IV. *On Certain Linear Differential Equations of Astronomical Interest.*

By Prof. H. F. BAKER, *F.R.S.*

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Preface.

Part II. of the present paper was written, very much in the form in which it is now presented, in the summer of 1913, and began with the remark in § 11, which appears to disprove a statement made by Poincaré in regard to the convergence of an astronomical series. It was laid aside partly because a good deal of the work is only of the nature of elementary algebra, partly because the matrix notation employed does not seem to find favour in its application to differential equations. Various circumstances have, however, led me to take up the matter again, and my original conviction that the method of Part II. is of importance has been strengthened by comparing it with the less formal methods which, for the sake of introducing the subject, I have followed in Part I. I hope, therefore, that the following exposition may be thought worth while. Part III. has only the value of a concluding remark.

The table of contents above may serve to give an idea of the scope and arrangement of the paper.

PART I.

§ 1. Consider a linear differential equation

$$U\frac{d^2X}{d\tau^2}+V\frac{dX}{d\tau}+WX = 0,$$

where U, V, W are power series in a small quantity, λ, of the forms

$$U = u+\lambda u_1+\lambda^2 u_2+\ldots,$$

$$V = v+\lambda v_1+\lambda^2 v_2+\ldots,$$

$$W = \lambda w_1+\lambda^2 w_2+\ldots,$$

in which each of u_r, v_r, w_r is a linear function of

$$\zeta^r, \zeta^{r-2}, \zeta^{r-4}, \ldots, \zeta^{4-r}, \zeta^{2-r}, \zeta^{-r},$$

ζ denoting e^τ. Thus each of u_{2n}, v_{2n}, w_{2n} will contain a term independent of ζ; we speak of these as the absolute terms. It is important that W contains no term in λ^0; and it is assumed that the quantity v/u, which is independent of ζ, is not a positive or negative integer, and that u, v are not both zero.

We prove that if the absolute terms in W, that is the absolute terms in

$$w_2, w_4, w_6, \ldots,$$

be suitably determined, the differential equation possesses a solution of the form

$$X = 1+\lambda\phi_1+\lambda^2\phi_2+\ldots,$$

wherein ϕ_r is a linear function of ζ^r, ζ^{r-2}, ζ^{r-4}, ..., ζ^{4-r}, ζ^{2-r}, ζ^{-r}, and this is a periodic solution. Its period is $2\pi i$; we can, however, if we wish, express the same result with a period 2π by writing $\tau = it$.

For the substitution of the assumed form for X requires the identity

$$(u+\Sigma\lambda^n u_n)\,\Sigma\lambda^n\phi''_n+(v+\Sigma\lambda^n v_n)\,\Sigma\lambda^n\phi'_n+\Sigma\lambda^n w_n\,(1+\Sigma\lambda^n\phi_n) = 0,$$

which, equating the coefficient of λ^n to zero, will be true if

$$u\phi''_n+u_1\phi''_{n-1}+\ldots+u_{n-1}\phi''_1 +v\phi'_n+v_1\phi'_{n-1}+\ldots+v_{n-1}\phi'_1 +w_1\phi_{n-1}+w_2\phi_{n-2}+\ldots+w_{n-1}\phi_1+w_n = 0.$$

In particular for $n = 1$

$$u\phi''_1+v\phi'_1+w_1 = 0.$$

If herein we suppose

$$\phi_1 = A_1\zeta+A_{-1}\zeta^{-1},\quad w_1 = c_1\zeta+c_{-1}\zeta^{-1},$$

u, v, c_1, c_{-1} being given constants, we obtain

$$u\,(A_1\zeta+A_{-1}\zeta^{-1})+v\,(A_1\zeta-A_{-1}\zeta^{-1})+c_1\zeta+c_{-1}\zeta^{-1} = 0,$$

which is satisfied by

$$A_1 = -\frac{c_1}{u+v},\quad A_{-1} = -\frac{c_{-1}}{u-v}.$$

For $n = 2$ the condition is

$$u\phi''_2+v\phi'_2+u_1\phi''_1+v_1\phi'_1+w_1\phi_1+w_2 = 0.$$

Writing

$$u_1 = a_1\zeta+a_{-1}\zeta^{-1},\quad v_1 = b_1\zeta+b_{-1}\zeta^{-1},\quad w_1 = c_1\zeta+c_{-1}\zeta^{-1},\quad w_2 = c_2\zeta^2+c_{-2}\zeta^{-2}+C_2,$$

and assuming a form

$$\phi_2 = A_2\zeta^2+A_{-2}\zeta^{-2},$$

the condition becomes

$$4u\,(A_2\zeta^2+A_{-2}\zeta^{-2})+2v\,(A_2\zeta^2-A_{-2}\zeta^{-2}) +(a_1\zeta+a_{-1}\zeta^{-1})(A_1\zeta+A_{-1}\zeta^{-1})+(b_1\zeta+b_{-1}\zeta^{-1})(A_1\zeta-A_{-1}\zeta^{-1}) +(c_1\zeta+c_{-1}\zeta^{-1})(A_1\zeta+A_{-1}\zeta^{-1})+c_2\zeta^2+c_{-2}\zeta^{-2}+C_2 = 0;$$

equating the coefficients of ζ^2, ζ^{-2}, ζ^0 to zero, we obtain

$$(4u+2v)\,A_2 = -a_1A_1-b_1A_1-c_1A_1-c_2,$$
$$(4u-2v)\,A_{-2} = -a_{-1}A_{-1}+b_{-1}A_{-1}-c_{-1}A_{-1}-c_{-2},$$
$$C_2 = -a_1A_{-1}-a_{-1}A_1+b_1A_{-1}-b_{-1}A_1-c_1A_{-1}-c_{-1}A_1,$$

which, as v/u is not 2 or -2, determine A_2, A_{-2}, and C_2, the last being expressible by means of the given coefficients of u, v, u_1, v_1, w_1.

Proceeding similarly with the general value of n, we at once reach the conclusion stated, the absolute term in w_n being determined in terms of the coefficients in

$$u, u_1, \ldots, u_{n-1}, \quad v, v_1, \ldots, v_{n-1}, \quad w_1, w_2, \ldots, w_{n-1}.$$

§ 2. Now consider an equation

$$A\frac{d^2x}{d\tau^2}+2B\frac{dx}{d\tau}+Cx = 0,$$

where, with $\zeta = e^\tau$, A, B, C have the forms

$$A = a_0+\lambda(a_1\zeta+a_{-1}\zeta^{-1})+\lambda^2(a_2\zeta^2+a_{-2}\zeta^{-2}+a_{20})+\ldots,$$

$$B = b_0+\lambda(b_1\zeta+b_{-1}\zeta^{-1})+\lambda^2(b_2\zeta^2+b_{-2}\zeta^{-2}+b_{20})+\ldots,$$

$$C = c_0+\lambda(c_1\zeta+c_{-1}\zeta^{-1})+\lambda^2(c_2\zeta^2+c_{-2}\zeta^{-2}+c_{20})+\ldots,$$

which are periodic functions of τ, with period $2\pi i$, capable of being arranged as power series in a parameter λ, the coefficient of λ^r being a linear function of $\zeta^r, \zeta^{r-2}, \ldots, \zeta^{2-r}, \zeta^{-r}$.

In accordance with the well-known theory of linear differential equations with periodic coefficients, we substitute

$$x = e^{\kappa\tau}X,$$

where κ is a constant, and so obtain a differential equation

$$AX''+2(\kappa A+B)X'+(A\kappa^2+2B\kappa+C)X = 0,$$

which, when κ is properly chosen, is to be satisfied by a periodic function X. That this is possible follows at once from § 1, as we now explain.

First we can draw some inference as to the form of κ. For compare the original differential equation in x with the equation obtained from it by changing the sign of λ in each of the series A, B, C. It is clear that the new differential equation may equally be obtained from the original equation by change of τ into $\tau+\pi i$, which changes ζ into $-\zeta$; this latter change, however, only multiplies the factor $e^{\kappa\tau}$ by the constant $e^{i\pi\kappa}$; the factors $e^{\kappa\tau}$ appropriate to the two independent solutions of the new differential equation are thus the same, in their aggregate, as the factors for the original equation. Thus the change of the sign of λ changes the two factors $e^{\kappa\tau}$ appropriate to the two independent solutions of the original differential equation among themselves, either by leaving both unaltered or by interchanging them. Assuming that κ is capable of expression as a power series in λ,

$$\kappa = \kappa_0+\kappa_1\lambda+\kappa_2\lambda^2+\ldots,$$

the case in which each κ is unaltered by change of the sign of λ is the case in which only even powers enter in this series. The case in which the two values of κ are interchanged by change of the sign of λ may arise when the differential equation is such that for $\lambda = 0$ the two values of κ are equal or differ by an integer; in this case $e^{\kappa'\tau}/e^{\kappa\tau}$ is a periodic function for $\lambda = 0$, and the factors $e^{\kappa'\tau}$, $e^{\kappa\tau}$ do not individualise the functions with which they are associated.

In the present case, the equation reduces when $\lambda = 0$, to

$$a_0 \frac{d^2x}{d\tau^2} + 2b_0 \frac{dx}{d\tau} + c_0 x = 0,$$

which, if a_0 is not zero, has the two solutions $e^{\sigma\tau}$, $e^{\sigma'\tau}$, where σ, σ' have the values

$$[-b_0 \pm (b_0^2 - a_0c_0)^{\frac{1}{2}}]/a_0.$$

Thus if we suppose not only that a_0 is other than zero, but also that

$$2\,(b_0^2 - a_0c_0)^{\frac{1}{2}}/a_0$$

is not zero or a positive or negative integer, we can assume

$$\kappa = \sigma + \kappa_2\lambda^2 + \kappa_4\lambda^4 + \ldots.$$

Then putting

$$X = 1 + \lambda\phi_1 + \lambda^2\phi_2 + \ldots,$$

where ϕ_r is a linear function of ζ^r, ζ^{r-2}, ..., ζ^{2-r}, ζ^{-r}, the differential equation for X can be compared with that of § 1. In the present case there is an unknown quantity κ entering into the coefficient $A\kappa + B$ of $dX/d\tau$, but it will be seen that in the equations obtained by taking the successive powers of λ, each unknown coefficient in κ in this $A\kappa + B$ is determined at an earlier stage as entering in the coefficient $A\kappa^2 + 2B\kappa + C$, and so enters as a known coefficient. We have

$$\begin{aligned}
A\kappa + B = {} & [a_0 + \lambda\,(a_1\zeta + a_{-1}\zeta^{-1}) + \lambda^2\,(a_2\zeta^2 + a_{-2}\zeta^{-2} + a_{20}) + \ldots]\,[\sigma + \kappa_2\lambda^2 + \ldots] \\
& + b_0 + \lambda\,(b_1\zeta + b_{-1}\zeta^{-1}) + \lambda^2\,(b_2\zeta^2 + b_{-2}\zeta^{-2} + b_{20}) + \ldots \\
= {} & a_0\sigma + b_0 + \lambda\,[\sigma\,(a_1\zeta + a_{-1}\zeta^{-1}) + b_1\zeta + b_{-1}\zeta^{-1}] \\
& + \lambda^2\,[\sigma\,(a_2\zeta^2 + a_{-2}\zeta^{-2} + a_{20}) + a_0\kappa_2 + b_2\zeta^2 + b_{-2}\zeta^{-2} + b_{20}] \\
& + \ldots,
\end{aligned}$$

and similarly,

$$\begin{aligned}
A\kappa^2 + 2B\kappa + C = {} & a_0\sigma^2 + 2b_0\sigma + c_0 \\
& + \lambda\,[\sigma^2\,(a_1\zeta + a_{-1}\zeta^{-1}) + 2\sigma\,(b_1\zeta + b_{-1}\zeta^{-1}) + c_1\zeta + c_{-1}\zeta^{-1}] \\
& + \lambda^2\,[\sigma^2\,(a_2\zeta^2 + a_{-2}\zeta^{-2} + a_{20}) + 2\sigma\,(b_2\zeta^2 + b_{-2}\zeta^{-2} + b_{20}) + 2\kappa_2\,(a_0\sigma + b_0) \\
& \qquad + c_2\zeta^2 + c_{-2}\zeta^{-2} + c_{20}] \\
& + \ldots,
\end{aligned}$$

the absolute term in the coefficient of λ^4 in this being

$$2\kappa_4(a_0\sigma+b_0)+a_0\kappa_2^2+2a_{20}\kappa_2\sigma+a_{40}\sigma^2+2b_{20}\kappa_2+2b_{40}\sigma+c_{40}.$$

Thus, as in § 1, we first put

$$a_0\sigma^2+2b_0\sigma+c_0 = 0,$$

assuming, as in § 1 it was assumed that v/u is not an integer, that

$$2(a_0\sigma+b_0)/a_0, \quad \text{or} \quad 2(a_0c_0-b_0^2)^{\frac{1}{2}}/a_0,$$

is not zero or integral; then the absolute term in the coefficient of λ^2 determines $\kappa_2(a_0\sigma+b_0)$, and hence κ_2, and the absolute term in the coefficient of λ^4 similarly determines κ_4.

The excepted case in which κ contains odd as well as even powers of λ we may leave aside at present.

§ 3. We may apply the preceding to the much discussed* equation

$$\frac{d^2x}{d\theta^2}+(\sigma^2+2\lambda k_1\cos\theta+2\lambda^2k_2\cos 2\theta+\ldots)x = 0.$$

When $\lambda = 0$ we have the two factors $e^{i\sigma t}$, $e^{-i\sigma t}$, and the general case is that in which $e^{2i\sigma t}$ has not the period, 2π, of the coefficients in the differential equation, that is, when 2σ is not an integer. First assume this to be so. Then writing

$$x = e^{i\kappa\theta}X$$

we obtain

$$X''+2i\kappa X'+(\sigma^2-\kappa^2+2\lambda k_1\cos\theta+2\lambda^2k_2\cos 2\theta+\ldots)X = 0.$$

Herein assume

$$\kappa = \sigma+\kappa_2\lambda^2+\kappa_4\lambda^4+\ldots, \qquad X = 1+\lambda\phi_1+\lambda^2\phi_2+\ldots,$$

* For this differential equation the following list of references may be useful, though it is probably far from complete:—MATHIEU, 'Louville's J.,' XIII. (1868), p. 137; HILL, 'Coll. Math. Works,' I., p. 255 ('Acta Math.,' VIII. (1886)); ADAMS, 'Coll. Scientific Papers,' I., p. 186, II., pp. 65, 86; TISSERAND, 'Méc. Cél.,' t. III., Ch. I.; POINCARÉ, 'Méth. Nouv.,' t. II., Ch. XVII.; FORSYTH, 'Linear Differential Equations' (1902), p. 431; RAYLEIGH, 'Scientific Papers,' vol. III. (1902), p. 1; LINDEMANN, 'Math. Annal.,' Bd. XXII. (1883), p. 117; LINDSTEDT, 'Astr. Nachr.,' 2503 (1883); LINDSTEDT, 'Mémoires de l'Acad. de St. Petersbourg,' t. XXI., No. 4; BRUNS, 'Astr. Nachr.,' 2533, 2553 (1883); CALLANDREAU, 'Astr. Nachr.,' 2547 (1883); CALLANDREAU, 'Ann. Observ.,' Paris, XXII. (1896); TISSERAND, 'Bull. Astr.,' t. IX. (1892); STIELTJES, 'Astr. Nachr.,' 2602, 2609 (1884); HARZER, 'Astr. Nachr.,' 2850, 2851 (1888); MOULTON and MACMILLAN, 'Amer. J.,' XXXIII. (1911); MOULTON, 'Rendic. Palermo,' XXXII. (1911); MOULTON, 'Math. Ann.,' LXXIII. (1913); WHITTAKER, 'Cambridge Congress' (1912), I., p. 366; WHITTAKER, YOUNG and MILNE, 'Edinburgh Math. Soc.,' XXXII., 1913–14; INCE, 'Monthly Not.,' Roy. Astr. Soc., LXXV. (1915); POINCARÉ, 'Bull. Astr.,' XVII. (1900).

where ϕ_r is an integral polynomial of order r in ζ and ζ^{-1}, the quantity $e^{i\theta}$ being denoted by ζ. Then we have

$$\Sigma\lambda^n\phi''_n + 2i(\sigma+\kappa_2\lambda^2+\ldots)\Sigma\lambda^n\phi'_n + [-2\sigma\kappa_2\lambda^2 - (2\sigma\kappa_4+\kappa_2^2)\lambda^4+\ldots+2\lambda k_1\cos\theta+\ldots][1+\Sigma\lambda^n\phi_n] = 0.$$

The terms in λ give

$$\phi''_1 + 2i\sigma\phi'_1 + k_1(\zeta+\zeta^{-1}) = 0,$$

which, if we denote $(\sigma+r)^2-\sigma^2$ or $r(2\sigma+r)$ by u_r, so that the result of substituting ζ^r for ϕ in $\phi''+2i\sigma\phi'$ is $-u_r\zeta^r$, leads to

$$\phi_1 = k_1\left(\frac{\zeta}{u_1}+\frac{\zeta^{-1}}{u_{-1}}\right).$$

The terms in λ^2 give

$$\phi''_2 + 2i\sigma\phi'_2 + k_1(\zeta+\zeta^{-1})\phi_1 + k_2(\zeta^2+\zeta^{-2}) - 2\sigma\kappa_2 = 0,$$

which, if we write

$$\phi_2 = A_2\zeta^2 + A_{-2}\zeta^{-2},$$

leads to

$$A_2 = \frac{1}{u_2}\left(k_2+\frac{k_1^2}{u_1}\right), \qquad A_{-2} = \frac{1}{u_{-2}}\left(k_2+\frac{k_1^2}{u_{-1}}\right)$$

and

$$\kappa_2 = \frac{k_1^2}{2\sigma}\left(\frac{1}{u_1}+\frac{1}{u_{-1}}\right) = -\frac{k_1^2}{\sigma(4\sigma^2-1)}.$$

By the terms in λ^3, λ^4, we similarly find the coefficients in

$$\phi_3 = A_3\zeta^3 + A_{-3}\zeta^{-3} + B_1\zeta + B_{-1}\zeta^{-1},$$

$$\phi_4 = A_4\zeta^4 + A_{-4}\zeta^{-4} + B_2\zeta^2 + B_{-2}\zeta^{-2},$$

and also

$$\kappa_4 = -\frac{60\sigma^4-35\sigma^2+2}{4\sigma^3(\sigma^2-1)(4\sigma^2-1)^3}k_1^4 + \frac{3}{2\sigma(\sigma^2-1)(4\sigma^2-1)}k_1^2k_2 - \frac{1}{4\sigma(\sigma^2-1)}k_2^2.$$

If we change the notation, writing $\theta = 2t$, $2\sigma = n$, so that the differential equation becomes

$$\frac{d^2x}{dt^2} + [n^2+8\lambda k_1\cos 2t + 8\lambda^2 k_2\cos 4t+\ldots]x = 0$$

and

$$\zeta = e^{2it}, \quad x = e^{2i\kappa t}X,$$

we have

$$\kappa = \tfrac{1}{2}n - \frac{2k_1^2\lambda^2}{n(n^2-1)} + \lambda^4\left\{-2\frac{15n^4-35n^2+8}{n^3(n^2-4)(n^2-1)^3}k_1^4 + \frac{12k_1^2k_2}{n(n^2-4)(n^2-1)} - \frac{2k_2^2}{n(n^2-4)}\right\}+\ldots$$

It is clear that κ is essentially real so long as this series converges.

As an immediate application take the equation in BROWN's 'Lunar Theory,' p. 107,

$$\frac{d^2x}{dt^2}+n^2x\left\{1+\tfrac{3}{2}m^2-\tfrac{9}{32}m^4+\left(3m^2+\tfrac{19}{2}m^3+\tfrac{137}{6}m^4\right)\cos 2\xi+\tfrac{33}{8}m^4\cos 4\xi\right\}=0,$$

where

$$\xi=(n-n')\,t+\epsilon-\epsilon',\quad m=n'/n.$$

Put

$$m_1=\frac{m}{1-m}=\frac{n'}{n-n'},\quad m=\frac{m_1}{1+m_1},\quad n\,dt=(1+m_1)\,d\xi;$$

then the equation becomes

$$\frac{d^2x}{d\xi^2}+x\left\{1+2m_1+\tfrac{5}{2}m_1^2-\tfrac{9}{32}m_1^4+m_1^2\left(3+\tfrac{19}{2}m_1+\tfrac{40}{3}m_1^2\right)\cos 2\xi+\tfrac{33}{8}m_1^4\cos 4\xi\right\}=0,$$

which is of the form above, ξ replacing t. We may then take

$$\lambda=\frac{m_1^2}{8},\quad n^2=1+2m_1+\tfrac{5}{2}m_1^2-\tfrac{9}{32}m_1^4,\quad k_1=3+\tfrac{19}{2}m_1+\tfrac{40}{3}m_1^2,\quad k_2=33.$$

Here m_1 is a small quantity and

$$\frac{\lambda^2}{n^2-1}=\frac{m_1^4}{64\,(2m_1+\ldots)}=\frac{m_1^3}{128}(1+\ldots)$$

is of the order m_1^3, while similarly $\lambda^4/(n^2-1)^3$ is of the order m_1^5. Also

$$\begin{aligned}n&=(1+m_1)\left\{1+\tfrac{3}{2}m_1^2(1-2m_1+3m_1^2)-\tfrac{9}{32}m_1^4\right\}^{\frac{1}{2}}\\&=(1+m_1)\left(1+\tfrac{3}{4}m_1^2-\tfrac{3}{2}m_1^3+\tfrac{117}{64}m_1^4\right).\end{aligned}$$

Thus

$$\begin{aligned}\kappa&=\tfrac{1}{2}n\left\{1-\frac{4k_1^2\lambda^2}{n^2(n^2-1)}\right\}\\&=\tfrac{1}{2}(1+m_1)\left(1+\tfrac{3}{4}m_1^2-\tfrac{57}{32}m_1^3+\tfrac{123}{128}m_1^4\right),\end{aligned}$$

which is easily seen to agree with the result given by BROWN, or by ADAMS, 'Coll. Works,' I., p. 187, when we take account of the fact that

$$2i\kappa\xi=2i\kappa(n-n')\,t=2i\kappa\,\frac{n-n'}{n}\,nt,$$

so that, in terms of the quantity denoted by g,

$$\kappa=\tfrac{1}{2}(1+m_1)\,g.$$

This example is chiefly useful here as calling attention to the fact that n^2, while not exactly equal to 1, is near to it, and consequently the factor $\lambda/(n^2-1)$ is only small of the first order in m_1. The same weakness occurs in the terms in ζ^{-1}, ..., in the solution.

