MATHEMATICAL RECREATIONS AND ESSAYS

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MATHEMATICAL RECREATIONS AND ESSAYS

BY

W. W. ROUSE BALL,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

SEVENTH EDITION.

MACMILLAN AND CO., LIMITED
ST MARTIN'S STREET, LONDON
1917

First Edition, February, 1892.
Second Edition, May, 1892.
Third Edition, 1896.
Fourth Edition, 1905.
Fifth Edition, 1911.
Sixth Edition, 1914.
Seventh Edition, 1917.

PREFACE.

THE earlier part of this book contains an account of certain Mathematical Recreations: this is followed by some Essays on subjects most of which are directly concerned with historical mathematical problems. I hasten to add that the conclusions are of no practical use, and that most of the results are not new. If therefore the reader proceeds further he is at least forewarned. At the same time I think I may say that many of the questions discussed are interesting, not a few are associated with the names of distinguished mathematicians, while hitherto several of the memoirs quoted have not been easily accessible to English readers. A great deal of new matter has been added since the work was first issued in 1892, but insertions made since 1911, when the book was stereotyped, have had to be placed where room for them could best be found.

The book is divided into two parts, but in both parts I have excluded questions which involve advanced mathematics.

The *First Part* now consists of ten chapters, in which are described various problems and amusements of the kind usually termed *Mathematical Recreations.* Several of the questions mentioned in the first five chapters are of a somewhat trivial character, and had they been treated in any standard English work to which I could have referred the reader, I should have left them out: in the absence of such a work, I thought it better to insert them and trust to the judicious reader to omit them altogether or to skim them as he feels inclined. I may add that in discussing problems where the complete solutions are long or intricate I have been generally content to indicate memoirs or books in which the methods are set out at length, and to give a few illustrative examples. In several cases I have also stated problems which still await solution.

The *Second Part* now consists of twelve chapters, mostly dealing with *Historical Questions*. To make room for my discussion of Calculating Prodigies and Arithmetical Calculating Machines I have, in this (the seventh) edition, cut out the essay on the History of the Mathematical Tripos. It is with some hesitation that I have inserted papers on String Figures, Astrology, and Ciphers, but I think they may be interesting to my readers, even though the subjects are only indirectly connected with Mathematics.

I have inserted detailed references, as far as I know them, to the sources of the various questions and solutions given; also, wherever I have given only the result of a theorem, I have tried to indicate authorities where a proof may be found. In general, unless it is stated otherwise, I have taken the references direct from the original works; but, in spite of considerable time spent in verifying them, I dare not suppose that they are free from all errors or misprints. I shall be grateful for notices of additions or corrections which may occur to any of my readers.

W. W. ROUSE BALL.

TRINITY COLLEGE, CAMBRIDGE.
July, 1917.

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PART I.

Mathematical Recreations.

"*Les hommes ne sont jamais plus ingénieux que dans l'invention des jeux; l'esprit s'y trouve à son aise....Après les jeux qui dépendent uniquement des nombres viennent les jeux où entre la situation.... Après les jeux où n'entrent que le nombre et la situation viendraient les jeux où entre le mouvement....Enfin il serait à souhaiter qu'on eût un cours entier des jeux, traités mathématiquement.*" (Leibnitz: letter to De Montmort, July 29, 1715.)

CHAPTER I.

ARITHMETICAL RECREATIONS.

I commence by describing some arithmetical recreations. The interest excited by statements of the relations between numbers of certain forms has been often remarked, and the majority of works on mathematical recreations include several such problems, which, though obvious to any one acquainted with the elements of algebra, have to many who are ignorant of that subject the same kind of charm that mathematicians find in the more recondite propositions of higher arithmetic. I devote the bulk of this chapter to these elementary problems.

Before entering on the subject, I may add that a large proportion of the elementary questions mentioned here are taken from one of two sources. The first of these is the classical *Problèmes plaisans et délectables*, by Claude Gaspar Bachet, sieur de Méziriac, of which the first edition was published in 1612 and the second in 1624: it is to the edition of 1624 that the references hereafter given apply. Several of Bachet's problems are taken from the writings of Alcuin, Pacioli di Burgo, Tartaglia, and Cardan, and possibly some of them are of oriental origin, but I have made no attempt to add such references. The other source to which I alluded above is Ozanam's *Récréations mathématiques et physiques*. The greater portion of the original edition, published in two volumes at Paris in 1694, was a compilation from the works of Bachet, Mydorge, and Leurechon: this part is excellent, but the same cannot be said of the additions due to Ozanam. In the *Biographie Universelle* allusion is made to subsequent editions issued in 1720, 1735, 1741, 1778, and 1790; doubtless these references are correct, but the following editions, all of which I have seen, are the only ones of which I have any knowledge. In 1696 an edition was issued at Amsterdam. In 1723—six years after the death of Ozanam—one was issued in three volumes, with a supplementary fourth volume, containing, among other things, an appendix on puzzles. Fresh editions were issued in 1741, 1750 (the second volume of which bears the date 1749), 1770, and 1790. The edition of 1750 is said to have been corrected by Montucla on condition that his name should not be associated with it; but the edition of 1790 is the earliest one in which reference is made to these corrections, though the editor is referred to only as Monsieur M***. Montucla expunged most of what was actually incorrect in the older editions, and added several historical notes, but unfortunately his scruples prevented him from striking out the accounts of numerous trivial experiments and truisms which overload the work. An English translation of the original edition appeared in 1708, and I believe ran through four editions, the last of them being published in Dublin in 1790. Montucla's revision of 1790 was translated by C. Hutton, and editions of this were issued in 1803, in 1814, and (in one volume) in 1840: my references are to the editions of 1803 and 1840.

I proceed to enumerate some of the elementary questions connected with numbers which for nearly three centuries have formed a large part of most compilations of mathematical amusements. They are given here largely for their historical—not for their arithmetical—interest; and perhaps a mathematician may well omit this chapter.

Many of these questions are of the nature of tricks or puzzles, and I follow the usual course and present them in that form. I may note however that most of them are not worth proposing, even as tricks, unless either the method employed is disguised or the result arrived at is different from that expected; but, as I am not writing on conjuring, I refrain from alluding to the means of disguising the operations indicated, and give merely a bare enumeration of the steps essential to the success of the method used, though I may recall the fundamental rule that no trick, however good, will bear immediate repetition, and that, if it is necessary to appear to repeat it, a different method of obtaining the result should be used.

TO FIND A NUMBER SELECTED BY SOME ONE. There are innumerable ways of finding a number chosen by some one, provided the result of certain operations on it is known. I confine myself to methods typical of those commonly used. Any one acquainted with algebra will find no difficulty in framing new rules of an analogous nature.

*First Method**. (i) Ask the person who has chosen the number to treble it. (ii) Enquire if the product is even or odd: if it is even, request him to take half of it; if it is odd, request him to add unity to it and then to take half of it. (iii) Tell him to multiply the result of the second step by 3. (iv) Ask how many integral times 9 divides into the latter product: suppose the answer to be n. (v) Then the number thought of was $2n$ or $2n+1$, according as the result of step (i) was even or odd.

The demonstration is obvious. Every even number is of the form $2n$, and the successive operations applied to this give (i) $6n$, which is even; (ii) $\frac{1}{2}6n=3n$; (iii) $3\times 3n=9n$; (iv) $\frac{1}{9}9n=n$; (v) $2n$. Every odd number is of the form $2n+1$, and the successive operations applied to this give (i) $6n+3$, which is odd; (ii) $\frac{1}{2}(6n+3+1)=3n+2$; (iii) $3(3n+2)=9n+6$; (iv) $\frac{1}{9}(9n+6)=n+$ a remainder; (v) $2n+1$. These results lead to the rule given above.

Second Method†. Ask the person who has chosen the number to perform in succession the following operations. (i) To multiply the number by 5. (ii) To add 6 to the product. (iii) To multiply the sum by 4. (iv) To add 9 to the product. (v) To multiply the sum by 5. Ask to be told the result of the last operation: if from this product 165 is subtracted, and

* Bachet, *Problèmes*, Lyons, 1624, problem I, p. 53.

† A similar rule was given by Bachet, problem IV, p. 74.

then the remainder is divided by 100, the quotient will be the number thought of originally.

For let n be the number selected. Then the successive operations applied to it give (i) $5n$; (ii) $5n+6$; (iii) $20n+24$; (iv) $20n+33$; (v) $100n+165$. Hence the rule.

*Third Method**. Request the person who has thought of the number to perform the following operations. (i) To multiply it by any number you like, say, a. (ii) To divide the product by any number, say, b. (iii) To multiply the quotient by c. (iv) To divide this result by d. (v) To divide the final result by the number selected originally. (vi) To add to the result of operation (v) the number thought of at first. Ask for the sum so found: then, if ac/bd is subtracted from this sum, the remainder will be the number chosen originally.

For, if n was the number selected, the result of the first four operations is to form nac/bd; operation (v) gives ac/bd; and (vi) gives $n+(ac/bd)$, which number is mentioned. But ac/bd is known; hence, subtracting it from the number mentioned, n is found. Of course a, b, c, d may have any numerical values it is liked to assign to them. For example, if $a=12$, $b=4$, $c=7$, $d=3$ it is sufficient to subtract 7 from the final result in order to obtain the number originally selected.

Fourth Method†. Ask some one to select a number less than 90. Request him to perform the following operations. (i) To multiply it by 10, and to add any number he pleases, a, which is less than 10. (ii) To divide the result of step (i) by 3, and to mention the remainder, say, b. (iii) To multiply the quotient obtained in step (ii) by 10, and to add any number he pleases, c, which is less than 10. (iv) To divide the result of step (iii) by 3, and to mention the remainder, say d, and the third digit (from the right) of the quotient; suppose this digit is e. Then, if the numbers a, b, c, d, e are known, the original number can be at once determined. In fact, if the number is $9x+y$, where $x \not> 9$ and $y \not> 8$, and if r is the

* Bachet, problem v, p. 80.

† *Educational Times*, London, May 1, 1895, vol. XLVIII, p. 234.

remainder when $a - b + 3(c - d)$ is divided by 9, we have $x = e$, $y = 9 - r$.

The demonstration is not difficult. Suppose the selected number is $9x + y$. Step (i) gives $90x + 10y + a$. Let $y + a = 3n + b$, then the quotient obtained in step (ii) is $30x + 3y + n$. Step (iii) gives $300x + 30y + 10n + c$. Let $n + c = 3m + d$, then the quotient obtained in step (iv) is $100x + 10y + 3n + m$, which I will denote by Q. Now the third digit in Q must be x, because, since $y \not> 8$ and $a \not> 9$, we have $n \not> 5$; and since $n \not> 5$ and $c \not> 9$, we have $m \not> 4$; therefore $10y + 3n + m \not> 99$. Hence the third or hundreds digit in Q is x.

Again, from the relations $y + a = 3n + b$ and $n + c = 3m + d$, we have $9m - y = a - b + 3(c - d)$: hence, if r is the remainder when $a - b + 3(c - d)$ is divided by 9, we have $y = 9 - r$. [This is always true, if we make r positive; but if $a - b + 3(c - d)$ is negative, it is simpler to take y as equal to its numerical value; or we may prevent the occurrence of this case by assigning proper values to a and c.] Thus x and y are both known, and therefore the number selected, namely $9x + y$, is known.

*Fifth Method**. Ask any one to select a number less than 60. Request him to perform the following operations. (i) To divide it by 3 and mention the remainder; suppose it to be a. (ii) To divide it by 4, and mention the remainder; suppose it to be b. (iii) To divide it by 5, and mention the remainder; suppose it to be c. Then the number selected is the remainder obtained by dividing $40a + 45b + 36c$ by 60.

This method can be generalized and then will apply to any number chosen. Let $a', b', c', \ldots$ be a series of numbers prime to one another, and let p be their product. Let n be any number less than p, and let $a, b, c, \ldots$ be the remainders when n is divided by $a', b', c', \ldots$ respectively. Find a number A which is a multiple of the product $b'c'd' \ldots$ and which exceeds by unity a multiple of a'. Find a number B which is a multiple of $a'c'd' \ldots$ and which exceeds by unity a multiple

* Bachet, problem VI, p. 84: Bachet added, on p. 87, a note on the previous history of the problem.

of b', and similarly find analogous numbers C, D,.... Rules for the calculation of A, B, C,... are given in the theory of numbers, but in general, if the numbers a', b', c',... are small, the corresponding numbers A, B, C, ... can be found by inspection. I proceed to show that n is equal to the remainder when $Aa + Bb + Cc + \ldots$ is divided by p.

Let $N = Aa + Bb + Cc + \ldots$, and let $M(x)$ stand for a multiple of x. Now $A = M(a') + 1$, therefore $Aa = M(a') + a$. Hence, if the first term in N, that is Aa, is divided by a', the remainder is a. Again, B is a multiple of $a'c'd'$.... Therefore Bb is exactly divisible by a' Similarly Cc, Dd,... are each exactly divisible by a'. Thus every term in N, except the first, is exactly divisible by a'. Hence, if N is divided by a', the remainder is a. Also if n is divided by a', the remainder is a.

Therefore $$N - n = M(a').$$
Similarly $$N - n = M(b'),$$
$$N - n = M(c'),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

But a', b', c',... are prime to one another.

$$\therefore\ N - n = M(a'b'c'\ldots) = M(p),$$

that is, $$N = M(p) + n.$$

Now n is less than p, hence if N is divided by p, the remainder is n.

The rule given by Bachet corresponds to the case of $a' = 3$, $b' = 4$, $c' = 5$, $p = 60$, $A = 40$, $B = 45$, $C = 36$. If the number chosen is less than 420, we may take $a' = 3$, $b' = 4$, $c' = 5$, $d' = 7$, $p = 420$, $A = 280$, $B = 105$, $C = 336$, $D = 120$.

To find the result of a series of operations performed on any number (*unknown to the operator*) without asking any questions. All rules for solving such problems ultimately depend on so arranging the operations that the number disappears from the final result. Four examples will suffice.

*First Example**. Request some one to think of a number. Suppose it to be n. Ask him (i) to multiply it by any number you please, say, a; (ii) then to add, say, b; (iii) then to divide

* Bachet, problem VIII, p. 102.

the sum by, say, c. (iv) Next, tell him to take a/c of the number originally chosen; and (v) to subtract this from the result of the third operation. The result of the first three operations is $(na + b)/c$, and the result of operation (iv) is na/c: the difference between these is b/c, and therefore is known to you. For example, if $a = 6$, $b = 12$, $c = 4$, then $a/c = 1\frac{1}{2}$, and the final result is 3.

*Second Example**. Ask A to take any number of counters that he pleases: suppose that he takes n counters. (i) Ask some one else, say B, to take p times as many, where p is any number you like to choose. (ii) Request A to give q of his counters to B, where q is any number you like to select. (iii) Next, ask B to transfer to A a number of counters equal to p times as many counters as A has in his possession. Then there will remain in B's hands $q(p+1)$ counters: this number is known to you; and the trick can be finished either by mentioning it or in any other way you like.

The reason is as follows. The result of operation (ii) is that B has $pn + q$ counters, and A has $n - q$ counters. The result of (iii) is that B transfers $p(n-q)$ counters to A: hence he has left in his possession $(pn+q) - p(n-q)$ counters, that is, he has $q(p+1)$.

For example, if originally A took any number of counters, then (if you chose p equal to 2), first you would ask B to take twice as many counters as A had done; next (if you chose q equal to 3) you would ask A to give 3 counters to B; and then you would ask B to give to A a number of counters equal to twice the number then in A's possession; after this was done you would know that B had $3(2+1)$, that is, 9 left.

This trick (as also some of the following problems) may be performed equally well with one person, in which case A may stand for his right hand and B for his left hand.

Third Example. Ask some one to perform in succession the following operations. (i) Take any number of three digits, in which the difference between the first and last digits exceeds unity. (ii) Form a new number by reversing the order of

* Bachet, problem XIII, p. 123: Bachet presented the above trick in a form, somewhat more general, but less effective in practice.

the digits. (iii) Find the difference of these two numbers. (iv) Form another number by reversing the order of the digits in this difference. (v) Add together the results of (iii) and (iv). Then the sum obtained as the result of this last operation will be 1089.

An illustration and the explanation of the rule are given below.

(i)	237	$100a+10b+c$
(ii)	732	$100c+10b+a$
(iii)	495	$100(a-c-1)\ +\ 90+(10+c-a)$
(iv)	594	$100(10+c-a)+\ 90+(a-c-1)$
(v)	1089	$900\quad +180+9$

The result depends only on the radix of the scale of notation in which the number is expressed. If this radix is r, the result is $(r-1)(r+1)^2$; thus if $r=10$, the result is 9×11^2, that is, 1089.

*Fourth Example**. The following trick depends on the same principle. Ask some one to perform in succession the following operations. (i) To write down any sum of money less than £12, in which the difference between the number of pounds and the number of pence exceeds unity. (ii) To *reverse* this sum, that is, to write down a sum of money obtained from it by interchanging the numbers of pounds and pence. (iii) To find the difference between the results of (i) and (ii). (iv) To reverse this difference. (v) To add together the results of (iii) and (iv). Then this sum will be £12. 18*s*. 11*d*.

For instance, take the sum £10. 17*s*. 5*d*.; we have

	£	*s*.	*d*.
(i)	10	17	5
(ii)	5	17	10
(iii)	4	19	7
(iv)	7	19	4
(v)	12	18	11

* *Educational Times Reprints*, 1890, vol. LIII, p. 78.

The following analysis explains the rule, and shows that the final result is independent of the sum written down initially.

	£	s.	d.
(i)	a	b	c
(ii)	c	b	a
(iii)	$a-c-1$	19	$c-a+12$
(iv)	$c-a+12$	19	$a-c-1$
(v)	11	38	11

Mr J. H. Schooling has used this result as the foundation of a slight but excellent conjuring trick. The rule can be generalized to cover any system of monetary units.

Problems involving Two Numbers. I proceed next to give a couple of examples of a class of problems which involve two numbers.

*First Example**. Suppose that there are two numbers, one even and the other odd, and that a person A is asked to select one of them, and that another person B takes the other. It is desired to know whether A selected the even or the odd number. Ask A to multiply his number by 2, or any even number, and B to multiply his by 3, or any odd number. Request them to add the two products together and tell you the sum. If it is even, then originally A selected the odd number, but if it is odd, then originally A selected the even number. The reason is obvious.

Second Example†. The above rule was extended by Bachet to any two numbers, provided they are prime to one another and one of them is not itself a prime. Let the numbers be m and n, and suppose that n is exactly divisible by p. Ask A to select one of these numbers, and B to take the other. Choose a number prime to p, say q. Ask A to multiply his number by q, and B to multiply his number by p. Request them to add the products together and state the sum. Then A originally selected m or n, according as this result is not or is divisible by p. The numbers, $m=7$, $n=15$, $p=3$, $q=2$, will illustrate the rest.

Problems depending on the Scale of Notation. Many of the rules for finding two or more numbers depend on the

* Bachet, problem IX, p. 107. † Bachet, problem XI, p. 113.

fact that in arithmetic an integral number is denoted by a succession of digits, where each digit represents the product of that digit and a power of ten, and the number is equal to the sum of these products. For example, 2017 signifies $(2 \times 10^3) + (0 \times 10^2) + (1 \times 10) + 7$; that is, the 2 represents 2 thousands, *i.e.* the product of 2 and 10^3, the 0 represents 0 hundreds, *i.e.* the product of 0 and 10^2; the 1 represents 1 ten, *i.e.* the product of 1 and 10, and the 7 represents 7 units. Thus every digit has a local value. The application to tricks connected with numbers will be understood readily from three illustrative examples.

*First Example**. A common conjuring trick is to ask a boy among the audience to throw two dice, or to select at random from a box a domino on each half of which is a number. The boy is then told to recollect the two numbers thus obtained, to choose either of them, to multiply it by 5, to add 7 to the result, to double this result, and lastly to add to this the other number. From the number thus obtained, the conjurer subtracts 14, and obtains a number of two digits which are the two numbers chosen originally.

For suppose that the boy selected the numbers a and b. Each of these is less than ten—dice or dominoes ensuring this. The successive operations give (i) $5a$; (ii) $5a + 7$; (iii) $10a + 14$; (iv) $10a + 14 + b$. Hence, if 14 is subtracted from the final result, there will be left a number of two digits, and these digits are the numbers selected originally. An analogous trick might be performed in other scales of notation if it was thought necessary to disguise the process further.

Second Example†. Similarly, if three numbers, say, a, b, c, are chosen, then, if each of them is less than ten, they can be

* Some similar questions were given by Bachet in problem XII, p. 117; by Oughtred or Leake in the *Mathematicall Recreations*, commonly attributed to the former, London, 1653, problem XXXIV; and by Ozanam, part I, chapter X. Probably the *Mathematicall Recreations* were compiled by Leake, but as the work is usually catalogued under the name of W. Oughtred, I shall so describe it: it is founded on the similar work by J. Leurechon, otherwise known as H. van Etten, published in 1626.

† Bachet gave some similar questions in problem XII, p. 117.

found by the following rule. (i) Take one of the numbers, say, a, and multiply it by 2. (ii) Add 3 to the product. (iii) Multiply this by 5, and add 7 to the product. (iv) To this sum add the second number, b. (v) Multiply the result by 2. (vi) Add 3 to the product. (vii) Multiply by 5, and, to the product, add the third number, c. The result is $100a+10b+c+235$. Hence, if the final result is known, it is sufficient to subtract 235 from it, and the remainder will be a number of three digits. These digits are the numbers chosen originally.

*Third Example**. The following rule for finding the age of a man born in the 19th century is of the same kind. Take the tens digit of the year of birth; (i) multiply it by 5; (ii) to the product add 2; (iii) multiply the result by 2; (iv) to this product add the units digit of the birth-year; (v) subtract the sum from 120. The result is the man's age in 1916.

The algebraic proof of the rule is obvious. Let a and b be the tens and units digits of the birth-year. The successive operations give (i) $5a$; (ii) $5a+2$; (iii) $10a+4$; (iv) $10a+4+b$; (v) $120-(10a+b)$, which is his age in 1916. The rule can be easily adapted to give the age in any specified year.

Fourth Example†. Another such problem but of more difficulty is the determination of all numbers which are integral multiples of their reversals. For instance, among numbers of four digits, $8712=4\times 2178$ and $9801=9\times 1089$ possess this property.

OTHER PROBLEMS WITH NUMBERS IN THE DENARY SCALE. I may mention here two or three other problems which seem to be unknown to most compilers of books of puzzles.

First Problem. The first of them is as follows. Take any number of three digits: reverse the order of the digits: subtract the number so formed from the original number: then, if the last digit of the difference is mentioned, all the digits in the difference are known.

* A similar question was given by Laisant and Perrin in their *Algèbre*, Paris, 1892; and in *L'Illustration* for July 13, 1895.

† *L'Intermédiaire des Mathématiciens*, Paris, vol. XV, 1908, pp. 228, 278; vol. XVI, 1909, p. 34; vol. XIX, 1912, p. 128.

For suppose the number is $100a+10b+c$, that is, let a be the hundreds digit of the number chosen, b be the tens digit, and c be the units digit. The number obtained by reversing the digits is $100c+10b+a$. The difference of these numbers is equal to $(100a+c)-(100c+a)$, that is, to $99(a-c)$. But $a-c$ is not greater than 9, and therefore the remainder can only be 99, 198, 297, 396, 495, 594, 693, 792, or 891—in each case the middle digit being 9 and the digit before it (if any) being equal to the difference between 9 and the last digit. Hence, if the last digit is known, so is the whole of the remainder.

Second Problem. The second problem is somewhat similar and is as follows. (i) Take any number; (ii) reverse the digits; (iii) find the difference between the number formed in (ii) and the given number; (iv) multiply this difference by any number you like to name; (v) cross out any digit except a nought; (vi) read the remainder. Then the sum of the digits in the remainder subtracted from the next highest multiple of nine will give the figure struck out. This is clear since the result of operation (iv) is a multiple of nine, and the sum of the digits of every multiple of nine is itself a multiple of nine. This and the previous problem are typical of numerous analogous questions.

Empirical Problems. There are also numerous empirical problems, such as the following. With the ten digits, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, express numbers whose sum is unity: each digit being used only once, and the use of the usual notations for fractions being allowed. With the same ten digits express numbers whose sum is 100. With the nine digits, 9, 8, 7, 6, 5, 4, 3, 2, 1, express four numbers whose sum is 100. To the making of such questions there is no limit, but their solution involves little or no mathematical skill.

Four Digits Problem. I suggest the following problem as being more interesting. With the digits 1, 2, ... n, express the consecutive numbers from 1 upwards as far as possible, say to p: four and only four digits, all different, being used in each number, and the notation of the denary scale (including decimals), as also algebraic sums, products, and positive integral powers, being allowed. If the use of the symbols for square roots and factorials (repeated if desired a finite number of times) is also permitted, the range can be extended considerably, say to q consecutive integers. If $n=4$, we have $p=88$, $q=264$; if $n=5$, $p=231$, $q=790$; with the four digits 0, 1, 2, 3, $p=36$, $q=40$.

Four Fours Problem. Another traditional recreation is, with the ordinary arithmetic and algebraic notation, to express the consecutive numbers from 1 upwards as far as possible in terms of four "4's." Everything turns on what we mean by ordinary notation. If (α) this is taken to admit only the use of the denary scale (ex. gr. numbers like 44), decimals, brackets, and the symbols for addition, subtraction, multiplication and division, we can thus express every number up to 22 inclusive. If (β) also we grant the use of the symbol for square root (repeated if desired a finite number of times) we can get to 30; but note that though by its use a number like $.\dot{2}$ can be expressed by one "4," we cannot for that reason say that $\cdot 2$ is so expressible. If (γ) further we permit the use of symbols for factorials we can express every number to 112. Finally, if (δ) we sanction the employment of integral indices expressible by a "4" or "4's" and allow the symbol for a square root to be used an infinite number of times we can get to 156; but if (ϵ) we concede the employment of integral indices and the use of sub-factorials* we can get to 877. These interesting problems are typical of a class of similar questions† for four (or n) "2's," "3's," &c.

Problems with a series of things which are numbered. Any collection of things numbered consecutively lend themselves to easy illustrations of questions depending on elementary properties of numbers. As examples I proceed to enumerate a few familiar tricks. The first two of these are commonly shown by the use of a watch, the last four may be exemplified by the use of a pack of playing cards.

* Sub-factorial n is equal to $n!(1-1/1!+1/2!-1/3!+\ldots\pm 1/n!)$. On the use of this for the four "4's" problem, see the *Mathematical Gazette*, May, 1912.

† With four "2's," we can under α get to 27 and under γ to 36; with four "3's," under α to 21 and under γ to 43; with four "5's," under α to 30, under γ to 36, and with the use of indices to 67; and with four "9's" under α to 12, under β to 37, under γ to 66, and with the use of indices to 132.

*First Example**. The first of these examples is connected with the hours marked on the face of a watch. In this puzzle some one is asked to think of some hour, say, m, and then to touch a number that marks another hour, say, n. Then if, beginning with the number touched, he taps each successive hour marked on the face of the watch, going in the opposite direction to that in which the hands of the watch move, and reckoning to himself the taps as m, $(m+1)$, &c., the $(n+12)$th tap will be on the hour he thought of. For example, if he thinks of V and touches IX, then, if he taps successively IX, VIII, VII, VI, ..., going backwards and reckoning them respectively as 5, 6, 7, 8, ..., the tap which he reckons as 21 will be on the V.

The reason of the rule is obvious, for he arrives finally at the $(n+12-m)$th hour from which he started. Now, since he goes in the opposite direction to that in which the hands of the watch move, he has to go over $(n-m)$ hours to reach the hour m: also it will make no difference if in addition he goes over 12 hours, since the only effect of this is to take him once completely round the circle. Now $(n+12-m)$ is always positive, since n is positive and m is not greater than 12, and therefore if we make him pass over $(n+12-m)$ hours we can give the rule in a form which is equally valid whether m is greater or less than n.

Second Example. The following is another well-known watch-dial problem. If the hours on the face are tapped successively, beginning at VII and proceeding backwards round the dial to VI, V, &c., and if the person who selected the number counts the taps, beginning to count from the number of the hour selected (thus, if he selected X, he would reckon the first tap as the 11th), then the 20th tap as reckoned by him will be on the hour chosen.

For suppose he selected the nth hour. Then the 8th tap is on XII and is reckoned by him as the $(n+8)$th; and the tap

* Bachet, problem xx, p. 155; Oughtred or Leake, *Mathematicall Recreations*, London, 1653, p. 28.

which he reckons as $(n+p)$th is on the hour $(20-p)$. Hence, putting $p=20-n$, the tap which he reckons as 20th is on the hour n. Of course the hours indicated by the first seven taps are immaterial: obviously also we can modify the presentation by beginning on the hour VIII and making 21 consecutive taps, or on the hour IX and making 22 consecutive taps, and so on.

Third Example. The following is another simple example. Suppose that a pack of n cards is given to some one who is asked to select one out of the first m cards and to remember (but not to mention) what is its number from the top of the pack; suppose it is actually the xth card in the pack. Then take the pack, reverse the order of the top m cards (which can be easily effected by shuffling), and transfer y cards, where $y < n - m$, from the bottom to the top of the pack. The effect of this is that the card originally chosen is now the $(y+m-x+1)$th from the top. Return to the spectator the pack so rearranged, and ask that the top card be counted as the $(x+1)$th, the next as the $(x+2)$th, and so on, in which case the card originally chosen will be the $(y+m+1)$th. Now y and m can be chosen as we please, and may be varied every time the trick is performed; thus any one unskilled in arithmetic will not readily detect the method used.

*Fourth Example**. Place a card on the table, and on it place as many other cards from the pack as with the number of pips on the card will make a total of twelve. For example, if the card placed first on the table is the five of clubs, then seven additional cards must be placed on it. The court cards may have any values assigned to them, but usually they are reckoned as tens. This is done again with another card, and thus another pile is formed. The operation may be repeated either only three or four times or as often as the pack will permit of such piles being formed. If finally there are p such piles, and if the number of cards left over is r, then the sum of the number of pips on the bottom cards of all the piles will be $13(p-4)+r$.

For, if x is the number of pips on the bottom card of a pile,

* A particular case of this problem was given by Bachet, problem XVII, p. 138.

the number of cards in that pile will be $13-x$. A similar argument holds for each pile. Also there are 52 cards in the pack; and this must be equal to the sum of the cards in the p piles and the r cards left over.

$$\therefore (13-x_1)+(13-x_2)+\ldots+(13-x_p)+r=52,$$
$$\therefore 13p-(x_1+x_2+\ldots+x_p)+r=52,$$
$$\therefore x_1+x_2+\ldots+x_p=13p-52+r$$
$$=13(p-4)+r.$$

More generally, if a pack of n cards is taken, and if in each pile the sum of the pips on the bottom card and the number of cards put on it is equal to m, then the sum of the pips on the bottom cards of the piles will be $(m+1)p+r-n$. In an écarté pack $n=32$, and it is convenient to take $m=15$.

Fifth Example. It may be noticed that cutting a pack of cards never alters the relative position of the cards provided that, if necessary, we regard the top card as following immediately after the bottom card in the pack. This is used in the following trick*. Take a pack, and deal the cards face upwards on the table, calling them one, two, three, &c. as you put them down, and noting in your own mind the card first dealt. Ask some one to select a card and recollect its number. Turn the pack over, and let it be cut (not shuffled) as often as you like. Enquire what was the number of the card chosen. Then, if you deal, and as soon as you come to the original first card begin (silently) to count, reckoning this as one, the selected card will appear at the number mentioned. Of course, if all the cards are dealt before reaching this number, you must turn the cards over and go on counting continuously.

Sixth Example. Here is another simple question of this class. Remove the court cards from a pack. Arrange the remaining 40 cards, faces upwards, in suits, in four lines thus. In the first line, the 1, 2, ... 10, of suit A; in the second line, the 10, 1, 2, ... 9, of suit B; in the third line, the 9, 10, 1, ... 8, of suit C; in the last line, the 8, 9, 10, 1, ... 7, of suit D. Next take up, face upwards, the first card of line 1, put below it the

* Bachet, problem xix, p. 152.

first card of line 2, below that the first card of line 3, and below that the first card of line 4. Turn this pile face downwards. Next take up the four cards in the second column in the same way, turn them face downwards, and put them below the first pile. Continue this process until all the cards are taken up. Ask someone to mention any card. Suppose the number of pips on it is n. Then if the suit is A, it will be the $4n$th card in the pack; if the suit is B, it will be the $(4n+3)$th card; if the suit is C, it will be the $(4n+6)$th card; and if the suit is D, it will be the $(4n+9)$th card. Hence by counting the cards, cyclically if necessary, the card desired can be picked out. It is easy to alter the form of presentation, and a full pack can be used if desired. The explanation is obvious.

Medieval Problems in Arithmetic. Before leaving the subject of these elementary questions, I may mention a few problems which for centuries have appeared in nearly every collection of mathematical recreations, and therefore may claim what is almost a prescriptive right to a place here.

First Example. The following is a sample of one class of these puzzles. A man goes to a tub of water with two jars, of which one holds exactly 3 pints and the other 5 pints. How can he bring back exactly 4 pints of water? The solution presents no difficulty.

*Second Example**. Here is another problem of the same kind. Three men robbed a gentleman of a vase, containing 24 ounces of balsam. Whilst running away they met a glass-seller, of whom they purchased three vessels. On reaching a place of safety they wished to divide the booty, but found that their vessels contained 5, 11, and 13 ounces respectively. How could they divide the balsam into equal portions? Problems like this can be worked out only by trial.

Third Example†. The next of these is a not uncommon

* Some similar problems were given by Bachet, Appendix, problem III, p. 206; problem IX, p. 233: by Oughtred or Leake in the *Mathematicall Recreations*, p. 22: and by Ozanam, 1803 edition, vol. I, p. 174; 1840 edition, p. 79. Earlier instances occur in Tartaglia's writings.

† Bachet, problem XXII, p. 170.

game, played by two people, say A and B. A begins by mentioning some number not greater than (say) six, B may add to that any number not greater than six, A may add to that again any number not greater than six, and so on. He wins who is the first to reach (say) 50. Obviously, if A calls 43, then whatever B adds to that, A can win next time. Similarly, if A calls 36, B cannot prevent A's calling 43 the next time. In this way it is clear that the key numbers are those forming the arithmetical progression 43, 36, 29, 22, 15, 8, 1; and whoever plays first ought to win.

Similarly, if no number greater than m may be added at any one time, and n is the number to be called by the victor, then the key numbers will be those forming the arithmetical progression whose common difference is $m+1$ and whose smallest term is the remainder obtained by dividing n by $m+1$.

The same game may be played in another form by placing p coins, matches, or other objects on a table, and directing each player in turn to take away not more than m of them. Whoever takes away the last coin wins. Obviously the key numbers are multiples of $m+1$, and the first player who is able to leave an exact multiple of $(m+1)$ coins can win. Perhaps a better form of the game is to make that player lose who takes away the last coin, in which case each of the key numbers exceeds by unity a multiple of $m+1$.

Another variety* consists in placing p counters in the form of a circle, and allowing each player in succession to take away not more than m of them which are in unbroken sequence: m being less than p and greater than unity. In this case the second of the two players can always win.

These games are simple, but if we impose on the original problem the restriction that each player may not add the same number more than (say) three times, the analysis becomes by no means easy. I have never seen this extension described in print, and I will enunciate it at length. Suppose that each player is given eighteen cards, three of them marked 6, three marked 5, three marked 4, three

* S. Loyd, *Tit-Bits*, London, July 17, Aug. 7, 1897.

marked 3, three marked 2, and three marked 1. They play alternately; *A* begins by playing one of his cards; then *B* plays one of his, and so on. He wins who first plays a card which makes the sum of the points or numbers on all the cards played exactly equal to 50, but he loses if he plays a card which makes this sum exceed 50. The game can be played by noting the numbers on a piece of paper, and it is not necessary to use cards.

Thus suppose they play as follows. *A* takes a 4, and scores 4; *B* takes a 3, and scores 7; *A* takes a 1, and scores 8; *B* takes a 6, and scores 14; *A* takes a 3, and scores 17; *B* takes a 4, and scores 21; *A* takes a 4, and scores 25; *B* takes a 5, and scores 30; *A* takes a 4, and scores 34; *B* takes a 4, and scores 38; *A* takes a 5, and scores 43. *B* can now win, for he may safely play 3, since *A* has not another 4 wherewith to follow it; and if *A* plays less than 4, *B* will win the next time. Again, suppose they play thus. *A*, 6; *B*, 3; *A*, 1; *B*, 6; *A*, 3; *B*, 4; *A*, 2; *B*, 5; *A*, 1; *B*, 5; *A*, 2; *B*, 5; *A*, 2; *B*, 3. *A* is now forced to play 1, and *B* wins by playing 1.

A slightly different form of the game has also been suggested. In this there are put on the table an agreed number of cards, say, for example, the four aces, twos, threes, fours, fives, and sixes of a pack of cards—twenty-four cards in all. Each player in turn takes a card. The score at any time is the sum of the pips on all the cards taken, whether by *A* or *B*. He wins who first selects a card which makes the score equal, say, to 50, and a player who is forced to go beyond 50 loses.

Thus, suppose they play as follows. *A* takes a 6, and scores 6; *B* takes a 2, and scores 8; *A* takes a 5, and scores 13; *B* takes a 2, and scores 15; *A* takes a 5, and scores 20; *B* takes a 2, and scores 22; *A* takes a 5, and scores 27; *B* takes a 2, and scores 29; *A* takes a 5, and scores 34; *B* takes a 6, and scores 40; *A* takes a 1, and scores 41; *B* takes a 4, and scores 45; *A* takes a 3, and scores 48; *B* now must take 1, and thus score 49; and *A* takes a 1, and wins.

In these variations the object of each player is to get to one of the key numbers, provided there are sufficient available remaining numbers to let him retain the possession of each subsequent key number. The number of cards used, the points on them, and the number to be reached can be changed at will; and the higher the number to be reached, the more difficult it is to forecast the result and to say whether or not it is an advantage to begin.

Fourth Example. Here is another problem, more difficult and less well known. Suppose that m counters are divided into n heaps. Two players play alternately. Each, when his turn comes, may select any one heap he likes, and remove from it all the counters in it or as many of them as he pleases. That player loses who has to take up the last counter.

To solve it we may proceed thus*. Suppose there are a_r counters in the rth heap. Express a_r in the binary scale, and denote the coefficient of 2^p in it by d_{rp}. Do this for each heap, and let S_p be the sum of the coefficients of 2^p thus determined. Thus $S_p = d_{1p} + d_{2p} + d_{3p} + \ldots$. Then either S_0, $S_1, S_2, \ldots$ are all even, which we may term an A arrangement, or they are not all even, which we may term a B arrangement.

It will be easily seen that if one player, P, has played so as to get the counters in any A arrangement (except that of an even number of heaps each containing one counter), he can force a win. For the next move of his opponent, Q, must bring the counters to a B arrangement. Then P can make the next move to bring the counters again to an A arrangement, other than the exceptional one of an even number of heaps each containing only one counter. Finally this will leave P a winning position, *ex. gr.* two heaps each containing 2 counters.

If that player wins who takes the last counter, the rule is easier. For if one player P has played so as to get the counters in any B arrangement, he can force a win, since the next move of his opponent Q must bring the counters to an A arrangement. Then P can make the next move to bring the counters again to a B arrangement. Finally this will leave P a winning position.

Fifth Example. The following medieval problem is somewhat more elaborate. Suppose that three people, P, Q, R,

* From a letter to me by Mr R. K. Morcom, August 2, 1910.

select three things, which we may denote by *a*, *e*, *i* respectively, and that it is desired to find by whom each object was selected*.

Place 24 counters on a table. Ask P to take one counter, Q to take two counters, and R to take three counters. Next, ask the person who selected *a* to take as many counters as he has already, whoever selected *e* to take twice as many counters as he has already, and whoever selected *i* to take four times as many counters as he has already. Note how many counters remain on the table. There are only six ways of distributing the three things among P, Q, and R; and the number of counters remaining on the table is different for each way. The remainders may be 1, 2, 3, 5, 6, or 7. Bachet summed up the results in the mnemonic line *Par fer* (1) *César* (2) *jadis* (3) *devint* (5) *si grand* (6) *prince* (7). Corresponding to any remainder is a word or words containing two syllables: for instance, to the remainder 5 corresponds the word *devint*. The vowel in the first syllable indicates the thing selected by P, the vowel in the second syllable indicates the thing selected by Q, and of course R selected the remaining thing.

Extension. M. Bourlet, in the course of a very kindly notice† of the second edition of this work, gave a much neater solution of the above question, and has extended the problem to the case of n people, $P_0, P_1, P_2, \ldots, P_{n-1}$, each of whom selects one object, out of a collection of n objects, such as dominoes or cards. It is required to know which domino or card was selected by each person.

Let us suppose the dominoes to be denoted or marked by the numbers $0, 1, \ldots, n-1$, instead of by vowels. Give one counter to P_1, two counters to P_2, and generally k counters to P_k. Note the number of counters left on the table. Next ask the person who had chosen the domino 0 to take as many counters as he had already, and generally whoever had chosen the domino h to take n^h times as many dominoes as he had already: thus if P_h had chosen the domino numbered h,

* Bachet, problem xxv, p. 187.

† *Bulletin des Sciences Mathématiques*, Paris, 1893, vol. xvii, pp. 105—107.

he would take $n^h k$ counters. The total number of counters taken is $\Sigma n^h k$. Divide this by n, then the remainder will be the number on the domino selected by P_0; divide the quotient by n, and the remainder will be the number on the domino selected by P_1; divide this quotient by n, and the remainder will be the number on the domino selected by P_2; and so on. In other words, if the number of counters taken is expressed in the scale of notation whose radix is n, then the $(h+1)$th digit from the right will give the number on the domino selected by P_h.

Thus in Bachet's problem with 3 people and 3 dominoes, we should first give one counter to Q, and two counters to R, while P would have no counters; then we should ask the person who had selected the domino marked 0 or a to take as many counters as he had already, whoever had selected the domino marked 1 or e to take three times as many counters as he had already, and whoever had selected the domino marked 2 or i to take nine times as many counters as he had already. By noticing the original number of counters, and observing that 3 of these had been given to Q and R, we should know the total number taken by P, Q, and R. If this number were divided by 3, the remainder would be the number of the domino chosen by P; if the quotient were divided by 3 the remainder would be the number of the domino chosen by Q; and the final quotient would be the number of the domino chosen by R.

Exploration Problems. Another common question is concerned with the maximum distance into a desert which could be reached from a frontier settlement by the aid of a party of n explorers, each capable of carrying provisions that would last one man for a days. The answer is that the man who reaches the greatest distance will occupy $na/(n+1)$ days before he returns to his starting point. If in the course of their journey they may make depôts, the longest possible journey will occupy $\frac{1}{2}a(1+\frac{1}{2}+\frac{1}{3}+\ldots+1/n)$ days.

The Josephus Problem. Another of these antique problems consists in placing men round a circle so that if every mth man is killed, the remainder shall be certain specified individuals. Such problems can be easily solved empirically.

Hegesippus* says that Josephus saved his life by such a device. According to his account, after the Romans had captured Jotapat, Josephus and forty other Jews took refuge in a cave. Josephus, much to his disgust, found that all except himself and one other man were resolved to kill themselves, so as not to fall into the hands of their conquerors. Fearing to show his opposition too openly he consented, but declared that the operation must be carried out in an orderly way, and suggested that they should arrange themselves round a circle and that every third person should be killed until but one man was left, who must then commit suicide. It is alleged that he placed himself and the other man in the 31st and 16th place respectively.

The medieval question was usually presented in the following form. A ship, carrying as passengers 15 Turks and 15 Christians, encountered a storm, and, in order to save the ship and crew, one-half of the passengers had to be thrown into the sea. Accordingly the passengers were placed in a circle, and every ninth man, reckoning from a certain point, was cast overboard. It is desired to find an arrangement by which all the Christians should be saved †. In this case we must arrange the men thus: $C\,C\,C\,C\,T\,T\,T\,T\,T\,C\,C\,T\,C\,C\,C\,T\,C\,T\,T\,C\,C\,T\,T\,T\,C\,T\,T\,C\,C\,T$, where C stands for a Christian and T for a Turk. The order can be recollected by the positions of the vowels in the following line: *From numbers' aid and art, never will fame depart*, where a stands for 1, e for 2, i for 3, o for 4, and u for 5. Hence the order is o Christians, u Turks, &c.

If every tenth man were cast overboard, a similar mnemonic line is *Rex paphi cum gente bona dat signa serena.* An oriental setting of this decimation problem runs somewhat as follows. Once upon a time, there lived a rich farmer who had 30 children, 15 by his first wife who was dead, and 15 by his second wife. The latter woman was eager that her eldest son should inherit the property. Accordingly one day she said to him, "Dear Husband,

* *De Bello Judaico*, bk. III, chaps. 16—18.

† Bachet, problem XXIII, p. 174. The same problem had been previously enunciated by Tartaglia.

you are getting old. We ought to settle who shall be your heir. Let us arrange our 30 children in a circle, and counting from one of them remove every tenth child until there remains but one, who shall succeed to your estate." The proposal seemed reasonable. As the process of selection went on, the farmer grew more and more astonished as he noticed that the first 14 to disappear were children by his first wife, and he observed that the next to go would be the last remaining member of that family. So he suggested that they should see what would happen if they began to count backwards from this lad. She, forced to make an immediate decision, and reflecting that the odds were now 15 to 1 in favour of her family, readily assented. Who became the heir?

In the general case n men are arranged in a circle which is closed up as individuals are picked out. Beginning anywhere, we continually go round, picking out each mth man until only r are left. Let one of these be the man who originally occupied the pth place. Then had we begun with $n+1$ men, he would have originally occupied the $(p+m)$th place when $p+m$ is not greater than $n+1$, and the $(p+m-n-1)$th place when $p+m$ is greater than $n+1$. Thus, provided there are to be r men left, their original positions are each shifted forwards along the circle m places for each addition of a single man to the original group*.

Now suppose that with n men the last survivor $(r=1)$ occupied originally the pth place, and that with $(n+x)$ men the last survivor occupied the yth place. Then, if we confine ourselves to the lowest value of x which makes y less than m, we have $y=(p+mx)-(n+x)$.

Based on this theorem we can, for any specified value of n, calculate rapidly the position occupied by the last survivor of the company. In effect, Tait found the values of n for which a man occupying a given position p, which is less than m, would be the last survivor, and then by repeated applications of the proposition, obtained the position of the survivor for intermediate values of n.

* P. G. Tait, *Collected Scientific Papers*, Cambridge, vol. II. 1900, pp. 432—435.

For instance, take the Josephus problem in which $m = 3$. Then we know that the final survivor of 41 men occupied originally the 31st place. Suppose that when there had been $(41 + x)$ men, the survivor occupied originally the yth place. Then, if we consider only the lowest value of x which makes y less than m, we have $y = (31 + 3x) - (41 + x) = 2x - 10$. Now, we have to take a value of x which makes y positive and less than m, that is, in this case equal to 1 or 2. This is $x = 6$ which makes $y = 2$. Hence, had there been 47 men the man last chosen would have originally occupied the second place. Similarly had there been $(47 + x)$ men the man would have occupied originally the yth place, where, subject to the same conditions as before, we have $y = (2 + 3x) - (47 + x) = 2x - 45$. If $x = 23$, $y = 1$. Hence, with 70 men the man last chosen would have occupied originally the first place. Continuing the process, it is easily found that if n does not exceed 2000000 the last man to be taken occupies the first place when $n = 4, 6, 9, 31, 70, 105, 355, 799, 1798, 2697, 9103, 20482, 30723, 69127, 155536, 233304, 349956, 524934$, or 787401; and the second place when $n = 2, 3, 14, 21, 47, 158, 237, 533, 1199, 4046, 6069, 13655, 46085, 103691, 1181102$, or 1771653. From these results, by repeated applications of the proposition, we find, for any intermediate values of n, the position originally occupied by the man last taken. Thus with 1000 men, the 604th place; with 100000 men, the 92620th place; and with 1000000 men, the 637798th place are those which would be selected by a prudent mathematician in a company subjected to trimation.

Similarly if a set of 100 men were subjected to decimation, the last to be taken would be the man originally in the 26th place. Hence, with 227 men the last to be taken would be the man originally in the first place.

Modifications of the original problem have been suggested. For instance*, let 5 Christians and 5 Turks be arranged round a circle thus, $TCTCCTCTCT$. Suppose that, if beginning at the ath man, every hth man is selected, all the Turks will be picked out for punishment; but if beginning at the bth man,

* H. E. Dudeney, *Tit-Bits*, London, Oct. 14 and 28, 1905.

every kth man is selected, all the Christians will be picked out for punishment. The problem is to find a, b, h, and k.

I suggest as a similar problem, to find an arrangement of c Turks and c Christians arranged in a circle, so that if beginning at a particular man, say the first, every hth man is selected, all the Turks will be picked out, but if, beginning at the same man, every kth man is selected, all the Christians will be picked out. This makes an interesting question because it is conceivable that the operator who picked out the victims might get confused and take k instead of h, or vice versa, and so consign all his friends to execution instead of those whom he had intended to pick out. The problem is, for any given value of c, to find an arrangement of the men and the corresponding suitable values of h and k. Obviously if $c=2$, then for an arrangement like $T\,C\,C\,T$ a solution is $h=4$, $k=3$. If $c=3$, then for an arrangement like $T\,C\,T\,C\,C\,T$ a solution is $h=7$, $k=8$. If $c=4$, then for an arrangement like $T\,C\,T\,T\,C\,T\,C\,C$ a solution is $h=9$, $k=5$; and so on. Is it possible to give similar arrangements for higher numbers?

ADDENDUM.

Note. *Page* 13. Solutions of the ten digit problems are

$$35/70+148/296=1, \text{ or } \cdot 0123\dot{4}+\cdot 9876\dot{5}=1;$$

and

$$50+49+1/2+38/76=100.$$

A solution of the nine digit problem is

$$1\cdot 23\dot{4}+98\cdot 76\dot{5}=100, \text{ or } 97+8/12+4/6+5/3=100;$$

but if an algebraic sum is permissible a neater solution is

$$123-45-67+89=100,$$

where the digits occur in their natural order.

Note. *Page* 18. There are several solutions of the division of 24 ounces under the conditions specified. One of these solutions is as follows:

The vessels can contain	24 oz.	13 oz.	11 oz.	5 oz.
Their contents originally are...	24 ...	0 ...	0 ...	0 ...
First, make their contents	0 ...	8 ...	11 ...	5 ...
Second, make their contents ...	16 ...	8 ...	0 ...	0 ...
Third, make their contents ...	16 ...	0 ...	8 ...	0 ...
Fourth, make their contents ...	3 ...	13 ...	8 ...	0 ...
Fifth, make their contents......	3 ...	8 ...	8 ...	5 ...
Lastly, make their contents ...	8 ...	8 ...	8 ...	0 ...

Note. *Pages* 26—27. The simplest solution of the five Christians and five Turks problem is $a=1$, $h=11$, $b=9$, $k=29$.

CHAPTER II.

ARITHMETICAL RECREATIONS CONTINUED.

I devote this chapter to the description of some arithmetical fallacies, a few additional problems, and notes on one or two problems in higher arithmetic.

ARITHMETICAL FALLACIES. I begin by mentioning some instances of demonstrations* leading to arithmetical results which are obviously impossible. I include algebraical proofs as well as arithmetical ones. Some of the fallacies are so patent that in preparing the first and second editions I did not think such questions worth printing, but, as some correspondents expressed a contrary opinion, I give them for what they are worth.

First Fallacy. One of the oldest of these—and not a very interesting specimen—is as follows. Suppose that $a = b$, then

$$ab = a^2. \quad \therefore\ ab - b^2 = a^2 - b^2. \quad \therefore\ b(a-b) = (a+b)(a-b).$$

$$\therefore\ b = a + b. \quad \therefore\ b = 2b. \quad \therefore\ 1 = 2.$$

* Of the fallacies given in the text, the first and second are well known; the third is not new, but the earliest work in which I recollect seeing it is my *Algebra*, Cambridge, 1890, p. 430; the fourth is given in G. Chrystal's *Algebra*, Edinburgh, 1889, vol. II, p. 159; the sixth is due to G. T. Walker, and, I believe, has not appeared elsewhere than in this book; the seventh is due to D'Alembert; and the eighth to F. Galton. It may be worth recording (i) that a mechanical demonstration that $1=2$ was given by R. Chartres in *Knowledge*, July, 1891; and (ii) that J. L. F. Bertrand pointed out that a demonstration that $1=-1$ can be obtained from the proposition in the Integral Calculus that, if the limits are constant, the order of integration is indifferent; hence the integral to x (from $x=0$ to $x=1$) of the integral to y (from $y=0$ to $y=1$) of a function ϕ should be equal to the integral to y (from $y=0$ to $y=1$) of the integral to x (from $x=0$ to $x=1$) of ϕ, but if $\phi=(x^2-y^2)/(x^2+y^2)^2$, this gives $\frac{1}{4}\pi = -\frac{1}{4}\pi$.

Second Fallacy. Another example, the idea of which is due to John Bernoulli, may be stated as follows. We have $(-1)^2 = 1$. Take logarithms, $\therefore\ 2\log(-1) = \log 1 = 0$. $\therefore\ \log(-1) = 0$. $\therefore\ -1 = e^0$. $\therefore\ -1 = 1$.

The same argument may be expressed thus. Let x be a quantity which satisfies the equation $e^x = -1$. Square both sides,

$$\therefore\ e^{2x} = 1. \quad \therefore\ 2x = 0. \quad \therefore\ x = 0. \quad \therefore\ e^x = e^0.$$

But $e^x = -1$ and $e^0 = 1$, $\therefore\ -1 = 1$.

The error in each of the foregoing examples is obvious, but the fallacies in the next examples are concealed somewhat better.

Third Fallacy. As yet another instance, we know that

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots.$$

If $x = 1$, the resulting series is convergent; hence we have

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \ldots.$$

$$\therefore\ 2\log 2 = 2 - 1 + \tfrac{2}{3} - \tfrac{1}{2} + \tfrac{2}{5} - \tfrac{1}{3} + \tfrac{2}{7} - \tfrac{1}{4} + \tfrac{2}{9} - \ldots.$$

Taking those terms together which have a common denominator, we obtain

$$\begin{aligned} 2\log 2 &= 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} \ldots \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots\ldots\ldots \\ &= \log 2. \end{aligned}$$

Hence $\qquad 2 = 1.$

Fourth Fallacy. This fallacy is very similar to that last given. We have

$$\begin{aligned} \log 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots \\ &= (1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots) - (\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \ldots) \\ &= \{(1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots) + (\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \ldots)\} - 2(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \ldots) \\ &= \{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\} - (1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots) \\ &= 0. \end{aligned}$$

Fifth Fallacy. We have

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}.$$

Hence $\qquad \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)(-1)},$

therefore, $(\sqrt{-1})^2 = \sqrt{1}$, that is, $-1 = 1$.

Sixth Fallacy. The following demonstration depends on the fact that an algebraical identity is true whatever be the symbols used in it, and it will appeal only to those who are familiar with this fact. We have, as an identity,

$$\sqrt{x-y} = i\sqrt{y-x} \quad \text{.....................(i)},$$

where i stands either for $+\sqrt{-1}$ or for $-\sqrt{-1}$. Now an identity in x and y is necessarily true whatever numbers x and y may represent. First put $x=a$ and $y=b$,

$$\therefore \sqrt{a-b} = i\sqrt{b-a} \quad \text{..................(ii)}.$$

Next put $x=b$ and $y=a$,

$$\therefore \sqrt{b-a} = i\sqrt{a-b} \quad \text{..................(iii)}.$$

Also since (i) is an identity, it follows that in (ii) and (iii) the symbol i must be the same, that is, it represents $+\sqrt{-1}$ or $-\sqrt{-1}$ in both cases. Hence, from (ii) and (iii), we have

$$\sqrt{a-b}\sqrt{b-a} = i^2\sqrt{b-a}\sqrt{a-b},$$

$$\therefore 1 = i^2,$$

that is, $$1 = -1.$$

Seventh Fallacy. The following fallacy is due to D'Alembert*. We know that if the product of two numbers is equal to the product of two other numbers, the numbers will be in proportion, and from the definition of a proportion it follows that if the first term is greater than the second, then the third term will be greater than the fourth: thus, if $ad=bc$, then $a:b=c:d$, and if in this proportion $a>b$, then $c>d$. Now if we put $a=d=1$ and $b=c=-1$ we have four numbers which satisfy the relation $ad=bc$ and such that $a>b$; hence, by the proposition, $c>d$, that is, $-1>1$, which is absurd.

Eighth Fallacy. The mathematical theory of probability leads to various paradoxes: of these one specimen† will suffice. Suppose three coins to be thrown up and the fact whether each comes down head or tail to be noticed. The probability that all three coins come down head is clearly $(1/2)^3$, that is, is $1/8$;

* *Opuscules Mathématiques*, Paris, 1761, vol. I, p. 201.

† See *Nature*, Feb. 15, March 1, 1894, vol. XLIX, pp. 365—366, 413.

similarly the probability that all three come down tail is 1/8; hence the probability that all the coins come down alike (*i.e.* either all of them heads or all of them tails) is 1/4. But, of three coins thus thrown up, at least two must come down alike: now the probability that the third coin comes down head is 1/2 and the probability that it comes down tail is 1/2, thus the probability that it comes down the same as the other two coins is 1/2: hence the probability that all the coins come down alike is 1/2. I leave to my readers to say whether either of these conflicting conclusions is right, and, if so, which is correct.

Arithmetical Problems. To the above examples I may add the following standard questions, or recreations.

The first of these questions is as follows. Two clerks, A and B, are engaged, A at a salary commencing at the rate of (say) £100 a year with a rise of £20 every year, B at a salary commencing at the same rate of £100 a year with a rise of £5 every half-year, in each case payments being made half-yearly; which has the larger income? The answer is B; for in the first year A receives £100, but B receives £50 and £55 as his two half-yearly payments and thus receives in all £105. In the second year A receives £120, but B receives £60 and £65 as his two half-yearly payments and thus receives in all £125. In fact B will always receive £5 a year more than A.

Another simple arithmetical problem is as follows. A hymn-board in a church has four grooved rows on which the numbers of four hymns chosen for the service are placed. The hymn-book in use contains 700 hymns. What is the smallest number of single figured numerals which must be kept in stock so that the numbers of any four different hymns selected can be displayed? How will the result be affected if an inverted 6 can be used for a 9? The answers are 86 and 81.

As another question take the following. A man bets $1/n$th of his money on an even chance (say tossing heads or tails with a penny): he repeats this again and again, each time betting $1/n$th of all the money then in his possession. If, finally, the number of times he has won is equal to the number of times he has lost, has he gained or lost by the transaction? He has, in fact, lost.

Here is another simple question to which not unfrequently I have received incorrect answers. One tumbler is half-full of wine, another is half-full of water: from the first tumbler a teaspoonful of wine is taken out and poured into the tumbler containing the water: a teaspoonful of the mixture in the second tumbler is then transferred to the first tumbler. As the result of this double transaction, is the quantity of wine removed from the first tumbler greater or less than the quantity of water removed from the second tumbler? In my experience the majority of people will say it is greater, but this is not the case.

Here is another paradox dependent on the mathematical theory of probability. Suppose that a player at bridge or whist asserts that an ace is included among the thirteen cards dealt to him, and let p be the probability that he has another ace among the other cards in his hand. Suppose, however, that he asserts that the ace of hearts is included in the thirteen cards dealt to him, then the probability, q, that he has another ace among the other cards in his hand is greater than was the probability p in the first case. For, if r is the probability that when he has one ace it is the ace of hearts, we have $p = r \cdot q$, and since p, q, r are proper fractions, we must have q greater than p, which at first sight appears to be absurd.

Permutation Problems. Many of the problems in permutations and combinations are of considerable interest. As a simple illustration of the very large number of ways in which combinations of even a few things can be arranged, I may note that there are 500,291833 different ways in which change for a sovereign can be given in current coins*, including therein the obsolescent double-florin, and crown; also that as many as 19,958400 distinct skeleton cubes can be formed with twelve differently coloured rods of equal length†; while there are no less than $(52!)/(13!)^4$, that is, 53644,737765,488792,839237,440000

* *The Tribune*, September 3, 1906.

† *Mathematical Tripos*, Cambridge, Part I, 1894.

possible different distributions of hands at bridge or whist with a pack of fifty-two cards.

Voting Problems. As a simple example on combinations I take the cumulative vote as affecting the representation of a minority. If there are p electors each having r votes of which not more than s may be given to one candidate, and n men are to be elected, then the least number of supporters who can secure the election of a candidate must exceed $pr/(ns+r)$.

The Knights of the Round Table. A far more difficult permutation problem consists in finding as many arrangements as possible of n people in a ring so that no one has the same two neighbours more than once. It is a well-known proposition that n persons can be arranged in a ring in $(n-1)!/2$ different ways. The number of these arrangements in which all the persons have different pairs of neighbours on each occasion cannot exceed $(n-1)(n-2)/2$, since this gives the number of ways in which any assigned person may sit between every possible pair selected from the rest. But in fact it is always possible to determine $(n-1)(n-2)/2$ arrangements in which no one has the same two neighbours on any two occasions.

Solutions for various values of n have been given. Here for instance (if $n=8$) are 21 arrangements* of eight persons. Each arrangement may be placed round a circle, and no one has the same two neighbours on any two occasions.

1.2.3.4.5.6.7.8; 1.2.5.6.8.7.4.3; 1.2.7.8.4.3.5.6;
1.3.5.2.7.4.8.6; 1.3.7.4.6.8.2.5; 1.3.8.6.2.5.7.4;
1.4.2.6.3.8.5.7; 1.4.3.8.7.5.6.2; 1.4.5.7.6.2.3.8;
1.5.6.4.3.7.8.2; 1.5.7.3.8.2.6.4; 1.5.8.2.4.6.3.7;
1.6.2.7.5.3.8.4; 1.6.3.5.8.4.2.7; 1.6.4.8.2.7.3.5;
1.7.4.2.5.8.6.3; 1.7.6.3.2.4.5.8; 1.7.8.5.6.3.4.2;
1.8.2.3.7.6.4.5; 1.8.4.5.3.2.7.6; 1.8.6.7.4.5.2.3.

The methods of determining these arrangements are lengthy, and far from easy.

* Communicated to me by Mr E. G. B. Bergholt, May, 1906, see *The Secretary* and *The Queen*, August, 1906. Mr Dudeney had given the problem for the case when $n=6$ in 1905, and informs me that the problem has been solved by Mr E. D. Bewley when n is even, and that he has a general method applicable when n is odd. Various memoirs on the subject have appeared in the mathematical journals.

The Ménage Problem*. Another difficult permutation problem is concerned with the number x of possible arrangements of n married couples, seated alternately man and woman, round a table, the n wives being in assigned positions, and the n husbands so placed that a man does not sit next to his wife.

The solution involves the theory of discordant permutations†, and is far from easy. I content myself with noting the results when n does not exceed 10. When $n=3$, $x=1$; when $n=4$, $x=2$; when $n=5$, $x=13$; when $n=6$, $x=80$; when $n=7$, $x=579$; when $n=8$, $x=4738$; when $n=9$, $x=43387$; and when $n=10$, $x=439792$.

Bachet's Weights Problem‡. It will be noticed that a considerable number of the easier problems given in the last chapter either are due to Bachet or were collected by him in his classical *Problèmes*. Among the more difficult problems proposed by him was the determination of the least number of weights which would serve to weigh any integral number of pounds from 1 lb. to 40 lbs. inclusive. Bachet gave two solutions: namely, (i) the series of weights of 1, 2, 4, 8, 16, and 32 lbs.; (ii) the series of weights of 1, 3, 9, and 27 lbs.

If the weights may be placed in only one of the scale-pans, the first series gives a solution, as had been pointed out in 1556 by Tartaglia§.

