MOIVRE

THE DOCTRINE OF CHANCES:

OR,

A Method of Calculating the Probability of Events in Play.

By *A. De Moivre.* F. R. S.

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TO

Sir Isaac Newton, *Kt. President of the* Royal Society.

SIR

THE greatest help I have receiv'd in writing upon this Subject having been from your Incomparable Works, especially your Method of Series; I think it my Duty publickly to acknowledge, that the Improvements I have made in the matter here treated of, are principally derived from your self. The great benefit which has accrued to me in this respect, requires my share in the general Tribute of Thanks due to you from the Learned World: But one advantage, which is more particularly my own, is the Honour I have frequently had of being admitted to your private Conversation, wherein the doubts I have had upon any Subject relating to *Mathematics*, have been resolved by you with the greatest Humanity and Condescention. Those

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Marks of your Favour are the more valuable to me, becauſe I had no other pretence to them, but the earneſt deſire of underſtanding your ſublime and univerſally uſeful Speculations. I ſhould think my ſelf very happy, if, having given my Readers a Method of calculating the Effects of Chance, as they are the reſult of Play, and thereby fix'd certain Rules, for eſtimating how far ſome ſort of Events may rather be owing to Deſign than Chance, I could by this ſmall Eſſay excite in others a deſire of proſecuting theſe Studies, and of learning from your Philoſophy how to collect, by a juſt Calculation, the Evidences of exquiſite Wiſdom and Deſign, which appear in the *Phenomena* of Nature throughout the Univerſe. I am, with the utmoſt Reſpect,

Sir,

Your moſt Humble,

and Obedient Servant,

A. De Moivre.

PREFACE.

'TIS now about Seven Years, since I gave a Specimen in the Philosophical Transactions, of what I now more largely treat of in this Book. The occasion of my then undertaking this Subject was chiefly owing to the Desire and Encouragement of the Honourable Mr. Francis Robartes, who, upon occasion of a French Tract, called, L'Analyse des jeux de Hazard, which had lately been Published, was pleased to propose to me some Problems of much greater difficulty than any he had found in that Book; which having solved to his Satisfaction, he engaged me to Methodise those Problems, and to lay down the Rules which had led me to their Solution. After I had proceeded thus far, it was enjoined me by the Royal Society, to communicate to them what I had discovered on this Subject, and thereupon it was ordered to be published in the Transactions, not as a matter relating only to Play, but as containing some general Speculations not unworthy to be considered by the Lovers of Truth.

I had not at that time read any thing concerning this Subject, but Mr. Huygens's Book, de Ratiociniis in Ludo Aleæ, and a little English Piece (which was properly a translation of it) done by a very ingenious Gentleman, who, tho' capable of carrying the matter a great deal farther, was contented to follow his Original; adding only to it the computation of the Advantage of the Setter in the Play called Hazard, and some few things more. As for the French Book, I had run it over but cursorily, by reason I had observed that the Author chiefly insisted on the Method of Huygens, which I was absolutely re-

ſolved to reject, as not ſeeming to me to be the genuine and natural way of coming at the Solution of Problems of this kind. However, had I allowed my ſelf a little more time to conſider it, I had certainly done the Juſtice to its Author, to have owned that he had not only illuſtrated Huygens's *Method by a great variety of well choſen Examples, but that he had added to it ſeveral curious things of his own Invention.*

Tho' I have not followed Mr. Huygens *in his Method of Solution, 'tis with very great pleaſure that I acknowledge the Obligations I have to him; his Book having Settled in my Mind the firſt Notions of this Doctrine, and taught me to argue about it with certainty.*

I had ſaid in my Specimen, that Mr. Huygens *was the firſt who had Publiſhed the Rules of this Calculation, intending thereby to do juſtice to that great Man; but what I then ſaid was miſinterpreted, as if I had deſigned to wrong ſome Perſons who had conſidered this matter before him, and a paſſage was cited againſt me out of* Huygens's *Preface, in which he ſaith,* Sciendum vero quod jam pridem, inter Præſtantiſſimos totâ Galliâ Geometras, Calculus hic fuerit agitatus; ne quis indebitam mihi primæ Inventionis gloriam hac in re tribuat. *But what follows immediately after, had it been minded, might have cleared me from any Suſpicion of injuſtice. The words are theſe* Cæterum illi difficillimis quibuſque Quæſtionibus ſe invicem exercere Soliti, methodum Suam quiſque occultam retinuere, adeo ut a primis elementis hanc materiam evolvere mihi neceſſe fuerit. *By which it appears, that tho'* Mr. Huygens *was not the firſt who had applied himſelf to thoſe ſorts of Queſtions, he was nevertheleſs the firſt who had publiſhed Rules for their Solution; which is all that I affirmed.*

Since the printing of my Specimen, Mr. de Monmort, *Author of the* Analyſe des jeux de Hazard, *Publiſhed a Second Edition of that Book, in which he has particularly given many proofs of his ſingular Genius, and extraordinary Capacity; which Teſtimony I give both to Truth, and to the Friendſhip with which he is pleaſed to Honour me.*

Such a Tract as this is may be uſeful to ſeveral ends; the firſt of which is, that there being in the World ſeveral inquiſitive Perſons, who are deſirous to know what foundation they *go upon, when they engage in Play, whether from a motive of Gain, or barely Diverſion, they may, by the help of this or the like Tract, gratifie their curioſity, either by taking the pains to underſtand what is here Demonſtrated, or elſe making uſe of the concluſions, and taking it for granted that the Demonſtrations are right.*

Another uſe to be made of this Doctrine of Chances is, that it may ſerve in Conjunction with the other parts of the Mathematicks, as a fit introduction to the Art of Reaſoning; it being known by experience that nothing can contribute more to the attaining of that Art, than the conſideration of a long Train of Conſequences, rightly deduced from undoubted Principles, of which this Book affords many Examples. To this may be added, that ſome of the Problems about Chance having a great appearance of Simplicity, the Mind is eaſily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Sence; which generally proving otherwiſe, and the Miſtakes occaſioned thereby being not unfrequent, 'tis preſumed that a Book of this Kind, which teaches to diſtinguiſh Truth from what ſeems ſo nearly to reſemble it, will be look'd upon as a help to good Reaſoning.

Among the ſeveral Miſtakes that are committed about Chance, one of the moſt common and leaſt ſuſpected, is that which relates to Lotterys. *Thus, ſuppoſing a Lottery wherein the proportion of the Blanks to the Prizes is as five to one; 'tis very natural to conclude that therefore five Tickets are requiſite for the Chance of a Prize; and yet it may be proved Demonſtratively, that four Tickets are more then ſufficient for that purpoſe, which will be confirmed by often repeated Experience. In the like manner, ſuppoſing a Lottery wherein the proportion of the Blanks to the Prizes is as thirty nine to* One, *(ſuch as was the Lottery of* 1710) *it may be proved, that in twenty eight Tickets, a Prize is as likely to be taken as not; which tho' it may ſeem to contradict the common Notions, is nevertheleſs grounded upon infallible Demonſtration.*

When the Play of the Royal Oak was in uſe, ſome Perſons who loſt conſiderably by it, had their Loſſes chiefly occaſioned by an Argument of which they could not perceive the Fallacy. The Odds againſt any particular Point of the Ball were one and Thirty to One, which intituled the Adventurers, in caſe they

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were winners, to have thirty two Stakes returned, including their own; instead of which they having but eight and Twenty, it was very plain that on the Single account of the disadvantage of the Play, they lost one eighth part of all the Money they play'd for. But the Master of the Ball maintained that they had no reason to complain; since he would undertake that any particular point of the Ball should come up in two and Twenty Throws; of this he would offer to lay a Wager, and actually laid it when required. The seeming contradiction between the Odds of one and thirty to One, and Twenty two Throws for any Chance to come up, so perplexed the Adventurers, that they begun to think the Advantage was on their side; for which reason they play'd on and continued to lose.

The Doctrine *of* Chances *may likewise be a help to cure a Kind of Superstition, which has been of long standing in the World,* viz. *that there is in Play such a thing as* Luck, *good or bad. I own there are a great many judicious people, who without any other Assistance than that of their own reason, are satisfied, that the Notion of* Luck *is meerly Chimerical; yet I conceive that the ground they have to look upon it as such, may still be farther inforced from some of the following Considerations.*

If by saying that a Man has had good Luck, nothing more was meant than that he has been generally a Gainer at play, the Expression might be allowed as very proper in a short way of speaking: But if the Word good Luck *be understood to signifie a certain predominant quality, so inherent in a Man, that he must win whenever he Plays, or at least win oftner than lose, it may be denied that there is any such thing in nature.*

The Asserters of Luck *are very sure from their own Experience, that at some times they have been very Lucky, and that at other times they have had a prodigious run of* ill Luck *against them, which whilst it continued obliged them to be very cautious in engaging with the fortunate; but how Chance should produce those extraordinary Events, is what they cannot conceive; They would be glad for Instance to be Satisfied, how they could lose Fifteen Games together at* Piquet, *if* ill Luck *had not strangely prevailed against them. But if they will be pleased to consider the Rules delivered in this Book, they will see that tho' the Odds against their losing so many times together be very great*

great, viz. 32767 *to* 1, *yet that the Poſſibility of it is not deſtroy'd by the greatneſs of the Odds, there being One Chance in* 32768 *that it may ſo happen, from whence it follows, that it was ſtill poſſible to come to paſs without the Intervention of what they call* Ill Luck.

Beſides, This Accident of loſing Fifteen times together at Piquet, *is no more to be imputed to ill Luck, than the Winning with one ſingle Ticket the Higheſt Prize, in a Lottery of* 32768 *Tickets, is to be imputed to good Luck, ſince the Chances in both Caſes are perfectly equal. But if it be ſaid that Luck has been concerned in this latter Caſe, the Anſwer will be eaſy; for let us ſuppoſe Luck not exiſting, or at leaſt let us ſuppoſe its Influence to be ſuſpended, yet the Higheſt Prize muſt fall into ſome Hand or other, not by Luck, (for by the Hypotheſis that has been laid aſide) but from the meer Neceſſity of its falling ſomewhere.*

Thoſe who contend for Luck, may, if they pleaſe, alledge other Caſes at Play, much more unlikely to happen than the Winning or Loſing Fifteen Games together, yet ſtill their Opinion will never receive any Addition of Strength from ſuch Suppoſitions: For, by the Rules of Chance, a time may be computed, in which thoſe Caſes may as probably happen as not; nay, not only ſo, but a time may be computed in which there may be any proportion of Odds for their ſo happening.

But ſuppoſing that Gain and Loſs were ſo fluctuating, as always to be diſtributed equally, whereby Luck would certainly be annihilated; would it be reaſonable in this Caſe to attribute the Events of Play to Chance alone? I think, on the contrary, it would be quite otherwiſe, for then there would be more reaſon to ſuſpect that ſome unaccountable Fatality did Rule in it: Thus, If two Perſons play at Croſs *and* Pile, *and Chance alone be ſuppos'd to be concern'd in regulating the fall of the Piece, is it probable that there ſhould be an Equality of Heads and Croſſes? It is Five to Three that in four times there will be an inequality; 'tis Eleven to Five in ſix,* 93 *to* 35 *in Eight, and about* 12 *to* 1 *in a hundred times: Wherefore Chance alone by its Nature conſtitutes the Inequalities of Play, and there is no need to have recourſe to Luck to explain them.*

Further, The ſame Arguments which explode the Notion of Luck, may, on the other ſide, be uſeful in ſome Caſes to eſtabliſh a due compariſon between Chance and Deſign: We may imagine Chance and Deſign to be as it were in Competition with each other, for the pro-

duction of some sorts of Events, and may calculate what Probability there is, that those Events should be rather owing to one than to the other. To give a familiar Instance of this, Let us suppose that two Packs of Piquet-Cards being sent for, it should be perceived that there is, from Top to Bottom, the same Disposition of the Cards in both Packs; Let us likewise suppose that, some doubt arising about this Disposition of the Cards, it should be questioned whether it ought to be attributed to Chance, or to the Maker's Design: In this case the Doctrine of Combination decides the Question, since it may be proved by its Rules, that there are the Odds of above 26313083 *Millions of Millions of Millions of Millions to One, that the Cards were designedly set in the Order in which they were found.*

From this last Consideration we may learn, in many Cases, how to distinguish the Events which are the effect of Chance, from those which are produc'd by Design: The very Doctrine that finds Chance where it really is, being able to prove by a gradual Increase of Probability, till it arrive at Demonstration, that where Uniformity, Order and Constancy reside, there also reside Choice and Design.

Lastly, *One of the Principal Uses to which this* Doctrine of Chances *may be apply'd, is the discovering of some Truths, which cannot fail of pleasing the Mind, by their Generality and Simplicity; the Admirable Connexion of its Consequences will increase the Pleasure of the Discovery; and the seeming Paradoxes wherewith it abounds, will afford very great matter of Surprize and Entertainment to the Inquisitive. A very remarkable Instance of this nature may be seen in the prodigious Advantage which the repetition of Odds will amount to; Thus, Supposing I play with an Adversary who allows me the Odds of* 43 *to* 40, *and agrees with me to play till* 100 *Stakes are won or lost on either side, on condition that I give him an Equivalent for the Gain I am intitled to by the Advantage of my Odds; the Question is what Equivalent I am to give him, on supposition we play a Guinea a Stake: The Answer is* 99 *Guineas and above* 18 *Shillings, which will seem almost incredible, considering the smalness of the Odds of* 43 *to* 40. *Now let the Odds be in any Proportion given, and let the Number of Stakes to be played for be never so great, yet one General Conclusion will include all the possible Cases, and the application of it to Numbers may be wrought in less than a Minutes time.*

I have explain'd, in my Introduction *to the following Treatise, the chief Rules on which the whole Art of Chances depends; I have* *done it in the plainest manner that I could think of, to the end it might be (as much as possible) of General Use. I flatter my self that those who are acquainted with Arithmetical Operations, will, by the help of the* Introduction *alone, be able to solve a great Variety of Questions depending on Chance: I wish, for the Sake of some Gentlemen who have been pleased to subscribe to the printing of my Book, that I could every where have been as plain as in the* Introduction; *but this was hardly practicable, the Invention of the greatest part of the Rules being intirely owing to* Algebra; *yet I have, as much as possible, endeavour'd to deduce from the Algebraical Calculation several practical Rules, the Truth of which may be depended upon, and which may be very useful to those who have contented themselves to learn only common Arithmetick.*

'Tis for the Sake of those Gentlemen that I have enlarged my first Design, which was to have laid down such Precepts only as might be sufficient to deduce the Solution of any difficult Problem relating to my Subject: And for this reason I have (towards the latter end of the Book) given the Solution, in Words at length, of some easy Problems, which might else have been made Corollaries or Consequences of the Rules before deliver'd: The single Difficulty which may occur from Pag. 155 *to the end, being only an Algebraical Calculation belonging to the* 49th *Problem, to explain which fully would have required too much room.*

On this Occasion, I must take notice to such of my Readers as are well vers'd in Vulgar Arithmetick, that it would not be difficult for them to make themselves Masters, not only of all the Practical Rules in this Book, but also of more useful Discoveries, if they would take the small Pains of being acquainted with the bare Notation of Algebra, *which might be done in the hundredth part of the Time that is spent in learning to read* Short-hand.

One of the Principal Methods I have made use of in the following Treatise, has been the Doctrine of Combinations, *taken in a Sence somewhat more extensive, than as it is commonly understood. The Notion of Combinations being so well fitted to the Calculation of Chance, that it naturally enters the Mind whenever any Attempt is made towards the Solution of any Problem of that kind. It was this which led me in course to the Consideration of the Degrees of Skill in the Adventurers at Play, and I have made use of it in most parts of this Book, as one of the* Data *that enter the Question; it being so far from perplexing the Calculation, that on*

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the contrary it is rather a Help and an Ornament to it: It is true, that this Degree of Skill is not to be known any other way than from Observation; but if the same Observation constantly recur, 'tis strongly to be presumed that a near Estimation of it may be made: However, to make the Calculation more precise, and to avoid causing any needless Scruples to those who love Geometrical Exactness, it will be easy, in the room of the Word Skill, *to substitute a Greater or Less Proportion of Chances among the Adventurers, so as each of them may be said to have a certain Number of Chances to win one single Game.*

The General Theorem invented by Sir Isaac Newton, *for raising a Binomial to any Power given, facilitates infinitely the Method of* Combinations, *representing in one View the Combination of all the Chances, that can happen in any given Number of Times. 'Tis by the help of that Theorem, joined with some other Methods, that I have been able to find practical Rules for the solving a great Variety of difficult Questions, and to reduce the Difficulty to a single Arithmetical Multiplication, whereof several Instances may be seen in the* 21st *Page of this Book.*

Another Method I have made use of is that of Infinite Series, *which in many cases will solve the Problems of Chance more naturally than Combinations. To give the Reader a Notion of this, we may suppose two Men at Play throwing a Die, each in their Turns, and that he be to be reputed the Winner who shall first throw an Ace: It is plain, that the Solution of this Problem cannot so properly be reduced to Combinations, which serve chiefly to determine the proportion of Chances between the Gamesters, without any regard to the Priority of Play. 'Tis convenient therefore to have recourse to some other Method, such as the following. Let us suppose that the first Man, being willing to Compound with his Adversary for the Advantage he is intitled to from his first Throw, should ask him what Consideration he would allow to yield it to him; it may naturally be supposed that the Answer would be one Sixth part of the Stake, there being but Five to One against him, and that this Allowance would be thought a just Equivalent for yielding his Throw: Let us likewise suppose the Second Man to require in his Turn to have one Sixth part of the remaining Stake for the Consideration of his Throw; which being granted, and the first Man's Right returning in course, he may claim again one Sixth part of the Remainder, and so on alternately, till the whole Stake be exhausted:*

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But this not being to be done till after an infinite number of Shares be thus taken on both Sides, it belongs to the Method of Infinite Series *to assign to each Man what proportion of the Stake he ought to take at first, so as to answer exactly that fictitious Division of the Stake* in infinitum; *by means of which it will be found, that the Stake ought to be Divided between the contending Parties into two parts, respectively proportional to the two Numbers 6 and 5. By the like Method it would be found that if there were Three or more Adventurers playing on the conditions above described, each Man, according to the Situation he is in with respect to Priority of Play, might take as his Due such part of the Stake, as is expressible by the corresponding Term of the proportion of 6 to 5, continued to so many Terms as there are Gamesters; which in the case of Three Gamesters, for Instance, would be the Numbers 6, 5 and* $4\frac{1}{6}$, *or their Proportionals 36, 30, and 25.*

Another Advantage of the Method of Infinite Series is, that every Term of the Series includes some particular Circumstance wherein the Gamesters may be found, which the other Methods do not; and that a few of its Steps are sufficient to discover the Law of its Process. The only Difficulty which attends this Method, being that of Summing up so many of its Terms as are requisite for the Solution of the Problem proposed: But it will be found by experience, that in the Series resulting from the consideration of most Cases relating to Chance, the Terms of it will either constitute a Geometric Progression, which by the Known Methods is easily Summable; or else some other sort of Progression, whose nature consists in this, that every Term of it has to a determinate number of the preceding Terms, each being taken in order, some constant relation; in which case I have contrived some easie Theorems, not only for finding the Law of that relation, but also for finding the Sums required; as may be seen in several places of this Book, but particularly from page 127 to page 134. I hope the Reader will excuse my not giving the Demonstrations of some few things relating to this Subject, especially of the two Theorems contained in page 134 and 154, and of the Method of Approximation contained in page 149 and 150; whereby the Duration of Play is easily determined with the help of a Table of Natural Sines: *Those Demonstrations are omitted purposely to give an occasion to the Reader to exercise his own Ingenuity. In the mean Time, I have deposited them with the* Royal Society, *in order to be Published when it shall be thought requisite.*

A Third Advantage of the Method of Infinite Series is, that the Solutions derived from it have a certain Generality and Elegancy, which ſcarce any other Method can attain to; thoſe Methods being always perplexed with Various unknown Quantities, and the Solutions obtained by them terminating commonly in particular Caſes.

There are other Sorts of Series, *which tho' not properly infinite, yet are called Series, from the Regularity of the Terms whereof they are compoſed; thoſe Terms following one another with a certain uniformity, which is always to be defined. Of this nature is the Theorem given by Sir* Iſaac Newton, *in the Fifth* Lemma *of the third Book of his* Principles, *for drawing a Curve through any given number of Points; of which the Demonſtration, as well as of other things belonging to the ſame Subject, may be deduced from the firſt Propoſition of his* Methodus Differentialis, *printed with ſome other of his Tracts, by the care of my Intimate Friend, and very skilful Mathematician,* Mr. W. Jones. *The abovementioned Theorem being very uſeful in Summing up any number of Terms whoſe laſt Differences are equal (Such as are the Numbers called Triangular, Pyramidal,* &c. *the Squares, the Cubes, or other Powers of Numbers in Arithmetic Progreſſion) I have ſhewn in many places of this Book how it might be applicable to theſe Caſes. I hope it will not be taken amiſs that I have aſcribed the Invention of it to its proper Author, tho' 'tis poſſible ſome Perſons may have found ſomething like it by their own Sagacity.*

After having dwelt ſome time upon Various Queſtions depending on the general Principle of Combinations, as laid down in my Introduction, and upon ſome others depending on the Method of Infinite Series, I proceed to treat of the Method of Combinations properly ſo called, which I ſhew to be eaſily deducible from that more general Principle which hath been before explained: Where it may be obſerved, that altho' the Caſes it is applyed to are particular, yet the way of reaſoning, and the conſequences derived from it, are general; that Method of Arguing about generals by particular Examples, being in my opinion very convenient for eaſing the Reader's Imagination.

Having explained the common Rules of Combination, and given a Theorem which may be of uſe for the Solution of ſome Problems relating to that Subject, I lay down a new Theorem, which is properly a contraction of the former, whereby ſeveral Queſtions of Chance are reſolved with wonderful eaſe, tho' the Solution might ſeem at firſt ſight to be of inſuperable difficulty.

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It is by the Help of that Theorem so contracted, that I have been able to give a compleat Solution of the Problems of Pharaon *and* Baſſete, *which was never done till now: I own that ſome great Mathematicians have before me taken the pains of calculating the Advantage of the* Banker, *in any circumſtance either of Cards remaining in his Hands, or of any number of times that the Card of the* Ponte *is contained in the Stock: But ſtill the curioſity of the Inquiſitive remained unſatisfied; The Chief Queſtion, and by much the moſt difficult, concerning* Pharaon *or* Baſſete, *being what it is that the Banker gets* per Cent *of all the Money adventured at thoſe Games, which now I can certainly anſwer is very near Three* per Cent *at* Pharaon, *and Three fourths* per Cent *at* Baſſete, *as may be ſeen in my* XXIII *Problem, where the preciſe Advantage is calculated.*

In the 24th and 25th Problems, I explain a new ſort of Algebra, *whereby ſome Queſtions relating to Combinations are ſolved by ſo eaſy a Proceſs, that their ſolution is made in ſome meaſure an immediate conſequence of the Method of Notation. I will not pretend to ſay that this new* Algebra *is abſolutely neceſſary to the Solving of thoſe Queſtions which I make to depend on it, ſince it appears by Mr.* De Monmort's *Book, that both he and Mr.* Nicholas Bernoully *have ſolved, by another Method, many of the caſes therein propoſed: But I hope I ſhall not be thought guilty of too much Confidence, if I aſſure the Reader, that the Method I have followed has a degree of Simplicity, not to ſay of Generality, which will hardly be attained by any other Steps than by thoſe I have taken.*

The 29th Problem, propoſed to me, amongſt ſome others, by the Honourable Mr. Francis Robartes, *I had ſolved in my Tract* De menſura Sortis; *It relates, as well as the 24th and 25th, to the Method of Combinations, and is made to depend on the ſame Principle; When I began for the firſt time to attempt its Solution, I had nothing elſe to guide me but the common Rules of Combinations, ſuch as they had been delivered by Dr.* Wallis *and others; which when I endeavoured to apply, I was Surprized to find that my calculation ſwelled by degrees to an Intolerable bulk: For this reaſon I was forced to turn my Views Another way, and to try whether the ſolution I was ſeeking for might not be deduced from ſome eaſier conſiderations; whereupon I happily fell upon the Method I have been mentioning, which as it led me to a very great Simplicity in the Solution, ſo I look upon it to be an Improvement made to the Method of Combinations.*

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The 30th Problem is the reverse of the preceding; It contains a very remarkable Method of Solution, the Artifice of which consists in changing an Arithmetic Progression of Numbers into a Geometric one; this being always to be done when the Numbers are large, and their Intervals small. I freely acknowledge that I have been indebted long ago for this useful Idea, to my much respected Friend, That Excellent Mathematician Doctor Halley, *Secretary to the* Royal Society, *whom I have seen practice the thing on an other occasion: For this and other Instructive Notions readily imparted to me, during an uninterrupted Friendship of five and Twenty years, I return him my very hearty Thanks.*

The 32d Problem, having in it a Mixture of the two Methods of Combinations and Infinite Series, may be proposed for a pattern of Solution, in some of the most difficult cases that may occurr in the Subject of Chance, and on this occasion I must do that Justice to Mr. Nicholas Bernoully, *the Worthy Professour of Mathematics at* Padua, *to own he had sent me the Solution of this Problem before mine was Published; which I had no sooner received, but I communicated it to the* Royal Society, *and represented it as a Performance highly to be commended: Whereupon the Society order'd that his Solution should be Printed; which was accordingly done some time after in the* Philosophical Transactions, *Numb.* 341. *where mine was also inserted.*

The Problems which follow relate chiefly to the Duration of Play, or to the Method of determining what number of Games may probably be played out by two Adversaries, before a certain number of Stakes agreed on between them be won or lost on either side. This Subject affording a very great Variety of Curious Questions, of which every one has a degree of Difficulty peculiar to it self, I thought it necessary to divide it into several distinct Problems, and to illustrate their Solution with proper Examples.

Tho' these Questions may at first sight seem to have a very great degree of difficulty, yet I have some reason to believe, that the Steps I have taken to come at their Solution, will easily be followed by those who have a competent skill in Algebra, *and that the chief Method of proceeding therein will be understood by those who are barely acquainted with the Elements of that Art.*

When I first began to attempt the general Solution of the Problem concerning the Duration of Play, there was nothing extant that could give me any light into that Subject; for altho' Mr.

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de Monmort, *in the firſt Edition of his Book, gives the Solution of this Problem, as limited to three Stakes to be won or loſt, and farther limited by the Suppoſition of an Equality of Skill between the Adventurers; yet he having given no Demonſtration of his Solution, and the Demonſtration when diſcovered being of very little uſe towards obtaining the general Solution of the Problem, I was forced to try what my own Enquiry would lead me to, which having been attended with Succeſs, the reſult of what I found was afterwards publiſhed in my* Specimen *before mentioned.*

All the Problems which in my Specimen *related to the Duration of Play, have been kept entire in the following Treatiſe; but the Method of Solution has received ſome Improvements by the new Diſcoveries I have made concerning the Nature of thoſe Series which reſult from the Conſideration of the Subject; however, the Principles of that Method having been laid down in my* Specimen *I had nothing now to do, but to draw the Conſequences that were naturally deducible from them.*

Mr. de Monmort, *and Mr.* Nicholas Bernoully, *have each of them ſeparately given the Solution of my* XXXIX*th Problem, in a Method differing from mine, as may be ſeen in Mr.* de Monmort'*s ſecond Edition of his Book. Their Solutions, which in the main agree together, and vary little more than in the form of Expreſſion, are extreamly beautiful; for which reaſon I thought the Reader would be well pleaſed to ſee their Method explained by me, in ſuch a manner as might be apprehended by thoſe who are not ſo well verſed in the nature of Symbols: In which matter I have taken ſome Pains, thereby to teſtify to the World the juſt Value I have for their Performance.*

The 43d *Problem having been propoſed to me by Mr.* Thomas Woodcock, *a Gentleman whom I infinitely reſpect, I attempted its Solution with a very great deſire of obtaining it; and having had the good Fortune to ſucceed in it, I returned him the Solution a few Days after he was pleaſed to propoſe it. This Problem is in my Opinion one of the moſt curious that can be propos'd on this Subject; its Solution containing the Method of determining, not only that Advantage which reſults from a Superiority of Chance, in a Play confined to a certain number of Stakes to be won or loſt by either Party, but alſo that which may reſult from an unequality of Stakes; and even compares thoſe two Advantages together, when the Odds of Chance being on one ſide, the Odds of Money are on the other.*

Before I make an end of this Discourse, I think my self obliged to take Notice, that some Years after my Specimen *was printed, there came out a Tract upon the Subject of Chance, being a Posthumous Work of Mr.* James Bernoully, *wherein the Author has shewn a great deal of Skill and Judgment, and perfectly answered the Character and great Reputation he hath so justly obtained. I wish I were capable of carrying on a Project he had begun, of applying the Doctrine of Chances to* Oeconomical *and* Political *Uses, to which I have been invited, together with Mr* de Monmort, *by Mr.* Nicholas Bernoully *: I heartily thank that Gentleman for the good Opinion he has of me; but I willingly resign my share of that Task into better Hands, wishing that either he himself would prosecute that Design, he having formerly published some successful Essays of that Kind, or that his Uncle, Mr.* John Bernoully, *Brother to the Deceased, could be prevailed upon to bestow some of his Thoughts upon it; he being known to be perfectly well qualified in all Respects for such an Undertaking.*

Due Care having been taken to avoid the *Errata* of the Press, we hope there are no other than these two,

VIZ.

Pag. 35. Lin. 35. for $n - 3$ read $n - 1$.
Pag. 36. Lin. 2. for $n - 3$ read $n - 2$.

The

The DOCTRINE OF CHANCES.

INTRODUCTION.

THE Probability of an Event is greater, or less, according to the number of Chances by which it may Happen, compar'd with the number of all the Chances, by which it may either Happen or Fail.

Thus, If an Event has 3 Chances to Happen, and 2 to Fail; the Probability of its Happening may be estimated to be $\frac{3}{5}$, and the Probability of its Failing $\frac{2}{5}$.

Therefore, if the Probability of Happening and Failing are added together, the Sum will always be equal to Unity.

If the Probabilities of Happening and Failing are unequal, there is what is commonly call'd Odds for, or againſt, the Happening or Failing; which Odds are proportional to the number of Chances for Happening or Failing.

The Expectation of obtaining any Thing, is eſtimated by the Value of that Thing multiplied by the Probability of obtaining it.

Thus, Suppoſing that *A* and *B* Play together; that *A* has depoſited 5 *l*, and *B* 3 *l*; that the number of Chances which *A* has to win is 4, and that the number of Chances which *B* has to win is 2: Since the whole Sum depoſited is 8, and that the Probability which *A* has of getting it, is $\frac{4}{6}$; it follows, that the Expectation of *A* upon the whole Sum depoſited will be $\frac{8}{1}\times\frac{4}{6} = 5\frac{1}{3}$; and for the ſame reaſon, the Expectation of *B* will be $\frac{8}{1}\times\frac{2}{6} = 2\frac{2}{3}$.

The Risk of loſing any Thing, is eſtimated by the Value of that Thing multiplied by the Probability of loſing it.

If from the reſpective Expectations, which the Gameſters have upon the whole Sum depoſited; the particular Sums they depoſit, that is their own Stakes, be ſubtracted, there will remain the Gain, if the difference is poſitive, or the Loſs, if the difference is negative.

Thus, If from $5\frac{1}{3}$ the Expectation of *A*, 5 which is his own Stake be ſubtracted, there will remain $\frac{1}{3}$ for his Gain; likewiſe if from $2\frac{1}{3}$ the Expectation of *B*, 3 which is his own Stake be ſubtracted, there will remain $-\frac{1}{3}$ for his Gain, or $\frac{1}{3}$ for his Loſs.

Again, If from the reſpective Expectations, which either Gameſter has upon the Sum depoſited by his Adverſary, the Risk of loſing what he himſelf depoſits, be ſubtracted, there will likewiſe remain his Gain or Loſs.

Thus, In the preceding Caſe, the Stake of *B* being 3, and the Probability which *A* has of winning it being $\frac{4}{6}$, the Expectation of *A* upon that Stake is $\frac{3}{1}\times\frac{4}{6} = \frac{12}{6} = 2$. Moreover the Stake of *A* being 5, and the Probability of loſing it being $\frac{2}{6}$, the Risk which *A* runs of loſing his own Stake is $\frac{5}{1}\times\frac{2}{6} = \frac{10}{6} = 1\frac{2}{3}$. Therefore, if from the

Ex-

Expectation 2, the Risk $1\frac{2}{3}$, be ſubtracted, there will remain $\frac{1}{3}$, as before, for the Gain of *A*; and by the ſame way of arguing, the Loſs of *B* will be found to be $\frac{1}{3}$.

N. B. Tho' the Gain of one is the Loſs of the other, yet it will be convenient to look for them ſeverally, that one Operation may be a Proof of the other.

If there is a certain number of Chances by which the poſſeſſion of a Sum can be ſecur'd; and alſo a certain number of Chances by which it may be loſt; that Sum may be Inſured for that part of it, which ſhall be to the whole, as the number of Chances there is to loſe it, to the number of all the Chances.

Thus, If there are 19 Chances to ſecure the poſſeſſion of 1000 *l*, and 1 Chance to loſe it, the Inſurance Money may be found by this Proportion.

As 20 is to 1, ſo is 1000 to 50; therefore 50 is the Sum that ought to be given, in this Caſe, to Inſure 1000.

If two Events have no dependence on each other, ſo that p be the number of Chances by which the firſt may Happen, and q the number of Chances by which it may Fail; and likewiſe that r be the number of Chances by which the ſecond may Happen, and s the number of Chances by which it may Fail: Multiply $p + q$ by $r + s$, and the Product $pr + qr + ps + qs$ will contain all the Chances, by which the Happening, or Failing of the Events may be varied amongſt one another.

Therefore, If *A* and *B* Play together, on condition that if both Events Happen, *A* ſhall win, and *B* loſe; the Odds that *A* ſhall be the winner, are as pr to $qr + ps + qs$; for the only Term in which both p and r occur is pr; therefore the Probability of *A*'s winning is $\frac{pr}{pr + qr + ps + qs}$.

But if *A* holds that either one or the other will Happen; the Odds of *A*'s winning are as $pr + qr + ps$ to qs; for ſome of the Chances that are favourable to *A*, occur in every one of the Terms pr, qr, ps.

Again, If *A* holds that the firſt will Happen, and the ſecond Fail; the Odds are as ps to $pr + qr + qs$.

From

From what has been ſaid, it follows, that if a Fraction expreſſes the Probability of an Event, and another Fraction the Probability of another Event, and thoſe two Events are independent; the Probability that both thoſe Events will Happen, will be the Product of thoſe two Fractions.

Thus, Suppoſe I have two Wagers depending, in the firſt of which I have 3 to 2 the beſt of the Lay, and in the ſecond 7 to 4, what is the Probability I win both Wagers?

The Probability of winning the firſt is $\frac{3}{5}$, that is the number of Chances I have to win, divided by the number of all the Chances; the Probability of winning the ſecond is $\frac{7}{11}$: Therefore multiplying theſe two Fractions together, the Product will be $\frac{21}{55}$, which is the Probability of winning both Wagers. Now this Fraction being ſubtracted from 1, the remainder is $\frac{34}{55}$, which is the Probability I do not win both Wagers: Therefore the Odds againſt me are 34 to 21.

2° If I would know what the Probability is of winning the firſt, and loſing the ſecond, I argue thus; The Probability of winning the firſt is $\frac{3}{5}$, the Probability of loſing the ſecond is $\frac{4}{11}$: Therefore multiplying $\frac{3}{5}$ by $\frac{4}{11}$, the Product $\frac{12}{55}$ will be the Probability of my winning the firſt, and loſing the ſecond; which being ſubtracted from 1, there will remain $\frac{43}{55}$, which is the Probability I do not win the firſt, and at the ſame time loſe the ſecond.

3° If I would know what the Probability is of winning the ſecond, and at the ſame time loſing the firſt; I ſay thus, the Probability of winning the ſecond is $\frac{7}{11}$, the Probability of loſing the firſt is $\frac{2}{5}$. Therefore multiplying theſe two Fractions together, the Product $\frac{14}{55}$ is the Probability I win the ſecond, and alſo loſe the firſt.

4° If I would know what the Probability is of loſing both Wagers; I ſay, the Probability of loſing the firſt is $\frac{2}{5}$, and the Probability of loſing the ſecond $\frac{4}{11}$; therefore the Probability of loſing them both is $\frac{8}{55}$, which being ſubtracted from 1, there remains $\frac{47}{55}$; therefore the Odds againſt loſing both Wagers is 47 to 8.

This

This way of reaſoning is plain, and is of very great extent, being applicable to the Happening or Failing of as many Events as may fall under conſideration. Thus, if I would know what the Probability is of miſſing an Ace 4 times together with a common Die, I conſider the miſſing of the Ace 4 times, as the Failing of 4 different Events; now the Probability of miſſing the firſt is $\frac{5}{6}$, the Probability of miſſing the ſecond is alſo $\frac{5}{6}$, the third $\frac{5}{6}$, the fourth $\frac{5}{6}$; therefore the Probability of miſſing the Ace 4 times together is $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{625}{1296}$; which being ſubtracted from 1, there will remain $\frac{671}{1296}$ for the Probability of throwing it once or oftner in 4 times; therefore the Odds of throwing an Ace in 4 times is 671 to 625.

But if the flinging of an Ace was undertaken in 3 times, 'tis plain that the Probability of miſſing it 3 times would be $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{216}$, which being ſubtracted from 1, there will remain $\frac{91}{216}$ for the Probability of throwing it once or oftner in 3 times; therefore the Odds againſt throwing it in 3 times are 125 to 91.

Again, ſuppoſe we wou'd know the Probability of throwing an Ace once in 4 throws and no more: Since the Probability of throwing it the firſt time is $\frac{1}{6}$, and the Probability of miſſing it the other three times is $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}$, it follows that the Probability of throwing it the firſt time, and miſſing it afterwards three times ſucceſſively, is $\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{1296}$; but becauſe it is poſſible to hit it in every throw as well as the firſt, it follows, that the Probability of throwing it once in 4 throws, and miſſing the other three times, is $\frac{4\times125}{1296}=\frac{500}{1296}$; which being ſubtracted from 1, there will remain $\frac{796}{1296}$ for the Probability of not throwing it once and no more in 4 times; therefore if one undertakes to throw an Ace once and no more in 4 times, he has 500 to 796 the worſt of the Lay, or 5 to 8 very near.

Suppoſe two Events are ſuch, that one of them has twice as many Chances to come up as the other; what is the Probability that the Event which has the greater number of Chances to come up, does not Happen twice before the other Happens once; which is the Caſe of flinging Seven with two Dice, before four once; ſince the number of Chances are as 2 to 1, the Probability of the firſt Happening before the ſecond is $\frac{2}{3}$, but the Probability of its Happening twice before it, is but $\frac{2}{3} \times \frac{2}{3}$ or $\frac{4}{9}$; therefore 'tis 5 to 4, Seven does not come up twice, before Four once.

But if it was demanded what muſt be the proportion of the Facilities of the coming up of two Events, to make that which has the moſt Chances, to come up twice, before the other comes up once; the anſwer is 12 to 5 very near, (and this proportion may be determined yet with greater exactneſs) for if the proportion of the Chances is 12 to 5, it follows, that the Probability of throwing the firſt before the ſecond is $\frac{12}{17}$, and the Probability of throwing it twice, is $\frac{12}{17} \times \frac{12}{17}$ or $\frac{144}{289}$; therefore the Probability of not doing it is $\frac{145}{289}$; therefore the Odds againſt it are as 145 to 144, which comes very near a proportion of Equality.

What we have ſaid hitherto concerning two or more Events, relates only to thoſe which have no dependency on each other; as for thoſe that have a dependency, the manner of arguing about them will be a little alter'd: But to know in what the nature of this dependency conſiſts, I ſhall propoſe the two following eaſy Problems.

Suppoſe there is a heap of 13 Cards of one colour, and another heap of 13 Cards of another colour; what is the Probability, that taking one Card at a venture out of each heap, I ſhall take out the two Aces?

The Probability of taking the Ace out of the firſt heap is $\frac{1}{13}$, the Probability of taking the Ace out of the ſecond is alſo $\frac{1}{13}$; therefore the Probability of taking out both Aces is $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$, which being ſubtracted from 1, there will remain $\frac{168}{169}$, therefore the Odds againſt me are 168 to 1.

But ſuppoſe that out of one ſingle heap of 13 Cards of one colour, I ſhould undertake to take out, firſt the Ace, ſecondly the Two; tho' the Probability of taking out the Ace be $\frac{1}{13}$, and the Probability of taking out the Two be likewiſe $\frac{1}{13}$, yet the Ace being ſuppoſed as taken out already, there will remain only 12 Cards in the heap, which will make the Probability of of taking out the Two to be $\frac{1}{12}$, therefore the Probability of taking out the Ace, and then the Two, will be $\frac{1}{13} \times \frac{1}{12}$. And upon this way of reaſoning may the whole Doctrine of Combinations be grounded, as will be ſhewn in its place.

It is plain that in this laſt Queſtion, the two Events propoſed have on each other a dependency of Order, which dependency conſiſts in this, that one of the Events being ſuppoſed as having Happened, the Probability of the other's Happening is thereby alter'd; whereas in the firſt Queſtion, the taking of the Ace out of the firſt heap does not alter the Probability of taking the Ace out of the ſecond; therefore the Independency of Events conſiſts in this, that the Happening of one does not alter the degree of Probability of the other's Happening.

We have ſeen already how to determine the Probability of the Happening of as many Events as may be aſſigned, and the Failing of as many others as may be aſſigned likewiſe, when thoſe Events are independent: We have ſeen alſo how to determine the Happening of two Events, or as many as may be aſſigned when they are Dependent.

But how to determine in the caſe of Events dependent, the Happening of as many as may be aſſigned, and at the ſame time the Failing of as many as may likewiſe be aſſigned, is a diſquiſition of a higher nature, and will be ſhewn afterwards.

If the Events in queſtion are n in number, and are ſuch as have the ſame number a of Chances by which they may Happen, and likewiſe the ſame number b of Chances by which they may Fail, raiſe $a+b$ to the Power n.

And if A and B play together, on condition that if either one or more of the Events in queſtion do Happen, A ſhall win, and B loſe; the Probability of A's winning will be $\frac{\overline{a+b}^n - b^n}{\overline{a+b}^n}$, and that of B's winning will be $\frac{b^n}{\overline{a+b}^n}$; for when $a+b$ is actually raiſed to the Power n, the only Term in which a does not occur is the laſt b^n; therefore all the Terms but the laſt are favourable to A.

Thus, if $n=3$; raiſing $a+b$ to the Cube $a^3+3aab+3abb+b^3$. all the Terms but b^3 will be favourable to A; and therefore the

the Probability of *A's* winning will be $\frac{a^3+3aab+3abb}{\overline{a+b}|^3}$ or $\frac{\overline{a+b}|^3-b^3}{\overline{a+b}|^3}$; and the Probability of *B's* winning will be $\frac{b^3}{\overline{a+b}|^3}$.

But if *A* and *B* play, on condition that if either two or more of the Events in queſtion do Happen, *A* ſhall win; but in caſe one only Happens or none, *B* ſhall win; the Probability of *A's* winning will be $\frac{\overline{a+b}|^n-nab^{n-1}-b^n}{\overline{a+b}|^n}$; for the only two Terms in which aa does not occur are the two laſt, *viz*, nab^{n-1} and b^n. And ſo of the reſt.

The

The Solution of ſeveral ſorts of Problems, deduced from the Rules laid down in the Introduction.

PROBLEM I.

SUppoſe A *to hold a ſingle Die, and to lay with* B, *that in* 8 *throws he ſhall fling Two Aces or more: What is his Probability of winning, or what are the Odds for or againſt him?*

SOLUTION.

BEcauſe there is one ſingle Chance for *A*, and five againſt him, let a be made $= 1$, and $b = 5$; again becauſe the number of throws is 8, let n be made $= 8$, and the Probability of *A*'s winning will be $\frac{\overline{a+b}^n - b^n - nab^{n-1}}{\overline{a+b}^n} = \frac{663991}{1679616}$. Therefore the Probability of his loſing will be $\frac{1015625}{1679616}$, and the Odds againſt him will be as 1015625 to 663991, or as 3 to 2, very near.

PROBLEM II.

TWO *Men* A *and* B *playing a Set together, in each Game of the Set the number of Chances which* A *has to win is* 3, *and the number of Chances which* B *has to win is* 2: *Now after ſome Games are over,* A *wants* 4 *Games of being up, and* B 6: *It is required in this circumſtance to determine the Probabilities which either has of winning the Set.*

SOLUTION.

BEcauſe *A* wants 4 Games of being up, and *B* 6; it follows, that the Set will be ended in 9 Games at moſt, which is the ſum of the Games wanting between them; therefore let $a+b$ be raiſed to the $9th$ Power, *viz.* $a^9+9a^8b+36a^7bb+84a^6b^3+126a^5b^4+126a^4b^5+84a^3b^6+36aab^7+9ab^8+b^9$; take for A all the Terms in which a has 4 or more dimenſions, and for B all the Terms in which b has 6 or more dimenſions; and the Proportion of the Odds will be, as $a^9+9a^8b+36a^7bb+84a^6b^3+126a^5b^4+126a^4b^5$, to $84a^3b^6+36aab^7+9ab^8+b^9$. Let now a be expounded by 3, and b by 2; and the Odds that A wins the Set will be found as 1759077 to 194048, or very near as 9 to 1.

And generally, ſuppoſing that p and q are the number of Games reſpectively wanting; raiſe $a+b$ to the Power $p+q-1$, then take for A, a number of Terms equal to q, and for B, a number of Terms equal to p.

REMARKS.

1. In this Problem, if inſtead of ſuppoſing that the Chances which the Gameſters have each time to get a Game are in the proportion of a to b, we ſuppoſe the Skill of the Gameſters to be in that proportion, the Solution of the Problem will be the ſame: We may compare the Skill of the Gameſters to the number of Chances they have to win. Whether the number of Chances which A and B have of getting a Game, are in a certain Proportion, or whether their Skill be in that proportion, is the ſame thing.

2. The preceding Problem might be ſolved without Algebra, by the bare help of the Arithmetical Principles which we have laid down in the Introduction, but the method will be longer: Yet for the ſake of thoſe who are not acquainted with Algebraical computation, I ſhall ſet down the Method of proceeding in like caſes.

