 0$ such that for $|s| \leq \delta$, $\left|\exp\left(-\frac{s}{2}\right) - 1\right| < \varepsilon$ and $|\cosh \sqrt{s} - 1| < \varepsilon$. For μ sufficiently small, $\mu^2 t_0 < \delta$ and we write

$$\int_0^\infty w_3(s)\, ds = \int_0^{\mu^2 t_0} w_3(s)\, ds + \int_{\mu^2 t_0}^{\delta} w_3(s)\, ds + \int_\delta^\infty w_3(s)\, ds ;$$

we denote these three integrals by I_1, I_2, and I_3. We note that $I_1 \leq t_0 \cosh \mu\sqrt{t_0+1}$ and $I_3 \leq \delta^{-1}(\alpha+\varepsilon) \int_0^\infty \exp\left(-\frac{s}{2}\right) \cosh \sqrt{s+\mu^2}\, ds$; by the dominated convergence theorem, this latter bound has a finite limit as $\mu \to 0$. Hence,

$$\lim_{\mu\to 0} \frac{I_1 + I_3}{\log \mu^{-2}} = 0 .$$

We note that

$$(\alpha-\varepsilon)\exp\left(-\frac{s}{2}\right)\cosh \mu\sqrt{t_0+1} \int_{\mu^2 t_0}^{\delta} s^{-1} ds \leq I_2 \leq (\alpha+\varepsilon) \cosh\sqrt{\delta+\mu^2} \int_{\mu^2 t_0}^{\delta} s^{-1} ds .$$

Thus,

$$(\alpha-\varepsilon)(1-\varepsilon) \leq \lim_{\mu\to 0} \frac{I_2}{\log \mu^{-2}} \leq (\alpha+\varepsilon)(1+\varepsilon).$$

Letting $\varepsilon \to 0$, we see that $ET(1) \sim \alpha \log \mu^{-2}$ as $\mu \to 0$.

Finally, we fix $c < 1$ and again write T, T_0, and β for $T(c)$, $T_0(c)$, and $\beta(c)$, respectively. We seek to show that $\lim_{\mu\to 0} |ET - ET_0| = 0$. Using the known densities, we note that

$$|ET - ET_0| \leq \int_0^\infty t \left| \exp\left(-\frac{\mu^2 t}{2}\right) \cosh (\mu c\sqrt{t+1}) - 1 \right| f_0(t)\, dt \;.$$

Clearly, as $\mu \to 0$, the integrand above converges to zero for each t. Also, since $\cosh (\mu c\sqrt{t+1}) \leq \exp (|\mu| c\sqrt{t+1})$, it is not hard to show that for μ sufficiently small the integrand is bounded by $Ktf_0(t)$ for some constant K, independent of μ and t. Since $ET_0(c) < \infty$ for $c < 1$, this function is integrable and, therefore, the desired result follows from the dominated convergence theorem. |||

CHAPTER 3

NORMALLY DISTRIBUTED RANDOM VARIABLES

On a probability space $(\Omega, \mathcal{F}, P)$, we consider the sequence $\{X_n\}_{n=1}^{\infty}$ of i. i. d. random variables, each having a normal distribution with finite mean μ and finite, positive variance σ^2. As usual, we say that X_1 has a $N(\mu,\sigma^2)$ distribution. For any positive constant c, we define the stopping time $N = N(c)$ by

$$N = N(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

It is our goal to obtain bounds for EN in terms of μ.

We begin with a lemma that is useful here and in later work when we study random variables with more general distributions.

Lemma 3.1. For a real parameter $\mu \neq 0$, let $f_\mu(x) = e^{-\alpha x+\gamma\sqrt{x}}\, x^{-\beta}$ where $\alpha = \alpha(\mu) > 0$ and $\gamma = \gamma(\mu)$ are such that $\lim_{\mu\to 0} \alpha(\mu) = 0$ and $\lim_{\mu\to 0} \gamma(\mu)^2/\alpha(\mu) = K_1$, a constant; β is a constant with $\beta < 1$. Then, for each $\mu \neq 0$,

$$\sum_1^{\infty} f_\mu(n) \leq \int_0^{\infty} f_\mu(x)dx + K(\mu)$$

where the integral is finite and $\lim_{\mu\to 0} K(\mu) = 0$.

Proof. We fix $\mu \neq 0$ and write f for f_μ. Clearly, $f \in C^{\infty}(0,\infty)$, $f \geq 0$, $\lim_{x\to 0^+} f(x) = +\infty$, and $\lim_{x\to +\infty} f(x) = 0$. Also, an elementary computation

shows that the roots of the equation $f'(x) = 0$ are the points

$$x = \frac{1}{\alpha}\left(\frac{\gamma}{4\sqrt{\alpha}} \pm \frac{1}{4}\sqrt{\frac{\gamma^2}{\alpha} - 16\beta}\right)^2 .$$

Thus, if $\gamma^2 \leq 16\alpha\beta$, the function f is decreasing and

$$\sum_1^\infty f(n) \leq \int_0^\infty f(x)dx.$$

On the other hand, if $\gamma^2 > 16\alpha\beta$, then $f'(x) = 0$ at two points, say x_1 and x_2 with $x_1 < x_2$. Then f is a piecewise monotone function, decreasing for $x < x_1$ and $x > x_2$ and increasing for $x_1 < x < x_2$. If $[x]$ denotes the greatest integer $\leq x$, we see that when $[x_2] > [x_1] + 1$ that

$$\sum_1^\infty f(n) = \sum_1^{[x_1]} f(n) + \sum_{[x_1]+1}^{[x_2]-1} f(n) + f([x_2]) + f([x_2]+1) + \sum_{[x_2]+2}^{\infty} f(n)$$

$$\leq \int_0^{[x_1]} f(x)dx + \int_{[x_1]+1}^{[x_2]} f(x)dx + \int_{[x_2]+1}^{\infty} f(x)dx + f([x_2]) + f([x_2]+1)$$

$$\leq \int_0^\infty f(x)dx + 2f(x_2).$$

It is easily seen that the same bound holds even if $[x_2] = [x_1]$ or $[x_2] = [x_1] + 1$.

Thus, in any case, $\sum_1^\infty f(n) \leq \int_0^\infty f(x)dx + K(\mu)$ where

$$K(\mu) = \begin{cases} 0 & \gamma^2 \leq 16\alpha\beta \\ 2f(x_2) & \gamma^2 > 16\alpha\beta \end{cases} .$$

We note that

$$f(x_2) = \left[\exp\left(-\left(\frac{\gamma}{4\sqrt{\alpha}} + \frac{1}{4}\sqrt{\frac{\gamma^2}{\alpha} - 16\beta}\right)^2 + \frac{\gamma}{\sqrt{\alpha}}\left(\frac{\gamma}{4\sqrt{\alpha}} + \frac{1}{4}\sqrt{\frac{\gamma^2}{\alpha} - 16\beta}\right)\right)\right]$$

$$\times\ \alpha^\beta\left(\frac{\gamma}{4\sqrt{\alpha}} + \frac{1}{4}\sqrt{\frac{\gamma^2}{\alpha} - 16\beta}\right)^{-2\beta} .$$

Since $\gamma(\mu)^2/\alpha(\mu) \to K_1$, it follows that $f(x_2) \sim \text{constant} \cdot \alpha(\mu)^\beta$ as $\mu \to 0$, and, therefore, $\lim_{\mu\to 0} K(\mu) = 0$. Finally, the value of $\int_0^\infty f(x)dx$ is known explicitly [10, p. 337] and is finite for $\alpha > 0$ and $\beta < 1$. |||

Theorem 3.1. If $\{X_n\}_{n=1}^\infty$ is a sequence of i. i. d. random variables with a common $N(\mu, \sigma^2)$ distribution and if $N(c)$ is the stopping time defined above, then, for each $c > 1$, there exist positive constants A and B, independent of μ, such that

$$A\mu^{-2[1-\beta(c)]} \leq EN(c) \leq B\mu^{-2[1-\beta(c)]}$$

for all μ sufficiently small. For $c = 1$, there exist positive constants A_1 and B_1 such that

$$A_1 \log \mu^{-2} \leq EN(1) \leq B_1 \log \mu^{-2}$$

for all μ sufficiently small.

Proof. We first fix $c > 1$ and write N and β for $N(c)$ and $\beta(c)$ respectively. Recall that $EN = \sum_0^\infty P(N > n)$. For $n \geq 1$, we define the set $C \subset R^n$ by $C = \{(x_1,\ldots,x_n): |\sum_1^k x_i| \leq c\sigma\sqrt{k},\ k = 1,\ldots,n\}$. Then

$$\begin{aligned} P(N > n) &= P\left(|\sum_1^k X_i| \leq c\sigma\sqrt{k},\ k = 1,\ldots,n\right) \\ &= \int \cdots \int_C (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum_1^n (x_i-\mu)^2}{2\sigma^2}\right) dx_1\ldots dx_n \\ &= \exp\left(-\frac{n\mu^2}{2\sigma^2}\right)(2\pi\sigma^2)^{-n/2} \int \cdots \int_C \exp\left(-\frac{\sum_1^n x_i^2}{2\sigma^2} + \frac{\mu}{\sigma^2}\sum_1^n x_i\right) dx_1\ldots dx_n. \end{aligned}$$

On the set C, $|\sum_1^n x_i| \leq c\sigma\sqrt{n}$. Thus, if $\{X_n'\}$ is a sequence of i. i. d.

random variables with a common $N(0, \sigma^2)$ distribution and if we define the stopping time N_0 by

$$N_0 = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i'\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then, for all $n \geq 1$,

$$P(N > n) \leq \exp\left(-\frac{n\mu^2}{2\sigma^2} + \frac{|\mu|c\sqrt{n}}{\sigma}\right) P(N_0 > n) \tag{1}$$

and

$$P(N > n) \geq \exp\left(-\frac{n\mu^2}{2\sigma^2} - \frac{|\mu|c\sqrt{n}}{\sigma}\right) P(N_0 > n) . \tag{2}$$

If we define the stopping time $N_0{}^*$ by

$$N_0{}^* = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i'\right| \geq c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then we note that $N_0{}^* = N_0$ a. s.; hence, in particular, $P(N_0 > n) = P(N_0{}^* > n)$. Now,

$$\begin{aligned} P(N_0{}^* > n) &\geq P\left(\sum_1^k |X_i'| < c\sigma\sqrt{k},\ k = 1,\ldots,n\right) \\ &\geq P\left(|X_j'| < c\sigma(\sqrt{j} - \sqrt{j-1}),\ j = 1,\ldots,n\right) \\ &= \prod_1^n P\left(|X_j'| < c\sigma(\sqrt{j} - \sqrt{j-1})\right) > 0. \end{aligned}$$

Hence, it follows from Theorem 1.1 that $P(N_0{}^* > n) \sim \alpha n^{-\beta}$ as $n \to \infty$ where α and β are positive constants, the latter being independent of the distribution of the $\{X_n'\}$. Thus, there exists a positive integer n_0 such that for $n \geq n_0$,

(3) $$\frac{\alpha}{2} n^{-\beta} \leq P(N_0{}^* > n) \leq \frac{3\alpha}{2} n^{-\beta}.$$

It follows from (1) and (3) that

(4) $$EN \leq n_0 + \sum_{n_0}^{\infty} \frac{3\alpha}{2} \exp\left(-\frac{n\mu^2}{2\sigma^2} + \frac{|\mu| c\sqrt{n}}{\sigma}\right) n^{-\beta}.$$

With $\alpha(\mu) = \mu^2/2\sigma^2$ and $\gamma(\mu) = |\mu| c/\sigma$, we see that the hypotheses of Lemma 3.1 hold. Hence,

$$EN \leq n_0 + \frac{3\alpha}{2}\left[\int_0^{\infty} \exp\left(-\frac{x\mu^2}{2\sigma^2} + \frac{|\mu| c\sqrt{x}}{\sigma}\right) x^{-\beta} dx + K(\mu)\right]$$

where $K(\mu) \to 0$ as $\mu \to 0$. A change of variable yields the inequality

$$EN \leq n_0 + \frac{3\alpha}{2} K(\mu) + \mu^{-2(1-\beta)}\left(\frac{3\alpha}{2}\right) \int_0^{\infty} \exp\left(-\frac{y}{2\sigma^2} + \frac{c\sqrt{y}}{\sigma}\right) y^{-\beta}\, dy \,.$$

As noted in the previous lemma, the integral above is finite since $\beta = \beta(c) < 1$. Thus, the above bound for EN is $0(\mu^{-2(1-\beta)})$ and, therefore, there exists a positive constant B, for example

$$B = 3\alpha \int_0^{\infty} \exp\left(-\frac{y}{2\sigma^2} + \frac{c\sqrt{y}}{\sigma}\right) y^{-\beta}\, dy \,,$$

such that $EN \leq B\mu^{-2(1-\beta)}$ for all μ sufficiently small. We note that the constant B depends on the value of the integral and, hence, on c.

Next, using (2) and (3), we see that

$$EN \geq \frac{\alpha}{2} \sum_{n_0}^{\infty} \exp\left(-\frac{n\mu^2}{2\sigma^2} - \frac{|\mu| c\sqrt{n}}{\sigma}\right) n^{-\beta} \,.$$

Since the terms of this series are decreasing, we replace the sum by an integral and make a change of variables to obtain

(5) $$EN \geq \mu^{-2(1-\beta)} \frac{\alpha}{2} \int_{\mu^2 n_0}^{\infty} \exp\left(-\frac{y}{2\sigma^2} - \frac{c\sqrt{y}}{\sigma}\right) y^{-\beta}\, dy \,.$$

Thus,

$$\lim_{\mu\to 0} \mu^{2(1-\beta)} EN \geq \frac{\alpha}{2}\int_0^\infty \exp\left(-\frac{y}{2\sigma^2} - \frac{\sigma\sqrt{y}}{\sigma}\right) y^{-\beta}\, dy$$

where, once again, this integral is finite. Hence, if, for example,

$$A = \frac{\alpha}{4}\int_0^\infty \exp\left(-\frac{y}{2\sigma^2} - \frac{\sigma\sqrt{y}}{\sigma}\right) y^{-\beta}\, dy,$$

then $EN \geq A\mu^{-2(1-\beta)}$ for all μ sufficiently small.

Finally, we consider $\sigma = 1$. Since $\beta(1) = 1$, we obtain from (4) that

$$EN(1) \leq n_0 + \frac{3\alpha}{2}\sum_{n_0}^{\infty} \exp\left(-\frac{n\mu^2}{2\sigma^2} + \frac{|\mu|\sqrt{n}}{\sigma}\right) n^{-1} .$$

Without loss of generality, we may assume that $n_0 \geq 2$ and can replace the sum above by one starting at $n = 2$ without decreasing the bound. Just as in Lemma 3.1, we can show that such a sum is bounded by

$$\int_1^\infty \exp\left(-\frac{x\mu^2}{2\sigma^2} + \frac{|\mu|\sqrt{x}}{\sigma}\right) x^{-1}\, dx + K(\mu)$$

where $K(\mu) \to 0$ as $\mu \to 0$. Again, a change of variables yields

$$EN(1) \leq n_0 + \frac{3\alpha}{2}K(\mu) + \frac{3\alpha}{2}\int_{\mu^2}^\infty \exp\left(-\frac{y}{2\sigma^2} + \frac{\sqrt{y}}{\sigma}\right) y^{-1}\, dy .$$

For $\mu^2 < 1$, we see that

$$EN(1) \leq n_0 + \frac{3\alpha}{2}K(\mu) + \frac{3\alpha}{2}\left[\int_1^\infty \exp\left(-\frac{y}{2\sigma^2} + \frac{\sqrt{y}}{\sigma}\right) y^{-1}dy + e^{\sigma-1}\int_{\mu^2}^1 y^{-1}dy\right].$$

This bound is $O(\log \mu^{-2})$ and, so, there is a positive constant B_1 such that $EN(1) \leq B_1 \log \mu^{-2}$ for all μ sufficiently small.

For the lower bound when $\sigma = 1$, (5) becomes

$$EN(1) \geq \frac{\alpha}{2}\int_{\mu^2 n_0}^\infty \exp\left(-\frac{y}{2\sigma^2} - \frac{\sqrt{y}}{\sigma}\right) y^{-1}dy .$$

For μ such that $\mu^2 n_0 < 1$,

$$EN(1) \geq \frac{\alpha}{2} \exp\left(-\frac{1}{2\sigma^2} - \frac{1}{\sigma}\right) \int_{\mu^2 n_0}^{1} y^{-1}dy + \frac{\alpha}{2} \int_{1}^{\infty} \exp\left(-\frac{y}{2\sigma^2} - \frac{\sqrt{y}}{\sigma}\right) y^{-1}dy \ .$$

Thus, there exists a positive constant A_1 (e.g., $A_1 = \frac{\alpha}{4} \exp\left(-\frac{1}{2\sigma^2} - \frac{1}{\sigma}\right)$) such that $EN(1) \geq A_1 \log \mu^{-2}$ for all μ sufficiently small. |||

Now, of course, we would like to obtain the same type of bound for EN when the sequence $\{X_n\}$ has a more general distribution. First, we present an important preliminary result in the next chapter. Then, in Chapter 5, we are able to use essentially the techniques of the previous theorem to obtain the desired result for a wider class of sequences.

CHAPTER 4

UNIFORM BOUNDS FOR TAIL PROBABILITIES

We now wish to direct our attention to more general sequences of i. i. d. random variables having a common nonzero mean μ. We again wish to study the stopping time N determined by the first exit of the random sum from a square root boundary. To obtain information on the behavior of EN as $\mu \to 0$, we study the probabilities $P(N > n)$.

Breiman [3] obtained the asymptotic behavior of these probabilities for the zero mean case. In studying the nonzero mean case, we will be led to consider collections of sequences of i. i. d. random variables indexed by the parameter μ. For each μ, the corresponding sequence will have a zero mean. We will need to extend Breiman's result and to obtain bounds for the tail probabilities which hold uniformly for sufficiently small values of μ. These bounds will then enable us to study EN for various sequences of random variables with nonzero mean.

To make the required extension, all that is really necessary is to closely examine Breiman's proof for a single sequence and make some appropriate alterations. For ease of reference, the first section is devoted to an examination of some of the details of this proof. In fact, a different approach is taken at one point in order to avoid an inaccuracy in the original proof. In Section 2, we first make the extension so that the results hold for a general class of distribution

functions. Then, at the end of this section, we will obtain the result on which the subsequent work depends.

1. *The Proof for a Single Sequence*

We begin by considering a sequence $\{X_n\}_{n=1}^{\infty}$ of i. i. d. random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with $EX_1 = 0$, $EX_1{}^2 = \sigma^2 > 0$, and $E|X_1|^3 < \infty$. For any $c > 0$, we define the stopping time

$$N^* = N^*(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| \geq c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

Breiman obtained the asymptotic behavior of $P(N^* > n)$ as $n \to \infty$ and his results have been stated in Theorem 1.1. Here, we review the methods used.

First, we introduce some notation. Let t_0 be a fixed, arbitrary positive number and for $t > 0$, let $m = [e^{2t}]$ and $n = [e^{2(t+t_0)}]$ where $[x]$ denotes the greatest integer less than or equal to x. Then, denoting $\sum_1^k X_i$ by S_k, we define $P(\gamma, t)$ and $Q(\gamma, \eta, t, t_0)$ by

$$P(\gamma, t) = P[S_m < \gamma\sigma\sqrt{m};\ N^*(c) > m]$$

$$Q(\gamma, \eta, t, t_0) =$$

$$P(S_n < \gamma\sigma\sqrt{n};\ |S_k| < c\sigma\sqrt{k},\ k = m,\ldots,n \mid S_m = \eta\sigma\sqrt{m}).$$

Also, if $\{Y(t), t \geq 0\}$ is the Uhlenbeck process [i.e., $Y(t) = W(e^{2t})/e^t$ where $W(t)$ is normalized Brownian motion], we define $Q(\gamma, \eta)$ by

$$Q(\gamma, \eta) = Prob[Y(t_0) < \gamma;\ |Y(t)| < c,\ 0 \leq t \leq t_0 \mid Y(0) = \eta].$$

Breiman proves in his Proposition 1 that there is a constant D such

that

(1) $$\sup_{\gamma,\eta} |Q(\gamma, \eta, t, t_0) - Q(\gamma, \eta)| \leq De^{-t/16}.$$

We also denote the Laplace transform of $P(\gamma, t)$ by $\hat{P}(\gamma, s)$; i.e.,

$\hat{P}(\gamma, s) = \int_0^\infty e^{-st} P(\gamma, t)\, dt$ where s is a complex number.