Bachet, however, assumed that any weight might be placed in either of the scale-pans. In this case the second series gives the least possible number of weights required. His reasoning is as follows. To weigh 1 lb. we must have a 1 lb. weight. To weigh 2 lbs. we must have in addition either a 2 lb. weight or a 3 lb. weight; but, whereas with a 2 lb. weight we can weigh 1 lb., 2 lbs., and 3 lbs., with a 3 lb. weight we can weigh 1 lb., $(3-1)$ lbs., 3 lbs., and $(3+1)$ lbs. Another weight of 9 lbs. will enable us to weigh all weights from 1 lb. to 13 lbs.; and we get thus a greater range than is obtainable with any

* *Théorie des Nombres*, by E. Lucas, Paris, 1891, pp. 215, 491—495.

† See P. A. MacMahon, *Combinatory Analysis*, vol. I, Cambridge, 1915, pp. 253—256.

‡ Bachet, Appendix, problem v, p. 215.

§ *Trattato de' numeri e misure*, Venice, 1556, vol. II, bk. I, chap. XVI, art. 32.

weight less than 9 lbs. Similarly weights of 1, 3, 9, and 27 lbs. suffice for all weights up to 40 lbs., and weights of 1, 3, 3^2, ..., 3^{n-1} lbs. enable us to weigh any integral number of pounds from 1 lb. to $(1 + 3 + 3^2 + \dots 3^{n-1})$ lbs., that is, to $\frac{1}{2}(3^n - 1)$ lbs.

To determine the arrangement of the weights to weigh any given mass we have only to express the number of pounds in it as a number in the ternary scale of notation, except that in finding the successive digits we must make every remainder either 0, 1, or -1: to effect this a remainder 2 must be written as $3-1$, that is, the quotient must be increased by unity, in which case the remainder is -1. This is explained in most text-books on algebra.

Bachet's argument does not prove that his result is unique or that it gives the least possible number of weights required. These omissions have been supplied by Major MacMahon, who has discussed the far more difficult problem (of which Bachet's is a particular case) of the determination of all possible sets of weights, not necessarily unequal, which enable us to weigh any integral number of pounds from 1 to n inclusive, (i) when the weights may be placed in only one scale-pan, and (ii) when any weight may be placed in either scale-pan. He has investigated also the modifications of the results which are necessary when we impose either or both of the further conditions (a) that no other weighings are to be possible, and (b) that each weighing is to be possible in only one way, that is, is to be unique*.

The method for case (i) consists in resolving $1 + x + x^2 + \dots + x^n$ into factors, each factor being of the form $1 + x^a + x^{2a} + \dots + x^{ma}$; the number of solutions depends on the composite character of $n + 1$. The method for case (ii) consists in resolving the expression $x^{-n} + x^{-n+1} + \dots + x^{-1} + 1 + x + \dots + x^{n-1} + x^n$ into factors, each factor being of the form $x^{-ma} + \dots + x^{-a} + 1 + x^a + \dots + x^{ma}$; the number of solutions depends on the composite character of $2n + 1$.

* See his article in the *Quarterly Journal of Mathematics*, 1886, vol. XXI, pp. 367—373. An account of the method is given in *Nature*, Dec. 4, 1890, vol. XLII, pp. 113—114.

Bachet's problem falls under case (ii), $n=40$. MacMahon's analysis shows that there are eight such ways of factorizing $x^{-40}+x^{-39}+\ldots+1+\ldots+x^{39}+x^{40}$. First, there is the expression itself in which $a=1$, $m=40$. Second, the expression is equal to $(1-x^{81})/x^{40}(1-x)$, which can be resolved into the product of $(1-x^3)/x(1-x)$ and $(1-x^{81})/x^{39}(1-x^3)$; hence it can be resolved into two factors of the form given above, in one of which $a=1$, $m=1$, and in the other $a=3$, $m=13$. Third, similarly, it can be resolved into two such factors, in one of which $a=1$, $m=4$, and in the other $a=9$, $m=4$. Fourth, it can be resolved into three such factors, in one of which $a=1$, $m=1$, in another $a=3$, $m=1$, and in the other $a=9$, $m=4$. Fifth, it can be resolved into two such factors, in one of which $a=1$, $m=13$, and in the other $a=27$, $m=1$. Sixth, it can be resolved into three such factors, in one of which $a=1$, $m=1$, in another $a=3$, $m=4$, and in the other $a=27$, $m=1$. Seventh, it can be resolved into three such factors, in one of which $a=1$, $m=4$, in another $a=9$, $m=1$, and in the other $a=27$, $m=1$. Eighth, it can be resolved into four such factors, in one of which $a=1$, $m=1$, in another $a=3$, $m=1$, in another $a=9$, $m=1$, and in the other $a=27$, $m=1$.

These results show that there are eight possible sets of weights with which any integral number of pounds from 1 to 40 can be weighed subject to the conditions (ii), (a), and (b). If we denote p weights each equal to w by w^p, these eight solutions are 1^{40}; $1, 3^{13}$; $1^4, 9^4$; $1, 3, 9^4$; $1^{13}, 27$; $1, 3^4, 27$; $1^4, 9, 27$; $1, 3, 9, 27$. The last of these is Bachet's solution: not only is it that in which the least number of weights are employed, but it is also the only one in which all the weights are unequal.

Problems in Higher Arithmetic. Many mathematicians take a special interest in the theorems of higher arithmetic: such, for example, as that every prime of the form $4n+1$ and every power of it is expressible as the sum of two squares*, and that the first and second powers can be expressed thus in only one way. For instance, $13=3^2+2^2$, $13^2=12^2+5^2$, $13^3=46^2+9^2$,

* Fermat's *Diophantus*, Toulouse, 1670, bk. III, prop. 22, p. 127; or Brassinne's *Précis*, Paris, 1853, p. 65.

and so on. Similarly $41=5^2+4^2$, $41^2=40^2+9^2$, $41^3=236^2+115^2$, and so on. Propositions such as the one just quoted may be found in text-books on the theory of numbers and therefore lie outside the limits of this work, but there are one or two questions in higher arithmetic involving points not yet quite cleared up which may find a place here. I content myself with the facts and shall not give the mathematical analysis.

PRIMES. The first of these is concerned with the possibility of determining readily whether a given number is prime or not. No test applicable to all numbers is known, though of course we can get tests for numbers of certain forms. It is difficult to believe that a problem which has completely baffled all modern mathematicians could have been solved in the seventeenth century, but it is interesting to note that in 1643, in answer to a question in a letter whether the number 100895,598169 was a prime, Fermat replied at once that it was the product of 898423 and 112303, both of which were primes. How many mathematicians to-day could answer such a question with ease?

MERSENNE'S NUMBERS*. Another illustration, confirmatory of the opinion that Fermat or some of his contemporaries had a test by which it was possible to find out whether numbers of certain forms were prime, may be drawn from Mersenne's *Cogitata Physico-Mathematica* which was published in 1644. In the preface to that work it is asserted that in order that 2^p-1 may be prime, the only values of p, not greater than 257, which are possible are 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257. To these numbers 89 and 107 must be added. Some years ago I gave reasons for thinking that the number 67 is a misprint for 61. With these corrections the statement appears to be true, and it has now been verified for all except fourteen values of p: namely, 101, 103, 109, 137, 139, 149, 157, 167, 193, 199, 227, 229, 241, and 257. Of these values, Mersenne asserted that $p=257$ makes 2^p-1 a prime, and that the other values

* For references, see chapter XVI below.

make 2^p-1 a composite number. The demonstration of the prime character of 2^p-1 when $p=127$ has not been published: and the verification in this case has not been corroborated by independent work.

Mersenne's result could not have been obtained empirically, and it is impossible to suppose that it was worked out for every case; hence it would seem that whoever first enunciated it was acquainted with certain theorems in higher arithmetic which have not been re-discovered.

PERFECT NUMBERS*. The theory of *perfect numbers* depends directly on that of Mersenne's Numbers. A number is said to be perfect if it is equal to the sum of all its integral subdivisors. Thus the subdivisors of 6 are 1, 2, and 3; the sum of these is equal to 6; hence 6 is a perfect number.

It is probable that all perfect numbers are included in the formula $2^{p-1}(2^p-1)$, where 2^p-1 is a prime. Euclid proved that any number of this form is perfect; Euler showed that the formula includes all even perfect numbers; and there is reason to believe—though a rigid demonstration is wanting—that an odd number cannot be perfect. If we assume that the last of these statements is true, then every perfect number is of the above form. It is easy to establish that every number included in this formula (except when $p=2$) is congruent to unity to the modulus 9, that is, when divided by 9 leaves a remainder 1; also that either the last digit is a 6 or the last two digits are 28.

Thus, if $p=2, 3, 5, 7, 13, 17, 19, 31, 61$, then by Mersenne's rule the corresponding values of 2^p-1 are prime; they are 3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951; and the corresponding perfect numbers are 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, and 2658455991569831744654692615953842176.

* On the theory of perfect numbers, see bibliographical references by H. Brocard, *L'Intermédiaire des Mathématiciens*, Paris, 1895, vol. II, pp. 52—54; and 1905, vol. XII, p. 19. The first volume of the second edition of the French translation of this book contains (pp. 280—294) a summary of the leading investigation on Perfect Numbers, as also some remarks on Amicable Numbers.

EULER'S BIQUADRATE THEOREM *. Another theorem, believed to be true but as yet unproved, is that the arithmetical sum of the fourth powers of three numbers cannot be the fourth power of a number; in other words, we cannot find values of x, y, z, v, which satisfy the equation $x^4 + y^4 + z^4 = v^4$. The proposition is not true of an algebraical sum, for Euler gave more than one solution of the equation $x^4 + y^4 = z^4 + v^4$, for instance, $x = 542$, $y = 103$, $z = 359$, $v = 514$.

GOLDBACH'S THEOREM. Another interesting problem in higher arithmetic is the question whether there are any even integers which cannot be expressed as a sum of two primes. Probably there are none. The expression of all even integers not greater than 5000 in the form of a sum of two primes has been effected †, but a general demonstration that all even integers can be so expressed is wanting.

LAGRANGE'S THEOREM ‡. Another theorem in higher arithmetic which, as far as I know, is still unsolved, is to the effect that every prime of the form $4n - 1$ is the sum of a prime of the form $4n + 1$ and of double another prime also of the form $4n + 1$; for example, $23 = 13 + 2 \times 5$. Lagrange, however, added that it was only by induction that he arrived at the result.

FERMAT'S THEOREM ON BINARY POWERS. Fermat enriched mathematics with a multitude of new propositions. With one exception all these have been proved or are believed to be true. This exception is his *theorem on binary powers*, in which he asserted that all numbers of the form $2^m + 1$, where $m = 2^n$, are primes §, but he added that, though he was convinced of the truth of this proposition, he could not obtain a valid demonstration.

* See Euler, *Commentationes Arithmeticae Collectae*, Petrograd, 1849, vol. I, pp. 473—476; vol. II, pp. 450—456.

† *Transactions of the Halle Academy* (*Naturforschung*), vol. LXXII, Halle, 1897, pp. 5—214: see also *L'Intermédiaire des Mathématiciens*, 1903, vol. X, and 1904, vol. XI.

‡ *Nouveaux Mémoires de l'Académie Royale des Sciences*, Berlin. 1775, p. 356.

§ Letter of Oct. 18, 1640, *Opera*, Toulouse, 1679, p. 162: or Brassinne's *Précis*, p. 143.

It may be shown that $2^m + 1$ is composite if m is not a power of 2, but of course it does not follow that $2^m + 1$ is a prime if m is a power of 2, say, 2^n. As a matter of fact the theorem is not true. In 1732 Euler* showed that if $n = 5$ the formula gives 4294,967297, which is equal to $641 \times 6{,}700417$: curiously enough, these factors can be deduced at once from Fermat's remark on the possible factors of numbers of the form $2^m \pm 1$, from which it may be shown that the prime factors (if any) of $2^{32} + 1$ must be primes of the form $64n + 1$.

During the last thirty years it has been shown† that the resulting numbers are composite when $n = 6, 7, 8, 9, 11, 12, 18, 23, 36, 38$ and 73. The digits in the last of these numbers are so numerous that, if the number were printed in full with the type and number of pages used in this book, many more volumes would be required than are contained in all the public libraries in the world. I believe that Eisenstein asserted that the number of primes of the form $2^m + 1$, where $m = 2^n$, is infinite: the proof has not been published, but perhaps it might throw some light on the general theory.

FERMAT'S LAST THEOREM. I pass now to another assertion made by Fermat which hitherto has not been proved. This, which is sometimes known as *Fermat's Last Theorem*, is to the effect‡ that no integral values of x, y, z can be found to satisfy the equation $x^n + y^n = z^n$, if n is an integer greater than 2.

* *Commentarii Academiae Scientiarum Petropolitanae*, Petrograd, 1738, vol. VI, p. 104; see also *Novi Comm. Acad. Sci. Petrop.*, Petrograd, 1764, vol. IX, p. 101: or *Commentationes Arithmeticae Collectae*, Petrograd, 1849, vol. I, pp. 2, 357.

† For the factors and bibliographical references, see A. J. C. Cunningham and A. E. Western, *Transactions of the London Mathematical Society*, 1903, series 2, vol. I, p. 175; and J. C. Morehead and A. E. Western, *Bulletin of the American Mathematical Society*, 1909, vol. XVI, pp. 1—6.

‡ Fermat's enunciation will be found in his edition of *Diophantus*, Toulouse, 1670, bk. II, qu. 8, p. 61; or Brassinne's *Précis*, Paris, 1853, p. 53. For bibliographical references, see the article on the theory of numbers in the *Encyclopédie des Sciences Mathématiques*: considerable additions are embodied in the French translation of this book which is therefore generally preferable to the German original. See also *L'Intermédiaire des Mathématiciens*, Paris, 1908, vol. XV, p. 234.

This proposition has acquired extraordinary celebrity from the fact that no general demonstration of it has been given, but there is no reason to doubt that it is true.

Fermat seems to have discovered its truth first* for the case $n=3$, and then for the case $n=4$. His proof for the former of these cases is lost, but that for the latter is extant†, and a similar proof for the case of $n=3$ was given by Euler‡. These proofs depend upon showing that, if three integral values of x, y, z can be found which satisfy the equation, then it will be possible to find three other and smaller integers which also satisfy it: in this way finally we show that the equation must be satisfied by three values which obviously do not satisfy it. Thus no integral solution is possible. It would seem that this method is inapplicable except when $n=3$ and $n=4$.

Fermat's discovery of the general theorem was made later. A demonstration can be given on the assumption that every number can be resolved in one and only one way into the product of primes and their powers. This assumption is true of real numbers, but it is not necessarily true when complex factors are admitted. For instance 21 can be expressed as the product of 3 and 7, or of $4+\sqrt{(-5)}$ and $4-\sqrt{(-5)}$, or of $1+2\sqrt{(-5)}$ and $1-2\sqrt{(-5)}$. It is possible that Fermat made some such erroneous supposition, though it is perhaps more probable that he discovered a rigorous demonstration. At any rate he asserts definitely that he had a valid proof—demonstratio mirabilis sane—and the fact that no theorem on the subject which he stated he had proved has been subsequently shown to be false must weigh strongly in his favour; the more so because in making the one incorrect statement in his writings (namely, that about binary powers) he added that he could not obtain a satisfactory demonstration of it.

It must be remembered that Fermat was a mathematician of quite the first rank who had made a special study of the theory of numbers. The subject is in itself one of peculiar

* See a Letter from Fermat quoted in my *History of Mathematics*, London, chapter xv.

† Fermat's *Diophantus*, note on p. 339; or Brassinne's *Précis*, p. 127.

‡ Euler's *Algebra* (English trans. 1797), vol. II, chap. xv, p. 247.

interest and elegance, but its conclusions have little practical importance, and since his time it has been discussed by only a few mathematicians, while even of them not many have made it their chief study. This is the explanation of the fact that it took more than a century before some of the simpler results which Fermat had enunciated were proved, and thus it is not surprising that a proof of the theorem which he succeeded in establishing only towards the close of his life should involve great difficulties.

In 1823 Legendre* obtained a proof for the case of $n=5$; in 1832 Lejeune Dirichlet† gave one for $n=14$; and in 1840 Lamé and Lebesgue‡ gave proofs for $n=7$.

The proposition appears to be true universally, and in 1849 Kummer§, by means of ideal primes, proved it to be so for all numbers except those (if any) which satisfy three conditions. The proof is complicated and difficult, and there can be little doubt is based on considerations unknown to Fermat. It is not known whether any number can be found to satisfy these conditions. It was shown a considerable time ago, that there is no number less than 100 which does so, and recently the limit has been extended by L. E. Dickson‖. But mere numerical verifications have little value; no one doubts the truth of the theorem, and its interest lies in the fact that we have not yet succeeded in obtaining a rigorous general demonstration of it. The general problem was also attacked on other lines by Sophie Germain, who showed that it was true for all numbers except those (if any) which satisfied certain defined conditions. I may add that to prove the truth of the proposition when n is greater than 4, obviously it is sufficient to confine

* Reprinted in his *Théorie des Nombres*, Paris, 1830, vol. II, pp. 361—368: see also pp. 5, 6.

† *Crelle's Journal*, 1832, vol. IX, pp. 390—393.

‡ *Liouville's Journal*, 1841, vol. V, pp. 195—215, 276—279, 348—349.

§ References to Kummer's Memoirs are given in Smith's Report to the British Association on the Theory of Numbers, London, 1860.

‖ See *L'Intermédiaire des Mathématiciens*, Paris, 1908, vol. XV, pp. 247—248; *Messenger of Mathematics*, Cambridge, 1908, vol. XXXVIII, pp. 14—32; and *Quarterly Journal of Mathematics*, Cambridge, 1908, vol. XL, pp. 27—45.

ourselves to cases where n is a prime. A prize * of 100,000 marks has been offered for a general proof, to be given before 2007.

Naturally there has been much speculation as to how Fermat arrived at the result. The modern treatment of higher arithmetic is founded on the special notation and processes introduced by Gauss, who pointed out that the theory of discrete magnitude is essentially different from that of continuous magnitude, but until the end of the last century the theory of numbers was treated as a branch of algebra, and such proofs by Fermat as are extant involve nothing more than elementary geometry and algebra, and indeed some of his arguments do not involve any symbols. This has led some writers to think that Fermat used none but elementary algebraic methods. This may be so, but the following remark, which I believe is not generally known, rather points to the opposite conclusion. He had proposed, as a problem to the English mathematicians, to show that there was only one integral solution of the equation $x^2 + 2 = y^3$: the solution evidently being $x = 5$, $y = 3$. On this he has a note † to the effect that there was no difficulty in finding a solution in rational fractions, but that he had discovered an entirely new method—sane pulcherrima et subtilissima—which enabled him to solve such questions in integers. It was his intention to write a work ‡ on his researches in the theory of numbers, but it was never completed, and we know but little of his methods of analysis. I venture however to add my private suspicion that continued fractions played a not unimportant part in his researches, and as strengthening this conjecture I may note that some of his more recondite results—such as the theorem that a prime of the form $4n + 1$ is expressible as the sum of two squares—may be established with comparative ease by properties of such fractions.

* *L'Intermédiaire des Mathématiciens*, vol. xv, pp. 217—218, for references and details.

† Fermat's *Diophantus*, bk. vi, prop. 19, p. 320; or Brassinne's *Précis*, p. 122.

‡ Fermat's *Diophantus*, bk. iv, prop. 31, p. 181; or Brassinne's *Précis*, p. 82.

CHAPTER III.

GEOMETRICAL RECREATIONS.

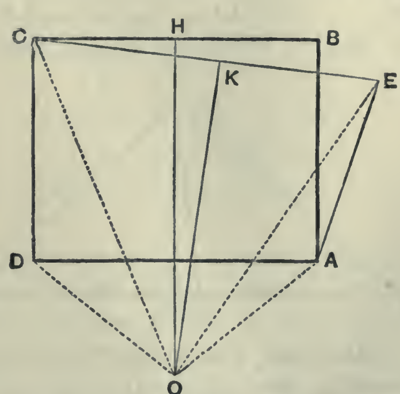
In this chapter and the next one I propose to enumerate certain geometrical questions the discussion of which will not involve necessarily any considerable use of algebra or arithmetic. Unluckily no writer like Bachet has collected and classified problems of this kind, and I take the following instances from my note-books with the feeling that they represent the subject but imperfectly. Most of this chapter is devoted to questions which are of the nature of formal propositions: the next chapter contains a description of various trivial puzzles and games, which the older writers would have termed geometrical.

In accordance with the rule I laid down for myself in the preface, I exclude the detailed discussion of theorems which involve advanced mathematics. Moreover (with one or two exceptions) I exclude any mention of the numerous geometrical paradoxes which depend merely on the inability of the eye to compare correctly the dimensions of figures when their relative position is changed. This apparent deception does not involve the conscious reasoning powers, but rests on the inaccurate interpretation by the mind of the sensations derived through the eyes, and I do not consider such paradoxes as coming within the domain of mathematics.

Geometrical Fallacies. Most educated Englishmen are acquainted with the series of logical propositions in geometry associated with the name of Euclid, but it is not known so generally that these propositions were supplemented originally by certain exercises. Of such exercises Euclid issued three series: two containing easy theorems or problems, and the third consisting of geometrical fallacies, the errors in which the student was required to find.

The collection of fallacies prepared by Euclid is lost, and tradition has not preserved any record as to the nature of the erroneous reasoning or conclusions; but, as an illustration of such questions, I append a few demonstrations, leading to obviously impossible results. Perhaps they may amuse any one to whom they are new. I leave the discovery of the errors to the ingenuity of my readers.

First Fallacy. To prove that a right angle is equal to an angle which is greater than a right angle.* Let $ABCD$ be a rectangle. From A draw a line AE outside the rectangle, equal to AB or DC and making an acute angle with AB, as



indicated in the diagram. Bisect CB in H, and through H draw HO at right angles to CB. Bisect CE in K, and through K draw KO at right angles to CE. Since CB and CE are not parallel the lines HO and KO will meet (say) at O. Join OA, OE, OC, and OD.

The triangles ODC and OAE are equal in all respects. For, since KO bisects CE and is perpendicular to it, we have

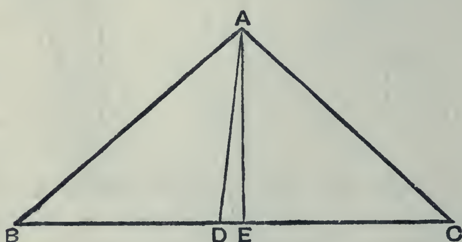
* I believe that this and the fourth of these fallacies were first published in this book. They particularly interested Mr C. L. Dodgson; see the *Lewis Carroll Picture Book*, London, 1899, pp. 264, 266, where they appear in the form in which I originally gave them.

$OC = OE$. Similarly, since HO bisects CB and DA and is perpendicular to them, we have $OD = OA$. Also, by construction, $DC = AE$. Therefore the three sides of the triangle ODC are equal respectively to the three sides of the triangle OAE. Hence, by Euc. I. 8, the triangles are equal; and therefore the angle ODC is equal to the angle OAE.

Again, since HO bisects DA and is perpendicular to it, we have the angle ODA equal to the angle OAD.

Hence the angle ADC (which is the difference of ODC and ODA) is equal to the angle DAE (which is the difference of OAE and OAD). But ADC is a right angle, and DAE is necessarily greater than a right angle. Thus the result is impossible.

*Second Fallacy**. *To prove that a part of a line is equal to the whole line.* Let ABC be a triangle; and, to fix our ideas, let us suppose that the triangle is scalene, that the angle B is



acute, and that the angle A is greater than the angle C. From A draw AD making the angle BAD equal to the angle C, and cutting BC in D. From A draw AE perpendicular to BC.

The triangles ABC, ABD are equiangular; hence, by Euc. VI. 19,

$$\triangle ABC : \triangle ABD = AC^2 : AD^2.$$

Also the triangles ABC, ABD are of equal altitude; hence, by Euc. VI. 1,

$$\triangle ABC : \triangle ABD = BC : BD,$$

$$\therefore AC^2 : AD^2 = BC : BD.$$

$$\therefore \frac{AC^2}{BC} = \frac{AD^2}{BD}.$$

* See a note by M. Coccoz in *L'Illustration*, Paris, Jan. 12, 1895.

Hence, by Euc. II. 13,

$$\frac{AB^2+BC^2-2BC.BE}{BC}=\frac{AB^2+BD^2-2BD.BE}{BD};$$

$$\therefore \frac{AB^2}{BC}+BC-2BE=\frac{AB^2}{BD}+BD-2BE.$$

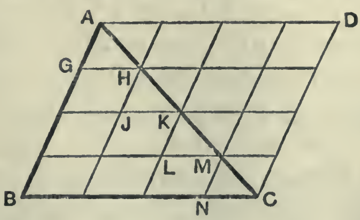
$$\therefore \frac{AB^2}{BC}-BD=\frac{AB^2}{BD}-BC.$$

$$\therefore \frac{AB^2-BC.BD}{BC}=\frac{AB^2-BC.BD}{BD}.$$

$$\therefore BC=BD,$$

a result which is impossible.

*Third Fallacy**. *To prove that the sum of the lengths of two sides of any triangle is equal to the length of the third side.*



Let ABC be a triangle. Complete the parallelogram of which AB and BC are sides. Divide AB into $n+1$ equal parts, and through the points so determined draw n lines parallel to BC. Similarly, divide BC into $n+1$ equal parts, and through the points so determined draw n lines parallel to AB. The parallelogram $ABCD$ is thus divided into $(n+1)^2$ equal and similar parallelograms.

I draw the figure for the case in which n is equal to 3, then, taking the parallelograms of which AC is a diagonal, as indicated in the diagram, we have

$$AB+BC=AG+HJ+KL+MN$$
$$+GH+JK+LM+NC.$$

A similar relation is true however large n may be. Now let n increase indefinitely. Then the lines AG, GH, &c. will

* *The Canterbury Puzzles*, by H. E. Dudeney, London, 1907, pp. 26—28.

get smaller and smaller. Finally the points G, J, L, ... will approach indefinitely near the line AC, and ultimately will lie on it; when this is the case the sum of AG and GH will be equal to AH, and similarly for the other similar pairs of lines. Thus, ultimately,

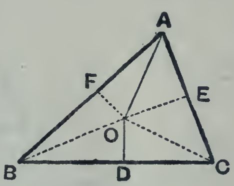
$$AB + BC = AH + HK + KM + MC$$
$$= AC,$$

a result which is impossible.

Fourth Fallacy. To prove that every triangle is isosceles. Let ABC be any triangle. Bisect BC in D, and through D draw DO perpendicular to BC. Bisect the angle BAC by AO.

First. If DO and AO do not meet, then they are parallel. Therefore AO is at right angles to BC. Therefore $AB = AC$.

Second. If DO and AO meet, let them meet in O. Draw OE perpendicular to AC. Draw OF perpendicular to AB. Join OB, OC.



Let us begin by taking the case where O is inside the triangle, in which case E falls on AC and F on BC.

The triangles AOF and AOE are equal, since the side AO is common, angle OAF = angle OAE, and angle OFA = angle OEA. Hence $AF = AE$. Also, the triangles BOF and COE are equal. For since OD bisects BC at right angles, we have $OB = OC$; also, since the triangles AOF and AOE are equal, we have $OF = OE$; lastly, the angles at F and E are right angles. Therefore, by Euc. I. 47 and I. 8, the triangles BOF and COE are equal. Hence $FB = EC$.

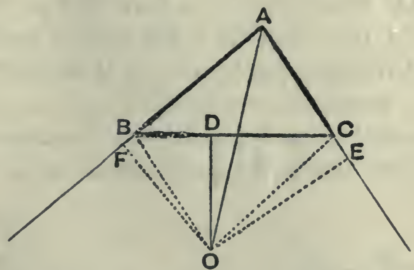
Therefore $AF + FB = AE + EC$, that is, $AB = AC$.

The same demonstration will cover the case where DO and AO meet at D, as also the case where they meet outside BC but so near it that E and F fall on AC and AB and not on AC and AB produced.

Next take the case where DO and AO meet outside the triangle, and E and F fall on AC and AB produced. Draw

OE perpendicular to AC produced. Draw OF perpendicular to AB produced. Join OB, OC.

Following the same argument as before, from the equality of the triangles AOF and AOE, we obtain $AF = AE$; and, from the equality of the triangles BOF and COE, we obtain $FB = EC$. Therefore $AF - FB = AE - EC$, that is, $AB = AC$.



Thus in all cases, whether or not DO and AO meet, and whether they meet inside or outside the triangle, we have $AB = AC$: and therefore every triangle is isosceles, a result which is impossible.

*Fifth Fallacy**. *To prove that* $\pi/4$ *is equal to* $\pi/3$. On the hypothenuse, BC, of an isosceles right-angled triangle, DBC, describe an equilateral triangle ABC, the vertex A being on the same side of the base as D is. On CA take a point H so that $CH = CD$. Bisect BD in K. Join HK and let it cut CB (produced) in L. Join DL. Bisect DL at M, and through M draw MO perpendicular to DL. Bisect HL at N, and through N draw NO perpendicular to HL. Since DL and HL intersect, therefore MO and NO will also intersect; moreover, since BDC is a right angle, MO and NO both slope away from DC and therefore they will meet on the side of DL remote from A. Join OC, OD, OH, OL.

The triangles OMD and OML are equal, hence $OD = OL$. Similarly the triangles ONL and ONH are equal, hence $OL = OH$. Therefore $OD = OH$. Now in the triangles OCD and OCH, we have $OD = OH$, $CD = CH$ (by construction), and

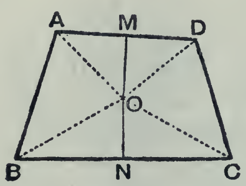
* This ingenious fallacy is due to Captain Turton: it appeared for the first time in the third edition of this work.

OC common, hence (by Euc. I. 8) the angle OCD is equal to the angle OCH. Hence the angle BCD is equal to the angle BCH, that is, $\pi/4$ is equal to $\pi/3$, which is absurd.

*Sixth Fallacy**. *To prove that, if two opposite sides of a quadrilateral are equal, the other two sides must be parallel.* Let $ABCD$ be a quadrilateral such that AB is equal to DC. Bisect AD in M, and through M draw MO at right angles to AD. Bisect BC in N, and draw NO at right angles to BC.

If MO and NO are parallel, then AD and BC (which are at right angles to them) are also parallel.

If MO and NO are not parallel, let them meet in O; then O must be either inside the quadrilateral as in the left-hand



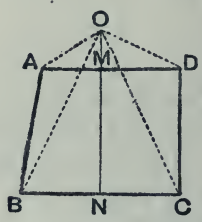


diagram or outside the quadrilateral as in the right-hand diagram. Join OA, OB, OC, OD.

Since OM bisects AD and is perpendicular to it, we have $OA = OD$, and the angle OAM equal to the angle ODM. Similarly $OB = OC$, and the angle OBN is equal to the angle OCN. Also by hypothesis $AB = DC$, hence, by Euc. I. 8, the triangles OAB and ODC are equal in all respects, and therefore the angle AOB is equal to the angle DOC.

Hence in the left-hand diagram the sum of the angles AOM, AOB is equal to the sum of the angles DOM, DOC; and in the right-hand diagram the difference of the angles AOM, AOB is equal to the difference of the angles DOM, DOC; and therefore in both cases the angle MOB is equal to the angle MOC, *i.e.* OM (or OM produced) bisects the angle BOC. But the angle NOB is equal to the angle NOC, *i.e.* ON bisects the angle BOC; hence OM and ON coincide in direction.

* *Mathesis*, October, 1893, series 2, vol. III, p. 224.

Therefore AD and BC, which are perpendicular to this direction, must be parallel. This result is not universally true, and the above demonstration contains a flaw.

Seventh Fallacy. The following argument is taken from a text-book on electricity, published in 1889 by two distinguished mathematicians, in which it was presented as valid. A given vector OP of length l can be resolved in an infinite number of ways into two vectors OM, MP, of lengths l', l'', and we can make l'/l'' have any value we please from nothing to infinity. Suppose that the system is referred to rectangular axes Ox, Oy; and that OP, OM, MP make respectively angles θ, θ', θ'' with Ox. Hence, by projection on Oy and on Ox, we have

$$l \sin\theta = l' \sin\theta' + l'' \sin\theta'',$$

$$l \cos\theta = l' \cos\theta' + l'' \cos\theta''.$$

$$\therefore \tan\theta = \frac{n \sin\theta' + \sin\theta''}{n \cos\theta' + \cos\theta''},$$

where $n = l'/l''$. This result is true whatever be the value of n. But n may have any value (*ex. gr.* $n = \infty$, or $n = 0$), hence $\tan\theta = \tan\theta' = \tan\theta''$, which obviously is impossible.

*Eighth Fallacy**. Here is a fallacious investigation of the value of π: it is founded on well-known quadratures. The area of the semi-ellipse bounded by the minor axis is (in the usual notation) equal to $\frac{1}{2}\pi ab$. If the centre is moved off to an indefinitely great distance along the major axis, the ellipse degenerates into a parabola, and therefore in this particular limiting position the area is equal to two-thirds of the circumscribing rectangle. But the first result is true whatever be the dimensions of the curve.

$$\therefore \tfrac{1}{2}\pi ab = \tfrac{2}{3}a \times 2b,$$

$$\therefore \pi = 8/3,$$

a result which obviously is untrue.

Ninth Fallacy. Every ellipse is a circle. The focal distance of a point on an ellipse is given (in the usual notation) in terms

* This was communicated to me by Mr R. Chartres.

of the abscissa by the formula $r = a + ex$. Hence $dr/dx = e$. From this it follows that r cannot have a maximum or minimum value. But the only closed curve in which the radius vector has not a maximum or minimum value is a circle. Hence, every ellipse is a circle, a result which obviously is untrue.

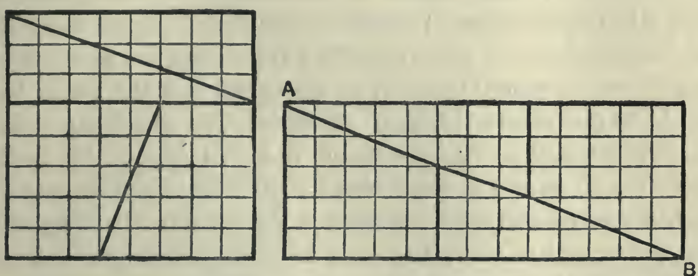
Geometrical Paradoxes. To the above examples I may add the following questions, which, though not exactly fallacious, lead to results which at a hasty glance appear impossible.

First Paradox. The first is a problem, sent to me by Mr W. Renton, to rotate a plane lamina (say, for instance, a sheet of paper) through four right angles so that the effect is equivalent to turning it through only one right angle.

Second Paradox. As in arithmetic, so in geometry, the theory of probability lends itself to numerous paradoxes. Here is a very simple illustration. A stick is broken at random into three pieces. It is possible to put them together into the shape of a triangle provided the length of the longest piece is less than the sum of the other two pieces (*cf.* Euc. I. 20), that is, provided the length of the longest piece is less than half the length of the stick. But the probability that a fragment of a stick shall be half the original length of the stick is 1/2. Hence the probability that a triangle can be constructed out of the three pieces into which the stick is broken would appear to be 1/2. This is not true, for actually the probability is 1/4.

Third Paradox. The following example illustrates how easily the eye may be deceived in demonstrations obtained by actually dissecting the figures and re-arranging the parts. In fact proofs by superposition should be regarded with considerable distrust unless they are supplemented by mathematical reasoning. The well-known proofs of the propositions Euclid I. 32 and Euclid I. 47 can be so supplemented and are valid. On the other hand, as an illustration of how deceptive a non-mathematical proof may be, I here mention the familiar paradox that a square of paper, subdivided like a chessboard into 64 small squares, can be cut into four pieces which being put together form a figure containing 65 such small squares*. This is effected by cutting the original square into four pieces in the manner indicated by the thick lines in the first figure. If these four pieces are put together in the shape of a rectangle in the way shown in the second figure it will appear as if this rectangle contains 65 of the small squares.

This phenomenon, which in my experience non-mathematicians find perplexing, is due to the fact that the edges of the four pieces of paper, which in the second figure lie along



the diagonal AB, do not coincide exactly in direction. In reality they include a small lozenge or diamond-shaped figure, whose area is equal to that of one of the 64 small squares in the original square, but whose length AB is much greater than its breadth. The diagrams show that the angle between the two sides of this lozenge which meet at A is $\tan^{-1}\frac{2}{5} - \tan^{-1}\frac{3}{8}$, that is, is $\tan^{-1}\frac{1}{46}$, which is less than $1\frac{1}{4}°$. To enable the eye to distinguish so small an angle as this the dividing lines in the first figure would have to be cut with extreme accuracy and the pieces placed together with great care.

This paradox depends upon the relation $5 \times 13 - 8^2 = 1$. Similar results can be obtained from the formulae

$$13 \times 34 - 21^2 = 1,\ 34 \times 89 - 55^2 = 1, \ldots;$$

or from the formulae

$$5^2 - 3 \times 8 = 1,\ 13^2 - 8 \times 21 = 1,\ 34^2 - 21 \times 55 = 1, \ldots.$$

* I do not know who discovered this paradox. It is given in various modern books, but I cannot find an earlier reference to it than one in the *Zeitschrift für Mathematik und Physik*, Leipzig, 1868, vol. XIII, p. 162. Some similar paradoxes were given by Ozanam, 1803 edition, vol. I, p. 299.

These numbers are obtained by finding convergents to the continued fraction

$$1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots.$$

Dissection Problems. The above paradoxes naturally suggest the consideration of dissection problems. An excellent typical example is to cut a square into 20 equal triangles, and conversely to construct a square of 20 such triangles.

There is an interesting historical example of such a problem. Two late Latin writers, Victorinus and Fortunatianus, describe an Archimedean toy composed of 14 ivory polygons which fitted exactly into a square box, and they suggest that the puzzle was to fit the pieces into the box. A recent discovery* has shown that its association with the name of Archimedes is due to the fact that he gave a construction for dividing a square into 14 such pieces (namely, 11 triangles, 2 scalene quadrilaterals, and one pentagon) so that the area of each piece is a rational fraction of the area of the square. His construction is as follows: let $ABCD$ be the square, and E, F, G, H, the mid-points of the sides AB, BC, CD, DA. Draw HB, HF, HC, and let J, K, L be the mid-points of these lines: draw AKC cutting HB in M, and let N be the mid-point of AM, and P the mid-point of BF. Draw BN. Draw AP cutting HB in Q. Draw PJ. Draw BL, and produce it to cut DC in R. Draw FL cutting AC in S. Draw LG. Rub out the lines AQ and BL. The remaining lines will give a division as required, each figure being an integral multiple of 1/48th of the square. Why Archimedes propounded so peculiar a division it is impossible to guess, but no doubt the problem has a history of which we are ignorant.

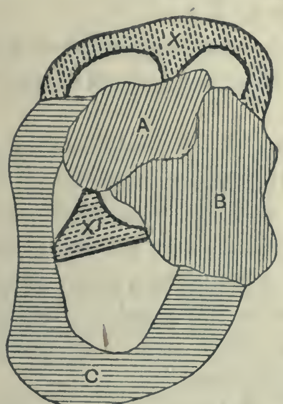
Colouring Maps. I proceed next to mention the geometrical proposition that *not more than four colours are necessary in order to colour a map of a country (divided into districts)*

* H. Suter, *Zeitschrift für Mathematik und Physik, Abhandlungen zur Gesch. der Math.* 1899, vol. XLIV, pp. 491—499.

in such a way that no two contiguous districts shall be of the same colour. By contiguous districts are meant districts having a common *line* as part of their boundaries: districts which touch only at points are not contiguous in this sense.

The problem was mentioned by A. F. Möbius* in his Lectures in 1840, but it was not until Francis Guthrie† communicated it to De Morgan about 1850 that attention was generally called to it: it is said that the fact had been familiar to practical map-makers for a long time previously. Through De Morgan the proposition then became generally known; and in 1878 Cayley‡ recalled attention to it by stating that he could not obtain a rigorous proof of it.

Probably the following argument, though not a formal demonstration, will satisfy the reader that the result is true.



Let A, B, C be three contiguous districts, and let X be any other district contiguous with all of them. Then X must lie either wholly outside the external boundary of the area ABC or wholly inside the internal boundary, that is, it must occupy a position either like X or like X'. In either case there is no possible way of drawing another area Y which shall be contiguous with A, B, C, and X. In other words, it is possible to draw on a plane four areas which are contiguous, but it is not possible to draw five such areas. If A, B, C are not contiguous, each with the other, or if X is not contiguous with A, B, and C, it is not necessary to colour them all differently, and thus the most unfavourable case is that already treated. Moreover any

* *Leipzig Transactions* (*Math.-phys. Classe*), 1885, vol. XXXVII, pp. 1—6.

† See *Proceedings of the Royal Society of Edinburgh*, July 19, 1880, vol. X, p. 728.

‡ *Proceedings of the London Mathematical Society*, 1878, vol. IX, p. 148, and *Proceedings of the Royal Geographical Society*, London, 1879, N.S., vol. I, pp. 259—261, where some of the difficulties are indicated.

of the above areas may diminish to a point and finally disappear without affecting the argument.

That we may require at least four colours is obvious from the above diagram, since in that case the areas A, B, C, and X would have to be coloured differently.

A proof of the proposition involves difficulties of a high order, which as yet have baffled all attempts to surmount them. This is partly due to the fact that if, using only four colours, we build up our map, district by district, and assign definite colours to the districts as we insert them, we can always contrive the addition of two or three fresh districts which cannot be coloured differently from those next to them, and which accordingly upset our scheme of colouring. But by starting afresh, it would seem that we can always re-arrange the colours so as to allow of the addition of such extra districts.

The argument by which the truth of the proposition was formerly supposed to be demonstrated was given by A. B. Kempe* in 1879, but there is a flaw in it.

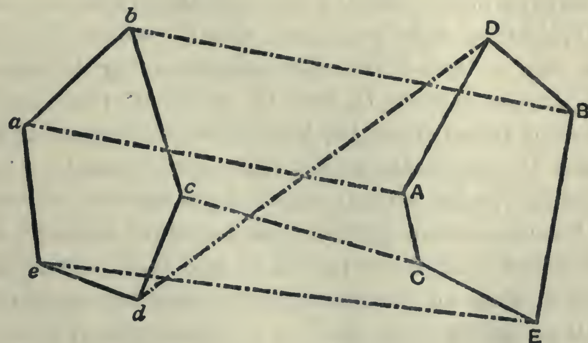
In 1880, Tait published a solution† depending on the theorem that if a closed network of lines joining an even number of points is such that three and only three lines meet at each point then three colours are sufficient to colour the lines in such a way that no two lines meeting at a point are of the same colour; a closed network being supposed to exclude the case where the lines can be divided into two groups between which there is but one connecting line.

This theorem may be true, if we understand it with the limitation that the network is in one plane and that no line

* He sent his first demonstration across the Atlantic to the *American Journal of Mathematics*, 1879, vol. II, pp. 193—200; but subsequently he communicated it in simplified forms to the London Mathematical Society, *Transactions*, 1879, vol. X, pp. 229—231, and to *Nature*, Feb. 26, 1880, vol. XXI, pp. 399—400. The flaw in the argument was indicated in articles by P. J. Heawood in the *Quarterly Journal of Mathematics*, London, 1890, vol. XXIV, pp. 332—338; and 1897, vol. XXXI, pp. 270—285.

† *Proceedings of the Royal Society of Edinburgh*, July 19, 1880, vol. X, p. 729; *Philosophical Magazine*, January, 1884, series 5, vol. XVII, p. 41; and *Collected Scientific Papers*, Cambridge, vol. II, 1890, p. 93.

meets any other line except at one of the vertices, which is all that we require for the map theorem; but it has not been proved. Without this limitation it is not correct. For instance the accompanying figure, representing a closed network in three dimensions of 15 lines formed by the sides of two pentagons and the lines joining their corresponding angular points, cannot be coloured as described by Tait. If the figure is in three dimensions, the lines intersect only at the ten vertices of the network. If it is regarded as being in two dimensions, only the ten angular points of the pentagons are treated as vertices of the network, and any other point of intersection of



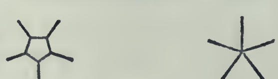
the lines is not regarded as such a vertex. Expressed in technical language the difficulty is this. Petersen* has shown that a graph (or network) of the 2*n*th order and third degree and without offshoots (or *feuilles*) can be resolved into three graphs of the 2*n*th order and each of the first degree, or into two graphs of the 2*n*th order one being of the first degree and one of the second degree. Tait assumed that the former resolution was the only one possible. The question is whether the limitations mentioned above exclude the second resolution.

Assuming that the theorem as thus limited can be established, Tait's argument that four colours will suffice for a map is divided into two parts and is as follows.

* See J. Petersen of Copenhagen, *L'Intermédiaire des Mathématiciens*, vol. v, 1898, pp. 225—227; and vol. VI, 1899, pp. 36—38. Also *Acta Mathematica*, Stockholm, vol. xv, 1891, pp. 193—220.

First, suppose that the boundary lines of contiguous districts form a closed network of lines joining an even number of points such that three and only three lines meet at each point. Then if the number of districts is $n+1$, the number of boundaries will be $3n$, and there will be $2n$ points of junction; also by Tait's theorem, the boundaries can be marked with three colours β, γ, δ so that no two like colours meet at a point of junction. Suppose this done. Now take four colours, A, B, C, D, wherewith to colour the map. Paint one district with the colour A; paint the district adjoining A and divided from it by the line β with the colour B; the district adjoining A and divided from it by the line γ with the colour C; the district adjoining A and divided from it by the line δ with the colour D. Proceed in this way so that a line β always separates the colours A and B, or the colours C and D; a line γ always separates A and C, or D and B; and a line δ always separates A and D, or B and C. It is easy to see that, if we come to a district bounded by districts already coloured, the rule for crossing each of its boundaries will give the same colour: this also follows from the fact that, if we regard β, γ, δ as indicating certain operations, then an operation like δ may be represented as equivalent to the effect of the two other operations β and γ performed in succession in either order. Thus for such a map the problem is solved.

In the second case, suppose that at any point four or more boundaries meet, then at any such point introduce a small district as indicated below: this will reduce the problem to the first case. The small district thus introduced may be



coloured by the previous rule; but after the rest of the map is coloured this district will have served its purpose, it may be then made to contract without limit to a mere point and will disappear leaving the boundaries as they were at first.

Although a proof of the four-colour theorem is still wanting, no one has succeeded in constructing a plane map which requires more than four tints to colour it, and there is no reason to doubt the correctness of the statement that it is not necessary to have more than four colours for any plane map. The number of ways in which such a map can be coloured with four tints has been also considered*, but the results are not sufficiently interesting to require mention here.

I believe that in the corresponding question with solids in space of three dimensions not more than six tints are required to colour the exposed surfaces, but I have never seen any attempt to prove this extension of the problem.

Physical Configuration of a Country. As I have been alluding to maps, I may here mention that the theory of the representation of the physical configuration of a country by means of lines drawn on a map was discussed by Cayley and Clerk Maxwell†. They showed that a certain relation exists between the number of hills, dales, passes, &c. which can co-exist on the earth or on an island. I proceed to give a summary of their nomenclature and conclusions.

All places whose heights above the mean sea level are equal are on the same level. The locus of such points on a map is indicated by a *contour-line.* Roughly speaking, an island is bounded by a contour-line. It is usual to draw the successive contour-lines on a map so that the difference between the heights of any two successive lines is the same, and thus the closer the contour-lines the steeper is the slope, but the heights are measured dynamically by the amount of work to be done to go from one level to the other and not by linear distances.

A contour-line in general will be a closed curve. This curve may enclose a region of elevation: if two such regions

* See A. C. Dixon, *Messenger of Mathematics*, Cambridge, 1902–3, vol. xxxii, pp. 81—83.

† Cayley on '*Contour and Slope Lines*,' *Philosophical Magazine*, London, October, 1859, series 4, vol. xviii, pp. 264—268; *Collected Works*, vol. iv, pp. 108—111. J. Clerk Maxwell on '*Hills and Dales*,' *Philosophical Magazine*, December, 1870, series 4, vol. xl, pp. 421—427; *Collected Works*, vol. ii, pp. 233—240.

meet at a point, that point will be a crunode (*i.e.* a real double point) on the contour-line through it, and such a point is called a *pass.* The contour-line may enclose a region of depression: if two such regions meet at a point, that point will be a crunode on the contour-line through it, and such a point is called a *fork* or bar. As the heights of the corresponding level surfaces become greater, the areas of the regions of elevation become smaller, and at last become reduced to points: these points are the *summits* of the corresponding mountains. Similarly as the level surface sinks the regions of depression contract, and at last are reduced to points: these points are the *bottoms,* or immits, of the corresponding valleys.

Lines drawn so as to be everywhere at right angles to the contour-lines are called *lines of slope.* If we go up a line of slope generally we shall reach a summit, and if we go down such a line generally we shall reach a bottom: we may come however in particular cases either to a pass or to a fork. Districts whose lines of slope run to the same summit are *hills.* Those whose lines of slope run to the same bottom are *dales.* A *watershed* is the line of slope from a summit to a pass or a fork, and it separates two dales. A *watercourse* is the line of slope from a pass or a fork to a bottom, and it separates two hills.

If $n+1$ regions of elevation or of depression meet at a point, the point is a multiple point on the contour-line drawn through it; such a point is called a pass or a fork of the nth order, and must be counted as n separate passes (or forks). If one region of depression meets another in several places at once, one of these must be taken as a fork and the rest as passes.

Having now a definite geographical terminology we can apply geometrical propositions to the subject. Let h be the number of hills on the earth (or an island), then there will be also h summits; let d be the number of dales, then there will be also d bottoms; let p be the whole number of passes, p_1 that of single passes, p_2 of double passes, and so on; let f be the whole number of forks, f_1 that of single forks, f_2 of double forks, and so on; let w be the number of watercourses, then there will be also w watersheds. Hence, by the theorems of Cauchy and Euler,

$$h = 1 + p_1 + 2p_2 + \dots,$$

$$d = 1 + f_1 + 2f_2 + \dots,$$

and

$$w = 2(p_1 + f_1) + 3(p_2 + f_2) + \dots.$$

These results can be extended to the case of a multiply-connected closed surface.

ADDENDUM.

Note. Page 52. The required rotation of the lamina can be effected thus. Suppose that the result is to be equivalent to turning it through a right angle about a point O. Describe on the lamina a square $OABC$. Rotate the lamina successively through two right angles about the diagonal OB as axis and through two right angles about the side OA as axis, and the required result will be attained.

CHAPTER IV.

GEOMETRICAL RECREATIONS CONTINUED.

Leaving now the question of formal geometrical propositions, I proceed to enumerate a few games or puzzles which depend mainly on the relative position of things, but I postpone to chapter x the discussion of such amusements of this kind as necessitate any considerable use of arithmetic or algebra. Some writers regard draughts, solitaire, chess, and such like games as subjects for geometrical treatment in the same way as they treat dominoes, backgammon, and games with dice in connection with arithmetic: but these discussions require too many artificial assumptions to correspond with the games as actually played or to be interesting.

The amusements to which I refer are of a more trivial description, and it is possible that a mathematician may like to omit this chapter. In some cases it is difficult to say whether they should be classified as mainly arithmetical or geometrical, but the point is of no importance.

Statical Games of Position. Of the innumerable statical games involving geometry of position I shall mention only three or four.

Three-in-a-row. First, I may mention the game of three-in-a-row, of which noughts and crosses, one form of merrilees, and go-bang are well-known examples. These games are played on a board—generally in the form of a square containing n^2 small squares or cells. The common practice is for one player to place a white counter or piece or to make a cross on each small square or cell which he occupies: his opponent similarly uses black counters or pieces or makes a nought on each cell which he occupies. Whoever first gets three (or any other assigned number) of his pieces in three adjacent cells and in a straight line wins. There is no difficulty in giving the complete analysis for boards of 9 cells and of 16 cells: but it is lengthy and not particularly interesting. Most of these games were known to the ancients*, and it is for that reason I mention them here.

Three-in-a-row. Extension. I may, however, add an elegant but difficult extension which has not previously found its way, so far as I am aware, into any book of mathematical recreations. The problem is to place n counters on a plane so as to form as many rows as possible, each of which shall contain three and only three counters†.

It is easy to arrange the counters in a number of rows equal to the integral part of $(n-1)^2/8$. This can be effected by the following construction. Let P be any point on a cubic. Let the tangent at P cut the curve again in Q. Let the tangent at Q cut the curve in A. Let PA cut the curve in B, QB cut it in C, PC cut it in D, QD cut it in E, and so on. Then the counters must be placed at the points P, Q, A, B, ... Thus 9 counters can be placed in 8 such rows; 10 counters in 10 rows; 15 counters in 24 rows; 81 counters in 800 rows; and so on.

If however the point P is a pluperfect point of the nth order on the cubic, then Sylvester proved that the above construction gives a number of rows equal to the integral part of $(n-1)(n-2)/6$. Thus 9 counters can be arranged in 9 rows, which is a well-known and easy puzzle; 10 counters in 12 rows; 15 counters in 30 rows; and so on.

Even this however is an inferior limit and may be exceeded—for instance, Sylvester stated that 9 counters can be placed in 10 rows, each containing three counters; I do not know how he placed them, but one way of so arranging them is

* Becq de Fouquières, *Les Jeux des Anciens*, second edition, Paris, 1873, chap. XVIII.

† *Educational Times Reprints*, 1868, vol. VIII, p. 106; *Ibid.* 1886, vol. XLV, pp. 127—128.

by putting them at points whose coordinates are (2, 0), (2, 2), (2, 4), (4, 0), (4, 2), (4, 4), (0, 0), (3, 2), (6, 4); another way is by putting them at the points (0, 0), (0, 2), (0, 4), (2, 1), (2, 2), (2, 3), (4, 0), (4, 2), (4, 4); more generally, the angular points of a regular hexagon and the three points (at infinity) of intersection of opposite sides form such a group, and therefore any projection of that figure will give a solution. At present it is not possible to say what is the maximum number of rows of three which can be formed from n counters placed on a plane.

Extension to p-in-a-row. The problem mentioned above at once suggests the extension of placing n counters so as to form as many rows as possible, each of which shall contain p and only p counters. Such problems can be often solved immediately by placing at infinity the points of intersection of some of the lines, and (if it is so desired) subsequently projecting the diagram thus formed so as to bring these points to a finite distance. One instance of such a solution is given above.

As examples I may give the arrangement of 10 counters in 5 rows, each containing 4 counters; the arrangement of 16 counters in 15 rows, each containing 4 counters; the arrangement of 18 counters in 9 rows, each containing 5 counters; and the arrangement of 19 counters in 10 rows, each containing 5 counters. These problems I leave to the ingenuity of my readers.

Tesselation. Another of these statical recreations is known as tesselation, and consists in the formation of geometrical designs or mosaics covering a plane area by the use of tiles of given geometrical forms.

If the tiles are regular polygons, the resulting forms can be found by analysis. For instance, if we confine ourselves to the use of like tiles each of which is a regular polygon of n sides, we are restricted to the use of equilateral triangles, squares, or hexagons. For suppose that to fill the space round a point where one of the angles of the polygon is situated we require m polygons. Each interior angle of the polygon is equal to $(n-2)\pi/n$. Hence $m(n-2)\pi/n = 2\pi$. Therefore $(m-2)(n-2) = 4$. Now from the nature of the problems m is greater than 2, and so is n. If $m = 3$, $n = 6$. If $m > 3$, then $n < 6$, and since $n > 2$, we have in this case only to consider the values $n=3$, $n=4$, and $n=5$. If $n=3$ we have $m=6$. If $n=4$ we have $m=4$. If $n=5$, m is non-integral, and this is impossible. Thus the only solutions are $m=3$ and $n=6$, $m=4$ and $n=4$, $m=6$ and $n=3$*.

If, however, we allow the use of unlike equilateral tiles (triangles, squares, &c.), we can construct numerous geometrical designs covering a plane area; though it is impossible to do so by the use of such starred concave polygons†. If at each point the same number and kind of polygons are used, analysis similar to the above shows that we can get six possible superposable arrangements, namely when the polygons are (i) 3-sided, 12-sided, 12-sided; (ii) 4-sided, 6-sided, 12-sided; (iii) 4-sided, 8-sided, 8-sided; (iv) 3-sided, 3-sided, 6-sided, 6-sided; (v) 3-sided, 4-sided, 4-sided, 6-sided; (vi) 3-sided, 3-sided, 3-sided, 4-sided, 4-sided.

The use of colours introduces new considerations. One formation of a pavement by the employment of square tiles of two colours is illustrated by the common chess-board; in this the cells are coloured alternately white and black. Another variety of a pavement made with square tiles of two colours was invented by Sylvester‡, who termed it anallagmatic. In the ordinary chess-board, if any two rows or any two columns are placed in juxtaposition, cell to cell, the cells which are side by side are either all of the same colour or all of different colours. In an anallagmatic arrangement, the cells are so coloured (with two colours) that when any two columns or any two rows are placed together side by side, half the cells next to one another are of the same colour and half are of different colours.

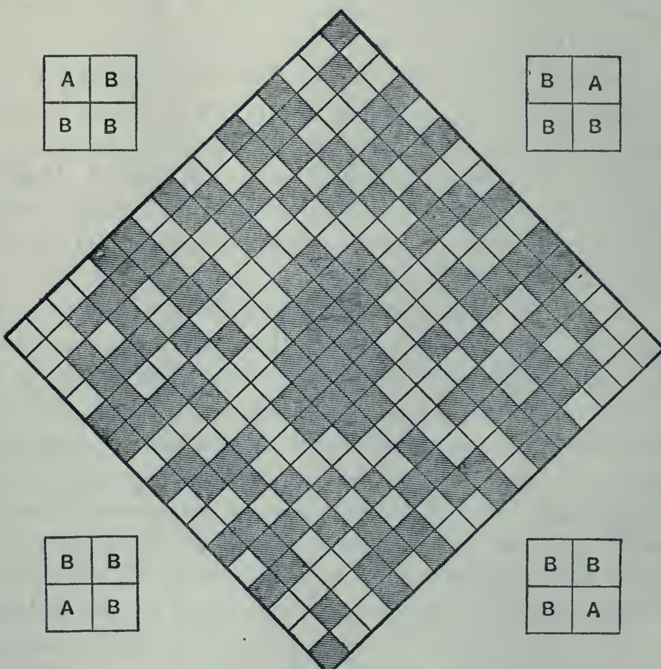
Anallagmatic pavements composed of m^2 cells or square tiles can be easily constructed by the repeated use of the four elementary anallagmatic arrangements given in the angular

* Monsieur A. Hermann has proposed an analogous theorem for polygons covering the surface of a sphere.

† On this, see the second edition of the French translation of this work, Paris, 1908, vol. II, pp. 26—37.

‡ See *Mathematical Questions from the Educational Times*, London, vol. x, 1868, pp. 74—76; vol. LVI, 1892, pp. 97—99. The results are closely connected with theorems in the theory of equations.

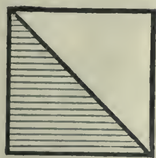
spaces of the accompanying diagram. In these fundamental forms A represents one colour and B the other colour. The diamond-shaped figure in the middle of the diagram represents an anallagmatic pavement of 256 tiles which is symmetrical about its diagonals. In half the rows and half the columns each line has 10 white tiles and 6 black tiles, and in the remaining rows and columns each line has 6 white tiles and



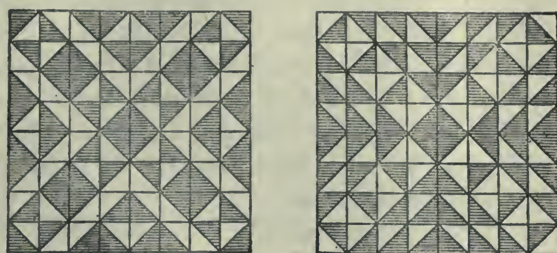
An Anallagmatic Isochromatic Pavement.

10 black tiles. Such an arrangement, where the difference between the number of white and black tiles used in each line is constant, and equal to $\sqrt{m}$, is called isochromatic. If m is odd or oddly even, it is impossible to construct anallagmatic boards which are isochromatic.

Interesting problems can also be proposed when the tiles are triangular, whether equilateral or isosceles right-angled. Two equal isosceles right-angled tiles of different colours can be put together so as to make a square tile as shown in the margin



We can arrange four such tiles in no less than 256 different ways, making 64 distinct designs. With the use of more tiles the number of possible designs increases with startling rapidity*. I content myself with giving two illustrations of designs of pavements constructed with sixty-four such tiles, all exactly alike.



Examples of Tesselated Pavements.

If more than two colours are used, the problems become increasingly difficult. As a simple instance take sixteen square tiles, the upper half of each being yellow, red, pink, or blue, and the lower half being gold, green, black, or white, no two tiles being coloured alike. Such tiles can be arranged in the form of a square so that in each vertical, horizontal, and diagonal line there shall be 8 colours and no more; or so that there shall be 6 colours and no more; or 5 colours and no more; or 4 colours and no more.

Colour-Cube Problem. As an example of a recreation analogous to tesselation I will mention the colour-cube problem†. Stripped of mathematical technicalities the problem may be enunciated as follows. A cube has six faces, and if six colours are chosen we can paint each face with a different colour. By permuting the order of the colours we can obtain

* On this, see Lucas, *Récréations Mathématiques*, Paris, 1882–3, vol. II, part 4; hereafter I shall refer to this work by the name of the author.

† By P. A. MacMahon; an abstract of his paper, read before the London Mathematical Society on Feb. 9, 1893, was given in *Nature*, Feb. 23, 1893, vol. XLVII, p. 406.

thirty such cubes, no two of which are coloured alike. Take any one of these cubes, K, then it is desired to select eight out of the remaining twenty-nine cubes, such that they can be arranged in the form of a cube (whose linear dimensions are double those of any of the separate cubes) coloured like the cube K, and placed so that where any two cubes touch each other the faces in contact are coloured alike.

Only one collection of eight cubes can be found to satisfy these conditions. These eight cubes can be determined by the following rule. Take any face of the cube K: it has four angles, and at each angle three colours meet. By permuting the colours cyclically we can obtain from each angle two other cubes, and the eight cubes so obtained are those required. A little consideration will show that these are the required cubes, and that the solution is unique.

For instance suppose that the six colours are indicated by the letters a, b, c, d, e, f. Let the cube K be put on a table, and to fix our ideas suppose that the face coloured f is at the bottom, the face coloured a is at the top, and the faces coloured b, c, d, and e front respectively the east, north, west, and south points of the compass. I may denote such an arrangement by $(f; a; b, c, d, e)$. One cyclical permutation of the colours which meet at the north-east corner of the top face gives the cube $(f; c; a, b, d, e)$, and a second cyclical permutation gives the cube $(f; b; c, a, d, e)$. Similarly cyclical permutations of the colours which meet at the north-west corner of the top face of K give the cubes $(f; d; b, a, c, e)$ and $(f; c; b, d, a, e)$. Similarly from the top south-west corner of K we get the cubes $(f; e; b, c, a, d)$ and $(f; d; b, c, e, a)$: and from the top south-east corner we get the cubes $(f; e; a, c, d, b)$ and $(f; b; e, c, d, a)$.

The eight cubes being thus determined it is not difficult to arrange them in the form of a cube coloured similarly to K, and subject to the condition that faces in contact are coloured alike; in fact they can be arranged in two ways to satisfy these conditions. One such way, taking the cubes in the numerical order given above, is to put the cubes 3, 6, 8, and 2 at the SE, NE, NW, and SW corners of the bottom face; of course each placed with the colour f at the bottom, while 3 and 6 have the colour b to the east, and 2 and 8 have the colour d to the west: the cubes 7, 1, 4, and 5 will then form the SE, NE, NW, and SW corners of the top face; of course each placed with the colour a at the top, while 7 and 1 have the colour b to the east, and 5 and 4 have the colour d to the west. If K is not given, the difficulty of the problem is increased. Similar puzzles in two dimensions can be made.