In order to which, it is neceſſary to know, that when a Queſtion ſeems ſomewhat difficult, it will be uſeful to ſolve at firſt a Queſtion of the like nature, that has a greater degree of ſimplicity than the caſe propoſed in the Queſtion given; the Solution of which caſe being obtained, it will be a ſtep to aſcend to a caſe a little more compounded, till at laſt the caſe propoſed may be attained to.

Therefore, to begin with the ſimpleſt caſe, we may ſuppoſe that A wants 1 Game of being up, and B 2; and that the

the number of Chances to win a Game are equal; in which caſe the Odds that *A* will be up before *B*, may be determined as follows.

Since *B* wants 2 Games of being up and *A* 1, 'tis plain that *B* muſt beat *A* twice together to win; but the Probability of his beating him once is $\frac{1}{2}$, therefore the Probability of his beating him twice together is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; ſubtract $\frac{1}{4}$ from 1, there remains $\frac{3}{4}$, which is the Probability which *A* has of winning once before *B* twice, therefore the Odds are as 3 to 1.

By the ſame way of arguing 'twill be found, that if *A* wants 1, and *B* 3, the Odds will be as 7 to 1, and the Probability of winning, $\frac{7}{8}$ and $\frac{1}{8}$ reſpectively. If *A* wants 1, and *B* 4, the Odds will be as 15 to 1, &c.

Again, ſuppoſe *A* wants 2, and B 3, what are the Odds that *A* is up before *B*?

Let the whole Stake depoſited between *A* and *B* be 1; now conſider that if *B* wins the firſt Game, *B* and *A* will have an equality of Chances, in which caſe the Expectation of *B* will be $\frac{1}{2}$; but the Probability of his winning the firſt Game is $\frac{1}{2}$, therefore the Expectation of *B* upon the Stake, ariſing from the Probability of beating *A* the firſt time, will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

But if *B* loſes the firſt Game, then he will want 3 of being up, and *A* but 1; in which caſe the Expectation of *B* will be $\frac{1}{8}$, but the Probability of that circumſtance is $\frac{1}{2}$, therefore the Expectation of *B* ariſing from the Probability of his loſing the firſt time is $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$.

Therefore the Expectation of *B* upon the Stake 1, will be $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$, which being ſubtracted from 1, there remains $\frac{11}{16}$ for the Expectation of *A*; therefore the Odds are as 11 to 5.

And thus proceeding gradually, it will be eaſy to compoſe the following Table.

A TABLE.

A TABLE of the ODDS for any number of Games wanting, from 1 to 6.

Games wanting.	1	2	1	3	1	4	1	5	1	6
Probabilities of Winning.	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{1}{8}$	$\frac{15}{16}$	$\frac{1}{16}$	$\frac{31}{32}$	$\frac{1}{32}$	$\frac{63}{64}$	$\frac{1}{64}$
Games wanting.	2	3	2	4	2	5	2	6		
Probabilities of Winning.	$\frac{11}{16}$	$\frac{5}{16}$	$\frac{26}{32}$	$\frac{6}{32}$	$\frac{57}{64}$	$\frac{7}{64}$	$\frac{120}{128}$	$\frac{8}{128}$		
Games wanting.	3	4	3	5	3	6				
Probabilities of Winning.	$\frac{42}{64}$	$\frac{22}{64}$	$\frac{99}{128}$	$\frac{29}{128}$	$\frac{219}{256}$	$\frac{37}{256}$				
Games wanting.	4	5	4	6						
Probabilities of Winning.	$\frac{163}{256}$	$\frac{93}{256}$	$\frac{382}{512}$	$\frac{130}{512}$						
Games wanting.	5	6								
Probabilities of Winning.	$\frac{638}{1024}$	$\frac{386}{1024}$								

And by the ſame way of proceeding, it would be eaſy to compoſe other Tables, for expreſſing the Probabilities which *A* and *B* have of winning the Set, when each wants a given number of Games of being up, and when the proportion of the Chances

Chances by which each of them may get a Game is as 2 to 1, or varies at pleaſure; but the Algebraic method explained in this Problem anſwering all that variety, 'tis needleſs to inſiſt upon it.

PROBLEM III.

IF A *and* B *play with ſingle Bowls, and ſuch be the Skill of* A *that he knows by Experience he can give* B 2 *Games out of* 3: *What is the proportion of their Skill, or what are the Odds that* A *may get any one Game aſſigned?*

SOLUTION.

LET the proportion of the Odds be as z to 1: Now ſince *A* can give *B* 2 Games out of 3, therefore *A* can, upon an equality of Play, undertake to win 3 Games together: Let therefore $z+1$ be raiſed to the Cube, *viz.* $z^3+3zz+3z+1$; therefore the Probabilities of winning will be, as z^3 to $3zz+3z+1$; but theſe Probabilities are equal, by ſuppoſition; therefore $z^3=3zz+3z+1$, or $2z^3=z^3+3zz+3z+1$. and extracting the Cube Root on both ſides, $z\sqrt[3]{2}=z+1$; therefore $z=\frac{1}{\sqrt[3]{2}-1}$, and conſequently the Odds that *A* may get any one Game aſſigned are as $\frac{1}{\sqrt[3]{2}-1}$ to 1, or as 1 to $\sqrt[3]{2}-1$, that is in this caſe as 50 to 13 very near.

PROBLEM IV.

IF A *can without advantage or diſadvantage give* B 1 *Game out of* 3; *what are the Odds that* A *ſhall take any one Game aſſigned? Or what is the proportion of the Chances they have to win any one Game aſſigned? Or what is the proportion of their Skill?*

SOLUTION.

LET the proportion be as z to 1; now ſince *A* can give *B* 1 Game out of 3; therefore *A* can, upon an equality of Play, undertake to get 3 Games before *B* gets 2; let therefore $z+1$ be raiſed to the *4th* Power, whoſe Index 4 is the Sum of the Games wanting between them leſs by 1; this Power will be $z^4+4z^3+6zz+4z+1$; therefore the Probabilities of winning the Set will be as z^4+4z^3 to $6zz+4z+1$: But theſe Probabilities are equal by Hypotheſis, ſince *A* and *B* are ſuppoſed to play without advantage or diſadvantage; therefore $z^4+4z^3=6zz+4z+1$, which Equation being ſolved, z will be found to be 1.6 very near; wherefore the proportion of the Odds will be as 1.6 to 1, or as 8 to 5.

PROBLEM V.

TO find in how many Trials an Event will Probably Happen, or how many Trials will be requiſite to make it indifferent to lay on its Happening or Failing; ſuppoſing that a *is the number of Chances for its Happening in any one Trial, and* b *the number of Chances for its Failing.*

SOLUTION.

LET x be the number of Trials; therefore by what has been already demonſtrated in the Introduction $\overline{a+b}|^x-b^x=b^x$, or $\overline{a+b}|^x=2b^x$; therefore $x=\dfrac{Log\ 2}{Log:\overline{a+b}-Log:b}$.

Moreover, let us reaſſume the Equation $\overline{a+b}|^x=2b^x$ and making $a, b :: 1, q$, the Equation will be changed into this $\overline{1+\frac{1}{q}}|^x=2$: let therefore $1+\frac{1}{q}$ be raiſed actually to the Power x by Sir *Iſaac Newton*'s Theorem, and the Equation will be $1+\frac{x}{q}+\frac{x\times\overline{x-1}}{1\times 2qq}+\frac{x\times\overline{x-1}\times\overline{x-2}}{1\times2\times3q^3}$ &c. $=2$. In this Equation, if $q=1$, then will x be likewiſe $=1$; if q be infinite, then will x alſo be infinite. Suppoſe q infinite, then the Equation will be reduced to $1+\frac{x}{q}+\frac{xx}{2qq}+\frac{x^3}{6qq}$ &c. $=2$: But the firſt part of this Equation is the number whoſe Hyperbolic Logarithm is $\frac{x}{q}$, therefore $\frac{x}{q}=\text{Log}: 2$: But the Hyperbolic Logarithm of 2 is 0.693 or nearly 0.7; Wherefore $\frac{x}{q}=$ 0.7, and $x=0.7q$ very near.

Thus we have aſſigned the very narrow limits within which the Ratio of x to q is comprehended; for it begins with

with Unity, and terminates at laſt in the Ratio of 10 to 7, very near.

But x ſoon Converges to the limit $0.7q$, ſo that this proportion may be aſſumed in all caſes, let the Value of q be what it will.

Some uſes of this Propoſition will appear by the following Examples.

EXAMPLE I.

LET it be propoſed to find in how many Throws one may undertake, with an equality of Chance, to fling two Aces with two Dice.

The number of Chances upon two Dice is 36, out of which there is but 1 Chance for two Aces; therefore the number of Chances againſt it is 215: Multiply 35 by 0.7, and the product 24.5 will ſhew that the number of Throws requiſite to that effect will be between 24 and 25.

EXAMPLE II.

TO find in how many Throws of three Dice, one may undertake to fling three Aces.

The number of all the Chances upon 3 Dice is 216, out of which there is but 1 Chance for 3 Aces, and 215 againſt it. Therefore let 215 be multiplied by 0.7, and the product 150.5 will ſhew that the number of Chances requiſite to that effect will be 150, or very near it.

EXAMPLE III.

IN a Lottery whereof the number of Blanks is to the number of Prizes as 39 to 1, (ſuch as was the Lottery of 1710;) To find how many Tickets one muſt take, to make it an equal Chance for one or more Prizes.

Multiply 39 by 0.7, and the product 27.3 will ſhow that the number of Tickets requiſite to that effect will be 27, or 28 at moſt.

Likewiſe, in a Lottery whereof the number of Blanks is to the number of Prizes; as 5 to 1, multiply 5 by 0.7 and the pro-

product 3.5 will ſhow, that there is more than an equality of Chance in 4 Tickets for one or more Prizes, but ſomething leſs than an equality in 3.

REMARK.

In a Lottery whereof the Blanks are to the Prizes as 39 to 1, if the number of Tickets in all was but 40, this proportion would be altered, for 20 Tickets would be a ſufficient number for the Expectation of the ſingle Prize; it being evident that the Prize may be as well among the Tickets which are taken as among thoſe that are left behind.

Again, if the number of Tickets was 80, ſtill preſerving the proportion of 39 Blanks to 1 Prize, and conſequently ſuppoſing 78 Blanks to 2 Prizes, this proportion would ſtill be altered: For by the Doctrine of Combinations, whereof we are to treat afterwards, it will appear that the Probability of taking one Prize or both in 20 Tickets would be but $\frac{139}{316}$, and the Probability of taking none would be $\frac{177}{316}$; Wherefore the Odds againſt taking any Prize would be as 177 to 139, or very near as 9 to 7.

And by the ſame Doctrine of Combinations it will be found that 23 Tickets would not be quite ſufficient for the Expectation of a Prize in this Lottery; but that 24 would rather be two many; ſo that one might with advantage lay an even Wager of taking a Prize in 24 Tickets.

If the proportion of 39 to 1 be oftner repeated, the number of Tickets requiſite for a Prize will ſtill increaſe with that repetition: Yet let the proportion of 39 to 1 be repeated never ſo many times, nay an infinite number of times, the number of Tickets requiſite for a Prize will never exceed $\frac{7}{10}$ of 39, that is about 27 or 28.

Therefore if the proportion of the Blanks to the Prizes be often repeated, as it uſually is in Lotteries; the number of Tickets requiſite for one Prize or more, will always be found by taking $\frac{7}{10}$ of the proportion of the Blanks to the Prizes.

LEM-

LEMMA.

TO find how many Chances there are upon any number of Dice, each of them of the same given number of Faces, to throw any given number of Points.

SOLUTION.

LET $p+1$ be the number of Points given, n the number of Dice, f the number of Faces in each Die: Make $p-f=q$; $q-f=r$; $r-f=s$; $s-f=t$ &c. and the number of Chances will be

$$
\begin{aligned}
&+\frac{p}{1}\times\frac{p-1}{2}\times\frac{p-2}{3}\ \&c.\\
&-\frac{q}{1}\times\frac{q-1}{2}\times\frac{q-2}{3}\ \&c.\times\frac{n}{1}\\
&+\frac{r}{1}\times\frac{r-1}{2}\times\frac{r-2}{3}\ \&c.\times\frac{n}{1}\times\frac{n-1}{2}\\
&-\frac{s}{1}\times\frac{s-1}{2}\times\frac{s-2}{3}\ \&c.\times\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\\
&+\ \&c.
\end{aligned}
$$

which Series ought to be continued till some of the Factors in each Product become either $=0$, or Negative.

N. B. So many Factors are to be taken in each of the Products $\frac{p}{1}\times\frac{p-1}{2}\times\frac{p-2}{3}$ &c. $\frac{q}{1}\times\frac{q-1}{2}\times\frac{q-2}{3}$ &c. as there are Units in $n-1$.

Thus, for Example, let it be required to find how many Chances there are for throwing Sixteen Points with Four Dice.

$$
\begin{aligned}
&+\frac{15}{1}\times\frac{14}{2}\times\frac{13}{3} &&=+455\\
&-\frac{9}{1}\times\frac{8}{2}\times\frac{7}{3}\times\frac{4}{1} &&=-336\\
&+\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}\times\frac{4}{1}\times\frac{3}{2} &&=+\ \ 6
\end{aligned}
$$

But $455-336+6=125$; therefore One Hundred and Twenty Five is the number of Chances required.

 Again,

Again, let it be required to find the number of Chances for throwing ſeven and Twenty Points with Six Dice.

$$+\frac{26}{1}\times\frac{25}{2}\times\frac{24}{3}\times\frac{23}{4}\times\frac{22}{5} = +\ 65780$$
$$-\frac{19}{1}\times\frac{18}{2}\times\frac{17}{3}\times\frac{16}{4}\times\frac{15}{5}\times\frac{6}{1} = -\ 93204$$
$$+\frac{13}{1}\times\frac{12}{2}\times\frac{11}{3}\times\frac{10}{4}\times\frac{9}{5}\times\frac{6}{1}\times\frac{5}{2} = +\ 30030$$
$$-\frac{8}{1}\times\frac{7}{2}\times\frac{6}{3}\times\frac{5}{4}\times\frac{4}{5}\times\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3} = -\ 1120$$

Therefore $65780 - 93204 + 30030 - 1120 = 1666$ is the number required.

Let it be required to find the number of Chances for throwing Fifteen Points with Six Dice.

$$+\frac{14}{1}\times\frac{13}{2}\times\frac{12}{3}\times\frac{11}{4}\times\frac{10}{5} = +\ 2002$$
$$-\frac{8}{1}\times\frac{7}{2}\times\frac{6}{3}\times\frac{5}{4}\times\frac{4}{5}\times\frac{6}{1} = -\ 336$$

But $2002 - 336 = 1666$, which is the number required.

Corol. All the Points equally diſtant from the extreams, that is, from the leaſt and greateſt number of Points that are upon the Dice, have the ſame number of Chances by which they may be produced; wherefore if the number of Points given be nearer to the greater Extream than to the leſs, let the number of Points given be ſubtracted from the ſum of the Extreams, and work with the remainder, and the Operation will be ſhorten'd.

Thus, if it be required to find the number of Chances for throwing 27 Points with 6 Dice: Let 27 be ſubtracted from 42 the ſum of the Extreams 6 and 36, and the Remainder being 15, it may be concluded that the number of Chances for throwing 27 Points is the ſame as for throwing 15.

Let it now be required to find in how many throws of 6 Dice one may undertake to throw 15 Points.

The number of Chances for throwing 15 Points being 1666; and the number of Chances for Failing being 44990; divide 44990 by 1666, the Quotient will be 27; Multiply 27 by by 0.7, and the Product 18.9 will ſhew that the number of throws requiſite to that effect is very near 19.

PROBLEM VI.

To find how many Trials are neceſſary to make it Probable, that an Event will Happen twice, ſuppoſing that a *is the number of Chances for its Happening at any one Trial, and* b *the number of Chances for its Failing.*

SOLUTION.

LET x be the number of Trials: Therefore by what has been already Demonſtrated, it will appear that $\overline{a+b}|^{x} = 2b^{x} + 2axb^{x-1}$; or making $a, b :: 1, q$; $\overline{1+\frac{1}{q}}|^{x} = 2 + \frac{2x}{q}$. Now if q be ſuppoſed $= 1$, x will be found $= 3$; and if q be ſuppoſed infinite, and alſo $\frac{x}{q} = z$, we ſhall have $z = \text{Log}: 2 + \text{Log}: 1 + z$; in which Equation the value of z will be found $= 1.678$ very nearly. Therefore the value of x will always be between the Limits $3\,q$ and $1.678q$. But x will ſoon converge to the laſt of theſe Limits; therefore if x be not very ſmall, it may in all caſes be ſuppoſed $= 1.678\,q$. Yet if there be any Suſpicion that the Value of x thus taken is too little, ſubſtitute this Value in the Original Equation $\overline{1+\frac{1}{q}}|^{x} = 2 + \frac{2x}{q}$, and note that Errour. If it be worth taking notice of, then increaſe a little the value of x, and ſubſtitute again this new value in the room of x in the aforeſaid Equation; and noting the new Errour, the value of x may be ſufficiently corrected by applying the Rule which the Arithmeticians call double Falſe Poſition.

EXAMPLE I.

TO find in how many throws of Three Dice one may undertake to throw Three Aces twice.

The number of all the Chances upon Three Dice being 216, out of which there is but one Chance for Three Aces, and 215 againſt it; Multiply 215 by 1.678, and the Product 360.7 will ſhew that the number of throws requiſite to that effect will be 360 or very near it.

EX-

EXAMPLE II.

TO find in how many throws of Six Dice one may undertake to throw Fifteen Points twice.

The number of Chances for throwing Fifteen Points is 1666, the number of Chances for Miſſing 44990; let 44990 be divided by 1666, the Quotient will be 27 very near: Wherefore the proportion of Chances for Throwing and Miſſing Fifteen Points are as 1 to 27 reſpectively; Multiply 27 by 1.678, and the Product 45.3 will ſhew that the number of throws requiſite to that effect will be 45 nearly.

EXAMPLE III.

IN a Lottery whereof the number of Blanks is to the number of Prizes as 39 to 1: To find how many Tickets muſt be taken, to make it as Probable that two or more Benefits will be taken as not.

Multiply 39 by 1.678, and the Product 65.4 will ſhew that no leſs than 65 Tickets will be requiſite to that effect; tho' one might undertake upon an Equality of Chance to have one at leaſt in 28.

PROBLEM VII.

TO find how many Trials are neceſſary to make it Probable that an Event will Happen Three, Four, Five, &c. *times; ſuppoſing that* a *is the number of Chances for its Happening in any one Trial, and* b *the number of Chances for its Failing.*

SOLUTION.

LET x be the number of Trials requiſite, then ſuppoſing, as before $a, b :: 1, q$, we ſhall have the Equation $\overline{1+\frac{1}{q}}^{x} = 2 \times \overline{1+\frac{x}{q}+\frac{x}{1}\times\frac{x-1}{2qq}}$, in the caſe of the triple Event; or $\overline{1+\frac{1}{q}}^{x} = 2\times\overline{1+\frac{x}{q}+\frac{x}{1}\times\frac{x-1}{2qq}+\frac{x}{1}\times\frac{x-1}{2}\times\frac{x-2}{3qq}}$, in the caſe of the quadruple Event: And the Law of the continuation of theſe Equations is manifeſt. Now in the firſt Equation if q be ſuppoſed $= 1$, then will x be $= 5$. If q be ſup-

ſuppoſed infinite or pretty large in reſpect to Unity; then the aforeſaid Equation, making $\frac{x}{q} = z$, will be changed into this, $z = \text{Log. } 2 + \text{Log. } \overline{1 + z + \frac{1}{2}zz}$; wherein z will be found nearly $= 2.675$. Wherefore x will always be between $5\,q$ and $2.675\,q$.

Likewiſe in the ſecond Equation, if q be ſuppoſed $= 1$, then will x be $= 7\,q$; but if x be ſuppoſed infinite, or pretty large in reſpect to Unity, then $z = \text{Log. } 2 + \text{Log. } \overline{1 + z + \frac{1}{2}zz + \frac{1}{6}z^3}$; whence z will be found nearly $= 3.6719$; Wherefore x will be between $7\,q$ and $3.6719\,q$.

If theſe Equations were continued, it would be found that the Limits of z converge continually to the proportion of two to one.

A TABLE of the Limits.

The Value of x will always be

For a ſingle Event, between	$1q$	and	$0.693q$.
For a double Event, between	$3q$	and	$1.678q$.
For a triple Event, between	$5q$	and	$2.675q$.
For a quadruple Event, between	$7q$	and	$3.672q$.
For a quintuple Event, between	$9q$	and	$4.670q$.
For a ſextuple Event, between	$11q$	and	$5.668q$.

If the number of Events contended for, as well as the number q be pretty large in reſpect to Unity; the number of Trials requiſite for thoſe Events to Happen n times, will be $\frac{2n-1}{2}q$ or barely nq.

PROBLEM VIII.

THree *Gameſters* A, B, C, *play together on this condition, that he ſhall win the Set who has ſooneſt got a certain number of Games; the proportion of the Chances which each of them has to get any one Game aſſigned, or, which is the ſame thing, the proportion of their Skill, being reſpectively as* a, b, c. *Now after they have played ſome time, they find themſelves in this circumſtance, that* A *wants One Game of being up,* B *Two Games, and* C *Three; the whole Stake between them being ſuppoſed* 1: *What is the Expectation of each?*

SOLUTION.

IN the Circumſtance the Gameſters are in, the Set will be ended in Four Games at moſt; let therefore $a + b + c$ be raiſed to the fourth Power, and it will be $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 + 4a^3c + 12aabc + 4b^3c + 6aacc + 12abcc + 6bbcc + 4ac^3 + 12acbb + 4bc^3 + c^4$.

The Terms $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc$, wherein the Dimenſions of a are equal to or greater than the number of Games which *A* wants, wherein alſo the Dimenſions of b and c are leſs than the number of Games which *B* and *C* want reſpectively, are intirely Favourable to *A*, and are part of the Numerator of his Expectation.

In the ſame manner the Terms $b^4 + 4b^3c + 6bbcc$ are intirely Favourable to *B*.

And likewiſe the Terms $4bc^3 + c^4$ are intirely Favourable to *C*.

The reſt of the Terms are common, as Favouring partly one of the Gameſters, partly one or both of the other: Wherefore theſe Terms are ſo to be divided into their parts, that the parts Favouring each Gameſter may be added to his Expectation.

Take therefore all the Terms which are common, *viz.* $6aabb + 4ab^3 + 12abcc + 4ac^3$, and divide them actually into their parts; that is 1°. $6aabb$ into $aabb$, $abab$, $abba$, $baab$, $baba$, $bbaa$. Out of theſe Six parts, one part only, *viz.* $bbaa$ will be found to Favour *B*, for 'tis only in this Term that two Dimenſions of b are placed before one ſingle Dimenſion of a, and therefore the other Five parts belong to *A*; let therefore $5aabb$ be added to the Expectation of *A*, and $1aabb$ to the Expectation of *B*. 2° Divide $4ab^3$ into its parts, $abbb$, $babb$, $bbab$, $bbba$. Of theſe parts there are two belonging to *A*, and the other two to *B*; let therefore $2ab^3$ be added to the Expectation of each. 3° Divide $12abbc$ into its parts; and eight of them will be found Favourable to *A*, and four to *B*; let therefore $8abbc$ be added to the Expectation of *A*, and $4abbc$ to the Expectation of *B*. 4° Divide $4ac^3$ into its parts, three of which will be found Favourable to *A*, and one to *C*; Add therefore $3ac^3$ to the Expectation of *A*, and $1ac^3$ to the Expectation of *C*. Hence the Numerators of the ſeveral Expectations of *A*, *B*, *C*, will be reſpectively. 1.

1. $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc + 5aabb + 2ab^3 + 8abbc + 3ac^3$.
2. $b^4 + 4b^3c + 6bbcc + 1aabb + 2ab^3 + 4abcc$.
3. $4bc^3 + 1c^4 + 1ac^3$.

The common Denominator of all their Expectations being $\overline{a+b+c}|^4$.

Therefore if a, b, c are in a proportion of equality, the Odds of winning will be respectively as 57, 18, 6.

If n be the number of all the Games that are wanting, p the number of the Gamesters, a, b, c, d, &c. the proportion of the Chances which each Gamester has respectively to win any one Game assigned; let $a+b+c+d$ &c. be raised to the Power $n+1-p$, then proceed as before.

PROBLEM IX.

TWO *Gamesters*, A *and* B, *each having* 12 *Counters, play with three Dice, on condition, that if* 11 *Points come up,* B *shall give one Counter to* A; *if* 14, A *shall give one Counter to* B; *and that he shall be the winner who shall soonest get all the Counters of his adversary: What are the Probabilities that each of them has of winning?*

SOLUTION.

LET the number of Counters which each of them have be $=p$; and let a and b be the number of Chances they have respectively for getting a Counter each cast of the Dice: I say that the Probabilities of winning are respectively as a^p to b^p; or because in this case $p=12$, $a=27$, $b=15$ as 27^{12} to 15^{12}, or as 9^{12} to 5^{12}, or as 282429536481 to 244140625; which is the proportion assigned by *M. Huygens*, but without any Demonstration:

Or more generally.

Let p be the number of the Counters of A, and q the number of the Counters of B; and let the proportion of the Chances be as a to b. I say that the proportions of the Probabilities which A has to get all the Counters of his adversary will be as $a^q \times \overline{a^p - b^p}$ to $b^p \times \overline{a^q - b^q}$.

DEMON-

DEMONSTRATION.

LET it be ſuppoſed that A has the Counters E, F, G, H &c. whoſe number is p, and that B has the Counters I, K, L &c. whoſe number is q: Moreover let it be ſuppoſed that the Counters are the thing play'd for, and that the Value of each of them is to the Value of the following as a to b, in ſuch a manner that the laſt Counter of A to the firſt Counter of B, be ſtill in that proportion. This being ſuppoſed, A and B, in every circumſtance of their Play, may lay down two ſuch Counters as may be proportional to the number of Chances each has to get a ſingle Counter; for in the beginning of the Play A may lay down the Counter H which is the loweſt of his Counters, and B the Counter I which is his higheſt; but $H, I :: a, b$, therefore A and B play upon equal Terms. If A win of B, then A may lay down the Counter I which he has juſt got of his adverſary, and B the Counter K; but $I, K :: a, b$, therefore A and B ſtill play upon equal Terms. But if A loſe the firſt time, then A may lay down the Counter G, and B the Counter H, which he but now got of his adverſary; but $G, H :: a, b$, and therefore they ſtill Play upon equal Terms as before. So that as long as they Play together, they Play without advantage, or diſadvantage, and conſequently the Probabilities of winning are reciprocal to the Sums which they expect to win, that is, are proportional to the Sums they reſpectively have before the Play begins. Whence the Probability which A has of winning all the Counters of B, is to the Probability which B has of winning all the Counters of A, as the Sum of the Terms E, F, G, H whoſe number is p, to the Sum of the Terms I, K, L whoſe number is q; that is, as $a^q \times \overline{a^p - b^p}$ to $b^p \times \overline{a^q - b^q}$: As will eaſily appear if thoſe Terms which are in Geometric Progreſſion are actually ſummed up by the known methods. Now the Probabilities of winning are not influenced by the ſuppoſition here made, of each Counter being to the following in the proportion of a to b; and therefore when thoſe Counters are ſuppoſed of equal Value, or rather of no Value, but ſerve only to mark the number of Stakes won or loſt on either ſide, the Probabilities of winning will be the ſame as we have aſſigned.

RE-

REMARK I.

IF p and q, or either of them are large numbers, 'twill be convenient to work by Logarithms.

Thus, If *A* and *B* play a Guinea a Stake, and the number of Chances which *A* has to win each ſingle Stake be 43, but the number of Chances which *B* has to win it be 40; and they oblige themſelves to play till ſuch time as 100 Stakes are won and loſt.

From the Logarithm of $43 = 1.6334685$
Subtract the Logarithm of $40 = 1.6020600$
Difference $= 0.0314085$

Multiply this Difference by the number of Stakes to be play'd off, *viz.* 100; the Product will be 3.1048500, to which anſwers, in the Tables of Logarithms, the number 1383; wherefore the Odds that *A* ſhall win before *B* are 1383 to 1.

Now in all circumſtances wherein *A* and *B* venture an equal Sum; the ſum of the numbers expreſſing the Odds, is to their difference, as the Money play'd for, is to the Gain of the one, and the Loſs of the other.

Therefore Multiplying 1382, difference of the numbers expreſſing the Odds, by 100, which is the ſum ventured by each Man, and dividing the product by 1384 ſum of the numbers expreſſing the Odds; the Quotient will be 99 Guineas, and about $18^{sh.} - 4\frac{1}{2}^{d}$, which conſequently is to be eſtimated as the Gain of *A*.

REMARK II.

IF the number of Stakes which are to be won and loſt be unequal, but the number of Chances to win and loſe be equal; the Probabilities of winning will be reciprocally proportional to the number of Stakes to be won.

Thus, If *A* ventures Ten Stakes to win One; the Odds that he wins One before he loſes Ten will be as 10 to 1.

But ten Chances to win One, and One Chance to loſe ten, makes the Play perfectly equal.

Therefore he that ventures many Stakes to win but few, has by it neither advantage nor diſadvantage.

PROBLEM X.

TWO Gameſters, A *and* B *lay by* 24 *Counters, and play with Three Dice, on this condition; that if* 11 *Points come up,* A *ſhall take one Counter out of the heap; if* 14, B *ſhall take out one, and he ſhall be reputed to win, who ſhall ſooneſt get* 12 *Counters. What are the Probabilities of their winning?*

This Problem differs from the preceding in this, that the play will be at an end in 23 Caſts of the Dice at moſt, (that is of thoſe Caſts which are favourable either to *A* or *B*:) Whereas in the preceding caſe, the Counters paſſing continually from one Hand to the other, it will often Happen that *A* and *B* will be in ſome of the ſame circumſtances they were in before, which will make the length of the play unlimited.

SOLUTION.

TAking a and b in the proportion of the Chances that there are to throw 11 and 14, let $a+b$ be raiſed to the 23*d* Power, that is to ſuch Power as is denoted by the number of all the Counters wanting one: Then ſhall the 12 firſt Terms of that Power be to the 12 laſt in the ſame proportion as are the reſpective Probabilities of winning.

PROBLEM XI.

THree Perſons A, B, C *out of a heap of* 12 *Counters, whereof four are White and Eight Black, draw blindfold one Counter at a time in this manner;* A *begins to draw;* B *follows* A; C *follows* B; *then* A *begins again; and they continue to draw in the ſame order, till one of them, who is to be reputed to win, draws the firſt White one. What are the Probabilities of their winning?*

SOLU-

SOLUTION.

LET n be the number of all the Counters, a the number of White ones, b the number of Black ones, and 1 the whole Stake or the ſum play'd for.

1° Since *A* has a Chances for a White Counter, and b Chances for a Black one, it follows that the Probability of his winning is $\frac{a}{a+b}$ or $\frac{a}{n}$; Therefore the Expectation he has upon the Stake 1 ariſing from the circumſtance he is in when he begins to draw is $\frac{a}{n} \times 1 = \frac{a}{n}$. Let it therefore be agreed amongſt the adventurers that *A* ſhall have no Chance for a White Counter, but that he ſhall be reputed to have had a Black one, which ſhall actually be taken out of the heap, and that he ſhall have the ſum $\frac{a}{n}$ paid him out of the Stake for an Equivalent. Now $\frac{a}{n}$ being taken out of the Stake, there will remain $1 - \frac{a}{n} = \frac{n-a}{n} = \frac{b}{n}$.

2° Since *B* has a Chances for a White Counter, and that the number of remaining Counters is $n-1$, his Probability of winning will be $\frac{a}{n-1}$. Whence his Expectation upon the remaining Stake $\frac{b}{n}$, ariſing from the circumſtance he is now in, will be $\frac{ab}{n \times \overline{n-1}}$. Suppoſe it therefore agreed that *B* ſhall have the ſum $\frac{ab}{n \times \overline{n-1}}$ paid him out of the Stake, and that a Black Counter be likewiſe taken out of the heap. This being done, the remaining Stake will be $\frac{b}{n} - \frac{ab}{n \times \overline{n-1}}$, or $\frac{nb-b-ab}{n \times \overline{n-1}}$; but $nb-ab=bb$; Wherefore the remaining Stake is $\frac{b \times \overline{b-1}}{n \times \overline{n-1}}$.

3° Since *C* has a Chances for a White Counter, and that the number of remaining Counters is $n-2$, his Probability of winning will be $\frac{a}{n-2}$: And therefore his Expectation upon the remaining Stake, ariſing from the circumſtance he is now in, will be $\frac{b \times \overline{b-1} \times a}{n \times \overline{n-1} \times \overline{n-2}}$ which we will likewiſe ſuppoſe to be paid him out of the Stake.

4° *A* may have out of the remainder $\frac{b \times \overline{b-1} \times \overline{b-2} \times a}{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}$; and ſo of the reſt till the whole Stake be exhauſted.

There-

Therefore having written the following general Series, *viz.* $\frac{a}{n}+\frac{b}{n-1}P+\frac{b-1}{n-2}Q+\frac{b-2}{n-3}R+\frac{b-3}{n-4}S$ &c. wherein P, Q, R, S &c. denote the preceding Terms, take as many Terms of this Series as there are Units in $b+1$, (for ſince b repreſents the number of Black Counters, the number of drawings cannot exceed $b+1$) then take for A the firſt, fourth, ſeventh &c. Terms; for B take the ſecond, fifth, eighth &c. Terms; for C the third, ſixth &c. and the ſums of thoſe Terms will be the reſpective Expectations of A, B, C; or becauſe the Stake is fix'd, theſe ſums will be proportional to their reſpective Probabilities of winning.

Now to apply this to the preſent caſe, make $n=12$, $a=4$, $b=8$, and the general Series will become

$$\frac{4}{12}+\frac{8}{11}P+\frac{7}{10}Q+\frac{6}{9}R+\frac{5}{8}S+\frac{4}{7}T+\frac{3}{6}V+\frac{2}{5}X$$

$+\frac{1}{4}Y$: Or multiplying the whole by 495, to take away the Fractions, the Series will be

$165+120+84+56+35+20+10+4+1$.

Therefore aſſign to A $165+56+10=231$; to B $120+35+4=159$; to C $84+20+1=105$, and their Probabilities of winning will be as 231, 159, 105, or as 77, 53, 35.

If there be never ſo many Gameſters A, B, C, D &c. whether they take every one of them one Counter or more; or whether the ſame or a different number of Counters; the Probabilities of winning may be determined by the ſame general Series.

REMARK I.

THE preceding Series may in any particular caſe be ſhorten'd; for if a is $=1$, then the Series will be

$\frac{1}{n}\times\overline{1+1+1+1+1+1+1}$ &c.

Hence it may be obſerved, that if the whole number of Counters be exactly diviſible by the number of perſons concerned in the Play, and that there be but one ſingle White Counter in the whole, there will be no advantage or diſadvantage to any one of them from the ſituation he is in, in reſpect to the order of drawing.

If

If $a = 2$. then the Series will be

$$\frac{2}{n \times \overline{n-1}} \times \overline{n-1} + \overline{n-2} + \overline{n-3} + \overline{n-4} + \overline{n-5} \text{ \&c.}$$

If $a = 3$. then the Series will be

$$\frac{3}{n \times \overline{n-1} \times \overline{n-2}} \times \overline{n-1} \times \overline{n-2} + \overline{n-2} \times \overline{n-3} + \overline{n-3} \times \overline{n-4} \text{ \&c.}$$

If $a = 4$. then the Series will be

$$\frac{4}{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}} \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \text{ \&c.}$$

Wherefore rejecting the common Multiplicators; the several Terms of these Series taken in due order will be Proportional to the several Expectations of any number of Gamesters. Thus in the case of this Problem where n is $= 12$, and $a = b$; the Terms of the Series will be

For *A*.	For *B*.	For *C*.
$11 \times 10 \times 9 = 990$	$10 \times 9 \times 8 = 720$	$9 \times 8 \times 7 = 504$
$8 \times 7 \times 6 = 336$	$7 \times 6 \times 5 = 210$	$6 \times 5 \times 4 = 120$
$5 \times 4 \times 3 = 60$	$4 \times 3 \times 2 = 24$	$3 \times 2 \times 1 = 6$
1386	954	630

Hence it follows, that the Probabilities of winning will be respectively as 1386, 954, 630; or dividing all by 18, as 77, 53, 35, as had been before determined.

REMARK II.

BUT if the Terms of the Series are many, it will be convenient to sum them up, by means of the following method, whose Demonstration may be had from the *Methodus Differentialis* of Sir *Isaac Newton*, printed in his *Analysis*.

Subtract every Term, but the first, from every following Term, and let the remainders be called *first Differences*; subtract in like manner every first difference from the following, and let the remainders be called *second Differences*; subtract again every one of these second differences from that which follows, and call the remainders *third Differences*; and so on, till the last differences become equal. Let the first Term be called a, the second b; the first of the first differences d', the firſt of the ſecond differences d'', the firſt of the third differences d''' &c. and let the number of Terms which follow the firſt be x, then will the ſum of all thoſe Terms be

$$a + xb + \frac{x}{1} \times \frac{x-1}{2} d' + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} d'' + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} d''' \text{ \&c.}$$

N. B. If the numbers whoſe ſums are to be taken are the Products of two numbers, the ſecond differences will be equal; if they are the Products of three, the third differences will be equal, and ſo on. Therefore the number of Terms, which are to be taken after the firſt, is to exceed only by Unity the number of Factors that enter the compoſition of every Term.

It may alſo be obſerved, that if thoſe numbers are decreaſing, it will be convenient to invert their order, and make that the firſt which was the laſt.

Thus, ſuppoſing the number of all the Counters to be 100, and the number of White ones 4: Then the number of all the Terms belonging to *A*, *B*, *C* will be 97, the laſt of which $3 \times 2 \times 1$ will belong to *A*, ſince 97 being divided by 3, the remainder is 1. Therefore beginning from the loweſt Term $3 \times 2 \times 1$, and taking every third Term, as alſo the differences of thoſe Terms, we ſhall have the following Scheme

$3 \times 2 \times 1 = 6$			
$6 \times 5 \times 4 = 120$			
	384		
$9 \times 8 \times 7 = 504$		432	
	816		162
$12 \times 11 \times 10 = 1320$		594	
	1410		
$15 \times 14 \times 13 = 2730$			
&c.			

From whence the Values of a, b, d', d'', d''', in the general Theorem, will be found to be reſpectively 6, 120, 384, 432, 162; and conſequently the ſum of all thoſe Terms will be

6 +

$$6 + x \times 120 + \frac{x}{1} \times \frac{x-1}{2} \times 384 + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times 432$$
$$+ \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times 162, \text{ or}$$
$$6 + 31\tfrac{1}{2}x + 50\tfrac{1}{4}xx + 31\tfrac{1}{2}x^3 + 6\tfrac{3}{4}x^4, \text{ or}$$
$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{3x+1} \times \overline{3x+4}.$$

In like manner it will be found, that the ſum of all the Terms which belong to *B*, the laſt of which is $5 \times 4 \times 3$ is

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{3x+5} \times \overline{3x+8}.$$

And alſo that the ſum of all the Terms belonging to *C*, the laſt of which is $4 \times 3 \times 2$, is

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{9xx+27x+16}.$$

Now x in each caſe repreſents the number of Terms wanting one, which belong ſeverally to *A*, *B*, *C*; wherefore making $x+1=p$, their ſeveral Expectations will be reſpectively proportional to

$$p \times \overline{p+1} \times \overline{3p-2} \times \overline{3p+1}$$
$$p \times \overline{p+1} \times \overline{3p+2} \times \overline{3p+5}$$
$$p \times \overline{p+1} \times \overline{9pp+9p-2}.$$

Again, the number of all the Terms which belong to them all being 97, and *A* being to take firſt, it follows, that p in the firſt caſe is $= 33$, in the other two $= 32$.

Therefore the ſeveral Expectations of *A*, *B*, *C* will be reſpectively proportional to 41225, 39592, 38008.

If the number of all the Counters were 500, and the number of the White ones ſtill 4; then the number of all the Terms repreſenting the Expectations of *A*, *B*, *C* would be 497. Now this number being divided by 3, the Quotient is 165, and the Remainder 2: From whence it follows, that the laſt Term $3 \times 2 \times 1$ will belong to *B*, the laſt but one $4 \times 3 \times 2$ to *A*, and the laſt but two to *C*; it follows alſo, that for *B*, and *A*, p muſt be interpreted by 166, but for *C* by 165.

The

The GAME of BASSETE.

RULES of the PLAY.

THE Dealer, otherwife called the *Banker*, holds a Pack of 52 Cards, and having fhuffled them, he turns the whole Pack at once, fo as to difcover the laft Card; after which he lays down by Couples all the Cards.

The Setter, otherwife called the *Ponte*, has 13 Cards in his hand, one of every fort, from the King to the Ace, which 13 Cards are called a *Book*; out of this Book he takes one Card or more at pleafure, upon which he lays a Stake.

The Ponte may at his choice, either lay down his Stake before the Pack is turned, or immediately after it is turned; or after any number of Couples are drawn.

The firft cafe being particular fhall be calculated by it felf; but the other two are comprehended under the fame Rules.

Suppofing the Ponte to lay down his Stake after the Pack is turned, I call 1, 2, 3, 4, 5 &c. the places of thofe Cards which follow the Card in view, either immediately after the Pack is turned, or after any number of Couples are drawn.

If the Card upon which the Ponte has laid a Stake comes out in any odd place, except the firft, he wins a Stake equal to his own.

If the Card upon which the Ponte has laid a Stake comes out in any even place except the fecond, he lofes his Stake.

If the Card of the Ponte comes out in the firft place, he neither wins nor lofes, but takes his own Stake again.

If the Card of the Ponte comes out in the fecond place, he does not lofe his whole Stake, but only a part of it, *viz*, a half; which to make the calculation more general we will call y. In this cafe the Ponte is faid to be *Faced*.

When the Ponte chufes to come in after any number of Couples are down; if his Card happens to be but once in the Pack, and is the very laft of all, there is an exception from the general Rule: for tho' it comes out in an odd place which fhould intitle him to win a Stake equal to his own, yet he neither wins nor lofes from that circumftance, but takes back his own Stake.

PRO-

PROBLEM XII.

TO *Eſtimate at* Baſſete *the loſs of the* Ponte *under any circumſtances of Cards remaining in the Stock, when he lays his Stake, and of any number of times that his Card is repeated in it.*

The Solution of this Problem containing Four Caſes, *viz.* of the Ponts Card being once, twice, three or four times in the Stock; we will give the Solution of all theſe Caſes ſeverally.

SOLUTION of the firſt Caſe.

THe *Ponte* has the following Chances to win or loſe, according to the place his Card is in.

1	1 Chance for winning	0
2	1 Chance for loſing	y
3	1 Chance for winning	1
4	1 Chance for loſing	1
5	1 Chance for winning	1
6	1 Chance for loſing	1
*	1 Chance for winning	0

It appears by this Scheme that he has as many Chances to win 1 as to loſe 1, and that there are two Chances for neither winning nor loſing, *viz.* the firſt and laſt, and therefore that his only Loſs is upon account of his being *Faced*: From which 'tis plain that the number of Cards covered by that which is in view being called n, his Loſs will be $\frac{y}{n}$, or $\frac{1}{2n}$ ſuppoſing $y=\frac{1}{2}$.

SOLUTION of the ſecond Caſe.

By the firſt Remark belonging to the XI*th* Problem it appears that the Chances which the Ponte has to win or loſe are proportional to the numbers, $n-1$, $n-2$, $n-3$ &c. Therefore his Chances for winning and loſing may be expreſſed by the following Scheme.

1	$n-1$	Chances for winning	0
2	$n-2$	Chances for losing	y
3	$n-3$	Chances for winning	1
4	$n-4$	Chances for losing	1
5	$n-5$	Chances for winning	1
6	$n-6$	Chances for losing	1
7	$n-7$	Chances for winning	1
8	$n-8$	Chances for losing	1
9	$n-9$	Chances for winning	1
*	1	Chance for losing	1

Now setting aside the first and second number of Chances, it will be found that the difference between the 3*d* and 4*th* is $=1$, and that the difference between the 5*th* and 6*th* is $=1$. The difference between the 7*th* and 8*th* also is $=1$, and so on. But the number of differences is $\frac{n-3}{2}$, and the sum of all the Chances is $\frac{n}{1}\times\frac{n-1}{2}$. Wherefore the Gain of the Ponte is $\frac{n-3}{n\times\overline{n-1}}$; but his Loss upon account of the *Face* is $\overline{n-2}\times y$ divided by $\frac{n}{1}\times\frac{n-1}{2}$, or $\frac{\overline{2n-4}\times y}{n\times\overline{n-1}}$: Hence it may be concluded that his Loss upon the whole is

$\frac{\overline{2n-4}\times y-\overline{n-3}}{n\times\overline{n-1}}$, or $\frac{1}{n\times\overline{n-1}}$ supposing $y=\frac{1}{2}$.

That the number of Differences is $\frac{n-3}{2}$ will be made evident from two considerations.

First, the Series $n-3$, $n-4$, $n-5$ &c. decreases in Arithmetic Progression, the difference of its Terms being Unity, and the last Term also Unity, therefore the number of its Terms is equal to the first Term $n-3$: But the number of Differences is one half of the number of Terms, therefore the number of Differences will be $\frac{n-3}{2}$.

Secondly, It appears by the XI*th* Problem, that the number of all the Terms including the two first is always $b+1$; But b in this case is $=2$. Therefore the number of all the Terms is $n-1$, from which excluding the two first, the number of remaining Terms will be $n-3$, and consequently the number of Differences will be $\frac{n-3}{2}$.

That the sum of all the Terms is $\frac{n}{1}\times\frac{n-1}{2}$, is evident also from two different considerations. First,

First, In any Arithmetic Progreſſion whereof the firſt Term is $n-1$, the difference Unity, and the laſt Term alſo Unity, the ſum of the Progreſſion will be $\frac{n}{1}\times\frac{n-1}{2}$.