More generally, for bounded functions f defined on $[-c, c]$, we write $P(f, t) = \int_{-c}^{c} f(\gamma)P(d\gamma, t)$. Writing I_c for the indicator function of $[-c, c]$, we note that $P(I_c, t) = P(N^* > [e^{2t}])$. We let $\hat{P}(f, s)$ denote the Laplace transform of $P(f, t)$. These Laplace transforms are analytic in some half plane of $\mathbb{C}$; in particular, we assume that $\{s: Re\ s > s_0\}$ is the maximal half plane of analyticity of $\hat{P}(I_c, \cdot)$. It follows from Fubini's theorem that for bounded f, $\hat{P}(f, s) = \int_{-c}^{c} f(\gamma)\ \hat{P}(d\gamma, s)$ for all s with $Re\ s > s_0$. We wish to show that $\hat{P}(I_c, \cdot)$ has a pole at $s = s_0$, but that for some $\delta > 0$, this is the only singularity in $\{s: Re\ s > s_0 - \delta\}$. In doing this, Breiman made an error which we are able to avoid; however, the argument needed is a lengthy one.

In an effort to obtain a series expression for $\hat{P}(I_c, s)$, Breiman is led to consider solutions of the differential equation

(2) $$\psi''(x) - x\psi'(x) = \lambda\psi(x)$$

with boundary conditions $\psi(-c) = \psi(c) = 0$. The transformation $\phi(x) = \exp\left(-\frac{x^2}{4}\right) \psi(x)$ changes (2) into the self-adjoint Sturm-Liouville system

$$\phi''(x) + (\tfrac{1}{2} - \tfrac{1}{4}x^2)\phi(x) = \lambda\phi(x)$$
(3)
$$\phi(-c) = \phi(c) = 0 .$$

Much is known about the eigenvalues and eigenfunctions of such a system. (The facts that we state here may be found, for instance, in Coddington and Levinson [5, Chapter 7].) The eigenvalues $\{\lambda_k\}$ of the system (3) can be ordered so that $\lambda_0 > \lambda_1 > \lambda_2 > \ldots$; also, $\lim_{k\to\infty} \lambda_k = -\infty$. In this particular case, the functions $e^{-x^2/4} M(\lambda/2, 1/2, x^2/2)$ are known to be even solutions of (3) ([1, p. 686]) where $M(a, b, z)$ is the confluent hypergeometric function; it follows from the boundary conditions that $\lambda_0 = -2\beta(c)$ where β is the function discussed in the Appendix and used in the previous chapters. The eigenfunctions $\{\phi_k\}$ can be chosen so that they form an orthonormal set; i.e., $\int_{-c}^{c} \phi_i(x)\, \phi_j(x)\, dx = \delta_{ij}$. Of course, the eigenvalues of (2) are the same as those of (3) and if for any f, g in $L^2(-c, c)$, we define the inner product $(f, g) = \int_{-c}^{c} \exp\left(-\frac{x^2}{2}\right) f(x)\, g(x)\, dx$, then the orthogonality relation becomes $(\psi_i, \psi_j) = \delta_{ij}$.

A further property of the eigenfunctions is that they form a complete orthonormal set for $L^2(-c, c)$; i.e., for any $f \in L^2(-c, c)$, the series $\sum_1^\infty (f,\psi_j)\psi_j$ converges to f in the L^2 norm. More fundamentally, if $f \in C^2[-c, c]$ and satisfies the boundary conditions [i.e., $f(-c) = f(c) = 0$], then the above series converges uniformly to f on $[-c, c]$.

We will also need to know how these eigenvalues and eigen-

functions behave. Tricomi [14, pp. 169-177] gives asymptotic expressions for eigenvalues and eigenfunctions of general Sturm-Liouville systems. Applying his results to the case considered here, we find that

$$|\lambda_n|^{\frac{1}{2}} = \frac{\pi}{2c}(n+1) + \frac{1}{n}\left(\frac{c^3}{3} - \frac{c}{2\pi}\right) + 0(n^{-2}),$$

$$\phi_n(x) = \frac{1}{\sqrt{c}}\left\{\sin\left[(n+1)\pi\left(\frac{x+c}{2c}\right)\right] - \frac{1}{n}\,T(x)\,\cos\left[(n+1)\pi\left(\frac{x+c}{2c}\right)\right]\right\} + 0(n^{-2})$$

where $T(x) = \frac{c}{12}(x+c)^3 - \frac{c^2}{8\pi}(x+c)^2 + c^3\left(\frac{1}{4\pi} - \frac{1}{3}\right)(x+c)$. He also obtains an expression for the derivative of the eigenfunctions which becomes

$$\phi_n'(x) = \frac{1}{\sqrt{c}}\left\{(n+1)\,\cos\left[(n+1)\pi\left(\frac{x+c}{2c}\right)\right] + T(x)\,\sin\left[(n+1)\pi\left(\frac{x+c}{2c}\right)\right]\right\} + 0(n^{-1}).$$

Expressions for the eigenfunctions of (2) are obtained by noting that $\psi_n(x) = \exp\left(\frac{x^2}{4}\right)\phi_n(x)$. We note that, in particular, $|\lambda_n| = 0(n^2)$ and, hence, $\sum_0^\infty |\lambda_n|^{-1} < \infty$. Also, for large n, $\psi_n(x) = 0(1)$ and $\psi_n'(x) = 0(n)$ uniformly for $x \in [-c, c]$.

Now, for all $f \in C^2$ with $f(-c) = f(c) = 0$, we can write $f(x) = \sum_0^\infty (f,\psi_j)\psi_j(x)$ where the series converges uniformly. Using this representation, Breiman shows that for such f,

$$\hat{P}(f, s) = \sum_0^\infty \frac{1}{e^{st_0} - e^{\lambda_j t_0}}(f,\psi_j)\int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s) \tag{4}$$

for $Re\ s > s_0$, $s \neq \lambda_j$, $j \geq 0$, where

$$I(\gamma, s) = e^{st_0} \int_0^{t_0} e^{-st} P(\gamma, t)\, dt$$

(5)

$$+ \int_0^{\infty} \int_{-c}^{c} [Q(\gamma, \eta, t, t_0) - Q(\gamma, \eta)] e^{-st} P(d\eta, t) dt .$$

It follows from (1) that $I(\gamma, s)$ is analytic in $\{s: Re\ s > s_0 - 1/16\}$. Integrating by parts and using the Schwarz inequality and (2), we obtain

$$\left|\int_{-c}^{c} \psi_j(\gamma) I(d\gamma, s)\right| \leq K_1 \sup_{\gamma} |I(\gamma, s)| \left(\int_{-c}^{c} \exp\left(-\frac{\gamma^2}{2}\right) [\psi_j'(\gamma)]^2 d\gamma\right)^{\frac{1}{2}}$$

$$\leq K_1 \sup_{\gamma} |I(\gamma, s)| |\lambda_j|^{\frac{1}{2}}$$

where K_1 is a constant depending only on c. Further, if $s = x + iy$, then, using (1), we see that

(6) $$\left|\int_{-c}^{c} \psi_j(\gamma) I(d\gamma, s)\right| \leq K_1[t_0 e^{|x|t_0} + D\hat{P}(I_c, x + 1/16)] |\lambda_j|^{\frac{1}{2}}.$$

At this point, Breiman claims that the sum of $|(1, \psi_j)||\lambda_j|^{\frac{1}{2}}$ is absolutely convergent and, therefore, f can be replaced by 1 in (4) to obtain a series expansion for $\hat{P}(I_c, s)$. However, if $\sum|(1, \psi_j)||\lambda_j|^{\frac{1}{2}} < \infty$, then certainly $\sum|(1, \psi_j)| < \infty$, and, so, $S(x) = \sum_0 (1, \psi_j)\psi_j(x)$ is uniformly convergent and continuous on $[-c, c]$ and $S(-c) = S(c) = 0$. Since $1 \in L^2(-c, c)$, $\sum_0 (1, \psi_j)\psi_j$ converges to 1 in L^2 norm and, hence, $S(x) = 1$ a. e.; but this is a contradiction since both S and 1 are continuous but do not agree at $-c$ and c.

To avoid this difficulty, we choose a sequence of "nice" functions which converge to 1 on $(-c, c)$ and use (4) to obtain the desired expression for $\hat{P}(I_c, s)$. Specifically, for each $n \geq 1$, we can

choose functions $f_n \in C^2[-c, c]$, $f_n(-c) = f_n(c) = 0$, $f_n = 1$ on $[-c + \frac{1}{n}, c - \frac{1}{n}]$, and $|f_n'| \leq 2n$. Then, $f_n(x) \to 1$ for all $x \in (-c, c)$. We note that we can write (4) with f_n in place of f. We denote the half plane of analyticity of $\hat{P}(I_c, s)$ by E; i.e., $E = \{s: Re\ s > s_0\}$. Also, we define $E' = \{s: Re\ s > s_0 - 1/16\}$; we recall that $I(\gamma, s)$ is analytic in E'. We now prove several lemmas which we use to obtain the relationship between s_0 and the eigenvalues.

Lemma 4.1. For all $s \in E$, $\lim_{n\to\infty} \hat{P}(f_n, s) = \hat{P}(I_c, s)$.

Proof. By the bounded convergence theorem, $\lim_{n\to\infty} P(f_n, t) = P(I_c, t)$ since $P(\{-c, c\}, t) = 0$ for all $t \in (0, \infty)$. Also, $|P(f_n, t)| \leq P(I_c, t)$ and, so, by the dominated convergence theorem, $\lim_{n\to\infty} \hat{P}(f_n, s) = \hat{P}(I_c, s)$ for all $s \in E$. |||

Next, we note that it follows easily from the definition of the inner product and the bounded convergence theorem that $\lim_{n\to\infty} (f_n, \psi_j) = (1, \psi_j)$ for each $j \geq 0$. Hence, it is clear that each term on the right side of (4) with f replaced by f_n converges to the corresponding term with f replaced by 1 for all $s \in E'$, $s \notin \{\lambda_j, j \geq 0\}$. However, we must also show that the tails of the series converge as $n \to \infty$. Let $\Lambda_k = \{\lambda_i, i = k+1, k+2, \ldots\}$ for each $k \geq 0$. Then, for each $k \geq 0$ and each $n \geq 1$, we define the function $g_{k,n}$ on $E' \setminus \Lambda_k$ by

$$g_{k,n}(s) = \sum_{k+1}^{\infty} \frac{1}{e^{st_0} - e^{\lambda_j t_0}} (f_n, \psi_j) \int_{-c}^{c} \psi_j(\gamma) I(d\gamma, s).$$

Lemma 4.2. The function $g_{k,n}$ is analytic on $E' \setminus \Lambda_k$.

Proof. Each term of the series is analytic in $E' \setminus \Lambda_k$ and, so, it

suffices to prove that the series converges uniformly on each compact subset of $E'\setminus\Lambda_k$. Since the only limit of the eigenvalues is at $-\infty$, there are at most a finite number of points of Λ_k in E'. Let K be a compact subset of $E'\setminus\Lambda_k$. Then, for $s \in K$, $|e^{st_0}-e^{\lambda_j t_0}|^{-1} \leq M_1$, a constant independent of s and j. Since $\hat{P}(I_c, \cdot)$ is analytic on E, it is bounded on the compact set $K^+ = \{s + 1/16: s \in K\} \subset E$. Thus, using (6), we find that for $s \in K$,

$$|(f_n, \psi_j) \int_{-c}^{c} \psi_j(\gamma)\, I(d\gamma, s)| \leq M_2|(f_n, \psi_j)||\lambda_j|^{\frac{1}{2}}$$

where M_2 is a constant independent of s and j.

Thus, to prove that $g_{k,n}$ is analytic it remains to show that $\sum|(f_n, \psi_j)||\lambda_j|^{\frac{1}{2}} < \infty$. By the Schwarz inequality, this sum is less than or equal to $\left(\sum|(f_n, \psi_j)\lambda_j|^2\right)^{\frac{1}{2}}\left(\sum|\lambda_j|^{-1}\right)^{\frac{1}{2}}$. If L is the operator defined by $L = \psi'' - x\psi'$, then, since $f_n \in C^2$ and satisfies the boundary conditions of (2), it follows that $(f_n, \psi_j)\psi_j = (f_n, L\psi_j) = (Lf_n, \psi_j)$. These are the Fourier coefficients of the continuous (hence, square integrable) function Lf_n and so $\sum|(Lf_n, \psi_j)|^2 < \infty$. We remarked previously that $\sum|\lambda_j|^{-1} < \infty$. This yields the desired conclusion. |||

Lemma 4.3. $\lim_{n\to\infty} e^{-st_0} \int_{-c}^{c} [1 - f_n(\gamma)]I(d\gamma, s) = 0$ uniformly on compact subsets of E'.

Proof. Let K be a compact subset of E'. $I(\gamma, s)$ is given in (5) as the sum of two terms and we consider each one separately. For the first, we must examine $\int_{-c}^{c} [1 - f_n(\gamma)] \int_0^{t_0} e^{-st}P(d\gamma, t)dt$. Using the definition of f_n, we see that for $s = x + iy$ and $A_n = (-c, -c + \frac{1}{n}) \cup (c - \frac{1}{n}, c)$,

$$\left|\int_{-c}^{c} [1 - f_n(\gamma)] \int_0^{t_0} e^{-st} P(d\gamma,\ t)dt\right| \leq \int_0^{t_0} e^{-xt} P(A_n,\ t)dt$$

$$\leq e^{|x|t_0}\ t_0 \sum_{1}^{[e^{2t_0}]} P\left(\frac{S_j}{\sqrt{j}} \in A_n\right).$$

On K, $e^{|x|t_0}$ is bounded and $P\left(\frac{S_j}{\sqrt{j}} \in A_n\right) \to 0$ as $n \to \infty$ for each j. Thus,

$$\lim_{n\to\infty} \int_{-c}^{c} [1 - f_n(\gamma)] \int_0^{t_0} e^{-st} P(d\gamma,\ t)dt = 0 \text{ uniformly on } K.$$

Now, considering the second term in the definition of $I(\gamma,\ s)$, we write $J(\gamma,\ s) = \int_0^{\infty} \int_{-c}^{c} [Q(\gamma,\ \eta,\ t,\ t_0) - Q(\gamma,\ \eta)] e^{-st} P(d\eta,\ t)dt$. Then we must show that $e^{-st_0} \int_{-c}^{c} [1 - f_n(\gamma)] J(d\gamma,\ s) =$ $e^{-st_0} \int_{A_n} [1 - f_n(\gamma)] J(d\gamma,\ s) \to 0$ uniformly on K. Since $J(-c,\ s) = 0$ and $|f_n'| \leq 2n$, an integration by parts yields

$$(7) \qquad \left|e^{-st_0} \int_{-c}^{-c+\frac{1}{n}} [1 - f_n(\gamma)] J(d\gamma,\ s)\right| \leq 2ne^{-xt_0} \int_{-c}^{-c+\frac{1}{n}} |J(\gamma,\ s)|d\gamma$$

for $s = x + iy$. We note that as $\gamma \downarrow -c$, both $Q(\gamma,\ \eta,\ t,\ t_0)$ and $Q(\gamma,\ \eta)$ converge to zero. Also, for all $s \in K$, there is an x_0 such that $Re\ s \geq x_0 > s_0 - 1/16$, and, therefore,

$$|J(\gamma,\ s)| \leq \int_0^{\infty} \int_{-c}^{c} |Q(\gamma,\ \eta,\ t,\ t_0) - Q(\gamma,\ \eta)| e^{-x_0 t} P(d\eta,\ t)dt.$$

The integral on the right goes to zero as $\gamma \downarrow -c$ by the dominated convergence theorem and, so, $\lim_{\gamma\downarrow -c} J(\gamma,\ s) = 0$ uniformly for $s \in K$. Since e^{-xt_0} is bounded on K, it follows from (7) that

$$\lim_{n\to\infty} e^{-st_0} \int_{-c}^{-c+\frac{1}{n}} [1 - f_n(\gamma)] J(d\gamma,\ s) = 0$$

uniformly on K. Letting $J_1(\gamma, s) = J(\gamma, s) - J(c, s)$, a similar argument shows that

$$\lim_{n\to\infty} e^{-st_0} \int_{c-\frac{1}{n}}^{c} [1 - f_n(\gamma)]J_1(d\gamma, s) = 0$$

uniformly on K. Combining these results, we obtain

$$\lim_{n\to\infty} e^{-st_0} \int_{-c}^{c} [1 - f_n(\gamma)]I(d\gamma, s) = 0$$

uniformly for $s \in K$. |||

Lemma 4.4. For each $k \geq 0$, $\lim_{n\to\infty} g_{k,n}(s)$ exists uniformly on compact subsets of $E' \setminus \Lambda_k$.

Proof. We write $(e^{st_0} - e^{\lambda_j t_0})^{-1} = e^{-st_0}[1 + e^{\lambda_j t_0}(e^{st_0} - e^{\lambda_j t_0})^{-1}]$. Then

$$g_{k,n}(s) = G_n(s) + e^{-st_0} \sum_{k+1}^{\infty} \frac{e^{\lambda_j t_0}}{e^{st_0} - e^{\lambda_j t_0}} (f_n, \psi_j) \int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s)$$

where $G_n(s) = e^{-st_0} \sum_{k+1}^{\infty} (f_n, \psi_j) \int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s)$ which, in turn, equals $-e^{-st_0} \sum_{k+1}^{\infty} (f_n, \psi_j) \int_{-c}^{c} \psi_j'(\gamma)I(\gamma, s)\, d\gamma$. As before, we can show that $\sum_{k+1}^{\infty} \int_{-c}^{c} |(f_n, \psi_j)||\psi_j'(\gamma)I(\gamma, s)|\, d\gamma < \infty$, thus permitting an interchange of the sum and integral. Hence,

$$-G_n(s) = e^{-st_0} \int_{-c}^{c} \left[\sum_{k+1}^{\infty} (f_n, \psi_j)\psi_j'(\gamma)\right] I(\gamma, s)\, d\gamma.$$

Since $\psi_j'(\gamma) = O(j)$, $|(f_n, \psi_j)\psi_j'(\gamma)| \leq M_1 |(Lf_n, \psi_j)| j^{-1}$ and, so, again $\sum_{k+1}^{\infty} (f_n, \psi_j)\psi_j'(\gamma)$ converges uniformly. Thus, this series is the derivative of the function $\sum_{k+1}^{\infty} (f_n, \psi_j)\psi_j(\gamma) = f_n(\gamma) - \sum_{0}^{k} (f_n, \psi_j)\psi_j(\gamma)$.

Therefore,

$$G_n(s) = e^{-st_0}\left(\int_{-c}^{c} f_n(\gamma)I(d\gamma, s) - \sum_{0}^{k}(f_n, \psi_j)\int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s)\right).$$

Now, let K be a fixed compact subset of $E'\backslash\Lambda_k$. It follows from Lemma 4.3 and the remarks concerning the convergence of (f_n, ψ_j) that $\lim_{n\to\infty} G_n(s)$ exists uniformly for $s \in K$. To complete the proof of the lemma, it remains to show that $\lim_{n\to\infty} h_n(s)$ exists uniformly for $s \in K$ where

$$h_n(s) = e^{-st_0} \sum_{k+1}^{\infty} \frac{e^{\lambda_j t_0}}{e^{st_0} - e^{\lambda_j t_0}}(f_n, \psi_j)\int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s).$$

We will denote the above series with f_n replaced by 1 by $h_0(s)$.

