H. Zhao's **MindMap** of **Galaxy** & **Accretion** Physics **Common Equations** & **Concepts/Examples**

I. Poisson Eq.: $\frac{\nabla\cdot[\nabla\Phi(t,\mathbf{X})]}{4\pi G} = \overbrace{\rho(t,\mathbf{X}) = m^{DM}n^{DM} + \rho^{gas}(t,\mathbf{X}) + ... + \int_\infty d\mathbf{v}_*^3}^{grav.density.=\sum_{p=gas,*}^{DM}\rho^p(t,\mathbf{X})=\sum_p m^p n_p} \underbrace{\overbrace{[m_* f_*(t,\mathbf{X},\mathbf{v}_*)]}^{O(10^{10}M_\odot)/[10kpc.100km/s]^3}}_{\to 0\ if.(X,v_*)\to\infty}$, **E.g.,**

$$\overbrace{\ddot{\mathbf{x}}^* = \underbrace{-\nabla_{\mathbf{x}^*}\Phi(t,\mathbf{x}_*)}_{\mathbf{g}}}^{Stellar.Eq.of.Motion} = \nabla_{\mathbf{x}_*}\int_\infty \frac{\overbrace{[d\mathbf{X}^3\rho(t,\mathbf{X})]}^{dM(\mathbf{X})} G}{|\mathbf{x}_* - \mathbf{X}|}, \quad \overbrace{\frac{v_{cir}^2}{r}}^{shell.accel.} = |\mathbf{g}| = \frac{d}{dr}\overbrace{\{\int_\infty^r 4\pi G\rho(r_1)r_1 dr_1 - \frac{GM(r)}{r}\}}^{\Phi(r)=-v_{esc}^2/2} = \frac{4\pi G}{(4\pi r^2)}\overbrace{\int_0^r \rho(r_1)4\pi r_1^2 dr_1}^{enclosed.\sum_p M^p(r)}.$$

II. Mass Conservation Eq.: Viscous flow in disc (of $2H$ thick) or on particle m^p (of Bondi size $2B \equiv \frac{2Gm^p}{\bar{\mathbf{v}}^2+\sigma^2}$):

$$\overbrace{cst}^{steady} = -\oint \rho_p \overbrace{d\mathbf{A}}^{O(4\pi B^2)} \cdot\bar{\mathbf{v}}_p = \overbrace{(-2\pi R v_R)}^{visc}\overbrace{\Sigma_{disc}^{gas}}^{\int_{-H}^{H}dZ\rho_{gas}} = \overbrace{\frac{\partial}{\partial t}\int_{\in\mathbf{A}}\rho_p d^3\mathbf{x}}^{\dot{M}_p} \approx \frac{M_p}{t_{visc}^{dyf,rlx}} \approx \overbrace{(2\pi B\sqrt{\bar{\mathbf{v}}_p^2+\sigma^2})}^{visc}\overbrace{(2B\rho_p)}^{surf.dens.}$$

III. Momentum (Jeans) Eqs. of a ^{p}opulation from integrated **6D CBE**: $\frac{1}{\rho_p}\int\{\mathbf{v}_p\frac{d[f_p m_p]}{dt} - \bar{\mathbf{v}}_p\frac{d[f_p m_p]}{dt}\}d^3\mathbf{v} = 0.$

$$\overbrace{(\frac{\partial}{\partial t} + \sum_{j=1}^{3}v_j^p\frac{\partial}{\partial x_j})v_i^p}^{flow.accel.\dot{v}_i^p\sim O(\overline{\dot\psi^2}R)} \underbrace{=}_{EoM} \overbrace{\frac{-\partial\Phi(t,\mathbf{x})}{\partial x_i}}^{g_i\sim O(-GM/R^2)} \underbrace{\overbrace{-}^{pressure}}_{balance} \sum_{j=1}^{3}\frac{\partial}{\rho^p\partial x_j}\overbrace{[\underbrace{\rho^p(t,\mathbf{x})}_{\int_\infty m_p f_p d^3\mathbf{v}}\ \underbrace{\sigma_{ji}^p(t,\mathbf{x})}_{O(c_s^2)}]}^{\int_\infty d\mathbf{v}^3(\mathbf{v}_j-\bar{\mathbf{v}}_j^p)(\mathbf{v}_i-\bar{\mathbf{v}}_i^p)m_p f_p} - \underbrace{v_i^p\overbrace{[\dot{m}_p/m_p]}^{1/t|_{visc}^{dyn.fric,relax.}\ m_p=M_{gas}}}_{snow.plough},$$

E.g.: **No relaxation** of Sun's angular momentum in 10^9 stress-free harmonic **periods** $\frac{2\pi}{\kappa}\sim\frac{2\pi}{\nu}\sim\frac{2\pi(R_0\sim 10kpc)}{v_{cir}\sim 200km/s}$.

$$\overbrace{[{}^{\ddot{R}/(R_0-R)}_{\ddot{Z}/(0-Z)}]_{Z\to 0}^{R\to R_0} = [{}^{\kappa^2}_{\nu^2}]}^{restoring.freq.^2\sim O(G\rho)} \equiv [{}^{\partial_R^2}_{\partial_Z^2}]\overbrace{[\Phi(R,Z) + (R\dot\psi)^2/2]}^{eff.pot.}, \quad \overbrace{J_z m_p = R(\dot\psi R)M_\odot = Rv_\psi M_\odot}^{looporbit}$$