Secondly, the Series $\frac{2}{n\times n-1}\times\overline{n-1+n-2+n-3}$ &c. belonging to the preceding Problem, expreſſes the ſum of the Probabilities of winning, which belong to the ſeveral Gameſters in the caſe of two White Counters, when the number of all the Counters is n. It therefore expreſſes likewiſe the ſum of the Probabilities of winning, which belong to the Ponte or Banker in the preſent caſe: But this ſum muſt always be equal to Unity, it being a certainty that the Ponte or Banker muſt win; ſuppoſing therefore that $n-1$, $n-2$, $n-3$ &c. is $= S$. we ſhall have the Equation $\frac{2S}{n\times n-1}=1$. Therefore $S=\frac{n}{1}\times\frac{n-1}{2}$.

SOLUTION of the third Caſe.

By the firſt Remark of the XI*th* Problem it appears that the Chances which the Ponte has to win and loſe, may be expreſſed by the following Scheme.

1	$n-1\times n-2$	for winning	o
2	$n-2\times n-3$	for loſing	y
3	$n-3\times n-4$	for winning	1
4	$n-4\times n-5$	for loſing	1
5	$n-5\times n-6$	for winning	1
6	$n-6\times n-7$	for loſing	1
7	$n-7\times n-8$	for winning	1
8	$n-8\times n-9$	for loſing	1
*	2×1	for winning	1

Setting aſide the firſt, ſecond and laſt number of Chances; it will be found that the difference between the 3*d* and 4*th* is $2n-8$; the difference between the 5*th* and 6*th* $2n-12$, the difference between the 7*th* and 8*th* $2n-16$. &c. Now theſe differences conſtitute an Arithmetic Progreſſion, whereof the firſt Term is $2n-8$, the common difference 4, and the laſt Term 6, being the difference between 4×3 and 3×2. Wherefore the ſum of this Progreſſion is $\frac{n-3}{1}\times\frac{n-5}{2}$, to which adding the laſt Term 2×1, which is favourable to the

Ponte

Ponte, the ſum total will be $\frac{n-3}{1}\times\frac{n-3}{2}$. But the ſum of all the Chances is $\frac{n}{1}\times\frac{n-1}{1}\times\frac{n-3}{3}$; as may be concluded from the firſt Remark of the preceding Problem: Therefore the Gain of the Ponte is $\frac{3\times\overline{n-3}\times\overline{n-3}}{2\times n\times\overline{n-1}\times\overline{n-2}}$. But his Loſs upon account of the Face is $\frac{6\times\overline{n-2}\times\overline{n-3}\times y}{2\times n\times\overline{n-1}\times\overline{n-2}}$. Conſequently his Loſs upon the whole will be $\frac{6\times\overline{n-2}\times\overline{n-3}\times y-3\times\overline{n-3}\times\overline{n-3}}{2\times n\times\overline{n-1}\times\overline{n-2}}$ or $\frac{3n-9}{2\times n\times\overline{n-1}\times\overline{n-2}}$, Suppoſing $y=\frac{1}{2}$.

SOLUTION of the fourth Caſe.

The Chances of the Ponte may be expreſſed by the following Scheme.

1	$\overline{n-1}\times\overline{n-2}\times\overline{n-3}$	for winning	0
2	$\overline{n-2}\times\overline{n-3}\times\overline{n-4}$	for loſing	y
3	$\overline{n-3}\times\overline{n-4}\times\overline{n-5}$	for winning	1
4	$\overline{n-4}\times\overline{n-5}\times\overline{n-6}$	for loſing	1
5	$\overline{n-5}\times\overline{n-6}\times\overline{n-7}$	for winning	1
6	$\overline{n-6}\times\overline{n-7}\times\overline{n-8}$	for loſing	1
7	$\overline{n-7}\times\overline{n-8}\times\overline{n-9}$	for winning	1
*	$3\times\quad 2\times\quad 1$	for loſing	1

Setting aſide the firſt and ſecond numbers of Chances, and taking the differences between the 3*d* and 4*th*, 5*th* and 6*th*; 7*th* and 8*th*, the laſt of theſe differences will be found to be 18. Now if the number of theſe differences be p, and we begin from the laſt 18, their ſum, from the ſecond Remark of the preceding Problem, will be collected to be $p\times\overline{p+1}\times\overline{4p+5}$: And the number p in this caſe being $\frac{n-5}{2}$, the ſum of theſe differences will be $\frac{n-5}{2}\times\frac{n-3}{2}\times\frac{2n-5}{1}$. But the ſum of all the Chances is $\frac{n}{1}\times\frac{n-1}{1}\times\frac{n-2}{1}\times\frac{n-3}{4}$; wherefore the Gain of the Ponte is $\frac{\overline{n-5}\times\overline{n-3}\times\overline{2n-5}}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$; now his Loſs upon account of the Face is $\frac{\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times 4y}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$ and therefore his Loſs upon the whole is $\frac{4\times\overline{n-2}\times\overline{n-4}\times y-\overline{n-5}\times\overline{2n-5}}{n\times\overline{n-1}\times\overline{n-2}}$ or $\frac{3n-9}{n\times\overline{n-1}\times\overline{n-2}}$, making $y=\frac{1}{2}$.

There

There ſtill remains the ſingle Caſe to be conſidered, *viz.* what the Loſs of the Ponte is, when he lays a Stake before the Pack is turned up; but there will be no difficulty in it after what we have ſaid, the difference between this Caſe and the reſt being only that he may be Faced by the firſt Card diſcovered, which will make his Loſs to be $\frac{3n-6}{n \times \overline{n-1} \times \overline{n-3}}$, that is, about $\frac{1}{866}$ part of his Stake.

Thoſe who are deſirous to try, by a kind of Mechanical Operation, the truth of the Rules which have been given for determining the Loſs of the Ponte in any Caſe, may do it in the following manner. Suppoſe for Inſtance it were required to find the Loſs of the Ponte when his Card is twice in the Stock, and there are five Cards remaining in the hands of the Banker beſide the Card in View. Let them be diſpoſed according to this Scheme.

1,	2,	3,	4,	5
*	*	·	·	·
*	·	*	·	·
*	·	·	*	·
*	·	·	·	*
·	*	*	·	·
·	*	·	*	·
·	*	·	·	*
·	·	*	*	·
·	·	*	·	*
·	·	·	*	*

Where the places filled with Aſteriſcs ſhew all the Various Poſitions which the Ponte's Card may obtain; it is evident that the Ponte has four Chances for neither winning nor loſing, three Chances for the Face or for loſing $\frac{1}{2}$, two Chances for winning 1, and one Chance for loſing 1; and conſequently that his Loſs is $\frac{1}{2}$ to be diſtributed into 10 parts, the number of all the Chances being 10, which will make his Loſs to be $\frac{1}{20}$. Likewiſe if the number of Cards that are covered by the firſt were ſeven, it would be found that the Ponte would have ſix Chances for neither winning nor loſing, five Chances for the Face, four Chances for winning 1, three Chances for loſing 1, two Chances again for winning 1, and one Chance for loſing 1, which would make his Loſs to be $\frac{1}{42}$. And the like may be done for any other Caſe whatſoever.

From what has been ſaid, a Table may eaſily be compoſed, ſhewing the ſeveral Loſſes of the Ponte in whatever circumſtance he may happen to be.

A TABLE for *BASSETE.*

N	1	2	3	4
52	* * *	* * *	* * *	866
51	* * *	* * *	1735	867
49	98	2352	1602	801
47	94	2162	1474	737
45	90	1980	1351	675
43	86	1806	1234	617
41	82	1640	1122	561
39	78	1482	1015	507
37	74	1332	914	457
35	70	1190	818	409
33	66	1056	727	363
31	62	930	642	321
29	58	812	562	281
27	54	702	487	243
25	50	600	418	209
23	46	506	354	177
21	42	420	295	147
19	38	342	242	121
17	34	272	194	97
15	30	210	151	75
13	26	156	114	57
11	22	110	82	41
9	18	72	56	28
7	14	42	35	17

The Uſe of this Table will be beſt explained by one or two Examples.

Exam-

EXAMPLE I.

LET it be propoſed to find the Loſs of the Ponte when there are 26 Cards remaining in the Stock, and his Card is twice in it.

In the Column *N* find the number 25, which is leſs by one than the number of Cards remaining in the Stock: Over againſt it, and under the number 2, which is at the head of the ſecond Column, you will find 600; which is the Denominator of a Fraction whoſe Numerator is Unity, and which ſhews that his Loſs in that circumſtance is one part in ſix hundred of his Stake.

EXAMPLE II.

TO find the Loſs of the Ponte when there is eight Cards remaining in the Stock, and his Card is three times in it.

In the Column *N* find the number 7, leſs by one than the number of Cards remaining in the Stock : Over againſt 7, and under the number 3 in the third Column, you will find 35; which denotes that his Loſs is one part in thirty five of his Stake.

Corollary I. 'Tis plain from the conſtruction of the Table, that the fewer Cards are in the Stock, the greater is the Loſs of the Ponte.

Corollary II. The leaſt Loſs of the Ponte, under the ſame circumſtances of Cards remaining in the Stock, is when his Card is but twice in it; the next greater when three times; ſtill greater when four times, but his greateſt Loſs when 'tis but once.

If the Loſs upon the Face were varied, 'tis plain that in all the like circumſtances, the Loſs of the Ponte would vary accordingly, but it would be eaſie to compoſe other Tables to anſwer that Variation, ſince the quantity y, which has been aſſumed to repreſent that Loſs may be interpreted at pleaſure. For inſtance, when the Loſs upon the Face is $\frac{1}{2}$, it has been found in the Caſe of 7 Cards covered remaining in the Stock, and the Card of the Ponte being twice in it, that his Loſs would be $\frac{1}{42}$, but upon ſuppoſition of its being $\frac{2}{3}$, it will be found to be $\frac{4}{63}$.

The

The GAME of PHARAON.

THE Calculation for *Pharaon* is much like the preceding, the reaſonings about it being the ſame; therefore I think it will be ſufficient to lay down the Rules of the Play, and the Scheme of the Calculation.

RULES of the PLAY.

Firſt, The Banker holds a Pack of 52 Cards.

Secondly, He draws the Cards one after the other, laying them alternately to his right and left hand.

Thirdly, the Ponte may at his choice ſet one or more Stakes upon one or more Cards either before the Banker has begun to draw the Cards, or after he has drawn any number of Couples, which are commonly called *Pulls*.

Fourthly, The Banker wins the Stake of the Ponte, when the Card of the Ponte comes out in an odd place on his right hand; but loſes as much to the Ponte when it comes out in an even place on his left hand.

Fifthly, The Banker wins half the Ponte's Stake, when in the ſame Pull the Card of the Ponte comes out twice.

Sixthly, When the Card of the Ponte, being but once in the Stock, happens to be the laſt, the Ponte neither wins nor loſes.

Seventhly, The Card of the Ponte being but twice in the Stock, and the two laſt Cards happening to be his Cards, he then loſes his whole Stake.

PROBLEM XIII.

TO *Find at* Pharaon *the Gain of the Banker, in any Circumſtance of Cards remaining in the Stock, and of the number of times that the Ponte's Card is contained in it.*

This Problem, containing four Caſes, that is, when the Card of the Ponte is once, twice, three or four times in the Stock; we ſhall give the Solution of theſe four Caſes ſeverally.

SOLU-

SOLUTION of the firſt Caſe.

The Banker has the following number of Chances for winning and loſing, *viz.*

1	1	Chance for winning	1
2	1	Chance for loſing	1
3	1	Chance for winning	1
4	1	Chance for loſing	1
5	1	Chance for winning	1
*	1	Chance for loſing	0

Therefore the Gain of the Banker is $\frac{1}{n}$. Suppoſing n to be the number of Cards in the Stock.

SOLUTION of the ſecond Caſe.

The Banker has the following Chances for winning and loſing, *viz.*

1	$n-2$	Chances for winning	1
	1	Chance for winning	y
2	$n-2$	Chances for loſing	1
3	$n-4$	Chances for winning	1
	1	Chance for winning	y
4	$n-4$	Chances for loſing	1
5	$n-6$	Chances for winning	1
	1	Chance for winning	y
6	$n-6$	Chances for loſing	1
7	$n-8$	Chances for winning	1
	1	Chance for winning	y
8	$n-8$	Chances for loſing	1
*	1	Chance for winning	1

Therefore the Gain of the Banker is $\frac{\overline{n-2}\times y+2}{n\times\overline{n-1}}$, or $\frac{\frac{1}{2}n+1}{n\times\overline{n-1}}$ ſuppoſing $y=\frac{1}{2}$.

The only thing that deſerves to be explained here, is this; how it comes to paſs that whereas at *Baſſete* the firſt number of Chances for winning was repreſented by $n-1$, here 'tis repreſented by $n-2$. To anſwer this it muſt be remember'd, that according to the Law of this Play, if the Ponte's Card comes out in an odd place, the Banker is not thereby entitled to the Ponte's whole Stake: For if it ſo happens that his Card comes out again immediately after, the Banker wins but one half of it. Therefore the number $n-1$ is divided into two parts $n-2$ and 1, whereof the firſt is proportional to the Probability which the Banker has for winning the whole Stake of the Ponte; and the ſecond is proportional to the Probability of his winning the half of it.

SOLUTION of the third Caſe.

The number of Chances which the Banker has for winning, and loſing are as follow;

1	$\overline{n-2} \times \overline{n-3}$	Chances for winning	1
	$2 \times \overline{n-2}$	Chances for winning	y
2	$\overline{n-2} \times \overline{n-3}$	Chances for loſing	1
3	$\overline{n-4} \times \overline{n-5}$	Chances for winning	1
	$2 \times \overline{n-4}$	Chances for winning	y
4	$\overline{n-4} \times \overline{n-5}$	Chances for loſing	1
5	$\overline{n-6} \times \overline{n-7}$	Chances for winning	1
	$2 \times \overline{n-6}$	Chances for winning	y
6	$\overline{n-6} \times \overline{n-7}$	Chances for loſing	1
7	$\overline{n-8} \times \overline{n-9}$	Chances for winning	1
	$2 \times \overline{n-8}$	Chances for winning	y
*	2×1	Chances for loſing	1

Therefore the Gain of the Banker is $\frac{3y}{2 \times \overline{n-1}}$, or $\frac{3}{4 \times \overline{n-1}}$ ſuppoſing $y = \frac{1}{2}$.

The number of Chances for the Banker to win is divided into two parts, whereof the firſt expreſſes the Chances he has for winning the whole Stake of the Ponte, and the ſecond for winning the half thereof.

Now

Now for determining exactly these two parts, it may be considered, that in the first Pull the number of Chances for the first Card to be the Ponte's is $n-1 \times n-2$; also that the number of Chances for the second to be the Ponte's but not the first, is $n-2 \times n-3$: Wherefore the number of Chances for the first to be the Ponte's and not the second, is likewise $n-2 \times n-3$. Hence it follows, that if from the number of Chances for the first Card to be the Ponte's, *viz.* from $n-1 \times n-2$ there be subtracted the number of Chances for the first to be the Ponte's and not the second, *viz.* $n-2 \times n-3$, there will remain the number of Chances for both first and second Cards to be the Ponte's, *viz.* $2 \times n-2$ and so for the rest.

SOLUTION of the fourth Case.

The number of Chances which the Banker has for winning and losing, are as follows;

1	$n-2 \times n-3 \times n-4$	Chances for winning	1
	$3 \times n-2 \times n-3$	Chances for winning	y
2	$n-2 \times n-3 \times n-4$	Chances for losing	1
3	$n-4 \times n-5 \times n-6$	Chances for winning	1
	$3 \times n-4 \times n-5$	Chances for winning	y
4	$n-4 \times n-5 \times n-6$	Chances for losing	1
5	$n-6 \times n-7 \times n-8$	Chances for winning	1
	$3 \times n-6 \times n-7$	Chances for winning	y
6	$n-6 \times n-7 \times n-8$	Chances for losing	1
7	$n-8 \times n-9 \times n-10$	Chances for winning	1
	$3 \times n-8 \times n-9$	Chances for winning	y
8	$n-8 \times n-9 \times n-10$	Chances for losing	1
*	$2 \times 1 \times 0$	Chances for winning	1
	$3 \times 2 \times 1$	Chances for winning	1
	$2 \times 1 \times 0$	Chances for losing	1

Therefore the Gain of the Banker is $\frac{2n-5}{n-1 \times n-3} y$, or $\frac{2n-5}{2 \times n-1 \times n-3}$ supposing $y = \frac{1}{2}$.

A TA-

A TABLE for *PHARAON.*

N	1	2	3	4
52	* * *	* * *	* * *	50
50	* * *	94	65	48
48	48	90	62	46
46	46	86	60	44
44	44	82	57	42
42	42	78	54	40
40	40	74	52	38
38	38	70	49	36
36	36	66	46	34
34	34	62	44	32
32	32	58	41	30
30	30	54	38	28
28	28	50	36	26
26	26	46	33	24
24	24	42	30	22
22	22	38	28	20
20	20	34	25	18
18	18	30	22	16
16	16	26	20	14
14	14	22	17	12
12	12	18	14	10
10	10	14	12	8
8	8	11	9	6

The numbers of the foregoing Table, as well as those of the Table for *Baſſete*, are ſufficiently exact to give at firſt view an Idea of the advantage of the Banker in all circumſtances: But if an abſolute degree of exactneſs be required, it will be eaſily obtained from the Rules given at the end of each Caſe.

PRO-

PROBLEM XIV.

IF A, B, C *throw in their turns a regular Ball, having four White Faces and eight Black ones; and he be to be reputed to win who ſhall firſt bring up one of the White Faces: It is demanded what is the proportion of their reſpective Probabilities of winning?*

SOLUTION.

THE method of reaſoning in this Problem is exactly the ſame with that which we made uſe of in the Solution of the XI*th* Problem: But whereas the different throws of the Ball do not diminiſh the number of its Faces; in the room of the Quantities $b-1$, $b-2$, $b-3$ &c. $n-1$, $n-2$, $n-3$ &c. employed in the Solution of the aforeſaid Problem, we muſt ſubſtitute b and n reſpectively, and the Series belonging to that Problem will be changed into the following, *viz.*

$$\frac{a}{n} + \frac{ab}{nn} + \frac{abb}{n^3} + \frac{ab^3}{n^4} + \frac{ab^4}{n^5} + \frac{ab^5}{n^6} \text{ \&c.}$$

which is to be continued infinitely: Then taking every third Term thereof, the reſpective Expectations of A, B, C will be expreſſed by the three following Series.

$$\frac{a}{n} + \frac{ab^3}{n^4} + \frac{ab^6}{n^7} + \frac{ab^9}{n^{10}} + \frac{ab^{12}}{n^{13}} \text{ \&c.}$$
$$\frac{ab}{nn} + \frac{ab^4}{n^5} + \frac{ab^7}{n^8} + \frac{ab^{10}}{n^{11}} + \frac{ab^{13}}{n^{14}} \text{ \&c.}$$
$$\frac{abb}{n^3} + \frac{ab^5}{n^6} + \frac{ab^8}{n^9} + \frac{ab^{11}}{n^{12}} + \frac{ab^{14}}{n^{15}} \text{ \&c.}$$

But the Terms of which each Series is compounded are in Geometric Progreſſion, and the Ratio of each Term to the following the ſame in each of them; Wherefore the Sums of theſe Series are in the ſame proportion as their firſt Terms, *viz.* as $\frac{a}{n}$, $\frac{ab}{nn}$, $\frac{abb}{n^3}$ or as nn, bn, bb; that is, in the preſent Caſe, as 144, 96, 64, or as 9, 6, 4. Hence the reſpective Probabilities of winning will be likewiſe as the numbers 9, 6, 4.

Corollary I. If there be any other number of Gameſters A, B, C, D &c. playing on the ſame conditions as above;

take as many Terms in the Ratio of n to b as there are Gamesters, and those Terms will respectively denote the several Expectations of each Gamester.

Corollary II. If there be any number of Gamesters A, B, C, D &c. playing on the same conditions as above; with this difference only, that all the Faces of the Ball are mark'd by particular Figures, 1, 2, 3, 4 &c. and that a certain number p of those Faces shall intitle A to be the winner; and that likewise any other number of them, as q, r, s, t &c. shall respectively intitle B, C, D, E &c. to be winners: Make $n-p=a$, $n-q=b$, $n-r=c$, $n-s=d$, $n-t=e$ &c. then in the following Series,

$$\frac{p}{n} + \frac{qa}{nn} + \frac{rab}{n^3} + \frac{sabc}{n^4} + \frac{tabcd}{n^5} \text{ &c.}$$

the Terms taken in due order shall represent the several Probabilities of winning.

For if the Law of the Play be such, that every Man having once play'd in his turn, shall begin again regularly in the same manner, and that continually till such time as one of them wins: Then take as many Terms of the Series as there are Gamesters, and those Terms shall represent the respective Probabilities of winning.

And if it were the Law of the Play, that every Man should play several times together, for instance twice: Then taking for A the two first Terms, for B the two following, and so on; each Couple of Terms shall represent their respective Probabilities of winning; observing that now p and q are equal, as also r and s.

But if the Law of the Play should be Irregular, then you must take for each Man as many Terms of the Series as will answer that Irregularity, and continue the Series till such time as it gives a sufficient Approximation.

Yet, if at any time the Law of the Play having been Irregular should afterwards recover its Regularity, the Probabilities of winning will (with the help of this Series) be determined by finite expressions.

Thus, if it should be the Law of the Play, that two Men A and B, having play'd irregularly for ten times together, should afterwards play alternately each in his turn: Distribute the ten first Terms of the Series between them, according

ing to their order of playing; and having ſubtracted the ſum of thoſe Terms from Unity, divide the remainder of it between them, in the proportion of the two following Terms, which add reſpectively to the ſhares they had before: Then ſhall the two parts of Unity which *A* and *B* have thus obtained, be proportional to their reſpective Probabilities of winning.

Of Permutations and Combinations.

Permutations are the Changes which ſeveral things can receive in the different Orders in which they may be placed, being conſidered as taken two and two, three and three, four and four, &c.

Combinations are the various Conjunctions which ſeveral things may receive without any reſpect to Order, being taken two and two, three and three, four and four, &c.

LEMMA.

If the Probability that an Event ſhall Happen be $\frac{1}{r}$, *and if that Event being ſuppoſed to have Happened, the Probability of anothers Happening be* $\frac{1}{s}$; *the Probability of both Happening will be* $\frac{1}{r} \times \frac{1}{s}$ or $\frac{1}{rs}$. *This having been already Demonſtrated in the Introduction, will not require any farther proof.*

PROBLEM XV.

ANY number of Things a, b, c, d, e, f *being given, out of which Two are taken as it happens: To find the Probability that any one of them, as* a, *ſhall be the firſt taken, and any other, as* b, *the ſecond.*

SOLUTION.

THE number of Things in this Example being Six, it follows that the Probability of taking *a* in the firſt place is $\frac{1}{6}$: Let *a* be conſidered as taken, then the Probability of taking *b* will be $\frac{1}{5}$; wherefore the Probability of taking firſt *a* and then *b* is $\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$.

Corol-

Corollary. Since the taking of a in the first place and b in the second, is but one single Case of those by which Six Things may change their Order, being taken two and two; it follows, that the number of Changes or Permutations of Six Things taken two and two must be 30.

Generally, Let n be any number of Things; the Probability of taking a in the first place and b in the second, will be $\frac{1}{n \times n-1}$; and the number of Permutations of those Things taken two by two will be $n \times n-1$.

PROBLEM XVI.

A*NY number* n *of Things* a, b, c, d, e, f *being given, out of which Three are taken as it Happens: To find the Probability that* a *shall be the first taken,* b *the second and* c *the third.*

SOLUTION.

THe Probability of taking a in the first place is $\frac{1}{6}$: Let a be considered as taken; the Probability of taking b will be $\frac{1}{5}$: Suppose both a and b taken, the Probability of taking c will be $\frac{1}{4}$. Wherefore the Probability of taking first a, then b, and thirdly c, will be $\frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{120}$.

Corollary. Since the taking of a in the first place, b in the second, and c in the third, is but one single Case of those by which Six Things may change their Order, being taken three and three; it follows, that the number of Changes or Permutations of Six Things, taken three and three, must be $6 \times 5 \times 4 = 120$.

Generally, If n be any number of Things; the Probability of taking a in the first place, b in the second and c in the third will be $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$. And the number of Permutations of three Things will be $n \times n-1 \times n-2$.

General *COROLLARY.*

The number of Permutations of n Things, out of which as many are taken together as there are Units in p, will be $n \times n-1 \times n-2 \times n-3$, &c. continued to so many Terms as there are Units in p.

Thus,

Thus, the number of Permutations of Six Things taken four and four, will be $6\times5\times4\times3 = 360$. Likewiſe the number of Permutations of Six Things taken all together will be $6\times5\times4\times3\times2\times1 = 720$.

PROBLEM XVII.

TO *Find the Probability that any number of Things, whereof ſome are repeated ſeveral times, ſhall all be taken in any Order propoſed: For Inſtance, that* aabbbcccc *ſhall be taken in the Order wherein they are written.*

SOLUTION.

THe Probability of taking a in the firſt place is $\frac{2}{9}$: Suppoſing one a to be taken; the Probability of taking the other is $\frac{1}{8}$. Let now the two firſt Letters be ſuppoſed to be taken, the Probability of taking b will be $\frac{3}{7}$: Let this alſo be ſuppoſed taken, the Probability of taking another b will be $\frac{2}{6}$: Let this likewiſe be ſuppoſed taken, the Probability of taking the third b will be $\frac{1}{5}$; after which there remaining nothing but the Letter c, the Probability of taking it becomes a certainty, and conſequently is equal to Unity. Wherefore the Probability of taking all thoſe Letters in the Order given is $\frac{2}{9}\times\frac{1}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}$.

Corollary. Therefore the number of Permutations which the Letters *aabbbcccc* may receive, being taken all together will be $\frac{9\times8\times7\times6\times5}{2\times1\times3\times2\times1}$.

Generally. The number of Permutations which any number n of Things may receive, being taken all together, whereof the firſt ſort is repeated p times, the ſecond q times, the third r times, the fourth s times, &c. will be the Series $n\times n-1\times n-2\times n-3\times n-4$, &c. continued to ſo many Terms as there are Units in $p+q+r$ or $n-s$, divided by the Product of the following Series, *viz.* $p\times p-1\times p-2$, &c. $\times q\times q-1\times q-2$, &c. $\times r\times r-1\times r-2$, &c. whereof the firſt muſt be continued to ſo many Terms as there are Units in p; the ſecond, to ſo many Terms as three are Units in q; the third, to ſo many Terms as there are Units in r &c.

PROBLEM XVIII.

A*NY number of Things* a, b, c, d, e, f *being given: To find the Probability that, in taking two of them as it may Happen, both* a *and* b *ſhall be taken independently, or without any regard to Order.*

SOLUTION.

THE Probability of taking a or b in the firſt place will be $\frac{2}{6}$; ſuppoſe one of them taken, as for Inſtance a, then the Probability of taking b will be $\frac{1}{5}$. Wherefore the Probability of taking both a and b will be $\frac{2}{6} \times \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$.

Corollary. The taking of both a and b is but one ſingle Caſe of all thoſe by which Six Things may be combined two and two; wherefore the number of Combinations of Six Things taken two and two will be $\frac{6}{1} \times \frac{5}{2} = 15$.

Generally. The number of Combinations of n Things, taken two and two, will be $\frac{n}{1} \times \frac{n-1}{2}$.

PROBLEM XIX.

A*NY number of Things* a, b, c, d, e, f *being given: To find the Probability, that in taking three of them as it Happens, they ſhall be any three propoſed, as* a, b, c; *no reſpect being had to Order.*

SOLUTION.

THE Probability of taking either a, or b, or c in the firſt place will be $\frac{3}{6}$. Suppoſe one of them as a to be taken, then the Probability of taking b, or c in the ſecond place will be $\frac{2}{5}$. Again let either of them taken, as ſuppoſe b; then the Probability of taking c in the third place will be $\frac{1}{4}$; wherefore the Probability of taking the three Things propoſed, *viz.* a, b, c will be $\frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}$.

Corol-

Corollary. The taking of a, b, c is but one ſingle Caſe of all thoſe by which Six Things may be combined three and three; wherefore the number of Combinations of Six Things taken three and three will be $\frac{6}{1} \times \frac{5}{2} \times \frac{4}{3} = 20$.

Generally. The number of Combinations of n Things combined according to the number p, will be $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \overline{p-4}}$ &c. Both Numerator and Denominator being continued to ſo many Terms as there are Units in p.

PROBLEM XX.

TO find what Probability there is, that in taking as it Happens Seven Counters out of Twelve, whereof four are White and eight Black, three of them ſhall be White ones.

SOLUTION.

FIrſt, Find the number of Chances for taking three White ones out of four, which will be $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} = 4$.

Secondly, Find the number of Chances for taking four Black ones out of eight: Theſe Chances will be found to be $\frac{8 \times 7 \times 6 \times 5}{1 \quad 2 \quad 3 \quad 4} = 70$.

Thirdly, Becauſe every one of the preceding Chances may be joined with every one of the latter, it follows, that the number of Chances for taking three White ones and four Black ones, will be $4 \times 70 = 280$.

Fourthly, Find the number of Chances for taking four White ones out of four, which will be found to be $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} = 1$.

Fifthly, Find the number of Chances for taking three Black ones out of eight, which will be $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} = 56$.

Sixthly, Multiply theſe two laſt numbers together, and the Product 56 will ſhew that there are 56 Chances for taking four White ones and three Black ones; which is a Caſe not expreſſed in the Problem, yet is implyed: For he who undertakes to take three White Counters out of eight, is reputed to be a winner tho' he takes four; unleſs the contrary be expreſly ſtipulated.

Seventhly,

Seventhly, Wherefore the number of Chances for taking three White Counters will be $280 + 56 = 336$.

Eighthly, Seek the number of all the Chances for taking seven Counters out of twelve, which will be found to be $\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 792$.

Lastly, Divide the preceding number 336 by the last 792, and the Quotient $\frac{336}{792}$, or $\frac{14}{33}$ will be the Probability required.

Corollary. Let n be the number of all the Counters, a the number of White ones, b the number of Black ones, c the number of Counters to be taken out of the number n; then the number of Chances for taking none of the White ones, or one single White, or two White ones and no more, or three White ones and no more, or four White ones and no more, &c. will be exprest as follows.

$$\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4} \text{ \&c.} \times \frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} \times \frac{a-3}{4} \text{ \&c.}$$

The number of Terms wherein b enters being always equal to $c-a$, and the whole number of Terms equal to c.

But the number of all the Chances for taking a certain number c of Counters out of the number n, with one or more White ones, or without any, will be

$$\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{n-6}{7} \times \frac{n-7}{8} \text{ \&c.}$$

which Series must be continued to so many Terms as there are Units in c.

REMARK.

IF the numbers n and c were large, such as 40000 and 8000, the foregoing method would seem impracticable, by reason of the vast number of Terms to be taken in both Series, whereof the first is to be divided by the second: Tho' if those Terms were actually set down, a great many of them being common Divisors, might be expunged out of both Series. However to avoid the trouble of setting down so many Terms, it will be convenient to use the following Theorem, which is a contraction of that Method.

Let therefore n be the number of all the Counters, a the number of White ones, c the number of Counters to be taken

out

out of the number n, p the number of White Counters to be taken precisely in the number c: Then making $n-c=d$. I say that the Probability of taking precisely the number p of White Counters will be

$$\frac{\overline{c\times c-1\times c-2}\ \&c.\ \times\overline{d\times d-1\times d-2}\ \&c.\ \times\overline{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\&c.}}{n\times n-1\times n-2\times n-3\times n-3\times n-4\times n-5\times n-6\ \&c.}$$

Here it is to be observed, that the Numerator consists of three Series, which are to be Multiplied together; whereof the first contains as many Terms as there are Units in p, the second as many as there are Units in $a--p$, the third as many as there are Units p: And the Denominator as many as there are Units in a.

PROBLEM XXI.

IN A Lottery consisting of 40000 *Tickets, among which are Three particular Benefits: What is the Probability that taking* 8000 *of them, one or more of the Three particular Benefits shall be amongst them?*

SOLUTION.

FIrst in the Theorem belonging to the Remark of the foregoing Problem, having substituted respectively, 8000, 40000, 32000, 3 and 1, in the room of c, n, d, a and p; it will appear, that the Probability of taking precisely one of the Three particular Benefits will be

$\frac{8000\times 32000\times 31999\times 3}{40000\times 39999\times 39998}$, or $\frac{48}{125}$ nearly.

Secondly, c, n, d, a being interpreted as before, let us suppose $p=2$. Hence the Probability of taking precisely Two of the particular Benefits will be found to be

$\frac{8000\times 7999\times 32000\times 3}{40000\times 39999\times 39998}$, or $\frac{12}{125}$ very near.

Thirdly, Making $p=3$. The Probability of taking all the Three particular Benefits will be found to be

$\frac{8000\times 7999\times 7998}{40000\times 39998\times 39998}$, or $\frac{1}{125}$ very near.

Wherefore the Probability of taking one or more of the Three particular Benefits will be $\frac{48+12+1}{125}$, or $\frac{61}{125}$ very near.

N. B. Theſe three Operations might be contracted into one, by inquiring what the Probability is, that none of the particular Benefits may be taken; for then it will be found to be $\frac{32000 \times 31999 \times 31998}{40000 \times 39999 \times 39998} = \frac{64}{125}$ nearly; which being ſubtracted from 1, the Remainder $1 - \frac{64}{125}$, or $\frac{61}{125}$ will ſhew the Probability required.

PROBLEM XXII.

TO Find how many Tickets ought be taken in a Lottery conſiſting of 40000, *among which there are Three particular Benefits, to make it as Probable that one or more of thoſe Three may be taken as not.*

SOLUTION.

LET the number of Tickets requiſite to be taken be x: It will follow therefore from the Theorem belonging to the Remark of the XX*th*. Problem, that the Probability of not taking amongſt them any of the particular Benefits will be $\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times \frac{n-x-2}{n-2}$. But this Probability is $= \frac{1}{2}$, ſince the Probability of the contrary is $\frac{1}{2}$ by Hypotheſis; whence it follows that $\frac{n-x}{1} \times \frac{n-x-1}{2} \times \frac{n-x-2}{3} = \frac{1}{2}$. This Equation being ſolved, the value of x will be found to be nearly 8252.

N. B. The Factors, whereof both Numerator and Denominator are compoſed being in Arithmetic Progreſſion, and the difference being very ſmall in reſpect of n; thoſe Terms may be conſidered as being in Geometric Progreſſion, wherefore the Cube of the middle Term $\frac{n-x-1}{n-1}$ may be ſuppoſed equal to the Product of thoſe Terms; from whence will ariſe the Equation $\frac{\overline{n-x-1}|^3}{\overline{n-1}|^3} = \frac{1}{2}$ or $\frac{\overline{n-x}|^3}{n^3} = \frac{1}{2}$ (neglecting Unity in both Numerator and Denominator) and

con-

consequently x will be found to be $n \times \overline{1-\sqrt[3]{\frac{1}{2}}}$, but n is $= 40000$, and $1-\sqrt[3]{\frac{1}{2}} = 0.2063$; Therefore $x = 8252$.

In the Remark belonging to the V*th* Problem, a Rule was given for finding the number of Tickets that were to be taken to make it as Probable that one or more of the Benefits should be taken, as not; but in that Rule it is supposed that the proportion of the Blanks to the Prizes was often repeated, as it usually is in Lotteries: Now in the Case of the present Problem, the particular Benefits being but Three in all, the remaining Tickets are to be considered as Blanks in respect of them; from whence it follows, that the proportion of the number of Blanks to one Prize being very near as 13332 to 1, and that proportion being repeated but three times in the whole number of Tickets, the Rule there given would not have been sufficiently exact in this Case; to supply which it was thought necessary to give the Solution of this Problem.

PROBLEM XXIII.

TO *Find at* Pharaon, *how much it is that the Banker gets* per Cent *of all the Money that is adventured*,

HYPOTHESIS.

I Suppose, *First*, that there is but one single Ponte: *Secondly*, That he lays his Money upon one single Card at a time: *Thirdly*, That he begins to take a Card in the beginning of the Game: *Fourthly*, That he continues to take a new Card after the laying down of every Pull: *Fifthly*, That when there remains but Six Cards in the Stock, he ceases to take a Card.

SOLUTION.

WHEN at any time the Ponte lays a new Stake upon a Card taken as it Happens out of his Book, let the number of Cards that are already laid down by the Banker be supposed equal to x.

Now

Now in this circumftance, the Card taken by the Ponte has either paft four times, or three times, or twice, or once, or not at all.

Firft, If it has paffed four times, he can be no lofer upon that account.

Secondly, If it has paffed three times, then his Card is once in the Stock; now the number of Cards remaining in the Stock being $n-x$, it follows by the firft Cafe of the XIII*th* Problem that the lofs of the Ponte will be $\frac{1}{n-x}$: But by the Remark belonging to the XX*th* Problem, the Probability that his Card has paffed three times precifely in x Card, is $\frac{x\times\overline{x-1}\times\overline{x-2}\times\overline{n-x}\times 4}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$. Now fuppofing the Denominator equal to s, Multiply the lofs he will fuffer (if he has that Chance) by the Probability of having it, and the Product $\frac{x\times\overline{x-1}\times\overline{x-2}\times 4}{s}$, will be his abfolute lofs in that circumftance.

Thirdly, If it has paffed twice, his lofs by the fecond Cafe of the XIII*th* Problem will be $\frac{\frac{1}{2}n-\frac{1}{2}x+1}{\overline{n-x}\times\overline{n-x-1}}$, but the Probability that his Card has paffed twice in x Cards, is by the Remark of the XX*th* Problem, $\frac{x\times\overline{x-1}\times\overline{n-x}\times\overline{n-x-1}\times 6}{s}$; wherefore Multiplying the lofs he will fuffer (if he has that Chance) by the Probability of his having it, the Product $\frac{x\times\overline{x-1}\times\overline{\frac{1}{2}n-\frac{1}{2}x+1}\times 6}{s}$ will be his abfolute lofs in that circumftance.

Fourthly, If it has paffed once, his lofs Multiplyed into the Probability that it has paffed, will make his abfolute lofs to be $\frac{x\times\overline{n-x}\times\overline{n-x-2}\times 3}{s}$.

Fifthly, If it has not yet paffed, his lofs Multiplyed into the Probability that it has not paffed, will make his abfolute lofs to be $\frac{\overline{n-x}\times\overline{n-x-2}\times\overline{2n-2x-5}}{s}$.

Now the Sum of all thefe loffes of the Ponte's will be $\frac{n^3-\frac{9}{2}nn+5n-3x-\frac{5}{2}xx+3x^3}{s}$, and this is the lofs he fuffers by venturing a new Stake after any number of Cards x are paft.

But

But the number of Pulls which at any time are laid down, is always one half of the number of Cards that are paſt; wherefore calling t the number of thoſe Pulls, the Loſs of the Ponte may be expreſſed thus, $\frac{n^3 - \frac{9}{2}nn + 5n - 6t - 6tt + 24t^3}{s}$.

Let now p be the number of Stakes which the Ponte adventures; let alſo the Loſs of the Ponte be divided into two parts, *viz.* $\frac{n^3 - \frac{9}{2}nn + 5n}{s}$, and $\frac{-6t - 6tt + 24t^3}{s}$.

And ſince he adventures a Stake p times; it follows, that the firſt part of his Loſs will be $\frac{pn^3 - \frac{9}{2}pnn + 5pn}{s}$.

In order to find the ſecond part, let t be interpreted ſucceſſively by 0, 1, 2, 3 &c. to the laſt Term $p-1$; Then in the room of $6t$ we ſhall have a ſum of numbers in Arithmetic Progreſſion to be Multiplyed by 6; in the room of $6tt$ we ſhall have a ſum of Squares whoſe Roots are in Arithmetic Progreſſion to be Multiplyed by 6; and in the room of $24t^3$ we ſhall have a ſum of Cubes whoſe Roots are in Arithmetic Progreſſion to be Multiplyed by 24: Theſe ſeveral ſums being collected, according to the II*d* Remark on the XI*th* Problem, will be found to be $\frac{6p^4 - 14p^3 + 6pp + 2p}{s}$, and therefore the whole Loſs of the Ponte will be

$$\frac{pn^3 - \frac{9}{2}pnn + 5pn + 6p^4 - 14p^3 + 6pp + 2p}{s}.$$

Now this being the Loſs which the Ponte ſuſtains by adventuring the ſum p, each Stake being ſuppoſed equal to Unity, it follows, that the Loſs *per Cent* of the Ponte, or the Gain *per Cent* of the Banker is $\frac{n^3 - \frac{9}{2}nn + 5n + 6p^3 - 14pp + 6p + 2}{s} \times 100$, or $\frac{2n-5}{2 \times \overline{n-1} \times \overline{n-3}} + \frac{\overline{p-1} \times \overline{6pp - 8p - 2}}{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}} \times 100$. Let now n be interpreted by 52, and p by 23; and the Gain of the Banker will be found to be 2.99251, that is 2 *l.* 19 Sh. 10*d.* *per Cent.*

By the ſame Method of arguing, it will be found that the Gain *per Cent* of the Banker, at *Baſſete*, will be

$\frac{3n-9}{n \times \overline{n-1} \times \overline{n-2}} + \frac{4p \times \overline{p-1} \times \overline{p-2}}{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}} \times 100$. Let n be interpreted by 51, and p by 23; and the foregoing expreſſion will become 0.790582, or 15$^{sh.}$ 9 *d.* half-penny. The consideration of the first Stake, which is adventured before the Pack is turned, being here omitted as being out of the general Rule: But if that Case be taken in, and the Ponte adventures 100*l.* in 24 Stakes, the Gain of the Banker will be diminished, and becomes only 0.76245, that is, 15 $^{sh.}$ 3 *d.* very near: And this is to be estimated, as the gain *per Cent* of the Banker when he takes but half Face.

Now whether the Ponte takes one Card at a time or several Cards, the Gain *per Cent* of the Banker continues the same: Whether the Ponte keeps constantly to the same Stake, or some times doubles or triples it, the Gain *per Cent* is still the same: Whether there be but one single Ponte or several, his Gain *per Cent* is not thereby altered. Wherefore the Gain *per Cent* of the Banker of all the Money that is adventured at *Pharaon* is 2 *l.* 19$^{sh.}$ 10 *d.* and at *Bassete* 15$^{sh.}$ 3*d.*

PROBLEM XXIV.

Supposing A *and* B *to play together, the Chances they have respectively to win being as* a *to* b, *and* B *obliging himself to Set to* A, *so long as* A *wins without interruption: What is the Advantage that* A *gets by his Hand?*

SOLUTION.

First, If A and B each Stake One, the Gain of A on the first Game is $\frac{a-b}{a+b}$.

Secondly, His Gain on the second Game will also be $\frac{a-b}{a+b}$, provided he should happen to win the first: But the Probability of *A's* winning the first Game is $\frac{a}{a+b}$. Wherefore his Gain on the second Game will be $\frac{a}{a+b} \times \frac{a-b}{a+b}$.

Thirdly, His Gain on the third Game, after winning the two first, will be likewise $\frac{a-b}{a+b}$: But the Probability of *A's* winning the two first Games is $\frac{aa}{\overline{a+b}|^2}$; Wherefore his Gain on the third

third Game, when it is eſtimated before the Play begins, is $\frac{aa}{\overline{a+b}|^2} \times \frac{a-b}{a+b}$ &c.

Fourthly, Wherefore the Gain of the Hand of *A* is an infinite Series, *viz.* $1 + \frac{a}{a+b} + \frac{aa}{\overline{a+b}|^2} + \frac{a^3}{\overline{a+b}|^3} + \frac{a^4}{\overline{a+b}|^4}$ &c. to be Multiplyed by $\frac{a-b}{a+b}$. But the ſum of that infinite Series is $\frac{a+b}{b}$; Wherefore the Gain of the Hand of *A* is $\frac{a+b}{b} \times \frac{a-b}{a+b} = \frac{a-b}{b}$.

Corollary I. If *A* has the advantage of the Odds, and *B* Sets his Hand out, the Gain of *A* is the difference of the numbers expreſſing the Odds divided by the leſſer. Thus if *A* has the Odds of Five to Three, then his Gain will be $\frac{5-3}{3} = \frac{2}{3}$.

Corollary II. If *B* has the Diſadvantage of the Odds, and *A* Sets his Hand out, the Loſs of *B* will be the difference of the number expreſſing the Odds divided by the greater: Thus if *B* has but Three to Five of the Game, his Loſs will be $\frac{2}{5}$.

Corollary III. If *A* and *B* do mutually engage to Set to one-another as long as either of them wins without interruption, the Gain of *A* will be found to be $\frac{aa-bb}{ab}$: That is the ſum of the numbers expreſſing the Odds Multiplyed by their difference, the product of that Multiplication being divided by the Product of the numbers expreſſing the Odds. Thus if the Odds were as Five to Three, the ſum of 5 and 3 is 8, and the difference 2; Multiply 8 by 2, and the Product 16 being divided by 15 (Product of the number expreſſing the Odds) the Quotient will be $\frac{16}{15}$, or $1\frac{1}{15}$; which therefore will be the Gain of *A*.

PROBLEM XXV.

A*NY given number of Letters* a, b, c, d, e, f *&c. all of them different, being taken promiſcuouſly, as it Happens: To find the Probability that ſome of them ſhall be found in their places,*

according

according to the rank they obtain in the Alphabet; and that others of them ſhall at the ſame time be found out of their places.

SOLUTION.

