NBS

Technical Note 191

Tables Describing Small-Sample Properties of the Mean, Median, Standard Deviation, and Other Statistics In Sampling From Various Distributions

CHURCHILL EISENHART, LOLA S. DEMING
AND CELIA S. MARTIN

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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ISSUED JUNE 14, 1963

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FOREWORD

As part of a continuing program of research into statistical methods appropriate for measurement and calibration programs in the physical sciences and engineering, the Statistical Engineering Laboratory of the National Bureau of Standards conducts studies of the properties that would be exhibited by frequently-used statistical techniques if they were applied to data obeying a variety of probability distributions. This note makes generally available the tables that were described in three related papers, presented before a joint session of the American Mathematical Society and the Institute of Mathematical Statistics in Madison, Wisconsin, on September 7, 1948. Copies of these tables were distributed to persons present at this session; and copies of some of them have been made available to various other persons from time to time during the intervening years. These tables were not submitted for formal publication heretofore because each represented unfinished portions of a larger study that subsequently evolved in a manner different from that originally contemplated. The tables are published now, for convenient reference, accompanied by the (slightly edited) brief descriptions of them that appeared as Abstracts in the Annals of Mathematical Statistics, Vol. 19, pp. 598-600 (1948). During the intervening years more accurate values have become available for the standard deviation of the median in small samples from the normal and double-exponential distributions; the final columns of Tables 1b and 2b were revised accordingly.

March 1963

Churchill Eisenhart, Chief
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ABSTRACT

This note includes a collection of tables useful for study of the sampling distributions of some frequently-used statistics, with brief discussions of their construction and use. (1) The probability level $P(\epsilon, n)$ of any continuous parent distribution corresponding to level ϵ of the distribution of the median. (2) Probability points of certain sample statistics for samples from six distributions: normal and double-exponential (mean, median), rectangular (mean, median, midrange), Cauchy, Sech, $Sech^2$ (median). In all the above tables, the sample size $n = 3(2)15(10)95$ and the probability levels are $\epsilon = .001, .005, .01, .025, .05, .10, .20, .25$. Together with the tables listed under (2) are given the values of certain ratios useful for comparing the various statistics. (3) Probability that the standard deviation of a normal distribution will be underestimated by the sample standard deviation s and by unbiased estimators of σ based on s, on the mean deviation, and on the sample range. Divisors are given for obtaining the corresponding "median unbiased" estimators.

Table of
THE PROBABILITY POINTS OF THE DISTRIBUTION OF THE MEDIAN IN RANDOM SAMPLES FROM ANY CONTINUOUS POPULATION

Churchill Eisenhart, Lola S. Deming, and Celia S. Martin

The abscissa of the ε-probability point* of the distribution of the median in random samples of size $n = 2m+1 (m \geq 0)$ from any continuous population is identical with the abscissa of the corresponding $P_{\epsilon,n}$-probability point of the parent distribution, where $P_{\epsilon,n}$ is determined by

$$\sum_{k=\frac{1}{2}(n+1)}^{n} C_k^n P_{\epsilon,n}^k (1-P_{\epsilon,n})^{n-k} = \epsilon , \quad (0 \leq \epsilon \leq 1) \qquad (1)$$

From (1) it follows that

$$P_{1-\epsilon,n} = 1-P_{\epsilon,n} \qquad (2)$$

and that

$$P_{\epsilon,2m+1} = x_\epsilon(m+1,\ m+1) = \frac{F_\epsilon(m+1,\ m+1)}{1 + F_\epsilon(m+1,\ m+1)} = \frac{1}{1 + \exp[-2z_\epsilon(m+1,\ m+1)]} , \qquad (3)$$

where $x_\epsilon(\nu_1,\nu_2)$, $F_\epsilon(\nu_1,\nu_2)$, and $z_\epsilon(\nu_1,\nu_2)$ denote the ε-probability points* of the incomplete-beta-function distribution, Snedecor's F-distribution and Fisher's z-distribution, for $\nu_1(=2q)$ and $\nu_2(=2p)$ degrees of freedom, respectively. The foregoing results are certainly not new: Harry S. Pollard implicitly utilized the first equality on the extreme left of (3) in his doctoral dissertation at the University of Wisconsin in 1933 (see Annals of Mathematical Statistics, vol. 5 (1934), p. 250), and John H. Curtiss has given the generalization of (1) appropriate to the case of the 'rth position' in random samples from any continuous population (see American Mathematical Monthly, vol. 50 (1943), p. 103) and utilized (3) explicitly to obtain the 5% point of the distribution of the median in random samples of size $n=23$. The aim of the present paper is to give these results somewhat greater publicity--they are hardly well known. To this end a table is given of the values of $P_{\epsilon,n}$ to 5 significant figures for ϵ = 0.001, 0.005, 0.01, 0.025, 0.05, 0.10, 0.20, 0.25 and n = 3(2)15(10)95, together with values of $1/\sqrt{n}$ for use in interpolation.

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*On this page and on page 2, an "ε-probability point" denotes a value exceeded with probability 1-ε. Elsewhere in this volume "ε" denotes the right-tail probability.

Probability Points of Distribution of Median Related to Probability Points of Parent Distribution

The abscissa of the ϵ-probability point of the distribution of the median in random samples of size n from any continuous distribution is identical with the abscissa of the $P_{\epsilon,n}$-probability point of the parent distribution. The values of $P_{\epsilon,n}$ that correspond to the values of ϵ and n shown as column and row designations, respectively, are given in the body of the table to five significant figures.