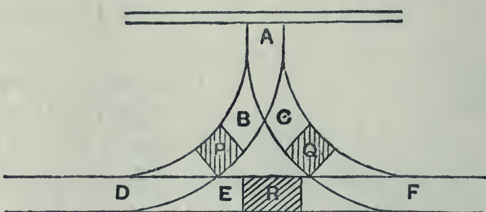
TANGRAMS. The formation of designs by means of seven pieces of wood, namely, a square, a rhombus, and five triangles, known as tans, of fixed traditional shapes, is one of the oldest amusements in the East. Many hundreds of figures representing men, women, birds, beasts, fish, houses, boats, domestic objects, designs, &c. can be made, but the recreation is not mathematical, and I reluctantly content myself with a bare mention of it.

DYNAMICAL GAMES OF POSITION. Games which are played by moving pieces on boards of various shapes—such as merrilees, fox and geese, solitaire, backgammon, draughts, and chess—present more interest. In general, possible movements of the pieces are so numerous that mathematical analysis is not practicable, but in a few games the possible movements are sufficiently limited as to permit of mathematical treatment; one or two of these are given later: here I shall confine myself mainly to puzzles and simple amusements.

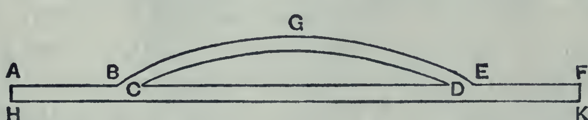
Shunting Problems. The first I will mention is a little puzzle which I bought some years ago and which was described as the "Great Northern Puzzle." It is typical of a good many problems connected with the shunting of trains, and though it rests on a most improbable hypothesis, I give it as a specimen of its kind.

The puzzle shows a railway, DEF, with two sidings, DBA and FCA, connected at A. The portion of the rails at A which is common to the two sidings is long enough to permit of a single wagon, like P or Q, running in or out of it; but is too short to contain the whole of an engine, like R. Hence, if an engine runs up one siding, such as DBA, it must come back the same way.

Initially a small block of wood, P, coloured to represent a wagon, is placed at B; a similar block, Q, is placed at C; and a longer block of wood, R, representing an engine, is placed at E. The problem is to use the engine R to interchange the wagons P and Q, without allowing any flying shunts.



Another shunting puzzle, on sale in the streets in 1905, under the name of the "Chifu-Chemulpo Puzzle," is made as follows. A loop-line BGE connects two points B and E on a railway track AF, which is supposed blocked at both ends, as shown in the diagram. In the model, the track AF is 9 inches long, $AB = EF = 1\frac{5}{6}$ inches, and $AH = FK = BC = DE = \frac{1}{4}$ inch.



On the track and loop are eight wagons, numbered successively 1 to 8, each one inch long and one-quarter of an inch broad, and an engine, e, of the same dimensions. Originally the wagons are on the track from A to F and in the order 1, 2, 3, 4, 5, 6, 7, 8, and the engine is on the loop. The construction and the initial arrangement ensure that at any one time there cannot be more than eight vehicles on the track. Also if eight vehicles are on it only the penultimate vehicle at either end can be moved on to the loop, but if less than eight are on the track then the last two vehicles at either end can be moved on to the loop. If the points at each end of the loop-line are clear, it will hold four, but not more than four, vehicles. The object is to reverse the order of the wagons on the track, so that from A to F they will be numbered successively 8 to 1; and to do this by means which will involve as few transferences of the engine or a wagon to or from the loop as is possible. Twenty-six moves are required, and there is more than one solution in 26 moves.

Other shunting problems are not uncommon, but these two examples will suffice.

Ferry-Boat Problems. Everybody is familiar with the story of the showman who was travelling with a wolf, a goat, and a basket of cabbages; and for obvious reasons was unable to leave the wolf alone with the goat, or the goat alone with the cabbages. The only means of transporting them across a river was a boat so small that he could take in it only one of them at a time. The problem is to show how the passage could be effected*.

A somewhat similar problem is to arrange for the passage of a river by three men and three boys who have the use of a boat which will not carry at one time more than one man or two boys. Fifteen passages are required†.

Problems like these were proposed by Alcuin, Tartaglia, and other medieval writers. The following is a common type of such questions. Three‡ beautiful ladies have for husbands three men, who are young, gallant, and jealous. The party are travelling, and find on the bank of a river, over which they have to pass, a small boat which can hold no more than two persons. How can they cross the river, it being agreed that, in order to avoid scandal, no woman shall be left in the society of a man unless her husband is present? Eleven passages are required. With two married couples five passages are required. The similar problem with four married couples is insoluble.

Another similar problem is the case of n married couples who have to cross a river by means of a boat which can be rowed by one person and will carry $n-1$ people, but not more, with the condition that no woman is to be in the society of a man unless her husband is present. Alcuin's problem given

* Ozanam, 1803 edition, vol. I, p. 171; 1840 edition, p. 77.

† H. E. Dudeney, *The Tribune*, October 4, 1906.

‡ Bachet, Appendix, problem IV, p. 212.

above is the case of $n=3$. Let y denote the number of passages from one bank to the other which will be necessary. Then it has been shown that if $n=3$, $y=11$; if $n=4$, $y=9$; and if $n>4$, $y=7$.

The following analogous problem is due to the late E. Lucas*. To find the smallest number x of persons that a boat must be able to carry in order that n married couples may by its aid cross a river in such a manner that no woman shall remain in the company of any man unless her husband is present; it being assumed that the boat can be rowed by one person only. Also to find the least number of passages, say y, from one bank to the other which will be required. M. Delannoy has shown that if $n=2$, then $x=2$, and $y=5$. If $n=3$, then $x=2$, and $y=11$. If $n=4$, then $x=3$, and $y=9$. If $n=5$, then $x=3$, and $y=11$. And finally if $n>5$, then $x=4$, and $y=2n-1$.

M. De Fonteney has remarked that, if there was an island in the middle of the river, the passage might be always effected by the aid of a boat which could carry only two persons. If there are only two or only three couples the island is unnecessary, and the case is covered by the preceding method. His solution, involving $8n-8$ passages, is as follows. The first nine passages will be the same, no matter how many couples there may be: the result is to transfer one couple to the island and one couple to the second bank. The result of the next eight passages is to transfer one couple from the first bank to the second bank, this series of eight operations must be repeated as often as necessary until there is left only one couple on the first bank, only one couple on the island, and all the rest on the second bank. The result of the last seven passages is to transfer all the couples to the second bank. It would however seem that if n is greater than 3, we need not require more than $6n-7$ passages from land to land.

M. G. Tarry has suggested an extension of the problem, which still further complicates its solution. He supposes that

* Lucas, vol. I, pp. 15—18, 237—238.

each husband travels with a harem of m wives or concubines; moreover, as Mohammedan women are brought up in seclusion, it is reasonable to suppose that they would be unable to row a boat by themselves without the aid of a man. But perhaps the difficulties attendant on the travels of one wife may be deemed sufficient for Christians, and I content myself with merely mentioning the increased anxieties experienced by Mohammedans in similar circumstances.

Geodesics. Geometrical problems connected with finding the shortest routes from one point to another on a curved surface are often difficult, but geodesics on a flat surface or flat surfaces are in general readily determinable.

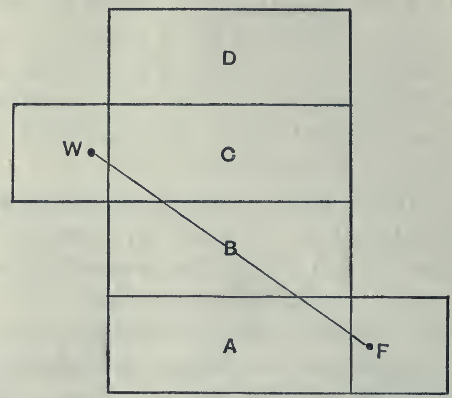
I append one instance*, though I should have hesitated to do so, had not experience shown that some readers do not readily see the solution. It is as follows. A room is 30 feet long, 12 feet wide, and 12 feet high. On the middle line of one of the smaller side walls and one foot from the ceiling is a wasp. On the middle line of the opposite wall and 11 feet from the ceiling is a fly. The wasp catches the fly by crawling all the way to it: the fly, paralysed by fear, remaining still. The problem is to find the shortest route that the wasp can follow.

To obtain a solution we observe that we can cut a sheet of paper so that, when folded properly, it will make a model to scale of the room. This can be done in several ways. If, when the paper is again spread out flat, we can join the points representing the wasp and the fly by a straight line lying wholly on the paper we shall obtain a geodesic route between them. Thus the problem is reduced to finding the way of cutting out the paper which gives the shortest route of the kind.

Here is the diagram corresponding to a solution of the above question, where A represents the floor, B and D the longer side-walls, C the ceiling, and W and F the positions on the two smaller side-walls occupied initially by the wasp and fly.

* This is due to Mr H. E. Dudeney. I heard a similar question propounded at Cambridge in 1903, but I first saw it in print in the *Daily Mail*, London, February 1, 1905.

In the diagram the square of the distance between *W* and *F* is $(32)^2 + (24)^2$; hence the distance is 40 feet.



Problems with Counters placed in a row. Numerous dynamical problems and puzzles may be illustrated with a box of counters, especially if there are counters of two colours. Of course coins or pawns or cards will serve equally well. I proceed to enumerate a few of these played with counters placed in a row.

First Problem with Counters. The following problem must be familiar to many of my readers. Ten counters (or coins) are placed in a row. Any counter may be moved over two of those adjacent to it on the counter next beyond them. It is required to move the counters according to the above rule so that they shall be arranged in five equidistant couples.

If we denote the counters in their initial positions by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, we proceed as follows. Put 7 on 10, then 5 on 2, then 3 on 8, then 1 on 4, and lastly 9 on 6. Thus they are arranged in pairs on the places originally occupied by the counters 2, 4, 6, 8, 10.

Similarly by putting 4 on 1, then 6 on 9, then 8 on 3, then 10 on 7, and lastly 2 on 5, they are arranged in pairs on the places originally occupied by the counters 1, 3, 5, 7, 9.

If two superposed counters are reckoned as only one, solutions analogous to those given above will be obtained by putting 7 on 10, then 5 on 2, then 3 on 8, then 1 on 6, and lastly 9 on 4; or by putting 4 on 1, then 6 on 9, then 8 on 3, then 10 on 5, and lastly 2 on 7*.

There is a somewhat similar game played with eight counters, but in this case the four couples finally formed are not equidistant. Here the transformation will be effected if we move 5 on 2, then 3 on 7, then 4 on 1, and lastly 6 on 8. This form of the game is applicable equally to $(8+2n)$ counters, for if we move 4 on 1 we have left on one side of this couple a row of $(8+2n-2)$ counters. This again can be reduced to one of $(8+2n-4)$ counters, and in this way finally we have left eight counters which can be moved in the way explained above.

A more complete generalization would be the case of n counters, where each counter might be moved over the m counters adjacent to it on to the one beyond them. For instance we may place twelve counters in a row and allow the moving a counter over three adjacent counters. By such movements we can obtain four piles, each pile containing three counters. Thus, if the counters be numbered consecutively, one solution can be obtained by moving 7 on 3, then 5 on 10, then 9 on 7, then 12 on 8, then 4 on 5, then 11 on 12, then 2 on 6, and then 1 on 2. Or again we may place sixteen counters in a row and allow the moving a counter over four adjacent counters on to the next counter available. By such movements we can get four piles, each pile containing four counters. Thus, if the counters be numbered consecutively, one solution can be obtained by moving 8 on 3, then 9 on 14, then 1 on 5, then 16 on 12, then 7 on 8, then 10 on 7, then 6 on 9, then 15 on 16, then 13 on 1, then 4 on 15, then 2 on 13, and then 11 on 6.

Second Problem with Counters. Another problem†, of a somewhat similar kind, is of Japanese origin. Place four florins (or white counters) and four halfpence (or black counters) alternately in a line in contact with one another. It is required

* Note by J. Fitzpatrick to a French translation of the third edition of this work, Paris, 1898.

† *Bibliotheca Mathematica*, 1896, series 3, vol. VI, p. 323; P. G. Tait, *Philosophical Magazine*, London, January, 1884, series 5, vol. XVII, p. 39; or *Collected Scientific Papers*, Cambridge, vol. II, 1890, p. 93.

in four moves, each of a pair of two contiguous pieces, without altering the relative position of the pair, to form a continuous line of four halfpence followed by four florins.

This can be solved as follows. Let a florin be denoted by a and a halfpenny by b, and let $\times\times$ denote two contiguous blank spaces. Then the successive positions of the pieces may be represented thus:

Initially	$\times \times a\, b\, a\, b\, a\, b\, a\, b.$
After the first move . .	$b\, a\, a\, b\, a\, b\, a \times \times b.$
After the second move .	$b\, a\, a\, b \times \times a\, a\, b\, b.$
After the third move . .	$b \times \times b\, a\, a\, a\, a\, b\, b.$
After the fourth move .	$b\, b\, b\, b\, a\, a\, a\, a \times \times.$

The operation is conducted according to the following rule. Suppose the pieces to be arranged originally in circular order, with two contiguous blank spaces, then we always move to the blank space for the time being that pair of coins which occupies the places next but one and next but two to the blank space on one assigned side of it.

A similar problem with $2n$ counters—n of them being white and n black—will at once suggest itself, and, if n is greater than 4, it can be solved in n moves. I have however failed to find a simple rule which covers all cases alike, but solutions, due to M. Delannoy, have been given* for the four cases where n is of the form $4m$, $4m+2$, $4m+1$, or $4m+3$; in the first two cases the first $\frac{1}{2}n$ moves are of pairs of dissimilar counters and the last $\frac{1}{2}n$ moves are of pairs of similar counters; in the last two cases, the first move is similar to that given above, namely, of the penultimate and antepenultimate counters to the beginning of the row, the next $\frac{1}{2}(n-1)$ moves are of pairs of dissimilar counters, and the final $\frac{1}{2}(n-1)$ moves are of similar counters.

The problem is also capable of solution if we substitute the restriction that at each move the pair of counters taken up must be moved to one of the two ends of the row instead of the condition that the final arrangement is to be continuous.

* *La Nature*, June, 1887, p. 10.

Tait suggested a variation of the problem by making it a condition that the two coins to be moved shall also be made to interchange places; in this form it would seem that five moves are required; or, in the general case, $n+1$ moves are required.

Problems on a Chess-board with Counters or Pawns. The following three problems require the use of a chess-board as well as of counters or pieces of two colours. It is more convenient to move a pawn than a counter, and if therefore I describe them as played with pawns it is only as a matter of convenience and not that they have any connection with chess. The first is characterized by the fact that in every position not more than two moves are possible; in the second and third problems not more than four moves are possible in any position. With these limitations, analysis is possible. I shall not discuss the similar problems in which more moves are possible.

*First Problem with Pawns**. On a row of seven squares on a chess-board 3 white pawns (or counters), denoted in the diagram by "*a*"s, are placed on the 3 squares at one end, and 3 black pawns (or counters), denoted by "*b*"s, are placed on the 3 squares at the other end—the middle square being left vacant. Each piece can move only in one direction; the "*a*" pieces can move from left to right, and the "*b*" pieces from right to left. If the square next to a piece is unoccupied, it can move on

a	*a*	*a*		*b*	*b*	*b*

to that; or if the square next to it is occupied by a piece of the opposite colour and the square beyond that is unoccupied, then it can, like a queen in draughts, leap over that piece on to the unoccupied square beyond it. The object is to get all the white pawns in the places occupied initially by the black pawns and vice versa.

The solution requires 15 moves. It may be effected by moving first a white pawn, then successively two black pawns, then three white pawns, then three black pawns, then three white pawns, then two black pawns, and then one white pawn. We can express this solution by saying that if we number the

* Lucas, vol. II, part 5, pp. 141—143.

cells (a term used to describe each of the small squares on a chess-board) consecutively, then initially the vacant space occupies the cell 4 and in the successive moves it will occupy the cells 3, 5, 6, 4, 2, 1, 3, 5, 7, 6, 4, 2, 3, 5, 4. Of these moves, six are simple and nine are leaps.

Similarly if we have n white pawns at one end of a row of $2n+1$ cells, and n black pawns at the other end, they can be interchanged in $n(n+2)$ moves, by moving in succession 1 pawn, 2 pawns, 3 pawns, ... $n-1$ pawns, n pawns, n pawns, n pawns, $n-1$ pawns, ... 2 pawns, and 1 pawn—all the pawns in each group being of the same colour and different from that of the pawns in the group preceding it. Of these moves $2n$ are simple and n^2 are leaps.

*Second Problem with Pawns**. A similar game may be played on a rectangular or square board. The case of a square board containing 49 cells, or small squares, will illustrate this sufficiently: in this case the initial position is shown in the annexed diagram where the "*a*"s denote the pawns or pieces

a	*a*	*a*	*a*	*b*	*b*	*b*
a	*a*	*a*	*a*	*b*	*b*	*b*
a	*a*	*a*	*a*	*b*	*b*	*b*
a	*a*	*a*		*b*	*b*	*b*
a	*a*	*a*	*b*	*b*	*b*	*b*
a	*a*	*a*	*b*	*b*	*b*	*b*
a	*a*	*a*	*b*	*b*	*b*	*b*

of one colour, and the "*b*"s those of the other colour. The "*a*" pieces can move horizontally from left to right or vertically down, and the "*b*" pieces can move horizontally from right to left or vertically up, according to the same rules as before.

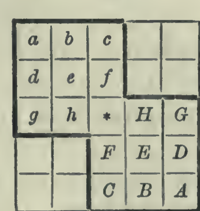
The solution reduces to the preceding case. The pieces in the middle column can be interchanged in 15 moves. In the course of these moves every one of the seven cells in that column is at some time or other vacant, and whenever that

* Lucas, vol. II, part 5, p. 144.

is the case the pieces in the row containing the vacant cell can be interchanged. To interchange the pieces in each of the seven rows will require 15 moves. Hence to interchange all the pieces will require $15 + (7 \times 15)$ moves, that is, 120 moves.

If we place $2n(n+1)$ white pawns and $2n(n+1)$ black pawns in a similar way on a square board of $(2n+1)^2$ cells, we can transpose them in $2n(n+1)(n+2)$ moves: of these $4n(n+1)$ are simple and $2n^2(n+1)$ are leaps.

Third Problem with Pawns. The following analogous problem is somewhat more complicated. On a square board of 25 cells, place eight white pawns or counters on the cells



denoted by small letters in the annexed diagram, and eight black pawns or counters on the cells denoted by capital letters: the cell marked with an asterisk (*) being left blank. Each pawn can move according to the laws already explained—the white pawns being able to move only horizontally from left to right or vertically downwards, and the black pawns being able to move only horizontally from right to left or vertically upwards. The object is to get all the white pawns in the places initially occupied by the black pawns and vice versa. No moves outside the dark line are permitted.

Since there is only one cell on the board which is unoccupied, and since no diagonal moves and no backward moves are permitted, it follows that at each move not more than two pieces of either colour are capable of moving. There are however a very large number of empirical solutions. In previous editions I have given a symmetrical solution in 48 moves, but the following, due to Mr H. E. Dudeney, is effected in 46 moves:

*Hhg*Ffc*CBHh*GDFfehbag*GABHEFfdg*Hhbc*CFf*GHh** the letters indicating the cells from which the pieces are successively moved. It will be noticed that the first twenty-three moves lead to a symmetrical position, and that the next twenty-two moves can be at once obtained by writing the first twenty-two moves in reverse order and interchanging small and capital letters. Similar problems with boards of various shapes can be easily constructed.

Probably, were it worth the trouble, the mathematical theory of games such as that just described might be worked out by the use of Vandermonde's notation, described later in chapter VI, or by the analogous method employed in the theory of the game of solitaire*.

Problems on a Chess-board with Chess-pieces. There are several mathematical recreations with chess-pieces, other than pawns. Some of these are given later in chapter VI.

GEOMETRICAL PUZZLES WITH RODS, ETC. Another species of geometrical puzzles, to which here I will do no more than allude, are made of steel rods, or of wire, or of wire and string. Numbers of these are often sold in the streets of London for a penny each, and some of them afford ingenious problems in the geometry of position. Most of them could hardly be discussed without the aid of diagrams, but they are inexpensive to construct, and in fact innumerable puzzles on geometry of position can be made with a couple of stout sticks and a ball of string, or with only a box of matches: several examples are given in various recent English works. Most of them exemplify the difficulty of mentally realizing the effect of geometrical alterations in a figure unless they are of the simplest character.

PARADROMIC RINGS. The fact just stated is illustrated by the familiar experiment of making *paradromic rings* by cutting a paper ring prepared in the following manner.

* On the theory of the solitaire, see Reiss, '*Beiträge zur Theorie des Solitär-Spiels*,' *Crelle's Journal*, Berlin, 1858, vol. LIV, pp. 344—379; and Lucas, vol. I, part V, pp. 89—141.

Take a strip of paper or piece of tape, say, for convenience, an inch or two wide and at least nine or ten inches long, rule a line in the middle down the length AB of the strip, gum one end over the other end B, and we get a ring like a section of a cylinder. If this ring is cut by a pair of scissors along the ruled line we obtain two rings exactly like the first, except that they are only half the width. Next suppose that the end A is twisted through two right angles before it is gummed to B (the result of which is that the back of the strip at A is gummed over the front of the strip at B), then a cut along the line will produce only one ring. Next suppose that the end A is twisted once completely round (*i.e.* through four right angles) before it is gummed to B, then a similar cut produces two interlaced rings. If any of my readers think that these results could be predicted off-hand, it may be interesting to them to see if they can predict correctly the effect of again cutting the rings formed in the second and third experiments down their middle lines in a manner similar to that above described.

The theory is due to J. B. Listing* who discussed the case when the end A receives m half-twists, that is, is twisted through $m\pi$, before it is gummed to B.

If m is even we obtain a surface which has two sides and two edges, which are termed paradromic. If the ring is cut along a line midway between the edges, we obtain two rings, each of which has m half-twists, and which are linked together $\frac{1}{2}m$ times.

If m is odd we obtain a surface having only one side and one edge. If this ring is cut along its mid-line, we obtain only one ring, but it has $2m$ half-twists, and if m is greater than unity it is knotted.

ADDENDUM.

Note. *Page* 64. One method of arranging 16 counters in 15 lines, as stated in the text, is as follows. Draw a regular re-entrant pentagon vertices A_1, A_2, A_3, A_4, A_5, and centre O. The sides intersect in five

* *Vorstudien zur Topologie, Die Studien*, Göttingen, 1847, part x.

points $B_1, \dots B_5$. These latter points may be joined so as to form a smaller regular re-entrant pentagon whose sides intersect in five points $C_1, \dots C_5$. The 16 points indicated are arranged as desired (*The Canterbury Puzzles*, 1907, p. 140).

An arrangement of 18 counters in 9 rows, each containing 5 counters, can be obtained thus. From one angle, A of an equilateral triangle $AA'A''$, draw lines AD, AE inside the triangle making any angles with AA'. Draw from A' and A'' lines similarly placed in regard to $A'A''$ and $A''A$. Let $A'D'$ cut $A''E''$ in F, and $A'E'$ cut $A''D''$ in G. Then AFG is a straight line. The 3 vertices of the triangle and the 15 points of intersection of AD, AE, AF, with the similar pencils of lines drawn from A', A'', will give an arrangement as required.

An arrangement of 19 counters in 10 rows, each containing 5 counters, can be obtained by placing counters at the 19 points of intersection of the 10 lines $x = \pm a$, $x = \pm b$, $y = \pm a$, $y = \pm b$, $y = \pm x$: of these points two are at infinity.

Note. Page 69. The Great Northern Shunting Problem is effected thus. (i) R pushes P into A. (ii) R returns, pushes Q up to P in A, couples Q to P, draws them both out to F, and then pushes them to E. (iii) A is now uncoupled, R takes Q back to A, and leaves it there. (iv) R returns to P, takes P back to C, and leaves it there. (v) R running successively through F, D, B comes to A, draws Q out, and leaves it at B.

Note. Page 70. One solution of the Chifu-Chemulpo Puzzle is as follows. Move successively wagons 2, 3, 4 up, *i.e.* on to the loop line. [Then push 1 along the straight track close to 5; this is not a "move."] Next, move 4 down, *i.e.* on to the straight track and push it along to 1. Next, move 8 up, 3 down to the end of the track and keep it there temporarily, 6 up, 2 down, *e* down, 3 up, 7 up. [Then push 5 to the end of the track and keep it there temporarily.] Next, move 7 down, 6 down, 2 up, 4 up. [Then push *e* along to 1.] Next, move 4 down to the end of the track and keep it there temporarily, 2 down, 5 up, 3 down, 6 up, 7 up, 8 down to the end of the track, *e* up, 5 down, 6 down, 7 down. In this solution we moved *e* down to the track at one end, then shifted it along the track, and finally moved it up to the loop from the other end of the track. We might equally well move *e* down to the track at one end, and finally move it back to the loop from the same end. In this solution the pieces successively moved are 2, 3, 4, 4, *e*, 8, 7, 3, 2, 6, 5, 5, 6, 3, 2, 7, 2, 5, 6, 3, 7, *e*, 8, 5, 6, 7.

CHAPTER V.

MECHANICAL RECREATIONS.

I proceed now to enumerate a few questions connected with mechanics which lead to results that seem to me interesting from a historical point of view or paradoxical. Problems in mechanics generally involve more difficulties than problems in arithmetic, algebra, or geometry, and the explanations of some phenomena—such as those connected with the flight of birds—are still incomplete, while the explanations of many others of an interesting character are too difficult to find a place in a non-technical work. Here I exclude all transcendental mechanics, and confine myself to questions which, like those treated in the preceding chapters, are of an elementary character. The results are well-known to mathematicians.

I assume that the reader is acquainted with the fundamental ideas of kinematics and dynamics, and is familiar with the three Newtonian laws; namely, first that a body will continue in its state of rest or of uniform motion in a straight line unless compelled to change that state by some external force: second, that the change of momentum per unit of time is proportional to the external force and takes place in the direction of it: and third, that the action of one body on another is equal in magnitude but opposite in direction to the reaction of the second body on the first. The first and second laws state the principles required for solving any question on the motion of a particle under the action of given forces. The third law supplies the additional principle required for the solution of problems in which two or more particles influence one another.

MOTION. The difficulties connected with the idea of *motion* have been for a long time a favourite subject for paradoxes, some of which bring us into the realm of the philosophy of mathematics.

Zeno's Paradoxes on Motion. One of the earliest of these is the remark of Zeno to the effect that since an arrow cannot move where it is not, and since also it cannot move where it is (that is, in the space it exactly fills), it follows that it cannot move at all. This is sometimes presented in the form that at each instant a flying arrow occupies a fixed position; but occupying a fixed position at a given instant means that it is then at rest. Hence the arrow is at rest at every instant of its flight, and therefore is not in motion. The usual answer is that the very idea of the motion of the arrow implies the passage from where it is to where it is not.

Zeno also asserted that the idea of motion was itself inconceivable, for what moves must reach the middle of its course before it reaches the end. Hence the assumption of motion presupposes another motion, and that in turn another, and so ad infinitum. His objection was in fact analogous to the biological difficulty expressed by Swift:—

"So naturalists observe, a flea hath smaller fleas that on him prey.
And these have smaller fleas to bite 'em. And so proceed ad infinitum."

Or as De Morgan preferred to put it

"Great fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so ad infinitum.
And the great fleas themselves, in turn, have greater fleas to go on;
While these have greater still, and greater still, and so on."

Achilles and the Tortoise. Zeno's paradox about Achilles and the tortoise is known even more widely. The assertion was that if Achilles ran ten times as fast as a tortoise, yet if the tortoise had (say) 1000 yards start it could never be overtaken. To establish this, Zeno argued that when Achilles had gone the 1000 yards, the tortoise would still be 100 yards in front of him; by the time he had covered these 100 yards, it would still be 10 yards in front of him; and so on for ever. Thus Achilles would get nearer and nearer to the tortoise but would never overtake it. Zeno regarded this as confirming his view that the popular idea of motion is self-contradictory.

The fallacy lies in the use of the word "never." The argument shows that during the time occupied by the motion described Achilles will not reach the tortoise. It does not deal with what happens after that time, and in fact Achilles would then overtake and pass the tortoise. Probably Zeno would have stated that the argument and explanation alike rest on the assumption, which he would not have admitted, that space and time are infinitely divisible.

Zeno's Paradox on Time. Zeno seems further to have contended that while, to an accurate thinker, the notion of the infinite divisibility of time was impossible, it was equally impossible to think of a minimum measure of time. For suppose, he argued, that τ is the smallest conceivable interval, and suppose that three horizontal lines composed of three consecutive spans abc, $a'b'c'$, $a''b''c''$ are placed so that a, a', a'' are vertically over one another, as also b, b', b'' and c, c', c''. Imagine the second line moved as a whole one span to the right in the time τ, and simultaneously the third line moved as a whole one span to the left. Then b, a', c'' will be vertically over one another. And in this duration τ (which by hypothesis is indivisible) a' must have passed vertically over the space $a''b''$ and the space $b''c''$. Hence the duration is divisible, contrary to the hypothesis.

The Paradox of Tristram Shandy. Mr Russell has enunciated* a paradox somewhat similar to that of Achilles and the Tortoise, save that the intervals of time considered get longer and longer during the course of events. Tristram Shandy, as we know, took two years writing the history of the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that he could never finish the work, however long he lived. But had he lived long enough, and not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten. For if he wrote the events of the first day in the first year, he would write the

* B. A. W. Russell, *Principles of Mathematics*, Cambridge, 1903, vol. I, p. 358.

events of the nth day in the nth year, hence in time the events of any assigned day would be written, and therefore no part of his biography would remain unwritten. This argument might be put in the form of a demonstration that the part of a magnitude may be equal to the whole of it.

Questions, such as those given above, which are concerned with the continuity of space and time involve difficulties of a high order. Many of the resulting perplexities are due to the assumption that the number of things in a collection of them is greater than the number in a part of that collection. This is axiomatic for a finite number of things, but must not be assumed as being necessarily true of infinite collections.

Angular Motion. A non-mathematician finds additional difficulties in the idea of angular motion. For instance, there is a well-known proposition on motion in an equiangular spiral which shows that a body, moving with uniform velocity and as slowly as we please, may in a finite time whirl round a fixed point an infinite number of times. To a non-mathematician the result seems paradoxical if not impossible.

The demonstration is as follows. The equiangular spiral is the trace of a point P, which moves along a line OP, the line OP turning round a fixed point O with uniform angular velocity while the distance of P from O decreases with the time in geometrical progression. If the radius vector rotates through four right angles we have one convolution of the curve. All convolutions are similar, and the length of each convolution is a constant fraction, say $1/n$th, that of the convolution immediately outside it. Inside any given convolution there are an infinite number of convolutions which get smaller and smaller as we get nearer the pole. Now suppose a point Q to move uniformly along the spiral from any point towards the pole. If it covers the first convolution in a seconds, it will cover the next in a/n seconds, the next in a/n^2 seconds, and so on, and will finally reach the pole in

$$(a + a/n + a/n^2 + a/n^3 + \dots\dots\dots)$$

seconds, that is, in $an/(n-1)$ seconds. The velocity is uniform, and yet in a finite time, Q will have traversed an infinite number of convolutions and therefore have circled round the pole an infinite number of times*.

Simple Relative Motion. Even if the philosophical difficulties suggested by Zeno are settled or evaded, the mere idea of relative motion has been often found to present difficulties, and Zeno himself failed to explain a simple phenomenon involving the principle. As one of the easiest examples of this kind, I may quote the common question of how many trains going from B to A a passenger from A to B would meet and pass on his way, assuming that the journey either way takes $4\frac{1}{2}$ hours and that the trains start from each end every hour. The answer is 9. Or again, take two pennies, face upwards on a table and edges in contact. Suppose that one is fixed and that the other rolls on it without slipping, making one complete revolution round it and returning to its initial position. How many revolutions round its own centre has the rolling coin made? The answer is 2.

Laws of Motion. I proceed next to make a few remarks on points connected with the laws of motion.

The first law of motion is often said to define *force*, but it is in only a qualified sense that this is true. Probably the meaning of the law is best expressed in Clifford's phrase, that force is "the description of a certain kind of motion"—in other words it is not an entity but merely a convenient way of stating, without circumlocution, that a certain kind of motion is observed.

It is not difficult to show that any other interpretation lands us in difficulties. Thus some authors use the law to justify a definition that force is that which moves a body or changes its motion; yet the same writers speak of a steam-engine moving a train. It would seem then that, according to them, a steam-engine is a force. That such statements are current may be fairly reckoned among mechanical paradoxes.

* The proposition is put in this form in J. Richard's *Philosophie des Mathématiques*, Paris, 1903, pp. 119—120.

The idea of force is difficult to grasp. How many people, for instance, could predict correctly what would happen in a question as simple as the following? A rope (whose weight may be neglected) hangs over a smooth pulley; it has one end fastened to a weight of 10 stone, and the other end to a sailor of weight 10 stone, the sailor and the weight hanging in the air. The sailor begins steadily to climb up the rope; will the weight move at all; and, if so, will it rise or fall? In fact, it will rise.

It will be noted that in the first law of motion it is asserted that, unless acted on by an external force, a body in motion continues to move (i) with uniform velocity, and (ii) in a straight line.

The tendency of a body to continue in its state of rest or of uniform motion is called its *inertia*. This tendency may be used to explain various common phenomena and experiments. Thus, if a number of dominoes or draughts are arranged in a vertical pile, a sharp horizontal blow on one of those near the bottom will send it out of the pile, and those above will merely drop down to take its place—in fact they have not time to change their relative positions before there is sufficient space for them to drop vertically as if they were a solid body. On this principle depends the successful performance of numerous mechanical tricks and puzzles.

The statement about inertia in the first law may be taken to imply that a body set in rotation about a principal axis passing through its centre of mass will continue to move with a uniform angular velocity and to keep its axis of rotation fixed in direction. The former of these statements is the assumption on which our measurement of time is based as mentioned below in chapter XX. The latter assists us to explain the motion of a projectile in a resisting fluid. It affords the explanation of why the barrel of a rifle is grooved; and why, similarly, anyone who has to throw a flat body of irregular shape (such as a card) in a given direction usually gives it a rapid rotatory motion about a principal axis. Elegant illustrations of the fact just mentioned are afforded by a good many of the tricks of acrobats, though the full explanation of most of them also introduces other considerations. Thus it is a common feat to toss on to the top surface of an umbrella a penny so that it alights on its edge, and then, by turning round the stick of the umbrella rapidly, to cause the coin to rotate. By twisting the umbrella at the proper rate, the coin can be made to appear stationary and standing upright, though the umbrella is moving away underneath it, while by diminishing or increasing the angular velocity of the umbrella the penny can be made to run forwards or backwards. This is not a difficult trick to execute: it was introduced by Japanese conjurers.

The tendency of a body in motion to continue to move in a straight line is sometimes called its *centrifugal force.* Thus, if a train is running round a curve, it tends to move in a straight line, and is constrained only by the pressure of the rails to move in the required direction. Hence it presses on the outer rail of the curve. This pressure can be diminished to some extent both by raising the outer rail, and by putting a guard rail, parallel and close to the inner rail, against which the wheels on that side also will press.

An illustration of this fact occurred in a little known incident of the American civil war*. In the spring of 1862 a party of volunteers from the North made their way to the rear of the Southern armies and seized a train, intending to destroy, as they passed along it, the railway which was the main line of communication between various confederate corps and their base of operations. They were however detected and pursued. To save themselves, they stopped on a sharp curve and tore up some rails so as to throw the engine which was following them off the line. Unluckily for themselves they were ignorant of dynamics and tore up the inner rails of the curve, an operation which did not incommode their pursuers, who were travelling at a high speed.

The second law gives us the means of measuring mass, force, and therefore *work.* A given agent in a given time can do only a definite amount of work. This is illustrated by the

* *Capturing a Locomotive* by W. Pittenger, London, 1882, p. 104.

fact that although, by means of a rigid lever and a fixed fulcrum, any force however small may be caused to move any mass however large, yet what is gained in power is lost in speed—as the popular phrase runs.

Montucla* inserted a striking illustration of this principle founded on the well-known story of Archimedes who is said to have declared to Hiero that, were he but given a fixed fulcrum, he could move the world. Montucla assumed that a man could work incessantly at the rate of 116 foot-lbs. per second, which is a very high estimate. On this assumption it would take over three billion centuries, *i.e.* 3×10^{14} years, before a particle whose mass was equal to that of the earth was moved as much as one inch against gravity at the surface of the earth: to move it one inch along a horizontal plane on the earth's surface would take about 6,000 centuries.

Stability of Equilibrium. It is known to all those who have read the elements of mechanics that the centre of gravity of a body, which is resting in equilibrium under its own weight, must be vertically above its base: also, speaking generally, we may say that, if every small displacement has the effect of raising the centre of gravity, then the equilibrium is stable, that is, the body when left to itself will return to its original position; but, if a displacement has the effect of lowering the centre of gravity, then for that displacement the equilibrium is unstable; while, if every displacement does not alter the height above some fixed plane of the centre of gravity, then the equilibrium is neutral. In other words, if in order to cause a displacement work has to be done against the forces acting on the body, then for that displacement the equilibrium is stable, while if the forces do work the equilibrium is unstable.

A good many of the simpler mechanical toys and tricks afford illustrations of this principle.

Magic Bottles†. Among the most common of such toys are the small bottles—trays of which may be seen any day in the streets of London—which keep always upright, and cannot

* Ozanam, 1803 edition, vol. II, p. 18; 1840 edition, p. 202.

† Ozanam, 1803 edition, vol. II, p. 15; 1840 edition, p. 201.

be upset until their owner orders them to lie down. Such a bottle is made of thin glass or varnished paper fixed to the plane surface of a solid hemisphere or smaller segment of a sphere. Now the distance of the centre of gravity of a homogeneous hemisphere from the centre of the sphere is three-eighths of the radius, and the mass of the glass or varnished paper is so small compared with the mass of the lead base that the centre of gravity of the whole bottle is still within the hemisphere. Let us denote the centre of the hemisphere by C, and the centre of gravity of the bottle by G.

If such a bottle is placed with the hemisphere resting on a horizontal plane and GC vertical, any small displacement on the plane will tend to raise G, and thus the equilibrium is stable. This may be seen also from the fact that when slightly displaced there is brought into play a couple, of which one force is the reaction of the table passing through C and acting vertically upward, and the other the weight of the bottle acting vertically downward at G. If G is below C, this couple tends to restore the bottle to its original position.

If there is dropped into the bottle a shot or nail so heavy as to raise the centre of gravity of the whole above C, then the equilibrium is unstable, and, if any small displacement is given, the bottle falls over on to its side.

Montucla says that in his time it was not uncommon to see boxes of tin soldiers mounted on lead hemispheres, and when the lid of the box was taken off the whole regiment sprang to attention.

In a similar way we may explain how to balance a pencil in a vertical position, with its point resting on the top of one's finger, an experiment which is described in nearly every book of puzzles*. This is effected by taking a penknife, of which one blade is opened through an angle of (say) 120°, and sticking the blade in the pencil so that the handle of the penknife is below the finger. The centre of gravity is thus brought below the point of support, and a small displacement given to the

* *Ex. gr.* Oughtred, *Mathematicall Recreations*, p. 24; Ozanam, 1803 edition, vol. II, p. 14; 1840 edition, p. 200.

pencil will raise the centre of gravity of the whole: thus the equilibrium is stable.

Other similar tricks are the suspension of a bucket over the edge of a table by a couple of sticks, and the balancing of a coin on the edge of a wine-glass by the aid of a couple of forks*—the sticks or forks being so placed that the centre of gravity of the whole is vertically below the point of support and its depth below it a maximum.

The toy representing a horseman, whose motion continually brings him over the edge of a table into a position which seems to ensure immediate destruction, is constructed in somewhat the same way. A wire has one end fixed to the feet of the rider; the wire is curved downwards and backwards, and at the other end is fixed a weight. When the horse is placed so that his hind legs are near the edge of the table and his fore-feet over the edge, the weight is under his hind feet. Thus the whole toy forms a pendulum with a curved instead of a straight rod. Hence the farther it swings over the table, the higher is the centre of gravity raised, and thus the toy tends to return to its original position of equilibrium.

An elegant modification of the prancing horse was brought out at Paris in 1890 in the shape of a toy made of tin and in the figure of a man†. The legs are pivoted so as to be movable about the thighs, but with a wire check to prevent too long a step, and the hands are fastened to the top of a ∩-shaped wire weighted at its ends. If the figure is placed on a narrow sloping plank or strip of wood passing between the legs of the ∩, then owing to the ∩-shaped wire any lateral displacement of the figure will raise its centre of gravity, and thus for any such displacement the equilibrium is stable. Hence, if a slight lateral disturbance is given, the figure will oscillate and will rest alternately on each foot: when it is supported by one foot the other foot under its own weight moves forwards, and thus the figure will walk down the plank though with a slight reeling motion.

* Oughtred, p. 30; Ozanam, 1803 edition, vol. II, p. 12; 1840 edition, p. 199.

† *La Nature*, Paris, March, 1891.

Columbus's Egg. The toy known as Columbus's egg depends on the same principle as the magic bottle, though it leads to the converse result. The shell of the egg is made of tin and cannot be opened. Inside it and fastened to its base is a hollow truncated tin cone, and there is also a loose marble inside the shell. If the egg is held properly, the marble runs inside the cone and the egg will stand on its base, but so long as the marble is outside the cone, the egg cannot be made to stand on its base.

*Cones running up hill**. Another common experiment, which produces the optical effect of a body moving by itself up an inclined plane, also depends on the tendency of a body to take a position so that its centre of gravity is as low as possible. Uusually the experiment is performed as follows. Arrange two sticks in the shape of a V, with the apex on a table and the two upper ends resting on the top edge of a book placed on the table. Take two equal cones fixed base to base, and place them with the curved surfaces resting on the sticks near the apex of the V, the common axis of the cones being horizontal and parallel to the edge of the book. Then, if properly arranged, the cones will run up the plane formed by the sticks.

The explanation is obvious. The centre of gravity of the cones moves in the vertical plane midway between the two sticks and it occupies a lower position as the points of contact on the sticks get farther apart. Hence as the cone rolls up the sticks its centre of gravity descends.

Test of Internal Structure. Here is another simple experiment of a somewhat different character. Suppose two balls constructed of equal size and weight, one of lead and hollow in the middle, the other of copper and solid; and suppose that both spheres are gilt so that in weight, appearance, and elasticity they are indistinguishable. How can we tell which of them is solid? The answer is by allowing them to roll down a rough inclined plane, side by side.

Perpetual Motion. The idea of making a machine which once set going would continue to go for ever by itself has been

* Ozanam, 1803 edition, vol. II, p. 49; 1840 edition, p. 216.

the ignis fatuus of self-taught mechanicians in much the same way as the quadrature of the circle has been that of self-taught geometricians.

Now the obvious meaning of the third law of motion is that a force is only one aspect of a stress, and that whenever a force is caused another equal and opposite one is brought also into existence—though it may act upon a different body, and thus be immaterial for the particular problem considered. The law however is capable of another interpretation*, namely, that the rate at which an agent does work (that is, its action) is equal to the rate at which work is done against it (that is, its reaction). If it is allowable to include in the reaction the rate at which kinetic energy is being produced, and if work is taken to include that done against molecular forces, then it follows from this interpretation that the work done by an agent on a system is equivalent to the total increase of energy, that is, the power of doing work. Hence in an isolated system the total amount of energy is constant. If this is granted, then since friction and some molecular dissipation of energy cannot be wholly prevented, it must be impossible to construct in an isolated system a machine capable of perpetual motion.

I do not propose to describe in detail the various machines for producing perpetual motion which have been suggested†, but the machine described below will serve to illustrate one of the assumptions commonly made by these inventors.

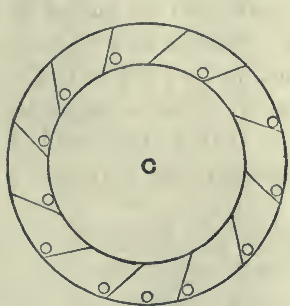
The machine to which I refer consists of two concentric vertical wheels in the same plane, and mounted on a horizontal axle through their centre, C. The space between the wheels is divided into compartments by spokes inclined at a constant angle to the radii to the points whence they are drawn, and each compartment contains a heavy bullet. This will be clear from the diagram. Apart from these bullets, the wheels would be in equilibrium. Each bullet tends to turn the wheels round their axle, and the moment which measures this tendency is

* Newton's *Principles*, last paragraph of the Scholium to the Laws of Motion.

† Several of them have been described in H. Dirck's *Perpetuum Mobile*, London, 1861, 2nd edition, 1870.

the product of the weight of the bullet and its distance from the vertical through *C*.

The idea of the constructors of such machines was that, as the bullet in any compartment would roll under gravity to the lowest point of the compartment, the bullets on the right-hand side of the wheel in the diagram would be farther from the vertical through *C* than those on the left. Hence the sum of



the moments of the weights of the bullets on the right would be greater than the sum of the moments of those on the left. Thus the wheels would turn continually in the same direction as the hands of a watch. The fallacy in the argument is obvious.

Another large group of machines for producing perpetual motion depended on the use of a magnet to raise a mass which was then allowed to fall under gravity. Thus, if the bob of a simple pendulum was made of iron, it was thought that magnets fixed near the highest points which were reached by the bob in the swing of the pendulum would draw the bob up to the same height in each swing and thus give perpetual motion, but the inventors omitted to notice that the bob of the pendulum would gradually get magnetised.

Of course it is only in isolated systems that the total amount of energy is constant, and, if a source of external energy can be obtained from which energy is continually introduced into the system, perpetual motion is, in a sense, possible; though even here materials would ultimately wear out. Streams, wind, the solar heat, and the tides are among the more obvious of such sources.

There was at Paris in the latter half of the eighteenth century a clock which was an ingenious illustration of such perpetual motion*. The energy which was stored up in it to maintain the motion of the pendulum was provided by the expansion of a silver rod. This expansion was caused by the daily rise of temperature, and by means of a train of levers it wound up the clock. There was a disconnecting apparatus, so that the contraction due to a fall of temperature produced no effect, and there was a similar arrangement to prevent over-winding. I believe that a rise of eight or nine degrees Fahrenheit was sufficient to wind up the clock for twenty-four hours.

By utilizing the rise and fall of the barometer, James Cox, a London jeweller of the eighteenth century, produced, in an analogous way, a clock† which ran continuously without winding up.

I have in my possession a watch which produces the same effect by somewhat different means. Inside the case is a steel weight, and if the watch is carried in a pocket this weight rises and falls at every step one takes, somewhat after the manner of a pedometer. The weight is raised by the action of the person who has it in his pocket in taking a step, and in falling it winds up the spring of the watch. On the face is a small dial showing the number of hours for which the watch is wound up. As soon as the hand of this dial points to fifty-six hours, the train of levers which winds up the watch disconnects automatically, so as to prevent over-winding the spring, and it reconnects again as soon as the watch has run down eight hours. The watch is an excellent time-keeper, and a walk of about a couple of miles is sufficient to wind it up for twenty-four hours.

* Ozanam, 1803 edition, vol. II, p. 105; 1840 edition, p. 238.

† A full description of the mechanism will be found in the *English Mechanic*, April 30, 1909, pp. 288—289.

MODELS. I may add here the observation, which is well known to mathematicians, but is a perpetual source of disappointment to ignorant inventors, that it frequently happens that an accurate model of a machine will work satisfactorily while the machine itself will not do so.

One reason for this is as follows. If all the parts of a model are magnified in the same proportion, say m, and if thereby a line in it is increased in the ratio $m:1$, then the areas and volumes in it will be increased respectively in the ratios $m^2:1$ and $m^3:1$. For example, if the side of a cube is doubled then a face of it will be increased in the ratio $4:1$ and its volume will be increased in the ratio $8:1$.

Now if all the linear dimensions are increased m times then some of the forces that act on a machine (such, for example, as the weight of part of it) will be increased m^3 times, while others which depend on area (such as the sustaining power of a beam) will be increased only m^2 times. Hence the forces that act on the machine and are brought into play by the various parts may be altered in different proportions, and thus the machine may be incapable of producing results similar to those which can be produced by the model.

The same argument has been adduced in the case of animal life to explain why very large specimens of any particular breed or species are usually weak. For example, if the linear dimensions of a bird were increased n times, the work necessary to give the power of flight would have to be increased no less than n^7 times*. Again, if the linear dimensions of a man of height 5 ft. 10 in. were increased by one-seventh his height would become 6 ft. 8 in., but his weight would be increased in the ratio $512:343$ (*i.e.* about half as much again), while the cross sections of his legs, which would have to bear this weight, would be increased only in the ratio $64:49$; thus in some respects he would be less efficient than before. Of course the increased dimensions, length of limb, or size of muscle might be of greater advantage than the relative loss of strength; hence the problem of what are the most efficient

* Helmholtz, *Gesammelte Abhandlungen*, Leipzig, 1881, vol. I, p. 165.

proportions is not simple, but the above argument will serve to illustrate the fact that the working of a machine may not be similar to that of a model of it.

Leaving now these elementary considerations I pass on to some other mechanical questions.

Sailing quicker than the Wind. As a kinematical paradox I may allude to the possibility of *sailing quicker than the wind blows,* a fact which strikes many people as curious.

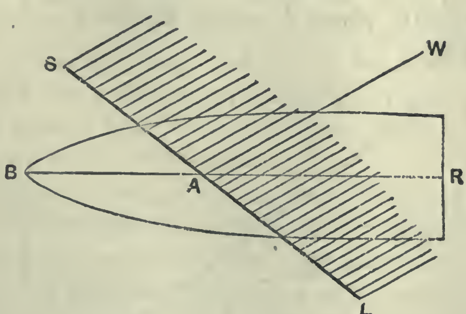
The explanation* depends on the consideration of the velocity of the wind relative to the boat. Perhaps, however, a non-mathematician will find the solution simplified if I consider first the effect of the wind-pressure on the back of the sail which drives the boat forward, and second the resistance to motion caused by the sail being forced through the air.

When the wind is blowing against a plane sail the resultant pressure of the wind on the sail may be resolved into two components, one perpendicular to the sail (but which in general is not a function only of the component velocity in that direction, though it vanishes when that component vanishes) and the other parallel to its plane. The latter of these has no effect on the motion of the ship. The component perpendicular to the sail tends to move the ship in that direction. This pressure, normal to the sail, may be resolved again into two components, one in the direction of the keel of the boat, the other in the direction of the beam of the boat. The former component drives the boat forward, the latter to leeward. It is the object of a boat-builder to construct the boat on lines so that the resistance of the water to motion forward shall be as small as possible, and the resistance to motion in a perpendicular direction (*i.e.* to leeward) shall be as large as possible; and I will assume for the moment that the former of these resistances may be neglected, and that the latter is so large as to render motion in that direction impossible.

Now, as the boat moves forward, the pressure of the air on the front of the sail will tend to stop the motion. As

* Ozanam, 1803 edition, vol. III, pp. 359, 367; 1840 edition, pp. 540, 543.

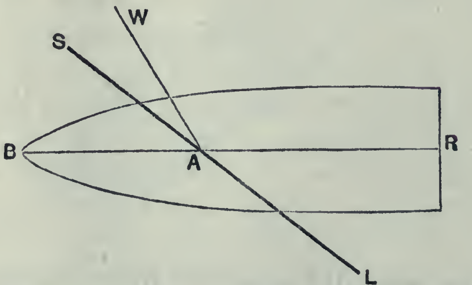
long as its component normal to the sail is less than the pressure of the wind behind the sail and normal to it, the resultant of the two will be a force behind the sail and normal to it which tends to drive the boat forwards. But as the velocity of the boat increases, a time will arrive when the pressure of the wind is only just able to balance the resisting force which is caused by the sail moving through the air. The velocity of the boat will not increase beyond this, and the motion will be then what mathematicians describe as "steady."



In the accompanying figure, let BAR represent the keel of a boat, B being the bow, and let SAL represent the sail. Suppose that the wind is blowing in the direction WA with a velocity u; and that this direction makes an angle θ with the keel, *i.e.* angle $WAR = \theta$. Suppose that the sail is set so as to make an angle α with the keel, *i.e.* angle $BAS = \alpha$, and therefore angle $WAL = \theta + \alpha$. Suppose finally that v is the velocity of the boat in the direction AB.

I have already shown that the solution of the problem depends on the relative directions and velocities of the wind and the boat; hence to find the result reduce the boat to rest by impressing on it a velocity v in the direction BA. The resultant velocity of v parallel to BA and of u parallel to WA will be parallel to SL, if $v \sin \alpha = u \sin (\theta + \alpha)$; and in this case the resultant pressure perpendicular to the sail vanishes.

Thus, for steady motion we have $v \sin \alpha = u \sin (\theta + \alpha)$. Hence, whenever $\sin (\theta + \alpha) > \sin \alpha$, we have $v > u$. Suppose, in the same way as before we have $v \sin \alpha = u \sin(\theta + \alpha)$, or $v \sin \alpha = u \sin \phi$, where $\phi = \text{angle } WAS = \pi - \theta - \alpha$. Hence $v = u \sin \phi \operatorname{cosec} \alpha$.



Let w be the component velocity of the boat in the teeth of the wind, that is, in the direction AW. Then we have $w = v \cos BAW = v \cos(\alpha + \phi) = u \sin \phi \operatorname{cosec} \alpha \cos(\alpha + \phi)$. If α is constant, this is a maximum when $\phi = \frac{1}{4}\pi - \frac{1}{2}\alpha$; and, if ϕ has this value, then $w = \frac{1}{2}u(\operatorname{cosec} \alpha - 1)$. This formula shows that w is greater than u, if $\sin \alpha < \frac{1}{3}$. Thus, if the sails can be set so that α is less than $\sin^{-1} \frac{1}{3}$, that is, rather less than 19° 29′, and if the wind has the direction above assigned, then the component velocity of the boat in the face of the wind is greater than the velocity of the wind.

The above theory is curious, but it must be remembered that in practice considerable allowance has to be made for the fact that no boat for use on water can be constructed in which the resistance to motion in the direction of the keel can be wholly neglected, or which would not drift slightly to leeward if the wind was not dead astern Still this makes less difference than might be thought by a landsman. In the case of boats sailing on smooth ice the assumptions made are substantially correct, and the practical results are said to agree closely with the theory.

BOAT MOVED BY A ROPE. There is a form of boat-racing, occasionally used at regattas, which affords a somewhat curious illustration of certain mechanical principles. The only thing supplied to the crew is a coil of rope, and they have, without leaving the boat, to propel it from one point to another as rapidly as possible. The motion is given by tying one end of the rope to the after thwart, and giving the other end a series of violent jerks in a direction parallel to the keel. I am told that in still water a pace of two or three miles an hour can be thus attained.

The chief cause for this result seems to be that the friction between the boat and the water retards all relative motion, but it is not great enough to affect materially motion caused by a sufficiently big impulse. Hence the usual movements of the crew in the boat do not sensibly move the centre of gravity of themselves and the boat, but this does not apply to an impulsive movement, and if the crew in making a jerk move their centre of gravity towards the bow n times more rapidly than it returns after the jerk, then the boat is impelled forwards at least n times more than backwards: hence on the whole the motion is forwards.

MOTION OF FLUIDS and MOTION IN FLUIDS. The theories of *motion of fluids* and *motion in fluids* involve considerable difficulties. Here I will mention only one or two instances—mainly illustrations of Hauksbee's Law.

Hauksbee's Law. When a fluid is in motion the pressure is less than when it is at rest*. Thus, if a current of air is

* See Besant, *Hydromechanics*, Cambridge, 1867, art. 149, where however it is assumed that the pressure is proportional to the density. Hauksbee was the earliest writer who called attention to the problem, but I do not know who first explained the phenomenon; some references to it are given by Willis, *Cambridge Philosophical Transactions*, 1830, vol. III, pp. 129—140.

moving in a tube, the pressure on the sides of the tube is less than when the air is at rest—and the quicker the air moves the smaller is the pressure. This fact was noticed by Hauksbee nearly two centuries ago. In an elastic perfect fluid in which the pressure is proportional to the density, the law connecting the pressure, p, and the steady velocity, v, is $p = \Pi\alpha^{-v^2}$, where Π and α are constants: the establishment of the corresponding formula for gases where the pressure is proportional to a power of the density presents no difficulty.

The principle is illustrated by a twopenny toy, on sale in most toy-shops, called the *pneumatic mystery*. This consists of a tube, with a cup-shaped end in which rests a wooden ball. If the tube is held in a vertical position, with the mouthpiece at the upper end and the cup at the lower end, then, if anyone blows hard through the tube and places the ball against the cup, the ball will remain suspended there. The explanation is that the pressure of the air below the ball is so much greater than the pressure of the air in the cup that the ball is held up.

The same effect may be produced by fastening to one end of a tube a piece of cardboard having a small hole in it. If a piece of paper is placed over the hole and the experimenter blows through the tube, the paper will not be detached from the card but will bend so as to allow the egress of the air.

An exactly similar experiment, described in many text-books on hydromechanics, is made as follows. To one end of a straight tube a plane disc is fitted which is capable of sliding on wires projecting from the end of the tube. If the disc is placed at a small distance from the end, and anyone blows steadily into the tube, the disc will be drawn towards the tube instead of being blown off the wires, and will oscillate about a position near the end of the tube.

In the same way we may make a tube by placing two books on a table with their backs parallel and an inch or so apart and laying a sheet of newspaper over them. If anyone blows steadily through the tube so formed, the paper will be sucked in instead of being blown out.

The following experiment is explicable by the same argument. On the top of a vertical axis balance a thin horizontal rod. At each end of this rod fasten a small vertical square or sail of thin cardboard—the two sails being in the same plane. If anyone blows close to one of these squares and in a direction parallel to its plane, the square will move towards the side on which one is blowing, and the rod with the two sails will rotate about the axis.

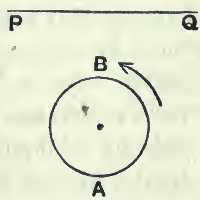
The experiments above described can be performed so as to illustrate Hauksbee's Law; but unless care is taken other causes will be also introduced which affect the phenomena: it is however unnecessary for my purpose to go into these details.

Cut on a Tennis-Ball. Racquet and court-tennis players know that if a strong cut is given to a ball it can be made to rebound off a vertical wall and then (without striking the floor or any other wall) return and hit the wall again. This affords another illustration of Hauksbee's Law. The effect had been noticed by Isaac Newton, who, in his letter to Oldenburg, February, 1672, N.S., says "I remembered that I had often seen "a tennis-ball struck with an oblique racket describe such "a curve line. For, a circular as well as a progressive motion "being communicated to it by that stroke, its parts on that side "where the motions conspire, must press and beat the contiguous "air more violently than on the other; and there excite a "reluctancy and re-action of the air proportionably greater. "And...globular bodies," thus acquiring "a circulating motion... "ought to feel the greater resistance from the ambient aether "on that side where the motions conspire, and thence be con-"tinually bowed to the other."

The question was discussed by Magnus in 1837 and by Tait in various papers from 1887 to 1896. The explanation* is that the cut causes the ball to rotate rapidly about an axis through

* See Magnus on '*Die Abweichung der Geschosse*' in the *Abhandlungen der Akademie der Wissenschaften*, Berlin, 1852, pp. 1—23; Lord Rayleigh, '*On the irregular flight of a tennis ball*,' *Messenger of Mathematics*, Cambridge, 1878, vol. VII, pp. 14—16; and P. G. Tait, *Transactions Royal Society, Edinburgh*, vol. XXVII, 1893; or *Collected Scientific Papers*, Cambridge, vol. II, 1900, pp. 356—387, and references therein.

its centre of figure, and the friction of the surface of the ball on the air produces a sort of whirlpool. This rotation is in addition to its motion of translation. Suppose the ball to be spherical and rotating about an axis through its centre perpendicular to the plane of the paper in the direction of the arrow-head, and at the same time moving through still air from left to right parallel to PQ. Any motion of the ball perpendicular to PQ will be produced by the pressure of the air on the surface of the ball, and this pressure will, by Hauksbee's Law, be greatest where the velocity of the air relative to the ball is least, and vice versa. To find the velocity of the air relative to the ball we may reduce the centre of the ball to rest, and suppose a stream of air to impinge on the surface of the ball moving with a velocity equal and opposite to that of the centre of the ball.



The air is not frictionless, and therefore the air in contact with the surface of the ball will be set in motion by the rotation of the ball and will form a sort of whirlpool rotating in the direction of the arrow-head in the figure. To find the actual velocity of this air relative to the ball we must consider how the motion due to the whirlpool is affected by the motion of the stream of air parallel to QP. The air at A in the whirlpool is moving against the stream of air there, and therefore its velocity is retarded: the air at B in the whirlpool is moving in the same direction as the stream of air there, and therefore its velocity is increased. Hence the relative velocity of the air at A is less than at B, and, since the pressure of the air is greatest where the velocity is least, the pressure of the air on the surface of the ball at A is greater than on that at B. Hence the ball is forced by this pressure in the direction from the line PQ, which we may suppose to represent the section of the vertical wall in a racquet-court. In other words, the ball tends to move at right angles to the line in which its centre is moving and in the direction in which the surface of the front of the ball is being carried by the rotation. Sir J. J. Thomson has pointed out that if we consider the direction in which the nose (or foremost point) of the ball is travelling, we may sum up the results by saying that the ball always follows its nose. Lord Rayleigh has shown that the line of action of resulting force on the ball is perpendicular to the plane containing the direction (m) of motion of the centre of the ball and the axis (s) of spin, and its magnitude varies directly as the velocity of translation, the velocity of spin, and the sine of the angle between the lines m and s.

In the case of a lawn tennis-ball, the shape of the ball is altered by a strong cut, and this introduces additional complications.

Spin on a Cricket-Ball. The curl of a cricket-ball in its flight through the air, caused by a spin given by the bowler in delivering the ball, is explained by the same reasoning.

Thus suppose the ball is delivered in a direction lying in a vertical plane containing the middle stumps of the two wickets. A spin round a horizontal axis parallel to the crease in a direction which the bowler's umpire would describe as positive, namely, counter clock-wise, will, in consequence of the friction of the air, cause it to drop, and therefore decrease the length of the pitch. A spin in the opposite direction will cause it to rise, and therefore lengthen the pitch. A spin round a vertical axis in the positive direction, as viewed from above, will make it curl sideways in the air to the left, that is, from leg to off. A spin in the opposite direction will make it curl to the right. A spin given to the ball round the direction of motion of the centre of the ball will not sensibly affect the motion through the air, though it would cause the ball, on hitting the ground, to break. Of course these various kinds of spin can be combined.

Flight of Golf-Balls. The same argument explains the effect of the spin given to a golf-ball by impact with the club. Here the motion takes place for a longer interval of time, and additional complications are introduced by the fact that the velocities of translation and rotation are retarded at different rates and that usually there will be some wind blowing in a cross direction. But generally we may say that the effect of the under-cut given by a normal stroke is to cause the ball to rise, and therefore to lengthen the carry. Also, if a wind is blowing across the line to the hole from right to left, a drive, if the player desires a long carry and has sufficient command over his club, should be pulled; but an approach shot, if it is desired that the ball should fall dead, should be sliced because then the ball as soon as it meets the wind will tend to fall dead. Conversely, if the wind is blowing across the course from left to right, the drive should be sliced if a long carry is desired, and an approach shot should be pulled if it is desired that the ball should fall dead.

The questions involving the application of Hauksbee's Law are easy as compared with many of the problems in fluid motion. The analysis required to attack most of these problems is beyond the scope of this book, but one of them may be worth mentioning even though no explanation is given.

The Theory of the Flight of Birds. A mechanical problem of great interest is the explanation of the means by which birds are enabled to fly for considerable distances with no (perceptible) motion of the wings. Albatrosses, to take an instance of special difficulty, have been known to follow for some days ships sailing at the rate of nine or ten knots, and sometimes for considerable periods there is no motion of the wings or body which can be detected, while even if the bird moved its wings it is not easy to understand how it has the muscular energy to propel itself so rapidly and for such a length of time. Of this phenomenon various explanations* have been suggested. Notable among these are Sir Hiram Maxim's of upward air-currents, Lord Rayleigh's of variations of the wind velocity at different heights above the ground, Dr S. P. Langley's of the

* See G. H. Bryan in the *Transactions of the British Association* for 1896, vol. LXVI, pp. 726—728.

incessant occurrence of gusts of wind separated by lulls, and Dr Bryan's of vortices in the atmosphere.

