THE ELEMENTS OF ANALYTICAL GEOMETRY.

By ELIAS LOOMIS, LL.D.,

PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY IN YALE COLLEGE,

AND AUTHOR OF "A COURSE OF MATHEMATICS."

REVISED EDITION.

NEW YORK ·:· CINCINNATI ·:· CHICAGO

AMERICAN BOOK COMPANY

W. P. I.

PREFACE.

The stereotype plates of my Elements of Analytical Geometry having become so much worn by long-continued use that it was thought desirable to renew them, I have improved the opportunity to make a thorough revision of the work. In doing this, it has been thought best to extend considerably the plan of the work, and accordingly I have not merely added a third part on Geometry of three dimensions, but have introduced new matter in nearly every section of the book. I have aimed to illustrate every portion of the subject, as far as practicable, by numerical examples, generally of the simplest kind, the main object being to make sure that the student understands the meaning of the formulæ which he has learned. In making this revision I have been favored with the constant assistance of Prof. H. A. Newton, who has carefully examined every portion of the volume, and to whom I am indebted for numerous suggestions both as to the plan and execution of the work. It is hoped that the volume in its present form will be found adapted to the wants of mathematical students in our colleges and higher schools; and that, if any should desire to prosecute this subject further, they will find this volume a good introduction to larger and more difficult treatises.

CONTENTS.

PART I.

DETERMINATE GEOMETRY.

SECTION I.

APPLICATION OF ALGEBRA TO GEOMETRY.

SECTION II.

CONSTRUCTION OF ALGEBRAIC EXPRESSIONS.

PART II.

INDETERMINATE GEOMETRY.

SECTION I.

CO-ORDINATES OF A POINT.

SECTION II.

THE STRAIGHT LINE.

SECTION III.

TRANSFORMATION OF CO-ORDINATES.

SECTION IV.

THE CIRCLE.

SECTION V.

THE PARABOLA.

SECTION VI.

THE ELLIPSE.

SECTION VII.

THE HYPERBOLA.

SECTION VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

SECTION IX.

LINES OF THE THIRD AND HIGHER ORDERS.

SECTION X.

TRANSCENDENTAL CURVES.

PART III.

GEOMETRY OF THREE DIMENSIONS.

SECTION I.

OF POINTS IN SPACE.

SECTION II.

THE STRAIGHT LINE IN SPACE.

SECTION III.

OF THE PLANE IN SPACE.

SECTION IV.

OF SURFACES OF REVOLUTION.

SECTION V.

GENERAL EQUATION OF THE SECOND DEGREE BETWEEN THREE VARIABLES.

APPENDIX.

ANALYTICAL GEOMETRY.

PART I.

DETERMINATE GEOMETRY.

SECTION I.

APPLICATION OF ALGEBRA TO GEOMETRY.

1. We have seen in Geometry (pages 40, 69, and 162) that all geometrical magnitudes, including angles, lines, surfaces, and solids, may be expressed either exactly or approximately by numbers, and for this purpose it is only necessary to take one of these magnitudes as the unit of measure. If we denote by a, b, and c the number of linear units contained in the adjacent edges of a rectangular parallelopiped, then will ab, ac, bc denote the magnitude of three of its faces, and abc will denote its volume.

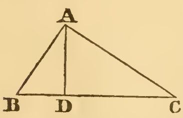
2. In like manner, every geometrical magnitude may be represented by algebraic symbols, and the relations between different magnitudes, or different parts of the same figure, may also be denoted by symbols. We may then operate upon these representatives by the known methods of Algebra, and thus deduce relations before unknown; and since the operations are generally very much abridged by the use of algebraic symbols, the algebraic method has many advantages over the geometrical. This method is applicable either to the solution of problems or to the demonstration of theorems.

3. Geometrical problems may be divided into two classes: *determinate* and *indeterminate.* Determinate problems are those in which the number of independent equations is equal to the number of unknown quantities, and therefore the unknown quantities can have but a finite number of values. Indeterminate problems are those in which the number of independent equations is less than the number of unknown quantities involved, and therefore the unknown quantities may have an infinite number of values.

4. If it is required to determine the magnitude of certain lines from the knowledge of several other lines connected with the former in the same figure, we first draw a figure which represents all the parts of the problem, both those which are given and those which are required to be found. We denote both the known and unknown parts of the figure, or as many of them as may be necessary, by convenient symbols. We then observe the relations which the several parts of the figure bear to each other, from which, by the aid of the proper theorems in Geometry, we derive as many independent equations as there are unknown quantities employed. By solving these equations we obtain expressions for the unknown quantities in terms of the known quantities.

If a theorem is to be demonstrated, we express by algebraic equations the relations which exist between the different parts of the figure, and then transform these equations in such a manner as to deduce an equation which expresses the theorem sought.

5. In order to illustrate these principles, let it be required to deduce the various properties of a right-angled triangle from the principles that *two equiangular triangles have their homologous sides proportional*, and that the perpendicular drawn from the right angle of a right-angled triangle to the hypothenuse divides the whole triangle into similar triangles.



Let the triangle ABC be right angled at A; from A draw AD perpendicular to BC, and let us put $BC=a$, $AC=b$, $AB=c$, $AD=h$, $BD=m$, and $DC=n$. Then, by similar triangles, we have the proportions

$$\left.\begin{array}{l} a:b::b:n \\ a:c::c:m \\ a:c::b:h \end{array}\right\} \text{whence we deduce} \left\{\begin{array}{ll} b^2=an. & (1) \\ c^2=am. & (2) \\ bc=ah. & (3) \end{array}\right.$$

Moreover, we have $a=m+n.$ (4)

These four equations involve the various properties of right-angled triangles, and these properties may be deduced by suitable transformations of these equations.

1st. Equations (1) and (2), or rather the proportions from which they are deduced, show that *each side about the right angle is a mean proportional between the entire hypothenuse and its adjacent segment.*

2d. By adding equations (1) and (2) member to member, we obtain

$$b^2+c^2=am+an=a(m+n);$$

whence, from equation (4), we obtain $b^2+c^2=a^2$; that is, *the square of the hypothenuse is equal to the sum of the squares of the other two sides of the triangle.*

3d. By multiplying equations (1) and (2) member by member, we obtain

$$b^2c^2=a^2mn.$$

But from equation (3) we have also $b^2c^2=a^2h^2$.

Hence $a^2mn=a^2h^2$, or $h^2=mn$; that is,

$$m:h::h:n,$$

or, *the perpendicular drawn from the vertex of the right angle upon the hypothenuse is a mean proportional between the two segments of the hypothenuse.*

4th. By dividing equation (1) by equation (2) member by member, we obtain

$$\frac{b^2}{c^2}=\frac{an}{am}; \text{ or } b^2:c^2::n:m;$$

that is, *the squares described upon the sides about the right angle are proportional to the segments of the hypothenuse.*

Thus we see that every equation deduced from the equations (1), (2), (3), and (4), when translated into geometrical language, is a geometrical theorem.

6. The four equations of the preceding article contain six quantities, of which, when a certain number are given, it may be required to deduce the values of the other quantities.

Suppose we have given the hypothenuse BC, and the perpendicular AD, and it is required to determine the other two sides of the triangle, as also the two segments of the hypothenuse.

We have already found $b^2+c^2=a^2$,
and from equation (3) we have $2bc=2ah$.
By adding and subtracting successively, we obtain

$$(b+c)^2=a^2+2ah;$$

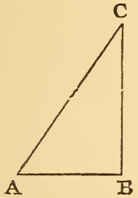
and $$(b-c)^2=a^2-2ah.$$

whence $b+c=\sqrt{a^2+2ah}$; $\quad b-c=\sqrt{a^2-2ah}$.
Knowing the sum and difference of the two sides b and c, by a well-known principle (Alg., p. 89) we obtain
the greater side $b=\frac{1}{2}\sqrt{a^2+2ah}+\frac{1}{2}\sqrt{a^2-2ah}$,
the less side $c=\frac{1}{2}\sqrt{a^2+2ah}-\frac{1}{2}\sqrt{a^2-2ah}$.

Since a, b, and c are now known quantities, the two segments are given by equations (1) and (2).

The preceding principles will be further illustrated by the following examples:

Ex. 1. *The base and sum of the hypothenuse and perpendicular of a right-angled triangle are given, to find the perpendicular.*



Let ABC be the proposed triangle, right angled at B. Represent the base AB by b, the perpendicular BC by x, and the sum of the hypothenuse and perpendicular by s; then the hypothenuse will be represented by $s-x$.

By Geom., B. IV., Pr. 11, $\overline{AB}^2+\overline{BC}^2=\overline{AC}^2$;

or, $$b^2+x^2=(s-x)^2=s^2-2sx+x^2.$$

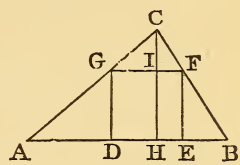
Hence $$b^2=s^2-2sx,$$

or, $$x=\frac{s^2-b^2}{2s};$$

that is, in any right-angled triangle, the perpendicular is equal to the square of the sum of the hypothenuse and perpendicular, diminished by the square of the base, and divided by twice the sum of the hypothenuse and perpendicular.

Thus, if the base is 3 feet, and the sum of the hypothenuse and perpendicular 9 feet, the expression $\frac{s^2-b^2}{2s}$ becomes $\frac{9^2-3^2}{2\times 9}=4$, the perpendicular.

Ex. 2. *The base and altitude of any triangle are given, and it is required to find the side of the inscribed square.*



Let ABC represent the given triangle, and suppose the inscribed square DEFG to be drawn. Represent the base AB by b, the perpendicular CH by h, and the side of the inscribed square by x; then will CI be represented by $h-x$.

Then, because GF is parallel to the base AB, we have by similar triangles (Geom., B. IV., Pr. 16),

$$AB : GF :: CH : CI;$$

that is, $$b : x :: h : h-x,$$

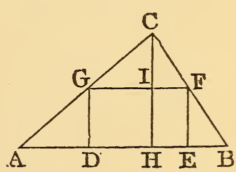
whence $$bh-bx=hx;$$

or, $$x=\frac{bh}{b+h};$$

that is, the side of the inscribed square is equal to the product of the base and height divided by their sum.

Thus, if the base of the triangle is 12 feet, and the altitude 6 feet, the side of the inscribed square is found to be 4 feet.

Ex. 3. *The base and altitude of any triangle are given, and it is required to inscribe within it a rectangle whose sides shall have to each other a given ratio.*



Let ABC be the given triangle, and suppose the required rectangle to be inscribed within it. Represent the base AB by b, the altitude CH by h, the altitude of the rectangle DG by x, and its base DE by y; also let $x : y :: 1 : n$; or $y=nx$.

Then, because the triangle CGF is similar to the triangle CAB, we have

$$AB : GF :: CH : CI;$$

that is, $$b : y :: h : h - x;$$

whence $$bh - bx = hy.$$

But, since $y = nx$, we have

$$bh - bx = hnx;$$

whence $$x = \frac{bh}{b + nh}.$$

If we suppose n equals unity, in which case the rectangle becomes a square, the preceding result becomes identical with that in Example 2.

Ex. 4. *It is required to divide a straight line in extreme and mean ratio; that is, into two parts such that one of them shall be a mean proportional between the whole line and the other part.*

Suppose the problem to be solved, and that C is such a point of the line AB that we have the proportion

$$AB : AC :: AC : CB.$$

Put $AB = a$, $AC = x$, whence $CB = a - x$.

The preceding proportion will then become

$$a : x :: x : a - x;$$

whence $$x^2 = a^2 - ax,$$

which equation, being solved, gives

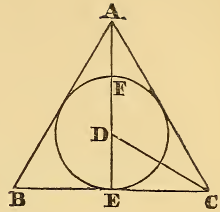
$$x = -\frac{a}{2} \pm \sqrt{a^2 + \frac{a^2}{4}}.$$

Of these two values obtained by the solution of the equation, the first is the only one which satisfies the enunciation of the problem; for the second is numerically greater than a, and therefore can not represent a *part* of the given line. We shall consider hereafter the geometrical signification of this equation.

Ex. 5. *It is required to determine the side of an equilateral triangle described about a circle whose diameter is given.*

Suppose ABC to be the required triangle described about a circle whose diameter is given. Draw AE perpendicular to BC, and join DC. Represent FE by d, and CE by x. The two triangles ACE, CDE are similar, for each contains a right angle, and the angle CAE is equal to the angle DCE. Hence we have the proportion

$$AC : EC :: DC : DE.$$



But AC is double of EC; therefore DC is double of DE, or is equal to d.

Now $$DC^2 - DE^2 = EC^2,$$

or $$d^2 - \frac{d^2}{4} = x^2,$$

whence $$x = \tfrac{1}{2} d\sqrt{3},$$

or $$2x = d\sqrt{3};$$

that is, the side of the circumscribed triangle is equal to the diameter of the circle multiplied by the square root of 3.

Ex. 6. Given the base b and the difference d between the hypothenuse and perpendicular of a right-angled triangle, to find the perpendicular.

Ans. $\dfrac{b^2 - d^2}{2d}$.

Ex. 7. Given the hypothenuse h of a right-angled triangle, and the ratio of the base to the perpendicular, as m to n, to find the perpendicular.

Ans. $\dfrac{nh}{\sqrt{m^2+n^2}}$.

Ex. 8. Given the diagonal d of a rectangle, and the perimeter $4p$, to find the lengths of the sides.

Ans. $p \pm \sqrt{\dfrac{d^2}{2} - p^2}$.

Ex. 9. If the diagonal of a rectangle be 10 feet, and its perimeter 28 feet, what are the lengths of the sides?

Ans.

Ex. 10. From any point within an equilateral triangle, perpendiculars are drawn to the three sides. It is required to find the sum, s, of these perpendiculars.

Ans. s = altitude of the triangle.

Ex. 11. Given the lengths of three perpendiculars, a, b, and

c, drawn from a certain point in an equilateral triangle to the three sides, to find the length of the three sides.

$$Ans.\ \frac{2(a+b+c)}{\sqrt{3}}.$$

Ex. 12. Given the difference, d, between the diagonal of a square and one of its sides, to find the length of the sides.

$$Ans.\ d+d\sqrt{2}.$$

Ex. 13. In a right-angled triangle, the lines a and b, drawn from the acute angles to the middle of the opposite sides, are given, to find the lengths of the sides.

$$Ans.\ 2\sqrt{\frac{4b^2-a^2}{15}},\ \text{and}\ 2\sqrt{\frac{4a^2-b^2}{15}}.$$

Ex. 14. In a right-angled triangle, having given the hypothenuse (a), and the difference between the base and perpendicular ($2d$), to determine the two sides.

$$Ans.\ \sqrt{\frac{a^2-2d^2}{2}}+d,\ \text{and}\ \sqrt{\frac{a^2-2d^2}{2}}-d.$$

Ex. 15. Having given the area (c) of a rectangle inscribed in a triangle whose base is (b) and altitude (a), to determine the height of the rectangle.

$$Ans.\ \frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{ac}{b}}.$$

Ex. 16. Having given the ratio of the two sides of a triangle, as m to n, together with the segments of the base, a and b, made by a perpendicular from the vertical angle, to determine the sides of the triangle.

$$Ans.\ m\sqrt{\frac{a^2-b^2}{m^2-n^2}},\ \text{and}\ n\sqrt{\frac{a^2-b^2}{m^2-n^2}}.$$

Ex. 17. Having given the base of a triangle ($2a$), the sum of the other two sides ($2s$), and the line (c) drawn from the vertical angle to the middle of the base, to find the sides of the triangle.

$$Ans.\ s\pm\sqrt{a^2+c^2-s^2}.$$

Ex. 18. Having given the two sides (a) and (b) about the vertical angle of a triangle, together with the line (c) bisecting that angle and terminating in the base, to find the segments of the base.

Ans. $a\sqrt{\frac{ab-c^2}{ab}}$, and $b\sqrt{\frac{ab-c^2}{ab}}$.

Ex. 19. The sum of the two legs of a right-angled triangle is s, and the perpendicular let fall from the right angle upon the hypothenuse is a. What is the hypothenuse of the triangle?

Ans. $\sqrt{s^2+a^2}-a$.

Ex. 20. Determine the radii of three equal circles, described in a given circle, which touch each other, and also the circumference of the given circle whose radius is R.

Ans. $R(2\sqrt{3}-3)$.

SECTION II.

CONSTRUCTION OF ALGEBRAIC EXPRESSIONS.

7. The construction of an algebraic expression consists in finding a geometrical figure which may be considered as representing that expression; that is, a figure in which the parts shall have the same geometrical relation as that indicated in the algebraic expression.

The *elementary expressions*, to which algebraic expressions admitting of geometrical construction may in general be reduced, are the following, viz.:

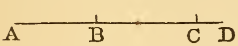
$$x=a-b+c-d, \text{ etc.}, \quad x=ab, \quad x=abc, \quad x=\frac{ab}{c},$$

$$x=\sqrt{ab}, \quad x=\sqrt{a^2+b^2}, \quad x=\sqrt{a^2-b^2};$$

where a, b, c, etc., express the number of linear units contained in the given lines.

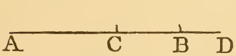
Problem I. *To construct the expression* $x=a+b$.

The symbols a and b, being supposed to stand for numerical quantities, may be represented by lines. The length of a line is determined by comparing it with some known standard, as an inch or a foot. If the line AB contains the standard unit



a times, then AB may be taken to represent a. So, also, if BC contains the standard unit b times, then BC may be taken to represent b. Therefore, in order to construct the expression $a+b$, draw an indefinite line AD. From the point A lay off a distance AB equal to a, and from B lay off a distance BC equal to b; then AC will be a right line representing $a+b$.

Problem II. *To construct the expression* $x=a-b$.



Draw the indefinite line AD. From the point A lay off a distance AB equal to a, and from B lay off a distance BC, in the direction toward A, equal to b; then will AC be the difference between AB and BC; consequently, it may be taken to represent the expression $a-b$.

Problem III. *To construct the expression*

$$x=a-b+c-d+e.$$

This expression may be written

$$x=a+c+e-(b+d).$$

To obtain an expression for $a+c+e$, draw an indefinite line AX, and from A set off $AB=a$, from B set off $BC=c$, from C set off $CD=e$; then $AD=a+c+e$.

Then set off from D toward A, $DE=b$; from E set off $EF=d$; then $DF=b+d$.

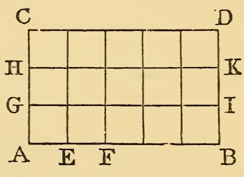
Hence $AF=a+c+e-(b+d)=x$.

In a similar manner we may construct any algebraic expression consisting of a series of letters connected together by the signs + and −.

In like manner we may construct the expressions $x=3a$, $x=5b$, etc.

Problem IV. *To construct the expression* $x=ab$.

Let ABDC be a rectangle of which the side AB contains the standard unit a times, and the side AC contains the same unit b times. If through the points E, F, etc., we draw lines parallel to AC, and through the points G, H, etc., we draw lines parallel to AB, the rectangle will be divided into square units. In the first row, AGIB, there are a square units; in the second row, GHKI, there are also a square units; and there are as many rows as there are units in AC. Therefore the rectangle ABDC contains $a \times b$ square units, or the rectangle may be considered as representing the expression ab.



An algebraic expression of two dimensions may therefore be represented by a *surface*.

Problem V. *To construct the expression* $x=abc$.

Let there be a rectangular parallelopiped whose three adjacent edges contain the standard unit respectively a, b, and c times; then, dividing the solid by planes parallel to its sides, we may prove that the number of solid units in the figure is $a\times b\times c$, and consequently the parallelopiped may be considered as representing the expression abc.

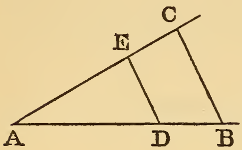
An algebraic expression of three dimensions may therefore be represented by a *solid*.

Problem VI. *To construct the expression* $x=\frac{ab}{c}$.

From this equation we derive the proportion

$$c:a::b:x;$$

that is, x is a fourth proportional to the three given lines c, a, and b.



To obtain an expression for x, draw two lines, AB, AC, making any angle with each other. From A, upon the line AB, lay off a distance $AD=c$, and $AB=a$, and upon the line AC lay off a distance $AE=b$. Join DE, and through B draw BC parallel to DE; then will AC be equal to x.

For, by similar triangles, we have

$$AD:AB::AE:AC,$$

or

$$c:a::b:AC.$$

Hence

$$AC=\frac{ab}{c}=x.$$

The expression $x=\frac{a^2}{c}$, or $x=\frac{a\times a}{c}$, may be constructed in the same manner, since x is a fourth proportional to the three lines c, a, and a.

Problem VII. *To construct the expression* $x=\frac{abc}{de}$.

This expression can be put under the form

$$x=\frac{ab}{d}\times\frac{c}{e}.$$

First find a fourth proportional m to the three quantities d, a, and b, as in Prob. VI. This gives us $m=\frac{ab}{d}$. The proposed expression then becomes $\frac{mc}{e}$, which may be constructed in a similar manner.

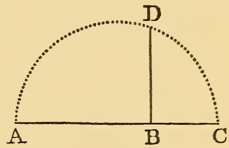
In like manner more complicated expressions may be constructed; as $\frac{a^3b^2c}{d^2f^2g}$.

Problem VIII. *To construct the expression* $x=\sqrt{ab}$.

The expression $\sqrt{ab}$ denotes a mean proportional between a and b; for we have

$$x^2=a\times b;\ \text{or}\ a:x::x:b.$$

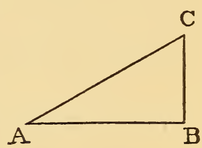
To construct this expression, draw an indefinite straight line, and upon it set off AB$=a$, and BC$=b$. On AC as a diameter, describe a semicircumference, and from B draw BD perpendicular to AC, meeting the circumference in D; then BD is a mean proportional between AB and BC (Geom., Bk. IV., Prop. 23, Cor.). Hence BD is a line representing the expression $\sqrt{ab}=x$.



Problem IX. *To construct the expression* $x=\sqrt{a^2+b^2}$.

This expression represents the hypothenuse of a right-angled triangle, of which a and b are the two sides about the right angle.

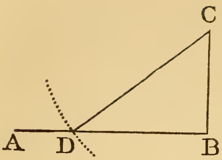
Draw the line AB, and make it equal to a; from B draw BC perpendicular to AB, and make it equal to b. Join AC, and it will represent the value of $\sqrt{a^2+b^2}$; since AC^2=AB^2+BC^2 (Geom., Bk. IV., Prop. 11).



Problem X. *To construct the expression* $x=\sqrt{a^2-b^2}$.

This expression represents one of the sides of a right-angled triangle, of which a represents the hypothenuse, and b the remaining side.

Draw an indefinite line AB; at B draw BC perpendicular to AB, and make it equal to b. With C as a centre, and a



radius equal to a, describe an arc of a circle cutting AB in D; then will BD represent the expression $\sqrt{a^2-b^2}$. For

$$BD^2=DC^2-BC^2=a^2-b^2.$$

Whence $$BD=\sqrt{a^2-b^2}=x.$$

Problem XI. *To construct the expression* $x=\sqrt{a^2+b^2-c^2}$.

Put $a^2+b^2=d^2$, and construct d as in Prob. IX.; then we shall have

$$x=\sqrt{d^2-c^2},$$

which may be constructed as in Prob. X.

In the same manner we may construct the expression

$$x=\sqrt{a^2-b^2+c^2-d^2+e^2-}, \text{ etc.}$$

By methods similar to the preceding the following expressions may be constructed:

1. $x=\sqrt{a^2+ab}$.
2. $x=\sqrt{ab+cd}$.
3. $x=\sqrt{\frac{abc}{d}}$.
4. $x=\sqrt{a^2-bc}$.
5. $x=a^2+ab$.
6. $x=\frac{a^3}{b^2}$.

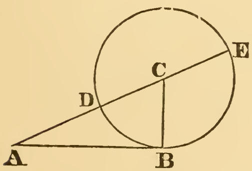
Problem XII. *To construct the roots of the four forms of equations of the second degree* (Alg., Art. 277).

In the equation $$x^2\pm px=\pm q,$$

x^2 and px represent surfaces (Prob. IV.); q must therefore represent a surface. We will suppose this surface transformed into a square (k^2), and, to avoid misapprehension, will write the general equation of the second degree

$$x^2\pm px=\pm k^2.$$

First form. The first form $x^2+px=k^2$ gives for x the two values $x=-\frac{p}{2}+\sqrt{\frac{p^2}{4}+k^2}$ and $x=-\frac{p}{2}-\sqrt{\frac{p^2}{4}+k^2}$.



Draw the line AB, and make it equal to k. From B draw BC perpendicular to AB, and make it equal to $\frac{p}{2}$. Join A and C; then, as in Prob. IX., AC will represent the value of $\sqrt{\frac{p^2}{4}+k^2}$.

With C as a centre, and CB as a radius, describe a circle cutting AC in D, and AC produced in E. For the first value of x the radical is positive, and is set off from A toward C; then $-\frac{p}{2}$ is set off from C to D, and AD, which equals

$$\sqrt{\frac{p^2}{4}+k^2}-\frac{p}{2},$$

represents the first value of x, measured from A to D.

For the second value of x we begin at E, and set off EC equal to $-\frac{p}{2}$; we then set off the minus radical from C to A; then EA, measured from E to A, represents the second value of x.

Second form. The second form $x^2-px=k^2$ gives for x the two values

$$x=\frac{p}{2}+\sqrt{\frac{p^2}{4}+k^2} \quad \text{and} \quad x=\frac{p}{2}-\sqrt{\frac{p^2}{4}+k^2}.$$

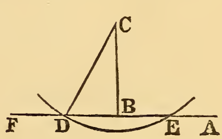
Construct as before $AC=\sqrt{\frac{p^2}{4}+k^2}$; then from C lay off CE equal to $\frac{p}{2}$, and the first value of x will be represented by AE, measured from A to E.

From D lay off DC equal to $\frac{p}{2}$; then from C in a contrary direction lay off CA equal to $\sqrt{\frac{p^2}{4}+k^2}$, and the second value of x will be represented by DA, measured from D to A.

Third form. The third form $x^2+px=-k^2$ gives for x the two values

$$x=-\frac{p}{2}+\sqrt{\frac{p^2}{4}-k^2} \quad \text{and} \quad x=-\frac{p}{2}-\sqrt{\frac{p^2}{4}-k^2}.$$

Draw an indefinite line FA, and from any point, as A, set off a distance $AB=-\frac{p}{2}$. We set off this line to the left, because $\frac{p}{2}$ is

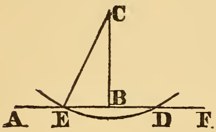


negative. At B draw BC perpendicular to FA, and make it equal to k. From C as a centre, with a radius equal to $\frac{p}{2}$, describe an arc of a circle cutting the line FA in D and E. Join CD, and we shall have BD or BE equal to $\sqrt{\frac{p^2}{4}-k^2}$.

The first value of x will be represented by $-AB+BE$, which is equal to $-AE$. The second value of x will be represented by $-AB-BD$, which is equal to $-AD$; so that both of the roots are negative, and are measured from A toward the left.

Fourth form. The fourth form $x^2-px=-k^2$ gives for x the two values

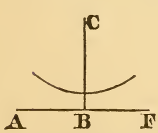
$$x=\frac{p}{2}+\sqrt{\frac{p^2}{4}-k^2} \quad \text{and} \quad x=\frac{p}{2}-\sqrt{\frac{p^2}{4}-k^2}.$$



Set off AB equal to $\frac{p}{2}$ from A toward the right. We set it off toward the right because $\frac{p}{2}$ is positive. Then construct the radical part of the value of x as for the third form. To AB we add BD, which gives AD for the first value of x; and from AB we subtract BE, which gives AE for the second value of x. Both values are positive, and are measured from A toward the right.

Equal roots. If the radius CE be taken equal to CB, that is, if k is equal to $\frac{p}{2}$, the arc described with the centre C will not cut the line AF, but will touch it at the point B, the two points D and E will unite, the radical part of the value of x becomes zero, and the two values of x become equal to each other.

Imaginary roots. If the radius of the circle described with the centre C be taken less than CB, it will not meet the line AF. In this case k^2 is numerically greater than $\frac{p^2}{4}$, and the radical part of the value of x becomes imaginary.



8. Every algebraic expression admitting of geometrical construction must have all its terms *homogeneous* (Alg., Art. 33); that is, each term must be of the *same degree.* The degree of any monomial expression is the number of its literal factors. If, however, the expression have a literal divisor, its degree is the number of literal factors in the numerator diminished by the number in the denominator. Thus the expressions x, $\frac{ab}{c}$, $\frac{abc}{de}$ are of the first degree; the expressions x^2, $\frac{a^2b}{c}$, $\frac{abcd}{ef}$ are of the second degree. In order that an algebraic expression may admit of geometrical construction, each term must either be of the first degree, and so represent a line; or, secondly, each must be of the second degree, and so represent a surface; or, thirdly, each must be of the third degree, and denote a solid, since dissimilar geometrical magnitudes can neither be added together nor subtracted from each other.

It may, however, happen that an expression really admitting of geometrical construction appears to be not homogeneous; but this result arises from the circumstance that the geometrical unit of length, having been represented in the calculation by the numeral unit 1, disappears from all algebraic expressions in which it is either a factor or a divisor. To render these results homogeneous, it is only necessary to restore this factor or divisor which represents unity.

Thus, suppose we have an equation of the form

$$x=ab+c.$$

If we put l to represent the unit of measure for lines, we may change it into the homogeneous equation

$$lx=ab+cl,$$

or

$$x=\frac{ab}{l}+c,$$

which is easily constructed geometrically.

Suppose the expression to be constructed to be of the form

$$x=\frac{a^2+3b-2}{b-2c+3}.$$

Since one of the terms of the numerator is of the second degree, each of the other terms of the numerator should be made of the same degree, and each term in the denominator should be made of the first degree; so that, introducing the linear unit l, the expression to be constructed is

$$x=\frac{a^2+3lb-2l^2}{b-2c+3l}.$$

The denominator of this fraction may be constructed by Prob. III. If we represent the denominator by m, the expression may be written

$$x=\frac{a^2}{m}+\frac{3lb}{m}-\frac{2l^2}{m},$$

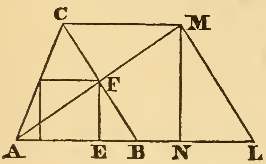
each of which terms may be constructed by Prob. VI.

The following examples will show how an algebraic solution of a problem may be converted into a geometrical solution.

Problem XIII. *Having given the base and altitude of any triangle, it is required to find the side of the inscribed square by a geometrical construction.*

We have found, on page 13, the side of the inscribed square to be equal to $\frac{bh}{b+h}$; that is, it is a fourth proportional to $b+h$, b and h.

In order to construct this expression, produce the base AB until BL is equal to the altitude h; through L draw LM parallel to BC, meeting CM drawn through C parallel to AB. Join AM, and let it meet BC in F; draw FE perpendicular to AB, and it will be the required line. Draw MN perpendicular to AL.



By similar triangles we have

$$AL:AB::LM:BF::MN:FE;$$

that is, $$b+h:b::h:FE;$$

whence $$FE=\frac{bh}{b+h}=x;$$

and therefore EF is equal to a side of the inscribed square.

Example 3, page 13, may be constructed in a similar manner by laying off BL equal to nh.

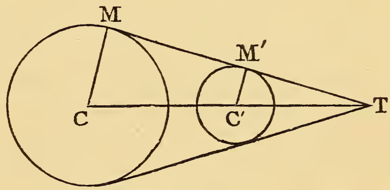
Problem XIV. *It is required to draw a straight line tangent to two given circles situated in the same plane.*

Since the two circles are given both in extent and position, we know their radii and the distance between their centres.

Let C, C′ be the centres of the two circles, CM, C′M′ their radii. Denote the radius CM of the first circle by r, that of the second C′M′ by r', and the distance between their centres CC′ by a. Suppose that MM′ is the required tangent; produce this line to meet CC′ produced in T, and denote the distance CT by x.

There are two cases:

Case First. When the tangent does not pass between the circles.



Draw the radii CM, C′M′ to the points of tangency; the angles CMT, C′M′T will be right angles, and the triangles CMT, C′M′T will be similar. Hence we shall have the proportion

$$CM : C'M' :: CT : C'T,$$

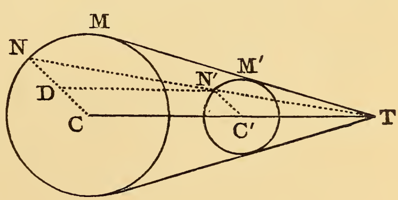
or
$$r : r' :: x : x - a;$$

whence
$$rx - ra = r'x,$$

and
$$x = \frac{ar}{r - r'};$$

from which we see that CT or x is a fourth proportional to $r - r'$, a and r.

To obtain x by a geometrical construction, through the centres C and C′ draw any two parallel radii CN, C′N′, on the same side of CC′. Through N and N′ draw the line NN′, and produce it to meet CC′ produced in T. CT will be the line represented by x.



For through N′ draw N′D parallel to CT; then ND will represent $r-r'$, and N′D will be equal to a; and by similar triangles we have DN : DN′ :: CN : CT,

or $r-r' : a :: r : \mathrm{CT}$;

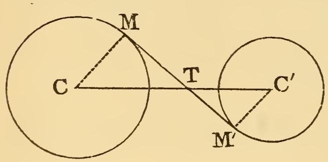
whence $$\mathrm{CT}=\frac{ar}{r-r'}=x.$$

Therefore a line drawn from T, tangent to one of the circles, will also be tangent to the other; and, since two tangent lines can be drawn from the point T, we see that this first case of the proposed problem admits of two solutions.

Cor. If we suppose the radius r of the first circle to remain constant, and the smaller radius r' to increase, the difference $r-r'$ will diminish; and, since the numerator ar remains constant, the value of x will increase; which shows that the nearer the two circles approach to equality, the more distant is the point of intersection of the tangent line with the line joining the centres. When the two radii r and r' become equal, the denominator becomes zero, the value of x becomes infinite, and the two tangents are parallel.

If we suppose r' to increase so as to become greater than r, the value of x becomes negative, which shows that the point T is on the left of the two circles.

Case Second. When the tangent passes between the circles.

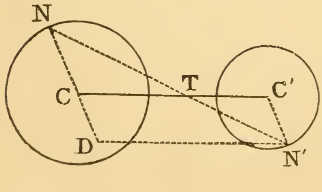


In this case, as in the other, the lines CM and C′M′ are parallel; hence the triangles CMT, C′M′T are similar, and we have the proportion

CM : C′M′ :: CT : C′T,

or $r : r' :: x : a-x$;

whence $$x=\frac{ar}{r+r'}.$$



To construct this expression, through the centres C and C′ draw any two parallel radii CN, C′N′, lying on different sides of CC′; join the points NN′, and through T, where this line intersects CC′, draw a line tangent to one of the circles. It will be a tangent to the other.

For through N′ draw N′D parallel to CC′, and meeting CN produced in D. From the similar triangles NCT, NDN′ we have the proportion

$$\mathrm{ND} : \mathrm{DN}' :: \mathrm{NC} : \mathrm{CT},$$

or

$$r+r' : a :: r : \mathrm{CT};$$

whence

$$\mathrm{CT} = \frac{ar}{r+r'} = x.$$

Cor. The value of x is positive for all values of r and r'; when $r=r'$, the value of x reduces to $\frac{a}{2}$.

If each circle is wholly exterior to the other, there may therefore be two exterior tangents and two interior tangents, in which case the problem admits of *four* solutions.

If the two circles touch each other externally, the two interior tangents unite in one, and the problem admits but *three* solutions.

If the two circles cut each other, the interior tangents are impossible, and the problem admits but *two* solutions.

If the two circles touch each other internally, the two exterior tangents unite in one, and the problem admits but *one* solution.

If one circle is wholly interior to the other, no tangent line can be drawn, and *no* solution of the problem is possible.

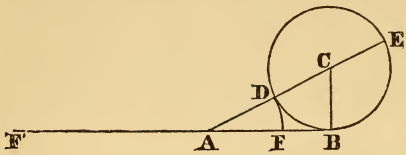
The general values of x already found undergo changes corresponding to the changes here supposed in the position of the two circles.

Problem XV. *To divide a straight line in extreme and mean ratio.*

We have found, in Example 4, page 14,

$$x = -\frac{a}{2} \pm \sqrt{a^2 + \frac{a^2}{4}}.$$

To construct the first value of x, make $\mathrm{AB} = a$; at B erect the perpendicular $\mathrm{BC} = \frac{a}{2}$, and join AC.



Then, as in Prob. 9, page 21,

$$AC=\sqrt{a^2+\frac{a^2}{4}}.$$

From C as a centre, with a radius $CB=\frac{a}{2}$, describe a circumference cutting AC in D and AC produced in E. From AC take $CD=\frac{a}{2}$, and we have

$$AD=AC-CD=\sqrt{a^2+\frac{a^2}{4}}-\frac{a}{2}.$$

To construct the second value of x.

From E set off EC towards the left equal to $\frac{a}{2}$, and from C also towards the left set off CA equal to $\sqrt{a^2+\frac{a^2}{4}}$. Then EA, measured from E to A, will represent

$$x=-\frac{a}{2}-\sqrt{a^2+\frac{a^2}{4}}.$$

With A as a centre, and AD as a radius, describe the arc DF. The line AB will be divided in the required ratio at F, and AF will be the greater part.

The second value of $x=-AE$ is numerically greater than AB. It can not, then, form a part of AB, and is not an answer to the question in the form here proposed.

Each value of x may, however, be regarded as the solution of the more general problem, "Two points A and B being given, to find, on the indefinite line that passes through them, a third point F, such that the distance AF shall be a mean proportional between the distances AB and BF." To this problem there are evidently two solutions, F on the right of A being one of the points, and F′ on the left of A is the other.

9. From the preceding examples we perceive that the solution of a geometrical problem by the aid of Algebra consists of *three* principal parts:

1*st.* To translate the problem into algebraic language, or to reduce it to an equation.

2*d.* To solve the equation or equations.

3*d.* To construct geometrically the algebraic expressions obtained.

Frequently it becomes necessary to add a fourth part, whose object is the *discussion of the problem*, or an examination of all the circumstances relating to it.

PART II.

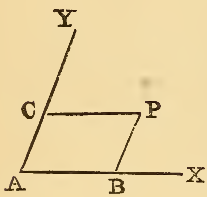
INDETERMINATE GEOMETRY.

SECTION I.

CO-ORDINATES OF A POINT.

10. The object of the second branch of Analytical Geometry is to determine the algebraic equations by which known lines and curves may be represented, and from these equations to deduce their geometrical properties; and conversely, having given the equations, to determine the lines and curves which they represent.

11. *To determine the position of a point in a plane.* The position of a point in a plane may be denoted by means of its distances from two given lines which intersect one another.



Thus, let AX, AY be two assumed straight lines which intersect in any angle at A, and let P be any point in the same plane; then, if we draw PB parallel to AY, and PC parallel to AX, the position of the point P will be determined by means of the distances PB and PC.

The two lines AX, AY, to which the position of the point P is referred, are called *axes*, and their point of intersection, A, is called the *origin*. The distance AB, or its equal CP, is called the *abscissa* of the point P; and BP, or its equal AC, is called the *ordinate* of the same point. Hence the axis AX is called the *axis of abscissas*, and AY is called the *axis of ordinates.*

The abscissa and ordinate of a point, when spoken of together, are called the *co-ordinates* of the point, and the two axes are called *axes of co-ordinates*, or *co-ordinate axes.*

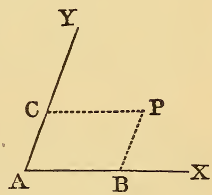
A system of axes may be either *rectangular* or *oblique;* that is, the angle YAX may be either a right angle or an oblique angle. Rectangular axes are ordinarily most convenient, and will generally be employed in this treatise.

An abscissa is usually denoted by the letter x, and an ordinate by the letter y; and hence the axis of abscissas is often called the axis of x, and the axis of ordinates the axis of y.

The abscissa of any point is its distance from the axis of ordinates measured on a line parallel to the axis of abscissas.

The ordinate of any point is its distance from the axis of abscissas measured on a line parallel to the axis of ordinates.

12. *Equations of a point.* The position of a point may be determined when its co-ordinates are known. For, suppose the abscissa of the point P is equal to 5, and its ordinate is equal to 4. Then, to determine the position of the point P, from the origin A lay off on the axis of abscissas a distance AB equal to 5 units of length, and through B draw a line parallel to the axis of ordinates. On this line lay off a distance BP equal to 4 units of length, and P will be the point required.



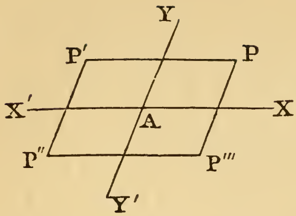
So, if $x=a$ and $y=b$, measure off AB equal to a units, and draw BP parallel to AY, and equal to b units.

Hence, in order to determine the position of a point, we need only have the two equations

$$x=a, y=b,$$

in which a and b are given. These equations are therefore called *the equations of a point.*

13. *Signs of the co-ordinates.* It is however necessary, in order to determine the position of a point, that not only the absolute values of a and b should be given, but also the *signs* of these quantities. If the axes are produced through the origin to X′ and Y′, it is obvious that the abscissas reckoned in the direction AX′ ought not to have the same sign as those reckoned in the opposite direction AX, nor should the ordinates measured in the direction AY′ have the same sign as those



measured in the opposite direction AY; for if there were no distinction in this respect, the position of a point as determined by its equations would be ambiguous. Thus the equations of the point P would equally belong to the points P′, P″, P‴, provided the absolute lengths of the co-ordinates of these points were equal to those of P. This ambiguity is avoided by regarding the co-ordinates which are measured in one direction as *plus*, and those in the opposite direction as *minus*. It has been agreed to regard those abscissas which fall on the right of the axis YAY′ as positive, and hence those which fall on the left must be considered negative. So also it has been agreed to consider those ordinates which are above the axis XAX′ as positive, and hence those which fall below it must be considered negative.

14. *Equations of a point in each of the four angles.* The angle YAX is called the *first angle;* YAX′ the *second angle;* Y′AX′ the *third angle;* and Y′AX the *fourth angle.*

The following, therefore, are the equations of a point in each of the four angles:

For the point P in the first angle, $x=+a,\ y=+b$.
" P′ " second angle, $x=-a,\ y=+b$.
" P″ " third angle, $x=-a,\ y=-b$.
" P‴ " fourth angle, $x=+a,\ y=-b$.

If the point be situated on the axis AX, the equation $y=b$ becomes $y=0$, so that the equations

$$x=\pm a,\ y=0$$

denote a point in the axis of abscissas at the distance a from the origin.

If the point be situated on the axis AY, the equation $x=a$ becomes $x=0$, so that the equations

$$x=0,\ y=\pm b$$

denote a point on the axis of ordinates at the distance b from the origin.

If the point be common to both axes, that is, if it be at the origin, its position will be denoted by the equations

$$x=0,\ y=0.$$

The point P, whose co-ordinates are x, y, is often called the point (x, y); thus a point for which $x=a$, $y=b$ is called the point (a, b). Hitherto the letters a and b have been supposed to stand for positive numbers, but they may also be used to represent negative numbers.

Ex. 1. Indicate by a figure the position of the point whose equations are $x=+4$, $y=-3$.

Ex. 2. Indicate by a figure the position of the point whose equations are $x=-2$, $y=+7$.

Ex. 3. Indicate by a figure the position of the point $0, -5$.

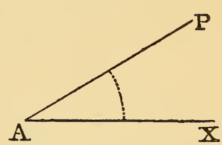
Ex. 4. Indicate by a figure the position of the point $-8, 0$.

Ex. 5. Indicate by a figure the position of the point $-3, -2$.

Ex. 6. Draw a triangle the co-ordinates of whose angular points are $3, 4$; $-3, -4$; $-1, 0$.

15. *Polar co-ordinates.* The position of a point may also be denoted by means of the distance and direction of the proposed point from a given point.

Thus, if A be a known point, and AX be a known direction, the position of the point P will be determined when we know the distance AP, and the angle PAX.

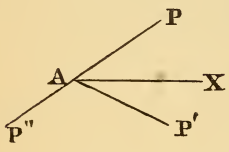


Thus, if we denote the distance AP by r, and the angle PAX by θ, the position of P is determined if r and θ are known.

The assumed point A is called the *pole*; the distance of P from A is called the *radius vector*; the line AX is called the *initial line*; and the radius vector, together with its angle of inclination to the initial line, are called the *polar co-ordinates* of the point. The point whose polar co-ordinates are r and θ is sometimes called the point r, θ.

16. *Unit for the measure of angles.* The unit commonly employed in Trigonometry for measuring angles is the ninetieth part of a right angle, called a *degree;* but a different unit is sometimes more convenient. Since angles at the centre of a circle are proportional to the arcs on which they stand, we may employ the arc to measure the angle which it subtends, and it is convenient to take as the unit of measure the arc which is equal to the radius of the circle. Since the circumference of a circle whose radius is unity is 2π, the measure of four right angles will accordingly be 2π; the measure of one right angle will be $\frac{\pi}{2}$; the measure of an angle of 45° will be $\frac{\pi}{4}$, etc.

17. *Negative values of polar co-ordinates.* The position of any point might be expressed by positive values of the polar co-ordinates r and θ, since there is here no ambiguity corresponding to that arising from the four angles of the figure in Art. 13. It is, however, sometimes convenient to admit the use



of negative angles, and in this case an angle XAP′ is considered negative when it is measured in the direction corresponding to the motion of the hands of a watch; and an angle is considered positive when it is measured in the opposite direction, as XAP.

The same direction may be represented either by a negative angle or by a positive angle. Thus, if the angle XAP′ be half a right angle, the direction AP′ may be denoted either by $-\frac{\pi}{4}$ or $+\frac{7\pi}{4}$.

We also sometimes admit negative as well as positive values of the radius vector. Thus, suppose the co-ordinates of P to be a and $\frac{\pi}{4}$; that is, let $XAP=\frac{\pi}{4}$, and $AP=a$; if we produce PA to P″, making $AP''=AP$, then P″ may be determined by saying that its co-ordinates are $-a$ and $\frac{\pi}{4}$. The radius vector is considered positive when it is measured in the direction of the extremity of the arc measuring the variable angle; it is considered negative when it is measured in the opposite direction.

Thus the co-ordinates

r and $\frac{\pi}{4}$ represent the point P.

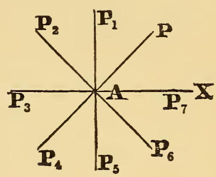
$-r$ and $\frac{\pi}{2}$ " P_5.

$-r$ and $\frac{3\pi}{4}$ " P_6.

$-r$ and π " P_7.

$-r$ and $\frac{5\pi}{4}$ " P.

$-r$ and $-\frac{3\pi}{4}$ " P.



Thus the same point P is denoted either by the co-ordinates r and $\frac{\pi}{4}$, or $-r$ and $\frac{5\pi}{4}$, or $-r$ and $-\frac{3\pi}{4}$.

Ex. 1. Indicate by a figure the position of the point whose co-ordinates are $a, 15°$, where $a=1$ inch.

Ex. 2. Indicate by a figure the position of the point $2a, 40°$.

Ex. 3. Indicate by a figure the position of the points

$$-a, 45°;\ -a, -135°;\ 3a, \frac{3\pi}{4};\ 5a, \frac{7\pi}{4};\ 2a \sin.\frac{\pi}{6}, \frac{\pi}{6}.$$

18. *Implicit equations of a point.* The position of a point may be determined not only *explicitly* by co-ordinates, but *implicitly* by means of simultaneous equations which these co-ordinates satisfy. For if we have two simultaneous equations between two variables, we can find the values of these variables by the methods of Algebra, and these values are the co-ordinates of known points.

Ex. 1. Thus, suppose we have the equations

$$\left.\begin{aligned} 2x+3y&=12, \\ 3x-2y&=5, \end{aligned}\right\}$$

we find $x=3$, and $y=2$.

In this and the following examples the pupil should draw the figure representing the problem.

Ex. 2. Determine the point whose co-ordinates satisfy the equations

$$\left.\begin{array}{l}5x-4y=9,\\ 7x-5y=15.\end{array}\right\}$$

Ans. $x=5$, and $y=4$.

Ex. 3. Determine the point whose co-ordinates satisfy the equations

$$\left.\begin{array}{l}\dfrac{x}{a}+\dfrac{y}{b}=3,\\[2ex] \dfrac{x}{b}+\dfrac{y}{a}=3.\end{array}\right\}$$

Ans. $x=y=\dfrac{3ab}{a+b}$.

Ex. 4. Determine the points whose co-ordinates satisfy the equations

$$\left.\begin{array}{l}x+y=4(x-y),\\ x^2+y^2=34.\end{array}\right\}$$

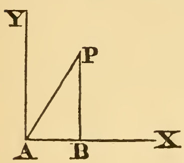
Ans. (5, 3), and (−5, −3).

Ex. 5. Determine the points whose co-ordinates satisfy the equations

$$\left.\begin{array}{l}x^2+xy=40,\\ xy-y^2=6.\end{array}\right\}$$

Ans. (5, 3), (−5, −3), $(4\sqrt{2}, \sqrt{2})$, and $(-4\sqrt{2}, -\sqrt{2})$.

19. *To find the distance of any point from the origin in terms of the co-ordinates of that point.*



Case First. Let the co-ordinates be rectangular.

We have $AP^2=AB^2+BP^2=x^2+y^2$;

therefore $AP=\sqrt{x^2+y^2}$.

Ex. 1. Determine the distance from the origin to the point whose co-ordinates are $x=3a$, $y=4a$.

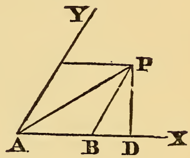
Ans. $AP=\sqrt{9a^2+16a^2}=5a$.

Ex. 2. Determine the distance of the point $-2b, 3b$, from the origin. *Ans.* $b\sqrt{13}$.

Ex. 3. Determine the distance from the origin to the point $a \sin.\beta, a \cos.\beta$. *Ans.* a.

Ex. 4. Determine the distance of the point $5a, -3a$, from the origin.

20. *Case Second.* When the co-ordinates are oblique.



From P draw PD perpendicular to AX; then (Geom., B. IV., Prop. 13)

$$AP^2 = AB^2 + BP^2 + 2AB \,.\, BD.$$

But by Trig., Art. 41,

$$R : \cos. PBD :: PB : BD.$$

Hence $BD = PB \cos. PBD$ (radius being unity).

Therefore $AP^2 = AB^2 + BP^2 + 2AB \,.\, PB \cos. PBD.$

But $PBD = YAX$, which we will represent by ω.

Hence $AP = (x^2 + y^2 + 2xy \cos. \omega)^{\frac{1}{2}}.$

In the following examples we will suppose the axes to be inclined at an angle of 60°.

Ex. 1. Determine the distance from the origin to the point $3a, 4a$. *Ans.* $AP = (9a^2 + 16a^2 + 24a^2 \times \frac{1}{2})^{\frac{1}{2}} = a\sqrt{37}.$

Ex. 2. Determine the distance from the origin to the point $-2b, 3b$. *Ans.* $AP = b\sqrt{7}.$

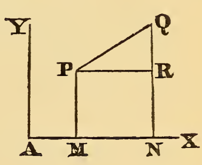
Ex. 3. Determine the distance from the origin to the point $a \sin. \beta, a \cos. \beta$. *Ans.* $a(1 + \frac{1}{2} \sin. 2\beta)^{\frac{1}{2}}.$

Note. $\sin. 2A = 2 \sin. A \cos. A$ (Trig., Art. 73).

Ex. 4. Determine the distance from the origin to the point $5a, -3a$.

21. *To find the distance between two given points.*

Case First. Let the axes be rectangular.



Let P and Q be the two points, and represent the co-ordinates of P by x_1, y_1, and those of Q by x_2, y_2.

Draw PR parallel to the axis of x, cutting QN in R.

Then $PQ^2 = PR^2 + RQ^2.$

But $PR = MN = AN - AM = x_2 - x_1,$

and $QR = QN - PM = y_2 - y_1.$

Therefore $PQ^2=(x_2-x_1)^2+(y_2-y_1)^2$,

and $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

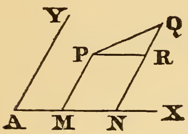
Ex. 1. Determine the distance between the point 3, 4, and the point 4, 3. *Ans.* $PQ^2=(3-4)^2+(4-3)^2 \therefore PQ=\sqrt{2}$.

Ex. 2. Determine the distance between the point $-3, 4$, and the point $4, -3$. *Ans.* $7\sqrt{2}$.

Ex. 3. Determine the distance between the point 2, 2, and the point $-2, -2$. *Ans.* $4\sqrt{2}$.

Ex. 4. Determine the distance between the point $2a$, 0, and the point $0, -2a$. *Ans.* $2a\sqrt{2}$.

Ex. 5. Determine the distance between the point $-2a, 2a$, and the point $4a, -6a$.



22. *Case Second.* Let the axes be inclined at an angle ω.

Then, as in Art. 20,

$$PQ^2=PR^2+RQ^2+2PR\,.\,RQ\cos.YAX,$$

or $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+2(x_2-x_1)(y_2-y_1)\cos.\omega}$.

Ex. 1. Determine the distance between the point 0, 3, and the point 4, 0.

Ans. $PQ^2=4^2+3^2-2.4.3\cos.\omega=25-24\cos.\omega$,

and $PQ=\sqrt{25-24\cos.\omega}$.

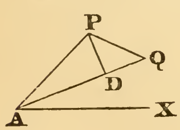
Ex. 2. Determine the distance between the point 0, 3, and the point -4, 0. *Ans.* $\sqrt{25+24\cos.\omega}$.

Ex. 3. Determine the distance between the point $2, -2$, and the point $-2, 2$. *Ans.* $8\sin.\frac{\omega}{2}$.

Note. $2\sin.^2A=1-\cos.2A$ (Trig., Art. 74).

Ex. 4. Determine the distance between the point a, 0, and the point $0, a$. *Ans.* $2a\sin.\frac{\omega}{2}$.

23. *Case Third.* Let the co-ordinates be polar.



Let P and Q be the two given points; represent AP by r_1, and AQ by r_2; also PAX by θ_1 and QAX by θ_2.

From P draw PD perpendicular to AQ.