Sample Size n	One-tail Probability Level ϵ of the Median 0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.25	$\frac{1}{\sqrt{n}}$
	Corresponding Probability Level $P_{\epsilon,n}$ of Parent Distribution								
3	.018370	.041400	.058903	.094299	.13535	.19580	.28714	.32635	.577350
5	.047552	.082829	.10564	.14663	.18926	.24664	.32660	.35944	.447214
7	.076655	.11770	.14227	.18405	.22532	.27860	.35009	.37885	.377964
9	.10252	.14606	.17097	.21201	.25137	.30097	.36609	.39196	.333333
11	.12493	.16931	.19398	.23379	.27125	.31772	.37787	.40158	.301511
13	.14431	.18870	.21288	.25135	.28705	.33086	.38700	.40902	.277350
15	.16117	.20514	.22873	.26586	.29999	.34152	.39436	.41409	.258199
25	.22065	.26074	.28141	.31306	.34139	.37514	.41725	.43352	.200000
35	.25722	.29359	.31201	.33989	.36457	.39369	.42973	.44358	.169031
45	.28247	.31534	.33258	.35774	.37987	.40586	.43786	.45013	.149071
55	.30123	.33217	.34760	.37071	.39094	.41462	.44369	.45482	.134840
65	.31585	.34482	.35920	.38067	.39942	.42132	.44815	.45840	.124035
75	.32766	.35498	.36850	.38864	.40621	.42666	.45168	.46125	.115470
85	.33746	.36338	.37617	.39520	.41176	.43103	.45458	.46358	.108465
95	.34577	.37047	.38264	.40072	.41644	.43471	.45702	.46553	.102598

Tables of THE ARITHMETIC MEAN AND THE MEDIAN IN SMALL SAMPLES FROM THE NORMAL AND CERTAIN NON-NORMAL POPULATIONS

Churchill Eisenhart, Lola S. Deming, and Celia S. Martin

Let $\bar{x}_{\epsilon,n}$ and $\tilde{x}_{\epsilon,n}$ denote the abscissae of the one-tail ϵ-probability points of the arithmetic mean and the median, more specifically, the abscissae exceeded with probability $\underline{\epsilon}$ by the mean and the median, respectively, in random samples of size n ($= 2m+1$) from any specified population, and let $\sigma_{\bar{x}_n}$ and $\sigma_{\tilde{x}_n}$ denote the standard deviations of the mean and the median in such samples, respectively. The following symmetrical populations with zero location parameters and unit scale parameters are considered in this paper:

Type	Probability Density Function	
normal (Gaussian)	$\frac{1}{\sqrt{2x}} e^{-\frac{1}{2}x^2}$	, $-\infty \leq x \leq \infty$
double-exponential (Laplace)	$\frac{1}{2} e^{-\lvert x \rvert}$	, $-\infty \leq x \leq \infty$
rectangular (uniform)	1	, $-\frac{1}{2} \leq x \leq \frac{1}{2}$
Cauchy	$\frac{1}{x} \frac{1}{1+x^2}$	, $-\infty \leq x \leq \infty$
sech	$\frac{1}{x}$ sech x	, $-\infty \leq x \leq \infty$
sech² (derivative of "logistic")	$\frac{1}{2}$ sech² x	, $-\infty \leq x \leq \infty$

Using the basic table relating probability points of the distribution of the median to probability points of the parent distribution, given in Churchill Eisenhart, Lola S. Deming, and Celia S. Martin, "The probability points of the distribution of the median in random samples from any continuous population", values of $\tilde{x}_{\epsilon,n}$ for random samples from each of the above distributions have been evaluated, and are tabulated to 5 decimal places in the present paper, for $n = 3(2)15(10)95$ and $\epsilon = 0.001$, 0.005, 0.01, 0.025, 0.05, 0.10, 0.20, 0.25.

In the case of the *normal distribution*, values of $\bar{x}_{\epsilon,n}$ to 5 decimal places are given also for the aforementioned combinations of ϵ and n. Comparison of the values of $\tilde{x}_{\epsilon,n}$ and $\bar{x}_{\epsilon,n}$ gives precise numerical meaning to the well-known lesser accuracy of the median as an estimator of the center of a normal population for samples of *any* odd size $n = 2m+1 > 1$. Values of the ratio $R_{\epsilon,n} = \tilde{x}_{\epsilon,n}/\bar{x}_{\epsilon,n}$ are given also for this case (normal population), to 4 decimal places for the above combinations of ϵ and n, together with the best available values of $\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n}$ for $n = 3(2)15(10)55$. When $0 < \epsilon \le 0.025$, the ratio $R_{\epsilon,n}$ exceeds the ratio $\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n}$, showing that the 'tails' of the exact distribution of the median are 'longer' than the tails of the normal distribution with the same mean and standard deviation; and, when $0.05 \le \epsilon \le 0.25$, the ratio $R_{\epsilon,n}$ is less than $\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n}$. (A theoretical argument shows that the point of equality is close to the 0.042-probability point.)