It now seems reasonably certain that the second and third of these sources of energy account for at least a portion of the observed phenomena. The effect of the third cause may be partially explained by noting that the centre of gravity of the bird with extended wings is slightly below the aeroplane or wing surface, so that the animal forms a sort of parachute. The effect of a sudden gust of wind upon such a body is that the aeroplane is set in motion more rapidly than the suspended mass, causing the structure to heel over so as to receive the wind on the under surface of the aeroplane, and this lifts the suspended mass giving it an upward velocity. When the wind falls the greater inertia of the mass carries it on upwards causing the aeroplane to again present its under side to the air; and if while the parachute is in this position the wind is still blowing from the side, the suspended mass is again lifted. Thus the more the bird is blown about, the more it rises in the air; actually birds in flight are carried up by a sudden side gust of wind as we should expect from this theory.

The fact that the bird is in motion tends also to keep it up, for it has been recently shown that a horizontal plane under the action of gravity falls to the ground more slowly if it is travelling through the air with horizontal velocity than it would do if allowed to fall vertically, hence the bird's forward motion causes it to fall through a smaller height between successive gusts of wind than it would do if it were at rest. Moreover it has been proved experimentally that the horse-power required to support a body in horizontal flight by means of an aeroplane is less for high than for low speeds: hence when a side-wind (that is, a wind at right angles to the bird's course) strikes the bird, the lift is increased in consequence of the bird's forward velocity.

Curiosa Physica. When I was writing the first edition of these *Recreations*, I put together a chapter, following this one, on "Some Physical Questions," dealing with problems such as, in the Theory of Sound, the explanation of the fact that in some of Captain Parry's experiments the report of a cannon, when fired, travelled so much more rapidly than the sound of the human voice that observers heard the report of the cannon when fired before that of the order to fire it*: in the Kinetic Theory of Gases, the complications in our universe that might be produced by "Maxwell's demon"†: in the Theory of Optics, the explanation of the Japanese "magic mirrors,"‡ which reflect the pattern on the back of the mirror, on which the light does not fall: to which I might add the theory of the "spectrum top," by means of which a white surface, on which some black lines are drawn, can be moved so as to give the impression§ that the lines are coloured (red, green, blue, slate, or drab), and the curious fact that the colours change with the direction of rotation: it has also been recently shown that if two trains of waves, whose lengths are in the ratio $m-1:m+1$, be superposed, then every mth wave in the system will be big—thus the current opinion that every ninth wave in the open sea is bigger than the other waves may receive scientific confirmation. There is no lack of interesting and curious phenomena in physics, and in some branches, notably in electricity and magnetism, the difficulty is rather one of selection, but I felt that the connection with mathematics was in general either too remote or too technical to justify the insertion of such a collection in a work on elementary mathematical recreations, and therefore I struck out the chapter. I mention the fact now partly to express the hope that some physicist will one day give us a collection of the kind, partly to suggest these questions to those who are interested in such matters.

* The fact is well authenticated. Mr Earnshaw (*Philosophical Transactions*, London, 1860, pp. 133—148) explained it by the acceleration of a wave caused by the formation of a kind of bore, a view accepted by Clerk Maxwell and most physicists, but Sir George Airy thought that the explanation was to be found in physiology; see Airy's *Sound*, second edition, London, 1871, pp. 141, 142.

† See *Theory of Heat*, by J. Clerk Maxwell, second edition, London, 1872, p. 308.

‡ See a memoir by W. E. Ayrton and J. Perry, *Proceedings of the Royal Society of London*, part I, 1879, vol. XXVIII, pp. 127—148.

§ See letters from Mr C. E. Benham and others in *Nature*, 1894-5; and a paper read by Prof. G. D. Liveing before the Cambridge Philosophical Society, November 26, 1894.

CHAPTER VI.

CHESS-BOARD RECREATIONS.

A chess-board and chess-men lend themselves to recreations many of which are geometrical. The problems are, however, of a distinct type, and sufficiently numerous to deserve a chapter to themselves. A few problems which might be included in this chapter have been already considered in chapter IV.

The ordinary chess-board consists of 64 small squares, known as cells, arranged as shown below in 8 rows and 8 columns. Usually the cells are coloured alternately white and black, or white and red. The cells may be defined by the numbers 11, 12, &c., where the first digit denotes the number of the column,

18	28	38	48	58	68	78	88
17	27	37	47	57	67	77	87
16	26	36	46	56	66	76	86
15	25	35	45	55	65	75	85
14	24	34	44	54	64	74	84
13	23	33	43	53	63	73	83
12	22	32	42	52	62	72	82
11	21	31	41	51	61	71	81

and the second digit the number of the row—the two digits representing respectively the abscissa and ordinate of the mid-points of the cells. I use this notation in the following pages. A generalized board consists of n^2 cells arranged in n rows and

n columns. Most of the problems which I shall describe can be extended to meet the case of a board of n^2 cells.

The usual chess-pieces are Kings, Queens, Bishops, Knights, and Rooks or Castles; there are also Pawns. I assume that the moves of these pieces are known to the reader.

With the game itself and with chess problems of the usual type I do not concern myself. Particular positions of the pieces may be subject to mathematical analysis, but in general the moves open to a player are so numerous as to make it impossible to see far ahead. Probably this is obvious, but it may emphasize how impossible it is to discuss the theory of the game effectively if I add that it has been shown that there may be as many as 197299 ways of playing the first four moves, and nearly 72000 different positions at the end of the first four moves (two on each side), of which 16556 arise when the players move pawns only*.

Relative Value of Pieces. The first question to which I will address myself is the determination of the relative values of the different chess-pieces†.

If a piece is placed on a cell, the number of cells it commands depends in general on its position. We may estimate the value of the piece by the average number of cells which it commands when placed in succession on every cell of the board. This is equivalent to saying that the value of a piece may be estimated by the chance that if it and a king are put at random on the board, the king will be in check: if no other restriction is imposed this is called a simple check. On whatever cell the piece is originally placed there will remain 63 other cells on which the king may be placed. It is equally probable that it may be put on any one of them. Hence the chance that it will be in check is 1/63 of the average number of cells commanded by the piece.

* *L'Intermédiaire des Mathématiciens*, Paris, December, 1903, vol. x, pp. 305—308: also *Royal Engineers Journal*, London, August—November, 1889; or *British Association Transactions*, 1890, p. 745.

† H. M. Taylor, *Philosophical Magazine*, March, 1876, series 5, vol. I, pp. 221—229.

A rook put on any cell commands 14 other cells. Wherever the rook is placed there will remain 63 cells on which the king may be placed, and on which it is equally likely that it will be placed. Hence the chance of a simple check is 14/63, that is, 2/9. Similarly on a board of n^2 cells the chance is $2(n-1)/(n^2-1)$, that is, $2/(n+1)$.

A knight when placed on any of the 4 corner cells like 11 commands 2 cells. When placed on any of the 8 cells like 12 and 21 it commands 3 cells. When placed on any of the 4 cells like 22 or any of the 16 boundary cells like 13, 14, 15, 16, it commands 4 cells. When placed on any of the 16 cells like 23, 24, 25, 26, it commands 6 cells. And when placed on any of the remaining 16 middle cells it commands 8 cells. Hence the average number of cells commanded by a knight put on a chess-board is $(4 \times 2 + 8 \times 3 + 20 \times 4 + 16 \times 6 + 16 \times 8)/64$, that is, 336/64. Accordingly if a king and a knight are put on the board, the chance that the king will be in simple check is $336/64 \times 63$, that is 1/12. Similarly on a board of n^2 cells the chance is $8(n-2)/n^2(n+1)$.

A bishop when placed on any of the ring of 28 boundary cells commands 7 cells. When placed on any ring of the 20 cells next to the boundary cells, it commands 9 cells. When placed on any of the 12 cells forming the next ring, it commands 11 cells. When placed on the 4 middle cells it commands 13 cells. Hence, if a king and a bishop are put on the board the chance that the king will be in simple check is $(28 \times 7 + 20 \times 9 + 12 \times 11 + 4 \times 13)/64 \times 63$, that is, 5/36. Similarly on a board of n^2 cells, when n is even, the chance is $2(2n-1)/3n(n+1)$. When n is odd the analysis is longer, owing to the fact that in this case the number of white cells on the board differs from the number of black cells. I do not give the work, which presents no special difficulty.

A queen when placed on any cell of a board commands all the cells which a bishop and a rook when placed on that cell would do. Hence, if a king and a queen are put on the board, the chance that the king will be in simple check is $2/9 + 5/36$, that is, 13/36. Similarly on a board of n^2 cells, when n is even, the chance is $2(5n-1)/3n(n+1)$.

On the above assumptions the relative values of the rook, knight, bishop, and queen are 16, 6, 10, 26. According to Staunton's *Chess-Player's Handbook* the actual values, estimated empirically, are in the ratio of 548, 305, 350, 994; according to Von Bilguer the ratios are 540, 350, 360, 1000—the value of a pawn being taken as 10.

There is considerable discrepancy between the above results as given by theory and practice. It has been, however, suggested that a better test of the value of a piece would be the chance that when it and a king were put at random on the board it would check the king without giving the king the opportunity of taking it. This is called a safe check as distinguished from a simple check.

Applying the same method as above, the chances of a safe check work out as follows. For a rook the chance of a safe check is $(4 \times 12 + 24 \times 11 + 36 \times 10)/64 \times 63$, that is, $1/6$; or on a board of n^2 cells is $2(n-2)/n(n+1)$. For a knight all checks are safe, and therefore the chance of a safe check is $1/12$; or on a board of n^2 cells is $8(n-2)/n^2(n+1)$. For a bishop the chance of a safe check is $364/64 \times 63$, that is, $13/144$; or on a board of n^2 cells, when n is even, is $2(n-2)(2n-3)/3n^2(n+1)$. For a queen the chance of a safe check is $1036/64 \times 63$, that is, $37/144$; or on a board of n^2 cells, is $2(n-2)(5n-3)/3n^2(n+1)$, when n is even.

On this view the relative values of the rook, knight, bishop, queen are 24, 12, 13, 37; while, according to Staunton, experience shows that they are approximately 22, 12, 14, 40, and according to Von Bilguer, 18, 12, 12, 33.

The same method can be applied to compare the values of combinations of pieces. For instance the value of two bishops (one restricted to white cells and the other to black cells) and two rooks, estimated by the chance of a simple check, are respectively 35/124 and 37/93. Hence on this view a queen in general should be more valuable than two bishops but less valuable than two rooks. This agrees with experience.

An analogous problem consists in finding the chance that two kings, put at random on the board, will not occupy adjoining cells, that is, that neither would (were such a move possible) check the other. The chance is 43/48, and therefore the chance that they will occupy adjoining cells is 5/48. If three kings are put on the board, the chance that no two of them occupy adjoining cells is 1061/1488. The corresponding chances* for a board of n^2 cells are $(n-1)(n-2)(n^2+3n-2)/n^2(n^2-1)$ and $(n-1)(n-2)(n^4+3n^3-20n^2-30n+132)/n^2(n^2-1)(n^2-2)$.

THE EIGHT QUEENS PROBLEM†. One of the classical problems connected with a chess-board is the determination of the number of ways in which eight queens can be placed on a chess-board—or more generally, in which n queens can be placed on a board of n^2 cells—so that no queen can take any other. This was proposed originally by Franz Nauck in 1850.

In 1874 Dr S. Günther‡ suggested a method of solution by means of determinants. For, if each symbol represents the corresponding cell of the board, the possible solutions for a board of n^2 cells are given by those terms, if any, of the determinant

$$\begin{vmatrix} a_1 & b_2 & c_3 & d_4 & \cdots\cdots & \\ \beta_2 & a_3 & b_4 & c_5 & \cdots\cdots & \\ \gamma_3 & \beta_4 & a_5 & b_6 & \cdots\cdots & \\ \delta_4 & \gamma_5 & \beta_6 & a_7 & \cdots\cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a_{2n-3} & b_{2n-2} \\ \cdots & \cdots & \cdots & \cdots & \beta_{2n-2} & b_{2n-1} \end{vmatrix}$$

in which no letter and no suffix appears more than once.

The reason is obvious. Every term in a determinant contains one and only one element out of every row and out of every column: hence any term will indicate a position on the board in which the queens cannot take one another by moves rook-wise. Again in the above determinant the letters and suffixes are so arranged that all the same letters and all the

* *L'Intermédiaire des Mathématiciens*, Paris, 1897, vol. IV, p. 6, and 1901, vol. VIII, p. 140.

† On the history of this problem see W. Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chap. ix.

‡ Grunert's *Archiv der Mathematik und Physik*, 1874, vol. LVI, pp. 281—292.

same suffixes lie along bishop's paths: hence, if we retain only those terms in each of which all the letters and all the suffixes are different, they will denote positions in which the queens cannot take one another by moves bishop-wise. It is clear that the signs of the terms are immaterial.

In the case of an ordinary chess-board the determinant is of the 8th order, and therefore contains 8!, that is, 40320 terms, so that it would be out of the question to use this method for the usual chess-board of 64 cells or for a board of larger size unless some way of picking out the required terms could be discovered.

A way of effecting this was suggested by Dr J. W. L. Glaisher* in 1874, and so far as I am aware the theory remains as he left it. He showed that if all the solutions of n queens on a board of n^2 cells were known, then all the solutions of a certain type for $n+1$ queens on a board of $(n+1)^2$ cells could be deduced, and that all the other solutions of $n+1$ queens on a board of $(n+1)^2$ cells could be obtained without difficulty. The method will be sufficiently illustrated by one instance of its application.

It is easily seen that there are no solutions when $n=2$ and $n=3$. If $n=4$ there are two terms in the determinant which give solutions, namely, $b_2c_5\gamma_3\beta_6$ and $c_3\beta_2b_6\gamma_5$. To find the solutions when $n=5$, Glaisher proceeded thus. In this case, Günther's determinant is

$$\begin{vmatrix} a_1 & b_2 & c_3 & d_4 & e_5 \\ \beta_2 & a_3 & b_4 & c_5 & d_6 \\ \gamma_3 & \beta_4 & a_5 & b_6 & c_7 \\ \delta_4 & \gamma_5 & \beta_6 & a_7 & b_8 \\ \epsilon_5 & \delta_6 & \gamma_7 & \beta_8 & a_9 \end{vmatrix}$$

To obtain those solutions (if any) which involve a_9 it is sufficient to append a_9 to such of the solutions for a board of 16 cells as do not involve a. As neither of those given above involves an a we thus get two solutions, namely, $b_2c_5\gamma_3\beta_6a_9$ and $c_3\beta_2b_6\gamma_5a_9$.

* *Philosophical Magazine*, London, December, 1874, series 4, vol. XLVIII, pp. 457—467.

The solutions which involve a_1, e_5 and ϵ_6 can be written down by symmetry. The eight solutions thus obtained are all distinct; we may call them of the first type.

The above are the only solutions which can involve elements in the corner squares of the determinant. Hence the remaining solutions are obtainable from the determinant

$$\begin{vmatrix} 0 & b_2 & c_3 & d_4 & 0 \\ \beta_2 & a_3 & b_4 & c_5 & d_6 \\ \gamma_3 & \beta_4 & a_5 & b_6 & c_7 \\ \delta_4 & \gamma_5 & \beta_6 & a_7 & b_8 \\ 0 & \delta_6 & \gamma_7 & \beta_8 & 0 \end{vmatrix}$$

If, in this, we take the minor of b_2 and in it replace by zero every term involving the letter b or the suffix 2 we shall get all solutions involving b_2. But in this case the minor at once reduces to $d_6 a_5 \delta_4 \beta_8$. We thus get one solution, namely, $b_2 d_6 a_5 \delta_4 \beta_8$. The solutions which involve β_2, δ_4, δ_6, β_8, b_8, d_6, and d_4 can be obtained by symmetry. Of these eight solutions it is easily seen that only two are distinct: these may be called solutions of the second type.

Similarly the remaining solutions must be obtained from the determinant

$$\begin{vmatrix} 0 & 0 & c_3 & 0 & 0 \\ 0 & a_3 & b_4 & c_5 & 0 \\ \gamma_3 & \beta_4 & a_5 & b_6 & c_7 \\ 0 & \gamma_5 & \beta_6 & a_7 & 0 \\ 0 & 0 & \gamma_7 & 0 & 0 \end{vmatrix}$$

If, in this, we take the minor of c_3, and in it replace by zero every term involving the letter c or the suffix 3, we shall get all the solutions which involve c_3. But in this case the minor vanishes. Hence there is no solution involving c_3, and therefore by symmetry no solutions which involve γ_3, γ_7, or c_7. Had there been any solutions involving the third element in the first or last row or column of the determinant we should have described them as of the third type.

Thus in all there are ten and only ten solutions, namely, eight of the first type, two of the second type, and none of the third type.

Similarly, if $n=6$, we obtain no solutions of the first type, four solutions of the second type, and no solutions of the third type; that is, four solutions in all. If $n=7$, we obtain sixteen solutions of the first type, twenty-four solutions of the second type, no solutions of the third type, and no solutions of the fourth type; that is, forty solutions in all. If $n=8$, we obtain sixteen solutions of the first type, fifty-six solutions of the second type, and twenty solutions of the third type, that is, ninety-two solutions in all.

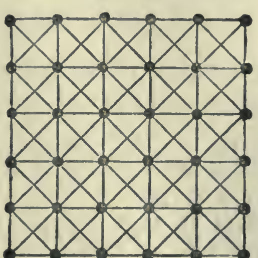
It will be noticed that all the solutions of one type are not always distinct. In general, from any solution seven others can be obtained at once. Of these eight solutions, four consist of the initial or fundamental solution and the three similar ones obtained by turning the board through one, two, or three right angles; the other four are the reflexions of these in a mirror: but in any particular case it may happen that the reflexions reproduce the originals, or that a rotation through one or two right angles makes no difference. Thus on boards of 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2 cells there are respectively 1, 2, 1, 6, 12, 46, 92 fundamental solutions; while altogether there are respectively 2, 10, 4, 40, 92, 352, 724 solutions.

The following collection of fundamental solutions may interest the reader. Each position on the board of the queens is indicated by a number, but as necessarily one queen is on each column I can use a simpler notation than that explained on page 109. In this case the first digit represents the number of the cell occupied by the queen in the first column reckoned from one end of the column, the second digit the number in the second column, and so on. Thus on a board of 4^2 cells the solution 3142 means that one queen is on the 3rd square of the first column, one on the 1st square of the second column, one on the 4th square of the third column, and one on the 2nd square of the fourth column. If a fundamental solution gives rise to only four solutions the number which indicates it is placed in curved brackets, ( ); if it gives rise to only two solutions the number which indicates it is placed in square brackets, [ ]; the other fundamental solutions give rise to eight solutions each.

On a board of 4^2 cells there is 1 fundamental solution: namely, [3142].

On a board of 5^2 cells there are 2 fundamental solutions: namely, 14253, [25314]. It may be noted that the cyclic solutions 14253, 25314, 31425, 42531, 53142 give five superposable arrangements by which five white queens, five black queens, five red queens, five yellow queens, and five blue queens can be put simultaneously on the board so that no queen can be taken by any other queen of the same colour.

On a board of 6^2 cells there is 1 fundamental solution: namely, (246135). The four solutions are superposable. The puzzle for this case is sold in the streets of London for a penny, a small wooden board being ruled in the manner shown in the diagram and having holes drilled in it at the points marked by dots. The object is to put six pins into the holes so that no two are connected by a straight line.



On a board of 7^2 cells there are 6 fundamental solutions: namely, 1357246, 3572461, (5724613), 4613572, 3162574, (2574136). It may be noted that the solution 1357246 gives by cyclic permutations seven superposable arrangements.

On a board of 8^2 cells there are 12 fundamental solutions: namely, 25713864, 57138642, 71386425, 82417536, 68241753, 36824175, 64713528, 36814752, 36815724, 72418536, 26831475, (64718253). The arrangement in this order is due to Mr Oram. It will be noticed that the 10th, 11th, and 12th solutions somewhat resemble the 4th, 6th, and 7th respectively. The 6th solution is the only one in which no three queens are in a straight line. It is impossible* to find eight superposable solutions; but we can in five typical ways pick out six solutions which can be superposed, and to some of these it is possible to add 2 sets of 7 queens, thus filling 62 out of the 64 cells with 6 sets of 8 queens and 2 sets of 7 queens, no one of which can take another of the same set. Here is such a solution: 16837425, 27368514, 35714286, 41586372, 52473861, 68241753, 73625140, 04152637. Similar superposition problems can be framed for boards of other sizes.

On a board of 9^2 cells there are 46 fundamental solutions, and on a board of 10^2 cells there are 92 fundamental solutions; these were given by Dr A. Pein†. On a board of 11^2 cells there are 341 fundamental solutions; these have been given by Dr T. B. Sprague‡.

On any board empirical solutions may be found with but little difficulty, and Mr Derrington has constructed the following table of solutions:

2.4.1.3	for a board of		4^2	cells
2.4.1.3.5	,,	,,	5^2	,,
2.4.6.1.3.5	,,	,,	6^2	,,
2.4.6.1.3.5.7	,,	,,	7^2	,,
2.4.6.8.3.1.7.5	,,	,,	8^2	,,
2.4.1.7.9.6.3.5.8	,,	,,	9^2	,,
2.4.6.8.10.1.3.5.7.9	,,	,,	10^2	,,
2.4.6.8.10.1.3.5.7.9.11	,,	,,	11^2	,,
2.4.6.8.10.12.1.3.5.7.9.11	,,	,,	12^2	,,
2.4.6.8.10.12.1.3.5.7.9.11.13	,,	,,	13^2	,,
9.7.5.3.1.13.11.6.4.2.14.12.10.8	,,	,,	14^2	,,
15.9.7.5.3.1.13.11.6.4.2.14.12.10.8	,,	,,	15^2	,,
2.4.6.8.10.12.14.16.1.3.5.7.9.11.13.15	,,	,,	16^2	,,
2.4.6.8.10.12.14.16.1.3.5.7.9.11.13.15.17	,,	,,	17^2	,,
2.4.6.8.10.12.14.16.18.1.3.5.7.9.11.13.15.17	,,	,,	18^2	,,
2.4.6.8.10.12.14.16.18.1.3.5.7.9.11.13.15.17.19		,,	19^2	,,
12.10.8.6.4.2.20.18.16.14.9.7.5.3.1.19.17.15.13.11		,,	20^2	,,
21.12.10.8.6.4.2.20.18.16.14.9.7.5.3.1.19.17.15.13.11		,,	21^2	,,

* See G. T. Bennett, *The Messenger of Mathematics*, Cambridge, June, 1909, vol. XXXIX, pp. 19—21.

† *Aufstellung von n Königinnen auf einem Schachbrett von n^2 Feldern*, Leipzig, 1889.

‡ *Proceedings of the Edinburgh Mathematical Society*, vol. XVII, 1898-9, pp. 43—68.

and so on. The rule is obvious except when n is of the form $6m+2$ or $6m+3$.

MAXIMUM PIECES PROBLEM*. The Eight Queens Problem suggests the somewhat analogous question of finding the maximum number of kings—or more generally of pieces of one type—which can be put on a board so that no one can take any other, and the number of solutions possible in each case.

In the case of kings the number is 16; for instance, one solution is when they are put on the cells 11, 13, 15, 17, 31, 33, 35, 37, 51, 53, 55, 57, 71, 73, 75, 77. For queens, it is obvious that the problem is covered by the analysis already given, and the number is 8. For bishops the number is 14, the pieces being put on the boundary cells; for instance one solution is when they are put on the cells 11, 12, 13, 14, 15, 16, 17, 81, 82, 83, 84, 85, 86, 87, there are 256 fundamental solutions. For knights the number is 32; for instance, they can be put on all the white or on all the black cells, and there are 2 fundamental solutions. For rooks it is obvious that the number is 8, and there are in all 8! solutions.

MINIMUM PIECES PROBLEM*. Another problem of a somewhat similar character is the determination of the minimum number of kings—or more generally of pieces of one type—which can be put on a board so as to command or occupy all the cells.

For kings the number is 9; for instance, they can be put on the cells 11, 14, 17, 41, 44, 47, 71, 74, 77. For queens the number is 5; for instance, they can be put on the cells 18, 35, 41, 76, 82. For bishops the number is 8; for instance, they can be put on the cells 41, 42, 43, 44, 45, 46, 47, 48. For knights the number is 12; for instance, they can be put on the cells 26, 32, 33, 35, 36, 43, 56, 63, 64, 66, 67, and 73—constituting four triplets arranged symmetrically. For rooks the number is 8, and the solutions are obvious.

* Mr H. E. Dudeney has written on these problems in the *Weekly Dispatch*.

For queens the problem has been also discussed for a board of n^2 cells where n has various values*. One queen can be placed so as to command all the cells when $n=2$ or 3, and there is only 1 fundamental solution. Two queens are required when $n=4$; and there are 3 fundamental solutions, namely, when they are placed on the cells 11 and 33, or on the cells 12 and 42, or on the cells 22, 23: these give 12 solutions in all. Three queens are required when $n=5$; and there are 37 fundamental solutions, giving 186 solutions in all. Three queens are also required when $n=6$, but there is only 1 fundamental solution, namely, when they are put on the cells 11, 35, and 53, giving 4 solutions in all. Four queens are required when $n=7$, one solution is when they are put on the cells 12, 26, 41, 55.

Jaenisch proposed also the problem of the determination of the minimum number of queens which can be placed on a board of n^2 cells so as to command all the unoccupied cells, subject to the restriction that no queen shall attack the cell occupied by any other queen. In this case three queens are required when $n=4$, for instance, they can be put on the cells 11, 23, 42; and there are 2 fundamental solutions, giving 16 solutions in all. Three queens are required when $n=5$, for instance, they can be put on the cells 11, 24, 43, or on the cells 11, 34, 53; and there are 2 fundamental solutions in all. Four queens are required when $n=6$, for instance, when they are put on the cells 13, 36, 41, 64; and there are 17 fundamental solutions. Four queens are required when $n=7$, and there is only 1 fundamental solution, namely, that already mentioned, when they are put on the cells 12, 26, 41, 55, which gives 8 solutions in all. Five queens are required when $n=8$, and there are no less than 91 fundamental solutions; for instance, one is when they are put on the cells 11, 23, 37, 62, 76.

I leave to any of my readers who may be interested in such questions the discussion of the corresponding problems for the

* C. F. de Jaenisch, *Applications de l'Analyse Mathématique au Jeu des Échecs*, Petrograd, 1862, Appendix, p. 244 *et seq.*; see also *L'Intermédiaire des Mathématiciens*, Paris, 1901, vol. VIII, p. 88.

other pieces*, and of the number of possible solutions in each case.

A problem of the same nature would be the determination of the minimum number of queens (or other pieces) which can be placed on a board so as to protect one another and command all the unoccupied cells. For queens the number is 5; for instance, they can be put on the cells 24, 34, 44, 54 and 84. For bishops the number is 10; for instance, they can be put on the cells 24, 25, 34, 35, 44, 45, 64, 65, 74, and 75. For knights the number is 14; for instance, they can be put on the cells 32, 33, 36, 37, 43, 44, 45, 46, 63, 64, 65, 66, 73, and 76: the solution is semi-symmetrical. For rooks the number is 8, and a solution is obvious. I leave to any who are interested in the subject the determination of the number of solutions in each case.

In connexion with this class of problems, I may mention two other questions, to which Captain Turton first called my attention, of a somewhat analogous character.

The first of these is to place eight queens on a chess-board so as to command the fewest possible squares. Thus, if queens are placed on cells 21, 22, 62, 71, 73, 77, 82, 87, eleven cells on the board will not be in check; the same number can be obtained by other arrangements. Is it possible to place the eight queens so as to leave more than eleven cells out of check? I have never succeeded in doing so, nor in showing that it is impossible to do it.

The other problem is to place *m* queens (*m* being less than 5) on a chess-board so as to command as many cells as possible. For instance, four queens can be placed in several ways on the board so as to command 58 cells besides those on which the queens stand, thus leaving only 2 cells which are not commanded; for instance, queens may be placed on the cells 35, 41,

* The problem for knights was discussed in *L'Intermédiaire des Mathématiciens*, Paris, 1896, vol. III, p. 58; 1897, vol. IV, pp. 15—17, 254; 1898, vol. V, pp. 87, 230—231.

76, and 82. Analogous problems with other pieces will suggest themselves.

There are endless similar questions in which combinations of pieces are involved. For instance, if queens are put on the cells 35, 41, 76, and 82 they command or occupy all but two cells, and these two cells may be commanded or occupied by a queen, a king, a rook, a bishop, or a pawn. If queens are put on the cells 22, 35, 43, and 54 they command or occupy all but three cells, and two of these three cells may be commanded by a knight which occupies the third of them.

RE-ENTRANT PATHS ON A CHESS-BOARD. Another problem connected with the chess-board consists in moving a piece in such a manner that it shall move successively on to every possible cell once and only once.

Knight's Re-Entrant Path. I begin by discussing the classical problem of a knight's tour. The literature* on this subject is so extensive that I make no attempt to give a full account of the various methods for solving the problem, and I shall content myself by putting together a few notes on some of the solutions I have come across, particularly on those due to De Moivre, Euler, Vandermonde, Warnsdorff, and Roget.

On a board containing an even number of cells the path may or may not be re-entrant, but on a board containing an odd number of cells it cannot be re-entrant. For, if a knight begins on a white cell, its first move must take it to a black cell, the next to a white cell, and so on. Hence, if its path passes through all the cells, then on a board of an odd number of cells the last move must leave it on a cell of the same colour as that on which it started, and therefore these cells cannot be connected by one move.

* For a bibliography see A. van der Linde, *Geschichte und Literatur des Schachspiels*, Berlin, 1874, vol. II, pp. 101—111. On the problem and its history see a memoir by P. Volpicelli in *Atti della Reale Accademia dei Lincei*, Rome, 1872, vol. XXV, pp. 87—162: also *Applications de l'Analyse Mathématique au Jeu des Échecs*, by C. F. de Jaenisch, 3 vols., Petrograd, 1862–3; and General Parmentier, *Association Française pour l'avancement des Sciences*, 1891, 1892, 1894.

The earliest solutions of which I have any knowledge are those given at the beginning of the eighteenth century by De Montmort and De Moivre*. They apply to the ordinary chess-board of 64 cells, and depend on dividing (mentally) the board into an inner square containing sixteen cells surrounded by an outer ring of cells two deep. If initially the knight is placed on a cell in the outer ring, it moves round that ring always in the same direction so as to fill it up completely—only going into the inner square when absolutely necessary. When the outer ring is filled up the order of the moves required for filling the remaining cells presents but little difficulty. If initially the knight is placed on the inner square the process must be reversed. The method can be applied to square and rectangular boards of all sizes. It is illustrated sufficiently by De Moivre's solution which is given below, where the numbers

34	49	22	11	36	39	24	1
21	10	35	50	23	12	37	40
48	33	62	57	38	25	2	13
9	20	51	54	63	60	41	26
32	47	58	61	56	53	14	3
19	8	55	52	59	64	27	42
46	31	6	17	44	29	4	15
7	18	45	30	5	16	43	28

De Moivre's Solution.

30	21	6	15	28	19
7	16	29	20	5	14
22	31	8	35	18	27
9	36	17	26	13	4
32	23	2	11	34	25
1	10	33	24	3	12

Euler's Thirty-six Cell Solution.

indicate the order in which the cells are occupied successively. I place by its side a somewhat similar re-entrant solution, due to Euler, for a board of 36 cells. If a chess-board is used it is convenient to place a counter on each cell as the knight leaves it.

The earliest serious attempt to deal with the subject by

* They were sent by their authors to Brook Taylor who seems to have previously suggested the problem. I do not know where they were first published; they were quoted by Ozanam and Montucla, see Ozanam, 1803 edition, vol. I, p. 178; 1840 edition, p. 80.

mathematical analysis was made by Euler* in 1759: it was due to a suggestion made by L. Bertrand of Geneva, who subsequently (in 1778) issued an account of it. This method is applicable to boards of any shape and size, but in general the solutions to which it leads are not symmetrical and their mutual connexion is not apparent.

Euler commenced by moving the knight at random over the board until it has no move open to it. With care this will leave only a few cells not traversed: denote them by *a*, *b*, ... His method consists in establishing certain rules by which these vacant cells can be interpolated into various parts of the circuit, and by which the circuit can be made re-entrant.

The following example, mentioned by Legendre as one of exceptional difficulty, illustrates the method. Suppose that

55	58	29	40	27	44	19	22
60	39	56	43	30	21	26	45
57	54	59	28	41	18	23	20
38	51	42	31	8	25	46	17
53	32	37	*a*	47	16	9	24
50	3	52	33	36	7	12	15
1	34	5	48	*b*	14	*c*	10
4	49	2	35	6	11	*d*	13

Figure i.

22	25	50	39	52	35	60	57
27	40	23	36	49	58	53	34
24	21	26	51	38	61	56	59
41	28	37	48	3	54	33	62
20	47	42	13	32	63	4	55
29	16	19	46	43	2	7	10
18	45	14	31	12	9	64	5
15	30	17	44	1	6	11	8

Figure ii.

Example of Euler's Method.

we have formed the route given in figure i above; namely, 1, 2, 3, ..., 59, 60; and that there are four cells left untraversed, namely, *a*, *b*, *c*, *d*.

We begin by making the path 1 to 60 re-entrant. The cell 1 commands a cell *p*, where *p* is 32, 52, or 2. The cell 60 commands a cell *q*, where *q* is 29, 59, or 51. Then, if any of these values of *p* and *q* differ by unity, we can make the route

* *Mémoires de Berlin* for 1759, Berlin, 1766, pp. 310—337; or *Commentationes Arithmeticae Collectae*, Petrograd, 1849, vol. I, pp. 337—355.

re-entrant. This is the case here if $p=52$, $q=51$. Thus the cells 1, 2, 3, ..., 51; 60, 59, ..., 52 form a re-entrant route of 60 moves. Hence, if we replace the numbers 60, 59, ..., 52 by 52, 53, ..., 60, the steps will be numbered consecutively. I recommend the reader who wishes to follow the subsequent details of Euler's argument to construct this square on a piece of paper before proceeding further.

Next, we proceed to add the cells *a*, *b*, *d* to this route. In the new diagram of 60 cells formed as above the cell *a* commands the cells there numbered 51, 53, 41, 25, 7, 5, and 3. It is indifferent which of these we select: suppose we take 51. Then we must make 51 the last cell of the route of 60 cells, so that we can continue with *a*, *b*, *d*. Hence, if the reader will add 9 to every number on the diagram he has constructed, and then replace 61, 62, ..., 69 by 1, 2, ..., 9, he will have a route which starts from the cell occupied originally by 60, the 60th move is on to the cell occupied originally by 51, and the 61st, 62nd, 63rd moves will be on the cells *a*, *b*, *d* respectively.

It remains to introduce the cell *c*. Since *c* commands the cell now numbered 25, and 63 commands the cell now numbered 24, this can be effected in the same way as the first route was made re-entrant. In fact, the cells numbered 1, 2, ..., 24; 63, 62, ..., 25, *c* form a knight's path. Hence we must replace 63, 62, ..., 25 by the numbers 25, 26, ..., 63, and then we can fill up *c* with 64. We have now a route which covers the whole board.

Lastly, it remains to make this route re-entrant. First, we must get the cells 1 and 64 near one another. This can be effected thus. Take one of the cells commanded by 1, such as 28, then 28 commands 1 and 27. Hence the cells 64, 63, ..., 28; 1, 2, ..., 27 form a route; and this will be represented in the diagram if we replace the cells numbered 1, 2, ..., 27 by 27, 26, ..., 1.

The cell now occupied by 1 commands the cells 26, 38, 54, 12, 2, 14, 16, 28; and the cell occupied by 64 commands the cells 13, 43, 63, 55. The cells 13 and 14 are consecutive, and therefore the cells 64, 63, ..., 14; 1, 2, ..., 13 form a route.

Hence we must replace the numbers 1, 2, ..., 13 by 13, 12, ..., 1, and we obtain a re-entrant route covering the whole board, which is represented in the second of the diagrams given above. Euler showed how seven other re-entrant routes can be deduced from any given re-entrant route.

It is not difficult to apply the method so as to form a route which begins on one given cell and ends on any other given cell.

Euler next investigated how his method could be modified so as to allow of the imposition of additional restrictions.

An interesting example of this kind is where the first 32 moves are confined to one-half of the board. One solution of this is delineated below. The order of the first 32 moves

58	43	60	37	52	41	62	35
49	46	57	42	61	36	53	40
44	59	48	51	38	55	34	63
47	50	45	56	33	64	39	54
22	7	32	1	24	13	18	15
31	2	23	6	19	16	27	12
8	21	4	29	10	25	14	17
3	30	9	20	5	28	11	26

Euler's Half-board Solution.

50	45	62	41	60	39	54	35
63	42	51	48	53	36	57	38
46	49	44	61	40	59	34	55
43	64	47	52	33	56	37	58
26	5	24	1	20	15	32	11
23	2	27	8	29	12	17	14
6	25	4	21	16	19	10	31
3	22	7	28	9	30	13	18

Roget's Half-board Solution.

can be determined by Euler's method. It is obvious that, if to the number of each such move we add 32, we shall have a corresponding set of moves from 33 to 64 which would cover the other half of the board; but in general the cell numbered 33 will not be a knight's move from that numbered 32, nor will 64 be a knight's move from 1.

Euler however proceeded to show how the first 32 moves might be determined so that, if the half of the board containing the corresponding moves from 33 to 64 was twisted through two right angles, the two routes would become united and re-entrant. If x and y are the numbers of a cell reckoned from two consecutive sides of the board, we may call the cell whose distances are respectively x and y from the opposite sides a complementary cell. Thus the cells (x, y) and $(9-x, 9-y)$ are complementary, where x and y denote respectively the column and row occupied by the cell. Then in Euler's solution the numbers in complementary cells differ by 32: for instance, the cell (3, 7) is complementary to the cell (6, 2), the one is occupied by 57, the other by 25.

Roget's method, which is described later, can be also applied to give half-board solutions. The result is indicated above. The close of Euler's memoir is devoted to showing how the method could be applied to crosses and other rectangular figures. I may note in particular his elegant re-entrant symmetrical solution for a square of 100 cells.

The next attempt of any special interest is due to Vandermonde*, who reduced the problem to arithmetic. His idea was to cover the board by two or more independent routes taken at random, and then to connect the routes. He defined the position of a cell by a fraction x/y, whose numerator x is the number of the cell from one side of the board, and whose denominator y is its number from the adjacent side of the board; this is equivalent to saying that x and y are the co-ordinates of a cell. In a series of fractions denoting a knight's path, the differences between the numerators of two consecutive fractions can be only one or two, while the corresponding differences between their denominators must be two or one respectively. Also x and y cannot be less than 1 or greater than 8. The notation is convenient, but Vandermonde applied it merely to obtain a particular solution of the problem for a board of 64 cells: the method by which he effected this is analogous to that established by Euler, but it is applicable only to squares of an even order. The route that he arrives at is defined in his notation by the following fractions: 5/5, 4/3, 2/4, 4/5, 5/3, 7/4, 8/2, 6/1, 7/3, 8/1, 6/2, 8/3, 7/1, 5/2, 6/4, 8/5, 7/7, 5/8, 6/6, 5/4, 4/6, 2/5, 1/7, 3/8, 2/6, 1/8, 3/7, 1/6, 2/8, 4/7, 3/5, 1/4, 2/2, 4/1, 3/3, 1/2, 3/1, 2/3, 1/1, 3/2, 1/3, 2/1, 4/2, 3/4, 1/5, 2/7, 4/8, 3/6,

* *L'Histoire de l'Académie des Sciences* for 1771, Paris, 1774, pp. 566—574.

4/4, 5/6, 7/5, 8/7, 6/8, 7/6, 8/8, 6/7, 8/6, 7/8, 5/7, 6/5, 8/4, 7/2, 5/1, 6/3.

The path is re-entrant but unsymmetrical. Had he transferred the first three fractions to the end of this series he would have obtained two symmetrical circuits of thirty-two moves joined unsymmetrically, and might have been enabled to advance further in the problem. Vandermonde also considered the case of a route in a cube.

In 1773 Collini* proposed the exclusive use of symmetrical routes arranged without reference to the initial cell, but connected in such a manner as to permit of our starting from it. This is the foundation of the modern manner of attacking the problem. The method was re-invented in 1825 by Pratt†, and in 1840 by Roget, and has been subsequently employed by various writers. Neither Collini nor Pratt showed skill in using this method. The rule given by Roget is described later.

One of the most ingenious of the solutions of the knight's path is that given in 1823 by Warnsdorff‡. His rule is that the knight must be always moved to one of the cells from which it will command the fewest squares not already traversed. The solution is not symmetrical and not re-entrant; moreover it is difficult to trace practically. The rule has not been proved to be true, but no exception to it is known: apparently it applies also to all rectangular boards which can be covered completely by a knight. It is somewhat curious that in most cases a single false step, except in the last three or four moves, will not affect the result.

Warnsdorff added that when, by the rule, two or more cells are open to the knight, it may be moved to either or any of them indifferently. This is not so, and with great ingenuity two or three cases of failure have been constructed, but it would require exceptionally bad luck to happen accidentally on such a route.

* *Solution du Problème du Cavalier au Jeu des Échecs*, Mannheim, 1773.

† *Studies of Chess*, sixth edition, London, 1825.

‡ H. C. Warnsdorff, *Des Rösselsprunges einfachste und allgemeinste Lösung*, Schmalkalden, 1823: see also Jaenisch, vol. II, pp. 56—61, 273—289.

The above methods have been applied to boards of various shapes, especially to boards in the form of rectangles, crosses, and circles *.

All the more recent investigations impose additional restrictions: such as to require that the route shall be re-entrant, or more generally that it shall begin and terminate on given cells.

The simplest solution with which I am acquainted is due to De Lavernède, but is more generally associated with the name of Roget whose paper in 1840 attracted general notice to it †. It divides the whole route into four circuits, which can be combined so as to enable us to begin on any cell and terminate on any other cell of a different colour. Hence, if we like to select this last cell at a knight's move from the initial cell, we obtain a re-entrant route. On the other hand, the rule is applicable only to square boards containing $(4n)^2$ cells: for example, it could not be used on the board of the French *jeu des dames*, which contains 100 cells.

Roget began by dividing the board of 64 cells into four quarters. Each quarter contains 16 cells, and these 16 cells can be arranged in 4 groups, each group consisting of 4 cells which form a closed knight's path. All the cells in each such path are denoted by the same letter *l*, *e*, *a*, or *p*, as the case may be. The path of 4 cells indicated by the consonants *l* and the path indicated by the consonants *p* are diamond-shaped: the paths indicated respectively by the vowels *e* and *a* are square-shaped, as may be seen by looking at one of the four quarters in figure i below.

Now all the 16 cells on a complete chess-board which are marked with the same letter can be combined into one circuit, and wherever the circuit begins we can make it end on any other cell in the circuit, provided it is of a different colour to the initial cell. If it is indifferent on what cell the circuit terminates we may make the circuit re-entrant, and

* See *ex. gr.* T. Ciccolini's work *Del Cavallo degli Scacchi*, Paris, 1836.

† J. E. T. de Lavernède, *Mémoires de l'Académie Royale du Gard*, Nimes, 1839, pp. 151—179. P. M. Roget, *Philosophical Magazine*, April, 1840, series 3, vol. XVI, pp. 305—309; see also the *Quarterly Journal of Mathematics* for 1877, vol. XIV, pp. 354—359; and the *Leisure Hour*, Sept. 13, 1873, pp. 587—590, and Dec. 20, 1873, pp. 813—815.

in this case we can make the direction of motion round each group (of 4 cells) the same. For example, all the cells marked *p* can be arranged in the circuit indicated by the successive numbers 1 to 16 in figure ii below. Similarly all the cells marked *a* can be combined into the circuit indicated by the numbers 17 to 32; all the *l* cells into the circuit 33 to 48; and all the *e* cells into the circuit 49 to 64. Each of the circuits indicated above is symmetrical and re-entrant. The consonant and the vowel circuits are said to be of opposite kinds.

l	*e*	*a*	*p*	*l*	*e*	*a*	*p*
a	*p*	*l*	*e*	*a*	*p*	*l*	*e*
e	*l*	*p*	*a*	*e*	*l*	*p*	*a*
p	*a*	*e*	*l*	*p*	*a*	*e*	*l*
l	*e*	*a*	*p*	*l*	*e*	*a*	*p*
a	*p*	*l*	*e*	*a*	*p*	*l*	*e*
e	*l*	*p*	*a*	*e*	*l*	*p*	*a*
p	*a*	*e*	*l*	*p*	*a*	*e*	*l*

Roget's Solution (i).

34	51	32	15	38	53	18	3
31	14	35	52	17	2	39	54
50	33	16	29	56	37	4	19
13	30	49	36	1	20	55	40
48	63	28	9	44	57	22	5
27	12	45	64	21	8	41	58
62	47	10	25	60	43	6	23
11	26	61	46	7	24	59	42

Roget's Solution (ii).

The general problem will be solved if we can combine the four circuits into a route which will start from any given cell, and terminate on the 64th move on any other given cell of a different colour. To effect this Roget gave the two following rules.

First. If the initial cell and the final cell are denoted the one by a consonant and the other by a vowel, take alternately circuits indicated by consonants and vowels, beginning with the circuit of 16 cells indicated by the letter of the initial cell and concluding with the circuit indicated by the letter of the final cell.

Second. If the initial cell and the final cell are denoted both by consonants or both by vowels, first select a cell, *Y*, in the same circuit as the final cell, *Z*, and one move from it, next select a cell, *X*, belonging to one of the opposite circuits and one move from *Y*. This is always possible. Then, leaving out the cells Z and Y, it always will be possible, by the rule already given, to travel from the initial cell to the cell X in 62 moves, and thence to move to the final cell on the 64th move.

In both cases however it must be noticed that the cells in each of the first three circuits will have to be taken in such an order that the circuit does not terminate on a corner, and it may be desirable also that it should not terminate on any of the border cells. This will necessitate some caution. As far as is consistent with these restrictions it is convenient to make these circuits re-entrant, and to take them and every group in them in the same direction of rotation.

As an example, suppose that we are to begin on the cell numbered 1 in figure ii above, which is one of those in a *p* circuit, and to terminate on the cell numbered 64, which is one of those in an *e* circuit. This falls under the first rule: hence first we take the 16 cells marked *p*, next the 16 cells marked *a*, then the 16 cells marked *l*, and lastly the 16 cells marked *e*. One way of effecting this is shown in the diagram. Since the cell 64 is a knight's move from the initial cell the route is re-entrant. Also each of the four circuits in the diagram is symmetrical, re-entrant, and taken in the same direction, and the only point where there is any apparent breach in the uniformity of the movement is in the passage from the cell numbered 32 to that numbered 33.

A rule for re-entrant routes, similar to that of Roget, has been given by various subsequent writers, especially by De Polignac* and by Laquière†, who have stated it at much greater length. Neither of these authors seems to have been aware of Roget's theorems. De Polignac, like Roget, illustrates the rule by assigning letters to the various squares in the way explained above, and asserts that a similar rule is applicable to all even squares.

* *Comptes Rendus*, April, 1861; and *Bulletin de la Société Mathématique de France*, 1881, vol. IX, pp. 17—24.

† *Bulletin de la Société Mathématique de France*, 1880, vol. VIII, pp. 82—102, 132—158.

Roget's method can be also applied to two half-boards, as indicated in the figure given above on page 126.

The method which Jaenisch gives as the most fundamental is not very different from that of Roget. It leads to eight forms, similar to that in the diagram printed below, in which the sum of the numbers in every column and every row is 260; but although symmetrical it is not in my opinion so easy to reproduce as that given by Roget. Other solutions, notably those by Moon and by Wenzelides, were given in former editions of this work. The two re-entrant routes printed below, each covering 32 cells, and together covering the board, are remarkable as constituting a magic square*.

63	22	15	40	1	42	59	18
14	39	64	21	60	17	2	43
37	62	23	16	41	4	19	58
24	13	38	61	20	57	44	3
11	36	25	52	29	46	5	56
26	51	12	33	8	55	30	45
35	10	49	28	53	32	47	6
50	27	34	9	48	7	54	31

Jaenisch's Solution.

15	20	17	36	13	64	61	34
18	37	14	21	60	35	12	63
25	16	19	44	5	62	33	56
38	45	26	59	22	55	4	11
27	24	39	6	43	10	57	54
40	49	46	23	58	3	32	9
47	28	51	42	7	30	53	2
50	41	48	29	52	1	8	31

Two Half Board Solutions.

It is as yet impossible to say how many solutions of the problem exist. Legendre† mentioned the question, but Minding‡ was the earliest writer to attempt to answer it. More recent investigations have shown that on the one hand the number of possible routes is less§ than the number of combinations of 168 things taken 63 at a time, and on the other hand is greater than 31,054144—since this latter number is the number of re-entrant paths of a particular type‖.

* See A. Rilly, *Le Problème du Cavalier des Échecs*, Troyes, 1905.

† *Théorie des Nombres*, Paris, 2nd edition, 1830, vol. II, p. 165.

‡ *Cambridge and Dublin Mathematical Journal*, 1852, vol. VII, pp. 147—156; and *Crelle's Journal*, 1853, vol. XLIV, pp. 73—82.

§ Jaenisch, vol. II, p. 268.

‖ *Bulletin de la Société Mathématique de France*, 1881, vol. IX, pp. 1—17.

Analogous Problems. Similar problems can be constructed in which it is required to determine routes by which a piece moving according to certain laws (*ex. gr.* a chess-piece such as a king, &c.) can travel from a given cell over a board so as to occupy successively all the cells, or certain specified cells, once and only once, and terminate its route in a given cell. Euler's method can be applied to find routes of this kind: for instance, he applied it to find a re-entrant route by which a piece that moved two cells forward like a castle and then one cell like a bishop would occupy in succession all the black cells on the board.

King's Re-Entrant Path. As one example here is a re-entrant tour of a king which moves successively to every cell

61	62	63	64	1	2	3	4
60	11	58	57	8	7	54	5
12	59	10	9	56	55	6	53
13	14	15	16	49	50	51	52
20	19	18	17	48	47	46	45
21	38	23	24	41	42	27	44
37	22	39	40	25	26	43	28
36	35	34	33	32	31	30	29

King's Magic Tour on a Chess-Board.

of the board. I give it because the numbers indicating the cells successively occupied form a magic square. Of course this also gives a solution of a re-entrant route of a queen covering the board.

Rook's Re-Entrant Path. There is no difficulty in constructing re-entrant tours for a rook which moves successively to every cell of the board. For instance, if the rook starts from the cell 11 it can move successively to the cells 18, 88, 81, 71, 77, 67, 61, 51, 57, 47, 41, 31, 37, 27, 21, and so back to 11: this is a symmetrical route. Of course this also gives a solution of a re-entrant route for a king or a queen covering the board.

If we start from any of the cells mentioned above, the rook takes sixteen moves. If we start from any cell in the middle of one of these moves, it will take seventeen moves to cover this route, but I believe that in most cases wherever the initial cell be chosen sixteen moves will suffice, though in general the route will not be symmetrical. On a board of n^2 cells it is possible to find a route by which a rook can move successively from its initial cell to every other cell once and only once. Moreover* starting on any cell its path can be made to terminate, if n be even, on any other cell of a different colour, and, if n be odd, on any other cell of the same colour.

Bishop's Re-Entrant Path. As yet another instance, a bishop can traverse all the cells of one colour on the board in seventeen moves if the initial cell is properly chosen†; for instance, starting from the cell 11, it may move successively to the cells 55, 82, 71, 17, 28, 46, 13, 31, 86, 68, 57, 48, 15, 51, 84, 66, 88. One more move will bring it back to the initial cell. From the nature of the case, it must traverse some cells more than once.

Miscellaneous Problems. We may construct numerous such problems concerning the determination of routes which cover the whole or part of the board subject to certain conditions. I append a few others which may tax the ingenuity of those not accustomed to such problems.

Routes on a Chess-Board. One of the simplest is the determination of the path taken by a rook, placed in the cell 11, which moves, one cell at a time, to the cell 88, so that in the course of its path it enters every cell once and only once. This can be done, though I have seen good mathematicians puzzled to effect it. A hasty reader is apt to misunderstand the conditions of the problem.

Another simple problem of this kind is to move a queen from the cell 33 to the cell 66 in fifteen moves entering every cell once

* *L'Intermédiaire des Mathématiciens*, Paris, 1901, vol. VIII, pp. 153—154.

† H. E. Dudeney, *The Tribune*, Dec. 3, 1906.

and only once, and never crossing its own track or entering a cell more than once*.

A somewhat similar, but more difficult, question is the determination of the greatest distance which can be travelled by a queen starting from its own square in five consecutive moves, subject to the condition that it never crosses its own track or enters a cell more than once†. In calculating the distance it may be assumed that the paths go through the centres of the cells. If the length of the side of a cell is one inch, the distance exceeds 33·97 inches.

Another familiar problem can be enunciated as follows. Construct a rectangular board of mn cells by ruling $m+1$ vertical lines and $n+1$ horizontal lines. It is required to know how many routes can be taken from the top left-hand corner to the bottom right-hand corner, the motion being along the ruled lines and its direction being always either vertically downwards or horizontally from left to right. The answer is the number of permutations of $m+n$ things, of which m are alike of one kind and n are alike of another kind: this is equal to $(m+n)!/m!n!$. Thus on a square board containing 16 cells (*i.e.* one-quarter of a chess-board), where $m=n=4$, there are 70 such routes; while on a common chess-board, where $m=n=8$, there are no less than 12870 such routes. A rook, moving according to the same law, can travel from the top left-hand cell to the bottom right-hand cell in $(m+n-2)!/(m-1)!(n-1)!$ ways. Similar theorems can be enunciated for a parallelepiped.

Another question of this kind is the determination of the number of closed routes through mn points arranged in m rows and n columns, following the lines of the quadrilateral net-work, and passing once and only once through each point‡.

Guarini's Problem. One of the oldest European problems connected with the chess-board is the following which was

* H. E. Dudeney, *The Tribune*, Oct. 3, 1906.

† *Ibid.*, Oct. 2, 1906.

‡ See C. F. Sainte-Marie in *L'Intermédiaire des Mathématiciens*, Paris, vol. XI, March, 1904, pp. 86—88.

propounded in 1512. It was quoted by Lucas in 1894, but I believe has not been published otherwise than in his works and the earlier editions of this book. On a board of nine cells, such as that drawn below, the two white knights are placed on the

a	*C*	*d*
D		*B*
b	*A*	*c*

two top corner cells (a, d), and the two black knights on the two bottom corner cells (b, c): the other cells are left vacant. It is required to move the knights so that the white knights shall occupy the cells b and c, while the black shall occupy the cells a and d. The solution is obvious.

Queens' Problem. Another problem consists in placing sixteen queens on a board so that no three are in a straight line *. One solution is to place them on the cells 15, 16, 25, 26, 31, 32, 41, 42, 57, 58, 67, 68, 73, 74, 83, 84. It is of course assumed that each queen is placed on the middle of its cell.

Latin Squares. Another problem of the chess-board type is the determination of the number x_n of Latin Squares of any assigned order n: a Latin Square of the nth order being defined as a square of n^2 cells (in n rows and n columns) in which n^2 letters consisting of n "a's," n "b's,"..., are arranged in the cells so that the n letters in each row and each column are different. The general theory is difficult †, but it may amuse my readers to verify the following results for some of the lower values of n: $x_2 = 2$, $x_3 = 6$, $x_4 = 576$, $x_5 = 149760$. Clearly x_n is a multiple of $n!\,(n-1)!$

* H. E. Dudeney, *The Tribune*, November 7, 1906.

† See P. A. MacMahon, *Combinatory Analysis*, Cambridge, 1915–16, vol. I, pp. 246—263; vol. II, pp. 323—326.

CHAPTER VII.

MAGIC SQUARES.

A *Magic Square* consists of a number of integers arranged in the form of a square, so that the sum of the numbers in every row, in every column, and in each diagonal is the same. If the integers are the consecutive numbers from 1 to n^2 the square is said to be of the nth order, and it is easily seen that in this case the sum of the numbers in any row, column, or diagonal is equal to $\frac{1}{2}n(n^2+1)$: this number may be denoted by N. Unless otherwise stated, I confine my account to such magic squares, that is, to squares formed with consecutive integers from 1 upwards. The same rules however cover similar problems with n^2 numbers in arithmetical progression.

Thus the first 16 integers, arranged in either of the forms given in figures i and ii below, represent magic squares of the

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure i.

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

Figure ii.

fourth order, the sum of the numbers in any row, column, or diagonal being 34. Similarly figure iii on page 140, figure vii on page 143, figure xii on page 153, and figure xvi on page 157 represent magic squares of the fifth order; figure xi on page 150 represents a magic square of the sixth order; figure xviii on page 160 represents a magic square of the seventh order, and figures xxii and xxiii on pages 163, 164 represent magic squares of the eighth order.

The formation of these squares is an old amusement, and in times when mystical philosophical ideas were associated with particular numbers it was natural that such arrangements should be deemed to possess magical properties. Magic squares of an odd order were constructed in India before the Christian era according to a law of formation which is explained hereafter. Their introduction into Europe appears to have been due to Moschopulus, who lived at Constantinople in the early part of the fifteenth century, and enunciated two methods for making such squares. The majority of the medieval astrologers and physicians were much impressed by such arrangements. In particular the famous Cornelius Agrippa (1486—1535) constructed magic squares of the orders 3, 4, 5, 6, 7, 8, 9, which were associated respectively with the seven astrological "planets": namely, Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon. He taught that a square of one cell, in which unity was inserted, represented the unity and eternity of God; while the fact that a square of the second order could not be constructed illustrated the imperfection of the four elements, air, earth, fire, and water; and later writers added that it was symbolic of original sin. A magic square engraved on a silver plate was sometimes prescribed as a charm against the plague, and one, namely, that represented in figure i on the last page, is drawn in the picture of Melancholy, painted in 1514 by Albert Dürer: the numbers in the middle cells of the bottom row give the date of the work. Such charms are still worn in the East.

The development of the theory was at first due mainly to French mathematicians. Bachet gave a rule for the construction of any square of an odd order in a form substantially equivalent to one of the rules given by Moschopulus. The formation of magic squares, especially of even squares, was considered by Frénicle and Fermat. The theory was continued by Poignard, De la Hire, Sauveur, D'Ons-en-bray, and Des Ourmes. Ozanam included in his work an essay on magic squares which was amplified by Montucla. Like most algebraical problems, the construction of magic squares attracted the attention of Euler, but he did not advance the general theory. In 1837 an elaborate work on the subject was compiled by B. Violle, which is useful as containing numerous illustrations. I give the references in a footnote*.

I shall confine myself to establishing rules for the construction of squares subject to no conditions beyond those given in the definition. I shall commence by giving rules for the construction of a square of an odd order, and then shall proceed to similar rules for one of an even order.

It will be convenient to use the following terms. The spaces or small squares occupied by the numbers are called *cells*. The diagonal from the top left-hand cell to the bottom right-hand cell is called the *leading diagonal* or *left diagonal*. The diagonal from the top right-hand cell to the bottom left-hand cell is called the *right diagonal*.

MAGIC SQUARES OF AN ODD ORDER. I proceed to give three methods for constructing *odd magic squares*, but for simplicity I shall apply them to the formation of squares of the fifth order; though exactly similar proofs will apply equally to any odd square.

* For a sketch of the history of the subject and its bibliography see S. Günther's *Geschichte der mathematischen Wissenschaften*, Leipzig, 1876, chapter iv; and W. Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chapter xii. The references in the text are to Bachet, *Problèmes plaisans*, Lyons, 1624, problem XXI, p. 161: Frénicle, *Divers Ouvrages de Mathématique par Messieurs de l'Académie des Sciences*, Paris, 1693, pp. 423—483; with an appendix (pp. 484—507), containing diagrams of all the possible magic squares of the fourth order, 880 in number: Fermat, *Opera Mathematica*, Toulouse, 1679, pp. 173—178; or Brassinne's *Précis*, Paris, 1853, pp. 146—149: Poignard, *Traité des Quarrés Sublimes*, Brussels, 1704: De la Hire, *Mémoires de l'Académie des Sciences* for 1705, Paris, 1706, part I, pp. 127—171; part II, pp. 364—382: Sauveur, *Construction des Quarrés Magiques*, Paris, 1710: D'Ons-en-bray, *Mémoires de l'Académie des Sciences* for 1750, Paris, 1754, pp. 241—271: Des Ourmes, *Mémoires de Mathématique et de Physique* (French Academy), Paris, 1763, vol. IV, pp. 196—241: Ozanam and Montucla, *Récréations*, part I, chapter xii: Euler, *Commentationes Arithmeticae Collectae*, St Petersburg, 1849, vol. II, pp. 593—602: Violle, *Traité Complet des Carrés Magiques*, 3 vols, Paris, 1837-8.

*De la Loubère's Method**. If the reader will look at figure iii he will see one way in which such a square containing 25 cells can be constructed. The middle cell in the top row is occupied by 1. The successive numbers are placed in their natural order in a diagonal line which slopes upwards to the right, except that (i) when the top row is reached the next number is written in the bottom row as if it came immediately above the top row; (ii) when the right-hand column is reached, the next number is written in the left-hand column, as if it immediately succeeded the right-hand column; and (iii) when a cell which has been filled up already, or when the top right-hand square is reached, the path of the series drops to the row vertically below it and then continues to mount again. Probably a glance at the diagram in figure iii will make this clear.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure iii.

15 + 2	20 + 4	0 + 1	5 + 3	10 + 5
20 + 3	0 + 5	5 + 2	10 + 4	15 + 1
0 + 4	5 + 1	10 + 3	15 + 5	20 + 2
5 + 5	10 + 2	15 + 4	20 + 1	0 + 3
10 + 1	15 + 3	20 + 5	0 + 2	5 + 4

Figure iv.

The reason why such a square is magic can be explained best by expressing the numbers in the scale of notation whose radix is 5 (or n, if the magic square is of the order n), except that 5 is allowed to appear as a unit-digit and 0 is not allowed to appear as a unit-digit. The result is shown in figure iv. From that figure it will be seen that the method of construction ensures that every row and every column shall contain one and only one of each of the unit-digits 1, 2, 3, 4, 5, the sum of which is 15; and also one and only one of each of the radix-digits 0, 5, 10, 15, 20, the sum of which is 50. Hence, as

* De la Loubère, *Du Royaume de Siam* (Eng. Trans.), London, 1693, vol. II, pp. 227—247. De la Loubère was the envoy of Louis XIV to Siam in 1687–8, and there learnt this method.

far as rows and columns are concerned, the square is magic. Moreover if the square is odd, each of the diagonals will contain one and only one of each of the unit-digits 1, 2, 3, 4, 5. Also the leading diagonal will contain one and only one of the radix-digits 0, 5, 10, 15, 20, the sum of which is 50; and if, as is the case in the square drawn above, the number 10 is the radix-digit to be added to the unit-digits in the right diagonal, then the sum of the radix-digits in that diagonal is also 50. Hence the two diagonals also possess the magical property.

And generally if a magic square of an odd order n is constructed by De la Loubère's method, every row and every column must contain one and only one of each of the unit-digits $1, 2, 3, \ldots, n$; and also one and only one of each of the radix-digits $0, n, 2n, \ldots, n(n-1)$. Hence, as far as rows and columns are concerned, the square is magic. Moreover each diagonal will either contain one and only one of the unit-digits or will contain n unit-digits each equal to $\frac{1}{2}(n+1)$. It will also either contain one and only one of the radix-digits or will contain n radix-digits each equal to $\frac{1}{2}n(n-1)$. Hence the two diagonals will also possess the magical property. Thus the square will be magic.

I may notice here that, if we place 1 in any cell and fill up the square by De la Loubère's rule, we shall obtain a square that is magic in rows and in columns, but it will not in general be magic in its diagonals.

It is evident that other squares can be derived from De la Loubère's square by permuting the symbols properly. For instance, in figure iv, we may permute the symbols 1, 2, 3, 4, 5 in 5! ways, and we may permute the symbols 0, 5, 15, 20 in 4! ways. Any one of these 5! arrangements combined with any one of these 4! arrangements will give a magic square. Hence we can obtain 2880 magic squares of the fifth order of this kind, though only 720 of them are really distinct. Other squares can however be deduced, for it may be noted that from any magic square, whether even or odd, other magic squares of the same order can be formed by the mere interchange of the row and the column which intersect in a cell on a diagonal with the row and the column which intersect in the complementary cell of the same diagonal.