We first show that these series converge uniformly for all $s \in K$ and all $n \geq 0$. We note that for $m > k$ and $s \in K$,

$$\left|e^{-st_0} \sum_{m+1}^{\infty} \frac{e^{\lambda_j t_0}}{e^{st_0} - e^{\lambda_j t_0}}(f, \psi_j)\int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s)\right|$$
$$\leq K_3\|f\|_2 \sum_{m+1}^{\infty} e^{\lambda_j t_0}|\lambda_j|^{\frac{1}{2}}$$
$$\leq K_4 \sum_{m+1}^{\infty} e^{\lambda_j t_0}|\lambda_j|^{\frac{1}{2}}$$

where $f = f_n$ or 1, $\|\cdot\|_2$ denotes the L^2 norm, and K_3 and K_4 are constants independent of $s \in K$ and n. Recalling that $|\lambda_j| \sim \left(\frac{\pi}{2c}\right)^2 j^2$, we see that $\sum e^{\lambda_j t_0}|\lambda_j|^{\frac{1}{2}} < \infty$. Hence, given $\varepsilon > 0$, there exists an integer $m_0 = m_0(\varepsilon)$ such that for $m \geq m_0$,

$$\left|e^{-st_0} \sum_{m+1}^{\infty} \frac{e^{\lambda_j t_0}}{e^{st_0} - e^{\lambda_j t_0}}(f, \psi_j)\int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s)\right| < \varepsilon$$

for all $s \in K$ and $f = f_n$ or 1. It then follows immediately that $\lim_{n\to\infty} h_n(s) = h_0(s)$ uniformly on K. |||

We note that because of the uniform convergence, the limit function, say g_k, of the $g_{k,n}$ is analytic in the domain $E'\backslash\Lambda_k$. Also, g_k is seen to be the function

$$(8)\qquad \begin{aligned} g_k(s) = e^{-st_0}\Bigg[I(c, s) - \sum_0^k (1, \psi_j)\int_{-c}^{c} \psi_j(\gamma) I(d\gamma, s) \\ + \sum_{k+1}^{\infty} \frac{e^{\lambda_j t_0}}{e^{st_0} - e^{\lambda_j t_0}}(1, \psi_j)\int_{-c}^{c} \psi_j(\gamma) I(d\gamma, s)\Bigg]. \end{aligned}$$

We now seek to locate s_0 in relation to the eigenvalues.

Proposition 4.1. If for any $k \geq 0$, $\int_{-c}^{c} \psi_j(\gamma) I(d\gamma, \lambda_j) = 0$, $0 \leq j < k$, then $s_0 \leq \lambda_k$. Equality holds if $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) \neq 0$.

Proof. Starting with (4) with f replaced by f_n and taking limits as $n \to \infty$, we obtain by applying the previous lemmas that for all $s \in E$, $s \neq \lambda_j$,

$$(9)\qquad \begin{aligned} \hat{P}(I_c, s) = \sum_0^{k-1} \frac{1}{e^{st_0} - e^{\lambda_j t_0}}(1, \psi_j)\int_{-c}^{c} \psi_j(\gamma) I(d\gamma, s) \\ + \frac{1}{e^{st_0} - e^{\lambda_k t_0}}(1, \psi_k)\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, s) + g_k(s) \end{aligned}$$

where g_k is analytic in $E'\backslash\Lambda_k$. By hypothesis, the finite sum on the right is analytic in E' and, so, in fact, (9) holds for all $s \in E\backslash\Lambda_{k-1}$. Now, if $\lambda_k < s_0$, then the second term on the right of (9) is analytic in $\{s\colon Re\ s > s_0 - \delta\}$ for some $\delta > 0$. Thus, the right side of (9) is an analytic function on $\{s\colon Re\ s > s_0 - \delta\}$ and, as such, defines an analytic continuation for $\hat{P}(I_c, s)$. However, the latter is the Laplace transform of the nonnegative function $P(I_c, t)$; therefore, it cannot be analytically continued across its axis of convergence (see [15, p.

58]). This contradiction implies that $s_0 \leq \lambda_k$.

Next, (4) with f replaced by ψ_k yields

$$\int_{-c}^{c} \psi_k(\gamma)\hat{P}(d\gamma, s) = \frac{1}{e^{st_0} - e^{\lambda_k t_0}} \int_{-c}^{c} \psi_k(\gamma) I(d\gamma, s)$$

for all $s \in E$, $s \neq \lambda_k$. Now, suppose $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) \neq 0$ and assume $s_0 < \lambda_k$. Then the left side above is analytic in E, the right side has a pole at $\lambda_k \in E$, and the two functions are equal in every small neighborhood of λ_k. Thus, we again have a contradiction and, so, $s_0 = \lambda_k$ if $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) \neq 0$. |||

Now, there are two possibilities; i.e., either $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) = 0$ for all $k \geq 0$ or there exists an integer $k \geq 0$ such that $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) \neq 0$. These possibilities are exactly what determines how $P(N^* > n)$ behaves for large n. We obtain immediately

Corollary 4.1. If $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) = 0$ for all $k \geq 0$, then there exists an $n > 0$ such that $P(N^* > n) = 0$.

Proof. By the previous proposition, $s_0 \leq \lambda_k$ for all k and, therefore, $s_0 = -\infty$. It is in this case that Breiman shows that $P(N^* > n) = 0$ for some integer n. |||

On the other hand, suppose now that k is the smallest non-negative integer such that $\int_{-c}^{c} \psi_k(\gamma) I(d\gamma, \lambda_k) \neq 0$. Then, applying Proposition 4.1 to (4), we find that for all $f \in C^2$ with $f(-c) = f(c) = 0$ and for all $s \in \{s\colon Re\ s > \lambda_k\}$

$$\hat{P}(f, s) = \frac{1}{e^{st_0} - e^{\lambda_k t_0}}(f, \psi_k)\int_{-c}^{c} \psi_k(\gamma)I(d\gamma, s) + h_f(s)$$

where h_f is analytic in $\{s: Re\ s > \lambda_k - \delta\}$ for some $\delta > 0$. This expression shows that $\hat{P}(f, s) \sim \alpha/(s-\lambda_k)$ as $s \to \lambda_k^+$ where $\alpha = t_0^{-1} e^{-\lambda_k t_0}(f, \psi_k) \int_{-c}^{c} \psi_k(\gamma)I(d\gamma, \lambda_k)$. $\hat{P}(f, s)$ is the Laplace transform of $P(f, t)$ and, so, by a Tauberian theorem due to Karamata ([15, p. 192]), it follows that $\int_0^t e^{-\lambda_k u} P(f, u)\, du \sim \alpha t$ as $t \to \infty$. For $f \geq 0$, this integral is nonnegative for each t and, so, $\alpha \geq 0$. Consequently, (f, ψ_k) must be of constant sign for all $f \in C^2$, $f(-c) = f(c) = 0$, $f \geq 0$. Therefore, ψ_k must be of constant sign and the only such eigenfunction is $\psi_0 > 0$ on $(-c, c)$. Thus, we have proved

Corollary 4.2. If $s_0 = \lambda_k$ for some $k \geq 0$, then, in fact, $s_0 = \lambda_0$.

Now it is possible to rewrite (9) as

$$\hat{P}(I_c, s) = \frac{1}{e^{st_0} - e^{\lambda_0 t_0}}(1, \psi_0)\int_{-c}^{c} \psi_0(\gamma)I(d\gamma, s) + h(s). \tag{10}$$

This equality holds for all $s \in \{s: Re\ s > \lambda_0\}$ and we note that h is analytic in $\{s: Re\ s > \lambda_0 - \delta\}$ for some $\delta > 0$. It follows from (8) that

$$\begin{aligned} h(s) = e^{-st_0}\Bigg[& I(c, s) - (1, \psi_0)\int_{-c}^{c} \psi_0(\gamma)I(d\gamma, s) \\ & + \sum_{1}^{\infty} \frac{e^{\lambda_j t_0}}{e^{st_0} - e^{\lambda_j t_0}}(1, \psi_j)\int_{-c}^{c} \psi_j(\gamma)I(d\gamma, s)\Bigg]. \end{aligned} \tag{11}$$

We now wish to make use of the following modification of a Tauberian theorem due to Ikehara.

Proposition 4.2. If $\phi(t)$ is a nonnegative, nonincreasing function for $t \in [0, \infty)$, $\phi(0) < \infty$, if the integral $\Phi(s) = \int_0^\infty e^{-st}\,\phi(t)\,dt$ $(s = x + iy)$ converges for $Re\ s > x_0$ with $x_0 < 0$, and if for some constant α and some function $g(y)$,

$$\lim_{x \to x_0^+} \left[\Phi(s) - \frac{\alpha}{s-x_0}\right] = g(y)$$

uniformly on every finite interval $|y| \le y_0$, then $\lim_{t\to\infty} e^{-x_0 t}\,\phi(t) = \alpha$.

Ikehara's Theorem asserts the same conclusion for a function ϕ that is nonnegative and nondecreasing and for $x_0 = 1$. A proof of this theorem is given by Widder [15, pp. 233-236] and by Doetsch [6, pp. 216-222]. Only slight modifications of these proofs are necessary to obtain a proof for Proposition 4.2. Hence, the proof will not be given here.

We may note, in particular, that if $\Phi(s)$ is analytic in $\{s: Re\ s > x_0 - \delta\}$ for some $\delta > 0$ except for a first order pole at $s = x_0$, then the constant α is just the residue at that pole; i.e., $\alpha = \lim_{s\to x_0} (s-x_0)\Phi(s)$. We see from (10) that $\hat{P}(I_c, \cdot)$ is analytic in $\{s: Re\ s > \lambda_0 - \delta\}$ for some $\delta > 0$ except possibly for poles at the points $s = \lambda_0 \pm 2n\pi i/t_0$. However, $\hat{P}(I_c, \cdot)$ is independent of the parameter t_0 and, so, the only singularity of $\hat{P}(I_c, \cdot)$ in $\{s: Re\ s > \lambda_0 - \delta\}$ is the first order pole at $s = \lambda_0$. Applying Proposition 4.2, we find that $\lim_{t\to\infty} e^{-\lambda_0 t}P(I_c, t) = \alpha$ where $\alpha = t_0^{-1}e^{-\lambda_0 t_0}(1, \psi_0)\int_{-c}^{c} \psi_0(\gamma)I(d\gamma, \lambda_0) > 0$. As noted above, $\lambda_0 = -2\beta(c)$ and $P(I_c, t) = P(N^* > [e^{2t}])$. It follows immediately that $P(N^* > n) \sim$

$cn^{-\beta(c)}$ as $n \to \infty$. This, together with Corollary 4.1, is the content of Breiman's result which we have stated previously as Theorem 1.1.

2. *The Uniform Result*

Let $\mathcal{C}$ be a class of probability distribution functions such that each $F \in \mathcal{C}$ satisfies the following properties:

$$\text{(i)} \int_{-\infty}^{\infty} x \, dF(x) = 0$$

$$\text{(ii)} \; m < \sigma_F^2 = \int_{-\infty}^{\infty} x^2 \, dF(x)$$

$$\text{(iii)} \int_{-\infty}^{\infty} |x|^3 \, dF(x) \leq M < \infty$$

(iv) F is continuous except possibly at zero.

We remark that m and M are positive constants which are independent of F.

With each $F \in \mathcal{C}$, we may associate a sequence $\{X_n\}_{n=1}^{\infty}$ of i. i. d. random variables defined on some probability space $(\Omega_F, \mathcal{F}_F, P_F)$ such that $X_1 = X_1(F)$ has distribution function F. For this sequence and for any $c > 0$, the stopping time $N_F = N_F(c)$ is defined by

$$N_F = N_F(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma_F\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

If we define $N_F^* = N_F^*(c)$ by

$$N_F^* = N_F^*(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| \geq c\sigma_F\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then, since the only possible atom of the distribution function F

occurs at zero, it follows that $N_F = N_F^*$ a. s. $[P_F]$. Thus, in particular, $P_F(N_F > n) = P_F(N_F^* > n)$. Hence, Theorem 1.1 may be applied to the sequence $\{X_n\}$ to obtain the fact that either (a) for some n, $P_F(N_F > n) = 0$, or (b) $P_F(N_F > n) \sim \alpha_F n^{-\beta(c)}$ as $n \to \infty$ where $\beta(c)$ is the same for all $F \in \mathcal{C}$.

Henceforth, we will be interested only in the case $c \geq 1$. For such c, Chow, Robbins, and Teicher [4] proved a theorem which implies that $EN_F = +\infty$. Thus, (a) is excluded for $c \geq 1$. We now seek to show that (b) holds uniformly for all $F \in \mathcal{C}$; i.e., that $\lim_{n\to\infty} n^{\beta(c)} P_F(N_F > n) = \alpha_F$ uniformly for $F \in \mathcal{C}$.

Just as above, we fix an arbitrary $t_0 > 0$ and for $t > 0$, we define $m = [e^{2t}]$ and $n = [e^{2(t+t_0)}]$. For each $F \in \mathcal{C}$, we define $P_F(\gamma, t)$, $Q_F(\gamma, \eta, t, t_0)$, $P_F(I_c, t)$, and $\hat{P}_F(I_c, s)$ as above; e.g., $P_F(\gamma, t) = P_F[S_m(F) < \gamma\sigma_F\sqrt{m};\ N_F > m]$ where $S_m(F) = \sum_1^m X_i(F)$. First, we prove that (1) holds uniformly for $F \in \mathcal{C}$.

Lemma 4.5. There exists a constant D such that for all $F \in \mathcal{C}$,

$$\sup_{\gamma,\eta} |Q_F(\gamma, \eta, t, t_0) - Q(\gamma, \eta)| \leq De^{-t/16}.$$

Proof. Prohorov [11, p. 202] obtains an estimate of the rate at which the measure P on $C[0, 1]$ converges to the Wiener measure, where P is the measure generated by the functions in $C[0, 1]$ which are formed by linearly interpolating between the values of N independent random variables. For the case considered by Breiman in his Proposition 1, the upper bound ρ_N for the distance between these two measures is of the form $\rho_N = kN^{-1/8}(\log N)^2$ where $N = n - m$ and k is a constant, independent of N but dependent on F. A careful analysis of Prohorov's

work shows that k depends on σ_F^2 and $E|X_1(F)|^3$. However, since for all $F \in \mathcal{C}$, $\sigma_F^2 > m$ and $E|X_1(F)|^3 \leq M$, we can replace ρ_N by $\rho_N' = k'N^{-1/8}(\log N + k'')^2$ where k' and k'' are constants independent of N and $F \in \mathcal{C}$. Then, Breiman's proof shows that there is a constant D, now independent of $F \in \mathcal{C}$, such that $\sup_{\gamma,\eta} |Q_F(\gamma, \eta, t, t_0) - Q(\gamma, \eta)| \leq De^{-t/16}$ for all $F \in \mathcal{C}$. |||

Next, we recall that in the first section we showed that if $P_F(N_F > n) > 0$ for all n, then $\hat{P}_F(I_c, s)$ is analytic in $\{s\colon Re\ s > \lambda_0\}$; i.e., the half plane of analyticity is the same for all $F \in \mathcal{C}$. Also, we obtained the asymptotic behavior of $P_F(N_F > n)$ using Proposition 4.2. Now we are interested in not a single function ϕ, but rather a certain family of functions indexed by the distribution functions $F \in \mathcal{C}$. In general, we consider a family of functions, say $\{\phi_w, w \in \mathcal{W}\}$ where $\mathcal{W}$ is an arbitrary index set. In order to show that the conclusion of Proposition 4.2 holds uniformly for all ϕ_w, $w \in \mathcal{W}$, we require that the corresponding α_w and g_w satisfy certain restrictions. More precisely, we prove

Proposition 4.3. Let $\{\phi_w, w \in \mathcal{W}\}$ be a family of nonnegative, nonincreasing functions on $[0, \infty)$ for which $\sup \{\phi_w(0)\colon w \in \mathcal{W}\} < \infty$. Suppose that for every $w \in \mathcal{W}$, the integral $\Phi_w(s) = \int_0^\infty e^{-st}\ \phi_w(t)\ dt$ $(s = x + iy)$ converges for $x > x_0$ where $x_0 < 0$ is independent of w. Suppose also that there are constants α_w and functions g_w such that for each $w \in \mathcal{W}$

$$\lim_{x \to x_0^+} \left(\Phi_w(s) - \frac{\alpha_w}{s - x_0}\right) = g_w(y)$$

uniformly on every finite interval $|y| \leq y_0$. If $\{\alpha_w, w \in \mathcal{W}\}$ are uniformly bounded and if $\{g_w, w \in \mathcal{W}\}$ are uniformly bounded and equicontinuous on every finite interval, then $\lim e^{-x_0 t} \phi_w(t) = \alpha_w$ uniformly for $w \in \mathcal{W}$.

Proof. Let us define the functions

$$a_w(t) = \begin{cases} e^{-x_0 t} \phi_w(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$A_w(t) = \begin{cases} \alpha_w & t \geq 0 \\ 0 & t < 0 , \end{cases}$$

and for any $\lambda > 0$, the functions

$$k_\lambda(x) = 2\lambda \frac{1}{\sqrt{2\pi}} \left(\frac{\sin \lambda x}{\lambda x}\right)^2$$

$$K_\lambda(x) = \begin{cases} 1 - \left|\frac{x}{2\lambda}\right| & |x| \leq 2\lambda \\ 0 & |x| > 2\lambda . \end{cases}$$

It is a well-known result of harmonic analysis that $K_\lambda(x)$ is the Fourier transform of $k_\lambda(x)$.

Following Widder's proof of Ikehara's Theorem as adapted to Proposition 4.2, we arrive at the identity

$$(12) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_\lambda(x-t)[a_w(t) - A_w(t)]dt = \frac{1}{2\pi} \int_{-2\lambda}^{2\lambda} K_\lambda(y) e^{-ixy} g_w(y)\, dy$$

for each $w \in \mathcal{W}$ and each $\lambda > 0$. The hypotheses on the functions g_w imply, by Arzela's Theorem, that $\{g_w, w \in \mathcal{W}\}$ is a compact set. Using this fact together with the Riemann-Lebesgue Theorem, we see that the limit as $x \to \infty$ of the right side of (12) is zero uniformly for $w \in \mathcal{W}$. Further, since the α_w are uniformly bounded,

$$\lim_{x \to \infty} \frac{\alpha w}{\sqrt{2\pi}} \int_0^{\infty} k_\lambda(x-t)\, dt = \alpha_w$$

uniformly for $w \in \mathcal{W}$. Hence, for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_\lambda(x-t)\, a_w(t)\, dt = \alpha_w$$

uniformly for $w \in \mathcal{W}$. Writing $\delta(2x) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin x}{x}\right)^2$, we note that $k_\lambda(x) = 2\lambda\delta(2\lambda x)$ and, so, a change of variables yields

$$\text{(13)} \qquad \lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\, a_w\left(x - \frac{t}{2\lambda}\right) dt = \alpha_w$$

uniformly for $w \in \mathcal{W}$.

Now, following Doetsch's proof of Ikehara's theorem, we let $\varepsilon > 0$ be given. Then, there is an $x_1 = x_1(\varepsilon, \lambda)$ such that for $x \geq x_1$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\, a_w\left(x - \frac{1}{\sqrt{\lambda}} - \frac{t}{2\lambda}\right) dt < \alpha_w + \varepsilon/2.$$

Since the integrand is nonnegative, the region of integration can be reduced to $[-2\sqrt{\lambda}, 2\sqrt{\lambda}]$. We note that $a_w(y)e^{x_0 y} = \phi_w(y)$ is nonincreasing and, therefore, on the interval $[-2\sqrt{\lambda}, 2\sqrt{\lambda}]$, $a_w\left(x - \frac{1}{\sqrt{\lambda}} - \frac{t}{2\lambda}\right) \geq a_w(x) \exp\left(-x_0\left(-\frac{1}{\sqrt{\lambda}} - \frac{t}{2\lambda}\right)\right) \geq a_w(x) \exp\left(x_0 \frac{2}{\sqrt{\lambda}}\right)$ since $x_0 < 0$. Thus, for $x \geq x_1$,

$$a_w(x) < (\alpha_w + \varepsilon/2) \exp\left(-x_0 \frac{2}{\sqrt{\lambda}}\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-2\sqrt{\lambda}}^{2\sqrt{\lambda}} \delta(t)\, dt\right)^{-1}$$

Since $\int_{-\infty}^{\infty} \delta(t)\, dt = \sqrt{2\pi}$, for λ sufficiently large, the right side is less than $\alpha_w + \varepsilon$. Hence, fixing such a λ, it follows that for all $x \geq x_1(\varepsilon, \lambda)$, $a_w(x) < \alpha_w + \varepsilon$.