In the case of the *double-exponential distribution*, values of $\bar{x}_{\epsilon,n}$ are given to 4 decimal places for $n = 3(2)11$, and $\epsilon = 0.005, 0.01, 0.025, 0.05, 0.10, 0.25$, for comparison with the corresponding values of $\tilde{x}_{\epsilon,n}$. It is found that when $n = 3$, $\bar{x}_{\epsilon,3} < \tilde{x}_{\epsilon,3}$ for $\epsilon = 0.005, 0.01$, and 0.025, indicating that in random samples of 3 from a double-exponential distribution the arithmetic mean furnishes narrower confidence limits for the center of the distribution at the 0.95, 0.98, and 0.99 levels of confidence. When $n=5$, the mean is 'better' at the .98 and .99 levels of confidence. For all other combinations of ϵ and n (≥ 3), the median is 'better'.

In the case of the *rectangular distribution*, values of $\bar{x}_{\epsilon,n}$ are tabulated to 4 decimals for $n = 3(2)9$, and values of $\check{x}_{\epsilon,n}$, the ϵ-probability point of the mid-range in samples of n, for $n = 3(2)15(10)95$, in each instance for $\epsilon = 0.001, 0.005, 0.01, 0.025, 0.05, 0.10, 0.25$. The superiority of the mid-range over the mean and the median, well-known but here exhibited numerically for the first time, is truly amazing.

TABLE 1a

Probability Points of the Distributions of the Mean and the Median in Random Samples from a Normal Population

Let $\bar{x}_{\epsilon,n}$ and $\tilde{x}_{\epsilon,n}$ denote the abscissae of the one-tail ϵ-probability points of the mean and the median, that is, the abscissae exceeded with probability ϵ by the mean and the median, respectively, in random samples of size n from a normal population with zero mean and unit standard deviation. Values of $\bar{x}_{\epsilon,n}$ and $\tilde{x}_{\epsilon,n}$ are given in the body of the table to five decimal places, for the values of ϵ and n shown as column and row designations, respectively.

Sample Size n		One-tail Probability ϵ: 0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.25
3	$\bar{x}$	1.78415	1.48716	1.34312	1.13159	0.94966	0.73990	0.48591	0.38942
	$\tilde{x}$	2.08864	1.73467	1.56405	1.31474	1.10145	0.85672	0.56176	0.45001
5		1.38199	1.15195	1.04037	0.87652	0.73560	0.57313	0.37638	0.30164
		1.66907	1.38629	1.25005	1.05100	0.88063	0.68510	0.44932	0.35996
7		1.16800	0.97357	0.87928	0.74080	0.62170	0.48438	0.31810	0.25493
		1.42794	1.18656	1.07018	0.90004	0.75435	0.58701	0.38508	0.30850
9		1.03008	0.85861	0.77545	0.65332	0.54828	0.42718	0.28054	0.22483
		1.26732	1.05348	0.95034	0.79947	0.67018	0.52161	0.34223	0.27421
11		0.93174	0.77664	0.70142	0.59095	0.49594	0.38640	0.25376	0.20337
		1.15069	0.95690	0.86332	0.72642	0.60904	0.47408	0.31108	0.24926
13		0.85708	0.71441	0.64521	0.54360	0.45620	0.35544	0.23342	0.18707
		1.06115	0.88270	0.79647	0.67025	0.56202	0.43754	0.28715	0.23007
15		0.79789	0.66508	0.60066	0.50606	0.42470	0.33090	0.21731	0.17415
		0.98966	0.82340	0.74304	0.62538	0.52443	0.40832	0.26797	0.21473
25		0.61805	0.51517	0.46527	0.39199	0.32897	0.25631	0.16832	0.13490
		0.77000	0.64107	0.57866	0.48720	0.40867	0.31827	0.20893	0.16742
35		0.52234	0.43539	0.39322	0.33129	0.27803	0.21662	0.14226	0.11401
		0.65194	0.54293	0.49016	0.41276	0.34627	0.26971	0.17706	0.14190
45		0.46066	0.38398	0.34679	0.29217	0.24520	0.19104	0.12546	0.10055
		0.57552	0.47936	0.43280	0.36451	0.30582	0.23821	0.15640	0.12533
55		0.41669	0.34732	0.31368	0.26428	0.22179	0.17280	0.11348	0.09095
		0.52087	0.43393	0.39181	0.32997	0.27687	0.21568	0.14162	0.11349
65		0.38330	0.31949	0.28855	0.24310	0.20402	0.15896	0.10439	0.08366
		0.47934	0.39934	0.36060	0.30372	0.25485	0.19852	0.13034	0.10447
75		0.35683	0.29743	0.26862	0.22632	0.18993	0.14798	0.09718	0.07788
		0.44638	0.37191	0.33583	0.28287	0.23731	0.18488	0.12142	0.09729
85		0.33518	0.27939	0.25233	0.21259	0.17841	0.13900	0.09129	0.07316
		0.41941	0.34944	0.31556	0.26579	0.22302	0.17375	0.11410	0.09142
95		0.31705	0.26427	0.23868	0.20109	0.16876	0.13148	0.08635	0.06920
		0.39677	0.33061	0.29855	0.25148	0.21101	0.16440	0.10794	0.08651

TABLE 1b

Values of the Ratio of the ϵ-probability Point of the Median to the ϵ-probability Point of the Arithmetic Mean, and of the Ratio of the Standard Deviation of the Median to that of the Mean, in Random Samples from a Normal Population.