§ 4. In the equation considered by HILL ('Coll. Works,' I., p. 268) the ratio $4k_1\lambda/(n^2-1)$ is about $(2{\cdot}785)^{-1}$, and there is a term slightly greater than $4k_1\lambda\,(2{\cdot}785)^{-r}$ arising in the terms in λ^{r+1} in the series for κ, in which $4k_1\lambda$ is about $\dot{0}{\cdot}5704$; and the series fails absolutely in cases in which n is an integer. Then the assumption that κ is a power series in λ^2, and the terms in X which are independent of λ, must be modified, for reasons above given. The series when n is an integer has been considered by TISSERAND, 'Bull. Astr.,' IX., 1892; modifying his procedure, so as to include the case when n is near to 1 as well as that in which $n = 1$, we may write, in accordance with the suggestion of such examples as that above quoted,

$$n^2 = 1+4\lambda h_1+4\lambda^2 h_2+\ldots,$$

and then, denoting $e^{2irt}+e^{-2irt}$ by w_r, consider the equation

$$\frac{d^2x}{dt^2}+[1+4\lambda\,(h_1+k_1w_1)+4\lambda^2\,(h_2+k_2w_2)+\ldots]\,x = 0.$$

By the changes

$$\tau = 2it, \quad \zeta = e^{\tau},$$

$$-ix = e^{-i(1+2q)t}\,[\mathrm{U}-\mathrm{V}\zeta], \quad \frac{dx}{dt} = e^{-i(1+2q)t}\,[\mathrm{U}+\mathrm{V}\zeta],$$

the differential equation may be replaced by the pair

$$\frac{d\mathrm{U}}{d\tau}-q\mathrm{U} = -\phi\,(\mathrm{U}-\mathrm{V}\zeta), \quad \frac{d\mathrm{V}}{d\tau}-q\mathrm{V} = -\phi\,(\mathrm{U}\zeta^{-1}-\mathrm{V}),$$

where

$$w_r = \zeta^r+\zeta^{-r},$$

$$\phi = \lambda\,(h_1+k_1w_1)+\lambda^2\,(h_2+k_2w_2)+\ldots.$$

Assuming here

$$q = \lambda q_1+\lambda^2 q_2+\ldots,$$

$$\mathrm{U} = 1+\lambda u_1+\lambda^2 u_2+\ldots, \quad \mathrm{V} = \mathrm{B}\,(1+\lambda v_1+\lambda^2 v_2+\ldots),$$

in which B is a constant, and u_r, v_s are polynomials in ζ and ζ^{-1}, we find, equating coefficients of like powers of λ,

$$-\frac{du_r}{d\tau}+q_1u_{r-1}+q_2u_{r-2}+\ldots+q_r = \mathrm{H}_r,$$

$$-\frac{dv_r}{d\tau}+q_1v_{r-1}+q_2v_{r-2}+\ldots+q_r = \mathrm{K}_r,$$

in which

$$\mathrm{H}_r = (h_1+k_1w_1)\,(u_{r-1}-\zeta\mathrm{B}v_{r-1})+(h_2+k_2w_2)\,(u_{r-2}-\zeta\mathrm{B}v_{r-2})+\ldots+(h_r+k_rw_r)\,(1-\zeta\mathrm{B}),$$

$$\mathrm{K}_r = \zeta^{-1}\mathrm{B}^{-1}\mathrm{H}_r.$$

In these equations, as u_r, v_s are to be polynomials in ζ, ζ^{-1}, the absolute terms, those involving ζ^0, must vanish. For $r = 1$ this gives

$$h_1 - q_1 = k_1 B, \quad h_1 + q_1 = k_1 B^{-1}.$$

We thus write, using hyperbolic functions,

$$h_1 = k_1 ch\alpha, \qquad q_1 = k_1 sh\alpha, \qquad B = e^{-\alpha}.$$

With these we find at once by integration the values of u_1, v_1, save for the absolute terms in these, which we denote by P_1, Q_1 respectively. The conditions for these are to be found by considering the absolute terms in the equations for $r = 2$; and so on continually. In general, when we have found

$$u_1, v_1, u_2, v_2, \ldots, u_{r-1}, v_{r-1},$$

and have found u_r, v_r, save for their absolute terms, P_r, Q_r, we find, on taking the absolute terms in the equations which involve $du_{r+1}/d\tau$ and $dv_{r+1}/d\tau$, and adding and subtracting these terms, that the two quantities

$$k_1 sh\alpha\,(P_r - Q_r) - h_{r+1}, \qquad k_1 ch\alpha\,(P_r - Q_r) - q_{r+1}$$

are thereby expressed in terms of known quantities. It is at once seen that there would be no loss of generality in putting P_1, P_2, P_3, ... all zero. Carrying out the work, and writing M_r for $P_r - Q_r$, we obtain

$$q = k_1 sh\alpha\lambda + (M_1 k_1 ch\alpha - k_1^2 sh2\alpha)\lambda^2$$
$$+ \{\tfrac{1}{2}M_1^2 k_1 e^{\alpha} - 2M_1 k_1^2 ch2\alpha + k_1^3 sh\alpha\,(6ch^2\alpha - \tfrac{11}{4}) + k_1 k_2 sh\alpha + k_1 ch\alpha\,(M_2 - M_1 P_1)\}\,\lambda^3 + \ldots,$$

where

$$h_1 = k_1 ch\alpha,$$
$$h_2 = M_1 k_1 sh\alpha - \tfrac{1}{2}k_1^2 ch2\alpha,$$
$$h_3 = \tfrac{1}{2}M_1^2 k_1 e^{\alpha} - M_1 k_1^2 sh2\alpha + k_1^3 ch\alpha\,(2sh^2\alpha - \tfrac{1}{4}) + k_1 k_2 ch\alpha + k_1 sh\alpha\,(M_2 - M_1 P_1).$$

Also

$$-ixe^{i(1+2q)t} = 1 - e^{-\alpha}\zeta + \lambda W_1 + \lambda^2 W_2 + \ldots$$

in which

$$W_1 = \tfrac{1}{2}k_1\zeta^{-1} + P_1 - k_1 sh\alpha + (-P_1 + M_1 - k_1 sh\alpha)\,e^{-\alpha}\zeta - \tfrac{1}{2}k_1 e^{-\alpha}\zeta^2,$$

$$W_2 = \tfrac{1}{6}\zeta^{-2}(k_2 + \tfrac{1}{2}k_1^2) + \zeta^{-1}[\tfrac{1}{2}P_1 k_1 - \tfrac{1}{2}k_2 e^{-\alpha} + k_1^2(\tfrac{1}{4}e^{-\alpha} - sh\alpha)]$$
$$+ P_2 - P_1 k_1 sh\alpha - M_1 k_1 ch\alpha + k_1^2 sh\alpha\,(ch\alpha + e^{\alpha})$$
$$- \zeta e^{-\alpha}[P_2 - M_2 + P_1 k_1 sh\alpha + M_1 k_1 e^{-\alpha} - k_1^2 sh\alpha\,(ch\alpha + e^{-\alpha})]$$
$$+ \zeta^2 e^{-\alpha}[-\tfrac{1}{2}P_1 k_1 + \tfrac{1}{2}M_1 k_1 + \tfrac{1}{2}k_2 e^{\alpha} - k_1^2(sh\alpha + \tfrac{1}{4}e^{\alpha})]$$
$$- \tfrac{1}{6}\zeta^3 e^{-\alpha}(k_2 + \tfrac{1}{2}k_1^2).$$

If from these formulæ we determine M_1 and $M_2-M_1P_1$ in terms of h_2 and h_3 we find for q,

$$q = k_1 sh\alpha\lambda + H_2 cth\alpha \, . \, \lambda^2$$
$$+\lambda^3\left[-\frac{H_2^2}{2k_1 sh^3\alpha}-\frac{2k_1H_2ch^2\alpha}{sh\alpha}+\frac{h_3ch\alpha-k_1k_2-\frac{1}{4}k_1^3(2sh^2\alpha-1)}{sh\alpha}\right]+\ldots,$$

where

$$H_2 = h_2-\tfrac{1}{2}k_1^2(2sh^2\alpha-1).$$

This formulæ is apparently unsatisfactory when $sh\alpha$ is small, or n^2-1 nearly equal to $4\lambda k_1$. In fact, the series is of the form

$$a+\frac{b}{2a}\lambda+\frac{4ca^2-b^2}{8a^3}\lambda^2+\frac{8a^4d-4a^2bc+b^3}{16a^5}\lambda^3+\ldots,$$

whose square has a form in which we can put $a = 0$. On squaring, we have

$$q^2 = (h_1^2-k_1^2)\lambda^2+2h_1H_2\lambda^3+\lambda^4(H_2^2-4h_1^2H_2+2h_1h_3-2k_1^2k_2-h_1^2k_1^2+\tfrac{3}{2}k_1^4)+\ldots,$$

wherein

$$H_2 = h_2-h_1^2+\tfrac{3}{2}k_1^2,$$

and this form is appropriate when $\alpha = 0$ or $h_1 = k_1$. In particular, when $h_2 = h_3 = \ldots = 0$, but h_1 is not zero, this gives

$$q^2 = (h_1^2-k_1^2)\lambda^2+h_1(3k_1^2-2h_1^2)\lambda^3+[5(h_1^2-k_1^2)^2-\tfrac{5}{4}k_1^4-2k_1^2k_2]\lambda^4+\ldots,$$

a formula reproducing the former if $h_1+h_2\lambda+h_3\lambda^2$ be put for h_1. It will be seen in Part II. of this paper why the form of q^2 is comparatively so simple.

Brief reference may be made to another way in which we may use the foregoing equations, regarding h_1, h_2, h_3, ... not as given constants but as quantities to be determined to simplify the result; this has been adopted by Prof. WHITTAKER (' Proc. Math. Soc.,' Edinburgh, XXXII., 1913–14) who chooses as his condition that no terms in ζ^0, ζ^1 shall occur in W_1, W_2, ..., in the expression for x. This can be secured by taking

$$P_1 = k_1 sh\alpha,\quad M_1 = 2k_1 sh\alpha,\quad P_2 = 0,\quad Q_2 = 0,\ \ldots.$$

From our present point of view a more natural procedure is to take $P_1 = 0 = Q_1 = P_2 = Q_2 = \ldots.$ Then we obtain

$$n^2 = 1+\lambda k_1 ch\beta-\tfrac{1}{2}\lambda^2k_1^2ch2\beta+\lambda^3[k_1^3ch\beta(2sh^2\beta-\tfrac{1}{4})+k_1k_2ch\beta]+\ldots,$$

where we have written β in place of α, as this argument is now supposed to be determined, from this equation, corresponding to a given value of n^2. When β is so determined, q is given by

$$q = k_1\lambda sh\beta-k_1^2\lambda^2sh2\beta+\lambda^3[k_1^3sh\beta(6ch^2\beta-\tfrac{11}{4})+k_1k_2sh\beta]+\ldots,$$

an equation which does not contain $sh\beta$ in its denominator. With a view to the comparison of this method with the two others given in the present paper we consider two examples. First, for the equation

$$\frac{d^2x}{dt^2}+[1+4\lambda k_1 w_1+4\lambda^2 k_2 w_2+\ldots]\,x = 0,$$

for which n^2 is actually unity, we should determine β so that

$$0 = k_1 ch\beta-\tfrac{1}{2}k_1^2 ch2\beta+k_1^3 ch\beta\,(2sh^2\beta-\tfrac{1}{4})+k_1k_2 ch\beta+\ldots,$$

where we have replaced λ by 1. This gives approximately

$$ch\beta = -\tfrac{1}{2}k_1\,(1+\tfrac{7}{4}k_1^2-k_2),\quad sh\beta = i\,(1-\tfrac{1}{8}k_1^2),$$

and hence

$$q = ik_1\,(1-\tfrac{15}{8}k_1^2+k_2+\ldots),$$

while the value for β, substituted for α, gives the series for x. We may remark that for the equation

$$\frac{d^2x}{dt^2}+(1+8k_1\cos 2t)\,x = 0,$$

TISSERAND ('Bull. Astr.,' IX., 1892, p. 102) finds

$$q = ik_1\left(1-\tfrac{15}{8}k_1^2+\frac{9719}{2^7\,.\,3^2}k_1^4+\ldots\right).$$

As a further example take

$$\frac{d^2x}{dt^2}+x\,[1+4k_1\,(1+w_1)+4k^2w_2+\ldots] = 0,$$

which, as will appear, is an interesting equation. Then β is to be found from

$$k_1 = k_1 ch\beta-\tfrac{1}{2}k_1^2 ch2\beta+k_1^3 ch\beta\,(2sh^2\beta-\tfrac{1}{4})+k_1k_2 ch\beta+\ldots,$$

so that

$$ch\beta = 1+\tfrac{1}{2}k_1+\tfrac{5}{4}k_1^2-k_2+\tfrac{7}{8}k_1^3-\tfrac{5}{2}k_1k_2+\ldots,$$

$$sh\beta = (k_1)^{\frac{1}{2}}\left(1+\tfrac{11}{8}k_1-\frac{k_2}{k_1}+\ldots\right),$$

and hence

$$q = (k_1)^{\frac{1}{2}}\,(k_1-\tfrac{5}{8}k_1^2-k_2+\ldots).$$

In both these examples the value found for q follows at once from the general formula above given for q^2, of which a further deduction is found below in Part II. In the last example the value found for β gives a solution for x in a series involving $(k_1)^{\frac{1}{2}}$. It will be seen in Part II. that when x involves $(k_1)^{\frac{1}{2}}$, it is in a very simple way, and the case seems better treated as there explained. The occurrence of $(k_1)^{\frac{1}{2}}$ in the value of q, in certain cases, is a particular case of POINCARÉ's theorem, 'Méth. Nouv.,' I., § 79, p. 219. The phenomenon presents itself, however, as a consequence of the use of elliptic functions in TISSERAND's theory of the small planets; see TISSERAND, 'Méc. Cél.,' IV., p. 426 (or 'Bull. Astr.,' IV.).

§ 5. A very important question in regard to the differential equation under discussion is whether q is real or not, since upon this depends the conventional stability of the secondary oscillation determined by the differential equation. We have remarked above (§ 3) that when n is not an integer, and $k_1\lambda$, $k_2\lambda^2$, ... are small enough to render the series there obtained convergent, the value of q is necessarily real. The cases in which n is an integer and $k_2 = 0 = k_3 = \ldots$ have been discussed by TISSERAND, 'Bull. Astr.,' IX., 1892, who obtains the result that the motion is unstable for $n = 1$ or $n = 2$, that is for the equations

$$\frac{d^2x}{dt^2} + [1 + 4\lambda k_1 w_1] x = 0, \quad \frac{d^2x}{dt^2} + [4 + 4\lambda k_1 w_1] x = 0,$$

when λ is small enough, but stable for greater integer values of n. The formula for q^2, given in the earlier part of § 4 preceding, shows that for cases in which

$$n^2 = 1 + 4h_1\lambda$$

the motion is stable provided

$$(h_1/k_1)^2 > 1,$$

the values of $ch\alpha$ and $sh\alpha$ being then both real. It shows further that it is stable for

$$h_1 = \pm k_1 = \text{positive}$$

provided λ be small enough. The critical equation is thus

$$\frac{d^2x}{dt^2} + x[1 + 4k_1(1 + w_1) + 4k_2 w_2 + \ldots] = 0,$$

the other sign of k_1 being obtainable by changing t into $t + \frac{\pi}{2}$.

§ 6. We proceed now to the case when $n = 2$.

If in the equation

$$\frac{d^2x}{dt^2} + x[m^2 + 4\lambda(h_1 + k_1 w_1) + 4\lambda^2(h_2 + k_2 w_2) + \ldots] = 0,$$

in which m is an integer, we put

$$\tau = 2it, \quad \zeta = e^{\tau},$$

$$U = \tfrac{1}{2}e^{\frac{1}{2}m\tau + q\tau}\left(\frac{dx}{dt} - imx\right), \quad V = \tfrac{1}{2}e^{-\frac{1}{2}m\tau + q\tau}\left(\frac{dx}{dt} + imx\right),$$

we obtain

$$\frac{dU}{d\tau}-qU = -\frac{\phi}{m}(U-V\zeta^{m}), \quad \frac{dV}{d\tau}-qV = -\frac{\phi}{m}(U\zeta^{-m}-V),$$

where

$$\phi = \lambda\,(h_1+k_1w_1)+\lambda^2\,(h_2+k_2w_2)+\ldots,$$
$$w_r = \zeta^r+\zeta^{-r}.$$

We may then further substitute

$$W = U\zeta^{-m}-V, \quad U_1 = mU,$$

leading to

$$\frac{dU_1}{d\tau}-qU_1 = -\phi\zeta^{m}W,$$

$$\frac{dW}{d\tau}-qW = -\zeta^{-m}U_1,$$

where

$$x = \frac{i}{m}\,e^{\frac{1}{2}m\tau-q\tau}W.$$

These equations can be solved by writing

$$q = \lambda q_1+\lambda^2 q_2+\ldots,$$

$$U_1 = 1+\lambda u_1+\lambda^2 u_2+\ldots, \quad W = \frac{A+\zeta^{-m}}{m}+\lambda w_1+\lambda^2 w_2+\ldots,$$

where A is a constant, and u_1, u_2, ..., w_1, w_2, ... are polynomials in ζ, ζ^{-1}.

For $m = 2$, in particular, we find that if $h_1 = 0$, the quantity A is required, and determined in the course of the work, and $q_1 = 0$. But if h_1 is not zero, we must take $A = 0$, and obtain $q_1 = \frac{1}{2}h_1$, the succeeding q_2, q_3, ... being real. In fact, as far as λ^3,

$$q = \tfrac{1}{2}h_1\lambda-(\tfrac{1}{8}h_1^2+\tfrac{1}{3}k_1^2-\tfrac{1}{2}h_2)\,\lambda^2+\left\{\frac{h_1^3}{16}+\tfrac{11}{18}h_1k_1^2-\frac{h_1h_2}{4}+\frac{h_3}{2}-\frac{(k_1^2-k_2)^2}{4h_1}\right\}\lambda^3+\ldots,$$

which gives

$$q^2 = \tfrac{1}{4}h_1^2\lambda^2-h_1\,(\tfrac{1}{8}h_1^2+\tfrac{1}{3}k_1^2-\tfrac{1}{2}h_2)\,\lambda^3$$
$$+\{\tfrac{5}{64}h_1^4+\tfrac{25}{36}h_1^2k_1^2-\tfrac{5}{36}k_1^4+\tfrac{1}{2}k_1^2k_2-\tfrac{1}{4}k_2^2-h_2\,(\tfrac{3}{8}h_1^2+\tfrac{1}{3}k_1^2)+\tfrac{1}{4}h_2^2+\tfrac{1}{2}h_1h_3\}\,\lambda^4+\ldots.$$

We know, as is shown in Part II. of this paper, that the form of q^2 is valid even when $h_1 = 0$. Then we have

$$q^2 = \tfrac{1}{4}\lambda^4\,(h_2+k_2-\tfrac{5}{3}k_1^2)\,(h_2-k_2+\tfrac{1}{3}k_1^2)+\ldots,$$

which, when $h_2 = 0$, is only positive, provided

$$5k_1^2 > 3k_2 > k_1^2.$$

The case discussed by TISSERAND is that in which $k_2 = k_3 = \ldots = 0$. Then

$$q = \frac{k_1^2\lambda^2}{6}\sqrt{(-5)},$$

and the quantity A in the formula for W, or x, is found to be $\frac{1}{3}(-2\pm\sqrt{-5})$.

When $m = 3$, for the equation

$$\frac{d^2x}{dt^2}+(9+8\lambda k\cos 2t)\,x = 0,$$

we find A = 0, and

$$q = -\frac{k^2\lambda^2}{12}-\frac{269}{64\,.\,27\,.\,5}k^4\lambda^4+\ldots,$$

$$\mathrm{U}_1 = 1-\tfrac{1}{3}\lambda k(\zeta-\zeta^{-1})+\ldots,\quad \mathrm{W} = \tfrac{1}{3}\zeta^{-3}+\tfrac{1}{3}\lambda k(\tfrac{1}{4}\zeta^{-4}-\tfrac{1}{2}\zeta^{-2}+\tfrac{1}{5})+\ldots.$$

The question of the reality of q, in cases where $k_2 = 0 = k_3 = \ldots$, is discussed by POINCARÉ, 'Méth. Nouv.,' II. (1893), p. 243, and by CALLANDREAU, 'Ann. Observ.,' Paris, XXII. (1896), p. 23. So far the results are :—

(1) For the equation at the bottom of p. 135 (§ 3) q is real when n^2 is not an integer, provided the series obtained converges.

(2) This condition does not however include, for instance, the case when n^2 is near to unity. For q is imaginary, for the equation

$$\frac{d^2x}{dt^2}+[n^2+8k_1\cos 2t+\ldots]\,x = 0,$$

if $(n^2-1)^2 < (4k_1)^2$. It is real if $(n^2-1)^2 > (4k_1)^2$, and real if n^2-1 is positive and equal to $\pm 4k_1$. This has been proved here.

(3) q may be real when n is just greater than 2, when $k_1, k_2, \ldots$ are small enough. This has been proved here.

(4) q is real when n is any integer greater than 2, if $k_2 = k_3 = \ldots = 0$, but imaginary when $n = 1$ or $n = 2$. This result is given by TISSERAND and CALLANDREAU, as above.*

[*October 30, 1915.*—It may be worth adding, in connexion with the numerical results given in § 6, that the equation

$$\frac{d^2x}{dt^2}+c\sin t\,.\,x = 0,$$

in which c is small, is solved by

$$x = e^{i\lambda t}\mathrm{U},$$

* See the note at the conclusion of § 21 (p. 184).

in which, as far as c^3,

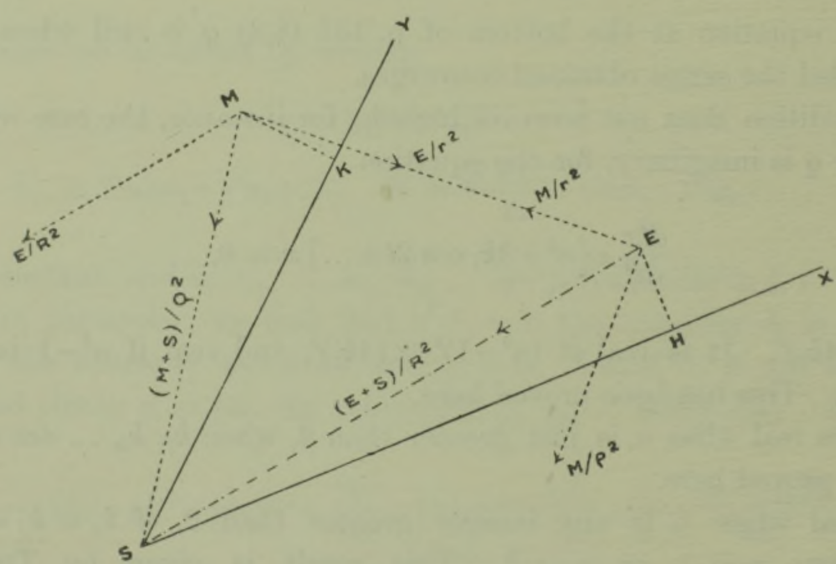
$$\lambda = \frac{c}{\sqrt{2}}\left(1+\tfrac{25}{32}c^2\right),$$

and, as far as c^4,

$$U = 1+c\sin t+c^2\left(i\sqrt{2}\cos t-\tfrac{1}{8}\cos 2t\right)+c^3\left(\tfrac{25}{16}\sin t+\frac{3i}{8\sqrt{2}}\sin 2t-\frac{\sin 3t}{144}\right)$$
$$+c^4\left(\frac{31}{8\sqrt{2}}\,i\cos t-\tfrac{53}{144}\cos 2t-\frac{11}{432\sqrt{2}}\,i\cos 3t+\frac{\cos 4t}{4608}\right).\Big]$$

§ 7. We pass now to the consideration of a pair of simultaneous differential equations arising in the consideration of the stability of the motion of three particles occupying the angular points of an equilateral triangle moving under their mutual gravitation.