Also, (13) implies that there exists an $x_2 = x_2(\varepsilon, \lambda)$ such that for $x \geq x_2$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\, a_w\left(x + \frac{1}{\sqrt{\lambda}} - \frac{t}{2\lambda}\right) dt > \alpha_w - \varepsilon/2.$$

For $t < x_1$, we note that $a_w(t) \leq e^{-x_0 t}\, \phi_w(t) \leq e^{-x_0 x_1}\, \phi_w(0)$ and for

$t \geq x_1$, $a_w(t) < \alpha_w + \varepsilon$. By hypothesis, $\phi_w(0)$ and α_w are uniformly bounded and so $a_w(t) \leq K$, a constant which is independent of $w \in \mathcal{W}$. Also, as before, we find that for $t \in [-2\sqrt{\lambda}, 2\sqrt{\lambda}]$, $a_w\left(x + \frac{1}{\sqrt{\lambda}} - \frac{t}{2\lambda}\right) \leq a_w(x) \exp\left(-x_0 \frac{2}{\sqrt{\lambda}}\right)$. Hence, for $x \geq x_2$,

$$a_w(x) > \left[(\alpha_w - \varepsilon/2) - \frac{K}{\sqrt{2\pi}}\left(\int_{-\infty}^{-2\sqrt{\lambda}} \delta(t)\, dt + \int_{2\sqrt{\lambda}}^{\infty} \delta(t)\, dt\right)\right]$$

$$\cdot \exp\left(x_0 \frac{2}{\sqrt{\lambda}}\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-2\sqrt{\lambda}}^{2\sqrt{\lambda}} \delta(t)\, dt\right)^{-1}.$$

Again, for fixed λ sufficiently large and $x \geq x_2(\varepsilon, \lambda)$, $a_w(x) > \alpha_w - \varepsilon$. Therefore, since x_1 and x_2 are independent of $w \in \mathcal{W}$, we conclude that $\lim_{x\to\infty} a_w(x) = \alpha_w$ uniformly for $w \in \mathcal{W}$. |||

We now seek to apply this proposition with $\mathcal{W} = \mathcal{C}$ ($w = F \in \mathcal{C}$) and $\phi_F(t) = P_F(I_c, t)$. Clearly, $\sup \{\phi_F(0), F \in \mathcal{C}\} \leq 1 < \infty$ and we have already remarked that the Laplace integral $\Phi_F(s) = \hat{P}_F(I_c, s)$ converges for s with $Re\ s > \lambda_0 = -2\beta(c)$. As noted previously, $\hat{P}_F(I_c, \cdot)$ is independent of the parameter t_0 and it follows from (10) applied to each $F \in \mathcal{C}$ that $\hat{P}_F(I_c, \cdot)$ is analytic in $\{s: Re\ s > \lambda_0 - \delta\}$ for some $\delta > 0$ except for a first order pole at $s = \lambda_0$. With this in mind, we prove several lemmas which will enable us to apply the previous proposition.

Lemma 4.6. For any fixed $F \in \mathcal{C}$,

$$\lim_{x\to\lambda_0^+} \left(\hat{P}_F(I_c, s) - \frac{\alpha_F}{s-\lambda_0}\right) = g_F(y) \qquad (s = x + iy)$$

uniformly on every finite interval $|y| \leq y_0$, where α_F and g_F are given by

$$\alpha_F = t_0^{-1} e^{-\lambda_0 t_0} B_F(\lambda_0) \tag{14}$$

$$(15)\qquad g_F(y) = \begin{cases} e^{-\lambda_0 t_0}\,\dfrac{B_F(\lambda_0+iy)}{e^{iyt_0}-1} - \dfrac{\alpha_F}{iy} + h_F(\lambda_0+iy) & y \neq 0 \\ e^{-\lambda_0 t_0}[t_0^{-1}B_F{}'(\lambda_0) - \tfrac{1}{2}B_F(\lambda_0)] + h_F(\lambda_0) & y = 0 \end{cases}$$

where $B_F(s) = (1, \psi_0)\int_{-c}^{c} \psi_0(\gamma)I_F(d\gamma, s)$ and $h_F(s)$ is the function given in (11) applied to $F \in \mathcal{C}$.

Proof. We denote the residue of $\hat{P}_F(I_c, \cdot)$ at the pole $s = \lambda_0$ by α_F. If we write $G_F(s) = \hat{P}_F(I_c, s) - \alpha_F/(s-\lambda_0)$, then G_F is analytic in $\{s\colon Re\ s > \lambda_0 - \delta\}$. Hence, for $g_F(y) = G_F(\lambda_0+iy)$, the uniform convergence on finite intervals is an immediate consequence of the analyticity of G_F. Straightforward computations show that α_F and g_F are given explicitly by (14) and (15) respectively. We remark that the analyticity of G_F implies that g_F is well-defined at $y = 2n\pi/t_0$. |||

Lemma 4.7. For each $\gamma \in [-c, c]$, for each $k \geq 0$, and for $s = x + iy$, $0 \geq x > \lambda_0 - 1/16$, $|I_F{}^{(k)}(\gamma, s)| \leq 2^k t_0{}^{k+1} + D|\hat{P}_F{}^{(k)}(I_c, x + 1/16)|$ where D is a constant which is independent of $F \in \mathcal{C}$.

Proof. We recall that $I_F(\gamma, \cdot)$ is analytic in $\{s\colon Re\ s > \lambda_0 - 1/16\}$. It is given explicitly in (5) (when applied to each $F \in \mathcal{C}$) and a simple calculation shows

$$I_F{}^{(k)}(\gamma, s) = e^{st_0}\sum_0^k \binom{k}{j} t_0{}^j \int_0^{t_0} (-t)^{k-j}\, e^{-st}\, P_F(\gamma, t)\, dt$$

$$+ \int_0^\infty \int_{-c}^{c} [Q_F(\gamma, \eta, t, t_0) - Q(\gamma, \eta)](-t)^k e^{-st} P_F(d\eta, t)dt.$$

Thus, using Lemma 4.5, we see

$$|I_F{}^{(k)}(\gamma, s)| \leq 2^k t_0{}^{k+1} + \int_0^\infty t^k e^{-(x + 1/16)t} P_F(I_c, t)dt;$$

the integral above is exactly $|\hat{P}_F^{(k)}(I_c, x + 1/16)|$ as desired. |||

Lemma 4.8. For each $k \geq 0$ and each $s \in \{s: Re\ s > \lambda_0\}$ with $s = x + iy$, $|\hat{P}_F^{(k)}(I_c, s)| \leq M_k(x)$, a real-valued function which is independent of F.

Proof. First of all, if $x = Re\ s > 0$, then $|\hat{P}_F^{(k)}(I_c, s)| \leq \int_0^\infty t^k e^{-xt} P_F(I_c, t)\, dt \leq k!\, x^{-k-1}$. This is the desired bound for $Re\ s > 0$.

Now, for $Re\ s \leq 0$, we use the explicit expression for $\hat{P}_F(I_c, s)$ obtained from (10) and (11) applied to $F \in \mathcal{C}$. First, we obtain the bound for $k = 0$. We note that $|e^{st_0} - e^{\lambda_j t_0}| \geq e^{xt_0} - e^{\lambda_0 t_0}$ for all $j \geq 0$ and $x = Re\ s > \lambda_0$. Also, integrating by parts and recalling that $\psi_j{}' = O(j)$, we obtain, using the previous lemma, that $\left|\int_{-c}^{c} \psi_j(\gamma) I_F(d\gamma, s)\right| \leq K[t_0 + D\hat{P}_F(I_c, x + 1/16)]j$ where K is a constant depending only on c. As seen previously, $|(1, \psi_j)| \leq \|1\|_2 \|\psi_j\|_2 = \|1\|_2$ and $\sum j e^{\lambda_j t_0} < \infty$. Thus,

$$|\hat{P}_F(I_c, s)| \leq \left(K'(e^{xt_0} - e^{\lambda_0 t_0})^{-1} + e^{-xt_0}[K'' + K'''(e^{xt_0} - e^{\lambda_0 t_0})^{-1}]\right) \cdot [t_0 + \hat{DP}_F(I_c, x + 1/16)],$$

the constants being independent of F. If $x + 1/16 > 0$, then $\hat{P}_F(I_c, x + 1/16) \leq (x + 1/16)^{-1}$ and it follows that $\hat{P}_F(I_c, s)$ is bounded independently of F. Otherwise, we repeat the same process applied to $\hat{P}_F(I_c, x + 1/16)$ to obtain

$$|\hat{P}_F(I_c, s)| \leq K_1(x) + K_2(x)\, \hat{P}_F(I_c, x + 2/16).$$

Continuing in this manner, we obtain for some $m = m(x) > 0$,

$$|\hat{P}_F(I_c, s)| \leq J(x) + K(x)\, \hat{P}_F(I_c, x + m/16)$$

where $x + m/16 > 0$ and J and K are functions of x which do not depend on F. Hence, $|\hat{P}_F(I_c, s)| \leq J(x) + K(x)(x + m/16)^{-1}$, a function which is independent of F.

To obtain the desired bounds for higher derivatives, we differentiate $\hat{P}_F(I_c, s)$ [as given by (10) and (11)] and use essentially the same method as for $k = 0$. We note that the infinite series of (11) is uniformly convergent on compact subsets of $\{s : Re\ s > \lambda_0\}$ and, therefore, can be differentiated term by term. |||

Lemma 4.9. $\sup\{\alpha_F : F \in \mathcal{C}\} < \infty$.

Proof. Since $\alpha_F = t_0^{-1} e^{-\lambda_0 t_0} (1, \psi_0) \int_{-c}^{c} \psi_0(\gamma) I_F(d\gamma, \lambda_0)$, it follows as before from Lemma 4.7 that $\alpha_F \leq K(\lambda_0)[t_0 + DP_F(I_c, \lambda_0 + 1/16)]$. Hence, by Lemma 4.8, $\alpha_F \leq K(\lambda_0)[t_0 + DM_0(\lambda_0 + 1/16)]$; since this bound is independent of F, the desired conclusion follows. |||

Lemma 4.10. For any $y_0 > 0$, the family $\{g_F, F \in \mathcal{C}\}$ is uniformly bounded and equicontinuous on $[-y_0, y_0]$.

Proof. Let $y_0 > 0$ be fixed. We note that since $\hat{P}_F(I_c, \cdot)$ is independent of the parameter t_0, so is $\alpha_F = \lim_{s \to \lambda_0} (s-\lambda_0)\hat{P}_F(I_c, s)$. Hence, since $g_F(y) = G_F(\lambda_0+iy)$ where $G_F(s) = \hat{P}_F(I_c, s) - \alpha_F/(s-\lambda_0)$, g_F is also independent of t_0. Therefore, since t_0 was arbitrary, it may be chosen so that $2\pi/t_0 > y_0$ without affecting the g_F's. Now, g_F is given explicitly in (15); we write $g_F(y) = u_F(y) + h_F(\lambda_0+iy)$ where

$$u_F(y) = e^{-\lambda_0 t_0} \left(\frac{B_F(\lambda_0+iy)-B_F(\lambda_0)}{e^{iyt_0}-1} + B_F(\lambda_0) \left(\frac{1}{e^{iyt_0}-1} - \frac{1}{iyt_0} \right) \right)$$

and h_F is as before.

First, we show that $g_F(y)$ is bounded independently of $F \in \mathcal{C}$ and $y \in [-y_0, y_0]$. We write the first term of u_F as $([B_F(\lambda_0+iy)-B_F(\lambda_0)]/y) \cdot [(e^{iyt_0}-1)/y]^{-1}$. The second term of this product is continuous [when it is extended to equal $(it_0)^{-1}$ at $y = 0$] on $|y| \leq y_0 < 2\pi/t_0$ and, hence, is bounded. If we regard $B_F(\lambda_0+iy)$ as a function from $[-y_0, y_0]$ to $\mathbb{C}$, we see by the mean value theorem that for $y \neq 0$, $|[B_F(\lambda_0+iy)-B_F(\lambda_0)]/y| \leq |B_F'(\lambda_0+iy^*)|$, $0 < |y^*| < |y|$, while for $y = 0$, the first term is defined to be $iB_F'(\lambda_0)$. Thus, using Lemmas 4.7 and 4.8, we see that the modulus of the first term is bounded independently of F and y. In a similar manner, the second term of u_F is uniformly bounded and so, therefore, is u_F. The fact that $h_F(\lambda_0+iy)$ is uniformly bounded follows by using again Lemmas 4.7 and 4.8. Thus, g_F is bounded independently of F and y.

To show that $\{g_F, F \in \mathcal{C}\}$ is equicontinuous on $[-y_0, y_0]$, it suffices to show that the derivatives g_F' are uniformly bounded. Computing these derivatives, we find, using the mean value theorem, that they may be bounded by expressions involving $I_F^{(k)}(\gamma, \lambda_0+iy)$, $k = 0, 1, 2$. Hence, it follows from Lemmas 4.7 and 4.8 that g_F' are bounded independently of F and y. Therefore, $\{g_F, F \in \mathcal{C}\}$ is an equicontinuous family on $[-y_0, y_0]$. |||

Theorem 4.1. Let $\mathcal{C}$ be a class consisting of distribution functions which satisfy properties (i)-(iv). For $F \in \mathcal{C}$, let P_F be the probability measure induced on the space $(\Omega_F, \mathcal{F}_F)$ by F and for $c > 0$, let $N_F(c)$ be the stopping time defined above. Then, $\lim_{n\to\infty} n^{\beta(c)} P_F[N_F(c) > n] = \alpha_F$ uniformly for $F \in \mathcal{C}$ where $\beta(c)$ is a constant which is independent of F and α_F is the constant defined by (14).

Proof. It follows from the previous lemmas that we may apply Proposition 4.3 to the family of functions $\{P_F(I_c, \cdot), F \in \mathcal{C}\}$ to obtain

$$(16) \qquad \lim_{t\to\infty} e^{2\beta(c)t} P_F(I_c, t) = \alpha_F$$

uniformly for $F \in \mathcal{C}$. We noted previously that $P_F(I_c, t) = P_F(N_F(c) > [e^{2t}])$. Hence, (16) is clearly equivalent to the desired conclusion. |||

As mentioned at the beginning of this chapter, we actually need to apply this uniform result to a collection of distribution functions indexed by the parameter μ. For each real μ in some neighborhood of zero, say $|\mu| < \mu_0$, let $\{Y_n(\mu)\}_{n=1}^{\infty}$ be a sequence of i. i. d. random variables defined on the probability space $(\Omega_\mu, \mathcal{F}_\mu, P_\mu)$ with common distribution function G_μ which satisfies properties (i)-(iv). We will write $\sigma(\mu)^2$ for $\sigma_{G_\mu}{}^2$. Further, we assume that as $\mu \to 0$, G_μ converges weakly to G_0 (denoted by $G_\mu \xrightarrow{w} G_0$); i.e., $Y_1(\mu)$ converges in distribution to $Y_1(0)$. Writing σ^2 for $\sigma(0)^2$, we note that $G_\mu \xrightarrow{w} G_0$ and $E|Y_1(\mu)|^3 \leq M$ together imply that $\lim_{\mu\to 0} \sigma(\mu)^2 = \sigma^2$. Examples of collections of sequences satisfying these properties will arise in our subsequent work.

Now, for each μ and for every $c > 0$, we define the stopping time $M_\mu = M_\mu(c)$ on the space $(\Omega_\mu, \mathcal{F}_\mu, P_\mu)$ by

$$M_\mu = M_\mu(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n Y_i(\mu)\right| > c\sigma(\mu)\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

We note that M_μ corresponds to N_{G_μ} defined previously. We prove one additional lemma before the final theorem of this chapter.

Lemma 4.11. $\lim_{\mu\to 0} P_\mu(M_\mu > n) = P_0(M_0 > n)$ for all $n \geq 1$.

Proof. Let $n \geq 1$ be a fixed integer. Defining the sets $B_\mu \subset R^n$ by

$B_\mu = \{(y_1,\ldots,y_n): |\sum_1^k y_i| \leq c\sigma(\mu)\sqrt{k},\ k = 1,\ldots,n\}$, we can write

$$P_\mu(M_\mu > n) = \int \cdots \int_{B_\mu} dG_\mu(y_n) \ldots dG_\mu(y_1).$$

Since G_0 is continuous except possibly at zero, the distribution of

$\sum_1^k Y_i(0)$ can have an atom only possibly at zero. Thus,

$$\int \cdots \int_{\partial B_0} dG_0(y_n) \ldots dG_0(y_1) = 0$$

where ∂B_0 denotes the boundary of B_0. Since $G_\mu \xrightarrow{w} G_0$, it follows that

$$\lim_{\mu\to 0} \int \cdots \int_{B_0} dG_\mu(y_n) \ldots dG_\mu(y_1) = \int \cdots \int_{B_0} dG_0(y_n) \ldots dG_0(y_1)$$

$$= P_0(M_0 > n).$$

Therefore, it remains to show that

$$\lim_{\mu\to 0} \left| \int \cdots \int_{B_\mu} dG_\mu(y_n) \ldots dG_\mu(y_1) - \int \cdots \int_{B_0} dG_\mu(y_n) \ldots dG_\mu(y_1) \right| = 0.$$

Denoting this absolute value by $\Delta(\mu)$ and writing $S_k(\mu) = \sum_1^k Y_i(\mu)$, we see that

$$\Delta(\mu) \leq P_\mu[|S_k(\mu)| \leq c\sigma(\mu)\sqrt{k},\ 1 \leq k \leq n;\ |S_j(\mu)| > c\sigma\sqrt{j} \text{ for some } j \leq n]$$
$$+ P_\mu[|S_k(\mu)| \leq c\sigma\sqrt{k},\ 1 \leq k \leq n;\ |S_j(\mu)| > c\sigma(\mu)\sqrt{j} \text{ for some } j \leq n],$$

and, therefore, that

$$\Delta(\mu) \le \sum_{1}^{n} P_\mu[c\sigma\sqrt{k} < |S_k(\mu)| \le c\sigma(\mu)\sqrt{k}] \tag{17}$$
$$+ \sum_{1}^{n} P_\mu[c\sigma(\mu)\sqrt{k} < |S_k(\mu)| \le c\sigma\sqrt{k}].$$

Now, for any $\varepsilon > 0$, we may choose μ sufficiently small so that $|\sigma(\mu) - \sigma| < \varepsilon$. For such μ, (17) may be written

$$\Delta(\mu) \le \sum_{1}^{n} P_\mu[c\sigma\sqrt{k} < |S_k(\mu)| \le c(\sigma+\varepsilon)\sqrt{k}] \tag{18}$$
$$+ \sum_{1}^{n} P_\mu[c(\sigma-\varepsilon)\sqrt{k} < |S_k(\mu)| \le c\sigma\sqrt{k}].$$

Recall that $G_\mu \xrightarrow{w} G_0$ as $\mu \to 0$ and that G_0 is continuous except possibly at zero. If we first let $\mu \to 0$ and then let $\varepsilon \to 0$ in (18), it follows that $\lim_{\mu\to 0} \Delta(\mu) = 0$ as desired. |||

Before proceeding, we should remark that in order to insure the convergence asserted in this lemma for every $n \ge 1$ and every $c > 0$, something like the hypothesis that G_0 is continuous is necessary. For example, consider the sequences of i. i. d. random variables $\{X_n(\mu)\}_{n=1}^{\infty}$, $|\mu| < 1$, where $Prob[X_1(\mu) = 1+\mu] = \frac{1-\mu}{2}$ and $Prob[X_1(\mu) = -1+\mu] = \frac{1+\mu}{2}$. Then, $EX_1(\mu) = 0$, $EX_1(\mu)^2 = \sigma(\mu)^2 = 1-\mu^2$, and $E|X_1(\mu)|^3 = 1-\mu^4 \le 1$. Also, if F_μ is the distribution function of $X_1(\mu)$, then it is clear that $F_\mu \xrightarrow{w} F_0$ as $\mu \to 0$; however, F_0 is not continuous at ± 1. If the stopping time $M_\mu'(c)$ is defined by

$$M_\mu'(c) = \text{least } n \ge 1 \text{ such that } \left|\sum_{1}^{n} X_i(\mu)\right| > c\sigma(\mu)\sqrt{n}$$

or $+\infty$ if no such n exists,

then, for $c = 1$,

$$P_\mu[M_\mu'(1) > 1] = \begin{cases} \frac{1+\mu}{2} & \mu > 0 \\ 1 & \mu = 0 \\ \frac{1-\mu}{2} & \mu < 0 . \end{cases}$$

Thus, $\lim_{\mu\to 0} P_\mu[M_\mu'(1) > 1] = \frac{1}{2} \neq P_0[M_0'(1) > 1]$.

Finally, we prove the theorem which we will need in our subsequent work.