By Geom., Bk. IV., Prop. 12,

$$PQ^2 = AP^2 + AQ^2 - 2AQ \times AD.$$

But $\quad AD = AP \cos. PAQ$ (radius being unity).

Hence $\quad PQ^2 = AP^2 + AQ^2 - 2AP \times AQ \times \cos. PAQ$

$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos. (\theta_1 - \theta_2),$$

and $\quad PQ = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos. (\theta_1 - \theta_2)}.$

Ex. 1. Determine the distance between the point $2a, 30°$, and the point $a, 60°$. *Ans.* $PQ^2 = 4a^2 + a^2 - 4a^2 \times \frac{1}{2}\sqrt{3}$,

and $PQ = a\sqrt{5 - 2\sqrt{3}}$.

Ex. 2. Determine the distance between the point $a, 0°$, and the point $b, 30°$. *Ans.* $PQ^2 = a^2 + b^2 - 2ab \times \frac{1}{2}\sqrt{3}$,

and $PQ = \sqrt{a^2 + b^2 - ab\sqrt{3}}$.

Ex. 3. Determine the distance between the point a, θ, and the point $-a, -\theta$.

Ans. $PQ^2 = a^2 + a^2 + 2a^2 \cos. 2\theta = 2a^2(1 + \cos. 2\theta)$,

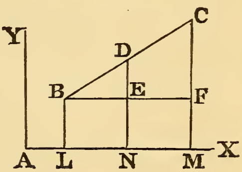
and $PQ = 2a \cos. \theta$.

Note. $2 \cos.^2 A = 1 + \cos. 2A$ (Trig., Art. 74).

Ex. 4. Determine the distance between the point a, θ, and the point $a, -\theta$. *Ans.* $2a \sin. \theta$.

24. *To find the co-ordinates of the point which bisects the straight line joining two given points.*

Let D be the point required, AN, DN its co-ordinates, and let DN cut BF in E.



Then

$$AN = AL + LN = AL + BE = AL + \tfrac{1}{2}BF;$$

that is, $\quad AN = x_1 + \dfrac{x_2 - x_1}{2} = \dfrac{x_1 + x_2}{2}.$

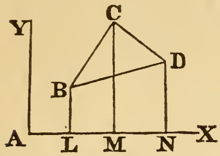
In like manner, $\quad DN = \dfrac{y_1 + y_2}{2}.$

Ex. 1. Determine the co-ordinates of the point of bisection of the line joining the point $-1, 1$, with the point $3, -5$.

Ans. $x = 1, \; y = -2$.

Ex. 2. Determine the co-ordinates of the point of bisection of the line joining the point $3, -3$, with the point $5, -5$.

25. *To find the area of a triangle whose angular points are given.*



Let BCD be the triangle, and let the co-ordinates of B, C, D be $x_1\ y_1$, $x_2\ y_2$, $x_3\ y_3$ respectively.

The area BCD

$$=\text{BCML}+\text{CDNM}-\text{BDNL}.$$

But $\quad \text{BCML}=\frac{1}{2}\text{LM}(\text{BL}+\text{CM})=\frac{1}{2}(x_2-x_1)(y_2+y_1).$

So also $\quad \text{CDNM}=\frac{1}{2}(x_3-x_2)(y_3+y_2),$

and $\quad \text{BDNL}=\frac{1}{2}(x_3-x_1)(y_3+y_1).$

Therefore the area BCD =

$$=\tfrac{1}{2}\{(x_2-x_1)(y_2+y_1)+(x_3-x_2)(y_3+y_2)-(x_3-x_1)(y_3+y_1)\}$$
$$=\tfrac{1}{2}(x_1y_3+x_2y_1+x_3y_2-x_1y_2-x_2y_3-x_3y_1).$$

Ex. 1. Determine the area of the triangle whose angular points are $\quad 3, 4;\ -3, -4;\ 0, 4.$ *Ans.* 12.

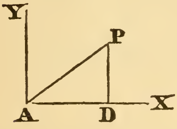
Ex. 2. Determine the area of the triangle whose angular points are $\quad 0, 0;\ 1, 2;\ 2, 1.$ *Ans.* $\frac{3}{2}$.

Ex. 3. Determine the area of the triangle whose angular points are $\quad a, 0;\ -a, 0;\ 0, b.$ *Ans.* ab.

Ex. 4. Determine the area of the triangle whose angular points are $\quad 1, 1;\ -1, 2;\ -1, 1.$

26. *To convert the rectangular co-ordinates of a point into polar co-ordinates, and vice versa.*

Let x and y denote the co-ordinates of P referred to the rect-



angular axes AX and AY. Also, let r and θ denote the polar co-ordinates of P, the pole being at the origin A, and AX being the initial line. Draw PD perpendicular to AX. Then, by Trig., Art. 41,

$$\text{AD}=\text{AP}\cos.\text{PAD}, \quad \text{or } x=r\cos.\theta;$$

also
$$\text{PD}=\text{AP}\sin.\text{PAD}, \quad \text{or } y=r\sin.\theta,$$

which equations enable us to deduce the rectangular co-ordinates of a point from the polar co-ordinates.

Again, $AD^2+PD^2=AP^2$, or $x^2+y^2=r^2$,

and $AD:R::PD:\text{tang. } PAD$, or $\frac{y}{x}=\text{tang. } \theta$,

which equations enable us to deduce the polar co-ordinates of a point from the rectangular co-ordinates.

Ex. 1. Find the polar co-ordinates of the point whose rectangular co-ordinates are $x=1, y=1$, and indicate the point by a figure. *Ans.* $r=\sqrt{2}$, $\theta=45°$.

Ex. 2. Find the polar co-ordinates of the points whose rectangular co-ordinates are

(1) $x=-1,\ y=+2.$
(2) $x=-1,\ y=-2.$
(3) $x=+1,\ y=-2.$

Ex. 3. Find the rectangular co-ordinates of the point whose polar co-ordinates are $r=3$, $\theta=\frac{\pi}{3}$. *Ans.* $x=\frac{3}{2}$, $y=\frac{3}{2}\sqrt{3}$.

Ex. 4. Find the rectangular co-ordinates of the points whose polar co-ordinates are

(1) $r=+3,\ \theta=-\frac{\pi}{3}.$

(2) $r=-3,\ \theta=+\frac{\pi}{3}.$

(3) $r=-3,\ \theta=-\frac{\pi}{3}.$

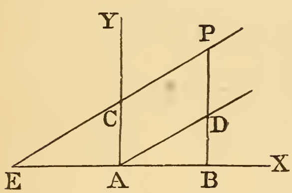
SECTION II.

THE STRAIGHT LINE.

28. *Definition. The equation of a line is the equation which expresses the relation between the two co-ordinates of every point of that line.*

Hence, if any point be taken upon the line, and the values of x and y for that point be substituted in the equation, the equation will be satisfied; and conversely, if the co-ordinates of any point whatever of the plane satisfy the equation of a line, that point will be on the line.

29. *To find the equation to a straight line referred to rectangular axes.*



Let A be the origin of co-ordinates, AX and AY be rectangular axes, and let PC be any straight line whose equation is required to be determined. Take any point P in the given line, and draw PB parallel to AY; then will PB be the ordinate and AB the abscissa of the point P. From A draw AD parallel to CP, meeting the line BP in D.

Let $\qquad AB = x,$

$\qquad BP = y,$

$\qquad$ tang. PEX or $DAX = m,$

and $\qquad$ AC or $DP = c.$

Then, by Trigonometry, Theorem II., Art. 42,

$$AB : BD :: \text{radius} : \text{tang. } DAX;$$

that is, $\qquad x : BD :: 1 : m,$

or $\qquad BD = mx.$

But $\qquad BP = BD + DP;$

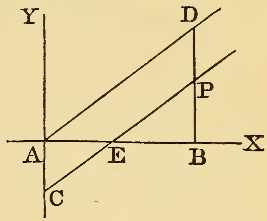
that is, $\qquad y = mx + c.$

Hence the equation to a straight line referred to rectangular axes is

$$y=mx+c;$$

where x and y are the co-ordinates of any point of the line, m represents the tangent of the angle which the line makes with the axis of abscissas, and c the distance from the origin at which it intersects the axis of ordinates.

30. *Signs of* m *and* c.



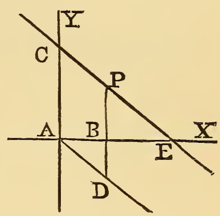
If the line CP cuts the axis of ordinates below the origin, then c or AC will be negative.

In that case, $BP=BD-DP$;

or,

$$y=mx-c.$$

The angle which the line makes with the axis of abscissas is supposed to be measured from the axis AX around the circle by the left. If the line CP is directed downward toward the right, as in the annexed figure, the line makes either an obtuse angle, CEX, with the axis of abscissas, or the negative acute angle CEA, the tangent of either of which angles is negative (Trig., Art. 69).



In this case we have

$$AB:BD::\text{radius}:\text{tang. }DAX,$$

or

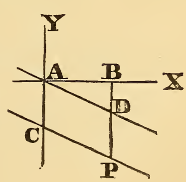
$$x:BD::\quad 1\quad :m.$$

The tangent of DAX being negative, BD is also negative.

But

$$BP=-BD+DP,$$

and the equation becomes $y=-mx+c$, where it must be observed that the minus sign applies only to the quantity m, and not to x, for the sign of x depends upon its direction from the origin A.



If the line CP is directed downward toward the right, and cuts the axis of ordinates below the origin, then c is negative as well as m; and since $BP=-BD-DP$, the equation becomes

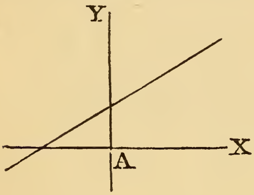
$$y=-mx-c.$$

It is to be remembered that the symbols x, y, m, and c may stand for negative numbers, and therefore the single equation

$$y=mx+c$$

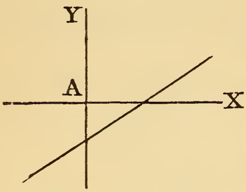
may represent any line whatever.

31. *Four different positions of a line.* There may, therefore, be four positions of the proposed line, and these positions are indicated by the signs of m and c in the general equation.

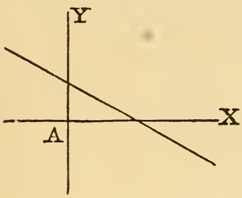


1. Let the line cut the axis of X to the left of the origin, and the axis of Y above it; then m and c are both positive, and the equation is

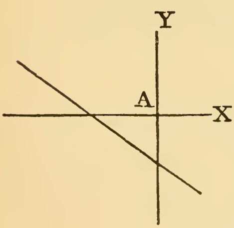
$$y=+mx+c.$$



2. If the line cuts the axis of X to the right of the origin, and the axis of Y below it, then m will still be positive, but c will be negative, and the equation becomes $y=+mx-c.$



3. If the line cuts the axis of X to the right of the origin, and the axis of Y above it, then m becomes negative and c positive. In this case, therefore, the equation is $y=-mx+c.$



4. If the line cuts the axis of X to the left of the origin, and the axis of Y below it, then both m and c will be negative, so that the equation becomes

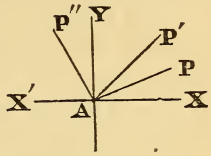
$$y=-mx-c.$$

If we suppose the straight line to pass through the origin A, then c will become zero, and the general equation becomes

$$y=mx,$$

which is the equation of a straight line passing through the origin.

32. *Direction of a line indicated.* It will be seen that the *direction* of the proposed line is indicated by the symbol m. If m is very small and positive, the line whose equation is $y=mx$ takes the position AP, near the axis AX. As m increases the line changes its position, and when $m=1$ the line makes an angle of 45° with AX. As the value of m increases the line approaches AY, and coincides with it when m becomes infinite.



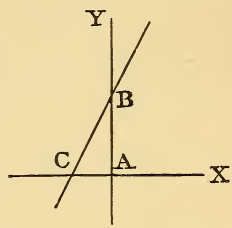
If m is negative and very large, the line assumes the position AP″, and as m decreases the line moves toward AX′, and when $m=-1$ the line bisects the angle YAX′. When m becomes zero, the line coincides again with the axis of abscissas.

So, also, if the point P is supposed to travel round A through the third and fourth quadrants, the value of m will be positive in the third quadrant and negative in the fourth.

Ex. 1. Let it be required to draw the line whose equation is

$$y=2x+4.$$

Draw the co-ordinate axes AX, AY. Now if in this equation we suppose $x=0$, the value of y will designate the point in which the line intersects the axis of ordinates, for this is the only point of the line whose abscissa is zero. This supposition will give $y=4$.



Hence, if we take AB$=4$, B will be one point of the required line.

Again, if in the proposed equation we suppose $y=0$, the value of x which is found from the equation will designate the point in which the line intersects the axis of abscissas, for that is the only point of the line whose ordinate is zero. This supposition will give

$$2x=-4,$$

or

$$x=-2.$$

Hence, if we lay off from A toward the left a distance AC $=2$, C will be a second point of the proposed line. Draw the straight line BC, and produce it indefinitely both ways; it will be the line whose equation is $y=2x+4$.

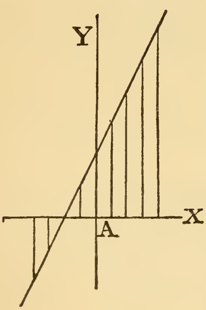
The student should regard every algebraic equation in this treatise as expressing some geometrical truth, and he should accustom himself to express these truths in appropriate geometrical language. Thus the equation $y=2x+4$ expresses the truth that *any ordinate of a certain straight line is equal to twice the corresponding abscissa, increased by four.*

So also the general equation of a straight line, $y=mx+c$, expresses the truth that *any ordinate of any straight line is equal to some multiple of the corresponding abscissa, increased by a constant number.*

33. *Any number of points of a line determined.* When the equation of a line is given, we may, if desired, determine any number of points of the line by assuming particular values for x, and computing the corresponding values of y.

Thus, if in the equation $y=2x+4$ we suppose

$x=1$,	we find	$y=6$.	$x=-1$,	we find	$y=2$.
$x=2$,	"	$y=8$.	$x=-2$,	"	$y=0$.
$x=3$,	"	$y=10$.	$x=-3$,	"	$y=-2$.
$x=4$,	"	$y=12$, etc.	$x=-4$,	"	$y=-4$, etc.



In order to represent all these values by a figure, set off on the axis of abscissas lines equal to 1, 2, 3, etc., both to the right and left of A; then erect a perpendicular from each of these points, and make it equal to the corresponding value of y, setting it off above AX if the ordinate be positive, but below AX if negative. The required line must pass through all the points thus determined.

34. *Variables and constants.* In the equation $y=mx+c$, m and c remain unchanged so long as we consider the same straight line; they are therefore called *constant quantities*, or *constants*. But x and y may have an *indefinite* number of

values, since we may ascribe to one of them, as x, any value we please, and find from the equation the corresponding value of y. x and y are therefore called *variable quantities*, or *variables.*

35. *Meaning of the equation of a line.* The equation of a line may be regarded as a statement of some geometrical proposition respecting that line.

Thus the equation

$$y=2x+10$$

may be regarded as the algebraic statement of the proposition, *any ordinate of a certain line is always equal to twice its corresponding abscissa increased by ten.*

36. *Equation to a line parallel to one of the axes.* If in the equation $y=mx+c$ we suppose $m=0$, the line will be parallel to the axis of X, and the equation becomes

$$y=0 \,.\, x+c,$$

or

$$y=c.$$

This is then the equation of a line parallel to the axis of X. If c is positive, the line is above the axis of X; if negative, it is below it.

So also $x=\pm a$ is the equation to a straight line parallel to the axis of Y.

Examples. Construct the lines of which the following are the equations:

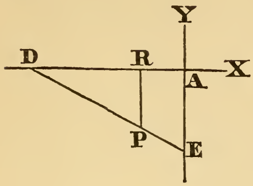
1. $y=2x+3$.	4. $y=-2x-5$.	7. $y=5$.
2. $y=3x-7$.	5. $y=3x$.	8. $y=-2$.
3. $y=-x+2$.	6. $y=x$.	9. $y=-x$.

37. *Every equation of the first degree containing two variables represents a straight line.*

Every equation of the first degree containing two variables can be reduced to the form

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}=0,$$

in which A, B, and C may be positive or negative. We shall now prove that every equation of this form represents a straight line.



In this equation put $y=0$, and we have $x=-\frac{C}{A}$, which represents the point D where the line intersects the axis of X. Again, put $x=0$, and we have $y=-\frac{C}{B}$, which represents the point E where the line intersects the axis of Y. We have thus determined two points in the line which this equation represents.

Let P be any other point of the line or curve represented by the given equation. We are to prove that P is on the straight line passing through the points D and E.

Since P is supposed to be on the line represented by the given equation, its co-ordinates must satisfy this equation; and representing its co-ordinates by x and y, we shall have

$$Ax+By+C=0,$$

whence

$$y=\frac{-C-Ax}{B}=PR.$$

Now $-\frac{C}{A}:-\frac{C}{B}::-\frac{C}{A}-x:\frac{-C-Ax}{B}$, identically.

But these several terms are equal to those of the proportion

$$AD:AE::DR:PR;$$

that is, PR is a fourth proportional to the three lines AD, AE, and DR; that is, P lies on the straight line joining D and E, and the equation $Ax+By+C=0$ represents that straight line.

If either A, B, or C be negative, the same demonstration will apply with a slight change of the figure.

This equation always represents *some* straight line, and may be made to represent *any* one by giving appropriate values to A, B, and C.

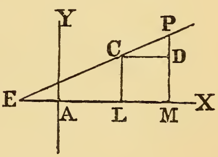
If in this equation $A=0$, then the line is parallel to the axis of x; if $B=0$, the line is parallel to the axis of y; if $C=0$, the line passes through the origin.

Examples. Draw the straight lines represented by the following equations:

1. $x+y+10=0$.	6. $2y=3x-5$.	11. $x=2y$.
2. $x+y=10$.	7. $y=4-x$.	12. $x=4$.
3. $x+y=0$.	8. $2x=y+7$.	13. $y=2$.
4. $2x+3y=0$.	9. $\frac{x}{2}+\frac{y}{3}=1$.	14. $4x-3y=1$.
5. $4x+3y=1$.	10. $y-3=2(x-2)$.	15. $x-2y=-4$.

38. *To find the equation to a straight line which passes through a given point.*

When a point P is not completely determined, its co-ordinates are denoted by the variables x and y; but when the position of a point is completely known, the co-ordinates are generally denoted by the letters a, b, or by x, y, with suffixes, as x_1, y_1, x_2, y_2; or by x and y with accents, as x', y', x'', y'', etc.



Let PCE be the straight line, C the given point whose co-ordinates are x_1, y_1, and P any point of the line whose co-ordinates are x and y. Draw the ordinates CL, PM; also draw CD parallel to AX.

Now $\qquad$ $PD=y-y_1$,

and $\qquad$ $CD=x-x_1$.

But $\qquad$ CD : PD : : radius : tang. PCD.

Hence $\frac{PD}{CD}=$tang. PCD, which we will represent by m.

That is, $\qquad \frac{y-y_1}{x-x_1}=m$, or $y-y_1=m(x-x_1)$,

which is the equation of a straight line passing through a given point P.

Since the coefficient m, which fixes the direction of the line, is not determined, there may be an infinite number of straight lines drawn through a given point. This is also apparent from the figure.

39. *Line passing through a given point and parallel to a given line.* If it be required that the line shall pass through a given point, and make a given angle with the axis of X, then

m becomes a known quantity, and if we put m' for the tangent of the given angle we shall have

$$y-y_1=m'(x-x_1),$$

which is the equation of a straight line passing through a given point, and making a given angle with the axis of X.

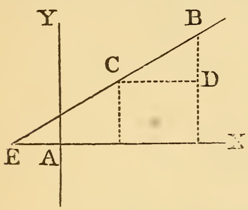
Ex. 1. Draw a line through the point whose abscissa is 5 and ordinate 3, making an angle with the axis of abscissas whose tangent is equal to 2, and give the equation of the line.

Ans. The equation is $y-2x+7=0$.

Ex. 2. Find the equation to the straight line which passes through the point (a, b), and makes an angle of 30° with the axis of X. *Ans.* $x-a=(y-b)\sqrt{3}$.

Ex. 3. Find the equation to the line which passes through the point (4, 4), and makes an angle of 45° with the axis of X.

40. *To find the equation to the straight line which passes through two given points.*



Let B and C be the two given points, the co-ordinates of B being x_2 and y_2, and the co-ordinates of C being x_1 and y_1. Then, since the general equation for every point in the required line is

$$y=mx+c, \qquad (1)$$

it follows that when the variable abscissa x becomes x_1, then y will become y_1; hence

$$y_1=mx_1+c. \qquad (2)$$

Also, when the variable abscissa x becomes x_2, then y becomes y_2, and hence $y_2=mx_2+c. \qquad (3)$

By combining these three equations we may eliminate m and c.

If we subtract equation (2) from equation (1), we obtain

$$y-y_1=m(x-x_1). \qquad (4)$$

Also, if we subtract equation (2) from equation (3), we obtain

$$y_2-y_1=m(x_2-x_1),$$

from which we find

$$m=\frac{y_2-y_1}{x_2-x_1}.$$

Substituting this value of m in equation (4), we have

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1),$$

which is the equation of the line passing through the two given points B and C.

It is evident from the figure that $\frac{y_2-y_1}{x_2-x_1}$ denotes the tangent of the angle BCD or BEX.

If the origin be one of the proposed points (x_2, y_2), then $x_2=0$ and $y_2=0$, and the equation becomes

$$y=\frac{y_1}{x_1}x,$$

which is the equation to a straight line passing through the origin and through a given point.

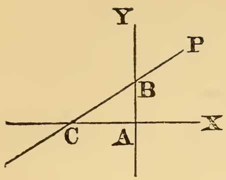
Ex. 1. Find the equation to the straight line which passes through the two points whose co-ordinates are $x_1=7$, $y_1=4$, and $x_2=5$, $y_2=3$, and determine the angle which it makes with the axis of abscissas.

Ex. 2. Find the equation to the straight line which passes through the two points $x_1=2$, $y_1=3$, and $x_2=4$, $y_2=5$.

Ex. 3. Find the equations to the straight lines which pass through the following pairs of points:

(1)	$x_1=3, y_1=4;$	and	$x_2=1, y_2=2.$
(2)	$x_1=5, y_1=6;$	"	$x_2=-1, y_2=0.$
(3)	$x_1=1, y_1=2;$	"	$x_2=2, y_2=-4.$
(4)	$x_1=4, y_1=-2;$	"	$x_2=-3, y_2=-5.$
(5)	$x_1=3, y_1=-2;$	"	$x_2=0, y_2=0.$
(6)	$x_1=2, y_1=5;$	"	$x_2=0, y_2=-7.$
(7)	$x_1=0, y_1=1;$	"	$x_2=1, y_2=-1.$
(8)	$x_1=0, y_1=-a;$	"	$x_2=0, y_2=-b.$
(9)	$x_1=a, y_1=b;$	"	$x_2=a, y_2=-b.$
(10)	$x_1=a, y_1=-b;$	"	$x_2=-a, y_2=-b.$

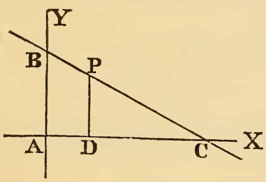
41. *Definition.* The distance from the origin to the point where a line intersects the axis of X is called the *intercept* on the axis of X; and the distance from the origin to the point



where a line intersects the axis of Y is called the intercept on the axis of Y.

Thus, in the annexed figure, AB and AC are the intercepts of the line PC on the two axes.

42. *To find the equation to a straight line in terms of its intercepts on the two axes.*



Let B and C be the points where the straight line meets the axes of y and x respectively. Suppose $AC=a$, and $AB=b$. Let P be any point in the line, and let x and y be its co-ordinates. Draw PD parallel to AY. Then, by similar triangles, we have

$$AB:DP::AC:DC;$$

that is,
$$b:y::a:a-x,$$

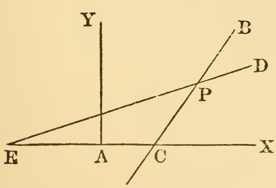
whence
$$\frac{x}{a}+\frac{y}{b}=1,$$

which is the equation to a straight line in terms of its intercepts a and b.

Ex. 1. Find the equation to a straight line which cuts off intercepts on the axes of x and y equal to 3 and -5 respectively.

Ex. 2. Find the equation to a straight line which cuts off the intercepts -4 and 2.

43. *To find the angle included between two given straight lines.*



Let BC and DE be any two lines intersecting each other in P. Let the equation to the line BC be

$$y=m_1x+c_1,$$

and the equation to the line DE be

$$y=m_2x+c_2;$$

then m_1 will be the tangent of the angle BCX, and m_2 the tangent of the angle DEX. Now, because PCX is the exterior angle of the triangle PEC, it is equal to the sum of the angles CPE and PEC; that is, the angle EPC is equal to the difference of the angles PCX and PEX, or

$$\text{EPC}=\text{PCX}-\text{PEX},$$

whence $\quad\text{tang. EPC}=\text{tang. (PCX}-\text{PEX)},$

which, by Trig., Art. 77,

$$=\frac{\text{tang. PCX}-\text{tang. PEX}}{1+\text{tang. PCX}\times\text{tang. PEX}}$$

$$=\frac{m_1-m_2}{1+m_1m_2},$$

which denotes the tangent of the angle included between the two given lines.

44. *To determine the co-ordinates of the point of intersection of two given straight lines.*

Let the equation to one line be

$$y=m_1x+c_1, \tag{1}$$

and the equation to the other

$$y=m_2x+c_2. \tag{2}$$

Since the co-ordinates of every point on a line must satisfy its equation, the co-ordinates of the point through which both the lines pass will satisfy both equations; we must, therefore, find the values of x and y from (1) and (2) regarded as simultaneous equations. We thus obtain

$$x=\frac{c_1-c_2}{m_2-m_1}, \text{ and } y=\frac{c_1m_2-c_2m_1}{m_2-m_1},$$

which are the co-ordinates of the point of intersection of the two lines.

Ex. 1. Find the angle included between the lines $x+y=1$ and $y=x+2$; also find the co-ordinates of the point of intersection.

Ans. $90°$, $x=-\frac{1}{2}$, $y=\frac{3}{2}$.

Ex. 2. Find the angle between the lines $x+3y=1$ and $x-2y=1$; also the co-ordinates of the point of intersection.

Ans. $45°$, $x=1$, $y=0$.

Ex. 3. Find the angle between the lines $x+y\sqrt{3}=0$ and

$x - y\sqrt{3} = 2$; also the co-ordinates of the point of intersection.

Ans. $60°$, $x = 1$, $y = -\frac{\sqrt{3}}{3}$.

Ex. 4. Find the angle between the lines $3y - x = 0$ and $2x + y = 1$; also the co-ordinates of the point of intersection.

Ans. $81° \, 52'$, $x = \frac{3}{7}$, $y = \frac{1}{7}$.

Ex. 5. Find the angle between the lines $3y - 2x + 1 = 0$ and $3x - y = 0$; also the co-ordinates of the point of intersection.

Ex. 6. Find the angle between the lines $x + y - 3 = 0$ and $x + y = 2$; also the co-ordinates of the point of intersection.

45. *To find the equation to the straight line which passes through a given point, and is perpendicular to a given straight line.*

Let x_1y_1 be the co-ordinates of the given point, and

$$y = mx + c$$

the equation to the given line. The form of the equation to a line through (x_1y_1) (Art. 38) is

$$y - y_1 = m_1(x - x_1).$$

The tangent of the angle between the two lines is (Art. 43)

$$\frac{m - m_1}{1 + mm_1}.$$

If the angle of intersection of the two lines be a right angle, its tangent must be infinite, and the denominator $1 + mm_1$ must become zero, so that we must have

$$m_1 = -\frac{1}{m}.$$

Hence the required equation is

$$y - y_1 = -\frac{1}{m}(x - x_1),$$

which is the equation to the straight line passing through the point (x_1y_1), and perpendicular to the line $y = mx + c$.

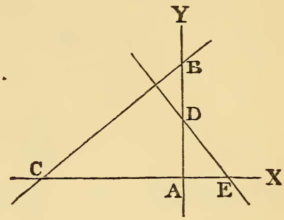
46. *Condition of perpendicularity.* We conclude from the last article that $$y = -\frac{x}{m} + c_1$$

represents a line perpendicular to the line

$$y = mx + c.$$

The condition by which two straight lines are shown to be at right angles to each other may also be determined as follows:

Let BC be a given line, and let tang. BCX $= m$.



Let DE be perpendicular to BC, and let tang. DEX $= m_1$; then

$$\text{tang. DEX} = -\text{tang. DEA},$$
$$= -\text{cotang. BCA};$$

that is, $m_1 = -\frac{1}{m}$. (Trig., Art. 28.)

Hence we see that *when two lines are perpendicular to each other, the tangents of the angles which they make with either axis are the reciprocals of each other, and have contrary signs.*

Ex. 1. Find the equation to the line which passes through the origin, and is perpendicular to the line $x + y = 2$.

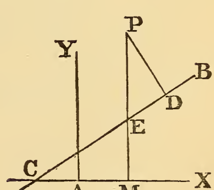
Ans. $y = x$.

Ex. 2. Find the equation to the line which passes through the point $x_1 = 2, y_1 = -4$, and is perpendicular to the line $3y + 2x - 1 = 0$. *Ans.* $2y = 3x - 14$.

Ex. 3. Find the equation to the line which passes through the point (8, 4), and is perpendicular to the line whose equation is $y = 2x - 16$.

Ex. 4. Find the equation to the line which passes through the point $(-1, 3)$, and is perpendicular to the line $3x + 4y + 2 = 0$.

47. *To find the perpendicular distance of a given point from a given straight line.*



Let P be the given point, whose co-ordinates are $x_1 y_1$, and let BC be the given straight line whose equation is

$$y = mx + c.$$

From P draw PD perpendicular to BC, and PM perpendicular to AX, cutting BC in

E. Now, since the above equation applies to every point of BC, it must apply to E; that is,

$$EM = mx_1 + c.$$

The perpendicular $\quad PD = PE \sin. PED.$

But $\quad PE = PM - ME = y_1 - mx_1 - c,$

and $\quad \sin. PED = \sin. CEM = \cos. ECM = \frac{1}{\sec. ECM} =$

$$= \frac{1}{\sqrt{1 + (\text{tang. } ECM)^2}} = \frac{1}{\sqrt{1+m^2}}.$$

Therefore $\quad PD = \frac{y_1 - mx_1 - c}{\sqrt{1+m^2}},$

which equation expresses the distance from the given point $(x_1 y_1)$ to the given straight line.

If the point P be at the origin, then $x_1 = 0$, $y_1 = 0$, and we have

$$PD = \frac{-c}{\sqrt{1+m^2}},$$

which equation expresses the distance of the proposed line from the origin.

Ex. 1. Find the perpendicular distance of the point 2, 3 from the line $x + y = 1$. *Ans.* $2\sqrt{2}$.

Ex. 2. Find the distance of the point -1, 3 from the line $3x + 4y + 2 = 0$. *Ans.* $\frac{11}{5}$.

Ex. 3. Find the distance of the point 0, 1 from the line $x - 3y = 1$. *Ans.* $\frac{2\sqrt{10}}{5}$.

Ex. 4. Find the distance of the point 3, 0 from the line $\frac{x}{2} + \frac{y}{3} = 1$. *Ans.* $\frac{3}{\sqrt{13}}$.

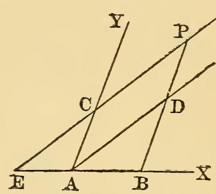
Ex. 5. Find the distance of the point $1, -2$ from the line $x + y - 3 = 0$. *Ans.* $2\sqrt{2}$.

Ex. 6. Find the distance of the origin of co-ordinates from the line $\frac{x}{2} + \frac{y}{3} = 1$. *Ans.* $\frac{6}{\sqrt{13}}$.

Ex. 7. Find the distance of the point $3, -5$ from the line $2x-8y+7=0$.

Ex. 8. Find the distance of the point 8, 4 from the line $y=2x-16$.

48. *To find the equation to a straight line referred to oblique axes.*



Let A be the origin of co-ordinates; let AX, AY be the oblique axes, and let PC be any straight line whose equation is required to be determined. Take any point P in the given line, and draw PB parallel to AY; then will PB be the ordinate and AB the abscissa of the point P. Through the origin draw a line AD parallel to CP, meeting the line BP in D.

Denote the inclination of the axes by ω, and the angle DAX by a. Since PB is parallel to AY, the angle ADB is equal to DAY; that is, equal to $\omega - a$.

Let x, y be the co-ordinates of P, and represent AC or DP by c.

Then, by Trig., Art 49,

$$\text{BD} : \text{AB} :: \sin. a : \sin. (\omega - a).$$

Hence $$\text{BD} = \frac{x \sin. a}{\sin. (\omega - a)}.$$

But $$\text{BP} = \text{BD} + \text{DP}.$$

Hence $$y = \frac{x \sin. a}{\sin. (\omega - a)} + c,$$

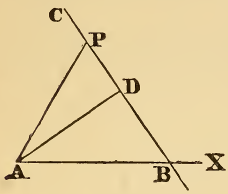
which is the equation to a straight line referred to oblique axes.

If we put m for $\frac{\sin. a}{\sin. (\omega - a)}$, the equation becomes

$$y = mx + c,$$

which is of the same form as the equation referred to rectangular axes, Art. 29. The meaning of c is the same as in Art. 29; but the factor m denotes the ratio of the sine of the inclination of the line to the axis of X, to the sine of its inclination to the axis of Y. When the axes are at right angles to each other, m becomes the tangent of a.

49. *To find the polar equation to a straight line.*



Let BC be any straight line, and P any point in it. Let A be the pole, AX the initial line, and let AD be drawn from A perpendicular to BC. Let AD $=p$, the angle DAX $=a$, and let the polar co-ordinates of P be r, θ; then we shall have

$$AD = AP \cos. PAD;$$

that is, $$p = r \cos. (\theta - a),$$

or $$r = p \sec. (\theta - a),$$

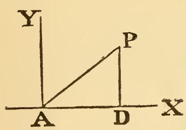
which is the polar equation to a straight line.

If AD be taken for the initial line, then $a=0$, and the equation becomes $$r = p \sec. \theta,$$

which is the equation to a right line *perpendicular to the initial line.*

To trace a right line by its polar equation, we find its intercept on the initial line by making $\theta = 0$. Then from the pole as a centre, with a radius equal to p, describe a circle, and draw a tangent to this circle from the point first determined; this tangent line will be the line required.

50. *To find the polar equation to a line passing through the pole.*



Let x and y denote the co-ordinates of P referred to rectangular axes; also let r and θ denote the polar co-ordinates of P, the pole being at the origin A, and AX being the initial line.

Then, as in Art. 26, $$x = r \cos. \theta,$$

and $$y = r \sin. \theta.$$

Substituting these values in the equation

$$y = mx,$$

we have $$r \sin. \theta = mr \cos. \theta;$$

therefore $$\text{tang. } \theta = m;$$

that is, $\theta = \text{a constant},$

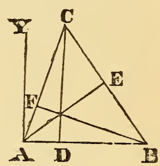
which is the polar equation to a straight line passing through the pole.

Examples. Draw the straight lines represented by the equations

1. $r \cos. \left(\theta - \frac{\pi}{6}\right) = 1.$
2. $r \cos. \left(\theta - \frac{\pi}{4}\right) = 4.$
3. $r \cos. \left(\theta - \frac{\pi}{3}\right) = 8.$
4. $\theta = \frac{\pi}{3}.$
5. $\theta = \frac{\pi}{2}.$
6. $\theta = 0.$

51. The following examples are designed to show how the preceding principles may be applied to the solution of geometrical problems.

To determine whether the perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a point.



Let ABC be any triangle, and let AE, BF, CD be perpendiculars from A, B, and C upon the opposite sides.

Let A be the origin of co-ordinates; let AB be the axis of abscissas, and AY, perpendicular to AB, the axis of ordinates. Let the co-ordinates of C be $x_1 y_1$, and those of B be x_2, 0.

Now if the abscissa of the point where AE and BF intersect is equal to AD, the intersection of these lines must be on CD. Since each of these lines passes through a given point and is perpendicular to a given line, its equation will be given by Art. 45; but we must first find the equations to the lines AC, BC, to which they are perpendicular.

Since AC passes through the origin and the given point C, its equation is (Art. 40)

$$y = \frac{y_1}{x_1} x; \tag{1}$$

and since BF passes through a given point B(x_2, 0), and is perpendicular to (1), its equation is (Art. 45)

$$y=-\frac{x_1}{y_1}(x-x_2). \tag{2}$$

Also, since BC passes through the point B$(x_2, 0)$ and the point C$(x_1 y_1)$, its equation is (Art. 40)

$$y=\frac{y_1}{x_1-x_2}(x-x_2); \tag{3}$$

and since AE passes through the origin (0, 0), and is perpendicular to (3), its equation is

$$y=-\frac{x_1-x_2}{y_1}x. \tag{4}$$

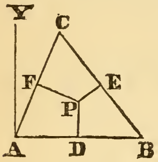
At the point where (2) and (4) intersect, their ordinates must be identical. Hence we may equate their values, and we have

$$\frac{x_1}{y_1}(x-x_2)=\frac{x_1-x_2}{y_1}x;$$

whence $x=x_1$;

that is, x, the abscissa of the intersection of BF, AE, is equal to x_1, the abscissa of the point C; hence the perpendicular CD passes through that intersection, and the three perpendiculars meet in a point.

52. *To determine whether the three perpendiculars through the middle points of the sides of a triangle meet in a point.*



Let ABC be any triangle, and let D, E, F be the middle points of its sides. Let P be the point where two of the perpendiculars EP, FP meet; now if the abscissa of P is equal to AD, the intersection of the lines EP, FP must be in the perpendicular drawn from D.

Represent the point C by $(x_1 y_1)$, and the point B by $(x_2, 0)$.

The co-ordinates of F are $\frac{x_1}{2}, \frac{y_1}{2}$ (Art. 24), and the co-ordinates of E are

$$\frac{x_1+x_2}{2} \text{ and } \frac{y_1}{2}.$$

Now the equation to AC, passing through the origin and the point $x_1 y_1$, is

$$y=\frac{y_1}{x_1}x, \tag{1}$$

and the equation to FP, which passes through F$\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$, and is perpendicular to (1), is

$$y-\frac{y_1}{2}=-\frac{x_1}{y_1}\left(x-\frac{x_1}{2}\right). \tag{2}$$

The equation to BC, passing through the point $(x_2, 0)$ and $(x_1 y_1)$,

is
$$y=\frac{y_1}{x_1-x_2}(x-x_2), \tag{3}$$

and the equation to EP, which passes through E$\left(\frac{x_1+x_2}{2}, \frac{y_1}{2}\right)$, and is perpendicular to (3), is

$$y-\frac{y_1}{2}=-\frac{x_1-x_2}{y_1}\left(x-\frac{x_1+x_2}{2}\right). \tag{4}$$

At the point where (2) and (4) intersect, their ordinates must be identical; and equating their values, we have

$$\frac{x_1}{y_1}\left(x-\frac{x_1}{2}\right)=\frac{x_1-x_2}{y_1}\left(x-\frac{x_1+x_2}{2}\right),$$

which gives
$$x=\frac{x_2}{2};$$

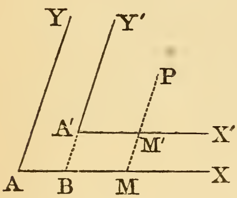
that is, x, the abscissa of the intersection of EP and FP, is equal to $\frac{x_2}{2}$, which is the abscissa of the point D; hence the perpendicular from D passes through that intersection, and the three perpendiculars meet in a point.

SECTION III.

TRANSFORMATION OF CO-ORDINATES.

53. When a line is represented by an equation with reference to any system of axes, we can always transform that equation into another which shall equally represent the line, but with reference to a new system of axes chosen at pleasure. This is called the transformation of co-ordinates, and may consist either in altering the relative position of the axes without changing the origin; or changing the origin without disturbing the relative position of the axes; or we may change both the direction of the axes and the position of the origin.

54. *To change the origin from one point to another without altering the direction of the axes.*



Let AX, AY be the primitive axes, and let A′X′, A′Y′ be the new axes, respectively parallel to the preceding.

Let AB, A′B, the co-ordinates of the new origin referred to the old axes, be represented by a and b; let the co-ordinates of any point P referred to the primitive axes be x and y, and the co-ordinates of the same point referred to the new axes be x' and y'. Then we shall have

$$AM = AB + BM = AB + A'M',$$

or

$$x = a + x'.$$

Also,

$$PM = MM' + PM' = BA' + PM',$$

or

$$y = b + y'.$$

Hence, to find the equation to any line when the origin is changed, the new axes remaining parallel to the old, we must substitute in the equation to the line, $a+x'$ for x, and $b+y'$ for y.

These formulas are equally true for rectangular and oblique co-ordinates.

Ex. 1. Find what the equation $2x+3y=8$ becomes when the origin is transferred to a point whose co-ordinates are $a=3$, $b=1$. *Ans.* $2x'+3y'=-1$.

Ex. 2. Find what the equation $y+2x=-5$ becomes when the origin is changed to the point (2, 1).

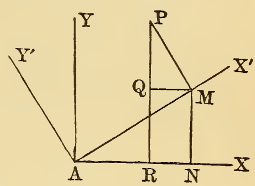
Ans. $y'+2x'=-10$.

Ex. 3. Find what the equation $y=3x-7$ becomes when the origin is changed to the point $(-2, -3)$. *Ans.* $y'=3x'-10$.

Ex. 4. Find what the equation $y^2+4y-4x+8=0$ becomes when the origin is changed to the point $(1, -2)$.

Ans. $y'^2=4x'$.

55. *To change the direction of the axes without changing the origin, both systems being rectangular.*



Let AX, AY be the primitive axes, and AX′, AY′ be the new axes, both systems being rectangular. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes. Denote the angle XAX′ by θ. Through P draw PR parallel to AY, and PM parallel to AY′. From M draw MN parallel to AY, and MQ parallel to AX.

Then $\qquad x=AR=AN-NR=AN-MQ.$

Also $\qquad AN=AM\cos. XAX'=x'\cos.\theta,$

and $\qquad MQ=PM\sin. MPQ=y'\sin.\theta.$

Hence $\qquad x=x'\cos.\theta-y'\sin.\theta.$

Also $\qquad y=PR=QR+PQ=MN+PQ.$

But $\qquad MN=AM\sin. MAX=x'\sin.\theta,$

and $\qquad PQ=PM\cos. MPQ=y'\cos.\theta.$

Hence $\qquad y=x'\sin.\theta+y'\cos.\theta.$

Hence, to find the equation to any line when referred to the new axes, we must substitute in the equation to the line $x'\cos.\theta-y'\sin.\theta$ for x, and $x'\sin.\theta+y'\cos.\theta$ for y.

Ex. 1. Find what the equation $x+y=10$ becomes when the axes are moved through an angle of 45°.

Note. $\sin. 45° = \cos. 45° = \frac{\sqrt{2}}{2}.$

Here
$$x = \frac{x'}{2}\sqrt{2} - \frac{y'}{2}\sqrt{2},$$
$$y = \frac{x'}{2}\sqrt{2} + \frac{y'}{2}\sqrt{2}.$$

By substitution, the given equation becomes $x' = 5\sqrt{2}$ *Ans.*

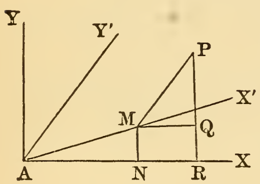
Ex. 2. Find what the equation $y=3x-6$ becomes when the axes are moved through an angle of 45°.

Ans. $2y' = x' - 3\sqrt{2}.$

Ex. 3. Find what the equation $y^2 - x^2 = 6$ becomes when the axes are moved through an angle of 45°. *Ans.* $x'y' = 3.$

Ex. 4. Find what the equation $\frac{x}{2} + \frac{y}{3} = 1$ becomes when the axes are moved through an angle of 45°.

56. *To transform an equation from rectangular to oblique co-ordinates.*



Let AX, AY be the primitive axes, and AX′, AY′ be the new axes. Let P be any point; x, y its co-ordinates referred to the old axes; x', y' its co-ordinates referred to the new axes. Through P draw PR parallel to AY, and PM parallel to AY′. Draw also MN parallel to AY, and MQ parallel to AX. Denote the angle XAX′ by α, and the angle XAY′ by β.

Then $x = AR = AN + NR = AN + MQ.$

But $AN = AM \cos. XAX' = x' \cos. \alpha,$

and $MQ = PM \cos. PMQ = y' \cos. \beta.$

Hence $x = x' \cos. \alpha + y' \cos. \beta.$

Also $y = PR = QR + PQ = MN + PQ.$

But $MN = AM \sin. XAX' = x' \sin. \alpha,$

and $PQ = PM \sin. PMQ = y' \sin. \beta.$

Hence $y = x' \sin. \alpha + y' \sin. \beta.$

Hence, if we wish to pass from rectangular to oblique axes, we must substitute in the equation to the line, $x'\cos.\alpha+y'\cos.\beta$ for x, and $x'\sin.\alpha+y'\sin.\beta$ for y.

If the origin be changed at the same time to a point whose co-ordinates referred to the primitive system are m and n, these equations will become

$$x=m+x'\cos.\alpha+y'\cos.\beta.$$
$$y=n+x'\sin.\alpha+y'\sin.\beta.$$

In the following examples the origin and the axis of X are supposed to remain unchanged.

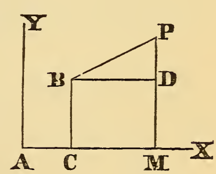
Ex. 1. Transform the equation $y=4-x$ from rectangular to oblique co-ordinates, the new axes being inclined to one another at an angle of 45°. *Ans.* $x'+y'\sqrt{2}=4$.

Ex. 2. Transform the equation $y=3x$ from rectangular to oblique co-ordinates, the new axes being inclined to one another at an angle of 45°. *Ans.* $3x'+y'\sqrt{2}=0$.

Ex. 3. Transform the equation $y=4-x$ from rectangular to oblique co-ordinates, the new axes being inclined to one another at an angle of 60°. *Ans.* $y'(\sqrt{3}+1)+2x'=8$.

Ex. 4. Transform the equation $2x=3y+6$ from rectangular to oblique co-ordinates, the new axes being inclined to one another at an angle of 60°. *Ans.* $2x'+y'(1-\frac{3}{2}\sqrt{3})=6$.

57. *To transform an equation from rectangular to polar co-ordinates.*



Let AX, AY be the rectangular axes; let B be the pole; and let BD, the initial line, be parallel to AX.

Let P be any point; x, y its co-ordinates referred to the rectangular axes; ρ, θ its polar co-ordinates. Draw PM, BC parallel to AY, and let a, b be the co-ordinates of B referred to the primitive axes.

Now $\quad AM=AC+CM=AC+BD.$

But $\quad BD=BP\cos.PBD=\rho\cos.\theta.$

Hence $\quad x=a+\rho\cos.\theta.$

Also $PM = DM + PD = BC + PD.$

But $PD = BP \sin. PBD = \rho \sin. \theta.$

Hence $y = b + \rho \sin. \theta.$

Hence, to transform the equation to any line from rectangular to polar co-ordinates, we must substitute in the equation to the line, $a + \rho \cos. \theta$ for x, and $b + \rho \sin. \theta$ for y.

In the following examples the pole is supposed to coincide with the origin, and the initial line with the axis of X.

Ex. 1. Transform the equation $x^2 + y^2 = 9$ from rectangular to polar co-ordinates. *Ans.* $\rho^2(\cos.^2\theta + \sin.^2\theta) = 9$, or $\rho = 3$.

Ex. 2. Transform the equation $xy = 4$ from rectangular to polar co-ordinates.

Note. $\text{Sin. } 2\theta = 2 \sin. \theta \cos. \theta$ (Trig., Art. 73).

Ans. $\rho^2 \sin. 2\theta = 8$.

Ex. 3. Transform the equation $x^2 + y^2 = mx$ from rectangular to polar co-ordinates. *Ans.* $\rho = m \cos. \theta$.

Ex. 4. Transform the equation $x^2 - y^2 = 3$ from rectangular to polar co-ordinates.

Note. $\text{Cos. } 2\theta = \cos.^2\theta - \sin.^2\theta$ (Trig., Art. 73).

Ans. $\rho^2 \cos. 2\theta = 3$.

58. To transform an equation from oblique to rectangular axes, find the values of x' and y' from the formulas of Art. 56.

To transform an equation from polar to rectangular co-ordinates, deduce the values of ρ and θ from the equations of Art. 57. These values are

$$\rho^2 = (x-a)^2 + (y-b)^2,$$

and

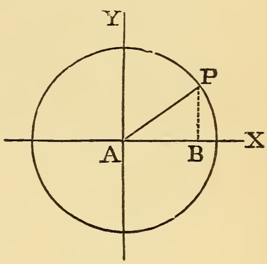
$$\text{tang. } \theta = \frac{y-b}{x-a}.$$

SECTION IV.

THE CIRCLE.

59. *Definition.* A circle is a plane figure bounded by a line, all the points of which are equally distant from a point within called the centre. The line which bounds the circle is called its circumference. A radius of a circle is a straight line drawn from the centre of the circle to the circumference.

60. *To find the equation to a circle referred to rectangular axes when the origin of co-ordinates is at the centre.*



Let A be the centre of the circle, and P any point on its circumference. Let r be the radius of the circle, and x, y the co-ordinates of P. Then, by Geom., Bk. IV., Pr. 11,

$$AB^2 + BP^2 = AP^2;$$

or,

$$x^2 + y^2 = r^2,$$

which is the equation required.

61. *Points of intersection with the axes.* If we wish to determine the points where the curve cuts the axis of X, we must put

$$y = 0,$$

for this is the property of all points situated on the axis of abscissas. On this supposition, we have

$$x = \pm r,$$

which shows that the curve cuts the axis of abscissas in two points on different sides of the origin, and at a distance from it equal to the radius of the circle.

To determine the points where the curve cuts the axis of ordinates, we make $x = 0$, and we find

$$y = \pm r,$$

which shows that the curve cuts the axis of ordinates in two points on different sides of the origin, and at a distance from it equal to the radius of the circle.

62. *Curve traced through intermediate points.* If we wish to trace the curve through the intermediate points, we reduce the equation to the form

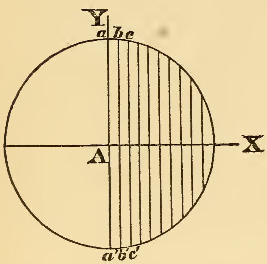
$$y = \pm\sqrt{r^2 - x^2},$$

from which we may compute the value of y corresponding to any assumed value of x.

Example. Trace the curve whose equation is $x^2 + y^2 = 100$.

By assuming for x different values from 0 to 11, etc., we obtain the corresponding values of y as given below.

When $x=0$, $y=\pm 10$.	When $x=6$, $y=\pm 8.00$.
$x=1$, $y=\pm 9.95$.	$x=7$, $y=\pm 7.14$.
$x=2$, $y=\pm 9.80$.	$x=8$, $y=\pm 6.00$.
$x=3$, $y=\pm 9.54$.	$x=9$, $y=\pm 4.36$.
$x=4$, $y=\pm 9.16$.	$x=10$, $y=\pm 0.00$.
$x=5$, $y=\pm 8.66$.	$x=11$, y is imaginary.

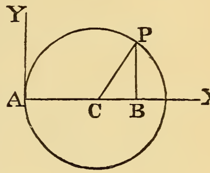


When $x=0$, y will equal ± 10, which gives two points, a and a', one above and the other below the axis of X. When $x=1$, $y=\pm 9.95$, which gives the points b and b'. When $x=2$, $y=\pm 9.80$, which gives the points c and c', etc. If we suppose x greater than 10, the value of y will be imaginary, which shows that the curve does not extend from the centre beyond the value $x=10$.

If x is negative, we shall in like manner obtain points in the third and fourth quadrants, and the curve will not extend to the left beyond the value $x=-10$.

Since every value of x furnishes two equal values of y with contrary signs, it follows that the curve is symmetrical above and below the axis of X.

63. *To find the equation to a circle when the origin is on the circumference, and the axis of X passes through the centre.*



Let the origin of co-ordinates be at A, a point on the circumference of the circle, and let the axis of X pass through the centre. Let r be the radius of the circle, and let x, y be the co-ordinates of P, any point on the circumference. Then CB will be represented by $x-r$.

Now $$CB^2+BP^2=CP^2,$$
or $$(x-r)^2+y^2=r^2,$$
whence $$y^2=2rx-x^2,$$
which is the equation required.

64. *Points of intersection with the axes.* If we wish to determine where the curve cuts the axis of X, we make $y=0$, and we find $$x(2r-x)=0.$$

This equation is satisfied by supposing $x=0$, or $2r-x=0$, from the last of which equations we find $x=2r$. The curve, therefore, cuts the axis of abscissas in two points, one at the origin, and the other at a distance from it equal to $2r$.

To determine where the curve meets the axis of ordinates, we make $x=0$, which gives
$$y=0,$$
which shows that the curve meets the axis of ordinates in but one point, viz., the origin.

65. *Curve traced through intermediate points.* In order to trace the curve through intermediate points, we reduce the equation to the form
$$y=\pm\sqrt{2rx-x^2},$$
from which we may compute the value of y corresponding to any assumed value of x, as in Art. 62.

Ex. 1. Trace the curve whose equation is $y^2=10x-x^2$.

By assuming for x different values from 0 to 11, etc., we obtain the corresponding values of y as given on the next page.

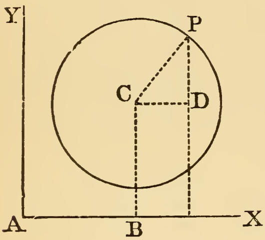
When $x=0$, $y=0$.	When $x=6$, $y=4.90$.
$x=1$, $y=3$.	$x=7$, $y=4.58$.
$x=2$, $y=4$.	$x=8$, $y=4$.
$x=3$, $y=4.58$.	$x=9$, $y=3$.
$x=4$, $y=4.90$.	$x=10$, $y=0$.
$x=5$, $y=5$.	$x=11$, y is imaginary.

These values may be represented by a figure as in Art. 62.