Let $\bar{x}_{\epsilon,n}$ and $\tilde{x}_{\epsilon,n}$ denote the abscissae of the one-tail ϵ-probability points of the arithmetic mean and the median, respectively, and let $\sigma_{\bar{x}_n}$ and $\sigma_{\tilde{x}_n}$ denote the standard deviations of the mean and the median, respectively, in random samples of size n(= 2m+1) from a normal population with zero mean and unit standard deviation. Values of the ratio $R_{\epsilon,n} = \tilde{x}_{\epsilon,n} / \bar{x}_{\epsilon,n}$ are given in the body of the table to 4 decimal places, for the values of ϵ and n shown as column and row designations, respectively. Values of the ratio $\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n}$ are given also, for purposes of comparison.

Sample Size n	One-tail Probability ϵ 0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.25	$\frac{\sigma_{\tilde{x}_n}}{\sigma_{\bar{x}_n}}$
3	1.1707	1.1664	1.1645	1.1619	1.1598	1.1579	1.1561	1.1556	1.1602 1/
5	1.2077	1.2034	1.2015	1.1991	1.1972	1.1954	1.1938	1.1933	1.1976 2/
7	1.2226	1.2188	1.2171	1.2150	1.2134	1.2119	1.2106	1.2101	1.2137 3/
9	1.2303	1.2270	1.2255	1.2237	1.2223	1.2211	1.2199	1.2196	1.2227 3/
11	1.2350	1.2321	1.2308	1.2292	1.2281	1.2269	1.2259	1.2256	1.2283 3/
13	1.2381	1.2356	1.2344	1.2330	1.2320	1.2310	1.2302	1.2299	1.2322 3/
15	1.2403	1.2380	1.2370	1.2358	1.2348	1.2340	1.2331	1.2330	1.2351 3/
25	1.2459	1.2444	1.2437	1.2429	1.2423	1.2417	1.2413	1.2411	1.2424 4/
35	1.2481	1.2470	1.2465	1.2459	1.2454	1.2451	1.2446	1.2446	1.2456 4/
45	1.2493	1.2484	1.2480	1.2476	1.2472	1.2469	1.2466	1.2464	1.2473 4/
55	1.2500	1.2494	1.2491	1.2486	1.2483	1.2481	1.2480	1.2478	1.2484 4/
65	1.2506	1.2499	1.2497	1.2494	1.2491	1.2489	1.2486	1.2487	1.2492 4/
75	1.2510	1.2504	1.2502	1.2499	1.2495	1.2494	1.2494	1.2492	1.2497 4/
85	1.2513	1.2507	1.2506	1.2502	1.2500	1.2500	1.2499	1.2496	1.2501 4/
95	1.2514	1.2510	1.2508	1.2506	1.2504	1.2504	1.2500	1.2501	1.2505 4/

1/ Exact value = $\sqrt{3(1 - \frac{\sqrt{3}}{x})}$, from H. L. Jones, Ann. Math. Stat. 19, 270-273(1948).

2/ Exact value = $\sqrt{5\left[1 - \frac{10\sqrt{3}}{x}\left(1 - \frac{3}{x}\arctan\sqrt{\frac{5}{3}}\right)\right]}$, due to H. G. Landau (see Paul H. Jacobsen, J. Amer. Stat. Assoc. 42, footnote on p. 580, 1947).

3/ From D. Teichroew, Ann. Math. Stat. 27(June 1956), Table II, p. 417.

4/ Approximate values from M. M. Siddiqui, Ann. Math. Stat. 33(March 1962), eq. 14, p. 164.

TABLE 2a

Probability Points of the Distribution of the Mean and the Median in Random Samples from a Double-Exponential Population

Let $\bar{x}_{\epsilon,n}$ and $\tilde{x}_{\epsilon,n}$ denote the abscissae of the one-tail ϵ-probability points of the mean and the median, that is, the abscissae <u>exceeded</u> with probability <u>ϵ</u> by the mean and the median, respectively, in random samples of size n from a double-exponential population with zero location and unit scale parameters. Values of $\bar{x}_{\epsilon,n}$ to four decimal places and of $\tilde{x}_{\epsilon,n}$ to five decimal places are given in the body of the table for the values of ϵ and n shown as column and row designations, respectively.