E.g.:If BH tide or gas-selfgravity beats spin $\propto J_z^2$, **density** $G\rho\propto\kappa^2$ **or sound** $c_s^2\sim\sigma^2\equiv\frac{P}{\rho^p}\propto T(\mathbf{x})\propto(\frac{L}{Area})^{\frac{1}{4}}$,

$$1 \geq Q_{Edd,tide}^{Jn,cnf} \equiv \{\frac{\overbrace{c_s\kappa_{gas\ disc}}^{\sqrt{\kappa^2 R.c^2/R}}}{G[\pi\Sigma_{disc}^{gas}]}, \frac{\overbrace{c_s\sqrt{G\pi\rho_{gas}}}^{c_s\kappa}}{G[\pi\rho_{gas}r]} \equiv \frac{r_{Jn}}{r}, \frac{c[\frac{\sigma_e/m_p}{4\pi r^2c^2}L]}{G[\frac{m_{BH}}{r^2}]} \equiv \frac{L/c^2}{L_{Edd}/c^2}, \frac{J_z^2/R^3}{G[\frac{m_{BH}}{R^2}]} \equiv \frac{R_{cnf}}{R}, \frac{(GM_\odot R_\odot)/R_\odot^3}{\frac{Gm_{BH}}{R^2} - \frac{Gm_{BH}}{(R+R_\odot)^2}} \equiv \frac{R^3}{R_{tide}^3}\}.$$

E.g.: Virial$_{jj}$ tensor is space-time-averaged work in $\overline{\mathbf{v_j x_j}}^{\mathbf{p=*}}_{\mathbf{p=gas}}$ moment of **Collision-nixed Boltzmann Eq.**: $\frac{1}{N_p}\int_\infty \mathbf{x}_j d^3\mathbf{x}\int_\infty \mathbf{v}_j d^3\mathbf{v}\ \int_0^T\frac{dt}{T|_{\to\infty}}\{\frac{\partial f^p}{\partial t} + \nabla_\mathbf{x}\cdot[f^p\mathbf{v}] - \nabla_\mathbf{v}\cdot[f^p\nabla_\mathbf{x}\Phi]\} = 0 \leftarrow \frac{\overline{\mathbf{x_j v_j}}^p|_0^T}{T|_{T\to\infty}} = \overline{\mathbf{v_j v_j}}^\mathbf{p} - \overline{\mathbf{x_j}\partial_{\mathbf{x_j}}\mathbf{\Phi}}^\mathbf{p}$

$$\overbrace{M_*\sum_{j=1,3}\overline{\mathbf{v_j^2}}^*}^{2T} = \overbrace{M_*\overline{\mathbf{x}\cdot\nabla_\mathbf{x}\mathbf{\Phi}}^*}^{M_*\overline{\mathbf{v_{cir}^2}}^*} \overbrace{=}^{selfgravity} \overbrace{\underbrace{M}_{\approx M_*}\overline{[-\Phi/2]}}^{W} = \frac{1}{2}\int\int_\infty\overbrace{[\rho(\mathbf{x})d^3\mathbf{x}]}^{dM_*(\mathbf{x})}[\overbrace{\rho(\mathbf{X})}^{\approx m_* n_*}\ d^3\mathbf{X}]/|\mathbf{x}-\mathbf{X}|$$

E.g.: a BH cluster $M = N_p m_p = 10^3\times 10^6 M_\odot = \int m_p f_p d\mathbf{x}^3 d\mathbf{v}^3$, $f_p(\mathbf{x},\mathbf{v}) \propto |1+\frac{E-\frac{J^2}{2B^2}\to 0.5|v_r^2+(1-\frac{r^2}{B^2})(v_\theta^2+v_\psi^2)|+\Phi(|\mathbf{x}|)}{[GM/B]\sim\mathbf{Virial}\sim[c_s\sim\sigma\sim v_{esc}\sim 60kms^{-1}\sim v_{cir}]^2}|^{-0.5}$ is hydrostatic: $Q^{Jn} = \frac{B}{1kpc}\sim 1$, $\bar{\mathbf{v}}^p \equiv \frac{\int\mathbf{v}f_p dv_r dv_\theta dv_\psi}{\int f_p dv_r dv_\theta dv_\psi} = 0$, uniform $\rho_p(|\mathbf{x}|) = \int_0^{-\Phi-GM/B} f_p\ 4\pi\tilde{v}d(\tilde{v}^2/2)/(1-\frac{r^2}{B^2}) = \frac{N_p m_p}{(4\pi B^3/3)} = \frac{d}{r^2 dr}[\frac{r^2 d\Phi}{4\pi G dr}]$, **anisotropic dispersion** $\sigma_\theta^2 = \sigma_\psi^2 = \frac{GM}{2B}, \sigma_r^2 \equiv \frac{GM}{2B}(1-\frac{r^2}{B^2}) = -\Phi(r) - \frac{GM}{B}$ from **P.E./J.E.** $\partial_r\Phi = \frac{\sigma_\theta^2+\sigma_\psi^2-2\sigma_r^2}{r} - \frac{\partial_r(\rho\sigma_r^2)}{\rho}$. **Virial** $\overline{(0.5v_{esc})^2} = \overline{|0.5\Phi|} = \overline{v_{cir}^2} = \overline{\sigma_\theta^2+\sigma_\psi^2+\sigma_r^2} = \frac{3GM}{5B}$. **Jeans/Bondi radius** $B(t)$ doubles once BHs relax into $500\times(2\times 10^6 M_\odot)$ pairs of size $2B = \frac{2Gm_p}{2\sigma^2}\sim 2pc$ by wake-dragging/accretion each $t_{rlx}^{dyf}\sim\frac{10^3\times B}{c_s}\sim\frac{10^3}{\sqrt{G\rho_p}}\sim$ Hubble Time, $m_p\sim 10^6 M_\odot e^{\frac{t}{10Gyr}}$.