Theorem 4.2. For each real μ such that $|\mu| < \mu_0$, let $\{Y_n(\mu)\}_{n=1}^{\infty}$ be a sequence of i. i. d. random variables defined on a probability space $(\Omega_\mu, \mathfrak{F}_\mu, P_\mu)$ with common distribution function G_μ such that G_μ satisfies properties (i)-(iv) and $G_\mu \xrightarrow{w} G_0$ as $\mu \to 0$. If for any positive c,

$$M_\mu(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n Y_i(\mu)\right| > c\sigma(\mu)\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then, for each $c \geq 1$, there exists a positive integer n_0 and positive constants a_1 and a_2, all independent of μ, such that

$$a_1 n^{-\beta(c)} \leq P_\mu[M_\mu(c) > n] \leq a_2 n^{-\beta(c)}$$

for all $n \geq n_0$ and all μ sufficiently small.

Proof. Let us consider the class of distribution functions $\{G_\mu, |\mu| < \mu_0\}$. We noted that $M_\mu(c)$ corresponds to $N_{G_\mu}(c)$ and, therefore, by Theorem 4.1,

$$\lim_{n\to\infty} n^{\beta(c)} P_\mu[M_\mu(c) > n] = \alpha_\mu \tag{19}$$

uniformly for $|\mu| < \mu_0$ where we have written α_μ for the constant α_{G_μ}. If we take the limit of both sides of (19) as $\mu \to 0$, the uniform

convergence justifies interchanging the limits and, so, by Lemma 4.11, $\lim_{\mu\to 0} \alpha_\mu = \alpha_0$.

Let $\varepsilon > 0$ be given such that $\alpha_0 - \varepsilon > 0$. Then, for all μ sufficiently small, $\alpha_0 - \varepsilon/2 < \alpha_\mu < \alpha_0 + \varepsilon/2$. Also, there exists an n_0, independent of μ, such that for all $n \geq n_0$ and all μ with $|\mu| \leq \mu_0$,

$$\alpha_\mu - \varepsilon/2 \leq n^{\beta(c)} P_\mu[M_\mu(c) > n] \leq \alpha_\mu + \varepsilon/2.$$

Hence, for all μ sufficiently small and $n \geq n_0$,

$$(20) \qquad \alpha_0 - \varepsilon \leq n^{\beta(c)} P_\mu[M_\mu(c) > n] \leq \alpha_0 + \varepsilon.$$

Taking $a_1 = \alpha_0 - \varepsilon$ and $a_2 = \alpha_0 + \varepsilon$, (20) is exactly the desired result. |||

CHAPTER 5

RANDOM VARIABLES HAVING MOMENT GENERATING FUNCTIONS AND CONTINUOUS DISTRIBUTIONS

We have just obtained uniform bounds on the tail probabilities $P(M_\mu > n)$ for a stopping time M_μ associated with a certain collection of distribution functions indexed by a parameter μ. In that case, each distribution was assumed to have a zero mean. What we wish to consider now are certain sequences of i. i. d. random variables of the form $\{X_n\}$ where $X_n = \bar{X}_n + \mu$, $E\bar{X}_1 = 0$. In this chapter we will determine the behavior of EN_μ as $\mu \to 0$ where N_μ is a stopping time defined on the sequence $\{X_n\}$.

To do this, we write $P(N_\mu > n) = \int_B dP_\mu^n$ where P_μ^n denotes the joint distribution of $X_1, \ldots, X_n$. In order to apply the results of Chapter 4, we will define a new distribution function G_μ and an associated sequence $\{Y_n\}$ such that G_μ has zero mean and is absolutely continuous with respect to F_μ, the distribution function of X_1. In fact, if we assume that the moment generating function $\phi_\mu(t)$ of F_μ exists, then we can write

$$(1) \qquad dP_\mu^n(x) = [\phi_\mu(-h)]^n \exp\left(h \sum_1^n x_i\right) dQ_\mu^n(x)$$

where $x = (x_1, \ldots, x_n)$, Q_μ^n is the joint distribution of $Y_1, \ldots, Y_n$, and h is such that $\phi_\mu'(-h) = 0$. The assumption of the existence of a moment generating function is rather strong; however, it is this

condition that guarantees that the density in (1) is such that it can be bounded on the set B in an appropriate manner. The desired result then follows using the techniques employed in the proof of Theorem 3.1 for the normally distributed case.

We begin by investigating a certain collection of distribution functions. For each real μ in some neighborhood of zero, say $|\mu| < \mu_1'$, let F_μ be a probability distribution function satisfying the following properties:

$$\text{(i')} \int_{-\infty}^{\infty} x \, dF_\mu(x) = m(\mu) \neq 0 \text{ for } \mu \neq 0, \; m(0) = 0$$

$$\text{(ii')} \int_{-\infty}^{\infty} [x-m(\mu)]^2 \, dF_\mu(x) = \sigma(\mu)^2 < \infty, \; \sigma(\mu)^2 > 0$$

$$\text{(iii')} \; \phi_\mu(t) = \int_{-\infty}^{\infty} e^{tx} \, dF_\mu(x) \leq M_1 < \infty \text{ for } |t| \leq 2r$$

(iv') F_μ is continuous.

In (iii'), M_1 and r are positive constants which are independent of μ. Further, we shall assume that $F_\mu \xrightarrow{w} F_0$ as $\mu \to 0$. An example of such a collection is $\{F_\mu\}$ where $F_\mu(x) = F(x-\mu)$ and F is a continuous distribution function having zero mean and a finite moment generating function in a neighborhood of zero. This is exactly the case we wish to consider; however, with the more general hypotheses, we are able to obtain a result (Theorem 5.1) that will be useful not only in the special case but also later in Chapter 6 as well.

We will assume, without loss of generality, that the moment generating functions ϕ_μ are finite in an open interval containing $[-2r, 2r]$ so that $\phi_\mu \in C^\infty[-2r, 2r]$. We recall that $\phi_\mu^{(k)}(t) =$

$\int_{-\infty}^{\infty} x^k e^{tx} dF_\mu(x)$. An important consequence of the boundedness assumption in (iii') is given in

Lemma 5.1. If $\phi_\mu(t) \leq M_1$ for $|t| \leq 2r$ and $|\mu| < \mu_1'$, then for each $k \geq 0$, the family $\{\phi_\mu{}^{(k)}, |\mu| < \mu_1'\}$ is equicontinuous on $[-r, r]$.

Proof. By the mean value theorem, we can write $\phi_\mu{}^{(k-1)}(t_2) - \phi_\mu{}^{(k-1)}(t_1) = \phi_\mu{}^{(k)}(t^*)(t_2-t_1)$, $k \geq 1$, where $-2r \leq t_1 < t^* < t_2 \leq 2r$. Hence, for each $k \geq 1$, in order to prove that $\{\phi_\mu{}^{(k-1)}, |\mu| < \mu_1'\}$ is equicontinuous, it suffices to show that $\{\phi_\mu{}^{(k)}, |\mu| < \mu_1'\}$ is uniformly bounded on $[-r, r]$.

We note that $|\phi_\mu{}^{(k)}(t)| \leq \int_{-\infty}^{\infty} |x|^k e^{tx} dF_\mu(x)$. Also, the function $|x|^k e^{-r|x|}$ is bounded on the real line and, so, for some constant K, $|x|^k \leq Ke^{r|x|}$. Thus, $|\phi_\mu{}^{(k)}(t)| \leq K[\phi_\mu(t-r) + \phi_\mu(t+r)] \leq 2KM_1$ for $|t| \leq r$. This is the desired uniform bound. |||

We also make note of several consequences of the assumption that $F_\mu \xrightarrow{w} F_0$. First, just as in the previous lemma, we can show that $\int_{-\infty}^{\infty} |x|^3 dF_\mu(x) \leq 2KM_1$ for some constant K. Hence, it follows that $m(\mu) \to m(0) = 0$ and $\sigma(\mu)^2 \to \sigma(0)^2$ as $\mu \to 0$. Further, we may write $\phi_\mu(t) = \int_0^{\infty} e^{tx} dF_\mu(x) - \int_0^{\infty} e^{-tx} dF_\mu(-x)$; i.e., ϕ_μ may be regarded as the difference of two Laplace transforms. As such, since ϕ_μ is bounded for $|t| \leq 2r$, it follows from the continuity theorem for Laplace transforms [9, p. 433] that $F_\mu \xrightarrow{w} F_0$ implies that $\lim_{\mu\to 0} \phi_\mu(t) = \phi_0(t)$ for each t, $|t| \leq 2r$.

Now, for each μ, $|\mu| < \mu_1'$, there exists a probability space

$(\Omega_\mu, \mathfrak{F}_\mu, P_\mu)$ and a sequence of i. i. d. random variables $\{X_n(\mu)\}_{n=1}^{\infty}$ with common distribution function F_μ. We define the stopping time $N_\mu = N_\mu(c)$ on the space $(\Omega_\mu, \mathfrak{F}_\mu, P_\mu)$ for all positive c by

$$N_\mu = N_\mu(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i(\mu)\right| > c\sigma(\mu)\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

We wish to obtain bounds for $P_\mu(N_\mu > n)$ for large n which hold uniformly for all μ sufficiently small. To do this, we wish to define a collection of distribution functions, indexed by the parameter μ, which satisfy the conditions of Theorem 4.2.

We note that $\phi_0(0) = 1$, $\phi_0'(0) = m(0) = 0$, and $\phi_0''(0) = \sigma(0)^2 > 0$ [implying $\phi_0''(t) > 0$ for all t]; hence, it follows that $\phi_0(t) > 1$ for all $t \neq 0$, $|t| \leq r$. Thus, since $\lim_{\mu\to 0} \phi_\mu(t) = \phi_0(t)$ for $|t| \leq 2r$, we may choose $|\mu|$ sufficiently small so that $\phi_\mu(-r) \geq 1$ and $\phi_\mu(r) \geq 1$. Since $\phi_\mu(0) = 1$ and $\phi_\mu''(t) > 0$, we conclude that for all $|\mu|$ sufficiently small, there is a unique number $h(\mu)$, $|h(\mu)| \leq r$, such that $\phi_\mu'[-h(\mu)] = 0$; of course, $h(0) = 0$. We will write h for $h(\mu)$ when no confusion will arise. Henceforth, we will only consider μ in a neighborhood of zero for which $h(\mu)$ as defined above exists, say $|\mu| < \mu_1$ where, without loss of generality, $\mu_1 \leq \mu_1'$.

We now make use of a transformation due to Esscher [7] to define a new function G_μ; i.e., for $|\mu| < \mu_1$,

$$G_\mu(y) = \frac{1}{\phi_\mu(-h)} \int_{-\infty}^{y} e^{-hx}\, dF_\mu(x).$$

It is easy to see that G_μ is a continuous probability distribution function. For each μ, $|\mu| < \mu_1$, there exists a probability space

$(\Omega_\mu', \mathcal{F}_\mu', P_\mu')$ and a sequence of i. i. d. random variables $\{Y_n(\mu)\}_{n=1}^\infty$ with common distribution function G_μ. We note, in particular, that $G_0 \equiv F_0$; hence, we will assume $(\Omega_0', \mathcal{F}_0', P_0')$ is the space $(\Omega_0, \mathcal{F}_0, P_0)$ and that $Y_n(0) \equiv X_n(0)$. We now show that the collection $\{G_\mu, |\mu| < \mu_1\}$ and the associated sequences $\{\{Y_n(\mu)\}_{n=1}^\infty, |\mu| < \mu_1\}$ satisfy the conditions of Theorem 4.2.

We denote $EY_1(\mu)^2$ by $\tau(\mu)^2$ and write τ^2 when no confusion will arise. Also, we write σ^2 for $\sigma(0)^2 = \tau(0)^2$. For future reference, we let $\nu(\mu) = -\log \phi_\mu(-h)$. The next lemma gives the limiting behavior of $h(\mu)$, $\tau(\mu)^2$, and $\nu(\mu)$.

Lemma 5.2. As $\mu \to 0$, $h(\mu) \sim m(\mu)/\sigma^2$, $\tau(\mu)^2 \to \sigma^2$, and $\nu(\mu) \sim m(\mu)^2/2\sigma^2$.

Proof. First, we show that $\lim_{\mu\to 0} h(\mu) = 0$. Let $\varepsilon > 0$ be given. Then, since for each fixed t, $|t| \leq 2r$, $\lim_{\mu\to 0} \phi_\mu(t) = \phi_0(t)$ and $\phi_0(t) > 1$ for $t \neq 0$, there exists a $\delta > 0$ such that for $|\mu| < \delta$, $\phi_\mu(-\varepsilon) \geq 1$ and $\phi_\mu(\varepsilon) \geq 1$. Since $\phi_\mu(0) = 1$ and the minimum of ϕ_μ occurs at $-h(\mu)$ and ϕ' is strictly increasing, it follows that $-\varepsilon < -h(\mu) < \varepsilon$ for $|\mu| < \delta$; i.e., $\lim_{\mu\to 0} h(\mu) = 0$.

To obtain the asymptotic behavior of $h(\mu)$, we express $\phi_\mu'(t)$ as the power series

$$\phi_\mu'(t) = m(\mu) + [\sigma(\mu)^2 + m(\mu)^2]t + \sum_2^\infty \frac{t^n}{n!} \int_{-\infty}^{\infty} x^{n+1}\, dF_\mu(x).$$

By the definition of $h(\mu)$, $\phi_\mu'[-h(\mu)] = 0$ and, therefore,

$$m(\mu) = h\cdot[\sigma(\mu)^2 + m(\mu)^2] - h^2 \sum_2^\infty \frac{(-h)^{n-2}}{n!} \int_{-\infty}^{\infty} x^{n+1}\, dF_\mu(x).$$

The series of higher-order terms above has an absolute value not

greater than the integral $\int_{-\infty}^{\infty} |x|^3 e^{|hx|}\, dF_\mu(x)$ which, as in Lemma 5.1, is bounded independently of μ. Hence, $\lim_{\mu\to 0} m(\mu)/h(\mu) = \sigma^2$ or $h(\mu) \sim m(\mu)/\sigma^2$ as $\mu \to 0$.

Next, $\tau(\mu)^2 = EY_1(\mu)^2 = \int_{-\infty}^{\infty} y^2\, dG_\mu(y) = \dfrac{1}{\phi_\mu(-h)} \int_{-\infty}^{\infty} y^2 e^{-hy}\, dF_\mu(y)$.

Thus, $\tau(\mu)^2 = \phi_\mu''[-h(\mu)]/\phi_\mu[-h(\mu)]$. Since both $\{\phi_\mu, |\mu| < \mu_1\}$ and $\{\phi_\mu'', |\mu| < \mu_1\}$ are equicontinuous families, it follows easily from the fact that $h(\mu) \to 0$ that $\phi_\mu''(-h) \to \phi_0''(0) = \sigma^2$ and $\phi_\mu(-h) \to \phi_0(0) = 1$; hence, $\lim_{\mu\to 0} \tau(\mu)^2 = \sigma^2$.

Finally, we consider $\nu(\mu) = -\log \phi_\mu(-h)$. We write

(2) $$\frac{-\log \phi_\mu(-h)}{m(\mu)^2} = \frac{\log \phi_\mu(-h)}{\phi_\mu(-h)-1} \quad \frac{\phi_\mu(-h)-1}{-h^2} \quad \frac{h^2}{m(\mu)^2} .$$

Also, we express $\phi_\mu(-h)$ as the power series

$$\phi_\mu(-h) = 1 - hm(\mu) + \frac{[\sigma(\mu)^2+m(\mu)^2]}{2} h^2 + \sum_3^\infty \frac{(-h)^n}{n!} \int_{-\infty}^{\infty} x^n dF_\mu(x).$$

Now, since $\lim_{\mu\to 0} \dfrac{\log(1+x)}{x} = 1$, the limit of the first term on the right in (2) as $\mu \to 0$ is 1. From the power series expansion, the limit of the second term as $\mu \to 0$ is $\sigma^2 - \sigma^2/2 = \sigma^2/2$. By the first part of the proof, $h^2/m(\mu)^2 \to 1/\sigma^4$ as $\mu \to 0$. Thus, $\lim_{\mu\to 0} [-\log \phi_\mu(-h)/m(\mu)^2] = 1/2\sigma^2$, and, so, $\nu(\mu) \sim m(\mu)^2/2\sigma^2$ as desired. |||

Let $\psi_\mu(t)$ be the moment generating function of $Y_1(\mu)$. From the definition of G_μ, it follows that $\psi_\mu(t) = \phi_\mu(t-h)/\phi_\mu(-h)$ and, so, ψ_μ is finite for $|t| \le r$.

Lemma 5.3. $G_\mu \xrightarrow{w} F_0$ as $\mu \to 0$.

Proof. It follows from the equicontinuity of $\{\phi_\mu, |\mu| < \mu_1\}$ that

$\phi_\mu(t-h) \to \phi_0(t)$ as $\mu \to 0$; hence, $\lim_{\mu\to 0} \psi_\mu(t) = \phi_0(t)$ for all t, $|t| \le r$. Therefore, $G_\mu \xrightarrow{w} F_0$. |||

Lemma 5.4. For all μ sufficiently small, say $|\mu| < \mu_0$, (a) $EY_1(\mu) = 0$, (b) $0 < m_0 \le EY_1(\mu)^2 < \infty$, and (c) $E|Y_1(\mu)|^3 \le M$ where m_0 and M are positive constants which are independent of μ.

Proof. For (a), $EY_1(\mu) = \psi_\mu'(0) = \phi_\mu'(-h)/\phi_\mu(-h) = 0$. Next, we recall that $EY_1(\mu)^2 = \tau(\mu)^2 + \sigma^2$ and $0 < \sigma^2 < \infty$; therefore, for any m_0, $0 < m_0 < \sigma^2$, we will have $m_0 \le EY_1(\mu)^2 < \infty$ for $|\mu|$ sufficiently small. Finally, from the definition of G_μ, we see that

$$E|Y_1(\mu)|^3 = \frac{1}{\phi_\mu(-h)} \int_{-\infty}^{\infty} |y|^3 e^{-hy}\, dF_\mu(y). \tag{3}$$

Since $\phi_\mu(-h) \to 1$, it follows that $\phi_\mu(-h) \ge \frac{1}{2}$ for $|\mu|$ sufficiently small. Also, as in Lemma 5.1, the integral in (3) can be shown to be bounded by $2KM_1$ where K is a constant, independent of μ, and M_1 is the bound in (iii'). Thus, for $M = 4KM_1$, $E|Y_1(\mu)|^3 \le M$ for $|\mu|$ sufficiently small. |||

The preceding lemmas show that $\{G_\mu, |\mu| < \mu_0\}$ satisfy the hypotheses of Theorem 4.2. Hence, if we define, for any positive d, the stopping time $M_\mu(d)$ on the space $(\Omega_\mu, \mathfrak{F}_\mu, P_\mu)$ by

$$M_\mu(d) = \text{least } n \ge 1 \text{ such that } \left|\sum_1^n Y_i(\mu)\right| > d\tau(\mu)\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then it follows that for each $d \ge 1$ there exists a positive integer $n_0 = n_0(d)$ and there exist positive constants a_1 and a_2 [$a_1 = a_1(d)$, $a_2 = a_2(d)$], all independent of μ, such that

(4) $$a_1 n^{-\beta(d)} \leq P_\mu[M_\mu(d) > n] \leq a_2 n^{-\beta(d)}$$

for all $n \geq n_0$ and for all μ sufficiently small.

We are now able to obtain the desired bounds for $P_\mu[N_\mu(c) > n]$.

Theorem 5.1. Let $\{F_\mu, |\mu| < \mu_0\}$ be a class of distribution functions such that each function in the class satisfies properties (i')-(iv') and $F_\mu \xrightarrow{w} F_0$ as $\mu \to 0$. For $|\mu| < \mu_0$, let P_μ be the probability measure induced on the space $(\Omega_\mu, \mathcal{F}_\mu)$ by F_μ and for $c > 0$, let $N_\mu(c)$ be the stopping time defined above. Then, for each $c > 1$ and any $\delta > 0$ such that $c-\delta > 1$, there exist positive constants $a_1 = a_1(c)$ and $a_2 = a_2(c)$ and a positive integer n_0, all independent of μ, such that

(5) $$\begin{aligned} a_1[\phi_\mu(-h)]^n e^{-|h|(c-\delta)\tau\sqrt{n}} n^{-\beta(c-\delta)} &\leq P_\mu[N_\mu(c) > n] \\ &\leq a_2[\phi_\mu(-h)]^n e^{|h|(c+\delta)\tau\sqrt{n}} n^{-\beta(c+\delta)} \end{aligned}$$

for all $n \geq n_0$ and all μ sufficiently small where $h = h(\mu)$ and $\tau = \tau(\mu)$ are as defined above. The upper bound also holds for $c = 1$ and any $\delta > 0$.