Sample Size n		One-tail Probability ϵ 0.001	0.005	0.010	0.025	0.050	0.100	0.200	0.250
1	$\bar{x}$	6.21461	4.60517	3.91202	2.99573	2.30259	1.60944	0.91629	0.69315
	$\tilde{x}$	6.21461	4.60517	3.91202	2.99573	2.30259	1.60944	0.91629	0.69315
3			2.3533	2.0577	1.6562	1.3368	0.9978		0.4940
		3.30389	2.49133	2.13872	1.66814	1.30674	0.93751	0.55464	0.42664
5			1.7559	1.5511	1.2666	1.0363	0.7863		0.3998
		2.35278	1.79783	1.55457	1.22670	0.97149	0.70668	0.42587	0.33006
7			1.4566	1.2933	1.0640	0.8764	0.6699		0.3443
		1.87529	1.44647	1.25688	0.99940	0.79709	0.58483	0.35642	0.27747
9			1.2703	1.1315	0.9352	0.7732	0.5935		0.3068
		1.58455	1.23059	1.07312	0.85797	0.68768	0.50760	0.31173	0.24345
11			1.1405	1.0180	0.8440	0.6997	0.5385		0.2793
		1.38685	1.08288	0.94685	0.76018	0.61157	0.45344	0.28006	0.21920
13	$\tilde{x}$	1.24264	0.97445	0.85388	0.68776	0.55495	0.41291	0.25618	0.20084
15		1.13215	0.89092	0.78207	0.63164	0.51086	0.38120	0.23734	0.18635
25		0.81803	0.65108	0.57480	0.46821	0.38158	0.28731	0.18092	0.14267
35		0.66468	0.53242	0.47157	0.38599	0.31589	0.23904	0.15145	0.11973
45		0.57104	0.45937	0.40773	0.33480	0.27478	0.20860	0.13271	0.10507
55		0.50673	0.40896	0.36356	0.29919	0.24605	0.18725	0.11948	0.09471
65		0.45934	0.37159	0.33073	0.27268	0.22459	0.17122	0.10948	0.08687
75		0.42263	0.34255	0.30517	0.25195	0.20774	0.15862	0.10163	0.08067
85		0.39316	0.31916	0.28457	0.23522	0.19417	0.14843	0.09523	0.07563
95		0.36883	0.29984	0.26751	0.22135	0.18287	0.13993	0.08988	0.07143

TABLE 2b

Values of Certain Ratios Useful for Judging the Normality of the Mean and the Median in Random Samples of Size n from a Double-Exponential Population.

Let $\bar{x}_{\epsilon,n}$ and $\tilde{x}_{\epsilon,n}$ denote the abscissae of the one-tail ϵ-probability points of the arithmetic mean and the median, respectively, and let $\sigma_{\bar{x}_n}$ and $\sigma_{\tilde{x}_n}$ denote the standard deviations of the mean and the median, respectively, in random samples of size n (= 2m+1) from a double-exponential population with zero location and unit scale parameters. Values of the ratios $\bar{R}_{\epsilon,n} = \bar{x}_{\epsilon,n}/(K_\epsilon/\sqrt{n})$ and $\tilde{R}_{\epsilon,n} = \tilde{x}_{\epsilon,n}/(K_\epsilon/\sqrt{n})$, where K_ϵ is the standardized normal deviate exceeded with probability ϵ, are given in the body of the table to 3 and 4 decimal places, respectively, for the values of ϵ and n shown as column and row designations, respectively. Some values of $\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n}$ are given also to 4 decimals. ($\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n} \to 0.25$ as $n \to \infty$.)

Sample Size n		One-tail probability ϵ 0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.25	$\sigma_{\tilde{x}_n}/\sigma_{\bar{x}_n}$
1	$\bar{R}_{\epsilon,n}$	2.0110	1.7878	1.6816	1.5285	1.3999	1.2559	1.0887	1.0277	1.0000
	$\tilde{R}_{\epsilon,n}$	2.0110	1.7878	1.6816	1.5285	1.3999	1.2559	1.0887	1.0277	
3			1.582	1.532	1.464	1.408	1.349		1.269	0.9789[1]
		1.8518	1.6752	1.5924	1.4742	1.3760	1.2671	1.1414	1.0956	
5			1.524	1.491	1.445	1.409	1.372		1.325	0.9370[1]
		1.7025	1.5607	1.4942	1.3995	1.3207	1.2330	1.1315	1.0942	
7			1.496	1.471	1.436	1.410	1.383		1.351	~0.90[1]
		1.6056	1.4857	1.4294	1.3491	1.2821	1.2074	1.1205	1.0884	
9			1.479	1.459	1.431	1.410	1.389		1.365	0.8827[2]
		1.5383	1.4332	1.3839	1.3132	1.2542	1.1883	1.1112	1.0828	
11			1.469	1.451	1.428	1.411	1.394		1.373	
		1.4885	1.3943	1.3499	1.2864	1.2332	1.1735	1.1036	1.0778	
13	$\tilde{R}_{\epsilon,n}$	1.4499	1.3640	1.3234	1.2652	1.2165	1.1617	1.0975	1.0736	
15		1.4189	1.3396	1.3020	1.2482	1.2029	1.1520	1.0922	1.0701	
25		1.3236	1.2638	1.2354	1.1944	1.1599	1.1209	1.0749	1.0576	
35		1.2725	1.2229	1.1993	1.1651	1.1362	1.1035	1.0646	1.0502	
45		1.2396	1.1963	1.1757	1.1459	1.1206	1.0919	1.0578	1.0450	
55		1.2161	1.1775	1.1590	1.1321	1.1094	1.0836	1.0529	1.0413	
65		1.1984	1.1631	1.1462	1.1217	1.1008	1.0771	1.0488	1.0384	
75		1.1844	1.1517	1.1361	1.1132	1.0938	1.0719	1.0458	1.0358	
85		1.1730	1.1423	1.1278	1.1064	1.0883	1.0678	1.0432	1.0338	
95		1.1633	1.1346	1.1208	1.1008	1.0836	1.0643	1.0409	1.0322	

1/ From M. M. Siddiqui, Ann. Math. Stat. 33 (March 1962), Table 5, p. 165.

2/ From E. L. Crow in a private communication, April 1963.