LET the number of all the Letters be $= n$; let the number of thoſe that are to be in their places be $= p$, and the number of thoſe that are to be out of their places $= q$. Suppoſe for Brevity ſake $\frac{1}{n} = r$, $\frac{1}{n \times n-1} = s$, $\frac{1}{n \times n-1 \times n-2} = t$, $\frac{1}{n \times n-1 \times n-2 \times n-3} = v$ &c. then let all the Quantities $1, r, s, t, v$ &c. be written down with Signs alternately poſitive and negative, beginning at 1, if p be $= 0$; at r, if $p = 1$; at s, if $p = 2$ &c. Prefix to theſe Quantities the reſpective Coefficients of a Binomial Power, whoſe Index is $= q$: This being done, thoſe Quantities taken all together will expreſs the Probability required; thus the Probability that in Six Letters taken promiſcuouſly, two of them, *viz.* a and b ſhall be in their places, and three of them, *viz.* c, d, e out of their places, will be

$$\frac{1}{6 \times 5} - \frac{3}{6 \times 5 \times 4} + \frac{3}{6 \times 5 \times 4 \times 3} - \frac{1}{6 \times 5 \times 4 \times 3 \times 2} = \frac{11}{720},$$

And the Probability that a ſhall be in its place, and b, c, d, e out of their places, will be

$$\frac{1}{6} - \frac{4}{6 \times 5} + \frac{6}{6 \times 5 \times 4} - \frac{4}{6 \times 5 \times 4 \times 3} + \frac{1}{6 \times 5 \times 4 \times 3 \times 2} = \frac{53}{720}.$$

The Probability that a ſhall be in its place, and b, c, d, e, f out of their places, will be

$$\frac{1}{6} - \frac{5}{6 \times 5} + \frac{10}{6 \times 5 \times 4} - \frac{10}{6 \times 5 \times 4 \times 3} + \frac{5}{6 \times 5 \times 4 \times 3 \times 2} - \frac{1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{44}{720}, \text{ or } \frac{11}{180}.$$

The Probability that a, b, c, d, e, f ſhall be all diſplaced is,

$$1 - \frac{6}{6} + \frac{15}{6 \times 5} - \frac{20}{6 \times 5 \times 4} + \frac{15}{6 \times 5 \times 4 \times 3} - \frac{6}{6 \times 5 \times 4 \times 3 \times 2} + \frac{1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}, \text{ or } 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{265}{720} = \frac{53}{144}.$$

Hence

Hence it may be concluded that the Probability that one or more of them will be found in their places is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144}$; and that the Odds that one or more of them will be ſo found are as 91 to 53.

N. B. So many Terms of this laſt Series are to be taken as there are Units in n.

DEMONSTRATION.

THE number of Chances for the Letter a to be in the firſt place contains the number of Chances, by which a being in the firſt place, b may be in the ſecond, or out of it: This is an Axiom of common Senſe, of the ſame degree of Evidence as that the Whole is equal to all its Parts.

From this it follows, that if from the number of Chances that there are for a to be in the firſt place, there be ſubſtracted the number of Chances that there are for a to be in the firſt place, and b at the ſame time in the ſecond, there will remain the number of Chances, by which a being in the firſt place, b may be excluded the ſecond.

For the ſame reaſon it follows, that if from the number of Chances that there are for a and b to be reſpectively in the firſt and ſecond places, there be ſubtracted the number of Chances by which a, b and c may be reſpectively in the firſt, ſecond and third places; there will remain the number of Chances by which a being in the firſt and b in the ſecond, c may be excluded the third place: And ſo of the reſt.

Let $+ a'$ denote the Probability that a ſhall be in the firſt place, and let $- a'$ denote the Probability of its being out of it. Likewiſe let the Probabilities that b ſhall be in the ſecond place or out of it be reſpectively expreſt by $+ b''$ and $- b''$.

Let the Probability that, a being in the firſt place, b ſhall be in the ſecond, be expreſt by $a' + b''$: Likewiſe let the Probability that a being in the firſt place, b ſhall be excluded the ſecond, be expreſt by $a' - b''$.

Generally. Let the Probability there is, that as many as are to be in their proper places, ſhall be ſo, and at the ſame time that as many others as are to be out of their proper places ſhall be ſo found, be denoted by the particular Probabilities of their being in their proper places, or out of them, written all together: So that for Inſtance $a' + b'' + c''' - d'''' - e'''''$ may denote the Probability that a, b and c ſhall be in their proper places, and that at the ſame time both d and e ſhall be excluded their proper places.

Now to be able to derive a proper concluſion by vertue of this Notation, it is to be obſerved, that of the Quantities which are here conſidered, thoſe from which the Subtraction is to be made, are indifferently compoſed of any number of Terms connected by $+$ and $-$; the Quantities which are to be ſubtracted do exceed by one Term thoſe from which the ſubtraction is to be made; the reſt of the Terms being alike and their ſigns alike: And the remainder will contain all the Quantities that are alike with their own ſigns, and alſo the Quantity Exceeding, but with its ſign varied.

It having been demonſtrated in what we have ſaid of Permutations and Combinations, that $a' = \frac{1}{n}$, $a' + b'' = \frac{1}{n \times n-1}$, $a' + b'' + c''' = \frac{1}{n \times n-1 \times n-2}$, let $\frac{1}{n}$, $\frac{1}{n \times n-1}$ &c. be reſpectively called r, s, t, v &c. This being ſuppoſed, we may come to the following concluſions.

$$\begin{array}{ll}
 & b'' = r \\
 & b'' + a' = s \\
\text{Therefore} & b'' - a' = r - s
\end{array}$$

$$\begin{array}{lll}
 & c''' + b'' = s & \text{for the ſame reaſon that } a' + b'' = s \\
 & c''' + b'' + a' = t & \\
2^{\circ}\ \text{Theref.} & c''' + b'' - a' = s - t &
\end{array}$$

$$\begin{array}{lll}
 & c''' - a' = r - s & \text{By the firſt Concluſion.} \\
 & c''' - a' + b'' = s - t & \text{By the } 2d. \\
3^{\circ}\ \text{Theref.} & c''' - a' - b'' = r - 2s + t &
\end{array}$$

$$\begin{array}{ll}
 & d'''' + c''' + b'' = t \\
 & d'''' + c''' + b'' + a' = v \\
4^{\circ}\ \text{Theref.} & d'''' + c''' + b'' - a' = t - v
\end{array}$$

$$\begin{array}{lll}
 & d'''' + c''' - a' = s - t & \text{By the } 2d. \text{ Concluſion.} \\
 & d'''' + c''' - a' + b'' = t - v & \text{By the } 4th. \\
5^{\circ}\ \text{Theref.} & d'''' + c''' - a' - b'' = s - 2t + v &
\end{array}$$

$$d'''' - b'' - a' \quad = r - 2s + t \quad \text{By the 3}d\text{. Conc.}$$
$$d'''' - b'' - a' + c''' = \quad s - 2t + v \quad \text{By the 5}th\text{.}$$

6° Theref. $d'''' - b'' - a' - c''' = r - 3s + 3t - v$

By the ſame proceſs, if no Letter be particularly aſſigned to be in its place, the Probability that ſuch of them as are aſſigned may be out of their places will likewiſe be found thus.

$$-a' \quad = 1 - r \quad \text{For } + a' \text{ and } - a' \text{ together make [Unity.}$$
$$-a' + b'' = \quad r - s$$

7° Theref. $-a' - b'' = 1 - 2r + s$

$$-a' - b'' \quad = 1 - 2r + s \quad \text{By the 7}th\text{. Conc.}$$
$$-a' - b'' + c'' = \quad r - 2s + t \quad \text{By the 3}d\text{. Conc.}$$

8° Theref. $-a' - b'' - c'' = 1 - 3r - 3s - t$

Now examining carefully all the foregoing Concluſions, it will be perceived, that when the Queſtion runs barely upon the diſplacing any given number of Letters without requiring that any other ſhould be in its place, but leaving it wholly indifferent, then the vulgar Algebraic Quantities which lie on the right hand of the Equations, begin conſtantly with Unity: It will alſo be perceived, that when one ſingle Letter is aſſigned to be in its place, then thoſe Quantities begin with r; and that when two Letters are aſſigned to be in their places, they begin with s, and ſo on. Moreover 'tis obvious, that theſe Quantities change their ſigns alternately, and that the Numerical Coefficients which are prefixt to them are thoſe of a Binomial Power, whoſe Index is equal to the number of Letters which are to be diſplaced.

PROBLEM XXVI.

ANY given number of different Letters a, b, c, d, e, f *&c. being each of them repeated a certain number of times, and taken promiſcuouſly as it Happens: To find the Probability that of ſome of thoſe Sorts, ſome one Letter of each may be found in its proper place, and at the ſame time that of ſome other Sorts, no one Letter be found in its place.*

SOLU-

SOLUTION.

SUppose n be the number of all the Letters, l the number of times that each Letter is repeated, and consequently $\frac{n}{l}$ the number of Sorts: Suppose also that p be the number of Sorts that are to have one Letter of each in its place; and q the number of Sorts of which no one Letter is to be found in its place. Let now the prescriptions given in the preceding Problem be followed in all respects, saving that r must here be made $= \frac{l}{n}$, $s = \frac{ll}{n \times \overline{n-1}}$, $t = \frac{l^3}{n \times \overline{n-1} \times \overline{n-2}}$ &c. and the Solution of any particular Case of the Problem will be obtained.

Thus if it were required to find the Probability that no Letter of any sort shall be in its place, the Probability thereof would be

$$1 - qr + \frac{q}{1} \times \frac{q-1}{2} s - \frac{q}{1} \times \frac{q-1}{2} \times \frac{q-2}{3} t \text{ \&c.}$$

But in this particular Case q would be equal to $\frac{n}{l}$, wherefore the foregoing Series might be changed into this, *viz.*

$$\frac{1}{2} \times \frac{\overline{n-l}}{n-1} - \frac{1}{6} \times \frac{\overline{n-l} \times \overline{n-2l}}{n-1 \times n-2} + \frac{1}{24} \times \frac{\overline{n-l} \times \overline{n-2l} \times \overline{n-3l}}{n-1 \times n-2 \times n-3},$$

&c.

Corollary I. From hence it follows, that the Probability that one or more Letters indeterminately taken may be in their places will be

$$1 - \frac{1}{2} \times \frac{\overline{n-l}}{n-1} + \frac{1}{6} \times \frac{\overline{n-l \times n-2l}}{n-1 \times n-2} - \frac{1}{24} \times \frac{\overline{n-l \times n-2l \times n-3l}}{n-1 \times n-2 \times n-3}$$

&c.

Corollary II. The Probability that two or more Letters indeterminately taken may be in their places will be exprest as follows,

$$\frac{1}{2} \times \frac{\overline{n-l}}{n-1} - \frac{2}{1 \times 3} \times \frac{\overline{n-2l}}{n-2} A + \frac{3}{2 \times 4} \times \frac{\overline{n-3l}}{n-3} B - \frac{4}{3 \times 5} \times \frac{\overline{n-4l}}{n-4} C + \frac{5}{4 \times 6} \times \frac{\overline{n-5l}}{n-5} D \text{ \&c.}$$

Corollary III. The Probability that three or more Letters indeterminately taken may be in their places will be as follows,

$$\frac{1}{6} \times \frac{\overline{n-l} \times \overline{n-2l}}{\overline{n-1} \times \overline{n-2}} - \frac{3}{1 \times 4} \times \frac{\overline{n-3l}}{n-3} A + \frac{4}{2 \times 5} \times \frac{\overline{n-4l}}{n-4} B$$
$$- \frac{5}{3 \times 6} \times \frac{\overline{n-5l}}{n-5} C + \frac{6}{4 \times 7} \times \frac{\overline{n-6l}}{n-6} D \text{ \&c.}$$

Corollary IV. The Probability that four or more Letters, indeterminately taken, may be in their places will be thus expreſt,

$$\frac{1}{24} \times \frac{\overline{n-l}}{n-1} \times \frac{\overline{n-2l}}{n-2} \times \frac{\overline{n-3l}}{n-3} - \frac{4}{1 \times 5} \times \frac{\overline{n-4l}}{n-4} A + \frac{5}{2 \times 6} \times \frac{\overline{n-5l}}{n-5} B$$
$$- \frac{6}{3 \times 7} \times \frac{\overline{n-6l}}{n-7} C \text{ \&c.}$$

The Law of the continuation of theſe Series being manifeſt, it will be eaſy to reduce them all to one general Series.

From what we have ſaid it follows, that in a common Pack of 52 Cards, the Probability that one of the four Aces may be in the firſt place; one of the four Duces in the ſecond; or one of the four Traes in the third; or that ſome one of any other ſort may be in its place (making 13 different places in all) will be expreſt by the Series exhibited in the firſt Corollary.

It follows likewiſe, that if there be two Packs of Cards, and that the Order of the Cards in one of the Packs be the Rule whereby to eſtimate the rank which the Cards of the ſame Suite and Name are to obtain in the other; the Probability that one Card or more, in one of the Packs, may be found in the ſame Poſition as the like Card in the other Pack, will be expreſt by the Series belonging to the firſt Corollary, making $n = 52$ and $l = 1$: Which Series will in this Caſe be $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720}$ &c. whereof 52 Terms ought to be taken.

If the Terms of the foregoing Series are joined by couples, the Series will become,

$$\frac{1}{2} + \frac{1}{2 \times 4} + \frac{1}{2 \times 3 \times 4 \times 6} + \frac{1}{2 \times 3 \times 4 \times 5 \times 6 \times 8} + \frac{1}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 10}$$

&c. of which 26 Terms ought to be taken.

But by reaſon of the great Convergency of the aforeſaid Series, a few of its Terms will give a ſufficient approximation

tion in all Cafes required; as appears by the following Operation,

$$\frac{1}{2} = 0.500000$$
$$\frac{1}{2\times4} = 0.125000$$
$$\frac{1}{2\times3\times4\times6} = 0.006944+$$
$$\frac{1}{2\times3\times4\times5\times6\times8} = 0.000174+$$
$$\frac{1}{2\times3\times4\times5\times6\times7\times8\times10} = 0.000002+$$
$$\text{Sum} = 0.632129+$$

Wherefore the Probability that one or more like Cards in two different Packs may obtain the fame Pofition, will be in all Cafes very near 0.632; and the Odds that this will Happen once or oftner, as 632 to 368, or as 12 to 7 very near.

But the Odds that two or more like Cards in two different Packs will not obtain the fame Pofition, are very nearly as 736 to 264 or 14 to 5.

Corollary V. If *A* and *B*, each holding a Pack of Cards, pull them out at the fame time one after another, on condition that every time two like Cards are pulled out, *A* fhall give *B* a Guinea; and it were required to find what confideration *B* ought to give *A* to Play on thofe terms: The Anfwer will be, One Guinea, let the number of Cards be what it will.

Corollary VI. If the number of Packs be given, the Probability that any given number of circumftances may Happen in them all, or in any of them, will be found eafily by our method. Thus, if the number of the Packs be *k*, the Probability that one Card or more of the fame Sute and Name, in every one of the Packs, may be in the fame Pofition, will be expreft as follows.

$$\frac{1}{n^{k-2}} - \frac{1}{2\times\overline{n\times n-1}|^{k-2}} + \frac{1}{6\times\overline{n\times n-1\times n-2}|^{k-2}} - \frac{1}{24\times\overline{n\times n-1\times n-2\times n-3}|^{k-2}}$$

&c.

PRO-

PROBLEM XXVII.

If A *and* B *play together, each with a certain number of Bowls* $= n$: *What are their reſpective Probabilities of winning, ſuppoſing that each of them want a certain number of Games of being up?*

SOLUTION.

First, the Probability that ſome Bowl of B may be nearer the Jack than any Bowl of A is $\frac{1}{2}$.

Secondly, Suppoſing one of his Bowls nearer the Jack than any Bowl of A, the number of his remaining Bowls is $n-1$, and the number of all the Bowls remaining between them is $2n-1$: Wherefore the Probability that ſome other of his Bowls may be nearer the Jack than any Bowl of A will be $\frac{n-1}{2n-1}$, from whence it follows, that the Probability of his winning two Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

Thirdly, Suppoſing two of his Bowls nearer the Jack than any Bowl of A, the Probability that ſome other of his Bowls may be nearer the Jack than any Bowl of A will be $\frac{n-2}{2n-2}$; Wherefore the Probability of winning three Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$: The continuation of which proceſs is manifeſt.

Fourthly, The Probability that one ſingle Bowl of B ſhall be nearer the Jack than any Bowl of A is $\frac{1}{2} - \frac{1}{2} \times \frac{n-1}{2n-1}$, or $\frac{1}{2} \times \frac{n}{2n-1}$; For, if from the Probability that one or more of his Bowls may be nearer the Jack than any Bowl of A, there be ſubtracted the Probability that two or more may be nearer, there remains the Probability of one ſingle Bowl of B being nearer: In this Caſe B is ſaid to Win one Bowl at an End.

Fifthly, The Probability that two Bowls of B, and not more, may be nearer the Jack than any Bowl of A, will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$, in which Caſe B is ſaid to win two Bowls at an End.

Sixthly,

Sixthly, The Probability that B may win three Bowls at an End will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2} \times \frac{n}{2n-3}$. The procefs whereof is manifeft.

The Reader may obferve, that the foregoing Expreffions might be reduced to fewer Terms; but leaving them unreduced, the Law of the procefs is thereby made more confpicuous.

Let it carefully be obferv'd, when we mention henceforth the Probability of winning two Bowls, that the Senfe of it ought to be extended to two Bowls or more; and that when we mention the winning two Bowls at an End, it ought to be taken in the common acceptation of two Bowls only: The like being to be obferved in other Cafes.

This Preparation being made; fuppofe, *Firft*, that A wants one Game of being up, and B two; and let it be required, in that circumftance, to determine their Probabilities of winning.

Let the whole Stake between them be fuppofed $= 1$. Then either A may win a Bowl, or B win one Bowl at an End, or B may win two Bowls.

In the firft Cafe B lofes his Expectation.

In the fecond Cafe he becomes intitled to $\frac{1}{2}$ of the Stake. But the Probability of this Cafe is $\frac{1}{2} \times \frac{n}{2n-1}$: wherefore his Expectation arifing from that part of the Stake he will be intitled to, if this Cafe fhould Happen, and from the Probability of its Happening, will be $\frac{1}{4} \times \frac{n}{2n-1}$.

In the third Cafe B wins the whole Stake 1. But the Probability of this Cafe is $\frac{1}{2} \times \frac{n-1}{2n-1}$: wherefore the Expectation of B upon that account is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

From this it follows that the whole Expectation of B is $\frac{1}{4} \times \frac{n}{2n-1} + \frac{1}{2} \times \frac{n-1}{2n-1}$ or $\frac{\frac{3}{4}n - \frac{1}{2}}{2n-1}$, or $\frac{3n-2}{8n-4}$; which being fubtracted from Unity, the remainder will be the Expectation of A, *viz.* $\frac{5n-2}{8n-4}$. It may therefore be concluded, that the Probabilities which A and B have of winning are refpectively as $5n-2$ to $3n-2$.

'Tis remarkable, that the fewer the Bowls are, the greater is the proportion of the Odds; for if A and B play with fingle ſingle Bowls, the proportion will be as 3 to 1; if they play with two Bowls each, the proportion will be as 2 to 1; if with three Bowls each, the proportion will be as 13 to 7: yet let the number of Bowls be never ſo great, that proportion will not deſcend ſo low as 5 to 3.

Secondly, Suppoſe *A* wants one Game of being up, and *B* three; then either *A* may win a Bowl, or *B* win one Bowl at an End, or two Bowls at an End, or three Bowls.

In the firſt Caſe *B* loſes his Expectation.

If the ſecond Caſe Happens, then *B* will be in the circumſtance of wanting but two to *A's* one; in which Caſe his Expectation will be $\frac{3n-2}{8n-4}$, as it has been before determined: but the Probability that this Caſe may Happen is $\frac{1}{2} \times \frac{n}{2n-1}$; wherefore the Expectation of *B*, ariſing from the proſpect of this Caſe, will be $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4}$.

If the third Caſe Happen, then *B* will be intitled to one half of the Stake: but the Probability of its Happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$; wherefore the Expectation of *B* ariſing from the Proſpect of this Caſe is $\frac{1}{4} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$, or $\frac{1}{8} \times \frac{n}{2n-1}$.

If the fourth Caſe Happen, then *B* wins the whole Stake 1: but the Probability of its Happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$, or $\frac{1}{4} \times \frac{n-2}{2n-1}$; wherefore the Expectation of *B* ariſing from the proſpect of this Caſe will be found to be $\frac{1}{4} \times \frac{n-2}{2n-1}$.

From this it follows, that the whole Expectation of *B* will be $\frac{9nn-13n+4}{8 \times \overline{2n-1}|^2}$; which being ſubtracted from Unity, the remainder will be the Expectation of *A*, *viz.* $\frac{23nn-19n+4}{8 \times \overline{2n-1}|^2}$. It may therefore be concluded, that the Probabilities which *A* and *B* have of winning are reſpectively as $23nn-19n+4$ to $9nn-13n+4$.

N. B. If *A* and *B* play only with One Bowl each, the Expectation of *B* deduced from the foregoing Theorem would be found $= 0$. which we know from other principles ought to be $= \frac{1}{8}$. The reaſon of which is that the Caſe of winning Two Bowls at an End, and the Caſe of winning

Three Bowls at an End, enter this conclusion, which Cases do not belong to the supposition of playing with single Bowls: wherefore excluding those two Cases, the Expectation of B will be found to be $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4} = \frac{1}{8}$, which will appear if n be made $= 1$. Yet the Expectation of B, in the Case of two Bowls, would be rightly determined, tho' the Case of winning Three Bowls at an End enters it: The reason of which is, that the Probability of winning Three Bowls at an End is $=$, $\frac{1}{4} \times \frac{n-2}{2n-1}$, which in the Case of Two Bowls becomes $= 0$, so that the general Expression is not thereby disturbed.

After what we have said, it will be easy to extend this way of Reasoning to any circumstance of Games wanting between A and B; by making the Solution of each simpler Case subservient to the Solution of that which is immediately more compound.

Having given formerly the Solution of this Problem, proposed to me by the Honourable *Frances Robarts*, in the *Philosophical Transactions* Number 339; I there said, by way of Corollary, that if the proportion of Skill in the Gamesters were given, the Problem might also be Solved; since which time *Mr de Monmort*, in the second Edition of a Book by him Published upon the subject of Chance, has thought it worth his while to Solve this Problem as it is extended to the consideration of the Skill, and to carry his Solution to a very great number of Cases, giving also a Method by which it may still be carried farther: I very willingly acknowledge his Solution to be extreamly good, and own that he has in this, as well as in a great many other things, shewn himself entirely master of the doctrine of Combinations, which he has employed with very great Industry and Sagacity.

The Solution of this Problem, as it is restrained to an equality of Skill, was in my *Specimen* deduced from the Method of Combinations; but the Solution which is given of it in this place, is deduced from a Principle which has more of simplicity in it, being that by the help of which I have Demonstrated the Doctrine of Permutations and Combinations: Wherefore to make it as familiar as possible, and to shew its vast extent, I shall now apply it to the general Solution

of

of this Problem, taking in the consideration of the Skill of the Gamesters.

But before I proceed I think it necessary to define what I call Skill: *viz.* That it is the proportion of Chances which the Gamesters may be supposed to have for winning a single Game with one Bowl each.

PROBLEM XXVIII.

I*F* A *and* B, *whose proportion of Skill is as* a *to* b, *play together, each with a certain number of Bowls: What are their respective Probabilities of winning, supposing each of them to want a certain number of Games of being up?*

SOLUTION.

F*Irst*, The Chance of B for winning one single Bowl being b, and the number of his Bowls being n, it follows that the sum of all his Chances is nb; and for the same reason the sum of all the Chances of A is na: wherefore the sum of all the Chances for winning one Bowl or more is $na+nb$; which for brevity sake we may call s. From whence it follows, that the Probability which B has of winning one Bowl or more is $\frac{nb}{s}$.

Secondly, Supposing one of his Bowls nearer the Jack than any of the Bowls of A, the number of his remaining Chances is $\overline{n-1}\times b$; and the number of Chances remaining between them is $s-b$: wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of A will be $\frac{\overline{n-1}\times b}{s-b}$: From whence it follows, that the Probability of his winning Two Bowls or more is $\frac{nb}{s}\times\frac{\overline{n-1}\times b}{s-b}$.

Thirdly, Supposing Two of his Bowls nearer the Jack than any of the Bowls of A, the number of his remaining Chances is $\overline{n-2}\times b$; and the number of Chances remaining between them is $s-2b$: wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of A will be $\frac{\overline{n-2}\times b}{s-2b}$. From whence it follows, that the

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Probability of his winning Three Bowls or more is $\frac{nb}{s} \times \frac{\overline{n-1}\times b}{s-b} \times \frac{\overline{n-2}\times b}{s-2b}$; the continuation of which procefs is manifeft.

Fourthly, If from the Probability which *B* has of winning One Bowl or more, there be fubtracted the Probability which he has of winning Two or more, there will remain the Probability of his winning One Bowl at an End: Which therefore will be found to be $\frac{nb}{s} - \frac{nb}{s} \times \frac{\overline{n-1}\times b}{s-b}$ or $\frac{nb}{s} \times \frac{s-nb}{s-b}$ or $\frac{nb}{s} \times \frac{an}{s-b}$.

Fifthly, For the fame reafon as above, the Probability which *B* has of winning Two Bowls at an End will be found to be $\frac{nb}{s} \times \frac{\overline{n-1}\times b}{s-b} \times \frac{an}{s-2b}$.

Sixthly, And for the fame reafon likewife, the Probability which *B* has of winning Three Bowls at an End will be found to be $\frac{nb}{s} \times \frac{\overline{n-1}\times b}{s-b} \times \frac{\overline{n-2}\times b}{s-2b} \times \frac{an}{s-3b}$. The continuation of which procefs is manifeft.

N. B. The fame Expectations which denote the Probability of any circumftance of *B*, will denote likewife the Probability of the like circumftance of *A*, only changing *b* into *a* and *a* into *b*.

Thefe Things being premifed, Suppofe *Firft*, that each of them wants one Game of being up; 'tis plain that the Expectations of *A* and *B* are refpectively $\frac{an}{s}$ and $\frac{bn}{s}$. Let this Expectation of *B* be called *P*.

Secondly, Suppofe *A* wants One Game of being up and *B* Two, and let the Expectation of *B* be required: Then either *A* may win a Bowl, or *B* win One Bowl at an End, or *B* win Two Bowls.

If the firft Cafe Happens, *B* lofes his Expectation.

If the fecond Happens, he gets the Expectation *P*; but the Probability of this Cafe is $\frac{nb}{s} \times \frac{an}{s-b}$: wherefore the Expectation of *B* arifing from the poffibility that it may fo Happen is $\frac{nb}{s} \times \frac{an}{s-b} \times P$.

If

If the third Cafe Happens, he gets the whole Stake 1; but the Probability of this Cafe is $\frac{nb}{s} \times \frac{n-b}{s-b}$, wherefore the Expectation of *B* arifing from the Probability of this Cafe is $\frac{nb}{s} \times \frac{n-b}{s-b} \times 1$.

From which it follows that the whole Expectation of *B* will be $\frac{nb}{s} \times \frac{an}{s-b} P + \frac{nb}{s} \times \frac{n-b}{s-b}$. Let this Expectation be called *Q*.

Thirdly, Suppofe *A* to want One Game of being up, and *B* Three. Then either *B* may win One Bowl at an End, in which Cafe he gets the Expectation *Q*; or Two Bowls at an End, in which Cafe he gets the Expectation *P*; or Three Bowls in which Cafe he gets the whole Stake 1. Wherefore the Expectation of *B* will be found to be $\frac{nb}{s} \times \frac{an}{s-b} \times Q$

$$+ \frac{nb}{s} \times \frac{\overline{n-1} \times b}{s-b} \times \frac{an}{s-2b} \times P + \frac{nb}{s} \times \frac{\overline{n-1} \times b}{s-b} \times \frac{\overline{n-2} \times b}{s-2b}.$$

An infinite number of thefe Theorems may be formed in the fame manner, which may be continued by infpection, having well obferved how each of them is deduced from the preceding.

If the number of Bowls were unequal, fo that *A* had *m* Bowls and *B n* Bowls; Suppofing $ma + nb = s$, other Theorems might be found to anfwer that inequality: And if that inequality fhould not be conftant, but vary at pleafure; other Theorems might alfo be formed to anfwer that Variation of inequality, by following the fame way of arguing. And if Three or more Gamefters were to play together under any circumftance of Games wanting, and of any given proportion of Skill, their Probabilities of winning might be determined after the fame manner.

PROBLEM XXIX.

TO *find the Expectation of* A *when with a Die of any given number of Faces he undertakes to fling any determinate number of them in any given number of Cafts.*

SOLUTION.

LET $p+1$ be the number of all the Faces in the Die, n the number of Caſts, f the number of Faces which he undertakes to fling.

The number of Chances for an Ace to come up once or more in any number of Caſts n, is $\overline{p+1}|^n - p^n$: As has been proved in the Introduction.

Let the Duce, by thought, be expunged out of the Die, and thereby the number of its Faces reduced to p, then the number of Chances for the Ace to come up will at the ſame time be reduced to $p^n - \overline{p-1}|^n$. Let now the Duce be reſtored, and the number of Chances for the Ace to come up without the Duce, will be the ſame as if the Duce were expunged. But if from the number of Chances for the Ace to come up with or without the Duce, *viz.* from $\overline{p+1}|^n - p^n$ be ſubtracted the number of Chances for the Ace to come up without the Duce, *viz.* $p^n - \overline{p-1}|^n$, there will remain the number of Chances for the Ace and Duce to come up once or more, which conſequently will be $\overline{p+1}|^n - 2\times p^n + \overline{p-1}|^n$.

By the ſame way of arguing it will be proved, that the number of Chances for the Ace and Duce to come up without the Trae will be $p^n - 2\times\overline{p-1}|^n + \overline{p-2}|^n$, and conſequently, that the number of Chances for the Ace, the Duce and Trae to come up once or more, will be the difference between $\overline{p+1}|^n - 2\times p^n + \overline{p-1}|^n$ and $p^n - 2\times\overline{p-1}|^n + \overline{p-2}|^n$; which therefore is $\overline{p+1}|^n - 3\times p^n + 3\times\overline{p-1}|^n + \overline{p-2}|^n$.

Again it may be proved that the number of Chances for the Ace, the Duce, the Trae and Quater to come up, is $\overline{p+1}|^n - 4\times p^n + 6\times\overline{p-1}|^n - 4\times\overline{p-2}|^n + \overline{p-3}|^n$; the continuation of which Proceſs is manifeſt.

Wherefore if all the Powers $\overline{p+1}|^n$, p^n, $\overline{p-1}|^n$, $\overline{p-2}|^n$, $\overline{p-3}|^n$ &c. with the Signs alternately Poſitive and Negative, be written in Order, and to thoſe Powers there be prefixt the reſpective Coefficients of a Binomial raiſed to the Power f; the ſum of all thoſe Terms will be the Numerator of the Expectation of *A*, of which the Denominator will be $\overline{p+1}|^n$.

EXAM-

EXAMPLE I.

LET Six be the number of Faces in the Die, and let *A* undertake in Eight Casts to fling both an Ace and a Duce: Then his Expectation will be $\frac{6^8 - 2 \times 5^8 + 4^8}{6^8}$ $= \frac{964502}{1680216} = \frac{4}{7}$ nearly.

EXAMPLE II.

IF *A* undertake with a common Die to fling all the Faces in 12 Casts, his Expectation will be found to be $\frac{6^{12} - 6 \times 5^{12} + 15 \times 4^{12} - 20 \times 3^{12} + 15 \times 2^{12} - 6 \times 1^{12}}{6^{12}} = \frac{10}{23}$ nearly.

EXAMPLE III.

IF *A* with a Die of 36 Faces undertake to fling two given Faces in 43 Casts; or, which is the same thing, if with two common Dice he undertake in 43 Casts to fling Two Aces at one time, and Two Sixes at another time, his Expectation will be $\frac{36^{43} - 2 \times 35^{43} + 34^{43}}{36^{43}} = \frac{49}{100}$ nearly.

N. B. The parts of which these Expectations are compounded, are easily obtained by the help of a Table of Logarithms.

PROBLEM XXX.

TO *find in how many Trials it will be probable that* A *with a Die of any given number of Faces shall throw any proposed number of them.*

SOLUTION.

LET $p+1$ be the number of Faces in the Die, and f the number of Faces which are to be thrown. Divide the Logarithm of $\frac{1}{1-\sqrt[f]{\frac{1}{2}}}$ by the Logarithm of $\frac{p+1}{p}$, and the

the Quotient will expreſs nearly the number of Trials requiſite, to make it as probable that the propoſed Faces may be thrown as not.

DEMONSTRATION.

SUppoſe Six to be the number of Faces which are to be thrown, and n the number of Trials: Then by what has been demonſtrated in the preceding Problem, the Expectation of A will be,

$$\frac{\overline{p+1}^n - 6\times p^n + 15\times\overline{p-1}^n - 20\times\overline{p-2}^n + 15\times\overline{p-3}^n - 6\times\overline{p-4}^n + \overline{p-5}^n}{\overline{p+1}^n}$$

Let it be ſuppoſed that the Terms $\overline{p+1}$, p, $p-1$, $p-2$ &c. are in Geometric Progreſſion (which ſuppoſition will very little err from the truth, eſpecially if the proportion of p to 1 be not very ſmall). Let now r be written inſtead of $\frac{p+1}{p}$, and then the Expectation of A will be changed into $1 - \frac{6}{r^n} + \frac{15}{r^{2n}} - \frac{20}{r^{3n}} + \frac{15}{r^{4n}} - \frac{6}{r^{5n}} + \frac{1}{r^{6n}}$, or $\overline{1 - \frac{1}{r^n}}^6$. But this Expectation of A ought to be made equal to $\frac{1}{2}$, ſince by ſuppoſition he has an equal Chance to win or loſe: Hence will ariſe the Equation $\overline{1 - \frac{1}{r^n}}^6 = \frac{1}{2}$ or $r^n = \frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, from which it may be concluded that $n \times \text{Log.}\ r$, or $n \times \text{Log.}\ \frac{p+1}{p} = \text{Log.}\ \frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, and conſequently that n is equal to the Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$ divided by the Logarithm of $\frac{6}{5} = \frac{p+1}{p}$. And the ſame Demonſtration will hold in any other Caſe.

EXAMPLE I.

TO find in how many Trials A may with equal Chance undertake to throw all the Faces of a common Die.

The

The Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}} = 0.9621753$; the Logarithm of $\frac{p+1}{p}$ or $\frac{6}{5} = 0.0791812$: Wherefore $n = \frac{0.9621753}{0.0791812} = 12+$. From hence it may be concluded that in 12 Caſts *A* has the worſt of the Lay, and in 13 the beſt of it.

EXAMPLE II.

TO find in how many Trials, *A* may with equal Chance, with a Die of Thirty-ſix Faces, undertake to throw Six determinate Faces; or, in how many Trials he may with a Pair of common Dice undertake to throw all the Doublets.

The Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$ being 0.9621753, and the Logarithm of $\frac{p+1}{p}$ or $\frac{36}{35}$ being 0.0122345; it follows that the number of Caſts requiſite to that effect is $\frac{0.9621753}{0.0122345}$ or 79 nearly.

But if it were the Law of the Play, that the Doublets muſt be thrown in a given Order, and that any Doublet Happening to be thrown out of its turn ſhould go for nothing; then the throwing of the Six Doublets would be like the throwing of the two Aces Six times; to produce which effect the number of Caſts requiſite would be found by Multiplying 35 by 5.668, as appears from our VII*th*. Problem, and conſequently would be about 198.

N. B. the Fraction $\frac{1}{1-\sqrt[f]{\frac{1}{2}}}$ may be reduced to $\frac{\sqrt[f]{2}}{\sqrt[f]{2}-1}$ which will Facilitate the taking of its Logarithm.

PROBLEM XXXI.

IF A, B, C *Play together on the following conditions*; *Firſt*, *that they ſhall each of them Stake* 1 l. *Secondly*, *that* A *and* B *ſhall begin the Play*; *Thirdly*, *that the Loſer ſhall yield his place to the third Man*, *which is to be obſerved conſtantly afterwards*; *Fourthly*, *that the Loser shall be fined a certain Sum* p, *which is to serve to increase the common Stock; Lastly, that he shall Win the whole Sum deposited at first, and increased by the several Fines, who shall first beat the other two successively: 'Tis demanded what is the Advantage of* A *and* B, *whom we suppose to begin the Play.*

SOLUTION.

LEt BA signifie that B beats A, and AC that A beats C; and let always the first Letter denote the Winner, and the second the Loser.

Let us suppose that B beats A the first time: Then let us inquire what the Probability is that the Set shall be ended in any given number of Games; and also what is the Probability which each Gamester has of winning the Set in that given number of Games.

First, If the Set be ended in two Games, B must necessarily be the winner; for by Hypothesis he wins the first time: Which may be expressed as follows.

$$\begin{array}{r|l} 1 & BA \\ \hline 2 & BC \end{array}$$

Secondly, If the Set be ended in Three Games, C must be the winner; as appears by the following Scheme.

$$\begin{array}{r|l} 1 & BA \\ \hline 2 & CB \\ 3 & CA \end{array}$$

Thirdly, If the Set be ended in Four Games, A must be the winner; as appears by this Scheme.

$$\begin{array}{r|l} 1 & BA \\ \hline 2 & CB \\ 3 & AC \\ 4 & AB \end{array}$$

Fourthly, If the Set be ended in Five Games, B must be the winner; which is thus expressed,

$$\begin{array}{r|l} 1 & BA \end{array}$$

1	*B A*
2	*C B*
3	*A C*
4	*B A*
5	*B C*

Fifthly, If the Set be ended in Six Games, *C* muſt be the winner; as will appear by ſtill following the ſame Proceſs, thus,

1	*B A*
2	*C B*
3	*A C*
4	*B A*
5	*C B*
6	*C A*

And this Proceſs recurring continually in the ſame Order needs not be proſecuted any farther.

Now the Probability that the firſt Scheme ſhall take place is $\frac{1}{2}$, in conſequence of the ſuppoſition that *B* beats *A* the firſt time; it being an equal Chance whether *B* beat *C*, or *C* beat *A*.

And the Probability that the ſecond Scheme ſhall take place is $\frac{1}{4}$: For the Probability of *C* beating *B* is $\frac{1}{2}$, and that being ſuppoſed, the Probability of his beating *A* will alſo be $\frac{1}{2}$; wherefore the Probability of *B* beating *C*, and then *A*, will be $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$.

And from the ſame conſiderations the Probability that the Third Scheme ſhall take place is $\frac{1}{8}$: and ſo on.

Hence it will be eaſie to compoſe a Table of the Probabilities which *B*, *C*, *A* have of winning the Set in any given number of Games; and alſo of their Expectations: Which Expectations are the Probabilities of winning Multiplyed by the Stock Three depoſited at firſt, and increaſed ſucceſſively by the ſeveral Fines.

TABLE *of the Probabilities, &c.*

	B	C	A
2	$\frac{1}{2}\times\overline{3+2p}$	. . . . . .	. . . . . .
3	. . . . . .	$\frac{1}{4}\times\overline{3+3p}$	. . . . . .
4	. . . . . .	. . . . . .	$\frac{1}{8}\times\overline{3+4p}$
5	$\frac{1}{16}\times\overline{3+5p}$	. . . . . .	. . . . . .
6	. . . . . .	$\frac{1}{32}\times\overline{3+6p}$	. . . . . .
7	. . . . . .	. . . . . .	$\frac{1}{64}\times\overline{3+7p}$
8	$\frac{1}{128}\times\overline{3+8p}$	. . . . . .	. . . . . .
9	. . . . . .	$\frac{1}{256}\times\overline{3+9p}$	. . . . . .
10	. . . . . .	. . . . . .	$\frac{1}{512}\times\overline{3+10p}$
11	$\frac{1}{1024}\times\overline{3+11p}$	. . . . . .	. . . . . .
&c.			

Now the ſeveral Expectations of *B*, *C*, *A* may be ſummed up by the following Lemma.

LEMMA.

$\frac{n}{b} + \frac{n+d}{bb} + \frac{n+2d}{b^3} + \frac{n+3d}{b^4} + \frac{n+4d}{b^5}$ &c. *Ad infinitum* is equal to $\frac{n}{b-1} + \frac{d}{\overline{b-1}^2}$.

Let the Expectations of *B* be divided into two Series, *viz.*

$$\frac{3}{2} + \frac{3}{16} + \frac{3}{128} + \frac{3}{1024} \text{ \&c.}$$

$$+ \frac{2p}{2} + \frac{5p}{16} + \frac{8p}{128} + \frac{11p}{1024} \text{ \&c.}$$

The firſt Series conſtitutes a Geometric Progreſſion continually decreaſing, whoſe ſum will be found to be $\frac{12}{7}$.

The ſecond Series may be reduced to the form of the Series in our Lemma, and may be thus expreſt,

$\frac{p}{2}\times$

$\frac{p}{2} \times \frac{2}{1} + \frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$ &c. Wherefore dividing the whole by $\frac{p}{2}$, and laying aside the Term 2, we shall have the Series $\frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$, &c. which has the same form as the Series of the Lemma, and may be compared with it: Let therefore n be made $= 5$, $d = 3$ and $b = 8$, and the sum of this Series will be $\frac{5}{7} + \frac{3}{49}$, or $\frac{38}{49}$; to this adding the first Term 2, which had been laid aside, the new sum will be $\frac{136}{49}$; and that being Multiplied by $\frac{p}{2}$, the Product will be $\frac{68}{49}p$, which is the sum of the second Series expressing the Expectations of B: From hence it may be concluded, that all the Expectations of B contained in both the abovementioned Series will be equal to $\frac{12}{7} + \frac{68}{49}p$.

And by the help of the foregoing Lemma it will be found likewise that all the Expectations of C will be equal to $\frac{6}{7} + \frac{48}{49}p$.

It will also be found that all the Expectations of A will be $= \frac{3}{7} + \frac{31}{49}p$.

Hitherto we have determined the several Expectations of the Gamesters, upon the sum by them deposited at first, as also upon the Fines by which the common Stock is increased: It remains now to Estimate the several Risks of their being Fined; that is to say, the sum of the Probabilities of their being Fined multiplyed by the respective Quantities of the Fine.

Now after the supposition made of A being beat the first time, by which he is obliged to lay down his Fine p, B and C have an equal Chance of being Fined after the second Game, which makes the Risk of each to be $= \frac{1}{2}p$, as appears by the following Scheme.

$$\frac{BA}{CB} \text{ or } \frac{BA}{BC}$$

In the like manner, it will be found that both C and A have one Chance in four for their being Fined after the Third Game, and consequently that the Risk of each is $\frac{1}{4}p$, according to the following Scheme.

$$\begin{array}{ccc} \underline{BA} & & \underline{BA} \\ CB & \text{or} & CB \\ AC & & CA \end{array}$$

And by the like Proceſs it will be found that the Risk of B and C after the fourth Game is $\frac{1}{8}p$.

Hence it will be eaſie to compoſe the following Table which expreſſes the Risks of each Gameſter.

TABLE *of* RISKS.

	B	C	A
2	$\frac{1}{2}p$	$\frac{1}{2}p$	- - - - - - -
3	- - - - - - -	$\frac{1}{4}p$	$\frac{1}{4}p$
4	$\frac{1}{8}p$	- - - - - - -	$\frac{1}{8}p$
5	$\frac{1}{16}p$	$\frac{1}{16}p$	- - - - - - -
6	- - - - - - -	$\frac{1}{32}p$	$\frac{1}{32}p$
7	$\frac{1}{64}p$	- - - - - - -	$\frac{1}{64}p$
8	$\frac{1}{128}p$	$\frac{1}{128}p$	- - - - - - -
9	- - - - - - -	$\frac{1}{256}p$	$\frac{1}{256}p$
&c.			

In the Column belonging to B, if the vacant places were filled up, and the Terms $\frac{1}{4}p$, $\frac{1}{32}p$, $\frac{1}{256}p$ &c. were Interpoled, the Sum of the Risks of B would compoſe one uninterrupted Geometric Progreſſion, whoſe Sum would be $= p$; But the Terms interpoled conſtitute a Geometric Progreſſion whoſe Sum is $= \frac{2}{7}p$: Wherefore, if from p there be ſubtracted $\frac{2}{7}p$, there will remain $\frac{5}{7}p$ for the Sum of the Risks of B.

In like manner it will be found that the Sum of the Risks of C will be $= \frac{6}{7}p$.

And

And the Sum of the Risks of *A*, after his being Fined the firſt time, will be $= \frac{3}{7}p$.

Now if from the ſeveral Expectations of the Gameſters there be ſubtracted each Man's Stake, as alſo the Sum of his Risks, there will remain the clear Gain or Loſs of each of them.

Wherefore, from the Expectations of B	$= \frac{12}{7} + \frac{68}{49}p$
Subtracting *firſt* his Stake	$= 1$
Then the Sum of his Risks	$= \frac{5}{7}p$
There remains the clear Gain of B	$= \frac{5}{7} + \frac{33}{49}p$

Likewiſe, from the Expectations of C	$= \frac{6}{7} + \frac{48}{49}p$
Subtracting *firſt* his Stake	$= 1$
Then the Sum of his Risks	$= \frac{6}{7}p$
There remains the clear Gain of C	$= -\frac{1}{7} + \frac{6}{49}p$

In like manner, from the Expectation of A	$= \frac{3}{7} + \frac{31}{49}p$
Subtracting, *Firſt*, his Stake	$= 1$
Secondly, the Sum of his Risks	$= \frac{3}{7}p$
Laſtly, the Fine p due to the Stock by the Loſs of the firſt Game	$= p$
There remains the clear Gain of A	$= -\frac{4}{7} - \frac{39}{49}p$

But we have ſuppoſed in the beginning of the Game that *A* was beat; whereas *A* had the ſame Chance to beat *B*, as *B* had to beat him: Wherefore dividing the Sum of the Gains of *B* and *A* into two equal Parts, each part will be $\frac{1}{14} - \frac{3}{49}p$; which conſequently muſt be reputed to be as the Gain of each of them.

Corollary I. The Gain of *C* being $-\frac{1}{7} + \frac{6}{49}p$, Let that be made $= 0$. Then p will be found $= \frac{7}{6}$. If therefore the Fine has the ſame proportion to each Man's Stake as 7 has to 6, the Gameſters play all upon equal Terms: But if the Fine bears a leſs proportion to the Stake than 7 to to 6, C has the diſadvantage: Thus, Suppoſing $p = 1$, his Loſs would be $\frac{1}{49}$. But if the Fine bears a greater proportion to the Stake than 7 to 6, C has the Advantage.

Corollary II. If the Stake were conſtant, that is, if there were no Fines, then the Probabilities of winning would be reſpectively proportional to the Expectations; wherefore ſuppoſing $p = 0$, the Expectations of the Gameſters, or their Probabilities of winning, will be as $\frac{12}{7}$, $\frac{6}{7}$, $\frac{3}{7}$, or, as 4, 2, 1: But the increaſe of the Stock cauſes no alteration in the Probabilities of winning, and conſequently thoſe Probabilities are, in the Caſe of this Problem, as 4, 2, 1; whereof the firſt belongs to B after his beating A the firſt time; the ſecond to C, and the third to A: Wherefore 'tis Five to Two, before the Play begins, that either A or B wins the Set; and Five to Four that one of them, that ſhall be fixt upon, wins it.