Ex. 2. Trace the circle $x^2+y^2=10y$.

Ex. 3. Trace the circle $x^2+y^2=-10x$.

66. *To find the equation to the circle referred to any rectangular axes.*



Let C be the centre of the circle, and P any point on its circumference. Let r be the radius of the circle; a and b the co-ordinates of C; x, y the co-ordinates of P. From C and P draw lines perpendicular to AX, and draw CD parallel to AX. Then

$$CD^2+DP^2=CP^2;$$

that is,
$$(x-a)^2+(y-b)^2=r^2,$$
which is the equation required.

67. *Varieties in the equation to the circle.* If in the equation $(x-a)^2+(y-b)^2=r^2$ we suppose $a=0$ and $b=0$, the centre of the circle becomes the origin of co-ordinates, and the equation becomes

$$x^2+y^2=r^2 \text{ (as in Art. 60).}$$

If we suppose $a=r$ and $b=0$, the axis of X becomes a diameter, and the origin is at its extremity, and the equation becomes
$$(x-r)^2+y^2=r^2,$$
whence
$$y^2=2rx-x^2 \text{ (as in Art. 63).}$$

If we suppose $a=0$ and $b=r$, the axis of Y becomes a diameter, and the origin is at its extremity, and the equation becomes
$$x^2+(y-r)^2=r^2,$$
whence
$$x^2=2ry-y^2.$$

68. *General equation to the circle.* Expanding the general equation to the circle referred to rectangular axes, we have

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0;$$

and hence it appears that the general equation to the circle is of the form

$$x^2+y^2+Ax+By+C=0,$$

where A, B, and C are constant quantities, any one or more of which in particular cases may be equal to zero. The equation

$$Ax^2+Ay^2+Bx+Cy+D=0$$

may be reduced to this form by dividing by A, and is therefore the most general form that the equation can assume when the co-ordinates are rectangular.

69. *To determine the circle represented by an equation.* If we can reduce an equation to the form

$$x^2+y^2+Ax+By+C=0,$$

we may determine the circle it represents; for, adding $\frac{A^2+B^2}{4}$ to both sides of the equation, and transposing C, we have

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2=\frac{A^2+B^2}{4}-C.$$

By comparing this equation with that of Art. 66, we perceive that it represents a circle, the co-ordinates of whose centre are $-\frac{A}{2}, -\frac{B}{2}$, and whose radius is

$$\left(\frac{A^2+B^2}{4}-C\right)^{\frac{1}{2}} \text{ or } \tfrac{1}{2}(A^2+B^2-4C)^{\frac{1}{2}}.$$

If $A^2+B^2<4C$, the radius becomes imaginary, and the equation can represent no real curve.

Ex. 1. Determine the co-ordinates of the centre, and the radius of the circle denoted by the equation $x^2+y^2+4x-8y-5=0$.

This equation may be reduced to the form

$$(x+2)^2+(y-4)^2=25.$$

Hence the co-ordinates of the centre are $-2, 4$, and the radius is 5.

Ex. 2. Determine the co-ordinates of the centre and the radius of the circle denoted by the equation $x^2+y^2+4y-4x-1=0$. *Ans.* Co-ordinates 2, -2, radius 3.

Ex. 3. Determine the co-ordinates of the centre and the radius of the circle denoted by the equation $x^2+y^2+6x-4y-36=0$. *Ans.* Co-ordinates -3, 2, radius 7.

Ex. 4. Determine the co-ordinates of the centre and the radius of the circle denoted by the equation $x^2+y^2-3x-4y+4=0$. *Ans.* Co-ordinates $\frac{3}{2}$, 2, radius $\frac{3}{2}$.

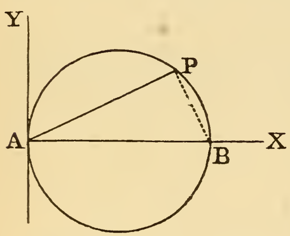
Ex. 5. Determine the co-ordinates of the centre and the radius of the circle denoted by the equation $x^2+y^2-2a(x-y)=c^2$.

Ans. Co-ordinates a, $-a$, radius $(2a^2+c^2)^{\frac{1}{2}}$.

Ex. 6. Find the equation to the circle whose radius is 9, and co-ordinates of the centre -1, 5.

Ex. 7. Find the equation to the circle whose radius is $5a$, and co-ordinates of the centre $3a$, $4a$.

70. *To find the polar equation to a circle when the origin is on the circumference, and the initial line is a diameter.*



Let A be the pole situated on the circumference of the circle; let AX, passing through the centre, be the initial line, and let P be any point on the circumference. Let r be the radius of the circle, and let ρ and θ be the polar co-ordinates of P.

The equation of the circle referred to rectangular axes (Art. 63) is
$$y^2=2rx-x^2.$$

To transform this equation from rectangular to polar co-ordinates (Art. 59), we must substitute for x, ρ cos. θ; and for y, ρ sin. θ.

Making this substitution, we obtain
$$\rho^2 \sin.^2\theta=2r\rho \cos.\theta-\rho^2 \cos.^2\theta;$$
or, by transposition,
$$\rho^2(\sin.^2\theta+\cos.^2\theta)=2r\rho \cos.\theta.$$

But $\sin.^2\theta + \cos.^2\theta$ is equal to unity.
Hence, dividing by ρ, we obtain

$$\rho = 2r \cos. \theta,$$

which is the polar equation of the circle.

71. *Points of the circle determined.* When $\theta=0$, $\cos. \theta=1$, and we have

$$\rho = 2r = AB.$$

As θ increases from 0 to 90°, the radius vector determines all the points in the semi-circumference BPA; and when $\theta = 90°$, $\cos. \theta = 0$, and ρ becomes zero.

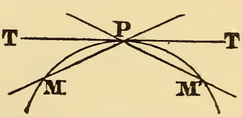
From $\theta=90°$ to $\theta=180°$ the radius vector is negative, and is measured into the fourth quadrant, determining all the points in the semi-circumference below the axis of abscissas. From $\theta=180°$ to $\theta=360°$ the circumference is described a second time.

Ex. 1. The polar co-ordinates of P are $\rho=10$, $\theta=45°$; determine the radius of the circle.

Ex. 2. The radius of a circle is 5 inches, and $\rho=8$ inches; determine the value of θ.

Ex. 3. The radius of a circle is 5 inches, and $\theta=60°$; determine the radius vector.

72. *Definition.* Let two points be taken on a curve, and a secant line be drawn through them; let the first point remain fixed, while the second point moves on the curve toward the first until it coincides with it; when the two points coincide, the secant line becomes a *tangent* to the curve.



Suppose a straight line MP to intersect a curve in two points, M and P, and let the line turn about the fixed point P until it comes into the position PM′. The second point of intersection, which at first was on the *left* of P, is now found on the *right* of P; hence, in the movement of the straight line from the position MP to the position PM′, there must have been one position in which the point M coincided with P. In this position, represented by the line TT′, the line is said to be a *tangent* to the curve.

This definition of a tangent suggests a method of finding its equation which is applicable to all curves.

73. *To find the equation to the tangent at any point of a circle.*

Let the equation to the circle be $x^2+y^2=r^2$.

Let x', y' be the co-ordinates of the point on the circle at which the tangent is drawn, and x'', y'' the co-ordinates of an adjacent point on the circle. The equation to the secant line passing through the points x', y' and x'', y'' (Art. 40) is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1)$$

Now, since the points x', y' and x'', y'' are both on the circumference of the circle, we must have

$$x'^2+y'^2=r^2=x''^2+y''^2,$$

or

$$y''^2-y'^2=x'^2-x''^2,$$

whence

$$\frac{y''-y'}{x''-x'}=-\frac{x''+x'}{y''+y'}.$$

Substituting this value in equation (1), we obtain

$$y-y'=-\frac{x''+x'}{y''+y'}(x-x'), \qquad (2)$$

which is the equation to the secant line passing through the two given points.

Now when the point x', y' coincides with the point x'', y'', we have $x'=x''$, and $y'=y''$; hence equation (2) becomes

$$y-y'=-\frac{x'}{y'}(x-x'), \qquad (3)$$

which is the equation to the tangent at the point x', y', where x and y are the co-ordinates of any point of the tangent line.

Clearing of fractions and transposing, we obtain

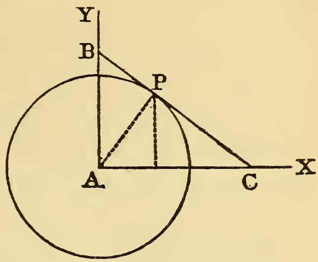
$$xx'+yy'=x'^2+y'^2,$$

or

$$xx'+yy'=r^2,$$

which is the simplest form of the equation to the tangent line.

74. *Points where the tangent cuts the axes.* To determine the point in which the tangent intersects the axis of X, we make $y=0$, which gives



$$xx'=r^2,$$

or
$$x=\frac{r^2}{x'}=AC,$$

since x is AC when $y=0$.

To determine the point in which the tangent intersects the axis of Y, we make $x=0$, which gives

$$yy'=r^2,$$

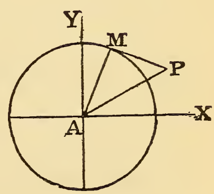
or
$$y=\frac{r^2}{y'}=AB.$$

Ex. 1. On a circle whose radius is 6 inches, a tangent line is drawn through the point whose ordinate is 4 inches; determine where the tangent line meets the two axes; also the angle which the tangent line makes with the axis of X.

Ex. 2. Find the point on the circumference of a circle whose radius is 5 inches, from which, if a radius and a tangent line be drawn, they will form with the axis of X a triangle whose area is 35 inches.

75. *To find the length of the tangent drawn to the circle from a given point.*

Let P be a point without the circle from which a tangent line PM is drawn. Draw the radius AM, and join AP. Let the co-ordinates of P be x, y. Then we have



$$PM^2=AP^2-AM^2.$$

But
$$AP^2=x^2+y^2 \text{ (Art. 19)}.$$

Hence
$$PM=(x^2+y^2-r^2)^{\frac{1}{2}},$$

which denotes the length of the tangent line from the point x, y.

If $x^2+y^2>r^2$, or the point P be *without* the circle, the tangent PM will be real; if $x^2+y^2=r^2$, or the point P be *on* the circle, the length of the tangent becomes zero; if $x^2+y^2<r^2$, or

the point P be *within* the circle, the tangent is imaginary; but the quantity $r^2-x^2-y^2$ represents the product of the segments of the chord drawn through P.

Ex. 1. Find the length of the tangent drawn from the point $-7, +5$, to a circle whose radius is 4.

Ex. 2. Find the length of the tangent drawn from the point $-3, -6$, to a circle whose radius is 5.

76. *Definition.* The *normal* at any point of a curve is a straight line drawn through that point perpendicular to the tangent to the curve at that point.

77. *To find the equation to the normal at any point of a circle.*

Let the equation to the circle be $x^2+y^2=r^2$, and let x', y' be the co-ordinates of the point on the circle through which the normal is drawn.

We have found (Art. 73, Eq. 3) that the equation to the tangent at the point x', y' is

$$y-y'=-\frac{x'}{y'}(x-x'),$$

where $-\frac{x'}{y'}$ denotes the tangent of the angle which the tangent line makes with the axis of X. Hence (Art. 46) the equation to the normal will be

$$y-y'=\frac{y'}{x'}(x-x'),$$

which, after reduction, becomes

$$y=\frac{y'}{x'}x,$$

and this is the equation to the normal passing through the given point.

We have found (Art. 40) that $y=\frac{y'}{x'}x$ is the equation to a straight line passing through the origin and through a given point; hence the normal at any point of a circle passes through the centre.

78. *To determine the co-ordinates of the points of intersection of a straight line with a circle.*

Let the equation to the circle be

$$x^2+y^2=r^2, \qquad (1)$$

and the equation to the straight line be

$$y=mx+c. \qquad (2)$$

Since the co-ordinates of every point on a line must satisfy its equation, the co-ordinates of the points through which both of the given lines pass must satisfy both equations. We may therefore regard (1) and (2) as simultaneous equations containing but two unknown quantities, and we may hence determine the values of x and y. By substitution in equation (1) we obtain

$$x^2+m^2x^2+2cmx+c^2=r^2,$$

or

$$(1+m^2)x^2+2cmx=r^2-c^2,$$

an equation of the second degree which may be solved by completing the square. We thus find

$$x=\frac{-cm\pm\sqrt{r^2(1+m^2)-c^2}}{1+m^2};$$

and since x has two values, we conclude that there will be two points of intersection.

If $r^2(1+m^2)=c^2$, the two values of x become equal, and the straight line will touch the circle. If $r^2(1+m^2)$ is less than c^2, the straight line will not meet the circle.

Ex. 1. Find the co-ordinates of the points in which the circle whose equation is $x^2+y^2=25$ is intersected by the line whose equation is $x+y=1$.

Ans. $\begin{cases} x=4, \text{ and } y=-3, \\ \text{or } x=-3, \text{ and } y=4. \end{cases}$

Ex. 2. Find the co-ordinates of the points in which the circle whose equation is $x^2+y^2=25$ is intersected by the line whose equation is $x+y=5$.

Ans. $\begin{cases} x=5, \text{ and } y=0, \\ \text{or } x=0, \text{ and } y=5. \end{cases}$

Ex. 3. Find the co-ordinates of the points in which the circle whose equation is $x^2+y^2=65$ is intersected by the line whose equation is $3x+y=25$.

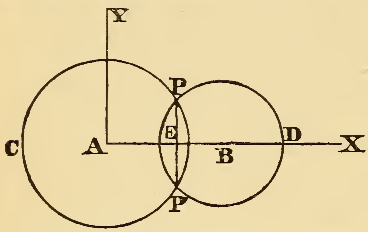
Ans. $\begin{cases} x=7, \text{ and } y=4, \\ \text{or } x=8, \text{ and } y=1. \end{cases}$

Ex. 4. Find the points in which the line $y=5x+2$ intersects the circle $y^2+x^2-4y-13x=9$.

Ans. $\begin{cases} x=1, \text{ and } y=7, \\ \text{or } x=-\frac{1}{2}, \text{ and } y=-\frac{1}{2}. \end{cases}$

Ex. 5. Find the points in which the line $y=3x+2$ cuts the circle $$y^2+x^2-4x+4y=7.$$

79. *To find the co-ordinates of the points of intersection of two circumferences.*



Let CPP′, DPP′ be two circumferences which intersect in P and P′. Let A and B be the centres of the circles, r and r' their radii, and let AB, the distance between their centres, be denoted by d. Assume the line AB as the axis of X, and let AY be drawn perpendicular to AX for the axis of Y.

The equation to the circle CPP′ is

$$x^2+y^2=r^2. \qquad (1)$$

The equation to DPP′, the co-ordinates of whose centre are $(d, 0)$ (Art. 66), is

$$(x-d)^2+y^2=r'^2. \qquad (2)$$

Since the co-ordinates of every point of a circumference must satisfy the equation of the circle, the co-ordinates of the points through which both circumferences pass must satisfy both equations. We may therefore regard (1) and (2) as simultaneous equations involving but two unknown quantities, and hence we may determine the values of x and y. Subtracting equation (2) from equation (1), we obtain

$$2xd-d^2=r^2-r'^2,$$

whence $$x=\frac{r^2-r'^2+d^2}{2d}.$$

Substituting this value of x in equation (1), we have

$$y^2=r^2-\left\{\frac{r^2-r'^2+d^2}{2d}\right\}^2,$$

whence $$y=\pm\frac{1}{2d}\sqrt{4d^2r^2-(r^2-r'^2+d^2)^2},$$
which gives the ordinates of the points of intersection of the two circles.

The double sign of y shows that the two points of intersection have the same abscissa AE, but two ordinates numerically the same and with contrary signs. Hence, when two circumferences cut each other, the line joining their centres is perpendicular to the common chord, and divides it into two equal parts.

Ex. 1. Find the co-ordinates of the points of intersection of the two circumferences
$$x^2+y^2=25, \text{ and } x^2+y^2+14x=-13.$$
Ans. $x=-2.714$; $y=\pm 4.199$.

Ex. 2. Find the co-ordinates of the points of intersection of the two circumferences $x^2+y^2=6$, and $x^2+y^2-8x=-8$.
Ans. $x=1.75$; $y=\pm 1.714$.

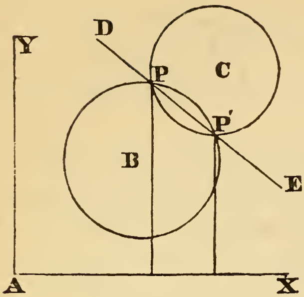
Ex. 3. Find the co-ordinates of the points of intersection of the two circumferences
$$x^2+y^2-2x-4y=1, \text{ and } x^2+y^2-4x-6y=-5.$$

80. *To find the equation to the straight line which passes through the points of intersection of two circles which cut each other.*

Let the equations of the two circumferences, whose centres are at B and C, be severally
$$x^2+y^2+ax+by+c=0, \quad (1)$$
and
$$x^2+y^2+a'x+b'y+c'=0; \quad (2)$$
it is required to find the equation of the straight line passing through the points P and P′ where these circumferences intersect.

Since the co-ordinates of the points P and P′ satisfy each of the above equations, we may treat them as simultaneous equations containing two unknown quantities.

Subtracting equation (2) from equation (1), we have
$$(a-a')x+(b-b')y+c-c'=0. \quad (3)$$



Since this is an equation of the first degree between x and y, it is the equation of a straight line (Art. 37); and since it must be satisfied by the co-ordinates of the two points P and P′, it must be the equation of the straight line DE passing through those points, and is, therefore, the equation required.

If we combine equation (3) with the equation of either circle, we shall obtain the values of the co-ordinates of the points of intersection as in Art. 79.

In general, if we have any two equations of curves, and we add or subtract those equations as in the process of elimination in Algebra, we obtain a new equation, which is the equation of a new line or curve which passes through the points of intersection of the first two curves.

81. *To find the equation to a circle which passes through three given points.*

We have found (Art 68) that the general equation to the circle is

$$x^2+y^2+Ax+By+C=0,$$

where A, B, and C are constant for a given circle, but vary for different circles; so that when A, B, and C are known, the circle is fully determined.

If the three points $x'y'$, $x''y''$, $x'''y'''$ are on the circumference of a circle, the co-ordinates of each of these points must satisfy the equation of that circle. If then we substitute the values of x', y' in the general equation, we shall obtain an equation which expresses the relation between the coefficients A, B, and C. So also, if we substitute successively the values of $x''y''$ and $x'''y'''$, we shall obtain two other equations expressing the relations between the same coefficients. We shall then have three simultaneous equations expressing the relations between the three quantities A, B, and C, from which the values of these quantities can be determined.

Ex. 1. Find the equation to the circle which passes through the three points 1, 2; 1, 3; and 2, 5; also the co-ordinates of the centre and the radius of the circle.

Substituting these values successively in the general equation of the circle, we have

$$A+2B+C+5=0,$$
$$A+3B+C+10=0,$$
$$2A+5B+C+29=0,$$

from which we find $A=-9$; $B=-5$; $C=14$.

Hence the equation to the circle is $x^2+y^2-9x-5y+14=0$.

Hence the co-ordinates of the centre are $\frac{9}{2}, \frac{5}{2}$; and the radius is $\frac{5}{2}\sqrt{2}$.

Ex. 2. Find the equation to the circle which passes through the three points $2, -3$; $3, -4$; and $-2, -1$; also the co-ordinates of the centre and the radius of the circle.

Ans. Eq., $x^2+y^2+8x+20y+31=0$; co-ordinates, $-4, -10$; radius $=\sqrt{85}$.

Ex. 3. Find the equation to the circle which passes through the origin and through the points 2, 3 and 3, 4; also the co-ordinates of the centre and radius of circle.

Ans. Eq., $x^2+y^2-23x+11y=0$; co-ordinates, $\frac{23}{2}, -\frac{11}{2}$; radius $=\frac{5}{2}\sqrt{26}$.

Ex. 4. Find the equation of the circle which passes through the three points $-4, -4$; $-4, -2$; $-2, +2$; also the co-ordinates of the centre and radius of circle.

Ans. Eq., $x^2+y^2-6x+6y-32=0$; co-ordinates, $3, -3$; radius, $5\sqrt{2}$.

Ex. 5. Find the equation of the circle which passes through the points $-2, -4$; 2, 2; 4, 4; also the co-ordinates of the centre and radius of circle.

Ans. Eq., $x^2+y^2-42x+30y+16=0$; co-ordinates, $21, -15$; radius $=5\sqrt{26}$.

Ex. 6. Find the equation of the circle which passes through the origin and cuts off lengths 6, 8 from the axes; also the co-ordinates of the centre and radius of circle.

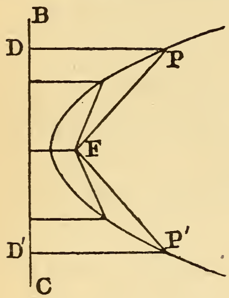
Ans. Eq., $x^2+y^2-6x-8y=0$; co-ordinates, 3, 4; radius, 5.

SECTION V.

THE PARABOLA.

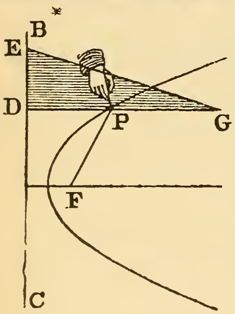
82. A *parabola* is a plane curve every point of which is equally distant from a fixed point and a fixed straight line.

The fixed point is called the *focus* of the parabola, and the fixed straight line is called the *directrix*.



Thus, if a straight line BC, and a point F without it be fixed in position, and the point P be supposed to move in such a manner that PF, its distance from the fixed point, is always equal to PD, its perpendicular distance from the fixed line, the point P will describe a parabola of which F is the focus and BC the directrix.

83. From the definition of a parabola the curve may be described mechanically by means of a ruler, a square, and a cord.



Let BC be a ruler whose edge coincides with the directrix of the parabola, and let DEG be a square. Take a cord whose length is equal to DG, and attach one extremity of it at G and the other at the focus F. Then slide the side of the square DE along the ruler BC, and at the same time keep the cord continually stretched by means of the point of a pencil, P, in contact with the square; the pencil will trace out a portion of a parabola. For, in every position of the square,

$$PF+PG=PD+PG,$$

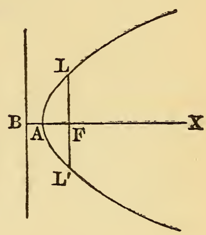
and hence

$$PF=PD;$$

that is, the point P is always equally distant from the focus F and the directrix BC.

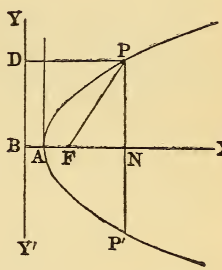
If the square be turned over, and moved on the other side of the point F, the other part of the same parabola may be described.

84. A straight line drawn through the focus perpendicular to the directrix is called the *axis of the parabola.* The *vertex* of the axis is the point in which it intersects the curve. The chord drawn through the focus of a parabola at right angles to the axis is called the *latus rectum.*



Thus, in the figure, BX is the axis of the parabola, A is the vertex of the axis, and LL′ is the latus rectum.

85. *To find the equation to the parabola referred to rectangular axes.*



Take the directrix YY′ as the axis of ordinates, and BX, drawn perpendicular to it through the focus, as the axis of abscissas. Let $BF=2a$. By the definition,

$$FP=PD=BN.$$

Therefore $FP^2=BN^2$,

or $FN^2+PN^2=BN^2$;

that is, $(x-2a)^2+y^2=x^2$,

or
$$y^2=4a(x-a), \qquad (1)$$

which is the equation to the parabola.

If in this equation we put $y=0$, we have $x=a$, which shows that the curve cuts the axis at a point A which bisects BF.

The equation will be simplified if we put the origin at A. Let $x'=AN$; then $x=x'+a$; and, since the axis of abscissas remains unchanged, $y=y'$.

By substitution, equation (1) becomes

$$y'^2=4ax'.$$

We may suppress the accents if we remember that the origin is now at A; thus we have

$$y^2=4ax, \qquad (2)$$

which is the equation to the parabola referred to its vertex as origin, and the axis of the parabola is the axis of X.

86. *To trace the form of the parabola from its equation.*

Since $y^2=4ax$, or $x=\frac{y^2}{4a}$, x can not be negative; that is, the curve lies wholly on the positive side of the axis of y.

Since $$y^2=4ax,\ y=\pm 2(ax)^{\frac{1}{2}};$$
therefore, since this equation is unaltered if we write $-y$ for y, to every point P on the curve on one side of the axis of X, there corresponds another point P′ on the other side, such that P′N=PN. Hence the curve is symmetrical with respect to the axis of X.

Again, if $x=0$, $y=0$, and has no other value; therefore the curve does not meet either axis at any other point besides the origin.

Also, the greater the value we give to x, the greater values we get for y; and when x is infinite, y is infinite; hence the curve goes off to an infinite distance on each side of the axis of X.

87. *To find the distance of any point on the curve from the focus.*

The distance of any point on the curve from the focus is equal to the distance of the same point from the directrix. Hence $$FP=PD=BA+AN,$$
or $$FP=a+x.$$

88. *To find the length of the latus rectum.*

In the equation
$$y^2=4ax,$$
put $$x=a;$$
then $$y^2=4a^2,$$
and $$y=\pm 2a,$$
or the latus rectum LL′$=4a$ (see figure in Art. 84).

If we convert the equation $y^2=4ax$ into a proportion, we shall have $$x:y::y:4a;$$

that is, *the latus rectum is a third proportional to any abscissa and its corresponding ordinate.*

89. *The squares of ordinates to the axis are to each other as their corresponding abscissas.*

Designate any two ordinates by y', y'', and the corresponding abscissas by x', x''; then we shall have

$$y'^2 = 4ax',$$
$$y''^2 = 4ax''.$$

Hence $$y'^2 : y''^2 :: 4ax' : 4ax'' :: x' : x''.$$

Ex. 1. The equation of a parabola is $y^2 = 4x$. What is the abscissa corresponding to the ordinate 7? *Ans.* $12\frac{1}{4}$.

Ex. 2. The equation of a parabola is $y^2 = 18x$. What is the ordinate corresponding to the abscissa 7?

Ans. $\pm\sqrt{126}$.

Ex. 3. The equation of a parabola is $y^2 = 10x$. What is the ordinate corresponding to the abscissa 3?

90. *To trace the form of the parabola by means of points.*

If we reduce the equation of the parabola to the form

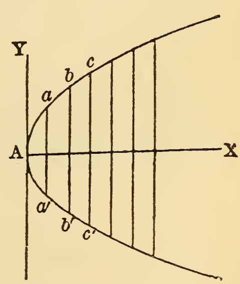
$$y = \pm 2\sqrt{ax},$$

we may compute the values of y corresponding to any assumed value of x.

Ex. 1. Trace the curve whose equation is $y^2 = 4x$.

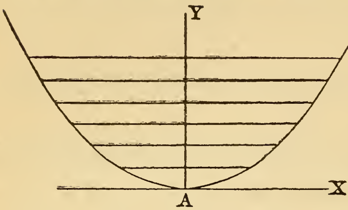
By assuming for x different values from 0 to 5, etc., we obtain the corresponding values of y as given below.

When $x=0$, $y=0$.
" $x=1$, $y=\pm 2$.
" $x=2$, $y=\pm 2.828$.
" $x=3$, $y=\pm 3.464$.
" $x=4$, $y=\pm 4$.
" $x=5$, $y=\pm 4.472$.



The first point (0, 0) is the origin; the point (1, +2) is represented by a in the figure; the point (1, −2) by a' in the figure; the point (2, +2.828) by b; the point (2, −2.828) by b', etc.

Ex. 2. Trace the curve whose equation is $y^2=18x$.



Ex. 3. Trace the curve whose equation is $x^2=9y$.

The curve will be of the form exhibited in the annexed figure, and is evidently a parabola whose axis is the axis of Y.

Ex. 4. Trace the curve whose equation is $y^2=-3x$.

91. *To find the equation to the tangent at any point of a parabola.*

Let the equation to the parabola be $y^2=4ax$.

Let x', y' be the co-ordinates of the point on the curve at which the tangent is drawn, and x'', y'' the co-ordinates of an adjacent point on the curve. The equation to the secant line passing through the points x', y' and x'', y'' (Art. 40) is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1)$$

Now, since the points x', y' and x'', y'' are both on the parabola, we must have $y'^2=4ax'$,

and $y''^2=4ax''$.

Hence $y''^2-y'^2=4a(x''-x')$,

or $$\frac{y''-y'}{x''-x'}=\frac{4a}{y''+y'}.$$

Substituting this value in equation (1), the equation of the secant line becomes

$$y-y'=\frac{4a}{y''+y'}(x-x'). \qquad (2)$$

The secant will become a tangent when the two points coincide, in which case $y'=y''$.

Equation (2) will then become

$$y-y'=\frac{2a}{y'}(x-x'), \qquad (3)$$

which is the equation to a tangent at the point x', y'.

Clearing of fractions and transposing, we obtain

$$yy'=2a(x-x')+y'^2,$$

$$yy' = 2ax - 2ax' + 4ax',$$

or
$$yy' = 2a(x + x'),$$

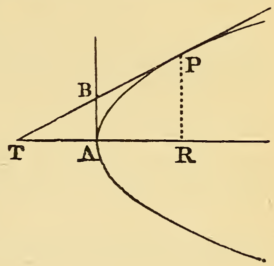
which is the simplest form of the equation to the tangent line.

92. *Points where the tangent cuts the axes.* To determine the point in which the tangent intersects the axis of X, we make $y = 0$, which gives

$$0 = 2a(x + x');$$

that is, $x = -x'$,

or $AT = -AR$.



To determine the point in which the tangent intersects the axis of Y, we make $x = 0$, which gives

$$y = \frac{2ax'}{y'} = \frac{y'^2}{2y'} = \frac{y'}{2};$$

that is, $AB = \frac{1}{2} PR$.

93. *Definition.* A *subtangent* to a parabola is that part of the axis intercepted between a tangent and ordinate drawn to the point of contact. Thus TR is the subtangent corresponding to the tangent PT.

From Art. 92 we see that *the subtangent to the axis is bisected by the curve.*

94. The preceding property enables us to draw a tangent to the curve through a given point. Let P be the given point; from P draw PR perpendicular to the axis, and make $AT = AR$. Draw a line through P and T, and it will be a tangent to the parabola at P.

95. *To find the equation to a tangent to the parabola in terms of the tangent of the angle it makes with the axis.*

In the equation of a tangent line,

$$y - y' = \frac{2a}{y'}(x - x') \text{ (Art. 91, Eq. 3)},$$

$\frac{2a}{y'}$ represents the trigonometrical tangent of the angle which the tangent line makes with the axis of the parabola (Art. 38). If we represent this tangent by m, we shall have

$$\frac{2a}{y'}=m, \text{ and } \frac{y'}{2}=\frac{a}{m}. \qquad (1)$$

The equation to a tangent line to the parabola (Art. 91) is

$$yy'=2a(x+x'),$$

whence

$$y=\frac{2a}{y'}x+\frac{2ax'}{y'},$$

$$=\frac{2a}{y'}x+\frac{4ax'}{2y'};$$

or

$$y=\frac{2a}{y'}x+\frac{y'}{2}.$$

Hence, substituting equation (1), we have

$$y=mx+\frac{a}{m},$$

which is the equation to a tangent line.

Hence the straight line whose equation is

$$y=mx+\frac{a}{m},$$

touches the parabola whose equation is $y^2=4ax$.

Ex. 1. Find the equation of a tangent to the parabola $y^2=18x$ at the point $x'=2, y'=6$.

Ex. 2. Find the equation of a tangent to the parabola $y^2=4x$, and parallel to the right line whose equation is $y=5x+1$.

Ex. 3. On a parabola whose equation is $y^2=10x$, a tangent line is drawn through the point whose ordinate is 8. Determine where the tangent line meets the two axes of reference.

Ex. 4. On a parabola whose latus rectum is 10 inches, a tangent line is drawn through the point whose ordinate is 6 inches, the origin being at the vertex of the axis. Determine where the tangent line meets the two axes of reference.

Ex. 5. Find the angle which the tangent line in the last example makes with the axis of X.

Ex. 6. On a parabola whose latus rectum is 10 inches, find the point from which a tangent line must be drawn in order that it may make an angle of 35° with the axis of the parabola.

96. *Definitions.* The term *normal* is often used to denote that part of the normal line (Art. 76) which is included between the curve and the axis of abscissas.

A *subnormal* is the portion of the axis intercepted between the normal and the ordinate drawn from the same point of the curve.

97. *To find the equation to the normal at any point of a parabola.*

Let x', y' be the co-ordinates of the given point.

The equation to a straight line passing through this point (Art. 38) is $$y-y'=m(x-x');$$
and, since this line must be perpendicular to the tangent whose equation is

$$y-y'=\frac{2a}{y'}(x-x') \text{ (Art. 91, Eq. 3)},$$

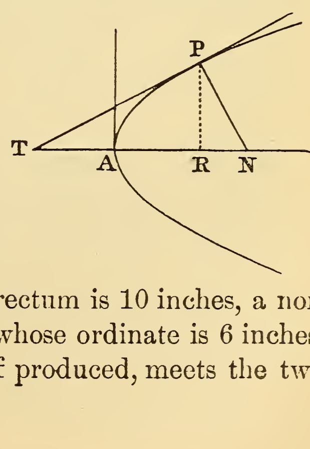
we have $$m=-\frac{y'}{2a} \text{ (Art. 45)}.$$

Hence the equation to the normal is

$$y-y'=-\frac{y'}{2a}(x-x').$$

98. *Point where the normal cuts the axis of x.* To find the point in which the normal intersects the axis of abscissas, make $y=0$ in the equation to the normal, and we have, after reduction,

$$x-x'=2a.$$

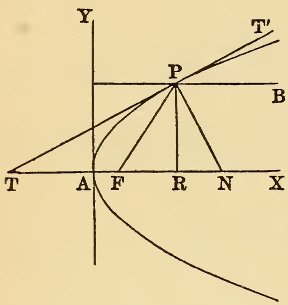


But x is equal to the distance AN, and x' to AR; hence $x-x'$ is equal to RN, which is equal to $2a$; that is, *the subnormal is constant, and is equal to half the latus rectum.*

Ex. 1. On a parabola whose latus rectum is 10 inches, a normal line is drawn through the point whose ordinate is 6 inches. Determine where the normal line, if produced, meets the two axes of reference.

Ex. 2. Find the point on the curve of a parabola whose latus rectum is 10 inches, from which, if a tangent be drawn, and also an ordinate to the axis of X, they will form with the axis a triangle whose area is 36 inches.

99. *If a tangent to the parabola cuts the axis produced, the points of contact and intersection are equally distant from the focus.*



Let PT be a tangent to the parabola at P, and let PF be the radius vector drawn to the point of contact.

We have found (Art. 92)

$$TA = AR.$$

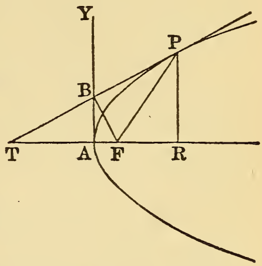
Hence $TF = AR + AF = FP$ (Art. 87); that is, the distance from the focus to the point where the tangent cuts the axis, is equal to the distance from the focus to the point where the tangent touches the curve.

100. *A tangent to the curve makes equal angles with the radius vector and with a line drawn through the point of contact parallel to the axis.*

Let TT′ touch the parabola at P, and let BP be drawn through P parallel to AX; then the angle BPT′ is equal to the angle ATP. But since $TF = PF$, the angle FTP is equal to the angle FPT. Hence FPT is equal to BPT′, or the two lines FP and BP are equally inclined to the tangent.

101. If a ray of light, proceeding in the direction BP, be incident on the parabola at P, it will be reflected to F on account of the equal angles BPT′ and FPT. In like manner, all rays coming in a direction parallel to the axis, and incident on the curve, will converge to F. Also, if a portion of the curve revolves round its axis so as to form a hollow concave mirror, all rays from a distant luminous point in the direction of the axis will be concentrated in F. Thus, if a parabolic mirror be held with its axis pointing to the sun, an intense heat and a brilliant light will be found at the focus.

102. *If from the focus of a parabola a straight line be drawn perpendicular to any tangent, it will intersect this tangent on the tangent at the vertex.*



Let the tangent PT be drawn, and from the focus F let FB be drawn perpendicular to it; the point B will fall on the axis AY, which touches the curve at A (Art. 86).

Since the triangle PFT is isosceles, the line FB, drawn perpendicular to the base PT, will pass through its middle point; and since AT = AR (Art. 92), the line AY, which is parallel to PR, also passes through the middle point of PT; that is, the line FB intersects PT in the same point with AY.

Since the triangle FBT is right angled at B, we have

$$FB^2 = FA \times FT = FA \times FP,$$

or *the perpendicular from the focus to any tangent is a mean proportional between the distances of the focus from the vertex and the point of contact.*

103. *To determine the co-ordinates of the points of intersection of a straight line with a parabola.*

Let the equation to the parabola be

$$y^2 = 4ax, \qquad (1)$$

and the equation to the straight line be

$$y = mx + c. \qquad (2)$$

As in Art. 78, we may regard (1) and (2) as simultaneous equations, containing but two unknown quantities. By substitution in equation (1), we obtain

$$my^2 = 4ay - 4ac.$$

Completing the square, we obtain

$$y = \frac{2a}{m} \pm \frac{2}{m}(a^2 - amc)^{\frac{1}{2}};$$

and, since y has two values, we conclude that there will be two points of intersection.

If $a=mc$, the two values of y become equal, and the straight line will touch the parabola. If $a-mc$ is negative, the straight line will not meet the parabola.

Ex. 1. Find the co-ordinates of the points in which the parabola whose equation is $y^2=4x$ is intersected by the line whose equation is $y=2x-5$.

Ans. $y=4.3166$, or -2.3166; $x=4.6583$, or 1.3417.

Ex. 2. Find the co-ordinates of the points in which the parabola whose equation is $y^2=18x$ is intersected by the line whose equation is $y=2x-5$.

Ans. $y=12.5777$, or -3.5777; $x=8.7888$, or 0.7111.

Ex. 3. Find whether the parabola whose equation is $y^2=16x$ is intersected by the line whose equation is $y=2x+2$, and, if there is a point of contact, determine its co-ordinates.

Ex. 4. Find whether the parabola whose equation is $y^2=16x$ is intersected by the line whose equation is $y=2x+5$.

104. *To determine the co-ordinates of the points of intersection of a circle and parabola.*

If the centre of the circle is not restricted in position, there may be four points of intersection, corresponding to an equation of the fourth degree, which can not generally be resolved by quadratics. If, however, the centre of the circle is upon the axis of the parabola, the several points of intersection may be easily found.

Let the equation to the parabola be

$$y^2=4ax,$$

and the equation to the circle be

$$x^2+y^2=r^2;$$

then, by substitution, we have

$$x^2+4ax=r^2,$$

and
$$x=-2a\pm(4a^2+r^2)^{\frac{1}{2}},$$

where x has two values, but one of them is negative, and gives imaginary values for y. There will, therefore, be but two real points of intersection. These have the same abscissa, and their ordinates will differ only in sign.

Ex. 1. Find the co-ordinates of the points in which the parabola whose equation is $y^2=4x$ is intersected by the circle whose equation is $x^2+y^2=64$. *Ans.* $x=6.2462$; $y=\pm 4.9985$.

Ex. 2. Find the co-ordinates of the points in which the parabola whose equation is $y^2=18x$ is intersected by the circle whose equation is $x^2+y^2=32x-40$.

$$Ans. \begin{cases} x=4, \text{ or } 10; \\ y=\pm 6\sqrt{2}, \text{ or } \pm 6\sqrt{5}. \end{cases}$$

Construct the two curves, and show the points of intersection.

Ex. 3. Find the co-ordinates of the points in which the parabola whose equation is $y^2=2x$ is intersected by the circle whose equation is $x^2+y^2=6x+5$.

105. *To transform the equation to the parabola into another referred to oblique axes, and so that the equation shall preserve the same form.*

The formulas for passing from rectangular to oblique axes (Art. 56) are
$$x=m+x' \cos. a+y' \cos. \beta,$$
$$y=n+x' \sin. a+y' \sin. \beta.$$

Substituting these values in the equation $y^2=4ax$, and arranging the terms, we have

$$y'^2 \sin.^2\beta+x'^2 \sin.^2 a+2x'y' \sin. a \sin. \beta+$$
$$2(n \sin. \beta-2a \cos. \beta)y'+n^2-4am=2(2a \cos. a-n \sin. a)x',$$

which is the equation to the parabola referred to any oblique axes.

In order that this equation may be of the form $y^2=4ax$, we must have the following conditions:

1st. There must be no absolute term; hence $n^2-4am=0$.

2d. There must be no term containing x'^2; hence $\sin.^2 a=0$.

3d. There must be no term containing $x'y'$; hence $\sin. a \sin. \beta=0$.

4th. There must be no term containing y'; hence $n \sin. \beta-2a \cos. \beta=0$.

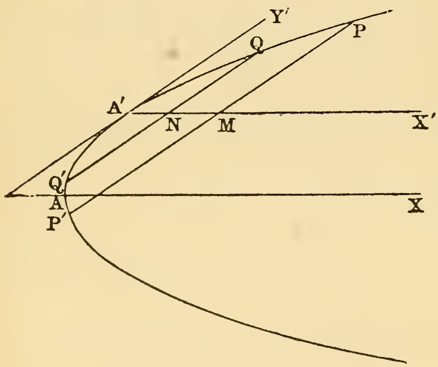
These equations contain four arbitrary constants, m, n, α, β; it is therefore possible to assign such values to the constants as to satisfy the four equations, and thus reduce the new equation of the parabola to the proposed form.

Since the equation $y^2 = 4ax$ becomes $n^2 = 4am$ by substituting the co-ordinates of the new origin for x and y, it follows that the first condition, $n^2 - 4am = 0$, requires *the new origin to be on the curve.*

The second condition, $\sin.^2\alpha = 0$, requires the *new axis of* x *to be parallel to the axis of the parabola.*

The third condition, $\sin.\ \alpha \sin.\ \beta = 0$, is satisfied by the second, without introducing any new condition.

Since $\frac{2a}{y'}$ or $\frac{2a}{n}$ (Art. 95) has been found to represent the tangent of the angle which the tangent line makes with the axis of the parabola, the fourth condition, $n \sin.\ \beta - 2a \cos.\ \beta = 0$, or $\frac{\sin.\ \beta}{\cos.\ \beta} = \text{tang.}\ \beta = \frac{2a}{n}$, requires that *the new axis of* y *shall be tangent to the curve at the origin.*



If, therefore, the curve is referred to any tangent line A′Y′, and a line A′X′ drawn through the point of contact parallel to the axis, the equation becomes

$$y'^2 \sin.^2\beta = 2(2a \cos.\ \alpha - n \sin.\ \alpha)x';$$

or, since $\sin.\ \alpha = 0$,

and $\cos.\ \alpha = 1$,

we have
$$y'^2 = \frac{4a}{\sin.^2\beta}x'.$$

If we represent $\frac{4a}{\sin.^2\beta}$ by $4a'$, and omit the accents of the variables, we shall have

$$y^2 = 4a'x,$$

which is the equation required.

106. Since the preceding problem furnished four arbitrary constants, m, n, a, β, and required but three independent conditions (the second and third being but one), we may assign any value at pleasure to either of them except a; that is, *the new origin may be placed any where on the curve.*

107. From the equation of Art. 105, we find

$$y = \pm\sqrt{4a'x},$$

which shows that for every positive value of x there are two values of y equal numerically but having opposite signs, and these two values, taken together, form a chord PP′ parallel to the axis of Y, which chord is bisected by the axis of X at M. So, also, the parallel chord QQ′ is bisected by the axis of X at N. Hence *a straight line parallel to the axis of the parabola bisects all chords parallel to the tangent at its extremity.*

108. *Definition.* A *diameter* of a parabola is a straight line drawn through any point of the curve parallel to the axis of the parabola. The *vertex* of the diameter is the point in which it meets the curve.

109. The equation of Art. 105, $y^2 = 4a'x$, is called the equation of the parabola referred to a tangent line, and the diameter drawn through the point of contact; and, since the new axis of Y is a tangent to the curve at the origin, *a diameter bisects all chords parallel to the tangent at its extremity.*

The equation $$y^2 = 4a'x$$
shows that for oblique axes *the squares of the ordinates are proportional to the corresponding abscissas*, which is a generalization of the property proved in Art. 89.

110. *To determine the value of the coefficient of* x *in the equation* $y^2 = 4a'x$.

From Art. 105 we have

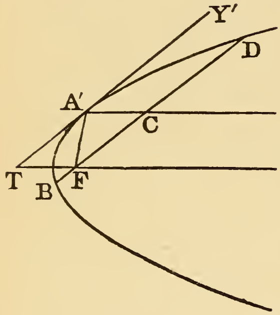
$$\frac{\sin.\beta}{\cos.\beta} = \frac{2a}{n},$$

whence $$n \sin.\beta = 2a \cos.\beta,$$
and $$n^2 \sin.^2\beta = 4a^2 \cos.^2\beta,$$
$$= 4a^2(1 - \sin.^2\beta),$$
$$= 4a^2 - 4a^2 \sin.^2\beta.$$

Therefore $$\sin.^2\beta = \frac{4a^2}{n^2 + 4a^2}.$$

But from the equation to the curve referred to the original axes we have $$n^2 = 4am;$$

therefore $$\sin.^2\beta = \frac{4a^2}{4am + 4a^2} = \frac{a}{m+a},$$



and $$\frac{a}{\sin.^2\beta} = a + m = a'.$$

But m represents the abscissa of the new origin referred to the original axes; hence
$$a + m = \text{FA}' \text{ (Art. 87)} = a',$$
or the coefficient of x in the equation $y^2 = 4a'x$ is *four times the distance from the focus to the new origin.*

111. *To determine the length of the chord drawn through the focus parallel to the new axis of ordinates.*

If through the focus F the line BD be drawn parallel to the new axis of Y, then, calling x and y the co-ordinates of the point D, we have
$$x = \text{A}'\text{C} = \text{TF} = \text{A}'\text{F (Art. 99)} = a' \text{ (Art. 110)}.$$
But, by Art. 105, $$y^2 = 4a'x;$$
hence $$y^2 = 4a' \times a' = 4a'^2,$$
or $$y = 2a',$$
and $$2y = 4a';$$
that is, *the coefficient* $4a'$ *is the double ordinate passing through the focus and corresponding to the diameter which passes through the origin.*

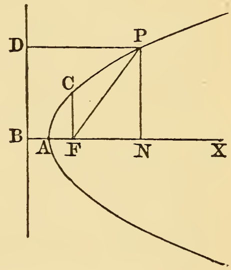
112. *Definition.* The *parameter* of any diameter is the double ordinate which passes through the focus.

From Art. 110 we see that *the parameter of any diameter is equal to four times the distance from the vertex of that diameter to the focus.*

In the equation $y^2=4a'x$, $4a'$ is the parameter of the diameter passing through the origin.

The parameter to the axis is called the principal parameter, or latus rectum (Art. 84).

113. *To find the polar equation to the parabola, the focus being the pole.*



Let the axis be the initial line. Represent FP by ρ, and PFN by θ.

Then $\quad PF=PD=BF+FN,$

or $\quad \rho=2a+\rho\cos.\theta;$

whence $\quad \rho(1-\cos.\theta)=2a,$

or $$\rho=\frac{2a}{1-\cos.\theta},$$

which is the polar equation to the parabola.

114. If $\theta=180°$, then $\cos.\theta=-1$, and the value of ρ becomes

$$\rho=\frac{2a}{1+1}=a=FA.$$

If $\theta=90°$, then $\cos.\theta=0$, and the value of ρ becomes

$$\rho=2a=FC.$$

If $\theta=0$, then $\cos.\theta=1$, and we have

$$\rho=\frac{2a}{0}=\infty;$$

the radius vector takes the direction AX, and does not meet the curve at a finite distance.

115. If the variable angle be measured from the point A toward the *right*, then we must substitute for θ, $180°-\theta$, in which case $\cos.\theta=-\cos.\theta'$, and we have

$$\rho=\frac{2a}{1+\cos.\theta'}.$$

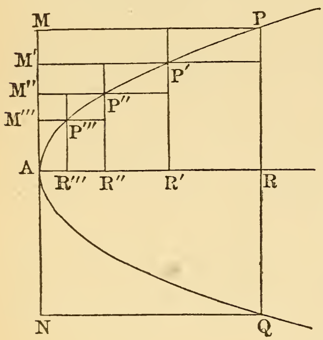
Ex. 1. What is the polar equation of a parabola whose latus rectum is 10, the pole being at the focus; and what is the length of the radius vector for $\theta=60°$?

Ex. 2. The latus rectum of a parabola is 8 inches, and $\theta=135°$; what is the radius vector?

Ex. 3. The latus rectum of a parabola is 6 inches, and the radius vector is 10 inches; determine the value of θ.

Ex. 4. The radius vector of a parabola is 25 inches and $\theta=135°$; what is the latus rectum?

116. *To determine the area of a segment included between an arc of a parabola and a chord perpendicular to the axis.*



Let PAQ be a segment of a parabola, bounded by the curve PAQ, and the chord PQ perpendicular to the axis AR. It is required to determine its area.

Inscribe in the semi-parabola PAR a polygon PP′P″.... AR, and through the points P, P′, P″, etc., draw parallels to AR and PR, forming the interior rectangles P′R, P″R′, etc., and the corresponding exterior rectangles P′M, P″M′, etc.

Designate the former by P, P′, P″, etc.; the latter by p, p', p'', etc., and the corresponding co-ordinates by x, y, x', y', etc. We shall then have

the rectangle $\quad P'R=P'R'\times RR'$,

or $\quad P=y'(x-x')$.

Also the rectangle $\quad P'M=P'M'\times MM'$,

or $\quad p=x'(y-y')$.

Whence
$$\frac{P}{p}=\frac{y'(x-x')}{x'(y-y')}. \qquad (1)$$

But, since the points P, P′, etc., are on the curve, we have
$$y^2=4ax,\ y'^2=4ax';$$

whence
$$x-x'=\frac{y^2-y'^2}{4a},\ \text{and}\ x'=\frac{y'^2}{4a}.$$

Substituting these values in equation (1), we obtain

$$\frac{\mathrm{P}}{p}=\frac{y'(y^2-y'^2)}{y'^2(y-y')}=\frac{y+y'}{y'}=1+\frac{y}{y'}.$$

In the same manner we find

$$\frac{\mathrm{P}'}{p'}=1+\frac{y'}{y''},$$

$$\frac{\mathrm{P}''}{p''}=1+\frac{y''}{y'''}, \text{ etc.}$$

If now we suppose the vertices of the polygons P, P′, P″, etc., to be so placed that the ordinates shall be in geometrical progression, we shall have

$$\frac{y}{y'}=\frac{y'}{y''}=\frac{y''}{y'''}, \text{ etc.},$$

so that each interior rectangle has to its corresponding exterior rectangle the ratio of $1+\frac{y}{y'}$ to 1.

Therefore, by composition,

$$\frac{\mathrm{P}+\mathrm{P}'+\mathrm{P}''+, \text{ etc.},}{p+p'+p''+, \text{ etc.},}=1+\frac{y}{y'};$$

that is, the sum of all the interior rectangles is to the sum of all the exterior rectangles as $1+\frac{y}{y'}$ to 1.

When the points P, P′, P″, etc., are taken indefinitely near, the ratio $\frac{y}{y'}$ approaches indefinitely near to a ratio of equality; the sum of the interior rectangles converges to the area of the interior parabolic segment APR, and the sum of the exterior rectangles to the area of the exterior parabolic segment AMP. Designating the former by S, and the latter by s, we have

$$\frac{\mathrm{S}}{s}=1+1=2,$$

or

$$\mathrm{S}=2s=\tfrac{2}{3}(\mathrm{S}+s).$$

But $\mathrm{S}+s$ is equal to the area of the rectangle AMPR; hence the parabolic segment APR is two thirds of the rectangle AMPR, or the segment PAQ is two thirds of the rectangle PMNQ. Hence *the area of a parabolic segment cut off by a* *double ordinate to the axis is two thirds of the circumscribing rectangle.*

Ex. 1. Determine the area of the parabolic segment cut off by a double ordinate whose length is 24 inches, the latus rectum being 8 inches.

Ex. 2. The area of a parabolic segment cut off by a double ordinate to the axis is 96, and the corresponding abscissa is 6. Determine the equation to the curve.

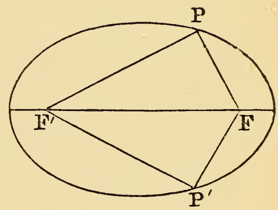
117. By a demonstration like that of the preceding article, it may be also shown that the area of a parabolic segment cut off by the double ordinate of *any diameter* is two thirds of the circumscribing parallelogram.

Example. Prove that if two tangents are drawn at the extremities of any chord of a parabola, the segment cut off from the parabola is two thirds of the triangle formed by the chord and the two tangents.

SECTION VI.

THE ELLIPSE.

118. An *ellipse* is a plane curve traced out by a point which moves in such a manner that the *sum* of its distances from two fixed points is always the same. The two fixed points are called the *foci* of the ellipse.



Thus, if F and F′ are two fixed points, and if the point P moves about F in such a manner that the sum of its distances from F and F′ is always the same, the point P will describe an ellipse, of which F and F′ are the foci. The distance of the point P from either focus is called the *focal distance*, or the *radius vector*.

119. *Description of the curve.* From the definition of an ellipse the curve may be described mechanically. Thus, take a thread whose length is greater than the distance FF′, and fasten one of its extremities at F, the other at F′. Place the point of a pencil, P, against the thread, and slide it so as to keep the thread constantly stretched; the point of the pencil will describe an ellipse. For in every position of P we shall have FP+F′P equal to the fixed length of the thread; that is, equal to a constant quantity.

120. *Definitions.* The *centre* of the ellipse is the middle point of the straight line joining the foci.

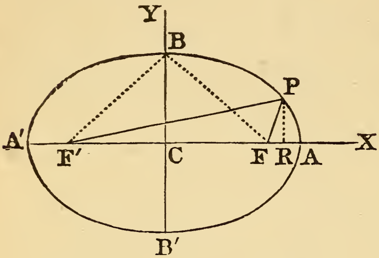
A *diameter* is any straight line passing through the centre, and terminated on both sides by the curve.