TABLE 3a

Probability Points of the Distribution of the Mean, the Median, and the Mid-Range in Random Samples from a Rectangular Population

Let $\bar{x}_{\epsilon,n}$, $\tilde{x}_{\epsilon,n}$ and $\check{x}_{\epsilon,n}$ denote the abscissae of the one-tail ϵ-probability points of the mean, the median, and the mid-range, that is, the abscissae <u>exceeded</u> with probability $\underline{\epsilon}$ by the mean, the median, and the mid-range, respectively, in random samples of size n from a rectangular population with zero location and unit scale parameters. Values of $\bar{x}_{\epsilon,n}$ to four decimal places and of $\tilde{x}_{\epsilon,n}$ and $\check{x}_{\epsilon,n}$ to five decimal places are given in the body of the table for the values of ϵ and n shown as column and row designations, respectively.

Sample Size n		One-tail Probability ϵ 0.001	0.005	0.010	0.025	0.050	0.100	0.200	0.250
1	$\bar{x}$	0.49900	0.49500	0.49000	0.47500	0.45000	0.40000	0.30000	0.25000
	$\tilde{x}$	0.49900	0.49500	0.49000	0.47500	0.45000	0.40000	0.30000	0.25000
3	$\bar{x}$	0.4394	0.3964	0.3695	0.3229	0.2769	0.2189		0.1176
	$\tilde{x}$	0.48163	0.45860	0.44110	0.40570	0.36465	0.30420	0.21286	0.17365
	$\check{x}$	0.43700	0.39228	0.36428	0.31580	0.26792	0.20760	0.13160	0.10315
5		0.3691	0.3194	0.2926	0.2508	0.2131	0.1679		0.0894
		0.45245	0.41717	0.39436	0.35337	0.31074	0.25336	0.17340	0.14056
		0.35573	0.30095	0.27135	0.22536	0.18452	0.13761	0.08372	0.06472
7		0.3200	0.2732	0.2492	0.2125	0.1799	0.1413		0.0750
		0.42334	0.38230	0.35773	0.31595	0.27468	0.22140	0.14991	0.12115
		0.29422	0.24103	0.21407	0.17408	0.14016	0.10270	0.06135	0.04714
9		0.2858	0.2426	0.2207	0.1877	0.1585	0.1243		0.0658
		0.39748	0.35394	0.32903	0.28799	0.24863	0.19903	0.13391	0.10804
		0.24934	0.20026	0.17626	0.14156	0.11287	0.08187	0.04840	0.03706
11	$\tilde{x}$	0.37507	0.33069	0.30602	0.26621	0.22875	0.18228	0.12213	0.09842
	$\check{x}$	0.21581	0.17103	0.14964	0.11920	0.09443	0.06806	0.03996	0.03053
13		0.35569	0.31130	0.28712	0.24865	0.21295	0.16914	0.11300	0.09098
		0.19000	0.14915	0.12993	0.10291	0.08116	0.05822	0.03403	0.02596
15		0.33883	0.29486	0.27127	0.23414	0.20001	0.15848	0.10564	0.08501
		0.16960	0.13218	0.11478	0.09052	0.07115	0.05087	0.02963	0.02258
25		0.27935	0.23926	0.21859	0.18694	0.15861	0.12486	0.08275	0.06648
		0.11005	0.08412	0.07243	0.05646	0.04400	0.03117	0.01799	0.01367
35		0.24278	0.20641	0.18799	0.16011	0.13543	0.10631	0.07027	0.05642
		0.08134	0.06164	0.05288	0.04102	0.03184	0.02247	0.01292	0.00980
45		0.21753	0.18416	0.16742	0.14226	0.12013	0.09414	0.06214	0.04987
		0.06450	0.04864	0.04163	0.03220	0.02494	0.01757	0.01008	0.00764
55		0.19877	0.16783	0.15240	0.12929	0.10906	0.08538	0.05631	0.04518
		0.05342	0.04016	0.03433	0.02651	0.02050	0.01442	0.00826	0.00626
65		0.18415	0.15518	0.14080	0.11933	0.10058	0.07868	0.05185	0.04160
		0.04559	0.03420	0.02920	0.02252	0.01740	0.01223	0.00700	0.00530
75		0.17234	0.14502	0.13150	0.11136	0.09379	0.07334	0.04832	0.03875
		0.03976	0.02978	0.02541	0.01958	0.01512	0.01062	0.00607	0.00460
85		0.16254	0.13662	0.12382	0.10480	0.08824	0.06897	0.04542	0.03642
		0.03525	0.02637	0.02249	0.01732	0.01336	0.00938	0.00536	0.00406
95		0.15423	0.12953	0.11736	0.09928	0.08356	0.06529	0.04298	0.03447
		0.03166	0.02366	0.02017	0.01552	0.01197	0.00840	0.00480	0.00363

TABLE 4a

Probability Points of the Distribution of the Median in Random Samples from a Cauchy Population

Let $\tilde{x}_{\epsilon,n}$ denote the abscissa of the one-tail ϵ-probability point of the median, that is, the abscissa exceeded with probability ϵ by the median in random samples of size n from a Cauchy population with zero location and unit scale parameters. Values of $\tilde{x}_{\epsilon,n}$ are given in the body of the table to five decimal places for the values of ϵ and n shown as column and row headings respectively.