Corollary III. If the proportion of Skill between the Gameſters A, B, C be as a, b, c reſpectively, and that the reſpective Probabilities of winning, in any number of Games after the firſt, wherein B is Suppoſed to beat A, be denoted by B', B'''', B^{VII}, B^{X} &c, C'', C^{V}, C^{VIII}, C^{XI} &c. A''', A^{VI}, A^{IX}, A^{XII}, &c. it will be found, by the bare inſpection of the Schemes belonging to the Solution of the foregoing Problem, that

$$B' = \frac{b}{b+c}$$

$$C'' = \frac{c}{c+b} \times \frac{c}{c+a}$$

$$A''' = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{a}{a+b}$$

$$B'''' = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{b}{b+a} \times \frac{b}{b+c}$$

$$C^{V} = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{b}{b+a} \times \frac{c}{c+b} \times \frac{c}{c+a}$$

$$A^{VI} = \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{b}{b+a} \times \frac{c}{c+b} \times \frac{a}{a+c} \times \frac{a}{a+b}.$$

&c.

Let $\frac{b}{b+a} \times \frac{c}{c+b} \times \frac{a}{a+c}$ be made $= m$; then it will plainly appear that the ſeveral Probabilities of winning will compoſe each of them a Geometric Progreſſion, for

$B''' =$

$B^{IIII} = m\ B^{I}$	$C^{V} = m\ C^{I}$	$A^{VI} = m\ A^{III}$
$B^{VII} = m\ B^{IIII}$	$C^{VIII} = m\ C^{V}$	$A^{IX} = m\ A^{VI}$
$B^{X} = m\ B^{VII}$	$C^{XI} = m\ C^{VIII}$	$A^{XII} = m\ A^{IX}$
&c.	&c.	&c.

Hence a Table of Expectations and Risks may eaſily be formed as above; and the reſt of the Solution carried on by following exactly the ſteps of the former.

When the Solution is brought to its concluſion, it will be neceſſary to make an allowance for the ſuppoſition made that *B* beats *A* the firſt time, which may be done thus,

Let *P* be the Gain of *B*, when expreſt by the Quantities *a*, *b*, *c*, and *Q* the Gain of *A*, when expreſt by the ſame: Change *a* into *b* and *b* into *a*, in the Quantity *Q*; then the Quantity reſulting from this Change will be the Gain of *B*, in caſe he be ſuppoſed to loſe the firſt Game. Let this Quantity therefore be called *R*, and then the Gain of *B*, to be eſtimated before the Play begins, will be $\frac{bP + aR}{b + a}$.

PROBLEM XXXII.

IF Four Gameſters A, B, C, D *Play on the conditions of the foregoing Problem, and he be to be reputed the Winner, who ſhall beat the other Three ſucceſſively: What is the Advantage of* A *and* B, *whom we Suppoſe to begin the Play?*

SOLUTION.

LET *BA* denote, as in the preceding Problem, that *B* beats *A*, and *AC* that *A* beats *C*; and generally let the firſt Letter always denote the Winner and the ſecond the Loſer.

Let it be Suppoſed alſo that *B* beats *A* the firſt time: Then let it be inquired what is the Probability that the Play ſhall be ended in any given number of Games; as alſo what is the Probability which each Gameſter has of winning the Set in that given number of Games.

Z *Firſt*,

First, If the Set be ended in Three Games, *B* muſt neceſſarily be the winner: Since by Hypotheſis he beats *A* the firſt Game, which is expreſſed as follows,

1	*BA*
2	*BC*
3	*BD*

Secondly, If the Set be ended in Four Games, *C* muſt be the winner; as it thus appears.

1	*BA*
2	*CB*
3	*CD*
4	*CA*

Thirdly, If the Set be ended in Five Games, *D* will be the winner; for which he has two Chances, as it appears by the following Scheme.

1	*BA*		*BA*
2	*CB*		*BC*
3	*DC*	or	*DB*
4	*DA*		*DA*
5	*DB*		*DC*

Fourthly, If the Set be ended in Six Games, *A* will be the winner; and he has three Chances for it, which are thus collected,

1	*BA*	*BA*	*BA*
2	*CB*	*CB*	*BC*
3	*DC*	*CD*	*DB*
4	*AD*	*AC*	*AD*
5	*AB*	*AB*	*AC*
6	*AC*	*AD*	*AB*

Fifthly, If the Set be ended in Seven Games, then *B* will have three Chances to be the winner, and *C* will have two; thus,

1 | *BA*

1	*BA*	*BA*	*BA*	*BA*	*BA*
2	*CB*	*CB*	*CB*	*BC*	*BC*
3	*DC*	*DC*	*CD*	*DB*	*DB*
4	*AD*	*DA*	*AC*	*AD*	*DA*
5	*BA*	*BD*	*BA*	*CA*	*CD*
6	*BC*	*BC*	*BD*	*CB*	*CB*
7	*BD*	*BA*	*BC*	*CD*	*CA*

Sixthly, If the Set be ended in Eight Games, then *D* will have two Chances to be the Winner, *C* will have three, and *B* alſo three, thus

1	*BA*	*BA*	*BA*	*BA*	*BA*	*BA*	*BA*	*BA*
2	*CB*	*CB*	*CB*	*CB*	*CB*	*BC*	*BC*	*BC*
3	*DC*	*DC*	*DC*	*CD*	*CD*	*DB*	*DB*	*DB*
4	*AD*	*AD*	*DA*	*AC*	*AC*	*AD*	*AD*	*DA*
5	*BA*	*AB*	*BD*	*BA*	*AB*	*CA*	*AC*	*CD*
6	*CB*	*CA*	*CB*	*DB*	*DA*	*BC*	*BA*	*BC*
7	*CD*	*CD*	*CA*	*DC*	*DC*	*BD*	*BD*	*BA*
8	*CA*	*CB*	*CD*	*DA*	*DB*	*BA*	*BC*	*BD*

Let now the Letters by which the winners are denoted be written in Order, prefixing to them the Numbers which expreſs their ſeveral Chances for winning; in this manner,

3	1 B
4	1 C
5	2 D
6	3 A
7	3 B + 2 C
8	3 C + 2 D + 3 B
9	3 D + 2 A + 3 C + 3 D + 2 A
10	3 A + 2 B + 3 D + 3 A + 2 B + 3 A + 2 C + 3 D
&c.	

Then

Then Examining the formation of theſe Letters, it will appear; *Firſt*, that the Letter *B* is always found ſo many times in any Rank, as the Letter *A* is found in the two preceding Ranks: *Secondly*, that *C* is found ſo many times in any Rank, as *B* is found in the preceding Rank, and *D* in the Rank before that. *Thirdly*, that *D* is found ſo many times in each Rank, as *C* is found in the preceding, and *B* in the Rank before that: And *Fourthly*, that *A* is found ſo many times in each, as *D* is found in the preceding Rank, and *C* in the Rank before that.

From whence it may be concluded, that the Probability which the Gameſter *B* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *A* has of winning it one Game ſooner, together with $\frac{1}{4}$ of the Probability which *A* has of winning it two Games ſooner.

The Probability which *C* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *B* has of winning it one Game ſooner, together with $\frac{1}{4}$ of the Probability which *D* has of winning it two Games ſooner.

The Probability which *D* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *C* has of winning it one Game ſooner, and alſo $\frac{1}{4}$ of the Probability which *B* has of winning it two Games ſooner.

The Probability which *A* has of winning the Set, in any given number of Games, is $\frac{1}{2}$ of the Probability which *D* has of winning it one Game ſooner, and alſo $\frac{1}{4}$ of the Probability which *C* has of winning it two Games ſooner.

Theſe things being obſerved, it will be eaſie to compoſe a Table of the Probabilities which *B*, *C*, *D*, *A* have of winning the Set in any given number of Games; as alſo of their Expectations, which will be as follows.

A TA-

TABLE *of the Probabilities*, &c.

		B	C	D	A
′	3	$\frac{1}{4}\times\overline{4+3p}$	- - - - - -	- - - - - -	- - - - - -
″	4	- - - - - -	$\frac{1}{8}\times\overline{4+4p}$	- - - - - -	- - - - - -
‴	5	- - - - - -	- - - - - -	$\frac{2}{16}\times\overline{4+5p}$	- - - - - -
⁗	6	- - - - - -	- - - - - -	- - - - - -	$\frac{3}{32}\times\overline{4+6p}$
v	7	$\frac{3}{64}\times\overline{4+7p}$	$\frac{2}{64}\times\overline{4+7p}$	- - - - - -	- - - - - -
v′	8	$\frac{3}{128}\times\overline{4+8p}$	$\frac{3}{128}\times\overline{4+8p}$	$\frac{2}{128}\times\overline{4+8p}$	- - - - - -
v″	9	- - - - - -	$\frac{2}{256}\times\overline{4+9p}$	$\frac{6}{256}\times\overline{4+9p}$	$\frac{4}{256}\times\overline{4+9p}$
v‴	10	$\frac{4}{512}\times\overline{4+10p}$	$\frac{2}{512}\times\overline{4+10p}$	$\frac{6}{512}\times\overline{4+10p}$	$\frac{9}{512}\times\overline{4+10p}$
′x	11	$\frac{13}{1024}\times\overline{4+11p}$	$\frac{10}{1024}\times\overline{4+11p}$	$\frac{2}{1024}\times\overline{4+11p}$	$\frac{9}{1024}\times\overline{4+11p}$
x	12	$\frac{18}{2048}\times\overline{4+12p}$	$\frac{19}{2048}\times\overline{4+12p}$	$\frac{14}{2048}\times\overline{4+12p}$	$\frac{4}{2048}\times\overline{4+12p}$
&c.	&c.				

The Terms whereof each Column of this Table is composed, being not eaſily ſummable by any of the known Methods, it will be convenient, in order to find their Sums, to uſe the following *Analyſis*.

Let $B' + B'' + B''' + B'''' + B^{v} + B^{v'}$ &c. repreſent the reſpective Probabilities which B has of winning the Set, in any number of Games, anſwering to 3, 4, 5, 6, 7, 8 &c. and let the ſum of theſe Probabilities *Ad infinitum* be ſuppoſed $= y$.

In the ſame manner, let $C' + C'' + C''' + C'''' + C^{v} + C^{v'}$ &c. repreſent the Probabilities which C has of winning, which ſuppoſe $= z$.

Let the like Probabilities which D has of winning be repreſented by $D' + D'' + D''' + D'''' + D^{v} + D^{v'}$ &c. which ſuppoſe $= v$.

Laſtly, Let the Probabilities which A has of winning be repreſented by $A' + A'' + A''' + A'''' + A^{v} + A^{v'}$ &c. which ſuppoſe $= x$.

Now from the Obſervations ſet down before the Table of Probabilities, it will follow, that

 $B' =$

$$B' = B'$$
$$B'' = B''$$
$$B''' = \frac{1}{2}A'' + \frac{1}{4}A'$$
$$B'''' = \frac{1}{2}A''' + \frac{1}{4}A''$$
$$B^{V} = \frac{1}{2}A'''' + \frac{1}{4}A'''$$
$$B^{VI} = \frac{1}{2}A^{V} + \frac{1}{4}A''''$$
&c.

From which Scheme we may deduce the Equation following, $y = \frac{1}{4} + \frac{3}{4}x$: For the Sum of the Terms in the first Column is equal to the Sum of the Terms in the other two. But the Sum of the Terms in the first Column is y by Hypothesis; wherefore y ought to be made equal to the Sum of the Terms in the other two Columns.

In order to find the Sum of the Terms of the second Column, I argue thus,

$A' + A'' + A''' + A'''' + A^{V} + A^{VI}$ is $= x$ by Hypoth.
or $A'' + A''' + A'''' + A^{V} + A^{VI}$ is $= x - A'$
and $\frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A'''' + \frac{1}{2}A^{V} + \frac{1}{2}A^{VI}$ is $= \frac{1}{2}x - \frac{1}{2}A'$

Then adding $B' + B''$ on both sides of the last Equation, we shall have

$$B' + B'' + \frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A'''' + \frac{1}{2}A^{V} + \frac{1}{2}A^{VI}\ \&c.$$
$$= \frac{1}{2}x - \frac{1}{2}A' + B' + B''.$$

But $A' = 0$, $B' = \frac{1}{4}$, $B'' = 0$, as appears from the Table: Wherefore the Sum of the Terms of the second Column is equal to $\frac{1}{2}x + \frac{1}{4}$.

The Sum of the Terms of the third Column is $\frac{1}{4}x$ by Hypothesis; and consequently the Sum of the Terms in the second and third Columns is $= \frac{3}{4}x + \frac{1}{4}$. From whence it follows that the Equation $y = \frac{1}{4} + \frac{3}{4}x$ had been rightly determined.

In

In the ſame manner, if we write

$$\begin{array}{lcl} C' & = & C' \\ C'' & = & C'' \\ C''' & = & \frac{1}{2}B'' + \frac{1}{4}D' \\ C'''' & = & \frac{1}{2}B''' + \frac{1}{4}D'' \\ C^{V} & = & \frac{1}{2}B'''' + \frac{1}{4}D''' \\ C^{VI} & = & \frac{1}{2}B^{V} + \frac{1}{4}D'''' \\ \&c. & & \end{array}$$

By a reaſoning like the former we ſhall at length come at the Equation $z = \frac{1}{2}y + \frac{1}{4}v$.

So likewiſe if we write

$$\begin{array}{lcl} D' & = & D' \\ D'' & = & D'' \\ D''' & = & \frac{1}{2}C'' + \frac{1}{4}B' \\ D'''' & = & \frac{1}{2}C''' + \frac{1}{4}B'' \\ D^{V} & = & \frac{1}{2}C'''' + \frac{1}{4}B''' \\ D^{VI} & = & \frac{1}{2}C^{V} + \frac{1}{4}B'''' \\ \&c. & & \end{array}$$

We ſhall deduce the Equation $v = \frac{1}{2}z + \frac{1}{4}y$.

Laſtly, if after the ſame manner we write

$$\begin{array}{lcl} A' & = & A' \\ A'' & = & A'' \\ A''' & = & \frac{1}{2}D'' + \frac{1}{4}C' \\ A'''' & = & \frac{1}{2}D''' + \frac{1}{4}C'' \\ A^{V} & = & \frac{1}{2}D'''' + \frac{1}{4}C''' \\ A^{VI} & = & \frac{1}{2}D^{V} + \frac{1}{4}C'''' \\ \&c. & & \end{array}$$

We ſhall obtain the Equation $x = \frac{1}{2}v + \frac{1}{4}z$.

Now,

Now these Four Equations being resolved, it will be found that

$$B' + B'' + B''' + B'''' + B^{v} + B^{vi} \text{ \&c.} = y = \frac{56}{149},$$
$$C' + C'' + C''' + C'''' + C^{v} + C^{vi} \text{ \&c.} = z = \frac{36}{149},$$
$$D' + D'' + D''' + D'''' + D^{v} + D^{vi} \text{ \&c.} = v = \frac{32}{149},$$
$$A' + A'' + A''' + A'''' + A^{v} + A^{vi} \text{ \&c.} = x = \frac{25}{149}.$$

These Values being once found, let b, c, d, a, which are commonly employed to denote known Quantities, be respectively substituted in the room of them; to the end that the Letters y, z, v, x may now be employed to denote other unknown Quantities.

Hitherto we have been determining the Probabilities of winning: But in order to find the Expectations of the Gamesters, each Term of the Series expressing these Probabilities, is to be multiplyed by the respective Terms of the following Series; $4+3p$, $4+4p$, $4+5p$, $4+6p$, &c.

The first part of each Product being no more than a Multiplication by 4, the sums of all the first parts of those Products are only the sums of the Probabilities multiplied by 4; and consequently are $4b$, $4c$, $4d$, and $4a$ respectively.

But to find the Sums of the other parts,

Let $3\,B'p + 4\,B''p + 5\,B'''p + 6\,B''''p$ &c. be $= p\,y$,

$$3\,C'p + 4\,C''p + 5\,C'''p + 6\,C''''p \text{ \&c.} = p\,z,$$
$$3\,D'p + 4\,D''p + 5\,D'''p + 6\,D''''p \text{ \&c.} = p\,v,$$
$$3\,A'p + 4\,A''p + 5\,A'''p + 6\,A''''p \text{ \&c.} = p\,x,$$

Now Since

$$\begin{aligned} 3\,B' &= 3\,B' \\ 4\,B'' &= 4\,B'' \\ 5\,B''' &= \tfrac{5}{2}A'' + \tfrac{5}{4}A' \\ 6\,B'''' &= \tfrac{6}{2}A''' + \tfrac{6}{4}A'' \\ 7\,B^{v} &= \tfrac{7}{2}A'''' + \tfrac{7}{4}A''' \\ 8\,B^{vi} &= \tfrac{8}{2}A^{v} + \tfrac{8}{4}A'''' \end{aligned}$$

&c.

It

It follows, that $y = \frac{3}{4} + \frac{3}{4}x + a$. For the first Column is $= y$, by *Hypothesis*.

Again, $3A' + 4A'' + 5A''' + 6A'''' + 7A^v$ &c. $= x$ by *Hypothesis*.

But $A' + A'' + A''' + A'''' + A^v$ &c. has been found $= a$,

Wherefore adding these two Equations together, we shall have $4A' + 5A'' + 6A''' + 7A'''' + 8A^v$ &c. $= x + a$.
or $\frac{4}{2}A' + \frac{5}{2}A'' + \frac{6}{2}A''' + \frac{7}{2}A'''' + \frac{8}{2}A^v$ &c. $= \frac{1}{2}x + \frac{1}{2}a$.

Now the Terms of this last Series, together with $3B' + 4B''$, compose the second Column: But $3B' = \frac{3}{4}$, and $4B'' = 0$, as appears from the Table. Consequently the sum of the Terms of the second Column is $= \frac{3}{4} + \frac{1}{2}x + \frac{1}{2}a$.

By the same Method of proceding, it will be found, that the sum of the Terms of the third Column is $= \frac{1}{4}x + \frac{1}{2}a$.

From whence it follows that $y = \frac{3}{4} + \frac{1}{2}x + \frac{1}{2}a + \frac{1}{4}x + \frac{1}{2}a$, or $y = \frac{3}{4} + \frac{3}{4}x + a$.

In the same manner if we write

$$\begin{aligned}
3C' &= 3C' \\
4C'' &= 4C' \\
5C''' &= \tfrac{5}{2}B'' + \tfrac{5}{4}D' \\
6C'''' &= \tfrac{6}{2}C''' + \tfrac{6}{4}D'' \\
7C^v &= \tfrac{7}{2}B'''' + \tfrac{7}{4}D''' \\
8C^{vi} &= \tfrac{8}{2}B^v + \tfrac{8}{4}D''''
\end{aligned}$$
&c.

We shall from thence deduce the Equation $z = \frac{1}{2}y + \frac{1}{2}b + \frac{1}{4}v + \frac{1}{2}d$.

So likewiſe in the ſame manner, if we write

$$\begin{aligned} 3\,D' &= 3\,D' \\ 4\,D'' &= 4\,D'' \\ 5\,D''' &= \tfrac{5}{2}\,C' + \tfrac{5}{4}\,B' \\ 6\,D'''' &= \tfrac{6}{2}\,C''' + \tfrac{6}{4}\,B'' \\ 7\,D^{v} &= \tfrac{7}{2}\,D'''' + \tfrac{7}{4}\,B''' \\ 8\,D^{vi} &= \tfrac{8}{2}\,D^{v} + \tfrac{8}{4}\,B'''' \end{aligned}$$

&c.

Laſtly, if after the ſame manner we write

$$\begin{aligned} 3\,A' &= 3\,A' \\ 4\,A'' &= 4\,A'' \\ 5\,A''' &= \tfrac{5}{2}\,D' + \tfrac{5}{4}\,C' \\ 6\,A'''' &= \tfrac{6}{2}\,D''' + \tfrac{6}{4}\,C'' \\ 7\,A^{v} &= \tfrac{7}{2}\,D'''' + \tfrac{7}{4}\,C''' \\ 8\,A^{vi} &= \tfrac{8}{2}\,D^{v} + \tfrac{8}{4}\,C'''' \end{aligned}$$

&c.

We ſhall deduce the two following Equations, *viz.* $v = \frac{1}{2}z + \frac{1}{2}c + \frac{1}{4}y + \frac{1}{2}b$. And $x = \frac{1}{2}v + \frac{1}{2}d + \frac{1}{4}z + \frac{1}{2}c$.

Now the foregoing Equations being Solved, and the values of b, c, d, a reſtored, it will be found that $y = \frac{45536}{22201}$, $z = \frac{38724}{22201}$, $v = \frac{37600}{22201}$, $x = \frac{33547}{22201}$.

From which we may conclude, that the ſeveral Expectations of B, C, D, A are reſpectively, *Firſt*, $4 \times \frac{56}{149} + \frac{45536}{22201}p$. *Secondly*, $4 \times \frac{36}{149} + \frac{38724}{22201}p$. *Thirdly*, $4 \times \frac{32}{149} + \frac{37600}{22201}p$. *Fourthly*, $4 \times \frac{25}{149} + \frac{33547}{22201}p$.

The

The Expectations of the Gamesters being thus found, it will be necessary to find the Risks of their being Fined, or otherwise what sum each of them ought justly to give to have their Fines Insured. In order to which, let us form so many Schemes for expressing the Probabilities of the Fines as are sufficient to find the Law of their Process.

And *First*, we may observe, that upon the supposition of *B* beating *A* the first Game, in consequence of which *A* is to be Fined, *B* and *C* have one Chance each for being Fined the second Game, as it thus appears

$$\begin{array}{c|cc} 1 & BA & BA \\ \cline{2-3} 2 & CB & BC \end{array}$$

Secondly, that *C* has one Chance in four for being Fined the third Game, *B* one Chance likewise, and *D* two; according to the following Scheme,

$$\begin{array}{c|cccc} 1 & BA & BA & BA & BA \\ \cline{2-5} 2 & CB & CB & BC & BC \\ 3 & DC & CD & DB & BD \end{array}$$

Thirdly, that *D* has two Chances in eight for being Fined the fourth Game, that *A* has three and *C* one; according to the following Scheme,

$$\begin{array}{c|cccccc} 1 & BA & BA & BA & BA & BA & BA \\ \cline{2-7} 2 & CB & CB & CB & CB & BC & BC \\ 3 & DC & DC & CD & CD & DB & DB \\ 4 & AD & DA & AC & CA & AD & DA \end{array}$$

N. B. The two Combinations *BA*, *BC*, *BD*, *AB*, and *BA*, *BC*, *BD*, *BA* are omitted in this Scheme, as being superfluous; their disposition shewing that the Set must have been ended in three Games, and consequently not affecting the Gamesters as to the Probability of their being Fined the fourth Game; Yet the number of all the Chances must be reckoned as being Eight; since the Probability of any one circumstance is but $\frac{1}{8}$.

These

These Schemes being continued, it will easily be perceived that the circumstances under which the Gamesters find themselves, in respect of their Risks of being Fined, stand related to one another in the same manner as were their Probabilities of Winning; from which consideration a Table of the Risks may easily be composed as follows.

A TABLE *of* RISKS &c.

		B	C	D	A
$'$	2	$\frac{1}{2}p$	$\frac{1}{2}p$	------	------
$''$	3	$\frac{1}{4}p$	$\frac{1}{4}p$	$\frac{2}{4}p$	------
$'''$	4	------	$\frac{1}{8}p$	$\frac{2}{8}p$	$\frac{3}{8}p$
$''''$	5	$\frac{3}{16}p$	$\frac{2}{16}p$	$\frac{2}{16}p$	$\frac{3}{16}p$
v	6	$\frac{6}{32}p$	$\frac{5}{32}p$	$\frac{2}{32}p$	$\frac{3}{32}p$
v'	7	$\frac{6}{64}p$	$\frac{8}{64}p$	$\frac{8}{64}p$	$\frac{4}{32}p$
v''	8	$\frac{7}{128}p$	$\frac{8}{128}p$	$\frac{14}{128}p$	$\frac{13}{128}p$
v'''	9	$\frac{17}{256}p$	$\frac{15}{256}p$	$\frac{14}{256}p$	$\frac{22}{256}p$
&c.	&c.				

Wherefore supposing $B' + B'' + B'''$ &c. $C' + C'' + C'''$ &c. $D' + D'' + D'''$ &c. $A' + A'' + A'''$ &c. to represent the several Probabilities; and supposing that the several sums of these Probabilities are respectively equal to y, x, z, v, we shall have the following Schemes and Equations

$$B' = B'$$
$$B'' = B''$$
$$B''' = \frac{1}{2}A'' + \frac{1}{4}A'$$
$$B'''' = \frac{1}{2}A''' + \frac{1}{4}A''$$
$$B^{v} = \frac{1}{2}A'''' + \frac{1}{4}A'''$$
$$B^{v'} = \frac{1}{2}A^{v} + \frac{1}{4}A''''$$
&c.

Hence $y = \frac{3}{4}' + \frac{3}{4}x$.

$C' =$

$$\begin{aligned}
C' &= C' \\
C'' &= C'' \\
C''' &= \tfrac{1}{2} B'' + \tfrac{1}{4} D' \\
C'''' &= \tfrac{1}{2} B''' + \tfrac{1}{4} D'' \\
C^{v} &= \tfrac{1}{2} B'''' + \tfrac{1}{4} D''' \\
C^{vi} &= \tfrac{1}{2} B^{v} + \tfrac{1}{4} D''''
\end{aligned}$$

&c.

Hence $z = \frac{1}{2} + \frac{1}{2} y + \frac{1}{4} v.$

$$\begin{aligned}
D' &= D' \\
D'' &= D'' \\
D''' &= \tfrac{1}{2} C'' + \tfrac{1}{4} B' \\
D'''' &= \tfrac{1}{2} C'' + \tfrac{1}{4} B'' \\
D^{v} &= \tfrac{1}{2} C'''' + \tfrac{1}{4} B''' \\
D^{vi} &= \tfrac{1}{2} C^{v} + \tfrac{1}{4} B''''
\end{aligned}$$

&c.

Hence $v = \frac{1}{4} + \frac{1}{2} z + \frac{1}{4} y.$

$$\begin{aligned}
A' &= A' \\
A'' &= A'' \\
A''' &= \tfrac{1}{2} D'' + \tfrac{1}{4} C' \\
A'''' &= \tfrac{1}{2} D''' + \tfrac{1}{4} C'' \\
A^{v} &= \tfrac{1}{2} D'''' + \tfrac{1}{4} C''' \\
A^{vi} &= \tfrac{1}{2} D^{v} + \tfrac{1}{4} C''''
\end{aligned}$$

&c.

Hence $x = \frac{1}{2} v + \frac{1}{4} z.$

The foregoing Equations being resolved, we shall have $y = \frac{243}{149}$, $z = \frac{252}{149}$, $v = \frac{224}{149}$, $x = \frac{175}{149}$.

Let every one of those Fractions be now multiplied by p, and the Products $\frac{243}{149}p$, $\frac{252}{149}p$, $\frac{224}{149}p$, $\frac{175}{149}p$ will express the respective Risks of B, C, D, A, or the sums they might justly give to have their Fines Insured.

But if from the several Expectations of the Gamesters there be subtracted, *First*, the sums by them deposited in the beginning of the Play, and *Secondly*, the Risks of their Fines, there will remain the clear Gain or Loss of each. Wherefore

From the Expectations of B	$=$	$4 \times \frac{56}{149}$	$+ \frac{45536}{22201}p,$
Subtracting his own Stake	$=$	1	
and also the sum of the Risks	$=$		$\frac{243}{149}p,$
There remains his clear Gain	$=$	$\frac{75}{149}$	$+ \frac{9329}{22201}p.$

From the Expectations of C	$=$	$4 \times \frac{36}{149}$	$+ \frac{38724}{22201}p,$
Subtracting his own Stake	$=$	1	
and also the Sum of his Risks	$=$		$\frac{252}{149}p,$
There remains his clear Gain	$=$	$-\frac{5}{149}$	$+ \frac{1176}{22201}p.$

From the Expectations of D	$=$	$4 \times \frac{32}{149}$	$+ \frac{37600}{22201}p,$
Subtracting his own Stake	$=$	1	
and also the sum of his Risks	$=$		$+ \frac{224}{149}p,$
There remains his clear Gain	$=$	$-\frac{21}{149}$	$+ \frac{4224}{22201}p.$

From the Expectations of A	$=$	$4 \times \frac{25}{149}$	$+ \frac{33547}{22201}p,$
Subtracting his own Stake	$=$	1	
and also the Sum of his Risks	$=$		$\frac{175}{194}p,$
Lastly, the Fine due to the Stock by the loss of the first Game.	$=$		$p,$
There remains his clear Gain	$=$	$-\frac{49}{149}$	$- \frac{14729}{22201}p.$

The

The foregoing Calculation being made upon the ſuppoſition of *B* beating *A* in the beginning of the Play, which ſuppoſition neither affects *C* nor *D*, it follows that the ſum of the Gains between *B* and *A* ought to be divided equally; and their ſeveral Gains will ſtand as follows,

$$\text{Gain of}\begin{cases} A = & \frac{13}{149} - \frac{2700}{22201}p, \\ B = & \frac{13}{149} - \frac{2700}{22201}p, \\ C = & -\frac{5}{149} + \frac{1176}{22201}p, \\ D = & -\frac{21}{149} + \frac{4224}{22201}p, \end{cases}$$

$$\text{Sum of the Gains} \quad = \quad 0 \qquad 0$$

If $\frac{13}{149} - \frac{2700}{22201}p$, which is the Gain of *A* or *B*, be made $= 0$; then p will be found $= \frac{1937}{2700}$: From which it follows, that if each Man's Stake be to the Fine in the proportion of 2700 to 1937, then *A* and *B* are in this caſe neither winners nor loſers; but *C* wins $\frac{1}{225}$, which *D* loſes.

And in the like manner may be found what the proportion between the Stake and the Fine ought to be, to make *C* or *D* play without Advantage or Diſadvantage; and alſo what this proportion ought to be, to make them play with any Advantage or Diſadvantage given.

Corollary I. A ſpectator *R* might at firſt in conſideration of the Sum $4+7p$ paid him in hand, undertake to furniſh the four Gameſters with Stakes, and to pay all their Fines.

Corollary II. If the proportion of Skill between the Gameſters be given, then their Gain or Loſs may be determined by the methods uſed in this and the preceding Problem.

Corollary III. If there be never ſo many Gameſters playing on the conditions of this Problem, and the proportion of Skill between them all be ſuppoſed equal, then the Probabilities of winning, or of being Fined, may be determined as follows.

Let

Let $\overline{B'}$, $\overline{C'}$, $\overline{D'}$, $\overline{E'}$, $\overline{F'}$, $\overline{A'}$ denote the Probabilities which B, C, D, E, F, A have of winning the Set, or of being Fined, in any number of Games; and let the Probabilities of winning or being Fined in any number of Games leſs by one than the preceding, be denoted by $\overline{B''}$ $\overline{C''}$ $\overline{D''}$ $\overline{E''}$ $\overline{F''}$ $\overline{A''}$: And ſo on. Then I ſay that,

$$\overline{B'} = \frac{1}{2}\overline{A''} + \frac{1}{4}\overline{A'''} + \frac{1}{8}\overline{A''''} + \frac{1}{16}\overline{A^{v}}$$
$$\overline{C'} = \frac{1}{2}\overline{B''} + \frac{1}{4}\overline{F'''} + \frac{1}{8}\overline{E''''} + \frac{1}{16}\overline{D^{v}}$$
$$\overline{D'} = \frac{1}{2}\overline{C''} + \frac{1}{4}\overline{B'''} + \frac{1}{8}\overline{F''''} + \frac{1}{16}\overline{E^{v}}$$
$$\overline{E'} = \frac{1}{2}\overline{D''} + \frac{1}{4}\overline{C'''} + \frac{1}{8}\overline{B''''} + \frac{1}{16}\overline{F^{v}}$$
$$\overline{F'} = \frac{1}{2}\overline{E''} + \frac{1}{4}\overline{D'''} + \frac{1}{8}\overline{C''''} + \frac{1}{16}\overline{B^{v}}$$
$$\overline{A'} = \frac{1}{2}\overline{F''} + \frac{1}{4}\overline{E'''} + \frac{1}{8}\overline{D''''} + \frac{1}{16}\overline{C^{v}}$$

Corollory IV. If the Terms A, B, C, D, E, F &c. of a Series be continually decreaſing, and that the Relation which each Term of the Series has to the ſame number of preceding ones be conſtantly expreſt by the ſame number of given Fractions $\frac{1}{p}$, $\frac{1}{q}$, $\frac{1}{r}$, $\frac{1}{s}$ &c. For Example, if E be equal to $\frac{1}{p}D + \frac{1}{q}C + \frac{1}{r}B$, and F be alſo equal to $\frac{1}{p}E + \frac{1}{q}D + \frac{1}{r}C$, and ſo on: Then I ſay that all the Terms *Ad infinitum* of ſuch Series as this, may be eaſily ſummed up, by following the ſteps of the Analyſis uſed in this Problem; of which ſeveral Inſtances will be given in the Problem relating to the duration of Play.

And if the Terms of ſuch Series be multiplyed reſpectively by any Series of Terms, whoſe laſt differences are equal, then the Series reſulting fiom this multiplication is exactly ſummable.

And if there be two ſuch Series or more, and the Terms of one be reſpectively multiplyed by the correſponding Terms of the other, then the Series reſulting from this multiplication will be exactly ſummable.

Laſtly, If there be ſeveral Series ſo related to one another, that each Term in the one may have to a certain number of terms in the other certain given porportions, and that the order of theſe proportions be conſtant and uniform, then will all thoſe Series be exactly ſummable. The

The foregoing Problem having been formerly Solved by me, and Printed in the *Philoſophical Tranſactions* N° 341. Dr. *Brook Taylor*, that Excellent Mathematician, Secretary to the *Royal Society*, and my Worthy Friend, ſoon after communicated to me a very Ingenious Method of his, for finding the Relations which the Probabilities of winning bear to one-another, in the caſe of an equality of Skill between the Gameſters. The Method is as follows.

Let *BACD* repreſent the four Gameſters; let alſo the two firſt Letters repreſent that *B* beats *A* the firſt Game, and the other two the order of Play.

This being ſuppoſed, the circumſtances of the Gameſters will be repreſented in the next Game by *BCDA* or *CBDA*.

Again, the two preceding Combinations will each of them produce two more Combinations for the Game following, ſo that the Combinations for that Game will be four in all, *viz. BDAC*, *DBAC*, and *CDAB*, *DCAB*; which may be fitly repreſented by the following Scheme.

BACD

BCDA *CBDA*

BDAC *DBAC* *CDAB* *DCAB*

It appears from this Scheme, that if the Combination *CBDA* Happens, which muſt be in the ſecond Game, then *B* will be in the ſame Circumſtance wherein *A* was the Game before; the conformity of which Circumſtances lies in this, that *B* is beat by one who was juſt come into Play when he engaged him. It appears likewiſe, that if the Combination *DBAC* Happens, which muſt be in the third Game, then *B* is again in the ſame Circumſtance wherein *A* was two Games before.

But the Probability of the firſt Circumſtance is $\frac{1}{2}$, and the Probability of the ſecond is $\frac{1}{4}$.

Wherefore the Probability which *B* has of winning the Set, in any number of Games taken from the beginning, is $\frac{1}{2}$ of the Probability which *A* has of winning it in the ſame number of Games wanting one, taken from the beginning; as alſo $\frac{1}{4}$ of the Probability which he has of winning in the ſame number of Games wanting two. From which it follows, that if the Probability which B has of winning the Set, in Five Games for inſtance, and the Probabilities which A has of winning it in Four and Three, be reſpectively denoted by B^{v}, $A^{''''}$, $A^{'''}$, we ſhall have the Equation, $B^{v} = \frac{1}{2} A^{''''} + \frac{1}{4} A^{'''}$, which is conformable to what we had found before. And from the Inſpection of the ſame Scheme may likewiſe be deduced the Relations of the Probabilities of winning, as they lye between the other Gameſters. And other Schemes of this nature for any number of Gameſters may eaſily be made in imitation of this, by which the Probabilities of winning or being Fined may be determined by bare Inſpection.

PROBLEM XXXIII.

TWO *Gameſters* A *and* B, *whoſe proportion of Skill is as* a *to* b, *each having a certain number of Pieces, play together on condition that as often as* A *Wins a Game,* B *ſhall give him one Piece, and that as often as* B *Wins a Game,* A *ſhall give him one Piece; and that they ceaſe not to Play till ſuch time as either one or the other has got all the Pieces of his adverſary. Now let us ſuppoſe two Spectators* R *and* S *to lay a Wager about the Ending of the Play, the firſt of them laying that the Play will be Ended in a certain number of Games which he aſſigns; the other laying to the contrary. What is the Probability that* S *has of Winning his Wager?*

SOLUTION.

CASE I.

LET Two be the number of Pieces which each Gameſter has, let alſo Two be the number of Games about which the Wager is laid: Now becauſe two is the number of Games contended for, let $a+b$ be raiſed to its Square, *viz.* $aa+2ab+bb$; and it is plain that the Term $2ab$ favours S, and that the other two are againſt him, and conſequently that the Probability he has of Winning is $\frac{2ab}{\overline{a+b}|^{2}}$.

CASE II.

CASE II.

LET Two be the number of Pieces of each Gameſter, but let Three be the number of Games upon which the Wager is laid: Then $a+b$ being raiſed to its Cube *viz.* $a^3+3aab+3abb+b^3$, it is plain that the two Terms a^3 and b^3 are contrary to *S*, ſince they denote the number of Chances for winning three times together; 'tis plain alſo that the other Terms $3aab+3abb$ are partly for him, partly againſt him. Let theſe Terms therefore be divided into their proper parts, *viz.* $3aab$ into aab, aba, baa, and $3abb$, into abb, bab, bba. Now out of theſe Six parts there are four which are favourable to *S*; *viz.* aba, baa, abb, bab or $2aab+2abb$; from whence it follows that the Probability which *S* has of winning his Wager will be $\frac{2aab+2abb}{\overline{a+b}|^3}$: Or dividing both Numerator and Denominator by $a+b$, it will be found to be $\frac{2ab}{\overline{a+b}|^2}$, which is the ſame as before.

CASE III.

LET Two be the number of Pieces of each Gameſter, and Four the number of Games upon which the Wager is laid: Let therefore $a+b$ be raiſed to the fourth Power, which is $a^4+4a^3b+6aabb+4ab^3+b^4$. The Terms $a^4+4a^3b+4ab^3+b^4$ are wholly againſt *S*, and the only Term $6aabb$ is partly for him, partly againſt him: Let this Term therefore be divided into its Parts, *viz.* $aabb$, $abab$, $abba$, $baab$, $baba$, $bbaa$; and Four of theſe Parts $abab$, $abba$, $baab$, $baba$, or $4aabb$ will be found to favours *S*; from which it follows, that his Probability of winning will be $\frac{4aabb}{\overline{a+b}|^4}$.

CASE IV.

IF Two be the number of Peices of each Gameſter, and Five the number of Games about which the Wager is laid; the Probability which *S* has of Winning his Wager will be found to be the ſame as in the preceding Caſe, *viz.* $\frac{4aabb}{\overline{a+b}|^4}$.

GENE-

GENERALLY.

Let Two be the number of Pieces of each Gamefter, and $2+d$ the number of Games about which R and S contend, and it will be found that the Probability which S has of Winning will be $\frac{\overline{2ab}|^{1+\frac{1}{2}d}}{\overline{a+b}|^{2+d}}$. But if d be an odd Number, fubftitute $d-1$ in the room of it.

CASE V.

LET Three be the number of Pieces of each Gamefter, and $3+d$ the number of Games upon which the Wager is laid; and the Probability which S has of Winning will be $\frac{\overline{3ab}|^{1+\frac{1}{2}d}}{\overline{a+b}|^{2+d}}$. But if d be an Odd Number, you are to fubftitute $d-1$ in the room of it.

CASE VI.

IF the number of Pieces of each Gamefter be more than Three, the Expectation of S, or the Probability there is that the Play will not be Ended in a given number of Games, may be determined in the following manner.

A General R U L E *for Determining what Probability there is that the Play will not be Ended in a given number of Games.*

LET n be the number of Pieces of each Gamefter; let alfo $n+d$ be the number of Games given. Raife $a+b$ to the Power n, then cut off the two extream Terms, and multiply the remainder by $aa+2ab+bb$: then cut off again the two Extreams, and multiply again the remainder by $aa+2ab+bb$, ftill rejecting the two Extreams, and fo on, making as many Multiplycations as there are Units in $\frac{1}{2}d$; Let the laft Product be the Numerator of a Fraction whofe Denominator is $\overline{a+b}|^{n+d}$, and that Fraction will exprefs the Proba-

Probability required, or the Expectation of S. Still obſerving that if d be an Odd Number, you write $d-1$ in the room of it.

EXAMPLE I.

LET Four be the number of Pieces of each Gameſter, and Ten the number of Games given: In this Caſe $n = 4$, and $n+d = 10$. Wherefore $d = 6$, and $\frac{1}{2} d = 3$. Let therefore $a+b$ be raiſed to the Fourth Power, and rejecting continually the Extreams, let three Multiplications be made by $aa + 2ab + bb$. thus,

$$
\begin{array}{l}
a^4) + 4a^3b + 6aabb + 4ab^3 (+ b^4 \\
\qquad aa + 2ab + bb \\
\hline
\quad 4a^5b) + 6a^4bb + 4a^3b^3 \\
\qquad\qquad + 8a^4bb + 12a^3b^3 + 8aab^4 \\
\qquad\qquad\qquad + 4a^3b^3 + 6aab^4 (+ 4ab^5 \\
\hline
\qquad\qquad 14a^4bb + 20a^3b^3 + 14aab^4 \\
\qquad\qquad\quad aa + 2ab + bb \\
\hline
\qquad 14a^6bb) + 20a^5b^3 + 14a^4b^4 \\
\qquad\qquad\quad + 28a^5b^3 + 40a^4b^4 + 28a^3b^5 \\
\qquad\qquad\qquad\qquad + 14a^4b^4 + 20a^3b^5 (+ 14aab^6 \\
\hline
\qquad\qquad\qquad 48a^5b^3 + 68a^4b^4 + 48a^3b^5 \\
\qquad\qquad\qquad\quad aa + 2ab + bb \\
\hline
\qquad\qquad\qquad 48a^7b^3) + 68a^6b^4 + 48a^5b^5 \\
\qquad\qquad\qquad\qquad + 96a^6b^4 + 136a^5b^5 + 96a^4b^6 \\
\qquad\qquad\qquad\qquad\qquad + 48a^5b^5 + 68a^4b^6 (+ 48a^3b^7 \\
\hline
\qquad\qquad\qquad\qquad 164a^6b^4 + 232a^5b^5 + 164a^4b^6. \\
\hline
\end{array}
$$

Wherefore the Probability that the Play will not be ended in Ten Games will be $\frac{164a^6b^4 + 232a^5b^5 + 164a^4b^6}{\overline{a+b}|^{10}}$, which expreſſion will be reduced to $\frac{560}{1024}$ or $\frac{35}{64}$, if there be an equality of Skill between the Gameſters. Now this Fraction being ſubtracted from Unity, the remainder will be $\frac{29}{64}$, which will expreſs the Probability of the Play Ending in

 Ten

Ten Games: And consequently it is 35 to 29, that two equal Gamesters playing together, there will not be Four Stakes lost on either side in Ten Games.

N. B. The foregoing Operation may be very much contracted by omitting the Letters a and b, and restoring them after the last Multiplication; which may be done in this manner. Make $n + \frac{1}{2}d - 1 = p$, and $\frac{1}{2}d + 1 = q$: Then annex to the respective Terms resulting from the last Multiplication the literal Products $a^p b^q$, $a^{p-1} b^{q+1}$, $a^{p-2} b^{q+2}$ &c. Thus in the foregoing Example, instead of the first Multiplicand $4a^3b + 6aabb + 4ab^3$, we might have taken only $4 + 6 + 4$, and instead of Multiplying Three times by $aa + 2ab + bb$, we might have Multiplyed only by $1 + 2 + 1$, which would have made the last Terms to have been $164 + 232 + 164$. Now since that n is $= 4$ and $d = 6$, p will be $= 6$, and $q = 4$; and consequently the literal Products to be annext to the Terms $164 + 232 + 164$ will be respectively a^6b^4, a^5b^5, a^4b^6, which will make the Terms resulting from the last Multiplication to be $164a^6b^4 + 232a^5b^5 + 164a^4b^6$, as they had been found before.

EXAMPLE II.

LET Five be the number of Pieces of each Gamester, and Ten the number of Games given. Let also the proportion of Skill between A and B be as Two to One.

Since n is $= 5$, and $n + d = 10$, it follows that d is $= 5$. Now d being an odd number must be lessened by Unity, and supposed $= 4$, so that $\frac{1}{2}d = 2$. Let therefore $a + b$ be raised to the fifth Power; and always rejecting the extreams, Multiply twice by $aa + 2ab + bb$, or rather by $1 + 2 + 1$; thus,

1) + 5 + 10 + 10 + 5 (+ 1	20 + 35 + 35 + 20
1 + 2 + 1	1 + 2 + 1
5) + 10 + 10 + 5	20) + 35 + 35 + 20
+ 10 + 20 + 20 + 10	40 + 70 + 70 + 40
+ 5 + 10 + 10 (+ 5	20 + 35 + 35 (+ 20
20 + 35 + 35 + 20	75 + 125 + 125 + 75

Now

Now to supply the literal Products that are wanting, let $n + \frac{1}{2}d - 1$ be made $= p$, and $\frac{1}{2}d + 1 = q$, then p will be $= 6$ and $q = 3$. Wherefore the Products to be annext, *viz.* $a^p b^q$, $a^{p-1} b^{q+1}$ &c. will become a^6b^3, a^5b^4, a^4b^5, a^3b^6; and consequently the Expectation of S will be found to be

$$\frac{75a^6b^3 + 125a^5b^4 + 125a^4b^5 + 75a^3b^6}{\overline{a+b}^{\,9}}.$$

N. B. When n is an odd number, as it is in this Case, the Expectation of S will always be divisible by $a+b$. Wherefore dividing both Numerator and Denominator by $a+b$, the foregoing Expression will be reduced to

$$\frac{75a^5b^3 + 50a^4b^4 + 75a^3b^5}{\overline{a+b}^{\,8}}, \text{ or } 25a^3b^3 \times \frac{3aa + 2ab + 3bb}{\overline{a+b}^{\,8}}$$

Let now a be interpreted by 2 and b by 1, and the Expectation of S will become $\frac{3800}{6561}$.