Bachet proposed* a similar rule. In this, we begin by placing 1 in the cell above the middle one, and then we write the successive numbers in a diagonal line sloping upwards to the right, subject to the condition that when the cases (i) and (ii) mentioned under De la Loubère's method occur the rules there given are followed, but when the case (iii) occurs the path of the series, instead of going on to the cell already occupied, is continued from one cell to the cell next but one vertically above it. If this cell is above the top row the path continues from the corresponding cell in one of the bottom two rows following the analogy of rule (i) in De la Loubère's method. Bachet's method leads ultimately to this arrangement; except that the rules are altered so as to make the line slope downwards. This method also gives 720 magic squares of the fifth order.

In the notation given later (see pp. 157, 158), De la Loubère's rule is equivalent to taking steps $a = -1$, $b = 1$, and cross-steps $x = 1$, $y = 0$. Bachet's form of it, as here enunciated, is equivalent to taking steps $a = -1$, $b = 1$, and cross-steps $x = -2$, $y = 0$.

De la Hire's Method†. I shall now give another rule for the formation of odd magic squares. To form an odd magic square of the order n by this method, we begin by constructing two subsidiary squares, one of the unit-digits, 1, 2, ..., n, and the other of multiples of the radix, namely, 0, n, $2n$, ..., $(n-1)n$. We then form the magic square by adding together the numbers in the corresponding cells in the two subsidiary squares.

De la Hire gave several ways of constructing such subsidiary squares. I select the following method (props. x and xiv of his memoir) as being the simplest, but I shall apply it to form a square of only the fifth order. It leads to the same results as the second of the two rules given by Moschopulus.

* Bachet, Problem xxi, p. 161.

† *Mémoires de l'Académie des Sciences* for 1705, part i, pp. 127—171.

The first of the subsidiary squares (figure v, below) is constructed thus. First, 3 is put in the top left-hand corner, and then the numbers 1, 2, 4, 5 are written in the other cells of the top line (in any order). Next, the number in each cell of the top line is repeated in all the cells which lie in a diagonal line sloping downwards to the right (see figure v) according to the rule (ii) in De la Loubère's method. The cells filled by the same number form a *broken diagonal*. It follows that every row and every column contains one and only one 1, one and only one 2, and so on. Hence the sum of the numbers in every row and in every column is equal to 15; also, since we placed 3, which is the average of these numbers, in the top left-hand corner, the sum of the numbers in the left diagonal is 15;

3	4	1	5	2
2	3	4	1	5
5	2	3	4	1
1	5	2	3	4
4	1	5	2	3

First Subsidiary Square.
***Figure* v.**

15	0	20	5	10
0	20	5	10	15
20	5	10	15	0
5	10	15	0	20
10	15	0	20	5

Second Subsidiary Square.
***Figure* vi.**

18	4	21	10	12
2	23	9	11	20
25	7	13	19	1
6	15	17	3	24
14	16	5	22	8

Resulting Magic Square.
***Figure* vii.**

and, since the right diagonal contains one and only one of each of the numbers 1, 2, 3, 4, and 5, the sum of the numbers in that diagonal also is 15.

The second of the subsidiary squares (figure vi) is constructed in a similar way with the numbers 0, 5, 10, 15, and 20, except that the mean number 10 is placed in the top right-hand corner; and the broken diagonals formed of the same numbers all slope downwards to the left. It follows that every row and every column in figure vi contains one and only one 0, one and only one 5, and so on; hence the sum of the numbers in every row and every column is equal to 50. Also the sum of the numbers in each diagonal is equal to 50.

If now we add together the numbers in the corresponding cells of these two squares, we shall obtain 25 numbers such that the sum of the numbers in every row, every column, and each diagonal is equal to $15 + 50$, that is, to 65. This is represented in figure vii. Moreover, no two cells in that figure contain the same number. For instance, the numbers 21 to 25 can occur only in those five cells which in figure vi are occupied by the number 20, but the corresponding cells in figure v contain respectively the numbers 1, 2, 3, 4, and 5; and thus in figure vii each of the numbers from 21 to 25 occurs once and only once. De la Hire preferred to have the cells in the subsidiary squares which are filled by the same number connected by a knight's move and not by a bishop's move; and usually his rule is enunciated in that form.

By permuting the numbers 1, 2, 4, 5 in figure v we get 4! arrangements, each of which combined with that in figure vi would give a magic square. Similarly by permuting the numbers 0, 5, 15, 20 in figure vi we obtain 4! squares, each of which might be combined with any of the 4! arrangements deduced from figure v. Hence altogether we can obtain in this way 144 magic squares of the fifth order.

There are various other methods by which odd magic squares of any order can be constructed, but most or all of them depend on the form of n. I content myself here with the two methods described above; later, when discussing pandiagonal squares, I shall mention another rule for odd magic squares whose order is higher than three, which permits us to place a selected number in any cell we like.

MAGIC SQUARES OF AN EVEN ORDER. The above methods are inapplicable to squares of an even order. I proceed to give two methods for constructing any *even magic square* of an order higher than two.

It will be convenient to use the following terms. Two rows which are equidistant, the one from the top, the other from the bottom, are said to be *complementary*. Two columns which are equidistant, the one from the left-hand side, the other from the right-hand side, are said to be *complementary*. Two cells in the same row, but in complementary columns, are said to be *horizontally related*. Two cells in the same column, but in complementary rows, are said to be *vertically related*. Two cells in complementary rows and columns are said to be *skewly related*; thus, if the cell b is horizontally related to the cell a, and the cell d is vertically related to the cell a, then the cells b and d are skewly related; in such a case if the cell c is vertically related to the cell b, it will be horizontally related to the cell d, and the cells a and c are skewly related: the cells a, b, c, d constitute an *associated group*, and if the square is divided into four equal quarters, one cell of an associated group is in each quarter.

A *horizontal interchange* consists in the interchange of the numbers in two horizontally related cells. A *vertical interchange* consists in the interchange of the numbers in two vertically related cells. A *skew interchange* consists in the interchange of the numbers in two skewly related cells. A *cross interchange* consists in the change of the numbers in any cell and in its horizontally related cell with the numbers in the cells skewly related to them; hence, it is equivalent to two vertical interchanges and two horizontal interchanges.

*First Method**. This method is the simplest with which I am acquainted. Begin by filling the cells of the square with the numbers $1, 2, \ldots, n^2$ in their natural order commencing (say) with the top left-hand corner, writing the numbers in each row from left to right, and taking the rows in succession from the top. I will commence by proving that a certain number of horizontal and vertical interchanges in such a square must make it magic, and will then give a rule by which the cells whose numbers are to be interchanged can be at once picked out.

First we may notice that the sum of the numbers in each diagonal is equal to N, where $N = n(n^2+1)/2$; hence the diagonals are already magic, and will remain so if the numbers therein are not altered.

Next, consider the rows. The sum of the numbers in the

* It seems to have been first enunciated in 1889 by W. Firth, but later was independently discovered by various writers: see the *Messenger of Mathematics, Cambridge*, September, 1893, vol. XXIII, pp. 65—69, and the *Monist*, Chicago, 1912, vol. XXII, pp. 53—81. I leave my account as originally written, though perhaps the procedure used by C. Planck in the latter paper is somewhat simpler.

xth row from the top is $N - n^2(n - 2x + 1)/2$. The sum of the numbers in the complementary row, that is, the xth row from the bottom, is $N + n^2(n - 2x + 1)/2$. Also the number in any cell in the xth row is less than the number in the cell vertically related to it by $n(n - 2x + 1)$. Hence, if in these two rows we make $n/2$ interchanges of the numbers which are situated in vertically related cells, then we increase the sum of the numbers in the xth row by $n \times n(n - 2x + 1)/2$, and therefore make that row magic; while we decrease the sum of the numbers in the complementary row by the same number, and therefore make that row magic. Hence, if in every pair of complementary rows we make $n/2$ interchanges of the numbers situated in vertically related cells, the square will be made magic in rows. But, in order that the diagonals may remain magic, either we must leave both the diagonal numbers in any row unaltered, or we must change both of them with those in the cells vertically related to them.

The square is now magic in diagonals and in rows, and it remains to make it magic in columns. Taking the original arrangement of the numbers (in their natural order) we might have made the square magic in columns in a similar way to that in which we made it magic in rows. The sum of the numbers originally in the yth column from the left-hand side is $N - n(n - 2y + 1)/2$. The sum of the numbers originally in the complementary column, that is, the yth column from the right-hand side, is $N + n(n - 2y + 1)/2$. Also the number originally in any cell in the yth column was less than the number in the cell horizontally related to it by $n - 2y + 1$. Hence, if in these two columns we had made $n/2$ interchanges of the numbers situated in horizontally related cells, we should have made the sum of the numbers in each column equal to N. If we had done this in succession for every pair of complementary columns, we should have made the square magic in columns. But, as before, in order that the diagonals might remain magic, either we must have left both the diagonal numbers in any column unaltered, or we must have changed both of them with those in the cells horizontally related to them.

It remains to show that the vertical and horizontal interchanges, which have been considered in the last two paragraphs, can be made independently, that is, that we can make these interchanges of the numbers in complementary columns in such a manner as will not affect the numbers already interchanged in complementary rows. This will require that in every column there shall be exactly $n/2$ interchanges of the numbers in vertically related cells, and that in every row there shall be exactly $n/2$ interchanges of the numbers in horizontally related cells. I proceed to show how we can always ensure this, if n is greater than 2. I continue to suppose that the cells are initially filled with the numbers $1, 2, \ldots, n^2$ in their natural order, and that we work from that arrangement.

A *doubly-even square* is one where n is of the form $4m$. If the square is divided into four equal quarters, the first quarter will contain $2m$ columns and $2m$ rows. In each of these columns take m cells so arranged that there are also m cells in each row, and change the numbers in these $2m^2$ cells and the $6m^2$ cells associated with them by a cross interchange. The result is equivalent to $2m$ interchanges in every row and in every column, and therefore renders the square magic.

One way of selecting the $2m^2$ cells in the first quarter is to divide the whole square into sixteen subsidiary squares each

a	*b*	*b*	*a*
b	*a*	*a*	*b*
b	*a*	*a*	*b*
a	*b*	*b*	*a*

Figure viii.

containing m^2 cells, which we may represent by the diagram above, and then we may take either the cells in the *a* squares or those in the *b* squares; thus, if every number in the eight *a* squares is interchanged with the number skewly related to it the resulting square is magic. A magic square of the eighth order, constructed in this way, is shown in figure xxiii on page 164.

Another way of selecting the $2m^2$ cells in the first quarter would be to take the first m cells in the first column, the cells 2 to $m+1$ in the second column, and so on, the cells $m+1$ to $2m$ in the $(m+1)$th column, the cells $m+2$ to $2m$ and the first cell in the $(m+2)$th column, and so on, and finally the $2m$th cell and the cells 1 to $m-1$ in the $2m$th column.

A *singly-even square* is one where n is of the form $2(2m+1)$. If the square is divided into four equal quarters, the first quarter will contain $2m+1$ columns and $2m+1$ rows. In each of these columns take m cells so arranged that there are also m cells in each row: as, for instance, by taking the first m cells in the first column, the cells 2 to $m+1$ in the second column, and so on, the cells $m+2$ to $2m+1$ in the $(m+2)$th column, the cells $m+3$ to $2m+1$ and the first cell in the $(m+3)$th column, and so on, and finally the $(2m+1)$th cell and the cells 1 to $m-1$ in the $(2m+1)$th column. Next change the numbers in these $m(2m+1)$ cells and the $3m(2m+1)$ cells associated with them by cross interchanges. The result is equivalent to $2m$ interchanges in every row and in every column. In order to make the square magic we must have $n/2$, that is, $2m+1$, such interchanges in every row and in every column, that is, we must have one more interchange in every row and in every column. This presents no difficulty; for instance, in the arrangement indicated above the numbers in the $(2m+1)$th cell of the first column, in the first cell of the second column, in the second cell of the third column, and so on, to the $2m$th cell in the $(2m+1)$th column may be interchanged with the numbers in their vertically related cells; this will make all the rows magic. Next, the numbers in the $2m$th cell of the first column, in the $(2m+1)$th cell of the second column, in the first cell of the third column, in the second cell of the fourth column, and so on, to the $(2m-1)$th cell of the $(2m+1)$th column may be interchanged with those in the cells horizontally related to them; and this will make the columns magic without affecting the magical properties of the rows.

It will be observed that we have implicitly assumed that m is not zero, that is, that n is greater than 2; also it would seem that, if $m = 1$ and therefore $n = 6$, then the numbers in the diagonal cells must be included in those to which the cross interchange is applied, but, if n is greater than 6, this is not necessary, though it may be convenient.

The construction of odd magic squares and of doubly-even magic squares is very easy. But though the rule given above for singly-even squares is not difficult, it is tedious of application. It is unfortunate that no more obvious rule—such, for instance, as one for bordering a doubly-even square—can be suggested for writing down instantly and without thought singly-even magic squares.

*De la Hire's Method**. I now proceed to give another way, due to De la Hire, of constructing any even magic square of an order higher than two.

In the same manner as in his rule for making odd magic squares, we begin by constructing two subsidiary squares, one of the unit-digits $1, 2, 3, \ldots, n$, and the other of the radix-digits $0, n, 2n, \ldots, (n-1)n$. We then form the magic square by adding together the numbers in the corresponding cells in the two subsidiary squares. Following the analogy of the notation used above, two numbers which are equidistant from the ends of the series $1, 2, 3, \ldots, n$ are said to be *complementary*. Similarly numbers which are equidistant from the ends of the series $0, n, 2n, \ldots, (n-1)n$ are said to be *complementary*.

For simplicity I shall apply this method to construct a magic square of only the sixth order, though an exactly similar method will apply to any even square of an order higher than the second.

The first of the subsidiary squares (figure ix) is constructed as follows. First, the cells in the leading diagonal are filled with the numbers 1, 2, 3, 4, 5, 6 placed in any order whatever that puts complementary numbers in complementary positions

* The rule is due to De la Hire (part 2 of his memoir) and is given by Montucla in his edition of Ozanam's work: I have used the modified enunciation of it inserted in Labosne's edition of Bachet's *Problèmes*, as it saves the introduction of a third subsidiary square. I do not know to whom the modification is due.

(*ex. gr.* in the order 2, 6, 3, 4, 1, 5, or in their natural order 1, 2, 3, 4, 5, 6). Second, the cells vertically related to these are filled respectively with the same numbers. Third, each of the remaining cells in the first vertical column is filled either with the same number as that already in two of them or with the complementary number (*ex. gr.* in figure ix with a "1" or a "6") in any way, provided that there are an equal number of each of these numbers in the column, and subject also to proviso (ii) mentioned in the next paragraph. Fourth, the cells horizontally related to those in the first column are filled with the complementary numbers. Fifth, the remaining cells in the second and third columns are filled in an analogous way

1	5	4	3	2	6
6	2	4	3	5	1
6	5	3	4	2	1
1	5	3	4	2	6
6	2	3	4	5	1
1	2	4	3	5	6

First Subsidiary Square.
Figure ix.

0	30	30	0	30	0
24	6	24	24	6	6
18	18	12	12	12	18
12	12	18	18	18	12
6	24	6	6	24	24
30	0	0	30	0	30

Second Subsidiary Square.
Figure x.

1	35	34	3	32	6
30	8	28	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

Resulting Magic Square.
Figure xi.

to that in which those in the first column were filled: and then the cells horizontally related to them are filled with the complementary numbers. The square so formed is necessarily magic in rows, columns, and diagonals.

The second of the subsidiary squares (figure x) is constructed as follows. First, the cells in the leading diagonal are filled with the numbers 0, 6, 12, 18, 24, 30 placed in any order whatever that puts complementary numbers in complementary positions. Second, the cells horizontally related to them are filled respectively with the same numbers. Third, each of the remaining cells in the first horizontal row is filled either with the same number as that already in two of them or with the complementary number (*ex. gr.* in figure x with a "0" or a "30") in any way, provided (i) that there are an equal number of each of these numbers in the row, and (ii) that if any cell in the first row of figure ix and its vertically related cell are filled with complementary numbers, then the corresponding cell in the first row of figure x and its horizontally related cell must be occupied by the same number*. Fourth, the cells vertically related to those in the first row are filled with the complementary numbers. Fifth, the remaining cells in the second and the third rows are filled in an analogous way to that in which those in the first row were filled: and then the cells vertically related to them are filled with the complementary numbers. The square so formed is necessarily magic in rows, columns, and diagonals.

It remains to show that proviso (ii) in the third step described in the last paragraph can be satisfied always. In a doubly-even square, that is, one in which n is divisible by 4, we need not have any complementary numbers in vertically related cells in the first subsidiary square unless we please, but even if we like to insert them they will not interfere with the satisfaction of this proviso. In the case of a singly-even square, that is, one in which n is divisible by 2, but not by 4, we cannot satisfy the proviso if any horizontal row in the first square has all its vertically related squares, other than the two squares in the diagonals, filled with complementary numbers. Thus in the case of a singly-even square it will be necessary in constructing the first square to take care in the third step that in every row at least one cell which is not in a diagonal shall have its vertically related cell filled with the same number as itself: this is always possible if n is greater than 2.

The required magic square will be constructed if in each cell we place the sum of the numbers in the corresponding cells of the subsidiary squares, figures ix and x. The result of this is given in figure xi. The square is evidently magic. Also every number from 1 to 36 occurs once and only once, for the numbers from 1 to 6 and from 31 to 36 can occur only in the top or the bottom rows, and the method of construction ensures that the

* The insertion of this step evades the necessity of constructing (as Montucla did) a third subsidiary square.

same number cannot occur twice. Similarly the numbers from 7 to 12 and from 25 to 30 occupy two other rows, and no number can occur twice; and so on. The square in figure i on page 137 may be constructed by the above rules; and the reader will have no difficulty in applying them to any other even square.

OTHER METHODS FOR CONSTRUCTING ANY MAGIC SQUARE. The above methods appear to me to be the simplest which have been proposed. There are however *two other methods*, of less generality, to which I will allude briefly in passing. Both depend on the principle that, if every number in a magic square is multiplied by some constant, and a constant is added to the product, the square will remain magic.

The *first method* applies only to such squares as can be divided into smaller magic squares of some order higher than two. It depends on the fact that, if we know how to construct magic squares of the mth and nth orders, we can construct one of the mnth order. For example, a square of 81 cells may be considered as composed of 9 smaller squares each containing 9 cells, and by filling the cells in each of these small squares in the same relative order and taking the small squares themselves in the same order, the square can be constructed easily. Such squares are called *Composite Magic Squares*.

The *second method*, which was introduced by Frénicle, consists in surrounding a magic square with a *Border*. Thus in figure xii on page 153 the inner square is magic, and it is surrounded with a border in such a way that the whole square is also magic. In this manner from the magic square of the 3rd order we can build up successively squares of the orders 5, 7, 9, &c., that is, any odd magic square. Similarly from the magic square of the 4th order we can build up successively any higher even magic square.

If we construct a magic square of the first n^2 numbers by bordering a magic square of $(n-2)^2$ numbers, the usual process is to reserve for the $4(n-1)$ numbers in the border the first $2(n-1)$ natural numbers and the last $2(n-1)$ numbers. Now the sum of the numbers in each line of a square of the order

$(n-2)$ is $\frac{1}{2}(n-2)\{(n-2)^2+1\}$, and the average is $\frac{1}{2}\{(n-2)^2+1\}$. Similarly the average number in a square of the nth order is $\frac{1}{2}(n^2+1)$. The difference of these is $2(n-1)$. We begin then by taking any magic square of the order $(n-2)$, and we add to every number in it $2(n-1)$; this makes the average number $\frac{1}{2}(n^2+1)$.

The numbers reserved for the border occur in pairs, n^2 and 1, n^2-1 and 2, n^2-2 and 3, &c., such that the average of each pair is $\frac{1}{2}(n^2+1)$, and they must be bordered on the square so that these numbers are opposite to one another. Thus the bordered square will be necessarily magic, provided that the sum of the numbers in two adjacent sides of the external border is correct. The arrangement of the numbers in the borders will be somewhat facilitated if the number n^2+1-p (which has to be placed opposite to the number p) is denoted by $\bar{p}$, but it is not worth while going into further details here.

1	2	19	20	23
18	16	9	14	8
21	11	13	15	5
22	12	17	10	4
3	24	7	6	25

Bordered Magic Square.
Figure xii.

Since a doubly-even square can be formed with ease, this method would provide a simple way of constructing a singly-even square if definite rules for bordering a square could be laid down. I am however unable to formulate precise instructions for this purpose, though it is possible to give directions which require only slight and easy empirical work. It will sufficiently illustrate the general method if I explain how the square in figure xii is constructed. A magic square of the third order is formed by De la Loubère's rule, and to every number in it 8 is added: the result is the inner square in figure xii. The numbers not used are 25 and 1, 24 and 2, 23 and 3, 22 and 4, 21 and 5, 20 and 6, 19 and 7, 18 and 8. The sum of each pair is 26, and obviously they must be placed at opposite ends of any line.

With a little patience a magic square of any order can be thus built up, border upon border, and of course it will have the property that, if each border is successively stripped off, the remaining square will still be magic. This is a method of construction constantly adopted by self-taught mathematicians.

I may add here (figure xiii) the following general solution of a magic square of the fourth order in which any numbers (not necessarily consecutive) are used*. There are eight independent quantities.

$V-v$	$Y+v+y$	$X+x-y$	$Z-x$
$Z+v-z$	X	Y	$V-v+z$
$Y-x+z$	V	Z	$X+x-z$
$X+x$	$Z-v-y$	$V-x+y$	$Y+v$

Figure xiii.

32768	4	2	4096
16	512	1024	128
256	32	64	2048
8	16384	8192	1

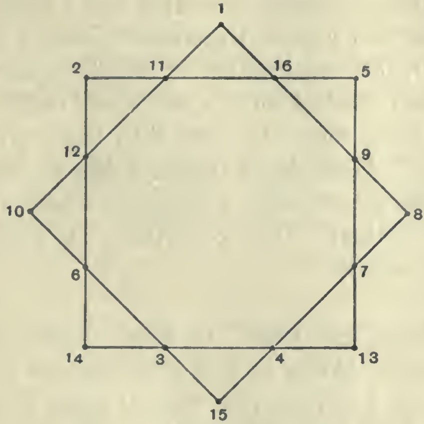
Figure xiv.

I may also mention that Montucla suggested the construction of squares whose cells are occupied by numbers such that the product of the numbers in each row, column, and diagonal is constant. The formation of such figures is immediately deducible from that of magic squares, for if the consecutive numbers in a magic square are replaced by consecutive powers of any number m the products of the numbers in every line will be magic. This is obvious, for if the numbers in any line of a square are a, a', a'', &c., such that Σa is constant for every line in the square, then Πm^a is also constant. For instance, figure xiv represents such a magic figure in which the product of the numbers in each line is 1,073,741,824; it is constructed from the magic square given in figure i on page 137.

Magic Stars. Some elegant magic constructions on star-shaped figures (pentagons, hexagons, &c.) may be noticed in passing, though I will not go into details. One instance will suffice. Suppose a re-entrant octagon is constructed by the

* E. Bergholt, *Nature*, London, vol. LXXXIII, May 26, 1910, pp. 368—369.

intersecting sides of two equal concentric squares. It is required to place the first 16 natural numbers on the corners and points of intersection of the sides so that the sum of the numbers on the corner of each square and the sum of the numbers on every



Magic Star.
Figure XV.

side of each square is equal to 34. Eighteen fundamental solutions exist. One of these is given above*.

There are magic circles, rectangles, crosses, diamonds, and other figures: also magic cubes, cylinders, and spheres. The theory of the construction of such figures is of no value, and I cannot spare the space to describe rules for forming them.

In the above sketch, two questions remain unsolved. One is the determination of a definite rule for bordering a square; such a rule might lead to a simpler method than that given above for forming oddly-even squares. The other is the determination of the number of magic squares of the fifth (or any higher) order. There is, in effect, only one magic square of the third order though by reflexions and rotations it can be presented in 9 forms. There are 880 magic squares of the fourth order, but by reflexions and rotations these can be presented in 7040 forms. De la Hire showed that,

* Communicated to me by Mr R. Strachey.

apart from mere reflexions and rotations, there were 57600 magic squares of the fifth order which could be formed by the methods he enumerated. Taking account of other methods, it would seem that the total number of magic squares of the fifth order is very large, perhaps exceeding half a million. Notwithstanding these two unsolved problems we may fairly say that the theory of the construction of magic squares as defined above has been worked out in sufficient detail—though not exhaustively, since methods other than those given above may be expounded. Accordingly attention has of late been chiefly directed to the construction of squares which, in addition to being magic, satisfy other conditions. I shall term such squares hyper-magic.

HYPER-MAGIC SQUARES. Of hyper-magic squares, I will deal only with the theory of Pan-Diagonal and of Symmetrical Squares, though I will describe without going into details what are meant by Doubly and Trebly Magic Squares.

Pandiagonal Squares. One of the earliest additional conditions to be suggested was that the square should be magic along the broken diagonals as well as along the two ordinary diagonals*. Such squares are called *Pandiagonal.* They are also known as Nasik, or perfect, or diabolic squares.

For instance, a magic pandiagonal square of the fourth order is represented in figure ii on page 137. In it the sum of the numbers in each row, column, and in the two diagonals is 34, as also is the sum of the numbers in the six broken diagonals formed by the numbers 15, 9, 2, 8, the numbers 10, 4, 7, 13, the numbers 3, 5, 14, 12, the numbers 6, 4, 11, 13, the numbers 3, 9, 14, 8, and the numbers 10, 16, 7, 1.

It follows from the definition that if a pandiagonal square be cut into two pieces along a line between any two rows or

* Squares of this type were mentioned by De la Hire, Sauveur, and Euler. Attention was again called to them by A. H. Frost in the *Quarterly Journal of Mathematics*, London, 1878, vol. xv, pp. 34—49, and subsequently their properties have been discussed by several writers. Besides Frost's papers I have made considerable use of a paper by E. McClintock in the *American Journal of Mathematics*, vol. xix, 1897, pp. 99—120.

any two columns, and the two pieces be interchanged, the new square so formed will be also pandiagonally magic. Hence it is obvious that by one vertical and one horizontal transposition of this kind any number can be made to occupy any specified cell.

Pandiagonal magic squares of an odd order can be constructed by a rule somewhat analogous to that given by De la Loubère, and described above. I proceed to give an outline of the method.

If we write the numbers in the scale of notation whose radix is n, with the understanding that the unit-digits run from 1 to n, it is evident, as in the corresponding explanation of why De la Loubère's rule gives a magic square, that all we have to do is to ensure that each row, column, and diagonal (whether broken or not) shall contain one and only one of each of the unit-digits, as also one and only one of each of the radix-digits.

7	20	3	11	24
13	21	9	17	5
19	2	15	23	6
25	8	16	4	12
1	14	22	10	18

Figure xvi.

5+2	15+5	0+3	10+1	20+4
10+3	20+1	5+4	15+2	0+5
15+4	0+2	10+5	20+3	5+1
20+5	5+3	15+1	0+4	10+2
0+1	10+4	20+2	5+5	15+3

Figure xvii.

This is seen to be the case in the square of the fifth order delineated above, figures xvi and xvii.

Let us suppose that we write the numbers consecutively, and proceed from cell to cell by steps, using the term *step* (a, b) to denote going a cells to the right and b cells up. Thus a step (a, b) will take us from any cell to the ath column to the right of it, to the bth row above it, to the $(b+a)$th diagonal above it sloping down to the right, and to the $(b-a)$th diagonal above it sloping down to the left. In all cases we have the convention, as in De la Loubère's rule, that the movements along lines are taken cyclically; thus a step $n+a$ is equivalent to a step a. Of course, also, if a means going a cells to the right, then $-a$ will mean going a cells to the left; thus if the bth upper line is outside the square we take it as equivalent to the $(n-b)$th lower line.

It is clear that a and b cannot be zero, and that if a and b are prime to n (that is, if each has no divisor other than unity which also divides n) we can make $n-1$ steps from any cell from which we start, before we come to a cell already occupied. Thus the first n numbers form a path which will give a different unit-digit in every row, column, and in one set of n diagonals; of the other diagonals, $n-1$ are empty, and one contains every unit-digit—thus they are constructed on magical lines. We must take some other step (h, k) from the cell n to get to an unoccupied cell in which we place the number $n+1$. Continuing the process with $n-1$ more steps (a, b) we get another series of n numbers in various cells. If h and k are properly selected this second series will not interfere with the first series, and the rows, columns, and diagonals, as thus built up, will continue to be constructed on magical lines provided h and k are chosen so that the same unit-digit does not appear more than once in any row, column, and diagonal. We will suppose that this can be done, and that another cross-step (h, k) of the same form as before enables us to continue filling in the numbers in compliance with the conditions, and that this process can be continued until the square is filled. If this is possible, the whole process will consist of n series of n steps, each series consisting of $n-1$ uniform steps (a, b) followed by one cross-step (h, k). The numbers inscribed after the n cross-steps will be $n+1$, $2n+1$, $3n+1$, ..., and these will be themselves connected by uniform steps (u, v), where $u=(n-1)a+k \equiv k-a$, mod. n, and $v=(n-1)b+k \equiv k-b$.

I proceed to investigate the conditions that a, b, h, and k must satisfy in order that the square can be constructed as above described with uniform steps (a, b) and (h, k). We notice at once that in order to secure the magic property in the rows and columns, we must have a and b prime to n; and to secure it in the diagonals, we must have a and b unequal and $b+a$ and $b-a$ prime to n. The leading numbers of the

n sequences of n numbers, namely 1, $n+1$, $2n+1$, ..., are connected by steps (u, v), where $u \equiv h-a$ and $v \equiv k-b$. Hence, if these are to fit in their places, we must also have u and v unequal, and u, v, $u+v$, and $u-v$ prime to n. Also a, b, u, and v cannot be zero. Lastly the cross-steps (h, k) must be so chosen that in no case shall a cross-step lead to a cell already occupied. This would happen, and therefore the rule would fail, if p steps (a, b) from any cell and q steps (u, v) from it, where p and q are each less than n, should lead to the same cell. Thus, to the modulus n, we cannot have $pa \equiv qu \equiv q(h-a)$, and at the same time $pb \equiv qv \equiv q(k-b)$.

It is impossible to satisfy these conditions if n is equal to 3 or to a multiple of 3. For a and b are to be unequal, not zero, and less than n, and $a+b$ is to be less than n and prime to n. Thus we cannot construct a pandiagonal square of the third order.

Next I will show that, if n is not a multiple of 3, these conditions are satisfied when $a=1$, $b=2$, $h=0$, $k=-1$, and therefore that in this case these values provide a particular solution of the general problem. It is at once obvious that in this case a and b are unequal, not zero, and prime to n, that $b+a$ and $b-a$ are prime to n, and that the corresponding relations for u and v are true. The remaining condition for the validity of a rule based on these particular steps is that it shall be impossible to find integral values of p and q each less than n, which will simultaneously make $p \equiv -q$, and $2p \equiv -3q$. This condition is satisfied. Hence, any odd pandiagonal square of an order which is not a multiple of 3 can be constructed by this rule. Thus, to form a pandiagonal square of the fifth order we may put 1 in any cell; proceed by four successive steps, like a knight's move, of one cell to the right and two cells up, writing consecutively numbers 2, 3, 4, 5 in each cell, until we come to a cell already occupied; then take one step, like a rook's move, one cell down, and so on until the square is filled. This is illustrated by the square delineated in figure xvi.

Further discussion of the general case depends on whether or not n is prime; here I will confine myself to the simpler alternative, and assume that n is prime: this will sufficiently illustrate the theory. From the above relations it follows that we cannot have $pqa(k-b) \equiv pqb(h-a)$, that is, $pq(ak-bh) \equiv 0$. Therefore $ak-bh$ cannot be a multiple of n, that is, it must be prime to n. If this condition is fulfilled, as well as the other conditions given above, each cross-step (h, k) can be made in due sequence, and the square can be constructed. The result that $ak-bh$ is prime to n shows that the cross-step (h, k) must be chosen so as to take us to an unoccupied ce'l not in the same row, column, or diagonal (broken or not) as the initial number. By noting this fact we can in general place any two given numbers in two assigned cells.

There are some advantages in having the cross-steps uniform with the other steps, since, as we shall see later, the square can then be written in a form symmetrical about the centre. This will be effected if we take $h=-b$, $k=a$. If n is prime our conditions are then satisfied if b be any number from 2 to $(n-1)/2$, if a be positive and less than b, and if a^2+b^2 be prime to n. We can, if we prefer, take $h=b$, $k=-a$; but it is not possible to take $h=a$ and $k=-b$, or $h=-a$ and $k=b$, since they make $u=0$ or $v=0$.

For instance, if we use a knight's move, we may take $a=1$, $b=2$. The square of the seventh order given below (figure xviii)

35	23	18	13	1	45	40
4	48	36	31	26	21	9
22	17	12	7	44	39	34
47	42	30	25	20	8	3
16	11	6	43	38	33	28
41	29	24	19	14	2	46
10	5	49	37	32	27	15

A Pandiagonal Symmetrical Square.
Figure xviii.

is constructed by this rule. But in the case of a square of the fifth order we cannot use a knight's move, since, if $a=1$, and $b=2$, we have $a^2+b^2=5$. Hence the use of a knight's move is not applicable when n is 5, or a multiple of 5.

The construction of singly-even pandiagonal squares (that is, those whose order is $4m+2$) is impossible, but that of doubly-even squares (that is, those whose order is $4m$) is possible.

Here is one way of constructing a doubly-even square. Suppose the order of the square is $4m$, and as before let us write the number in a cell in the scale $4m$, that is, as $4mp+r$, so that p and r are the radix and unit-digits, with the convention that r cannot be zero. Place $p_1, p_2, p_3, \ldots, p_{4m}$ in order in the cells in the bottom row. Proceeding from p_1 by steps $(2m, 1)$ fill up $2m$ cells with it. And proceed similarly with the other radix-digits. Next place $r_1, r_2, \ldots, r_{4m}$ in order in the cells in the first column. Proceeding from r_1 by steps $(1, 2m)$ fill up $2m$ cells with it. And proceed similarly with the other unit-digits. Then if we take for $r_1, r_2, \ldots, r_{4m}$, the values $1, 2, \ldots, 2m, 4m, 4m-1, \ldots, 2m+1$, and for $p_1, p_2, \ldots, p_{4m}$, the values $0, 1, \ldots, 2m-1, 4m-1, \ldots, 2m$, the square will be pandiagonally magic. I leave the demonstration to my readers. The resulting square in the case when $m=1$, $n=4$, and the p and r subsidiary squares are shown below. This is the square represented in figure ii on page 137.

3	2	0	1
0	1	3	2
3	2	0	1
0	1	3	2

Subsidiary p Square.
Figure xix.

3	2	3	2
4	1	4	1
2	3	2	3
1	4	1	4

Subsidiary r Square.
Figure xx.

12+3	8+2	0+3	4+2
0+4	4+1	12+4	8+1
12+2	8+3	0+2	4+3
0+1	4+4	12+1	8+4

Resulting Square.
Figure xxi.

The rows, columns, and all diagonals of pandiagonal squares possess the magic property. So also do a group of any n numbers connected cyclically by steps (c, d) provided the first two numbers of the group are such that when divided by n they have either different quotients or different remainders. Such groups include rows, columns, and diagonals as particular cases. Thus in the square delineated in figure ii on page 137 the numbers 1, 7, 10, 16 form a magic group whose sum is 34, connected cyclically by steps (1, 3). Again in the square delineated in figure xviii on page 160, 10, 30, 1, 28, 48, 19, 39 form a magic group connected cyclically by steps (2, 3).

Symmetrical Squares. It has been suggested that we might impose on the construction of a magic square of the order n the condition that the sum of any two numbers in cells geometrically symmetrical to the centre (*ex. gr.* 22 and 28 in figure xviii) shall be constant and equal to n^2+1. Such squares are called *Symmetrical.*

The construction of odd symmetrical squares of the order n, when n is prime to 3 and 5, involves no difficulty. We can begin by placing the mean number in the middle cell and work from that, either in both directions or forwards, making the number 1 follow after n^2; we can also effect the same result by constructing any pandiagonal square of the order n and then transposing a certain number of rows and columns. If the rule given above on page 160, where $a=1$, $b=2$, $h=2$, $k=-1$, be followed, this will lead to placing the number 1 in the $(n+3)/2$th cell of the top row: see, for instance, figure xviii.

Such a square must be symmetrical, for if we begin with the middle number $(n^2+1)/2$, which I will denote by m, in the middle cell, and work from it forwards with the numbers $m+1$, $m+2$, ..., and backwards with the numbers $m-1$, $m-2$, ..., the pairs of cells filled by the numbers $m+1$ and $m-1$, $m+2$ and $m-2$, &c., are necessarily situated symmetrically to the middle cell and the sum of each pair is $2m$. I believe this was first pointed out by McClintock.

The construction of doubly-even symmetrical pandiagonal squares is also possible, but the analysis is too lengthy for me to find room for it here.

In a symmetrical square any n such pairs of numbers together with the number in the middle cell will form a magic group. For instance in figure xviii, the group 32, 18, 36, 14, 47, 3, and 25 is magic. So also is the group 47, 3, 35, 15, 13, 37, and 25. Thus in a symmetrical pandiagonal square, even of a low order, there are hundreds of magic groups of n numbers whose sum is constant.

Doubly-Magic Squares. In another species of hyper-magic squares the problem is to construct a magic square of the nth order in such a way that if the number in each cell is replaced by its mth power the resulting square shall also be magic. Here for example (see figure xxii) is a magic square* of the eighth order, the sum of the numbers in each line being equal

5	31	35	60	57	34	8	30
19	9	53	46	47	56	18	12
16	22	42	39	52	61	27	1
63	37	25	24	3	14	44	50
26	4	64	49	38	43	13	23
41	51	15	2	21	28	62	40
54	48	20	11	10	17	55	45
36	58	6	29	32	7	33	59

A Doubly-Magic Square.
Figure xxii.

to 260, so constructed that if the number in each cell is replaced by its square the resulting square is also magic (the sum of the numbers in each line being equal to 11180).

Trebly-Magic Squares. The construction of squares which shall be magic for the original numbers, for their squares, and for their cubes has also been studied. I know of no square of this kind which is of a lower order than 128.

MAGIC PENCILS. Hitherto I have concerned myself with numbers arranged in lines. By reciprocating the figures composed of the points on which the numbers are placed we obtain a collection of lines forming pencils, and, if these lines be numbered to correspond with the points, the pencils will be magic†. Thus, in a magic square of the nth order, we arrange n^2 consecutive numbers to form $2n+2$ lines, each containing

* See M. Coccoz in *L'Illustration*, May 29, 1897. The subject has been studied by Messieurs G. Tarry, B. Portier, M. Coccoz and A. Rilly. More than 200 such squares have been given by the last-named in his *Étude sur les Triangles et les Carrés Magiques aux deux premiers degrés*, Troyes, 1901.

† See *Magic Reciprocals* by G. Frankenstein, Cincinnati, 1875.

n numbers so that the sum of the numbers in each line is the same. Reciprocally we can arrange n^2 lines, numbered consecutively to form $2n+2$ pencils, each containing n lines, so that in each pencil the sum of the numbers designating the lines is the same.

For instance, figure xxiii represents a magic square of 64

1	2	62	61	60	59	7	8
9	10	54	53	52	51	15	16
48	47	19	20	21	22	42	41
40	39	27	28	29	30	34	33
32	31	35	36	37	38	26	25
24	23	43	44	45	46	18	17
49	50	14	13	12	11	55	56
57	58	6	5	4	3	63	64

Figure xxiii.

consecutive numbers arranged to form 18 lines, each of 8 numbers. Reciprocally, figure xxiv represents 64 lines arranged to form 18 pencils, each of 8 lines. The method of construction is fairly obvious. The eight-rayed pencil, vertex O, is cut by two parallels perpendicular to the axis of the pencil, and all the points of intersection are joined cross-wise. This gives 8 pencils, with vertices $A, B, \ldots H$; 8 pencils, with vertices $A', \ldots H'$; one pencil with its vertex at O; and one pencil with its vertex on the axis of the last-named pencil.

The sum of the numbers in each of the 18 lines in figure xxiii is the same. To make figure xxiv correspond to this we must number the lines in the pencil A from left to right, 1, 9, ..., 57, following the order of the numbers in the first column of the square: the lines in pencil B must be numbered similarly to correspond to the numbers in the second column of the square, and so on. To prevent confusion in the figure I have not inserted the numbers, but it will be seen that the method of construction ensures that the sum of the 8 numbers which designate the lines in each of these 18 pencils is the same.

We can proceed a step further, if the resulting figure is cut by two other parallel lines perpendicular to the axis, and if all the points of their intersection with the cross-joins be joined cross-wise, these new cross-joins will intersect on the

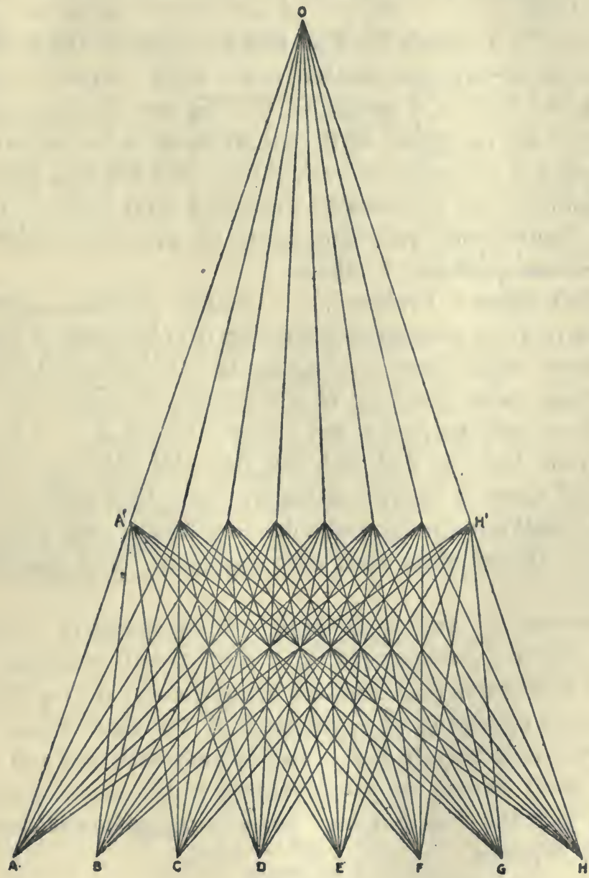


Figure xxiv.

axis of the original pencil or on lines perpendicular to it. The whole figure will now give 8^3 lines, arranged in 244 pencils each of 8 rays, and will be the reciprocal of a magic cube of the 8th order. If we reciprocate back again we obtain a representation in a plane of a magic cube.

MAGIC SQUARE PUZZLES. Many empirical problems, closely related to magic squares, will suggest themselves; but most of them are more correctly described as ingenious puzzles than as mathematical recreations. The following will serve as specimens.

*Magic Card Square**. The first of these is the familiar problem of placing the sixteen court cards (taken out of a pack) in the form of a square so that no row, no column, and neither of the diagonals shall contain more than one card of each suit and one card of each rank. The solution presents no difficulty, and is indicated in figure xxvi below. There are 72 fundamental solutions, each of which by reflexions and reversals produces 7 others.

Euler's Officers Problem†. A similar problem, proposed by Euler in 1779, consists in arranging, if it be possible, thirty-six officers taken from six regiments—the officers being in six groups, each consisting of six officers of equal rank, one drawn from each regiment; say officers of rank a, b, c, d, e, f, drawn from the 1st, 2nd, 3rd, 4th, 5th, and 6th regiments—in a solid square formation of six by six, so that each row and each file shall contain one and only one officer of each rank and one and only one officer from each regiment. The problem is insoluble.

Extension of Euler's Problem. More generally we may investigate the arrangement on a chess-board, containing n^2 cells, of n^2 counters (the counters being divided into n groups, each group consisting of n counters of the same colour and numbered consecutively $1, 2, \ldots, n$), so that each row and each column shall contain no two counters of the same colour or marked with the same number. Such arrangements are termed Eulerian Squares.

* Ozanam, 1723 edition, vol. IV, p. 434.

† Euler's *Commentationes Arithmeticae*, Petrograd, 1849, vol. II, pp. 302—361. See also a paper by G. Tarry in the *Comptes rendus* of the French Association for the Advancement of Science, Paris, 1900, vol. II, pp. 170—203; and various notes in *L'Intermédiaire des Mathématiciens*, Paris, vol. III, 1896, pp. 17, 90; vol. V, 1898, pp. 83, 176, 252; vol. VI, 1899, p. 251; vol. VII, 1900, pp. 14, 311.

For instance, if $n = 3$, with three red counters a_1, a_2, a_3, three white counters b_1, b_2, b_3, and three black counters c_1, c_2, c_3, we can satisfy the conditions by arranging them as in figure xxv below. If $n = 4$, then with counters a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4, we can arrange them as in figure xxvi below. A solution when $n = 5$ is indicated in figure xxvii.

a_1	b_2	c_3
b_3	c_1	a_2
c_2	a_3	b_1

Figure xxv.

a_1	b_2	c_3	d_4
c_4	d_3	a_2	b_1
d_2	c_1	b_4	a_3
b_3	a_4	d_1	c_2

Figure xxvi.

a_1	b_2	c_3	d_4	e_5
b_5	c_1	d_2	e_3	a_4
c_4	d_5	e_1	a_2	b_3
d_3	e_4	a_5	b_1	c_2
e_2	a_3	b_4	c_5	d_1

Figure xxvii.

The problem is soluble if n is odd; it is insoluble if n is of the form $2(2m+1)$. If solutions when $n = a$ and when $n = b$ are known, a solution when $n = ab$ can be written down at once. The theory is closely connected with that of magic squares and need not be here discussed further.

Reversible Magic Squares. The digits 0, 1, 2, 6, and 8, when turned upside down, can be read as 0, 1, 7, 9, and 8. This property has been utilized in constructing magic squares

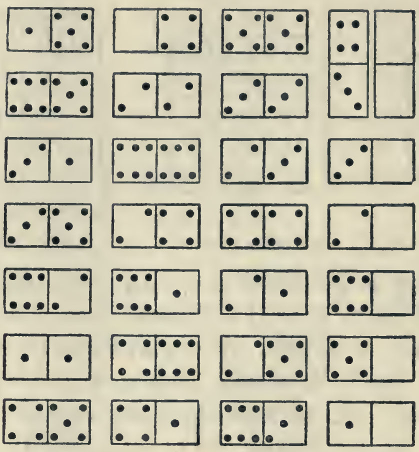
11	77	62	29
69	22	17	71
27	61	79	12
72	19	21	67

Reversible Magic Square.
Figure xxviii.

of non-consecutive numbers, which remain magic when the paper on which they are written is turned upside down. Here, for instance, figure xxviii*, is such a square in which only the digits 1, 2, 6 and their reversals are used.

* See *The Tribune*, April 29, 1907.

Magic Domino Squares. Analogous problems can be made with dominoes. An ordinary set of dominoes, ranging from double zero to double six, contains 28 dominoes. Each domino is a rectangle formed by fixing two small square blocks together side by side: of these 56 blocks, eight are

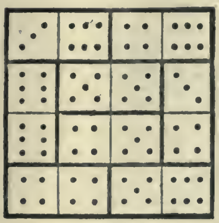


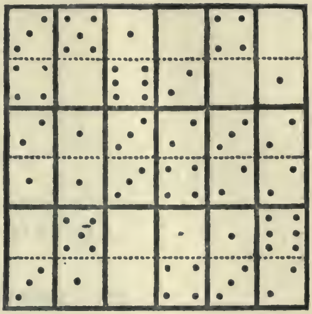
Magic Domino Square.
Figure xxix.

blank, on each of eight of them is one pip, on each of another eight of them are two pips, and so on. It is required to arrange the dominoes so that the 56 blocks form a square of 7 by 7 bordered by one line of 7 blank squares and so that the sum of the pips in each row, each column, and in the two diagonals of the square is equal to 24. A solution* is given above.

If we select certain dominoes out of the set and reject the others we can use them to make various magic puzzles. As instance, I give on the next page magic squares of this kind due to Mr Escott and Mr Dudeney. Numerous squares of this kind can be formed.

* See *L'Illustration*, July 10, 1897.





Magic Domino Squares.
Figure xxx.

*Magic Coin Squares**. There are somewhat similar questions concerned with coins. Here is one applicable to a square of the third order divided into nine cells, as in figure xxv above. If a five-shilling piece is placed in the middle cell c_1 and a florin in the cell below it, namely, in a_3, it is required to place the fewest possible current English coins in the remaining seven cells so that in each cell there is at least one coin, so that the total value of the coins in every cell is different, and so that the sum of the values of the coins in each row, column, and diagonal is fifteen shillings: it will be found that thirteen additional coins will suffice. Similar problems can be proposed with postage stamps.

ADDENDUM.

Note. Page 169. *Magic Coin Squares.* Taking the notation of figure xxv we must put a double-florin and a sixpence in cell a_1, two double-florins in cell b_2, a half-crown in cell c_3, a florin and a shilling in cell b_3, a crown and a florin in cell a_2, a crown and a half-crown in cell c_2, a crown and a sixpence in cell b_1.

* See *The Strand Magazine*, London, December, 1896, pp. 720, 721.

CHAPTER VIII.

UNICURSAL PROBLEMS.

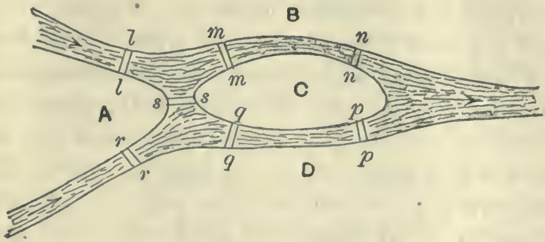
I propose to consider in this chapter some problems which arise out of the theory of unicursal curves. I shall commence with *Euler's Problem and Theorems*, and shall apply the results briefly to the theories of *Mazes* and *Geometrical Trees.* The reciprocal unicursal problem of the *Hamilton Game* will be discussed in the latter half of the chapter.

EULER'S PROBLEM. Euler's problem has its origin in a memoir* presented by him in 1736 to the St Petersburg Academy, in which he solved a question then under discussion as to whether it was possible from any point in the town of Königsberg to take a walk in such a way as to cross every bridge in it once and only once and return to the starting point.

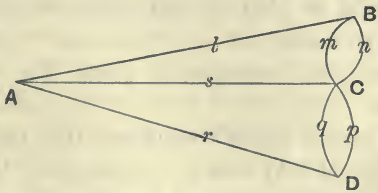
The town is built near the mouth of the river Pregel, which there takes the form indicated below and includes the island of Kneiphof. In the eighteenth century there were (and according to Baedeker there are still) seven bridges in the positions shown in the diagram, and it is easily seen that with such an arrangement the problem is insoluble. Euler however did not confine himself to the case of Königsberg, but discussed the general problem of any number of islands connected in any way by bridges. It is evident that the question

* '*Solutio problematis ad Geometriam situs pertinentis,*' *Commentarii Academiae Scientiarum Petropolitanae* for 1736, Petrograd, 1741, vol. VIII, pp. 128—140. This has been translated into French by M. Ch. Henry; see Lucas, vol. I, part 2, pp. 21—33.

will not be affected if we suppose the islands to diminish to points and the bridges to lengthen out. In this way we



ultimately obtain a geometrical figure or network. In the Königsberg problem this figure is of the shape indicated below, the areas being represented by the points *A*, *B*, *C*, *D*, and the bridges being represented by the lines *l*, *m*, *n*, *p*, *q*, *r*, *s*.



Euler's problem consists therefore in finding whether a given geometrical figure can be described by a point moving so as to traverse every line in it once and only once. A more general question is to determine how many strokes are necessary to describe such a figure so that no line is traversed twice: this is covered by the rules hereafter given. The figure may be either in three or in two dimensions, and it may be represented by lines, straight, curved, or tortuous, joining a number of given points, or a model may be constructed by taking a number of rods or pieces of string furnished at each end with a hook so as to allow of any number of them being connected together at one point.

The theory of such figures is included as a particular case in the propositions proved by Listing in his *Topologie**. I shall, however, adopt here the methods of Euler, and I shall begin by giving some definitions, as it will enable me to put the argument in a more concise form.

A *node* (or isle) is a point to or from which lines are drawn. A *branch* (or bridge or path) is a line connecting two consecutive nodes. An *end* (or hook) is the point at each termination of a branch. The *order* of a node is the number of branches which meet at it. A node to which only one branch is drawn is a *free* node or a free end. A node at which an even number of branches meet is an *even* node: evidently the presence of a node of the second order is immaterial. A node at which an odd number of branches meet is an *odd* node. A figure is closed if it has no free end: such a figure is often called a closed network.

A *route* consists of a number of branches taken in consecutive order and so that no branch is traversed twice. A closed route terminates at a point from which it started. A figure is described *unicursally* when the whole of it is traversed in one route.

The following are Euler's results. (i) In a closed network the number of odd nodes is even. (ii) A figure which has no odd node can be described unicursally, in a re-entrant route, by a moving point which starts from any point on it. (iii) A figure which has two and only two odd nodes can be described unicursally by a moving point which starts from one of the odd nodes and finishes at the other. (iv) A figure which has more than two odd nodes cannot be described completely in one route; to which Listing added the corollary that a figure which has $2n$ odd nodes, and no more, can be described completely in n separate routes. I now proceed to prove these theorems.

* *Die Studien*, Göttingen, 1847, part x. See also Tait on '*Listing's Topologie*,' *Philosophical Magazine*, London, January, 1884, series 5, vol. xvii, pp. 30—46; and *Collected Scientific Papers*, Cambridge, vol. ii, 1900, pp. 85—98. The problem was discussed by J. C. Wilson in his *Traversing of Geometrical Figures*, Oxford, 1905.

First. The number of odd nodes in a closed network is even.

Suppose the number of branches to be b. Therefore the number of hooks is $2b$. Let k_n be the number of nodes of the nth order. Since a node of the nth order is one at which n branches meet, there are n hooks there. Also since the figure is closed, n cannot be less than 2.

$$\therefore\quad 2k_2 + 3k_3 + 4k_4 + \ldots + nk_n + \ldots = 2b.$$

Hence $\qquad 3k_3 + 5k_5 + \ldots$ is even.

$$\therefore\quad k_3 + k_5 + \ldots \text{ is even.}$$

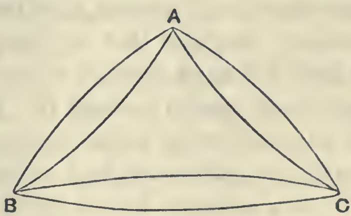
Second. A figure which has no odd node can be described unicursally in a re-entrant route.

Since the route is to be re-entrant it will make no difference where it commences. Suppose that we start from a node A. Every time our route takes us through a node we use up one hook in entering it and one in leaving it. There are no odd nodes, therefore the number of hooks at every node is even: hence, if we reach any node except A, we shall always find a hook which will take us into a branch previously untraversed. Hence the route will take us finally to the node A from which we started. If there are more than two hooks at A, we can continue the route over one of the branches from A previously untraversed, but in the same way as before we shall finally come back to A.

It remains to show that we can arrange our route so as to make it cover all the branches. Suppose each branch of the network to be represented by a string with a hook at each end, and that at each node all the hooks there are fastened together. The number of hooks at each node is even, and if they are unfastened they can be re-coupled together in pairs, the arrangement of the pairs being immaterial. The whole network will then form one or more closed curves, since now each node consists merely of two ends hooked together.

If this random coupling gives us one single curve then the proposition is proved; for starting at any point we shall go along every branch and come back to the initial point. But if this random coupling produces anywhere an isolated loop, L, then where it touches some other loop, M, say at the node P, unfasten the four hooks there (viz. two of the loop L and two of the loop M) and re-couple them in any other order: then the loop L will become a part of the loop M. In this way, by altering the couplings, we can transform gradually all the separate loops into parts of only one loop.

For example, take the case of three isles, A, B, C, each connected with both the others by two bridges. The most unfavourable way of re-coupling the ends at A, B, C would be to make ABA, ACA, and BCB separate loops. The loops ABA and ACA are separate and touch at A; hence we should re-couple the hooks at A so as to combine ABA and ACA into



one loop $ABACA$. Similarly, by re-arranging the couplings of the four hooks at B, we can combine the loop BCB with $ABACA$ and thus make only one loop.

I infer from Euler's language that he had attempted to solve the problem of giving a practical rule which would enable one to describe such a figure unicursally without knowledge of its form, but that in this he was unsuccessful. He however added that any geometrical figure can be described completely in a single route provided each part of it is described twice and only twice, for, if we suppose that every branch is duplicated, there will be no odd nodes and the figure is unicursal. In this case any figure can be described completely without knowing its form: rules to effect this are given below.

Third. A figure which has two and only two odd nodes can be described unicursally by a point which starts from one of the odd nodes and finishes at the other odd node.

This at once reduces to the second theorem. Let A and Z be the two odd nodes. First, suppose that Z is not a free end. We can, of course, take a route from A to Z; if we imagine the branches in this route to be eliminated, it will remove one hook from A and make it even, will remove two hooks from every node intermediate between A and Z and therefore leave each of them even, and will remove one hook from Z and therefore will make it even. All the remaining network is now even: hence, by Euler's second proposition, it can be described unicursally, and, if the route begins at Z, it will end at Z. Hence, if these two routes are taken in succession, the whole figure will be described unicursally, beginning at A and ending at Z. Second, if Z is a free end, then we must travel from Z to some node, Y, at which more than two branches meet. Then a route from A to Y which covers the whole figure exclusive of the path from Y to Z can be determined as before and must be finished by travelling from Y to Z.

Fourth. A figure having $2n$ *odd nodes, and no more, can be described completely in* n *separate routes,* n *being a positive number.*

If any route starts at an odd node, and if it is continued until it reaches a node where no fresh path is open to it, this latter node must be an odd one. For every time we enter an even node there is necessarily a way out of it; and similarly every time we go through an odd node we use up one hook in entering and one hook in leaving, but whenever we reach it as the end of our route we use only one hook. If this route is suppressed there will remain a figure with $2n-2$ odd nodes. Hence n such routes will leave one or more networks with only even nodes. But each of these must have some node common to one of the routes already taken and therefore can be described as a part of that route. Hence the complete passage will require n and not more than n routes. It follows, as stated by Euler, that, if there are more than two odd nodes, the figure cannot be traversed completely in one route.

The Königsberg bridges lead to a network with four odd nodes; hence, by Euler's fourth proposition, it cannot be described unicursally in a single journey, though it can be traversed completely in two separate routes.

The first and second diagrams figured below contain only even nodes, and therefore each of them can be described unicursally. The first of these is a regular re-entrant pentagon; the second is the so-called sign-manual of Mohammed, said to have been originally traced in the sand by the point of his scimetar without taking it off the ground or retracing any part of the figure—which, as it contains only even nodes, is possible. The third diagram is taken from Tait's article: it contains only two odd nodes, and therefore can be described unicursally if we start from one of them, and finish at the other.

The re-entrant pentagon, figured above, has some interest from having been used by the Pythagoreans as a sign—known as the triple triangle or pentagram star—by which they could recognize one another. It was considered symbolical of health, and probably the angles were denoted by the letters of the word *ὑγίεια*, the diphthong *ει* being replaced by a θ. Iamblichus, who is our authority for this, tells us that a certain Pythagorean, when travelling, fell ill at a roadside inn where he had put up for the night; he was poor and sick, but the landlord, who was a kind-hearted fellow, nursed him carefully and spared no trouble or expense to relieve his pains. However, in spite of all efforts, the student got worse. Feeling that he was dying and unable to make the landlord any pecuniary recompense, he asked for a board on which he inscribed the pentagram star; this he gave to his host, begging him to hang it up outside so that all passers by might see it, and assuring him that the result would recompense him for his charity. The scholar died and was honourably buried, and the board was duly exposed. After a considerable time had elapsed, a traveller one day riding by saw the sacred symbol; dismounting, he entered the inn, and after hearing the story, handsomely remunerated the landlord. Such is the anecdote, which if not true is at least well found.

As another example of a unicursal diagram I may mention the geometrical figure formed by taking a $(2n+1)$gon and joining every angular point to every other angular point. The edges of an octahedron also form a unicursal figure. On the other hand a chess-board, divided as usual by straight lines into 64 cells, has 28 odd nodes: hence it would require 14 separate pen-strokes to trace out all the boundaries without going over any more than once. Again, the diagram on page 117 has 20 odd nodes and therefore would require 10 separate pen-strokes to trace it out.

It is well known that a curve which has as many nodes as is consistent with its degree is unicursal.

I turn next to discuss in how many ways we can describe a unicursal figure, all of whose nodes are even*.

Let us consider first how the problem is affected by a path which starts from a node A of order $2n$ and returns to it, forming a closed loop L. If this loop were suppressed we should have a figure with all its nodes even, the node A being now of the order $2(n-1)$. Suppose the original figure can be described in N ways, and the reduced figure in N' ways. Then each of these N' routes passes $(n-1)$ times through A, and in any of these passages we could describe the loop L in either sense as a part of the path. Hence $N = 2(n-1)N'$.

Similarly if the node A on the original figure is of the order $2(n+l)$, and there are l independent closed loops which start from and return to A, we shall have

$$N = 2^l n(n+1)(n+2)\dots(n+l-1)N',$$

where N' is the number of routes by which the figure obtained by suppressing these l loops can be described.

* See G. Tarry, *Association Française pour l'Avancement des Sciences*, 1886, pp. 49—53.

By the use of these results, we may reduce any unicursal figure to one in which there are no closed loops of the kind above described. Let us suppose that in this reduced figure there are k nodes. We can suppress one of these nodes, say A, provided we replace the figure by two or more separate figures each of which has not more than $k-1$ nodes. For suppose that the node A is of the order $2n$. Then the $2n$ paths which meet at A may be coupled in n pairs in $1.3.5\ldots(2n-1)$ ways and each pair will constitute either a path through A, or (in the special case where both members of the pair abut on another node B) a loop from A. This path or loop will form a portion of the route through A in which this pair of paths are concerned. Hence the number of ways of describing the original figure is equal to the sum of the number of ways of describing $1.3.5\ldots(2n-1)$ separate simpler figures.

It will be seen that the process consists in successively suppressing node after node. Applying this process continually we finally reduce the figure to a number of figures without loops and in each of which there are only two nodes. If in one of these figures these nodes are each of the order $2n$ it is easily seen that it can be described in $2\times(2n-1)!$ ways.

We know that a figure with only two odd nodes, A and B, is unicursal if we start at A (or B) and finish at B (or A). Hence the number of ways in which it can be described unicursally will be the same as the number required to describe the figure obtained from it by joining A and B. For if we start at A it is obvious that at the B end of each of the routes which cover the figure we can proceed along BA to the node A whence we started.

This theory has been applied by Monsieur Tarry* to determine the number of ways in which a set of dominoes, running up to even numbers, can be arranged. This example will serve to illustrate the general method.

A domino consists of a small rectangular slab, twice as long as it is broad, whose face is divided into two squares,

* See the second edition of the French Translation of this work, Paris, 1908, vol. II, pp. 253—263; see also Lucas, vol. IV, pp. 145—150.

which are either blank or marked with 1, 2, 3 ... dots. An ordinary set contains 28 dominoes marked 6–6, 6–5, 6–4, 6–3, 6–2, 6–1, 6–0, 5–5, 5–4, 5–3, 5–2, 5–1, 5–0, 4–4, 4–3, 4–2, 4–1, 4–0, 3–3, 3–2, 3–1, 3–0, 2–2, 2–1, 2–0, 1–1, 1–0, and 0–0. Dominoes are used in various games in most, if not all, of which the pieces are played so as to make a line such that consecutive squares of adjacent dominoes are marked alike. Thus if 6–3 is on the table the only dominoes which can be placed next to the 6 end are 6–6, 6–5, 6–4, 6–2, 6–1, or 6–0. Similarly the dominoes 3–5, 3–4, 3–3, 3–2, 3–1, or 3–0, can be placed next to the 3 end. Assuming that the doubles are played in due course, it is easy to see that such a set of dominoes will form a closed circuit*. We want to determine the number of ways in which such a line or circuit can be formed.