Proof. Let $c > 1$ be fixed and choose $\delta > 0$ such that $c-\delta > 1$. As noted above, $\lim_{\mu\to 0} \tau(\mu)^2 = \lim_{\mu\to 0} \sigma(\mu)^2 = \sigma^2$ and, therefore, for μ sufficiently small, $|c\sigma(\mu)/\tau(\mu) - c| < \delta$. If we define the sets

$$B = \{(x_1,\ldots,x_n): |\sum_1^k x_i| \leq c\sigma(\mu)\sqrt{k},\ k = 1,\ldots,n\}$$

$$B^+ = \{(x_1,\ldots,x_n): |\sum_1^k x_i| \leq (c+\delta)\tau\sqrt{k},\ k = 1,\ldots,n\}$$

$$B^- = \{(x_1,\ldots,x_n): |\sum_1^k x_i| \leq (c-\delta)\tau\sqrt{k},\ k = 1,\ldots,n\}\ ,$$

then for all μ sufficiently small, $B^- \subset B \subset B^+$.

Now, $P_\mu[N_\mu(c) > n] = \int \cdots_B \int dF_\mu(x_n) \ldots dF_\mu(x_1)$, and, so,

(6) $$P_\mu[N_\mu(c) > n] = [\phi_\mu(-h)]^n \int \cdots_B \int e^{h\sum_1^n x_i} dG_\mu(x_n) \ldots dG_\mu(x_1).$$

Thus, for all μ sufficiently small, $P_\mu[N_\mu(c) > n]$ is bounded above by the right side of (6) with B replaced by B^+, and is bounded below by the right side with B replaced by B^-. Using the bounds for $\sum_1^n x_i$ on B^+ and B^- and the definition of $M_\mu(d)$, we obtain

$$[\phi_\mu(-h)]^n e^{-|h|(c-\delta)\tau\sqrt{n}} P_\mu[M_\mu(c-\delta) > n] \leq P_\mu[N_\mu(c) > n]$$
$$\leq [\phi_\mu(-h)]^n e^{|h|(c+\delta)\tau\sqrt{n}} P_\mu[M_\mu(c+\delta) > n].$$

Combining this and (4), we find that for $a_1 = a_1(c-\delta)$, $a_2 = a_2(c+\delta)$, and $n_0 = \max\{n_0(c-\delta), n_0(c+\delta)\}$, (5) holds for all $n \geq n_0$ and all μ sufficiently small.

We note that if $c = 1$, then $c+\delta > 1$ for any $\delta > 0$. Hence, the right-hand inequality of (4) may be applied for $d = c+\delta$ to obtain the desired upper bound as above. |||

In Chapter 3, we studied a stopping time defined on a sequence of normally distributed random variables with nonzero mean. Now, we wish to find bounds for the expectation of the same stopping time defined on a sequence with a more general distribution. Let us fix a distribution function F that is continuous and possesses a finite moment generating function in a neighborhood of zero. Further, if $\{\tilde{X}_n\}_{n=1}^\infty$ is a sequence of i. i. d. random variables defined on the probability space $(\Omega, \mathfrak{F}, P)$ with distribution function F, we assume that $E\tilde{X}_1 = 0$ and $E\tilde{X}_1^2 = \sigma^2 < \infty$, $\sigma^2 > 0$. For each nonzero μ, we may define the random variable $X_n = X_n(\mu)$ on $(\Omega, \mathfrak{F}, P)$ by setting $X_n = \tilde{X}_n + \mu$.

Then $\{X_n\}$ is a sequence of i. i. d. random variables with common mean μ and we obtain the following result.

Theorem 5.2. Let $\{X_n\}_{n=1}^{\infty}$ be the sequence of i. i. d. random variables defined by $X_n = \tilde{X}_n + \mu$ where $\tilde{X}_n$ has the distribution function F; i.e., $\{X_n\}$ is a sequence with common nonzero mean μ, finite, positive variance σ^2, a continuous distribution function, and a moment generating function Ee^{tX_1} which exists for t in a neighborhood of zero. If for any positive c,

$$N(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then for each $c > 1$ and for any $\varepsilon > 0$, there exist positive constants A and B, independent of μ, such that

$$A|\mu|^{-2[1-\beta(c)]+\varepsilon} \leq EN(c) \leq B|\mu|^{-2[1-\beta(c)]-\varepsilon}$$

for all sufficiently small μ. The upper bound is also valid for $c = 1$.

Proof. For each small $\mu \neq 0$, $X_n = \tilde{X}_n + \mu$ has a distribution function F_μ defined by $F_\mu(x) = F(x-\mu)$; i.e., F_μ is a translate of F. Of course, $\phi_\mu(t) = Ee^{tX_1} = e^{\mu t}\phi_0(t)$; hence, say for $|\mu| < 1$, the functions ϕ_μ are clearly uniformly bounded on any finite interval containing zero on which ϕ_0 is finite. Thus, $\{F_\mu, |\mu| < 1\}$ satisfies properties (i')-(iv') with $m(\mu) = \mu$ and $\sigma(\mu)^2 = \sigma^2$. Also, clearly, $F_\mu \xrightarrow{w} F_0$ as $\mu \to 0$.

Let $c > 1$ and $\varepsilon > 0$ be fixed. Since β is a continuous function of c (see Appendix), there exists a $\delta > 0$ such that $c-\delta > 1$ and, for all d for which $|c-d| \leq \delta$, $|\beta(c)-\beta(d)| < \varepsilon/2$; in particular, $\beta(c-\delta) < \beta(c) + \varepsilon/2$ and $\beta(c+\delta) > \beta(c) - \varepsilon/2$. Fixing such a δ, we may apply

Theorem 5.1.

First, we shall consider the upper bound. Since $c > 1$ and $\delta > 0$ have been fixed, we shall write N, β^+, and β^- for $N(c)$, $\beta(c+\delta)$, and $\beta(c-\delta)$, respectively. Then, from (5),

$$EN \leq n_0 + a_2 \sum_{n_0}^{\infty} [\phi_\mu(-h)]^n e^{|h|(c+\delta)\tau\sqrt{n}} n^{-\beta^+} \tag{7}$$

for all μ sufficiently small.

Recalling that $\nu(\mu) = -\log \phi_\mu(-h)$, we write $[\phi_\mu(-h)]^n = e^{-n\nu(\mu)}$. Let $f_\mu(x) = e^{-x\nu(\mu)} e^{|h|(c+\delta)\tau\sqrt{x}} x^{-\beta^+}$. By Lemma 5.2, $h \sim \mu/\sigma^2$ and $\nu(\mu) \sim \mu^2/2\sigma^2$; thus, $[|h|(c+\delta)\tau]^2/\nu(\mu) \to 2(c+\delta)^2$ as $\mu \to 0$. Also, we note that $\nu(\mu) > 0$ for μ sufficiently small and that $\beta^+ < 1$. Hence, it follows from (7), using Lemma 3.1, that for μ sufficiently small

$$EN \leq n_0 + a_2\left(\int_0^\infty f_\mu(x)dx + K(\mu)\right) \tag{8}$$

where $K(\mu) \to 0$ as $\mu \to 0$.

Performing the change of variable $y = \mu^2 x$, we see that

$$\int_0^\infty f_\mu(x)dx = \mu^{-2(1-\beta^+)} \int_0^\infty \exp\left(-\frac{y\nu(\mu)}{\mu^2} + \frac{|h|\tau}{|\mu|}(c+\delta)\sqrt{y}\right) y^{-\beta^+} dy.$$

For each $y \in [0, \infty)$, the integrand on the right converges to $\exp\left(-\frac{y}{2\sigma^2} + \frac{c+\delta}{\sigma}\sqrt{y}\right) y^{-\beta^+}$ as $\mu \to 0$; for sufficiently small μ, the integrand is dominated by $\exp\left(-\frac{y}{4\sigma^2} + \frac{2(c+\delta)}{\sigma}\sqrt{y}\right) y^{-\beta^+}$. The latter function is integrable, the value of the integral being given explicitly in [10, p. 337]. Hence, by the dominated convergence theorem,

$$\int_0^\infty f_\mu(x)dx \sim \mu^{-2(1-\beta^+)} \int_0^\infty \exp\left(-\frac{y}{2\sigma^2} + \frac{c+\delta}{\sigma}\sqrt{y}\right) y^{-\beta^+} dy\ , \tag{9}$$

the integral on the right having a finite value since $\beta^+ < 1$. Finally, combining (8) and (9), we find that there exists a positive constant B, for example

$$B = 2a_2 \int_0^\infty \exp\left(-\frac{y}{2\sigma^2} + \frac{c+\delta}{\sigma}\sqrt{y}\right) y^{-\beta^+}\,dy ,$$

such that for μ sufficiently small, $EN \le B\mu^{-2(1-\beta^+)}$. As noted above, $\beta^+ = \beta(c+\delta) > \beta(c) - \varepsilon/2$. Thus, $EN \le B|\mu|^{-2[1-\beta(c)]-\varepsilon}$ as desired.

We note in the case $c = 1$, the upper bound of Theorem 5.1 remains valid. Recalling that $\beta(1) = 1$, we are able to use the same methods to obtain $EN(1) \le B\mu^{-\varepsilon}$ for all μ sufficiently small.

For the lower bound in the case that $c > 1$ is fixed as before, we obtain for all μ sufficiently small,

$$\begin{aligned} EN &\ge a_1 \sum_{n_0}^{\infty} e^{-n\nu(\mu)} e^{-|h|(c-\delta)\tau\sqrt{n}}\, n^{-\beta^-} \\ &\ge a_1 \int_{n_0}^{\infty} e^{-x\nu(\mu)} e^{-|h|(c-\delta)\tau\sqrt{x}}\, x^{-\beta^-}\,dx \\ &= a_1 \mu^{-2(1-\beta^-)} \int_{\mu^2 n_0}^{\infty} \exp\left(-\frac{y\nu(\mu)}{\mu^2} - \frac{|h|\tau}{|\mu|}(c-\delta)\sqrt{y}\right) y^{-\beta^-}\,dy . \end{aligned}$$

Denoting this last term by $J(\mu)$, we find, again, by the dominated convergence theorem that

$$J(\mu) \sim \mu^{-2(1-\beta^-)} a_1 \int_0^\infty \exp\left(-\frac{y}{2\sigma^2} - \frac{c-\delta}{\sigma}\sqrt{y}\right) y^{-\beta^-}\,dy$$

as $\mu \to 0$, where, once more, the integral in this expression has a finite value since $\beta^- < 1$. Thus, there exists a constant A, e.g.

$$A = \frac{a_1}{2} \int_0^\infty \exp\left(-\frac{y}{2\sigma^2} - \frac{c-\delta}{\sigma}\sqrt{y}\right) y^{-\beta^-}\,dy,$$

such that for all μ sufficiently small, $EN \ge A\mu^{-2(1-\beta^-)}$, and, there-

fore, since $\beta^- = \beta(c-\delta) < \beta(c) + \varepsilon/2$, $EN \geq A|\mu|^{-2[1-\beta(c)]+\varepsilon}$. |||

We remark that the conclusion of this theorem is equivalent to the statement that

$$\log EN(c) \sim [1-\beta(c)] \log \mu^{-2}$$

as $\mu \to 0$.

CHAPTER 6

SOME GENERALIZATIONS

Breiman obtained the asymptotic behavior of certain tail probabilities for a stopping time defined on a sequence of i. i. d. random variables which were assumed to have a finite absolute third moment. In obtaining bounds for the expected value of this stopping time in the nonzero mean case, we have imposed additional restrictions on the common distribution of the sequence. We now seek to weaken some of these more restrictive conditions. Some results in that direction are presented in this final chapter.

If we drop the assumption that the random variables have a moment generating function, we can prove that the lower bound remains valid. Further, when the moment generating function exists, we can prove easily that the upper bound holds even when the common distribution function is not necessarily continuous.

1. *Lower Bound*

First, let us consider a probability distribution function F having the following properties:

$$\text{(i'')} \int_{-\infty}^{\infty} x \, dF(x) = 0$$

$$\text{(ii'')} \int_{-\infty}^{\infty} x^2 \, dF(x) = \sigma^2 < \infty, \; \sigma^2 > 0$$

$$\text{(iii")} \int_{-\infty}^{\infty} |x|^3 \, dF(x) < \infty$$

(iv") F is continuous.

It follows from (i") and (ii") that $0 < F(0) < 1$. For future reference, we identify the following three points:

$$x_1 = \inf \{x: F(x) > F(0)\}$$
$$x_2 = \sup \{x: F(x) < 1\} \text{ or } +\infty \text{ if } \{x: F(x) < 1\} = R^1$$
$$x_2' = \inf \{x: F(x) > 0\} \text{ or } -\infty \text{ if } \{x: F(x) > 0\} = R^1.$$

We note that $-\infty \leq x_2' < 0 \leq x_1 < x_2 \leq +\infty$.

For any $\mu \neq 0$, we define the distribution function F_μ by $F_\mu(x) = F(x-\mu)$; i.e., F_μ is a translate of F. If $\tilde{X}$ is a random variable defined on the probability space $(\Omega, \mathfrak{F}, P)$ having as its distribution function F, then F_μ is the distribution function of $\tilde{X}+\mu$; we write $X = X(\mu) = \tilde{X}+\mu$. Property (i") above implies that $E\tilde{X} = 0$ and, therefore, $EX = \mu$. We now wish to study a sequence of i. i. d. random variables with common distribution function F_μ. If N is the stopping time associated with such a sequence, we obtain a lower bound for $P(N > n)$ of the form $P(N' > n)P(N'' > n)$ where N' is a stopping time defined for a sequence of bounded random variables and N'' is a stopping time defined for a sequence of random variables with zero mean. We can then apply the results of the previous chapter to N' and Breiman's results to N'' to obtain the desired conclusion.

To get this type of bound, we first express $X = X(\mu)$ as the sum $Y + Z$ where Y is a truncation of X and Z is a random variable with zero mean. Although this is not difficult to do, we also wish to insure that $VarZ$ is small and that the truncation points converge as $\mu \to 0$.

The details are presented in the following two lemmas.

Lemma 6.1. For every α, $x_1 < \alpha < x_2$, there exists a corresponding $\alpha' < 0$ such that if

$$\tilde{Y} = \begin{cases} \tilde{X} & \alpha' < \tilde{X} \leq \alpha \\ 0 & \text{otherwise} , \end{cases}$$

then $E\tilde{Y} = E\tilde{X} = 0$. Further, given $\delta > 0$, α can be chosen so that $Var(\tilde{X} - \tilde{Y}) < \delta$.

Proof. Let α, $x_1 < \alpha < x_2$, be given. If we define the function g by

$$g(y) = \int_y^\alpha x \, dF(x),$$

then we note that g is continuous and nondecreasing for $y \in (-\infty, 0]$. Further, $g(0) > 0$ and $\lim_{y \to -\infty} g(y) < 0$. Defining $\alpha' = \sup \{y: g(y) = 0\}$ and $\tilde{Y}$ as above, it follows immediately that $E\tilde{Y} = 0$.

We remark that for each α, the definition of the corresponding α' insures that it is a unique function of α. It is also clear that as $\alpha \uparrow x_2$, $\alpha' \downarrow x_2'$. Thus, since

$$Var(\tilde{X} - \tilde{Y}) = \int_{x_2'}^{\alpha'} x^2 \, dF(x) + \int_\alpha^{x_2} x^2 \, dF(x)$$

and $\sigma^2 = \int_{x_2'}^{x_2} x^2 \, dF(x) < \infty$, it follows that given $\delta > 0$, α can be chosen sufficiently large so that $Var(\tilde{X} - \tilde{Y}) < \delta$. |||

Remark. If α lies in an interval of constancy of the function F, without loss of generality, we will henceforth assume that α is the left endpoint of such an interval; i.e., if $F(\alpha) = a$, then we choose (a possibly new) $\alpha = \inf \{x: F(x) = a\}$. Given such an α, we have just noted that there is a unique corresponding α'. We shall then speak of

the pair (α', α).

Lemma 6.2. For any $\delta > 0$, let the pair (α', α) be such that $Var(\tilde{X} - \tilde{Y}) < \delta$. Then for all μ with $|\mu|$ sufficiently small, there exist numbers $x_0 = x_0(\mu)$ and $x_0' = x_0'(\mu)$, $x_0' < 0 < x_0$, such that

(a) if

$$Y = Y(\mu) = \begin{cases} X(\mu) & x_0'(\mu) + \mu < X(\mu) \leq x_0(\mu) + \mu \\ 0 & \text{otherwise} \end{cases}$$

then $EY = EX = \mu$;

(b) $\lim_{\mu \to 0} x_0'(\mu) = \alpha'$ and $\lim_{\mu \to 0} x_0(\mu) = \alpha$;

(c) $Var(X - Y) < \delta$ for $|\mu|$ sufficiently small.

Proof. First, we consider the case where $\mu > 0$. For all such μ, let $x_0(\mu) \equiv \alpha$ and write

$$g_\mu(y) = \int_{y+\mu}^{\alpha+\mu} x \, dF_\mu(x).$$

The function g_μ is continuous and nondecreasing for $y \in (-\infty, 0]$. We note that

$$g_\mu(y) = \int_y^\alpha x \, dF(x) + \mu[F(\alpha) - F(y)].$$

Thus, for μ sufficiently small, $g_\mu(0) > \mu$ and $\lim_{y \to -\infty} g_\mu(y) < \mu$. Hence, letting $x_0'(\mu) = \sup \{y: g_\mu(y) = \mu\}$, we see that if

$$Y = \begin{cases} X & x_0 + \mu < X \leq x_0 + \mu \\ 0 & \text{otherwise ,} \end{cases}$$

then $EY = \mu = EX$.

We note that for the given pair (α', α) and each μ, $x_0'(\mu)$ is uniquely defined. Further $\mu = EY = \int_{x_0'}^{\alpha} (x + \mu) \, dF(x)$ and, so,

$$-\int_{\alpha'}^{x_0'} x\, dF(x) = \mu[1 - F(\alpha) + F(x_0')]$$

since $\int_{\alpha'}^{\alpha} x\, dF(x) = 0$. Noting that the right side above is greater than zero, we see that $x_0'(\mu) > \alpha'$ for all (small) $\mu > 0$. Also, since the right side is less than or equal to 2μ, it follows that

$$\lim_{\mu \to 0^+} \left(-\int_{\alpha'}^{x_0'} x\, dF(x) \right) = 0.$$

Recalling that α' was chosen so that it was the right endpoint of a possible interval of constancy, we conclude that $\lim_{\mu \to 0^+} x_0'(\mu) = \alpha'$.

Next, for $\mu < 0$, we choose $x_0'(\mu) \equiv \alpha'$. For $-\mu$ sufficiently small, we can show as above that

$$x_0(\mu) = \inf \left\{ y \colon \int_{\alpha'+\mu}^{y+\mu} x\, dF_\mu(x) = \mu \right\}$$

is well-defined and is the number desired. Also, as before, it follows that $\lim_{\mu \to 0^-} x_0(\mu) = \alpha$.

Finally, we see that

$$\begin{aligned} Var(X - Y) &= \int_{-\infty}^{x_0'} x^2\, dF_\mu(x) + \int_{x_0}^{\infty} x^2\, dF_\mu(x) \\ &= \int_{-\infty}^{x_0'} x^2\, dF(x) + \int_{x_0}^{\infty} x^2\, dF(x) - \mu^2[1-F(x_0)+F(x_0')]. \end{aligned}$$

It follows from the convergence of the truncation points that $\lim_{\mu \to 0} Var(X - Y) = Var(\tilde{X} - \tilde{Y}) < \delta$. Thus, for $|\mu|$ sufficiently small, $Var(X - Y) < \delta$. |||

Henceforth, we will always assume that a pair (α', α) has been chosen and is fixed. Later, we will introduce certain restrictions

that will determine how large α must be.