PROBLEM XXXIV.

THE *same Things being given as in the preceding Problem; to find the Expectation of R, or otherwise what the Probability is that the Play will be Ended in a given number of Games.*

SOLUTION.

FIrst, It is plain that if the Expectation of S, obtained by the preceding Problem, be subtracted from Unity, there will remain the Expectation of R.

Secondly, Since the Expectation of S decreases continually as the number of Games increases, and that the Terms we rejected in the former Problem being divided by $aa + 2ab + bb$ are the Decrement of his Expectation; it follows, that if those rejected Terms be divided continually by $\overline{a+b}^2$ they will be the Increment of the Expectation of R. Wherefore the Expectation of R may be expressed by means of those rejected Terms. Thus, in the second Example of the preceding Problem, the Expectation of R expressed by means of the rejected Terms will be found to be

$a^5 + b^5$

$$\frac{a^5+b^5}{\overline{a+b}|^5}+\frac{5a^6b+5ab^6}{\overline{a+b}|^7}+\frac{20a^7bb+20aab^7}{\overline{a+b}|^9} \text{ or}$$

$$\frac{a^5+b^5}{\overline{a+b}|^5}\times\overline{1+\frac{5ab}{\overline{a+b}|^2}+\frac{20aabb}{\overline{a+b}|^4}}.$$

In the like manner, if Six were the number of the Pieces of each Gamester, and the number of Games were Fourteen; it would be found that the Expectation of R would be

$$\frac{a^6+b^6}{\overline{a+b}|^6}\times\overline{1+\frac{6ab}{\overline{a+b}|^2}+\frac{27aabb}{\overline{a+b}|^4}+\frac{110a^3b^3}{\overline{a+b}|^6}+\frac{429a^4b^4}{\overline{a+b}|^8}}:$$

And if Seven were the number of the Pieces of each Gamester, and the number of Games given were Fifteen; then the Expectation of R would be found to be

$$\frac{a^7+b^7}{\overline{a+b}|^7}\times\overline{1+\frac{7ab}{\overline{a+b}|^2}+\frac{35aabb}{\overline{a+b}|^4}+\frac{154a^3b^3}{\overline{a+b}|^6}+\frac{637a^4b^4}{\overline{a+b}|^7}}.$$

N. B. The number of Terms of these Series will always be equal to $\frac{1}{2}d+1$, if d be an even number, or to $\frac{d+1}{2}$ if it be odd.

Thirdly, All the Terms of these Series have to one another certain Relations; which being once discovered, each Term of any Series resulting from any Case of this Problem, may be easily generated from the preceding ones.

Thus in the first of the two last foregoing Series, the Numerical Coefficient belonging to the Numerator of each Term, may be derived from the preceding ones, in the following manner. Let K, L, M be the Three last Coefficients, and let N be the Coefficient of the next Term wanted; then it will be found that N in that Series will constantly be equal to $6M-9L+2K$. Wherefore if the Term which would follow $\frac{429a^4b^4}{\overline{a+b}|^8}$, in the Case of Sixteen Games given, were desired; then make $M=429$, $L=110$, $K=27$, and the following Coefficient will be found 1638. From whence it appears that the Term it self would be $\frac{1638a^5b^5}{\overline{a+b}|^{10}}$.

Likewise, in the second of the two foregoing Series, if the Law by which each Term is related to the preceding ones were

were demanded, it might be thus found. Let K, L, M, be the Coefficients of the three last Terms, and N the Coefficient of the Term desired; then N will in that Series, constantly be equal to $7M - 14L + 7K$, or to $\overline{M - 2L + K} \times 7$. Now this Coefficient being obtained, the Term to which it belongs is formed immediately.

But if the general Law, by which each Coefficient is generated from the preceding ones, be demanded, it will be exprest as follows. Let n be the number of Pieces of each Gamester: Then each Coefficient contains

n times the last,
$- n \times \frac{n-3}{2}$ times the last but one,
$+ n \times \frac{n-4}{2} \times \frac{n-5}{3}$ times the last but two,
$- n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4}$ times the last but three,
$+ n \times \frac{n-6}{2} \times \frac{n-7}{3} \times \frac{n-8}{4} \times \frac{n-9}{5}$ times the last but four,
&c.

Thus the number of Pieces of each Gamester being Six, the first Term n would be $= 6$, the second Term $n \times \frac{n-3}{2}$ would be $= 9$, the third Term $n \times \frac{n-4}{2} \times \frac{n-5}{3}$ would be $= 2$; the rest of the Terms vanishing in this Case. Wherefore if K, L, M are the three last Coefficients, the Coefficient of the following Term will be $6M - 9L + 2K$.

Fourthly, The Coefficient of any Term of these Series may be found, independently from any relation they may have to the preceding ones: In order to which it is to be observed that each Term of these Series is proportional to the Probability of the Plays Ending in a certain number of Games precisely: Thus in the Series which expresses the Expectation of R, when each Gamester is supposed to have Six Pieces, *viz.*

$$\frac{a^6+b^6}{\overline{a+b}|^6} \times \overline{1 + \frac{6ab}{\overline{a+b}|^2} + \frac{27aabb}{\overline{a+b}|^4} + \frac{110a^3b^3}{\overline{a+b}|^6} + \frac{429a^4b^4}{\overline{a+b}|^8}},$$

the last Term, being multiplied by the common Multiplicator $\frac{a^6+b^6}{\overline{a+b}|^6}$ set down before the Series, that is the Product $\frac{429a^4b^4 \times \overline{a^6+b^6}}{\overline{a+b}|^{14}}$, denotes the Probability of the Plays

 End-

Ending in Fourteen Games precisely. Wherefore if that Term were desired which expresses the Probability of the Plays Ending in Twenty Games precisely, or in any number of Games denoted by $n+d$, I say that the Coefficient of that Term will be,

$\frac{1}{1} \times \frac{n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1$.

$- \frac{1}{1} \times \frac{3n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - n$.

$+ \frac{1}{1} \times \frac{5n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 2n$.

$- \frac{1}{1} \times \frac{7n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d + 1 - 3n$. &c.

Let now $n+d$ be supposed $= 20$, n being already supposed $= 6$, then the Coefficient demanded will be found from the general Rule to be,

$$\frac{1}{1} \times \frac{6}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{15}{6} \times \frac{14}{7} = 23256$$

$$- \frac{1}{1} \times \frac{18}{1} = 18$$

Wherefore the Coefficient demanded will be $23256 - 18 = 23238$: And then the Term it self to which this Coefficient does belong, will be $\frac{23238\,a^7b^7}{\overline{a+b}|^{14}}$. Consequently the Probability of the Plays Ending in Twenty Games precisely will be $\frac{a^6+b^6}{\overline{a+b}|^6} \times \frac{23238\,a^7b^7}{\overline{a+b}|^{14}}$.

Fifthly, By the help of the two Methods explained in this Problem (whereof the first is for finding the relation which any Term of the Series resulting from the Problem, has to a certain number of preceding ones; and the second for finding any Term of the Series independently from any other Term) together with the Method of summing up any given num-

number of Terms of these Series, (which shall be explained in its place); the Probability of the Plays Ending in any given number of Games, will be found much more readily than can be done by either of the two first Methods taken singly.

PROBLEM XXXV.

Supposing A *and* B *to play together, till such time as Four Stakes are Won or Lost on either side: What must be their proportion of Skill, to make it as Probable that the Play will be Ended in Four Games, as not?*

SOLUTION.

THE Probability of the Play Ending in Four Games, is by the preceding Problem $\frac{a^4+b^4}{\overline{a+b}|^4}$; Now because, by *Hypothesis*, it is to be an equal Chance whether the Play Ends or Ends not in Four Games; let this expression of the Probability be made equal to $\frac{1}{2}$; And we shall have this Equation $\frac{a^4+b^4}{\overline{a+b}|^4} = \frac{1}{2}$, which, making $b, a :: 1, z$, is reduced to $\frac{z^4+1}{\overline{z+1}|^4} = \frac{1}{2}$, or $z^4 - 4z^3 - 6zz - 4z + 1 = 0$. Let $12zz$ be added on both sides the Equation, then will $z^4 - 4z^3 + 6zz - 4z + 1$ be $= 12zz$; and extracting the square Root on both sides, it will be reduced to this *Quadratick* Equation $zz - 2z + 1 = z\sqrt{12}$, whose double Root is $z = 5,274$ and $\frac{1}{5,274}$. Wherefore whether the Skill of *A* be to that of *B* as 5.274 to 1, or as 1 to 5.274, there will be an equality of Chance for the Play to be Ended or not Ended in Four Games.

PROBLEM XXXVI.

Supposing that A *and* B *Play till such time as Four Stakes are Won or Lost: What must be their proportion of Skill, to make it a Wager of Three to One, that the Play will be Ended in Four Games?*

SOLU-

SOLUTION.

THE Probability of the Plays Ending in Four Games, ariſing from the number of Games Four, from the number of Stakes Four, and from the proportion of Skill is $\frac{a^4+b^4}{\overline{a+b}^4}$. The ſame Probability ariſing from the odds of Three to One, is $\frac{3}{4}$: Wherefore $\frac{a^4+b^4}{\overline{a+b}^4}=\frac{3}{4}$, and ſuppoſing $b, a :: 1, z$, the foregoing Equation will be changed into $\frac{z^4+1}{\overline{z+1}^4}=\frac{3}{4}$, or $z^4-12z^3-18zz-12z+1=0$. Let $56zz$ be added on both ſides the Equation, then we ſhall have $z^4-12z^3+38zz-12z+1=56zz$. And Extracting the ſquare Root on both ſides, we ſhall have $zz-6z+1=z\sqrt{56}$, the Roots of which Equation will be found 13.407 and $\frac{1}{13.407}$. Wherefore, whether the Skill of *A* be to that of *B* as 13.407 to 1, or as 1 to 13.407, 'tis a Wager of Three to One, that the Play will be ended in Four Games.

PROBLEM XXXVII.

SUppoſing that A *and* B *Play till ſuch time as Four Stakes are Won or Loſt; What muſt be their proportion of Skill, to make it an equal Wager that the Play will be Ended in Six Games?*

SOLUTION.

THE Probability of the Plays Ending in Six Games, ariſing from the given number Six, from the number of Stakes Four, and from the proportion of Skill, is $\frac{a^4+b^4}{\overline{a+b}^4}\times\overline{1+\frac{4ab}{\overline{a+b}^2}}$. The ſame Probability ariſing from an equality of Chance for Ending or not Ending in Six Games, is equal to $\frac{1}{2}$, from whence reſults the Equation $\frac{a^4+b^4}{\overline{a+b}^4}\times\overline{1+\frac{4ab}{\overline{a+b}^2}}=\frac{1}{2}$ which by making $b, a :: 1, z$ may be changed into the following, *viz.* $z^6+6z^5-13z^4-20z^3-13zz+6z+1=0$.

In

In this Equation, the Coefficients of the Terms equally diſtant from the Extreams being the ſame, let it be ſuppoſed that the Equation is generated from the Multiplication of two other Equations of the ſame nature, *viz.* $zz - yz + 1 = 0$, and $z^4 + pz^3 + qzz + pz + 1 = 0$. Now the Equation reſulting from the Multiplication of theſe two will be

$$z^6 \begin{array}{c}-y\\+p\end{array} z^5 \begin{array}{c}+1\\-py\\+q\end{array} z^4 + \begin{array}{c}2p\\-qy\end{array} z^3 \begin{array}{c}+1\\-py\\+q\end{array} zz \begin{array}{c}+p\\-y\end{array} z + 1 = 0,$$

which being compared with the firſt Equation, we ſhall have $p - y = 6$, $1 - py + q = -13$, $2p - qy = -20$.

From hence will be deduced a new Equation, *viz.* $y^3 + 6yy - 16y - 32 = 0$, one of whoſe Roots will be 2.9644; which being ſubſtituted in the Equation $zz - yz + 1 = 0$, we ſhall at laſt come to the Equation $zz - 2.9644z + 1 = 0$, of which the two Roots will be 2.576 and $\frac{1}{2.576}$. It follows therefore, that if the Skill of either Gameſter be to that of the other as 2.576 to 1; there will be an equal Chance for Four Stakes to be Loſt, or not to be Loſt, in Six Games.

Corollary. If the Coefficients of the Extream Terms of an Equation, and likewiſe the Coefficients of the other Terms equally diſtant from the Extreams, be the ſame, that Equation will be reducible to another, in which the Dimenſions of the higheſt Term will not exceed half the Dimenſions of the higheſt Term in the former.

PROBLEM XXXVIII.

S*Uppoſing* A *and* B, *whoſe proportion of Skill is as* a *to* b, *to Play together till ſuch time as* A *either Wins a certain number* q *of Stakes, or* B *ſome other number* p *of them: What is the Probability that the Play will not be Ended in a given number of Games?*

SOLUTION.

TAKE the Binomial $a+b$, and rejecting continually thoſe Terms in which the Dimenſions of the quantity a exceed the Dimenſions of the quantity b by q, rejecting alſo thoſe Terms in which the Dimenſions of the quantity b exceed the Dimenſions of the quantity a by p; multiply conſtantly the remainder by $a+b$, and make as many Multiplications, as there are Units in the given number of Games wanting one. Then ſhall the laſt Product be the Numerator of a Fraction expreſſing the Probability required; the Denominator of which Fraction always being the Binomial $a+b$ raiſed to that Power which is denoted by the given number of Games.

EXAMPLE.

LET p be $=3$, $q=2$, and let the given number of Games be $=7$. Let the following Operation be made according to the foregoing directions.

$$\begin{array}{c}
a+b \\
a+b \\
\hline
aa)+2ab+bb \\
a+b \\
\hline
2aab+3abb+(+b^3 \\
a+b \\
\hline
2a^3b)+5aabb+3ab^3 \\
a+b \\
\hline
5a^3bb+8aab^3+(3ab^4 \\
a+b \\
\hline
5a^4bb)+13a^3b^3+8aab^4 \\
a+b \\
\hline
13a^4b^3+21a^3b^4(+8aab^5. \\
\hline
\end{array}$$

From this Operation we may conclude, that the Probability that the Play will not be Ended in Seven Games is equal to $\frac{13a^4b^3+21a^3b^4}{\overline{a+b}|^7}$. Now if an Equality of Skill be ſuppoſed between A and B, the Expreſſion of this Probability

bility will be reduced to $\frac{13+21}{128}$ or $\frac{17}{64}$: Wherefore the Probability that the Play will End in Seven Games will be $\frac{47}{64}$; from which it follows that 'tis 47 to 17 that in Seven Games, either *A* wins two Stakes or *B* wins three.

PROBLEM XXXIX.

THE same Things being suppos'd as in the preceding Problem, to find the Probability of the Plays being Ended in a given number of Games.

SOLUTION.

First, If the Probability of the Plays not being Ended in the given number of Games be ſubtracted from Unity, there will remain the Probability of its being Ended in the ſame number of Games.

Secondly, This Probability may be expreſſed by means of the Terms rejected in the Operation belonging to the preceding Problem; Thus, if the number of Stakes be Three and Two, the Probability of the Plays being Ended in Seven Games may be expreſſed as follows.

$$\frac{aa}{\overline{a+b}|^2} \times \overline{1 + \frac{2ab}{\overline{a+b}|^2} + \frac{5aabb}{\overline{a+b}|^4}}.$$
$$+ \frac{b^3}{\overline{a+b}|^3} \times \overline{1 + \frac{3ab}{\overline{a+b}|^2} + \frac{8aabb}{\overline{a+b}|^4}}.$$

Suppoſing a and b both equal to Unity, the ſum of the firſt Series will be $= \frac{29}{64}$, and the ſum of the ſecond will be $= \frac{18}{64}$; which two ſums being added together, the aggregate $\frac{47}{64}$ expreſſes the Probability that in Seven Games either *A* ſhall win Two Stakes or *B* Three.

Thirdly, The Probability of the Plays being Ended in a certain number of Games, or ſooner, is always compoſed of a double Series, when the Stakes are unequal; which double Series is reduced to a ſingle one, in the Caſe of equality of Stakes.

The firſt Series always expreſſes the Probability there is that *A*, in a given number of Games, or ſooner, may win of *B* the num-

number q of Stakes, excluding the Probability there is, that B, before that time, may be in a circumſtance of winning the number p of Stakes. Both which Probabilities are not inconſiſtent together; for A in Fifteen Games, for Inſtance, or ſooner, may win Two Stakes of B, though B before that time may have been in a circumſtance of winning Three Stakes of A.

The ſecond Series always expreſſes the Probability there is that B, in that given number of Games or ſooner, may win of A a certain number p of Stakes, excluding the Probability there is that A, before that time, may win of B the number q of Stakes.

The firſt Terms of each Series may be repreſented reſpectively by the following Terms.

$$\frac{a^q}{\overline{a+b}|^q} \times \overline{1 + \frac{qab}{\overline{a+b}|^2} + \frac{q \times \overline{q+3} \times aabb}{1 \times 2 \times \overline{a+b}|^4} + \frac{q \times \overline{q+4} \times \overline{q+5} \times a^3b^3}{1 \times 2 \times 3 \times \overline{a+b}|^6}} \text{ \&c. and}$$

$$\frac{b^p}{\overline{a+b}|^p} \times \overline{1 + \frac{pab}{\overline{a+b}|^2} + \frac{p \times \overline{p+3} \times aabb}{1 \times 2 \times \overline{a+b}|^4} + \frac{p \times \overline{p+4} \times \overline{p+5} \times a^3b^3}{1 \times 2 \times 3 \times \overline{a+b}|^6}} \text{ \&c.}$$

Each of theſe Series continuing in that regularity, till ſuch time as there be a number p of Terms taken in the firſt, and a number q of Terms taken in the ſecond; after which the Law of the continuation breaks off.

Now in order to find any of the Terms following in either of theſe Series, proceed thus; let $p + q - 2$ be called l; let the Coefficient of the Term deſired be T; let alſo the Coefficients of the preceding Terms taken in an inverted order be S, R, Q, P &c. Then will T be equal to $lS - \frac{l-1}{1} \times \frac{l-2}{2} R + \frac{l-2}{1} \times \frac{l-3}{2} \times \frac{l-4}{3} Q - \frac{l-3}{1} \times \frac{l-4}{2} \times \frac{l-5}{3} \times \frac{l-6}{4} P$ &c. Thus if p be $=3$, and $q = 2$, then l will be $3 + 2 - 2 = 3$. Wherefore $lS - \frac{l-1}{1} \times \frac{l-2}{2} \times R$ would in this Caſe be equal to $3S - R$; which ſhews that the Coefficient of any Term deſired would be conſtantly three times the laſt, *minus* once the laſt but one.

To apply this, let it be required to find what Probability there is, that in Fifteen Games or ſooner, either A ſhall win two Stakes of B, or B three Stakes of A; or which is all one, to find what Probability there is, that the Play ſhall end in Fifteen Games or ſooner, A and B reſolving to Play, till ſuch time as A either wins three Stakes, or B two.

Let

Let Two and Three, in the two foregoing Series, be ſubſtituted reſpectively in the room of q and p; then the three firſt Terms of the firſt Series will be, ſetting aſide the common Multiplicator, $1 + \frac{2ab}{\overline{a+b}|^2} + \frac{5aabb}{\overline{a+b}|^4}$: Likewiſe the two firſt Terms of the ſecond will be $1 + \frac{3ab}{\overline{a+b}|^2}$. Now becauſe the Coefficient of any Term deſired in each Series, is reſpectively three times the laſt, *minus* once the laſt but one, it follows, that the next Coefficient in the firſt Series will be 13, and by the ſame rule the next to it 34, and ſo on. In the ſame manner the next Coefficient in the ſecond Series will be found to be 8, and the next to it 21, and ſo on. Wherefore, reſtoring the common Multiplicators, the two Series will be

$$\frac{a^2}{\overline{a+b}|^2} \times \overline{1 + \frac{2ab}{\overline{a+b}|^2} + \frac{5aabb}{\overline{a+b}|^4} + \frac{13a^3b^3}{\overline{a+b}|^6} + \frac{34a^4b^4}{\overline{a+b}|^8} + \frac{89a^5b^5}{\overline{a+b}|^{10}} + \frac{233a^6b^6}{\overline{a+b}|^{12}}}.$$

$$\frac{b^3}{\overline{a+b}|^3} \times \overline{1 + \frac{3ab}{\overline{a+b}|^2} + \frac{8aabb}{\overline{a+b}|^4} + \frac{21a^3b^3}{\overline{a+b}|^6} + \frac{55a^4b^4}{\overline{a+b}|^8} + \frac{144a^5b^5}{\overline{a+b}|^{10}} + \frac{377a^6b^6}{\overline{a+b}|^{12}}}.$$

If we ſuppoſe an equality of Skill between *A* and *B*, the ſum of the firſt Series will be $\frac{18778}{32768}$, the ſum of the ſecond will be $\frac{12393}{32768}$, and the aggregate of theſe two ſums will be $\frac{31171}{32768}$, which will expreſs the Probability of the Plays Ending in Fifteen Games or ſooner. This laſt Fraction being ſubtracted from Unity, there will remain $\frac{1597}{32768}$, which expreſſes the Probability of the Plays continuing for Fifteen Games at leaſt: Wherefore 'tis 31171 to 1597, or 39 to 2 nearly, that one of the two equal Gameſters, that ſhall be pitcht upon, ſhall in Fifteen Games, or ſooner, either win Two Stakes of his adverſary, or loſe Three to him.

N. B. The Index of the Denominator in the laſt Term of each Series, and the Index of the common Multiplicator prefixt to it, being added together, muſt either equal the number of Games given, or be leſs than it by Unity. Thus, in the firſt Series, the Index 12 of the Denominator of the laſt Term, and the Index 2 of the common Multiplicator being added together, the ſum is 14, which is leſs by Unity than the number of Games given. So likewiſe in the ſecond Series, the Index 12 of the Denominator of the laſt Term, and the Index 3 of the common Multiplicator being added together, the ſum is 15, which preciſely equals the number of Games given.

It is carefully to be obſerved, that theſe two Series taken together, expreſs the Expectation of one and the ſame Perſon, and not of two different Perſons; that is properly the Expectation of a ſpectator who lays a Wager that the Play will be Ended in a given number of Games. Yet in one Caſe they may expreſs the Expectations of two different Perſons: For Inſtance, of the Gameſters themſelves, provided that both Series be continued infinitely; for in that Caſe, the firſt Series infinitely continued will expreſs the Probability that the Gameſter A may ſooner win two Stakes of B, than that he may loſe three to him: Likewiſe the ſecond Series infinitely continued will expreſs the Probability that the Gameſter B may ſooner win three Stakes of A, than that he may loſe two to him. And it will be found, when we come to treat of the method of ſumming up theſe Series, that the firſt Series infinitely continued will be to the ſecond infinitely continued, in the proportion of $aa \times \overline{aa+ab+bb}$ to $b^3 \times \overline{a+b}$; that is, in the Caſe of an equality of Skill, as three to two; which is conformable to what we have ſaid in our IX*th*. Problem.

Fourthly, Any Term of theſe Series may be found independently from any of the preceding ones: For if a Wager be laid that A ſhall either win a certain number of Stakes denominated by q; or that B ſhall win a certain number of them denominated by p, and that the number of Games given be expreſſed by $q+d$; then I ſay that the Coefficient of any Term in the firſt Series, anſwering to that number of Games, will be

$$+\frac{1}{1} \times \frac{q}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4} \text{ \&c. conti-}$$

nued to ſo many Terms as there are Units in $\frac{1}{2}d+1$.

$-\frac{1}{1}$

$-\frac{1}{1}\times\frac{q+2p}{1}\times\frac{q+d-1}{2}\times\frac{q+d-2}{3}\times\frac{q+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d+1-p$.

$+\frac{1}{1}\times\frac{3q+2p}{1}\times\frac{q+d-1}{2}\times\frac{q+d-2}{3}\times\frac{q+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d+1-p-q$.

$-\frac{1}{1}\times\frac{3q+4p}{1}\times\frac{q+d-1}{2}\times\frac{q+d-2}{3}\times\frac{q+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d+1-2p-q$.

$+\frac{1}{1}\times\frac{5q+4p}{1}\times\frac{q+d-1}{2}\times\frac{q+d-2}{3}\times\frac{q+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d+1-2p-2q$.

$-\frac{1}{1}\times\frac{5q+6p}{1}\times\frac{q+d-1}{2}\times\frac{q+d-2}{3}\times\frac{q+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d+1-3p-2q$.

$+\frac{1}{1}\times\frac{7q+6p}{1}\times\frac{q+d-1}{2}\times\frac{q+d-2}{3}\times\frac{q+d-3}{4}$ &c. continued to so many Terms as there are Units in $\frac{1}{2}d+1-3p-3q$, &c.

And the same Law will hold for the other Series, calling $p+\delta$ the number of Games given, and changing q into p, and p into q, as also d into δ.

But when it Happens that d is an odd number, substitute $d-1$ in the room of it, and the like for δ.

PROBLEM XL.

I*F* A *and* B, *whose proportion of Skill is supposed as* a *to* b, *play together: What is the Probability that one of them, suppose* A, *may in a number of Games not exceeding a number given, win of* B *a certain number of Stakes? Leaving it wholly indifferent whether* B *before the expiration of those Games may or may not have been in a circumstance of winning the same, or any other number of Stakes of* A.

SOLUTION.

SUpposing n to be the number of Stakes which A is to win of B, and $n+d$ the given number of Games; let $a+b$ be raised to the Power whose Index is $n+d$: Then if d be an an odd Number, take ſo many Terms of that Power as there are Units in $\frac{d+1}{2}$; take alſo as many of the Terms next following as have been taken already, but prefix to them, in an inverted order, the Coefficients of the preceding Terms. But if d be an even number, take ſo many Terms of the ſaid Power as there are Units in $\frac{1}{2}d+1$; then take as many of the Terms next following as there are Units in $\frac{1}{2}d$, and prefix to them, in an inverted order, the Coefficients of the preceding Terms, omitting the laſt of them; and thoſe Terms taken all together will compoſe the Numerator of a Fraction expreſſing the Probability required, its Denominator being $\overline{a+b}|^{n+d}$.

EXAMPLE I.

SUppoſing the number of Stakes which A is to win to be Three, and the given number of Games to be Ten; let $a+b$ be raiſed to the tenth Power, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45aab^8+10ab^9+b^{10}$. Then by reaſon that n is $=3$ and $n+d=10$, it follows that d is $=7$, and $\frac{d+1}{2}=4$. Wherefore let the Four firſt Terms of the ſaid Power be taken, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3$, and let the Four Terms next following be taken likewiſe, without regard to their Coefficients; then prefix to them, in an Inverted order, the Coefficients of the preceding Terms: Thus the Four Terms following with their new Coefficients, will be $120a^6b^4+45a^5b^5+10a^4b^6+1a^3b^7$. And the Probability which A has of winning Three Stakes of B in Ten Games, or ſooner, will be expreſſed by the following Fraction,

$$\frac{a^{10}+10a^9b+45a^8bb+120a^7b^3+120a^6b^4+45a^5b^5+10a^4b^6+a^3b^7}{\overline{a+b}|^{10}}$$

which, in the Caſe of an equality of Skill between A and B, will be reduced to $\frac{352}{1024}$ or $\frac{11}{32}$.

E X-

EXAMPLE II.

SUpposing the number of Stakes which *A* is to win to be Four, and the given number of Games to be Ten; let $a+b$ be raised to the tenth Power, and by reason that n is $=4$, and $n+d=10$, it follows, that $d=6$ and $\frac{1}{2}d+1=4$; wherefore let the Four first Terms of the said Power be taken, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3$; take also Three of the Terms following, but prefix to them, in an invested order, the Coefficients of the Terms already taken, omitting the last of them. Hence the Three Terms following with their new Coefficients will be $45a^6b^4+10a^5b^5+1a^4b^6$. And the Probability which *A* has of winning Four Stakes of *B*, in Ten Games or sooner, will be expressed by the following Fraction

$$\frac{a^{10}+10a^9b+45a^8bb+120a^7b^3+45a^6b^4+10a^5b^5+1a^4b^6}{\overline{a+b}|^{10}}$$

which, in the Case of an equality of Skill between *A* and *B*, will be reduced to $\frac{232}{1024}$ or $\frac{29}{128}$.

Another SOLUTION.

SUpposing, as before, that n be the number of Stakes which *A* is to win, and that the given number of Games be $n+d$ the Probability which *A* has of winning will be expressed by the following Series, *viz.*

$$\frac{a^n}{\overline{a+b}|^n}\times\overline{1+\frac{nab}{\overline{a+b}|^2}+\frac{n\times\overline{n+3}\times aabb}{1\times2\times\overline{a+b}|^4}+\frac{n\times\overline{n+4}\times\overline{n+5}\times a^3b^3}{1\times2\times3\times\overline{a+b}|^6}}$$
$$\overline{+\frac{n\times\overline{n+5}\times\overline{n+6}\times\overline{n+7}\times a^4b^4}{1\times2\times3\times4\times\overline{a+b}|^8}}\ \&c.$$

which Series ought to be continued to so many Terms as there are Units in $\frac{1}{2}d+1$; always observing to substitute $d-1$ in the room of d, in case d be an odd number, or which is the same thing, taking so many Terms as there are Units in $\frac{d+1}{2}$.

Now fuppofing, as in the firft Example of the preceding Solution, that Three is the number of Stakes, and Ten the given number of Games, as alfo that there is an equality of Skill between A and B, the foregoing Series will become $\frac{1}{8} \times \overline{1 + \frac{3}{4} + \frac{9}{16} + \frac{28}{64}} = \frac{11}{32}$, as before.

REMARK.

MOnfieur *de Monmort*, in the Second Edition of his Book of Chances, having given a very handfom Solution of the Problem relating to the duration of Play, (which Solution is coincident with that of Monfieur *Nicolas Bernoully*, to be feen in that Book) and the Demonftration of it being very naturally deduced from our firft Solution of the foregoing Problem, I thought the Reader would be well pleafed to fee it transferred to this place.

Let it therefore be propofed to find the number of Chances there are, for A either to win Two Stakes of B, or for B to win Three of A in Fifteen Games.

The number of Chances required is expreffed by two Branches of Series; all the Series of the firft Branch taken together exprefs the number of Chances there are for A to win Two Stakes of B, exclufive of the number of Chances there are for B, before that time, to win Three Stakes of A. All the Series of the fecond Branch taken together exprefs the number of Chances there are for B to win Three Stakes of A, exclufive of the number of Chances there are for A, before that time, to win Two Stakes of B.

Firft Branch of SERIES.

$$a^{15}\ a^{14}b\ a^{13}b^2\ a^{12}b^3\ a^{11}b^4\ a^{10}b^5\ a^9b^6\ a^8b^7\ a^7b^8\ a^6b^9\ a^5b^{10}\ a^4b^{11}\ a^3b^{12}\ a^2b^{13}$$

$$1+15+105+455+1365+3003+5005+5005+3003+1365+455+105+15+1$$

$$-1-15-105-455-455-105-15-1$$

$$+1+15+15+1$$

Second Branch of SERIES.

$$b^{15}\ b^{14}a\ b^{13}a^2\ b^{12}a^3\ b^{11}a^4\ b^{10}a^5\ b^9a^6\ b^8a^7\ b^7a^8\ b^6a^9\ b^5a^{10}\ b^4a^{11}\ b^3a^{12}$$

$$1+15+105+455+1365+3003+5005+3003+1365+455+105+15+1$$

$$-1-15-105-455-1365-455-105-15-1$$

$$+1+15+1$$

The

The literal Quantities, which are commonly annext to the numerical ones, are here written on the top of them; which is done, to the end that each Series being contained in one line, the dependency they have upon one another, may thereby be made the more conſpicuous.

The firſt Series of the firſt Branch expreſſes the number of Chances there are for *A* to win Two Stakes of *B*, including the number of Chances there are for *B*, before the expiration of the Fifteen Games, to be in a circumſtance of winning Three Stakes of *A*; which number of Chances may be deduced from our foregoing Problem.

The ſecond Series of the firſt Branch is a part of the firſt, and expreſſes the number of Chances there are, for *B* to win Three Stakes of *A*, out of the number of Chances there are for *A* in the firſt Series, to win Two Stakes of *B*. It is to be obſerved about this Series, *Firſt*, that the Chances of *B* expreſſed by it are not reſtrained to Happen in any Order, that is, either before or after *A* has won Two Stakes of *B*. *Secondly*, that the literal Products belonging to it are the ſame with thoſe of the correſponding Terms of the firſt Series. *Thirdly*, that it begins and ends at an interval from the firſt and laſt Terms of the firſt Series equal to the number of Stakes which *B* is to win. *Fourthly*, that the numbers belonging to it are the numbers of the firſt Series repeated in order, and continued to one half of its Terms; after which thoſe numbers return in an inverted order to the end of that Series: Which is to be underſtood in caſe the number of its Terms ſhould Happen to be even, for if it ſhould Happen to be odd, then that order is to be continued to the greateſt half, after which the return is made by omitting the laſt number. *Fifthly*, that all the numbers of it are Negative.

The Third Series of the firſt Branch is a part of the ſecond, and expreſſes the number of Chances there are for *A* to win Two Stakes of *B*, out of the number of Chances there are in the ſecond Series, for *B*, to win Three Stakes of *A*; with this difference, that it begins and ends at an interval from the firſt and laſt Terms of the ſecond Series, equal to the number of Stakes which *A* is to win; and that the Terms of it are all Poſitive.

It

It is to be obſerved in general that, let the number of theſe Series be what it will, the Interval between the beginning of the firſt and the beginning of the ſecond, is to be equal to the number of Stakes which *B* is to win; and that the Interval between the beginning of the ſecond and the beginning of the third, is to be equal to the number of Stakes which *A* is to win; and that theſe Intervals recurr alternately in the ſame Order. It is to be obſerved likewiſe, that all theſe Series are alternately Poſitive and Negative.

All the Obſervations made upon the firſt Branch of Series belonging alſo to the ſecond, it would be needleſs to ſay any more of them.

Now the ſum of all the Series of the firſt Branch, being added to the ſum of all the Series of the ſecond, the aggregate of theſe ſums will be the Numerator of a Fraction expreſſing the Probability of the Plays terminating in the given number of Games; of which Fraction the Denominator is the Binomial $a+b$ raiſed to a Power, whoſe Index is equal to that given number of Games. Thus, ſuppoſing that, in the Caſe of this Problem, both a and b are equal to Unity, the ſum of the Series in the firſt Branch will be 18778, the ſum of the Series in the ſecond will be 12393; and the aggregate of both 31171: And the Fifteenth Power of 2 being 32768, it follows, that the Probability of the Plays terminating in Fifteen Games will be $\frac{31171}{32768}$, which being ſubtracted from Unity, the remainder will be $\frac{1597}{32768}$: From whence we may conclude, that 'tis a Wager of 31171 to 1597, that either *A* in Fifteen Games ſhall win Two Stakes of *B*, or *B* win Three Stakes of *A*: Which is conformable to what we had before found in our XXXIX*th*. Problem.

PROBLEM XLI.

T*O Find what Probability there is, that in a given number of Games,* A *may be winner of a certain number* q *of Stakes; and at ſome other time,* B *may likewiſe be winner of the number* p *of Stakes, ſo that both circumſtances may Happen.*

SOLU-

SOLUTION.

FIND, by our XL*th* Problem, the Probability which *A* has of winning, without any Limitation, the number *q* of Stakes: Find alſo by our XXXIV*th* Problem the Probability which *A* has of winning that number of Stakes before *B* may Happen to win the number *p*; then from the firſt Probability ſubtracting the ſecond, the remainder will expreſs the Probability there is, that both *A* and *B* may be in a circumſtance of winning, but *B* before *A*. In the like manner, from the probability which *B* has of winning without any Limitation, ſubtracting the Probability which he has of winning before *A*, the remainder will expreſs the Probability there is, that both *A* and *B* may be in a circumſtance of winning, but *A* before *B*. Wherefore adding theſe two remainders together, their ſum will expreſs the Probability required.

Thus, if it were required to find what Probability there is, that in Ten Games *A* may win Two Stakes, and that at ſome other time *B* may win Three. The firſt Series will be found to be,

$$\frac{aa}{\overline{a+b}|^2} \times \overline{1 + \frac{2ab}{\overline{a+b}|^2} + \frac{5aabb}{\overline{a+b}|^4} + \frac{14a^3b^3}{\overline{a+b}|^6} + \frac{42a^4b^4}{\overline{a+b}|^8}}.$$

The ſecond Series will likewiſe be found to be

$$\frac{aa}{\overline{a+b}|^2} \times \overline{1 + \frac{2ab}{\overline{a+b}|^2} + \frac{5aabb}{\overline{a+b}|^4} + \frac{13a^3b^3}{\overline{a+b}|^6} + \frac{34a^4b^4}{\overline{a+b}|^8}}.$$

The difference of theſe Series being $\frac{aa}{\overline{a+b}|^2} \times \overline{\frac{a^3b^3}{\overline{a+b}|^6} + \frac{8a^4b^4}{\overline{a+b}|^8}}$ expreſſes the firſt part of the Probability required, which, in the Caſe of an equality of Skill between the Gameſters, would be reduced to $\frac{3}{256}$.

The Third Series is as follows,

$$\frac{b^3}{\overline{a+b}|^3} \times \overline{1 + \frac{3ab}{\overline{a+b}|^2} + \frac{9aabb}{\overline{a+b}|^4} + \frac{28a^3b^3}{\overline{a+b}|^6}}.$$

The Fourth Series is

$$\frac{b^3}{\overline{a+b}|^3} \times \overline{1 + \frac{3ab}{\overline{a+b}|^2} + \frac{8aabb}{\overline{a+b}|^4} + \frac{21a^3b^3}{\overline{a+b}|^6}}.$$

The difference of theſe two Series being $\frac{b^3}{\overline{a+b}|^3} \times \overline{\frac{aabb}{\overline{a+b}|^4} + \frac{7a^3b^3}{\overline{a+b}|^6}}$

expreſſes the ſecond part of Probability required, which, in the Caſe of an equality of Skill, would be reduced to $\frac{11}{512}$. Wherefore the Probability required would in this Caſe be $\frac{3}{256} + \frac{11}{512} = \frac{17}{512}$. Whence it follows, that 'tis a Wager of 495 to 17, or of 29 to 1 very nearly, that in Ten Games, *A* and *B* may not both be in a circumſtance of winning, *viz.* *A* the number *q*, and *B* the number *p* of Stakes. But if, by the conditions of the Problem, it were left indifferent whether *A* or *B* ſhould win the Two Stakes or the Three, then the Probability required would be increaſed, and become as follows, *viz.*

$$\frac{aa+bb}{\overline{a+b}|^2} \times \overline{\frac{a^3b^3}{\overline{a+b}|^6} + \frac{8a^4b^4}{\overline{a+b}|^8}}$$

$$\frac{a^3+b^3}{\overline{a+b}|^3} \times \overline{\frac{aabb}{\overline{a+b}|^4} + \frac{7a^3b^3}{\overline{a+b}|^6}}.$$

which, in the Caſe of an equality of Skill between the Gameſters, would be the double of what it was before.

PROBLEM XLII.

TO Find what Probability there is, that in a given number of Games, A *may win the number* q *of Stakes; with this farther condition, that* B, *during that whole number of Games, may never have been winner of the number* p *of Stakes.*

SOLUTION.

FRom the Probability that *A* has to win without any limitation the number *q* of Stakes, ſubtract the Probability there is that both *A* and *B* may be winners, *viz.* *A* of the number *q*, and *B* of the number *p* of Stakes, and there will remain the Probability required.

But, if the conditions of the Problem were extended to this alternative, *viz.* that either *A* ſhould win the number *q* of Stakes, and *B* be excluded the winning of the number *p*; or that *B* ſhould win the number *p* of Stakes, and *A* be excluded the winning of the number *q*, the Probability that either the one or the other of theſe two Caſes may Happen, will eaſily be deduced from what we have ſaid.

LEM-

LEMMA I.

IN *any Series of Terms, whereof the first Differences are equal, the Third Term will be twice the Second,* minus *once the First; and the Fourth Term likewise will be twice the Third,* minus *once the Second: Each following Term being always related in the same manner to the two preceding ones. And as this relation is expressed by the two Numbers* 2 — 1, *I therefore call those Numbers the* Index *of that Relation.*

In any Series of Terms, whose second Differences are equal, the Fourth Term will be three times the Third, minus *three times the Second,* plus *once the First: And each Term in such a Series is always related in the same manner to the three next preceding ones, according to the Index* 3—3+1. *Thus, if there be a Series of Squares, such as* 4, 16, 36, 64, 100, *whose second differences are known to be equal when their Roots have equal Intervals, as they have in this Case, it will be found that the Fourth Term* 64 *is* $= 3\times36 - 3\times16 + 1\times4$, *and that the Fifth Term* 100 *is* $= 3\times64 - 3\times36 + 1\times16$. *In like manner, if there were a Series of Triangular numbers, such as* 3, 10, 21, 36, 55, *whose second Differences are known to be equal, when their sides have equal Intervals, as they have in this Case, it will be found that the Fourth Term is* $= 3\times21 - 3\times10 + 1\times3$, *and that the Fifth Term is* $= 3\times36 - 3\times21 + 1\times10$; *and so on.*

So likewise, if there were a Series of Terms whose Third Differences are equal, or whose Fourth Differences are $= 0$; *such as is a Series of Cubes or Pyramidal numbers, or any other Series of numbers generated by the Quantities* $ax^3 + bxx + cx + d$, *when* a, b, c, d *being constant Quantities,* x *is interpreted successively by the Terms of any Arithmetic Progression: Then it will be found that any Term of it is related to the Four next preceding ones, according to the following Index,* viz. $4 - 6 + 4 - 1$, *whose parts are the Coefficients of the Binomial* $a - b$ *raised to the fourth Power, the first Coefficient being omitted.*

And generally, if there be any Series of Terms whose last Differences are $= 0$. *Let the number denoting the rank of that difference be* n; *then the Index of the Relation of each Term to as many of the preceding ones as there are Units in* n, *will be expressed by the Coefficients of the Binomial* $a - b$ *raised to the Power* n, *omitting the first.*

But

But if the Relation of any Term of a Series to a constant number of preceding Terms, be expressed by any other Indices than those which are comprised under the foregoing general Law; or even if, those Indices remaining, any of their Signs + or — be changed, that Series of Terms will have none of its differences equal to Nothing.

LEMMA II.

IF in any Series, the Terms A, B, C, D, E, F *&c. be continually decreasing, and be so related to one another that each of them may have to the same number of preceding Terms a certain given Relation, always expressible by the same Index; I say, that the sum of all the Terms of that Series* ad infinitum *may always be obtained.*

First, *Let the Relation of each Term to the two preceding ones be expressed in this manner,* viz. *Let* C *be* $= mBr - nArr$; *and let* D *likewise be* $= mCr - nBrr$, *and so on: Then will the sum of that Infinite Series be equal to* $\frac{A + B - mrA}{1 - mr + nrr}$.

Thus, if it be proposed to find the sum of the following Series,

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ \end{array}$$

viz. $1r + 3rr + 5r^3 + 7r^4 + 9r^5 + 11r^6 + 13r^7$ *&c.* *whose Terms are related to one another in this manner,* viz. $C = 2rB - 1rrA$, $D = 2rC - 1rrB$ *&c. Let* m *and* n *be made respectively equal to* 2 *and* 1, *and these Numerical Quantities being Substituted, in the room of the literal ones, in the general Theorem, the sum of the Terms of the foregoing Series will be found to be equal to* $\frac{r + 3rr - 2rr}{1 - 2r + rr}$, *or to* $\frac{r + rr}{\overline{1 - r}|^2}$.

Let it be also proposed to find the sum of the following Series

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ 1r & +3rr & +4r^3 & +7r^4 & +11r^5 & +18r^6 & +29r^7 \text{ \&c.} \end{array}$$

whose Terms are related to one another in this manner, viz. $C = 1Br + 1Arr$, $D = 1Cr + 1Brr$ *&c. Let* m *and* n *be respectively made equal to* 1 *and* — 1, *and then that Series will be found equal to* $\frac{r + 3rr - rr}{1 - r - rr}$, or to $\frac{r + 2rr}{1 - r - rr}$.

DEMON-

DEMONSTRATION.

Let the following Scheme be written down, *viz.*

$$\begin{aligned} A &= A \\ B &= B \\ C &= mBr - nArr \\ D &= mCr - nBrr \\ E &= mDr - nCrr \\ F &= mEr - nDrr \end{aligned}$$

&c.

This being done, if the ſum of the Terms A, B, C, D, E, F &c. *ad infinitum*, compoſing the firſt Column, be ſuppoſed equal to x, then the ſum of the Terms of the other two Columns will be found thus: By *Hypotheſis*, $A + B + C + D + E$ &c. $= x$, or $B + C + D + E$ &c. $= x - A$; and Multiplying both ſides of this Equation by mr, it will follow that $mBr + mCr + mDr + mEr$ &c. is $= mrx - mrA$. Again, adding $A + B$ on both ſides, we ſhall have the ſum of the Terms of the ſecond Column, *viz.* $A + B + mBr + mCr + mDr$ &c. equal to $A + B + mrx - mrA$. The ſum of the Terms of the third Column will be found by bare inſpection to be $-nrrx$. But the ſum of the Terms contain'd in the firſt Column, is equal to the other two ſums contained in the other two Columns. Wherefore the following Equation will be had, *viz.* $x = A + B + mrx - mrA - nrrx$; from whence it follows that the value of x, or the ſum of all the Terms $A + B + C + D + E$ &c. will be equal to

$$\frac{A + B - mrA}{1 - mr + nrr}$$

Secondly, Let the Relation of each Term to the three next preceding ones be expreſſed as follows, *viz.* let D be $= mCr - nBrr + pAr^3$, and let E likewiſe be $= mDr - nCrr + pBr^3$, and ſo on: Then will the ſum of all the Terms $A + B + C + D + E$ &c. *ad infinitum*, be equal to

$$\frac{A + B + C - mrA + nrrA \atop - mrB}{1 - mr + nrr - pr^3}$$

L 1 To

To apply this Theorem, let it be propoſed to find the ſum of the following Series,

$$\begin{matrix} A & & B & & C & & D & & E & & F & & G \\ 4r & + & 16rr & + & 36r^3 & + & 64r^4 & + & 100r^5 & + & 144r^6 & + & 196r^7 \text{ \&c.} \end{matrix}$$

whoſe Numerical Quantities are related to one another according to the Index $3 - 3 + 1$, correſponding to $m - n + p$: Let therefore 3, 3, 1 be ſubſtituted in the room of m, n, p; let alſo $4r, -16rr, 36r^3$ be ſubſtituted in the room of A, B, C: Then the ſum of the Terms of the foregoing Series will be found equal to $\frac{4r + 16rr + 36r^3 - 12rr - 48r^3 + 12r^3}{1 - 3r + 3rr - r^3}$, or $\frac{4r + 4rr}{\overline{1-r}|^3}$.