Let us begin by considering the case of a set of 15 dominoes marked up to double-four. Of these 15 pieces, 5 are doubles. The remaining 10 dominoes may be represented by the sides and diagonals of a regular pentagon 01, 02, &c. The intersections of the diagonals do not enter into the representation,

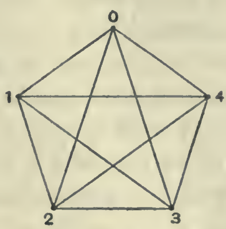


Figure A.

and accordingly are to be neglected. Omitting these from our consideration, the figure formed by the sides and diagonals of the pentagon has five even nodes, and therefore is unicursal. Any unicursal route (*ex. gr.* 0–1, 1–3, 3–0, 0–2, 2–3, 3–4, 4–1, 1–2, 2–4, 4–0) gives one way of arranging these 10 dominoes. Suppose there are *a* such routes. In any such route we may put each of the five doubles in any one of two positions (*ex. gr.*

* Hence if we remove one domino, say 5–4, we know that the line formed by the rest of the dominoes must end on one side in a 5 and on the other in a 4.

in the route given above the double-two can be put between 0–2 and 2–3 or between 1–2 and 2–4). Hence the total number of unicursal arrangements of the 15 dominoes is 2^5a. If we arrange the dominoes in a straight line, then as we may begin with any of the 15 dominoes, the total number of arrangements is $15 \,.\, 2^5 \,.\, a$.

We have next to find the number of unicursal routes of the pentagon delineated above in figure A. At the node 0 there are four paths which may be coupled in three pairs. If 0 1 and 0 2 are coupled, as also 0 3 and 0 4, we get figure B. If 0 1 and 0 3 are coupled, as also 0 2 and 0 4, we get figure C. If 0 1 and 0 4 are coupled, as also 0 2 and 0 3, we get figure D.

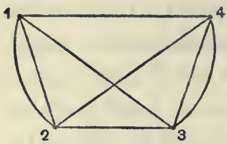


Figure B.

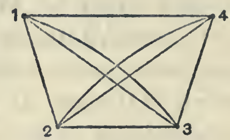


Figure C.

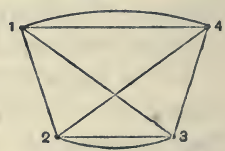


Figure D.

Let us denote the number of ways of describing figure B by b, of describing figure C by c, and so on. The effect of suppressing the node 0 in the pentagon A is to give us three quadrilaterals, B, C, D. And, in the above notation, we have $a = b + c + d$.

Take any one of these quadrilaterals, for instance D. We can suppress the node 1 in it by coupling the four paths which meet there in pairs. If we couple 1 2 with the upper of the paths 1 4, as also 1 3 with the lower of the paths 1 4, we get

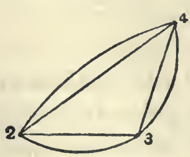


Figure E.

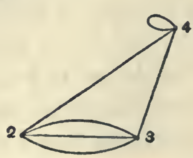


Figure F.

the figure E. If we couple 1 2 with the lower of the paths 1 4, as also 1 3 with the upper of the paths 1 4, we again get the figure E. If we couple 1 2 and 1 3, as also the two paths 1 4, we get the figure F. Then as above, $d = 2e + f$. Similarly $b = 2e + f$, and $c = 2e + f$. Hence $a = b + c + d = 6e + 3f$.

We proceed to consider each of the reduced figures E and F. First take E, and in it let us suppress the node 4. For simplicity of description, denote the two paths 0 2 by β and β', and the two paths 4 3 by γ and γ'. Then we can couple β and γ, as also β' and γ', or we can couple β and γ', as also β' and γ: each of these couplings gives the figure G. Or we can couple β and β', as also γ and γ': this gives the figure H. Thus $e = 2g + h$. Each of the figures G and H has only two nodes. Hence by the formulae given above, we have $g = 2.3.2 = 12$, and $h = 2.2.2 = 8$. Therefore $e = 2g + h = 32$. Next take the figure F. This has a loop at 4. If we suppress this

Figure G.

Figure H.

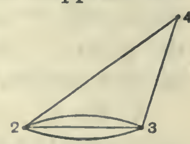


Figure J.

loop we get the figure J, and $f = 2j$. But the figure J, if we couple the two lines which meet at 4, is equivalent to the figure G. Thus $f = 2j = 2g = 24$. Introducing these results we have $a = 6e + 3f = 192 + 72 = 264$. And therefore $N = 15 . 2^5 . a = 126720$. This gives the number of possible arrangements in line of a set of 15 dominoes. In this solution we have treated an arrangement from right to left as distinct from one which goes from left to right: if these are treated as identical we must divide the result by 2. The number of arrangements in a closed ring is $2^5 a$, that is 8448.

We have seen that this number of unicursal routes for a pentagon and its diagonals is 264. Similarly the number for a heptagon is $h = 129976320$. Hence the number of possible arrangements in line of the usual set of 28 dominoes, marked up to double-six, is $28 . 3^7 . h$, which is equal to 7959229931520. The number of unicursal routes covering a polygon of nine sides is $n = 2^{17} . 3^{11} . 5^2 . 711 . 40787$. Hence the number of possible arrangements in line of a set of 45 dominoes marked up to double-eight is $48 . 4^9 . n$*.

* These numerical conclusions have also been obtained by algebraical analysis: see M. Reiss, *Annali di Matematica*, Milan, 1871, vol. v, pp. 63—120.

MAZES. Everyone has read of the labyrinth of Minos in Crete and of Rosamund's Bower. A few modern mazes exist here and there—notably one, a very poor specimen of its kind, at Hampton Court—and in one of these, or at any rate on a drawing of one, most people have at some time threaded their way to the interior. I proceed now to consider the manner in which any such construction may be completely traversed even by one who is ignorant of its plan.

The theory of the description of mazes is included in Euler's theorems given above. The paths in the maze are what previously we have termed branches, and the places where two or more paths meet are nodes. The entrance to the maze, the end of a blind alley, and the centre of the maze are free ends and therefore odd nodes.

If the only odd nodes are the entrance to the maze and the centre of it—which will necessitate the absence of all blind alleys—the maze can be described unicursally. This follows from Euler's third proposition. Again, no matter how many odd nodes there may be in a maze, we can always find a route which will take us from the entrance to the centre without retracing our steps, though such a route will take us through only a part of the maze. But in neither of the cases mentioned in this paragraph can the route be determined without a plan of the maze.

A plan is not necessary, however, if we make use of Euler's suggestion, and suppose that every path in the maze is duplicated. In this case we can give definite rules for the complete description of the whole of any maze, even if we are entirely ignorant of its plan. Of course to walk twice over every path in a labyrinth is not the shortest way of arriving at the centre, but, if it is performed correctly, the whole maze is traversed, the arrival at the centre at some point in the course of the route is certain, and it is impossible to lose one's way.

I need hardly explain why the complete description of such a duplicated maze is possible, for now every node is even, and hence, by Euler's second proposition, if we begin at the entrance we can traverse the whole maze; in so doing we shall at some point arrive at the centre, and finally shall emerge at the point from which we started. This description will require us to go over every path in the maze twice, and as a matter of fact the two passages along any path will be always made in opposite directions.

If a maze is traced on paper, the way to the centre is generally obvious, but in an actual labyrinth it is not so easy to find the correct route unless the plan is known. In order to make sure of describing a maze without knowing its plan it is necessary to have some means of marking the paths which we traverse and the direction in which we have traversed them —for example, by drawing an arrow at the entrance and end of every path traversed, or better perhaps by marking the wall on the right-hand side, in which case a path may not be entered when there is a mark on each side of it.

Of the various practical rules for threading a maze those enunciated by M. Trémaux seem to be the simplest*. These I proceed to explain. For brevity I shall describe a path or a node as old or new according as it has been traversed once before or not at all. Then the rules are (i) whenever you come to a new node, take any path you like; (ii) whenever you come by a new path to an old node or to the closed end of a blind alley, turn back along the path by which you have just come; (iii) whenever you come by an old path to an old node, take a new path, if there is one, but if not, an old path; (iv) of course a path traversed twice must not be entered. I should add that on emerging at any node then, of the various routes which are permitted by these rules, it will be convenient always to select that which lies next to one's right hand, or always that which lies next to one's left hand.

Few if any mazes of the type I have been considering (namely, a series of interlacing paths through which some route can be obtained leading to a space or building at the centre of the maze) existed in classical or medieval times. One class of what the ancients called mazes or labyrinths seems to have comprised any complicated building with numerous

* Lucas, vol. I, part iii, p. 47 *et seq.*

vaults and passages*. Such a building might be termed a labyrinth, but it is not what is now usually understood by the word. The above rules would enable anyone to traverse the whole of any structure of this kind. I do not know if there are any accounts or descriptions of Rosamund's Bower other than those by Drayton, Bromton, and Knyghton: in the opinion of some, these imply that the bower was merely a house, the passages in which were confusing and ill-arranged.

Another class of ancient mazes consisted of a tortuous path confined to a small area of ground and leading to a tree or shrine in the centre†. This is a maze in which there is no chance of taking a wrong turning; but, as the whole area can be occupied by the windings of one path, the distance to be traversed from the entrance to the centre may be considerable, even though the piece of ground covered by the maze is but small.

Figure i.

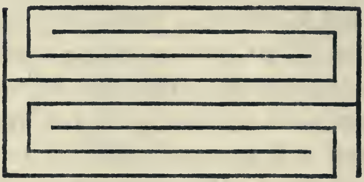


Figure ii.

The traditional form of the labyrinth constructed for the Minotaur is a specimen of this class. It was delineated on the reverses of the coins of Cnossus, specimens of which are not uncommon; one form of it is indicated in the accompanying diagram (figure i). The design really is the same as that

* For instance, see the descriptions of the labyrinth at Lake Moeris given by Herodotus, bk. ii, c. 148; Strabo, bk. xvii, c. 1, art. 37; Diodorus, bk. i, cc. 61, 66; and Pliny, *Hist. Nat.*, bk. xxxvi, c. 13, arts. 84—89. On these and other references see A. Wiedemann, *Herodots zweites Buch*, Leipzig, 1890, p. 522 *et seq*. See also Virgil, *Aeneid*, bk. v, c. v, 588; Ovid, *Met.*, bk. viii, c. 5, 159; Strabo, bk. viii, c. 6.

† On ancient and medieval labyrinths—particularly of this kind—see an article by Mr E. Trollope in *The Archaeological Journal*, 1858, vol. xv, pp. 216—235, from which much of the historical information given above is derived.

drawn in figure ii, as can be easily seen by bending round a circle the rectangular figure there given.

Mr Inwards has suggested* that this design on the coins of Cnossus may be a survival from that on a token given by the priests as a clue to the right path in the labyrinth there. Taking the circular form of the design shown above he supposed each circular wall to be replaced by two equidistant walls separated by a path, and thus obtained a maze to which the original design would serve as the key. The route thus indicated may be at once obtained by noticing that when a node is reached (*i.e.* a point where there is a choice of paths) the path to be taken is that which is next but one to that by which the node was approached. This maze may be also threaded by the simple rule of always following the wall on the right-hand side or always that on the left-hand side. The labyrinth may be somewhat improved by erecting a few additional barriers, without affecting the applicability of the above rules, but it cannot be made really difficult. This makes a pretty toy, but though the conjecture on which it is founded is ingenious it has no historical justification. Another suggestion is that the curved line on the reverse of the coins indicated the form of the rope held by those taking part in some rhythmic dance; while others consider that the form was gradually evolved from the widely prevalent svastika.

Copies of the maze of Cnossus were frequently engraved on Greek and Roman gems; similar but more elaborate designs are found in numerous Roman mosaic pavements†. A copy of the Cretan labyrinth was embroidered on many of the state robes of the later Emperors, and, apparently thence, was copied on to the walls and floors of various churches‡. At a later time in Italy and in France these mural and pavement decorations were developed into scrolls of great complexity, but consisting, as far as I know, always of a single line. Some of the best specimens now extant are on the walls of the

* *Knowledge*, London, October, 1892.

† See *ex. gr.* Breton's *Pompeia*, p. 303.

‡ Ozanam, *Graphia aureae urbis Romae*, pp. 92, 178.

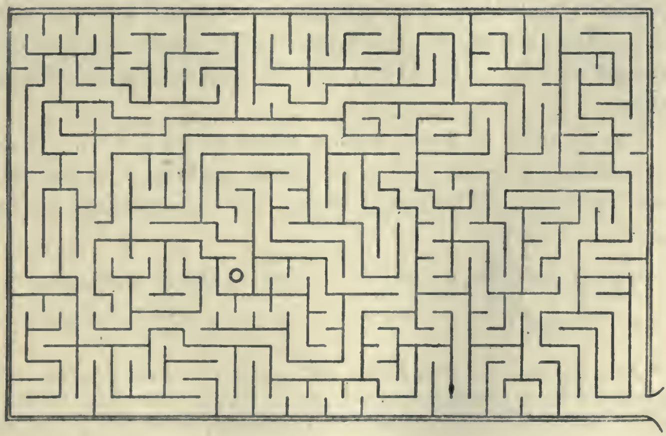
cathedrals at Lucca, Aix in Provence, and Poitiers; and on the floors of the churches of Santa Maria in Trastevere at Rome, San Vitale at Ravenna, Notre Dame at St Omer, and the cathedral at Chartres. It is possible that they were used to represent the journey through life as a kind of pilgrim's progress.

In England these mazes were usually, perhaps always, cut in the turf adjacent to some religious house or hermitage: and there are some slight reasons for thinking that, when traversed as a religious exercise, a *pater* or *ave* had to be repeated at every turning. After the Renaissance, such labyrinths were frequently termed Troy-Towns or Julian's Bowers. Some of the best specimens, which are still extant, or were so until recently, are those at Rockliff Marshes, Cumberland; Asenby, Yorkshire; Alkborough, Lincolnshire; Wing, Rutlandshire; Boughton-Green, Northamptonshire; Comberton, Cambridgeshire; Saffron Walden, Essex; and Chilcombe, near Winchester.

MAZE AT HAMPTON COURT.

The modern maze seems to have been introduced—probably from Italy—during the Renaissance, and many of the palaces and large houses built in England during the Tudor and the Stuart periods had labyrinths attached to them. Those adjoining the royal palaces at Southwark, Greenwich, and Hampton Court were well known from their vicinity to the capital. The last of these was designed by London and Wise in 1690, for William III, who had a fancy for such conceits: a plan of it is given in various guide-books. For the majority of the sight-seers who enter, it is sufficiently elaborate; but it is an indifferent construction, for it can be described completely by always following the hedge on one side (either the right hand or the left hand), and no node is of an order higher than three.

Unless at some point the route to the centre forks and subsequently the two forks reunite, forming a loop in which the centre of the maze is situated, the centre can be reached by the rule just given, namely, by following the wall on one side—either on the right hand or on the left hand. No labyrinth is worthy of the name of a puzzle which can be threaded in this way. Assuming that the path forks as described above, the more numerous the nodes and the higher their order the more difficult will be the maze, and the difficulty might be increased considerably by using bridges and tunnels so as to construct a labyrinth in three dimensions. In an ordinary garden and on a small piece of ground, often of an inconvenient shape, it is not easy to make a maze which fulfils these conditions. Here is a plan of one which I put up



in my own garden on a plot of ground which would not allow of more than 36 by 23 paths, but it will be noticed that none of the nodes are of a high order.

Geometrical Trees. Euler's original investigations were confined to a closed network. In the problem of the maze it was assumed that there might be any number of blind alleys in it, the ends of which formed free nodes. We may now progress one step further, and suppose that the network or closed part of the figure diminishes to a point. This last arrangement is known as a *tree*. The number of unicursal descriptions necessary to completely describe a tree is called the *base* of the ramification.

We can illustrate the possible form of these trees by rods, having a hook at each end. Starting with one such rod, we can attach at either end one or more similar rods. Again, on any free hook we can attach one or more similar rods, and so on. Every free hook, and also every point where two or more rods meet, are what hitherto we have called nodes. The rods are what hitherto we have termed branches or paths.

The theory of trees—which already plays a somewhat important part in certain branches of modern analysis, and possibly may contain the key to certain chemical and biological theories—originated in a memoir by Cayley*, written in 1856. The discussion of the theory has been analytical rather than geometrical. I content myself with noting the following results.

The number of trees with n given nodes is n^{n-2}. If A_n is the number of trees with n branches, and B_n the number of trees with n free branches which are bifurcations at least, then

$$(1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2}\ldots\ldots = 1 + A_1x + A_2x^2 + A_3x^3 + \ldots,$$

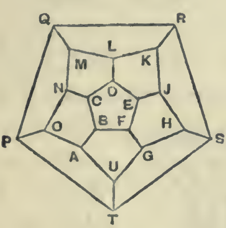
$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}\ldots\ldots = 1 + x + 2B_2x^2 + 2B_3x^3 + \ldots.$$

* *Philosophical Magazine*, March, 1857, series 4, vol. xiii, pp. 172—176; or *Collected Works*, Cambridge, 1890, vol. iii, no. 203, pp. 242—246: see also the paper on double partitions, *Philosophical Magazine*, November, 1860, series 4, vol. xx, pp. 337—341. On the number of trees with a given number of nodes, see the *Quarterly Journal of Mathematics*, London, 1889, vol. xxiii, pp. 376—378. The connection with chemistry was first pointed out in Cayley's paper on isomers, *Philosophical Magazine*, June, 1874, series 4, vol. xlvii, pp. 444—447, and was treated more fully in his report on trees to the British Association in 1875, *Reports*, pp. 257—305.

Using these formulae we can find successively the values of $A_1, A_2, \ldots$, and $B_1, B_2, \ldots$. The values of A_n when $n = 2, 3, 4, 5, 6, 7$, are 2, 4, 9, 20, 48, 115; and of B_n are 1, 2, 5, 12, 33, 90

I turn next to consider some problems where it is desired to find a route which will pass once and only once through each node of a given geometrical figure. This is the reciprocal of the problem treated in the first part of this chapter, and is a far more difficult question. I am not aware that the general theory has been considered by mathematicians, though two special cases—namely, the *Hamiltonian* (or Icosian) *Game* and the *Knight's Path on a Chess-Board*—have been treated in some detail.

THE HAMILTONIAN GAME. The Hamiltonian Game consists in the determination of a route along the edges of a regular dodecahedron which will pass once and only once through every angular point. Sir William Hamilton*, who invented this game—if game is the right term for it—denoted the twenty angular points on the solid by letters which stand for various towns. The thirty edges constitute the only possible paths. The inconvenience of using a solid is considerable, and the dodecahedron may be represented conveniently in



perspective by a flat board marked as shown in the first of the annexed diagrams. The second and third diagrams will answer our purpose equally well and are easier to draw.

* See *Quarterly Journal of Mathematics*, London, 1862, vol. v, p. 305; or *Philosophical Magazine*, January, 1884, series 5, vol. XVII, p. 42; also Lucas, vol. II, part vii.

The first problem is to go "all round the world," that is, starting from any town, to go to every other town once and only once and to return to the initial town; the order of the n towns to be first visited being assigned, where n is not greater than five.

Hamilton's rule for effecting this was given at the meeting in 1857 of the British Association at Dublin. At each angular point there are three and only three edges. Hence, if we approach a point by one edge, the only routes open to us are one to the right, denoted by r, and one to the left, denoted by l. It will be found that the operations indicated on opposite sides of the following equalities are equivalent,

$$lr^2l = rlr,\ rl^2r = lrl,\ lr^3l = r^2,\ rl^3r = l^2.$$

Also the operation l^5 or r^5 brings us back to the initial point: we may represent this by the equations

$$l^5 = 1,\ r^5 = 1.$$

To solve the problem for a figure having twenty angular points we must deduce a relation involving twenty successive operations, the total effect of which is equal to unity. By repeated use of the relation $l^2 = rl^3r$ we see that

$$1 = l^5 = l^2l^3 = (rl^3r)\,l^3 = \{rl^3\}^2 = \{r\,(rl^3r)\,l\}^2$$
$$= \{r^2l^3rl\}^2 = \{r^2\,(rl^3r)\,lrl\}^2 = \{r^3l^3rlrl\}^2.$$

Therefore $\{r^3l^3\,(rl)^2\}^2 = 1$ ....(i),

and similarly $\{l^3r^3\,(lr)^2\}^2 = 1$ ....(ii).

Hence on a dodecahedron either of the operations

$r\ r\ r\ l\ l\ l\ r\ l\ r\ l\ r\ r\ r\ l\ l\ l\ r\ l\ r\ l$ ... (i),

$l\ l\ l\ r\ r\ r\ l\ r\ l\ r\ l\ l\ l\ r\ r\ r\ l\ r\ l\ r$ ... (ii),

indicates a route which takes the traveller through every town. The arrangement is cyclical, and the route can be commenced at any point in the series of operations by transferring the proper number of letters from one end to the other. The point at which we begin is determined by the order of certain towns which is given initially.

Thus, suppose that we are told that we start from F and then successively go to B, A, U, and T, and we want to find a route from T through all the remaining towns which will end at F. If we think of ourselves as coming into F from G, the path FB would be indicated by l, but if we think of ourselves as coming into F from E, the path FB would be indicated by r. The path from B to A is indicated by l, and so on. Hence our first paths are indicated either by $l\,l\,l\,r$ or by $r\,l\,l\,r$. The latter operation does not occur either in (i) or in (ii), and therefore does not fall within our solutions. The former operation may be regarded either as the 1st, 2nd, 3rd, and 4th steps of (ii), or as the 4th, 5th, 6th, and 7th steps of (i). Each of these leads to a route which satisfies the problem. These routes are

$$F\,B\,A\,U\,T\,P\,O\,N\,C\,D\,E\,J\,K\,L\,M\,Q\,R\,S\,H\,G\,F,$$

and $$F\,B\,A\,U\,T\,S\,R\,K\,L\,M\,Q\,P\,O\,N\,C\,D\,E\,J\,H\,G\,F.$$

It is convenient to make a mark or to put down a counter at each corner as soon as it is reached, and this will prevent our passing through the same town twice.

A similar game may be played with other solids provided that at each angular point three and only three edges meet. Of such solids a tetrahedron and a cube are the simplest instances, but the reader can make for himself any number of plane figures representing such solids similar to those drawn on page 189. Some of these were indicated by Hamilton. In all such cases we must obtain from the formulae analogous to those given above cyclical relations like (i) or (ii) there given. The solution will then follow the lines indicated above. This method may be used to form a rule for describing any maze in which no node is of an order higher than three.

For solids having angular points where more than three edges meet—such as the octahedron where at each angular point four edges meet, or the icosahedron where at each angular point five edges meet—we should at each point have more than two routes open to us; hence (unless we suppress some of the edges) the symbolical notation would have to be extended before it could be applied to these solids. I offer the suggestion to anyone who is desirous of inventing a new game.

Another and a very elegant solution of the Hamiltonian dodecahedron problem has been given by M. Hermary. It consists in unfolding the dodecahedron into its twelve pentagons, each of which is attached to the preceding one by only one of its sides; but the solution is geometrical, and not directly applicable to more complicated solids.

Hamilton suggested as another problem to start from any town, to go to certain specified towns in an assigned order, then to go to every other town once and only once, and to end the journey at some given town. He also suggested the consideration of the way in which a certain number of towns should be blocked so that there was no passage through them, in order to produce certain effects. These problems have not, so far as I know, been subjected to mathematical analysis.

The problem of the knight's path on a chess-board is somewhat similar in character to the Hamiltonian game. This I have already discussed in chapter VI.

CHAPTER IX.

KIRKMAN'S SCHOOL-GIRLS PROBLEM.

The Fifteen School-Girls Problem—first enunciated by T. P. Kirkman, and commonly known as *Kirkman's Problem*—consists in arranging fifteen things in different sets of triplets. It is usually presented in the form that a school-mistress was in the habit of taking her girls for a daily walk. The girls were fifteen in number, and were arranged in five rows of three each so that each girl might have two companions. The problem is to dispose them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once.

In the general problem, here discussed, we require to arrange n girls, where n is an odd multiple of 3, in triplets to walk out for y days, where $y=(n-1)/2$, so that no girl will walk with any of her school-fellows in any triplet more than once.

The theory of the formation of all such possible triplets in the case of nine girls is comparatively easy, but the general theory involves considerable difficulties. Before describing any methods of solution, I will give briefly the leading facts in the history of the problem. For this and much of the material of this chapter I am indebted to O. Eckenstein. Detailed references to the authorities mentioned are given in the bibliography mentioned in the footnote*.

* The problem was first published in the *Lady's and Gentleman's Diary* for 1850, p. 48, and has been the subject of numerous memoirs. A bibliography of the problem by O. Eckenstein appeared in the *Messenger of Mathematics*, Cambridge, July, 1911, vol. XLI, pp. 33—36.

The question was propounded in 1850, and in the same year solutions were given for the cases when $n = 9$, 15, and 27; but the methods used were largely empirical.

The first writer to subject it to mathematical analysis was R. R. Anstice who, in 1852 and 1853, described a method for solving all cases of the form $12m + 3$ when $6m + 1$ is prime. He gave solutions for the cases when $n = 15$, 27, 39. Substantially, his process, in a somewhat simplified form, is covered by that given below under the heading Analytical Methods.

The next important advance in the theory was due to B. Peirce who, in 1860, gave cyclical methods for solving all cases of the form $12m + 3$ and $24m + 9$. But the processes used were complicated and partly empirical.

In 1871 A. H. Frost published a simple method applicable to the original problem when $n = 15$ and to all cases when n is of the form $2^{2m} - 1$. It has been applied to find solutions when $n = 15$ and $n = 63$.

In 1883 E. Marsden and A. Bray gave three-step cyclical solutions for 21 girls. These were interesting because Kirkman had expressed the opinion that this case was insoluble.

Another solution when $n = 21$, by T. H. Gill, was given in the fourth edition of this book in 1905. His method though empirical appears to be applicable to all cases, but for high values of n it involves so much preliminary work by trial and error as to be of little value.

A question on the subject which I propounded in the *Educational Times* in 1906, attracted the attention of L. A. Legros, H. E. Dudeney and O. Eckenstein, and I received from them a series of interesting and novel solutions. As illustrations of the processes used, Dudeney published new solutions for $n = 27$, 33, 51, 57, 69, 75, 87, 93, 111; and Eckenstein for $n = 27$, 33, 39, 45, 51, 57, 69, 75, 93, 99, 111, 123, 135.

I now proceed to describe some of the methods applicable to the problem. We can use cycles and combinations of them. I confine my discussion to processes where the steps of the cycles do not exceed three symbols at a time. It will be convenient to begin with the easier methods, where however a certain amount of arrangement has to be made empirically, and then to go on to the consideration of the more general method.

One-Step Cycles. As illustrating solutions by one-step cyclical permutations I will first describe Legros's method. Solutions obtained by it can be represented by diagrams, and their use facilitates the necessary arrangements. It is always applicable when n is of the form $24m+3$, and seems to be also applicable when n is of the form $24m+9$. Somewhat similar methods were used by Dudeney, save that he made no use of geometrical constructions.

We have $n=2y+1=24m+3$ or $n=2y+1=24m+9$. We may denote one girl by k, and the others by the numbers 1, 2, 3, ... $2y$. Place k at the centre of a circle, and the numbers 1, 2, 3, ... $2y$ at equidistant intervals on the circumference. Thus the centre of the circle and each point on its circumference will indicate a particular girl. A solution in which the centre of the circle is used to denote one girl is termed a central solution.

The companions of k are to be different on each day. If we suppose that on the first day they are 1 and $y+1$, on the second 2 and $y+2$, and so on, then the diameters through k will give for each day a triplet in which k appears. On each day we have to find $2(y-1)/3$ other triplets satisfying the conditions of the problem. Every triplet formed from the remaining $2y-2$ girls will be represented by an inscribed triangle joining the points representing these girls. The sides of the triangles are the chords joining these $2y-2$ points. These chords may be represented symbolically by [1], [2], [3], ... $[y-1]$; these numbers being proportional to the smaller arcs subtended. I will denote the sides of a triangle so represented by the letters p, q, r, and I will use the term triad or grouping to denote any group of p, q, r which determines the dimensions of an inscribed triangle. I shall place the numbers of a triad in square brackets. If p, q, r are proportional to the smaller arcs subtended, it is clear that if $p+q$ is less than y, we have $p+q=r$; and if $p+q$ is greater than y we have $p+q+r=2y$. If we like to use arcs larger than the semi-circumference we may confine ourselves to the relation $p+q=r$. In the geometrical methods described below, we usually first determine the dimensions of the triangles to be used in the solution, and then find how they are to be arranged in the circle.

If $(y-1)/3$ scalene triangles, whose sides are p, q, r, can be inscribed in the circle so that to each triangle corresponds an equal complementary triangle having its equal sides parallel to those of the first and with its vertices at free points, then the system of $2(y-1)/3$ triangles with the corresponding diameter will give an arrangement for one day. If the system be permuted cyclically $y-1$ times we get arrangements for the other $y-1$ days. No two girls will walk together twice, for each chord occupies a different position after each permutation, and as all the chords forming the $(y-1)/3$ triangles are unequal the same combination cannot occur twice. Since the triangles are placed in complementary pairs, one being y points in front of the other, it follows that after $y-1$ permutations we shall come to a position like the initial one, and the cycle will be completed. If the circle be drawn and the triangles cut out to scale, the arrangement of the triangles is facilitated. The method will be better understood if I apply it to one or two of the simpler cases.

The first case is that of three girls, a, b, c, walking out for one day, that is, $n=3, m=0, y=1$. This involves no discussion, the solution being $(a.\ b.\ c)$.

The next case is that of nine girls walking out for four days, that is, $n=9, m=0, y=4$. The first triplet on the first day is $(1.\ k.\ 5)$. There are six other girls represented by the points 2, 3, 4, 6, 7, 8. These points can be joined so as to form triangles, and each triangle will represent a triplet. We want to find one such triangle, with unequal sides, with its vertices at three of these points, and such that the triangle formed by the other three points will have its sides equal and parallel to the sides of the first triangle.

The sides of a triangle are p, q, r. The only possible values are 1, 2, 3, and they satisfy the condition $p+q=r$. If a triangle of this shape is placed with its vertices at the points 3, 4, 6, we can construct a complementary equal triangle, four points further on, having 7, 8, 2 for its vertices. All the points in the figure are now joined, and form the three triplets for the first day, namely (*k*. 1. 5), (3. 4. 6), (7. 8. 2). It is only necessary to rotate the figure one step at a time in order to obtain the triplets for the remaining three days. Another similar solution is obtained from the diameter (1. *k*. 5), and the triangles (2. 3. 8), (6. 7. 4). It is the reflection of the former solution.

Figure i.

The next case to which the method is applicable is when $n = 27$, $m = 1$, $y = 13$. Proceeding as before, the 27 girls must be arranged with one of them, *k*, at the centre and the other 26 on the circumference of a circle. The diameter (1. *k*. 14) gives the first triplet on the first day. To obtain the other triplets we have to find four dissimilar triangles which satisfy the conditions mentioned above. The chords used as sides of these triangles may be of the lengths represented symbolically by [1], [2], ... [12]. We have to group these lengths so that $p + q = r$ or $p + q + r = 2y$; if the first condition can be satisfied it is the easier to use, as the numbers are smaller. In this instance the triads [3, 8, 11], [5, 7, 12], [2, 4, 6], [1, 9, 10] will be readily found. Now if four triangles with their sides of these lengths can be arranged in a system so that all the vertices fall on the ends of different diameters (exclusive of the ends of the diameter 1, *k*, 14), it follows that the opposite ends of those diameters can be joined by chords giving a series of equal triangles, symmetrically placed, each having its sides parallel to those of a triangle of the first system. The following arrangement of triangles satisfies the conditions: (4. 11. 25), (5. 8. 23), (6. 7. 16), (9. 13. 15). The complementary system is (17. 24. 12), (18. 21. 10), (19. 20. 3), (22. 26. 2). These triplets with (k. 1. 14) give an arrangement for the first day; and, by rotating the system cyclically, the arrangements for the remaining 12 days can be found immediately.

I proceed to give one solution of this type for every remaining case where n is less than 100. From the result the triads or groupings used can be obtained. It is sufficient in each case to give an arrangement on the first day, since the arrangements on the following days are at once obtainable by cyclical permutations.

I take first the three cases, 33, 57, 81, where n is of the form $24m+9$. In these cases the arrangements on the other days are obtained by one-step cyclical permutations.

For 33 girls, a solution is given by the system of triplets (2. 11. 16), (4. 6. 10), (5. 13. 30), (7. 8. 19), (9. 28. 31), and the complementary system (18. 27. 32), (20. 22. 26), (21. 29. 14), (23. 24. 3), (25. 12. 15). These 10 triplets, together with that represented by (k. 1. 17), will give an arrangement for the first day.

For 57 girls, a possible arrangement of triplets is (18. 13. 50), (20. 11. 28), (21. 52. 3), (8. 10. 51), (4. 25. 26), (2. 6. 12), (7. 19. 33), (27. 43. 16), (37. 14. 17). These, with the 9 complementary triplets, and the diameter triplet (1. k. 29), give an arrangement for the first day.

For 81 girls an arrangement for the first day consists of the diameter triplet (1. k. 41), the 13 triplets (3. 35. 42), (4. 10. 29), (5. 28. 56), (6. 26. 39), (7. 15. 17), (8. 11. 32), (13. 27. 49), (14. 19. 30), (20. 37. 38), (21. 25. 52), (24. 36. 62), (18. 33. 63), (31. 40. 74), and the 13 complementary triplets.

I take next the three cases, 51, 75, 99, where n is of the form $24m+3$. In these cases the arrangements on the other days are obtained either by one-step or by two-step cyclical permutations.

For 51 girls, an arrangement for the first day consists of the diameter triplet (k. 1. 26), the 8 triplets (2. 9. 36), (4. 7. 25), (6. 10. 19), (8. 14. 22), (12. 17. 45), (13. 24. 48), (15. 16. 46), (18. 28. 30), and the 8 complementary triplets.

For 75 girls, an arrangement for the first day consists of the diameter triplet (k. 1. 38), the 12 triplets (2. 44. 55), (4. 11. 19), (5. 50. 66), (6. 52. 57), (8. 46. 58), (10. 59. 65), (12. 60. 64), (14. 24. 68), (16. 25. 72), (17. 3. 74), (33. 34. 73), (30. 32. 63), and the 12 complementary triplets.

For 99 girls an arrangement for the first day consists of the diameter triplet (1. k. 50), the 16 triplets (2. 17. 47), (3. 9. 68), (4. 44. 82), (5. 12. 75), (6. 32. 42), (7. 23. 97), (8. 21. 30), (15. 20. 76), (16. 35. 85), (18. 45. 62), (22. 40. 63), (25. 37. 92), (28. 29. 80), (34. 38. 59), (39. 41. 73), (46. 49. 60), and the 16 complementary triplets.

It is also possible to obtain, for numbers of the form $24m+3$, solutions which are uniquely two-step, but in these the complementary triangles are not placed symmetrically to each other. I give 27 girls as an instance, using the same triads as in the solution of this case given above. The triplets for the first day are (k. 1. 14), (2. 12. 3), (21. 5. 22), (20. 24. 26), (11. 15. 17), (8. 16. 19), (25. 7. 10), (6. 18. 13), and (23. 9. 4). From this the arrangements on the other days can be obtained by a two-step (but not by a one-step) cyclical permutation.

It is unnecessary to give more examples, or to enter on the question of how from one solution others can be deduced, or how many solutions of each case can be obtained in this way. The types of the possible triangles are found analytically, but their geometrical arrangement is empirical. The defect of this method is that it may not be possible to arrange a given grouping. Thus when $n = 27$, we easily obtain 24 different groupings, but two of them cannot be arranged geometrically to give solutions; and whether any particular grouping will give a solution can, in many cases, be determined only by long and troublesome empirical work. The same objection applies to the two-step and three-step methods which are described below.

Two-Step Cycles. The method used by Legros was extended by Eckenstein to cases where n is of the form $12m + 3$. When n is of this form and m is odd we cannot get sets of complementary triangles as is required in Legros's method; hence, to apply a similar method, we have to find $2(y-1)/3$ different dissimilar inscribed triangles having no vertex in common and satisfying the condition $p + q = r$ or $p + q + r = 2y$. These solutions are also central. Since there are $2y$ points on the circumference of the circle the permutations, if they are to be cyclical, must go in steps of two numbers at a time. In Legros's method we represented one triplet by a diameter. But obviously it will answer our purpose equally well to represent it by a triangle with k as vertex and two radii as sides, one drawn to an even number and the other to an odd number: in fact this will include the diameter as a particular case.

I begin by considering the case where we use the diameter (1. k. y) to represent one triplet on the first day. Here the chords used for sides of the triangles representing the other triplets must be of lengths [1], [2], ... [$y - 1$]. Also each given length must appear twice, and the two equal lines so represented must start one from an even number and the other from an odd number, so as to avoid the same combination of points occurring again when the system is rotated cyclically. Of course a vertex cannot be at the point 1 or y, as these points will be required for the diameter triplet (1. k. y).

These remarks will be clearer if we apply them to a definite example. I take as an instance the case of 15 girls. As before we represent 14 of them by equidistant points numbered 1, 2, 3, ... 14 on the circumference of a circle, and one by a point k at its centre. Take as one triplet the diameter (1. k. 8). Then the sides p, q, r may have any of the values [1], [2], [3], [4], [5], [6], and each value must be used twice. On examination it will be found that there are only two possible groupings, namely [1, 1, 2], [2, 4, 6], [3, 3, 6], [5, 5, 4], and [1, 2, 3], [1, 4, 5], [3, 5, 6], [2, 4, 6]. One of the solutions to which the first set of groupings leads is defined by the diameter (1. k. 8) and the four triplets (9. 10. 11), (4. 6. 14), (2. 5. 13), (3. 7. 12); see figure ii, below: of the four triangles used three are isosceles. The

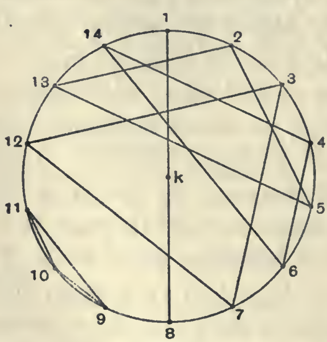


Figure ii.

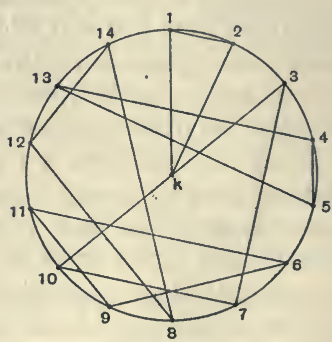


Figure iii.

second set leads to solutions defined by the triplets (k. 1. 8), (4. 5. 7), (13. 14. 9), (3. 6. 11), (10. 12. 2), or by the triplets (k. 1. 8), (6. 7. 9), (3. 4. 13), (5. 11. 14), (10. 12. 2): in these solutions all the triangles used are scalene. If any one of these three sets of triplets is rotated cyclically two steps at a time, we get a solution of the problem for the seven days required. Each of these solutions by reflection and inversion gives rise to three others.

Next, if we take (*k*. 1. 2) for one triplet on the first day we shall have the points 3, 4, ... 14 for the vertices of the four triangles denoting the other triplets on that day. The sides must be of the lengths [1], [2], ... [7], of which [2], ... [6] must be used not more than twice and [1], [7] must be used only once. The [1] used must start from an even number, for otherwise the chord denoted by it would, when the system was rotated, occupy the position joining the points 1 and 2, which has been already used. The only possible groupings are [2, 4, 6], [2, 3, 5], [3, 4, 7], [1, 5, 6]; or [2, 4, 6], [2, 5, 7], [1, 4, 5], [3, 3, 6]; or [2, 4, 6], [2, 5, 7], [1, 3, 4], [3, 5, 6]. Each of these groupings gives rise to various solutions. For instance the first grouping gives a set of triplets (*k*. 1. 2), (3. 7. 10), (4. 5. 13), (6. 9. 11), (8. 12. 14). From this by a cyclical two-step permutation we get a solution. This solution is represented in figure iii. If we take (*k*. 1. 4) or (*k*. 1. 6) as one triplet on the first day, we get other sets of solutions.

Solutions involving the triplets (*k*. 1. 2), (*k*. 1. 4), (*k*. 1. 6), (*k*. 1. 8), and other analogous solutions, can be obtained from the solutions illustrated in the above diagrams by re-arranging the symbols denoting the girls. For instance, if in figure iii, where all the triangles used are scalene, we replace the numbers 2, 8, 3, 13, 4, 6, 5, 11, 7, 9, 10, 14 by 8, 2, 13, 3, 6, 4, 11, 5, 9, 7, 14, 10, we get the solution (*k*. 1. 8), (4. 5. 7), etc., given above. Again, if in figure iii we replace the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, *k* by 12, 2, 10, 6, 1, 14, 5, 13, 9, 15, 4, 3, 11, 8, 7, we obtain a solution equivalent to that given by T. H. Gill and printed in the fourth edition of this book. In this the arrangement on the first day is (1. 6. 11), (2. 7. 12), (3. 8. 13), (4. 9. 14), (5. 10. 15). The arrangements on the other days are obtained as before by rotating the system so delineated round 7 as a centre two steps at a time. Gill's arrangement is thus presented in its canonical form as a central two-step cyclical solution.

I proceed to give one solution of this type for every remaining case where n is less than 100. In each case I give an arrangement on the first day; the arrangements for the other days can be got from it by a two-step cyclical permutation of the numbers.

In the case of 27 girls, one arrangement on the first day is (k. 1. 14), (19. 21. 20), (3. 9. 6), (13. 23. 18), (5. 17. 24), (7. 25. 16), (11. 15. 26), (10. 4. 2), (12. 22. 8).

In the case of 39 girls, one arrangement for the first day is (k. 1. 20), (35. 37. 36), (7. 13. 10), (19. 29. 24), (11. 25. 18), (3. 21. 12), (17. 33. 6), (15. 27. 2), (23. 31. 8), (5. 9. 26), (4. 16. 32), (14. 34. 38), (22. 28. 30).

In the case of 51 girls, one arrangement for the first day is (k. 1. 26), (21. 23. 22), (11. 17. 14), (35. 45. 40), (25. 39. 32), (15. 33. 24), (13. 41. 2), (5. 31. 18), (19. 49. 34), (27. 43. 10), (9. 47. 28), (29. 37. 8), (3. 7. 30), (4. 20. 46), (6. 36. 42), (12. 16. 44), (38. 48. 50).

In the case of 63 girls, one arrangement for the first day is (k. 1. 32), (57. 59. 58), (23. 29. 26), (37. 47. 42), (5. 53. 60), (17. 61. 8), (3. 43. 54), (7. 33. 20), (9. 39. 24), (27. 55. 10), (21. 45. 2), (11. 31. 52), (35. 51. 12), (13. 25. 50), (41. 49. 14), (15. 19. 48), (16. 18. 36), (6. 34. 38), (46. 56. 62), (22. 30. 44), (4. 28. 40).

In the case of 75 girls, one arrangement for the first day is (k. 1. 38), (23. 25. 24), (3. 9. 6), (29. 39. 34), (7. 21. 14), (51. 69. 60), (33. 55. 44), (11. 59. 72), (35. 65. 50), (37. 71. 54), (27. 63. 8), (15. 57. 36), (13. 41. 64), (43. 67. 18), (19. 73. 46), (31. 47. 2), (5. 17. 48), (53. 61. 20), (45. 49. 10), (30. 32. 52), (22. 26. 66), (56. 62. 70), (12. 28. 74), (16. 40. 58), (4. 42. 68).

In the case of 87 girls, one arrangement for the first day is (k. 1. 44), (61. 63. 62), (73. 79. 76), (35. 45. 40), (11. 83. 4), (25. 43. 34), (59. 81. 70), (7. 33. 20), (41. 71. 56), (23. 75. 6), (27. 65. 46), (13. 57. 78), (5. 51. 28), (3. 39. 64), (15. 47. 74), (9. 67. 38), (31. 55. 86), (19. 85. 52), (21. 37. 72), (17. 29. 66), (69. 77. 30), (49. 53. 8), (10. 12. 24), (22. 26. 84), (18. 54. 60), (2. 50. 80), (32. 42. 58), (14. 36. 82), (16. 48. 68).

In the case of 99 girls, one arrangement for the first day is (k. 1. 50), (47. 49. 48), (53. 59. 56), (55. 65. 60), (57. 71. 64), (23. 41. 32), (17. 39. 28), (63. 89. 76), (5. 35. 20), (3. 67. 84), (9. 69. 88), (29. 85. 8), (27. 79. 4), (25. 73. 98), (33. 77. 6), (21. 61. 90), (45. 81. 14), (51. 83. 18), (15. 43. 78), (13. 37. 74), (11. 31. 70), (75. 91. 34), (7. 19. 62), (87. 95. 42), (93. 97. 46), (58. 94. 96), (40. 44. 66), (10. 16. 26), (30. 38. 82), (68. 80. 2), (22. 36. 92), (24. 54. 72), (12. 52. 86).

This method may be also represented as a one-step cycle. For if we denote the girls by a point k at the centre of the circle, and points a_1, b_1, a_2, b_2, a_3, b_3, ... placed in that order on the circumference, we can re-write the solutions in the suffix notation, and then the cyclical permutation of the numbers denoting the suffixes is by one step at a time.

The one-step and two-step methods described above cover all cases except those where n is of the form $24m + 21$. These I have failed to bring under analogous rules, but we can solve them by recourse to the three-step cycles next described.

Three-Step Cycles. The fact that certain cases are soluble by one-step cycles, and others by two-step cycles, suggests the use of three-step cycles, and the fact that n is a multiple of 3 points to the same conclusion. On the other band, if we denote the n girls by 1, 2, 3, ... n, and make a cyclical permutation of three steps at a time (or if we denote the girls by $a_1, b_1, c_1, a_2, b_2, c_2, \ldots$, and make a cyclical permutation of the suffixes one step at a time), we cannot get arrangements for more than $n/3$ days. Hence there will remain $(n-1)/2 - n/3$ days, that is, $(n-3)/6$ days, for which we have to find other arrangements. In fact, however, we can arrange the work so that in addition to the cyclical arrangements for $n/3$ days we can find $(n-3)/6$ single triplets from each of which by a cyclical permutation of the numbers or suffixes an arrangement for one of these remaining days can be obtained; other methods are also sometimes available.

For instance take the case of 21 girls. An arrangement for the first day is (1. 4. 10), (2. 5. 11), (3. 6. 12), (7. 14. 18), (8. 15. 16), (9. 13. 17), (19. 20. 21). From this by cyclical permutations of the numbers three steps at a time, we can get arrangements for 7 days in all. The arrangement for the 8th day can be got from the triplet (1. 6. 11) by a three-step cyclical permutation of the numbers in it. Similarly the arrangement for the 9th day can be got from the triplet (2. 4. 12), and that for the 10th day from the triplet (3. 5. 10), by three-step cyclical permutations.

This method was first used by A. Bray in 1883, and was subsequently developed by Dudeney and Eckenstein. It gives a solution for every value of n except 15, but it is not so easy to use as the methods already described, partly because the solution is in two parts, and partly because the treatment varies according as n is of the form $18m + 3$, or $18m + 9$, or $18m + 15$. Most of the difficulties in using it arise in the case when n is of the form $18m + 15$.

The geometrical representation is sufficiently obvious. In the methods used by Legros and Eckenstein, previously described, the girls were represented by $2y$ equidistant points on the circumference of a circle and a point at its centre. It is evident that we may with equal propriety represent all the girls by symbols placed at equidistant intervals round the circumference of a circle: such solutions are termed non-central. The symbols may be 1, 2, 3, ... n, or letters a_1, b_1, c_1, a_2, b_2, c_2, ... Any triplet will be represented by a triangle whose sides are chords of the circle. The arrangement on any day is to include all the girls, and therefore the triangles representing the triplets on that day are $n/3$ in number, and as each girl appears in only one triplet no two triangles can have a common vertex.

The complete three-step solution will require the determination of a system of $(n-1)/2$ inscribed triangles. In the first part of the solution $n/3$ of these triangles must be selected to form an arrangement for the first day, so that by rotating this arrangement three steps at a time we obtain triplets for $n/3$ days in all. In the second part of the solution we must assure ourselves that the remaining $(n-3)/6$ triangles are such that from each of them, by a cyclical permutation of three steps at a time, an arrangement for one of the remaining $(n-3)/6$ days is obtainable.

As before we begin by tabulating the possible differences [1], [2], [3], ... $[(n-1)/2]$, whose values denote the lengths of the sides p, q, r of the possible triangles, also, we have either $p+q=r$ or $p+q+r=n$. From these values of p, q, r are formed triads, and in these triads each difference must be used three times and only three times. Triangles of these types must be then formed and placed in the circle so that the side denoting any assigned difference p must start once from a number of the form $3m$, once from a number of the form $3m+1$, and once from a number of the form $3m+2$. Also an isosceles triangle, one of whose sides is a multiple of three, cannot be used: thus in any particular triad a 3, 6, 9, ... cannot appear more than once. Save in some exceptional cases of high values of n, every triangle, one of whose sides is a multiple of 3, must be used in the first part of the solution. In the whole arrangement every possible difference will occur n times, and, since any two assigned numbers can occur together only once, each difference when added to a number must start each time from a different number. I will not go into further details as to how these triangles are determined, but I think the above rules will be clear if I apply them to one or two easy examples.

For 9 girls, the possible differences are [1], [2], [3], [4], each of which must be used three times in the construction of four triangles the lengths of whose sides p, q, r are such that $p+q=r$ or $p+q+r=9$. One possible set of triads formed from these numbers is [1, 2, 3], [1, 2, 3], [2, 3, 4], and [1, 4, 4]. Every triangle with a side of the length [3] must appear in the first part of the solution; thus the triplets used in the first part of the solution must be obtained from the first three of these triads. Hence we obtain as an arrangement for the first day the triplets (1. 3. 9), (2. 4. 7), (5. 6. 8). From this, three-step cyclical permutations give arrangements for other two days. The remaining triad [1, 4, 4] leads to a triplet (1. 2. 6) which, by a three-step cyclical permutation, gives an arrangement for the remaining day.

If we use the suffix notation, an arrangement for the first day is $a_1b_2a_3$, $b_1c_2b_3$, $c_1a_2c_3$. From this, by simple cyclical permutations of the suffixes, we get arrangements for the second and third days. Lastly, the triplet $a_1b_1c_1$ gives, by cyclical permutation of the suffixes, the arrangement for the fourth day, namely, $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$.

For 15 girls, the three-step process is inapplicable. The explanation of this is that two triads are required in the second part of the solution, and in neither of them may a 3 appear. The triads are to be formed from the differences [1], [2], ... [7], each of which is to be used three times, and the condition that in any particular triad only one 3 or one 6 may appear necessitates that six of the triads shall involve a 3 or a 6. Hence only one triad will be available for the second part of the solution.

I proceed to give one solution of this type for every remaining case where n is less than 100. I give some in the numerical, others in the suffix notation. The results will supply an indication of the process used.

First, I consider those cases where n is of the form $18m+3$. In these cases it is always possible to find $2m$ triads each repeated thrice, and one equilateral triad, and to use the equilateral and m triads in the first part of the solution, the 3 triplets representing any one of these m triads being placed in the circle at equal intervals from each other in the first day's arrangement. From this, three-step or one-step cyclical permutations give arrangements for $6m+1$ days in all. In the second part of the solution each of the m remaining triads is used thrice; it suffices to give the first triplet on each day, since from it the other triplets on that day are obtained by a three-step cyclical permutation.

For 21 girls an arrangement for the first day, for the first part of the solution, is (1. 4. 10), (8. 11. 17), (15. 18. 3), (2. 6. 7), (9. 13. 14), (16. 20. 21), (5. 12. 19). From this, three-step or one-step cyclical permutations give arrangements for 7 days in all. The first triplets used in the second part of the solution are (1. 3. 11), (2. 4. 12), (3. 5. 13). Each of these three triplets gives by a three-step cyclical permutation an arrangement for one of the remaining 3 days.

For 39 girls, arrangements for 13 days can be obtained from the following arrangement of triplets for the first day: (1. 4. 13), (14. 17. 26), (27. 30. 39), (2. 8. 23), (15. 21. 36), (28. 34. 10), (7. 11. 12), (20. 24. 25), (33. 37. 38), (3. 5. 19), (16. 18. 32), (29. 31. 6), (9. 22. 35). From each of the 6 triplets (1. 8. 18), (2. 9. 19), (3. 10. 20), (1. 9. 20), (2. 10. 21), (3. 11. 22), an arrangement for one of the remaining 6 days is obtainable.

For 57 girls, arrangements for 19 days can be obtained from the following arrangement of triplets for the first day: (1. 4. 25), (20. 23. 44), (39. 42. 6), (2. 8. 17), (21. 27. 36), (40. 46. 55), (3. 15. 33), (22. 34. 52), (41. 53. 14), (18. 19. 26), (37. 38. 45), (56. 57. 7), (13. 30. 35), (32. 49. 54), (51. 11. 16), (9. 29. 43), (28. 48. 5), (47. 10. 24), (12. 31. 50). From each of the 9 triplets (1. 3. 14), (2. 4. 15), (3. 5. 16), (1. 5. 30), (2. 6. 31), (3. 7. 32), (1. 11. 27), (2. 12. 28), (3. 13. 29), an arrangement for one of the remaining 9 days is obtainable.

For 75 girls, arrangements for 25 days can be obtained from the following arrangement of triplets for the first day: (1. 4. 10), (26. 29. 35), (51. 54. 60), (3. 15. 36), (28. 40. 61), (53. 65. 11), (2. 17. 41), (27. 42. 66), (52. 67. 16), (13. 31. 58), (38. 56. 8), (63. 6. 33), (14. 21. 22), (39. 46. 47), (64. 71. 72), (7. 18. 20), (32. 43. 45), (57. 68. 70), (5. 9. 37), (30. 34. 62), (55. 59. 12), (19. 24. 50), (44. 49. 75), (69. 74. 25), (23. 48. 73). From each of the 12 triplets (1. 11. 30), (2. 12. 31), (3. 13. 32), (1. 15. 35), (2. 16. 36), (3. 17. 37), (1. 17. 39), (2. 18. 40), (3. 19. 41), (1. 18. 41), (2. 19. 42), (3. 20. 43), an arrangement for one of the remaining 12 days is obtainable.

For 93 girls, arrangements for 31 days can be obtained from the following arrangement of triplets for the first day: (1. 76. 79), (32. 14. 17), (63. 45. 48), (13. 25. 55), (44. 56. 86), (75. 87. 24), (29. 50. 74), (60. 81. 12), (91. 19. 43), (20. 26. 59), (51. 57. 90), (82. 88. 28), (3. 30. 39), (34. 61. 70), (65. 92. 8), (27. 64. 71), (58. 2. 9), (89. 33. 40), (35. 36. 52), (66. 67. 83), (4. 5. 21), (11. 15. 37), (42. 46. 68), (73. 77. 6), (16. 18. 41), (47. 49. 72), (78. 80. 10),

(23. 31. 69), (54. 62. 7), (85. 93. 38), (22. 53. 84). From each of the 15 triplets (1. 14. 42), (2. 15. 43), (3. 16. 44), (1. 33. 44), (2. 34. 45), (3. 35. 46), (1. 11. 30), (2. 12. 31), (3. 13. 32), (1. 6. 41), (2. 7. 42), (3. 8. 43), (1. 15. 35), (2. 16. 36), (3. 17. 37), an arrangement for one of the remaining 15 days is obtainable by a three-step cyclical permutation.

Before leaving the subject of numbers of this type I give two other solutions of the case when $n=21$, one to illustrate the use of the suffix notation, and the other a cyclical solution which is uniquely three-step.

If we employ the suffix notation, the suffixes, with the type here used, are somewhat trying to read. Accordingly hereafter I shall write *a*1, *a*2, ..., instead of $a_1, a_2, \ldots$. In the case of 21 girls, an arrangement for the first day is (*a*1. *a*2. *a*4), (*b*1. *b*2. *b*4), (*c*1. *c*2. *c*4), (*a*3. *b*6. *c*5), (*b*3. *c*6. *a*5), (*c*3. *a*6. *b*5), (*a*7. *b*7. *c*7). From this, by one-step cyclical permutations of the suffixes, we get arrangements for the 2nd, 3rd, 4th, 5th, 6th and 7th days. The arrangement for the 8th day can be obtained from the triplet (*a*1. *b*2. *c*4) by permuting the suffixes cyclically one step at a time. Similarly the arrangement for the 9th day can be obtained from the triplet (*b*1. *c*2. *a*4) and that for the 10th day from the triplet (*c*1. *a*2. *b*4). Thus with seven suffixes we keep 7 for each symbol in one triplet, and every other triplet depends on one or other of only two arrangements, namely, (1. 2. 4), or (3. 6. 5). If the solution be written out at length the principle of the method used will be clear.

Cyclical solutions which are uniquely three-step can also be obtained for numbers of the form $18m+3$; in them the same triads can be used as before, but they are not placed at equal intervals in the circle. I give 21 girls as instance. The arrangements on the first 7 days can be obtained from the arrangement (1. 4. 10), (2. 20. 14), (15. 18. 3), (16. 17. 21), (8. 9. 13), (6. 7. 11), (5. 12. 19) by a three-step (but not by a one-step) cyclical permutation. From each of the triplets (1. 3. 11), (2. 4. 12), (3. 5. 13), an arrangement for one of the remaining 3 days is obtainable.

Next, I consider those cases where n is of the form $18m+9$. Here, regular solutions in the suffix notation can be obtained in all cases except in that of 27 girls, but if the same solutions are expressed in the numerical notation, the triads are irregular. Accordingly, except when $n=27$, it is better to use the suffix notation. I will deal with the case when $n=27$ after considering the cases when $n=45$, 63, 81, 99.

For 45 girls, an arrangement for the first day consists of the 5 triplets (*a*1. *a*12. *a*13), (*a*2. *a*9. *a*11), (*a*5. *a*10. *b*15), (*a*4. *b*3. *c*7), (*a*8. *b*6. *c*14), and the 10 analogous triplets, namely, (*b*1. *b*12. *b*13), (*c*1. *c*12. *c*13), (*b*2. *b*9. *b*11), (*c*2. *c*9. *c*11), (*b*5. *b*10. *c*15), (*c*5. *c*10. *a*15), (*b*4. *c*3. *a*7), (*c*4. *a*3. *b*7), (*b*8. *c*6. *a*14), (*c*8. *a*6. *b*14). From these, by one-step cyclical permutations of the suffixes, the arrangements for 15 days can be got. Each of the 2 triplets (*c*4. *b*3. *a*7), (*c*8. *b*6. *a*14), the 4 analogous triplets, namely, (*a*4. *c*3. *b*7), (*b*4. *a*3. *c*7), (*a*8. *c*6. *b*14), (*b*8. *a*6. *c*14), and the triplet (*a*1. *b*1. *c*1), gives, by a one-step cyclical permutation of the suffixes, an arrangement for one of the remaining 7 days.

For 63 girls, an arrangement for the first day consists of the 7 triplets (*a*1. *a*10. *a*9), (*a*5. *a*8. *a*3), (*a*4. *a*19. *a*15), (*a*7. *a*14. *b*21), (*a*20. *b*11. *c*12), (*a*16. *b*13. *c*18), (*a*17. *b*2. *c*6), and the 14 analogous triplets. From these, by one-step cyclical permutations of the suffixes, the arrangements for 21 days can be got. Each of the 10 triplets, consisting of the 3 triplets ($c20.\ b11.\ a12$), ($c16.\ b13,\ a18$), ($c17.\ b2.\ a6$), the 6 analogous triplets, and the triplet ($a1.\ b1.\ c1$), gives, by a one-step cyclical permutation of the suffixes, an arrangement for one of the remaining 10 days.

For 81 girls, an arrangement for the first day consists of the 9 triplets ($a5.\ a7.\ a8$), ($a3.\ a10.\ a14$), ($a11.\ a21.\ a26$), ($a4.\ a12.\ a25$), ($a9.\ a18.\ b27$), ($a22.\ b20.\ c19$), ($a24.\ b17.\ c13$), ($a16.\ b6.\ c1$), ($a23.\ b15.\ c2$), and the 18 analogous triplets. From these, by cyclical permutations of the suffixes, the arrangements for 27 days can be got. Each of the 13 triplets consisting of the 4 triplets ($c22.\ b20.\ a19$), ($c24.\ b17.\ a13$), ($c16.\ b6.\ a1$), ($c23.\ b15.\ a2$), the 8 analogous triplets, and the triplet ($a1.\ b1.\ c1$), gives, by a cyclical permutation of the suffixes, an arrangement for one of the remaining 13 days.

For 99 girls, an arrangement for the first day consists of the 11 triplets ($a1.\ a3.\ a10$), ($a2.\ a6.\ a20$), ($a4.\ a12.\ a7$), ($a8.\ a24.\ a14$), ($a16.\ a15.\ a28$), ($a11.\ a22.\ b33$), ($a32.\ b30.\ c23$), ($a31.\ b27.\ c13$), ($a29.\ b21.\ c26$), ($a25.\ b9.\ c19$), ($a17.\ b18.\ c5$), and the 22 analogous triplets. From these, by cyclical permutations of the suffixes, the arrangements for 33 days can be got. Each of the 16 triplets consisting of the 5 triplets ($c32.\ b30.\ a23$), ($c31.\ b27.\ a13$), ($c29.\ b21.\ a26$), ($c25.\ b9.\ a19$), ($c17.\ b18.\ a5$), the 10 analogous triplets, and the triplet ($a1.\ b1.\ c1$), gives, by a cyclical permutation of the suffixes, an arrangement for one of the remaining 16 days.

For 27 girls, an arrangement of triplets for the first day is (1. 7. 10), (14. 17. 8), (27. 6. 9), (13. 20. 25), (11. 23. 18), (12. 16. 24), (21. 2. 4), (15. 22. 26), (19. 3. 5). From this, three-step cyclical permutations give arrangements for 9 days in all. The first triplets on the remaining 4 days are (1. 2. 3), (1. 6. 20), (1. 11. 15), (1. 17. 27), from each of which, by a three-step cyclical permutation, an arrangement for one of those days is obtainable.

Regular solutions in the numerical notation can also be obtained for all values of n, except 9, where n is of the form $18m+9$. I give 27 girls as an instance. The first day's arrangement is (1. 2. 4), (10. 11. 13), (19. 20. 22), (8. 15. 23), (17. 24. 5), (26. 6. 14), (3. 25. 9), (12. 7. 18), (21. 16. 27); from this, arrangements for 9 days in all are obtained by one-step cyclical permutations. Each of the three triplets (1. 5. 15), (2. 6. 16), (3. 7. 17) gives an arrangement for one day by a three-step cyclical permutation. Finally the triplet (1. 10. 19), represented by an equilateral triangle, gives the arrangement on the last day by a one-step cyclical permutation.

Lastly, I consider those cases where n is of the form $18m+15$. As before, the solution is divided into two parts. In the first part, we obtain an arrangement of triplets for the first day, from which arrangements for $6m+5$ days are obtained by three-step cyclical permutations. In the second part, we obtain the first triplet on each of the remaining $3m+2$ days, from which the other triplets on that day are obtained by three-step cyclical permutations.

For 33 girls, an arrangement in the first part is (1. 13. 19), (23. 11. 5), (3. 15. 21), (2. 4. 7), (12. 14. 17), (28. 30. 33), (6. 16. 25), (10. 20. 29), (8. 18. 27), (24. 31. 32), (9. 22. 26). The triplets in the second part are (25. 32. 33), (26. 33. 1), (10. 23. 27), (11. 24. 28), (1. 12. 23).