The stability of this motion has been discussed by ROUTH ('Proc. Lond. Math. Soc.,' VI., 1875; 'Rigid Dynamics,' II., p. 61) in the case when the relative paths of the particles are circles.* In what follows we do not assume this.



The three particles being S, E, M, take an axis through S, say SX, rotating with angular velocity $\dot{\theta}$, the line SE being supposed to coincide very nearly with SX. Draw a perpendicular EH from E to SX, denote EH by y, and SH by $A+x$, where x, y will be considered small, their squares being neglected, but A is a variable finite quantity. Draw a second axes SY through S at a constant angle $\frac{\pi}{3}$ with SX, and

* The following references may be of use:—CHARLIER, 'Die Mechanik des Himmels,' and 'Astr. Nachr.,' 193, 15; STOCKWELL, 'Astron. Journ.,' 557 (1904); LINDERS, 'Arkiv for Mat.' (Stockholm), IV., No. 20; BROWN, 'Monthly Notices, R.A.S.,' LXXI. (1911), pp. 439, 492; HEINRICH, 'Astr. Nachr.,' 194, 12 (December, 1912); BLOCK, 'Arkiv for Mat.,' X., 4 (1914).

similarly, draw a perpendicular MK from M to SY; denote SK, KM by $A+\xi$ and η. If $R = SE$, $r = EM$, $\rho = MS$, we have, with proper conventions of sign,

$$R^2 = (A+x)^2+y^2, \qquad \rho^2 = (A+\xi)^2+\eta^2,$$

$$r^2 = [\tfrac{1}{2}(A+\xi)-\tfrac{1}{2}\eta\sqrt{3}-A-x]^2+[\tfrac{1}{2}(A+\xi)\sqrt{3}+\tfrac{1}{2}\eta-y]^2.$$

The accelerations of E, relatively to S, parallel to SX and parallel to HE, are, respectively,

$$-(E+S)\frac{A+x}{R^3}+M\frac{\tfrac{1}{2}(A+\xi)-\tfrac{1}{2}\eta\sqrt{3}-(A+x)}{r^3}-M\frac{\tfrac{1}{2}(A+\xi)-\tfrac{1}{2}\eta\sqrt{3}}{\rho^3},$$

$$-(E+S)\frac{y}{R^3}+M\frac{\tfrac{1}{2}(A+\xi)\sqrt{3}+\tfrac{1}{2}\eta-y}{r^3}-M\frac{\tfrac{1}{2}\eta+\tfrac{1}{2}(A+\xi)\sqrt{3}}{\rho^3};$$

the accelerations of M, relatively to S, parallel to SY and parallel to KM, are, respectively,

$$-(M+S)\frac{A+\xi}{\rho^3}-E\left(\frac{1}{2}\cdot\frac{\tfrac{1}{2}(A+\xi)-\tfrac{1}{2}\eta\sqrt{3}-(A+x)}{r^3}+\frac{\sqrt{3}}{2}\cdot\frac{\tfrac{1}{2}(A+\xi)\sqrt{3}+\tfrac{1}{2}\eta-y}{r^3}\right)$$
$$-E\frac{\tfrac{1}{2}(A+x)+\tfrac{1}{2}y\sqrt{3}}{R^3},$$

$$-(M+S)\frac{\eta}{\rho^3}-E\left(\frac{1}{2}\cdot\frac{\tfrac{1}{2}(A+\xi)+\tfrac{1}{2}\eta-y}{r^3}-\frac{\sqrt{3}}{2}\cdot\frac{\tfrac{1}{2}(A+\xi)-\tfrac{1}{2}\eta\sqrt{3}-A-x}{r^3}\right)$$
$$-E\cdot\frac{\tfrac{1}{2}y-\tfrac{1}{2}(A+x)\sqrt{3}}{R^3}.$$

If, then, in the equations of motion relatively to S, after expanding in powers of x, y, ξ, η, we equate the finite and the small parts, the squares of x, y, ξ, η being neglected, we obtain

$$\ddot{A}-A\dot{\theta}^2 = -\frac{\mu}{A^2}$$

and

$$A^2\dot{\theta} = \text{constant}, = h, \text{ say},$$

where

$$\mu = S+E+M, \qquad \dot{\theta} = \frac{d\theta}{dt}, \qquad \ddot{\theta} = \frac{d^2\theta}{dt^2}, \text{ \&c.},$$

together with

$$\ddot{X}-2\dot{\theta}\dot{Y}-\ddot{\theta}Y-\left(\dot{\theta}^2-\frac{\mu}{A^3}\right)X = \frac{3\sqrt{3}}{4A^3}\left[\frac{4S+E+M}{\sqrt{3}}X+(E-M)Y\right],$$

$$\ddot{Y}+2\dot{\theta}\dot{X}+\ddot{\theta}X-\left(\dot{\theta}^2-\frac{\mu}{A^3}\right)Y = \frac{3\sqrt{3}}{4A^3}[(E-M)X+\sqrt{3}(E+M)Y],$$

in which X, Y respectively denote $\xi - x$ and $\eta - y$, and also

$$\ddot{x} - 2\dot{\theta}\dot{y} - \ddot{\theta}y - \left(\dot{\theta}^2 - \frac{\mu}{A^3}\right)x = \frac{3\sqrt{3}M}{4A^3}\left[\frac{4\mu}{M\sqrt{3}}x + X\sqrt{3} + Y\right],$$

$$\ddot{y} + 2\dot{\theta}\dot{x} + \ddot{\theta}x - \left(\dot{\theta}^2 - \frac{\mu}{A^3}\right)y = \frac{3\sqrt{3}M}{4A^3}[X - Y\sqrt{3}].$$

The first equations have integrals expressible by

$$\frac{l}{A} = 1 + 2\lambda\cos\theta, \qquad \frac{h^2}{l} = \mu,$$

the point (A, θ) moving in an ellipse of eccentricity 2λ and semilatusrectum l. With these values the other equations are much simplified if we take θ, instead of the time t, as independent variable, as was pointed out to me by Mr. H. M. Garner, of St. John's College, Cambridge. With this change they become

$$\left.\begin{aligned}(1+2\lambda\cos\theta)(X''-2Y'-X)-4\lambda\sin\theta(X'-Y) &= aX+hY,\\ (1+2\lambda\cos\theta)(Y''+2X'-Y)-4\lambda\sin\theta(Y'+X) &= hX+bY,\end{aligned}\right\} \quad . \quad . \quad \text{(I.)}$$

where

$$a = \frac{8S-E-M}{4\mu}, \qquad h = \frac{3(E-M)\sqrt{3}}{4\mu}, \qquad b = \frac{-4S+5(E+M)}{4\mu},$$

and

$$X' = \frac{dX}{d\theta}, \qquad X'' = \frac{d^2X}{d\theta^2}, \text{ \&c.,}$$

together with

$$\left.\begin{aligned}(1+2\lambda\cos\theta)(x''-2y'-x)-4\lambda\sin\theta(x'-y)-2x &= \frac{3\sqrt{3}M}{4\mu}(X\sqrt{3}+Y),\\ (1+2\lambda\cos\theta)(y''+2x'-y)-4\lambda\sin\theta(y'+x)+y &= \frac{3\sqrt{3}M}{4\mu}(X-Y\sqrt{3}).\end{aligned}\right\} \quad . \text{ (II.)}$$

The first thing then is to solve the equations (I.), after which the right side in (II.) will be known. Considerable simplification can be introduced by change of notation; let $w = \exp\left(\frac{2\pi i}{3}\right)$, $w^2 = \exp\left(\frac{4\pi i}{3}\right)$,

$$A = \tfrac{1}{2}(a+b+2) = \tfrac{3}{2}, \quad H = \tfrac{1}{2}(a-b+2ih), \quad K = \tfrac{1}{2}(a-b-2ih),$$

so that

$$H = \tfrac{3}{2}\frac{S+wE+w^2M}{S+E+M}, \quad K = \tfrac{3}{2}\frac{S+w^2E+wM}{S+E+M}, \quad HK = \tfrac{9}{4}\left(1-\frac{m^2}{9}\right),$$

where

$$m^2 = 27\frac{SE+SM+EM}{(S+E+M)^2};$$

further

$$p = A+(HK)^{\frac{1}{2}} = \tfrac{3}{2}\left[1+\left(1-\frac{m^2}{9}\right)^{\frac{1}{2}}\right], \quad q = A-(HK)^{\frac{1}{2}} = \tfrac{3}{2}\left[1-\left(1-\frac{m^2}{9}\right)^{\frac{1}{2}}\right],$$

so that

$$p+q = 3, \quad pq = \tfrac{1}{4}m^2.$$

Also

$$u = (1+2\lambda\cos\theta)(X+iY), \quad v = (1+2\lambda\cos\theta)(X-iY),$$

whereby the equations (I.) become

$$\left.\begin{aligned}(1+2\lambda\cos\theta)(u''+2iu') &= Au+Hv,\\ (1+2\lambda\cos\theta)(v''-2iv') &= Ku+Av,\end{aligned}\right\} \quad \text{(I.)}'$$

in which $u' = du/d\theta$, &c., and then

$$\Phi = K^{\frac{1}{2}}u+H^{\frac{1}{2}}v, \quad i\Psi = K^{\frac{1}{2}}u-H^{\frac{1}{2}}v,$$

so that Φ, Ψ are both real, and

$$\Phi+i\Psi = 2K^{\frac{1}{2}}(1+2\lambda\cos\theta)(X+iY), \quad \Phi-i\Psi = 2H^{\frac{1}{2}}(1+2\lambda\cos\theta)(X-iY),$$

and the equations (I.) become

$$\left.\begin{aligned}(1+2\lambda\cos\theta)(\Phi''-2\Psi') &= p\Phi,\\ (1+2\lambda\cos\theta)(\Psi''+2\Phi') &= q\Psi,\end{aligned}\right\} \quad \text{(I.)}''$$

in which, beside the eccentricity 2λ, there are the two constants p, q, which are dependent upon the single number m.

The equations (II.), by means of the changes

$$U = (1+2\lambda\cos\theta)(x+iy), \quad V = (1+2\lambda\cos\theta)(x-iy),$$

become

$$\left.\begin{aligned}(1+2\lambda\cos\theta)(U''+2iU')-\tfrac{3}{2}(U+V) &= \frac{3M}{2\mu}(1-w^3)\,v,\\ (1+2\lambda\cos\theta)(V''-2iV')-\tfrac{3}{2}(U+V) &= \frac{3M}{2\mu}(1-w)\,u.\end{aligned}\right\} \quad \text{(II.)}'$$

Consider now the equations (I.)$''$. We know that the solutions are of the form

$$\Phi = Ce^{i\kappa\theta}F+C_1e^{i\kappa_1\theta}F_1+C_2e^{i\kappa_2\theta}F_2+C_3e^{i\kappa_3\theta}F_3,$$

$$\Psi = Ce^{i\kappa\theta}G+C_1e^{i\kappa_1\theta}G_1+C_2e^{i\kappa_2\theta}G_2+C_3e^{i\kappa_3\theta}G_3,$$

where C, C_1, C_2, C_3 are arbitrary constants, F, F_1, ..., G_2, G_3 are definite functions of period 2π, and κ, κ_1, κ_2, κ_3 are definite constants. When $\lambda = 0$, substituting in the equations

$$\Phi = e^{i\sigma\theta}, \quad \Psi = Pe^{i\sigma\theta},$$

X 2

we obtain

$$\sigma^2+p+2i\sigma P = 0, \quad (\sigma^2+q)P-2i\sigma = 0,$$

so that the values assumed by κ, κ_1, κ_2, κ_3, when $\lambda = 0$, are the roots of the equation

$$(\sigma^2+p)(\sigma^2+q)-4\sigma^2 = 0,$$

or

$$\sigma^4-\sigma^2+\tfrac{1}{4}m^2 = 0.$$

Thus

$$\sigma = \pm\{\tfrac{1}{2}(1+m)^{\frac{1}{2}}\pm(1-m)^{\frac{1}{2}}\},$$

and the four values are all imaginary when $m > 1$, and all real when $m < 1$. Supposing $S > E > M$, we find at once, from the formula for m, that the least possible value of $S/(S+E+M)$ in order that $m < 1$ is $0{\cdot}96147\ldots$, but this requires M to be very small; but if $S/(S+E+M)$ be greater than $0{\cdot}9618\ldots$, then m is certainly < 1 even if $E = M$. In our solar system the sun's mass is more than 99·8 per cent. of the mass of the whole system; thus if S in our problem were the sun, and E, M were any two planets of the system, the condition for $m < 1$ would be easily satisfied. We shall then suppose $m < 1$.

Now compare with the equations (I.)″ the equations

$$\left.\begin{aligned}(1-2\lambda\cos\theta)(\Phi''-2\Psi') &= p\Phi,\\ (1-2\lambda\cos\theta)(\Psi''+2\Phi') &= q\Psi,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

obtained from (I.)″ by change of the sign of λ. They can also be obtained from (I.)″ by changing θ into $\theta+\pi$. This last change shows that the characteristic constants κ belonging to the equations (III.) are the same as for (I.)″, while the former change shows that the values of κ proper to (III.) are obtained by changing the sign of λ in the constants κ appropriate for (I.)″. When m is such that the values of κ for $\lambda = 0$, namely, the four values of σ above, are all different, a change in the sign of λ cannot interchange the values of κ among themselves. Thus we infer that each κ is unaltered by changing the sign of λ; for two of the values of σ can only be equal when $m^2 = 1$. In the applications in view of which the question was first considered, S denotes the sun, E denotes either Jupiter, or another planet such as Mercury, while M is of negligible mass. When E is Jupiter we have

$$m^2 = 27\tfrac{1}{1050}/(1+\tfrac{1}{1050})^2 = 0{\cdot}0257, \quad \lambda = \tfrac{1}{2}(0{\cdot}05) = 0{\cdot}025,$$

and m^2/λ is nearly unity. When E is mercury

$$m^2 = 27/5{\cdot}10^6 = 0{\cdot}0000054, \quad \lambda = \tfrac{1}{2}(0{\cdot}2) = 0{\cdot}1,$$

and $m^2 = 5{\cdot}4\lambda^6$, $m = (2{\cdot}3)\lambda^3$, nearly. In either case we may regard m as small, and the four possible values of σ are approximately

$$\pm(1-\tfrac{1}{8}m^2), \quad \pm\tfrac{1}{2}m,$$

of which the first two correspond to a period nearly the same but slightly greater than that of E, and the last two correspond to a period $\frac{2}{m}$ times that of E. When E is Jupiter, this last is $\frac{25}{2}$ times the period of Jupiter, or nearly 150 years; when E is Mercury, this period is approximately 200 years. As m is small we have approximately

$$p = 3 - \tfrac{1}{12}m^2, \quad q = \tfrac{1}{12}m^2.$$

To neglect m^2 would be to neglect the ratio 27E/S; but we may remark in passing that if we put $q = 0$, $p = 3$, the equations give

$$\Psi' + 2\Phi = C, \text{ a constant,}$$

together with

$$\Phi'' + \frac{1 + 8\lambda \cos\theta}{1 + 2\lambda \cos\theta}\Phi = 2C,$$

of which the integration can be completed in finite terms. For it may be verified that the equation

$$(1 + 2\lambda \cos\theta)\Phi'' + (1 + 8\lambda \cos\theta)\Phi = 0$$

possesses the two integrals

$$\sin\theta (1 + 2\lambda \cos\theta),$$

$$\cos\theta - 2\lambda(1 + \sin^2\theta) - 4\lambda^2 \cos\theta + 8\lambda^3 \cos 2\theta + 12\lambda^2 \sin\theta (1 + 2\lambda\cos\theta)\psi,$$

where

$$\psi = \int \frac{d\theta}{1 + 2\lambda \cos\theta}.$$

§ 8. We consider briefly, first of all, what would be the application of the method of infinite determinants to the equations (I.)″, which we may now write, with x, y for Φ, Ψ, in the forms

$$(1 + 2\lambda \cos\theta)(x'' - 2y') = px,$$

$$(1 + 2\lambda \cos\theta)(y'' + 2x') = qy.$$

We should substitute

$$x = \sum_{-\infty}^{\infty} A_n e^{i(\kappa+n)\theta}, \qquad y = \sum_{-\infty}^{\infty} B_n e^{i(\kappa+n)\theta},$$

and equate to zero the coefficients of the various powers of $e^{i\theta}$. The substitution gives, if $\zeta = e^{i\theta}$,

$$[1 + \lambda(\zeta + \zeta^{-1})]\Sigma[A_n(\kappa+n)^2 + 2i(\kappa+n)B_n]\zeta^n + p\Sigma A_n\zeta^n = 0,$$

$$[1 + \lambda(\zeta + \zeta^{-1})]\Sigma[B_n(\kappa+n)^2 - 2i(\kappa+n)A_n]\zeta^n + q\Sigma B_n\zeta^n = 0,$$

and, denoting $\kappa + n$ by κ_n, we obtain for the unknown coefficients A_n, B_n the equations

$$\lambda(A_{n-1}\kappa^2_{n-1} + 2iB_{n-1}\kappa_{n-1}) + A_n(\kappa_n^2 + p) + 2iB_n\kappa_n + \lambda(A_{n+1}\kappa^2_{n+1} + 2iB_{n+1}\kappa_{n+1}) = 0,$$

$$\lambda(-2iA_{n-1}\kappa_{n-1} + B_{n-1}\kappa^2_{n-1}) - 2iA_n\kappa_n + B_n(\kappa_n^2 + q) + \lambda(-2iA_{n+1}\kappa_{n+1} + B_{n+1}\kappa^2_{n+1}) = 0$$

If we now write

$$P_n = A_n\kappa_n^2+2iB_n\kappa_n, \qquad Q_n = -2iA_n\kappa_n+B_n\kappa_n^2,$$

which are equivalent with

$$A_n = \frac{\kappa_n P_n-2iQ_n}{\kappa_n(\kappa_n^2-4)}, \qquad B_n = \frac{2iP_n+\kappa_n Q_n}{\kappa_n(\kappa_n^2-4)},$$

the equations may be replaced by

$$\left.\begin{array}{l} \lambda P_{n-1} \qquad +a_nP_n+b_nQ_n+\lambda P_{n+1} \qquad = 0, \\ \qquad \lambda Q_{n-1}+c_nP_n+d_nQ_n+ \qquad \lambda Q_{n+1} = 0, \end{array}\right\} \quad . \quad . \quad . \quad \text{(A)}$$

wherein

$$a_n = 1+\frac{p}{\kappa_n^2-4}, \qquad b_n = \frac{-2ip}{\kappa_n(\kappa_n^2-4)},$$

$$c_n = \frac{2iq}{\kappa_n(\kappa_n^2-4)}, \qquad d_n = 1+\frac{q}{\kappa_n^2-4},$$

so that

$$a_nd_n-b_nc_n = \frac{\kappa_n^4-\kappa_n^2+\frac{1}{4}m^2}{\kappa_n^2(\kappa_n^2-4)},$$

it being remembered that $p+q = 3$, $pq = \frac{1}{4}m^2$.

When we eliminate $P_{n-1}, \ldots, Q_{n+1}$ from the equations (A), we obtain an infinite determinant, which, leaving aside questions of convergence, we may denote by

$$\left|\begin{array}{cccccccccccc}
. & . & . & . & . & . & . & . & . & . & . & . \\
. & \lambda & . & a_{-1} & b_{-1} & \lambda & . & . & . & . & . & . \\
. & . & \lambda & c_{-1} & d_{-1} & . & \lambda & . & . & . & . & . \\
. & . & . & \lambda & . & a & b & \lambda & . & . & . & . \\
. & . & . & . & \lambda & c & d & . & \lambda & . & . & . \\
. & . & . & . & . & \lambda & . & a_1 & b_1 & \lambda & . & . \\
. & . & . & . & . & . & \lambda & c_1 & d_1 & . & \lambda & . \\
. & . & . & . & . & . & . & . & . & . & . & .
\end{array}\right| = 0.$$

The product of the diagonal determinants $a_nd_n-b_nc_n$ is here

$$\frac{\sin\pi(\kappa-\sigma_1)\,.\,\sin\pi(\kappa-\sigma_2)\,.\,\sin\pi(\kappa-\sigma_3)\,.\,\sin\pi(\kappa-\sigma_4)}{\sin^4\pi\kappa},$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are the four roots of $\sigma^4-\sigma^2+\frac{1}{4}m^2 = 0$, previously considered. In using this determinant to obtain a further approximation to κ it would seem appropriate to use a theorem* for the expression of a determinant of $2n$ rows and columns as a Pfaffian, a sum $1 . 3 . 5 \ldots (2n-1)$ terms, of which each term is a product of n factors, each factor being of the form

$$(12) = a_1b'_1 - a'_1b_1 + a_2b'_2 - a'_2b_2 + \ldots + a_nb'_n - a'_nb_n,$$

where the elements

$$a_1b_1,\ a_2b_2,\ \ldots,\ a_nb_n,$$
$$a'_1b'_1,\ a'_2b'_2,\ \ldots\ a'_nb'_n,$$

are the constituents of two rows of the determinant. For in this case the factors (12) are easily calculated. But we do not pursue this method.