Sample Size n	One-tail Probability ε							
	0.001	0.005	0.010	0.025	0.050	0.100	0.200	0.250
1	318.33578	63.65527	31.82011	12.70628	6.31376	3.07768	1.37638	1.00000
3	17.30838	7.64498	5.34252	3.27612	2.20827	1.41527	0.79017	0.60699
5	6.64405	3.75582	2.90172	2.01505	1.47884	1.02133	0.60581	0.47271
7	4.07192	2.58002	2.08635	1.53231	1.16846	0.83471	0.50917	0.40011
9	2.99675	2.02419	1.67921	1.27253	0.99142	0.72190	0.44740	0.35308
11	2.41571	1.69930	1.43259	1.10741	0.87466	0.64471	0.40369	0.31944
13	2.05250	1.48446	1.26538	0.99155	0.79063	0.58776	0.37071	0.29387
15	1.80326	1.33066	1.14344	0.90500	0.72659	0.54355	0.34462	0.27361
25	1.20378	0.93470	0.81984	0.66564	0.54408	0.41370	0.26599	0.21194
35	0.95563	0.75778	0.67041	0.55020	0.45315	0.34698	0.22442	0.17913
45	0.81429	0.65310	0.58052	0.47926	0.39640	0.30469	0.19774	0.15796
55	0.72065	0.58223	0.51906	0.43010	0.35669	0.27485	0.17877	0.14290
65	0.65306	0.53019	0.47364	0.39350	0.32693	0.25234	0.16435	0.13144
75	0.60136	0.48997	0.43835	0.36486	0.30348	0.23457	0.15298	0.12234
85	0.56019	0.45765	0.40991	0.34168	0.28454	0.22014	0.14367	0.11492
95	0.52638	0.43099	0.38637	0.32242	0.26871	0.20804	0.13585	0.10872

TABLE 5a

Probability Points of the Distribution of the Median in Random Samples from a Sech Population

Let $\tilde{x}_{\epsilon,n}$ denote the abscissa of the one-tail ϵ-probability point of the median, that is, the abscissa exceeded with probability ϵ by the median in random samples of size n from a sech population with zero location and unit scale parameters. Values of $\tilde{x}_{\epsilon,n}$ are given in the body of the table to five decimal places for the values of ϵ and n shown as column and row headings respectively.

Sample Size n	One-tail Probability ϵ							
	0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.250
1	6.44463	4.84674	4.15351	3.23678	2.54209	1.84273	1.12418	0.88137
3	3.54516	2.73149	2.37742	1.90235	1.53308	1.14683	0.72497	0.57480
5	2.59248	2.03373	1.78690	1.45036	1.18297	0.89638	0.57388	0.45667
7	2.11201	1.67655	1.48160	1.21255	0.99563	0.75954	0.48940	0.39014
9	1.81742	1.45439	1.29023	1.06158	0.87529	0.67051	0.43368	0.34613
11	1.61546	1.30047	1.15678	0.95532	0.78991	0.60679	0.39346	0.31424
13	1.46687	1.18611	1.05717	0.87539	0.72534	0.55830	0.36270	0.28980
15	1.35202	1.09700	0.97926	0.81258	0.67431	0.51982	0.33814	0.27030
25	1.01839	0.83444	0.74808	0.62429	0.52029	0.40272	0.26295	0.21039
35	0.84965	0.69936	0.62826	0.52566	0.43892	0.34037	0.22258	0.17819
45	0.74378	0.61383	0.55205	0.46259	0.38669	0.30016	0.19647	0.15731
55	0.66951	0.55353	0.49820	0.41783	0.34953	0.27151	0.17783	0.14242
65	0.61379	0.50805	0.45751	0.38399	0.32137	0.24974	0.16362	0.13106
75	0.56999	0.47222	0.42540	0.35721	0.29901	0.23247	0.15239	0.12204
85	0.53439	0.44302	0.39922	0.33535	0.28083	0.21840	0.14318	0.11467
95	0.50468	0.41865	0.37735	0.31708	0.26558	0.20656	0.13544	0.10850

TABLE 6a

Probability Points of the Distribution of the Median in Random Samples from a Sech^2 Population

Let $\tilde{x}_{\epsilon,n}$ denote the abscissa of the one-tail ϵ-probability point of the median, that is, the abscissa exceeded with probability ϵ by the median in random samples of size n from a sech^2 population with zero location and unit scale parameters. Values of $\tilde{x}_{\epsilon,n}$ are given in the body of the table to five decimal places for the values of ϵ and n shown as column and row headings, respectively.

Sample Size n	One-tail Probability ϵ 0.001	0.005	0.01	0.025	0.05	0.10	0.20	0.25
1	3.45338	2.64665	2.29756	1.83178	1.47222	1.09861	0.69315	0.54931
3	1.98925	1.57110	1.38558	1.13112	0.92723	0.70638	0.45466	0.36237
5	1.49863	1.20226	1.06804	0.88064	0.72741	0.55831	0.36180	0.28890
7	1.24434	1.00720	0.89828	0.74457	0.61746	0.47571	0.30932	0.24722
9	1.08477	0.88292	0.78938	0.65643	0.54566	0.42134	0.27451	0.21954
11	0.97328	0.79526	0.71218	0.59352	0.49414	0.38213	0.24930	0.19944
13	0.88997	0.72924	0.65383	0.54571	0.45488	0.35215	0.22997	0.18401
15	0.82477	0.67724	0.60775	0.50786	0.42367	0.32826	0.21451	0.17169
25	0.63094	0.52106	0.46874	0.39293	0.32855	0.25511	0.16704	0.13375
35	0.53023	0.43901	0.39537	0.33189	0.27779	0.21591	0.14148	0.11332
45	0.46612	0.38648	0.34827	0.29259	0.24505	0.19055	0.12493	0.10007
55	0.42072	0.34919	0.31480	0.26459	0.22168	0.17245	0.11310	0.09061
65	0.38645	0.32094	0.28942	0.24335	0.20394	0.15868	0.10407	0.08339
75	0.35939	0.29861	0.26933	0.22652	0.18983	0.14775	0.09694	0.07766
85	0.33732	0.28036	0.25292	0.21275	0.17835	0.13882	0.09109	0.07297
95	0.31884	0.26510	0.23918	0.20123	0.16870	0.13133	0.08617	0.06905

