Rob 501 - Mathematics for Robotics
Recitation #6

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1 Matrices

1. - $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^\top = A$.
   - $A \in \mathbb{R}^{n\times n}$ is skew symmetric if $A^\top = -A$.
   - $A \in \mathbb{R}^{n\times n}$ is orthogonal if $A^\top A = AA^\top = I$.
   - $A \in \mathbb{R}^{n\times n}$ is normal if $A^\top A = AA^\top$.
2. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$.
   - A is positive definite if $\forall\, x \neq 0,\ x^\top Ax > 0$. We denote it as $A \succ 0$.
   - A is positive semi-definite or non-negative definite if $\forall\, x \neq 0,\ x^\top Ax \geq 0$. We denote it as $A \succeq 0$.
   - A is negative definite if $\forall\, x \neq 0,\ x^\top Ax < 0$. We denote it as $A \prec 0$.
   - A is negative semi-definite or non-positive definite if $\forall\, x \neq 0,\ x^\top Ax \leq 0$. We denote it as $A \preceq 0$.
3. Given a real matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, $\Delta_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$ is called the leading principal minor of order k.
4. We have proven the following statement in the lecture. Given a symmetric matrix $P \in \mathbb{R}^{n\times n}$.
$$P \succeq 0 \iff (\exists\, N \in \mathbb{R}^{n\times n} : P = NN^\top)$$
5. Given a symmetric matrix $A \in \mathbb{R}^{m\times m}$. The following are equivalent (TFAE):
   - A is positive definite.
   - All the eigenvalues of A are positive, or the minimum eigenvalue of A is positive.
   - All the leading principal minors of A are positive definite.
   - All the leading principal minors of A have positive determinants.
6. Theorem 1: For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q^\top AQ = \Lambda$, where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix. In other words, any real symmetric matrix is orthogonally diagonalizable.
7. Matrix inversion lemma: Suppose the following matrix product are compatible and the inverses exists
$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$
   This is also called Sherman-Morrison-Woodbury formula or Woodbury matrix identity.

8. Other useful matrix identities: Given A, P, $Q \in \mathbb{R}^{n\times n}$, then

   - $(I + A)^{-1} = I - (I + A)^{-1}A$,
   - $(I + PQ)^{-1}P = P(I + QP)^{-1}$,

   if the inverses exist.

Ex:

(1) Which of the following matrices are orthogonally diagonalizable?

$$\text{a)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

(2) For those matrices that are normal in (1) find Q and Λ.

(3) Given $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $D = B^\top$. Is $(A + BCD)$ invertible?

(4) Given two vectors u, $v \in \mathbb{R}^n$, when is the matrix $(I + uv^\top)$ invertible?

2 Block Matrices / Partitioned Matrices

1. What is a block matrix? Examples.

2. Given two block matrices $A = \begin{bmatrix} A_{11} & \cdots & A_{1q} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pq} \end{bmatrix} \in \mathbb{R}^{m \times n}$, $B = \begin{bmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & \ddots & \vdots \\ B_{q1} & \cdots & B_{qr} \end{bmatrix} \in \mathbb{R}^{n \times l}$, where $A_{ij} \in \mathbb{R}^{m_i \times n_j}$ and $B_{jk} \in \mathbb{R}^{n_j \times l_k}$ are block matrices and $\sum_{i=1}^{p} m_i = m$, $\sum_{j=1}^{q} n_j = n$, $\sum_{k=1}^{r} l_k = l$.

   - $A^\top = \begin{bmatrix} A_{11}^\top & \cdots & A_{p1}^\top \\ \vdots & \ddots & \vdots \\ A_{1q}^\top & \cdots & A_{pq}^\top \end{bmatrix}$.
   - For $k \in \mathbb{R}$, $kA = \begin{bmatrix} kA_{11} & \cdots & kA_{1q} \\ \vdots & \ddots & \vdots \\ kA_{p1} & \cdots & kA_{pq} \end{bmatrix}$.
   - Suppose $C = AB$, then C can be partitioned as $C = \begin{bmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & \ddots & \vdots \\ C_{p1} & \cdots & C_{pr} \end{bmatrix} \in \mathbb{R}^{m \times l}$, where

$$C_{ik} = \sum_{j=1}^{q} A_{ij} B_{jk} \in \mathbb{R}^{m_i \times l_j}.$$

3. A block matrix of the form $A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_p \end{bmatrix}$ is called a block diagonal matrix.

   If $\forall\, 1 \le i \le p$, A_i is a square matrix, then $\det A = \prod_{i=1}^{p} \det A_i$, and the eigenvalues of A are the collection of the eigenvalues of A_i, i.e., $\{\lambda \,|\, Ax = \lambda x, x \neq 0\} = \{\lambda_i \,|\, \forall\, 1 \le i \le p,\ A_i y = \lambda_i y, y \neq 0\}$

4. Schur Complements:
   Given a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, D are square matrices, and A, D are invertible, then $D - CA^{-1}B$ is called Schur complement of A in M, and $A - BD^{-1}C$ is called Schur complement of D in M.

   Theorem:

   - If M is symmetric, i.e., $A = A^\top$, $D = D^\top$, $C = B^\top$, then the following are equivalent:
     - $M \succ 0$.
     - $A \succ 0$ and $D - B^\top A^{-1} B \succ 0$.
     - $D \succ 0$ and $A - BD^{-1}B^\top \succ 0$.
   - $\det(M) = \det(A) \det(D - CA^{-1}B) = \det(D) \det(A - BD^{-1}C)$

- Note that even if A, B, C, D are square matrices of the same size, $\det(M) \neq \det(AD - BC)$ in general.

Ex: Given $M = \begin{bmatrix} 7 & 6 & 4 \\ 6 & 6 & 4 \\ 4 & 4 & 4 \end{bmatrix}$. Is M positive definite?