Next, we consider several sequences of random variables and make use of their independence to get a lower bound for $P(N > n)$. First, we define three additional distribution functions. Let $H_\mu(x) = P(X \le x \mid x_0' < X \le x_0)$, $\overline{H}_\mu(x) = P(X \le x \mid X \le x_0' \text{ or } X > x_0)$, and let $\tilde{H}_\mu$ be the distribution function of the indicator function of the event $\{x_0' < X \le x_0\}$. Of course, these distribution functions may each be expressed in terms of F_μ.

Then, we can consider random variables X', X'', and γ (defined on some probability space) so that these random variables are mutually independent and have distribution functions H_μ, $\overline{H}_\mu$, and $\tilde{H}_\mu$, respectively. We again let Y denote the truncation of X corresponding to the pair (x_0', x_0) as in Lemma 6.2, and let $Z = X - Y$. If ξ_1 and ξ_2 are two random variables having the same distribution, then we write $\xi_1 \overset{D}{=} \xi_2$.

Lemma 6.3. $\gamma X' \overset{D}{=} Y$, $(1-\gamma)X'' \overset{D}{=} Z$, and $\gamma X' + (1-\gamma)X'' \overset{D}{=} X$.

Proof. The lemma is proved by simply calculating the distribution functions of $\gamma X'$ and $(1-\gamma)X''$ and that of their sum. Let Pr denote the probability measure on the space on which the given random variables are defined. Using the independence and the definitions of H_μ and $\tilde{H}_\mu$, we find that

$$\begin{aligned} Pr(\gamma X' \le x) &= Pr(\gamma X' \le x,\ \gamma = 1) + Pr(\gamma X' \le x,\ \gamma = 0) \\ &= P(X \le x,\ x_0' < X \le x_0) + I_{[0,\infty)}(x)[P(X \le x_0') + P(X > x_0)] \end{aligned}$$

where I_A denotes the indicator function of the set A. Recalling that Y is just the truncation of X, we see that $\gamma X' \overset{D}{=} Y$.

In the same manner, $(1-\gamma)X'' \overset{D}{=} Z$, and, also,

$\gamma X' + (1-\gamma)X'' \overset{D}{=} X.$ |||

Now, given the distribution functions F_μ, H_μ, $\overline{H}_\mu$, and $\tilde{H}_\mu$ as above, there exist, for each μ, probability spaces $(\Omega_i, \mathcal{F}_i, P_i)$, i = 1, 2, 3, 4, and sequences of i. i. d. random variables $\{X_n(\mu)\}$, $\{X_n'(\mu)\}$, $\{X_n''(\mu)\}$, and $\{\gamma_n(\mu)\}$ defined on the respective spaces such that $X_1(\mu) \overset{D}{=} X$, $X_n'(\mu) \overset{D}{=} X'$, $X_n''(\mu) \overset{D}{=} X''$, and $\gamma_n(\mu) \overset{D}{=} \gamma$. If the product space of the four spaces above is denoted by $(\Omega_\mu, \mathcal{F}_\mu, P_\mu)$, then we can assume that the four sequences of random variables are defined on the same probability space and that the sequences are mutually independent.

Also, we define the sequences of random variables $\{Y_n(\mu)\}$ and $\{Z_n(\mu)\}$ by

$$Y_n(\mu) = \begin{cases} X_n(\mu) & x_0' < X_n(\mu) \le x_0 \\ 0 & \text{otherwise} \end{cases}$$

and $Z_n(\mu) = X_n(\mu) - Y_n(\mu)$. Henceforth, we shall omit the dependence on μ when writing these random variables [e.g., we write X_n for $X_n(\mu)$].

We denote $VarZ_1 = Var(X_1 - Y_1)$ by $\delta_\alpha(\mu)^2$ and $VarX_1'$ by $\sigma_\alpha(\mu)^2$. It follows from Lemma 6.1 that given $\delta > 0$, α can be chosen sufficiently large so that $\delta_\alpha(0)^2 < \delta^2$. In addition, we prove

Lemma 6.4. $\lim\limits_{\alpha \to x_2} \sigma_\alpha(0)^2 = \sigma^2.$

Proof. For fixed α, $x_1 < \alpha < x_2$, $X_1'(0)$ has the distribution function H_0. Thus,

$$\sigma_\alpha(0)^2 = \frac{1}{F(\alpha)-F(\alpha')} \int_{\alpha'}^{\alpha} x^2 \, dF(x).$$

As noted in Lemma 6.1, $\alpha' \downarrow x_2'$ as $\alpha \uparrow x_2$ and, by definition,

$F(x_2') = 0$ and $F(x_2) = 1$ since F is continuous. Thus,

$$\lim_{\alpha \to x_2} \sigma_\alpha(0)^2 = \frac{1}{F(x_2)-F(x_2')} \int_{x_2'}^{x_2} x^2 \, dF(x)$$

$$= \int_{-\infty}^{\infty} x^2 \, dF(x) = \sigma^2. \qquad |||$$

Finally, we are prepared to introduce the stopping time whose first moment we wish to study. The i. i. d. sequence $\{X_n\}$ has a common continuous distribution function F_μ and $EX_n = \mu$, $VarX_n = \sigma^2$, and $E|X_n|^3 < \infty$. For this sequence, we define, for any positive number c, the stopping time $N = N(c)$ on the space $(\Omega_\mu, \mathcal{F}_\mu, P_\mu)$ by

$$N = N(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists.}$$

We now wish to obtain a lower bound for $EN(c)$ when $|\mu|$ is sufficiently small. Recalling that $EN(c) = \sum_0^\infty P_\mu[N(c) > n]$, we consider the probability $P_\mu[N(c) > n]$ for fixed n.

Lemma 6.5. For any η, $0 < \eta < c$,

$$P_\mu[N(c) > n] \geq P_\mu\left(\left|\sum_1^k X_i'\right| \leq (c-\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right) \tag{1}$$
$$\cdot\, P_\mu\left(\left|\sum_1^k Z_i\right| \leq \eta\sigma\sqrt{k},\ k = 1,\ldots,n\right).$$

Proof. Let η, $0 < \eta < c$, be fixed. Then, since $X_i \overset{D}{=} \gamma_i X_i' + (1-\gamma_i)X_i''$,

$$P_\mu(N > n) = P_\mu\left(\left|\sum_1^k X_i\right| \leq c\sigma\sqrt{k},\ k = 1,\ldots,n\right)$$

$$\geq P_\mu\left(\left|\sum_1^j \gamma_i X_i'\right| \leq (c-\eta)\sigma\sqrt{j},\ 1 \leq j \leq n;\ \left|\sum_1^k (1-\gamma_i)X_i''\right| \leq \eta\sigma\sqrt{k},\ 1 \leq k \leq n\right).$$

Letting

$$A_n = \{|\sum_1^k \gamma_i X_i'| \le (c-\eta)\sigma\sqrt{k},\ k = 1,\dots,n\},$$

$$B_n = \{|\sum_1^k (1-\gamma_i)X_i''| \le \eta\sigma\sqrt{k},\ k = 1,\dots,n\},$$

we may write $P_\mu(N > n) \ge P_\mu(A_n \cap B_n)$.

If $S_n = \{s = (s_1,\dots,s_n)\colon s_i = 0 \text{ or } 1,\ 1 \le i \le n\}$ and if Γ_n is the random vector $\Gamma_n = (\gamma_1,\dots,\gamma_n)$, then

$$P_\mu(A_n \cap B_n) = \sum_{s\in S_n} P_\mu(A_n \cap B_n \mid \Gamma_n = s)P_\mu(\Gamma_n = s).$$

For a particular term in this sum, suppose $s_0 = (s_1,\dots,s_n)$ where $s_{i_1} = s_{i_2} = \dots = s_{i_p} = 1$, $1 \le i_1 < i_2 < \dots < i_p \le n$, and $s_{j_1} = s_{j_2} = \dots = s_{j_q} = 0$, $1 \le j_1 < j_2 < \dots < j_q \le n$ with $p + q = n$. Let

$$C_1 = \{|\sum_{m=1}^k X_{i_m}'| \le (c-\eta)\sigma\sqrt{i_k},\ k = 1,\dots,p\},$$

$$C_0 = \{|\sum_{m=1}^k X_{j_m}'| \le \eta\sigma\sqrt{j_k},\ k = 1,\dots,q\}.$$

Then it follows from the mutual independence of $\{X_n'\}$, $\{X_n''\}$, and $\{\gamma_n\}$ that

$$\begin{aligned}P_\mu(A_n \cap B_n \mid \Gamma_n = s_0)P_\mu(\Gamma_n = s_0) &= P_\mu(A_n \cap B_n \cap \{\Gamma_n = s_0\})\\ &= P_\mu(C_1 \cap C_0 \cap \{\Gamma_n = s_0\})\\ &= P_\mu(C_1)P_\mu(C_0 \cap \{\Gamma_n = s_0\})\\ &= P_\mu(C_1)P_\mu(B_n \cap \{\Gamma_n = s_0\}).\end{aligned}$$

Now, we note that $i_k \ge k$ and, therefore,

$$P_\mu(C_1) \ge P_\mu\left(|\sum_{m=1}^k X_{i_m}'| \le (c-\eta)\sigma\sqrt{k},\ k = 1,\dots,p\right).$$

However, since the X_j' are identically distributed and $p \leq n$, the latter is not less than $P_\mu\left(\left|\sum_{m=1}^{k} X_m'\right| \leq (c-\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right)$. Hence,

$$
\begin{aligned}
P_\mu(N > n) &\geq P_\mu(A_n \cap B_n) \\
&= \sum_{s \in S_n} P_\mu(C_1)P_\mu(B_n \cap \{\Gamma_n = s\}) \\
&\geq P_\mu\left(\left|\sum_{m=1}^{k} X_m'\right| \leq (c-\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right)P_\mu(B_n).
\end{aligned}
$$

But $(1-\gamma_i)X_i'' \overset{D}{=} Z_i$ and, so, $P_\mu(B_n) = P_\mu\left(\left|\sum_1^k Z_i\right| \leq \eta\sigma\sqrt{k},\ k = 1,\ldots,n\right)$. |||

Theorem 6.1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i. i. d. random variables with $EX_1 = \mu \neq 0$, $VarX_1 = \sigma^2$, $E|X_1|^3 < \infty$, and a common distribution function which is continuous. Let $N(c)$ be the stopping time defined above. Then, for each $c > 1$ and any $\varepsilon > 0$ such that $\beta(c) + \varepsilon/2 < 1$, there exists a positive constant A, independent of μ, such that for all μ sufficiently small

$$EN(c) \geq A|\mu|^{-2[1-\beta(c)]+\varepsilon}.$$

Proof. Let us fix a number $c > 1$ [hence, $\beta(c) < 1$] and fix an $\varepsilon > 0$ such that $\beta(c) + \varepsilon/2 < 1$. Before proceeding, we place restrictions on the parameters η, δ, and α that we will need in the course of the proof. First, we fix $\eta > 0$ such that (a) $c-4\eta > 1$ and (b) $\beta(c-4\eta) < \beta(c) + \varepsilon/4$. Next, we fix $\delta > 0$ such that (c) $\eta\sigma/\delta > 1$ and (d) $\beta(\eta\sigma/\delta) < \varepsilon/4$ [recalling that $\beta(u) \to 0$ as $u \to +\infty$]. Finally, we fix α sufficiently large so that (e) $Var(\tilde{X} - \tilde{Y}) = \delta_\alpha(0)^2 < \delta^2$ and (f) $\sigma/\sigma_\alpha(0) > 1 - \eta/(c-\eta)$. Lemma 6.4 and the remark preceding it imply that such an α exists.

Now, $EN(c) = EN = \sum_0^{\infty} P_\mu(N > n)$ and, so, by Lemma 6.5, we study

$$P_\mu\left(\left|\sum_1^k Z_i\right| \le \eta\sigma\sqrt{k},\ k = 1,\ldots,n\right) \text{ and } P_\mu\left(\left|\sum_1^k X_i'\right| \le (c-\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right).$$

We consider, first, the sequence $\{Z_n\}$, recalling that $Z_n = Z_n(\mu) = X_n(\mu) - Y_n(\mu)$. We note that $EZ_n = 0$ and $VarZ_n = \delta_\alpha(\mu)^2$. In the proof of Lemma 6.2, we showed that $\lim_{\mu\to 0} \delta_\alpha(\mu)^2 = \delta_\alpha(0)^2$; hence, for $0 < m < \delta_\alpha(0)^2$, $\delta_\alpha(\mu)^2 \ge m$ for μ sufficiently small. Also, for $|\mu| < 1$, $E|Z_n|^3 \le E|X_n(\mu)|^3 \le \int_{-\infty}^{\infty} |x|^3\, dF(x) + 3\sigma^2 + 3\int_{-\infty}^{\infty} |x|\, dF(x) + 1$; i.e., $E|Z_n|^3$ is uniformly bounded for $|\mu| < 1$. In addition, if we denote the distribution function of Z_1 by $\overline{G}_\mu$, then $\overline{G}_\mu$ may be simply expressed in terms of F_μ and it follows from the fact that $x_0(\mu) \to \alpha$ and $x_0'(\mu) \to \alpha'$ as $\mu \to 0$ that $\overline{G}_\mu \xrightarrow{w} \overline{G}_0$. It is also easy to see that $\overline{G}_\mu$ is continuous except at zero.

If we now define the stopping time $N''(d)$ for $d > 0$ by

$$N''(d) = \text{least } n \ge 1 \text{ such that } \left|\sum_1^n Z_i\right| > d\delta_\alpha(\mu)\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then, by Theorem 4.2, for all $d \ge 1$, there is a positive constant $a''(d)$ and a positive integer n_1, both independent of μ, such that $P_\mu[N''(d) > n] \ge a''(d)n^{-\beta(d)}$ for all $n \ge n_1$ and for all μ sufficiently small.

Since α has been chosen so that $\delta_\alpha(0) < \delta$, it follows from Lemma 6.2 that $\delta_\alpha(\mu) < \delta$ for all μ sufficiently small. Thus,

$$P_\mu\left(\left|\sum_1^k Z_i\right| < \eta\sigma\sqrt{k},\ k = 1,\ldots,n\right) \ge P_\mu\left(\left|\sum_1^k Z_i\right| \le \frac{\eta\sigma}{\delta}\,\delta_\alpha(\mu)\sqrt{k},\ 1 \le k \le n\right);$$

the latter is just $P\,[N''(\eta\sigma/\delta) > n]$. Now, δ was chosen so that $\eta\sigma/\delta > 1$. Thus, for all $n \ge n_1$

(2) $$P_\mu\left(|\sum_1^k Z_i| \leq \eta\sigma\sqrt{k},\ k = 1,\ldots,n\right) \geq a''n^{-\beta(\eta\sigma/\delta)}$$

for all μ sufficiently small where $a'' = a''(\eta\sigma/\delta)$.

Next, we must consider the sequence $\{X_n'\}$. We recall that $X_1' = X_1'(\mu)$ has the continuous distribution function H_μ. Hence, $EX_1' = \mu/[F_\mu(x_0)-F_\mu(x_0')]$. Writing $m(\mu) = EX_1'$, we note that $m(\mu) \sim \mu/[F(\alpha)-F(\alpha')]$ as $\mu \to 0$ and $m(0) = 0$. Also, $VarX_1' = \sigma_\alpha(\mu)^2$, a finite, positive number. Since X_1' is bounded, the moment generating function, $\phi_\mu(t)$, of X_1' exists for all t R^1 and

$$\phi_\mu(t) = \int_{-\infty}^{\infty} e^{tx}\, dH_\mu(x) = \frac{1}{F_\mu(x_0)-F_\mu(x_0')}\int_{x_0'}^{x_0} e^{tx}\, dF_\mu(x).$$

Hence, $\phi_\mu(t) \leq e^{|t|[\alpha \vee (-\alpha')]}$ for μ sufficiently small; the latter function is bounded on any finite interval containing zero. Therefore, for μ sufficiently small, say $|\mu| < \mu_0$, the family $\{H_\mu,\ |\mu| < \mu_0\}$ satisfies properties (i')-(iv'). Further, as $\mu \to 0$, $\phi_\mu(t) \to \phi_0(t)$ and, so, $H_\mu \xrightarrow{w} H_0$.

As in Chapter 5, we let $h = h(\mu)$ be the unique root of $\phi_\mu'(t) = 0$ and let $\tau(\mu)^2 = \phi_\mu''(-h)/\phi_\mu(-h)$. We define the stopping time $N'(d)$ for $d > 0$ by

$$N'(d) = \text{least } n \geq 1 \text{ such that } |\sum_1^n X_i'| > d\sigma_\alpha(\mu)\sqrt{n}$$

or $+\infty$ if no such n exists.

Then, by Theorem 5.1, it follows that for each $d > 1$ and $\eta > 0$ such that $d-\eta > 1$, there exists a positive constant $a'(d)$ and a positive integer n_2, both independent of μ, such that

(3) $$P_\mu[N'(d) > n] \geq a'(d)[\phi_\mu(-h)]^n\, e^{-|h|(d-\eta)\tau\sqrt{n}}\, n^{-\beta(d-\eta)}$$

for all $n \geq n_2$ and all μ sufficiently small.

Now, by assumption (f), $(c-\eta)\sigma > (c-2\eta)\sigma_\alpha(0)$. Also, since $\sigma_\alpha(\mu)^2 \to \sigma_\alpha(0)^2$ as $\mu \to 0$, it follows that for all μ sufficiently small, $(c-2\eta)\sigma_\alpha(0) > (c-3\eta)\sigma_\alpha(\mu)$. Hence,

$$(4) \qquad P_\mu\left(\left|\sum_1^k X_i'\right| \leq (c-\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right) \geq P_\mu[N'(c-3\eta) > n].$$

Thus, combining (3) (with $d = c-3\eta$) and (4), we see that since by assumption (a), $c-4\eta > 1$, it follows that

$$(5) \qquad \begin{aligned} &P_\mu\left(\left|\sum_1^k X_i'\right| \leq (c-\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right) \\ &\qquad \geq a'[\phi_\mu(-h)]^n\, e^{-|h|(c-4\eta)\tau\sqrt{n}}\, n^{-\beta(c-4\eta)} \end{aligned}$$

for all $n \geq n_2$ and all μ sufficiently small where $a' = a'(c-3\eta)$.

Let $n_0 = \max\{n_1, n_2\}$. It follows from (1), (2), and (5) that

$$EN \geq a \sum_{n_0}^{\infty} e^{-n[-\log \phi_\mu(-h)]}\, e^{-|h|(c-4\eta)\tau\sqrt{n}}\, n^{-[\beta(c-4\eta)+\beta(\eta\sigma/\delta)]}$$

for all μ sufficiently small, where $a = a'a''$ is a positive constant. Assumptions (b) and (d) imply that $n^{-[\beta(c-4\eta)+\beta(\eta\sigma/\delta)]} > n^{-[\beta(c) + \varepsilon/2]}$. Recalling that by Lemma 5.2, $-\log \phi_\mu(-h) \sim m(\mu)^2/2\sigma_\alpha(0)^2$ as $\mu \to 0$, we see that $-\log \phi_\mu(-h)$ is positive for sufficiently small values of μ. Thus, replacing the above sum by an integral and writing m for $m(\mu)$, we obtain

$$\begin{aligned} EN &\geq m^{-2(1-[\beta(c) + \varepsilon/2])} \\ &\quad \cdot a\int_{m^2 n_0}^{\infty} \exp\left(-y\,\frac{-\log \phi_\mu(-h)}{m^2} - \frac{|h|(c-4\eta)\tau\sqrt{y}}{|m|}\right) y^{-[\beta(c) + \varepsilon/2]}\,dy. \end{aligned}$$

It follows from the dominated convergence theorem and the fact that $m = m(\mu) \sim \mu/[F(\alpha)-F(\alpha')]$ that this lower bound, say $J(\mu)$, is such that

$$J(\mu) \sim \mu^{-2(1-[\beta(c) + \varepsilon/2])} a_0 \int_0^\infty \exp\left(- \frac{y}{2\sigma_\alpha(0)^2} - \frac{c-4\eta}{\sigma_\alpha(0)} \sqrt{y}\right) y^{-[\beta(c) + \varepsilon/2]} dy.$$

This last integral is finite [10, p. 337] since we have chosen ε such that $\beta(c) + \varepsilon/2 < 1$. Hence, if

$$A = \frac{a_0}{2} \int_0^\infty \exp\left(- \frac{y}{2\sigma_\alpha(0)^2} - \frac{c-4\eta}{\sigma_\alpha(0)} \sqrt{y}\right) y^{-[\beta(c) + \varepsilon/2]} dy$$

for example, then for all μ sufficiently small $EN \geq A|\mu|^{-2[1-\beta(c)]+\varepsilon}$. |||

2. *Upper Bound*

The above theorem shows that for the lower bound to hold, it suffices to assume only the existence of an absolute third moment rather than that of a moment generating function. However, the common distribution function is still assumed to be continuous. On the other hand, the same type of upper bound is valid if we again assume that a moment generating function exists, but no longer require that the common distribution function be continuous. We prove

Theorem 6.2. Let $\{X_n\}_{n=1}^\infty$ be a sequence of i. i. d. random variables defined by $X_n = \tilde{X}_n + \mu$, where $\tilde{X}_n$ is such that $E\tilde{X}_1 = 0$, $Var\tilde{X}_1 = \sigma^2$, a finite, positive number, and $Ee^{tX_1} < \infty$ for t in a neighborhood of zero. If for any positive c,

$$N(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then for each $c \geq 1$ and any $\varepsilon > 0$, there exists a positive constant B, independent of μ, such that for all μ sufficiently small

$$EN(c) \leq B|\mu|^{-2[1-\beta(c)]-\varepsilon}.$$

Proof. Suppose that the sequence $\{\tilde{X}_n\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$. Without loss of generality, we may assume that on this same space there is a sequence of i. i. d. random variables $\{Y_n\}$ with a common $N(0, \delta^2)$ distribution, where $\delta > 0$ is arbitrary; further, we may assume that the sequences $\{\tilde{X}_n\}$ and $\{Y_n\}$ are independent, and, therefore, also, that $\{X_n\}$ and $\{Y_n\}$ are independent.