§ 9. Instead we proceed as follows. In the equations

$$[1+\lambda(\zeta+\zeta^{-1})]\,[x''-2y'] = px,$$
$$[1+\lambda(\zeta+\zeta^{-1})]\,[y''+2x'] = qy,$$

where $\zeta = e^{i\theta}$, write

$$x = e^{i\kappa\theta}\mathrm{X}, \qquad y = e^{i\kappa\theta}\mathrm{Y}, \qquad \kappa = \sigma + \kappa_2\lambda^2 + \kappa_4\lambda^4 + \ldots,$$

in which κ_2, κ_4, ... are certain functions of p, q to be determined. Then the equations become

$$\left.\begin{aligned}[1+\lambda(\zeta+\zeta^{-1})]\,[\mathrm{X}''-2\mathrm{Y}'+2i\kappa(\mathrm{X}'-\mathrm{Y})-\kappa^2\mathrm{X}] &= p\mathrm{X},\\ [1+\lambda(\zeta+\zeta^{-1})]\,[\mathrm{Y}''-2\mathrm{X}'+2i\kappa(\mathrm{Y}'+\mathrm{X})-\kappa^2\mathrm{Y}] &= q\mathrm{Y},\end{aligned}\right\}$$

which by the general theory are capable of periodic solution when κ is properly chosen. Put then

$$\mathrm{X} = 1+\lambda\phi_1+\lambda^2\phi_2+\ldots, \qquad \mathrm{Y} = \mathrm{P}(1+\lambda\psi_1+\lambda^2\psi_2+\ldots),$$

where P is a constant; the differential equations then take the forms

$$(1+\lambda w)(\mathrm{H}_0+\lambda\mathrm{H}_1+\lambda^2\mathrm{H}_2+\ldots) = p\mathrm{X},$$
$$(1+\lambda w)(\mathrm{K}_0+\lambda\mathrm{K}_1+\lambda^2\mathrm{K}_2+\ldots) = q\mathrm{Y},$$

w denoting $\zeta+\zeta^{-1}$. Comparing the coefficients of like powers of λ,

$$\mathrm{H}_0 = p, \qquad \mathrm{K}_0 = \mathrm{P}_q, \qquad \mathrm{H}_1+w\mathrm{H}_0 = p\phi_1, \qquad \mathrm{K}_1+w\mathrm{K}_0 = q\mathrm{P}\psi_1,$$

and, in general,

$$\mathrm{H}_n+w\mathrm{H}_{n-1} = p\phi_n, \qquad \mathrm{K}_n+w\mathrm{K}_{n-1} = \mathrm{P}q\psi_n,$$

so that

$$\mathrm{H}_1 = p(\phi_1-w), \qquad \mathrm{K}_1 = \mathrm{P}q(\psi_1-w),$$

* Proved in SCOTT-MATHEWS' 'Determinants' (1904), Chap. VIII, p. 99, § 19. Also in BAKER, 'Multiply-periodic Functions,' p. 314.

and, in general,

$$H_n = p\,[\phi_n - w\phi_{n-1} + w^2\phi_{n-2} - \dots + (-w)^n],$$

$$K_n = Pq\,[\psi_n - w\psi_{n-1} + w^2\psi_{n-2} - \dots + (-w)^n],$$

where H_n, K_n are the coefficients of λ^n respectively in

$$X'' - 2Y' + 2i\kappa(X' - Y) - \kappa^2 X,$$

$$Y'' + 2X' + 2i\kappa(Y' + X) - \kappa^2 Y.$$

In particular

$$H_0 = -2i\sigma P - \sigma^2, \quad K_0 = 2i\sigma - \sigma^2 P,$$

so that

$$\sigma^2 + p + 2i\sigma P = 0, \quad 2i\sigma = (\sigma^2 + q)\,P,$$

and, as previously,

$$\sigma^4 - \sigma^2 + \tfrac{1}{4}m^2 = 0,$$

while, if we write

$$P = \frac{\sigma^2 + p}{-2i\sigma}, \quad Q = \frac{\sigma^2 + q}{2i\sigma},$$

which are both pure imaginaries, we have $PQ = 1$.

Next

$$H_1 = \phi''_1 - 2P\psi'_1 + 2i\sigma(\phi'_1 - P\psi_1) - \sigma^2\phi_1,$$

$$K_1 = P\,[\psi''_1 + 2Q\phi'_1 + 2i\sigma(\psi'_1 + Q\phi_1) - \sigma^2\psi_1];$$

putting these respectively equal to $p(\phi_1 - w)$, $Pq(\psi_1 - w)$, we obtain two differential equations for ϕ_1 and ψ_1. If we assume

$$\phi_1 = A_1\zeta + A_{-1}\zeta^{-1}, \quad \psi_1 = B_1\zeta + B_{-1}\zeta^{-1},$$

and notice that

$$(\zeta^r)' = ir\zeta^r, \quad (\zeta^r)'' = -r^2\zeta^r,$$

we find, writing σ_n for $\sigma + n$,

$$A_1(\sigma_1^2 + p) + 2Pi\sigma_1 B_1 = p, \qquad A_{-1}(\sigma_{-1}^2 + p) + 2Pi\sigma_{-1}B_{-1} = p,$$

$$-A_1 \,.\, 2Qi\sigma_1 + (\sigma_1^2 + q)\,B_1 = q, \quad -A_{-1}2Qi\sigma_{-1} + (\sigma_{-1}^2 + q)\,B_{-1} = q.$$

If

$$\Delta_1 = \sigma_1^4 - \sigma_1^2 + \tfrac{1}{4}m^2,$$

these give

$$\Delta_1 A_1 = (\sigma_1^2 + q)\,p + \frac{\sigma_1}{\sigma}(\sigma^2 + p)\,q,$$

$$\Delta_1 B_1 = (\sigma_1^2 + p)\,q + \frac{\sigma_1}{\sigma}(\sigma^2 + q)\,p,$$

with similar equations for A_{-1}, B_{-1}.

Proceeding similarly to equate terms in λ^2, we find

$$\phi''_2 + 2i\sigma\phi'_2 - \sigma^2\phi_2 - 2P(\psi'_2 + i\sigma\psi_2) - 2\kappa_2(iP + \sigma) = p(\phi_2 - w\phi_1 + w^2),$$

$$\psi''_2 + 2i\sigma\psi'_2 - \sigma^2\psi_2 + 2Q(\phi'_2 + i\sigma\phi_2) - 2\kappa_2(-iQ + \sigma) = q(\psi_2 - w\psi_1 + w^2).$$

If herein we assume

$$\phi_2 = A_2\zeta^2 + A_{-2}\zeta^{-2} + H, \quad \psi_2 = B_2\zeta^2 + B_{-2}\zeta^{-2} + K,$$

and equate terms in ζ^2, ζ^{-2}, ζ^0, we obtain

$$A_2(\sigma_2^2+p) + 2Pi\sigma_2 B_2 = p(A_1 - 1), \qquad A_{-2}(\sigma^2_{-2}+p) + 2Pi\sigma_{-2}B_{-2} = p(A_{-1}-1),$$

$$-A_2 \,.\, 2Qi\sigma_2 + (\sigma_2^2+q)B_2 = q(B_1-1), \quad -A_{-2} \,.\, 2Qi\sigma_{-2} + (\sigma^2_{-2}+q)B_{-2} = q(B_{-1}-1),$$

and

$$(\sigma^2+p)(H-K) + 2\kappa_2(iP+\sigma) = p(A_1+A_{-1}-2),$$

$$-(\sigma^2+q)(H-K) + 2\kappa_2(-iQ+\sigma) = q(B_1+B_{-1}-2),$$

wherein the coefficients of $H-K$ and κ_2 have for determinant

$$(\sigma^2+p)(-iQ+\sigma) + (\sigma^2+q)(iP+\sigma),$$

which is

$$\sigma(1-m^2)^{\frac{1}{2}}$$

and is not zero. That H, K should not be determinable separately is obvious *à priori;* to regard H as zero would be equivalent to dividing X, Y by a power series in λ^2 with constant coefficients. We notice that the successive coefficients A_1, A_{-1}, ..., B_2, B_{-2} are all real. The value found for κ_2 is

$$\kappa_2 = pq\frac{7-6\sigma^2}{\sigma(1-2\sigma^2)(1-4\sigma^2)}.$$

A similar procedure can be continued. The differential equations for ϕ_3, ψ_3 can be solved by forms

$$\phi_3 = A_3\zeta^3 + A_{-3}\zeta^{-3} + H_1\zeta + H_{-1}\zeta^{-1}, \quad \psi_3 = B_3\zeta^3 + B_{-3}\zeta^{-3} + K_1\zeta + K_{-1}\zeta^{-1},$$

the differential equations for ϕ_4, ψ_4 by forms

$$\phi_4 = A_4\zeta^4 + A_{-4}\zeta^{-4} + M_2\zeta^2 + M_{-2}\zeta^{-2} + M,$$

$$\psi_4 = B_4\zeta^4 + B_{-4}\zeta^{-4} + N_2\zeta^2 + N_{-2}\zeta^{-2} + N,$$

and then the terms in ζ^0 will involve the unknown quantities

$$(\sigma^2+p)(M-N) + 2\kappa_4(iP+\sigma),$$

$$-(\sigma^2+q)(M-N) + 2\kappa_4(-iQ+\sigma),$$

from which κ_4 is found. And it serves as verification of the computation to see that κ_4 involves H, K only in the combination $H-K$, as it must in order to be determined without ambiguity.

The value found for x is of the form

$$x = e^{i\kappa\theta}[1 + (\lambda\zeta, \lambda\zeta^{-1}) + (\lambda^2\zeta^2, \lambda^2\zeta^{-2}, \lambda^2) + (\lambda^3\zeta^3, \lambda^3\zeta^{-3}, \lambda^3\zeta, \lambda^3\zeta^{-1}) + (\lambda^4\zeta^4, \lambda^4\zeta^{-4}, \lambda^4\zeta^2, \lambda^4\zeta^{-2}, \lambda^4) + \ldots],$$

or, say,

$$x = e^{i\kappa\theta}[u_0 + \lambda\zeta u_1 + \lambda\zeta^{-1}u_{-1} + \lambda^2\zeta^2 u_2 + \lambda^2\zeta^{-2}u_{-2} + \ldots],$$

where every one of u_0, u_1, u_{-1}, u_2, u_{-2}, ... is a power series in λ^2 with real coefficients, not generally vanishing with λ^2. And similarly for y.

§ 10. The interesting case of the preceding solution is that corresponding to the value of σ given by

$$\sigma = \tfrac{1}{2}[(1+m)^{\frac{1}{2}} - (1-m)^{\frac{1}{2}}], \quad = \tfrac{1}{2}m\left(1 + \frac{m^2}{8} + \ldots\right).$$

The quantity

$$\kappa_2 = pq\,\frac{7 - 6\sigma^2}{\sigma(1-2\sigma^2)(1-4\sigma^2)}$$

is then equal to

$$\tfrac{1}{2}m\left(7 + \tfrac{65}{8}m^2\right)$$

approximately, and $\kappa_2\lambda^2$ is of the order $m\lambda^2$. When $m^2 \propto \lambda$, this is of the order m^5 or $\lambda^{5/2}$; when $m \propto \lambda^3$, it is of the order $m^{5/3}$ or λ^5. Thus a very few terms of the preceding solutions would seem to be sufficient for practical cases.

PART II.

§ 11. A large part of the interest of POINCARÉ'S 'Methodés Nouvelles de la Mécanique Céleste' depends on his criticism of the convergence of the series used by astronomers, particularly those series in which the time enters only under trigonometrical signs. In t. II., p. 277, he refers to a linear differential equation

$$\frac{d^2x}{dt^2} + x(1+\psi) = 0,$$

in which ψ, for our purposes, may be supposed to have a form

$$\psi = 4a\cos ht + 4b\cos kt,$$

in which a, b are small. When h, k are commensurable the equation has periodic coefficients, and POINCARÉ makes the convergence of the series expressing the solution depend on this circumstance ('Méth. Nouv.,' t. I., p. 66). Considering the case in which h and k are incommensurable, and so ψ not periodic, and supposing a, b to have common small factor μ, he obtains formal solutions of the differential equation in sines and cosines, and says "les séries . . ., qu'on peut ordonner suivant les puissances de μ, ne sont plus convergentes" ('Méth Nouv.,' t. II., pp. 277, 278). On the contrary, I believe that the solution of the differential equation above, arranged as a power series in a and b, converges for all finite values of these parameters, and that this is a consequence of a general theory of linear differential equations considered in papers* published by me in 1902. As this theory is capable of application to many other differential equations, as will be illustrated below by application to the equation considered by G. W. HILL for the motion of the moon's perigee, I wish to deal with it here, repeating the argument in part.

§ 12. Consider any system of linear differential equations, the n^2 coefficients

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n, \qquad (i = 1, 2, \ldots, n),$$

u_{ij} being functions of t. If these are considered only for real values of t, the properties which we require to assume are that, along a certain range which we shall suppose to include $t = 0$, these functions u_{ij} are single-valued, limited, and capable of integration, the same being true of certain other functions derived from these by multiplications, and further, that certain infinite series, which we shall prove to be absolutely and uniformly convergent, are capable of differentiation, term by term. But in the majority of practical cases the coefficients u_{ij} may be looked upon as the values, when t is real, of functions of a complex variable t. In this case we suppose a star region to be defined by lines passing to infinity from certain points in the finite part of the plane, which we call the singular points; we suppose $t = 0$ not to be a singular point, and the lines may be straight continuations of the radii joining the origin to these singular points. Within this star region, bounded by the lines in question, the functions u_{ij} are supposed to be single-valued and capable of development by power series about every point, forming monogenic analytic functions in the usual sense. Taking then any region within this star region, we obtain solutions of the differential equations, with arbitrary values for $t = 0$, in the form of infinite series of functions, obtained by quadratures, which are proved to converge absolutely and uniformly within the region taken.

The method of forming these solutions is extremely simple, involving only integrations and multiplications, but the way in which the work is arranged, though often of great utility, does not seem yet to find common acceptance, and some words must be given to it.

* 'Proc. Lond. Math. Soc.,' XXXIV., 1902, p. 355; XXXV., 1902, p. 339. See also the same 'Proc.,' 2nd Series, II., p. 293, where it is explained that the same idea had already been used by PEANO and others. To me the method was independently suggested by the theory of continuous groups, 'Proc. Lond. Math. Soc.,' XXXIV., 1902, p. 91. POINCARÉ'S conclusions as to the convergence of astronomical series have been criticised by G. W. HILL, 'Coll. Works,' IV., p. 94; but the point there at issue is different from that considered here. In connexion with an example considered by POINCARÉ, *loc. cit.*, p. 279, see BRUNS, 'Astr. Nachr.,' No. 2606 (CIX., 1884), pp. 217, 218. Also BOREL, 'Théorie des Fonctions' (1898), p. 27; HARDY, 'Quart. Journ.,' XXXVI., p. 93: 'Proc. Lond. Math. Soc.,' III., p. 441, and the references there given.

The n^2 quantities u_{ij} can be arranged to form a square of n rows and n columns, the first suffix i denoting the row, and the second suffix j denoting the column in which a particular element u_{ij} is placed. This square is denoted by a single symbol, say u, and called a matrix. The symbol uv, formed from the two symbols u, v, written in a definite order, denotes then another matrix whose $(i, j)^{\text{th}}$ element has the value

$$u_{i1}v_{1j}+u_{i2}v_{2j}+\ldots+u_{in}v_{nj},$$

which is formed from the elements of the i^{th} *row* of the matrix u and those of the j^{th} *column* of the matrix v. This new matrix uv is called the product of u and v, taken in this order; it is generally different from vu. The symbol 1, when used for a matrix of an assigned number of rows and columns, denotes the matrix of which every element is zero except those in the diagonal, all of which have the same value, unity; it is easy to see that any matrix is unaltered by multiplication with the matrix unity of the same number of rows and columns. The symbol u^{-1} denotes the matrix such that the product $u^{-1}u$ is the matrix unity; in that case uu^{-1} is equal to $u^{-1}u$; the symbol u^{-1} is nugatory when the determinant formed with the elements of u is zero, and only then. In general, the determinant formed with the elements of u will be denoted by $|u|$. By the *sum*, $u+v$, of two matrices u, v, of the same number of rows and columns, is meant the matrix whose $(i, j)^{\text{th}}$ element is $u_{ij}+v_{ij}$, and, similarly, for the difference. Frequently we denote the aggregate of a row of n quantities, $x_1, x_2, \ldots, x_n$ by the single letter x; then if u be a matrix of n rows and columns, the symbol ux denotes a set of n quantities of which the i^{th} is

$$u_{i1}x_1+u_{i2}x_2+\ldots+u_{in}x_n.$$

By the differential coefficient of a matrix we mean the single matrix whose elements are the differential coefficients of the given one. In what follows, if the $(i, j)^{\text{th}}$ element of a matrix u be a function of t, we denote by Qu the matrix of which the $(i, j)^{\text{th}}$ element is the integral of u_{ij} taken in regard to t from $t = 0$ to $t = t$. If, for an instant this matrix Qu be denoted by v, the product matrix uv will be denoted by uQu, and the matrix $Q(uv)$, or $Q(uQu)$, will be denoted by $QuQu$. Similarly, $Q(u\,.\,QuQu)$ will be denoted by $QuQuQu$, and so on.

Now consider a matrix of which the $(i, j)^{\text{th}}$ element is the infinite series formed by the sum of the $(i, j)^{\text{th}}$ elements taken from the matrix unity (of the same number of rows and columns as u), the matrix Qu, the matrix $QuQu$, the matrix $QuQuQu$, and so on. This will be denoted by

$$\Omega(u) = 1+Qu+QuQu+QuQuQu+\ldots,$$

and the series on the right will be said to be uniformly and absolutely convergent when this property is proved to hold for each of the n^2 infinite series which constitute its elements.

Repeating now the demonstration given, 'Proc. Lond. Math. Soc.,' April 10, 1902, p. 354, let $u_{ij}^{(1)}$ denote the $(i, j)^{\text{th}}$ element of the matrix Qu, that is,

$$u_{ij}^{(1)} = \int_0^t u_{ij}\, dt\,;$$

similarly, let $u_{ij}^{(2)}$ denote the $(i, j)^{\text{th}}$ element of the matrix $QuQu$, namely,

$$u_{ij}^{(2)} = \int_0^t [u_{i1}u_{1j}^{(1)} + u_{i2}u_{2j}^{(1)} + \ldots + u_{in}u_{nj}^{(1)}]\, dt,$$

and so on. For the region chosen within the star region above explained, when the functions u_{ij} are functions of a complex variable, or for the range of values of t adopted when the elements u_{ij} are functions of a real variable, there will exist a real positive quantity M_{ij} not exceeded by the absolute value of u_{ij} for the values of t involved. Taking a path of integration limited to such values, from the origin $t = 0$ to $t = t$, this being a rectifiable curve of length s, let t_1 be an intermediate point of this path, the length of the path from the origin to t_1 being s_1. Then we have, considering absolute values,

$$|u_{ij}^{(1)}(t)| \lesseqgtr M_{ij} \int_0^s ds_1 \lesseqgtr sM_{ij},$$

and in particular

$$|u_{ij}^{(1)}(t_1)| \lesseqgtr s_1 M_{ij}.$$

Similarly,

$$|u_{ij}^{(2)}(t)| \lesseqgtr \int_0^s (M_{i1}s_1M_{1j} + \ldots + M_{in}s_1M_{nj})\, ds_1\,;$$

now if M denote the matrix whose $(i, j)^{\text{th}}$ element is M_{ij}, the $(i, j)^{\text{th}}$ element of the matrix M^2, formed by the product of M with itself, will be

$$M_{i1}M_{1j} + M_{i2}M_{2j} + \ldots + M_{in}M_{nj},$$

which we may denote by $(M^2)_{ij}$; thence

$$|u_{ij}^{(2)}(t)| \lesseqgtr (M^2)_{ij} \int_0^s s_1\, ds_1 \lesseqgtr \tfrac{1}{2}s^2 (M^2)_{ij},$$

and in particular

$$|u_{ij}^{(2)}(t_1)| \lesseqgtr \tfrac{1}{2}s_1^2 (M^2)_{ij}.$$

We can continue this process. The next step will be

$$|u_{ij}^{(3)}(t)| \lesseqgtr \int_0^s \tfrac{1}{2}s_1^2\, ds_1 [M_{i1}(M^2)_{1j} + \ldots + M_{in}(M^2)_{nj}],$$

$$\lesseqgtr \frac{s^3}{3\,!} (M^3)_{ij}.$$

Thus we see that each of the n^2 infinite series constituting the elements of the matrix

$$\Omega(u) = 1 + Qu + QuQu + \ldots$$

has terms whose moduli are respectively equal to, or less than, the real positive terms of the corresponding infinite series constituting the elements of the matrix

$$1 + sM + \frac{s^2}{2!}M^2 + \frac{s^3}{3!}M^3 + \ldots.$$

This last is, however, certainly convergent for all finite values of s, whatever be the (finite) values of the elements of the matrix M. For the case when the algebraic equation satisfied by M has unequal roots, its sum is given by the formula of 'Proc. Lond. Math. Soc.,' XXXIV., February 14, 1901, p. 114, which can be easily modified to meet the case of unequal roots.

Thus each of the elements of the matrix $\Omega(u)$ is an absolutely and uniformly convergent series; in the case when the elements u_{ij} are functions of the complex variable, as explained above, it follows that every element of the matrix $\Omega(u)$ is a function of the complex variable, and differentiation (and integration) of the series representing this element is permissible, term by term. For the case of real functions we introduce this as a condition.

Hence, if x^0 denote a row of n arbitrary values $x_1^0, x_2^0, \ldots, x_n^0$, the row of n quantities denoted by

$$x = \Omega(u)\, x^0$$

is at once seen to form a set of n integrals of the differential equations, reducing for $t = 0$ to the arbitrary values x^0, that is, x_i reducing to x_i^0. For if v denote any matrix, of n rows and columns, whose elements are differentiable functions of t, if x^0 denote a row of n constants, and y the set of n functions given by

$$y = vx^0,$$

that is,

$$y_i = v_{i1}x_1^0 + v_{i2}x_2^0 + \ldots + v_{in}x_n^0,$$

we have

$$\frac{dy_i}{dt} = \frac{dv_{i1}}{dt}x_1^0 + \ldots + \frac{dv_{in}}{dt}x_n^0,$$

which, if $\frac{dv}{dt}$ denote the matrix whose elements are the differential coefficients of the elements of v, we can denote by

$$\frac{dy}{dt} = \frac{dv}{dt}x^0.$$

Hence the equation

$$x = \Omega(u)\, x^0,$$

gives

$$\frac{dx}{dt} = \frac{d}{dt}(1+\mathrm{Q}u+\mathrm{Q}u\mathrm{Q}u+\ldots)\,x^0,$$

$$= \frac{d}{dt}[1+\mathrm{Q}u+\mathrm{Q}\,(u\mathrm{Q}u)+\mathrm{Q}\,(u\mathrm{Q}u\mathrm{Q}u)+\ldots]\,x^0,$$

$$= [u+u\mathrm{Q}u+u\mathrm{Q}u\mathrm{Q}u+\ldots]\,x^0,$$

$$= u\,[1+\mathrm{Q}u+\mathrm{Q}u\mathrm{Q}u+\ldots]\,x^0,$$

$$= u\Omega\,(u)\,x^0,$$

or

$$dx/dt = ux,$$

so that the functions $x = \Omega\,(u)\,x^0$ satisfy the differential equations. By the definition, $\mathrm{Q}u_{ij}$ reduces to zero for $t = 0$; hence $\Omega\,(u)$ reduces to its first term, the matrix unity, when $t = 0$; that is, $x = \Omega\,(u)\,x^0$ reduces to $x = x^0$ when $t = 0$.