For 51 girls, an arrangement in the first part is (17. 34. 51), (49. 16. 10), (11. 44. 50), (15. 33. 27), (19. 4. 46), (2. 38. 29), (21. 36. 45), (22. 25. 24), (5. 8. 7), (39. 42. 41), (43. 13. 20), (26. 47. 3), (9. 30. 37), (1. 14. 6), (35. 48. 40), (18. 31. 23), (28. 32. 12). The triplets in the second part are (29. 33. 13), (30. 34. 14), (1. 26. 12), (2. 27. 13), (3. 28. 14), (1. 11. 30), (2. 12. 31), (3. 13. 32).

For 69 girls, an arrangement in the first part is (23. 46. 69), (31. 34. 43), (11. 8. 68), (54. 57. 66), (58. 52. 28), (35. 29. 5), (45. 51. 6), (16. 1. 49), (62. 47. 26), (39. 24. 3), (4. 55. 42), (50. 32. 19), (27. 9. 65), (40. 67. 18), (17. 44. 64), (63. 21. 41), (10. 14. 15), (33. 37. 38), (56. 60. 61), (25. 53. 36), (2. 30. 13), (48. 7. 59), (22. 20. 12). The triplets in the second part are (23. 21. 13), (24. 22. 14), (1. 8. 33), (2. 9. 34), (3. 10. 35), (1. 17. 36), (2. 18. 37), (3. 19. 38), (1. 41. 15), (2. 42. 16), (3. 43. 17).

For 87 girls, an arrangement in the first part is (29. 58. 87), (76. 82. 73), (47. 53. 44), (15. 9. 18), (70. 1. 13), (41. 59. 71), (12. 30. 42), (4. 67. 31), (2. 26. 62), (60. 84. 33), (40. 79. 25), (11. 50. 83), (69. 21. 54), (43. 64. 63), (14. 35. 34), (72. 6. 5), (61. 16. 77), (32. 74. 48), (3. 45. 19), (28. 68. 66), (37. 86. 39), (10. 8. 57), (46. 80. 36), (22. 65. 75), (7. 17. 51), (85. 23. 78), (52. 20. 27), (49. 56. 81), (55. 38. 24). The triplets in the second part are (56. 39. 25), (57. 40. 26), (1. 5. 42), (2. 6. 43), (3. 7. 44), (1. 29. 6), (2. 30. 7), (3. 31. 8), (1. 9. 20), (2. 10. 21), (3. 11. 22), (1. 14. 36), (2. 15. 37), (3. 16. 38).

Before proceeding to the consideration of other methods I should add that it is also possible to obtain irregular solutions of cases where n is of any of these three forms. As an instance I give a three-step solution of 33 girls. A possible arrangement in the first part is (1. 4. 10), (14. 23. 26), (9. 15. 30), (3. 28. 33), (2. 6. 8), (11. 18. 27), (13. 24. 25), (5. 16. 20), (7. 17. 22), (21. 29. 31), (12. 19. 32). The triplets in the second part are (1. 2. 21), (1. 8. 30), (1. 3. 26), (1. 15. 20), (1. 17. 18). I describe this solution as irregular, since all, save one, of the triads used are different.

The Focal Method. Another method of attacking the problem, comparatively easy to use in practice, is applicable when n is of the form $24m+3p$, where $p=6q+3$. It is due to Eckenstein. Here it is convenient to use a geometrical representation by denoting $24m+2p$ girls by equidistant numbered points on the circumference of a circle, and the remaining p girls by lettered points placed inside the circle; these p points are termed foci. The solution is in two parts. In the first part, we obtain an order from which the arrangements for $12m+p$ days are deducible by a two-step cycle of the numbers: in none of these triplets does more than one focus appear. In the second part, we find the arrangements for the remaining $3q+1$ days; here the foci and the numbered points are treated separately, the former being arranged by any of the methods used for solving the case of $6q+3$ girls, while of the latter a typical triplet is used on each of those days, from which the remaining triplets on that day are obtained by cyclical permutations.

This method covers all cases except when $n=15$, 21, 39; and solutions by it for all values of n less than 200 have been written out. Sets of all the triplets required can be definitely determined. One way of doing this is by finding the primitive roots of the prime factors of $4m+2q+1$, though in the simpler cases the triplets can be written down empirically without much trouble. An advantage of this method is that solutions of several cases are obtained by the same work. Suppose that we have arranged suitable triangles in a circle, having on its circumference $12m+p$ or $3c$ equidistant points, and let y be the greatest integer satisfying the indeterminate equation $2x+4y+1=c$, where $x=0$ or $x=1$, and α the highest multiple of 6 included in $x+y$, then solutions of not less than $y+1-\alpha$ cases can be deduced. Thus from a 27 circle arrangement where $c=9$, $y=2$, $x=0$, $\alpha=0$, we can by this method deduce three solutions, namely when $n=57$, 69, 81; from a 39 circle arrangement where $c=13$, $y=3$, $x=0$, $\alpha=0$, we can deduce four solutions, namely when $n=81$, 93, 105, 117.

I have no space to describe the method fully, but I will give solutions for two cases, namely for 33 girls ($n=33$, $m=1$, $p=3$) where there are 3 foci, and for 51 girls ($n=51$, $m=1$, $p=9$) where there are 9 foci.

For 33 girls we have 3 foci which we may denote by a, b, c, and 30 points which we may denote by the numbers 1 to 30 placed at equidistant intervals on the circumference of a circle. Then if the arrangement on the first day is (a. 5. 10), (b. 20. 25), (c. 15. 30), (1. 2. 14), (16. 17. 29), (4. 23. 26), (19. 8. 11), (9. 7. 3), (24. 22. 18), (6. 27. 13), (21. 12. 28), a two-step cyclical permutation of the numbers gives arrangements for 15 days; that on the second day being (a. 7. 12), (b. 22. 27), &c. The arrangement on the 16th day is (a. b. c), (1. 11. 21), (2. 12. 22), (3. 13. 23), . . (10. 20. 30).

For 51 girls we have 9 foci which we may denote by a, b, c, *d, e, f, g, h, j*, and 42 points denoted by 1, 2, ... 42, placed at equidistant intervals on the circumference of a circle. Then if the arrangement on the first day is (*a*. 5. 6), (*b*. 26, 27), (*c*. 3. 10), (*d*. 24. 31), (*e*. 19. 34), (*f*. 40. 13), (*g*. 39. 16), (*h*. 18. 37), (*j*. 21. 42), (9. 11. 22), (30. 32. 1), (35. 41. 2), (14. 20. 23), (17. 29. 12), (38. 8. 33), (7. 15. 25), (28. 36. 4), a two-step cyclical permutation of the numbers gives arrangements for 21 days. Next, arrange the 9 foci in triplets by any of the methods already given so as to obtain arrangements for 4 days. From the numbers 1 to 42 we can obtain four typical triplets not already used, namely (1. 5. 21), (2. 6. 22), (3. 7. 23), (14. 28. 42). From each of these triplets we can, by a three-step cyclical permutation, obtain an arrangement of the 42 girls for one day, thus getting arrangements for 4 days in all. Combining these results of letters and numbers we obtain arrangements for the 4 days. Thus an arrangement for the first day would be (*a. c. j*), (*b. d. g*), (*e. f. h*), (1. 5. 21), (4. 8. 24), (7. 11. 27), (10. 14. 30), (13. 17. 33), (16. 20. 36), (19. 23. 39), (22. 26. 42), (25. 29. 3), (28. 32. 6), (31. 35. 9), (34. 38. 12), (37. 41. 15), (40. 2. 18). For the second day the corresponding arrangement would be (*d. f. c*), (*e. g. a*), (*h. j. b*), (2. 6. 22), (5. 9. 25), &c.

Analytical Methods. The methods described above, under the headings One-Step, Two-Step and Three-Step Cycles, involve some empirical work. It is true that with a little practice it is not difficult to obtain solutions by them when n is a low number, but the higher the value of n the more troublesome is the process and the more uncertain its success. A general arithmetical process has, however, been given by which it is claimed that *some* solutions for *any* value of n can be always obtained. Most of the solutions given earlier in this chapter can be obtained in this way.

The essential feature of the method is the arrangement of the numbers by which the girls are represented in an order such that definite rules can be laid down for grouping them in pairs and triplets so that the differences of the numbers in each pair or triplet either are all different or are repeated as often as

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may be required. The process depends on finding primitive roots of the prime factors of whatever number is taken as the base of the solution. When $(n-1)/2$ is prime, of the form $6u+1$, and is taken as base, the order is obtainable at once, and the rules for grouping the numbers are easy of application; owing to considerations of space I here confine myself to such instances, but similar though somewhat longer methods are applicable to all cases.

I use the geometrical representation already explained. We have $n=2y+1$, and y is a prime of the form $6u+1$. In forming the triplets we either proceed directly by arranging all the points in threes, or we arrange some of them in pairs and make the selection of the third point dependent on those of the two first chosen, leaving only a few triplets to be obtained otherwise. In the former case we have to arrange the numbers in triplets so that each difference will appear twice, and so that no two differences will appear together more than once. In the latter case we have to arrange the numbers so that the differences between the numbers in each pair comprise consecutive integers from 1 upwards and are all different. In both cases, we commence by finding a primitive root of y, say x. The residues to the modulus y of the $6u$ successive powers of x form a series of numbers, $e1, e2, e3, \ldots$, comprising all the integers from 1 to $6u$, and when taken in the order of the successive powers, they can be arranged in the manner required by definite rules.

I will apply the method to the case of 27 girls from which the general theory, in the restricted case where y is a prime of the form $6u+1$, will be sufficiently clear. In this case we have $n=27$, $y=13$, $u=2$, and $x=2$. I take 13 as the base of the analysis. I will begin by pairing the points, and this being so, it is convenient to represent the girls by a point k at the centre of a circle and points $a1$, $b1$, $a2$, $b2$, ... at equidistant intervals on the circumference. We reserve k, $a13$, $b13$ for one triplet, and we have to arrange the other 24 points so as to form 8 triangles of certain types. The residues are in the order 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, and these may be taken as the suffixes of the remaining 'a's and 'b's.

First arrange these residues in pairs so that every difference between the numbers in a pair occurs once. One rule, by which this can be effected, is to divide the residues into two equal sections and pair the numbers in the two sections. This gives (2. 11), (4. 9), (8. 5), (3. 10), (6. 7), (12. 1) as possible pairs. Another such rule is to divide the residue into six equal sections, and pair the numbers in the first and second sections, those in the third and fourth sections, and those in the fifth and sixth sections. This gives (2. 8), (4. 3), (6. 11), (12. 9), (5. 7), (10. 1) as possible pairs. Either arrangement can be used, but the first set of pairs leads only to scalene triangles. In none of the pairs of the latter set does the sum of the numbers in a pair add up to 13, and since this may allow the formation of isosceles as well as of scalene triangles, and thus increase the variety of the resulting solutions, I will use the latter set of pairs. We use these basic pairs as suffixes of the 'a's, and each pair thus determines two points of one of the triangles required. We have now used up all the 'a's. The third point associated with each of these six pairs of points must be a 'b,' and the remaining six 'b's must be such that they can be arranged in suitable triplets.

Next, then, we must arrange the $6u$ residues $e1, e2, e3, \ldots$ in possible triplets. To do this arrange them cyclically in triplets, for instance, as shown in the first column of the left half of the annexed table. We write in the second column the differences between the first and second numbers in each triplet, in the third column the differences between the second and third numbers in each triplet, and in the fourth column the differences between the third and first numbers in each triplet. If any of these differences d is greater than $3u$ we may replace it by the complementary number $y-d$: that this is permissible is obvious from the geometrical representation. By shifting cyclically the symbols in any vertical line in the first column we change these differences. We can, however, in this way always displace the second and third vertical lines in the first column so that the numbers in the second, third, and fourth columns include the numbers 1 to $3u$ twice over. This can be effected thus. If any term in the residue series is greater than $3u$ replace it by its complementary number $y-e$. In this way, from the residue series, we get a derivative series $d1$, $d2$, $d3$, ... such that any $3u$ consecutive terms comprise all the integers from 1 to $3u$. The first half of this series may be divided into three equal divisions thus: (1) $d1$, $d4$, $d7$, ...; (2) $d2$, $d5$, $d8$, ...; (3) $d3$, $d6$, $d9$, ... If the displacement is such that the first numbers in the second, third, and fourth columns are contained in different divisions, each difference must occur twice, and it will give a possible solution. Other possible regular arrangements give other solutions. Applying this to our case we have the residue series, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1. The derivative series is 2, 4, 5, 3, 6, 1, 2, 4, 5, 3, 6, 1. The three divisions are (i) 2, 3; (ii) 4, 6; (iii) 5, 1. The cyclical arrangement we started with and the consequent differences are shown in the left half of the accompanying table. A cyclical change

2. 4. 8	2	4	6	2. 6. 5	4	1	3
3. 6. 12	3	6	4	3. 9. 1	6	5	2
11. 9. 5	2	4	6	11. 7. 8	4	1	3
10. 7. 1	3	6	4	10. 4. 12	6	5	2

as described above of the vertical lines of the symbols in the first column gives the arrangement in the right half of the table. Here each difference occurs twice and accordingly this gives a possible arrangement of the triplets, namely, 2, 6, 5; 3, 9, 1; 11, 7, 8; 10, 4, 12. These are the suffixes of the 'b's. We have now to use six of these 'b's in connection with the basic 'a's already determined, keeping the other six 'b's for the remaining two triangles.

For instance we may obtain a scalene solution by taking as a suffix of the 'b' associated with any pair of 'a's, a number equal to the sum of the suffixes of the 'a's. We thus get as a solution ($a2$. $a8$. $b10$), ($a4$. $a3$. $b7$), ($a6$. $a11$. $b4$), ($a12$. $a9$. $b8$), ($a5$. $a7$. $b12$), ($a10$. $a1$. $b11$), ($b2$. $b6$. $b5$), ($b3$. $b9$. $b1$), (k. $a13$. $b13$). Or we might take as a suffix of the 'b' associated with any pair of 'a's, the number midway between the suffixes of the 'a's on the 13 circle. This gives the following solution, in which the first six triangles are isosceles: (a2. a8. b5), (a4. a3. b10), (a6. a11. b2), (a12. a9. b4), (a5. a7. b6), (a10. a1. b12), (b3. b9. b1), (b11. b7. b8), (k. a13. b13).

In the case of 27 girls, we may equally well represent the points by k at the centre of the circle and 26 equidistant points 1, 2,... 26 on the circumference. The points previously denoted by a and b with the suffix h are now denoted by the numbers $2h-1$ and $2h$. Hence the basic pairs (2. 8), (4. 3), ... become (3. 15), (7. 5), (11. 21), (23. 17), (9. 13), (19. 1), and the corresponding scalene arrangement for the first day is (3. 15. 20), (7. 5. 14), ... (k. 25. 26). From this by a two-step cyclical permutation of the numbers, an arrangement for 13 days can be got.

The case of 27 girls can also be treated by the direct formation of triplets. The triplets must be such that each difference is represented twice, but so that the groups of differences are different. There are analytical rules for forming such triplets somewhat analogous to those I have given for forming basic pairs, but their exposition would be lengthy, and I will not discuss them here. One set which will answer our purpose is (1. 12. 5), (2. 3. 10), (4. 6. 9), (8. 11. 7), giving respectively the differences [2, 6, 4], [1, 6, 5], [2, 3, 5], [3, 4, 1]. Now every difference d in a 13 circle will correspond to d or $13-d$ in a 26 circle, and every residue e in a 13 circle will correspond to e or $13+e$ in a 26 circle. Further, a triplet in the 26 circle must either have three even differences, or one even and two odd differences. Hence from the above sets we can get the following arrangement for the first day, (1. 25. 5) and (14. 12. 18) with differences [2, 6, 4], (15. 3. 10) and (2. 16. 23) with differences [12, 7, 5], (17. 6. 9) and (4. 19. 22) with differences [11, 3, 8], (21. 11. 20) and (8. 24. 7) with differences [10, 9, 1], and (k. 13. 26). From this by either a one-step or a two-step cyclical permutation of the numbers, an arrangement for 13 days can be got. I will not go into further details about the deduction of other similar solutions. A similar method is always applicable when n is of the form $24m+3$.

The process by pairing when y is a prime of the form $6u+1$ is extremely rapid. For instance, in the case of 15 girls we have $n=15$, $y=7$, $u=1$, $x=5$. The order of the residues is 5, 4, 6, 2, 3, 1. By our rule we can at once arrange basic pairs (5. 4), (6. 2), (3. 1). From these pairs we can obtain numerous solutions. Thus using scalene triangles as above explained, we get as an arrangement for the first day ($a5$, $a4$, $b2$), ($a6$. $a2$. $b1$), ($a3$. $a1$. $b4$), ($b3$. $b5$. $b6$), (k. $a7$. $b7$), from which by a one-step cyclical permutation of the numbers, arrangements for the seven days can be obtained. Using the basic pairs as bases of isosceles triangles, we get as an arrangement for the first day ($a5$. $a4$. $b1$), ($a6$. $a2$. $b4$), ($a3$. $a1$. $b2$), ($b3$. $b5$. $b6$), (k. $a7$. $b7$).

Again, take the case of 39 girls. Here we have $n=39$, $y=19$, $u=3$, $x=3$. The order of the residues is 3, 9, 8; 5, 15, 7; 2, 6, 18; 16, 10, 11; 14, 4, 12; 17, 13, 1. The basic pairs are (3. 5), (9. 15), (8. 7), (2. 16), &c. These are the suffixes of the 'a's. The possible triplets which determine what 'b's are to be associated with these, and what 'b's are to be left for the remaining three triangles, can be determined as follows: From the residue series we obtain the derivative series 3, 9, 8, 5, 4, 7, 2, 6, 1, &c. The divisions are (i) 3, 5, 2; (ii) 9, 4, 6; (iii) 8, 7, 1. A cyclical arrangement like that given above leads to the result in the left half of the annexed table which does not satisfy our condition. A cyclical displacement of the symbols in the vertical lines in the first column leads to the arrangement given in the right half of the table, and shows

3. 9. 8	6	2	5	3. 15. 18	7	3	4
5. 15. 7	9	8	2	5. 6. 11	1	5	6
2. 6. 18	4	7	3	2. 10. 12	8	2	9
16. 10. 11	6	1	5	16. 4. 1	7	3	4
14. 4. 12	9	8	2	14. 13. 8	1	5	6
17. 13. 1	6	7	3	17. 9. 7	8	2	9

that (3. 15. 18), (5. 6. 11), &c. are possible triplets. From these results numerous solutions can be deduced in the same way as above. For instance one solution is ($a3$. $a5$. $b4$),

(a9. a15. b12), (a8. a7. b17), (a2. a16. b9), (a6. a10. b8), (a18. a11. b5), (a14. a17. b6), (a4. a13. b18), (a12. a1. b16), (b3. b10. b1), (b2. b13. b7), (b14. b15. b11), (k. a19. b19).

In the case of 39 girls we may also extend the method used above by which for 27 girls we obtained the solution (1. 25. 5), (14. 12. 18), ... We thus get a solution for 39 girls as follows: (1. 25. 18), (14. 38. 31), (27. 12. 5); (15. 16. 10), (28. 29. 23), (2. 3. 36); (17. 19. 35), (30. 32. 9), (4. 6. 22); (21. 24. 33), (34. 37. 7), (8. 11. 20); (13. 26. 39). From this the arrangements for the first 13 days are obtained either by a one-step or a three-step cyclical permutation of the numbers. The single triplets from each of which an arrangement for one of the other six days is obtainable are (1. 5. 15), (2. 6. 16), (3. 7. 17); (1. 9. 20), (2. 10. 21), (3. 11. 22). From each of these the arrangement for one day is obtainable by a three-step cyclical permutation of the numbers.

These examples of the use of the Focal and Analytical Methods are given only by way of illustration, but they will serve to suggest the applications to other cases. When the number taken as base is composite, the formations of the series used in the Analytical Method may be troublesome, but the principle of the method is not affected, though want of space forbids my going into further details. Eckenstein, to whom the development of this method is mainly due, can, with the aid of a table of primitive roots and sets of numbers written on cards, within half an hour obtain a solution for any case in which n is less than 500, and can within one hour obtain a solution for any case in which n lies between 500 and 900.

Number of Solutions. The problems of 9 and of 15 girls have been subjected to an exhaustive examination. It has been shown that all solutions of the problem of 9 girls are reducible to one type, and that the number of independent solutions is 840, where, however, any arrangement on Monday, Tuesday, Wednesday and Thursday, and the same arrangement on (say) Monday, Tuesday, Thursday and Wednesday are regarded as identical. [If they are regarded as different the number of possible independent solutions is 20,160.] The total number of possible arrangements of the girls in triplets for four days is $(280)^4/4!$; hence the probability of obtaining a solution by a chance arrangement is about 1 in 300,000.

All solutions of the problem of 15 girls are reducible to one of eleven types, distinguished by the number of cycles required for expressing them. The number of independent solutions is said to be $65 \times (13!)$, but I do not vouch for the correctness of the result. The total number of ways in which the girls can walk out for a week in triplets is $(455)^7$; so the probability that any chance way satisfies the condition of the problem is very small.

Harison's Theorem. If we know solutions for Kirkman's Problem for $3l$ girls and for $3m$ girls we can find a solution for $3lm$ girls. The particular case of this when $l=1$ was established by Walecki and given in the earlier editions of this work. Harison's proof of the more general theorem is as follows:—

If the school-girls be denoted by the consecutive numbers from 1 to $3lm$ and the numbers be divided into $3l$ sets, each of m consecutive numbers, each of these sets can, by the method for the $3l$ problem, be divided in $(3l-1)/2$ collections of groups of three sets, so that every set shall be included once in the same group with every other set.

In one of these collections, each group of three sets (involving $3m$ numbers) can, by the method for the $3m$ problem, be arranged in triplets for $(3m-1)/2$ days, so as to have each number in each of the sets composing the group included once in the same triplet with every other number in the set to which it belongs and with every number in the other two sets in the group. This will give arrangements of all the numbers for that number of days.

In the remaining $(3l-3)/2$ collections, each group can be arranged in triplets for m days, so as to have each number in each of the sets composing the group included once in the same triplet with each number of the other two sets. In the first arrangement in each collection, the first triplet in each group is composed of the first number of each set. In the second arrangement, the first triplet is composed of the first number of the first set, the second number of the second set, and the last number of the third set. In every other arrangement the first triplet is formed from the first triplet in the next proceeding arrangement, by adding unity to the number of the second set, subtracting unity from the number of the third set, and leaving the number of the first set unchanged. In every arrangement the second and all subsequent triplets are formed cyclically from the next preceding triplet.

The arrangements thus made of the numbers in all the groups of all the collections will give arrangements of all the numbers for $(3lm-1)/2$ days, and will provide a solution of the problem.

Harison's Theorem provides solutions, alternative to those given above, for the cases when $n=27, 45, 63, 75, 81, 99$. Also we can, by it, from the solutions already given for all values of n less than 100, at once deduce solutions for the cases when $n=105, 117, 135, 147, 153, 165, 171, 189, 195$, &c.

Extension to n^2 Girls. Peirce suggested the corresponding problem of arranging n^2 girls in n groups, each group containing n girls, on $n+1$ days so that no two girls will be together in a group on more than one day. We may conveniently represent the girls by a point k at the centre of a circle and n^2-1 equidistant points, numbered 1, 2, 3, ... , on the circumference.

When $n=2$, we may arrange initially the 4 points in two pairs, one pair consisting of k and one of the points, say, 3, and the other pair of the remaining points (1. 2). These two pairs give the arrangement for the first day. From them, the solution for the other days is obtained by one-step cyclical permutations.

When $n=3$, we may arrange initially the 9 points in three triplets, namely, k and the ends of a diameter (k. 4. 8); a triangle (1. 2. 7); and the similar triangle (5. 6. 3) obtained by a four-step cyclical permutation. These three triplets give an arrangement for the first day. From them the solutions for the other days are obtained by one-step cyclical permutations.

When $n=4$, we may arrange initially the 16 points in four quartets, namely, k and three equidistant points, (k. 5. 10. 15);

a quadrilateral (1. 2. 4. 8); and the two similar quadrilaterals (6. 7. 9. 13) and (11. 12. 14. 3) obtained by five-step cyclical permutations. These four quartets give an arrangement for the first day. From them the solutions for the other days are obtained by one-step cyclical permutations.

When $n = 5$, we may arrange initially the 25 points in five quintets, namely, k and four equidistant points, (k. 6. 12. 18. 24); a pentagon (2. 3. 5. 13. 22); and the three similar pentagons (8. 9. 11. 19. 4), (14. 15. 17. 1. 10), (20. 21. 23. 7. 16) obtained by six-step cyclical permutations. These five quintets give an arrangement for the first day. From them the solutions for the other days are obtained by one-step cyclical permutations. There is a second solution (k. 6. 12. 18. 24), (1. 2. 15. 17. 22), &c.

Hitherto the case when $n = 6$ has baffled all attempts to find a solution.

When $n = 7$, we may initially arrange the 49 points in seven groups, namely (k. 8. 16. 24. 32. 40. 48), a group (1. 2. 5. 11. 31. 36. 38) and five similar groups obtained by successive additions of 8 to these numbers. There are three other solutions: in these the second group is either (2. 3. 17. 28. 38. 45. 47), or (3. 4. 6. 18. 23. 41. 45), or (3. 4. 14. 17. 26. 45. 47).

When $n = 8$ there are three solutions, due to Eckenstein. If the first group is (k. 9. 18. 27. 36. 45. 54. 63), the second group is either (1.2.4.8.16.21.32.42), or (2.3.16.22.24.50.55.62), or (3. 4. 7. 19. 24. 26. 32. 56): the other groups in each solution being obtained by successive addition of 9 to the numbers in the second group.

When n is composite no general method of attacking the

$k1$, $k2$, ... kn.	$k1$, $a1$, $b2$, $c3$, ...
$a1$, $a2$, ... an.	$k2$, $a2$, $b4$, $c6$, ...
$b1$, $b2$, ... bn,	$k3$, $a3$, $b6$, $c9$, ...
....	....
....	kn, an, bn, cn, ...
Arrangement on First Day	*Arrangement on Second Day*

problem has been discovered, though solutions for various particular cases have been given. But when n is prime, we can proceed thus. Denote the n^2 girls by the suffixed letters shown in the left half of the above table. Take this as giving the arrangement on the first day. Then on the second day we may take as an arrangement that shown in the right half of the table. From the arrangement on the second day, the arrangements for the other days are obtained by one-step cyclical permutations of the suffixes of a, b, &c.; the suffixes of k being unaltered.

Kirkman's Problem in Quartets. The problem of arranging $4m$ girls, where m is of the form $3n+1$, in quartets to walk out for $(4m-1)/3$ days, so that no girl will walk with any of her school-fellows in any quartet more than once has been attacked. Methods similar to those given above are applicable, and solutions for all cases where m does not exceed 31, have been written out. Analogous methods seem to be applicable to corresponding problems about quintets, sextets, &c.

Bridge Problem. Another analogous question is where we deal with arrangements in pairs instead of triplets. One problem of this kind is to arrange $4m$ members of a bridge club for $4m-1$ rubbers so that (i) no two members shall play together as partners more than once, and (ii) each member shall meet every other member as opponent twice. The general theory has been discussed by E. H. Moore and O. Eckenstein. A typical method for obtaining cyclic solutions is as follows. Denote the members by a point k at the centre of a circle and by $4m-1$ equidistant points, numbered 1, 2, 3, ..., on the circumference. We can join the points 2, 3, 4, ... by chords, and these chords with (k. 1) give possible partners at the m tables in the first rubber. A one-step cyclical permutation of the numbers will give the arrangements for the other rubbers if, in the initial arrangement, (i) the lengths of the chords representing every pair of partners are unequal and thus appear only once, and (ii) the lengths of the chords representing every pair of opponents appear only twice. Since the chords representing pairs of partners are unequal, their lengths are uniquely determined, but the selection of the chords is partly empirical. In the following examples for $m = 2, 3, 4$, I give an arrangement of the card tables for the first rubber: the arrangements for the subsequent rubbers being thence obtained by one-step cyclical permutations of the numbers. If $m = 2$, such an initial arrangement is (k. 1 against 5. 6) and (2. 4 against 3. 7). If $m = 3$, one such initial arrangement is (k. 1 against 5. 6), (2. 11 against 3. 9) and (4. 8 against 7. 10). If $m = 4$, one such initial arrangement is (k. 1 against 6. 11), (2. 3 against 5. 9), (4. 12 against 13. 15), and (7. 10 against 8. 14). There are also solutions by other methods.

Sylvester's Corollary. To the original theorem J. J. Sylvester added the corollary that the school of 15 girls could walk out in triplets on 91 days until every possible triplet had walked abreast once, and he published a solution in 1861.

The generalized problem of finding the greatest number of ways in which x girls walking in rows of a abreast can be arranged so that every possible combination of b of them may walk abreast once and only once has been solved for various cases. Suppose that this greatest number of ways is y. It is obvious that, if all the x girls are to walk out each day in rows of a abreast, then x must be an exact multiple of a and the number of rows formed each day is x/a. If such an arrangement can be made for z days, then we have a solution of the problem to arrange x girls to walk out in rows of a abreast for z days so that they all go out each day and so that every possible combination of b girls may walk together once and only once.

An example where the solution is obvious is if $x = 2n$, $a = 2$, $b = 2$, in which case $y = n(2n-1)$, $z = 2n-1$. If we take the case $x = 15$, $a = 3$, $b = 2$, we find $y = 35$; and it happens that these 35 rows can be divided into 7 sets, each of which contains all the symbols; hence $z = 7$. More generally, if $x = 5 \times 3^m$, $a = 3$, $b = 2$, we find $y = 3(x-1)/2x$, $z = (x-1)/2$. It will be noticed that in the solutions of the original fifteen school-girls problem and of Walecki's extension of it given above every possible pair of girls walk together once; hence we might infer that in these cases we could determine z as well as y.

The results of the last paragraph were given by Kirkman in 1850. In the same memoir he also proved that, if $x=9$, $a=3$, $b=3$, then $y=84$, $z=28$; and if $x=15$, $a=3$, $b=3$, $y=455$, $z=91$, but some of the extensions he gave are not correct. He showed, however, how 9 girls can be arranged to walk out 28 times (say 4 times a day for a week) so that in any day the same pair never are together more than once and so that at the end of the week each girl has been associated with every possible pair of her school-fellows. Three years later he attacked the problem when $x=2^n$, $a=4$, $b=3$, but his analysis is open to objection. In 1867 S. Bills showed that if $x=15$, $a=3$, $b=2$, then $y=35$: and the method used will give the value of y, if $x=2^n-1$, $a=3$, $b=2$. Shortly afterwards W. Lea showed that if $x=16$, $a=4$, $b=3$, then $y=140$. These writers did not confine their discussion to cases where x is an exact multiple of a.

Steiner's Problem. I should add that Kirkman's problem, but in a somewhat more general form, was proposed independently by J. Steiner in 1852, and, as enunciated by him, is known as *Steiner's Combinatorische Aufgabe.* In substance Steiner sought to find for what values of n it is possible to arrange n things in triplets, so that every pair of things is contained in one and only one triplet: any triplet forming part of such an arrangement is called a triad. Also, if n is a number for which such an arrangement can be formed, he asked whether there are other arrangements that cannot be obtained from it by permutations of the things. Other problems proposed by him are as follows. When such an arrangement of triads has been made, is it possible to arrange the n things in sets of four, called tetrads, so that no one of the triads is contained in any tetrad, but every triplet which is not a triad is contained in one and only one tetrad? And generally when an arrangement in k-ads has been made, is it possible to arrange the things in sets of $(k+1)$-ads so that no h-ad, where $h \not> k$, is obtained in a $(k+1)$-ad, and so that every k-let that is not or does not contain an h-ad is contained in one and only one $(k+1)$-ad?

CHAPTER X.

MISCELLANEOUS PROBLEMS.

I propose to discuss in this chapter the mathematical theory of a few common mathematical amusements and games. I might have dealt with them in the first four chapters, but, since most of them involve mixed geometry and algebra, it is rather more convenient to deal with them apart from the problems and puzzles which have been described already; the arrangement is, however, based on convenience rather than on any logical distinction.

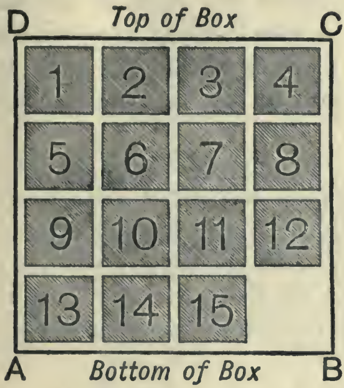
The majority of the questions here enumerated have no connection one with another, and I jot them down almost at random.

I shall discuss in succession the *Fifteen Puzzle*, the *Tower of Hanoï*, *Chinese Rings*, and some miscellaneous *Problems connected with a Pack of Cards.*

THE FIFTEEN PUZZLE*. Some years ago the so-called *Fifteen Puzzle* was on sale in all toy-shops. It consists of a shallow wooden box—one side being marked as the top—in the form of a square, and contains fifteen square blocks or counters numbered 1, 2, 3, ... up to 15. The box will hold just sixteen such counters, and, as it contains only fifteen, they can be moved about in the box relatively to one another. Initially they are put in the box in any order, but leaving the sixteenth

* There are two articles on the subject in the *American Journal of Mathematics*, 1879, vol. II, by Professors Woolsey Johnson and Storey; but the whole theory is deducible immediately from the proposition I give in the text.

cell or small square empty; the puzzle is to move them so that finally they occupy the position shown in the first of the annexed figures.



We may represent the various stages in the game by supposing that the blank space, occupying the sixteenth cell, is moved over the board, ending finally where it started.

The route pursued by the blank space may consist partly of tracks followed and again retraced, which have no effect on the arrangement, and partly of closed paths travelled round, which necessarily are cyclical permutations of an odd number of counters. No other motion is possible.

Now a cyclical permutation of n letters is equivalent to $n-1$ simple interchanges; accordingly an odd cyclical permutation is equivalent to an even number of simple interchanges. Hence, if we move the counters so as to bring the blank space back into the sixteenth cell, the new order must differ from the initial order by an even number of simple interchanges. If therefore the order we want to get can be obtained from this initial order only by an odd number of interchanges, the problem is incapable of solution; if it can be obtained by an even number, the problem is possible.

Thus the order in the second of the diagrams given above is deducible from that in the first diagram by six interchanges; namely, by interchanging the counters 1 and 2, 3 and 4, 5 and 6, 7 and 8, 9 and 10, 11 and 12. Hence the one can be deduced from the other by moving the counters about in the box.

If however in the second diagram the order of the last three counters had been 13, 15, 14, then it would have required seven interchanges of counters to bring them into the order given in the first diagram. Hence in this case the problem would be insoluble.

The easiest way of finding the number of simple interchanges necessary in order to obtain one given arrangement from another is to make the transformation by a series of cycles. For example, suppose that we take the counters in the box in any definite order, such as taking the successive rows from left to right, and suppose the original order and the final order to be respectively

1, 13, 2, 3, 5, 7, 12, 8, 15, 6, 9, 4, 11, 10, 14,
and 11, 2, 3, 4, 5, 6, 7, 1, 9, 10, 13, 12, 8, 14, 15.

We can deduce the second order from the first by 12 simple interchanges. The simplest way of seeing this is to arrange the process in three separate cycles as follows:—

1, 11, 8;	13, 2, 3, 4, 12, 7, 6, 10, 14, 15, 9;	5.
11, 8, 1;	2, 3, 4, 12, 7, 6, 10, 14, 15, 9, 13;	5.

Thus, if in the first row of figures 11 is substituted for 1, then 8 for 11, then 1 for 8, we have made a cyclical interchange of 3 numbers, which is equivalent to 2 simple interchanges (namely, interchanging 1 and 11, and then 1 and 8). Thus the whole process is equivalent to one cyclical interchange of 3 numbers, another of 11 numbers, and another of 1 number. Hence it is equivalent to $(2+10+0)$ simple interchanges. This is an even number, and thus one of these orders can be deduced from the other by moving the counters about in the box.

It is obvious that, if the initial order is the same as the required order except that the last three counters are in the order 15, 14, 13, it would require one interchange to put them in the order 13, 14, 15; hence the problem is insoluble.

If however the box is turned through a right angle, so as to make AD the top, this rotation will be equivalent to 13 simple interchanges. For, if we keep the sixteenth square always blank, then such a rotation would change any order such as

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,
to 13, 9, 5, 1, 14, 10, 6, 2, 15, 11, 7, 3, 12, 8, 4,

which is equivalent to 13 simple interchanges. Hence it will change the arrangement from one where a solution is impossible to one where it is possible, and vice versa.

Again, even if the initial order is one which makes a solution impossible, yet if the first cell and not the last is left blank it will be possible to arrange the fifteen counters in their natural order. For, if we represent the blank cell by b, this will be equivalent to changing the order

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, b,
to b, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15:

this is a cyclical interchange of 16 things and therefore is equivalent to 15 simple interchanges. Hence it will change the arrangement from one where a solution is impossible to one where it is possible, and vice versa.

So, too, if it were permissible to turn the 6 and the 9 upside down, thus changing them to 9 and 6 respectively, this would be equivalent to one simple interchange, and therefore would change an arrangement where a solution is impossible to one where it is possible.

It is evident that the above principles are applicable equally to a rectangular box containing mn cells or spaces and $mn-1$ counters which are numbered. Of course m may be equal to n. If such a box is turned through a right angle, and m and n are both even, it will be equivalent to $mn-3$ simple interchanges—and thus will change an impossible position to a possible one, and vice versa—but unless both m and n are even the rotation is equivalent to only an even number of interchanges. Similarly, if either m or n is even, and it is impossible to solve the problem when the last cell is left blank, then it will be possible to solve it by leaving the first cell blank.

The problem may be made more difficult by limiting the possible movements by fixing bars inside the box which will prevent the movement of a counter transverse to their directions. We can conceive also of a similar cubical puzzle, but we could not work it practically except by sections.

THE TOWER OF HANOÏ. I may mention next the ingenious puzzle known as the *Tower of Hanoï*. It was brought out in 1883 by M. Claus (Lucas).

It consists of three pegs fastened to a stand, and of eight circular discs of wood or cardboard each of which has a hole in the middle through which a peg can be passed. These discs are of different radii, and initially they are placed all on one peg, so that the biggest is at the bottom, and the radii of the successive discs decrease as we ascend: thus the smallest disc is at the top. This arrangement is called the *Tower*. The problem is to shift the discs from one peg to another in such a way that a disc shall never rest on one smaller than itself, and finally to transfer the tower (*i.e.* all the discs in their proper order) from the peg on which they initially rested to one of the other pegs.

The method of effecting this is as follows. (i) If initially there are n discs on the peg A, the first operation is to transfer gradually the top $n-1$ discs from the peg A to the peg B, leaving the peg C vacant: suppose that this requires x separate transfers. (ii) Next, move the bottom disc to the peg C. (iii) Then, reversing the first process, transfer gradually the $n-1$ discs from B to C, which will necessitate x transfers. Hence, if it requires x transfers of simple discs to move a tower of $n-1$ discs, then it will require $2x+1$ separate transfers of single discs to move a tower of n discs. Now with 2 discs it requires 3 transfers, *i.e.* 2^2-1 transfers; hence with 3 discs the number of transfers required will be $2(2^2-1)+1$, that is, 2^3-1. Proceeding in this way we see that with a tower of n discs it will require 2^n-1 transfers of single discs to effect the complete transfer. Thus the eight discs of the puzzle will require 255 single transfers. It will be noticed that every alternate move consists of a transfer of the smallest disc from one peg to another, the pegs being taken in cyclical order: further if the discs be numbered consecutively 1, 2, 3, ... beginning with the smallest, all those with odd numbers rotate in one direction, and all those with even numbers in the other direction.

De Parville gave an account of the origin of the toy which is a sufficiently pretty conceit to deserve repetition*. In the great temple at Benares, says he, beneath the dome which marks the centre of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish. Would that other writers were in the habit of inventing equally interesting origins for the puzzles they produce!

The number of separate transfers of single discs which the Brahmins must make to effect the transfer of the tower is $2^{64}-1$, that is, is 18,446744,073709,551615: a number which, even if the priests never made a mistake, would require many thousands of millions of years to carry out.

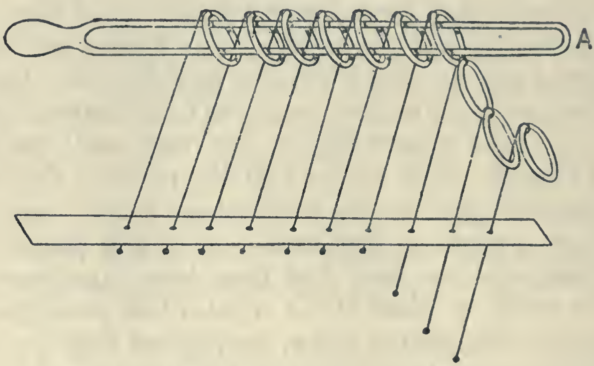
Chinese Rings†. A somewhat more elaborate toy, known as *Chinese Rings*, which is on sale in most English toy-shops,

* *La Nature*, Paris, 1884, part I, pp. 285—286.

† It was described by Cardan in 1550 in his *De Subtilitate*, bk. XV, paragraph 2, ed. Sponius, vol. III, p. 587; by Wallis in his *Algebra*, Latin edition, 1693, *Opera*, vol. II, chap. CXI, pp. 472—478; and allusion is made to it also in Ozanam's *Récréations*, 1723 edition, vol. IV, p. 439.

is represented in the accompanying figure. It consists of a number of rings hung upon a bar in such a manner that the ring at one end (say A) can be taken off or put on the bar at pleasure; but any other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the bar. The order of the rings cannot be changed.

Only one ring can be taken off or put on at a time. [In the toy, as usually sold, the first two rings form an exception to the rule. Both these can be taken off or put on together.



To simplify the discussion I shall assume at first that only one ring is taken off or put on at a time.] I proceed to show that, if there are n rings, then in order to disconnect them from the bar, it will be necessary to take a ring off or to put a ring on either $\frac{1}{3}(2^{n+1}-1)$ times or $\frac{1}{3}(2^{n+1}-2)$ times according as n is odd or even.

Let the taking a ring off the bar or putting a ring on the bar be called a *step*. It is usual to number the rings from the free end A. Let us suppose that we commence with the first m rings off the bar and all the rest on the bar; and suppose that then it requires $x-1$ steps to take off the next ring, that is, it requires $x-1$ additional steps to arrange the rings so that the first $m+1$ of them are off the bar and all the rest are on it. Before taking these steps we can take off the $(m+2)$th ring and thus it will require x steps from our initial position to remove the $(m+1)$th and $(m+2)$th rings.

Suppose that these x steps have been made and that thus the first $m+2$ rings are off the bar and the rest on it, and let us find how many additional steps are now necessary to take off the $(m+3)$th and $(m+4)$th rings. To take these off we begin by taking off the $(m+4)$th ring: this requires 1 step. Before we can take off the $(m+3)$th ring we must arrange the rings so that the $(m+2)$th ring is on and the first $m+1$ rings are off: to effect this, (i) we must get the $(m+1)$th ring on and the first m rings off, which requires $x-1$ steps, (ii) then we must put on the $(m+2)$th ring, which requires 1 step, (iii) and lastly we must take the $(m+1)$th ring off, which requires $x-1$ steps: these movements require in all $\{2(x-1)+1\}$ steps. Next we can take the $(m+3)$th ring off, which requires 1 step; this leaves us with the first $m+1$ rings off, the $(m+2)$th on, the $(m+3)$th off and all the rest on. Finally to take off the $(m+2)$th ring, (i) we get the $(m+1)$th ring on and the first m rings off, which requires $x-1$ steps, (ii) we take off the $(m+2)$th ring, which requires 1 step, (iii) we take the $(m+1)$th ring off, which requires $x-1$ steps: these movements require $\{2(x-1)+1\}$ steps.

Therefore, if when the first m rings are off it requires x steps to take off the $(m+1)$th and $(m+2)$th rings, then the number of additional steps required to take off the $(m+3)$th and $(m+4)$th rings is $1+\{2(x-1)+1\}+1+\{2(x-1)+1\}$, that is, is $4x$.

To find the whole number of steps necessary to take off an odd number of rings we proceed as follows.

To take off the first ring requires 1 step;

$\therefore$ to take off the first 3 rings requires 4 additional steps;

$\therefore$ " " 5 " " 4^2 " "

In this way we see that the number of steps required to take off the first $2n+1$ rings is $1+4+4^2+\ldots+4^n$, which is equal to $\frac{1}{3}(2^{2n+2}-1)$.

To find the number of steps necessary to take off an even number of rings we proceed in a similar manner.

To take off the first 2 rings requires 2 steps;
∴ to take off the first 4 rings requires 2×4 additional steps;
∴ " " " 6 " " 2×4^2 " "
In this way we see that the number of steps required to take off the first $2n$ rings is $2 + (2 \times 4) + (2 \times 4^2) + \ldots + (2 \times 4^{n-1})$, which is equal to $\frac{1}{3}(2^{2n+1} - 2)$.

If we take off or put on the first two rings in one step instead of two separate steps, these results become respectively 2^{2n} and $2^{2n-1} - 1$.

I give the above analysis because it is the direct solution of a problem attacked unsuccessfully by Cardan in 1550 and by Wallis in 1693, and which at one time attracted some attention.

I proceed next to give another solution, more elegant though rather artificial. This, which is due to Monsieur Gros*, depends on a convention by which any position of the rings is denoted by a certain number expressed in the binary scale of notation in such a way that a step is indicated by the addition or subtraction of unity.

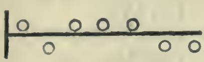
Let the rings be indicated by circles: if a ring is on the bar, it is represented by a circle drawn above the bar; if the ring is off the bar, it is represented by a circle below the bar. Thus figure i below represents a set of seven rings of which the first two are off the bar, the next three are on it, the sixth is off it, and the seventh is on it.

Denote the rings which are on the bar by the digits 1 or 0 alternately, reckoning from left to right, and denote a ring which is off the bar by the digit assigned to that ring on the bar which is nearest to it on the left of it, or by a 0 if there is no ring to the left of it.

Thus the three positions indicated below are denoted respectively by the numbers written below them. The position represented in figure ii is obtained from that in figure i by putting the first ring on to the bar, while the position represented in figure iii is obtained from that in figure i by taking the fourth ring off the bar.

* *Théorie du Baguenodier*, by L. Gros, Lyons, 1872. I take the account of this from Lucas, vol. I, part 7.

It follows that every position of the rings is denoted by a number expressed in the binary scale: moreover, since in going from left to right every ring on the bar gives a variation (that is, 1 to 0 or 0 to 1) and every ring off the bar gives a continuation, the effect of a step by which a ring is taken off or put on the bar is either to subtract unity from this number or to add unity to it. For example, the number denoting the position of the rings in figure ii is obtained from the number denoting that in figure i by adding unity to it. Similarly the number denoting the position of the rings in figure iii is obtained from the number denoting that in figure i by subtracting unity from it.



1101000

Figure i.

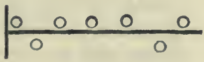


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Figure ii.

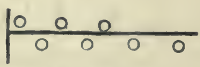

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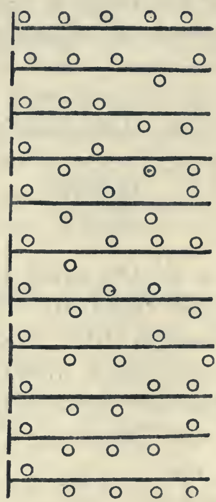
Figure iii.

The position when all the seven rings are off the bar is denoted by the number 0000000: when all of them are on the bar by the number 1010101. Hence to change from one position to the other requires a number of steps equal to the difference between these two numbers in the binary scale. The first of these numbers is 0: the second is equal to $2^6 + 2^4 + 2^2 + 1$, that is, to 85. Therefore 85 steps are required. In a similar way we may show that to put on a set of $2n + 1$ rings requires $(1 + 2^2 + \dots + 2^{2n})$ steps, that is, $\frac{1}{3}(2^{2n+2} - 1)$ steps; and to put on a set of $2n$ rings requires $(2 + 2^3 + \dots + 2^{2n-1})$ steps, that is, $\frac{1}{3}(2^{2n+1} - 2)$ steps.

I append a table indicating the steps necessary to take off the first four rings from a set of five rings. The diagrams in the middle column show the successive position of the rings after each step. The number following each diagram indicates that position, each number being obtained from the one above it by the addition of unity. The steps which are bracketed together can be made in one movement, and, if thus effected, the whole process is completed in 7 movements instead of 10 steps: this is in accordance with the formula given above.

Gros asserted that it is possible to take from 64 to 80 steps a minute, which in my experience is a rather high estimate. If we accept the lower of these numbers, it would be possible to take off 10 rings in less than 8 minutes; to take off 25 rings would require more than 582 days, each of ten hours work; and to take off 60 rings would necessitate no less than 768614,336404,564650 steps, and would require nearly 55000,000000 years work—assuming of course that no mistakes were made.



Initial position	10101
After 1st step	10110
„ 2nd „	10111
„ 3rd „	11000
„ 4th „	11001
„ 5th „	11010
„ 6th „	11011
„ 7th „	11100
„ 8th „	11101
„ 9th „	11110
„ 10th „	11111

PROBLEMS CONNECTED WITH A PACK OF CARDS. An ordinary pack of playing cards can be used to illustrate many questions depending on simple properties of numbers, or involving the relative position of the cards. In problems of this kind, the principle of solution generally consists in re-arranging the pack in a particular manner so as to bring the card into some definite position. Any such re-arrangement is a species of shuffling.

I shall treat in succession of problems connected with *Shuffling a Pack*, *Arrangements by Rows and Columns*, the *Determination of a Pair out of* $\frac{1}{2}n(n+1)$ *Pairs*, *Gergonne's Pile Problem*, and the game known as the *Mouse Trap*.

SHUFFLING A PACK. Any system of *shuffling a pack* of cards, if carried out consistently, leads to an arrangement which can be calculated; but tricks that depend on it generally require considerable technical skill.

Suppose for instance that a pack of n cards is shuffled, as is not unusual, by placing the second card on the first, the third below these, the fourth above them, and so on. The theory of this system of shuffling is due to Monge*. The following are some of the results and are not difficult to prove directly.

If the pack contains $6p+4$ cards, the $(2p+2)$th card will occupy the same position in the shuffled pack. For instance, if a complete pack of 52 cards is shuffled as described above, the 18th card will remain the 18th card.

Again, if a pack of $10p+2$ cards is shuffled in this way, the $(2p+1)$th and the $(6p+2)$th cards will interchange places. For instance, if an écarté pack of 32 cards is shuffled as described above, the 7th and the 20th cards will change places.

More generally, one shuffle of a pack of $2p$ cards will move the card which was in the x_0th place to the x_1th place, where $x_1 = \frac{1}{2}(2p + x_0 + 1)$ if x_0 is odd, and $x_1 = \frac{1}{2}(2p - x_0 + 2)$ if x_0 is even, from which the above results can be deduced. By repeated applications of the above formulae we can show that the effect of m such shuffles is to move the card which was initially in the x_0th place to the x_mth place, where

$$2^{m+1} x_m = (4p+1)(2^{m-1} \pm 2^{m-2} \pm \dots \pm 2 \pm 1) \pm 2x_0 + 2^m \pm 1,$$

the sign $\pm$ representing an ambiguity of sign.

Again, in any pack of n cards after a certain number of shufflings, not greater than n, the cards will return to their primitive order. This will always be the case as soon as the original top card occupies that position again. To determine

* Monge's investigations are printed in the *Mémoires de l'Académie des Sciences*, Paris, 1773, pp. 390—412. Among those who have studied the subject afresh I may in particular mention V. Bouniakowski, *Bulletin physico-mathématique de St Pétersbourg*, 1857, vol. xv, pp. 202—205, summarised in the *Nouvelles annales de mathématiques*, 1858, *Bulletin*, pp. 66—67; T. de St Laurent, *Mémoires de l'Académie de Gard*, 1865; L. Tanner, *Educational Times Reprints*, 1880, vol. XXXIII, pp. 73—75; M. J. Bourget, *Liouville's Journal*, 1882, pp. 413—434; and H. F. Baker, *Transactions of the British Association for* 1910, pp. 526—528.

the number of shuffles required for a pack of $2p$ cards, it is sufficient to put $x_m = x_0$ and find the smallest value of m which satisfies the resulting equation for all values of x_0 from 1 to $2p$. It follows that, if m is the least number which makes $4^m - 1$ divisible by $4p+1$, then m shuffles will be required if either $2^m + 1$ or $2^m - 1$ is divisible by $4p+1$, otherwise $2m$ shuffles will be required. The number for a pack of $2p+1$ cards is the same as that for a pack of $2p$ cards. With an écarté pack of 32 cards, six shuffles are sufficient; with a pack of 2^n cards, $n+1$ shuffles are sufficient; with a full pack of 52 cards, twelve shuffles are sufficient; with a pack of 13 cards ten shuffles are sufficient; while with a pack of 50 cards fifty shuffles are required; and so on.

W. H. H. Hudson* has also shown that, whatever is the law of shuffling, yet if it is repeated again and again on a pack of n cards, the cards will ultimately fall into their initial positions after a number of shufflings not greater than the greatest possible L.C.M. of all numbers whose sum is n.

For suppose that any particular position is occupied after the 1st, 2nd, ..., pth shuffles by the cards $A_1, A_2, \ldots, A_p$ respectively, and that initially the position is occupied by the card A_0. Suppose further that after the pth shuffle A_0 returns to its initial position, therefore $A_0 = A_p$. Then at the second shuffling A_2 succeeds A_1 by the same law by which A_1 succeeded A_0 at the first; hence it follows that previous to the second shuffling A_2 must have been in the place occupied by A_1 previous to the first. Thus the cards which after the successive shuffles take the place initially occupied by A_1 are $A_2, A_3, \ldots, A_p, A_1$; that is, after the pth shuffle A_1 has returned to the place initially occupied by it: and so for all the other cards $A_2, A_3, \ldots, A_{p-1}$.

Hence the cards $A_1, A_2, \ldots, A_p$ form a cycle of p cards, one or other of which is always in one or other of p positions in the pack, and which go through all their changes in p shufflings. Let the number n of the pack be divided into $p, q, r, \ldots$ such cycles, whose sum is n; then the L.C.M. of $p, q, r, \ldots$ is the

* *Educational Times Reprints*, London, 1865, vol. II, p. 105.

utmost number of shufflings necessary before all the cards will be brought back to their original places. In the case of a pack of 52 cards, the greatest L.C.M. of numbers whose sum is 52 is 180180.

ARRANGEMENTS BY ROWS AND COLUMNS. A not uncommon trick, which rests on a species of shuffling, depends on the obvious fact that if n^2 cards are arranged in the form of a square of n rows, each containing n cards, then any card will be defined if the row and the column in which it lies are mentioned.

This information is generally elicited by first asking in which row the selected card lies, and noting the extreme left-hand card of that row. The cards in each column are then taken up, face upwards, one at a time beginning with the lowest card of each column and taking the columns in their order from right to left—each card taken up being placed on the top of those previously taken up. The cards are then dealt out again in rows, from left to right, beginning with the top left-hand corner, and a question is put as to which row contains the card. The selected card will be that card in the row mentioned which is in the same vertical column as the card which was originally noted.

The trick is improved by allowing the pack to be cut as often as is liked before the cards are re-dealt, and then giving one cut at the end so as to make the top card in the pack one of those originally in the top row. For instance, take the

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure i.

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

Figure ii.

case of 16 cards. The first and second arrangements may be represented by figures i and ii. Suppose we are told that in figure i the card is in the third row, it must be either 9, 10, 11, 12: hence, if we know in which row of figure ii it lies, it is determined. If we allow the pack to be cut between the deals, we must secure somehow that the top card is either 1, 2, 3, or 4, since that will leave the cards in each row of figure ii unaltered though the positions of the rows will be changed.

DETERMINATION OF A SELECTED PAIR OF CARDS OUT OF $\frac{1}{2}n(n+1)$ GIVEN PAIRS*. Another common trick is to throw twenty cards on to a table in ten couples, and ask someone to select one couple. The cards are then taken up, and dealt out in a certain manner into four rows each containing five cards. If the rows which contain the given cards are indicated, the cards selected are known at once.

This depends on the fact that the number of homogeneous products of two dimensions which can be formed out of four things is 10. Hence the homogeneous products of two dimensions formed out of four things can be used to define ten things

Suppose that ten pairs of cards are placed on a table and someone selects one couple. Take up the cards in their

1	2	3	5	7
4	9	10	11	13
6	12	15	16	17
8	14	18	19	20

couples. Then the first two cards form the first couple, the next two the second couple, and so on. Deal them out in four rows each containing five cards according to the scheme shown above.

* Bachet, problem XVII, avertissement, p. 146 *et seq.*

The first couple (1 and 2) are in the first row. Of the next couple (3 and 4), put one in the first row and one in the second. Of the next couple (5 and 6), put one in the first row and one in the third, and so on, as indicated in the diagram. After filling up the first row proceed similarly with the second row, and so on.

Enquire in which rows the two selected cards appear. If only one line, the mth, is mentioned as containing the cards, then the required pair of cards are the mth and $(m+1)$th cards in that line. These occupy the clue squares of that line. Next, if two lines are mentioned, then proceed as follows. Let the two lines be the pth and the qth and suppose $q > p$. Then that one of the required cards which is in the qth line will be the $(q-p)$th card which is below the first of the clue squares in the pth line. The other of the required cards is in the pth line and is the $(q-p)$th card to the right of the second of the clue squares.

Bachet's rule, in the form in which I have given it, is applicable to a pack of $n(n+1)$ cards divided into couples, and dealt in n rows each containing $n+1$ cards; for there are $\frac{1}{2}n(n+1)$ such couples, also there are $\frac{1}{2}n(n+1)$ homogeneous products of two dimensions which can be formed out of n things. Bachet gave the diagrams for the cases of 20, 30, and 42 cards: these the reader will have no difficulty in constructing for himself, and I have enunciated the rule for 20 cards in a form which covers all the cases.

I have seen the same trick performed by means of a sentence and not by numbers. If we take the case of ten couples, then after collecting the pairs the cards must be dealt in four rows each containing five cards, in the order indicated by the sentence *Matas dedit nomen Cocis.* This sentence must be imagined as written on the table, each word forming one line. The first card is dealt on the *M.* The next card (which is the pair of the first) is placed on the second *m* in the sentence, that is, third in the third row. The third card is placed on the *a.* The fourth card (which is the pair of the third) is placed on the second *a*, that is, fourth in the first row. Each of the next two cards is placed on a *t*, and so on. Enquire in which rows the two selected cards appear. If two rows are mentioned, the two cards are on the letters common to the words that make these rows. If only one row is mentioned, the cards are on the two letters common to that row.

The reason is obvious: let us denote each of the first pair by an *a*, and similarly each of any of the other pairs by an *e*, *i*, *o*, *c*, *d*, *m*, *n*, *s*, or *t* respectively. Now the sentence *Matas dedit nomen Cocis* contains four words each of five letters; ten letters are used, and each letter is repeated only twice. Hence, if two of the words are mentioned, they will have one letter in common, or, if one word is mentioned, it will have two like letters.

To perform the same trick with any other number of cards we should require a different sentence.

The number of homogeneous products of three dimensions which can be formed out of four things is 20, and of these the number consisting of products in which three things are alike and those in which three things are different is 8. This leads to a trick with 8 trios of things, which is similar to that last given—the cards being arranged in the order indicated by the sentence *Lanata levete livini novoto*.

I believe that these arrangements by sentences are well-known, but I am not aware who invented them.

GERGONNE'S PILE PROBLEM. Before discussing Gergonne's theorem I will describe the familiar three pile problem, the theory of which is included in his results.

*The Three Pile Problem**. This trick is usually performed as follows. Take 27 cards and deal them into three piles, face upwards. By "dealing" is to be understood that the top card is placed as the bottom card of the first pile, the second card in the pack as the bottom card of the second pile, the third card as the bottom card of the third pile, the fourth card on the top of the first one, and so on: moreover I assume that throughout

* The trick is mentioned by Bachet, problem XVIII, p. 143, but his analysis of it is insufficient.

the problem the cards are held in the hand face upwards. The result can be modified to cover any other way of dealing.

Request a spectator to note a card, and remember in which pile it is. After finishing the deal, ask in which pile the card is. Take up the three piles, placing that pile between the other two. Deal again as before, and repeat the question as to which pile contains the given card. Take up the three piles again, placing the pile which now contains the selected card between the other two. Deal again as before, but in dealing note the middle card of each pile. Ask again for the third time in which pile the card lies, and you will know that the card was the one which you noted as being the middle card of that pile. The trick can be finished then in any way that you like. The usual method—but a very clumsy one—is to take up the three piles once more, placing the named pile between the other two as before, when the selected card will be the middle one in the pack, that is, if 27 cards are used it will be the 14th card.

The trick is often performed with 15 cards or with 21 cards, in either of which cases the same rule holds.

Gergonne's Generalization. The general theory for a pack of m^m cards was given by M. Gergonne*. Suppose the pack is arranged in m piles, each containing m^{m-1} cards, and that, after the first deal, the pile indicated as containing the selected card is taken up ath; after the second deal, is taken up bth; and so on, and finally after the mth deal, the pile containing the card is taken up kth. Then when the cards are collected after the mth deal the selected card will be nth from the top where

$$\text{if } m \text{ is even,} \quad n = km^{m-1} - jm^{m-2} + \ldots + bm - a + 1,$$
$$\text{if } m \text{ is odd,} \quad n = km^{m-1} - jm^{m-2} + \ldots - bm + a.$$

For example, if a pack of 256 cards (*i.e.* $m = 4$) was given, and anyone selected a card out of it, the card could be determined by making four successive deals into four piles of 64 cards each, and after each deal asking in which pile the

* Gergonne's *Annales de Mathématiques*, Nismes, 1813–4, vol. IV, pp. 276—283.

selected card lay. The reason is that after the first deal you know it is one of sixty-four cards. In the next deal these sixty-four cards are distributed equally over the four piles, and therefore, if you know in which pile it is, you will know that it is one of sixteen cards. After the third deal you know it is one of four cards. After the fourth deal you know which card it is.

Moreover, if the pack of 256 cards is used, it is immaterial in what order the pile containing the selected card is taken up after a deal. For, if after the first deal it is taken up ath, after the second bth, after the third cth, and after the fourth dth, the card will be the $(64d - 16c + 4b - a + 1)$th from the top of the pack, and thus will be known. We need not take up the cards after the fourth deal, for the same argument will show that it is the $(64 - 16c + 4b - a + 1)$th in the pile then indicated as containing it. Thus if $a = 3$, $b = 4$, $c = 1$, $d = 2$, it will be the 62nd card in the pile indicated after the fourth deal as containing it and will be the 126th card in the pack as then collected.

In exactly the same way a pack of twenty-seven cards may be used, and three successive deals, each into three piles of nine cards, will suffice to determine the card. If after the deals the pile indicated as containing the given card is taken up ath, bth, and cth respectively, then the card will be the $(9c - 3b + a)$th in the pack or will be the $(9 - 3b + a)$th card in the pile indicated after the third deal as containing it.

The method of proof will be illustrated sufficiently by considering the usual case of a pack of twenty-seven cards, for which $m = 3$, which are dealt into three piles each of nine cards.

Suppose that, after the first deal, the pile containing the selected card is taken up ath: then (i) at the top of the pack there are $a - 1$ piles each containing nine cards; (ii) next there are 9 cards, of which one is the selected card; and (iii) lastly there are the remaining cards of the pack. The cards are dealt out now for the second time: in each pile the bottom $3(a - 1)$ cards will be taken from (i), the next 3 cards from (ii), and the remaining $9 - 3a$ cards from (iii).

Suppose that the pile now indicated as containing the selected card is taken up bth: then (i) at the top of the pack are $9(b-1)$ cards; (ii) next are $9-3a$ cards; (iii) next are 3 cards, of which one is the selected card; and (iv) lastly are the remaining cards of the pack. The cards are dealt out now for the third time: in each pile the bottom $3(b-1)$ cards will be taken from (i), the next $3-a$ cards will be taken from (ii), the next card will be one of the three cards in (iii), and the remaining $8-3b+a$ cards are from (iv).