The diameter which passes through the foci is called the *transverse axis*, or the *major axis*.

The diameter which is perpendicular to the major axis is called the *conjugate axis*, or the *minor axis*.

The *latus rectum* is the chord drawn through one of the foci perpendicular to the major axis.

121. *To find the equation to the ellipse referred to its axes.*



Let F and F′ be the foci, and draw the rectangular axes CX, CY, the origin, C, being placed at the middle of FF′. Let P be any point of the curve, and draw PR perpendicular to CX.

Let $2c$ denote FF′, the constant distance between the foci, and $2a$ denote FP+F′P, the constant sum of the focal distances. Denote FP by r, F′P by r', and let x and y denote the co-ordinates of the point P.

Then, since

$$FP^2 = PR^2 + RF^2 = PR^2 + (CR - CF)^2,$$

we have $$r^2 = y^2 + (x-c)^2. \quad (1)$$

Also, $$PF'^2 = PR^2 + RF'^2 = PR^2 + (CR + CF)^2.$$

That is, $$r'^2 = y^2 + (x+c)^2. \quad (2)$$

Adding equations (1) and (2), we obtain

$$r^2 + r'^2 = 2(y^2 + x^2 + c^2); \quad (3)$$

and subtracting equation (1) from (2), we obtain

$$r'^2 - r^2 = 4cx,$$

which may be put under the form

$$(r'+r)(r'-r) = 4cx. \quad (4)$$

But from the definition of the ellipse we have

$$r' + r = 2a. \quad (5)$$

Dividing equation (4) by equation (5), we obtain

$$r' - r = \frac{2cx}{a}.$$

Combining the last two equations, we find

$$r' = a + \frac{cx}{a}, \quad (6)$$

$$r = a - \frac{cx}{a}. \quad (7)$$

Squaring these values, and substituting them in equation (3), we obtain

$$a^2+\frac{c^2x^2}{a^2}=c^2+x^2+y^2,$$

which may be reduced to the form

$$a^2y^2+(a^2-c^2)x^2=a^2(a^2-c^2), \qquad (8)$$

which is the equation to the ellipse.

This equation may, however, be put under a more convenient form.

Represent the line BC by b. In the two right-angled triangles BCF, BCF′, CF is equal to CF′, and BC is common to both triangles; hence BF is equal to BF′. But, by the definition of the ellipse, BF+BF′$=2a$; consequently BF$=a$.

Now $$BC^2=BF^2-CF^2;$$

that is, $$b^2=a^2-c^2. \qquad (9)$$

Substituting this value in equation (8), we obtain

$$a^2y^2+b^2x^2=a^2b^2, \qquad (10)$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad (11)$$

which is the equation of the ellipse referred to its axes.

This equation is sometimes written

$$y^2=\frac{b^2}{a^2}(a^2-x^2). \qquad (12)$$

122. *Points of intersection with the axes.* To determine where the curve cuts the axis of X, make $y=0$ in the equation of the ellipse, and we obtain

$$x=\pm a=\text{CA or CA}',$$

which shows that the curve cuts the axis of abscissas in two points, A and A′, at the same distance from the origin, the one being on the right, and the other on the left; and, since 2CA, or AA′, is equal to $2a$, it follows that *the sum of the two lines drawn from any point of an ellipse to the foci is equal to the major axis.*

If we make $x=0$ in the equation of the ellipse, we obtain

$$y=\pm b, =\text{CB or CB}',$$

which shows that the curve cuts the axis of Y in two points, B and B′, at the same distance from the origin.

123. *Curve traced through intermediate points.* If we wish to trace the curve through the intermediate points, we reduce the equation to the form

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2},$$

from which we may compute the value of y corresponding to any assumed value of x.

Example. Trace the curve whose equation is

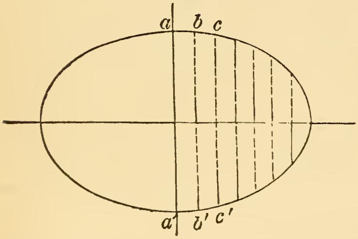
$$49y^2+36x^2=1764.$$

Solving the equation for y, we have

$$y=\pm\frac{6}{7}\sqrt{49-x^2}.$$

By assuming for x different values from 0 to 7, we obtain the corresponding values of y as given below.

When $x=0$, $y=\pm 6$.	When $x=4$, $y=\pm 4.92$.
$x=1$, $y=\pm 5.94$.	$x=5$, $y=\pm 4.20$.
$x=2$, $y=\pm 5.75$.	$x=6$, $y=\pm 3.09$.
$x=3$, $y=\pm 5.42$.	$x=7$, $y=\pm 0$.



When $x=0$, y will equal ± 6, which gives two points, a and a', one above and the other below the axis of X. When $x=1$, $y=\pm 5.94$, which gives the points b and b'. When $x=2$, $y=\pm 5.75$, which gives the points c and c', etc. If we suppose x greater than 7, the value of y will be imaginary, which shows that the curve does not extend from the centre beyond the value $x=7$.

If x is negative, we shall in like manner obtain points in the third and fourth quadrants, and the curve will not extend to the left beyond the value $x=-7$.

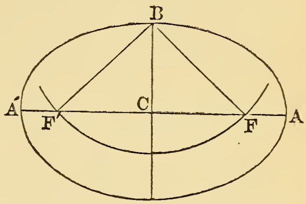
The ellipse is seen to be symmetrical above and below the axis of x, and also to the right and left of the axis of y.

124. *The circle is a particular case of the ellipse.* When b is made equal to a, the equation of the ellipse becomes

$$x^2+y^2=a^2,$$

which is the equation of a circle; hence *the circle may be regarded as an ellipse whose two axes are equal to each other.*

125. *To find the foci of an ellipse when the two axes are given.* Since BF or BF′ is equal to a (Art. 121), it follows that *the distance from either focus to the extremity of the minor axis is equal to half the major axis.*



If, then, from B, the extremity of the minor axis, with a radius equal to half the major axis, we describe an arc cutting the major axis AA′ in F and F′, the two points of intersection will be the foci of the ellipse.

126. *To find the length of the latus rectum.* According to Art. 121, Eq. 12,

$$y^2=\frac{b^2}{a^2}(a^2-x^2).$$

Suppose $x=c$, or CF; then

$$y^2=\frac{b^2}{a^2}(a^2-c^2),$$

where y is the ordinate at the point F. But by Art. 121, Eq. 9,

$$a^2-c^2=b^2;$$

hence we have

$$y^2=\frac{b^2}{a^2}\times b^2,$$

or

$$a:b::b:y,$$

and

$$2a:2b::2b:2y.$$

But $2y$ here represents the double ordinate drawn through the focus, and is called the latus rectum (Art. 120); hence *the latus rectum of any ellipse is a third proportional to the major and minor axes.*

127. *Equation of the ellipse in terms of the eccentricity.* The fraction $\frac{c}{a}$, which represents the ratio of CF to CA, or the distance from the centre to either focus, divided by half the major axis, is called the *eccentricity* of the ellipse. If we represent the eccentricity by e, then

$$\frac{c}{a}=e, \text{ or } c=ae.$$

But we have seen that $c^2=a^2-b^2$;

hence $$a^2-b^2=a^2e^2,$$

or $$\frac{b^2}{a^2}=1-e^2.$$

Making this substitution, the equation of the ellipse becomes

$$y^2=(1-e^2)(a^2-x^2),$$

which is the equation in terms of the eccentricity.

128. *To find the distance of any point on the curve from either focus.* Equations (6) and (7) of Art. 121 are

$$r'=a+\frac{cx}{a},$$

$$r=a-\frac{cx}{a}.$$

Substituting e for $\frac{c}{a}$, these equations become

$$r'=a+ex,$$
$$r=a-ex,$$

which equations represent the distance of any point on an ellipse from either focus.

Multiplying these values together, we obtain

$$rr'=a^2-e^2x^2,$$

which is the value of the product of the focal distances.

The equation of an ellipse may assume forms differing from those of Art. 121, in consequence of multiplication or division by a constant, or of transposition. Thus, $x^2+4y^2=7$; $y^2=25-2x^2$; $4(x^2+y^2)=7+x^2$, are equations of ellipses referred to the centre and axes.

Ex. 1. Trace the curve whose equation is $3x^2+5y^2=15$.

Ex. 2. In a given ellipse, half the sum of the focal distances is 4, and half the distance between the foci is 3; what is the equation to the ellipse?

Ex. 3. In a given ellipse, the sum of the focal distances is 10, and the difference between the squares of half that sum and half the distance between the foci is 16; what is the equation to the ellipse?

Ex. 4. What is the eccentricity of the ellipse whose equation is $9x^2+16y^2=144$?

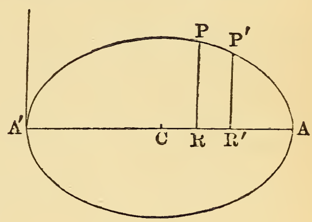
Ex. 5. Trace the curve whose equation is $x^2+4y^2=16$.

Ex. 6. Trace the curve whose equation is $3x^2+4y^2=120$.

Ex. 7. What are the eccentricities of the ellipses of examples 5 and 6?

129. *To find the equation of the ellipse when the origin is at the vertex of the major axis.* The equation of the ellipse when the origin is at the centre is

$$y^2=\frac{b^2}{a^2}(a^2-x^2). \qquad (1)$$



If the origin is placed at A′, the ordinates will have the same value as when the origin was at the centre, but the abscissas will be changed. If we represent the abscissas reckoned from A′ by x', we shall have

$$CR=A'R-A'C,$$

or

$$x=x'-a.$$

Substituting this value of x in equation (1), we have

$$y^2=\frac{b^2}{a^2}(2ax'-x'^2),$$

which is the equation of the ellipse referred to the vertex A′.

130. *Relation of ordinates to the major axis.* If the last equation be resolved into a proportion, we shall have

$$y^2:(2a-x)x::b^2:a^2.$$

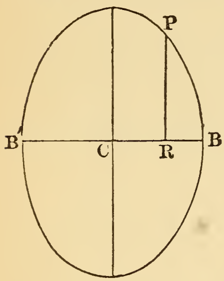
Now $2a$ represents the major axis AA′; and since x represents A′R, $2a-x$ will represent AR; therefore $(2a-x)x$ represents the product of the segments into which the major axis is divided by the ordinate PR. Hence we have, *the square of any ordinate to the major axis of an ellipse, is to the product of the segments into which it divides that axis, as the square of the minor axis, is to the square of the major axis.*

If we draw a second ordinate P′R′ to the major axis, we shall have $PR^2 : A'R \times RA :: b^2 : a^2 :: P'R'^2 : A'R' \times R'A$,

or $$PR^2 : P'R'^2 :: A'R \times RA : A'R' \times R'A;$$

that is, *the squares of ordinates to the major axis of an ellipse are to each other as the products of the segments into which they divide that axis.*

131. *Ordinates to the minor axis.* The equation to the ellipse, Art. 121, Eq. 10, may be put under the form



$$x^2 = \frac{a^2}{b^2}(b^2 - y^2),$$

or $$a^2 : b^2 :: x^2 : (b+y)(b-y).$$

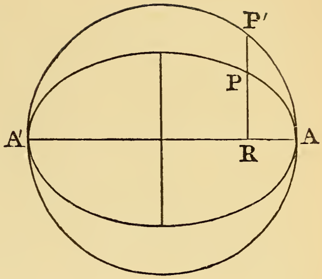
Now y represents CR; hence $b+y$ represents B′R, and $b-y$ represents BR. Also x represents PR, which may be called an ordinate to the minor axis. Hence we have *the square of any ordinate to the minor axis of an ellipse, is to the product of the segments into which it divides that axis, as the square of the major axis, is to the square of the minor axis.*

Example. The major axis of an ellipse is 12 inches, and the curve passes through the two points $x=4$, $y=0$, and $x=-4$, $y=0$; required the equation of the ellipse.

132. *An ordinate to the major axis of an ellipse is to the corresponding ordinate of the circumscribed circle, as the minor axis is to the major axis.*

Let a circle be described on AA′ as a diameter, and let the ordinate PR of the ellipse be produced to meet the circumference of the circle in P′.

The equation of the ellipse when the origin is at the centre (Art. 121, Eq. 12) is



$$y^2=\frac{b^2}{a^2}(a^2-x^2)=\frac{b^2}{a^2}(a-x)(a+x).$$

But $(a-x)(a+x)$ represents AR × A′R; hence we have

$$\frac{PR^2}{AR\times A'R}=\frac{b^2}{a^2}.$$

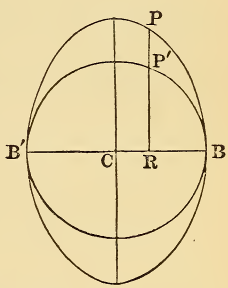
But $P'R^2=AR\times A'R$ (Geom., B. IV., Pr. 23, Cor.); hence

$$\frac{PR^2}{P'R^2}=\frac{b^2}{a^2},$$

or $$PR:P'R::b:a::2b:2a.$$

133. *An ordinate to the minor axis of an ellipse is to the corresponding ordinate of the inscribed circle, as the major axis is to the minor axis.*

Let a circle be described on BB′ as a diameter, and let the ordinate PR of the ellipse meet the circumference of the circle in P′.



The equation of the ellipse when the origin is at the centre is

$$x^2=\frac{a^2}{b^2}(b^2-y^2)=\frac{a^2}{b^2}(b-y)(b+y).$$

But $(b-y)(b+y)$ represents BR × B′R; hence we have

$$\frac{PR^2}{BR\times B'R}=\frac{a^2}{b^2}.$$

But $BR\times B'R=P'R^2$; hence

$$\frac{PR^2}{P'R^2}=\frac{a^2}{b^2},$$

or $$PR:P'R::a:b::2a:2b.$$

134. *To find the equation to the tangent at any point of an ellipse.*

Let the equation to the ellipse be $a^2y^2+b^2x^2=a^2b^2$.

Let x', y' be the co-ordinates of the point on the curve at which the tangent is drawn, and x'', y'' the co-ordinates of an adjacent point on the curve. The equation to the secant line passing through the points x', y' and x'', y'' (Art. 40) is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1)$$

Now, since the points x', y' and x'', y'' are both on the ellipse, we must have $a^2y'^2+b^2x'^2=a^2b^2$,

and $a^2y''^2+b^2x''^2=a^2b^2$;

therefore, by subtraction, $a^2(y''^2-y'^2)+b^2(x''^2-x'^2)=0$,

or $$\frac{y''-y'}{x''-x'}=-\frac{b^2}{a^2}\cdot\frac{x''+x'}{y''+y'}.$$

Substituting this value in equation (1), the equation of the secant line becomes

$$y-y'=-\frac{b^2}{a^2}\cdot\frac{x''+x'}{y''+y'}(x-x') \qquad (2)$$

The secant will become a tangent when the two points coincide, in which case

$$x'=x'', \text{ and } y'=y''.$$

Equation (2) will then become

$$y-y'=-\frac{b^2x'}{a^2y'}(x-x'), \qquad (3)$$

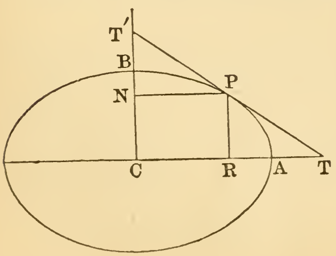
which is the equation to a tangent at the point x', y'.

Clearing this equation of fractions and transposing, we obtain $a^2yy'+b^2xx'=a^2y'^2+b^2x'^2$;

hence $a^2yy'+b^2xx'=a^2b^2$, (4)

which is the simplest form of the equation to the tangent line.

135 *Points where the tangent cuts the axes.* In equation (4) of the last article, x and y are co-ordinates of any point of the tangent line. Make $y=0$, in which case $x=\text{CT}$, and we have



$$b^2xx'=a^2b^2;$$

that is, $$x=\frac{a^2}{x'}.$$

But x' is CR; hence CR . CT = CA².

If from CT we subtract CR or x', we shall have the subtangent

$$RT = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

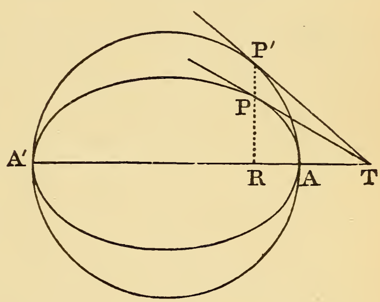
Since the subtangent is independent of the minor axis, it is the same for all ellipses which have the same major axis; and since the circle on the major axis may be considered as one of these ellipses, the subtangent is the same for an ellipse and its circumscribing circle.

To determine the point in which the tangent intersects the axis of Y, we make $x = 0$, which gives

$$y = \frac{b^2}{y'} = CT'.$$

Therefore CN . CT′ = CB².

136. *To draw a tangent to an ellipse through a given point.*



Let P be the given point on the ellipse. On AA′ describe a circle, and through P draw the ordinate PR, and produce it to meet the circumference of the circle in P′. Through P′ draw the tangent P′T, and from T, where the tangent to the circle meets the major axis produced, draw PT; it will be a tangent to the ellipse at P (Art. 135).

137. *To find the equation of a tangent line to the ellipse in terms of the tangent of the angle it makes with the major axis.*

In the equation of the tangent line (Art. 134, Eq. 3),

$$y - y' = -\frac{b^2 x'}{a^2 y'}(x - x'),$$

$-\frac{b^2 x'}{a^2 y'}$ represents the trigonometrical tangent of the angle which the tangent line makes with the major axis of the ellipse (Art. 40). If we represent this tangent by m, we shall have

$$-\frac{b^2x'}{a^2y'}=m.$$

The equation of the tangent line (Art. 134, Eq. 4) was reduced to the form

$$a^2yy'+b^2xx'=a^2b^2.$$

Hence $$y=-\frac{b^2xx'}{a^2y'}+\frac{b^2}{y'},$$

or $$y=mx+\frac{b^2}{y'}.$$

We wish then to express $\frac{b^2}{y'}$ in terms of m.

Now $$b^2x'=-a^2y'm,$$

and $$a^2y'^2+b^2x'^2=a^2b^2;$$

Therefore $$a^2y'^2+\frac{a^4y'^2m^2}{b^2}=a^2b^2.$$

Hence $$y'^2(a^2m^2+b^2)=b^4,$$

and $$\frac{b^2}{y'}=\pm\sqrt{a^2m^2+b^2}.$$

Hence the equation to the tangent may be written

$$y=mx\pm\sqrt{a^2m^2+b^2}.$$

Hence the straight line whose equation is

$$y=mx\pm\sqrt{a^2m^2+b^2},$$

touches the ellipse whose equation is $a^2y^2+b^2x^2=a^2b^2$.

Since m in this equation is indeterminate, it may assume successively any number of values. The corresponding straight lines will be a series of tangents to the ellipse. The double sign of the radical shows, moreover, that for any value of m there are two tangents to the ellipse parallel to each other.

Ex. 1. In an ellipse whose major axis is 50 inches, the abscissa of a certain point is 15 inches, and the ordinate 16 inches, the origin being at the centre. Determine where the tangent passing through this point meets the two axes produced.

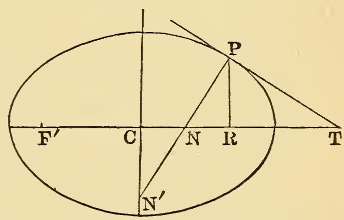
Ans. Distance from the centre on the axis of X, $=41\frac{2}{3}$ inches; on the axis of Y, $=25$ inches.

Ex. 2. Find the angle which the tangent line in the preceding example makes with the axis of X. *Ans.* 149° 3′.

Ex. 3. On an ellipse whose two axes are 50 and 40 inches, find the point from which a tangent line must be drawn in order that it may make an angle of 35° with the axis of X.

Ex. 4. Find the equations of the two lines which touch the ellipse $25y^2+16x^2=400$, and which make an angle of 135° with the axis of X. *Ans.* $y=-x\pm\sqrt{41}$.

138. *To find the equation to the normal at any point of an ellipse.*



The equation to a straight line passing through the point P, whose co-ordinates are x', y' (Art. 38), is

$$y-y'=m(x-x'); \qquad (1)$$

and, since the normal is perpendicular to the tangent, we shall have (Art. 45)

$$m=\frac{1}{-m'}.$$

But we have found for the tangent line, Art. 137,

$$m'=-\frac{b^2x'}{a^2y'}.$$

Hence

$$m=\frac{a^2y'}{b^2x'}.$$

Substituting this value in equation (1), we shall have for the equation of the normal line

$$y-y'=\frac{a^2y'}{b^2x'}(x-x'), \qquad (2)$$

where x and y are the general co-ordinates of the normal line, and x', y' the co-ordinates of the point of intersection with the ellipse.

139. *Points of intersection with the axes.* To find the point in which the normal cuts the major axis, make $y=0$ in equation (2), and we have, after reduction,

$$\text{CN, or } x=\frac{a^2-b^2}{a^2}x'.$$

If we subtract this value from CR, which is represented by x', we shall have the subnormal

$$NR = x' - \frac{a^2 - b^2}{a^2}x' = \frac{b^2x'}{a^2}.$$

To find the point in which the normal cuts the minor axis, make $x=0$ in equation (2), and we have

$$CN' = y = -\frac{a^2 - b^2}{b^2}y'.$$

140. *Distance from the focus to the foot of the normal.* If we put e^2 for $\frac{a^2-b^2}{a^2}$ (Art. 127), we shall have

$$CN = e^2x'.$$

If to this we add F′C, which equals c or ae (Art. 127), we have

$$F'N = ae + e^2x' = e(a + ex'),$$

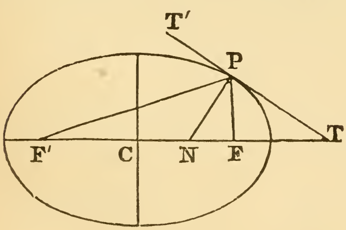
which is the distance from the focus to the foot of the normal.

Ex. 1. In an ellipse whose major axis is 50 inches, the abscissa of a certain point is 15 inches, and the ordinate 16 inches, the origin being at the centre. Determine where the normal line passing through this point meets the two axes.

Ans. Distance from the centre on the axis of X, $=5\frac{2}{5}$ inches; on the axis of Y, $=9$ inches.

Ex. 2. Find the point on the curve of an ellipse whose two axes are 50 and 40 inches, from which, if an ordinate and normal be drawn, they will form with the major axis a triangle whose area is 80 inches.

141. *The normal at any point of an ellipse bisects the angle formed by lines drawn from that point to the foci.*



Let PT be a tangent to an ellipse, and PF, PF′ be lines drawn from the point of contact to the foci. Draw PN bisecting the angle FPF′. Then, by Geom., Bk. IV., Pr. 17,

$$FP : F'P :: FN : F'N;$$

or by composition,

$$FP+F'P:FF'::F'P:F'N. \qquad (1)$$

But $$FP+F'P=2a.$$

Also $$FF'=2c=2ae \text{ (Art. 127)},$$

and $$F'P=a+ex \text{ (Art. 128)}.$$

Making these substitutions in proportion (1), we have

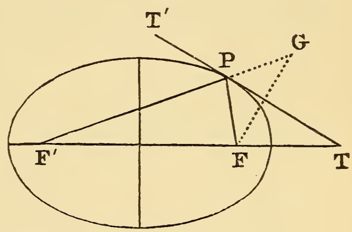
$$2a:2ae::a+ex:F'N.$$

Hence $$F'N=e(a+ex).$$

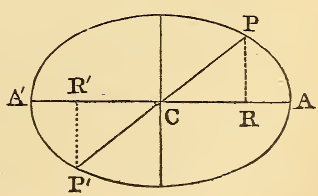
But by Art. 140, $e(a+ex)$ represents the distance from the focus F′ to the foot of the normal. Hence the line PN, which bisects the angle FPF′, is the normal.

142. *The radii vectores are equally inclined to the tangent.* Since PN is perpendicular to TT′, and the angle FPN is equal to the angle F′PN, therefore the angle FPT is equal to the angle F′PT′.

143. *Second method of drawing a tangent line to an ellipse.* Let P be the point through which the tangent line is to be drawn. Draw the radii vectores PF, PF′; produce PF′ to G, making PG equal to PF, and draw FG. Draw PT perpendicular to FG, and it will be the tangent required; for the angle FPT equals the angle GPT, which equals the vertical angle F′PT′.



144. *Every diameter of an ellipse is bisected at the centre.* Let PP′ be a straight line drawn through the centre of the ellipse, and terminated on both sides by the curve; it will be divided into two equal parts at the point C. Let x', y' be the co-ordinates of the point P, and x'', y'' those of the point P′.



Since the points P and P′ are on the curve, we shall have (Art. 121)

$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2),$$

and
$$y''^2 = \frac{b^2}{a^2}(a^2 - x''^2);$$

whence, by division,
$$\frac{y'^2}{y''^2} = \frac{a^2 - x'^2}{a^2 - x''^2}.$$

But, since the right-angled triangles CPR, CP′R′ are similar, we have

$$\frac{y'}{y''} = \frac{x'}{x''}.$$

Hence
$$\frac{x'^2}{x''^2} = \frac{a^2 - x'^2}{a^2 - x''^2}.$$

Clearing of fractions, we obtain

$$x'^2 = x''^2;$$

whence also we have
$$y'^2 = y''^2.$$

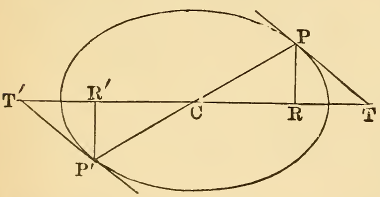
Consequently,
$$x'^2 + y'^2 = x''^2 + y''^2,$$

or
$$CP^2 = CP'^2;$$

that is,
$$CP = CP';$$

that is, PP′ is bisected in C.

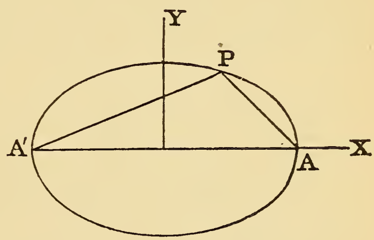
145. *Tangents to an ellipse at the extremities of a diameter are parallel to each other.*



In Art. 135 we found $CT = \frac{a^2}{x'}$, and similarly $CT' = \frac{a^2}{x''}$, where x' represents CR, the abscissa of the point P, and x'' represents CR′, the abscissa of the point P′. But we have found (Art. 144) that $x' = x''$; hence CT = CT′. The two triangles CPT, CP′T′, have therefore two sides, and the included angle of the one equal to two sides and the included angle of the other; hence the angle CPT = the angle CP′T′, and PT is parallel to P′T′.

Hence, if tangents are drawn through the vertices of any two diameters, they will form a parallelogram circumscribing the ellipse.

146. *If from any point in the curve, chords are drawn to the extremities of the major axis, the product of the tangents of the angles which they form with it, on the same side, is equal to* $-\frac{b^2}{a^2}$.



Let PA, PA′ be two chords drawn from the same point, P, on the ellipse to the extremities of the major axis.

The equation of the line PA, passing through the point A, whose co-ordinates are $x'=a$, $y'=0$ (Art. 38), is

$$y=m(x-a).$$

The equation of PA′, passing through the point A′, whose co-ordinates are $x''=-a$, $y''=0$, is

$$y=m'(x+a).$$

At the point of intersection, P, these equations are simultaneous, and, combining them together, we have

$$y^2=mm'(x^2-a^2). \qquad (1)$$

But, since the point P is on the curve, we must have at the same time

$$y^2=\frac{b^2}{a^2}(a^2-x^2)=-\frac{b^2}{a^2}(x^2-a^2). \qquad (2)$$

Comparing equations (1) and (2), we see that

$$mm'=-\frac{b^2}{a^2},$$

where m denotes the tangent of the angle PAX, and m' denotes the tangent of the angle PA′X.

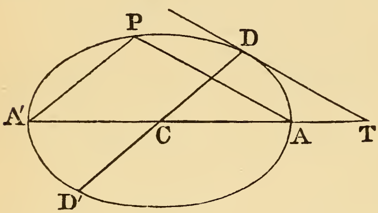
147. *Definition.* Two chords drawn from any point in the curve to the extremities of a diameter are called *supplementary chords.*

148. *Supplementary chords in a circle.* A circle may be considered as an ellipse whose two axes are equal to each other; hence, in a circle,

$$mm' = -1,$$

which shows that the supplementary chords are perpendicular to each other (Art. 46).

149. *If through one extremity of the major axis a chord be drawn parallel to a tangent line to the curve, the supplementary chord will be parallel to the diameter drawn through the point of contact, and conversely.*



Let DT be a tangent to the ellipse at the point D, and let the chord AP be drawn parallel to it; then will the supplementary chord A′P be parallel to the diameter DD′, which passes through the point of contact, D.

Let x', y' designate the co-ordinates of D. The equation of the line CD (Art. 31) gives

$$y' = m'x',$$

whence

$$m' = \frac{y'}{x'}.$$

But the tangent of the angle which the tangent line makes with the major axis (Art. 137) is

$$m = -\frac{b^2x'}{a^2y'}.$$

Multiplying together the values of m and m', we obtain

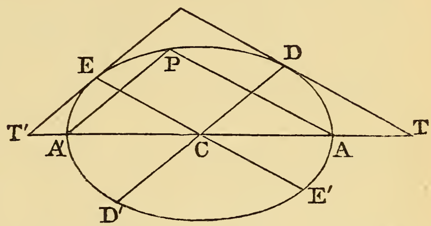
$$mm' = -\frac{b^2}{a^2},$$

which represents the product of the tangents of the angles which the lines CD and DT make with the major axis produced.

But, by Art. 146, the product of the tangents of the angles PAT and PA′T is also equal to $-\frac{b^2}{a^2}$. Hence, if AP is parallel to DT, A′P will be parallel to CD, and conversely.

150. *Definition.* Two diameters of an ellipse are said to be *conjugate* to one another when each is parallel to a tangent line drawn through the vertex of the other.

151. *Property of conjugate diameters.* Let DD′ be any diameter of an ellipse, and DT the tangent line drawn through its vertex, D, and let the chord AP be drawn parallel to DT; then, by Art. 149, the supplementary chord A′P is parallel to DD′. Let another tangent, ET′, be drawn parallel to A′P; it will also be parallel to DD′. Let the diameter EE′ be drawn through the point of contact, E; then, by Art. 149, A′P being parallel to T′E, the supplementary chord AP, and also its parallel DT, will be parallel to EE′. Hence each of the diameters DD′, EE′ is parallel to a tangent drawn through the vertex of the other, and by definition (Art. 150) they are conjugate to one another.



Since the conjugate diameters DD′, EE′ are parallel to the supplementary chords A′P, AP, by Art. 146, *the product of the tangents of the angles which conjugate diameters form with the major axis is equal to* $-\frac{b^2}{a^2}$.

Ex. 1. In an ellipse whose axes are 10 and 8, a chord drawn from one extremity of the major axis forms with that axis an angle whose tangent is 2; what angle does the supplementary chord form? *Ans.*

Ex. 2. In an ellipse whose axes are 12 and 8, a chord forms with the major axis an angle whose tangent is -3; what angle does the supplementary chord form? *Ans.*

Ex. 3. In an ellipse whose axes are 10 and 8, find the angles which supplementary chords drawn from the point $x=1$ form with the major axis. *Ans.*

Ex. 4. In an ellipse whose axes are 10 and 30, two conjugate diameters are equally inclined to the major axis. Find the angle between the two diameters.

152. *To determine the co-ordinates of the points of intersection of a straight line with an ellipse.*

Let the equation to the ellipse be

$$a^2y^2+b^2x^2=a^2b^2, \tag{1}$$

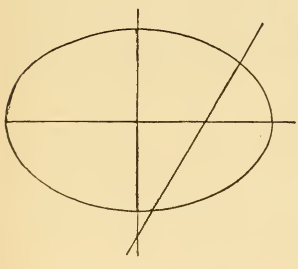
and the equation to a straight line be

$$y=mx+c. \tag{2}$$

If this line intersects the ellipse, then we may regard (1) and (2) as simultaneous equations containing but two unknown quantities. By substitution in equation (1) we obtain

$$(a^2m^2+b^2)x^2+2a^2cmx=(b^2-c^2)a^2,$$

the roots of which equation give the abscissas of the points where the straight line meets the curve, and the ordinates may be found from equation (2). Hence, if the roots be real, the straight line will cut the ellipse in two points, and it can not cut the ellipse in more than two points. If the roots are equal, the points of section coincide, and the line is then a tangent. If the roots are imaginary, the line falls entirely without the ellipse.



Ex. 1. Find the co-ordinates of the points in which the ellipse whose equation is $25y^2+16x^2=400$ is intersected by the line whose equation is $y=2x-5$.

Ans. $x=+3.7999$, or $+0.5104$;
$y=+2.5998$, or -3.9792.

Ex. 2. Find the co-ordinates of the points in which the ellipse whose equation is $49y^2+36x^2=1764$ is intersected by the line whose equation is $y=3x-7$, and draw a figure representing the several quantities.

153. *To determine the co-ordinates of the points of intersection of a circle and ellipse.*

If the centre of the circle is not restricted in position, there may be four points of intersection corresponding to an equation of the fourth degree. If, however, the centre of the circle is at one extremity of the major axis, there will be but two points of intersection, which may be easily found.

Let the equation to the ellipse be

$$y^2=\frac{b^2}{a^2}(2ax-x^2),$$

and the equation to the circle be

$$x^2+y^2=r^2;$$

then, by substitution, we obtain

$$r^2-x^2=\frac{b^2}{a^2}(2ax-x^2),$$

where x will be found to have two values, but one of them is negative, and gives imaginary values for y. There will, therefore, be but two points of intersection, both having the same abscissa, and the ordinates will differ only in sign.

Ex. 1. Find the co-ordinates of the points in which the ellipse whose equation is $y^2=\frac{16}{25}(10x-x^2)$ is intersected by the circle whose equation is $x^2+y^2=64$.

Ex. 2. Find the co-ordinates of the points in which the ellipse whose equation is $y^2=\frac{36}{49}(14x-x^2)$ is intersected by the circle whose equation is $x^2+y^2=100$.

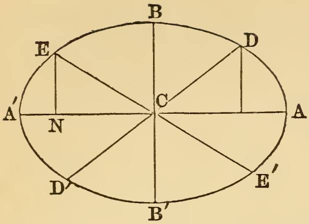
If the centre of the circle is upon either axis of the ellipse, there may be four points of intersection.

Ex. 3. Find the co-ordinates of the points where the ellipse $y^2=\frac{25}{100}(100-x^2)$ is intersected by the circles $x^2+y^2=64$; $y^2+(x-2)^2=64$, $y^2+(x-8)^2=64$, and $y^2+(x-20)^2=64$.

The first circle cuts the ellipse in four points, the second cuts it in three points, the third in two points, and the fourth does not cut the ellipse.

Ex. 4. Draw a figure representing these curves and their intersections.

154. *Having given the co-ordinates of one extremity of a diameter, to find those of either extremity of the diameter conjugate to it.*



Let AA′, BB′ be the axes of an ellipse; DD′, EE′ a pair of conjugate diameters. Let x', y' be the co-ordinates of D; then the equation to CD (Art. 40) is

$$y=\frac{y'}{x'}.x. \qquad (1)$$

Since the conjugate diameter EE′ is parallel to the tangent at D, the equation to EE′ (Art. 149) is

$$y=-\frac{b^2x'}{a^2y'}.x. \qquad (2)$$

To determine the co-ordinates of E and E′, we must combine the equation to EE′ with the equation to the ellipse, $a^2y^2+b^2x^2=a^2b^2$.

Substituting the value of y from equation (2), we have

$$\frac{b^4x'^2}{a^2y'^2}x^2+b^2x^2=a^2b^2.$$

Therefore $$(b^2x'^2+a^2y'^2)x^2=a^4y'^2,$$

or $$a^2b^2x^2=a^4y'^2;$$

whence $$x^2=\frac{a^2y'^2}{b^2},$$

and $$x=\pm\frac{ay'}{b}.$$

Taking the minus sign, in which case x is CN, and combining with equation (2), we have

$$y=\frac{bx'}{a}=\text{EN}.$$

We thus find the co-ordinates of the point E. The co-ordinates of the point E′ have the same values with contrary signs.

155. *The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes.*

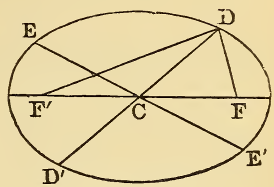
Let x', y' be the co-ordinates of D; then, by Art. 154,

$$\text{CD}^2+\text{CE}^2=x'^2+y'^2+\frac{a^2y'^2}{b^2}+\frac{b^2x'^2}{a^2},$$

$$=\frac{a^2y'^2+b^2x'^2}{b^2}+\frac{a^2y'^2+b^2x'^2}{a^2},$$

$$=a^2+b^2.$$

156. *The rectangle contained by the focal distances of any point on the ellipse is equal to the square of half the corresponding conjugate diameter.*



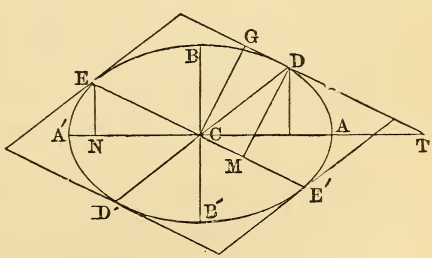
Let DD′, EE′ be a pair of conjugate diameters, and from D draw lines to the foci, F and F′. Represent the co-ordinates of D referred to rectangular axes by x', y'.

Then, since $CD^2 + CE^2 = a^2 + b^2$ (Art. 155), we have

$$\begin{aligned} CE^2 &= a^2 + b^2 - CD^2, \\ &= a^2 + b^2 - x'^2 - y'^2, \\ &= a^2 + b^2 - x'^2 - \frac{b^2}{a^2}(a^2 - x'^2), \\ &= a^2 - \left(1 - \frac{b^2}{a^2}\right)x'^2, \\ &= a^2 - e^2x'^2 \text{ (Art. 127)}, \\ &= DF \times DF' \text{ (Art. 128)}; \end{aligned}$$

that is, the product of the focal distances DF, DF′ is equal to the square of half EE′, which is the diameter conjugate to the diameter which passes through the point D.

157. *The parallelogram formed by drawing tangents through the vertices of two conjugate diameters is equal to the rectangle of the axes.*



Let DD′, EE′ be two conjugate diameters, and let D ED′E′ be a parallelogram formed by drawing tangents to the ellipse through the extremities of these diameters; the area of the parallelogram is equal to $AA' \times BB'$.

Draw DM perpendicular to EE′, and let the co-ordinates of D referred to rectangular axes be x', y'.

The area of the parallelogram DED′E′ is equal to 4CE . DM, which is equal to 4CE.CT sin. CTG, which is equal to 4CT. EN, because EC and DT are parallel.

But $\quad CT=\frac{a^2}{x'}$ (Art. 135), and $EN=\frac{bx'}{a}$ (Art. 154);

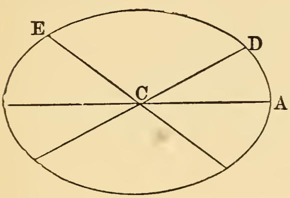
hence the parallelogram $DED'E'=4.\frac{a^2}{x'}.\frac{bx'}{a}=4ab=AA'\times BB'$.

Ex. 1. In an ellipse whose axes are 10 and 8, what is the length of a diameter which makes an angle of 45° with the axis of *x?* What is the length of its conjugate?

Ans.

Ex. 2. What is the altitude of the circumscribed parallelogram whose sides are parallel to the conjugate diameters of the preceding example? *Ans.*

158. *Equation to the ellipse referred to a pair of conjugate diameters as axes.*



Let CD, CE be two conjugate semi-diameters; take CD as the new axis of *x*, CE as that of *y*; let $DCA=\alpha$, $ECA=\beta$. Let x, y be the co-ordinates of any point of the ellipse referred to the original axes, and x', y' the co-ordinates of the same point referred to the new axes.

The equation of the ellipse referred to its centre and axes (Art. 121) is $\qquad a^2y^2+b^2x^2=a^2b^2.$

In order to pass from rectangular to oblique co-ordinates, the origin remaining the same, we must substitute for x and y in the equation of the curve (Art. 56) the values

$$x=x'\cos.\alpha+y'\cos.\beta,$$
$$y=x'\sin.\alpha+y'\sin.\beta.$$

Squaring these values of x and y, and substituting in the equation of the ellipse, we have

$$x'^2(a^2\sin.^2\alpha+b^2\cos.^2\alpha)+y'^2(a^2\sin.^2\beta+b^2\cos.^2\beta)+$$
$$2x'y'(a^2\sin.\alpha\sin.\beta+b^2\cos.\alpha\cos.\beta)=a^2b^2,$$

which is the equation of the ellipse when the oblique co-ordinates make any angles α, β with the major axis.

But, since CD, CE are conjugate semidiameters, we must have (Art. 151)

$$mm' = \text{tang. } a \text{ tang. } \beta = -\frac{b^2}{a^2},$$

whence $$a^2 \text{ tang. } a \text{ tang. } \beta + b^2 = 0.$$

Multiplying by cos. a cos. β, remembering that cos. a tang. a $=$ sin. a, we have

$$a^2 \sin. a \sin. \beta + b^2 \cos. a \cos. \beta = 0.$$

Hence the term containing $x'y'$ vanishes, and the equation becomes

$$x'^2(a^2 \sin.^2 a + b^2 \cos.^2 a) + y'^2(a^2 \sin.^2 \beta + b^2 \cos.^2 \beta) = a^2b^2, \quad (1)$$

which is the equation of the ellipse referred to conjugate diameters.

If in this equation we suppose $y' = 0$, we shall have

$$x'^2 = \frac{a^2b^2}{a^2 \sin.^2 a + b^2 \cos.^2 a}.$$

This is the value of CD^2, which we shall denote by a'^2.

If we suppose $x' = 0$, we shall have

$$y'^2 = \frac{a^2b^2}{a^2 \sin.^2 \beta + b^2 \cos.^2 \beta}.$$

This is the value of CE^2, which we shall denote by b'^2.

Dividing equation (1) by a^2b^2, and then substituting for the coefficients of x'^2 and y'^2 the equal values $\frac{1}{a'^2}$ and $\frac{1}{b'^2}$, we have for the equation to the ellipse referred to conjugate diameters

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1;$$

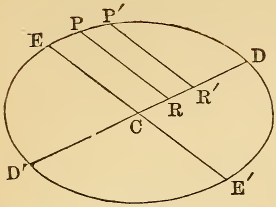
or, suppressing the accents of the variables, we have

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

159. *The square of any diameter is to the square of its conjugate, as the rectangle of the parts into which it is divided by any ordinate, is to the square of that ordinate.*

The equation of the ellipse referred to conjugate diameters may be put under the form

$$a'^2y^2 = b'^2(a'^2 - x^2).$$



This equation may be reduced to the proportion

$$a'^2 : b'^2 :: a'^2 - x^2 : y^2,$$

or $(2a')^2 : (2b')^2 :: (a'+x)(a'-x) : y^2.$

Now $2a'$ and $2b'$ represent the conjugate diameters DD′, EE′; and, since x represents CR, $a'+x$ will represent D′R, and $a'-x$ will represent DR; also PR represents y; hence

$$DD'^2 : EE'^2 :: DR \times RD' : PR^2.$$

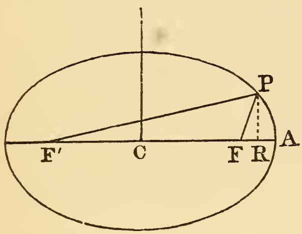
If we draw a second ordinate P′R′ to the diameter DD′, we shall have

$$PR^2 : DR \times RD' :: b'^2 : a'^2 :: P'R'^2 : DR' \times R'D',$$

or $$PR^2 : P'R'^2 :: DR \times RD' : DR' \times R'D';$$

that is, *the squares of any two ordinates to the same diameter are as the products of the parts into which they divide that diameter.*

160. *To find the polar equation to the ellipse, the pole being at one of the foci.*



1. Let F be the pole.

Let $FP = r$; angle $PFA = \theta$; then $FR = r \cos.\ \theta$.

By Art. 128, $r = a - ex$.

But $x = CR = CF + FR,$

$= ae + r \cos.\ \theta.$

Therefore $r = a - ae^2 - er \cos.\ \theta.$

Hence $$r(1 + e \cos.\ \theta) = a(1 - e^2),$$

or $$r = \frac{a(1-e^2)}{1 + e \cos.\ \theta},$$

which is the required equation when θ is measured from the radius to the nearer vertex.

2. Let F′ be the pole.

Let $F'P = r'$; $PF'A = \theta'$; then $F'R = r' \cos.\ \theta'$.

By Art. 128, $r' = a + ex.$

But $x = CR = F'R - F'C,$

$= r' \cos.\ \theta' - ae.$

Therefore $$r' = a + er' \cos. \theta' - ae^2.$$
Hence $$r'(1 - e \cos. \theta') = a(1 - e^2),$$
or $$r' = \frac{a(1-e^2)}{1 - e \cos. \theta'},$$
which is the required equation when θ' is measured from the radius to the remote vertex.

Ex. 1. The axes of an ellipse are 50 and 40 inches, and the radius vector is 12 inches. Determine the value of θ.

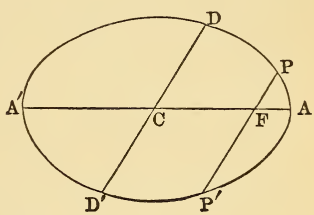
Ans. 56° 15′.

Ex. 2. The axes of an ellipse are 50 and 40 inches, and θ is equal to 36°. Determine the radius vector.

Ans. 10.771 inches.

Ex. 3. In an ellipse whose major axis is 50 inches, the radius vector is 12 inches, and θ is 36°. Determine the minor axis of the ellipse. *Ans.* 41.67 inches.

161. *Any chord which passes through the focus of an ellipse is a third proportional to the major axis and the diameter parallel to that chord.*



Let PP′ be a chord of an ellipse passing through the focus F, and let DD′ be a diameter parallel to PP′.

By Art. 160, $PF = r = \frac{a(1-e^2)}{1 + e \cos. \theta}$.

To find the value of FP′, we must substitute for θ, $180° + \theta$, and we obtain

$$FP' = r' = \frac{a(1-e^2)}{1 - e \cos. \theta}.$$

Hence $$PP' = r + r' = \frac{2a(1-e^2)}{1 - e^2 \cos.^2 \theta}. \qquad (1)$$

But, by Art. 158,

$$CD^2 = \frac{a^2 b^2}{a^2 \sin.^2 \theta + b^2 \cos.^2 \theta},$$

$$= \frac{a^2 b^2}{a^2 \sin.^2 \theta + (a^2 - a^2 e^2) \cos.^2 \theta} \text{ (Art 127)},$$

$$= \frac{a^2 b^2}{a^2 - a^2 e^2 \cos.^2 \theta},$$

$$=\frac{a^2(1-e^2)}{1-e^2\cos.^2\theta}. \qquad (2)$$

Comparing equations (1) and (2), we find

$$PP'=\frac{2CD^2}{a}=\frac{4CD^2}{2a};$$

that is, $$AA':DD'::DD':PP',$$

or PP′ is a third proportional to AA′ and DD′.

162. *Definition.* The *parameter* of any diameter is a third proportional to that diameter and its conjugate.

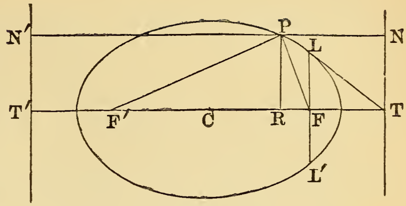
The parameter of the major axis is called the principal parameter, or *latus rectum*, and its value is $\frac{2b^2}{a}$ (Art. 126). The parameter of the minor axis is $\frac{2a^2}{b}$. The latus rectum is the double ordinate to the major axis passing through the focus (Art. 126). Now, since any focal chord is a third proportional to the major axis and the diameter parallel to that chord, and since the major axis is greater than any other diameter, it follows that *the major axis is the only diameter whose parameter is equal to the double ordinate passing through the focus.*

163. *Definition.* The *directrix* of an ellipse is a straight line perpendicular to the major axis produced, and intersecting it in the same point with the tangent drawn through one extremity of the latus rectum.

Thus, if LT be a tangent drawn through one extremity of the latus rectum LL′, meeting the major axis produced in T, and NT be drawn through the point of intersection perpendicular to the axis, it will be the directrix of the ellipse.

The ellipse has two directrices, one corresponding to the focus F, and the other to the focus F′.

164. *The distance of any point in an ellipse from either focus is to its distance from the corresponding directrix, as the eccentricity is to unity.*



Let F be one focus of an ellipse, NT the corresponding directrix; F′ the other focus, and N′T′ the corresponding directrix. Let P be any point on the ellipse; x, y its co-ordinates, the centre being the origin. Join PF, PF′, and draw NPN′ parallel to the major axis, and PR perpendicular to it.

By Art. 135, $$\text{CT}=\frac{a^2}{c}=\frac{a}{e}.$$

Hence, subtracting CR or x,

$$\text{RT}=\frac{a}{e}-x=\frac{a-ex}{e}.$$

But, by Art. 128, $r=\text{PF}=a-ex.$

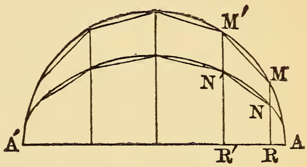
Hence $$e.\text{RT, or } e.\text{PN}=\text{PF};$$

or, $$\text{PF}:\text{PN}::e:1.$$

In like manner, we find that

$$\text{PF}':\text{PN}'::e:1.$$

165. *To find the area of an ellipse.*



On AA′, the major axis of an ellipse, let a semicircle be described, and within this semicircle inscribe a polygon, AMM′A′. From the vertices of this polygon draw ordinates to the major axis, and join the points in which they intersect the ellipse, thus forming a polygon ANN′A′, having the same number of sides.

Let Y, Y′, etc., denote the ordinates of the points M, M′, etc., and let y, y', etc., denote the ordinates of the points N, N′, etc., corresponding to the same abscissas x, x', etc.

The area of the trapezoid $\text{RMM}'\text{R}'=\frac{\text{Y}+\text{Y}'}{2}(x-x')$,

and the area of the trapezoid $\text{RNN}'\text{R}'=\frac{y+y'}{2}(x-x')$.

Hence $$\text{RMM}'\text{R}':\text{RNN}'\text{R}'::\text{Y}+\text{Y}':y+y'.$$

But, by Art. 132, $Y : y :: a : b;$
also $Y' : y' :: a : b.$
Whence $Y + Y' : y + y' :: a : b.$
Therefore $RMM'R' : RNN'R' :: a : b.$

In the same manner it may be proved that each of the trapezoids composing the polygon inscribed in the circle, is to the corresponding trapezoid of the polygon inscribed in the ellipse, as a is to b; hence the entire polygon inscribed in the circle is to the polygon inscribed in the ellipse, as a is to b; and this will be true whatever be the number of sides of the polygons.

If now the number of sides be indefinitely increased, the areas of the polygons will become equal to the areas of the semicircle and semi-ellipse respectively, and we shall have the first is to the second as a is to b; or, denoting the area of the circle by S, and that of the ellipse by s, we shall have

$$S : s :: a : b; \text{ that is, } s = \frac{b}{a}S.$$

But the area of a circle whose radius is a is represented by πa^2; hence $s = \pi ab;$
or *the area of an ellipse is equal to* π *times the rectangle described upon its semi-axes.*

166. Since $\pi ab = \sqrt{\pi^2 a^2 b^2} = \sqrt{\pi a^2 \times \pi b^2}$, we find that *the area of an ellipse is a mean proportional between the areas of its circumscribed and inscribed circles.*

Ex. 1. Determine the area of an ellipse whose two axes are 24 and 18 inches.

Ex. 2. The area of an ellipse is 40 square inches, and the latus rectum is 4 inches; required the axes of the ellipse.

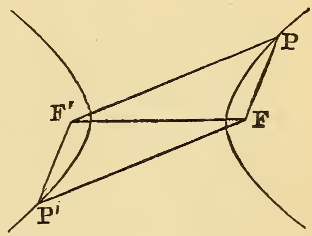
Ex. 3. The axes of an ellipse are 40 and 50; find the areas of the two parts into which it is divided by the latus rectum.

SECTION VII.

THE HYPERBOLA.

167. An *hyperbola* is a plane curve traced out by a point which moves in such a manner that the *difference* of its distances from two fixed points is always the same. The two fixed points are called the *foci* of the hyperbola.

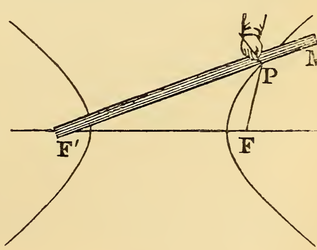
Thus, if F and F′ are two fixed points, and if the point P moves about F in such a manner that the difference of its distances from F and F′ is always the same, the point P will describe an hyperbola, of which F and F′ are the foci.



If the point P′ moves about F′ in such a manner that P′F − P′F′ is always equal to PF′ − PF, the point P′ will describe a second portion of the curve similar to the first. The two portions are called *branches* of the hyperbola.

The distance of the point P from either focus is called the *focal distance*, or the *radius vector*.

168. *Mechanical description of the curve.* From the definition of an hyperbola the curve may be described mechanically. Take any two points, as F and F′. Take a ruler longer than the distance FF′, and fix one of its extremities at the point F′ so that the ruler may be turned round this point in the plane of the paper. Take a thread shorter than the ruler, and fasten one end of it at F, and the other to the end M of the ruler. Then move the ruler on its



pivot at F′, while the thread is kept constantly stretched by a pencil pressed against the ruler; the curve described by the point of the pencil will be a portion of an hyperbola. For in every position of the ruler, the difference of the distances from the variable point P to the two fixed points F and F′ will always be the same, viz., the difference between the length of the ruler and the length of the thread.

If the ruler be turned and move on the other side of the point F, the other part of the same branch may be described.

Also, if one end of the ruler be fixed at F, and that of the thread at F′, the opposite branch of the hyperbola may be described.