In what follows we shall require a particular property of the matrix $\Omega\,(u)$, given in 'Proc. Lond. Math. Soc.,' XXXV., December 11, 1902, p. 339. If u, v be any two matrices of n rows and columns, of similar character to the u considered above, the property is expressed by

$$\Omega\,(u+v) = \Omega\,(u)\,\Omega\,\{[\Omega\,(u)]^{-1}\,v\Omega\,(u)\},$$

where $[\Omega\,(u)]^{-1}$ is the matrix inverse to $\Omega\,(u)$, defined above, such that $[\Omega\,(u)]^{-1}\,\Omega\,(u) = 1$. The theorem is nugatory when the determinant of $\Omega\,(u)$ is zero. It is only equivalent to saying that if in the system of linear differential equations

$$\frac{dx}{dt} = (u+v)\,x,$$

that is,

$$\frac{dx_i}{dt} = (u_{i1}+v_{i1})\,x_1+\ldots+(u_{in}+v_{in})\,x_n,$$

we introduce a set of n new dependent variables, denoted by z, by means of the equations

$$x = \Omega\,(u)\,z, \quad \text{or} \quad z = [\Omega\,(u)]^{-1}\,x,$$

then

$$dz/dt = [\Omega\,(u)]^{-1}\,v\Omega\,(u)\,z.$$

This follows at once from

$$(u+v)\,x = \frac{dx}{dt} = \frac{d}{dt}[\Omega\,(u)\,z] = \left[\frac{d}{dt}\,\Omega\,(u)\right]z+\Omega\,(u)\,\frac{dz}{dt}$$

$$= [u\Omega\,(u)]\,z+\Omega\,(u)\,\frac{dz}{dt} = u\Omega\,(u)\,z+\Omega\,(u)\,\frac{dz}{dt}$$

$$= ux+\Omega\,(u)\,\frac{dz}{dt},$$

which gives

$$\Omega(u)\frac{dz}{dt} = vx = v\Omega(u)z.$$

In what follows we shall generally write $\Omega^{-1}(u)$ in place of $[\Omega(u)]^{-1}$.

Another property to be noticed* is that the determinant of the matrix $\Omega(u)$ is equal to the exponential of the sum of the integrals from 0 to t of the diagonal elements of the matrix u. For, if Ω_{ij} denote the general element of $\Omega(u)$, the equation

$$\frac{d}{dt}\Omega(u) = u\Omega(u),$$

already remarked, is the aggregate of the equations

$$\frac{d\Omega_{ij}}{dt} = u_{i1}\Omega_{1j} + \ldots + u_{in}\Omega_{nj}.$$

Further, the differential coefficient of a determinant of n rows and columns can be written as a sum of n determinants, each of which is obtained from the original determinant by replacing the elements of one row respectively by their differential coefficients. Hence we at once see that, if Δ denote the determinant of $\Omega(u)$,

$$d\Delta/dt = (u_{11} + u_{22} + \ldots + u_{nn})\Delta,$$

which establishes the result in question.

In particular, if the sum of the diagonal elements of u,

$$u_{11} + u_{22} + \ldots + u_{nn},$$

be zero, the determinant of $\Omega(u)$ is independent of t, and is thus equal to unity. This result is of frequent application.

§ 13. After these introductory remarks we may at once show that the equation

$$\frac{d^2x}{dt^2} + x(1 + 4a\cos ht + 4b\cos kt) = 0,$$

to which reference has been made, is capable of solution as an absolutely and uniformly converging series in a, b, whatever h and k may be. It will be as simple, and of utility for other applications we wish to make, to take the equation

$$\frac{d^2x}{dt^2} + x(n^2 + \psi) = 0,$$

in which we may suppose n to be an integer.

* *Cf.* DARBOUX, 'Compt. Rend.,' XC. (1880), p. 526.

In this last equation, put

$$X = \tfrac{1}{2}e^{int}\left(\frac{dx}{dt} - inx\right), \qquad Y = \tfrac{1}{2}e^{-int}\left(\frac{dx}{dt} + inx\right),$$

leading to

$$-inx = Xe^{-int} - Ye^{int}, \qquad dx/dt = Xe^{-int} + Ye^{int};$$

then we have

$$\frac{dX}{dt} = -\frac{i\psi}{2n}e^{int}(Xe^{-int} - Ye^{int}),$$

$$\frac{dY}{dt} = -\frac{i\psi}{2n}e^{-int}(Xe^{-int} - Ye^{int}).$$

Writing

$$2it = \tau, \quad \zeta = e^{\tau},$$

these are

$$\frac{dX}{d\tau} = -\frac{\psi}{4n}(X - Y\zeta^{n}), \quad \frac{dY}{d\tau} = -\frac{\psi}{4n}(X\zeta^{-n} - Y),$$

or, say,

$$\frac{d}{d\tau}(X, Y) = -\frac{\psi}{4n}\begin{pmatrix} 1, & -\zeta^{n} \\ \zeta^{-n}, & -1 \end{pmatrix}(X, Y),$$

where, as is usual, the single quantity $-\frac{\psi}{4n}$, written before the matrix, is to be multiplied into every element of the matrix.

In particular, when $n = 1$, $\psi = 4a\cos ht + 4b\cos kt$,

$$\frac{d}{d\tau}(X, Y) = (ap + bq)(X, Y),$$

where p, q denote the matrices

$$p = \tfrac{1}{2}(\zeta^{\frac{1}{2}h} + \zeta^{-\frac{1}{2}h})\begin{pmatrix} -1, & \zeta \\ -\zeta^{-1}, & 1 \end{pmatrix}, \quad q = \tfrac{1}{2}(\zeta^{\frac{1}{2}k} + \zeta^{-\frac{1}{2}k})\begin{pmatrix} -1, & \zeta \\ -\zeta^{-1}, & 1 \end{pmatrix}.$$

Thus the solution is expressed by

$$(X, Y) = \Omega(ap + bq)(X^{0}, Y^{0}),$$

where $\Omega(ap + bq)$ is of the form

$$1 + aQp + bQq + a^{2}QpQp + ab(QpQq + QqQp) + b^{2}QqQq + \ldots,$$

and we have proved that this series is uniformly and absolutely convergent.

If we assume such a form of solution it is easy by successive steps to obtain the values of the coefficients independently of the method we have adopted. What is of present importance is that we have shown the series to be convergent, a fact which appears to be denied by POINCARÉ.

§ 14. Leaving aside this point, we pass on now to the application of the general method here explained to the computation of the integrals of particular differential equations with periodic coefficients, as, for instance, the equation for the motion of the lunar perigee, considered by G. W. HILL.

It is known from the general theory that the solutions of the n equations

$$dx_i/dt = u_{i1}x_1 + \ldots + u_{1n}x_n, \quad (i = 1, 2, \ldots, n),$$

in which $u_{i1}, \ldots, u_{in}$ are single-valued functions with a common period, say w, can be written, in the most general case, in the forms

$$x_i = A_1 e^{\lambda_1 t}\phi_{i1} + \ldots + A_n e^{\lambda_n t}\phi_{in},$$

wherein $A_1, \ldots, A_n$ are arbitrary constants, $\lambda_1, \ldots, \lambda_n$ are n definite constants, and the functions ϕ_{ij} are n^2 definite functions all with the period w. In many applications it is the constants $\lambda_1, \ldots, \lambda_n$ which it is of most importance to find; when these are all pure imaginaries, the motion* represented by the differential equations presents, beyond the fundamental period w, secondary oscillations of periods $2\pi/i\lambda_r$, and the motion is conventionally said to be stable.

We show first how this form of solution naturally arises from the point of view we have adopted.

Write $\Omega_0^t(u)$ in place of $\Omega(u)$, and for simplicity write only two rows and columns of the matrix, though the argument is quite general. Make the limitation, which, as is well known, does not cover all cases, that there exists a matrix of constants, h, of n rows and columns, whose inverse is denoted by h^{-1}, such that the complete matrix $\Omega_0^w(u)$ can be written in the form

$$\Omega_0^w(u) = h \begin{pmatrix} e^{ic_1 w}, & 0 \\ 0, & e^{ic_2 w} \end{pmatrix} h^{-1}$$

with only diagonal elements, here denoted by $e^{ic_1 w}$, $e^{ic_2 w}$, in the reduced matrix. This will be so, in the technical phraseology, if the matrix $\Omega_0^w(u)$ has linear invariant factors.† Then, from the definition of $\Omega(u)$,

$$\Omega_0^{w+t}(u) = \Omega_w^{w+t}(u) \,.\, \Omega_0^w(u),$$

while, as u has period w,

$$\Omega_w^{w+t}(u) = \Omega_0^t(u).$$

* Interesting physical examples are given by Lord RAYLEIGH, 'Collected Works,' III., p. 1.

† A proof of the general theorem for the reduction of a matrix, valid when this is of vanishing determinant, is given, 'Proc. Camb. Phil. Soc.,' XII. (1903), p. 65. The literature of this matter, which begins with SYLVESTER, 'Coll. Papers,' I., pp. 119, 139, 219, and WEIERSTRASS, 'Ges. Werke,' I., p. 233, is very wide. The reader may consult MUTH, 'Elementartheiler,' Leipzig, 1899.

Hence

$$\Omega_0^{w+t}(u) \,.\, h = \Omega_0^t(u) \,.\, h \begin{pmatrix} e^{ic_1 w}, & 0 \\ 0, & e^{ic_2 w} \end{pmatrix} h^{-1},$$

and so

$$\Omega_0^{w+t}(u) \,.\, h \begin{pmatrix} e^{-ic_1(w+t)}, & 0 \\ 0, & e^{-ic_2(w+t)} \end{pmatrix} = \Omega_0^t(u) \,.\, h \begin{pmatrix} e^{-ic_1 t}, & 0 \\ 0, & e^{-ic_2 t} \end{pmatrix}.$$

This shows that the matrix on the right has period w. Put then

$$P_0^t = \Omega_0^t(u)\, h \begin{pmatrix} e^{-ic_1 t}, & 0 \\ 0, & e^{-ic_2 t} \end{pmatrix} h^{-1},$$

which has period w, and is such that $P_0^w = P_0^0 = 1$. The matrix $\Omega_0^t(u)$ can therefore be written in the form

$$\Omega_0^t(u) = P_0^t \,.\, h \begin{pmatrix} e^{ic_1 t}, & 0 \\ 0, & e^{ic_2 t} \end{pmatrix} h^{-1},$$

which is the theorem in question.

We now compare this with the form of solution of the original differential equations by the method of successive approximation followed by LAGRANGE, LAPLACE, and others. We have

$$\begin{pmatrix} e^{ic_1 t}, & 0 \\ 0, & e^{ic_2 t} \end{pmatrix} = 1 + it \begin{pmatrix} c_1, & 0 \\ 0, & c_2 \end{pmatrix} + \frac{(it)^2}{2\,!} \begin{pmatrix} c_1^2, & 0 \\ 0, & c_2^2 \end{pmatrix} + \ldots;$$

thus

$$\Omega_0^t(u) = P_0^t + tP(h\gamma h^{-1}) + \frac{t^2}{2\,!} P(h\gamma^2 h^{-1}) + \ldots,$$

where P is written for P_0^t, and γ is written for

$$\begin{pmatrix} ic_1, & 0 \\ 0, & ic_2 \end{pmatrix}.$$

If then, as in LAPLACE, 'Méc. Cél.,' Liv. II., Ch. V., t. I., of the edition of 1878. p. 266, we obtain the solutions of the differential equations in the form

$$(P_0^t + tA + t^2B + \ldots)\, x^0,$$

where A, B are certain periodic matrices, and x^0 is a row of arbitrary constants, *we can obtain the constants ic_1, ic_2, which are the most important quantities in many applications, by taking the matrix* A, *which arises as the coefficient of t, and is equal in our notation to*

$$P_0^t(h\gamma h^{-1}),$$

putting therein $t = 0$, *so obtaining, say* A_0, *equal in our notation to* $h\gamma h^{-1}$, *and then solving the determinantal equation*

$$|A_0 - \lambda| = 0,$$

whose roots are ic_1 *and* ic_2. This process will be found to be equivalent to the general procedure explained by LAPLACE, in the passage above referred to, for bringing the time under trigonometrical signs. We have considered only the case of linear differential equations with periodic coefficients, and have supposed $\Omega_0^w(u)$ to have linear invariant factors; LAPLACE's method, if less definite, is of much wider application. An interesting exposition of the method in general is given by M. O. CALLANDREAU, 'Ann. de l'Observ. de Paris,' XXII., 1896, pp. 16, 20.

We may notice that

$$A_0 = h\begin{pmatrix} ic_1, & 0 \\ 0, & ic_2 \end{pmatrix}h^{-1}$$

gives

$$\Omega(A_0) = h\begin{pmatrix} e^{ic_1t}, & 0 \\ 0, & e^{ic_2t} \end{pmatrix}h^{-1},$$

so that we also have

$$\Omega(u) = P_0^t \,.\, \Omega(A_0)$$
$$= P_0^t(1 + tA_0 + \tfrac{1}{2}t^2A_0^2 + \ldots),$$

and the quantities e^{ic_1w}, e^{ic_2w} are the roots of the equation

$$|\Omega_0^w(u) - \rho| = 0.$$

§ 15. When the sum of the diagonal elements of the matrix u is zero, the determinant of $\Omega(u)$ is unity, as above remarked. In this case, when $n = 2$, the two quantities e^{ic_1w}, e^{ic_2w} are inverses and $c_2 = -c_1$. In this case the equation

$$|\Omega_0^w(u) - \rho| = 0$$

gives at once the value of $\cos cw$. This appears, however, a less advantageous way of determining c_1, c_2 than that explained above, as requiring greater approximation in the calculation of $\Omega_0^t(u)$, as will be seen in examples.

The fact that c_1, c_2 are equal and of opposite signs is a particular case of a well-known theorem for the variational equations arising in the general dynamical case, which is proved by POINCARÉ ('Méth. Nouv.,' I., 193). The following proof, though longer, appears more fundamental in character. The general dynamical equations being

$$\frac{dx_r}{dt} = \frac{\partial F}{\partial y_r}, \qquad \frac{dy_r}{dt} = -\frac{\partial F}{\partial x_r},$$

where it will be sufficient to suppose r to have the values 1, 2; let

$$x_r = \phi_r(t), \qquad y_r = \psi_r(t),$$

be a solution of these equations. Substitute in the differential equations

$$x_r = \phi_r(t) + \xi_r, \qquad y_r = \psi_r(t) + \eta_r,$$

and retain only first powers of the quantities ξ_r and η_r, which are supposed to be small. We thence obtain a system of linear differential equations of the form

$$\left(\frac{d\xi_1}{dt}, \frac{d\eta_1}{dt}, \frac{d\xi_2}{dt}, \frac{d\eta_2}{dt}\right) = \beta^{-1}\mathrm{A}\,(\xi_1, \eta_1, \xi_2, \eta_2),$$

where β is the skew-symmetrical matrix of constants given by

$$\beta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

(so that $\beta^{-1} = -\beta$), and A is a symmetrical matrix whose elements are functions of t. We then have the theorems following :—

(a) The roots of the determinantal equation for λ,

$$|\beta^{-1}\mathrm{A} - \lambda| = 0,$$

fall into pairs of equal roots of opposite sign ;

(b) The determinantal equation for ρ,

$$|\Omega(\beta^{-1}\mathrm{A}) - \rho| = 0,$$

is a reciprocal equation, unaltered by changing ρ into ρ^{-1}.

To express the proof we require a notation for the matrix obtained from a given matrix u by interchanging its rows with its columns, thus placing the element u_{ij} in the $(j, i)^{\text{th}}$ instead of the $(i, j)^{\text{th}}$ place. This transposed matrix may be denoted by trs (u) or by $\bar{u}$. It is easy also to show that

$$[\Omega(u)]^{-1} = \mathrm{trs}\,[\Omega(-\bar{u})].$$

Then (a) is immediate from the obvious relations among determinants expressed by

$$|\mathrm{A} - \beta\lambda| = |\bar{\mathrm{A}} - \bar{\beta}\lambda| = |\mathrm{A} + \beta\lambda|,$$

since $\bar{\mathrm{A}} = \mathrm{A}$, $\bar{\beta} = -\beta$.

For (b), since

$$\beta\Omega(u)\beta^{-1} = \Omega(\beta u\beta^{-1}), \qquad \text{trs}(\beta^{-1}\text{A}) = -\text{A}\beta^{-1},$$

we have the following transformations of matrices

$$[\Omega(\beta^{-1}\text{A})]^{-1} = \text{trs}[\Omega(\text{A}\beta^{-1})] = \text{trs}[\beta\Omega(\beta^{-1}\text{A})\beta^{-1}] = \beta^{-1}[\text{trs}\,\Omega(\beta^{-1}\text{A})]\beta,$$

and hence, writing $\Omega^{-1}(u)$ for $[\Omega(u)]^{-1}$, the following equations among determinants

$$|\Omega^{-1}(\beta^{-1}\text{A})-\rho| = |\text{trs}\,\Omega(\beta^{-1}\text{A})-\rho| = |\Omega(\beta^{-1}\text{A})-\rho|,$$

which establishes the result in question.

§ 16. In many dynamical applications the matrix A is a sum of two matrices

$$\text{A} = \alpha+\vartheta,$$

where α is a symmetrical matrix of real constants, and ϑ a symmetrical matrix whose elements are small. Suppose, further, that p, denoting a row of $2n$ real variables $p_1, p_2, \ldots$, the matrix α is such that the quadratic form

$$\Sigma\Sigma\alpha_{ij}p_ip_j$$

does not vanish unless every one of the $2n$ elements of p is zero, which requires that the determinant $|\alpha|$ is not zero. Then, if this quadratic form be denoted by αp^2, and if each of ξ and η be a row of $2n$ real quantities, the form

$$\alpha(\xi+i\eta)(\xi-in), \quad = \alpha\xi^2+i\alpha(\eta\xi-\xi\eta)+\alpha\eta^2, \quad = \alpha(\xi^2+\eta^2),$$

has the same property.

When this is so, it can be shown that the roots of the determinantal equation in ψ,

$$|\beta^{-1}\alpha-\psi| = 0,$$

are pure imaginaries, and that the invariant factors of the matrix $\beta^{-1}\alpha-\psi$ are linear. As the proof is not long it may be given here (*cf.* 'Proc. Lond. Math. Soc.,' XXXV., December 11, 1902, p. 380).

Let ψ satisfy the determinantal equation

$$|\alpha-\beta\psi| = 0;$$

as the determinant $|\alpha|$ is not zero, ψ cannot be zero. Then $2n$ quantities $x_1, x_2, \ldots$, whose aggregate is denoted by x, can be taken to satisfy the $2n$ linear equations

$$(\alpha-\beta\psi)x = 0.$$

If x_0 denote the row formed by the $2n$ quantities which are the conjugate complexes of those of x, we have in turn

$$\alpha xx_0 = \psi\beta xx_0, \qquad \bar{\alpha}x_0x = \psi\bar{\beta}x_0x, \qquad \alpha x_0x = -\psi\beta x_0x,$$

and, therefore, ψ_0 being the conjugate complex of ψ

$$\alpha x x_0 = -\psi_0 \beta x x_0.$$

Hence

$$\left(\frac{1}{\psi} + \frac{1}{\psi_0}\right) \alpha x x_0 = 0,$$

showing that $\psi + \psi_0 = 0$, which proves that ψ is a pure imaginary.

Writing λ for ψ^{-1}, the equations above are the same as

$$(\alpha^{-1}\beta - \lambda) x = 0 ;$$

we prove that the invariant factors are linear by showing* that it is not possible to find a row of $2n$ quantities $y_1, y_2, \ldots$, such that

$$(\alpha^{-1}\beta - \lambda) y = x.$$

For this would involve

$$(\beta - \alpha\lambda) y x_0 = \alpha x x_0,$$

of which the right side is real, so that, λ being a pure imaginary, either of these would be equal to

$$(\beta + \alpha\lambda) y_0 x ; \quad = (\bar{\beta} + \bar{\alpha}\lambda) x y_0 = (-\beta + \alpha\lambda) x y_0,$$

of which the last is zero in virtue of

$$(\alpha^{-1}\beta - \lambda) x = 0.$$

As $\alpha x x_0$ is not zero, the assumed equation for y is impossible, and the invariant factors are linear.

From this fact it follows that it is possible to find a matrix h such that

$$h^{-1}\beta^{-1}\alpha h = \begin{pmatrix} i\sigma_1 & 0 & 0 & 0 \\ 0 & -i\sigma_1 & 0 & 0 \\ 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \end{pmatrix},$$

where σ_1, σ_2 are real. Then the given differential equations, which are of the form

$$\left(\frac{d\xi_1}{dt}, \frac{d\eta_1}{dt}, \frac{d\xi_2}{dt}, \frac{d\eta_2}{dt}\right) = (\beta^{-1}\alpha + \beta^{-1}\vartheta)(\xi_1, \eta_1, \xi_2, \eta_2),$$

if transformed by the linear substitution

$$(\xi_1, \eta_1, \xi_2, \eta_2) = h(X_1, Y_1, X_2, Y_2)$$

* See, for example, 'Proc. Camb. Phil. Soc.,' XII. (1903), p. 65.

take the forms

$$\left(\frac{dX_1}{dt}, \frac{dY_1}{dt}, \frac{dX_2}{dt}, \frac{dY_2}{dt}\right) = (\sigma+\Theta)(X_1, Y_1, X_2, X_2),$$

where σ denotes the matrix above written, with only diagonal elements $i\sigma_1$, &c., and Θ is the matrix

$$\Theta = h^{-1}\beta^{-1}\vartheta h.$$

The solutions of these equations are then expressed by

$$(X_1, Y_1, X_2, Y_2) = \Omega(\sigma+\Theta)(X_1^0, Y_1^0, X_2^0, Y_2^0),$$

where X_1^0, Y_1^0, ..., are the initial values. Now, by a previously given formula,

$$\Omega(\sigma+\Theta) = \Omega(\sigma)\,\Omega[\Omega^{-1}(\sigma)\,\Theta\Omega(\sigma)],$$

where $\Omega(\sigma)$ has the simple form

$$\Omega(\sigma) = \begin{pmatrix} e^{i\sigma_1 t} & 0 & 0 & 0 \\ 0 & e^{-i\sigma_1 t} & 0 & 0 \\ 0 & 0 & e^{i\sigma_2 t} & 0 \\ 0 & 0 & 0 & e^{-i\sigma_2 t} \end{pmatrix};$$

the solution is thereby expressed in powers of the small quantities occurring in ϑ.