We fix $c \geq 1$ and write N for $N(c)$. Again, since $EN = \sum_0^\infty P(N > n)$, we consider $P(N > n)$ for n fixed. For any $\eta > 0$, we see that

$$P(N > n) = P\left(\left|\sum_1^k X_i\right| \leq c\sigma\sqrt{k},\ k = 1,\ldots,n\right)$$

(6)
$$= \frac{\left(\left|\sum_1^k X_i\right| \leq c\sigma\sqrt{k},\ 1\leq k\leq n;\ \left|\sum_1^j Y_i\right| \leq \eta\sigma\sqrt{j},\ 1\leq j\leq n\right)}{P\left(\left|\sum_1^j Y_i\right| \leq \eta\sigma\sqrt{j},\ 1\leq j\leq n\right)}$$

$$\leq \frac{P\left(\left|\sum_1^k (X_i+Y_i)\right| \leq (c+\eta)\sigma\sqrt{k},\ k = 1,\ldots,n\right)}{P\left(\left|\sum_1^j Y_i\right| \leq \eta\sigma\sqrt{j},\ j = 1,\ldots,n\right)}.$$

As noted in Chapter 3, it follows from Theorem 2 of Breiman that $P\left(\left|\sum_1^j Y_i\right| \leq \eta\sigma\sqrt{j},\ j = 1,\ldots,n\right) \sim a_1 n^{-\beta(\eta\sigma/\delta)}$ as $n \to \infty$ where a_1 is a positive constant. Thus, there exists an n_1 such that for $n \geq n_1$,

(7)
$$P\left(\left|\sum_1^j Y_i\right| \leq \eta\sigma\sqrt{j},\ j = 1,\ldots,n\right) \geq \frac{a_1}{2} n^{-\beta(\eta\sigma/\delta)}.$$

Now, let $Z_n = Z_n(\mu) = X_n + Y_n$. We note that $\{Z_n\}$ is a sequence of i. i. d. random variables with $EZ_1 = \mu$, $VarZ_1 = \sigma^2 + \delta^2$, and $Ee^{tZ_1} = \exp\left(\frac{t^2\delta^2}{2}\right) Ee^{tX_1} = \exp\left(\frac{t^2\delta^2}{2} + \mu t\right) Ee^{t\tilde{X}_1}$. Hence, for μ sufficiently small, say $|\mu| < \mu_0$, the moment generating functions $Ee^{tZ_1(\mu)}$ are uniformly bounded on a finite interval containing zero on which $Ee^{tX_1} < \infty$. Further, the distribution function, say G_μ, of Z_1 is the convolution of the distribution function of X_1 and that of Y_1; thus, G_μ is continuous and $G_\mu \xrightarrow{w} G_0$ as $\mu \to 0$. Therefore, the family $\{G_\mu, |\mu| < \mu_0\}$ is like those considered in Chapter 5.

We define the stopping time $N_\delta(d)$ for any $d > 0$ by

$$N_\delta(d) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n Z_i\right| > d\sqrt{\sigma^2+\delta^2}\sqrt{n}$$

or $+\infty$ if no such n exists.

We write $\phi_\mu(t) = Ee^{tZ_1}$ and again denote the unique root of the equation $\phi_\mu'(t) = 0$ by $-h = -h(\mu)$. Also, we write $\tau^2 = \tau(\mu)^2 = \phi_\mu''(-h)/\phi_\mu(-h)$. Then, it follows from Theorem 5.1 that for each $d \geq 1$ there exists a positive integer n_2 and a positive constant $a_2(d)$, both independent of μ, such that

$$(8) \qquad P[N_\delta(d) > n] \leq a_2(d)[\phi_\mu(-h)]^n e^{|h|(d+\eta)\tau\sqrt{n}} n^{-\beta(d+\eta)}$$

for all $n \geq n_2$ and all μ sufficiently small. Finally, we note also that

$$(9) \qquad P\left(\left|\sum_1^k Z_i\right| \leq (c+\eta)\sigma\sqrt{k},\ k = 1,\dots,n\right) \leq P[N_\delta(c+\eta) > n].$$

So far, δ and η have been arbitrary positive constants. Let $\varepsilon > 0$ be given. We choose $\eta > 0$ such that $\beta(c+2\eta) > \beta(c) - \varepsilon/4$. Then

we choose $\delta > 0$ such that $\beta(\eta\sigma/\delta) < \varepsilon/4$. These choices are possible by Proposition A.1.

Let $n_0 = \max\{n_1, n_2\}$. Then, we see from (6), (7), (8), and (9) that

$$EN \leq n_0 + \frac{2a_2}{a_1}\sum_{n_0}^{\infty} e^{-n[-\log \phi_\mu(-h)]}\, e^{|h|(c+2\eta)\tau\sqrt{n}}\, n^{-[\beta(c+2\eta)-\beta(\eta\sigma/\delta)]}$$

for all μ sufficiently small. We note that $\beta(c+2\eta)-\beta(\eta\sigma/\delta) > \beta(c) - \varepsilon/2$. Again, by Lemma 5.2, $-\log \phi_\mu(-h) \sim \mu^2/2(\sigma^2+\delta^2)$ as $\mu \to 0$ and, so, is positive for sufficiently small μ. Hence, using Lemma 3.1 and proceeding as in Theorem 5.2, we may replace the series by an integral to obtain for all sufficiently small μ,

$$EN \leq n_0 + \frac{2a_2}{a_1} K(\mu) + \mu^{-2(1-[\beta(c) - \varepsilon/2])}\frac{2a_2}{a_1}$$

$$\cdot \int^{\infty} \exp\left(-y\,\frac{-\log \phi_\mu(-h)}{\mu^2} + \frac{|h|(c+2\eta)\tau}{|\mu|}\sqrt{y}\right) y^{-[\beta(c) - \varepsilon/2]}dy$$

where $K(\mu) \to 0$ as $\mu \to 0$. Again, the integral term is asymptotic to $\mu^{-2(1-[\beta(c) - \varepsilon/2])}\frac{2a_1}{a_2}\int_0^{\infty} \exp\left(-\frac{y}{2(\sigma^2+\delta^2)} + \frac{c+2\eta}{\sqrt{\sigma^2+\delta^2}}\sqrt{y}\right) y^{-[\beta(c) - \varepsilon/2]}dy$ as $\mu \to 0$. Thus, for example, if

$$B = \frac{4a_2}{a_1}\int_0^{\infty} \exp\left(-\frac{y}{2(\sigma^2+\delta^2)} + \frac{c+2\eta}{\sqrt{\sigma^2+\delta^2}}\sqrt{y}\right) y^{-[\beta(c) - \varepsilon/2]}dy,$$

a finite positive constant, then $EN \leq B|\mu|^{-2[1-\beta(c)]-\varepsilon}$ for all μ sufficiently small. |||

Finally, we are also able to weaken the requirement that the random variables have a moment generating function; however, the bound we obtain is not as sharp.

Theorem 6.3. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i. i. d. random variables defined by $X_n = \tilde{X}_n + \mu$ where $\tilde{X}_1$ is such that $E\tilde{X}_1 = 0$, $Var\tilde{X}_1 = \sigma^2$, a finite, positive number, and $E|\tilde{X}_1|^{4\theta} < \infty$ for some $\theta > 1$. If for any $c > 0$,

$$N(c) = \text{least } n \geq 1 \text{ such that } \left|\sum_1^n X_i\right| > c\sigma\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then for each $c \geq 1$ there exists a positive constant B, independent of μ, such that for all μ sufficiently small

$$EN(c) \leq B\mu^{-2[1-\gamma(c)]}$$

where $\gamma(c) = \beta(c\sqrt{2})/2$.

Proof. Suppose that the sequence $\{\tilde{X}_n\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$. Without loss of generality, we may assume that on this same space, there is another sequence $\{\tilde{X}_n'\}$ of i. i. d. random variables such that $\tilde{X}_1'$ has the same distribution as $\tilde{X}_1$ and such that the sequences $\{X_n\}$ and $\{X_n'\}$ are independent. For $\mu \neq 0$, we write $X_n' = \tilde{X}_n' + \mu$ and let $N'(c)$ denote the stopping time $N(c)$ defined with respect to $\{X_n'\}$.

Fixing $c \geq 1$, we write N and N' for $N(c)$ and $N'(c)$ respectively. Since $EN = \sum_0^{\infty} P(N > n)$, we obtain bounds for the probabilities $P(N > n)$. First, we note

$$\begin{aligned} P(N > n)^2 &= P(N > n, N' > n) \\ &\leq P\left(\left|\sum_1^k (X_i - X_i')\right| \leq 2c\sigma\sqrt{k},\ k = 1,\ldots,n\right). \end{aligned} \tag{10}$$

Defining $Y_n = X_n - X_n'$, we see that $EY_n = 0$, $VarY_n = 2\sigma^2$, and $E|Y_n|^3 < \infty$. Thus, if we define the stopping time $N_0(d)$ for $d > 0$ by

$$N_0(d) = \text{least } n \geq 1 \text{ such that } |\sum_1^n Y_i| > d\sigma\sqrt{2}\sqrt{n}$$

$$\text{or } +\infty \text{ if no such } n \text{ exists,}$$

then, applying Theorem 2 of Breiman [3], we obtain for $d \geq 1$ the fact that $P[N_0(d) > n] \sim a'n^{-\beta(d)}$ as $n \to \infty$ where a' is a constant which depends on d and the distribution of Y_n. Noting that the right side of (10) equals $P[N_0(c\sqrt{2}) > n]$, we conclude that there exists a positive integer n_0, independent of μ, such that $P(N > n) \leq an^{-\beta(c\sqrt{2})/2}$ for all $n \geq n_0$ where a is a constant, independent of μ. We write $\gamma(c) = \beta(c\sqrt{2})/2$ and note that $\gamma(c) < \beta(c) \leq 1$.

Also, by Chebychev's inequality, $P(N > n) \leq n^{-2}EN^2$. Since $E|X_1|^{4\theta} < \infty$ for some $\theta > 1$, it follows from an inequality obtained by Woodroofe [16] that $\mu^2 EN^2 \leq dEN$ where d is a positive constant which is independent of μ. Thus, $P(N > n) \leq d\mu^{-2}n^{-2}EN$.

Now, for each $m > n_0$, we write

$$\begin{aligned} EN &\leq n_0 + \sum_{n_0}^{m} P(N > n) + \sum_{m+1}^{\infty} P(N > n) \\ &\leq n_0 + am^{1-\gamma(c)} + d\mu^{-2}m^{-1}EN. \end{aligned}$$

In particular, if we choose $m = [2d\mu^{-2}]$ where $[x]$ is the greatest integer less than or equal to x, then it follows that there exists a positive constant B $\left(\text{e.g., } B = 4[n_0 + a(2d)^{1-\gamma(c)}]\right)$, independent of μ, such that for all sufficiently small μ, $EN \leq B^{-2[1-\gamma(c)]}$. |||

APPENDIX

THE FUNCTION

In [3], Breiman first studies the stopping time $T_c^* = \inf\{t: |W(t)| \geq c\sqrt{t},\ t \geq 1\}$ where $\{W(t),\ t \geq 0\}$ is the standard Brownian motion process. He proves that as $t \to \infty$, $P[T_c^* > t \mid W(1) = 0] \sim \alpha t^{-\beta(c)}$ where $-2\beta(c)$ is the largest pole of

$$\Phi(\lambda) = e^{-c^2/4} \frac{D_{-\lambda}(0)+D_{-\lambda}(0)}{D_{-\lambda}(c)+D_{-\lambda}(-c)}$$

for $\lambda \in \mathbb{C}$, $Re\ \lambda > 0$. Here, $D_\lambda(z)$ is the parabolic cylinder function. Using certain standard identities (e.g., see [1, p. 687]), we see that $-\beta(c)$ is, in fact, the largest zero of the confluent hypergeometric function $M(\lambda, \frac{1}{2}, c^2/2)$ regarded as a function of λ. We remark that the function $M(a, b, z)$ is given by the power series

$$M(a, b, z) = \sum_0^\infty \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

where $(a)_0 = 1$ and $(a)_n = (a)(a+1)\ldots(a+n-1)$, $n \geq 1$. This series converges as long as b is not a negative integer.

Proposition A.1. The function $\beta(c)$ is a continuous, strictly decreasing function on $(1, \infty)$ with $0 < \beta(c) < 1$, $\beta(1) = 1$, and $\lim_{c\to\infty} \beta(c) = 0$. Also, β is right continuous at $c = 1$.

Proof. That $\beta(1) = 1$ and $\lim_{c\to\infty} \beta(c) = 0$ are included in Theorem 1 of Breiman's paper. The relevant properties of $M(\lambda, \frac{1}{2}, c^2/2)$ used below

are contained in [1, p. 504ff].

For a fixed c, the zeros of $M(\lambda, \frac{1}{2}, c^2/2)$ are all negative and, so, $\beta(c) > 0$. Also, $M(0, \frac{1}{2}, c^2/2) = 1$ and $M(-1, \frac{1}{2}, c^2/2) = 1 - c^2 < 0$ for all $c > 1$. Hence, since $M(\lambda, \frac{1}{2}, c^2/2)$ is a continuous function of λ, it must have a zero in the interval $(-1, 0)$; i.e., $\beta(c) < 1$ for $c > 1$.

Let c_1, c_2 be such that $1 \leq c_1 < c_2$. We note that for fixed λ, $-1 < \lambda < 0$, $M(\lambda, \frac{1}{2}, x)$ regarded as a function of x is strictly decreasing. Hence, for all $\lambda \in (-1, 0)$, $M(\lambda, \frac{1}{2}, c_1^2/2) > M(\lambda, \frac{1}{2}, c_2^2/2)$ and, so, $-\beta(c_1)$, the largest zero of $M(\lambda, \frac{1}{2}, c_1^2/2)$, is less than $-\beta(c_2)$. Thus, $\beta(c_1) > \beta(c_2)$ and β is strictly decreasing for $c \in [1, \infty)$.

Finally, we show that β is continuous. The monotonicity of β implies that it may have at worst only jump discontinuities. We fix $c_0 \geq 1$ and consider $\beta(c_0^+) = \lim_{\varepsilon \to 0^+} \beta(c_0+\varepsilon) \leq \beta(c_0)$. For fixed $x_0 > 0$, $M(\lambda, \frac{1}{2}, x_0)$ is strictly increasing for $\lambda \in (-1, 0)$. Thus, $M[-\beta(c_0^+), \frac{1}{2}, c_0^2/2] \geq M[-\beta(c_0), \frac{1}{2}, c_0^2/2] = 0$ and, so, there is an $x_1 \geq c_0^2/2$ such that $M[-\beta(c_0^+), \frac{1}{2}, x_1] = 0$ since $M(\lambda_0, \frac{1}{2}, x)$ is continuous and decreasing as a function of x. However, for all $\varepsilon > 0$, $M[-\beta(c_0+\varepsilon), \frac{1}{2}, x_1] > M[-\beta(c_0^+), \frac{1}{2}, x_1] = 0$ and, consequently, $x_1 < (c_0+\varepsilon)^2/2$. Hence, $x_1 = c_0^2/2$ and $-\beta(c_0^+)$ is a zero of $M(\lambda, \frac{1}{2}, c_0^2/2)$, implying $-\beta(c_0^+) < -\beta(c_0)$, the largest zero of this function. Therefore, $\beta(c_0^+) = \beta(c_0)$ and β is right continuous for all $c \geq 1$. A similar argument demonstrates that β is left continuous for $c > 1$ and, thus, β is continuous on $(1, \infty)$. |||

We make one additional remark about the function β. As noted above, for any $\lambda \in (-1, 0)$, $M(\lambda, \frac{1}{2}, x)$ is strictly decreasing as a

function of x. Hence, for $c < 1$, $M(\lambda, \frac{1}{2}, c^2/2) > M(\lambda, \frac{1}{2}, \frac{1}{2})$ and the latter is greater than zero for all $\lambda \in (-1, 0)$ since $\beta(1) = 1$. Also, $M(-1, \frac{1}{2}, c^2/2) = 1-c^2 > 0$. Thus, we conclude that for all $c < 1$, $\beta(c) > 1$.

Although the function β arose in conjunction with the study of the stopping time T_c^* for the Brownian motion process, we showed in Chapter 4 that this same $\beta(c)$ occurs in the study of the asymptotic behavior of $P[N(c) > n]$ where $N(c)$ is the analogous stopping time defined for certain sequences of i. i. d. random variables. Therefore, it is exactly this function β which plays such an important role throughout this paper.

BIBLIOGRAPHY

1. Abramowitz, M., and L. A. Stegun, eds. *Handbook of Mathematical Functions*. Washington, D.C.: National Bureau of Standards, 1964.

2. Blackwell, D., and D. Freedman. "A remark on the coin-tossing game." *Ann. Math. Statist.* 35 (1964), 1345-1347.

3. Breiman, L. "First exit times from a square root boundary." *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. II, Part 2. Berkeley and Los Angeles: University of California Press, 1967, 9-16.

4. Chow, Y. S., H. Robbins, and H. Teicher. "Moments of randomly stopped sums." *Ann. Math. Statist.* 36 (1965), 789-799.

5. Coddington, E. A., and N. Levison. *Theory of Ordinary Differential Equations*. New York: McGraw-Hill Book Company, Inc., 1955.

6. Doetsch, G. *Theorie und Anwendung der Laplace-Transformation*. Berlin: Verlag von Julius Springer, 1937.

7. Esscher, F. "On the probability function in the collective theory of risk." *Skand. Aktuarietidskrift* 15 (1932), 175-195.

8. Feller, W. "Statistical aspects of ESP." *Journal of Parapsychology* 4 (1940), 271-298.

9. ________. *An Introduction to Probability and Its Applications*, Vol. 2, 2nd ed. New York: John Wiley, 1971.

10. Gradshteyn, I. S., and I. M. Ryzhik. *Tables of Integrals, Series, and Products*, 4th ed. New York and London: Academic Press, 1965.

11. Prohorov, Yu. V. "Convergence of random processes and limit theorems in probability theory." *Theory of Probability and Its Applications* 1 (1956), 157-214.

12. Shepp, L. A. "Radon-Nikodym derivatives of Gaussian measures." *Ann. Math. Statist.* 37 (1966), 321-354.

13. ________. "A first passage problem for the Wiener process." *Ann. Math. Statist.* 38 (1967), 1912-1914.

14. Tricomi, F. G. *Differential Equations.* New York: Hafner Publishing Company, 1961.

15. Widder, D. V. *The Laplace Transform.* Princeton, N.J.: Princeton University Press, 1946.

16. Woodroofe, M. "On the first time $|S_n| > cn^{\frac{1}{2}}$." *Ann. Math. Statist.* 41 (1970), 2179-2183.

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