169. The *centre* of the hyperbola is the middle point of the straight line joining the foci.

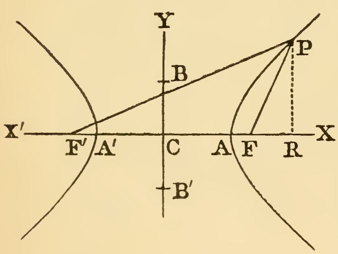
A *diameter* is any straight line passing through the centre, and terminated on both sides by opposite branches of an hyperbola.

The diameter which, when produced, passes through the foci, is called the *transverse axis.*

The *latus rectum* is the chord drawn through one of the foci perpendicular to the transverse axis.

170. *To find the equation to the hyperbola.*

Let F and F′ be the foci, and draw the rectangular axes CX, CY, the origin C being placed at the middle of FF′. Let P be any point of the curve, and draw PR perpendicular to CX.



Let $2c$ denote FF′, the constant distance between the foci, and let $2a$ denote F′P—FP, the constant difference of the focal distances. Denote PF by r, PF′ by r', and let x and y denote the co-ordinates of the point P.

Then, since $FP^2 = PR^2 + RF^2 = PR^2 + (CR - CF)^2$,

we have $$r^2=y^2+(x-c)^2. \qquad (1)$$
Also, $$PF'^2=PR^2+RF'^2=PR^2+(CR+CF)^2;$$
that is, $$r'^2=y^2+(x+c)^2. \qquad (2)$$

Adding equations (1) and (2), we obtain
$$r^2+r'^2=2(y^2+x^2+c^2); \qquad (3)$$
and subtracting equation (1) from (2), we obtain
$$r'^2-r^2=4cx,$$
which may be put under the form
$$(r'+r)(r'-r)=4cx. \qquad (4)$$
But, from the definition of the hyperbola, we have
$$r'-r=2a.$$
Substituting this value in equation (4), we obtain
$$r'+r=\frac{2cx}{a}.$$
Combining the last two equations, we find
$$r'=a+\frac{cx}{a}, \qquad (5)$$
$$r=-a+\frac{cx}{a}. \qquad (6)$$
Squaring these values, and substituting them in equation (3), we obtain
$$a^2+\frac{c^2x^2}{a^2}=c^2+x^2+y^2,$$
which may be reduced to the form
$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2), \qquad (7)$$
which is the equation to the hyperbola.

If we put $b^2=c^2-a^2$, the equation becomes
$$b^2x^2-a^2y^2=a^2b^2, \qquad (8)$$
or $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad (9)$$
which is the equation to the hyperbola referred to its centre and transverse axis.

This equation is sometimes written
$$y^2=\frac{b^2}{a^2}(x^2-a^2). \qquad (10)$$

The equation to the ellipse becomes the equation to the hyperbola by writing $-b^2$ for b^2; and we shall find that the hyperbola has many properties similar to those of the ellipse.

171. *Points of intersection with the axes.* To determine where the curve cuts the axis of X, make $y=0$ in the equation to the hyperbola, and we obtain

$$x=\pm a=\mathrm{CA} \text{ or } \mathrm{CA}',$$

which shows that the curve cuts the axis of abscissas in two points, A and A′, at the same distance from the origin, the one being on the right, and the other on the left; and, since 2CA, or AA′, is equal to $2a$, it follows that *the difference of the two lines drawn from any point of an hyperbola to the foci, is equal to the transverse axis.*

If we make $x=0$ in the equation of the hyperbola, we obtain

$$y=\pm b\sqrt{-1},$$

which shows that the hyperbola does not intersect the axis CY.

172. If with A or A′ as a centre, and a radius equal to CF, we describe a circle cutting the axis of y in two points, B and B′, we shall have

$$\mathrm{CB}^2=\mathrm{BA}^2-\mathrm{CA}^2$$
$$=c^2-a^2=b^2;$$

that is, $b=\mathrm{CB}$ or CB'.

The line BB′ thus determined is called the *conjugate axis* of the hyperbola.

173. *Figure of the hyperbola determined.* In equation (10), Art. 170, let x be numerically less than a; then the values of y are imaginary; therefore no point of the hyperbola is nearer the axis of y than $\pm a$.

Let x be numerically greater than a; then for each value of x there are two equal values of y with contrary signs.

As x increases, the values of y increase; and when x becomes indefinitely great, the value of y becomes so likewise.

The hyperbola therefore consists of two opposite branches, extending indefinitely to the right of A and to the left of A′, and symmetrically placed with respect to the axis XCX′.

174. *Other points of the curve determined.* If we wish to determine other points of the curve, we reduce the equation to the form

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2},$$

from which we may compute the value of y corresponding to any assumed value of x.

Ex. Trace the curve whose equation is $36x^2 - 49y^2 = 1764$.

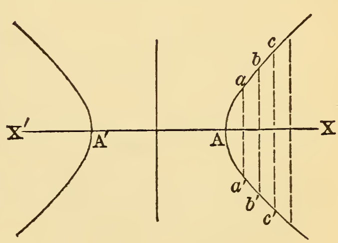
Solving the equation for y, we have

$$y = \pm\frac{6}{7}\sqrt{x^2 - 49}.$$

If x be assumed less than 7, the corresponding value of y is imaginary. If we assume for x different values from 7 upward, we obtain the corresponding values of y as given below.

When $x = 7$, $y = 0$.	When $x = 11$, $y = \pm\ 7.27$.
$x = 8$, $y = \pm 3.32$.	$x = 12$, $y = \pm\ 8.35$.
$x = 9$, $y = \pm 4.85$.	$x = 13$, $y = \pm\ 9.39$.
$x = 10$, $y = \pm 6.12$.	$x = 14$, $y = \pm 10.39$.

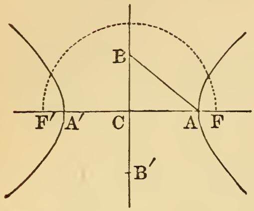
When $x = 7$, $y = 0$, which gives the point A. When $x = 8$, $y = \pm 3.32$, which gives two points, a and a', one above and the other below the axis of X; when $x = 9$, $y = \pm 4.85$, which gives the points b and b'; when $x = 10$, $y = \pm 6.12$, which gives the points c and c', etc.



If we ascribe to x a negative value, we shall obtain for y the same pair of values as when we ascribed to x the corresponding positive value. Hence the portion of the curve to the left of the axis of Y is similar to the portion to the right of it. Moreover, there is no point of the curve between the values $x = +7$ and $x = -7$.

175. *To find the foci of an hyperbola when the two axes are given.* Since $b^2=c^2-a^2$, we have

$$c^2 \text{ or } CF^2=a^2+b^2=AB^2;$$



that is, *the distance from the centre to either focus of an hyperbola is equal to the distance between the extremities of its axes.*

If, then, from the centre C, with a radius equal to the diagonal of the rectangle upon the semiaxes, we describe an arc cutting the transverse axis produced in F and F′, the two points of intersection will be the foci of the hyperbola.

176. *To find the length of the latus rectum.* According to Art. 170, eq. 10,

$$y^2=\frac{b^2}{a^2}(x^2-a^2).$$

Suppose $x=c$ or CF; then

$$y^2=\frac{b^2}{a^2}(c^2-a^2).$$

But $c^2-a^2=b^2$, Art. 172; hence we have

$$y^2=\frac{b^2}{a^2}\times b^2,$$

or $$a:b::b:y,$$

and $$2a:2b::2b:2y.$$

But $2y$ here represents the double ordinate drawn through the focus, and is called the latus rectum, Art. 169; hence *the latus rectum of any hyperbola is a third proportional to the transverse and conjugate axes.*

177. *Equation to the hyperbola in terms of the eccentricity.* The fraction $\frac{c}{a}$, which represents the ratio of CF to CA, or the distance from the centre to either focus divided by half the transverse axis, is called the *eccentricity* of the hyperbola. If we represent the eccentricity by e, then

$$\frac{c}{a}=e, \text{ or } c=ae.$$

But we have seen that $c^2=a^2+b^2$;

hence $a^2+b^2=a^2e^2$,

or $\frac{b^2}{a^2}=e^2-1$.

Making this substitution, the equation of the hyperbola becomes

$$y^2=(e^2-1)(x^2-a^2),$$

which is the equation in terms of the eccentricity.

178. *To find the distance of any point on the curve from either focus.* Equations (5) and (6) of Art. 170 are

$$r'=a+\frac{cx}{a},$$

$$r=-a+\frac{cx}{a}.$$

Substituting e for $\frac{c}{a}$, these equations become

$$r'=ex+a,$$

$$r=ex-a,$$

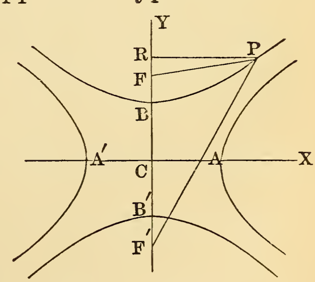
which equations represent the distance of any point on an hyperbola from either focus.

Multiplying these values together, we obtain

$$rr'=e^2x^2-a^2,$$

which is the value of the product of the focal distances.

179. *The conjugate hyperbola.* Suppose an hyperbola to be described whose foci F and F′ are at the same distance from the centre C as those of the curve hitherto described, but lie upon the axis CY instead of CX, and suppose the difference of the distances of any point on the new curve from the two foci is $2b$ instead of $2a$; then, retaining the same axes of reference as before, we shall have for the new position of F and F′,



$$FP^2=PR^2+RF^2=PR^2+(CR-CF)^2;$$

that is $$r^2=x^2+(y-c)^2.$$
Also, $$F'P^2=PR^2+F'R^2=PR^2+(CR+CF)^2;$$
that is, $$r'^2=x^2+(y+c)^2.$$
Proceeding as in Art. 170, we find
$$(c^2-b^2)y^2-b^2x^2=b^2(c^2-b^2).$$
Putting a^2 for c^2-b^2, the equation becomes
$$a^2y^2-b^2x^2=a^2b^2,$$
or $$y^2=\frac{b^2}{a^2}(x^2+a^2),$$
which is the equation to the new hyperbola.

In this equation, suppose $x=0$, and we have $y=\pm b$; that is, the curve passes through the points B and B′, and BB′ is the transverse axis of the new curve.

Suppose $y=0$, and we have $x=\pm a\sqrt{-1}$, which shows that the curve does not meet the axis of X, and AA′ is the conjugate axis of the new curve (Arts. 171 and 172).

This new hyperbola is called the hyperbola *conjugate* to the former. One hyperbola is therefore said to be conjugate to another, when *the transverse and conjugate axes of the one hyperbola are the conjugate and transverse axes of the other hyperbola.*

If $$y^2=\frac{b^2}{a^2}(x^2-a^2)$$
be the equation of any hyperbola, then
$$y^2=\frac{b^2}{a^2}(x^2+a^2)$$
is the equation to the hyperbola conjugate to the former. The latter equation may be deduced from the former by writing $-a^2$ for a^2, and $-b^2$ for b^2.

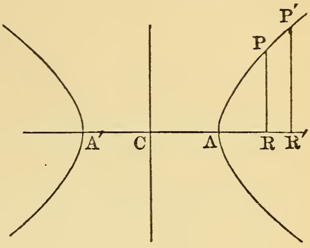
Ex. 1. Trace the curve whose equation is $3x^2-5y^2=15$.

Ex. 2. In a given hyperbola half the difference of the focal distances is 7, and half the distance between the foci is 9; what is the equation to the hyperbola?

Ex. 3. What is the eccentricity of the hyperbola whose equation is $9x^2-16y^2=144$?

Ex. 4. What is the equation of an hyperbola whose conjugate axis is 6 and the eccentricity $1\frac{1}{4}$?

180. *To find the equation of the hyperbola when the origin is at the vertex of the transverse axis.*



The equation of the hyperbola when the origin is at the centre is

$$y^2=\frac{b^2}{a^2}(x^2-a^2). \qquad (1)$$

If the origin is placed at A, the ordinates will have the same value as when the origin was at the centre, but the abscissas will be changed.

If we represent the abscissas reckoned from A by x', we shall have

$$\text{CR}=\text{AR}+\text{AC},$$

or

$$x=x'+a.$$

Substituting this value of x in equation (1), we have

$$y^2=\frac{b^2}{a^2}(2ax'+x'^2),$$

which is the equation of the hyperbola referred to the vertex A.

181. *Relation of ordinates to the transverse axis.* If the last equation be resolved into a proportion, we shall have

$$y^2:(2a+x)x::b^2:a^2.$$

Now $2a$ represents the transverse axis AA'; and since x represents AR, $2a+x$ will represent A'R; therefore $(2a+x)x$ represents the product of the distances from the foot of the ordinate PR to the vertices of the curve. Hence we have *the square of any ordinate to the transverse axis of an hyperbola, is to the product of its distances from the vertices of the curve, as the square of the conjugate axis is to the square of the transverse axis.*

If we draw a second ordinate P'R' to the transverse axis, we shall have

$$\text{PR}^2:\text{AR}\times\text{A'R}::b^2:a^2::\text{P'R'}^2:\text{AR'}\times\text{A'R'},$$

or

$$\text{PR}^2:\text{P'R'}^2::\text{AR}\times\text{A'R}:\text{AR'}\times\text{A'R'};$$

that is, *the squares of ordinates to the transverse axis of an hyperbola are to each other as the products of the distances from the foot of each ordinate to the vertices of the curve.*

182. *The equilateral hyperbola.* When b is made equal to a, the equation of the hyperbola becomes

$$y^2 = 2ax + x^2 \text{ (Art. 180)},$$

or

$$y^2 = x^2 - a^2 \quad \text{(Art. 170)}.$$

The hyperbola represented by these equations is called *equilateral* or *rectangular*, and is to the common hyperbola what the circle is to the ellipse.

Ex. 1. Trace the curve whose equation is $y^2 = x^2 - 16$.

Ex. 2. Trace the curve whose equation is $y^2 = x^2 + 16$.

Ex. 3. Trace the curve whose equation is $y^2 = 10x + x^2$.

183. *To find the equation to the tangent at any point of an hyperbola.*

Let the equation to the hyperbola be $a^2y^2 - b^2x^2 = -a^2b^2$.

Let x', y' be the co-ordinates of the point on the curve at which the tangent is drawn, and x'', y'' the co-ordinates of an adjacent point on the curve. The equation to the secant line passing through the points x', y' and x'', y'' (Art. 40) is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'). \qquad (1)$$

But, since the points x', y' and x'', y'' are both on the hyperbola, we must have

$$a^2y'^2 - b^2x'^2 = -a^2b^2,$$

and

$$a^2y''^2 - b^2x''^2 = -a^2b^2;$$

therefore, by subtraction,

$$a^2(y''^2 - y'^2) - b^2(x''^2 - x'^2) = 0,$$

or

$$\frac{y'' - y'}{x'' - x'} = \frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}.$$

Substituting this value in equation (1), the equation of the secant line becomes

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}(x - x'). \qquad (2)$$

The secant will become a tangent when the two points coincide, in which case, $x' = x''$, and $y' = y''$.

Equation (2) will then become

$$y - y' = \frac{b^2x'}{a^2y'}(x - x'), \qquad (3)$$

which is the equation to a tangent at the point x', y'.

Clearing this equation of fractions, and transposing, we obtain

$$a^2yy' - b^2xx' = a^2y'^2 - b^2x'^2;$$

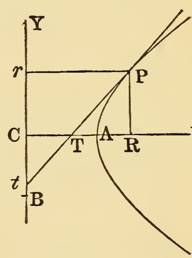
hence

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (4)$$

which is the simplest form of the equation to the tangent line.

In equation (3), $\frac{b^2x'}{a^2y'}$ represents the trigonometrical tangent of the angle which the tangent line makes with the transverse axis of the hyperbola.

184. *Points where the tangent cuts the axes.* To determine the point in which the tangent intersects the axis of X, we make $y=0$, which gives



$$b^2xx' = a^2b^2;$$

that is,

$$x = \frac{a^2}{x'},$$

which is equal to CT. Therefore

$$\text{CT} \times \text{CR} = \text{CA}^2.$$

If from CR or x' we subtract CT, we shall have the subtangent

$$\text{RT} = x' - \frac{a^2}{x'} = \frac{x'^2 - a^2}{x'}.$$

To determine the point in which the tangent intersects the axis of Y, we make $x=0$, which gives

$$y = -\frac{b^2}{y'},$$

which is equal to C*t*. Therefore $Ct \times Cr = \text{CB}^2$.

Hence it follows that *if a tangent and ordinate be drawn from the same point of an hyperbola meeting either axis produced, half of that axis will be a mean proportional between the distances of the two intersections from the centre.*

Ex. 1. In an hyperbola whose transverse axis is 32 inches, the abscissa of a certain point is 26 inches, and the ordinate 18 inches, the origin being at the centre. Determine where the tangent passing through this point meets the two axes produced.

Ex. 2. Find the angle which the tangent line in the preceding example makes with the axis of X.

185. *To find the equation to the normal at any point of an hyperbola.* The equation to a straight line passing through the point P, whose co-ordinates are x', y' (Art. 38), is

$$y-y'=m(x-x'); \qquad (1)$$

and since the normal is perpendicular to the tangent, we shall have (Art. 46)

$$m=\frac{1}{-m'}.$$

But we have found for the tangent line (Art. 183)

$$m'=\frac{b^2x'}{a^2y'};$$

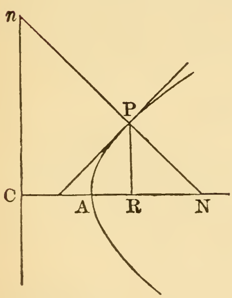
hence $$m=-\frac{a^2y'}{b^2x'}.$$

Substituting this value in equation (1), we shall have for the equation of the normal line

$$y-y'=-\frac{a^2y'}{b^2x'}(x-x'), \qquad (2)$$

where x and y are the general co-ordinates of the normal line, and x', y' the co-ordinates of the point of intersection with the hyperbola.

186. *Points of intersection with the axes.* To find the point in which the normal cuts the transverse axis, make $y=0$ in equation (2), and we have, after reduction,



$$\text{CN}=x=\frac{a^2+b^2}{a^2}\cdot x'.$$

If from CN we subtract CR, which is represented by x', we shall have the subnormal

$$\text{RN}=\frac{a^2+b^2}{a^2}\cdot x'-x'=\frac{b^2x'}{a^2}.$$

To find the point in which the normal cuts the axis of Y, make $x=0$ in equation (2), and we have, after reduction,

$$y=\frac{a^2+b^2}{b^2}\cdot y',$$

which equals Cn.

187. *Distance from the focus to the foot of the normal.* If we put e^2 for $\frac{a^2+b^2}{a^2}$ (Art. 177), we shall have

$$CN=e^2x'.$$

If to this we add F′C (see next figure), which equals c or ae, Art. 177, we have

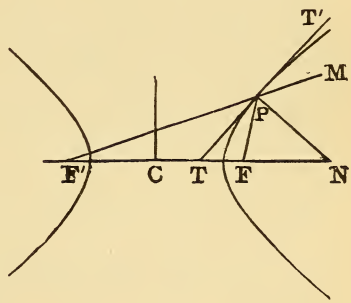
$$F'N=ae+e^2x'=e(a+ex'),$$

which is the distance from the focus to the foot of the normal.

Ex. In an hyperbola whose transverse axis is 32 inches, the abscissa of a certain point is 26 inches, and the ordinate 18 inches, the origin being at the centre. Determine where the normal line passing through this point meets the two axes.

188. *A tangent to the hyperbola bisects the angle contained by lines drawn from the point of contact to the foci.*

Let PT be a tangent line to the hyperbola, and PF, PF′ two lines drawn from the point of contact to the foci; then the angle FPT=F′PT.



For $CT=\frac{a^2}{x}$ (Art. 184),

and $CF=ae$ (Art. 177);

hence $FT=ae-\frac{a^2}{x}=\frac{a}{x}(ex-a)$,

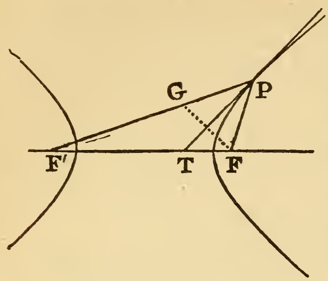
and $F'T=ae+\frac{a^2}{x}=\frac{a}{x}(ex+a)$.

Therefore $FT:F'T::ex-a:ex+a$,

$::PF:PF'$ (Art. 178).

Hence PT bisects the angle FPF′ (Geom., Bk. IV., Prop. 17).

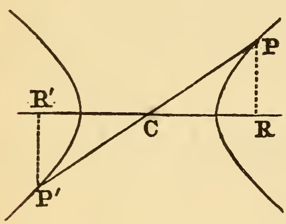
189. If F′P be produced to M, and the normal PN be drawn, it will bisect the exterior angle FPM. For, since PN is perpendicular to TT′, and the angle FPT is equal to F′PT or its vertical angle MPT′, therefore the angle FPN=MPN; or *the normal bisects the angle included by one radius vector and the other produced.*



190. *To draw a tangent to the hyperbola through a given point of the curve.* Let P be the given point; draw the radii vectores PF, PF′; on PF′ take PG equal to PF, and draw FG. Draw PT perpendicular to FG, and it will be the tangent required, for it bisects the angle FPF′.

191. *Every diameter of an hyperbola is bisected at the centre.* Let PP′ be a straight line drawn through the centre of the hyperbola, and terminated on both sides by the two branches of the curve; it will be divided into two equal parts at the point C. Let x', y' be the co-ordinates of the point P, and x'', y'' those of the point P′.

Since the points P and P′ are on the curve, we shall have



(Art. 170)

$$y'^2=\frac{b^2}{a^2}(x'^2-a^2),$$

and

$$y''^2=\frac{b^2}{a^2}(x''^2-a^2);$$

whence, by division,

$$\frac{y'^2}{y''^2}=\frac{x'^2-a^2}{x''^2-a^2}.$$

But, since the right-angled triangles CPR, CP′R′ are similar, we have

$$\frac{y'}{y''}=\frac{x'}{x''};$$

hence,

$$\frac{x'^2}{x''^2}=\frac{x'^2-a^2}{x''^2-a^2}.$$

Clearing of fractions, we obtain

$$x'^2=x''^2;$$

whence also we have $\quad y'^2=y''^2.$

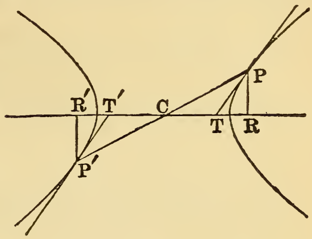
Consequently, $\quad x'^2+y'^2=x''^2+y''^2,$

or $\quad CP^2=CP'^2;$

that is, $\quad CP=CP';$

that is, PP′ is bisected in C.

192. *Tangents to an hyperbola at the extremities of a diameter are parallel to each other.*

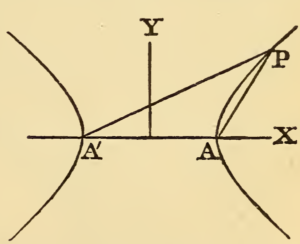


Let PP′ be a diameter of an hyperbola, and let PT, P′T′ be tangents drawn through its extremities; then is PT parallel to P′T′.

In Art. 184 we found $CT=\frac{a^2}{x'}$, and for the same reason $CT'=\frac{a^2}{x''}$, where x' represents CR, the abscissa of the point P, and x'' represents CR′, the abscissa of the point P′. But in Art. 191 we have found that $x'=x''$; hence $CT=CT'$. The two triangles CPT, CP′T′ have therefore two sides and the included angle of the one equal to two sides and the included angle of the other; hence the angle CPT = the angle CP′T′, and PT is parallel to P′T′.

193. *If from the extremities of the transverse axis two lines be drawn to meet on the curve, the product of the tangents of the angles which they form with that axis on the same side is equal to* $\frac{b^2}{a^2}$.

Let PA, PA′ be two lines drawn from the extremities of the transverse axis to the same point P on the hyperbola. The equation of the line PA passing through the point A, whose co-ordinates are $x'=a$, $y'=0$, Art. 38, is



$$y=m(x-a).$$

The equation of PA′ passing through the point A′, whose co-ordinates are $x''=-a$, $y''=0$, is

$$y=m'(x+a).$$

At the point of intersection, P, these equations are simultaneous, and, combining them together, we have

$$y^2=mm'(x^2-a^2). \qquad (1)$$

But, since the point P is on the curve, we must have at the same time

$$y^2=\frac{b^2}{a^2}(x^2-a^2) \text{ (Art. 170).} \qquad (2)$$

Comparing equations (1) and (2), we see that

$$mm'=\frac{b^2}{a^2},$$

where m denotes the tangent of the angle PAX, and m' denotes the tangent of the angle PA′X.

194. *Definition.* Two lines drawn from any point on the curve to the extremities of a diameter, are called *supplementary chords.*

195. *Supplementary chords in the equilateral hyperbola.* In the equilateral hyperbola $a=b$, and we have

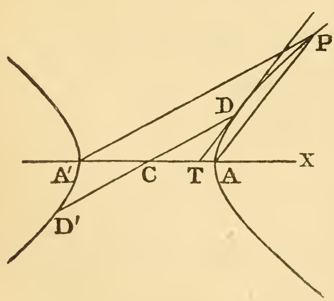
$$mm'=1,$$

or

$$m=\frac{1}{m'},$$

which shows that the angles formed by the supplementary chords with the transverse axis on the same side are complementary to each other (Trig., Art. 28).

196. *If through one extremity of the transverse axis, a chord be drawn parallel to a tangent line to the curve, the supplementary chord will be parallel to the diameter drawn through the point of contact, and conversely.*



Let DT be a tangent to the hyperbola at the point D, and let the chord AP be drawn parallel to it; then will the supplementary chord A′P be parallel to the diameter DD′, which passes through the point of contact, D.

Let x', y' denote the co-ordinates of D. The equation of the line CD (Art. 31) gives $y'=m'x'$;

whence $$m' = \frac{y'}{x'}.$$

But the tangent of the angle which the tangent line makes with the transverse axis (Art. 183) is

$$m = \frac{b^2x'}{a^2y'}.$$

Multiplying together the values of m and m', we obtain

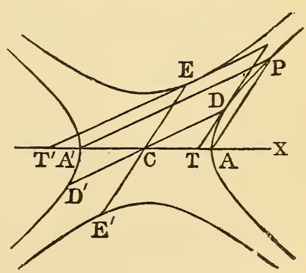
$$mm' = \frac{b^2}{a^2},$$

which represents the product of the tangents of the angles which the lines CD and DT make with the transverse axis.

But by Art. 193 the product of the tangents of the angles PAX, PA′X is also equal to $\frac{b^2}{a^2}$. Hence, if AP be parallel to DT, A′P will be parallel to CD, and conversely.

197. The last Proposition is also true when applied to a tangent to the conjugate hyperbola; that is, *if through one extremity of the transverse axis of an hyperbola a chord be drawn parallel to a tangent line to the conjugate hyperbola, the supplementary chord will be parallel to the diameter drawn through the point of contact, and conversely.*

Let ET′ be a tangent to the hyperbola which is conjugate to the former hyperbola, and let the chord A′P be drawn parallel to ET′, and through the point of contact, E, let the diameter EE′ be drawn; then will EE′ be parallel to the supplementary chord AP.



Let x'', y'' denote the co-ordinates of E. The equation of the line CE, Art. 31, gives $y'' = m''x''$;

whence $$m'' = \frac{y''}{x''}.$$

The equation of the conjugate hyperbola (Art. 179) is

$$a^2y^2 - b^2x^2 = a^2b^2;$$

and, proceeding as in Art. 183, we shall find that the tangent of the angle ET′C is

$$m=\frac{b^2x''}{a^2y''}.$$

Hence we have $$mm''=\frac{b^2}{a^2},$$

which represents the product of the tangents of the angles ECA and ET′A. But this has been found (Art. 196) equal to the product of the tangents of the angles DCX and DTX, or PAX. Hence, if A′P be parallel to ET′, AP will be parallel to CE, and conversely.

198. *Conjugate diameters.* Each of the diameters DD′, EE′ is thus seen to be parallel to a tangent line drawn through the vertex of the other diameter. Two diameters thus related are said to be *conjugate* to each other. Thus we see that *the product of the tangents of the angles which conjugate diameters form with the transverse axis is equal to* $\frac{b^2}{a^2}$.

199. *Of any two conjugate diameters, one meets the original hyperbola, and the other the conjugate hyperbola.*

Let $y=mx$ be the equation to any diameter, and let

$$a^2y^2-b^2x^2=-a^2b^2$$

be the equation to the hyperbola.

To determine the points in which the diameter intersects the curve, we must combine these two equations, and we have

$$(a^2m^2-b^2)x^2=-a^2b^2,$$

or $$x^2=\frac{a^2b^2}{b^2-a^2m^2}. \qquad (1)$$

In like manner, for the conjugate hyperbola we shall find

$$x^2=\frac{a^2b^2}{a^2m^2-b^2}. \qquad (2)$$

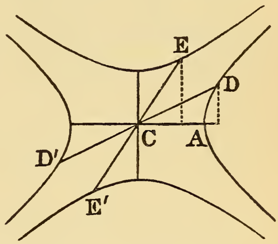
The values of x in equation (1) will be real as long as a^2m^2 is less than b^2, but imaginary when a^2m^2 is greater than b^2. In the former case the diameter intersects the curve, but in the latter it does not. The values of x in equation (2) are real when b^2 is less than a^2m^2, but imaginary when b^2 is greater than a^2m^2.

Now, in the case of conjugate diameters, we have

$$mm' = \frac{b^2}{a^2}, \text{ or } m^2m'^2 = \frac{b^4}{a^4}.$$

Hence, if m^2 be less than $\frac{b^2}{a^2}$, m'^2 will be greater than $\frac{b^2}{a^2}$; in this case the first diameter meets the original hyperbola, and the second the conjugate hyperbola. If m^2 be greater than $\frac{b^2}{a^2}$, m'^2 will be less than $\frac{b^2}{a^2}$; in this case the first diameter meets the conjugate hyperbola, and the second the original hyperbola.

200. *Having given the co-ordinates of one extremity of a diameter, to find those of either extremity of the diameter conjugate to it.*



Let AA′, BB′ be the axes of an hyperbola, DD′, EE′ a pair of conjugate diameters. Let x', y' be the co-ordinates of D; then the equation to CD (Art. 40) is

$$y = \frac{y'}{x'} \cdot x. \qquad (1)$$

Since the conjugate diameter EE′ is parallel to the tangent at D, the equation to EE′ (Art. 183) is

$$y = \frac{b^2x'}{a^2y'} \cdot x. \qquad (2)$$

To determine the co-ordinates of E and E′, we must combine the equation to EE′ with the equation to the conjugate hyperbola $a^2y^2 - b^2x^2 = a^2b^2$ (Art. 179).

Substituting the value of y from equation (2), we have, after reduction,

$$(b^2x'^2 - a^2y'^2)x^2 = a^4y'^2;$$

whence

$$x^2 = \frac{a^4y'^2}{a^2b^2} = \frac{a^2y'^2}{b^2},$$

or

$$x = \pm\frac{ay'}{b}.$$

Also, from equation (2) we have

$$y = \pm \frac{bx'}{a},$$

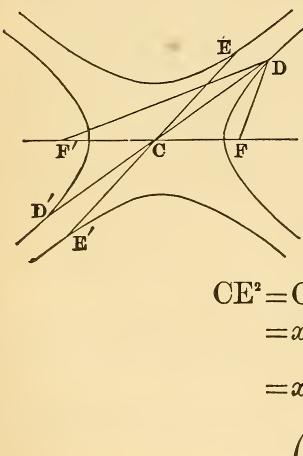
which are the co-ordinates of the points E and E′. The abscissa of E is positive, and that of E′ negative; hence the upper sign applies to E, and the lower to E′.

201. *The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.*

Let x', y' be the co-ordinates of D; then, by Art. 200,

$$\begin{aligned} CD^2 - CE^2 &= x'^2 + y'^2 - \frac{a^2y'^2}{b^2} - \frac{b^2x'^2}{a^2} \\ &= \frac{b^2x'^2 - a^2y'^2}{b^2} + \frac{a^2y'^2 - b^2x'^2}{a^2} \\ &= a^2 - b^2. \end{aligned}$$

202. *The rectangle contained by the focal distances of any point on the hyperbola is equal to the square of half the corresponding conjugate diameter.*



Let DD′, EE′ be a pair of conjugate diameters, and from D draw lines to the foci F and F′; then

$$DF \times DF' = CE^2.$$

Represent the co-ordinates of D by x', y'.

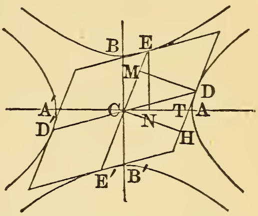
Then, since $CD^2 - CE^2 = a^2 - b^2$ (Art. 201), we have

$$\begin{aligned} CE^2 &= CD^2 - a^2 + b^2 \\ &= x'^2 + y'^2 - a^2 + b^2 \\ &= x'^2 + \frac{b^2}{a^2}(x'^2 - a^2) - a^2 + b^2 \\ &= \left(1 + \frac{b^2}{a^2}\right)x'^2 - a^2 \\ &= e^2x'^2 - a^2 \text{ (Art. 177)} \\ &= DF \times DF' \text{ (Art. 178)}; \end{aligned}$$

that is, the product of the focal distances DF, DF′ is equal to the square of half EE′, which is the diameter conjugate to that which passes through the point D.

203. *The parallelogram formed by drawing tangents through the vertices of two conjugate diameters is equal to the rectangle of the axes.*

Let DD′, EE′ be two conjugate diameters, and let DED′E′ be a parallelogram formed by drawing tangents to the hyperbola through the extremities of these diameters; the area of the parallelogram is equal to AA′ × BB′.



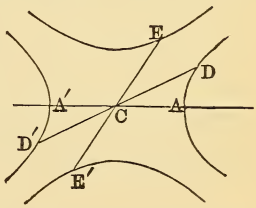
Draw DM perpendicular to EE′, and let the co-ordinates of D be x', y'.

The area of the parallelogram DED′E′ is equal to 4CE . DM, which is equal to 4CE . CT sin. CTH, which is equal to 4CT . EN, because EC and DT are parallel.

But $CT = \frac{a^2}{x'}$ (Art. 184), and $EN = \frac{bx'}{a}$ (Art. 200). Hence the parallelogram $DED'E' = 4\frac{a^2}{x'} \cdot \frac{bx'}{a} = 4ab = AA' \times BB'$.

204. *Equation to the hyperbola referred to any two conjugate diameters as axes.*

Let CD, CE be two conjugate semi-diameters; take CD as the new axis of x, CE as that of y; let $DCA = \alpha$, and $ECA = \beta$. Let x, y be the co-ordinates of any point of the hyperbola referred to the original axes, and x', y' the cordinates of the same point referred to the new axes.



The equation of the hyperbola referred to its centre and axes (Art. 170) is $a^2y^2 - b^2x^2 = -a^2b^2$.

In order to pass from rectangular to oblique co-ordinates, the origin remaining the same, we must substitute for x and y in the equation of the curve (Art. 56) the values

$$x = x' \cos. \alpha + y' \cos. \beta,$$
$$y = x' \sin. \alpha + y' \sin. \beta.$$

Squaring these values of x and y, and substituting in the equation of the hyperbola, we have

$$x'^2(a^2 \sin.^2\alpha - b^2 \cos.^2\alpha) + y'^2(a^2 \sin.^2\beta - b^2 \cos.^2\beta)$$
$$+ 2x'y'(a^2 \sin.\alpha \sin.\beta - b^2 \cos.\alpha \cos.\beta) = -a^2b^2,$$

which is the equation of the hyperbola when the oblique coordinates make any angles α, β with the transverse axis.

But, since CD, CE are conjugate semidiameters, we must have (Art. 198)

$$mm' = \text{tang.}\,\alpha \text{ tang.}\,\beta = \frac{b^2}{a^2},$$

whence $$a^2 \text{ tang.}\,\alpha \text{ tang.}\,\beta - b^2 = 0.$$

Multiplying by $\cos.\alpha \cos.\beta$, remembering that $\cos.\alpha \text{ tang.}\,\alpha = \sin.\alpha$, we have

$$a^2 \sin.\alpha \sin.\beta - b^2 \cos.\alpha \cos.\beta = 0.$$

Hence the term containing $x'y'$ vanishes, and the equation becomes

$$x'^2(a^2 \sin.^2\alpha - b^2 \cos.^2\alpha) + y'^2(a^2 \sin.^2\beta - b^2 \cos.^2\beta) = -a^2b^2,$$

which is the equation of the hyperbola referred to conjugate diameters.

If in this equation we suppose $y'=0$, we shall have

$$x'^2 = \frac{a^2b^2}{b^2 \cos.^2\alpha - a^2 \sin.^2\alpha}.$$

This is the value of CD^2, which we shall denote by a'^2.

If we suppose $x'=0$, we shall have

$$y'^2 = \frac{-a^2b^2}{a^2 \sin.^2\beta - b^2 \cos.^2\beta}.$$

Now, since we have supposed that the new axis of x meets the curve, we know that the new axis of y will *not* meet the curve (Art. 199), so that

$$\frac{-a^2b^2}{a^2 \sin.^2\beta - b^2 \cos.^2\beta}$$

is not a *positive* quantity. If we denote it by $-b'^2$, the equation to the hyperbola referred to conjugate diameters will be

$$b'^2x'^2 - a'^2y'^2 = a'^2b'^2;$$

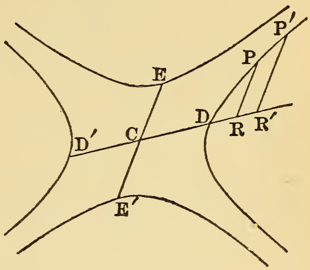
or, suppressing the accents on the variables,

$$b'^2x^2 - a'^2y^2 = a'^2b'^2,$$

or $$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1.$$

205. *The square of any diameter of an hyperbola is to the square of its conjugate, as the rectangle under any two segments of the diameter is to the square of the corresponding ordinate.*

Let DD′, EE′ be two conjugate diameters of an hyperbola, and from any point of the curve, as P, let PR be drawn parallel to EC, meeting the diameter DD′ produced in R.



The equation of the hyperbola referred to conjugate diameters may be put under the form

$$a'^2y^2 = b'^2(x^2 - a'^2).$$

This equation may be reduced to the proportion

$$a'^2 : b'^2 :: x^2 - a'^2 : y^2,$$

or $$(2a')^2 : (2b')^2 :: (x+a')(x-a') : y^2.$$

Now $2a'$ and $2b'$ represent the conjugate diameters DD′, EE′; and since x represents CR, $x+a'$ will represent D′R, and $x-a'$ will represent DR; also PR represents y. Hence

$$DD'^2 : EE'^2 :: DR \times RD' : PR^2.$$

If we draw a second ordinate P′R′ to the diameter DD′, we shall have $PR^2 : DR \times RD' :: b'^2 : a'^2 :: P'R'^2 : DR' \times R'D'$,

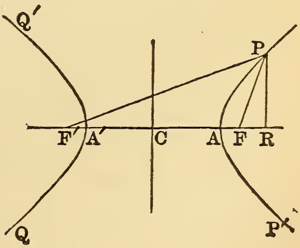
or $$PR^2 : P'R'^2 :: DR \times RD' : DR' \times R'D';$$

that is, *the squares of any two ordinates to the same diameter are proportional to the rectangles under the corresponding segments of the diameter.*

206. *To find the polar equation to the hyperbola, the pole being at one of the foci.*

1. Let F be the pole.

Let $FP = r$; the angle $AFP = \theta$; then $FR = r \cos. PFR = -r \cos. \theta$.



By Art. 178,

$$r = ex - a;$$

but $$x = CR = CF + FR$$
$$= ae - r \cos. \theta.$$

Therefore $r = ae^2 - er \cos.\theta - a.$

Hence $r(1 + e \cos.\theta) = a(e^2 - 1),$

or $$r = \frac{a(e^2-1)}{1+e\cos.\theta}, \qquad (1)$$

which is the equation required.

2. Let F′ be the pole.

Let F′P$=r'$; angle PF′A$=\theta'$; then F′R$=r' \cos.\theta'$.

By Art. 178, $r' = ex + a;$

but $x = \text{CR} = \text{F'R} - \text{F'C}$
$= r' \cos.\theta' - ae.$

Therefore $r' = er' \cos.\theta' - ae^2 + a.$

Hence $r'(1 - e\cos.\theta') = a(1 - e^2) = -a(e^2 - 1),$

or $$r' = \frac{-a(e^2-1)}{1-e\cos.\theta'}, \qquad (2)$$

which is the equation required.

207. *Form of the hyperbola traced.* The form of the hyperbola may be traced from its polar equation. In equation (1), suppose $\theta = 0$; then $r = a(e-1)$. If we measure off this length on the initial line from the pole F, we shall obtain the point A as one of the points of the curve.

While θ increases, $1 + e \cos.\theta$ diminishes, and r increases; and when $\theta = 90°$, $r = \frac{b^2}{a}$, which determines another point of the curve.

When θ becomes greater than 90°, $\cos.\theta$ becomes negative, and r continues to increase until $1 + e\cos.\theta = 0$, or $\cos.\theta = -\frac{1}{e}$, when r becomes infinite. Thus, while θ increases from 0 until $\cos.\theta = -\frac{1}{e}$, that portion of the curve is traced out which begins at A, and passes on through P to an indefinite distance from the origin.

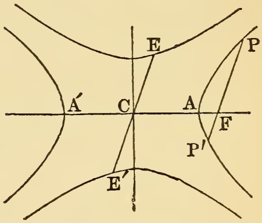
When $1 + e\cos.\theta$ becomes negative, r becomes negative, and we measure it in the direction opposite to that in which we should measure it, if it were positive. Thus, while θ increases to 180°, that portion of the curve is traced out which begins at an indefinite distance from C in the lower left-hand quadrant, and passes through Q to A′.

As θ increases from 180°, r continues negative, and increases numerically until $1+e\cos.\theta$ again becomes zero. Thus the branch of the curve is traced out which begins at A′, and passes on through Q′ to an indefinitely great distance from C.

As θ continues to increase, the value of $1+e\cos.\theta$ again becomes positive; r is again positive, and is at first indefinitely great, and then diminishes. Thus the portion of the curve is traced out which begins at an indefinitely great distance from C in the lower right-hand quadrant, and passes on through P′ to A. Thus both branches of the hyperbola are traced out by one complete revolution of the radius vector.

208. *Any chord which passes through the focus of an hyperbola is a third proportional to the transverse axis and the diameter parallel to that chord.*

Let PP′ be a chord of an hyperbola passing through the focus F, and let EE′ be a diameter parallel to PP′.



By Art. 206, $PF=\dfrac{a(e^2-1)}{1+e\cos.\theta}$.

To find the value of FP′, we must substitute for θ, $180°+\theta$, and we obtain

$$FP'=\frac{a(e^2-1)}{1-e\cos.\theta}.$$

Hence
$$PP'=\frac{2a(e^2-1)}{1-e^2\cos.^2\theta}.$$

Proceeding as in Art. 161, we find the value of CE^2 equal to

$$\frac{a^2(e^2-1)}{1-e^2\cos.^2\theta}.$$

Hence
$$PP'=\frac{2CE^2}{a}=\frac{4CE^2}{2a};$$

that is,
$$AA':EE'::EE':PP',$$

or PP′ is a third proportional to AA′ and EE′.

209. *Definition.* The parameter of any diameter is a third proportional to that diameter and its conjugate.

The parameter of the transverse axis is called the principal parameter, or *latus rectum*, and its value is $\frac{2b^2}{a}$ (Art. 176). The parameter of the conjugate axis is $\frac{2a^2}{b}$. The latus rectum is the double ordinate to the transverse axis passing through the focus (Art. 176). Now, since any focal chord is a third proportional to the transverse axis and the diameter parallel to that chord, and since the transverse axis is less than any other diameter of the same hyperbola, it follows that *the transverse axis is the only diameter whose parameter is equal to the double ordinate passing through the focus.*

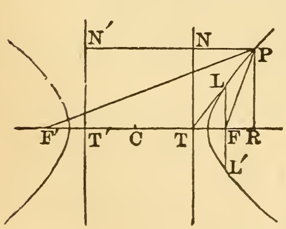
In the equilateral hyperbola $a=b$, and the latus rectum is equal to either of the axes of the curve.

210. *Definition.* The *directrix* of an hyperbola is a straight line perpendicular to the transverse axis, and intersecting it in the same point with the tangent to the curve at one extremity of the latus rectum.

Thus, if LT be a tangent drawn through one extremity of the latus rectum LL′, meeting the transverse axis in T, and NT be drawn through the point of intersection perpendicular to the axis, it will be the directrix of the hyperbola.

The hyperbola has two directrices, one corresponding to the focus F, and the other to the focus F′.

211. *The distance of any point in an hyperbola from either focus is to its distance from the corresponding directrix, as the eccentricity is to unity.*



Let F be one focus of an hyperbola, NT the corresponding directrix; F′ the other focus, and N′T′ the corresponding directrix. Let P be any point on the hyperbola, x', y' its co-ordinates, the origin being at the centre.

Join PF, PF′, and draw PNN′ parallel to the transverse axis, and PR perpendicular to it.

By Art. 184, $CT = \frac{a^2}{c} = \frac{a}{e}$;

hence $CR - CT = PN = x' - \frac{a}{e}$

$= \frac{ex' - a}{e}$.

But, by Art. 178, $r = ex' - a = PF$;

hence $e \, . \, PN = PF$,

or $PF : PN :: e : 1$.

In like manner we find that

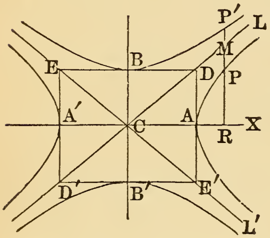
$PF' : PN' :: e : 1$.

212. *Conic sections compared.* In Art. 82 the parabola was defined to be a curve every point of which is equally distant from the focus and directrix, while in the ellipse and hyperbola these distances have been found to be in the ratio of the eccentricity to unity. In the ellipse, the eccentricity, being equal to $\frac{c}{a}$ (Art. 127), is *less* than unity, while in the hyperbola (Art. 177) it is *greater* than unity. In each of these curves the two distances have to each other a constant ratio. In the parabola this ratio is unity, in the ellipse it is less than unity, while in the hyperbola it is greater than unity. These curves, being the sections of a cone made by a plane in different positions, are called the *conic sections;* so that a conic section may be defined to be a curve traced out by a point which moves in such a manner that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line. If this ratio be *unity*, the curve is called a parabola; if *less* than unity, an ellipse; and if *greater* than unity, an hyperbola; and all the properties of these curves may be deduced from this definition.

ON THE ASYMPTOTES OF THE HYPERBOLA.

213. It was shown in Art. 199 that if a line drawn through the centre of an hyperbola meets the curve, m^2 must be less than $\frac{b^2}{a^2}$, or $m < \pm\frac{b}{a}$; and if the line meets the conjugate hyperbola, m^2 must be greater than $\frac{b^2}{a^2}$, or $m > \pm\frac{b}{a}$.

Let AA′, BB′ be the two axes of an hyperbola, and through the vertices A, A′, B, B′ let lines be drawn perpendicular to these axes; and let DD′, EE′, the diagonals of the rectangle thus formed, be indefinitely produced.



Then, since

$$\text{tang. DCX} = \frac{\text{DA}}{\text{AC}} = \frac{b}{a},$$

and

$$\text{tang. E}'\text{CX} = \frac{\text{E}'\text{A}}{\text{AC}} = -\frac{b}{a},$$

it follows that the lines CD, CE′ will never meet the curve at any finite distance from C.

The lines CD, CE′, indefinitely produced, are called *asymptotes* of the hyperbola.

214. *Definition.* An asymptote of any curve is a line which continually approaches the curve, coming indefinitely near to it, but meets it only at an infinite distance from the origin.

Since the lines DD′ and EE′ pass through the centre, and are inclined to the transverse axis at an angle whose tangent $= \pm\frac{b}{a}$, their equation will be

$$y = \pm\frac{b}{a}x.$$

215. *The diagonals of the rectangle formed by lines drawn through the extremities of the axes and perpendicular to the axes, are asymptotes to the curve, according to the definition of Art.* 214.

Let the equation to the hyperbola (Art. 170) be

$$y^2=\frac{b^2}{a^2}(x^2-a^2).$$

The equation to the line CL, the diagonal of the rectangle DED′E′, is

$$y=\frac{bx}{a}.$$

Let MPR be an ordinate meeting the hyperbola in P, and the straight line CL in M; then, if CR be denoted by x, we have

$$PR=\frac{b}{a}\sqrt{x^2-a^2},$$

and $$MR=\frac{bx}{a}.$$

Hence $$MP=\frac{b}{a}(x-\sqrt{x^2-a^2})$$

$$=\frac{b}{a}\cdot\frac{a^2}{x+\sqrt{x^2-a^2}}$$

$$=\frac{ab}{x+\sqrt{x^2-a^2}}.$$

If, then, the line MR be supposed to move from A parallel to itself, the value of x will continually increase, and the distance MP will continually diminish; and if we suppose the point P of the curve to recede to an infinite distance from the origin, MP will become zero.

In like manner the line CL′, whose equation is $y=-\frac{bx}{a}$, meets the curve below the transverse axis at an infinite distance from the origin.

216. *Asymptote to the conjugate hyperbola.* The line CL is also an asymptote to the conjugate hyperbola; for, let PR be produced to meet the conjugate hyperbola in P′; then (Art. 179)

$$P'R=\frac{b}{a}\sqrt{x^2+a^2}.$$

Hence $$P'M=\frac{b}{a}(\sqrt{x^2+a^2}-x)$$

$$=\frac{ab}{\sqrt{x^2+a^2}+x}.$$

Therefore, if CR or x be indefinitely increased, P′M will be indefinitely diminished, and hence CL is an asymptote to the conjugate hyperbola.

217. *An asymptote may be considered as a tangent to the hyperbola at a point infinitely distant from the centre.*

The equation to a tangent at any point x', y' of the curve (Art. 183) is
$$a^2yy'-b^2xx'=-a^2b^2,$$
or
$$y=\frac{b^2xx'}{a^2y'}-\frac{b^2}{y'}. \qquad (1)$$

Now
$$y'=\pm\frac{b}{a}\sqrt{x'^2-a^2}.$$

If x' becomes indefinitely great, then a^2 vanishes when compared with x'^2, and we have
$$y'=\pm\frac{b}{a}x'.$$

Substituting this value in equation (1), the equation to the tangent, when the point x', y' is infinitely distant, becomes
$$y=\pm\frac{b^2xx'}{a^2}\times\frac{a}{bx'}\pm\frac{ab^2}{bx'}$$
$$=\pm\frac{bx}{a}\pm\frac{ab}{x'}.$$

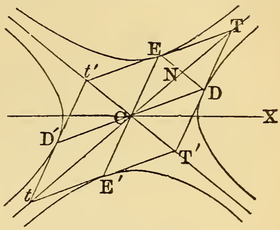
But when x' is infinite,
$$\frac{ab}{x'}=0;$$
hence
$$y=\pm\frac{bx}{a},$$
which is the equation to the asymptote (Art. 214). Hence the asymptote is a tangent to the curve at a point infinitely distant from the centre.

218. *The asymptotes are the diagonals of every parallelogram formed by drawing tangents through the vertices of two conjugate diameters.*

Let DED′E′ be a parallelogram formed by drawing tangents to the hyperbola through the vertices of two conjugate diameters DD′, EE′; the diagonals Tt, T′t' will be asymptotes of the curve.

Let x', y' be the co-ordinates of the point D; then the co-ordinates of E, the extremity of the conjugate diameter (Art. 200), are

$$\frac{ay'}{b} \text{ and } \frac{bx'}{a}.$$



Draw the diagonal DE, and it will bisect CT in N (Geom., Bk. I., Prop. 33). The co-ordinates of N are

$$\tfrac{1}{2}\left(x' + \frac{ay'}{b}\right) \text{ and } \tfrac{1}{2}\left(y' + \frac{bx'}{a}\right).$$

Hence we have

$$\text{tang. NCX} = \frac{y' + \frac{bx'}{a}}{x' + \frac{ay'}{b}}, \text{ which equals } \frac{b}{a}.$$

But $\frac{b}{a}$ is the tangent of the angle which the asymptote makes with the transverse axis (Art. 214); hence CT coincides with one of the asymptotes.

Also, since the diagonal DE passes through the points

$$x', y', \text{ and } \frac{ay'}{b}, \frac{bx'}{a},$$

the tangent of the angle which it makes with the transverse axis (Art. 40) is

$$\frac{y' - \frac{bx'}{a}}{x' - \frac{ay'}{b}}, \text{ which equals } -\frac{b}{a}.$$

But $-\frac{b}{a}$ is the tangent of the angle which the other asymptote makes with the transverse axis; hence DE is parallel to the other asymptote. And since DT′E′C is a parallelogram, DT′ = E′C, which equals EC; and since DT′ is parallel to EC, ED is parallel to CT′. Hence T′t' is the other asymptote.

219. Hence we see that *the line joining the extremities of two conjugate diameters is parallel to one asymptote, and is bisected by the other.*

Also, *if a tangent line be drawn at any point of an hyperbola, the part included between the asymptotes is equal to the parallel diameter.*

Moreover, if x and y are the co-ordinates of any point on the asymptote referred to two conjugate diameters, then we shall have

$$y : x :: b' : a',$$

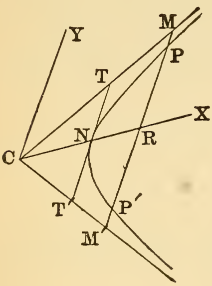
or

$$y = \frac{b'x}{a'},$$

which is therefore the equation to the asymptote referred to a pair of conjugate diameters.

220. *If any chord of the hyperbola be produced to meet the asymptotes, the parts included between the curve and the asymptotes will be equal.*

Let PP′ be any chord of the hyperbola, and let it be produced to meet the asymptotes in M and M′; then will PM be equal to P′M′.



Draw CY, the semidiameter to the conjugate hyperbola, parallel to PP′, and draw CX conjugate to CY; then PP′ is a double ordinate to CX, and is bisected in R.