The preceding work has wide applications; a particular case is that of the oscillations of a dynamical system about a state of steady motion, for which ϑ, and Θ, is zero.

[*October 30, 1915.*—To prevent misunderstanding, two remarks may be added to § 16. The condition that the quadratic form αx^2 should be positive, though sufficient, is not necessary in order that the roots of the determinantal equation $(\beta^{-1}\alpha-\psi) = 0$ should be pure imaginaries. For instance, if a, b, u, v be real positive constants, and H be a quadratic form

$$H = \tfrac{1}{2}a(y_1-nx_2)^2+\tfrac{1}{2}b(y_2-mx_1)^2-\frac{u^2}{2a}x_1^2-\frac{v^2}{2b}x_2^2,$$

the motion about $x_1 = 0$, $x_2 = 0$, $y_1 = 0$, $y_2 = 0$ expressed by the equations

$$\dot{x}_1 = \partial H/\partial y_1, \quad \dot{y}_1 = -\partial H/\partial x_1, \quad \dot{x}_2 = \partial H/\partial y_2, \quad \dot{y}_2 = -\partial H/\partial x_2,$$

is instantaneously stable if $ab(m-n)^2 > (u+v)^2$, the corresponding quartic equation having all its roots purely imaginary. This essentially is the case noticed by THOMSON and TAIT, 'Natural Philosophy,' I., pp. 395, 398, where the illustration is that of a gyrostat balanced on gimbals. A simple illustration is also that of the oscillations about steady motion of a weight suspended by a string of which the other end is made to describe uniformly a horizontal circle, in the case in which the string intersects the vertical drawn *downwards* from the centre of the circle described by its upper end. This motion is not, however, secularly stable when there is Dissipativity (THOMSON and TAIT, as above, p. 388); and, of course, not instantaneously stable, the roots of the corresponding quartic equation having real parts of which some are positive.

A second remark relates to the generality of the form of the differential equations used in the text. Equations such as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_r}\right)-\frac{\partial T}{\partial x_r}+\beta_{r1}\dot{x}_1+\ldots+\beta_{rn}\dot{x}_n+\frac{\partial F}{\partial \dot{x}_r}+\frac{\partial V}{\partial x_r}=Q_r,$$

where β_{rs} is a function of $x_1, \ldots, x_n$ capable of expression in terms of n functions $\beta_1, \ldots, \beta_n$ in the form

$$\beta_{rs}=\frac{\partial \beta_r}{\partial x_s}-\frac{\partial \beta_s}{\partial x_r}$$

are included in this form, with a slight modification due to the presence of the Dissipativity F, and the supposed non-conservative forces Q_r. For this it is only necessary to take

$$L = T+\beta_1\dot{x}_1+\ldots+\beta_n\dot{x}_n-V,$$

$$H = \dot{x}_1\frac{\partial L}{\partial \dot{x}_1}+\ldots+\dot{x}_n\frac{\partial L}{\partial \dot{x}_n}-L,$$

and to eliminate $\dot{x}_1, \ldots, \dot{x}_n$, in the familiar way, from the n equations

$$y_r=\frac{\partial L}{\partial \dot{x}_r}.$$

Then the final equations are

$$\dot{x}_r=\frac{\partial H}{\partial y_r},\quad \dot{y}_r=-\frac{\partial H}{\partial x_r}-\frac{\partial F}{\partial \dot{x}_r}+Q_r.$$

Particular illustrations are: (1) the equations of THOMSON and TAIT (as above), p. 392, for which the coefficients β_{rs} are constants. Then we may take $\beta_r = c_{r1}x_1+\ldots+c_{rn}x_n$, where the constant coefficients c_{rs} are in part arbitrary; (2) the equations of Lord KELVIN for liquid motions of ring-shaped solids, 'Collected Papers,' IV. (1910), p. 106; (3) the equations of motion of a system relatively to a rotating frame (LAMB, 'Hydrodynamics,' third edition (1906), p. 294. *Cf.* THOMSON and TAIT, as above, § 319, p. 307, and p. 319), for which we may take, if (ξ, η, ζ) be the co-ordinates of a point of the system relatively to the rotating frame,

$$\beta_r = w\Sigma m\left(\xi\frac{\partial \eta}{\partial x_r}-\eta\frac{\partial \xi}{\partial x_r}\right).$$

The equation of energy in general is at once seen to be

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} - \Sigma \dot{x}_r \frac{\partial F}{\partial \dot{x}_r} + \Sigma Q_r \dot{x}_r,$$

so that if H be explicitly independent of the time, the forces Q_r be absent, and F be a homogeneous quadratic function of $\dot{x}_1, \ldots, \dot{x}_n$,

$$\frac{dH}{dt} = -2F.]$$

§ 17. The simplicity of the formulation depends on the fact that the invariant factors of $\beta^{-1}\alpha - \psi$ are linear. We have obtained this by assuming that the form αp^2 only vanishes when every element of p is zero. But the invariant factors may be linear when this is not so, and the roots of the determinantal equation are not pure imaginaries. For instance, take HILL's equations for the motion of the moon, under certain limitations,

$$\frac{d^2x}{dt^2} - 2n\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n^2\right)x = 0, \quad \frac{d^2y}{dt^2} + 2n\frac{dx}{dt} + \frac{\mu y}{r^3} = 0.$$

Writing

$$F = -\frac{\mu}{r} - \tfrac{3}{2}n^2x^2 + \tfrac{1}{2}(X + ny)^2 + \tfrac{1}{2}(Y - nx)^2,$$

these are the same as

$$\frac{dx}{dt} = \frac{\partial F}{\partial X}, \quad \frac{dX}{dt} = -\frac{\partial F}{\partial x}, \quad \frac{dy}{dt} = \frac{\partial F}{\partial Y}, \quad \frac{dY}{dt} = -\frac{\partial F}{\partial y}.$$

The so-called moon of no quadratures is obtained by variation from the solution expressed by

$$x = \sigma, \quad X = 0, \quad y = 0, \quad Y = n\sigma,$$

where σ is given by $\mu = 3n^2\sigma^3$; this is a position of relative equilibrium. The matrix ϑ of the notation used above is zero; the matrix α is

$$\begin{pmatrix} -8n^2 & 0 & 0 & -n \\ 0 & 1 & n & 0 \\ 0 & n & 4n^2 & 0 \\ -n & 0 & 0 & 1 \end{pmatrix}.$$

In this case the quadratic form αp^2 is

$$-9n^2p_1^2 + (p_2 + 2np_3)^2 + (np_1 - p_4)^2$$

and vanishes when $p_1 = p_4 = 0$, $p_2 = -2np_3$. But the roots of the determinantal equation

$$|\beta^{-1}\alpha - \psi| = 0$$

are all different, and therefore the invariant factors are linear.

The roots are

$$\psi = \pm\, n\{(28)^{\frac{1}{2}}+1\}^{\frac{1}{2}}, \qquad \psi = \pm\, in\{(28)^{\frac{1}{2}}-1\}^{\frac{1}{2}},$$

of which only two are pure imaginaries; thus not every disturbed orbit is periodic.

§ 18. We pass on now to give the details of the application of the general method above explained to the computation of some particular cases.

A very simple case may be first given, merely as an example of the notation and method, since the results, once obtained, are easily verified.

Take the equations

$$2\frac{dx}{dt} = -x\cos t + y(1+\sin t),$$

$$2\frac{dy}{dt} = -x(1-\sin t) + y\cos t.$$

These may be written

$$\frac{d(x,\,y)}{dt} = \left[\tfrac{1}{2}\begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -\cos t, & \sin t \\ \sin t, & \cos t \end{pmatrix}\right](x,\,y),$$

or, say,

$$\frac{d(x,\,y)}{dt} = (u+v)(x,\,y),$$

where

$$u = \tfrac{1}{2}\begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \quad v = \tfrac{1}{2}\begin{pmatrix} -\cos t, & \sin t \\ \sin t, & \cos t \end{pmatrix}.$$

We have at once

$$(2u)^2 = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}\begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix} = -1,$$

and therefore

$$\begin{aligned}\Omega(u) &= 1 + ut + \frac{u^2}{2\,!}t^2 + \frac{u^3}{3\,!}t^3 + \ldots, \\ &= 1 - \frac{1}{2\,!}(\tfrac{1}{2}t)^2 + \frac{1}{4\,!}(\tfrac{1}{2}t)^4 - \ldots + 2u\left\{\tfrac{1}{2}t - \frac{1}{3\,!}(\tfrac{1}{2}t)^3 + \ldots\right\}, \\ &= \cos\tfrac{1}{2}t + \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}\sin\tfrac{1}{2}t, \\ &= \begin{pmatrix} \cos\tfrac{1}{2}t, & \sin\tfrac{1}{2}t \\ -\sin\tfrac{1}{2}t, & \cos\tfrac{1}{2}t \end{pmatrix}.\end{aligned}$$

This gives

$$\Omega^{-1}(u) = \begin{pmatrix} \cos \frac{1}{2}t, & -\sin \frac{1}{2}t \\ \sin \frac{1}{2}t, & \cos \frac{1}{2}t \end{pmatrix}.$$

Wherefore

$$\Omega^{-1}(u) \,.\, v \,.\, \Omega(u) = \tfrac{1}{2} \begin{pmatrix} \cos \frac{1}{2}t, & -\sin \frac{1}{2}t \\ \sin \frac{1}{2}t, & \cos \frac{1}{2}t \end{pmatrix} \begin{pmatrix} -\cos t, & \sin t \\ \sin t, & \cos t \end{pmatrix} \Omega(u)$$

$$= \tfrac{1}{2} \begin{pmatrix} -\cos \frac{1}{2}t, & \sin \frac{1}{2}t \\ \sin \frac{1}{2}t, & \cos \frac{1}{2}t \end{pmatrix} \Omega(u)$$

$$= \tfrac{1}{2} \begin{pmatrix} -1, & 0 \\ 0, & 1 \end{pmatrix}.$$

Denoting this by $\frac{1}{2}\sigma$, we find $\sigma^2 = 1$, and hence

$$\Omega\left[\Omega^{-1}(u)\, v\Omega(u)\right] = 1 + \tfrac{1}{2}\sigma t + \frac{\sigma^2}{2\,!}\left(\tfrac{1}{2}t\right)^2 + \frac{\sigma^3}{3\,!}\left(\tfrac{1}{2}t\right)^3 + \ldots,$$

$$= ch\tfrac{1}{2}t + \sigma sh\tfrac{1}{2}t,$$

$$= \begin{pmatrix} e^{-\frac{1}{2}t}, & 0 \\ 0, & e^{\frac{1}{2}t} \end{pmatrix}.$$

Thus the solution is

$$(x, y) = \Omega(u+v)\,(x^0, y^0) = \begin{pmatrix} \cos \frac{1}{2}t, & \sin \frac{1}{2}t \\ -\sin \frac{1}{2}t, & \cos \frac{1}{2}t \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}t}, & 0 \\ 0, & e^{\frac{1}{2}t} \end{pmatrix} (x^0, y^0),$$

namely,

$$x = x^0 e^{-\frac{1}{2}t} \cos \tfrac{1}{2}t + y^0 e^{\frac{1}{2}t} \sin \tfrac{1}{2}t,$$

$$y = -x^0 e^{-\frac{1}{2}t} \sin \tfrac{1}{2}t + y^0 e^{\frac{1}{2}t} \cos \tfrac{1}{2}t.$$

The period of the coefficients in the original equation is 2π. The functions $\cos \frac{1}{2}t$, $\sin \frac{1}{2}t$ have only the period 4π. To bring the result into the form given by the general theory we may write

$$x = x^0 e^{-\frac{1}{2}(1+i)t} \,.\, \tfrac{1}{2}\left(e^{it} + 1\right) + y^0 e^{\frac{1}{2}(1+i)t} \,.\, \frac{1}{2i}\left(1 - e^{-it}\right),$$

$$y = -x^0 e^{-\frac{1}{2}(1+i)t} \,.\, \frac{1}{2i}\left(e^{it} - 1\right) + y^0 e^{\frac{1}{2}(1+i)t} \,.\, \tfrac{1}{2}\left(1 + e^{-it}\right),$$

the so-called characteristic exponents being

$$\pm \tfrac{1}{2}(1+i).$$

§ 19. We now consider cases of the equations

$$\frac{d}{d\tau}(X, Y) = -\frac{\psi}{4n}\begin{pmatrix} 1, & -\zeta^n \\ \zeta^{-n}, & -1 \end{pmatrix}(X, Y);$$

these are derivable from the equation

$$\frac{d^2x}{dt^2}+(n^2+\psi)\,x = 0$$

by taking

$$X = \tfrac{1}{2}e^{int}\left(\frac{dx}{dt}-inx\right), \quad Y = \tfrac{1}{2}e^{-int}\left(\frac{dx}{dt}+inx\right), \quad \tau = 2it, \quad \zeta = e^{\tau},$$

leading to

$$x = \frac{i}{n}\,(Xe^{-int}-Ye^{int}).$$

As we wish particularly to illustrate the method of obtaining the characteristic exponents from the present point of view, we take first a case in which explicit terms in τ arise early in the method of successive approximation. We take namely $n = 1$, and suppose

$$\frac{\psi}{4} = \lambda h+2\lambda k_1 \cos 2t+2\lambda^2 k_2 \cos 4t+\ldots,$$

$$= \lambda h+\lambda k_1 w_1+\lambda^2 k_2 w_2+\ldots,$$

where λ is small, and w_r is used to denote $\zeta^r+\zeta^{-r}$.

Denoting $\frac{1}{4}\psi$ by ϕ, our differential equations are

$$\frac{d\,(X, Y)}{d\tau} = u\,(X, Y),$$

where

$$u = \begin{pmatrix} -\phi, & \phi\zeta \\ -\phi\zeta^{-1}, & \phi \end{pmatrix}.$$

The coefficients in these equations have period $2\pi i$; by what we have previously shown (§§ 14, 15), the solution is of the form

$$(X, Y) = Ph\begin{pmatrix} e^{-q\tau}, & 0 \\ 0, & e^{q\tau} \end{pmatrix}h^{-1}(X^0, Y^0),$$

where P is a matrix whose elements have the period $2\pi i$, h is a matrix of constants, and q is the constant which we particularly desire to find. As

$$x = i\,(Xe^{-it}-Ye^{it}),$$

this corresponds to characteristic factors $e^{\mp i(1+2q)t}$ for the original equation in t, whose coefficients have period π. The quantity q is to be found by determining the terms in

τ in the solution of the (X, Y) equations, and forming from this, after putting $\tau = 0$, a determinantal equation (§ 14).

We are to calculate in turn Qu, $QuQu$, &c., and arrange the result according to powers of λ. First we have

$$Qu = \begin{pmatrix} a_1, & b_1 \\ c_1, & d_1 \end{pmatrix},$$

where

$$a_1 = -\int_0^\tau \phi\, d\tau, \qquad b_1 = \int_0^\tau \phi\zeta\, d\tau,$$

$$c_1 = -\int_0^\tau \zeta^{-1}\phi\, d\tau, \quad d_1 = \int_0^\tau \phi\, d\tau\,;$$

thus, as ϕ is unaltered by changing the sign of τ, b_1 can be obtained from c_1 by changing the sign of τ, and similarly d_1 from a_1. This we denote by writing

$$b_1 = c'_1, \quad d_1 = a'_1.$$

Then

$$uQu = \begin{pmatrix} -\phi, & \zeta\phi \\ -\zeta^{-1}\phi, & \phi \end{pmatrix}\begin{pmatrix} a_1, & c'_1 \\ c_1, & a'_1 \end{pmatrix},$$

$$= \begin{pmatrix} -\phi a_1 + \zeta\phi c_1, & -\phi c'_1 + \zeta\phi a'_1 \\ -\zeta^{-1}\phi a_1 + \phi c_1, & -\zeta^{-1}\phi c'_1 + \phi a'_1 \end{pmatrix},$$

and hence

$$QuQu = \begin{pmatrix} a_2, & c'_2 \\ c_2, & a'_2 \end{pmatrix},$$

where

$$a_2 = \int_0^\tau \phi\,(-a_1 + \zeta c_1)\, d\tau, \qquad c'_2 = \int_0^\tau \phi\,(-c'_1 + \zeta a'_1)\, d\tau,$$

$$c_2 = \int_0^\tau \zeta^{-1}\phi\,(-a_1 + \zeta c_1)\, d\tau, \quad a'_2 = \int_0^\tau \phi\,(a'_1 - \zeta^{-1}c'_1)\, d\tau,$$

so that a'_2 is obtained from a_2 by changing the sign of τ throughout, and similarly c'_2 from c_2. In general, in passing from a term of $\Omega(u)$ involving r integrations to one involving $(r+1)$ integrations, we shall have a law expressible by

$$A_{r+1} = \int_0^\tau \phi\,(-A_r + \zeta C_r)\, d\tau, \quad C_{r+1} = \int_0^\tau \zeta^{-1}\phi\,(-A_r + \zeta C_r)\, d\tau,$$

and the new term, like that from which it is derived, will be of the form

$$\begin{pmatrix} A_{r+1}, & C'_{r+1} \\ C_{r+1}, & A'_{r+1} \end{pmatrix},$$

where A'_{r+1} is derived from A_{r+1} by change of the sign of τ, and similarly C'_{r+1} from C_{r+1}.

Thus, when in $\Omega(u)$ we pick out the coefficient of τ, as it occurs explicitly, independently of its occurrence in ζ, and in this coefficient put $\tau = 0$, we shall obtain a series of the form

$$\begin{pmatrix} \alpha_1, & -\gamma_1 \\ \gamma_1, & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2, & -\gamma_2 \\ \gamma_2, & -\alpha_2 \end{pmatrix} + \ldots,$$

where the first of these comes from Qu and involves terms in λ and higher powers, the second comes from $QuQu$ and involves terms in λ^2 and higher powers, and so on. And the equation for q will be of the form

$$\begin{vmatrix} \alpha_1+\alpha_2+\ldots-q, & -\gamma_1-\gamma_2-\ldots \\ \gamma_1+\gamma_2+\ldots, & -\alpha_1-\alpha_2-\ldots-q \end{vmatrix} = 0,$$

namely,

$$q^2 = (\alpha_1+\alpha_2+\ldots)^2-(\gamma_1+\gamma_2+\ldots)^2.$$

Further, if the part of Qu which is independent of explicit powers of τ, consisting of elements which are polynomials in ζ, ζ^{-1} and periodic with period $2\pi i$, be denoted by P_1, and similarly the periodic part of $QuQa$ be denoted by P_2, &c., then the periodic matrix P above spoken of will be

$$P = 1+P_1+P_2+\ldots.$$

Proceeding to the computation, retain first only to terms in λ. Then

$$a_1 = -\int_0^\tau \phi\, d\tau = -\lambda h\tau-\lambda k_1(\zeta-\zeta^{-1}),$$

$$c_1 = -\int_0^\tau \zeta^{-1}\phi\, d\tau = -\int_0^\tau \zeta^{-1}[\lambda h+\lambda k_1(\zeta+\zeta^{-1})]\, d\tau,$$

$$= \lambda h(\zeta^{-1}-1)+\lambda k_1\left(\frac{\zeta^{-2}-1}{2}-\tau\right).$$

Hence

$$\alpha_1 = -\lambda h, \quad \gamma_1 = -\lambda k,$$

and q is given by

$$q^2 = \lambda^2(h^2-k_1^2).$$

In the case when the differential equation is that considered by HILL, this gives at once a very near approximation, as he remarks, being equivalent to his formula

$$c^2 = 1+\{(\vartheta_0-1)^2-\vartheta_1^2\}^{\frac{1}{2}}$$

(HILL'S 'Collect. Works,' I., p. 260).