And in like manner the ſum of the Terms of the following Series, *viz.*

$r + 2rr + 5r^3 + 20r^4 + 72r^5 + 261r^6 + 947r^7$ &c. whoſe Numerical Quantities are related to one another according to the Index $3 + 2 + 1$, will by a proper ſubſtitution, be found to be equal to $\frac{r - rr - 3r^3}{1 - 3r - 2rr - r^3}$.

Thirdly, Let the Relation of each Term of a Series to four of the preceding Terms be expreſſed by means of the Index $m - n + p - q$, and the Sum of that Series will be

$$\frac{\begin{matrix} A + B + C + D - mrA + nrrA - pr^3A \\ - mrB + nrrB \\ - mrC \end{matrix}}{1 - mr + nrr - pr^3 + qr^4}$$

Fourthly, The Law of the continuation of theſe Theorems being manifeſt, they may be all eaſily comprehended under one general Rule.

Fifthly, If the correſponding Terms of any two or more Series, generated after the manner which we have above deſcribed, be multiplyed by one another, the new Series reſulting from that multiplication, will alſo be exactly ſummable: Thus, taking the two following Series, *viz.*

$r + 2rr + 3r^3 + 5r^4 + 8r^5 + 13r^6$ &c.

$r + 3rr + 4r^3 + 7r^4 + 11r^5 + 18r^6$ &c. in both of which

which each Numerical Quantity is the ſum of the two preceding ones; the Series reſulting from the multiplication of the correſponding Terms will be

$$rr + 6rr^4 + 12r^6 + 35r^8 + 88r^{10} + 234r^{12} \text{ \&c.}$$

in which each Numerical Quantity being related to the three preceding ones, according to the Index $2 + 2 - 1$, the ſum of that Series will be found to be $= \frac{rr + 4r^4 - 2r^6}{1 - 2rr - 2r^4 + r^6}$ as will appear, if in the room of $m - n + p$ there be ſubſtituted $2 + 2 - 1$, and rr be written inſtead of r.

When the Numerical Quantities belonging to the Terms of any Series are reſtrained to have their laſt differences equal to Nothing, then may the ſums of thoſe Series be alſo found by the following elegant Theorem, which has been communicated to me by *Mr. de Monmort*.

Let Ar be the firſt Term of the Series, and let the firſt, ſecond and third differences, &c. of the Numerical Quantities belonging to the Terms of the Series, be reſpectively equal to d' d'', d''', &c. Then will the ſum of the Series be equal to

$$\frac{Ar}{1-r} + \frac{rrd'}{\overline{1-r}|^2} + \frac{r^3d''}{\overline{1-r}|^3} + \frac{r^4d'''}{\overline{1-r}|^4} + \frac{r^5d''''}{\overline{1-r}|^5} \text{ \&c.}$$

Thus, if it were propoſed to find by this Theorem the ſum of the following Series, *viz.*

$$Ar + 16rr + 36r^3 + 64r^4 + 100r^5 \text{ \&c.}$$

It is plain that in this caſe A is $= 4$, $d' = 12$, $d'' = 8$, $d''' = 0$; and therefore that the ſum of this Series is equal to $\frac{4r}{1-r} + \frac{12rr}{\overline{1-r}|^2} + \frac{8r^3}{\overline{1-r}|^3}$, which is reduced to $\frac{4r + 4rr}{\overline{1-r}|^3}$.

REMARK.

OUr Method of ſumming up all the Terms which in theſe Series are related to one another according to conſtant Indices, may be extended to the finding of the ſum of any determinate number of thoſe Terms. Thus, if A, B, C, D be the firſt Terms of a Series, and V, X, Y, Z, be the laſt, then will the ſum of the Series be

$A -$

$$\frac{\begin{array}{l} A - mrA + nrrA - pr^3A + qr^4U \\ + B - mrB + nrrB - pr^3X + qr^4X \\ + C - mrC + nrrY - pr^3Y + qr^4Y \\ + D - mrZ + nrrZ - pr^3Z + qr^4Z \end{array}}{1 - mr + nrr - pr^3 + qr^4}$$

And if a general Theorem were desired, it might easily be formed from the inspection of the foregoing.

These Theorems are very useful for summing up readily those Series which express the Probability of the Plays being Ended in a given number of Games. For example, suppose it be required to find what Probability there is, that in Four and twenty Games, either A shall win Four Stakes of B, or B Four Stakes of A. The Series expressing that Probability is, from our XXXIV*th* Problem

$$\frac{a^4+b^4}{\overline{a+b}|^4} \times \overline{1 + \frac{4ab}{\overline{a+b}|^2} + \frac{14aabb}{\overline{a+b}|^4} + \frac{48a^3b^3}{\overline{a+b}|^6} + \frac{164a^4b^4}{\overline{a+b}|^8}} \text{ \&c.}$$

or, supposing an equality of Skill between the two Gamesters, $\frac{1}{8} \times \overline{1 + \frac{4}{4} + \frac{14}{16} + \frac{48}{64} + \frac{164}{256}}$ &c. which ought to be continued to eleven Terms independently from the common Multiplicator. Let this Series, whose Terms are related according to the Index $4 - 2$, be compared with the Theorem, making $A = 1$, $B = \frac{4}{4} = 1$, $m = 4$, $n = 2$. and neglecting the Terms C, D, V, X, the sum of the aforesaid Series will be found $= 8 + Y - 7Z$; which being multiplied by the common Multiplicator $\frac{1}{8}$ prefixt to it, the Probability required will be expressed by $1 + \frac{1}{8}Y - \frac{7}{8}Z$. Wherefore nothing remains to be done but to find the two last Terms Y and Z: But those two Terms, by our XXXIV*th*. Problem, will be found to be $\frac{76096}{2^{18}}$ and $\frac{259808}{2^{20}}$, or 0.2902, and 0.2477 nearly; which numbers being substituted respectively in the room of Y and Z the Probability required will be found to be equal to 0.8193 nearly. Let now this last number be subtracted from Unity, and the remainder being 0.1807, it follows, that 'tis a Wager of 82 to 18, or of 41 to 9 nearly, that in Twenty four Games or sooner, either A shall win four Stakes of B, or B four Stakes of A.

If

If the number of Stakes were Five, the ſum of the Terms of the Series belonging to that Caſe would alſo be expreſt by means of the two laſt Terms, ſuppoſing any given number of Games, or any proportion of Skill. If the number of Stakes were Six or Seven, the ſum of the Series belonging to thoſe Caſes would be expreſt by means of the three laſt Terms; If Eight or Nine, by means of the Four laſt Terms, and ſo on.

LEMMA III.

IF there be a Series of Numbers, as A, B, C, D, E &c. *whoſe Relation is expreſt by any conſtant Index, and there be another Series of Numbers, as* P, Q, R, S, T &c. *whoſe laſt Differences are equal to Nothing; and each Term of the firſt Series be Multiplied by each corresponding Term of the ſecond, I ſay that the Products* AP, BQ, CR, DS, ET &c. *conſtitute a Series of Terms, whoſe Relation may be expreſt by a conſtant Index. Thus if we take the Series* 1, 2, 8, 28, 100 &c. *whoſe Terms are related by the Index* $3+2$, *and each Term of that Series be reſpectively Multiplied by the correſponding Terms of an Arithmetic Progreſſion, ſuch as* 1, 3, 5, 7, 9 &c. *whoſe laſt Differences are equal to Nothing: Then it will be found that the Products* 1, 6, 40, 196, 900 &c. *conſtitute a Series of Numbers, each Term of which is Related to the preceding ones according to the Index* $6-5-12-4$. *Now the Rule for finding the Index of this Relation is as follows.*

Take the Index which expreſſes the Relation of the Terms in the firſt Series, and Multiply each Term of it by the correſponding Terms of the Literal Progreſſion r, rr, r^3 &c. *which being done, ſubtract the ſum of theſe Products from Unity; then let the remainder be raiſed to its Square, if the ſecond Series be compoſed of Terms in Arithmetic Progreſſion; or to its Cube, if it be compoſed of Terms whoſe third Differences are equal to Nothing; or to its fourth Power, if it be compoſed of Terms whoſe fourth Differences are equal to Nothing; and ſo on. Let that Power be ſubtracted from Unity, and the remainder, having cancelled the Letter* r, *will be the Index required. Thus in the foregoing Example, having taken the Index* $3+2$, *which belongs to the firſt Series, and Multiplied its Terms by* r *and* rr *reſpectively, let the Product* $3r+2rr$ *be* *be ſubtracted from Unity, and the* **Square** *of the remainder being* $1 - 6r + 5rr + 12r^3 + 4r^4$, *let that Square be alſo ſubtracted from Unity, then the remainder, having cancelled the Letter* r, *will be* 6 — 5 — 12 — 4, *which is the Index required.*

But in caſe neither of the two firſt Series have any of their laſt Differences equal to Nothing, yet if in both of them the Relation of their Terms be expreſſed by conſtant Indices, the third Series, reſulting from the Multiplication of the correſponding Terms of the two firſt Series, will alſo have its Terms related to one another according to a conſtant Index. Thus, taking the Series 1, 3, 5, 11, 21, 43 &c. *the Relation of whoſe Terms is expreſſed by means of the Index* 1+2, *and Multiplying its Terms by the correſponding Terms of the Series* 1, 2, 5, 13, 34, 89 &c. *the Relation of whoſe Terms is expreſſed by the Index* 3—1, *the Products will compoſe the Series* 1, 6, 25, 143, 714, 3827, *whoſe Terms are Related to one another according to the Index* 3 + 13 — 6 — 4.

Generally, *If the Index expreſſing the Relation of the Terms in the firſt Series be* m+n, *and the Index expreſſing the Relation of the Terms in the ſecond Series be* p+q; *then will the Index of the Relation, in the Series reſulting from the Multiplication of the correſponding Terms of the Two firſt Series, be expreſſed by the following Quantities,*

$$\textit{viz.}\quad mp + \begin{matrix} +mmq \\ npp \\ +2nq \end{matrix} + mnpq - nnqq.$$

But if it ſo Happen that p *be equal to* m, *and* q *to* n; *then the foregoing Theorem may be contracted, and the Index of the Relation may be expreſſed as follows,* viz. $\begin{matrix} mm + mmn \\ +n + nn \end{matrix} - n^3$ *ſo that the Relation of each Term to the preceding ones need not be extended, in this Caſe, to any more than three Terms.*

And in like manner other Theorems may be found, which may be extended farther, and at laſt be comprized under one general Rule.

PROBLEM XLIII.

Supposing A *and* B, *whose proportion of Skill is as* a *to* b, *to Play together, till* A *either wins the number* q *of Stakes, or loses the number* p *of them; and that* B *Sets at every Game the sum* G *to the sum* L: *It is required to find the Advantage, or Disadvantage of* A.

SOLUTION.

First, Let the number of Stakes to be won or lost on either side be equal, and let that number be p; let there be also an equality of Skill between the Gamesters: Then I say, the gain of A will be $pp \times \frac{G-L}{2}$, that is, the Square of the number of Stakes which either Gamester is to win or lose, Multiplied by one half of the Difference of the value of the Stakes. Thus, if A and B play till such time as Ten Stakes are won or lost, and B Setts a Guinea to Twenty Shillings; then the gain of A will be a hundred times half the Difference between a Guinea and Twenty Shillings, *viz.* 3 *l* — 15 shil.

Secondly, Let the number of Stakes be unequal, so that A be obliged either to win the number q of Stakes, or to lose the number p; let there be also an equality of Chance between A and B: Then I say, that the gain of A will be $pq \times \frac{G-L}{2}$; that is, the Product of the two numbers of Stakes, and one half of the Difference of the value of the Stakes Multiplied together. Thus, if A and B play together till such time as either A wins Eight Stakes, or loses Twelve; then the Gain of A will be the Product of the Three numbers 8, 12, 9, which makes 864 pence, or 3 *l* — 12 shil.

Thirdly, Let the number of Stakes be equal, but let the number of Chances to win a Game, or the Skill of the Gamesters be unequal, in the proportion of a to b. Then I say, that the gain of A will be $\frac{\overline{pa^p - pb^p}}{a^p + b^p} \times \frac{\overline{aG - bL}}{a - b}$.

Fourthly,

Fourthly, Let the number of Stakes be unequal, and let also the number of Chances be unequal: Then I say that the gain of A will be $\frac{qa^q \times \overline{a^p - b^p} - pb^p \times \overline{a^q - b^q}}{a^{p+q} - b^{p+q}} \times \frac{aG - bL}{a - b}$

DEMONSTRATION.

IN order to form a general Demonstration of these Rules, let us resolve some particular Cases of this Problem, and examine the process of their Solution: Let it therefore be proposed to find the gain of A in the Case of Four Stakes to be won or lost on either side, and of an equality of Chance between A and B to win a Game. There being an equality of Chance for A, every Game he plays, to win G or to lose L, it follows, that the gain of every Game he plays is to be reputed to be $\frac{G-L}{2}$. But it being uncertain whether any more Games than Four will be play'd, it follows, that the gain of the Tenth Game, for instance, to be estimated before the play begins, cannot be reputed to be $\frac{G-L}{2}$; for it would only be such provided the Play were not Ended before that Tenth Game: Wherefore the gain of the Tenth Game is the Quantity $\frac{G-L}{2}$ Multiplied by the Probability of the Plays not being Ended in Nine Games, or before, for the same reason, the gain of the Ninth Game is the Quantity $\frac{G-L}{2}$ Multiplied by the Probability that the Play will not be Ended in Eight Games: And likewise the gain of the Eighth Game is the Quantity $\frac{G-L}{2}$ Multiplied by the Probability that the Play will not be ended in Seven Games, and so on. From whence it may be concluded, that the gain of A, to be estimated before the Play begins, is the Quantity $\frac{G-L}{2}$ Multiplied by the sum of the Probabilities that the Play will not be Ended in 0, 1, 2, 3, 4, 5, 6, &c. Games *ad infinitum.*

Let those Probabilities be respectively called A', B', C', D', E', F', G', &c. Then, because the Probability of the Plays not being Ended in Five Games is equal to the Probability of its not Ending in Four, and that the Probability of its not Ending in Seven, is equal to the Probability of its not Ending

Ending in Six, it will follow, that the ſum of the Probabilities belonging to all the Even Games is equal to the ſum of the Probabilities belonging to all the Odd ones: We are therefore only to find the ſum of all the Even Terms, $A' + C' + E' + G'$ &c. and to double it afterwards.

Now it will appear, from our XXXIII*d* Problem, that theſe Terms conſtitute the following Series, *viz.*

$$\frac{1}{1} + \frac{4}{4} + \frac{14}{16} + \frac{48}{64} + \frac{164}{256} + \frac{560}{1024} + \frac{1912}{4096} \text{ \&c.}$$

In which Series, each Numerator being Related to the two preceding ones according to the Index $4 - 2$, and each Denominator being a Power of 4, it follows, that this Series may be compared with the firſt Theorem of our ſecond *Lemma*; by making the firſt and ſecond Terms A and B, uſed in that Theorem, to be reſpectively equal to 1 and $\frac{4}{4}$, making alſo the Quantities m, n, r reſpectively equal to 4, 2, $\frac{1}{4}$. Which being done, it will be found that the ſum of all thoſe Terms *ad infinitum* will be equal to 8.

We may therefore conclude that the ſum of all the Terms $A' + B' + C' + D' + E'$ &c. is equal to 16, and that the gain of A is equal to $16 \times \frac{G-L}{2}$.

But if the number of Chances which A and B have to win a Game, be in a proportion of inequality, then the ſum of the Series $A' + C' + E' + G' + I'$ &c. will be found thus: Let $\frac{ab}{\overline{a+b}|^2}$ be called r, and the Terms of that Series will be Related to one another as follows, *viz.* $E' = 4C'r - 2A'rr$, $G' = 4E'r - 2C'rr$, and ſo on. Let therefore 4, 2, 1, 1, be reſpectively ſubſtituted, in the firſt Theorem of our ſecond *Lemma*, in the room of m, n, A', B'; and the ſum of this Series will be found to be $\frac{2-4r}{1-4r+2rr}$; in which expreſſion, reſtoring the value of r, *viz.* $\frac{ab}{\overline{a+b}|^2}$, the ſum of the Series will become $\frac{\overline{2aa+2bb} \times \overline{a+b}|^2}{a^4+b^4}$, the double of which is the ſum of all the Terms $A' + B' + C' + D' + E'$ &c. But becauſe, in every Game, the Gameſter A has the number a of Chances to win G, and the number b of Chances to loſe L; it follows, that his gain in every Game is equal

 to

to $\frac{aG-bL}{a+b}$. From whence it may be concluded, that the Advantage of A, to be estimated before the Play begins, will be $\frac{4aa+4bb \times \overline{a+b}^2}{a^4+b^4} \times \frac{aG-bL}{a+b}$.

Before we proceed farther, we must observe, that the Series $A'+B'+C'+D'+E'+F'$ &c. which we have assumed to represent in general the Probabilities of the Plays not being Ended in 0, 1, 2, 3, 4, 5 &c. Games, whether the Stakes be equal or unequal, being divided into two parts, *viz.* $A'+C'+E'+G'$ &c. and $B'+D'+F'+H'$ &c. answering to 0, 2, 4, 6 &c. and 1, 3, 5, 7, &c. each Term of these two new Series will be related to the preceding ones, according to the same Law of Relation, as are the Terms of those Series which express the Probabilities of the Plays being Ended in a certain number of Games, under the like circumstances of Stakes to be won or Lost. The Law of which Relation is to be deduced from our XXXIV*th*, and XXXIX*th* Problems.

If the number of Stakes to be won or lost on either side be equal to Six, and the proportion of Chances to win a single Game be as a to b; then the Relation of each Term to the preceding ones, in the Series $A'+C'+E'+G'$ &c. will be expressed by the Index $6-9+2$. Wherefore to find the sum of these Terms, let the Quantities 6, 9, 2, 1, 1, 1, be respectively substituted, in the third Theorem of our second *Lemma*, in the room of m, n, p, A', B', C', and the sum of those Terms will be found to be

$$\frac{\begin{array}{c}1+1+1-6r+9rr\\ -6r\end{array}}{1-6r+9rr-2r^3} \text{ or } \frac{3-12r+9rr}{1-6r+9rr-2r^3}.$$

In which expression substituting $\frac{ab}{\overline{a+b}^2}$ in the room of r, the same will become $\frac{3a^4+3aabb+3b^4 \times \overline{a+b}^2}{a^6+b^6}$. From whence we may conclude, that the gain of A will be

$$\frac{6a^4+6aabb+6b^4 \times \overline{a+b}^2}{a^6+b^6} \times \frac{aG-bL}{a+b}.$$

Again,

Again, if the number of Stakes to be won or loſt on either ſide be Eight, it will be found, that the gain of A will be $\frac{8a^6 + 8a^4bb + 8aab^4 + 8b^6 \times \overline{a+b}|^2}{a^8 + b^8} \times \frac{aG - bL}{a+b}$.

But the Numerators of the foregoing Fractions being in Geometric Progreſſion, if thoſe Progreſſions be ſummed up, the gain of A, in the Caſe of Four Stakes to be won or loſt, may be expreſſed as follows,

viz. by the Fraction $\frac{4a^4 - 4b^4 \times \overline{a+b}|^2}{a^4 + b^4 \times aa-bb} \times \frac{aG - bL}{a+b}$; or dividing both Numerator and Denominator by $\overline{a+b}|^2$, the ſame may be expreſt by the Fraction $\frac{4a^4 - 4b^4}{a^4 + b^4} \times \frac{aG - bL}{a-b}$. His gain likewiſe, in the Caſe of Six Stakes to be won or loſt, will be expreſt by the Fraction $\frac{6a^6 - 6b^6}{a^6 + b^6} \times \frac{aG - bL}{a-b}$; and in the Caſe of Eight Stakes to be won or loſt, it will be expreſt by the Fraction $\frac{8a^8 - 8b^8}{a^8 + b^8} \times \frac{aG - bL}{a-b}$: So that we may conclude, that in any Caſe of an Even and equal number of Stakes denominated by p, the gain of A will be expreſt by the Fraction $\frac{pa^p - pb^p}{a^p + b^p} \times \frac{aG - bL}{a-b}$.

But if the number of Stakes be Odd and equal, as it is in the Caſe of Five Stakes to be won or loſt, then the two Series $A' + C' + E' + G' + I'$ &c. and $B' + D' + H'$ &c. will be unequal, and the exceſs of the firſt above the ſecond will be Unity. Wherefore to find the gain of A, in the Caſe of Five Stakes, having ſet aſide the firſt Term of the firſt Series, let all other the Terms be added together, by comparing them with thoſe that are employed in the firſt Theorem of our ſecond *Lemma*; which will be done thus. Since $C' = 1$, $E' = 1$, and $G' = 5E'r - 5C'rr$, let the numbers 1, 1, 5, 5, be reſpectively ſubſtituted in the aforeſaid Theorem, in the room of the Letters A', B', m, n; and

and the ſum of that Series will be found to be $\frac{2-5r}{1-5r+5rr}$: To the double of which adding Unity, which we had ſet aſide, it will appear that the ſum of the two Series together will be $\frac{5-5r+5rr}{1-5r+5rr}$; or writing $\frac{ab}{\overline{a+b}^2}$ in the room of r, $\frac{5a^4+5a^3b+5aabb+5ab^3+5b^4}{a^4-a^3b+aabb-ab^3+b^4}$. Now by reaſon that the Terms of both Numerator and Denominator of this laſt Fraction compoſe a Geometrick Progreſſion, the Numerator will be reduced to $\frac{5a^5-5b^5}{a-b}$, and the Denominator will be reduced to $\frac{a^5+b^5}{a+b}$. From whence it follows, that the ſum of theſe two Series will be $\frac{\overline{5a^5-5b^5}\times\overline{a+b}}{\overline{a^5+b^5}\times\overline{a-b}}$, and that the gain of A will be $\frac{\overline{5a^5-5b^5}}{a^5+b^5}\times\frac{\overline{aG-bL}}{a-b}$. If the gain of A be likewiſe inquired into, in the Caſe of Seven Stakes to be won or loſt, then it will be found to be $\frac{\overline{7a^7-7b^7}}{a^7+b^7}\times\frac{\overline{aG-bL}}{a-b}$. And the ſame form of expreſſion being conſtantly obſerved in all caſes wherein the number of Stakes is Odd and equal, we may conclude that if that number be denominated by p, then the gain of A will be $\frac{\overline{pa^p-pb^p}}{a^p+b^p}\times\frac{\overline{aG-bL}}{a-b}$. Now this expreſſion of the gain of A having been found to be the ſame in the Caſe of an Even number of Stakes, as it is now found in the Caſe of an Odd one; we may conclude, that it is general, and belongs to any equal number of Stakes whether Even or Odd.

If the number of Stakes be unequal, the Inveſtigation of the gain of A will be made in the ſame manner as it was in the Caſe of an equality of Stakes. Thus, let us ſuppoſe that the Play be to continue till ſuch time as either A wins Two Stakes, or B Three. In order therefore to find the gain of A, let the Series $A' + B' + C' + D' + E' + F'$ &c. be be divided into two parts, *viz.* $A' + C' + E'$ &c. and $B' + D' + F'$ &c. then it will appear, from our XXXIII*d*, and XXXIX*th* Problems, that $A' = 1$, $C' = \frac{2ab + bb}{\overline{a+b}|^2}$, $E' = 3\,C'r - 1\,A'rr$. Having now obtained the first Terms of the Series, and the Relation of each Term of it to the preceding ones; it will be easie to find the sum of all its Terms, by the help of the first Theorem of our second *Lemma*, making the Quantities, A, B, m, n therein employed to be respectively equal to 1, $\frac{2ab+bb}{\overline{a+b}|^2}$, 3, 1. This done, the sum of that Series will be found to be equal to

$$\frac{\overline{aa + ab + 2bb} \times \overline{a+b}|^2}{a^4 + a^3b + aabb + ab^3 + b^4}.$$

In like manner it will appear, that in the second Series B' is $= 1$, $D' = \frac{2aab + 3abb}{\overline{a+b}|^3}$, $F' = 3\,D'r - B'rr$; from whence the sum of all its Terms will be found to be $\frac{\overline{aa + 2ab + 2bb} \times \overline{a+b}|^2}{a^4 + a^3b + aabb + ab^3 + b^4}$: And both sums of those Series being added together, the aggregate of them will be $\frac{\overline{2a^3 + aab + 6abb + 3b^3} \times \overline{a+b}}{a^4 + a^3b + aab + ab^3 + b^4}$: But the Terms of this Denominator composing a Geometric Progression, whose sum is $\frac{a^5 - b^5}{a - b}$, the foregoing Fraction may be reduced to $\frac{\overline{2a^4 + 2a^3b - 2aabb - 3ab^3 - 3b^4} \times \overline{a+b}}{a^5 - b^5}$; which Fraction is still capable of a farther reduction; for the three first Terms of its Numerator compose a Geometric Progression, and the two last Terms may be considered as being in Geometric Progression, and consequently the Fraction may at last be reduced to $\frac{\overline{2aa \times \overline{a^3 - b^3} - 3b^3 \times \overline{aa - bb}} \times \overline{a+b}}{a^5 - b^5 \times \overline{a - b}}$, from which expression, the gain of A will be found to be

$$\frac{2aa \times \overline{a^3 - b^3} - 3b^3 \times \overline{aa - bb}}{a^5 - b^5} \times \frac{aG - bL}{a - b}.$$

By the ſame method of Proceſs, it will be eaſy to determine the gain of *A* under any other circumſtance of Stakes to be won or loſt: And if it be remembred always to ſum up thoſe Terms which are in Geometric Progreſſion, all the various expreſſions of the gain of *A*, calculated for differing numbers of Stakes, will appear to be uniform: From whence it may be collected by bare inſpection, that the gain of *A* is what we have aſſerted it to be, *viz.*

$$\frac{\overline{qa^q \times \overline{a^p - b^p} - pb^p \times \overline{a^q - b^q}}}{a^{p+q} - b^{p+q}} \times \frac{\overline{aG - bL}}{a - b}.$$

It is to be obſerved, *Firſt*, that if p and q be equal, the foregoing expreſſion may be reduced to $\frac{\overline{pa^p - pb^p}}{a^p + b^p} \times \frac{\overline{aG - bL}}{a - b}$, as will appear if both Numerator and Denominator be divided by $a^p - b^p$, having firſt ſubſtituted p in the room of q. *Secondly*, that if a and b be equal, the ſame expreſſion may be reduced to $pq \times \frac{\overline{G - L}}{2}$, which will appear if both Numerator and Denominator be divided by $\overline{a - b}^2$.

After I had Solved the foregoing Problem, I wrote word of it to Mr. *Nicolas Bernoully*, the preſent Profeſſour of Mathematics at *Padoua*, without acquainting him with my Solution: I only let him know in general that it was done by the Method of Infinite Series; whereupon he ſent me two different Solutions of that Problem: And as one of them has ſome Affinity with the Method of Series uſed all along in this Book, I ſhall tranſcribe it here in the Words of his Letter, " My Uncle has obſerved that this Problem may alſo " be Solved after the ſame manner as you have Solved the " Ninth Problem * of your Tract *de Menſura Sortis*, it be- " ing viſible that the Expectations of the Gameſters will re- " ceive no alteration whether it be ſuppoſed that the Pieces " which *A* and *B* Set every time to each other, are reſpective- " ly *L* and *G*, or whether it be ſuppoſed that thoſe Pieces " conſtitute the following Progreſſion, *viz.*

" $L, G, G + \frac{b}{a} \times \overline{G - L}, G + \frac{b}{a} \times \frac{bb}{aa} \times \overline{G - L}, G + \frac{b}{a} \times \frac{bb}{aa} \times \frac{b^3}{a^3} \times \overline{G - L}$ &c. the number of whoſe Terms is $p + q$, whereof

* *See the IXth Problem of this Book.*

the

" the firſt, whoſe number is p, denote the Pieces of A; and " the laſt, whoſe number is q, denote the Pieces of B: For in " either Caſe the gain of A will be $\frac{aG-bL}{a+b}$. Now it being " poſſible to find the ſum of any number of Terms of this Pro- " greſſion, it follows that the different values of all the Pieces " of each Gameſter may be obtained: Let therefore thoſe " values be denoted reſpectively by S and T; let alſo the " Probabilities of winning the number of Stakes agreed up- " on be called A and B reſpectively, which Probabilities are

" $\frac{a^{p+q}-a^q b^p}{a^{p+q}-b^{p+q}}$ and $\frac{a^q b^p - b^{p+q}}{a^{p+q}-b^{p+q}}$, ſuch as we had ſeverally

" derived them, your ſelf in your aforeſaid Problem, and I " in Mr. *Montmort*'s Book. This being ſuppoſed, the gain of A " will be found to be $AT - BS$, or $\frac{AG-bL}{a-b} \times \overline{Aq-Bp}$.

N. B. Tho' I may, accidentally, have given a uſeful Hint for that elegant Method of ſolving the foregoing Problem, yet I think it reaſonable to aſcribe it entirely to its proper Author; the Hint having been improv'd much beyond what I could have expected.

REMARK.

IT is to be obſerved, that the gain of A is not to be regulated by the equal Probability there is that the Play may, or may not be Ended in a certain number of Games. For inſtance, If two Gameſters having the ſame number of Chances to win a Game, deſign only to play untill ſuch time only as two Stakes are won or loſt; it is as Probable that the Play may be Ended in two Games as not, yet it cannot be concluded from thence, that the gain of A is to be eſtimated by the Product of the Number 2 by one half of the Difference of the Stakes: For it has been Demonſtrated that this gain will be Four times that half difference. In like manner, if the Play were to continue, till either A ſhould win Two Stakes, or B Three; it will be found, that it is as Probable that the Play may End in Four Games as not; and yet the gain of A is not to be eſtimated by the Product of the Number 4 by one half of the Difference of the Stakes; it having

ving been Demonſtrated that it is Six times that half Difference. To make this the more ſenſible, let us ſuppoſe that *A* and *B* are to Play till ſuch time as *A* either wins one Stake, or loſes Ten: It is plain, that in this Caſe it is as Probable that the Play may be Ended in One Game as not, and yet the gain of *A* will be found to be Ten times the Difference of the Stakes. From hence it is plain, that this gain is not to be eſtimated, by the equal Probability of the Plays Ending or not Ending in a certain number of Games, but by the Rules which have been preſcribed in this Problem.

PROBLEM XLIV.

IF A *and* B, *whoſe proportion of Skill is as* a *to* b, *reſolving to Play together till ſuch time as Four Stakes are won or loſt on either ſide, agree between themſelves, that the firſt Game that is play'd, they ſhall Set to each other the reſpective ſums* L *and* G; *that the ſecond Game they ſhall Set the ſums* 2 L *and* 2 G; *the third Game the ſums* 3 L *and* 3 G, *and ſo on; the Stakes increaſing continually in an Arithmetic Progreſſion: It is Demanded how the gain of* A *is to be eſtimated in this Caſe, before the Play begins.*

SOLUTION.

LET there be ſuppoſed a Time wherein the Number p of Games has been play'd; then *A* having the Number a of Chances to win the ſum $\overline{p+1} \times G$ in the next Game, and *B* having the Number b of Chances to win the ſum $\overline{p+1} \times L$; it is plain, that the gain of *A* in that circumſtance of Time will be $\overline{p+1} \times \overline{\frac{aG-bL}{a+b}}$. But this gain being to be eſtimated before the Play begins, it follows, that it ought to be eſtimated by the Quantity $\overline{p+1} \times \overline{\frac{aG-bL}{a+b}}$ multiplied by the reſpective Probability there is that the Play will not then be Ended; and therefore the whole gain of *A* is the ſum of the Probabilities of the Plays not Ending in 0, 1, 2, 3, 4, 5, 6 &c. Games *ad infinitum*, multiplied by the reſpective values of the Quantity $\overline{p+1} \times \overline{\frac{aG-bL}{a+b}}$, p being Interpreted

preted successively by the Terms of the Arithmetic Progression, 0, 1, 2, 3, 4, 5, 6 &c. Now let these Probabilities of the Plays not Ending be respectively called A', B', C' D', E', F', G' &c. Let also the Quantity $\frac{aG - bL}{a+b}$ be called S; and thence it will follow, that the gain of A will be $A'S + 2B'S + 3C'S + 4D'S + 5E'S + 6F'S$ &c. But in the Case of this Problem B' is equal to A', and D' is equal to C', and so on. Wherefore the gain of A may be expressed by the Series $S \times \overline{3A' + 7C' + 11E' + 15G' + 19I'}$ &c. But it appears, by our XXXIII*d* Problem, that the Terms A', C' E', G' are respectively equal to the following Quantities, *viz.* 1, 1, $\frac{4a^3b + 6aabb + 4ab^3}{\overline{a+b}|^4}$, $\frac{14a^4bb + 20a^3b^3 + 14aab^4}{\overline{a+b}|^6}$: Whence it follows, that the Terms $3A' + 7C' + 11E' + 15G'$ may be obtained: It appears also, from what we have observed in the preceding Problem, that the Relation of the Terms A', C', E' &c. may be expressed by the Index $4 - 2$; and by the Third *Lemma* prefixt to that Problem, that the Relation of the Terms $3A'$, $7C'$, $11E'$ &c. may be expressed by the Index $8 - 20 + 16 - 4$: And therefore substituting the Quantities $3A'$, $7C'$, $11E'$, $15G'$ in the room of the Quantities A, B, C, D, which we make use of in the Third Theorem of our second *Lemma*; substituting likewise the Quantities, 8, 20, 16, 4 in the room of m, n, p, q; and lastly substituting $\frac{ab}{\overline{a+b}|^2}$ in the room of r; the gain of A will be exprest by the following Quantities, *viz.*

$$S \times \frac{\overline{10a^6 + 24a^5b + 42a^4bb + 64a^3b^3 + 42aab^4 + 24ab^5 + 10b^6} \times \overline{a+b}|^2}{\overline{a^4 + b^4}|^2}$$

which, in the Case of an equal number of Chances to win a Stake, would be reduced to $216\,S$; and therefore if the Quantities G and L stand respectively for a Guinea and Twenty Shillings, which will make the value of S to be Nine pence, it follows, that the gain of A will in this Case be 8 *l* — 2 Shil.

Corollary I. If the Stakes were to Increaſe according to the proportion of the Terms of any of thoſe Series which we have deſcribed in our *Lemma's*, and that there were any given inequality in the number of Stakes to be won or loſt, the gain of *A* might ſtill be found.

Corollary II. There are ſome Caſes wherein the gain of *A* would be Infinite: Thus, if *A* and *B* were to Play till ſuch times as Four Stakes were won or loſt, and it were agreed between them to double their Stakes at every Game, the gain of *A* would in this Caſe be Infinite: Which conſequence may eaſily be deduced from what has been ſaid before.

PROBLEM XLV.

If A *and* B *reſolve to Play till ſuch time as* A *either wins a certain given number of Stakes, or that* B *wins the ſame, or ſome other given number of them: 'Tis required to find in how many Games it will be as Probable that the Play may be Ended as not?*

SOLUTION.

LET it be ſuppoſed that *A* and *B* are to play till ſuch time as either of them wins Three Stakes, and that there is an Equality of Skill between them. This being ſuppoſed, it will appear, from our XXXIV*th* Problem, that the Probability of the Plays continuing for an Indeterminate number of Games may be expreſt by the following Series, *viz.*

$\frac{\overline{a^3+b^3}}{\overline{a+b}|^3} \times \overline{1 + \frac{3ab}{\overline{a+b}|^2} + \frac{9aabb}{\overline{a+b}|^4} + \frac{27a^3b^3}{\overline{a+b}|^6}}$ &c. which, in the Caſe of an Equality of Skill between the Gameſters, will be reduced to this Series,

$\frac{2}{8} \times \overline{\overset{\text{III}}{1} + \overset{\text{V}}{\frac{3}{4}} + \overset{\text{VII}}{\frac{9}{16}} + \overset{\text{IX}}{\frac{27}{64}}}$ &c. whoſe Terms are reſpectively correſponding to the number of Games 3, 5, 7, 9 &c. Wherefore ſo many of thoſe Terms ought to be taken, as that their ſum being multiplied by the common Multiplicator

tor $\frac{2}{8}$ or $\frac{1}{4}$, the Product may be equal to the Fraction $\frac{1}{2}$, which Fraction denotes the equal Probability of an Events Happening or not Happening: But if two of thoſe Terms be taken, and that their ſum be Multiplied by $\frac{1}{4}$, the Product will be $\frac{7}{16}$; which being leſs than the Fraction $\frac{1}{2}$, it may be concluded that Five Games are too few to make it as Probable that the Play will be Ended in that number of Games as not; and that the Odds againſt its Ending in Five Games are 9 to 7. But if Three of thoſe Terms be taken, then their ſum being multiplied by the common Multiplicator $\frac{1}{4}$, the Product will be $\frac{37}{64}$; which exceeding the Fraction $\frac{1}{2}$, it may be concluded that Seven Games are too many; and that the Odds of the Play being Ended in Seven Games, or ſooner, are 37 to 27, or 4 to 3 very nearly.

N. B. It would be needleſs to inquire whether Six Games might not bring the Play to an equal Probability of Ending or not Ending; it having been obſerved before, that in the Caſe of an equality of Stakes to be play'd for, it is impoſſible that the Play ſhould End in an Even number of Games, if the number of Stakes be Odd; or that it ſhould End in an Odd number of Games, if the number of Stakes be Even.

In like manner, if the Play were to continue till Four Stakes be won or loſt on either ſide: Then taking the following Series, *viz.*

$$\overline{\frac{a^4+b^4}{\overline{a+b}|^4}} \times \overline{1 + \frac{4ab}{\overline{a+b}|^2} + \frac{14aabb}{\overline{a+b}|^4} + \frac{48a^3b^3}{\overline{a+b}|^6}}\ \text{\&c.}$$ which, upon

the ſuppoſition of an equality of Skill between the Gameſters, may be reduced to this, *viz.*

$$\frac{2}{16} \times \overline{\overset{\text{IV}}{1} + \overset{\text{VI}}{\frac{4}{4}} + \overset{\text{VIII}}{\frac{14}{16}} + \overset{\text{X}}{\frac{48}{64}} + \overset{\text{XII}}{\frac{164}{256}}}\ \text{\&c.}$$ let ſo many

of its Terms be tried, as will make the Product of their ſum multiplied by $\frac{2}{16}$, equal to the Fraction $\frac{1}{2}$, or as near it as poſſible. Now Five of thoſe Terms being tried, and their ſum being multiplied by $\frac{2}{16}$, or $\frac{1}{8}$, the Product will be $\frac{1092}{2048}$, which not differing much from $\frac{1}{2}$, it may be

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concluded that Twelve will be very near that number of Games, which will make the Probabilities of the Plays Ending or not Ending to be equal; the Odds for its Ending being only 1092 to 956, or 8 to 7 very nearly. But the Odds againſt its Ending in Ten Games, will be found to be 39 to 29, or 4 to 3 nearly.

By the ſame method of Proceſs, it will be found that Five Stakes will probably be won or loſt in about Seventeen Games: It being but the Odds of 11 to 10 nearly, that the Play will not be Ended in that number of Games, and 10 to 9 nearly, that it will be Ended in Nineteen.

It will alſo be found that Six Stakes will probably be won or loſt in about Twenty Six Games, there being but the Odds of 168 to 167 nearly, that the Play will not be Ended in that number of Games, and 25 to 22 nearly, that it will be Ended in Twenty Eight.

If the ſame Method of Trial be applied to any other number of Stakes, whether equal or unequal, and to any proportion of Skill, the number of Games required will always be found.

Yet if the number of Stakes were great, thoſe Trials would become tedious, notwithſtanding the Help that might be derived from our Second *Lemma*, whereby any number of Terms of thoſe Series which are employed in the Solution of this Problem, may be added together. For which reaſon it will be convenient to make ſome Trials of another nature, and to ſee whether, from the reſolution of ſome of the ſimpleſt Caſes of this Problem, any Analogy can be obſerved between the number of Stakes given, and the number of Games which determine the equal Probability of the Plays Ending or not Ending.

Now Mr *de Monmort* having with great Sagacity diſcovered that Analogy, in the Caſe of an equal and Odd number of Stakes, on ſuppoſition of an equality of Skill between the Gameſters, I thought the Reader would be well pleaſed to be acquainted with the Rule which he has given for that purpoſe, and which is as follows.

Let n be any Odd number of Stakes to be won or loſt on either ſide; let alſo $\frac{n+1}{2}$ be made equal to p: Then the Quantity $3pp-3p+1$ will denote a number of Games, wherein

wherein it will be more than an equal Chance that the Play will be Ended; thus, if the number of Stakes be Nineteen, then p will be 10, and the Quantity $3pp - 3p + 1$ will be 271, which ſhews that 'tis more than an equal Chance that the Play will be Ended in 271 Games.

The Author of this Rule owns that he has not been able to find another like it, for an Even number of Stakes; but I am of opinion, that tho' the ſame Rule, being applied to that Caſe, may not find the juſt number of Games wherein there will be more than an equal Probability of the Plays Ending, yet it will always find a number of Games, wherein it is very near an equal Wager that the Play will be Ended. Wherefore to make the Rule as extenſive as it may be, I would Chuſe to expreſs it by the number of Stakes whether Even or Odd, and make it $\frac{3}{4}nn$, which differs from his own, but by the ſmall Fraction $\frac{1}{4}$.

If any one has a mind to carry this ſpeculation ſtill farther, and to try whether ſome general Rule may not be diſcovered for determining, by a very near approximation, the number of Games requiſite to make it a Wager of any given proportion of Odds, that the Play will be Ended in that number of Games, whether the Skill of the Gameſters be equal or unequal; let him Solve ſeveral Caſes of this Problem in the following manner, which I take to be as expeditious as the nature of the Problem can admit of.

Upon a Diameter equal to Unity, if ſo be the Skill of the Gameſters be equal; or to the Quantity $\frac{4ab}{\overline{a+b}^2}$, if their Skill be in the proportion of a to b, let a Semicircle be deſcribed, which divide into ſo many equal parts as there are Stakes to be won or loſt on either ſide, ſuppoſing thoſe Stakes to be equal. From the Firſt, Third, Fifth, Seventh &c. Points of Diviſion, beginning from one extremity of the Diameter, let Perpendiculars fall upon that Diameter, which by their concourſe with it, ſhall determine the *verſed Sines* of ſo many Arcs, to be taken from the other extremity thereof. Let the greateſt of thoſe *verſed Sines* be called m, the next leſs p, the next to it q, the next s &c. Make alſo

$$\frac{\overline{1-p}\times\overline{1-q}\times\overline{1-s}}{\overline{m-p}\times\overline{m-q}\times\overline{m-s}}\ \&c. = A$$

$$\frac{\overline{1-q}\times\overline{1-s}\times\overline{1-m}}{\overline{p-q}\times\overline{p-s}\times\overline{p-m}}\ \&c. = B$$

$$\frac{\overline{1-s}\times\overline{1-m}\times\overline{1-p}}{\overline{q-s}\times\overline{q-m}\times\overline{q-p}}\ \&c. = C$$

$$\frac{\overline{1-m}\times\overline{1-p}\times\overline{1-q}}{\overline{s-m}\times\overline{s-p}\times\overline{s-q}}\ \&c. = D$$

&c.

then will the Probability of the Play's not Ending in a number of Games denominated by x, be expreſt by the Quantities

$m^{\frac{1}{2}x}A + p^{\frac{1}{2}x}B + q^{\frac{1}{2}x}C + s^{\frac{1}{2}x}D$ &c. if the number of Stakes be Even, or by the Quantities

$m^{\frac{x-1}{2}}A + p^{\frac{x-1}{2}}B + q^{\frac{x-1}{2}}C + s^{\frac{x-1}{2}}D$ &c. if the number of Stakes be Odd.

EXAMPLE I.

LET it be required to find what Odds there is, that in 40 Games there will be Four Stakes won or loſt on either ſide.

Having divided the Semicircle into Four equal parts, according to the abovementioned directions, the Quantity m will be the *Verſed Sine* of 135 Degrees, and the Quantity p will be the *Verſed Sine* of 45 Degrees, which by the help of a Table of *Sines* will readily be found to be 0.85355 and 0.14645 reſpectively. Moreover the Quantity A being equal to $\frac{1-p}{m-p}$, and the Quantity B to $\frac{1-q}{p-q}$, will be found to be 1.2071 and — 0.2071. From whence it follows, that the Probability of the Plays not Ending in Forty Games may be expres'd by the two following Products $\overline{0.85355}|^{20} \times 1.2071$ — $\overline{0.14645}|^{20} \times 0.2071$, of which the Second may be entirely neglected, as being inconſiderably little in reſpect of the

the firſt. Now the Logarithm of the firſt Product being $\overline{2}.7063225$, to which anſwers the number 0.05085, let that number be ſubtracted from Unity; and the remainder being 0.94915, I conclude that the Odds of the Plays Ending in Forty Games are as 94915 to 5085, or very near as 19 to 1.

EXAMPLE II.

LET it be required to find how many Games muſt be play'd, to make it a Wager of 100 to 1, that Four Stakes will be won or loſt on either ſide, in that number of Games.

Let x be the number of Games required: Then by the foregoing Example it will appear that we may have the Equation $\overline{0.85355}|^{\frac{1}{2}x} \times 1.2071 = \frac{1}{100}$, in which the value of x may eaſily be obtained by Logarithms; it being found by one ſingle Diviſion to be about 60.

If the Stakes be unequal, the Solution will conſiſt of two Series, in both which the Quantities m, p, q &c. will be of the ſame value, and will be determined likewiſe by a Table of *Sines*. In this Caſe the Semicircumference ought to be divided into as many equal parts as there are Units in the number of all the Stakes: Thus, if the Stakes were Four and Five, the Semicircumference ought to be divided into Nine equal parts: But then it is to be obſerved that the *verſed Sines* of thoſe Arcs, which, in the Caſe of Nine Stakes for each Gameſter, are alternately omitted, are thoſe which, in the Caſe of Four and Five, are to repreſent the Quantities m, p, q &c. It is to be obſerved alſo that the Quantities A, B, C, D &c. by which the Terms of the firſt Series are to be reſpectively Multiplied, will be found to differ from the Quantities A', B', C', D' &c. by which the Terms of the ſecond Series are alſo to be reſpectively Multiplied; and that both thoſe Series of Quantities may be determined by proper Theorems contrived for that purpoſe.