The equation to the hyperbola referred to CX, CY (Art. 204) is

$$y = \pm\frac{b'}{a'}\sqrt{x^2 - a'^2}, \qquad (1)$$

and the equation to the asymptotes (Art. 219) is

$$y = \pm\frac{b'}{a'}x. \qquad (2)$$

Now to the same abscissa CR there correspond (from eq. 1) two equal ordinates with opposite signs; hence we have

$$PR = P'R.$$

Also, from eq. 2, $\qquad MR = M'R.$

Therefore, by subtraction, $MP = M'P'$,

as was to be proved.

If the tangent line TT′ be drawn parallel to MM′, the triangles CTT′, CMM′ will be similar; and since MR is equal to

M′R, NT will be equal to NT′; that is, *the portion of a tangent included between the asymptotes is bisected at the point of contact.*

221. *If a straight line be drawn through any point on an hyperbola, the rectangle of the parts intercepted between that point and the asymptotes, will be equal to the square of the parallel semidiameter.*

Let a straight line drawn through the point P on the hyperbola meet the asymptotes in M and M′; then we have

$$\text{PM}\,.\,\text{PM}'=(\text{MR}-\text{PR})(\text{MR}+\text{PR})$$
$$=\text{MR}^2-\text{PR}^2.$$

But $\qquad \text{MR}^2=\frac{b'^2}{a'^2}x^2$ (Art. 219),

and $\qquad \text{PR}^2=\frac{b'^2}{a'^2}(x^2-a'^2)$ (Art. 204);

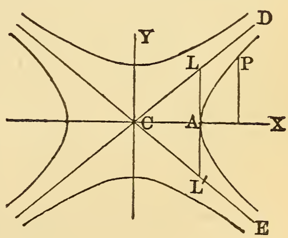
hence $\qquad \text{MR}^2-\text{PR}^2=\frac{b'^2}{a'^2}(x^2-x^2+a'^2)=b'^2;$

that is, $\qquad \text{PM}\,.\,\text{PM}'=b'^2,$

or, the rectangle of the parts PM and PM′ is equal to the square of the parallel semidiameter.

222. *To find the equation to the hyperbola referred to the asymptotes as axes.*

Let CX, CY be the original axes coinciding with the axes of the hyperbola, and let CD, CE be the new axes, inclined to CX on opposite sides of it at an angle β, such that tang. $\beta=\frac{b}{a}$ (Art. 213). Let x, y be the co-ordinates of a point P referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes.



The formulas for passing from rectangular to oblique co-ordinates, the origin remaining the same (Art. 56), are

$$x=x'\cos.\,a+y'\cos.\,\beta,$$
$$y=x'\sin.\,a+y'\sin.\,\beta.$$

But, since $\alpha=-\beta$, these equations become

$$x=(x'+y')\cos.\beta,$$
$$y=(y'-x')\sin.\beta.$$

Now $$\sin.\beta=\frac{\text{AL}}{\text{CL}}, \text{ and } \cos.\beta=\frac{\text{AC}}{\text{CL}};$$

also, $$\text{CL}^2=\text{CA}^2+\text{AL}^2=a^2+b^2.$$

Represent CL by c; then

$$\sin.\beta=\frac{b}{c}, \text{ and } \cos.\beta=\frac{a}{c}.$$

Therefore $$x=\frac{a(x'+y')}{c}, \text{ and } y=\frac{b(y'-x')}{c}.$$

Substitute these values in the equation to the hyperbola,

$$a^2y^2-b^2x^2=-a^2b^2,$$

and we have

$$a^2b^2(x'-y')^2-a^2b^2(x'+y')^2=-a^2b^2c^2,$$

or $$(x'-y')^2-(x'+y')^2=-c^2;$$

that is, $$4x'y'=c^2;$$

or, suppressing the accents,

$$xy=\frac{c^2}{4}=\frac{a^2+b^2}{4},$$

which is the equation of the hyperbola referred to the asymptotes as axes.

223. *Equation to the conjugate hyperbola.* The equation to the conjugate hyperbola referred to the same axes may be found by writing $-a^2$ for a^2, and $-b^2$ for b^2 (Art. 179). We shall then have

$$xy=-\frac{a^2+b^2}{4}.$$

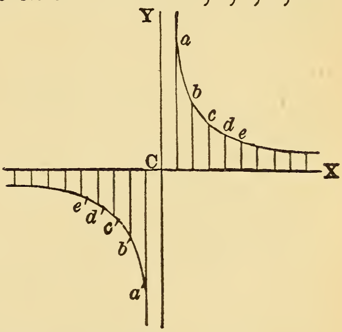
In the case of an equilateral hyperbola, the angle $\text{DCE}=90°$; that is, the asymptotes are perpendicular to each other. For all other hyperbolas the asymptotes make oblique angles with each other.

Ex. 1. Trace the curve whose equation referred to rectangular axes is $xy=10$.

We may assume any value for x, and the corresponding value of y may be found from the equation. Thus, if

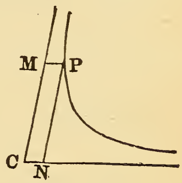
$x=1, y=10.$	$x=7, y=1.43.$
$x=2, y=5.$	$x=8, y=1.25.$
$x=3, y=3.33.$	$x=9, y=1.11.$
$x=4, y=2.5.$	$x=10, y=1.00.$
$x=5, y=2.$	$x=11, y=0.91.$
$x=6, y=1.66.$	$x=12, y=0.83.$

These values determine the points of the curves a, b, c, d, etc. If x is negative, y is also negative, and the points a', b', c', etc., will be determined in the third quadrant. As x increases indefinitely, y decreases, and the curve is unlimited in the direction of x positive, but continually approaches the axis of x without actually reaching it. The same is true for the direction of x negative, and for each direction of the axis of y.



Ex. 2. Trace the curve whose equation referred to oblique axes is $xy=-10$.

224. *Parallelogram on any abscissa and ordinate.* Let P be any point on the hyperbola, from which draw PM, PN parallel to the asymptotes, and represent these co-ordinates by x, y; then, by Art. 222,



$$xy=\frac{a^2+b^2}{4}.$$

If we multiply each member of this equation by sin. 2β, we shall have

$$xy \text{ sin. } 2\beta=\frac{a^2+b^2}{4} \text{ sin. } 2\beta, \qquad (1)$$

where 2β is the angle included by the asymptotes. The first member of this equation represents

$$\text{CM} \times \text{CN} \times \text{sin. MCN}. \qquad (2)$$

But CN $\times$ sin. MCN is the perpendicular from N upon the line

CM; hence expression (2) represents the area of the parallelogram CNPM.

Since $\sin. 2\beta = 2 \sin. \beta \cos. \beta = \frac{2ab}{a^2+b^2}$ (Art. 222),

the second member of eq. 1 reduces to $\frac{ab}{2}$.

Hence *the parallelogram* CNPM *described on the abscissa and ordinate of any point on the curve, is equal to* half the rectangle under the semiaxes, or *one eighth the rectangle under the axes.*

225. *To find the equation to the tangent at any point of an hyperbola when the curve is referred to its asymptotes as axes.*

Let x', y' be the co-ordinates of the point of contact, x'', y'' the co-ordinates of an adjacent point on the curve.

The equation to the secant line passing through these points is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'). \qquad (1)$$

Since the two given points are on the hyperbola, we have (Art. 222)

$$x'y'=\frac{c^2}{4}$$

$$x''y''=\frac{c^2}{4}.$$

Hence $$x'y'=x''y'', \text{ or } y''=\frac{x'y'}{x''}.$$

Therefore $$y''-y'=\frac{x'y'}{x''}-y'$$

$$=-\frac{y'}{x''}(x''-x');$$

whence $$\frac{y''-y'}{x''-x'}=-\frac{y'}{x''}.$$

By substitution, eq. (1) becomes

$$y-y'=-\frac{y'}{x''}(x-x'). \qquad (2)$$

If we suppose $x'=x''$, and $y'=y''$, the secant will become a tangent, and equation (2) will become

$$y-y'=-\frac{y'}{x'}(x-x'),$$

which is the equation to the tangent line.

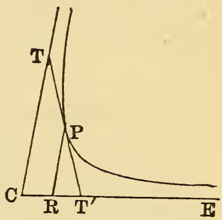
If we clear this equation of fractions, we shall have

$$yx'-x'y'=-xy'+x'y';$$

therefore $$yx'+xy'=2x'y'=\frac{a^2+b^2}{2},$$

which is the simplest form of the equation to the tangent line.

226. *Points of intersection with the axes.* To find where the tangent at x', y' meets the axis of abscissas, put $y=0$ in the equation to the tangent line, and we have



$$xy'=2x'y',$$

or $$x=2x';$$

that is, the abscissa CT′ of the point where the tangent meets the asymptote CE is double the abscissa CR of the point of tangency.

To find where the tangent cuts the axis of Y, put $x=0$ in the equation to the tangent line, and we have

$$yx'=2x'y',$$

or $$y=2y';$$

that is, CT is double of PR.

Also, because PR is parallel to CT, TT′ is double of PT, or the tangent TT′ is bisected in P; that is, *if a tangent line be drawn at any point of an hyperbola, the part intercepted between the asymptotes is bisected at the point of contact.*

SECTION VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

227. We have seen that the equations of the circle, the parabola, ellipse, and hyperbola are all of the second degree; we will now inquire whether any other curve is included in the general equation of the second degree.

The general equation of the second degree between two variables may be written

$$ax^2+bxy+cy^2+dx+ey+f=0, \qquad (1)$$

which contains the first and second powers of each variable, their product, and an absolute term.

We shall suppose the axes to be rectangular; for if they were oblique we might transform the equation to one referred to rectangular axes, and we should obtain an equation of the same degree as the above, and which could not, therefore, be more general than the one we have assumed.

228. *To remove certain terms from the general equation.* We wish, if possible, to cause certain terms of this equation to disappear. For this purpose we may change both the origin and direction of the co-ordinate axes, without assigning any particular values to the quantities which determine the position of the new axes. By this means, indeterminate quantities are introduced into the transformed equation, to which such values can afterwards be assigned as will cause certain of its terms to vanish. Instead of changing both the origin and direction of the co-ordinate axes at once, it is more convenient to effect these changes successively.

229. *The terms containing the first powers of x and y in the equation of the second degree, may in general be made to disappear by changing the origin of the co-ordinates.*

In order to effect this transformation, substitute for x and y in equation (1) the values

$$x = x' + h,$$
$$y = y' + k,$$

by which we pass from one system of axes to another system parallel to the first (Art. 54).

The result of this substitution is

$$ax'^2 + bx'y' + cy'^2 + (2ah + bk + d)x' + (2ck + bh + e)y' + ah^2 + bhk + ck^2 + dh + ek + f = 0.$$

Now, in order that the terms involving the first powers of x' and y' may disappear, we must have

$$2ah + bk + d = 0,$$

and $$2ck + bh + e = 0.$$

From these equations we obtain

$$h = \frac{2cd - be}{b^2 - 4ac}, \text{ and } k = \frac{2ae - bd}{b^2 - 4ac}.$$

These are the values of h and k which render the proposed transformation possible; hence, denoting the constant quantity

$$ah^2 + bhk + ck^2 + dh + ek + f$$

by f', the transformed equation becomes

$$ax'^2 + bx'y' + cy'^2 + f' = 0. \qquad (2)$$

When $b^2 - 4ac = 0$, the above values of h and k become infinitely great, and the proposed transformation is impossible.

If equation (2) is satisfied by any values x_1, y_1 of the variables, it is also satisfied by the values $-x_1, -y_1$. Hence the new origin of co-ordinates is the *centre* of the curve represented by equation (1).

Thus, if $b^2 - 4ac$ be not $= 0$, the curve represented by (1) has a *centre*, and its co-ordinates are h and k, the values of which are given above.

We may suppress the accents on the variables in equation (2), and write it $\quad ax^2 + bxy + cy^2 + f' = 0. \qquad (3)$

230. *The term containing xy in the general equation of the second degree may be taken away by changing the directions of the axes.*

For this purpose put

$$x = x' \cos.\theta - y' \sin.\theta,$$
$$y = x' \sin.\theta + y' \cos.\theta.$$

Substituting these values of x and y in equation (3), and arranging the result, we have

$$x'^2(a \cos.^2\theta + c \sin.^2\theta + b \sin.\theta \cos.\theta)$$
$$+ y'^2(a \sin.^2\theta + c \cos.^2\theta - b \sin.\theta \cos.\theta)$$
$$+ x'y'\{2(c-a) \sin.\theta \cos.\theta + b(\cos.^2\theta - \sin.^2\theta)\} + f' = 0. \quad (4)$$

Now, in order that the term involving $x'y'$ may become zero, we must have

$$2(c-a) \sin.\theta \cos.\theta + b(\cos.^2\theta - \sin.^2\theta) = 0.$$

But by Trig., Art. 73,

$$2 \sin.\theta \cos.\theta = \sin.2\theta; \text{ also } \cos.^2\theta - \sin.^2\theta = \cos.2\theta;$$

hence $$(c-a) \sin.2\theta + b \cos.2\theta = 0,$$

or $$\text{tang.}\,2\theta = \frac{b}{a-c}. \quad (5)$$

Since the tangent of an angle may have any magnitude from zero to infinity, this value of tang. 2θ is always possible, whatever be the values of a, b, and c; hence such a value of θ may always be found as shall remove the term involving $x'y'$ from equation (4), and the general equation is reduced to the form

$$Ax'^2 + By'^2 + f' = 0,$$

or, suppressing the accents on the variables, we have

$$Ax^2 + By^2 + f' = 0. \quad (6)$$

By solving this equation we have

$$x = \pm\sqrt{\frac{-f' - By^2}{A}},$$

from which we see that if A, B, and f' have the same sign, the quantity under the radical is negative, and equation (6) represents an imaginary curve.

If A and B have the same sign, and f' has the contrary sign, the equation represents an ellipse (Art. 121).

If A and B have different signs, the equation represents an hyperbola (Art. 170).

If $A = B$, the equation represents a circle (Art. 60).

If $f' = 0$, and A and B have the *same* sign, the equation can only be satisfied by the values $x=0$ and $y=0$; that is, the equation represents a point, viz., the origin.

If $f'=0$, and A and B have *different* signs, equation (6) reduces to

$$y=\pm x\sqrt{-\frac{A}{B}},$$

which represents two intersecting straight lines.

231. *To find the values of the coefficients* A *and* B *in equation* (6) *in terms of* a, b, *and* c.

Since $\quad A=a \cos.^2\theta+c \sin.^2\theta+b \sin.\theta \cos.\theta,$

and $\quad B=a \sin.^2\theta+c \cos.^2\theta-b \sin.\theta \cos.\theta,$

we have, by addition, observing that $\sin.^2\theta+\cos.^2\theta=1$,

$$A+B=a+c, \qquad (m)$$

and by subtraction, observing that $\cos.^2\theta-\sin.^2\theta=\cos. 2\theta$,

$$A-B=(a-c)\cos. 2\theta+b \sin. 2\theta.$$

Now, since $\sec. =\sqrt{1+\text{tang.}^2}$,

by eq. (5), $\quad \sec. 2\theta=\sqrt{1+\frac{b^2}{(a-c)^2}}=\frac{\sqrt{(a-c)^2+b^2}}{a-c};$

hence $\quad \cos. 2\theta=\frac{a-c}{\sqrt{b^2+(a-c)^2}},$

and $\quad \sin. 2\theta=\frac{b}{\sqrt{b^2+(a-c)^2}}.$

Hence we have

$$A-B=\frac{(a-c)^2}{\sqrt{b^2+(a-c)^2}}+\frac{b^2}{\sqrt{b^2+(a-c)^2}}$$

$$=\frac{b^2+(a-c)^2}{\sqrt{b^2+(a-c)^2}}=\pm\sqrt{b^2+(a-c)^2}. \qquad (n)$$

Adding and subtracting successively (m) and (n), we have

$$A=\tfrac{1}{2}\{a+c\pm\sqrt{b^2+(a-c)^2}\},$$

$$B=\tfrac{1}{2}\{a+c\mp\sqrt{b^2+(a-c)^2}\}.$$

Multiplying together these values of A and B, we have

$$A.B=\frac{(a+c)^2-b^2-(a-c)^2}{4}=\frac{4ac-b^2}{4}.$$

Hence A and B have the same sign or different signs according as $4ac-b^2$ is positive or negative.

232. *Particular case considered.* We will now consider the case in which b^2-4ac is zero. We can not in this case destroy the terms involving x and y by transferring the origin to the centre of the curve, as was done with the ellipse and hyperbola, but we may remove the term involving xy by changing the direction of the axes.

Let the equation be

$$ax^2+bxy+cy^2+dx+ey+f=0. \qquad (1)$$

Put

$$x=x' \cos.\theta-y' \sin.\theta,$$
$$y=x' \sin.\theta+y' \cos.\theta.$$

Substituting these values in equation (1), we have

$$\begin{aligned}&x'^2(a \cos.^2\theta+c \sin.^2\theta+b \sin.\theta \cos.\theta)\\&+y'^2(a \sin.^2\theta+c \cos.^2\theta-b \sin.\theta \cos.\theta)\\&+x'y'\{2(c-a) \sin.\theta \cos.\theta+b(\cos.^2\theta-\sin.^2\theta)\}\\&+x'(d \cos.\theta+e \sin.\theta)+y'(e \cos.\theta-d \sin.\theta)+f=0. \qquad (2)\end{aligned}$$

In order that the term involving $x'y'$ may become zero, we must have $2(c-a) \sin.\theta \cos.\theta+b(\cos.^2\theta-\sin.^2\theta)=0$;

whence, as in Art. 230, $\text{tang.}\, 2\theta=\dfrac{b}{a-c}$,

and the co-efficients of x'^2 and y'^2, as in Art. 231, are

$$\tfrac{1}{2}\{a+c\pm\sqrt{b^2+(a-c)^2}\}.$$

One of these coefficients must therefore vanish, since their product (Art. 231) is $\dfrac{4ac-b^2}{4}$, which, by hypothesis, $=0$. Suppose the coefficient of $x'^2=0$; if we suppress the accents on the variables, equation (2) will assume the form

$$Cy^2+Dx+Ey+f=0. \qquad (3)$$

Transposing and dividing by C, we have

$$y^2+\frac{Ey}{C}=-\frac{Dx}{C}-\frac{f}{C}.$$

Adding $\dfrac{E^2}{4C^2}$ to each member, we have

$$\left(y+\frac{E}{2C}\right)^2=-\frac{D}{C}\left(x-\frac{E^2}{4CD}+\frac{f}{D}\right). \qquad (4)$$

Put $l=-\dfrac{E}{2C}$, $M=-\dfrac{D}{C}$, and $n=\dfrac{E^2}{4CD}-\dfrac{f}{D}$, and equation (4) may be written $\qquad (y-l)^2=M(x-n).$

If now the origin be transferred to a point whose co-ordinates are

$$x=n,\ y=l,$$

we shall have, by writing $x+n$ for x, and $y+l$ for y,

$$y^2=\mathrm{M}x, \tag{5}$$

which is the equation to a parabola.

If in equation (3) $\mathrm{D}=0$, we have

$$\mathrm{C}y^2+\mathrm{E}y+f=0,$$

which gives
$$y=-\frac{\mathrm{E}}{2\mathrm{C}}\pm\frac{1}{2\mathrm{C}}\sqrt{\mathrm{E}^2-4\mathrm{C}f},$$

which represents two parallel straight lines, or one straight line, or an impossible locus, according as E^2 is greater, equal to, or less than $4\mathrm{C}f$.

233. *Conclusions.* Hence we arrive at the following results: The equation $ax^2+bxy+cy^2+dx+ey+f=0$ represents an ellipse, if b^2-4ac be *negative*, subject to three exceptions, in which it represents respectively a *circle*, a *point*, and an *imaginary curve* (Art. 230).

If b^2-4ac be positive, the equation represents an hyperbola, subject to one exception when it represents *two intersecting straight lines* (Art. 230).

If $b^2-4ac=0$, the equation represents a parabola, subject to three exceptions, in which it represents respectively *two parallel straight lines*, *one straight line*, and *an impossible locus* (Art. 232).

Ex. 1. Determine the form and situation of the curve represented by the equation

$$x^2-xy+y^2-2x-2y+2=0.$$

Here $b^2-4ac=-3$; hence the equation represents an ellipse (Art. 233).

In order to transfer the origin to the centre of the curve, we substitute $h+x'$ for x, and $k+y'$ for y. The values of h and k are given by the formulas of Art. 229,

$$h=\frac{-4-2}{-3}=+2;\ k=\frac{-4-2}{-3}=+2;$$

also,
$$f'=4-4+4-4-4+2=-2.$$

Hence the transformed equation is

$$x^2-xy+y^2-2=0.$$

Next, retaining the centre of the ellipse as the origin, we must find through what angle the axes must be turned in order that the term containing xy may vanish.

By Art. 230, tang. $2\theta=\dfrac{b}{a-c}=\dfrac{-1}{0}=$ infinity; hence $2\theta=90°$, and $\theta=45°$.

Also, by Art. 231, $A=\frac{1}{2}(2-\sqrt{1})=\frac{1}{2}$,

and $B=\frac{1}{2}(2+\sqrt{1})=\frac{3}{2}$.

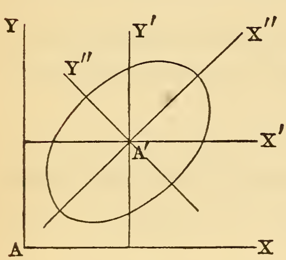
Therefore the equation to the ellipse referred to the new axes is

$$\frac{x^2}{2}+\frac{3y^2}{2}-2=0,$$

or $$x^2+3y^2=4.$$

The semiaxes are $\dfrac{2}{\sqrt{3}}$ and 2, and the axes are $\dfrac{4}{\sqrt{3}}$ and 4.

The annexed figure represents the form of the curve, and its position with respect to the different systems of axes, the co-ordinates of A′ being (2, 2), and the angle X′A′X″ being 45°.



$$x^2-xy+y^2-2x-2y+2=0$$

is the equation of the ellipse referred to the axes AX, AY.

$$x^2-xy+y^2-2=0$$

is the equation of the same ellipse referred to the axes A′X′, A′Y′.

$$x^2+3y^2=4$$

is the equation of the same ellipse referred to the axes A′X″, A′Y″.

Ex. 2. Determine the form and situation of the curve represented by the equation

$$x^2-6xy+y^2-6x+2y+5=0.$$

Here $b^2-4ac=36-4=32$; hence the equation represents an hyperbola.

By the formulas of Art. 229 we find

$$h=\frac{-12+12}{32}=0;\ k=\frac{4-36}{32}=-1;$$
$$f'=1-2+5=4.$$

Hence, when the origin is transferred to the point $(0, -1)$, the equation becomes $\quad x^2-6xy+y^2+4=0.$

In order that the term containing xy may vanish, we must have
$$\text{tang. } 2\theta=\frac{-6}{0}=\text{infinity.} \quad \text{Hence } \theta=45°.$$

Also, $\quad A=\tfrac{1}{2}(2-\sqrt{36})=-2,$

and $\quad B=\tfrac{1}{2}(2+\sqrt{36})=+4.$

Hence the transformed equation is
$$4y^2-2x^2+4=0.$$

The student should construct a figure showing the form and position of the curve with respect to the different axes of reference.

Ex. 3. Determine the form and situation of the curve represented by the equation
$$x^2-2xy+y^2-8x+16=0. \qquad (1)$$

Here $b^2-4ac=0$; hence the equation represents a parabola.

Substituting for x in eq. (1),
$$x' \cos.\theta-y' \sin.\theta,$$
and for y, $\quad x' \sin.\theta+y' \cos.\theta,$

we obtain an equation of the form
$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \qquad (2)$$
where
$$A=1-\sin.2\theta, \qquad D=-8\cos.\theta,$$
$$B=-2(\cos.^2\theta-\sin.^2\theta), \qquad E=\ 8\sin.\theta.$$
$$C=\ 1+\sin.2\theta, \qquad F=16.$$

Now, in order that B may vanish, we must have
$$\cos.\theta=\sin.\theta;\ \text{that is, } \theta=45°.$$

Making $\theta=45°$, equation (2) becomes
$$2y^2+y\frac{8}{\sqrt{2}}-x\frac{8}{\sqrt{2}}+16=0,$$
or
$$y^2+y\,.\,2\sqrt{2}-x\,.\,2\sqrt{2}+8=0,$$
which may be written
$$y^2+y\,.\,2\sqrt{2}+2=x\,.\,2\sqrt{2}-6,$$
or
$$(y+\sqrt{2})^2=2\sqrt{2}\left(x-\frac{3}{\sqrt{2}}\right).$$

If now we transfer the origin to a point whose co-ordinates are

$$x=\frac{3}{\sqrt{2}}, \text{ and } y=-\sqrt{2},$$

the equation to the curve will become

$$y^2=x\,.\,2\sqrt{2}.$$

The student should construct a figure showing the form and position of the curve with respect to the different axes of reference.

234. *Equation to the conic sections referred to the same axes and origin.* When the origin of co-ordinates is placed at the vertex of the major axis, the equation of the ellipse (Art. 129) is

$$y^2=\frac{b^2}{a^2}(2ax-x^2);$$

the equation to the hyperbola for a similar position of the origin (Art. 180) is

$$y^2=\frac{b^2}{a^2}(2ax+x^2);$$

the equation to the circle (Art. 63) is

$$y^2=2rx-x^2;$$

and the equation to the parabola (Art. 85) is

$$y^2=4ax.$$

These equations may all be reduced to the form

$$y^2=mx+nx^2.$$

In the ellipse, $m=\frac{2b^2}{a}$, and $n=\frac{-b^2}{a^2}$;

in the hyperbola, $m=\frac{2b^2}{a}$, and $n=\frac{b^2}{a^2}$;

in the parabola, $m=4a$, and $n=0$.

In each case m represents the latus rectum of the curve, and n the square of the ratio of the semiaxes. In the ellipse n is negative, in the hyperbola it is positive, and in the parabola it is zero.

The equation $y^2=mx+nx^2$ is the simplest form of the equation to the conic sections *taken collectively*, and referred to the same axes and origin.

235. *Miscellaneous Examples.*

Draw the curves of which the following are the equations:

Ex. 1. $x^2+2y^2=10$.

Ex. 2. $x^2-2y^2=10$.

Ex. 3. $x^2+3x=10y$.

Ex. 4. $xy+10y=40$.

Ex. 5. $3x^2+2y^2=18$.

Ex. 6. $3x^2+2y^2=-18$.

Ex. 7. $3x^2+2y^2=0$.

Ex. 8. $y^2=4(x-3)$.

Ex. 9. $3xy=5$.

Ex. 10. $3xy-x+2=0$.

Ex. 11. $5x^2+7y^2=11$.

Ex. 12. $3y^2-2y+4x=0$.

Ex. 13. $y^2+5y-9x+10=0$.

Ex. 14. $7x^2-11y^2=-50$.

SECTION IX.

LINES OF THE THIRD AND HIGHER ORDERS.

236. Lines of the third order have their equations of the form

$$ay^3+by^2x+cyx^2+dx^3+ey^2+fyx+gx^2+hy+kx+l=0.$$

Newton has shown that all lines of the third order are comprehended under some one of these four equations:

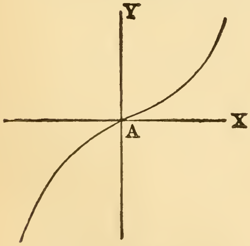
$$(1)\quad xy^2+ey=ax^3+bx^2+cx+d;$$
$$(2)\quad xy=ax^3+bx^2+cx+d;$$
$$(3)\quad y^2=ax^3+bx^2+cx+d;$$
$$(4)\quad y=ax^3+bx^2+cx+d;$$

in which a, b, c, d, e may be positive, negative, or evanescent, excepting those cases in which the equation would thus become one of an inferior order of curves.

The first equation comprehends seventy-three different species of curves, the second only one, the third five, and the fourth only one, making eighty different species of lines of the third order.

237. It is not proposed to attempt any general investigation of the equation of the third degree, but merely to select a few instances calculated to exhibit the properties of some of the more remarkable curves.

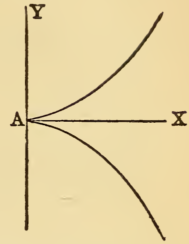
Ex. 1. Trace the curve whose equation is $6y=x^3$.



Suppose $x=0$, then $y=0$.
$x=\pm1$, " $y=\pm0.167$.
$x=\pm2$, " $y=\pm1.333$.
$x=\pm3$, " $y=\pm4.500$.
$x=\pm4$, " $y=\pm10.667$, etc.

Constructing these values, we obtain the figure annexed. This equation may be written more generally $ay=x^3$, and the curve is called the *cubical parabola*. It belongs to eq. (4), Art. 236.

Ex. 2. Trace the curve whose equation is $4y^2=x^3$.



Suppose $x=0$, then $y=0$.

$x=+1$, " $y=\pm 0.500$.

$x=+2$, " $y=\pm 1.414$.

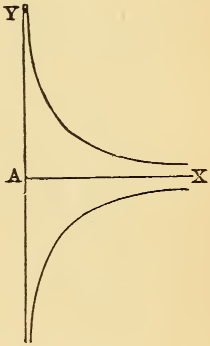
$x=+3$, " $y=\pm 2.598$.

$x=+4$, " $y=\pm 4.000$.

$x=+5$, " $y=\pm 5.590$.

If x is negative, y becomes imaginary. The curve is represented by the annexed figure, and is called the *semicubical parabola*. The equation in a more general form is $ay^2=x^3$, and belongs to eq. (3) of Art. 236.

Ex. 3. Trace the curve whose equation is $xy^2=10$.



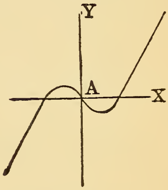
Suppose $y=0$, then $x=$ infinite.

" x is negative, " y is impossible.

" $y=\pm 1$, " $x=+10$, etc.

The curve is of the form represented in the annexed figure, and belongs to equation (1), Art. 236.

Ex. 4. Trace the curve $y=x^3-x$.



Suppose $x=0$, then $y=0$.

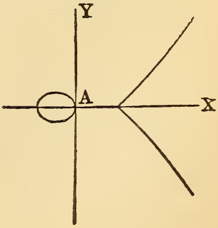
$x=\pm 0.5$, " $y=\mp 0.375$.

$x=\pm 1$, " $y=0$.

$x=\pm 2$, " $y=\pm 6$.

The curve is shown in the annexed figure, and belongs to eq. (4), Art. 236.

Ex. 5. Trace the curve $y^2=x^3-x$.



Suppose $x=0$, then $y=0$.

$x=\pm 1$, " $y=0$.

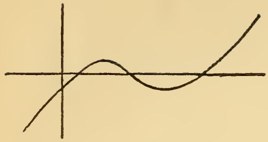
$x=+0.5$ " $y=$ impossible.

$x=-0.5$, " $y=\pm 0.612$.

$x=+2$, " $y=\pm 2.449$.

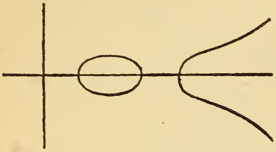
$x=+3$, " $y=\pm 4.899$.

The curve is shown in the annexed figure.



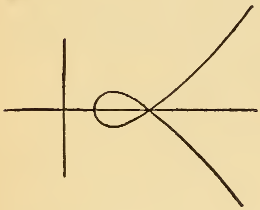
Ex. 6. Trace the curve whose equation is

$$10y = x^3 - 9x^2 + 23x - 15.$$



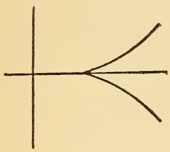
Ex. 7. Trace the curve whose equation is

$$10y^2 = x^3 - 11x^2 + 34x - 24.$$



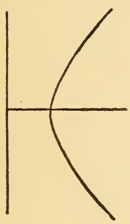
Ex. 8. Trace the curve whose equation is

$$10y^2 = x^3 - 9x^2 + 24x - 16.$$



Ex. 9. Trace the curve whose equation is

$$10y^2 = x^3 - 12x^2 + 48x - 64.$$



Ex. 10. Trace the curve whose equation is

$$10y^2 = x^3 + 3x^2 - 22x - 24.$$

Ex. 11. Trace the curve whose equation is

$$y = x^3 - 3x.$$

Ex. 12. Trace the curve whose equation is

$$y^2 = x^3 - 9x.$$

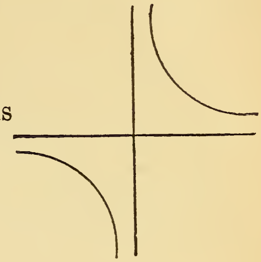
Ex. 13. Trace the curve whose equation is

$$y^2 = x^3 - x^2.$$

238. *Equations of the fourth degree.* The general equation of the fourth degree represents an immense variety of curve lines, the number of different species being estimated at more than 5000. The number of species of lines of the fifth and higher orders is so great as to preclude any attempt to enumerate them completely.

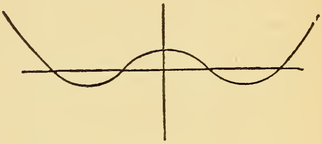
Ex. 1. Trace the curve whose equation is

$$yx^3 = 81.$$



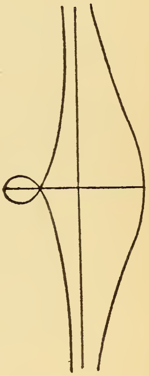
Ex. 2. Trace the curve whose equation is

$$27y = x^4 - 20x^2 + 64.$$



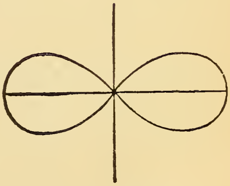
Ex. 3. Trace the curve whose equation is

$$x^2y^2 + x^4 + 6x^3 - 16x^2 - 150x = 225.$$



Ex. 4. Trace the curve whose equation is

$$x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2 = 16(x^2 - y^2).$$



SECTION X.

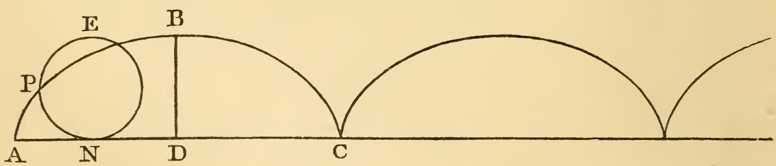
TRANSCENDENTAL CURVES.

239. *Equations classified.* Equations may be divided into two classes, *algebraic* and *transcendental.* An algebraic equation between two variables, x and y, is one which can be reduced to a finite number of terms involving only integral powers of x and y, and constant quantities. Equations which can not be thus reduced are called transcendental; for they can only be expanded into an infinite series of terms, in which the power of the variable increases without limit, and the equation transcends all finite orders.

240. *Curves classified.* Curves whose equations are transcendental are called *transcendental curves.* Among transcendental curves, the cycloid and the logarithmic curves are the most important. The logarithmic curve is useful in exhibiting the law of the diminution of the density of the atmosphere, and the cycloid in investigating the laws of the pendulum and the descent of heavy bodies toward the centre of the earth.

CYCLOID.

241. A *cycloid* is the curve described by a point in the circumference of a circle rolling in a straight line on a plane.

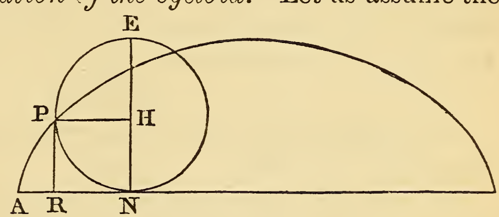


Thus, if the circle EPN be made to roll in a given plane upon a straight line AC, the point P of the circumference, which was in contact with A at the commencement of the motion, will in a revolution of the circle describe a curve ABC, which is called the cycloid. The circle EPN is called the *generating circle*, and P the *generating point.*

When the point P has arrived at C, having described the arc ABC, if it continue to move on, it will describe a second arc similar to the first, and so on indefinitely. As, however, in each revolution of the generating circle an equal curve is described, it is only necessary to examine the curve ABC, described in one revolution of the generating circle.

242. After the circle has made one revolution, every point of the circumference will have been in contact with AC, and the generating point will have arrived at C. The line AC is called the *base of the cycloid*, and is equal to the circumference of the generating circle. The line BD, drawn perpendicular to the base at its middle point, is called the *axis of the cycloid*, and is equal to the diameter of the generating circle.

243. *To find the equation of the cycloid.* Let us assume the point A as the origin of co-ordinates, and let us suppose that the generating point has described the arc AP. If N designates the point at which the generating circle touches the base, it is plain that the line AN will be equal to the arc PN. Through N draw the diameter EN, which will be perpendicular to the base of the cycloid. Through P draw PH parallel to the base, and PR perpendicular to it. Then PR will be equal to HN, which is the versed sine of the arc PN.



Let AR $=x$, and PR or HN $=y$; and let r represent the radius of the generating circle. By Geom., Bk. IV., Prop. 23, Cor.,

$$\mathrm{RN}=\mathrm{PH}=\sqrt{\mathrm{NH}\times\mathrm{HE}}=\sqrt{y(2r-y)}=\sqrt{2ry-y^2};$$

also,

$$\mathrm{AR}=\mathrm{AN}-\mathrm{RN}=\text{arc PN}-\mathrm{PH}.$$

The arc PN is the arc whose versed sine is HN or y.

Substituting the values of AR, AN, and RN, we have

$$x=(\text{the arc whose versed sine is } y)-\sqrt{2ry-y^2},$$

which is the equation of the cycloid.

244. *Another form of the equation.* It is sometimes convenient, in the equation of the cycloid, to employ the angle of rotation of the generating circle, or the angle subtended by the arc PN at the centre of the circle EPN. Let this angle be denoted by θ, and the radius of the circle by r; then

$$\text{the arc PN}=r\theta,$$

and
$$\text{AR or } x=r\theta-r\sin.\theta,$$

and
$$\text{HN or } y=r-r\cos.\theta.$$

If we eliminate θ from these two equations, we shall obtain the same value of x as given in Art. 243.

LOGARITHMIC CURVE.

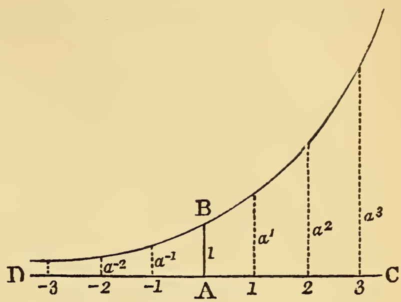
245. The logarithmic curve takes its name from the property that, when referred to rectangular axes, any abscissa is equal to the logarithm of the corresponding ordinate. The equation of the curve is therefore

$$x=\log. y.$$

If a represent the base of a system of logarithms, we shall have (Alg., Art. 394) $y=a^x$.

To examine the course of the curve, we find, when $x=0$, $y=a^0=1$; as x increases from 0 to ∞, y increases from 1 to ∞; as $-x$ increases to ∞, y decreases from 1 to 0. Draw AB perpendicular to DC, and make it equal to the linear unit; then the curve proceeding from B to the right of AB recedes from the axis of x, and on the left continually approaches that axis, which is therefore an asymptote.

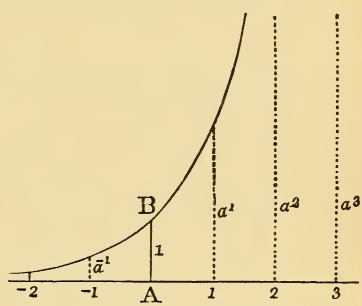
Any number of points of the curve may be determined from the equation $y=a^x$. Let AC be divided into portions each equal to AB. Let a be taken equal to the base of the given system of logarithms, for example 1.6, and let a^2, a^3, etc., correspond in length with the different powers of a. Then the distances from A to 1, 2, 3, etc., will represent the logarithms of a, a^2, a^3, etc.



The logarithms of numbers less than a unit are *negative*, and these are represented by portions of the line AD to the left of the origin.

246. In a similar manner we may construct the curve for any system of logarithms. Thus, for the Naperian system,

$a = 2.718.$
$a^2 = 7.389.$
$a^3 = 20.085.$
$a^{-1} = 0.368.$
$a^{-2} = 0.135$, etc.



If at the point A we erect an ordinate equal to unity, at the point 1 an ordinate equal to 2.718, at the point 2 an ordinate equal to 7.389, etc., at the point −1 an ordinate equal to 0.368, etc., the curve passing through the extremities of these ordinates will be the logarithmic curve for the Naperian base.

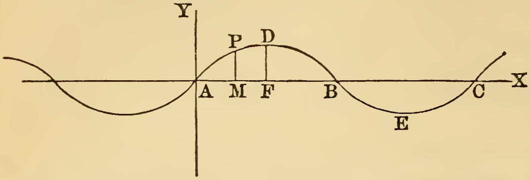
Ex. 1. Construct by points the logarithmic curve, the base being 10.

Ex. 2. Construct by points the logarithmic curve, the base being $\frac{1}{2}$.

CURVE OF SINES, TANGENTS, ETC.

247. If we conceive the circumference of a circle to be extended out in a right line, and at each point of this line a perpendicular ordinate to be erected equal to the sine of the corresponding arc, the curve line drawn through the extremity of each of these ordinates is called the *curve of sines*.

Draw a straight line ABC equal to the circumference of a given circle, and upon it lay off the lengths of several arcs, at every 10° for example, from 0° at A to 360° at C; from these points draw perpendicular ordinates equal to the sines of the corresponding arcs, upward or downward, according as the sine is positive or negative in that part of the circle; then draw a curve line ADBEC through the extremities of all these ordinates; it will be the curve of sines.



248. *To find the equation of the curve of sines.* Draw any ordinate PM. Let AM$=x$, and PM$=y$; then the equation is

$$y=\sin. x.$$

If r represent the radius of the given circle, then

$$y=r\sin. \frac{x}{r}.$$

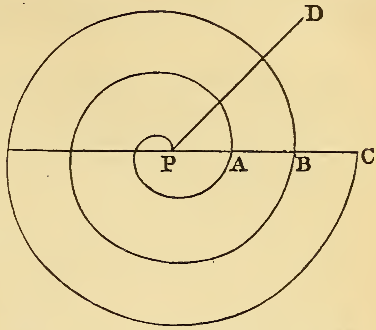
Since the sine is 0 when the arc is 0, the curve cuts the axis at A. Since the sine of 90° is a maximum, the highest point of the curve will be at D, where $y=r$. The curve cuts the axis again in B; from B, y increases negatively until it equals $-r$, and then decreases to 0, so that we have a second branch equal and similar to the first. Beyond C the values of y recur, and the curve continues the same course ad infinitum. Also, since $\sin. (-x)=-\sin. x$, there is a similar branch to the left of A.

In a similar manner may be drawn the curve of tangents, the curve of secants, etc.

SPIRALS.

249. *Definition.* If a right line be revolved uniformly in the same plane about one of its points as a centre, and if at the same time a second point travel along the line in accordance with some prescribed law, the latter point will generate a curve called a *spiral.*

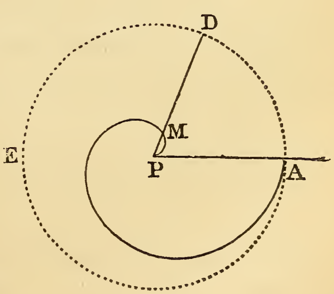
Thus, let PD be a straight line which revolves uniformly round the point P, starting from the position PC, and at the same time let a point move from P along the line PD according to some prescribed law; the point will trace out a curve line which commences at P, and after one revolution will arrive at a point A; after two revolutions it will arrive at a point B, and so on. The curve thus traced is called a *spiral.*



250. The fixed point P, about which the right line revolves, is called the *pole* of the spiral. The portion of the spiral generated while the straight line makes one revolution is called a *spire.* If the revolutions of the radius vector are continued indefinitely, the generating point will describe an unlimited spiral. It is assumed that the point does not, after a limited number of revolutions, describe again the previous curve, but that any straight line drawn through the pole of the spiral will cut the curve in an infinite number of points.

Instead of starting from the pole, the generating point may commence its motion at any distance from the pole; and instead of receding, it may move toward the pole.

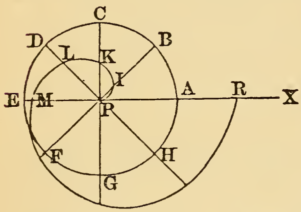
With P as a centre, and any convenient radius as PA, describe the circumference ADE; the angular motion of the radius vector about the pole may be measured by the arcs of this circle, estimated from A. It is generally convenient to make the radius of the measuring circle equal to the length of the radius vector at the end of one revolution of



the generating point, starting from the pole, but the measuring circle may have any magnitude.

251. *Spiral of Archimedes.* While the line PD revolves uniformly round the point P, let the generating point also move uniformly along the line PD; it will describe the spiral of Archimedes.

252. *To construct the spiral of Archimedes.* Let P be the pole, and PX the first position of the radius vector. With P as a centre, and any convenient radius, describe the measuring circle ACEG, and divide its circumference into any convenient number of equal parts, as, for example, eight. On PB set off PI any convenient distance; on PC set off PK=2PI; on PD set off PL=3PI, etc. The curve passing through the points I, K, L, M, etc., thus determined, will be the spiral of Archimedes, for the radii vectores are proportional to the arcs AB, AC, etc., of the measuring circle.



253. *To find the equation to the spiral of Archimedes.* From the definition of the curve, the radii vectores and the measuring arcs increase uniformly; that is, in the same ratio. Hence we have

PL : PR : : angle APD : four right angles.

Designate the radius vector PL by r, PR by b, and the variable angle by θ; then we shall have

$$r : b :: \theta : 2\pi;$$

whence $r=\frac{b\theta}{2\pi}$; or, putting $a=\frac{b}{2\pi}$, we have the equation

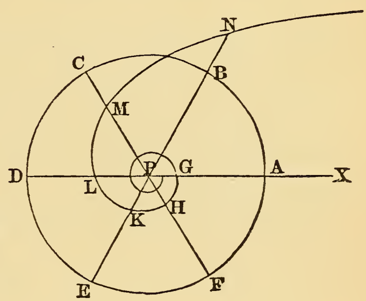
$$r=a\theta.$$

When the radius vector has made two revolutions, or $\theta=4\pi$, we have $r=2b$; that is, the curve cuts the axis PX at a distance equal to 2PR; after three revolutions it cuts the axis

PX at a distance equal to 3PR, etc. Hence the distance between any two consecutive spires, measured on a radius vector, is always the same.

254. *Hyperbolic spiral.* While the line PN revolves uniformly about P, let the generating point move along the line PN in such a manner that the radius vector shall be inversely proportional to the corresponding angle; it will describe the hyperbolic spiral.

255. *To find the equation to the hyperbolic spiral.* From the definition of the curve, the radius vector is inversely proportional to the measuring angle; hence we have



PG : PN :: angle APN : four right angles.

Designate the radius vector PN by r, PG by b, and the variable angle measured from the line PX by θ, and we shall have

$$b : r :: \theta : 2\pi.$$

Whence $r\theta = 2b\pi$;

or, putting $2b\pi = a$, we have

$$r\theta = a.$$

When $\theta = 0$, $r = \infty$; as θ increases, r decreases, at first very rapidly, but afterwards more uniformly. As θ may increase without limit, r may decrease indefinitely without actually becoming zero; hence, as the radius vector revolves, the curve continues to approach the pole, but reaches it only after an infinite number of revolutions. This curve is called the hyperbolic spiral from the similarity of its equation to that of the hyperbola referred to its asymptotes ($xy = c^2$), the product of the variables r and θ being equal to a constant quantity.

256. *To construct the hyperbolic spiral.* Let P be the pole, and PX the first position of the radius vector. With any convenient radius draw the measuring circle ABDE, and divide its circumference into any convenient number of equal parts AB, BC, CD, etc. On PB, produced if necessary, take any convenient distance, as PN; take PM equal to one half of PN, PL equal to one third of PN, PK equal to one fourth of PN, etc.; the curve passing through the points N, M, L, K, etc., will be an hyperbolic spiral, because the radii vectores are inversely proportional to the corresponding angles measured from PX.

257. *Logarithmic spiral.* While the line PA revolves uniformly about P, let the generating point move along PA in such a manner that the variable angle may be proportional to the logarithm of the radius vector; it will describe the logarithmic spiral.

The equation of the logarithmic spiral is

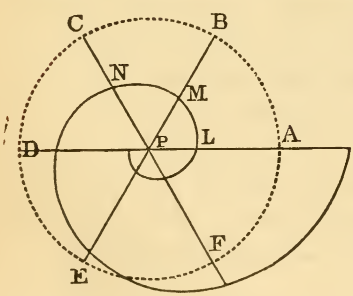
$$\theta = \log. \frac{r}{a},$$

or

$$r = ab^{\theta},$$

b being the base of the system of logarithms (Alg., Art. 394), and a any arbitrary constant.

258. *To construct the logarithmic spiral.* If we take $b=10$, the base of the common system of logarithms, the changes of r are so rapid that we can represent only a small arc of the curve. We will therefore assume $b=1.2$. When $\theta=0$, $r=a$, which determines the point L. When $\theta=1$, that is, 57°.3 (radius being unity), $r=1.2a$, which determines the point M. When



$\theta=2$, that is, 114°.6, $r=1.2^2a$, or $1.44a$, which determines the point N, etc. As θ increases, r also increases, but does not become infinite until θ becomes infinite.

If we suppose the radius vector to revolve in the negative direction from PA, when $\theta=-1$, $r=0.83a$, which determines another point of the curve. When $\theta=-2$, $r=0.69a$, etc. Hence we see that, as the radius vector revolves in the negative direction, it generates a portion of the spiral which slowly approaches the pole, but can not reach it until $\theta=-\infty$.

Thus we see that the logarithmic spiral makes an infinite series of convolutions around the pole P.

PART III.

GEOMETRY OF THREE DIMENSIONS.

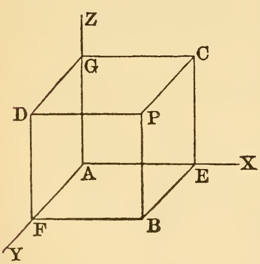
SECTION I.

OF POINTS IN SPACE.

259. Hitherto we have considered only points and lines situated in one plane, and we have seen that the position of a point in a plane may be denoted by its distances from two assumed fixed lines or axes situated in that plane. We have now to consider how the position of any point in space may be represented.

260. *To determine the position of a point in space.* Let three planes XAY, ZAX, ZAY, supposed to be of indefinite extent, be drawn perpendicular to each other, and let these planes intersect each other in the three straight lines AX, AY, AZ. Let P be any point in space whose position it is required to determine.

From the point P draw the line PB perpendicular to the plane XAY; draw PC perpendicular to the plane ZAX, and PD perpendicular to the plane ZAY; then the position of the point P is completely determined when these three perpendiculars are known.

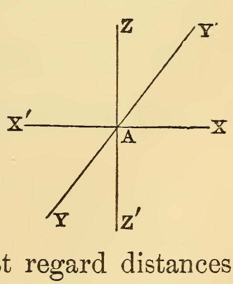


Let a, b, c represent these three perpendiculars. On AX take AE$=a$, on AY take AF$=b$, and on AZ take AG$=c$, and through the points E, F, and G let planes be drawn parallel to the three planes ZAY, ZAX, and XAY, forming the rectangular parallelopiped EFG.

Since the plane drawn through E is every where distant from the plane ZAY by a quantity equal to a, the point P must be somewhere in this plane; and since the plane drawn through F is every where distant from the plane ZAX by a quantity equal to b, the point P must be also in this plane. It must therefore be in the line BP, which is the common section of these two planes. Also, since the plane drawn through G is every where distant from the plane XAY by a quantity equal to c, the point P must be somewhere in this plane; it must therefore be at the intersection of this third plane with the line BP. Thus the position of the point P is completely determined.

261. *Definitions.* The three planes XAY, ZAX, ZAY, by reference to which the position of the point P has been determined, are called the *co-ordinate planes.* The first is designated as the plane XY, the second as the plane XZ, and the third as the plane YZ. The lines AX, AY, AZ, which are the intersections of these three planes, are called the *co-ordinate axes.* The first is called the axis of X, and distances parallel to it are denoted by x; the second is the axis of Y, and distances parallel to it are denoted by y; the third is the axis of Z, and distances parallel to it are denoted by z. The point A, in which the three axes intersect, is called the *origin* of co-ordinates. The equations of a point in space are therefore of the form $x=a, \quad y=b, \quad z=c.$

262. *Signs of the co-ordinates.* If the three co-ordinate planes be indefinitely produced, there will be formed about the point A eight solid angles, four above the horizontal plane XAY, and four below it. It is required to denote analytically in which of these angles the proposed point is situated. For this purpose, if we regard distances measured on AX to the right of A as *positive*, we must regard distances measured to the left of A as *negative.* So, also, y is regarded as positive when it is in *front* of the plane ZX, and negative



when it is *behind* that plane; and z is regarded as positive when it is *above* the plane XY, and negative when it is *below* that plane. Hence the equations of a point in each of these eight angles are as follows:

If $x=+a$, $y=+b$, $z=+c$, the point is in the angle ZXAY.
$x=+a$, $y=-b$, $z=+c$, " " " ZXAY′.
$x=-a$, $y=-b$, $z=+c$, " " " ZX′AY′.
$x=-a$, $y=+b$, $z=+c$, " " " ZX′AY.
$x=+a$, $y=+b$, $z=-c$, " " " Z′XAY.
$x=+a$, $y=-b$, $z=-c$, " " " Z′XAY′.
$x=-a$, $y=-b$, $z=-c$, " " " Z′X′AY′.
$x=-a$, $y=+b$, $z=-c$, " " " Z′X′AY.