If next we retain as far as λ^3, we have from

$$\phi = \lambda h + \lambda k_1 (\zeta^{-1}+\zeta) + \lambda^2 k_2 (\zeta^{-2}+\zeta^2) + \lambda^3 k_3 (\zeta^{-3}+\zeta^3),$$

$$-a_1 = \int_0^\tau \phi \, d\tau$$

$$= \lambda h\tau + \lambda k_1 (-\zeta^{-1}+\zeta) + \tfrac{1}{2}\lambda^2 k_2 (-\zeta^{-2}+\zeta^2) + \tfrac{1}{3}\lambda^3 k_3 (-\zeta^{-3}+\zeta^3),$$

$$c_1 = -\int_0^\tau \zeta^{-1}\phi \, d\tau$$

$$= \lambda h (\zeta^{-1}-1) + \lambda k_1 \left(\frac{\zeta^{-2}}{2} - \tfrac{1}{2} - \tau\right) + \lambda^2 k_2 \left(\frac{\zeta^{-3}}{3} + \tfrac{2}{3} - \zeta\right) + \lambda^3 k_3 \left(\frac{\zeta^{-4}}{4} + \tfrac{1}{4} - \frac{\zeta^2}{2}\right)$$

Hence

$$-a_1 + \zeta c_1$$

$$= \lambda h (\tau + 1 - \zeta) + \lambda k_1 \left(-\frac{\zeta^{-1}}{2} + \tfrac{1}{2}\zeta - \tau\zeta\right) + \lambda^2 k_2 \left(-\frac{\zeta^{-2}}{6} + \tfrac{2}{3}\zeta - \frac{\zeta^2}{2}\right) + \lambda^3 k_3 \left(-\frac{\zeta^{-3}}{12} + \frac{\zeta}{4} - \frac{\zeta^3}{6}\right).$$

Thus

$$\phi(-a_1 + \zeta c_1)$$

$$= \lambda^2 \{h^2 (\tau + 1 - \zeta) + hk_1 (\tau\zeta^{-1} + \tfrac{1}{2}\zeta^{-1} - 1 + \tfrac{3}{2}\zeta - \zeta^2) + k_1^2 (-\tfrac{1}{2}\zeta^{-2} - \tau - \tau\zeta^2 + \tfrac{1}{2}\zeta^2)\}$$

$$+ \lambda^3 \{hk_2 (\tau\zeta^{-2} + \tfrac{5}{6}\zeta^{-2} - \zeta^{-1} + \tfrac{2}{3}\zeta + \tau\zeta^2 + \tfrac{1}{2}\zeta^2 - \zeta^3)$$

$$+ k_1 k_2 (-\tfrac{2}{3}\zeta^{-3} - \tau\zeta^{-1} + \tfrac{1}{3}\zeta^{-1} + \tfrac{2}{3} - \zeta + \tfrac{2}{3}\zeta^3 - \tau\zeta^3)\}.$$

This gives

$$a_2 = \int_0^\tau \phi(-a_1 + \zeta c_1) \, d\tau$$

$$= \lambda^2 \{h^2 (\tfrac{1}{2}\tau^2 + \tau + 1 - \zeta) + hk_1 (-\tau\zeta^{-1} - \tfrac{3}{2}\zeta^{-1} - \tau + \tfrac{1}{2} + \tfrac{3}{2}\zeta - \tfrac{1}{2}\zeta^2)$$

$$+ k_1^2 (\tfrac{1}{4}\zeta^{-2} - \tfrac{1}{2}\tau^2 - \tfrac{3}{4} - \tfrac{1}{2}\tau\zeta^2 + \tfrac{1}{2}\zeta^2)\}$$

$$+ \lambda^3 \{hk_2 (-\tfrac{1}{2}\tau\zeta^{-2} - \tfrac{2}{3}\zeta^{-2} + \zeta^{-1} - \tfrac{2}{3} + \tfrac{2}{3}\zeta + \tfrac{1}{2}\tau\zeta^2 - \tfrac{1}{3}\zeta^3)$$

$$+ k_1 k_2 (\tfrac{2}{9}\zeta^{-3} + \tau\zeta^{-1} + \tfrac{2}{3}\zeta^{-1} + \tfrac{2}{3}\tau - \tfrac{1}{3} - \zeta + \tfrac{1}{3}\zeta^2 - \tfrac{1}{3}\tau\zeta^3 + \tfrac{1}{9}\zeta^3)\}.$$

Similarly,

$$c_2 = \int_0^\tau \zeta^{-1}\phi(-a_1 + \zeta c_1) \, d\tau$$

$$= \lambda^2 \{h^2 (-\tau\zeta^{-1} - 2\zeta^{-1} - \tau + 2) + hk_1 (-\tfrac{1}{2}\tau\zeta^{-2} - \tfrac{1}{2}\zeta^{-2} + \zeta^{-1} + \tfrac{3}{2}\tau + \tfrac{1}{2} - \zeta)$$

$$+ k_1^2 (\tfrac{1}{6}\zeta^{-3} + \tau\zeta^{-1} + \zeta^{-1} - \tfrac{8}{3} + \tfrac{3}{2}\zeta - \tau\zeta)\}$$

$$+ \lambda^3 \{hk_2 (-\tfrac{1}{3}\tau\zeta^{-3} - \tfrac{7}{18}\zeta^{-3} + \tfrac{1}{2}\zeta^{-2} + \tfrac{2}{3}\tau + \tfrac{8}{9} + \tau\zeta - \tfrac{1}{2}\zeta - \tfrac{1}{2}\zeta^2)$$

$$+ k_1 k_2 (\tfrac{1}{6}\zeta^{-4} + \tfrac{1}{2}\tau\zeta^{-2} + \tfrac{1}{12}\zeta^{-2} - \tfrac{2}{3}\zeta^{-1} - \tau - \tfrac{1}{2} + \tfrac{2}{3}\zeta - \tfrac{1}{2}\tau\zeta^2 + \tfrac{1}{4}\zeta^2)\}.$$

Forming now $-a_2 + \zeta c_2$ we obtain

$$\lambda^2 \{h^2 (-\tfrac{1}{2}\tau^2 - 2\tau - 3 - \tau\zeta + 3\zeta) + hk_1 (\tfrac{1}{2}\tau\zeta^{-1} + \zeta^{-1} + \tau + \tfrac{1}{2} + \tfrac{3}{2}\tau\zeta - \zeta - \tfrac{1}{2}\zeta^2)$$

$$+ k_1^2 (-\tfrac{1}{12}\zeta^{-2} + \tfrac{1}{2}\tau^2 + \tau + \tfrac{7}{4} - \tfrac{8}{3}\zeta + \zeta^2 - \tfrac{1}{2}\tau\zeta^2)\}$$

$$+ \lambda^3 \{hk_2 (\tfrac{1}{6}\tau\zeta^{-2} + \tfrac{5}{18}\zeta^{-2} - \tfrac{1}{2}\zeta^{-1} + \tfrac{2}{3} + \tfrac{2}{9}\zeta + \tfrac{2}{3}\tau\zeta + \tfrac{1}{2}\tau\zeta^2 - \tfrac{1}{2}\zeta^2 - \tfrac{1}{6}\zeta^3)$$

$$+ k_1 k_2 (-\tfrac{1}{18}\zeta^{-3} - \tfrac{1}{2}\tau\zeta^{-1} - \tfrac{7}{12}\zeta^{-1} - \tfrac{2}{3}\tau - \tfrac{1}{3} - \tau\zeta + \tfrac{1}{2}\zeta + \tfrac{1}{3}\zeta^2 - \tfrac{1}{6}\tau\zeta^3 + \tfrac{5}{36}\zeta^3)\}.$$

To find

$$a_3 = \int_0^\tau \phi(-a_2+\zeta c_2)\,d\tau, \qquad c_3 = \int_0^\tau \zeta^{-1}\phi(-a_2+\zeta c_2)\,d\tau$$

up to λ^3 it is sufficient to take $\phi = \lambda[h+k_1(\zeta^{-1}+\zeta)]$; so we obtain

$$\begin{aligned}
&\phi(-a_2+\zeta c_2)\\
&= \lambda^3\{h^3(-\tfrac{1}{2}\tau^2-2\tau-3-\tau\zeta+3\zeta)\\
&\quad +h^2k_1(-\tfrac{1}{2}\tau^2\zeta^{-1}-\tfrac{3}{2}\tau\zeta^{-1}-2\zeta^{-1}+\tfrac{7}{2}-\tfrac{1}{2}\tau^2\zeta-\tfrac{1}{2}\tau\zeta-4\zeta-\tau\zeta^2+\tfrac{5}{2}\zeta^2)\\
&\quad +hk_1^2(\tfrac{1}{2}\tau\zeta^{-2}+\tfrac{11}{12}\zeta^{-2}+\tau\zeta^{-1}+\tfrac{1}{2}\zeta^{-1}+\tfrac{1}{2}\tau^2+3\tau+\tfrac{7}{4}+\tau\zeta-\tfrac{8}{3}\zeta+\tau\zeta^2-\tfrac{1}{2}\zeta^3)\\
&\quad +k_1^3(-\tfrac{1}{12}\zeta^{-3}+\tfrac{1}{2}\tau^2\zeta^{-1}+\tau\zeta^{-1}+\tfrac{5}{3}\zeta^{-1}-\tfrac{8}{3}+\tfrac{1}{2}\tau^2\zeta+\tfrac{1}{2}\tau\zeta+\tfrac{11}{4}\zeta-\tfrac{8}{3}\zeta^2-\tfrac{1}{2}\tau\zeta^3+\zeta^3)\},
\end{aligned}$$

and hence

$$\begin{aligned}
a_3 &= \int_0^\tau \phi(-a_2+\zeta c_2)\,d\tau\\
&= \lambda^3h^3(-\tfrac{1}{6}\tau^3-\tau^2-3\tau-4-\tau\zeta+4\zeta)\\
&\quad +\lambda^3h^2k_1[(\tfrac{1}{2}\tau^2+\tfrac{5}{2}\tau+\tfrac{9}{2})\zeta^{-1}+\tfrac{7}{2}\tau-\tfrac{3}{2}+(-\tfrac{1}{2}\tau^2+\tfrac{1}{2}\tau-\tfrac{9}{2})\zeta+(-\tfrac{1}{2}\tau+\tfrac{3}{2})\zeta^2]\\
&\quad +\lambda^3hk_1^2\Big[(-\tfrac{1}{4}\tau-\tfrac{7}{12})\zeta^{-2}+(-\tau-\tfrac{3}{2})\zeta^{-1}+\tfrac{1}{36}\tau^3+\tfrac{3}{2}\tau^2+\tfrac{7}{4}\tau+\tfrac{37}{6}+(\tau-\tfrac{11}{3})\zeta\\
&\qquad\qquad +\left(\frac{\tau}{2}-\tfrac{1}{4}\right)\zeta^2-\tfrac{1}{6}\zeta^3\Big],\\
&\quad +\lambda^3k_1^3[\tfrac{1}{36}\zeta^{-3}+(-\tfrac{1}{2}\tau^2-2\tau-\tfrac{11}{3})\zeta^{-1}-\tfrac{8}{3}\tau+\tfrac{4}{3}+(\tfrac{1}{2}\tau^2-\tfrac{1}{2}\tau+\tfrac{13}{4})\zeta-\tfrac{4}{3}\zeta^2+(-\tfrac{1}{6}\tau+\tfrac{7}{18})\zeta^3].
\end{aligned}$$

Similarly,

$$\begin{aligned}
c_3 &= \int_0^\tau \zeta^{-1}\phi(-a_2+\zeta c_2)\,d\tau\\
&= \lambda^3h^3[(\tfrac{1}{2}\tau^2+3\tau+6)\zeta^{-1}+(-\tfrac{1}{2}\tau^2+3\tau-6)]\\
&\quad +\lambda^3h^2k_1[(\tfrac{1}{4}\tau^2+\tau+\tfrac{3}{2})\zeta^{-2}-\tfrac{7}{2}\zeta^{-1}+(-\tfrac{1}{6}\tau^3-\tfrac{1}{4}\tau^2-4\tau-\tfrac{3}{2})+(-\tau+\tfrac{7}{2})\zeta]\\
&\quad +\lambda^3hk_1^2[(-\tfrac{1}{6}\tau-\tfrac{13}{36})\zeta^{-3}-\tfrac{1}{2}(\tau+1)\zeta^{-2}+(-\tfrac{1}{2}\tau^2-4\tau-\tfrac{23}{4})\zeta^{-1}\\
&\qquad\qquad +\tfrac{1}{2}\tau^2-\tfrac{8}{3}\tau+\tfrac{283}{36}+\tau\zeta-\zeta-\tfrac{1}{4}\zeta^2]\\
&\quad +\lambda^3k_1^3[\tfrac{1}{48}\zeta^{-4}+(-\tfrac{1}{4}\tau^2-\tfrac{3}{4}\tau-\tfrac{29}{24})\zeta^{-2}+\tfrac{8}{3}\zeta^{-1}+\tfrac{1}{6}\tau^3+\tfrac{1}{4}\tau^2+\tfrac{11}{4}\tau+\tfrac{9}{16}-\tfrac{8}{3}\zeta+(-\tfrac{1}{4}\tau+\tfrac{5}{8})\zeta^2].
\end{aligned}$$

Picking out now the terms in τ, putting therein $\zeta = 1$, and using the notation previously explained, we have, up to λ^3,

$$\begin{aligned}
\alpha_1 &= -\lambda h, & \gamma_1 &= -\lambda k_1,\\
\alpha_2 &= \lambda^2(h^2-2hk_1-\tfrac{1}{2}k_1^2)+\lambda^3(\tfrac{4}{3}k_1k_2), & \gamma_2 &= \lambda^2(-2h^2+hk_1)+\lambda^3(\tfrac{4}{3}hk_2-k_1k_2),\\
\alpha_3 &= \lambda^3(-4h^3+6h^2k_1+2hk_1^2-\tfrac{16}{3}k_1^3), & \gamma_3 &= \lambda^3(6h^3-4h^2k_1-\tfrac{19}{3}hk_1^2+\tfrac{7}{4}k_1^3).
\end{aligned}$$

Hence

$$-(\alpha_1+\alpha_2+\alpha_3+\gamma_1+\gamma_2+\gamma_3)$$
$$= \lambda(h+k_1)+\lambda^2(h^2+hk_1+\tfrac{1}{2}k_1^2)+\lambda^3(-2h^3-2h^2k_1+\tfrac{13}{3}hk_1^2+\tfrac{43}{12}k_1^3-\tfrac{4}{3}hk_2-\tfrac{1}{3}k_1k_2)$$

and

$$-(\alpha_1+\alpha_2+\alpha_3)+\gamma_1+\gamma_2+\gamma_3$$
$$= \lambda(h-k_1)+\lambda^2(-3h^2+3hk_1+\tfrac{1}{2}k_1^2)+\lambda^3(10h^3-10h^2k_1-\tfrac{25}{3}hk_1^2+\tfrac{4}{3}hk_2+\tfrac{85}{12}k_1^3-\tfrac{7}{3}k_1k_2).$$

The product of these gives the value of q^2, namely,

$$q^2 = \lambda^2(h^2-k_1^2)-\lambda^3h(2h^2-3k_1^2)+\lambda^4[5(h^2-k_1^2)^2-\tfrac{5}{4}k_1^4-2k_1^2k_2].$$

This agrees with the value found above by a quite different method (§ 4).

The matrix of coefficients of τ, after ζ has been replaced by 1, is of the form

$$A_0 = \begin{pmatrix} \alpha, & -\gamma \\ \gamma, & -\alpha \end{pmatrix},$$

and its square is $(\alpha^2-\gamma^2)$ times the matrix unity. The matrix $\Omega_0^w(u)$ of § 14 is thus

$$\Omega_0^w(u) = 1+A_0w+\tfrac{1}{2}q^2w^2+\frac{1}{3!}q^2A_0w^3+\frac{1}{4!}q^4w^4+\ldots$$

or

$$\begin{pmatrix} C+\alpha S, & -\gamma S \\ \gamma S, & C-\alpha S \end{pmatrix},$$

where $C = ch(qw)$, $S = \dfrac{1}{q}sh(qw)$. From this it is easily seen that for the calculation of q the method we have followed is less laborious than to use the equation

$$|\Omega_0^w(u)-\rho| = 0.$$

The differential equation from which we have started is, to terms in k_2, if we suppose $\lambda = 1$,

$$\frac{d^2x}{dt^2}+(1+4h+8k_1\cos 2t+8k_2\cos 4t)x = 0.$$

If we compare this with the form considered by HILL (' Coll. Works,' I., pp. 246, 268), we have, with his numerical values,

$$h = 0{\cdot}03971\ \ 09848\ \ 99146,$$
$$k_1 = -0{\cdot}01426\ \ 10046\ \ 86726,$$
$$k_2 = 0{\cdot}00009\ \ 58094\ \ 99389.$$

§ 20. Consider now briefly the case of the equations

$$\frac{d}{d\tau}(\mathrm{X}, \mathrm{Y}) = \frac{\psi}{4n}\begin{pmatrix} -1, & \zeta^n \\ -\zeta^{-n}, & 1 \end{pmatrix}(\mathrm{X}, \mathrm{Y}),$$

in which $n = 2$. We suppose

$$\frac{\psi}{4n} = \lambda h + \lambda k_1 w_1 + \lambda^2 k_2 w_2 + \ldots, \quad = \phi, \text{ say},$$

$$w_r = \zeta^{-r} + \zeta^r,$$

$$u = \begin{pmatrix} -\phi, & \zeta^n\phi \\ -\zeta^{-n}\phi, & \phi \end{pmatrix}.$$

As in the case of $n = 1$,

$$\mathrm{Q}u = \begin{pmatrix} a_1, & c'_1 \\ c_1, & a'_1 \end{pmatrix},$$

where

$$a_1 = -\int_0^\tau \phi\, d\tau = -\lambda h\tau + \lambda k_1(\zeta^{-1} - \zeta) + \tfrac{1}{2}\lambda^2 k_2(\zeta^{-2} - \zeta^2),$$

and, for $n = 2$,

$$c_1 = -\int_0^\tau \zeta^{-2}\phi\, d\tau = \lambda h(\tfrac{1}{2}\zeta^{-2} - \tfrac{1}{2}) + \lambda k_1(\tfrac{1}{3}\zeta^{-3} + \zeta^{-1} - \tfrac{4}{3}) + \lambda^2 k_2(\tfrac{1}{4}\zeta^{-4} - \tfrac{1}{4} - \tau).$$

These give

$$\begin{aligned}\phi(-a_1 + \zeta^2 c_1) &= \lambda^2 h^2(\tau + \tfrac{1}{2} - \tfrac{1}{2}\zeta^2) + \lambda^2 h k_1[(\tau - \tfrac{1}{6})\zeta^{-1} + (\tau + 2)\zeta - \tfrac{4}{3}\zeta^2 - \tfrac{1}{2}\zeta^3] \\ &\quad + \lambda^2 k_1^2(-\tfrac{2}{3}\zeta^{-2} + \tfrac{4}{3} - \tfrac{4}{3}\zeta + 2\zeta^2 - \tfrac{4}{3}\zeta^3).\end{aligned}$$

As before

$$\mathrm{Q}u\mathrm{Q}u = \begin{pmatrix} a_2, & c'_2 \\ c_2, & a'_2 \end{pmatrix},$$

where

$$\begin{aligned}a_2 &= \int_0^\tau \phi(-a_1 + \zeta^2 c_1)\, d\tau \\ &= \lambda^2 h^2(\tfrac{1}{2}\tau^2 + \tfrac{1}{2}\tau - \tfrac{1}{4}\zeta^2 + \tfrac{1}{4}) + \lambda^2 h k_1(-\tau\zeta^{-1} - \tfrac{5}{6}\zeta^{-1} + \tfrac{2}{3} + \tau\zeta + \zeta - \tfrac{2}{3}\zeta^2 - \tfrac{1}{6}\zeta^3) \\ &\quad + \lambda^2 k_1^2(\tfrac{1}{3}\zeta^{-2} + \tfrac{4}{3}\tau + \tfrac{4}{9} - \tfrac{4}{3}\zeta + \zeta^2 - \tfrac{4}{9}\zeta^3)\end{aligned}$$

and

$$c_2 = \int_0^\tau \zeta^{-2}\phi(-a_1+\zeta^2 c_1)\,d\tau\,.$$

$$= \lambda^2 h^2\left(-\tfrac{1}{2}\tau\zeta^{-2}-\tfrac{1}{2}\zeta^{-2}+\tfrac{1}{2}-\frac{\tau}{2}\right)+\lambda^2 hk_1\left(-\tfrac{1}{3}\tau\zeta^{-3}-\tfrac{1}{18}\zeta^{-3}-\tau\zeta^{-1}-3\zeta^{-1}+\tfrac{32}{9}-\tfrac{4}{3}\tau-\tfrac{1}{2}\zeta\right)$$
$$+\lambda^2 k_1^{\,2}\left(\tfrac{1}{6}\zeta^{-4}-\tfrac{2}{3}\zeta^{-2}+\tfrac{4}{3}\zeta^{-1}+2\tau+\tfrac{1}{2}-\tfrac{4}{3}\zeta\right).$$

Picking out the coefficients of τ in these, and putting therein $\zeta = 1$, we have

$$a_1 = -\lambda h, \qquad \gamma_1 = -\lambda^2 k_2,$$
$$a_2 = \tfrac{1}{2}\lambda^2 h^2+\tfrac{4}{3}\lambda^2 k_1^{\,2}, \quad \gamma_2 = -\lambda^2 h^2-\tfrac{8}{3}\lambda^2 hk_1+2\lambda^2 k_1^{\,2},$$

and hence, to λ^3,

$$q^2 = (a_1+a_2)^2-(\gamma_1+\gamma_2)^2 = \left[\lambda h-\frac{\lambda^2}{2}(h^2+\tfrac{8}{3}k_1^{\,2})\right]^2-[\lambda^2(\quad)]^2,$$
$$= \lambda^2 h^2-\lambda^3 h\,(h^2+\tfrac{8}{3}k_1^{\,2}).$$

This agrees with a result previously found (§ 6), but fails to give the first term in q^2 if $h = 0$. When this is so it is necessary to take account of the terms in λ^3. By taking terms in λ^3 in a_1, c_1, we only obtain terms in $\phi(-a_1+\zeta^2 c_1)$ which involve λ^4. But the terms in λ^2 in a_1, c_1 which are written down give terms in λ^3 in $\phi(-a_1+\zeta^2 c_1)$, which are

$$\lambda^2 hk_2\left(\tfrac{1}{4}\zeta^{-2}+\tau\zeta^{-2}-\tfrac{1}{2}+\tfrac{3}{4}\zeta^2-\tfrac{1}{2}\zeta^4\right)+\lambda^3 k_1k_2\left(-\tfrac{11}{12}\zeta^{-3}+\tfrac{7}{4}\zeta^{-1}-\tfrac{4}{3}-\tfrac{5}{12}\zeta-\tau\zeta+\tfrac{9}{4}\zeta^3-\tau\zeta^3-\tfrac{4}{3}\zeta^4\right),$$

and hence the additional terms in a_2

$$\lambda^3 hk_2\left(-\tfrac{1}{2}\tau\zeta^{-2}-\tfrac{3}{8}\zeta^{-2}-\tfrac{1}{2}\tau+\tfrac{1}{8}+\tfrac{3}{8}\zeta^2-\tfrac{1}{8}\zeta^4\right)$$
$$+\lambda^3 k_1k_2\left(\tfrac{11}{36}\zeta^{-3}-\tfrac{7}{4}\zeta^{-1}-\tfrac{4}{3}\tau+\tfrac{7}{12}\zeta-\tau\zeta+\tfrac{31}{36}\zeta^3-\tfrac{1}{3}\tau\zeta^3-\tfrac{1}{3}\zeta^4+\tfrac{1}{3}\right),$$

and the additional terms in c_2

$$\lambda^3 hk_2\left(-\tfrac{1}{4}\tau\zeta^{-4}-\tfrac{1}{8}\zeta^{-4}+\tfrac{1}{4}\zeta^{-2}+\tfrac{3}{4}\tau+\tfrac{1}{8}-\frac{\zeta^2}{4}\right)$$
$$+\lambda^3 k_1k_2\left(\tfrac{11}{60}\zeta^{-5}-\tfrac{7}{12}\zeta^{-3}+\tfrac{2}{3}\zeta^{-2}+\tfrac{17}{12}\zeta^{-1}+\tau\zeta^{-1}+\tfrac{13}{4}\zeta-\tau\zeta-\tfrac{2}{3}\zeta^2-\tfrac{64}{15}\right).$$