Before I make an End of this Subject, I ſhall propoſe an Inquiry to be made by thoſe who have ſufficient leiſure to Try the foregoing Methods; which is, whether the number of Games, wherein it will be an equal Wager that the Play will

will be Ended, upon the ſuppoſition of an equal number n of Stakes to be won or loſt on either ſide; as alſo of the proportion of Skill expreſt by a and b, may not be determined very nearly by the following Expreſſion, *viz.*

$$\frac{na^n - nb^n}{a^n + b^n} \times \frac{aa + ab + bb}{aa - bb}.$$

PROBLEM XLVI.

IF A *and* B, *whoſe proportion of Skill is ſuppoſed equal, play together till Four Stakes be won or loſt on either ſide; and that* C *and* D, *whoſe proportion of Skill is alſo ſuppoſed equal, play likewiſe together till Five Stakes be won or loſt on either ſide: What is the Probability that the Play between* A *and* B *will be Ended in fewer Games than the Play between* C *and* D?

SOLUTION.

THE Probability of the Firſt Play's being Ended in any number of Games before the Second, is compounded of the Probability of the Firſt Play's being Ended in that number of Games, and of the Second's not being Ended with the Game immediately preceding: From whence it follows, that the Probability of the Firſt Plays Ending in an Indeterminate number of Games before the Second, is the ſum of all the Probabilities *ad Infinitum* of the Firſt Play's Ending, Multiplied by the reſpective Probabilities of the Second's not being Ended with the Game immediately preceding.

But it appears from our XXXIV*th* Problem, that the Probability of the firſt Play's Ending in an Indeterminate number of Games, may be expreſt by the following Series, *viz.*

$$\overset{\text{IV}}{\frac{1}{2^3}} + \overset{\text{VI}}{\frac{4}{2^5}} + \overset{\text{VIII}}{\frac{14}{2^7}} + \overset{\text{X}}{\frac{48}{2^9}} + \overset{\text{XII}}{\frac{164}{2^{11}}} + \overset{\text{XIV}}{\frac{560}{2^{13}}} \text{ \&c.}$$

It appears alſo, from our XXXIII*d* Problem, that the Probability of the Second Play's not Ending may be expreſt by the following Series, *viz.*

$$\overset{\text{III}}{\frac{4}{2^2}} + \overset{\text{V}}{\frac{15}{2^4}} + \overset{\text{VII}}{\frac{55}{2^6}} + \overset{\text{IX}}{\frac{200}{2^8}} + \overset{\text{XI}}{\frac{725}{2^{10}}} + \overset{\text{XIII}}{\frac{2625}{2^{12}}} \text{ \&c.}$$

Now

Now the Correſponding Terms of thoſe two Series being Multiplied together, the Products, ſuppoſing r equal to the Fraction $\frac{1}{16}$, will compoſe the following Series, *viz.*

$2r + 30rr + 385r^3 + 4800r^4 + 59400r^5$ &c. in which Series the Index of the Relation of each Numerical Quantity to the preceding ones, may be found by the help of our Third *Lemma*: For the Index of the Relation in the Numerator of the Firſt Series being $4 - 2$, and the Index of the Relation in the Numerator of the Second being $5 - 5$, which Relations are deduced from the XXXIV*th* Problem, it follows, that if in the Theorem of our Third *Lemma*, the Quantities $4, -2, 5, -5$, be reſpectively ſubſtituted in the room of the Quantities m, n, p, q, the Index of the Relation in the Third Series will be found to be $20 - 110 + 200 - 100$; wherefore all the Terms of this Series may be ſummed up by the Third Theorem of our Second *Lemma*, ſubſtituting the Quantities 20, 110, 200, 100 in the room of the Quantities m, n, p, q, therein employed; ſubſtituting alſo the Terms $2r$, $30rr$, $385r^3$, $4800r^4$ in the room of the Quantities A, B, C, D: For after thoſe Subſtitutions, the ſum of the Third Series will be found to be $\frac{2r - 10rr + 5r^3}{1 - 20r + 110rr - 200r^3 + 100r^4}$, which is reduced to $\frac{476}{723}$ by changing the Quantity r into its value $\frac{1}{16}$. Now ſubtracting the Fraction $\frac{476}{723}$ from Unity, the remainder will be the Fraction $\frac{247}{723}$, the Numerators of which two Fractions expreſs the Odds of the Firſt Plays Ending before the Second, which conſequently will be as 476 to 247, or 27 to 14 nearly.

If in the foregoing Problem, the Skill of the Gameſters had been in any proportion of inequality, the Problem might have been Solved with the ſame eaſe.

When in a Problem of this nature the number of Stakes to be loſt by either A or B, does not exceed the number Three, the Problem may be always readily Solved without the uſe of the Theorem inſerted in our Third *Lemma*; tho' the number of Stakes between C or D be never ſo great. For which reaſon, if any one has the curioſity to try, if from the Solution of ſeveral Caſes of this Problem, ſome Rule may not be diſcovered for Solving the ſame generally; it will be

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convenient he ſhould compare together the different Solutions, which may reſult from the ſuppoſition that the Stakes to be loſt by either A or B are Two or Three; and then the Caſe of the foregoing Problem may alſo be compared with all the reſt: Yet as theſe Trials might not perhaps be ſufficient to diſcover any Analogy between thoſe Solutions, I have thought fit to add a new Theorem in this place, whereby Four Caſes more of this Problem may be Solved, *viz.* When the number of Stakes to be loſt by A or B, and by C or D, are 4 and 6, 4 and 7, 5 and 6, 5 and 7: The Theorem being as follows.

If there be a Series of Terms whoſe Relation is expreſſed by the Index $l+m+n$, and there be likewiſe another Series of Terms whoſe Relation is expreſſed by the Index $p+q$; and the Corresponding Terms of thoſe two Series be Multiplied together: Then the Index of the Relation in the Third Series, reſulting from the Multiplication of their correſponding Terms, will be expreſſed by the Quantities.

$$\begin{array}{lllllll} & +2mq & +lmpq & +2lnqq & & & \\ lp & +llq & +np^3 & -mmqq & -mnpqq & +nnq^3. & \\ & +mpp & +3npq & +lnppq & & & \end{array}$$

It is to be obſerved, that altho' theſe ſorts of Theorems might be applicable to the finding of the Relation of thoſe Terms, which are the Products of the correſponding Terms of two different Series, both of which conſiſt of Terms whoſe laſt Differences are equal to nothing; yet there will be no neceſſity to uſe them for that purpoſe, that Relation being to be found much ſhorter, as follows.

Let e and f denote the rank of thoſe Differences which are reſpectively equal to nothing in each Series; then the Quantity $e+f-1$ will denote the rank of that Difference which is equal to nothing, in the Series reſulting from the Multiplication of the correſponding Terms of the other two; and conſequently the Relation of the Terms of this New Series will eaſily be obtained by our firſt *Lemma.*

After having given the Solution of ſeveral ſorts of Problems, each of them containing ſome degree of Difficulty not to be met with in any of the reſt; and having thereby laid a ſufficient foundation for ſolving the moſt intricate caſes that may occurr in this Subject of Chances, it might almoſt ſeem ſuperfluous to add any thing to this Tract: Yet conſidering that a Variety of Examples is the propereſt means of making Rules eaſy and familiar; and deſigning to be as uſeful as poſſible to thoſe of my Readers, who perhaps may not be ſo well verſed in Algebraical Calculations, I have choſe to fill up the remaining Pages of this Book, with ſome eaſy Problems relating to the Games which are moſt in uſe, ſuch as HAZARD, WHISK, PIQUET, *&c. and to enlarge a little more upon the Doctrine of* Combinations.

PROBLEM XLVII.

To find at HAZARD *the Advantage of the* Setter *upon all Suppoſitions of* Main *and* Chance.

SOLUTION.

LET the whole Money Play'd for be conſidered as a common Stake, upon which both the Setter and Caſter have their ſeveral Expectations; then let thoſe Expectations be determined in the following manner.

Firſt, Let it be ſuppoſed that the Main is *vii*; then if the Chance of the Caſter be *vi* or *viii*, it is plain that the Setter having Six Chances to win and Five to loſe, his Expectation will be $\frac{6}{11}$ of the Stake: But there being Ten Chances out

out of Thirty-ſix for the Chance to be *vi* or *viii*, it follows, that the Expectation of the Setter, reſulting from the Probability of the Chance being *vi* or *viii*, will be $\frac{10}{36}$ multiplyed by $\frac{6}{11}$, or $\frac{60}{11}$ to be divided by 36.

Secondly, If the Main being *vii*, the Chance ſhould Happen to be *v* or *ix*; then the Setter having Six Chances to win and Four to loſe, his Expectation will be $\frac{6}{10}$ or $\frac{3}{5}$ of the Stake: But there being Eight Chances in Thirty-ſix for the Chance to be *v* or *ix*, it follows, that the Expectation of the Setter, reſulting from the Probability of that Chance, will be $\frac{8}{36}$ multiplied by $\frac{3}{5}$, or $\frac{24}{5}$ to be divided by 36.

Thirdly, If the Main being *vii*, the Chance ſhould Happen to be *iv* or *x*; then the Setter having Six Chances to win and Three to loſe, his Expectation will be $\frac{6}{9}$ or $\frac{2}{3}$ of the Stake: But there being Six Chances out of Thirty-ſix for the Chance to be *iv* or *x*, it follows, that the Expectation of the Setter, reſulting from the Probability of that Chance, will be $\frac{6}{36}$ multiplied by $\frac{2}{3}$, or 4 divided by 36.

Fourthly, If the Main being *vii*, the Caſter ſhould Happen to throw *ii*, *iii*, or *xii*; then the Expectation of the Setter will be the whole Stake, for which there being Four Chances in Thirty-ſix, it follows, that the Expectation of the Setter, reſulting from the Probability of thoſe Caſes, will be $\frac{4}{36}$ of the Stake, or 4 divided by 36.

Laſtly, If the Main being *vii*, the Caſter ſhould Happen to throw *vii* or *xi*, the Setter loſes his Expectation.

From the Solution of the foregoing particular Caſes it follows, that the Main being *vii*, the Expectation of the Setter will be expreſt by the following Quantities, *viz.* $\dfrac{\frac{60}{11}+\frac{24}{5}+\frac{4}{1}+\frac{4}{1}}{36}$ which may be reduced to $\frac{251}{495}$. Now this Fraction being ſubtracted from Unity, to which the whole Stake is ſuppoſed equal, there will remain the Expectation of the Caſter *viz.* $\frac{244}{495}$.

But the Probabilities of winning being always proportional to the Expectations, on ſuppoſition of the Stake being fixt, it follows, that the Probabilities of winning for the Setter and

and Caſter are reſpectively Proportional to the two numbers 251 and 244, which properly denote the Odds of winning.

Now, if we ſuppoſe each Stake to be 1, or the whole Stake to be 2, the Gain of the Setter will be expreſt by the Fraction $\frac{7}{495}$, it being the Difference of the Odds divided by their Sum, which ſuppoſing each Stake to be a Guinea, will be about $3\,d : 2\frac{1}{2}\,f$.

By the ſame Method of Proceſs, it will be found that the Main being *vi* or *viii*, the Gain of the Setter will be $\frac{167}{7128}$, which is about $6\,d : \frac{1}{6}\,f$ in a Guinea.

It will alſo be found that the Main being *v* or *ix*, the Gain of the Setter will be $\frac{43}{2835}$, which is about $4d. : 2\frac{1}{9}\,f$ in a Guinea.

Coroll. 1. If each particular Gain made by the Setter, in the Caſe of any Main, be reſpectively Multiplied by the number of Chances there are for that Main to come up, and the Sum of the Products be divided by the number of all thoſe Chances, the Quotient will expreſs the Gain of the Setter before a Main is thrown: from whence it follows, that the Gain of the Setter, if he be reſolved to ſet upon the firſt Main, may be eſtimated to be $\frac{344}{2835} + \frac{1670}{7128} + \frac{42}{495}$ to be divided by 24; which being reduced will be $\frac{2}{109}$ very nearly, or about $4d. : 2\frac{1}{10}f$.

Coroll. 2. The Probability of no Main is to the Probability of a Main, as $109 + 2$ to $109 - 2$, or as 111 to 107.

Coroll. 3. The Loſs of the Caſter's hand, if each throw be for a Guinea, and he confine himſelf to hold it as long as he wins, will be $\frac{4}{111}$ or about $9d.$ in all, the Demonſtration of which may be deduced from our XXIV*th* Problem.

PROBLEM XLVIII.

IF Four Gameſters play at WHISK; *What are the Odds that any two of the Partners that are pitch'd upon, have not the Four Honours?*

SOLUTION.

FIrſt, ſuppoſe thoſe two Partners to have the Deal, and the laſt Card which is turn'd up to be an Honour.

From the ſuppoſition of theſe two Caſes, we are only to find what Probability the Dealers have of taking Three ſet Cards in Twenty five, out of a Stock containing Fifty one. To reſolve this the ſhorteſt way, recourſe muſt be had to the Theorem given in the *Corollary* of our XX*th* Problem, in which making the Quantities n, c, d, p, a, reſpectively equal to the numbers 51, 25, 26, 3, 3, the Probability required will be found to be $\frac{25 \times 24 \times 23}{51 \times 50 \times 49}$ or $\frac{92}{833}$.

Secondly, If the Card which is turn'd up be not an Honour, then we are to find what Probability the Dealers have, of taking Four given Cards in Twenty five out of a Stock containing Fifty one, which by the aforeſaid Theorem will be found to be $\frac{25 \times 24 \times 23 \times 22}{51 \times 50 \times 49 \times 48}$, or $\frac{253}{4998}$.

But the Probability of taking the Four Honours being to be eſtimated before the laſt Card is turn'd up; and there being Sixteen Chances in Fifty two, or Four in Thirteen for an Honour to turn up, and Nine in Thirteen againſt it; it follows, that the Fraction expreſſing the Probability of the Firſt Caſe ought to be Multiplied by 4; that the Fraction expreſſing the Probability of the Second ought to be Multiplied by 9; and that the ſum of thoſe Products ought to be divided by 13; which being done, the Quotient $\frac{115}{1666}$, or $\frac{2}{29}$ nearly, will expreſs the Probability required.

Corollary, By the help of the abovecited Theorem, the following Concluſions may eaſily be verified.

It is 27 to 2 nearly that the two Dealers have not the Four Honours.

It is 23 to 1 nearly that the two Eldeſt have not the Four Honours.

It is 8 to 1 nearly that neither one Side nor the other have the Four Honours.

It is 13 to 7 nearly that the two Dealers do not reckon Honours.

It is 20 to 7 nearly that the two Eldest do not reckon Honours.

It is 25 to 16 nearly that either one Side or the other do reckon Honours, or that the Honours are not equally divided.

PROBLEM XLIX.

Of RAFFLING.

IF *any number of Gamesters A, B, C, D* &c. *Play at* Raffles: *What is the Probability that the first of them having got his Chance wins the Money of the Play?*

SOLUTION.

IN order to Solve this Problem, it is necessary to have a Table ready compos'd, of all the Chances which there are in three Raffles, which Table is the following. Wherein

The first Column contains the number of Points which are supposed to have been thrown by *A* in three Raffles.

The second Column contains the number of Chances which *A* has to win if his Points be above *xxxi*, or the number of Chances he has to lose if they be either *xxxi* or below it.

The third Column contains the number of Chances which *A* has to lose, if his Points be above *xxxi*, or to win if they be either *xxxi* or below it.

The Fourth Column contains the number of Chances which he has for an equality of Chance.

The Construction of this Table easily flows from the consideration of the number of Chances which there are in a single Raffle; whereof *xviii* or *iii*, have 1 Chance; *xvii* or *iv*, 3 Chances; *xvi* or *v*, 6 Chances; *xv* or *vi*, 4 Chances; *xiv* or *vii*, 9 Chances; *xiii* or *viii*, 9 Chances; *xii* or *ix*, 7 Chances; *xi* or *x*, 9 Chances; which number of Chances being duly Combined will afford all the Chances of Three Raffles.

A TA-

A TABLE of all the CHANCES which are in three *Raffles*.

Points			Chances to win or lose.	Chances to win or lose.	Equality of Chance.
liv		*ix*	884735	0	1
liii		*x*	884726	1	9
lii		*xi*	884681	10	45
li		*xii*	884534	55	147
l		*xiii*	884165	202	369
xlxix		*xiv*	883400	571	765
xlviii		*xv*	881954	1336	1446
xlvii		*xvi*	879470	2782	2484
xlvi		*xvii*	875501	5266	3969
xlv		*xviii*	869632	9235	5869
xliv		*xix*	861199	15104	8433
xliii	or	*xx*	849706	23537	11493
xlii		*xxi*	834679	35030	15027
xli		*xxii*	815392	50057	19287
xl		*xxiii*	791506	69344	23886
xxxix		*xxiv*	762838	93230	28668
xxxviii		*xxv*	728971	121898	33867
xxxvii		*xxvi*	690100	155765	38871
xxxvi		*xxvii*	646929	194636	43171
xxxv		*xxviii*	599472	237807	47457
xxxiv		*xxix*	548865	285264	50607
xxxiii		*xxx*	496314	335871	52551
xxxii		*xxxi*	442368	388422	53946
				Sum	442368
					442368
					884736

This

This being once ſuppoſed, let it be required to find the Probability which *A* has of winning, when the number of his Points being *xl*, there is but one Gameſter *B* beſides himſelf.

Take the number 791506, which in the ſecond Column ſtands over againſt the number *xl*, to be found in the Firſt. Take alſo one half of the number which in the Fourth Column ſtands over againſt the ſaid Number *xl*, which half is 11943. Let theſe two Numbers *viz.* 791506 and 11943 be added together, and their Sum 803449 being divided by 884736, which is the Number of all the Chances, the Quotient, *viz.* $\frac{803449}{884736}$ will expreſs the Probability required.

Now this Fraction being Subtracted from Unity, and the remainder being $\frac{81287}{884736}$, it follows that the Numerators of theſe two Fractions, *viz.* 803449 and 81287 do expreſs the Odds of winning, which may be reduc'd to 89 and 9 nearly.

But if the Number of Points which *A* has thrown for his Chance being *xl* as above, there be two other Gameſters *B* and *C* beſides himſelf, the Probability which he has of winning will be found thus.

Take the Square of the Number ſet down over againſt *xl* in the ſecond Column, which Square is 626481748036. Take alſo the Product of that Number by the Number ſet down over againſt *xl* in the Fourth Column, which Product is 18905912316. Laſtly, take the third part of the Square of the Number ſet down in the Third Column, which third part will be 190180332, and let all thoſe numbers be added together: Then their Sum being divided by the Square of the whole Number of Chances, *viz.* by 782757789696, the Quotient $\frac{645577840684}{782757789696}$ will expreſs the Probability required; from whence it may be concluded that the Odds of winning are nearly as 33 to 7.

N. B. If ſome of the laſt figures in the Numbers of the foregoing Table be neglected, the Operation will be ſhortned, and a ſufficient Approximation obtain'd by help of the remaining Figures.

From what we have ſaid it follows, that *A* having *xl* for the number of his Points, has leſs advantage when he Plays againſt One than when he plays againſt Two: For ſuppoſing each Man's Stake be a Guinea, he has in the firſt Caſe 89 Chances for winning 1, and 9 Chances for loſing 1:

From whence it follows that his Gain is $\frac{89-9}{98}$ or $\frac{80}{98}$ which is about 17 *ſh.* 6 *d.*

But in the ſecond Caſe, ſuppoſing alſo each Man's Stake to be a Guinea, he has 33 Chances for winning 2, and 7 Chances for loſing 1:

Whence it appears, that his Gain in this Caſe is $\frac{2\times 33-7}{40}$ or $\frac{59}{40}$ which is about 1 *l.* — 12 *ſh.* — 8 *d.* But Note, that it is not to be concluded from this ſingle Inſtance, that the Gain of *A* will always increaſe with the number of Gameſters.

If the number of Gameſters be never ſo many, let p be their number, let a be the number of Chances which *A* has for winning when he has thrown his Chance, let m be the number of Chances which there are for an Equality of Chance between *A* and any of the other Gameſters; Laſtly, let the whole number of Chances be denoted by S: Then the Probability which *A* has of winning will be expreſſed by the following Series.

$$\frac{a^{p-1} + \frac{p-1}{2} m a^{p-2} + \frac{p-1}{2} \times \frac{p-2}{3} mm a^{p-3} + \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} m^3 a^{p-4} \&c.}{S^{p-1}}$$

which Series is compoſed of the Terms of the Binomial $\overline{a+m}|^{p-1}$ reduced into a Series, all its Terms being divided by 1, 2, 3, 4, 5 *&c.* reſpectively.

The foregoing Theorem may be uſeful, not only for ſolving any Caſe of the preſent Problem, but alſo an infinite Variety of other Caſes, in thoſe Games wherein there is no Advantage in the order of Play: And the Application of it to Numbers will be found eaſy, to thoſe who underſtand how to uſe Logarithms.

PRO-

PROBLEM L.

TO find what Probability there is, that any Number of Cards of each Suit may be contained in a given number of them taken out of a given Stock.

SOLUTION.

FIrst, Find the whole number of Chances there are for taking the given number of Cards out of the given Stock.

Secondly, Find all the particular Chances there are for taking each given number of Cards of each Suit out of the whole number of Cards belonging to that Suit.

Thirdly, Multiply all those particular Chances together; then divide the Product by the whole number of Chances, and the Quotient will express the Probability required.

Thus, If it be proposed to find the Probability of taking Four Hearts, Three Diamonds, Two Spades and One Club, in Ten Cards taken out of a Stock containing Thirty-two.

Find the whole number of Chances for taking ten Cards out of a Stock containing two and Thirty; which is properly Combining two and Thirty Cards Ten and Ten. To do this, write down all the Numbers from 32 inclusively to 22 exclusively, so as to have as many Terms as there are Cards to be Combined; then write under each of them respectively all the numbers from One to Ten inclusively; thus,

$$\frac{32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$$

Let all the numbers of the upper Row be Multiplied together; let also all the numbers of the lower Row be Multiplied together, then the first Product being divided by the second, the Quotient will express the whole number of Chances required, which will be 64512240.

By the like Operation the number of Chances for taking Four Hearts out of Eight, will be found to be $\frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$.

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The number of Chances for taking Three Diamonds out of Eight will alſo be found to be $\frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56$.

The number of Chances for taking Two Spades out of Eight will in the ſame manner be found to be $\frac{8 \times 7}{1 \times 2} = 28$.

Laſtly, The number of Chances for taking One Club out of Eight will be found to be $\frac{8}{1} = 8$.

Wherefore, Multiplying all theſe particular Chances together *viz.* 70, 56, 28, 8, the Product will be 878080; which being Divided by the whole number of Chances, the Quotient $\frac{878080}{64512240}$, or $\frac{2}{101}$ nearly, will expreſs the Probability required: From whence it follows, that the Odds againſt taking Four Hearts, Three Diamonds, Two Spades and One Club in Ten Cards, are very near 99 to 2.

It is to be obſerved that the Operations whereby the Number of Chances is determined, may always be contracted, except in the ſingle caſe of taking one Card only of a given Suit. Thus, If it were propoſed to ſhorten the Fraction $\frac{32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$, which determines the number of all the Chances belonging to the foregoing Problem: Let it be conſidered whether the Product of any two or more Terms of the Denominator, being Multiplied together, be equal to any one of the Terms of the Numerator; if ſo, all thoſe Terms may be expunged out of both Denominator and Numerator. Thus the Product of the three Numbers 2, 3, 4, which are in the Denominator, being equal to the Number 24, which is in the the Numerator, it follows, that the three Numbers 2, 3, 4 may be expunged out of the Denominator, and at the ſame time the Number 24 out of the Numerator. For the ſame reaſon the Numbers 5 and 6 may be expunged out of the Denominator, and the Number 30 out of the Numerator, which will reduce the Fraction to be

$$\frac{32 \times 31 \times \cancel{30} \times 29 \times 28 \times 27 \times 26 \times 25 \times \cancel{24} \times 23}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \cancel{6} \times 7 \times 8 \times 9 \times 10}$$

It ought likewiſe to be conſidered whether there be any of the remaining Numbers in the Denominator that Divide exactly any of the remaining Numbers of the Numerator. If

If ſo, thoſe Numbers are to be expunged out of the Denominator and Numerator, but the reſpective Quotients of the Terms of the Numerator divided by thoſe of the Denominator, are to be ſubſtituted in the room of thoſe Terms of the Numerator. Thus the Terms 7, 8, and 9 of the Denominator dividing exactly the Numbers 28, 32 and 27 of the Numerator, and the Quotients being 4, 4 and 3 reſpectively, all the Numbers 7, 8, 9, 28, 32, 27 ought to be expunged, and the Quotients 4, 4, and 3 ſubſtituted in the room of 28, 32, 27 reſpectively, in the following manner;

$$\frac{\overset{4}{\cancel{32}} \times 31 \times \cancel{30} \times 29 \times \overset{4}{\cancel{28}} \times \overset{3}{\cancel{27}} \times 26 \times 25 \times \cancel{24} \times 23}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \cancel{6} \times \cancel{7} \times \cancel{8} \times \cancel{9} \times 10}.$$

It ought alſo to be conſidered, whether the remaining Terms of the Denominator have any common Diviſor with any of the remaining Terms of the Numerator; if ſo, dividing thoſe Terms by their common Diviſors, the reſpective Quotients ought to be ſubſtituted in the room of the Terms of the Numerator. Thus, the only remaining Term in the Denominator, beſides Unity, being 10, which has a common Diviſor with one of the remaining Terms of the Numerator, *viz.* 25, and that common Diviſor being 5, let 10 and 25 be reſpectively divided by the common Diviſor 5, and let the reſpective Quotients 2 and 5 be ſubſtituted in the room of them, and the Fraction will be reduced to the following, *viz.*

$$\frac{\overset{4}{\cancel{32}} \times 31 \times \cancel{30} \times 29 \times \overset{4}{\cancel{28}} \times \overset{3}{\cancel{27}} \times 26 \times \overset{5}{\cancel{25}} \times \cancel{24} \times 23}{1 \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad \cancel{5} \quad \cancel{6} \quad \cancel{7} \quad \cancel{8} \quad \cancel{9} \quad \underset{2}{\cancel{10}}}.$$

Laſtly, Let the remaining Number 2 in the Denominator divide any of the Numbers of the Numerator which are diviſible by it, ſuch as 26, and let thoſe two Numbers be expunged; but let the Quotient of 26 by 2, *viz.* 13, be ſubſtituted in the room of 26: And then the Fraction, neglecting unity, which is the only Term remaining, may be reduced to $4 \times 31 \times 29 \times 4 \times 3 \times 13 \times 5 \times 23$, the Product of which Numbers is 64512240, as we have found it before.

The foregoing Solution being well underſtood, it will be eaſy to enlarge the Problem, and to find the Probability of taking at leaſt Four Hearts, Three Diamonds, Two Spades, and One Club, in Eleven Cards; the finding of which depends upon the four following Caſes, *viz.* taking

5 Hearts, 3 Diamonds, 2 Spades, 1 Club,
4 Hearts, 4 Diamonds, 2 Spades, 1 Club,
4 Hearts, 3 Diamonds, 3 Spades, 1 Club,
4 Hearts, 3 Diamonds, 2 Spades, 2 Clubs.

Now the number of Chances for the Firſt Caſe will be found to be 702464, for the Second 1097600, for the Third 1756160, for the Fourth 3073280; which Chances being added together, and their ſum divided by the whole number of Chances for taking Eleven Cards out of Thirty two, the Quotient will be $\frac{6628504}{129024480}$, which may be reduced to $\frac{5}{97}$ nearly.

From whence it may be concluded, that the Odds againſt the taking of Four Hearts, Three Diamonds, Two Spades, and One Club, in Eleven Cards, that is, ſo many at leaſt of every ſort, is about 92 to 5.

And by the ſame Method it would be eaſy to ſolve any other Caſe of the like nature, let the number of Cards be what it will.

PROBLEM LI.

TO find at PIQUET *the Probability which the Dealer has for taking One Ace or more in Three Cards, having none in his Hands.*

SOLUTION.

FRom the number of all the Cards, which are Thirty two, ſubtracting Twelve which are in the Dealers Hands, there remains Twenty, among which are the Four Aces.

From whence it follows, that the number of all the Chances for taking any three Cards in the Bottom, are the number of Combinations which Twenty Cards may afford, being taken Three and Three; which, by the Rule given in the preceding Problem, will be found to be $\frac{20 \times 19 \times 18}{1 \times 2 \times 3}$ or 1140.

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The number of all the Chances being thus obtained, find the number of Chances for taking one Ace precifely, with two other Cards; find next the number of Chances for taking Two Aces precifely with any other Card; Laftly, find the number of Chances for taking Three Aces: Then thefe Chances being added together, and their fum divided by the whole number of Chances, the Quotient will exprefs the Probability required.

But by the Directions given in the preceding Problem, it appears, that the number of Chances for taking One Ace precifely are $\frac{4}{1}$ or 4; and that the number of Chances for taking any two other Cards are $\frac{16 \times 15}{1 \times 2}$ or 120: From whence it follows, that the number of Chances for taking One Ace precifely with any two other Cards, is equal to 4×120 or 480.

In like manner it appears, that the number of Chances for taking Two Aces precifely is equal to $\frac{4 \times 3}{1 \times 2}$ or 6, and that the number of Chances for taking any other Card is $\frac{16}{1}$ or 16; from whence it follows, that the number of Chances for taking Two Aces precifely with any other Card is 6×16 or 96.

Laftly, It appears that the Number of Chances for taking Three Aces is equal to $\frac{4 \times 3 \times 2}{1 \times 2 \times 3}$ or 4.

Wherefore the Probability required will be found to be $\frac{480 + 96 + 4}{1140}$ or $\frac{580}{1140}$; which Fraction being fubtracted from Unity, the remainder, *viz.* $\frac{560}{1140}$ will exprefs the Probability of not taking an Ace in Three Cards: From whence it follows, that it is 580 to 560, or 29 to 28, that the Dealer takes One Ace or more in three Cards.

The preceding Solution may be very much contracted, by inquiring at firft what the Probability is of not taking an Ace in Three Cards, which may be done thus:

The number of Cards in which the Four Aces are contained being Twenty, and confequently the number of Cards out of which the Four Aces are excluded being Sixteen, it follows, that the number of Chances which there are for the taking Three Cards, among which no Ace fhall be found, is is the number of Combinations which Sixteen Cards may afford, being taken Three and Three; which number of Combinations by the preceding Problem will be found to be $\frac{16 \times 15 \times 14}{1 \times 2 \times 3}$ or 560.

But the number of all the Chances which there are for taking any Three Cards in Twenty, has been found to be 1140; from whence it follows, that the Probability of not taking an Ace in Three Cards is $\frac{560}{1140}$; and consequently that the Probability of taking One or more Aces in Three Cards is $\frac{580}{1140}$: The same as before.

In the like manner, if we would find the Probability which the Eldest has of taking One Ace or more in his Five Cards, he having none in his Hands; the severall Chances may be calculated as follows.

First, The number of Chances for taking One Ace and Four other Cards will be found to be 7280.

Secondly, The number of Chances for taking Two Aces and Three other Cards will be found to be 3360.

Thirdly, The number of Chances for taking Three Aces and two other Cards will be found to be 480.

Fourthly, The number of Chances for taking Four Aces and any other Card will be found to be 16.

Lastly, The number of Chances for taking any Five Cards will be found to be 15504.

Let the sum of all the particular Chances, *viz.* 7280 + 3360 + 480 + 16 or 11136, be divided by the sum of all the Chances, *viz.* by 15504, and the Quotient $\frac{11136}{15504}$ will express the Probability required.

Now the foregoing Fraction being subtracted from Unity, the remainder, *viz.* $\frac{4368}{15504}$ will express the Probability of not taking an Ace in Five Cards; wherefore the Odds of taking an Ace in Five Cards are 11136 to 4368, or 5 to 2 nearly.

But if the Probability of not taking an Ace in Five Cards be at first inquired into, the Work will be very much shortened; for it will be found to be $\frac{16 \times 15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4 \times 5}$ or 4368, to be divided by the whole number of Chances, *viz.* by 15504, which makes it as before, equal to $\frac{4368}{15504}$.

But

But ſuppoſe it were required to find the Probability which the Eldeſt has of taking an Ace and a King in Five Cards, he having none in his Hands: Let the following Chances be found, *Viz.*

1 | For One Ace, One King and Three other Cards.
2 | For One Ace, Two Kings and Two other Cards.
3 | For One Ace, Three Kings and any other Card.
4 | For One Ace and Four Kings.
5 | For Two Aces, One King and Two other Cards.
6 | For Two Aces, Two Kings and any other Card.
7 | For Two Aces and Three Kings.
8 | For Three Aces, One King and any other Card.
9 | For Three Aces and Two Kings.
10 | For Four Aces and One King.
11 | For taking any Five Cards in Twenty.

Among theſe Caſes, there being four Pairs that are alike, *viz.* the Second and Fifth, the Third and Eighth, the Fourth and Tenth, the Seventh and Ninth; it follows, that there are only Seven Caſes to be Calculated, whereof the Firſt, Sixth and Eleventh, are to be taken ſingly; but the Second, Third, Fourth and Seventh, to be doubled. Now the Operation is as follows.

The *Firſt* Caſe has $\frac{4}{1} \times \frac{4}{1} \times \frac{12 \times 11 \times 10}{1 \times 2 \times 3}$ or 3520 Chances.

The *Second*, $\frac{4}{1} \times \frac{4 \times 3}{1 \times 2} \times \frac{12 \times 11}{1 \times 2}$ or 1584, the double of which is 3168 Chances.

The *Third*, $\frac{4}{1} \times \frac{4 \times 3 \times 2}{1 \times 2 \times 3} \times \frac{12}{1}$ or 192, the double of which is 384 Chances.

The *Fourth*, $\frac{4}{1} \times \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4}$ or 4, the double of which is 8 Chances.

The *Sixth*, $\frac{4 \times 3}{1 \times 2} \times \frac{4 \times 3}{1 \times 2} \times \frac{12}{1}$ or 432 Chances.

The *Seventh*, $\frac{4 \times 3}{1 \times 2} \times \frac{4 \times 3 \times 2}{1 \times 2 \times 3}$ or 24, the double of which is 48 Chances.

The *Eleventh*, $\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5}$ or 15504, being the number of all the Chances for taking any Five Cards out of Twenty.

From whence it follows, that the Probability which the Eldeſt has for taking an Ace and a King in Five Cards, he having none in his Hands, will be expreſt by the Fraction

$$\frac{3520 + 3168 + 384 + 8 + 432 + 48}{15504} \text{ or } \frac{7560}{15504}.$$

Let this Fraction be ſubtracted from Unity, and the remainder being $\frac{7944}{15504}$, the Numerators of theſe two Fractions, *viz.* 7560 and 7944, will expreſs the proportion of Probability that there is, of taking or not taking an Ace and a King in Five Cards; which two numbers may be reduced nearly to the proportion of 20 to 21.

By the ſame Method of Proceſs, any Caſe relating to WHISK might be Calculated, tho' not ſo expeditiouſly, as by the Method explained in the *Corollary* of our XX*th* Problem: For which reaſon the Reader is deſired to have recourſe to the Method therein explained, when any other Caſe of the like nature happens to be propoſed.

PROBLEM LII.

To find the Probability of taking any number of Suits, in a given number of Cards taken out of a given Stock; without ſpecifying what number of Cards of each Suit ſhall be taken.

SOLUTION.

Suppoſe the number of Cards to be taken out of the given Stock to be Eight, the number of Suits to be Four, and the number of Cards in the Stock to be Thirty-two.

Let all the Variations that may happen, in taking One Card at leaſt of each Suit, be written down in order, as follows;

1,	1,	1,	5,
1,	1,	2,	4,
1,	1,	3,	3,
1,	2,	2,	3,
2,	2,	2,	2.

Then

Then ſuppoſing any particular Suits to be appropriated at pleaſure to the Numbers belonging to the Firſt Caſe, as if it were required, for Inſtance, to take One Heart, One Diamond, One Spade and Five Clubs; let the Probability of the ſame be inquired into, which, by our L*th* Problem, will be found to be $\frac{28672}{10518300}$; but the Problem not requiring the Suits to be confined to any number of Cards of each Sort, it follows, that this Probability ought to be increaſed in proportion to the number of Permutations, or Changes of Order, which Four Things may undergo, whereof Three are alike. Now this number of Permutations is Four, and conſequently the Probability of the Firſt Caſe, that is, of taking Three Cards of three different Suits, and five Cards of a Fourth Suit, in Eight Cards, will be the Fraction $\frac{28672}{10518300}$ multiplied by 4, or $\frac{114688}{10518300}$.

In the ſame manner the Probability of the Second Caſe, ſuppoſing it were confined to One Heart, One Diamond, Two Spades, and Four Clubs, would be found to be $\frac{125440}{10518300}$; which being multiplied by 12, *viz.* by the number of Permutations which Four Sorts may undergo, whereof Two are alike, and the other Two differing, it will follow, that the Probability of the Second Caſe, taken without any reſtriction, will be expreſſed by the Fraction $\frac{1505280}{10518300}$.

The Probability of the Third Caſe will likewiſe be found to be $\frac{1204224}{10518300}$.

The Probability of the Fourth will be found to be $\frac{4214784}{10518300}$.

Laſtly, The Probability of the Fifth will be found to be $\frac{614656}{10518300}$. Theſe Fractions being added together, their ſum, *viz.* $\frac{7653632}{10518300}$, will expreſs the Probability of taking the Four Suits in Eight Cards.

Let this laſt Fraction be ſubtracted from Unity, and the remainder being $\frac{2864668}{10518300}$, it follows that 'tis the Odds of 7653632 to 2864668, or 8 to 3 nearly, that the Four Suits may be taken in Eight Cards, out of a Stock containing Thirty-two.

The only difficulty remaining in this matter, is the finding readily the number of Permutations which any number of of Things may undergo, when either they be all different, or when ſome of them be alike. The Solution of which may be deduced from what we have ſaid in the *Corollary* of our XVII*th* Problem, and may be explained as follows, in words at length.

Let all the numbers that are from Unity to that number which expreſſes how many Things are to be Permuted, be written down in order; Multiply all thoſe Numbers together, and the Product of them all will expreſs the number of their Permutations, if they be all different. Thus the number of Permutations which Ten things are capable of, is the Product of all the Numbers $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$, which is equal to 3628800.

But if ſome of them be alike, as ſuppoſe Four of One ſort, Three of another, Two of a Third, and One of a Fourth, write down as before all the Nnmbers $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$; then write under them as many of thoſe Numbers as there are Things of the Firſt ſort that are alike, which in this Caſe being Four, write the Numbers $1 \times 2 \times 3 \times 4$, beginning at Unity, and following in order. Write alſo as many of thoſe Numbers as there are Things of the ſecond ſort that are alike, *viz.* $1 \times 2 \times 3$, ſtill beginning at Unity. In the ſame manner write as many more as there are Things of the Third ſort that are alike, *viz.* 1×2; and ſo on: Which being repreſented by the Fraction

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4 \times 1 \times 2 \times 3 \times 1 \times 2 \times 1},$$

let all the numbers of the upper Row be Multiplied together, let alſo all the numbers of the lower Row be Multiplied together, and the Firſt Product being divided by the Second, the Quotient 12600 will expreſs the number of Permutations required.

By this Method of Permutations, the Probability of throwing any determinate number of Faces of the like ſort, with any given number of Dice, may eaſily be found. Thus, ſuppoſe it were required to find the Probability of throwing an Ace, a Two, a Three, a Four, a Five, and a Six with ſix Dice. It is plain that there are as many Chances for doing it, as there are Changes or Permutations in the

Order

Order or Place of ſix different Things, ſuppoſe of the Six Letters *a*, *b*, *c*, *d*, *e*, *f*, which by the Rule above given would be 720, *viz.* the Product of the numbers 1, 2, 3, 4, 5, 6: For tho' the Dice are not conſidered as changing their Places, or as affording any Variation upon the ſcore of the different Situation they may have in Reſpect to one another, being thrown upon a Table; yet they ought to be conſidered as changing their Faces, which is equivalent to their changing of Place. Now the number of all the Chances upon Six Dice, being the number 6 Multiplied into it ſelf, as many times wanting one as there are Dice, *viz.* $6 \times 6 \times 6 \times 6 \times 6 \times 6$ or 46656, it follows, that the Probability required will be expreſt by the Fraction $\frac{720}{46656}$, and conſequently, that the Odds againſt throwing the Faces undertaken, will be $46656 - 720$ to 720, or 64 to 1 nearly.

In the ſame manner ſuppoſe it were required to find the Probability of throwing One Ace, Two Two's and Three Three's with Six Dice. The number of Chances for the doing it being equal to the number of Permutations which there are in the ſix Letters *abbccc*, it follows, by the Rule before delivered, that the number of thoſe Chances will be 60, *viz*, the Fraction $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 1 \times 2 \times 1 \times 2 \times 3}$; and conſequently that the Probability required will be $\frac{60}{46656}$, and the Odds againſt the doing it $46656 - 60$ to 60, or 776 to 1 nearly.

If it were required to find the Probability of throwing Two Aces, Two Two's and Two Three's with 6 Dice, the number of Chances for doing it being $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 1 \times 2 \times 1 \times 2}$ or 90, and the number of all the Chances upon Six Dice being 46656, it follows, that the Probability required will be expreſt by the Fraction $\frac{90}{46656}$.

Again, if it were required to find the Probability of throwing Three Aces and Three Sixes, the number of Chances for doing it being $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 3 \times 1 \times 2 \times 3}$ or 20, and the number of all the Chances 46656, the Probability required will be expreſt by the Fraction $\frac{20}{46656}$.

PROBLEM LIII.

TO *find at* HAZARD *the Chance of the Caſter, when the Main being given, he Throws to any given number of Points.*

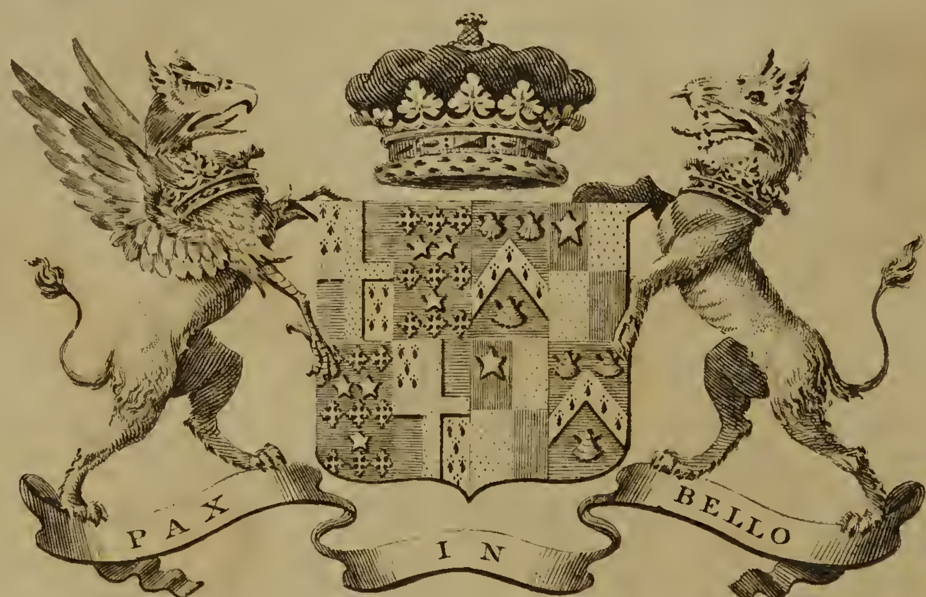
SOLUTION.

THis being eaſily reduced to our XLVII*th* Problem, it is thought ſufficient to exhibit the Solution of its different Caſes in the following Table, which ſhews the Odds for or againſt the Caſter.

Points Thrown to			
	MAIN V.	exactly	nearly
i	Againſt the Caſter	538 to 407	or 37 to 28.
ii	For the Caſter —	989 to 901	or 45 to 41.
iii	For the Caſter —	2293 to 1487	or 37 to 24.
iv	For the Caſter —	2293 to 1487	or 37 to 24.
v	Againſt the Caſter	2117 to 1663	or 14 to 11.
vi.	Againſt the Caſter	2467 to 1313	or 62 to 33.
	MAIN VI.		
i	Againſt the Caſter	2879 to 1873	or 83 to 54.
ii	Againſt the Caſter	2483 to 2269	or 58 to 53.
iii	For the Caſter —	2621 to 2131	or 16 to 13.
iv	For the Caſter —	2621 to 2131	or 16 to 13.
v	Againſt the Caſter	2483 to 2269	or 58 to 53.
vi.	Againſt the Caſter	2483 to 2269	or 58 to 53.
	MAIN VII.		
i	Againſt the Caſter	629 to 361	or 7 to 4.
ii	Againſt the Caſter	277 to 218	or 14 to 11.
iii	For the Caſter —	251 to 244	or 36 to 35.
iv	For the Caſter —	251 to 244	or 36 to 35.
v	For the Caſter —	601 to 389	or 20 to 13.
vi.	For the Caſter —	263 to 232	or 17 to 15.

Points Thrown to	
	MAIN VIII.
i	Againſt the Caſter 3275 to 1477 or 51 to 23.
ii	Againſt the Caſter 2483 to 2269 or 58 to 53.
iii	For the Caſter — 2621 to 2131 or 16 to 13.
iv	For the Caſter — 2621 to 2131 or 16 to 13.
v	For the Caſter — 2483 to 2269 or 58 to 53.
vi.	For the Caſter — 2665 to 2087 or 83 to 65.
	MAIN IX.
i	Againſt the Caſter 2467 to 1313 or 62 to 33.
ii	Againſt the Caſter 2117 to 1663 or 14 to 11.
iii	For the Caſter — 2293 to 1487 or 37 to 24.
iv	For the Caſter — 2293 to 1487 or 37 to 24.
v	For the Caſter — 989 to 901 or 45 to 41.
vi.	Againſt the Caſter 538 to 407 or 37 to 28.

F I N I S.



LEEDS.