263. *Co-ordinates of particular points.* If the point P be situated in the plane of xy, then its distance z from this plane is 0, and its equations will be

$$x=\pm a, \quad y=\pm b, \quad z=0.$$

If the point be situated in the plane of xz, then its distance y from this plane is 0, and its equations will be

$$x=\pm a, \quad y=0, \quad z=\pm c.$$

If the point be situated in the plane of yz, then its distance x from this plane is 0, and its equations will be

$$x=0, \quad y=\pm b, \quad z=\pm c.$$

If the point be situated on the axis of x, that is, on the intersection of the planes xy and xz, then its distance from each of these planes is 0, and its position will be expressed by the equations

$$x=\pm a, \quad y=0, \quad z=0.$$

So, also, if the point be situated on the axis of y, we shall have

$$x=0, \quad y=\pm b, \quad z=0;$$

and if it be situated on the axis of z, we shall have

$$x=0, \quad y=0, \quad z=\pm c.$$

If the point be at the origin, its position will be denoted by the equations

$$x=0, \quad y=0, \quad z=0.$$

Ex. 1. Indicate by a figure the position of the point whose equations are

$$x=+4, \quad y=-3, \quad z=-2.$$

Ex. 2. Indicate by a figure the position of the point whose equations are $x=-2, \quad y=+7, \quad z=+5.$

Ex. 3. Draw a triangle, the co-ordinates of whose angular points are

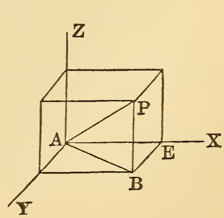
$$x=+3, \quad y=+4, \quad z=+2;$$
$$x=-3, \quad y=-4, \quad z=-2;$$
$$x=-1, \quad y=0, \quad z=+1.$$

264. *Projections.* If a perpendicular be let fall from any point P upon a given plane, the point in which this line meets the plane is called the projection of the point P on the plane. The projections of the point P (Art. 260) on the three co-ordinate planes are the points B, C, D.

The projection of any curve upon a given plane is the curve formed by projecting all of its points upon that plane. When the curve projected is a straight line, its projection on any one of the co-ordinate planes will also be a straight line, for all the points of the given line are comprised in the plane passing through this line and drawn perpendicular to the co-ordinate plane; and since the common section of any two planes is a straight line, the projections of the points must all lie in one straight line. This plane, which contains all the perpendiculars drawn from different points of the straight line, is called the *projecting plane.*

If the positions of any two projections of the point P are given, it will be sufficient to determine the point P; for a line drawn from either projection, perpendicular to the plane in which it is, necessarily passes through the point P, so that P will be at the intersection of two such perpendiculars. When two projections of a point are known, we can always determine the third.

265. *To find the distance of any point from the origin in terms of the co-ordinates of that point.* Let AX, AY, AZ be the rectangular axes, and P the given point. Let the co-ordinates of P be $AE=x$, $BE=y$, and $PB=z$.



The square on AP = the sum of the squares on AB and PB.

Also, the square on AB = the sum of the squares on AE and EB; that is, $AP^2 = AE^2 + EB^2 + PB^2$,

or $$AP^2 = x^2 + y^2 + z^2.$$

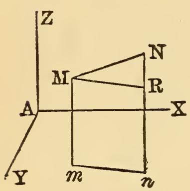
Ex. 1. Determine the distance from the origin to the point whose co-ordinates are

$$x = 2a, \quad y = -3a, \quad z = 6a.$$

Ex. 2. Determine the distance from the origin to the point whose co-ordinates are

$$x = -b, \quad y = -4b, \quad z = 8b.$$

266. *To find the distance between two given points in space.*



Let M and N be the two given points, their co-ordinates being respectively x, y, z, and x', y', z'. If the points M and N be projected on the plane of xy, the co-ordinates x, y of the projections m and n will be the same as those of the points M and N. Hence, for the distance mn we have (Art. 21)

$$mn^2 = (x - x')^2 + (y - y')^2.$$

Now, if MR be drawn parallel to mn, MRN will be a right angle, and hence $MN^2 = MR^2 + NR^2$

$$= MR^2 + (Nn - Rn)^2;$$

that is, $$MN = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2};$$

that is, *the distance between any two given points is the diagonal of a right parallelopiped, whose three adjacent edges are the differences of the parallel co-ordinates.*

Ex. 1. Determine the distance between the points

$$x = 3, \quad y = 4, \quad \text{and } z = -2,$$

and $$x = 4, \quad y = -3, \quad \text{and } z = 1.$$ *Ans.* $\sqrt{59}$.

Ex. 2. Determine the distance between the points

$$x = 2, \quad y = 2, \quad z = 1,$$

and $$x = -2, \quad y = -3, \quad z = 4.$$ *Ans.*

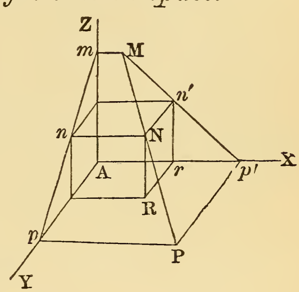
SECTION II.

THE STRAIGHT LINE IN SPACE.

267. A straight line may be regarded as the common section of two planes, and therefore its position will be known when the position of these planes is known; hence its position may be determined by the projecting planes, and the situation of the projecting planes is given by their intersections with the co-ordinate planes; that is, by the projections of the given line upon the co-ordinate planes.

268. *To find the equation of a straight line in space.*

Let $x = mz + a$



be the equation of a straight line Mp' in the plane of xz, and through this line let a plane be drawn perpendicular to the plane xz. Also, let

$$y = nz + b$$

be the equation of a line mp in the plane of yz, and through this line let a plane be drawn perpendicular to the plane yz. These two planes will intersect in a line MP, which will thus be completely determined. The two equations

$$x = mz + a, \qquad (1)$$

$$y = nz + b, \qquad (2)$$

taken together, may therefore be regarded as the equations of the line MP, and from these equations the line MP may be constructed; for, if a particular value be assigned to either variable in these equations, the values of the other two variables can be found, and these three quantities taken together will be the co-ordinates of a point of the required line.

Thus, suppose $n'r$ to be a value of z; this, with the corresponding value of x deduced from equation (1), will determine a point n', through which a line must be drawn perpendicular to the plane xz. The same value of z, with the corresponding value of y deduced from equation (2), will determine a point n, through which if Nn be drawn perpendicular to the plane yz, it will intersect the line Nn', since both lines are situated in the same plane, viz., a plane parallel to xy, and at a distance from it equal to z. The point N of the line MP is therefore determined, and in the same manner we may determine any number of points of this line. Hence the equations to the straight line MP are

$$x = mz + a, \tag{1}$$
$$y = nz + b. \tag{2}$$

269. *Interpretation of the constants in these equations.* In equation (1) m represents the tangent of the angle which the projection of the given line on the plane xz makes with the axis of z, and a represents the distance cut from the axis of X by the same projection (Art. 29).

In equation (2) n represents the tangent of the angle which the projection on the plane yz makes with the axis of z, and b is the distance cut from the axis of Y.

If we combine these two equations, and eliminate the variable z, we shall have

$$y - b = \frac{n}{m}(x - a),$$

which expresses the relation between the co-ordinates of the point R, which is the projection of the point N on the plane xy, and therefore this last equation is the equation of the line MP projected on the plane xy.

Ex. The equations of the projections of a straight line on the co-ordinate planes zx, zy are

$$x = 2z + 3, \quad y = 3z - 5;$$

required its equation on the plane xy. *Ans.* $2y = 3x - 19$.

270. *To determine the points where the co-ordinate planes are pierced by a given straight line.* At the point where a line pierces the plane xy the value of z must be 0. If we substitute this value of z in equations (1) and (2) of Art. 268, we shall find

$$x=a, \quad y=b;$$

hence a and b taken together are the co-ordinates of the point in which the given line pierces the plane xy.

In like manner, the co-ordinates of the point in which the line pierces the plane xz may be determined by putting $y=0$ in equation (2), and substituting the resulting expression for z in equation (1). In the same manner, the point where the line pierces the plane yz may be determined.

Ex. 1. Determine the points where the co-ordinate planes are pierced by the line whose equations are

$$x=2z+3,$$
$$y=3z-7.$$

Ex. 2. Determine the points where the co-ordinate planes are pierced by the line whose equations are

$$x=-2z-5,$$
$$y=-z+2.$$

271. *To find the equations of a straight line passing through a given point.* Let the co-ordinates of the given point be x', y', z', and let the equations to the straight line be

$$x=mz+a, \quad y=nz+b.$$

Now, since this line passes through the given point, we must have

$$x'=mz'+a,$$
$$y'=nz'+b;$$

hence we obtain

$$x-x'=m(z-z'),$$

and

$$y-y'=n(z-z'),$$

which are the equations sought, and characterize every straight line which can be drawn through the point x', y', z'. If the given point be the origin, then $x'=0$, $y'=0$, and $z'=0$, and the equations of a line passing through the origin are

$$x=mz, \quad y=nz.$$

272. *Equations of a straight line passing through two given points.* Let the co-ordinates of the given points be x', y', z', and x'', y'', z''; then the equations of the line passing through the first of these points are

$$\left.\begin{aligned} x-x'&=m(z-z'), \\ y-y'&=n(z-z'). \end{aligned}\right\} \qquad (1)$$

Since the line passes through the point x'', y'', z'', we must also have

$$x''-x'=m(z''-z'),$$

and

$$y''-y'=n(z''-z'),$$

from which we obtain the values of m and n, viz.:

$$m=\frac{x''-x'}{z''-z'}, \quad n=\frac{y''-y'}{z''-z'}.$$

These values of m and n, being substituted in equation (1), will furnish the equations of the line passing through both the given points. We have, therefore,

$$x-x'=\frac{x''-x'}{z''-z'}(z-z'),$$

$$y-y'=\frac{y''-y'}{z''-z'}(z-z').$$

If one of the points x'', y'', z'' be the origin, these equations become

$$x=\frac{x'}{z'}\cdot z,$$

$$y=\frac{y'}{z'}\cdot z.$$

Ex. 1. Find the equations to the straight line passing through the following points:

$$x'=3, \quad y'=-4, \quad z'=2,$$
$$x''=-5, \quad y''=6, \quad z''=3.$$

Ans. $x=-8z+19, \quad y=10z-24.$

Ex. 2. Find the equations to the straight line passing through the following points:

$$x'=4, \quad y'=-2, \quad z'=-3,$$
$$x''=0, \quad y''=1, \quad z''=-2.$$

Ans. $x=-4z-8, \quad y=3z+7.$

273. *To determine the conditions requisite for the intersection of two straight lines.* Two straight lines which are not parallel must meet if they are situated in the same plane, but this is not necessarily true for lines situated any where in space. In order that two lines may meet, there must be a particular relation among the constant quantities in their equations. In order to discover this relation, let the equations to the lines be

$$\left.\begin{array}{l} x=mz+a, \\ y=nz+b, \end{array}\right\} \text{ and } \left.\begin{array}{l} x=m'z+a', \\ y=n'z+b'. \end{array}\right\}$$

If these lines intersect, that is, have one point in common, the co-ordinates of this point must satisfy both sets of equations, or for this point the values of x, y, and z must be the same in all the equations. Since x of the one line equals x of the other, we have

$$(m-m')z+a-a'=0,$$

or

$$z=\frac{a'-a}{m-m'};$$

and since y of one line equals y of the other, we have

$$(n-n')z+b-b'=0,$$

or

$$z=\frac{b'-b}{n-n'}.$$

But z of the one line is equal to z of the other; hence

$$\frac{a'-a}{m-m'}=\frac{b'-b}{n-n'}.$$

Hence, when the lines intersect, the relation between the constants is given by the equation

$$(a'-a)(n-n')=(b'-b)(m-m'). \qquad (1)$$

Conversely, when this equation exists the two lines intersect.

The co-ordinates of the point of intersection may be determined by substituting in the expressions for x and y the value of z just found. They are

$$x=\frac{ma'-m'a}{m-m'}, \quad y=\frac{nb'-n'b}{n-n'},$$

$$z=\frac{a'-a}{m-m'}=\frac{b'-b}{n-n'}.$$

These values of x and y, with either value of z, will give a point of intersection when equation (1) is satisfied.

If $m=m'$, and $n=n'$, equation (1) is satisfied, and the values of x, y, and z become infinite. The point of intersection is then at an infinite distance; that is, *the two lines are parallel.*

But when $m=m'$, the projections of the two lines on the plane xz are parallel, and when $n=n'$ the projections on the plane yz are parallel. Hence, *if two right lines in space are parallel, their projections on the same co-ordinate plane will be parallel.*

274. *To find the equations of the straight line which passes through a given point and is parallel to a given line.* Let x', y', z' be the co-ordinates of the given point. The equations of the straight line passing through this point (Art. 271) are

$$x-x'=m(z-z'),$$

and

$$y-y'=n(z-z').$$

In order that this line may be parallel to a given line, its projections on the co-ordinate planes must be parallel to the projections of the former line (Art. 273); that is, they must cut the axis of z at the same angle. The quantities m and n therefore become known, and if we represent the tangents of the given angles by m' and n', we shall have

$$x-x'=m'(z-z'),$$

$$y-y'=n'(z-z'),$$

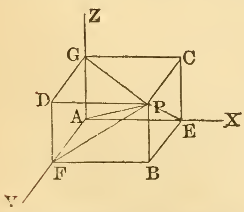
which are the equations of the required line.

Ex. Find the equation of a straight line which passes through the point $x'=3, \quad y'=-2, \quad z'=1,$

and is parallel to the line whose equations are

$$x=4z+5, \quad y=-z+3.$$

275. *To find the relation which exists among the angles which any straight line makes with the axes of co-ordinates.* Let α, β, and γ represent the angles which the straight line



makes with the axes of x, y, and z. From the origin, draw a line AP parallel to the proposed line; the angles which it makes with the co-ordinate axes will be the same as those made by the proposed line. In AP take any point P, and from it draw a line perpendicular to each of the co-ordinate planes. In the triangle APG, right-angled at G, we have $AG = AP \cos.\gamma$; also, in the triangle APF, right-angled at F, we have $AF = AP \cos.\beta$; and in the triangle APE, right-angled at E, we have $AE = AP \cos.\alpha$. But by Art. 265 we have

$$AE^2 + AF^2 + AG^2 = AP^2;$$

hence $$AP^2 \cos.^2\alpha + AP^2 \cos.^2\beta + AP^2 \cos.^2\gamma = AP^2;$$

or, dividing by AP^2, we have

$$\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1; \qquad (1)$$

that is, *the sum of the squares of the cosines of the angles which any straight line makes with the co-ordinate axes is equal to unity.*

If it is required to determine the value of each cosine, let

$$x = mz, \quad y = nz,$$

be the equations of the line AP (Art. 271). Then

$$\cos.\alpha = m \cos.\gamma, \text{ and } \cos.\beta = n \cos.\gamma.$$

Substituting these values in equation (1), we obtain

$$m^2 \cos.^2\gamma + n^2 \cos.^2\gamma + \cos.^2\gamma = 1,$$

whence $$\cos.\gamma = \frac{1}{\sqrt{m^2 + n^2 + 1}};$$

also, $$\cos.\alpha = \frac{m}{\sqrt{m^2 + n^2 + 1}},$$

and $$\cos.\beta = \frac{n}{\sqrt{m^2 + n^2 + 1}}.$$

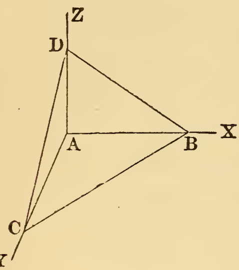
In these equations, m denotes the tangent of the angle which the projection of the proposed line upon the plane xz makes with the axis of z; and n denotes the tangent of the angle which the projection on the plane yz makes with the axis of z.

SECTION III.

OF THE PLANE IN SPACE.

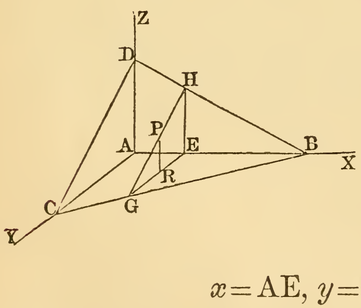
276. *The equation of a surface* is an equation which expresses the relation between the co-ordinates of every point of the surface.

Three points, not in the same straight line, are sufficient to determine the position of a plane (Geom., Bk. VII., Prop. 2, Cor. 1); hence, if we know the points where a plane BCD intersects the three co-ordinate axes, the position of the plane will be determined.



The intersections of any plane with the co-ordinate planes are called its *traces*. Thus BC is the trace of the plane BCD on the plane XY, BD is its trace on the plane ZX, and CD is its trace on the plane ZY.

277. *To find the equation to a plane.* Let AX, AY, AZ be three rectangular axes, and let BCD be the plane whose equation is required to be determined. Let the plane intersect the axes in the points B, C, D, and let AB be denoted by a, AC by b, and AD by c. Take any point P in the given plane, and through P draw the plane EGH parallel to the co-ordinate plane YZ, and cutting the given plane and the other co-ordinate planes in the triangle EGH. Draw PR perpendicular to the plane YX. Then will the co-ordinates of the point P be



$$x = \text{AE},\ y = \text{ER},\ \text{and}\ z = \text{PR}.$$

It is required to find an equation between these co-ordinates and the intercepts a, b, and c.

By similar triangles BAC, BEG, we have

$$BA : AC :: BE : EG,$$

or $$a : b :: a-x : EG.$$

Hence $$EG = b - \frac{bx}{a};$$

also, $$RG = b - y - \frac{bx}{a}.$$

Again, by similar triangles DAC, PRG, we have

$$DA : AC :: PR : RG,$$

or $$c : b :: z : b - y - \frac{bx}{a};$$

whence $$abz = abc - acy - bcx,$$

or $$bcx + acy + abz = abc, \qquad (1)$$

or $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \qquad (2)$$

which is the equation of a plane in terms of its intercepts on the three axes. This equation is similar in form to the equation of a straight line (Art. 42). If we represent the coefficients of x, y, and z in eq. (1) by A, B, and C, this equation assumes the form $$Ax + By + Cz + D = 0, \qquad (3)$$
being a complete equation of the first degree containing three variables, and this is the form in which the equation of a plane is usually written.

278. *Having given the equation of a plane, to determine the equations of its traces.* Let the equation of the plane be

$$Ax + By + Cz + D = 0;$$

then, for every point in this plane which is situated likewise in the plane of xy, that is, for every point in the trace on the plane of xy, we must have $z = 0$. Hence the equation of this trace is

$$Ax + By + D = 0. \qquad (1)$$

In like manner, for every point in the trace on the plane of xz, we must have $y = 0$; hence the equation of this trace is

$$Ax + Cz + D = 0. \qquad (2)$$

So also the equation of the trace on the plane of yz is

$$By + Cz + D = 0. \qquad (3)$$

If in equation (1) we make $y=0$, the resulting value of x, viz., $-\frac{D}{A}$, will be the distance from the origin to the point where the given plane meets the axis of x. If we make $x=0$, we have $y=-\frac{D}{B}$ for the distance from the origin to the point where the plane meets the axis of y. If in equation (2) we make $x=0$, we have $z=-\frac{D}{C}$ for the distance from the origin to the point where the plane meets the axis of z.

If $D=0$, the proposed plane must pass through the origin.

Ex. 1. Find the traces of the plane whose equation is

$$2x-5y+7z-10=0.$$

Ex. 2. Determine where the plane whose equation is

$$3x+4y+5z-60=0$$

meets the three co-ordinate axes.

Ans. $x=20$, $y=15$, $z=12$.

Ex. 3. Determine where the plane whose equation is

$$3x-4y+2z+12=0$$

meets the three co-ordinate axes.

279. *To find the equation of the plane which passes through three given points.* If in equation (2), Art. 277, we represent the coefficients of x, y, and z by M, N, and P, the equation of the plane will become

$$Mx+Ny+Pz=1. \qquad (1)$$

Let the co-ordinates of the three given points be

$$x', y', z'; \quad x'', y'', z''; \quad x''', y''', z'''.$$

Since the plane passes through the three given points, the co-ordinates of each of these points must satisfy the equation of the plane, so that we must have

$$Mx'+Ny'+Pz'=1,$$
$$Mx''+Ny''+Pz''=1,$$
$$Mx'''+Ny'''+Pz'''=1.$$

From these three equations the values of the three constants M, N, and P may be determined, and if these values are substituted in equation (1), we shall have the equation of a plane passing through the three given points.

Ex. 1. Find the equation of the plane passing through the three points

$$x'=1, \quad y'=-2, \quad z'=-3.$$
$$x''=2, \quad y''=1, \quad z''=0,$$
$$x'''=-2, \quad y'''=2, \quad z'''=-1.$$

Ans. $6x+11y-13z-23=0.$

Ex. 2. Find the equation of the plane passing through the three points

$$x'=3, \quad y'=2, \quad z'=4,$$
$$x''=0, \quad y''=4, \quad z''=1,$$
$$x'''=-2, \quad y'''=1, \quad z'''=0.$$

Ans. $11x-3y-13z+25=0.$

280. *To determine the conditions which must subsist in order that a straight line may be parallel to a plane.* Let the equations of the straight line be

$$x=mz+a, \quad y=nz+b,$$

and let the equation of the plane be

$$Ax+By+Cz+D=0.$$

If through the origin we draw a straight line parallel to the given line, its equations will be

$$x=mz, \quad y=nz;$$

and if through the origin we also draw a plane parallel to the given plane, its equation (Art. 278) will be

$$Ax+By+Cz=0.$$

Now, if the first line be parallel to the first plane, the line drawn through the origin must coincide with the plane drawn through the origin; hence the co-ordinates x and y of this straight line must satisfy the equation of the plane. If we substitute the values of x and y in the equation of the plane, we have

$$Amz+Bnz+Cz=0;$$

or, dividing by z, we have

$$Am+Bn+C=0,$$

which is the analytical condition that a right line shall be parallel to a plane.

281. *To determine the conditions which must subsist in order that two planes may be parallel.* Let the equations of the two planes be $Ax + By + Cz + D = 0$,

$$A'x + B'y + C'z + D' = 0.$$

The traces of these planes on either of the co-ordinate planes must be parallel, otherwise the two planes would meet. The equations of the traces on the plane of xz (Art. 278) are

$$z = -\frac{A}{C}x - \frac{D}{C}, \quad z = -\frac{A'}{C'}x - \frac{D'}{C'}.$$

If these traces are parallel, we must have

$$\frac{A}{C} = \frac{A'}{C'}.$$

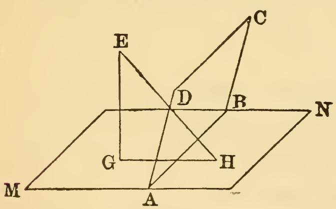
Comparing the traces on the other co-ordinate planes, we shall also find

$$\frac{B}{C} = \frac{B'}{C'}; \quad \frac{A}{B} = \frac{A'}{B'}.$$

The last equation could be derived from the two others, and hence the three equations express but two independent conditions.

282. *If a straight line be perpendicular to a plane, the projection of this line on either of the co-ordinate planes will be perpendicular to the trace of the given plane on that co-ordinate plane.*

Let MN be the co-ordinate plane, ABCD the proposed plane, EH the line perpendicular to it, and let GH be the projection of EH on the plane MN. The projecting plane EGH is perpendicular to MN; and since the line EH is in the plane EGH, the plane EGH is perpendicular to the plane BD (Geom., Bk. VII., Prop. 6). Hence the plane EGH is perpendicular to each of the planes MN and BD; it is therefore perpendicular to their common section AB (Geom., Bk. VII., Prop. 8). Hence AB, which is the trace of the given plane on the plane MN, is perpendicular to the plane EGH, and is therefore perpendicular to the line GH, which it



meets in that plane (Geom., Bk. VII., Def. 1); that is, GH, which is the projection of the given line EH, is perpendicular to AB, which is the trace of BD on the plane MN.

283. *To determine the conditions which must subsist in order that a straight line may be perpendicular to a plane.*

Let the equation of the plane be

$$Ax+By+Cz+D=0,$$

and let the equations of the projections of the straight line be

$$x=mz+a, \quad y=nz+b.$$

The equation of the trace of the plane on xz is

$$Ax+Cz+D=0,$$

or

$$x=-\frac{C}{A}z-\frac{D}{A}.$$

The equation of the trace on yz is

$$By+Cz+D=0,$$

or

$$y=-\frac{C}{B}z-\frac{D}{B}.$$

But since the projections of the line must be perpendicular to the traces of the plane (Art. 282), we shall have (Art. 46)

$$m=\frac{A}{C}, \text{ and } n=\frac{B}{C},$$

which are the conditions required.

284. *To find the equation of a plane that passes through a given point, and is perpendicular to a given straight line.*

Let x', y', z' be the co-ordinates of the given point, and let the equations to the given line be

$$x=mz+a, \text{ and } y=nz+b.$$

Also, let the equation of the plane be

$$Ax+By+Cz+D=0.$$

Since the point (x', y', z') is in this plane, we have

$$Ax'+By'+Cz'+D=0;$$

hence

$$A(x-x')+B(y-y')+C(z-z')=0,$$

which is the equation of any plane passing through the point (x', y', z'). But by Art. 283 we have

$$A = mC, \text{ and } B = nC;$$

hence $\quad mC(x-x')+nC(y-y')+C(z-z')=0,$

or $\quad m(x-x')+n(y-y')+(z-z')=0,$

which is the equation required.

285. *To find the equation of a straight line drawn from the origin perpendicular to a given plane, and determine its length.*

Let the equation of the given plane be

$$Ax+By+Cz+D=0. \qquad (1)$$

The equations of a line passing through the origin are

$$x=mz, \quad y=nz.$$

But if this line be perpendicular to the plane, we must have (Art. 283) $\quad m=\frac{A}{C}, \text{ and } n=\frac{B}{C};$

hence the equations of the line passing through the origin and perpendicular to the plane are

$$x=\frac{Az}{C}, \quad y=\frac{Bz}{C}. \qquad (2)$$

Those values of x, y, and z, which, when taken together, will satisfy equations (1) and (2) at the same time, must be the co-ordinates of a point common to the line and plane; therefore, by combining these equations, and deducing the values of x, y, and z, we shall obtain the co-ordinates of the point in which the line pierces the plane. The distance of this point from the origin may then be found by Art. 265.

If P represent the length of the perpendicular, we shall have

$$P=\pm\frac{D}{\sqrt{A^2+B^2+C^2}}.$$

Ex. 1. Find the equations of a straight line passing through the origin and perpendicular to the plane whose equation is

$$2x-4y+z-8=0.$$

Find, also, the point in which the line pierces the plane, and find the length of the perpendicular.

Ans. The equations of the line are $\quad x=2z, \quad y=-4z;$

it pierces the plane in the point $x=\frac{16}{21}$, $y=-\frac{32}{21}$, $z=\frac{8}{21}$;

and the length of the perpendicular is $\frac{8}{\sqrt{21}}$.

Ex. 2. Find the length of the perpendicular from the origin upon the plane whose equation is

$$2x+3y+4z-12=0. \qquad \textit{Ans. } \frac{12}{\sqrt{29}}.$$

286. *To find the equations of the intersection of two given planes.*

Let the equations of the two planes be

$$Ax+By+Cz+D=0,$$
$$A'x+B'y+C'z+D'=0.$$

If the given planes intersect, the co-ordinates of their line of intersection will satisfy at the same time the equations of both planes. If, therefore, we combine the two equations and eliminate z, we shall obtain an equation between x and y, which is the equation of the projection on the plane xy of the intersection of the planes.

In a similar manner we may find the equation of the projection of the intersection on the plane xz. But the equations to the projections of a line on two co-ordinate planes are the equations to the line itself; hence the two equations thus found are the required equations to the intersection.

Ex. Find the equations to the intersection of two planes of which the equations are

$$2x+5y-3z+6=0,$$
$$3x+y+z+4=0.$$

$$\textit{Ans. } \begin{cases} 13x+8z+14=0, \\ 13y-11z+10=0. \end{cases}$$

SECTION IV.

OF SURFACES OF REVOLUTION.

287. *Definitions.* A *solid of revolution* is a solid which may be generated by the revolution of a plane surface about a fixed axis.

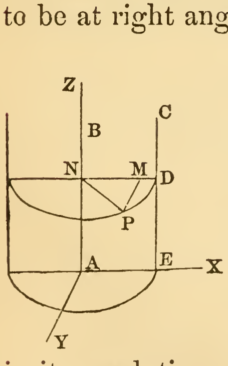
A *surface of revolution* is a surface which may be generated by the revolution of a line about a fixed axis.

The revolving line is called the *generatrix*, and the line about which it revolves is called the *axis of the surface* or *solid*, or the *axis of revolution.* The section made by a plane passing through the axis is called a *meridian section.*

It follows from the definition that every section made by a plane perpendicular to the fixed axis is a circle whose centre is in that axis.

288. The number of solids of revolution is unlimited, but those which are of most frequent use are the *cylinder*, *cone*, *sphere*, *spheroid*, *paraboloid*, and *hyperboloid.*

The equation to a surface of revolution is simplest when the axis of revolution coincides with one of the co-ordinate axes. In the following problems we shall suppose the axis of revolution to coincide with the axis of z, and the co-ordinate planes to be at right angles to each other.



289. *To find the equation to the surface of a right cylinder.* A right cylinder may be supposed to be generated by the revolution of a rectangle about one of its sides as an axis.

Let CE be one side of a rectangle, and let it revolve about the opposite side AB as an axis; it is plain that any point of CE, as D, in its revolution will describe the circumference of a circle.

Let AX, AY, AZ be the rectangular axes to which the cylinder is referred, having the origin at the centre of the base of the cylinder, and let the axis of z coincide with the axis of the cylinder.

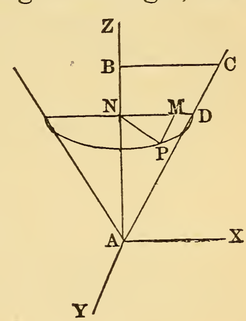
Let the co-ordinates of any point P on the surface be $AN=z$, $NM=x$, and $MP=y$; then the square on NP = the sum of the squares on NM and MP, or

$$PN^2=x^2+y^2.$$

But PN, which equals DN, is a constant quantity, and z may have any value whatever, so that the equation of a right cylinder is $\quad x^2+y^2=c^2$, z being *indeterminate.*

290. *To find the equation to the surface of a right cone.* A right cone may be supposed to be generated by the revolution of a right-angled triangle about one of its perpendicular sides as an axis, the hypothenuse generating the curved surface, and the remaining perpendicular side generating the base.

Let AC be the hypothenuse of a right-angled triangle, and let it be revolved about AB as an axis; then any point of AC, as D, in its revolution will describe the circumference of a circle.



Let the origin be placed at the vertex of the cone, and let the axis of z coincide with the axis of the cone; then, as in Art. 289, we shall have $PN^2=x^2+y^2$.

Let v represent the angle BAC, or the semiangle of the cone; then

$$NP=ND=AN \text{ tang. } CAB=AN \text{ tang. } v;$$

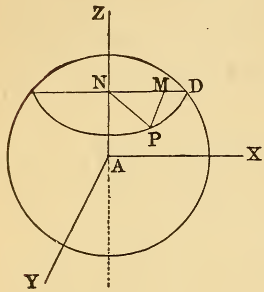
that is, $\quad x^2+y^2=z^2 \text{ tang.}^2 v,$

which is the equation of the surface of a right cone.

If the generatrix AC is of indefinite length, the whole surface generated consists of two symmetrical portions, each of indefinite extent, lying on opposite sides of the vertex. Each of these portions is called a *sheet* of the cone.

291. *To find the equation to the surface of a sphere.* The sphere is supposed to be generated by the revolution of a semicircle about its diameter.

If the centre of the sphere be at the origin of co-ordinates, then the co-ordinates of any point of the sphere, as P, are PM, MN, and AN, and we have



$$DN^2 = PN^2 = NM^2 + MP^2;$$

also,

$$AD^2 = AN^2 + ND^2 = NM^2 + MP^2 + AN^2.$$

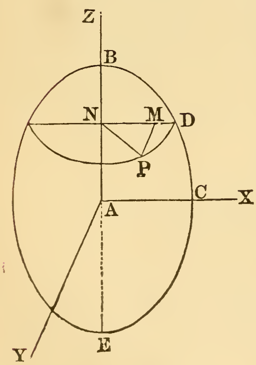
Hence, putting r for AD, the radius of the sphere, we have

$$x^2 + y^2 + z^2 = r^2,$$

which is the equation of the surface of a sphere.

292. *To find the equation to the surface of a prolate spheroid.* Spheroids are either prolate or oblate. A prolate spheroid is supposed to be generated by the revolution of an ellipse about its transverse axis. An oblate spheroid is supposed to be generated by the revolution of an ellipse about its conjugate axis.

Let BCE be an ellipse, and let it be revolved about its transverse axis; then any point of the circumference, as D, in its revolution will describe the circumference of a circle.



Let the origin be placed at the centre of the spheroid. The equation of an ellipse (Art. 121) is $a^2y^2 + b^2x^2 = a^2b^2$,

or
$$y^2 = \frac{a^2b^2 - b^2x^2}{a^2},$$

where x represents AN, which is now to be represented by z, and y represents ND, the radius of the circle described by the point D in its revolution.

Hence
$$ND^2 = \frac{a^2b^2 - b^2z^2}{a^2}.$$

But $$ND^2 = NP^2 = NM^2 + MP^2 = x^2 + y^2;$$

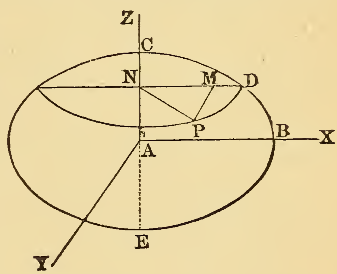
hence $$x^2 + y^2 = \frac{a^2b^2 - b^2z^2}{a^2},$$

or $$a^2(x^2 + y^2) + b^2z^2 = a^2b^2,$$

which is the equation of the surface of a prolate spheroid, where a is supposed to be greater than b.

293. *To find the equation to the surface of an oblate spheroid.*

Let the ellipse CBE be revolved about its conjugate axis CE; the point D in its revolution will describe the circumference of a circle. The equation of an ellipse is



$$x^2 = \frac{a^2b^2 - a^2y^2}{b^2},$$

where y represents AN, which is now to be represented by z, and x represents ND, the radius of the circle described by the point D in its revolution.

Hence $$ND^2 = \frac{a^2b^2 - a^2z^2}{b^2}.$$

But $$ND^2 = NP^2 = NM^2 + MP^2 = x^2 + y^2;$$

hence $$x^2 + y^2 = \frac{a^2b^2 - a^2z^2}{b^2},$$

or $$b^2(x^2 + y^2) + a^2z^2 = a^2b^2,$$

which is the equation of the surface of an oblate spheroid.

The equation of the prolate spheroid is sometimes written

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1,$$

and that of the oblate spheroid,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

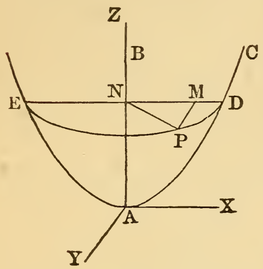
In both cases a is supposed greater than b.

If in the equation of either spheroid we make $a = b$, we shall have $$x^2 + y^2 + z^2 = r^2,$$

which is the equation of the surface of a sphere (Art. 291).

294. *To find the equation to the surface of a paraboloid.* A paraboloid is supposed to be generated by the revolution of a parabola about its axis.

Let EAC be a parabola, and let it be revolved about its axis AB; then any point on the curve, as D, in its revolution will describe the circumference of a circle. Let the origin be placed at the vertex of the parabola, and let the axis of the parabola be the axis of z.



The equation of a parabola (Art. 85) is

$$y^2 = 4ax,$$

where x represents AN, which is now to be represented by z, and y represents ND.

Hence $$ND^2 = 4az.$$

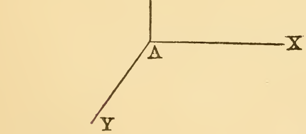
But $$ND^2 = NP^2 = NM^2 + MP^2 = x^2 + y^2;$$

hence we have $$x^2 + y^2 = 4az,$$

which is the equation of the surface of a paraboloid.

295. *To find the equation to the surface of an hyperboloid.* An hyperboloid is supposed to be generated by the revolution of an hyperbola about one of its axes.

1st. We will suppose the *hyperbola to revolve about its transverse axis.* Let CBD be an hyperbola, and let it be revolved about its transverse axis BE; then any point on the curve, as D, in its revolution will describe the circumference of a circle. Let the origin be placed at the centre of the hyperbola, and let the transverse axis of the hyperbola be the axis of z.



The equation of an hyperbola (Art. 170) is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

where x represents AN, which is now to be represented by z, and y represents ND.

Hence $$ND^2 = \frac{b^2}{a^2}(z^2 - a^2).$$

But $$ND^2 = NP^2 = NM^2 + MP^2 = x^2 + y^2;$$

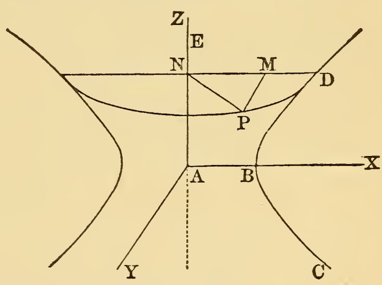
hence $$x^2 + y^2 = \frac{b^2}{a^2}(z^2 - a^2),$$

or $$b^2z^2 - a^2(x^2 + y^2) = a^2b^2,$$

which is the equation of the surface generated by revolving an hyperbola about its transverse axis. If we suppose both branches of the hyperbola to revolve, there will be generated two surfaces entirely symmetrical with respect to each other. This is therefore called the *hyperboloid of revolution of two sheets*, since it forms two surfaces entirely separate from each other.

If the asymptotes of the hyperbola also revolve around the transverse axis, they will describe the surface of a cone with two sheets. The surface of this cone will approach the surface of the hyperboloid, and will become tangent to it at an infinite distance from the centre.

2d. We will suppose the *hyperbola to revolve about its conjugate axis.* Let CBD be an hyperbola, and let it be revolved about its conjugate axis AE; then any point on the curve, as D, in its revolution will describe the circumference of a circle. Let the origin be placed at the centre of the hyperbola, and let the conjugate axis of the hyperbola be the axis of z.



The equation of the hyperbola is

$$x^2 = \frac{a^2y^2 + a^2b^2}{b^2},$$

where y represents AN, which is now to be represented by z, and x represents ND.

Hence $$ND^2 = \frac{a^2z^2 + a^2b^2}{b^2}.$$

But $$ND^2 = NP^2 = NM^2 + MP^2 = x^2 + y^2;$$

hence $$x^2 + y^2 = \frac{a^2z^2 + a^2b^2}{b^2},$$

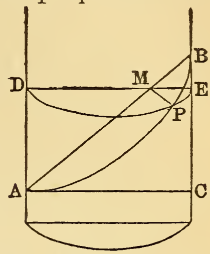
or $$a^2z^2-b^2(x^2+y^2)=-a^2b^2,$$
which is the equation of the surface generated by revolving an hyperbola about its conjugate axis. As both branches of the hyperbola are symmetrical with respect to the conjugate axis, each branch in its revolution will describe the same surface. This is therefore called the *hyperboloid of revolution of one sheet*, since it forms one uninterrupted surface.

The equations of the two hyperboloids of revolution are sometimes written $$\frac{z^2}{a^2}-\frac{x^2}{b^2}-\frac{y^2}{b^2}=1,$$
and $$-\frac{z^2}{b^2}+\frac{x^2}{a^2}+\frac{y^2}{a^2}=1,$$
where the minus sign in each case corresponds to an axis that does not meet the surface.

296. *To determine the curve which results from the intersection of a sphere with a plane.* Let d represent the distance of the intersecting plane from the centre of the sphere; let the origin be at the centre of the sphere, and let one of the co-ordinate planes, as the plane of xy, be parallel to the cutting plane; then every point in the intersecting plane will be given by the equation $z=d$, and we must have
$$x^2+y^2+d^2=r^2,$$
or $$x^2+y^2=r^2-d^2,$$
which represents all the points on the surface of the sphere which are also common to the plane. This equation represents a circle whose radius is $\sqrt{r^2-d^2}$. Hence every section of a sphere made by a plane is a circle.

Ex. A sphere whose radius is 10 inches is cut by a plane whose distance from the centre of the sphere is 6 inches. Determine the radius of the section.

297. *To determine the curve which results from the intersection of a right cylinder with a plane.* Every section of a right cylinder made by a plane parallel to the base is a circle; we will therefore suppose the section to be made by a plane inclined to the base. Let APB be such a section, and let ABC be a section of the cylinder through its axis, and perpendicular to the plane of the former section. Draw a plane perpendicular to the axis of the cylinder, intersecting the cylinder in a circle whose diameter is DE, and intersecting the first plane in PM, which will therefore be perpendicular both to AB and DE, and will be an ordinate common to the section and the circle.



Let AM$=x$, PM$=y$, AB$=2a$, AC$=2r$; then BM$=2a-x$.

We have $y^2=$DM . ME (Geom., Bk. IV., Prob. 23, Cor.); but by similar triangles we have

$$AB : AC :: AM : MD, \text{ whence } MD=\frac{rx}{a};$$

also $$AB : AC :: BM : ME, \text{ whence } ME=\frac{r}{a}(2a-x).$$

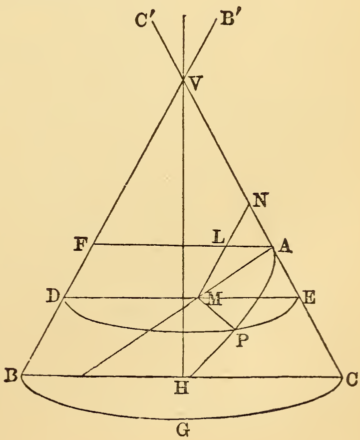
Whence $$y^2=\frac{r^2}{a^2}(2ax-x^2),$$

which is the equation of an ellipse (Art. 129).

Hence every section of a right cylinder made by a plane inclined to its base is an ellipse.

Ex. A right cylinder whose diameter is 10 inches, is cut by a plane making an angle of 30° with the axis of the cylinder. Determine the equation of the elliptic section.

298. *To determine the curve which results from the intersection of a right cone with a plane.* Let VBGC be a right cone, V the vertex, VH the axis, and BGC the circular base. Let AP be the line in which the cutting plane meets the surface of the cone, and let VBHC be a plane passing through the axis VH, and perpendicular to the cutting plane AMP. AM, the intersection of these planes,



is a straight line; and, since the curve is symmetrical with regard to it, it is called the axis of the conic section.

Let DPE be a section parallel to the base; it will be a circle, and DME, its intersection with the plane VBHC, will be a diameter.

Since the plane DPE and the plane PAM are both perpendicular to the plane VBHC, MP, the intersection of the two former, is perpendicular to the third plane, and therefore to every straight line in that plane. It is therefore perpendicular to DE and to AM. Draw AF parallel to DE, and ML parallel to VB, and let it meet VC in N.

Let $AM=x$, $PM=y$, $VA=a$; let the angle $CVH=\beta$, and the angle VAM, which is the inclination of the cutting plane to the side of the cone, $=\theta$; then the angle $AMN=180°-\theta-2\beta$.

Now AM : ME : : sin. AEM : sin. MAE, whence $ME=\dfrac{x \sin.\theta}{\cos.\beta}$;

also, $$DM=FL=AF-AL=2a \sin.\beta-AL, \text{ and}$$

AM : AL : : sin. ALM : sin. AML, whence $AL=\dfrac{x\sin.(\theta+2\beta)}{\cos.\beta}$;

therefore $$DM=2a\sin.\beta-x\frac{\sin.(\theta+2\beta)}{\cos.\beta}.$$

But by Geom., Bk. IV., Prob. 23,

$$MP^2=DM\,.\,ME;$$

hence $$y^2=\frac{x\sin.\theta}{\cos.\beta}\left\{2a\sin.\beta-\frac{x\sin.(\theta+2\beta)}{\cos.\beta}\right\}, \quad (1)$$

which is the equation of the curve resulting from the intersection of the cone by a plane.

Comparing this equation with the equation $y^2=mx+nx^2$ (Art. 234), which represents an ellipse, hyperbola, or parabola, according as n is negative, positive, or zero, we see that the section is an ellipse, hyperbola, or parabola according as the coefficient of the last term of the equation is negative, positive, or zero. In order to investigate these cases, we will suppose the cutting plane to turn about A, so as to make all possible angles with the side of the cone.

299. *Discussion of the equation to a conic section.* Equation (1) of Art. 298 will represent in succession every line which it is possible to cut from a given right cone by a plane, if we suppose β to remain unchanged, while all values are assigned to θ from 0 to 180°, and all values to a from 0 to infinity.

Case first. Let $\theta=0$; then equation (1) reduces to $y^2=0$. This is the equation to the straight line which is the axis of x, and we see from the figure that when $\theta=0$ the cutting plane becomes tangent to the cone, and the line AM coincides with AV. In this case *the section is said to be a straight line.* The same case occurs when $\theta=180°$.

Case second. Let $\theta+2\beta<180°$; then $\sin.(\theta+2\beta)$ will be positive; moreover, $\sin.\theta$ is positive so long as θ is supposed to be comprised between 0 and 180°, and $\cos.^2\beta$ is necessarily positive; hence $-\dfrac{\sin.\theta\ \sin.(\theta+2\beta)}{\cos.^2\beta}$ is negative, and equation (1) assumes the form $$y^2=mx-nx^2,$$ which is the equation of an *ellipse.* We see from the figure that in this case the angles VAM and AVF, or ANM, are together less than 180°; hence the lines VF and AM, if produced indefinitely towards the base of the cone, will meet; that is, the sectional plane cuts both sides of the cone. Hence *the section is an ellipse when the cutting plane meets both sides of the cone.* See fig. Art. 301.

Case third. In the preceding case the angle θ may be equal to the angle VAF, or $90°-\beta$, in which case $\theta+2\beta=90°+\beta$, and equation (1) reduces to $y^2=2ax\sin.\beta-x^2$, which is the equation of a circle (Art. 63). We see that in this case the cutting plane is parallel to the base, and hence *the ellipse becomes a circle* when the cutting plane is parallel to the base of the cone.

Case fourth. Let $\theta+2\beta=180°$; in this case, $\sin.(\theta+2\beta)=0$, and equation (1) becomes
$$y^2=2ax\sin.\theta\ \text{tang}.\beta,$$
which is the equation of a parabola (Art. 85). We see that in

this case $180° - \theta - 2\beta = 0$; that is, the angle AMN is zero, or the cutting plane is parallel to the side of the cone. Hence *the section becomes a parabola* when the cutting plane and the side of the cone make equal angles with the base (see fig., Art. 301).

Case fifth. Let $\theta + 2\beta > 180°$; then $\sin.(\theta + 2\beta)$ will be negative, and $-\sin.(\theta + 2\beta)$ will be positive, and equation (1) assumes the form $y^2 = mx + nx^2$,
which is the equation of an hyperbola. We see from the figure that in this case the angles VAM and ANM are together greater than 180°; hence the lines VB and AM, though produced indefinitely towards the base of the cone, will not meet, but if these lines are produced in the opposite direction they will meet; that is, the cutting plane intersects both cones, and the curve consists of two branches, one on the surface of each cone.

When $\theta = 180°$, the line AM produced returns to the same position which it had when $\theta = 0$; and when θ becomes greater than 180°, the line AM assumes the same positions already described. We therefore obtain all the possible positions of the line AM by supposing θ to be comprised between the limits 0 and 180°.

300. *Result of a change in the value of a.* The preceding results remain unchanged so long as a remains finite. When a becomes zero, the cutting plane passes through V, the vertex of the cone, and equation (1) becomes

$$y^2 = -\frac{\sin.\theta \sin.(\theta + 2\beta)}{\cos.^2\beta} x^2. \qquad (2)$$

This equation furnishes three cases:

Case first. Let $\theta + 2\beta < 180°$; then $-\sin.(\theta + 2\beta)$ will be *negative.* In this case equation (2) can only be satisfied when $x = 0$, $y = 0$, which are the equations of *a point.* A point is then to be regarded as a particular case of the ellipse. This case happens when the cutting plane, passing through the vertex V, occupies a position within the angle BVC′.

Case second. Let $\theta + 2\beta = 180°$; then $\sin.(\theta + 2\beta) = 0$, and equation (2) reduces to $y^2=0$. The section then becomes a straight line, or it may be regarded as a double line, since the equation may be written $y=\pm 0$. A straight line (or a double line) is then a particular case of the parabola.

Case third. Let $\theta+2\beta>180°$; then $-\sin.(\theta+2\beta)$ will be *positive*, and equation (2) assumes the form

$$y=\pm\frac{x}{\cos.\beta}\sqrt{-\sin.\theta\sin.(\theta+2\beta)},$$

which represents *two intersecting straight lines.* This case happens when the straight line AM, passing through the vertex V, meets BC between the points B and C. The cutting plane then meets the surface of the cone in two straight lines which pass through V. Two intersecting straight lines are then to be regarded as a particular case of the hyperbola.

301. *Results of the preceding discussion.* It appears from the preceding investigation that if a right cone be cut by a plane, the section will be

(1) A *parabola* when the plane makes an angle with the axis equal to half the vertical angle of the cone.

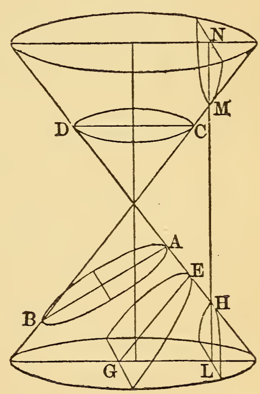
The particular case is a double line.

(2) An *ellipse* when the plane cuts only one sheet of the cone.

The particular cases are a point and a circle.

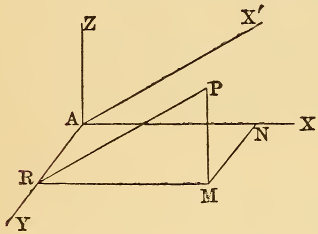
(3) An *hyperbola* when the plane cuts both sheets of the cone.

The particular case is two straight lines which intersect one another.



302. *To determine the curve which results from the intersection of any surface of revolution by a plane.* The sections of a surface made by the co-ordinate planes are called the principal sections of the surface, and the boundaries of the principal sections are called the *traces* of the surface on the co-ordinate planes. The equation to a trace is determined by putting the ordinate perpendicular to the plane of the trace $=0$ in the general equation (Art. 278). If, then, the cutting plane coincided with one of the co-ordinate planes, we could easily find the trace of the given surface upon that plane, and this would be the required curve of intersection. We may make the cutting plane coincide with one of the co-ordinate planes by a transformation of the co-ordinates. In the case of a surface of revolution, we may proceed as follows:

Through AZ, the axis of revolution, draw a plane perpendicular to the proposed section, and call this the plane xz, the origin being at A in the plane XAZ. Let AX′ represent the intersection of the cutting plane with the plane xz. The lines AX′ and AY will then be perpendicular to each other, and both will be in the cutting plane.



Let P be any point of the curve of intersection, and from P draw PM perpendicular to the plane xy, and from M draw MN perpendicular to AX. The co-ordinates of P referred to the primitive axes are

$$x = AN, \quad y = MN, \quad z = PM.$$

Let the point P be now referred to the two axes AX′, AY, which are in the plane of the given section. Through P draw PR perpendicular to AY, and join MR. The angle PRM, which we will denote by θ, is the angle which the cutting plane makes with the plane xy. The co-ordinates of P referred to the new axes are

$$x' = PR, \quad y' = AR = MN, \quad z' = 0.$$

In the right-angled triangle PMR we have

$$RM = AN = PR \cos. PRM, \quad \text{or } x = x' \cos. \theta,$$

$$PM = \quad PR \sin. PRM, \quad \text{or } z = x' \sin. \theta;$$

also we have $$MN = AR, \quad \text{or } y = y'.$$

If the origin be changed to a point in the plane xz whose co-ordinates are $x = a_{,}, \quad y = 0, \quad z = c_{,},$

these equations become $x=a_{,}+x' \cos.\theta,$

$$y=y',$$

$$z=c_{,}+x' \sin.\theta.$$

If these values be substituted for x, y, and z in the equation of the given surface, the result can only belong to points common to the surface and the cutting plane, and will therefore represent the required curve of intersection.

303. *To determine the curve of intersection of a plane and a prolate spheroid.*

The equation of the given surface (Art. 292) is

$$x^2+y^2+\frac{b^2}{a^2}z^2=b^2.$$

Substituting the values of x, y, and z found in Art. 302, this equation becomes

$$(a_{,}+x \cos.\theta)^2+y^2+\frac{b^2}{a^2}(c_{,}+x \sin.\theta)^2=b^2,$$

or
$$x^2(\cos.^2\theta+\frac{b^2}{a^2}\sin.^2\theta)+y^2+2x(\frac{b^2c_{,}}{a^2}\sin.\theta+a_{,}\cos.\theta)$$

$$=b^2-a_{,}^2-\frac{b^2c_{,}^2}{a^2}. \qquad (1)$$

Suppose now the origin to be placed on the surface of the spheroid, and in the plane xz. The section of the spheroid by the plane xz is equal to the generating ellipse; hence the coordinates of the origin must satisfy the equation of the ellipse; that is, we must have

$$a^2a_{,}^2+b^2c_{,}^2=a^2b^2,$$

or
$$b^2-a_{,}^2-\frac{b^2c_{,}^2}{a^2}=0.$$

The second member of equation (1) reduces therefore to zero, and the equation is of the form

$$y^2=mx-nx^2,$$

and therefore represents an ellipse. If $\theta=0$, the equation becomes
$$y^2=2ax-x^2,$$
which is the equation of a circle.

Hence every section of a prolate spheroid by a plane is an ellipse, except when the cutting plane is perpendicular to the axis of revolution, when the section becomes a circle.

The same is true of the sections of an oblate spheroid.

Ex. The two axes of a prolate spheroid are 8 and 6, and the spheroid is cut by a plane passing through the extremities of the axes, and perpendicular to their plane. Required the axes of the curve of intersection. *Ans.* 5 and $3\sqrt{2}$.

304. *To determine the curve of intersection of a plane and a paraboloid of revolution.* The equation of the given surface (Art. 294) is

$$x^2+y^2=4az.$$

Substituting the values of x, y, and z given in Art. 302, this equation becomes

$$(a_{,}+x\cos.\theta)^2+y^2=4a(c_{,}+x\sin.\theta),$$

or

$$x^2\cos.^2\theta+y^2+(2a_{,}\cos.\theta-4a\sin.\theta)x=4ac_{,}-a_{,}^2. \qquad (1)$$

Suppose now the origin to be placed on the surface of the paraboloid, and in the plane xz; the co-ordinates of the origin must satisfy the equation of the generating parabola, and we must have $a_{,}^2=4ac_{,}$, or $4ac_{,}-a_{,}^2=0$.

Equation (1) therefore reduces to the form

$$y^2=mx-nx^2,$$

and generally represents an ellipse. If $\theta=0$, the equation becomes

$$y^2=2ax-x^2,$$

which is the equation of a circle.