In finding the terms in λ^3 in a_3, c_3, it is sufficient to retain the terms λ^2 in a_2 and c_2. This gives for $\phi(-a_2+\zeta^2 c_2)$,

$$\lambda^3 h^3\left\{-\tfrac{1}{2}\tau^2-\tau-\tfrac{3}{4}+\tfrac{3}{4}\zeta^2-\tfrac{1}{2}\tau\zeta^2\right\}$$
$$+\lambda^3 h^2 k_1\left\{\zeta^{-1}\left(-\tfrac{1}{2}\tau^2-\tfrac{1}{3}\tau+\tfrac{1}{36}\right)-\tfrac{2}{3}+\zeta\left(-\tfrac{1}{2}\tau^2-\tfrac{7}{2}\tau-4\right)+\zeta^2\left(-\tfrac{4}{3}\tau+\tfrac{38}{9}\right)+\zeta^3\left(-\tfrac{1}{2}\tau+\tfrac{5}{12}\right)\right\}$$
$$-\lambda^3 hk_1^{\,2}\left\{\zeta^{-2}\left(\tfrac{2}{3}\tau+\tfrac{11}{18}\right)-\tfrac{2}{3}\zeta^{-1}-\tfrac{8}{3}\tau-\tfrac{13}{3}+\zeta\left(-\tfrac{4}{3}\tau+\tfrac{56}{9}\right)-\tfrac{29}{6}\zeta^2+\zeta^3\left(-\tfrac{4}{3}\tau+\tfrac{10}{3}\right)-\tfrac{1}{3}\zeta^4\right\}$$
$$+\lambda^3 k_1^{\,3}\left\{-\tfrac{1}{6}\zeta^{-3}-\zeta^{-1}\left(\tfrac{4}{3}\tau+\tfrac{23}{18}\right)+\tfrac{8}{3}+\zeta\left(\tfrac{2}{3}\tau-\tfrac{29}{18}\right)+\tfrac{16}{9}\zeta^2+\zeta^3\left(2\tau-\tfrac{1}{2}\right)-\tfrac{8}{9}\zeta^4\right\}.$$

For $a_3 = \int_0^\tau \phi\,(-a_2+\zeta^2 c_2)\,d\tau$, this leads to

$$\begin{aligned}
&\lambda^3 h^3 \{-\tfrac{1}{6}\tau^3-\tfrac{1}{2}\tau^2-\tfrac{3}{4}\tau-\tfrac{1}{2}+\zeta^2(-\tfrac{1}{4}\tau+\tfrac{1}{2})\} \\
&\quad +\lambda^3 h^2 k_1 \Big\{\zeta^{-1}(\tfrac{1}{2}\tau^2+\tfrac{4}{3}\tau+\tfrac{47}{36})-\tfrac{2}{3}\tau-\tfrac{22}{9}+\zeta(-\tfrac{1}{2}\tau^2-\tfrac{5}{2}\tau-\tfrac{3}{2}) \\
&\qquad\qquad +\zeta^2(-\tfrac{2}{3}\tau+\tfrac{22}{9})+\zeta^3\Big(-\frac{\tau}{6}+\tfrac{7}{36}\Big)\Big\} \\
&\quad +\lambda^3 h k_1^2 \{-\zeta^{-2}(\tfrac{1}{3}\tau+\tfrac{17}{36})+\tfrac{2}{3}\zeta^{-1}-\tfrac{4}{3}\tau^2-\tfrac{13}{3}\tau-\tfrac{703}{108}+\zeta(-\tfrac{4}{3}\tau+\tfrac{68}{9}) \\
&\qquad\qquad -\tfrac{29}{12}\zeta^2+\zeta^3(-\tfrac{4}{9}\tau+\tfrac{34}{27})-\tfrac{1}{12}\zeta^4\} \\
&\quad +\lambda^3 k_1^3 \{\tfrac{1}{18}\zeta^{-3}+\zeta^{-1}(\tfrac{4}{3}\tau+\tfrac{47}{18})+\tfrac{8}{3}\tau-\tfrac{2}{3}+\zeta(\tfrac{2}{3}\tau-\tfrac{41}{18})+\tfrac{8}{9}\zeta^2+\zeta^3(\tfrac{2}{3}\tau-\tfrac{7}{18})-\tfrac{2}{9}\zeta^4\}.
\end{aligned}$$

The terms in $c_3 = \int_0^\tau \zeta^{-2}\phi\,(-a_2+\zeta^2 c_2)\,d\tau$ are similarly

$$\begin{aligned}
&\lambda^3 h^3 \{\zeta^{-2}(\tfrac{1}{4}\tau^2+\tfrac{1}{4}\tau)+\tfrac{3}{4}\tau-\tfrac{1}{4}\tau^2\} \\
&\quad +\lambda^3 h^2 k_1 \Big\{\zeta^{-3}\Big(\tfrac{1}{6}\tau^2+\frac{2\tau}{9}+\tfrac{7}{108}\Big)+\tfrac{1}{3}\zeta^{-2}+\zeta^{-1}(\tfrac{1}{2}\tau^2+\tfrac{9}{2}\tau+\tfrac{17}{2}) \\
&\qquad\qquad -\tfrac{265}{27}-\tfrac{2}{3}\tau^2+\tfrac{38}{9}\tau+\zeta(-\tfrac{1}{2}\tau+\tfrac{11}{12})\Big\} \\
&\quad +\lambda^3 h k_1^2 \Big\{\zeta^{-4}\Big(-\frac{\tau}{6}-\tfrac{7}{36}\Big)+\tfrac{2}{9}\zeta^{-3}+\zeta^{-2}(\tfrac{4}{3}\tau+\tfrac{17}{6})+\zeta^{-1}(\tfrac{4}{3}\tau-\tfrac{44}{9}) \\
&\qquad\qquad -\tfrac{89}{36}-\tfrac{29}{6}\tau+\zeta(-\tfrac{4}{3}\tau+\tfrac{14}{3})-\tfrac{1}{6}\zeta^2\Big\} \\
&\quad +\lambda^3 k_1^3 \{\tfrac{1}{30}\zeta^{-5}+\zeta^{-3}(\tfrac{4}{9}\tau+\tfrac{31}{54})-\tfrac{4}{3}\zeta^{-2}+\zeta^{-1}(-\tfrac{2}{3}\tau+\tfrac{17}{18}) \\
&\qquad\qquad +\tfrac{368}{135}+\tfrac{16}{9}\tau+\zeta(2\tau-\tfrac{5}{2})-\tfrac{4}{9}\zeta^2\}.
\end{aligned}$$

It is easy to see that the terms in λ^3 in a_1, c_1 are respectively

$$\tfrac{1}{3}\lambda^3 k_3(\zeta^{-3}-\zeta^3) \quad \text{and} \quad \lambda^3 k_3(\tfrac{1}{5}\zeta^{-5}-\zeta+\tfrac{4}{5}),$$

neither of which contains τ. Thus up to λ^3 we have, in the preceding notation

$$\begin{aligned}
\alpha_1 &= -\lambda h, & \gamma_1 &= -\lambda^2 k_2, \\
\alpha_2 &= \tfrac{1}{2}\lambda^2(h^2+\tfrac{8}{3}k_1^2)-\lambda^3(hk_2+\tfrac{8}{3}k_1k_2), & \gamma_2 &= -\lambda^2(h^2+\tfrac{8}{3}hk_1-2k_1^2)+\tfrac{1}{2}\lambda^3 hk_2, \\
\alpha_3 &= -\lambda^3h^3-\tfrac{8}{3}\lambda^3h^2k_1-\tfrac{58}{9}\lambda^3hk_1^2+\tfrac{16}{3}\lambda^3k_1^3, & \gamma_3 &= \lambda^3h^3+\tfrac{76}{9}\lambda^3h^2k_1-\tfrac{11}{3}\lambda^3hk_1^2+\tfrac{32}{9}\lambda^3k_1^3.
\end{aligned}$$

Thus

$$\begin{aligned}
\alpha_1+\alpha_2+\alpha_3 &= -\lambda h+\tfrac{1}{2}\lambda^2(h^2+\tfrac{8}{3}k_1^2)+\lambda^3(-h^3-\tfrac{8}{3}h^2k_1-\tfrac{58}{9}hk_1^2+\tfrac{16}{3}k_1^3-hk_2-\tfrac{8}{3}k_1k_2), \\
\gamma_1+\gamma_2+\gamma_3 &= \lambda^2(-h^2-\tfrac{8}{3}hk_1+2k_1^2-k_2)+\lambda^3(h^3+\tfrac{76}{9}h^2k_1-\tfrac{11}{3}hk_1^2+\tfrac{32}{9}k_1^3+\tfrac{1}{2}hk_2).
\end{aligned}$$

This gives

$$q^2 = (\alpha_1+\alpha_2+\alpha_3)^2-(\gamma_1+\gamma_2+\gamma_3)^2$$
$$= \lambda^2h^2-\lambda^3h\,(h^2+\tfrac{8}{3}k_1^{\,2})+\lambda^4\,(\tfrac{5}{4}h^4+\tfrac{100}{9}h^2k_1^{\,2}-\tfrac{20}{9}k_1^{\,4}-k_2^{\,2}+4k_1^{\,2}k_2)$$

as far as terms in λ^4. This result is for the equation

$$\frac{d^2x}{dt^2}+(n^2+\psi)\,x = 0,$$

wherein $n = 2$ and

$$\frac{\psi}{8} = \lambda h+\lambda k_1\,(\zeta^{-1}+\zeta)+\lambda^2k_2\,(\zeta^{-2}+\zeta^2)+\ldots$$

and agrees with the result previously found (§ 6) when in this last we replace h, k_1, k_2 respectively by $2h$, $2k_1$, $2k_2$, as is necessary, taking account of the difference of notation for $\frac{\psi}{4}$ in the two cases. By an independent investigation for the case when

$$\frac{\psi}{8} = \lambda h_1+\lambda k_1.(\zeta^{-1}+\zeta)+\lambda^2h_2+\lambda^2k_2\,(\zeta^{-2}+\zeta^2)+\lambda^3h_3+\lambda^3k_3\,(\zeta^{-3}+\zeta^3)+\ldots$$

we have found (above, p. 142),

$$q^2 = h_1^{\,2}\lambda^2-h_1\,(h_1^{\,2}+\tfrac{8}{3}k_1^{\,2}-2h_2)\,\lambda^3$$
$$+\lambda^4\,\{(\tfrac{1}{2}h_1^{\,2}+\tfrac{4}{3}k_1^{\,2}-h_2)^2+h_1^{\,4}+\tfrac{88}{9}h_1^{\,2}k_1^{\,2}-2h_1^{\,2}h_2+2h_1h_3-(k_2-2k_1^{\,2})^2\}+\ldots,$$

which, replacing h by $h_1+h_2\lambda+h_3\lambda^2$, arises from the preceding result.

§ 21. Now consider the equations

$$\frac{d}{d\tau}(\mathrm{X},\,\mathrm{Y}) = u\,(\mathrm{X},\,\mathrm{Y}),$$

where

$$u = \begin{pmatrix} -\phi, & \zeta^n\phi \\ -\zeta^{-n}\phi, & \phi \end{pmatrix}$$

and n is not 1 or 2, but is an integer if u is a periodic matrix.

With

$$\phi = \lambda h+\lambda k_1\,(\zeta^{-1}+\zeta)+\lambda^2k_2\,(\zeta^{-2}+\zeta^2)+\ldots$$

we have, retaining only to λ^2,

$$a_1 = -\int_0^\tau\phi\,d\tau = -\lambda h\tau+\lambda k_1\,(\zeta^{-1}-\zeta)+\tfrac{1}{2}\lambda^2k_2\,(\zeta^{-2}-\zeta^2),$$

$$c_1 = -\int_0^\tau\zeta^{-n}\phi\,d\tau = -\int_0^\tau\left[\lambda h\zeta^{-n}+\lambda k_1\,(\zeta^{-n-1}+\zeta^{-n+1})+\lambda^2k_2\,(\zeta^{-n-2}+\zeta^{-n+2})\right]d\tau$$
$$= \frac{1}{n}\lambda h\,(\zeta^{-n}-1)+\lambda k_1\left(\frac{\zeta^{-n-1}}{n+1}+\frac{\zeta^{-n+1}}{n-1}-\frac{2n}{n^2-1}\right)+\lambda^2k_2\left(\frac{\zeta^{-n-2}}{n+2}+\frac{\zeta^{-n+2}}{n-2}-\frac{2n}{n^2-4}\right),$$

which lead to

$$\phi(-a_1+\zeta^n c_1) = \lambda^2 h^2\left(\tau+\frac{1-\zeta^n}{n}\right)$$
$$+\lambda^2 h k_1\left\{\zeta^{-1}\left(\tau-\frac{n^2-n-1}{n(n+1)}\right)+\zeta\left(\tau+\frac{n^2+n-1}{n(n-1)}\right)-\frac{\zeta^{n-1}}{n}-\frac{2n}{n^2-1}\zeta^n-\frac{\zeta^{n+1}}{n}\right\}$$
$$+\lambda^2 k_1^2\left\{-\frac{n}{n+1}\zeta^{-2}+\frac{2n}{n^2-1}+\frac{n}{n-1}\zeta^2-\frac{2n}{n^2-1}\zeta^{n-1}-\frac{2n}{n^2-1}\zeta^{n+1}\right\},$$

so that

$$a_2 = \int_0^\tau \phi(-a_1+\zeta^n c_1)\,d\tau$$
$$= \lambda^2 h^2\left(\tfrac{1}{2}\tau^2+\frac{\tau}{n}-\frac{\zeta^n-1}{n^2}\right)$$
$$+\lambda^2 h k_1\left\{-\zeta^{-1}\left(\tau+\frac{2n+1}{n(n+1)}\right)+\zeta\left(\tau+\frac{2n-1}{n(n-1)}\right)\right.$$
$$\left.+\frac{4}{n^2-1}-\frac{\zeta^{n-1}}{n(n-1)}-\frac{2}{n^2-1}\zeta^n-\frac{\zeta^{n+1}}{n(n+1)}\right\}$$
$$+\lambda^2 k_1^2\left\{\frac{n}{2(n+1)}\zeta^{-2}+\frac{2n}{n^2-1}\tau-\frac{n^2(n^2-5)}{(n^2-1)^2}\right.$$
$$\left.+\frac{n}{2(n-1)}\zeta^2-\frac{2n}{(n-1)(n^2-1)}\zeta^{n-1}-\frac{2n}{(n+1)(n^2-1)}\zeta^{n+1}\right\}.$$

Similarly,

$$c_2 = \int_0^\tau \zeta^{-n}\phi(-a_1+\zeta^n c_1)\,d\tau$$
$$= \lambda^2 h^2\left(-\frac{\tau\zeta^{-n}}{n}-\frac{2\zeta^{-n}}{n^2}+\frac{2}{n^2}-\frac{\tau}{n}\right)$$
$$+\lambda^2 h k_1\left\{-\frac{\zeta^{-n-1}}{n+1}\left(\tau-\frac{n^2-2n-1}{n(n+1)}\right)-\frac{\zeta^{-n+1}}{n-1}\left(\tau+\frac{n^2+2n-1}{n(n-1)}\right)\right.$$
$$\left.+\frac{\zeta^{-1}}{n}-\frac{2n\tau}{n^2-1}-\frac{\zeta}{n}+\frac{8n^2}{(n^2-1)^2}\right\}$$
$$+\lambda^2 k_1^2\left\{\frac{n}{(n+1)(n+2)}\zeta^{-n-2}-\frac{2}{n^2-1}\zeta^{-n}-\frac{n}{(n-1)(n-2)}\zeta^{-n+2}\right.$$
$$\left.+\frac{2n}{n^2-1}\zeta^{-1}-\frac{2n}{n^2-1}\zeta+\frac{8}{n^2-4}\right\}.$$

Thus we have, so far as terms in λ^2,

$$a_1 = -\lambda h, \qquad \gamma_1 = 0,$$

$$a_2 = \frac{\lambda^2 h^2}{n}+\lambda^2 k_1^2\frac{2n}{n^2-1}, \quad \gamma_2 = -\frac{2\lambda^2 h^2}{n}-\frac{4\lambda^2 h k_1 n}{n^2-1},$$

and so, to this approximation,

$$\pm q = \alpha_1 + \alpha_2 = -\lambda h + \frac{\lambda^2 h^2}{n} + \lambda^2 k_1^2 \cdot \frac{2n}{n^2-1}.$$

The characteristic factor is then $e^{i(n+2q)t}$, the differential equation being

$$\frac{d^2x}{dt^2} + (n^2 + 4n\lambda h + 8n\lambda k_1 \cos 2t + 8n\lambda^2 k_2 \cos 4t + \ldots)\, x = 0.$$

Thus q is always real, when λ is small enough, provided k_1 is not zero, even if h be zero. The result agrees with that found in § 6 for $n = 3$, if allowance be made for the change of notation.

[*December 1, 1915.*—Consider the differential equation differing from that just preceding only by the substitution of H for λh in the term $4n\lambda h$ of the coefficient of x, where H is supposed to be of the form $\lambda h_1 + \lambda^2 h_2 + \lambda^3 h_3 + \ldots$. The computation of q^2 proceeds then exactly as before. The formulæ for $\alpha_1 + \alpha_2 + \alpha_3$, $\gamma_1 + \gamma_2 + \gamma_3$, given above, p. 178, substituting H for λh, show that, for $n = 1$, q^2 is then of the form $(\mathrm{H} - \alpha_1)(\mathrm{H} - \alpha_2)\,\mathrm{Q}$, wherein Q is a power series in H, λk_1, $\lambda^2 k_2$, ..., reducing to 1 when $\mathrm{H} = 0$, $\lambda = 0$, and

$$\alpha_1 = -k_1\lambda - \tfrac{1}{2}k_1^2\lambda^2 + (\tfrac{1}{4}k_1^3 - k_1 k_2)\,\lambda^3 + \ldots,$$

$$\alpha_2 = \quad k_1\lambda - \tfrac{1}{2}k_1^2\lambda^2 - (\tfrac{1}{4}k_1^3 - k_1 k_2)\,\lambda^3 + \ldots.$$

The value of q^2 is positive, and the motion represented by the differential equation is stable, so long as H does not lie between these values. Similarly for $n = 2$, from the formulæ at the bottom of p. 181, the range in which q^2 is negative is when H lies between

$$-(\tfrac{2}{3}k_1^2 - k_2)\,\lambda^2 \quad \text{and} \quad (\tfrac{10}{3}k_1^2 - k_2)\,\lambda^2,$$

these being accurate as far as λ^3. Unless $\frac{2}{3}k_1^2 < k_2 < \frac{10}{3}k_1^2$, these limits are of opposite sign, and include $\mathrm{H} = 0$. This is the result given on p. 142 (save for a slight difference of notation). For $n = 3$, an analogous computation shows that q^2 is positive except when H is between

$$\tfrac{3}{4}k_1^2\lambda^2 - \mathrm{P}\lambda^3 \quad \text{and} \quad \tfrac{3}{4}k_1^2\lambda^2 + \mathrm{P}\lambda^3,$$

where

$$\mathrm{P} = \tfrac{9}{4}k_1^3 - 3k_1k_2 + k_3,$$

and this range does not include $\mathrm{H} = 0$ unless $k_1 = 0$. It would appear, from the formula above (p. 184), that the corresponding interval for greater integer values of n is between two quantities of the forms

$$\frac{2n}{n^2-1}\,k_1^2\lambda^2 + \mathrm{P}\lambda^3, \quad \frac{2n}{n^2-1}\,k_1^2\lambda^2 + \mathrm{Q}\lambda^3.$$

Mr. E. LINDSAY INCE, of Trinity College, Cambridge, following up the method of his paper referred to above (footnote, p. 134), has calculated numerical results for the case when k_1, k_2, ... have the values considered by G. W. HILL.]

PART III.

§ 22. I desire to add to the foregoing some very incomplete remarks in regard to a generalisation of which the work appears to be capable. The most important general result obtained is that when u is a periodic matrix, the matrix $\Omega(u)$ can be expressed as a periodic matrix P multiplied into a matrix involving quantities of the form $e^{\lambda\tau}$. One direction in which this result can be amplified is by extending the assumption we have made that the matrix $\Omega_0^{w}(u)$ has linear invariant factors. It is well enough understood what is the character of the modifications thereby introduced. A more important generalisation appears to be that the factorisation of the matrix $\Omega(u)$ does not in fact require that u be a periodic matrix. As an indication of the theorem consider an equation

$$\frac{d^2x}{dt^2}+\sigma^2x = x\left(ae^{i\kappa t}+be^{i\lambda t}+ce^{i\mu t}\right),$$

in which the constants κ, λ, μ are such that $\kappa+\lambda+\mu = 0$, but the ratio of two of them at least is irrational. For example, we might have $\kappa = \sqrt{2}+1$, $\mu = -\sqrt{2}+1$, $\mu = -2$. Then, assuming that there exists no identity of the form

$$\alpha\kappa+\beta\lambda+\gamma\mu\pm2\sigma = 0,$$

in which α, β, γ are positive integers, the equation would seem to have a solution of the form

$$x = e^{iqt}\mathrm{X},$$

where X is a series of positive and negative integral powers of $e^{i\kappa t}$, $e^{i\lambda t}$, $e^{i\mu t}$, $e^{i\sigma t}$, which may be arranged as a power series in a, b, c, and q is a series of the form

$$q = \sigma+\mathrm{A}_1abc+\mathrm{A}_2a^2b^2c^2+\ldots,$$

in which A_1, A_2, ... are constants. The differential equation has not periodic coefficients.

In a paper already far too often referred to, 'Proc. Lond. Math. Soc.,' XXXV., 1902, p. 353 *et seq.*, replacing the variable there called t by e^τ or ζ, it is shown (p. 365) for the equation system

$$\frac{dx}{d\tau} = (\mathrm{A}+\zeta\mathrm{V})\,x, \quad = ux, \text{ say},$$

in which A is a matrix of constants, and V a series of positive integral powers of ζ, that there is a factorisation of the matrix $\Omega(u)$, in the form $\mathrm{P}\Omega(\phi)\gamma$, where P is a matrix whose elements are power series in ζ, and $\Omega(\phi)$ is calculated in regard to τ from a matrix

$$\phi = \begin{pmatrix} \theta_1, & c_{12}(\zeta/\zeta_0)^{\theta_1-\theta_2}, & c_{13}(\zeta/\zeta_0)^{\theta_1-\theta_3}, & . & . \\ . & \theta_2, & c_{23}(\zeta/\zeta_0)^{\theta_2-\theta_3}, & . & . \\ . & . & . & . & . \end{pmatrix},$$

while γ is a matrix of constants. Here $\theta_1, \theta_2, \ldots, c_{12}, c_{13}, \ldots$ depend solely on the invariant factors of the matrix A.

This result is obtained from the form of the matrix u as expressible by powers of ζ, without reference to the question of periodicity. It would seem that the argument there employed is capable of modification, the integrations being performed in regard to τ (which is $\log t$ of the paper referred to), so as to lead to the general theorem here contemplated.