Table of
THE RELATIVE FREQUENCIES WITH WHICH CERTAIN ESTIMATORS OF THE STANDARD DEVIATION OF A NORMAL POPULATION TEND TO UNDERESTIMATE ITS VALUE

Churchill Eisenhart and Celia S. Martin

Let $x_1, x_2, \ldots, x_n$ denote a random sample of n independent observations from a normal population with mean μ and standard deviation σ. Common estimators of σ are

$$s_1 = \sum_{i=1}^{n} \sqrt{(x_i-\bar{x})^2/n}, \qquad s_2 = s_1\sqrt{n/(n-1)}, \qquad s_3 = s_1/c_2 ,$$

$$m_1 = \sqrt{\frac{\pi}{2}} \sum_{i=1}^{n} |x_i-\bar{x}|/n, \qquad m_2 = m_1\sqrt{n/(n-1)}, \qquad R_1 = (x_L-x_S)/d_2 = R/d_2$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$, x_L is the largest and x_S the smallest of the x's, $c_2\sigma = E(s_1)$, and $d_2\sigma = E(x_L-x_S)$, the symbol E( ) denoting "mathematical expectation (or mean value) of." A table is given that shows to 3 decimals the relative frequencies (probabilities) with which these estimators tend to underestimate σ when n = 2(1)10, 12, 15, 20, 24, 30, 40, 60, 120. The results show among other things that, for very small samples ($n \leq 10$) such as chemists and physicists commonly use, Bessel's formula for the probable error, which is based on s_2, has a marked downward bias in the probability sense (in addition to its known slight downward bias in the mean value sense), whereas Peter's formula, which is based on m_2, has only a slight downward bias in the probability sense and no bias in the mean value sense. Divisors are given by means of which "median estimators" of σ can be computed readily from the basic quantities

$$\sum_{i=1}^{n} (x_i-\bar{x})^2, \qquad \sum_{i=1}^{n} |x_i-\bar{x}|, \qquad (x_L-x_S),$$

that is, estimators that will over- and underestimate σ equally often in repeated use. Median estimators, like maximum likelihood estimators ("modal estimators") have the useful property that if $T_{\frac{1}{2}}$ is a median estimator of θ, then $f(T_{\frac{1}{2}})$ is a median estimator of $f(\theta)$, a property unfortunately not possessed by the customary "unbiased" ("mean") estimators.

The Relative Frequencies with which Certain Estimators of the Standard Deviation of a Normal Population Tend to Underestimate Its Value in Samples of Size n, Together with Divisors for Obtaining the Corresponding "Median" Estimators.

Sample Size n	Probability that σ will be underestimated by s_1	s_2	s_3	m_1	m_2	R_1	Divisors for "Median" Estimation of σ from $\Sigma(\ )^2$	$\Sigma\|\ \|$	R
2	.843	.683	.575	.741	.575	.575	0.4549	0.9538	0.9538
3	.777	.632	.544	.693	.545	.545	1.386	1.833	1.588
4	.739	.608	.531	.667	.537	.536	2.366	2.652	1.978
5	.713	.594	.527	.649	.532	.531	3.357	3.459	2.257
6	.694	.584	.524	.636	.529	.529	4.351	4.262	2.472
7	.679	.577	.521	.626	.527	.528	5.348	5.064	2.645
8	.667	.571	.519	.618	.525	.527	6.346	5.865	2.791
9	.658	.567	.518	.611	.523	.527	7.344	6.665	2.916
10	.650	.563	.517	.605	.522	.527	8.343	7.465	3.024
12	.637	.557	.515	.596	.520	.5265	10.34	9.063	3.207
15	.622	.550	.513	.586	.517	.527	13.34	11.460	3.422
20	.605	.543	.511	.574	.515	.527	18.34	15.452	3.686
24	.596	.539	.510	.568	.514		22.34	18.644	
30	.586	.525	.509	.560	.512		28.34	23.437	
40	.574	.530	.508	.552	.511		38.34	31.413	
60	.561	.524	.506	.542	.509		58.33	47.372	
120	.543	.517	.504	.530	.506		118.33	95.246	

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Radio Standards Physics. Frequency and Time Disseminations. Radio and Microwave Materials. Atomic Frequency and Time-Interval Standards. Radio Plasma. Microwave Physics.

Radio Standards Engineering. High Frequency Electrical Standards. High Frequency Calibration Services. High Frequency Impedance Standards. Microwave Calibration Services. Microwave Circuit Standards. Low Frequency Calibration Services.

Joint Institute for Laboratory Astrophysics - NBS Group (Univ. of Colo.).

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