If $\theta=90°$, the equation becomes

$$y^2=4ax,$$

which is the equation of a parabola. Hence the section of the paraboloid by a plane is a parabola, when the plane is parallel to the axis of revolution; it is a circle when the plane is perpendicular to this axis; and in all other positions of the cutting plane the section is an ellipse.

Ex. A paraboloid whose axis of revolution is $45\frac{5}{7}$, and its base, or greatest double ordinate, 32, is cut by a plane passing through the edge of the base, and meeting the opposite side of the solid at the height of 20 above the base. Required the axes of the section. *Ans.* 34.4 and 28.

305. *To determine the curve of intersection of a plane and an hyperboloid of revolution.* We will suppose the solid to be the *hyperboloid of two sheets* (Art. 295). The equation of the given surface is

$$x^2+y^2-\frac{b^2}{a^2}z^2=-b^2.$$

Substituting the values of x, y, and z given in Art. 302, this equation becomes

$$(a_{,}+x\cos.\theta)^2+y^2-\frac{b^2}{a^2}(c_{,}+x\sin.\theta)^2=-b^2,$$

or

$$x^2(\cos.^2\theta-\frac{b^2}{a^2}\sin.^2\theta)+y^2-2x(\frac{b^2c_{,}}{a^2}\sin.\theta-a_{,}\cos.\theta)$$
$$=\frac{b^2c_{,}^2}{a^2}-b^2-a_{,}^2. \qquad (1)$$

If we place the origin on the surface of the hyperboloid, and in the plane xz, the second member of this equation reduces to zero, and the equation is of the form

$$y^2=mx-nx^2.$$

If $\theta=0$, the equation becomes

$$y^2=2ax-x^2,$$

which is the equation of a circle.

If $\theta=90°$, the equation becomes

$$y^2=\frac{b^2}{a^2}(x^2+2c_{,}x),$$

which is the equation of an hyperbola.

If $\tan g.\,\theta=\frac{a}{b}$, the equation reduces to

$$y^2=2x(c_{,}\cos.\theta\ \text{cotang.}\,\theta-a_{,}\cos.\theta),$$

which is the equation of a parabola.

If $\text{tang.}^2\theta<\frac{a^2}{b^2}$, the curve is an ellipse; if $\text{tang.}^2\theta>\frac{a^2}{b^2}$, the curve is an hyperbola.

In every case the section of the hyperboloid by a plane is similar to the corresponding section of the cone formed by the revolution of the asymptotes of the hyperbola (Art. 295).

306. *Summary of the preceding results.* The equation to the surface of an oblate spheroid (Art. 293) may be written

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1, \qquad (1)$$

and that of a prolate spheroid,

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}=1. \qquad (2)$$

The equation to the surface of an hyperboloid of one sheet (Art. 295) is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=1, \qquad (3)$$

and that of an hyperboloid of two sheets is

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{z^2}{a^2}=-1. \qquad (4)$$

The equation to the surface of a right cone (Art. 290) is

$$x^2+y^2-z^2 \operatorname{tang.}^2 v=0;$$

if we divide by a^2, and put b^2 for $\frac{a^2}{\operatorname{tang.}^2 v}$, the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0. \qquad (5)$$

The equation to the surface of a paraboloid (Art. 294) is

$$x^2+y^2-4az=0;$$

if we divide by a^2, and put b for $\frac{a}{4}$, the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z}{b}=0. \qquad (6)$$

In each of these six equations the coefficients of x^2 and y^2 are equal, which shows that for each of these solids a section perpendicular to the axis of z is a circle.

307. *More general form of the preceding equations.* If we suppose the coefficients of x^2 and y^2 in either of these equations to be unequal, we shall have a new equation similar in form to the preceding, but representing a more complex surface. The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (1)$$

represents a surface similar in some respects to that of the spheroid, but its intersection with a plane perpendicular to the axis of z is an ellipse instead of a circle. All sections made by parallel planes are similar ellipses, and the surface is closed in every direction. This solid is called an *ellipsoid*, and has three unequal axes. When two of the axes are equal to each other it is called an ellipsoid of revolution, because it may be generated by the revolution of an ellipse about one of its axes.

The equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (2)$$

represents a surface like the hyperboloid of one sheet, except that the sections perpendicular to the axis of z are ellipses instead of circles.

So also the equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \qquad (3)$$

represents a surface like the hyperboloid of two sheets, but the sections perpendicular to the axis of z are ellipses.

The equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \qquad (4)$$

represents a conical surface, but the cone has an elliptic base instead of a circular one.

The equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0 \qquad (5)$$

represents a surface like the paraboloid of revolution, except that a section perpendicular to the axis of z is an ellipse instead of a circle. This solid is called an *elliptic paraboloid.*

Each of these surfaces may be conceived to be derived from the corresponding surface of revolution by increasing or diminishing the values of y in a constant ratio, in the same manner as oblate and prolate spheroids may be derived from the sphere by multiplying the values of y by a constant factor, or as the ellipse may be derived from the circle by multiplying the values of y by a constant factor.

308. *Surface of a cone asymptotic.* The conical surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

is asymptotic on the one side to the hyperboloid of one sheet whose equation is
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$
and on the other side to the hyperboloid of two sheets whose equation is
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1.$$

There is also a similar relation between the equations of two conjugate hyperbolas and the equation of their asymptotes. The equation of an hyperbola (Art. 170) may be written
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$
and the equation of its conjugate hyperbola (Art. 179) is
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$$
while the equation of their asymptotes (Art. 214) is
$$a^2y^2=b^2x^2,$$
or
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

SECTION V.

GENERAL EQUATION OF THE SECOND DEGREE BETWEEN THREE VARIABLES.

309. The general equation of the second degree between three variables is of the form

$$ax^2+bxy+cy^2+dz^2+exz+fyz+gx+hy+kz+l=0. \quad (1)$$

We may transform this equation into another, in which the axis of z remains unchanged, by employing the equations of transformation for plane co-ordinates (Art. 55), and we shall have

$$z=z'$$
$$x=x' \cos. \theta-y' \sin. \theta$$
$$y=x' \sin. \theta+y' \cos. \theta.$$

If we substitute these values of the variables in equation (1), the only terms in the resulting equation which can contain the product $x'y'$ will come from the three terms $ax^2+bxy+cy^2$. The term containing xy may therefore always be made to disappear from equation (1) by the method explained in Art. 230.

So, also, the term containing xz may always be made to disappear by a new transformation, in which the new axis of y remains unchanged; and in the same manner the term containing yz may be made to disappear. Hence equation (1) can always be transformed into an equation of the form

$$Ax^2+By^2+Cz^2+Dx+Ey+Fz+G=0. \quad (2)$$

If, in equation (2), neither A, B, nor C is zero, we may, as in Art. 229, cause the terms containing the first powers of x, y, and z to disappear by changing the origin of the co-ordinates, and the equation will be reduced to the form

$$Lx^2+My^2+Nz^2+P=0. \quad (3)$$

310. *Classification of the surfaces represented by the equation* (3). In discussing equation (3) we must suppose each of the coefficients to be either plus or minus, and we must also consider the case in which P reduces to zero. Now two of the coefficients L, M, and N must always have the same sign; we will suppose that L and M have the same sign, and will make these signs positive. We may then have the six following cases:

1. When N is plus and P minus. Equation (3) will then take the form $Lx^2+My^2+Nz^2-P=0.$

If we divide by P, and put $a^2=\frac{P}{L}$, $b^2=\frac{P}{M}$, and $c^2=\frac{P}{N}$, we shall have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which is the equation of the surface of an *ellipsoid* (Art. 307, eq. 1).

2. When N is plus and P plus. Equation (3) will then become $Lx^2+My^2+Nz^2+P=0,$
in which all the terms are positive. Hence the equation can not be satisfied for real values of the variables, and therefore the surface becomes *imaginary*.

3. When N is plus and P is zero. Equation (3) will then become $Lx^2+My^2+Nz^2=0,$
which can only be satisfied by the values

$$x=0, \qquad y=0, \qquad z=0;$$

and hence this supposition reduces the surface to a *point*, viz., the origin.

4. When N is minus and P is minus. Equation (3) will then become $Lx^2+My^2-Nz^2-P=0.$

If we divide by P, and put $a^2=\frac{P}{L}$, $b^2=\frac{P}{M}$, and $c^2=\frac{P}{N}$, we shall have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

which represents the surface of an *hyperboloid of one sheet* (Art. 307, eq. 2).

5. When N is minus and P is plus. Equation (3) becomes

$$Lx^2+My^2-Nz^2+P=0.$$

Substituting as before, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1,$$

which represents the surface of an *hyperboloid of two sheets* (Art. 307, eq. 3).

6. When N is minus and P is zero. Equation (3) becomes

$$Lx^2+My^2-Nz^2=0,$$

which by substitution becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

which represents the surface of *a cone having an elliptic base* (Art. 307, eq. 4).

311. *Particular cases of the general equation.* If both terms containing one variable, as z, are wanting from eq. (2), Art. 309, that is, if C and F are zero, all sections of the surface perpendicular to the axis of z are equal to each other, since the equation is independent of z. The common equation of these sections is $Ax^2+By^2+Dx+Ey+G=0$, which may represent either of the conic sections (Art. 233). This surface is called a *cylindrical surface*, and may be described either—

1. By the above-named conic section moving always parallel to itself and along a right line parallel to the axis of z, or

2. By a straight line which moves along the conic section, and in all of its positions is parallel to the axis of z.

The conic section is called the *base* of the cylinder, and the cylinder is called *circular*, *elliptic*, *hyperbolic*, or *parabolic*, according to the nature of the base.

When the equation $Ax^2+By^2+Dx+Ey+G=0$ represents two straight lines (Art. 233), the cylindrical surface becomes two planes, which may intersect or be parallel, or may coincide as a double plane.

When two of the three coefficients A, B, and C in eq. (2), Art. 309, are zero, as B and C, one of the corresponding terms Ey and Fz may be made to disappear by a transformation in which x remains unchanged, but the axes of y and z are changed in the plane yz, and the resulting equation is that of a cylinder, as above.

312. *Elliptic and hyperbolic paraboloids.* The only remaining case of eq. (2), Art. 309, is when two of the coefficients, as A and B, are finite, and the third is zero. The first powers of x and y can then be made to disappear by changing the origin of x and y, and the constant term may be made to disappear by changing the origin of z. The equation will then become

$$Ax^2+By^2+Fz=0,$$

which may be written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z}{c}=0.$$

If A and B have like signs, the surface is that of an *elliptic paraboloid;* if A and B have unlike signs, every cross section perpendicular to the axis of z becomes an hyperbola, and the surface is called an *hyperbolic paraboloid.*

313. *How an elliptic or hyperbolic paraboloid may be described.* A parabola may be regarded as the limiting case of an ellipse, one vertex of which is fixed, and the other is removed to an indefinitely great distance. So, also, the elliptic paraboloid may be regarded as an ellipsoid, one of whose axes has been indefinitely increased, while one vertex of that axis remains fixed.

The elliptic paraboloid may be regarded as described by one parabola moving upon another. Thus, let the plane of one parabola be at right angles to the plane of another; let the axes of the two parabolas coincide, and the concavities be turned in the *same* direction. Then, if one of the parabolas move so as to be always parallel to itself and to have its vertex upon the fixed parabola, the surface described by the movable parabola will be an *elliptic paraboloid.*

But if the concavities of the two parabolas are turned in *opposite* directions, the corresponding surface thus described will be an *hyperbolic paraboloid.*

314. *Section of a surface of the second degree by a plane.* Every intersection of a plane with a surface of the second degree is either a straight line or one of the conic sections.

For by one or two transformations of co-ordinates like those of Art. 309 we can refer the surface to a new system of co-ordinates, one of which, as z, will be parallel to the given intersecting plane. In these transformations it is evident that the degree of the equation can not be increased, since the values substituted for x, y, and z are always of the first degree. If now we substitute for z in the transformed equation the distance of the intersecting plane from the plane xy, we shall have an equation between x and y, which is the equation of the intersection of the plane and surface. The degree of this equation does not exceed the second, and therefore (Art. 233) the curve must be either a straight line or a conic section.

The conic section may, however, in special cases, break up into two lines, as shown in Art. 233.

APPENDIX.

ON THE GRAPHICAL REPRESENTATION OF NATURAL LAWS.

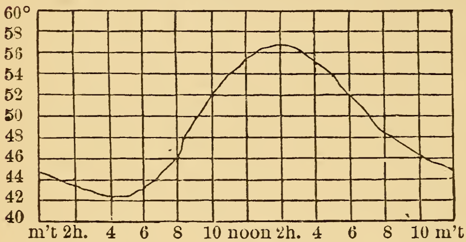
THE mutual dependence existing between any two or more variable quantities may be exhibited by means of curve lines. If, for example, we have a large collection of meteorological observations showing the temperature at any place for each hour of the day, the nature of the relations or laws expressed by these numbers may be represented by curve lines. Such a mode of representation frequently renders these laws perfectly obvious, and sometimes suggests relations which might easily have been overlooked in a large mass of figures arranged in tables. There is a great variety in the modes by which this representation may be effected. The following are some of the methods most frequently employed:

I. *Relations of two variables expressed by rectangular co-ordinates.* If on a horizontal line we set off distances proportional to the values of one of the two variables, regarding these as abscissas, and from the several points of division erect perpendiculars whose lengths are proportional to the corresponding values of the other variable, and then draw a continuous curve line through the extremities of these perpendiculars, this curve line may be regarded as representing the relation between the two variables. The cases of this nature most frequently occurring are those in which time is one of the variables, and this is usually laid off upon the axis of abscissas.

Ex. 1. *Diurnal change of temperature.* Let it be proposed to construct the curve which represents the relation between the different hours of the day and the corresponding mean temperature at a given place. The following table shows the mean temperature at New Haven for each hour of the day, as deduced from a long series of observations:

Hour.	Temp.	Hour.	Temp.	Hour.	Temp.	Hour.	Temp.
Midnight	45°.0	6 A.M.	43°.1	Noon.	55°.3	6 P.M.	52°.0
1 A.M.	44 .3	7 "	44 .6	1 P.M.	56 .1	7 "	50 .2
2 "	43 .6	8 "	46 .9	2 "	56 .5	8 "	48 .7
3 "	43 .1	9 "	49 .7	3 "	56 .3	9 "	47 .5
4 "	42 .7	10 "	52 .2	4 "	55 .4	10 "	46 .5
5 "	42 .6	11 "	54 .0	5 "	53 .9	11 "	45 .7

In order to represent these observations by a curve line, we draw upon a sheet of paper a horizontal line, and divide it into twenty-four equal parts, to represent the hours of the day, and through these points of division we draw a system of vertical lines. Upon each of these vertical lines we set off a distance proportional to the height of the thermometer for the corresponding hour, and then connect all the points thus determined by a continuous line. The curve thus formed represents the mean motion of the thermometer at New Haven for the different hours of the day, and, if constructed with proper care, and upon a scale of suitable size, may supply the place of the numbers from which it was derived, the temperatures being indicated by the numbers on the left of the diagram. In order to avoid confusion, the ordinates in the diagram have only been drawn for the alternate hours.



We readily perceive from the figure that on each day there is a maximum and a minimum of temperature, the maximum occurring generally about two hours after noon, and the minimum about an hour before the rising of the sun. We see, also, that the temperature is increasing during nine hours of the day, and decreasing during the remaining fifteen hours of the day.

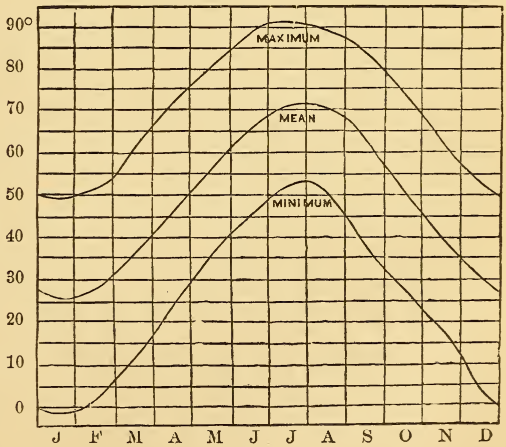
This curve readily shows us the two periods of the day when any given temperature is attained; as, for example, a temperature of 50°, 52°, etc. It also shows, not only the mean temperature at the hours of observation, but also for any time intermediate between these hours; as, for example, for each half hour, or quarter hour, etc.

Ex. 2. *Annual change of temperature.* In the same manner we may construct the curve representing the connection between the different months of the year and the corresponding temperature at a given place. We draw a horizontal line, and divide it into twelve equal parts, to represent the months of the year, and through these points of division draw a system of vertical lines, upon which we set off distances proportional to the heights of the thermometer for the corresponding months.

The following table shows the mean temperature of New Haven for each month of the year, as deduced from a long series of observations. It also shows the average maximum temperature of each month, and the average minimum temperature of each month:

	Mean Temp.	Maximum Temp.	Minimum Temp.		Mean Temp.	Maximum Temp.	Minimum Temp.
Jan....	26°.5	49°.6	−1°.0	July..	71°.7	90°.8	52°.8
Feb....	28 .1	51 .3	+1 .0	Aug..	70 .3	88 .6	50 .0
March.	36 .1	61 .6	10 .7	Sept..	62 .5	83 .6	37 .6
April..	46 .8	72 .6	25 .4	Oct...	51 .1	73 .2	26 .7
May...	57 .3	81 .3	35 .5	Nov..	40 .3	63 .2	17 .7
June..	67 .0	89 .3	45 .9	Dec..	30 .4	53 .1	4 .5

In the annexed figure, the middle curve line shows the mean temperature of each month of the year, according to the preceding observations, while the upper curve shows the average maximum temperature, and the lower curve the average minimum temperature for each month of the year.



These curves inform us that at New Haven the warmest months of the year are July and August, and the maximum for the year occurs near July 24th. The coldest month of the year is January, and the minimum for the year occurs near Jan. 21st. The difference between the minimum and the maximum for each month is greater in the cold months than in the warm months. Various other particulars respecting the connection between the temperature and the season of the year are also exhibited by the figure more palpably than by a column of numbers in a table.

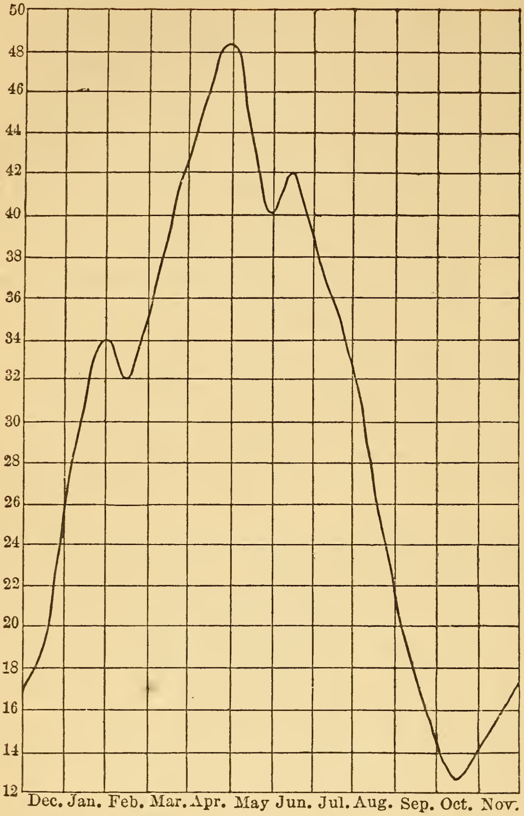
The same mode of representation may be employed to exhibit the relation between the height of the barometer and the hour of the day or the season of the year; also for the amount of vapor in the atmosphere, the force of the wind, the fall of rain or snow, the prevalence of cloud or fog, the intensity of atmospheric electricity, the declination or dip of the magnetic needle, or the intensity of terrestrial magnetism, or, indeed, any natural phenomenon which depends on the course of the sun.

Ex. 3. *Display of November meteors.* On the morning of Nov. 14, 1866, a remarkable display of meteors was witnessed in England, and the sudden increase, as well as the equally sudden decline in the number of meteors, is exhibited by a curve line much more strikingly than could be done by a simple numerical statement. For this purpose we draw a horizontal line, and divide it into equal parts, to represent the hours of observation, and through the points of division we draw a system of vertical lines. On these vertical lines we set off distances proportional to the number of meteors counted each minute, and through the points thus determined we draw a continuous curve line. The numbers on the left margin of the figure on the opposite page indicate the number of meteors visible each minute. From the diagram we perceive that before midnight the number of meteors did not exceed 5 per minute, but soon after midnight the number rapidly increased, and at 1h. 20m. exceeded 120 per minute; by 2 A.M. it had declined to 40 per minute, and by 3 A.M. to 10 per minute.

A similar mode of representation may be advantageously employed to exhibit the results of a large mass of observations, even though we have no previous knowledge of the laws which govern their changes. We may thus exhibit the influence of the day or the season of the year upon mortality; we may exhibit the average number of deaths at different ages; or we may exhibit the fluctuations in the price of any article of merchandise, as wheat, cotton, gold, etc.

Ex. 4. *Annual change in the depth of rivers.* The depth of the water in the Mississippi River fluctuates greatly with the season of the year. During the early part of autumn the water is usually lowest, and it is highest some time in the spring or the early part of summer. The figure on the following page shows the average result of twenty-three years of observations on the river at Natchez, Miss. The months are shown at the bottom of the figure, while the depth of water is indicated by the numbers on the left margin.

We see from this figure that the water is usually lowest in October, when its depth is only 12.5 feet. From this time the water rises pretty steadily to the first of May, when the depth amounts to 48.3 feet, from which time it declines pretty steadily

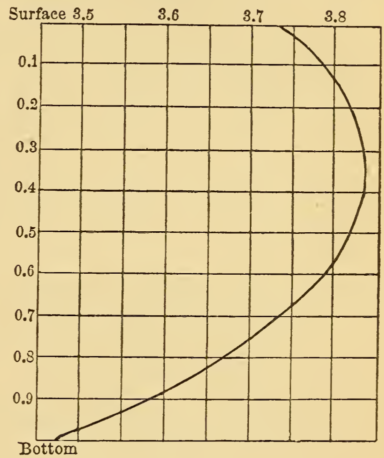


till the following October. There are, however, two smaller maxima which are well marked, viz., one about the 1st of February, and the other about the middle of June. These great fluctuations of the Mississippi are due not so much to an excess of rain near the time of maximum height as to the melting of the snow accumulated upon the numerous tributaries of this river.

Ex. 5. *Velocity of the current of a river at different depths.* It has been found by experiment that the velocity of the current in rivers varies sensibly with the depth. This may be shown by means of floats immersed to different depths in the water. The following is one mode of performing the experiment: A keg 15 inches high and 10 in diameter, without top or bottom, is ballasted with lead so as to sink and remain upright in the water; the keg is attached by a small cord to a mass of cork 8 inches square and 3 inches thick, and a small flag is supported by the cork, in order that it may be more readily observed at a distance. By varying the length of the cord, the keg may be made to sink to any required depth, and its size is so much greater than that of the surface-float that the latter does not sensibly affect the rate of movement.

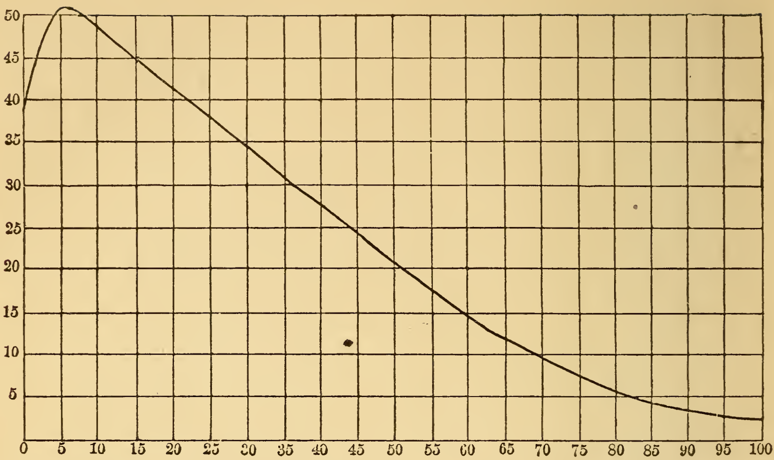
The apparatus being placed in the water, its rate of motion is determined by observers stationed on the bank of the river at known distances from each other, and watching the progress of the float by means of theodolites.

The curve line on the annexed figure shows the result of experiments made on the current of the Mississippi near New Orleans. The numbers on the left margin show the depth of the keg, expressed in *tenths* of the entire depth of the river, the mean depth of the water being 86 feet. The numbers at the top of the figure show the velocity of the current, expressed in miles and tenths of a mile per hour.



We see from the figure that the velocity at the surface is 3.74 miles per hour; the velocity increases as we descend, until we reach a depth about one third that of the river, where the velocity amounts to 3.84 miles per hour, while below this depth the velocity diminishes, and at the bottom of the river is reduced to 3.47 miles per hour.

Ex. 6. *Average duration of human life.* The average duration of life may be deduced from tables which show the number of deaths which occur each year out of a given number of individuals. If there were a million of births in the year 1770, and we had a record of the number of deaths out of this company for each year to the present time, we could construct a table showing the average duration of life for each age. The average duration of life for a person of a certain age is understood to be the average number of years which the survivors of that age should live. The duration of life is different in different countries. The curve line in the following figure shows the average duration of life as deduced from

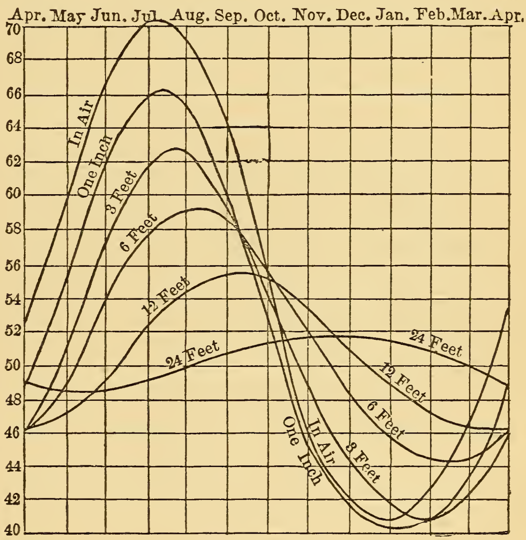


observations made at Carlisle, Eng. The numbers at the bottom of the figure show the age of the individual from 0 to 100 years, and the numbers on the left margin show the average duration of life. This average duration of the life of individuals after any specified age is called *the expectation of life.*

We see from the figure that the average duration of life for an infant just born is 38 years. If the child survives, its expectation of life increases for a few years, and attains its maximum at the age of 5, when the average duration of life is 51 years. After this age the average duration of life diminishes steadily and pretty uniformly until death. At the age of 25 the average duration of life is 38 years, at 50 it is 21 years, at 75 it is 7 years, and at 100 it is 2 years.

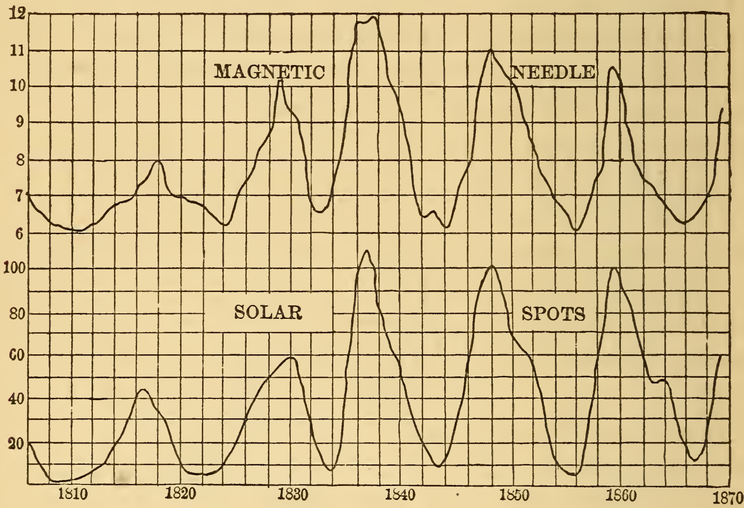
II. *Relations of several variables depending upon a common variable.* When we have several variable elements depending upon a common variable, we may graduate a horizontal line to represent successive values of the common variable, and then construct a number of curve lines to represent the changes in each of the other variables. A comparison of the different curves will show not only the relation of each variable to the common variable, but also the mutual relation of the several variables.

Ex. 1. *Temperature below the earth's surface.* Suppose we wish to discover how the diurnal and annual changes of temperature are modified by depth below the surface of the earth. For this purpose we require observations of temperature made at different depths below the earth's surface, and continued at least throughout an entire year. Such observations have been made at several places in Europe. Thermometers with very long stems have been buried at depths of 24, 12, 6, and 3 French feet, and 1 inch, and the observations have been continued for many years. The annexed figure presents a summary of such observations continued for 14 years at Greenwich, the months being given at the top of the figure, and the temperatures on the left margin.



We perceive from the figure that at a depth of about 6 feet the annual range of temperature is only about half what it is at the surface; at the depth of 12 feet the annual range of temperature is less than one third, and at the depth of 24 feet it is only one ninth what it is at the surface. We also perceive that the highest temperature of the year occurs later and later as we descend below the surface of the earth. At the depth of 12 feet the maximum temperature of the year occurs about the last of September, and the minimum about the last of March, while at the depth of 24 feet the maximum occurs about the first of December, and the minimum about the first of June.

Ex. 2. *Declination of the magnetic needle and the solar* *spots.* The surface of the sun often exhibits black spots of irregular form and variable size. The number of these spots varies greatly in different years; sometimes the sun is entirely free from them, and continues thus for months together, while some years the sun is never seen entirely free from spots. The curve in the lower part of the annexed figure presents a sum-



mary of observations of the spots for a period of 64 years, the dates being given at the bottom of the figure, while the frequency of the spots is exhibited on the left margin by a scale of numbers extending from 0 to 100. We readily perceive that the spots are subject to a certain periodicity, the number of the spots increasing for 5 or 6 years, and then decreasing for several years, showing alternate maxima and minima. The maxima occurred in 1817, 1830, 1837, 1848, and 1860, while the minima occurred in 1810, 1823, 1833, 1843, 1856, and 1867.

A magnetic needle, when freely suspended and carefully observed from hour to hour, exhibits a small daily oscillation varying from 5′ to 15′. The extent of this oscillation varies with the season of the year, and the mean annual range varies from one year to another. The curve in the upper part of the above figure shows the results of observations of the magnetic needle made in Europe for a period of 64 years, the dates being shown at the bottom of the figure, and the mean daily average of the needle being shown by numbers on the left margin, which represent minutes of arc.

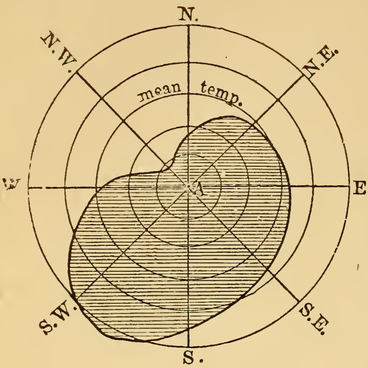
We see from the figure that the range of the needle, which was only 6′ in 1810, had increased to 8′ in 1818, had decreased again to about 6′ in 1824, and increased to 10′ in 1829, etc. In other words, the annual range of the magnetic needle shows alternate maxima and minima, and the times of these maxima correspond remarkably with the maxima of the solar spots, suggesting the idea that the two phenomena are dependent upon a common cause. Such a mode of representation by curve lines is well calculated to show the connection between two different classes of phenomena.

III. *Relations of two variables expressed by polar co-ordinates.* The relations between two variable elements may be expressed by means of polar co-ordinates, and this method is generally to be preferred when one of the variables denotes *direction*; for example, if one of the variables is the direction of the wind, and the other variable is the corresponding mean height of the barometer, or thermometer, or hygrometer. For example, suppose we wish to show the dependence of the temperature of the air upon the direction of the wind.

Ex. 1. *Influence of the wind on temperature.* From a comparison of several years of observations, it has been found that at New Haven the temperature of the air during the prevalence of winds from the eight principal points of the compass differs from the mean temperature of the year by the quantities shown in the annexed table:

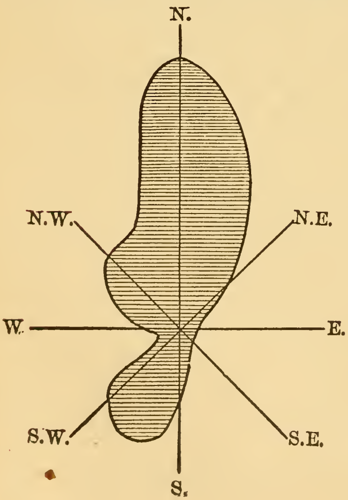
Wind.	Temperature.	Wind.	Temperature.
North........	−2°.7	South........	+3°.2
Northeast....	−0 .6	Southwest....	+4 .0
East.........	+0 .5	West.........	−1 .1
Southeast....	+1 .2	Northwest....	−4 .5

In order to represent these results by a curve line, we draw



eight radii inclined to each other in angles of 45°, to represent the directions of the wind. With the point A as a centre, we draw a series of equidistant circumferences, to represent differences of temperature, and then, having selected one of these to represent the mean temperature of New Haven, we set off upon the eight radii distances proportional to the numbers in the preceding table. When the numbers are negative, we set them off *towards* the centre of the circle; when they are positive, we set them off *from* the centre. The curve line passing through the eight points thus determined shows the influence of the wind's direction upon the temperature of the air. We perceive that the highest temperature accompanies a wind from S. 33° W., and the lowest temperature corresponds to a wind from the point N. 40° W., the mean difference in the temperature of these two winds being 8°.7.

Ex. 2. *Direction of the prevalent wind.* The prevalent wind at any station may be graphically represented by means of polar

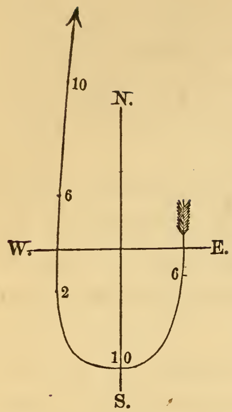


co-ordinates. Suppose we have a long series of observations of the wind from which we deduce the number of times the wind was observed to blow from the north point; also the number of times it blew from the northwest, the number of times from the west, and so on, for 8 or 16 points of the compass. We draw two lines at right angles to each other to represent the cardinal points, and also other lines to represent the intermediate directions. From the point of intersection we set off on these lines distances corresponding to the relative frequency of the winds from these different points of the compass. The curve line passing through the points thus determined may be regarded as showing the prevalent wind for that station.

The preceding figure shows the results of observations made during the month of January for several years at Wallingford, near New Haven. We see that the prevalent wind is almost exactly from the north, but that winds from the S.S.W. are also of frequent occurrence.

This mode of representation is valuable when we wish to exhibit the peculiarities of a large number of stations. The eye is thus able at a glance to detect characteristic peculiarities which might be easily overlooked in a large collection of numerical results.

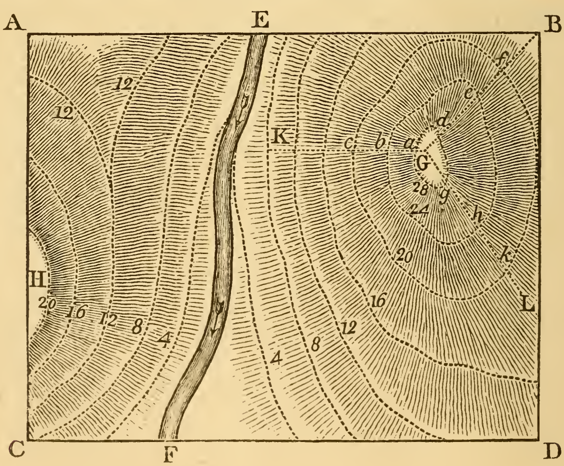
Ex. 3. *Diurnal change in the direction of the wind.* Another mode of representation, bearing some resemblance to the preceding, may be advantageously employed to denote the connection between the hour of the day and the corresponding direction of the wind. Suppose, from a long series of observations, we have determined the mean direction of the wind for each hour of the day. Having drawn two lines at right angles to each other to represent the cardinal points of the compass, we begin with the observation for the first hour, and draw a line of any convenient length to represent the wind's direction at that hour; from the extremity of this line we draw a line of the same length as before, to represent the wind's direction at the second hour, and in the same manner we set off the directions of the wind for each of the twenty-four hours. We thus construct a broken line, which may be regarded as representing the average progress of a particle of air for each hour of the day, supposing the wind's velocity to have been the same at all hours; or, if we have observations showing the wind's velocity for each hour, we may make the portions of the curve which represent the wind's direction for the different hours represent not only its direction, but, at the same time, its velocity.



The annexed figure shows the mean direction of the wind at New Haven for the different hours of the day during the month of August. We perceive that early in the morning the average direction of the wind for this month is from the north, while during the afternoon its average direction is from the south, and about 10 A.M. the wind veers from N. to S., going round by the east. This diurnal change in the wind's direction constitutes what is commonly known by the name of a "land and sea breeze." The change in the wind's direction for the other months of the year may be represented in a similar manner.

IV. *Contour lines and geographical distribution.* If it is required to represent upon a map the undulations in the surface of a tract of land, we suppose the surface of the ground to be intersected by a number of horizontal planes at equal distances from each other, and we delineate on paper the curve lines in which these planes intersect the surface.

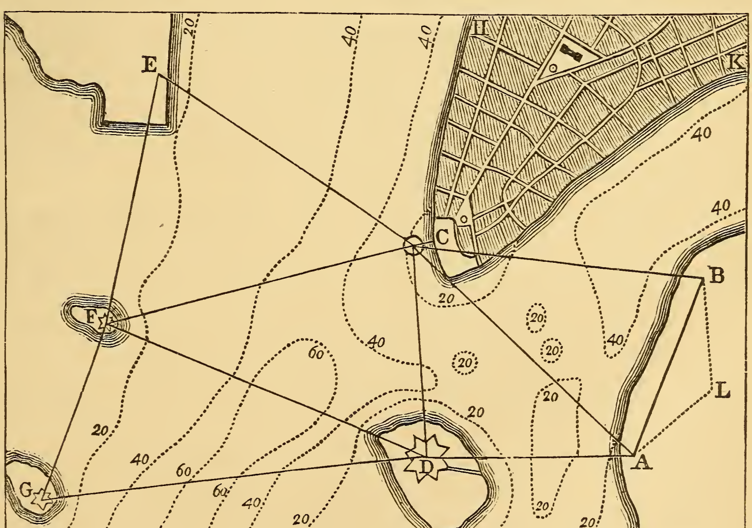
Ex. 1. *Survey of an undulating surface.* This method will be understood from the annexed figure, which represents a



tract of broken ground divided by a stream, EF. The ground is supposed to be intersected by a horizontal plane four feet above F, the lowest point of the tract, and this plane intersects the surface of the ground in the undulating lines marked 4, one on each side of the stream. A second horizontal plane is supposed to be drawn eight feet above F, and this intersects the surface of the ground in the lines marked 8. In like manner, other horizontal planes are drawn at distances of 12, 16, etc., feet above the point F. The projection of these lines upon paper shows at a glance the outline of the tract.

Ex. 2. *Depth of water in a harbor.* If we have soundings showing the depth of water at numerous points of a harbor, the results may be delineated on paper in a similar manner. We draw a curve line joining all those points where the depth of water is the same—for example, 10 feet. We draw another line connecting all those points where the depth of water is 20 feet; also other lines for 30 feet, 40 feet, etc.

The accompanying figure represents a portion of New York Harbor, and the dotted lines show depths of 20, 40, and 60 feet of water. We see that along the channel of the North River



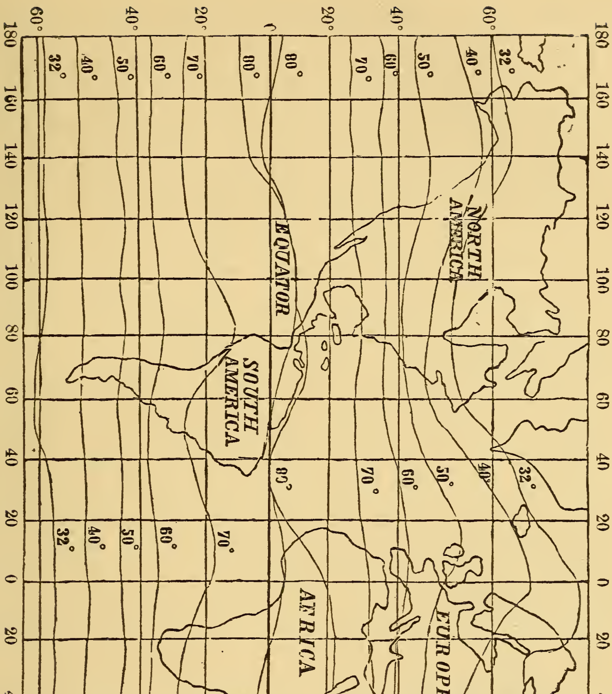
there is every where a depth of at least 40 feet, but in passing from the North River to East River there are obstructions where the depth of water is only 20 feet.

A similar principle is now very extensively employed to represent almost every variety of variable quantity depending upon geographical position. In many cases the representation is greatly assisted by variations in the depth of shading, or by varieties of color, etc. The following examples will afford some idea of this method.

Ex. 3. *Lines of equal mean temperature.* We draw upon a map of the earth a curve line connecting all those places whose mean temperature is the same—for example, 80°. As it may happen that we have no station whose observed temperature is exactly 80°, we select two adjacent stations, at one of which the temperature is a little less than 80°, and at the other a little greater; we then divide the interval between them in the same ratio as the differences between the observed temperatures and 80°. The point thus determined we call a point of 80° temperature. In the same manner we determine as many points of this line as practicable. Next we draw a line connecting all those places whose mean temperature is 70°, 60°, 50°, etc. The figure on the opposite page exhibits such a system of lines for nearly the entire globe. Maps of this kind, when carefully constructed, give a much clearer idea of the distribution of heat on the earth's surface than can be done by any system of numbers arranged in tables.

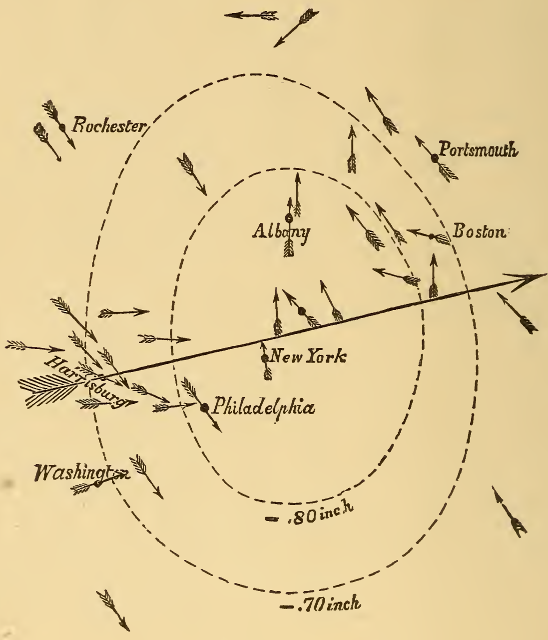
In like manner we may draw lines representing the mean temperature of different places for any month of the year, or we may draw lines to represent the temperatures observed for any given day and hour, thus enabling us to study the actual distribution of temperature at any instant of time.

Ex. 4. *Lines of equal atmospheric pressure.* We may draw upon a map of the earth a curve line connecting all those places where the mean pressure of the air, as shown by a barometer, is the same—for example, 30 inches. We may also draw lines connecting those places where the mean pressure is 29.9 inch-



es, also 29.8 inches, etc.; or we may draw lines connecting all those places where the pressure is the same at any given day and hour, thus enabling us readily to follow the daily fluctuations attending the progress of storms over the surface of the earth.

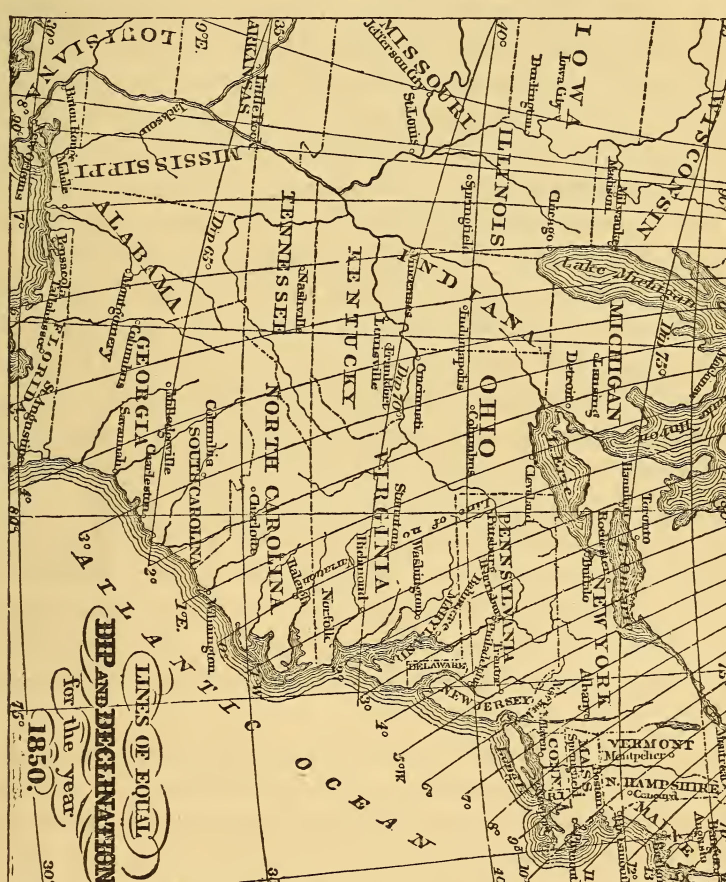
The annexed figure shows the state of the barometer and the direction of the wind as observed near the centre of a violent storm which prevailed in the neighborhood of New York February 16, 1842. The small oval line shows the area within



which the barometer sunk eight tenths of an inch below the mean, and the larger oval shows the area within which the barometer was depressed seven tenths of an inch. The long arrow represents the direction in which the storm advanced, while the short arrows show the observed direction of the wind at nearly forty different stations.

Ex. 5. *Lines of equal magnetic declination, dip, etc.* We may draw upon a map of the earth curve lines connecting all those places where the declination of the magnetic needle is the same, or where the dip of the magnetic needle is the same, or the earth's magnetic intensity is the same. Such lines give a far more distinct idea of the distribution of magnetism over the earth's surface than could be furnished by any amount of numerical results exhibited in a tabular form.

The annexed figure shows the lines of equal magnetic decli nation for a portion of the United States for the year 1850.



We perceive that the line of *no declination* passed through the centre of Lake Erie, and met the Atlantic near the middle of the coast of North Carolina. The line of 10 degrees west declination passed near Montreal, and the line of 8 degrees east declination passed near St. Louis. These lines show a small motion from year to year, and at present they all have a position westward of the positions represented on the map.

The map also shows the line of 65° magnetic dip, of 70°, and of 75° dip.

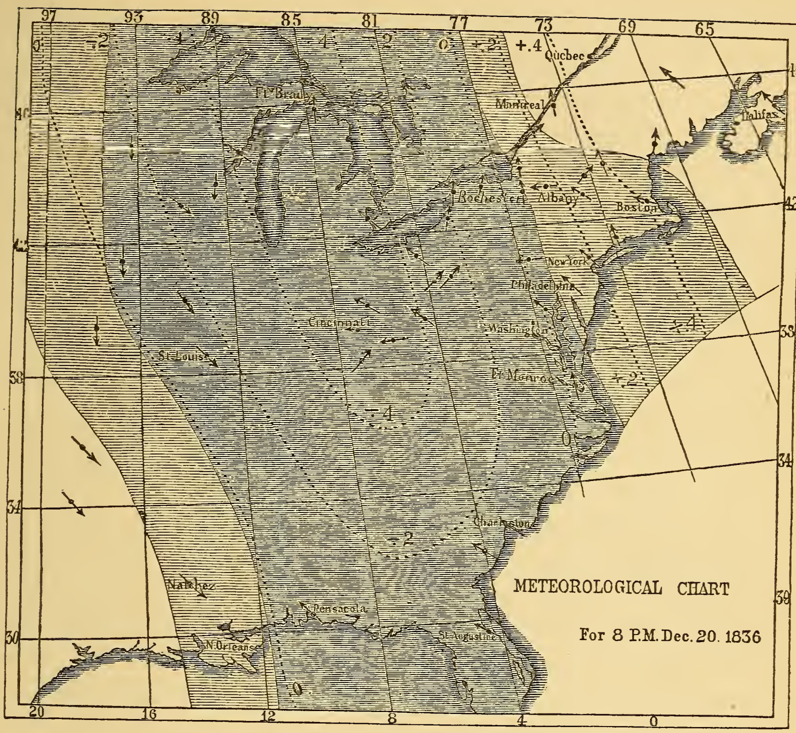
Ex. 6. *How the principal phenomena of a storm may be represented.* Winter storms in the United States are of great extent, sometimes exceeding 1000 miles in diameter. In order to represent the phenomena of such a storm, we require some suitable means of designating the area upon which rain or snow is falling; we wish to denote the region around the margin of the storm where clouds prevail without rain; and we wish to represent the region of clear sky which encircles the storm on every side. We wish also to represent the depression of the barometer within the storm area; also the state of the thermometer and the direction of the wind for each station of observation. The mode of accomplishing some of these objects will be understood from the figure on the opposite page, which represents the principal phenomena of a violent storm which was experienced in the United States December 20, 1836. The map represents the phenomena for 8 P.M.

The deeply shaded portion in the middle of the figure represents the area where rain or snow was falling; the lighter shade on the east and west margins of the rain represents the region where clouds prevailed without rain. Throughout the remaining portion of the United States, as far as the map extends, clear sky prevailed.

The dotted curve lines represent the state of the barometer. The inner curve shows the area where the barometer was depressed four tenths of an inch below the mean; the next curve shows where the barometer was two tenths of an inch below the mean; the next curve shows the barometer at its mean height; while farther eastward the barometer stood two tenths of an inch and four tenths of an inch above the mean.

The arrows show the directions of the wind as observed at a large number of stations.

A similar map, constructed for 8 A.M., December 21, would



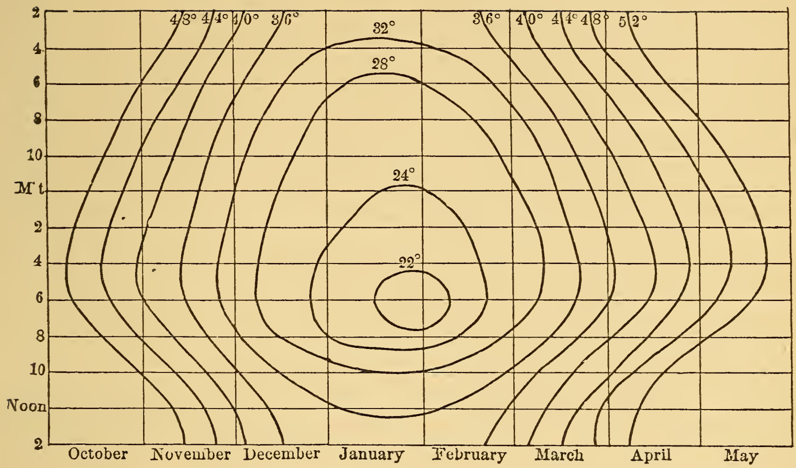
show not only that the storm had traveled eastward, but that important changes had taken place within the storm area.

This mode of representing the phenomena of a storm not merely compresses a vast amount of information within a small space, but it constitutes a powerful instrument of research, as it indicates a connection between the different classes of observations which might entirely escape notice if the comparisons were limited to a collection of observations arranged in a tabular form.

V. *Relations of three independent variables.* Since two coordinates are required to determine the position of a point on a plane, every point of a plane may be considered as corresponding to the known values of two of the variable elements. Take now three corresponding values of the three elements;

set off two of them as abscissa and ordinate on the given plane, and at the point thus determined erect a perpendicular whose length is proportional to the corresponding value of the third element. Proceed in the same manner with every three corresponding values of the three variables. The extremities of all these perpendiculars will be situated upon a curved surface which represents the law connecting the three variable elements. Suppose now a system of equidistant planes to be drawn parallel to the plane first assumed; these planes will intersect the curved surface in curve lines whose form will indicate the undulations of that surface. Let these curves be now projected on the plane first assumed, and we shall have on a single plane a system of curve lines which give a precise idea of the changes of the third variable corresponding to any given change of the other two variables.

Ex. *Temperature at any hour and for any month.* Let it be required to represent to the eye, by means of curve lines, the mean temperature of a given place for any hour of the day or any month of the year. We mark off on the axis of abscissas equal divisions to represent the months of the year, and on the axis of ordinates we set off, in like manner, twenty-four equal divisions to represent the hours of the day, and through these points of division we draw lines parallel to the co-ordinate axes. We are supposed to have a table, derived from observation, which shows us the mean temperature of the given place for each hour and each month of the year. We now select any temperature—for example, 32°—and find the two hours of each month at which that temperature occurs. At the intersection of the abscissa and ordinate corresponding to the given month and hour we place a point, and we do the same for each of the dates where the given temperature occurs. We join all these points by a continuous curve line, and we have a representation of the curve of 32°. In like manner we draw the curve of 30°, of 28°, etc., through the entire range of the observations. The figure on the opposite page shows the results of a long series of observations at New Haven.



Such a figure shows at a glance the mean temperature corresponding to any hour of either month of the year. If, for example, we desire to know the mean temperature of the month of January at 6 A.M., we find 6 A.M. on the left margin of the table, and follow along the corresponding horizontal line until we reach the middle of the month of January. The point falls nearly on the curve of 22°, which is therefore the temperature sought. In like manner we may find the temperature corresponding to any hour of any month of the year.

The same figure shows the season of the year and the hour of the day when the lowest temperature occurs. It also shows, for any season of the year, the two hours which have the same temperature; also, for any hour of the day, the two seasons of the year which have the same temperature. It also shows when the temperature changes most slowly, and when it changes most rapidly.

In a similar manner we may construct a system of curve lines representing the relation between any three independent variables.

THE END.