Hence, after this deal, as soon as the pile is indicated, it is known that the card is the $(9-3b+a)$th from the top of that pile. If the process is continued by taking up this pile as cth, then the selected card will come out in the place $9(c-1)+(8-3b+a)+1$ from the top, that is, will come out as the $(9c-3b+a)$th card.

Since, after the third deal, the position of the card in the pile then indicated is known, it is easy to notice the card, in which case the trick can be finished in some way more effective than dealing again.

If we put the pile indicated always in the middle of the pack we have $a=2$, $b=2$, $c=2$, hence $n=9c-3b+a=14$, which is the form in which the trick is usually presented, as was explained above on page 241.

I have shown that if a, b, c are known, then n is determined. We may modify the rule so as to make the selected card come out in any assigned position, say the nth. In this case we have to find values of a, b, c which will satisfy the equation $n=9c-3b+a$, where a, b, c can have only the values 1, 2, or 3.

Hence, if we divide n by 3 and the remainder is 1 or 2, this remainder will be a; but, if the remainder is 0, we must decrease the quotient by unity so that the remainder is 3, and this remainder will be a. In other words a is the smallest positive number (exclusive of zero) which must be subtracted from n to make the difference a multiple of 3.

Next let p be this multiple, *i.e.* p is the next lowest integer to $n/3$: then $3p=9c-3b$, therefore $p=3c-b$. Hence b is the smallest positive number (exclusive of zero) which must be added to p to make the sum a multiple of 3, and c is that multiple.

A couple of illustrations will make this clear. Suppose we wish the card to come out 22nd from the top, therefore $22 = 9c - 3b + a$. The smallest number which must be subtracted from 22 to leave a multiple of 3 is 1, therefore $a = 1$. Hence $22 = 9c - 3b + 1$, therefore $7 = 3c - b$. The smallest number which must be added to 7 to make a multiple of 3 is 2, therefore $b = 2$. Hence $7 = 3c - 2$, therefore $c = 3$. Thus $a = 1$, $b = 2$, $c = 3$.

Again, suppose the card is to come out 21st. Hence $21 = 9c - 3b + a$. Therefore a is the smallest number which subtracted from 21 makes a multiple of 3, therefore $a = 3$. Hence $6 = 3c - b$. Therefore b is the smallest number which added to 6 makes a multiple of 3, therefore $b = 3$. Hence $9 = 3c$, therefore $c = 3$. Thus $a = 3$, $b = 3$, $c = 3$.

If any difficulty is experienced in this work, we can proceed thus. Let $a = x + 1$, $b = 3 - y$, $c = z + 1$; then x, y, z may have only the values 0, 1, or 2. In this case Gergonne's equation takes the form $9z + 3y + x = n - 1$. Hence, if $n - 1$ is expressed in the ternary scale of notation, x, y, z will be determined, and therefore a, b, c will be known.

The rule in the case of a pack of m^m cards is exactly similar. We want to make the card come out in a given place. Hence, in Gergonne's formula, we are given n and we have to find $a, b, \ldots, k$. We can effect this by dividing n continually by m, with the convention that the remainders are to be alternately positive and negative and that their numerical values are to be not greater than m or less than unity.

An analogous theorem with a pack of lm cards can be constructed. C. T. Hudson and L. E. Dickson* have discussed the general case where such a pack is dealt n times, each time into l piles of m cards; and they have shown how the piles must be

* *Educational Times Reprints*, 1868, vol. IX, pp. 89—91; and *Bulletin* of the American Mathematical Society, New York, April, 1895, vol. I, pp. 184—186.

taken up in order that after the nth deal the selected card may be rth from the top.

The principle will be sufficiently illustrated by one example treated in a manner analogous to the cases already discussed. For instance, suppose that an écarté pack of 32 cards is dealt into four piles each of 8 cards, and that the pile which contains some selected card is picked up ath. Suppose that on dealing again into four piles, one pile is indicated as containing the selected card, the selected card cannot be one of the bottom $2(a-1)$ cards, or of the top $8-2a$ cards, but must be one of the intermediate 2 cards, and the trick can be finished in any way, as for instance by the common conjuring ambiguity of asking someone to choose one of them, leaving it doubtful whether the one he takes is to be rejected or retained.

THE MOUSE TRAP. TREIZE. I will conclude this chapter with the bare mention of another game of cards, known as the *Mouse Trap*, the discussion of which involves some rather difficult algebraic analysis.

It is played as follows. A set of cards, marked with the numbers 1, 2, 3, ..., n, is dealt in any order, face upwards, in the form of a circle. The player begins at any card and counts round the circle always in the same direction. If the kth card has the number k on it—which event is called a *hit*—the player takes up the card and begins counting afresh. According to Cayley, the player wins if he thus takes up all the cards, and the cards win if at any time the player counts up to n without being able to take up a card.

For example, if a pack of only four cards is used and these cards come in the order 3214, then the player would obtain the second card 2 as a hit, next he would obtain 1 as a hit, but if he went on for ever he would not obtain another hit. On the other hand, if the cards in the pack were initially in the order 1423, the player would obtain successively all four cards in the order 1, 2, 3, 4.

The problem may be stated as the determination of what hits and how many hits can be made with a given number of cards; and what permutations will give a certain number of hits in a certain order.

Cayley* showed that there are 9 arrangements of a pack of four cards in which no hit will be made, 7 arrangements in which only one hit will be made, 3 arrangements in which only two hits will be made, and 5 arrangements in which four hits will be made.

Prof. Steen† has investigated the general theory for a pack of n cards. He has shown how to determine the number of arrangements in which x is the first hit [Arts. 3—5]; the number of arrangements in which 1 is the first hit and x is the second hit [Art. 6]; and the number of arrangements in which 2 is the first hit and x the second hit [Arts. 7—8]; but beyond this point the theory has not been carried. It is obvious that, if there are $n-1$ hits, the nth hit will necessarily follow.

The French game of *treize* is very similar. It is played with a full pack of fifty-two cards (knave, queen, and king counting as 11, 12, and 13 respectively). The dealer calls out 1, 2, 3, ..., 13, as he deals the 1st, 2nd, 3rd, ..., 13th cards respectively. At the beginning of a deal the dealer offers to lay or take certain odds that he will make a hit in the thirteen cards next dealt.

* *Quarterly Journal of Mathematics*, 1878, vol. xv, pp. 8—10.

† *Ibid.*, vol. xv, pp. 230—241.

PART II.

Miscellaneous Essays.

'*No man of science should think it a waste of time to learn something of the history of his own subject; nor is the investigation of laborious methods now fallen into disuse, or of errors once commonly accepted, the least valuable of mental disciplines.*"

"*The most worthless book of a bygone day is a record worthy of preservation. Like a telescopic star, its obscurity may render it unavailable for most purposes; but it serves, in hands which know how to use it, to determine the places of more important bodies.*"

(DE MORGAN.)

CHAPTER XI.

CALCULATING PRODIGIES.

At rare intervals there have appeared lads who possess extraordinary powers of mental calculation*. In a few seconds they gave the answers to questions connected with the multiplication of numbers and the extraction of roots of numbers, which an expert mathematician could obtain only in a longer time and with the aid of pen and paper. Nor were their powers always limited to such simple problems. More difficult questions, dealing for instance with factors, compound interest, annuities, the civil and ecclesiastical calendars, and the solution of equations, were solved by some of them with facility as soon as the meaning of what was wanted had been grasped. In most cases these lads were illiterate, and usually their rules of working were of their own invention.

The performances were so remarkable that some observers held that these prodigies possessed powers differing in kind from those of their contemporaries. For such a view there is no foundation. Any lad with an excellent memory and a natural turn for arithmetic can, if he continuously gives his undivided attention to the consideration of numbers, and indulges in constant practice, attain great proficiency in mental arithmetic, and of course the performances of those that are specially gifted are exceptionally astonishing.

* Most of the facts about calculating prodigies have been collected by E. W. Scripture, *American Journal of Psychology*, 1891, vol. IV, pp. 1—59; by F. D. Mitchell, *Ibid.*, 1907, vol. XVIII, pp. 61—143; and G. E. Müller, *Zur Analyse der Gedächtnistätigkeit und des Vorstellungsverlaufes*, Leipzig, 1911. I have used these papers freely, and in some cases where authorities are quoted of which I have no first-hand information have relied exclusively on them. These articles should be consulted for bibliographical notes on the numerous original authorities.

In this chapter I propose to describe briefly the doings of the more famous calculating prodigies. It will be seen that their performances were of much the same general character, though carried to different extents, hence in the later cases it will be enough to indicate briefly peculiarities of the particular calculators.

I confine myself to self-taught calculators, and thus exclude the consideration of a few public performers who by practice, arithmetical devices, and the tricks of the showman have simulated like powers. I also concern myself only with those who showed the power in youth. As far as I know the only self-taught mathematician of advanced years whom I thus exclude is John Wallis, 1616—1703, the Savilian Professor at Oxford, who in middle-life developed, for his own amusement, his powers in mental arithmetic. As an illustration of his achievements, I note that on 22 December 1669 he, when in bed, occupied himself in finding (mentally) the integral part of the square root of 3×10^{40}; and several hours afterwards wrote down the result from memory. This fact having attracted notice, two months later he was challenged to extract the square root of a number of fifty-three digits; this he performed mentally, and a month later he dictated the answer which he had not meantime committed to writing. Such efforts of calculation and memory are typical of calculating prodigies.

One of the earliest of these prodigies, of whom we have records, was *Jedediah Buxton*, who was born in or about 1707 at Elmton, Derbyshire. Although a son of the village schoolmaster, his education was neglected, and he never learnt to write or cipher. With the exception of his power of dealing with large numbers, his mental faculties were of a low order: he had no ambition, and remained throughout his life a farm-labourer, nor did his exceptional skill with figures bring him any material advantage other than that of occasionally receiving small sums of money from those who induced him to exhibit his peculiar gift. He does not seem to have given public exhibitions. He died in 1772.

He had no recollection as to when or how he was first attracted by mental calculations, and of his performances in early life we have no reliable details. Mere numbers however seem always to have had a strange fascination for him. If the size of an object was stated, he began at once to compute how many inches or hair-breadths it contained; if a period of time was mentioned, he calculated the number of minutes in it; if he heard a sermon, he thought only of the number of words or syllables in it. No doubt his powers in these matters increased by incessant practice, but his ideas were childish, and do not seem to have gone beyond pride in being able to state accurately the results of such calculations. He was slow witted, and took far longer to answer arithmetical questions than most of these prodigies. The only practical accomplishment to which his powers led him was the ability to estimate by inspection the acreage of a field of irregular shape.

His fame gradually spread through Derbyshire. Among many questions put to him by local visitors were the following, which fairly indicate his powers when a young man:—How many acres are there in a rectangular field 351 yards long and 261 wide; answered in 11 minutes. How many cubic yards of earth must be removed in order to make a pond 426 feet long, 263 feet wide, and 2½ feet deep; answered in 15 minutes. If sound travels 1142 feet in one second, how long will it take to travel 5 miles; answered in 15 minutes. Such questions involve no difficulties of principle.

Here are a few of the harder problems solved by Buxton when his powers were fully developed. He calculated to what sum a farthing would amount if doubled 140 times: the answer is a number of pounds sterling which requires thirty-nine digits to represent it with 2*s.* 8*d.* over. He was then asked to multiply this number of thirty-nine digits by itself: to this he gave the answer two and a half months later, and he said he had carried on the calculation at intervals during that period. In 1751 he calculated how many cubic inches there are in a right-angled block of stone 23,145,789 yards long, 5,642,732 yards wide, and 54,965 yards thick; how many grains of corn would be required to fill a cube whose volume is 202,680,000,360 cubic miles; and how many hairs one inch long would be required to fill the same space—the dimensions of a grain and a hair being given. These problems involve high numbers, but are not intrinsically difficult, though they could not be solved mentally unless the calculator had a phenomenally good memory. In each case he gave the correct answer, though only after considerable effort. In 1753 he was asked to give the dimensions of a cubical cornbin, which holds exactly one quarter of malt. He recognized that to answer this required a process equivalent to the extraction of a cube root, which was a novel idea to him, but in an hour he said that the edge of the cube would be between $25\frac{1}{2}$ and 26 inches, which is correct: it has been suggested that he got this answer by trying various numbers.

Accounts of his performances were published, and his reputation reached London, which he visited in 1754. During his stay there he was examined by various members of the Royal Society, who were satisfied as to the genuineness of his performances. Some of his acquaintances took him to Drury Lane Theatre to see Garrick, being curious to see how a play would impress his imagination. He was entirely unaffected by the scene, but on coming out informed his hosts of the exact number of words uttered by the various actors, and of the number of steps taken by others in their dances.

It was only in rare cases that he was able to explain his methods of work, but enough is known of them to enable us to say that they were clumsy. He described the process by which he arrived at the product of 456 and 378: shortly it was as follows:—If we denote the former of these numbers by a, he proceeded first to find $5a =$ (say) b; then to find $20b =$ (say) c; and then to find $3c =$ (say) d. He next formed $15b =$ (say) e, which he added to d. Lastly he formed $3a$ which, added to the sum last obtained, gave the result. This is equivalent to saying that he used the multiplier 378 in the form $(5 \times 20 \times 3) + (5 \times 15) + 3$. Mitchell suggests that this may mean that Buxton counted by multiples of 60 and of 15, and thus reduced the multiplication to addition. It may be so, for it is difficult to suppose that he did not realize that successive multiplications by 5 and 20 are equivalent to a multiplication by 100, of which the result can be at once obtained. Of billions, trillions, &c., he had never heard, and in order to represent the high numbers required in some of the questions proposed to him he invented a notation of his own, calling 10^{18} a tribe and 10^{36} a cramp.

As in the case of all these calculators, his memory was exceptionally good, and in time he got to know a large number of facts (such as the products of certain constantly recurring numbers, the number of minutes in a year, and the number of hair-breadths in a mile) which greatly facilitated his calculations. A curious and perhaps unique feature in his case was that he could stop in the middle of a piece of mental calculation, take up other subjects, and after an interval, sometimes of weeks, could resume the consideration of the problem. He could answer simple questions when two or more were proposed simultaneously.

Another eighteenth-century prodigy was *Thomas Fuller*, a negro, born in 1710 in Africa. He was captured there in 1724, and exported as a slave to Virginia, U.S.A., where he lived till his death in 1790. Like Buxton, Fuller never learnt to read or write, and his abilities were confined to mental arithmetic. He could multiply together two numbers, if each contained not more than nine digits, could state the number of seconds in a given period of time, the number of grains of corn in a given mass, and so on—in short, answer the stock problems commonly proposed to these prodigies, as long as they involved only multiplications and the solutions of problems by rule of three. Although more rapid than Buxton, he was a slow worker as compared with some of those whose doings are described below.

I mention next the case of two mathematicians of note who showed similar aptitude in early years. The first of these was *André Marie Ampère*, 1775—1836, who, when a child some four years old, was accustomed to perform long mental calculations, which he effected by means of rules learnt from playing with arrangements of pebbles. But though always expert at mental arithmetic, and endowed with a phenomenal memory for figures, he did not specially cultivate this arithmetical power. It is more difficult to say whether *Carl Friedrich Gauss*, 1777—1855, should be reckoned among these calculating prodigies. He had, when three years old, taught himself some arithmetical processes, and astonished his father by correcting him in his calculations of certain payments for overtime; perhaps, however, this is only evidence of the early age at which his consummate abilities began to develop. Another remarkable case is that of *Richard Whately*, 1787—1863, afterwards Archbishop of Dublin. When he was between five or six years old he showed considerable skill in mental arithmetic: it disappeared in about three years. I soon, said he, "got to do the most difficult sums, always in my head, for I knew nothing of figures beyond numeration, nor had I any names for the different processes I employed. But I believe my sums were chiefly in multiplication, division, and the rule of three...I did these sums much quicker than any one could upon paper, and I never remember committing the smallest error. I was engaged either in calculating or in castle-building...morning, noon, and night... When I went to school, at which time the passion was worn off, I was a perfect dunce at ciphering, and so have continued ever since." The archbishop's arithmetical powers were, however, greater in after-life than he here allows.

The performances of *Zerah Colburn* in London, in 1812, were more remarkable. Colburn*, born in 1804, at Cabut, Vermont, U.S.A., was the son of a small farmer. While still less than six years old he showed extraordinary powers of mental calculation, which were displayed in a tour in America. Two years later he was brought to England, where he was repeatedly examined by competent observers. He could instantly give the product of two numbers each of four digits, but hesitated if both numbers exceeded 10,000. Among questions asked him at this time were to raise 8 to the 16th power; in a few seconds he gave the answer 281,474,976,710,656, which is correct. He was next asked to raise the numbers 2, 3,...9 to

* To the authorities mentioned by E. W. Scripture and F. D. Mitchell should be added *The Annual Register*, London, 1812, p. 507 *et seq.*

the 10th power: and he gave the answers so rapidly that the gentleman who was taking them down was obliged to ask him to repeat them more slowly; but he worked less quickly when asked to raise numbers of two digits like 37 or 59 to high powers. He gave instantaneously the square roots and cube roots (when they were integers) of high numbers, *e.g.*, the square root of 106,929 and the cube root of 268,336,125, such integral roots can, however, be obtained easily by various methods. More remarkable are his answers to questions on the factors of numbers. Asked for the factors of 247,483 he replied 941 and 263; asked for the factors of 171,395 he gave 5, 7, 59, and 83; asked for the factors of 36,083 he said there were none. He, however, found it difficult to answer questions about the factors of numbers higher than 1,000,000. His power of factorizing high numbers was exceptional and depended largely on the method of two-digit terminals described below. Like all these public performers he had to face buffoons who tried to make fun of him, but he was generally equal to them. Asked on one such occasion how many black beans were required to make three white ones, he is said to have at once replied "three, if you skin them"—this, however, has much the appearance of a pre-arranged show.

It was clear to observers that the child operated by certain rules, and during his calculations his lips moved as if he was expressing the process in words. Of his honesty there seems to have been no doubt. In a few cases he was able to explain the method of operation. Asked for the square of 4,395 he hesitated, but on the question being repeated he gave the correct answer, namely 19,395,025. Questioned as to the cause of his hesitation, he said he did not like to multiply four figures by four figures, but said he, "I found out another way; I multiplied 293 by 293 and then multiplied this product twice by the number 15." On another occasion when asked for the product of 21,734 by 543 he immediately replied 11,801,562; and on being questioned explained that he had arrived at this by multiplying 65,202 by 181. These remarks suggest that whenever convenient he factorized the numbers with which he was dealing.

In 1814 he was taken to Paris, but amid the political turmoil of the time his exhibitions fell flat. His English and American friends however raised money for his education, and he was sent in succession to the Lycée Napoleon in Paris and Westminster School in London. With education his calculating powers fell off, and he lost the frankness which when a boy had charmed observers. His subsequent career was diversified and not altogether successful. He commenced with the stage, then tried schoolmastering, then became an itinerant preacher in America, and finally a "professor" of languages. He wrote his own biography which contains an account of the methods he used. He died in 1840.

Contemporary with Colburn we find another instance of a self-taught boy, *George Parker Bidder*, who possessed quite exceptional powers of this kind. He is perhaps the most interesting of these prodigies because he subsequently received a liberal education, retained his calculating powers, and in later life analyzed and explained the methods he had invented and used.

Bidder was born in 1806 at Moreton Hampstead, Devonshire, where his father was a stone-mason. At the age of six he was taught to count up to 100, but though sent to the village school learnt little there, and at the beginning of his career was ignorant of the meaning of arithmetical terms and of numerical symbols. Equipped solely with this knowledge of counting he taught himself the results of addition, subtraction, and multiplication of numbers (less than 100) by arranging and re-arranging marbles, buttons, and shot in patterns. In after-life he attached great importance to such concrete representations, and believed that his arithmetical powers were strengthened by the fact that at that time he knew nothing about the symbols for numbers. When seven years old he heard a dispute between two of his neighbours about the price of something which was being sold by the pound, and to their astonishment remarked that they were both wrong, mentioning the correct price. After this exhibition the villagers delighted in trying to pose him with arithmetical problems.

His reputation increased and, before he was nine years old, his father found it profitable to take him about the country to exhibit his powers. A couple of distinguished Cambridge graduates (Thomas Jephson, then tutor of St John's, and John Herschel) saw him in 1817, and were so impressed by his general intelligence that they raised a fund for his education, and induced his father to give up the rôle of showman; but after a few months Bidder senior repented of his abandonment of money so easily earned, insisted on his son's return, and began again to make an exhibition of the boy's powers. In 1818, in the course of a tour young Bidder was pitted against Colburn and on the whole proved the abler calculator. Finally the father and son came to Edinburgh, where some members of that University intervened and persuaded his father to leave the lad in their care to be educated. Bidder remained with them, and in due course graduated at Edinburgh, shortly afterwards entering the profession of civil engineering in which he rose to high distinction. He died in 1878.

With practice Bidder's powers steadily developed. His earlier performances seem to have been of the same type as those of Buxton and Colburn which have been already described. In addition to answering questions on products of numbers and the number of specified units in given quantities, he was, after 1819, ready in finding square roots, cube roots, &c. of high numbers, it being assumed that the root is an integer, and later explained his method which is easy of application: this method is the same as that used by Colburn. By this time he was able also to give immediate solutions of easy problems on compound interest and annuities which seemed to his contemporaries the most astonishing of all his feats. In factorizing numbers he was less successful than Colburn and was generally unable to deal at sight with numbers higher than 10,000. As in the case of Colburn, attempts to be witty at his expense were often made, but he could hold his own. Asked at one of his performances in London in 1818, how many bulls' tails were wanted to reach to the moon, he immediately answered one, if it is long enough.

Here are some typical questions put to and answered by him in his exhibitions during the years 1815 to 1819—they are taken from authenticated lists which comprise some hundreds of such problems: few, if any, are inherently difficult. His rapidity of work was remarkable, but the time limits given were taken by unskilled observers and can be regarded as only approximately correct. Of course all the calculations were mental without the aid of books, pencil, or paper. In 1815, being then nine years old, he was asked:—If the moon be distant from the earth 123,256 miles, and sound travels at the rate of 4 miles a minute, how long would it be before the inhabitants of the moon could hear of the battle of Waterloo: answer, 21 days, 9 hours, 34 minutes, given in less than one minute. In 1816, being then ten years old, just learning to write, but unable to form figures, he answered questions such as the following:—What is the interest on £11,111 for 11,111 days at 5 per cent. a year: answer, £16,911. 11*s.*, given in one minute. How many hogsheads of cider can be made from a million of apples, if 30 apples make one quart: answer, 132 hogsheads, 17 gallons, 1 quart, and 10 apples over, given in 35 seconds. If a coach-wheel is 5 feet 10 inches in circumference, how many times will it revolve in running 800,000,000 miles: answer, 724,114,285,704 times and 20 inches remaining, given in 50 seconds. What is the square root of 119,550,669,121: answer 345,761, given in 30 seconds. In 1817, being then eleven years old, he was asked:—How long would it take to fill a reservoir whose volume is one cubic mile if there flowed into it from a river 120 gallons of water a minute: answered in 2 minutes. Assuming that light travels from the sun to the earth in 8 minutes, and that the sun is 98,000,000 miles off, if light takes 6 years 4 months travelling from the nearest fixed star to the earth, what is the distance of that star, reckoning 365 days 6 hours to each year and 28 days to each month—asked by Sir William Herschel: answer, 40,633,740,000,000 miles. In 1818, at one of his performances, he was asked:—If the pendulum of a clock vibrates the distance of 9¾ inches in a second of time, how many inches will it vibrate in 7 years 14 days

2 hours 1 minute 56 seconds, each year containing 365 days 5 hours 48 minutes 55 seconds: answer, 2,165,625,744$\frac{3}{4}$ inches, given in less than a minute. If I have 42 watches for sale and I sell the first for a farthing, and double the price for every succeeding watch I sell, what will be the price of the last watch: answer, £2,290,649,224. 10*s.* 8*d.* If the diameter of a penny piece is 1$\frac{3}{8}$ inches, and if the world is girdled with a ring of pence put side by side, what is their value sterling, supposing the distance to be 360 degrees, and a degree to contain 69·5 miles: answer, £4,803,340, given in one minute. Find two numbers, whose difference is 12, and whose product, multiplied by their sum, is equal to 14,560: answer, 14 and 26. In 1819, when fourteen years old, he was asked:—Find a number whose cube less 19 multiplied by its cube shall be equal to the cube of 6: answer, 3, given instantly. What will it cost to make a road for 21 miles 5 furlongs 37 poles 4 yards, at the rate of £123. 14*s.* 6*d.* a mile: answer, £2688. 13*s.* 9$\frac{3}{4}$*d.*, given in 2 minutes. If you are now 14 years old and you live 50 years longer and spend half-a-crown a day, how many farthings will you spend in your life: answer, 2,805,120, given in 15 seconds. Mr Moor contracted to illuminate the city of London with 22,965,321 lamps, the expense of trimming and lighting was 7 farthings a lamp, the oil consumed was $\frac{2}{9}$ths of a pint for every three lamps, and the oil cost 3*s.* 7$\frac{1}{2}$*d.* a gallon; he gained 16$\frac{1}{2}$ per cent. on his outlay: how many gallons of oil were consumed, what was the cost to him, and what was the amount of the contract: answer, he used 212,641 gallons of oil, the cost was £205,996. 16*s.* 1$\frac{3}{4}$*d.*, and the amount of the contract was £239,986. 13*s.* 2*d.* If the distance of the earth from the moon be 29,531,531$\frac{1}{4}$ yards, what is the weight of a thread which will extend that distance, supposing 7$\frac{11}{16}$ yards of it weigh $\frac{1}{16}$th part of a drachm: answer, 8 cwt. 1 qr. 13 lbs. 9 oz. 1 dr. and $\frac{13}{16}$ths of a drachm.

It should be noted that Bidder did not visualize a number like 984 in symbols, but thought of it in a concrete way as so many units which could be arranged in 24 groups of 41 each. It should also be observed that he, like Inaudi whom I mention later, relied largely on the auditory sense to enable him to recollect numbers. "For my own part," he wrote, in later life, "though much accustomed to see sums and quantities expressed by the usual symbols, yet if I endeavour to get any number of figures that are represented on paper fixed in my memory, it takes me a much longer time and a very great deal more exertion than when they are expressed or enumerated verbally." For instance suppose a question put to find the product of two numbers each of nine digits, if they were "read to me, I should not require this to be done more than once; but if they were represented in the usual way, and put into my hands, it would probably take me four times to peruse them before it would be in my power to repeat them, and after all they would not be impressed so vividly on my imagination."

Bidder retained his power of rapid mental calculation to the end of his life, and as a constant parliamentary witness in matters connected with engineering it proved a valuable accomplishment. Just before his death an illustration of his powers was given to a friend who talking of then recent discoveries remarked that if 36,918 waves of red light which only occupy one inch are required to give the impression of red, and if light travels at 190,000 miles a second, how immense must be the number of waves which must strike the eye in one second to give the impression of red. "You need not work it out," said Bidder, "the number will be 444,433,651,200,000."

Other members of the Bidder family have also shown exceptional powers of a similar kind as well as extraordinary memories. Of Bidder's elder brothers, one became an actuary, and on his books being burnt in a fire he rewrote them in six months from memory but, it is said, died of consequent brain fever; another was a Plymouth Brother and knew the whole Bible by heart, being able to give chapter and verse for any text quoted. Bidder's eldest son, a lawyer of eminence, was able to multiply together two numbers each of fifteen digits. Neither in accuracy nor rapidity was he equal to his father, but then he never steadily and continuously devoted himself to developing his abilities in this direction. He remarked that in his mental arithmetic he worked with pictures of the figures, and said "If I perform a sum mentally it always proceeds in a visible form in my mind; indeed I can conceive no other way possible of doing mental arithmetic": this it will be noticed is opposed to his father's method. Two of his children, one son and one daughter representing a third generation, have inherited analogous powers.

I mention next the names of *Henri Mondeux*, and *Vito Mangiamele*. Both were born in 1826 in humble circumstances, were sheep-herds, and became when children, noticeable for feats in calculation which deservedly procured for them local fame. In 1839 and 1840 respectively they were brought to Paris where their powers were displayed in public, and tested by Arago, Cauchy, and others. Mondeux's performances were the more striking. One question put to him was to solve the equation $x^3 + 84 = 37x$: to this he at once gave the answer 3 and 4, but did not detect the third root, namely, -7. Another question asked was to find solutions of the indeterminate equation $x^2 - y^2 = 133$: to this he replied immediately 66 and 67; asked for a simpler solution he said after an instant 6 and 13. I do not however propose to discuss their feats in detail, for there was at least a suspicion that these lads were not frank, and that those who were exploiting them had taught them rules which enabled them to simulate powers they did not really possess. Finally both returned to farm work, and ceased to interest the scientific world. If Mondeux was self-taught we must credit him with a discovery of some algebraic theorems which would entitle him to rank as a mathematical genius, but in that case it is inconceivable that he never did anything more, and that his powers appeared to be limited to the particular problems solved by him.

Johann Martin Zacharias Dase, whom I next mention, is a far more interesting example of these calculating prodigies. Dase was born in 1824 at Hamburg. He had a fair education, and was afforded every opportunity to develop his powers, but save in matters connected with reckoning and numbers he made little progress and struck all observers as dull. Of geometry and any language but German he remained ignorant to the end of his days. He was trustworthy and filled various small official posts in Germany. He gave exhibitions of his calculating powers in Germany, Austria, and England. He died in 1861.

When exhibiting in Vienna in 1840, he made the acquaintance of Strasznicky who urged him to apply his powers to scientific purposes. This Dase gladly agreed to do, and so became acquainted with Gauss, Schumacher, Petersen, and Encke. To his contributions to science I allude later. In mental arithmetic the only problems to which I find allusions are straightforward examples like the following:—Multiply 79,532,853 by 93,758,479: asked by Schumacher, answered in 54 seconds. In answer to a similar request to find the product of two numbers each of twenty digits he took 6 minutes; to find the product of two numbers each of forty digits he took 40 minutes; to find the product of two numbers each of a hundred digits he took 8 hours 45 minutes. Gauss thought that perhaps on paper the last of these problems could be solved in half this time by a skilled computator. Dase once extracted the square root of a number of a hundred digits in 52 minutes. These feats far surpass all other records of the kind, the only calculations comparable to them being Buxton's squaring of a number of thirty-nine digits, and Wallis' extraction of the square root of a number of fifty-three digits. Dase's mental work however was not always accurate, and once (in 1845) he gave incorrect answers to every question put to him, but on that occasion he had a headache, and there is nothing astonishing in his failure.

Like all these calculating prodigies he had a wonderful memory, and an hour or two after a performance could repeat all the numbers mentioned in it. He had also the peculiar gift of being able after a single glance to state the number (up to about 30) of sheep in a flock, of books in a case, and so on; and of visualizing and recollecting a large number of objects. For instance, after a second's look at some dominoes he gave the sum (117) of their points; asked how many letters were in a certain line of print chosen at random in a quarto page he instantly gave the correct number (63); shown twelve digits he had in half a second memorized them and their positions so as to be able to name instantly the particular digit occupying any assigned place. It is to be regretted that we do not know more of these performances. Those who are acquainted with the delightful autobiography of Robert-Houdin will recollect how he cultivated a similar power, and how valuable he found it in the exercise of his art.

Dase's calculations, when also allowed the use of paper and pencil, were almost incredibly rapid, and invariably accurate. When he was sixteen years old Strasznicky taught him the use of the familiar formula $\pi/4 = \tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{5}) + \tan^{-1}(\frac{1}{8})$, and asked him thence to calculate π. In two months he carried the approximation to 205 places of decimals, of which 200 are correct*. Dase's next achievement was to calculate the natural logarithms of the first 1,005,000 numbers to 7 places of decimals; he did this in his off-time from 1844 to 1847, when occupied by the Prussian survey. During the next two years he compiled in his spare time a hyperbolic table which was published by the Austrian Government in 1857. Later he offered to make tables of the factors of all numbers from 7,000 000 to 10,000,000 and, on the recommendation of Gauss, the Hamburg Academy of Sciences agreed to assist him so that he might have leisure for the purpose, but he lived only long enough to finish about half the work.

Truman Henry Safford, born in 1836 at Royalton, Vermont, U.S.A., was another calculating prodigy. He was of a somewhat different type for he received a good education, graduated in due course at Harvard, and ultimately took up astronomy in which subject he held a professional post. I gather that though always a rapid calculator, he gradually lost the exceptional powers shown in his youth. He died in 1901.

Safford never exhibited his calculating powers in public, and I know of them only through the accounts quoted by Scripture

* The result was published in *Crelle's Journal*, 1844, vol. XXVII, p. 198: on closer approximations and easier formulae, see below chapter XIII.

and Mitchell, but they seem to have been typical of these calculators. In 1842, he amused and astonished his family by mental calculations. In 1846, when ten years old, he was examined, and here are some of the questions then put to him:—Extract the cube root of a certain number of seven digits; answered instantly. What number is that which being divided by the product of its digits, the quotient is three, and if 18 be added the digits will be inverted: answer 24, given in about a minute. What is the surface of a regular pyramid whose slant height is 17 feet, and the base a pentagon of which each side is 33·5 feet: answer 3354·5558 square feet, given in two minutes. Asked to square a number of eighteen digits he gave the answer in a minute or less, but the question was made the more easy as the number consisted of the digits 365 repeated six times. Like Colburn he factorized high numbers with ease. In such examples his processes were empirical, he selected (he could not tell how) likely factors and tested the matter in a few seconds by actual division.

There are to-day four calculators of some note. These are *Ugo Zamebone*, an Italian, born in 1867; *Pericles Diamandi*, a Greek, born in 1868; *Carl Rückle*, a German; and *Jacques Inaudi*, born in 1867. The three first mentioned are of the normal type and I do not propose to describe their performances, but Inaudi's performances merit a fuller treatment.

Jacques Inaudi* was born in 1867 at Onorato in Italy. He was employed in early years as a sheep-herd, and spent the long idle hours in which he had no active duties in pondering on numbers, but used for them no concrete representations such as pebbles. His calculating powers first attracted notice about 1873. Shortly afterwards his elder brother sought his fortune as an organ grinder in Provence, and young Inaudi, accompanying him, came into a wider world, and earned a few coppers for himself by street exhibitions of his powers. His ability was exploited by showmen, and thus in 1880 he visited Paris

* See Charcot and Darboux, *Mémoires de l'Institut, Comptes Rendus*, 1892, vol. cxiv, pp. 275, 528, 578; and Binet, *Révue des deux Mondes*, 1892, vol. cxi, pp. 905—924.

where he gave exhibitions: in these he impressed all observers as being modest, frank, and straightforward. He was then ignorant of reading and writing: these arts he subsequently acquired.

His earlier performances were not specially remarkable as compared with those of similar calculating prodigies, but with continual practice he improved. Thus at Lyons in 1873 he could multiply together almost instantaneously two numbers of three digits. In 1874 he was able to multiply a number of six digits by another number of six digits. Nine years later he could work rapidly with numbers of nine or ten digits. Still later, in Paris, asked by Darboux to cube 27, he gave the answer in 10 seconds. In 13 seconds he calculated how many seconds are contained in 18 years 7 months 21 days 3 hours: and he gave immediately the square root of one-sixth of the difference between the square of 4801 and unity. He also calculated with ease the amount of wheat due according to the traditional story to Sessa who, for inventing chess, was to receive 1 grain on the first cell of a chess-board, 2 on the second, 4 on the third, and so on in geometrical progression.

He can find the integral roots of equations and integral solutions of problems, but proceeds only by trial and error. His most remarkable feat is the expression of numbers less than 10^5 in the form of a sum of four squares, which he can usually do in a minute or two; this power is peculiar to him. Such problems have been repeatedly solved at private performances, but the mental strain caused by them is considerable.

A performance before the general public rarely lasts more than 12 minutes, and is a much simpler affair. A normal programme includes the subtraction of one number of twenty-one digits from another number of twenty-one digits: the addition of five numbers each of six digits: the multiplying of a number of four digits by another number of four digits: the extraction of the cube root of a number of nine digits, and of the fifth root of a number of twelve digits: the determination of the number of seconds in a period of time, and the day of the week on which a given date falls. Of course the questions are put by members of the audience. To a professional calculator these problems are not particularly difficult. As each number is announced, Inaudi repeats it slowly to his assistant, who writes it on a blackboard, and then slowly reads it aloud to make sure that it is right. Inaudi then repeats the number once more. By this time he has generally solved the problem, but if he wants longer time he makes a few remarks of a general character, which he is able to do without interfering with his mental calculations. Throughout the exhibition he faces the audience: the fact that he never even glances at the blackboard adds to the effect.

It is probable that the majority of calculating prodigies rely on the speech muscles as well as on the eye and the ear to help them to recollect the figures with which they are dealing. It was formerly believed that they all visualized the numbers proposed to them, and certainly some have done so. Inaudi however trusts mainly to the ear and to articulation. Bidder also relied partly on the ear, and when he visualized a number it was not as a collection of digits but as a concrete collection of units divisible, if the number was composite, into definite groups. Rückle relies mainly on visualizing the numbers. So it would seem that there are different types of the memories of calculators. Inaudi can reproduce mentally the sound of the repetition of the digits of the number in his own voice, and is confused, rather than helped, if the numbers are shown to him in writing. The articulation of the digits of the number also seems necessary to enable him fully to exhibit his powers, and he is accustomed to repeat the numbers aloud before beginning to work on them—the sequence of sounds being important. A number of twenty-four digits being read to him, in 59 seconds he memorized the sound of it, so that he could give the sequence of digits forwards or backwards from selected points—a feat which Mondeux had taken 5 minutes to perform. Numbers of about a hundred digits were similarly memorized by Inaudi in 12 minutes, by Diamandi in 25 minutes, and by Rückle in under 5 minutes. This power is confined to numbers, and calculators cannot usually recollect a long sequence of letters. Numbers are ever before Inaudi: he thinks of little else, he dreams of them, and sometimes even solves problems in his sleep. His memory is excellent for numbers, but normal or subnormal for other things. At the end of a séance he can repeat the questions which have been put to him and his answers, involving hundreds of digits in all. Nor is his memory in such matters limited to a few hours. Once eight days after he had been given a question on a number of twenty-two digits, he was unexpectedly asked about it, and at once repeated the number. He has been repeatedly examined, and we know more of his work than of any of his predecessors, with the possible exception of Bidder.

Most of these calculating prodigies find it difficult or impossible to explain their methods. But we have a few analyses by competent observers of the processes used: notably one by Bidder on his own work; another by Colburn on his work; and others by Müller and Darboux on the work of Rückle and Inaudi respectively. That by Bidder is the most complete, and the others are on much the same general lines.

Bidder's account of the processes he had discovered and used is contained in a lecture* given by him in 1856 to the Institution of Civil Engineers. Before describing these processes there are two remarks of a general character which should, I think, be borne in mind when reading his statement. In the first place he gives his methods in their perfected form, and not necessarily in that which he used in boyhood: moreover it is probable that in practice he employed devices to shorten the work which he did not set out in his lecture. In the second place it is certain, in spite of his belief to the contrary, that he, like most of these prodigies, had an exceptionally good memory, which was strengthened by incessant practice. One example will suffice. In 1816, at a performance, a number was read to him backwards: he at once gave it in its normal form. An hour later he was asked if he remembered

* *Institution of Civil Engineers, Proceedings*, London, 1856, vol. xv, pp. 251—280. An early draft of the lecture is extant in MS.; the variations made in it are interesting, as showing the history of his mental development, but are not sufficiently important to need detailed notice here.

it: he immediately repeated it correctly. The number was:—
2,563,721,987,653,461,598,746,231,905,607,541,128,975,231.

Of the four fundamental processes, addition and subtraction present no difficulty and are of little interest. The only point to which it seems worth calling attention is that Bidder, in adding three or more numbers together, always added them one at a time, as is illustrated in the examples given below. Rapid mental arithmetic depended, in his opinion, on the arrangement of the work whenever possible, in such a way that only one fact had to be dealt with at a time. This is also noticeable in Inaudi's work.

The multiplication of one number by another was, naturally enough, the earliest problem Bidder came across, and by the time he was six years old he had taught himself the multiplication table up to 10 times 10. He soon had practice in harder sums, for, being a favourite of the village blacksmith, and constantly in the smithy, it became customary for the men sitting round the forge-fire to ask him multiplication sums. From products of numbers of two digits, which he would give without any appreciable pause for thought, he rose to numbers of three and then of four digits. Halfpence rewarded his efforts, and by the time he was eight years old, he could multiply together two numbers each of six digits. In one case he even multiplied together two numbers each of twelve digits, but, he says, "it required much time," and "was a great and distressing effort."

The method that he used is, in principle, the same as that explained in the usual text-books, except that he added his results as he went on. Thus to multiply 397 by 173 he proceeded as follows:—

We have	$100 \times 397 = 39700$,		
to this must be added	$70 \times 300 = 21000$	making	60,700,
" " " " "	$70 \times 90 = 6300$	"	67,000,
" " " " "	$70 \times 7 = 490$	"	67,490,
" " " " "	$3 \times 300 = 900$	"	68,390,
" " " " "	$3 \times 90 = 270$	"	68,660,
" " " " "	$3 \times 7 = 21$	"	68,681.

We shall underrate his rapidity if we allow as much as a second for each of these steps, but even if we take this low standard of his speed of working, he would have given the answer in 7 seconds. By this method he never had at one time more than two numbers to add together, and the factors are arranged so that each of them has only one significant digit: this is the common practice of mental calculators. It will also be observed that here, as always, Bidder worked from left to right: this, though not usually taught in our schools, is the natural and most convenient way. In effect he formed the product of $(100 + 70 + 3)$ and $(300 + 90 + 7)$, or $(a + b + c)$ and $(d + e + f)$ in the form $ad + ae \dots + ef$.

The result of a multiplication like that given above was attained so rapidly as to seem instantaneous, and practically gave him the use of a multiplication table up to 1000 by 1000. On this basis, when dealing with much larger numbers, for instance, when multiplying 965,446,371 by 843,409,133, he worked by numbers forming groups of 3 digits, proceeding as if 965, 446, &c., were digits in a scale whose radix was 1000: in middle life he would solve a problem like this in about 6 minutes. Such difficulty as he experienced in these multiplications seems to have been rather in recalling the result of the previous step than in making the actual multiplications.

Inaudi also multiplies in this way, but he is content if one of the factors has only one significant digit: he also sometimes makes use of negative quantities: for instance he thinks of 27×729 as $27\,(730 - 1)$; so, too, he thinks of 25×841 in the form $84100/4$: and in squaring numbers he is accustomed to think of the number in the form $a + b$, choosing a and b of convenient forms, and then to calculate the result in the form $a^2 + 2ab + b^2$.

In multiplying concrete data by a number Bidder worked on similar lines to those explained above in the multiplication of two numbers. Thus to multiply £14. 15*s.* $6\frac{3}{4}$*d.* by 787 he proceeded thus:

We have £(787) (14) = £11018. 0*s.* 0*d.*
to which we add (787) (15) shillings = £590. 5*s.* 0*d.* making £11608. 5*s.* 0*d.*
to which we add (787) (27) farthings = £22. 2*s.* $8\frac{1}{4}$*d.* making £11630. 7*s.* $8\frac{1}{4}$*d.*

Division was performed by Bidder much as taught in school-books, except that his power of multiplying large numbers at sight enabled him to guess intelligently and so save unnecessary work. This also is Inaudi's method. A division sum with a remainder presents more difficulty. Bidder was better skilled in dealing with such questions than most of these prodigies, but even in his prime he never solved such problems with the same rapidity as those with no remainder. In public performances difficult questions on division are generally precluded by the rules of the game.

If, in a division sum, Bidder knew that there was no remainder he often proceeded by a system of two-digit terminals. Thus, for example, in dividing (say) 25,696 by 176, he first argued that the answer must be a number of three digits, and obviously the left-hand digit must be 1. Next he had noticed that there are only 4 numbers of two digits (namely, 21, 46, 71, 96) which when multiplied by 76 give a number which ends in 96. Hence the answer must be 121, or 146, or 171, or 196; and experience enabled him to say without calculation that 121 was too small and 171 too large. Hence the answer must be 146. If he felt any hesitation he mentally multiplied 146 by 176 (which he said he could do "instantaneously") and thus checked the result. It is noticeable that when Bidder, Colburn, and some other calculating prodigies knew the last two digits of a product of two numbers they also knew, perhaps subconsciously, that the last two digits of the separate numbers were necessarily of certain forms. The theory of these two-digit arrangements has been discussed by Mitchell.

Frequently also in division, Bidder used what I will call a digital process, which *a priori* would seem far more laborious than the normal method, though in his hands the method was extraordinarily rapid: this method was, I think, peculiar to him. I define the digital of a number as the digit obtained by finding the sum of the digits of the original number, the sum of the digits of this number, and so on, until the sum is less than 10. Obviously the digital of a number is the same as the digital of the product of the digitals of its factors. Let us apply this in Bidder's way to see if 71 is an exact divisor of 23,141. The digital of 23,141 is 2. The digital of 71 is 8. Hence if 71 is a factor the digital of the other factor must be 7, since 7 times 8 is the only multiple of 8 whose digital is 2. Now the only number which multiplied by 71 will give 41 as terminal digits is 71. And since the other factor must be one of three digits and its digital must be 7, this factor (if any) must be 871. But a cursory glance shows that 871 is too large. Hence 71 is not a factor of 23,141. Bidder found this process far more rapid than testing the matter by dividing by 71. As another example let us see if 73 is a factor of 23,141. The digital of 23,141 is 2; the digital of 73 is 1; hence the digital of the other factor (if any) must be 2. But since the last two digits of the number are 41, the last two digits of this factor (if any) must be 17. And since this factor is a number of three digits and its digital is 2, such a factor, if it exists, must be 317. This on testing (by multiplying it by 73) is found to be a factor.

When he began to exhibit his powers in public, questions concerning weights and measures were, of course, constantly proposed to him. In solving these he knew by heart many facts which frequently entered into such problems, such as the number of seconds in a year, the number of ounces in a ton, the number of square inches in an acre, the number of pence in a hundred pounds, the elementary rules about the civil and ecclesiastical calendars, and so on. A collection of such data is part of the equipment of all calculating prodigies.

In his exhibitions Bidder was often asked questions concerning square roots and cube roots, and at a later period higher roots. That he could at once give the answer excited unqualified astonishment in an uncritical audience; if, however, the answer is integral, this is a mere sleight of art which anyone can acquire. Without setting out the rules at length, a few examples will illustrate his method.

He was asked to find the square root of 337,561. It is obvious that the root is a number of three digits. Since the given number lies between 500^2 or 250,000 and 600^2 or 360,000, the left-hand digit of the root must be a 5. Reflection had shown him that the only numbers of two digits, whose squares end in 61 are 19, 31, 69, 81, and he was familiar with this fact. Hence the answer was 519, or 531, or 569, or 581. But he argued that as 581 was nearly in the same ratio to 500 and 600 as 337,561 was to 250,000 and 360,000, the answer must be 581, a result which he verified by direct multiplication in a couple of seconds. Similarly in extracting the square root of 442,225, he saw at once that the left-hand digit of the answer was 6, and since the number ended in 225 the last two digits of the answer were 15 or 35, or 65 or 85. The position of 442,225 between $(600)^2$ and $(700)^2$ indicates that 65 should be taken. Thus the answer is 665, which he verified, before announcing it. Other calculators have worked out similar rules for the extraction of roots.

For exact cube roots the process is more rapid. For example, asked to extract the cube root of 188,132,517, Bidder saw at once that the answer was a number of three digits, and since $5^3 = 125$ and $6^3 = 216$, the left-hand digit was 5. The only number of two digits whose cube ends in 17 is 73. Hence the answer is 573. Similarly the cube root of 180,362,125 must be a number of three digits, of which the left-hand digit is a 5, and the two right-hand digits were either 65 or 85. To see which of these was required he mentally cubed 560, and seeing it was near the given number, assumed that 565 was the required answer, which he verified by cubing it. In general a cube root that ends in a 5 is a trifle more difficult to detect at sight by this method than one that ends in some other digit, but since 5^3 must be a factor of such numbers we can divide by that and apply the process to the resulting number. Thus the above number 180,362,125 is equal to $5^3 \times 1{,}442{,}897$ of which the cube root is at once found to be 5 (113), that is, 565.

For still higher exact roots the process is even simpler, and for fifth roots it is almost absurdly easy, since the last digit of the number is always the same as the last digit of the root. Thus if the number proposed is less than 10^{10} the answer consists of a number of two digits. Knowing the fifth powers to 10, 20,...90 we have, in order to know the first digit of the answer, only to see between which of these powers the number proposed lies, and the last digit being obvious we can give the answer instantly. If the number is higher, but less than 10^{15}, the answer is a number of three digits, of which the middle digit can be found almost as rapidly as the others. This is rather a trick than a matter of mental calculation.

In his later exhibitions, Bidder was sometimes asked to extract roots, correct to the nearest integer, the exact root involving a fraction. If he suspected this he tested it by "casting out the nines," and if satisfied that the answer was not an integer proceeded tentatively as best he could. Such a question, if the answer is a number of three or more digits, is a severe tax on the powers of a mental calculator, and is usually disallowed in public exhibitions.

Colburn's remarkable feats in factorizing numbers led to similar questions being put to Bidder, and gradually he evolved some rules, but in this branch of mental arithmetic I do not think he ever became proficient. Of course a factor which is a power of 2 or of 5 can be obtained at once, and powers of 3 can be obtained almost as rapidly. For factors near the square root of a number he always tried the usual method of expressing the number in the form $a^2 - b^2$, in which case the factors are obvious. For other factors he tried the digital method already described.

Bidder was successful in giving almost instantaneously the answers to questions about compound interest and annuities: this was peculiar to him, but his method is quite simple, and may be illustrated by his determination of the compound interest on £100 at 5 per cent. for 14 years. He argued that the simple interest amounted to £(14) (5), *i.e.* to £70. At the end of the first year the capital was increased by £5, the annual interest on this was 5*s.* or one crown, and this ran for 13 years, at the end of the second year another £5 was due, and the 5*s.* interest on this ran for 12 years. Continuing this argument he had to add to the £70 a sum of $(13 + 12 + \ldots + 1)$ crowns, *i.e.* (13/2) (14) (5) shillings, *i.e.* £22. 15*s.* 0*d.*, which, added to the

£70 before mentioned, made £92. 15*s.* 0*d.* Next the 5*s.* due at the end of the second year (as interest on the £5 due at the end of the first year) produced in the same way an annual interest of 3*d.* All these three-pences amount to (12/3) (13/2) (14) (3) pence, *i.e.* £4. 11*s.* 0*d.* which, added to the previous sum of £92. 15*s.* 0*d.*, made £97. 6*s.* 0*d.* To this we have similarly to add (11/4) (12/3) (13/2) (14) (3/20) pence, *i.e.* 12*s.* 6*d.*, which made a total of £97. 18*s.* 6*d.* To this again we have to add (10/5) (11/4) (12/3) (13/2) (14) (3/400) pence, *i.e.* 1*s.* 3*d.*, which made a total of £97. 19*s.* 9*d.* To this again we have to add (9/6) (10/5) (11/4) (12/3) (13/2) (14) (3/8000) pence, *i.e.* 1*d.*, which made a total of £97. 19*s.* 10*d.* The remaining sum to be added cannot amount to a farthing, so he at once gave the answer as £97. 19*s.* 10*d.* The work in this particular example did in fact occupy him less than one minute—a much shorter time than most mathematicians would take to work it by aid of a table of logarithms. It will be noticed that in the course of his analysis he summed various series.

In the ordinary notation, the sum at compound interest amounts to £$(1{\cdot}05)^{14} \times 100$. If we denote £100 by P and $\cdot 05$ by r, this is equal to $P(1+r)^{14}$ or $P(1+14r+91r^2+\ldots)$, which, as r is small, is rapidly convergent. Bidder in effect arrived by reasoning at the successive terms of the series, and rejected the later terms as soon as they were sufficiently small.

In the course of this lecture Bidder remarked that if his ability to recollect results had been equal to his other intellectual powers he could easily have calculated logarithms. A few weeks later he attacked this problem, and devised a mental method of obtaining the values of logarithms to seven or eight places of decimals. He asked a friend to test his accuracy, and in answer to questions gave successively the logarithms of 71, 97, 659, 877, 1297, 8963, 9973, 115249, 175349, 290011, 350107, 229847, 369353, to eight places of decimals; taking from thirty seconds to four minutes to make the various calculations. All these numbers are primes. The greater part of the answers were correct, but in a few cases there was an error, though generally of only one digit: such mistakes were at once corrected on his being told that his result was wrong. This remarkable performance took place when Bidder was over 50.

His method of calculating logarithms is set out in a paper* by W. Pole. It was, of course, only necessary for him to deal with prime numbers, and Bidder began by memorizing the logarithms of all primes less than 100. For a prime higher than this he took a composite number as near it as he could, and calculated the approximate addition which would have to be added to the logarithm: his rules for effecting this addition are set out by Pole, and, ingenious though they are, need not detain us here. They rest on the theorems that, to the number of places of decimals quoted, if $\log n$ is p, then $\log(n+n/10^2)$ is $p+\log 1{\cdot}01$, *i.e.* is $p+0{\cdot}0043214$, $\log(n+n/10^3)$ is $p+0{\cdot}00043407$, $\log(n+n/10^4)$ is $p+0{\cdot}0000434$, $\log(n+n/10^5)$ is $p+0{\cdot}0000043$, and so on.

The last two methods, dealing with compound interest and logarithms, are peculiar to Bidder, and show real mathematical skill. For the other problems mentioned his methods are much the same in principle as those used by other calculators, though details vary. Bidder, however, has set them out so clearly that I need not discuss further the methods generally used.

A curious question has been raised as to whether a law for the rapidity of the mental work of these prodigies can be found. Personally I do not think we have sufficient data to enable us to draw any conclusion, but I mention briefly the opinions of others. We shall do well to confine ourselves to the simplest case, that of the multiplication of a number of n digits by another number of n digits. Bidder stated that in solving such a problem he believed that the strain on his mind (which he assumed to be proportional to the time taken in answering the question) varied as n^4, but in fact it seems in his case according to this time test to have varied approximately as n^5. In 1855 he worked at least half as quickly again as in 1819, but the law of rapidity for different values of n is said to have been

* *Institution of Civil Engineers, Proceedings*, London, 1890—1891, vol. CIII, p. 250.

about the same. In Dase's case, if the time occupied is proportional to n^x, we must have x less than 3. From this, some have inferred that probably Dase's methods were different in character to those used by Bidder, and it is suggested that the results tend to imply that Dase visualized recorded numerals, working in much the same way as with pencil and paper, while Bidder made no use of symbols, and recorded successive results verbally in a sort of cinematograph way; but it would seem that we shall need more detailed observations before we can frame a theory on this subject.

The cases of calculating prodigies here mentioned, and as far as I know the few others of which records exist, do not differ in kind. In most of them the calculators were uneducated and self-taught. Blessed with excellent memories for numbers, self-confident, stimulated by the astonishment their performances excited, the odd coppers thus put in their pockets and the praise of their neighbours, they pondered incessantly on numbers and their properties; discovered (or in a few cases were taught) the fundamental arithmetical processes, applied them to problems of ever increasing difficulty, and soon acquired a stock of information which shortened their work. Probably *constant practice* and *undivided devotion to mental calculation* are essential to the maintenance of the power, and this may explain why a general education has so often proved destructive to it. The performances of these calculators are remarkable, but, in the light of Bidder's analysis, are not more than might be expected occasionally from lads of exceptional abilities.

CHAPTER XII.

ARITHMETICAL CALCULATING MACHINES.

The advantage of employing mechanical aids to calculation has been recognized from very early times. For instance, the abacus—common in Europe until the Renaissance and still employed in China—has long been widely used for ordinary arithmetical calculation; slips for facilitating multiplication and division, by which an operator escapes the necessity of knowing the multiplication table, were well known in the East before Napier introduced his "rods," and are familiar to students of the history of arithmetic; while slide-rules for arithmetical work, introduced a year or two after the publication of the earliest logarithmic tables, are in general use to-day in western Europe and America. Of late, however, more elaborate calculating machines have come into vogue in banks, offices, and shops*.

Calculating machines may be classified as addition machines; machines directly fitted for multiplication and division as well as addition and subtraction; difference machines; and machines for performing other analytical calculations such as solving equations and evaluating integrals. Of these I propose to consider only those constructed for executing elementary arithmetical processes, and of these latter I shall not further allude to the abacus whose form and use are described in many

* On modern Calculating Machines, see an article by F. J. W. Whipple printed in the *Handbook to the Napier Tercentenary Exhibition*, Edinburgh, 1914, pp. 69—135. On the use of the Abacus see C. G. Knott, *Ibid.*, pp. 136—154; and F. P. Barnard, *The Casting Counter*, Oxford, 1916, pp. 254—319.

text-books, or to the modern machines where plates are moved by hand over one another. Moreover I concern myself only with general principles and do not attempt to describe the mechanical devices by which the desired results are secured.

Addition Machines, in order to be efficient, must in general contain arrangements for showing the primary number on which the operation is to be performed, and if possible also the secondary number to be added to it; for adding to each digit of the primary number the requisite number of units shown by the corresponding digit of the secondary number; for automatically "carrying" whenever that operation is required; and for rapidly "clearing" the machine so that any other addition can be started on it afresh. The motive power is commonly supplied by turning a handle or pressing a key, and the arrangements of wheels and levers must be such that the final result can be obtained mechanically.

In the majority of modern machines these requirements are met thus: There are a series of wheels, each wheel having twenty faces marked successively with the digits 0, 1, 2, ... 9, 0, 1, ... 9. The wheels for the primary number are mounted on a single horizontal axis, and usually are covered by a frame with a slit or window cut in it through which one and only one face of each wheel can be seen. On turning any wheel through one twentieth of a complete revolution, called a step, the visible figure will be changed into that immediately above or below it. If two neighbouring wheels are geared together it is easy to see how the addition of unity to the digit 9 on any particular wheel which changes the 9 into 0 may also be used to rotate the wheel to the left of it through one digit, thus "carrying" 1, a process familiar to everyone in the mechanism of a gas meter, a speedometer, or watch with a second hand. A similar set of wheels is provided for the secondary number. The wheels are usually constructed with twenty faces rather than ten faces in order that half a revolution of the wheel may suffice to show all the right-hand digits of the various additions and the second half of the revolution be available for "carrying."

Suppose for instance that we want to add to the primary number 978 the secondary number 94. The machine is set so that the primary number appears through the window. In the earliest machines the operator first added 4 to the units by rotating the unit wheel through 4 steps, which made the number read 982, and then rotated the second wheel through 9 steps which gave the total result of 1072; originally these 13 steps were all done separately and by hand, and nothing appeared on the machine to show that 94 was the sum which had been added. In such a form the machine was of no practical use. In modern machines the secondary number 94 is set on a secondary set of wheels and is shown on a second window placed below the first window. Each wheel in the secondary set is connected with the corresponding wheel in the primary set. The whole 13 steps are then made by a single mechanical movement, such as turning a handle attached to the axis on which the primary wheels are set, and the result is shown on the face of the machine.

We can in various ways make a single half revolution of the axis give the unit digit of the sum of the corresponding digits of the primary and secondary numbers. One method is by the use of a "rocking segment"—the amplitude of the rock corresponding to each digit in the secondary number being determined by the key used to indicate that digit in the secondary window; this is used in Burrough's machine and in many Cash Registers. Another way is to use a "stepped reckoner" or a "rack" in which the gearing connecting the corresponding digits in the primary and secondary numbers varies according to the digit set in the secondary number; this is used in machines of the Thomas type, such as the Arithmometer and Saxonia. A third method is to make the setting of the digit in the secondary number push out in the gearing wheel a number of cogs corresponding to that digit; this is easy to effect and is used in the Brunsviga machine. By any of these methods the additions of all the corresponding digits in the primary and secondary numbers are made simultaneously. We have also to provide for the requisite "carryings," and these necessarily are done successively from right to left.

These are effected by another half-turn of the handle. The whole of these additions and carryings are thus performed smoothly by a single revolution of the handle. In the above description the primary number is replaced by the required sum. Practically it is often convenient to see on the machine the primary and secondary numbers as well as their sum, and clearly this can be easily effected without complicating the mechanism.

I have talked about applying motive power by turning a handle. I ought to add that in some machines it is obtained otherwise by the force exerted in pressing the knobs or keys used to record the digits in the secondary number; the former method is simpler and mechanically preferable, but it is alleged that the latter method is slightly more rapid. Originally all the wheels were geared in train. Consequently when unity was added to a number like 9999, where there were four carryings, considerable force had to be used to drive the machine, and a heavy strain was put on the mechanism. This defect was obviated by a way, invented by Roth, about 1840, of placing the wheels on the axis, and in machines worked by a handle the carrying is done by the second half-turn of the axis. In modern machines the expenditure of but little force is necessary.

It is important that the operations should be rapid, and accordingly the velocity of rotation of the wheels is great. All modern machines contain devices for stopping the wheels abruptly without straining the mechanism.

Various schemes have been proposed for clearing the machine by pressing a lever followed by a single turn of the handle, and some arrangement of the kind is necessary in practice. It is also necessary to be able to throw a single wheel out of gear so as to correct a mistake in setting a figure if one has been made. These mechanical problems have been solved in many ways.

There is no difficulty in extending the scope of the machine so as to enable numerous additions to be made at once. The mechanism by which the total sum is produced is called a Totaliser.

An addition machine can always be used to effect subtraction, multiplication, division, and commercial operations which involve them, since all these processes can be reduced to successive additions. It is not suited to operations in which we should usually employ logarithms or slide-rules, and there is no advantage in evolving complicated processes by which it can be used in such calculation.

The adaption of addition machines to direct *Subtraction* requires a device for reversing the motion of the secondary wheels. If the motive power is supplied by turning a handle this can be easily done. When the machine is driven by pressing keys the motion is always direct and it is usual to evade the difficulty by substituting for subtraction the addition of a number made up of the complementary digits. For instance, if we wish to subtract 723 from any number we can obtain the result by adding the complementary number 277 and then subtracting unity from the thousand digits. Preferably this is done by prefixing to the complementary number as many "9's" as the scope of the machine allows, in which case it is unnecessary to subtract anything from the visible result of the addition. Thus, if we are working with a machine which shows only six digits and we wish to subtract 723 from 8147 we can obtain the result by adding to it 999277, and the visible result will read 007424, the machine only showing six digits. Machines are made which on pressing a lever or using another row of keys set the complementary number directly, thus making the process of finding it entirely automatic.

Multiplication consists essentially in repeated additions. In the majority of arithmetical machines it is effected in this way, namely, by adding the number to itself the requisite number of times. For multipliers not exceeding 9 or 10 this suffices, but for higher numbers the process is not convenient. In order to multiply by a number like 72 we want to introduce subsidiary mechanism by which we can first multiply by 70 by the use of a lever moving the primary number one place to the left, followed by 7 turns of the handle and then multiply by 2 by making 2 turns of the handle, from which the final result would be obtainable. The problem of effecting this, without constant attention to the machine, has (I believe) so far baffled inventors, but various arrangements exist by which the result can be secured by some manipulation after a few revolutions of the handle.

The result of *Division* can always be obtained by successive subtraction, and for low divisors we may use this method. For divisors exceeding 9 or 10 this would be a tedious process and practically useless. If the divisors are large the older machines are rarely of use: if however they are employed the method is analogous to ordinary long division, and requires the operator to obtain the successive digits of the quotient by inspection. Unless he is very careful he will at some point probably make his quotient too large, and then have to retrace his steps which may not be easy. For this reason it is not uncommon to use a table of reciprocals, and thus substitute multiplication for division.

Recently, however, machines have been invented by which multiplication and division can be performed directly. One of these is Steiger's Millionaire Machine*. In this if we want to multiply 4 by 7, a marker is set say to 4, and a pointer to 7, and the product 28 is recorded after a single turn of the handle. During this turn there are two distinct operations; at the end of the first half-turn the 2 appears in the right place in the product and the product-carriage moves one place to the left; in the second half-turn the 8 appears to the right of the 2. This effect is secured by controlling the amplitude of the motion of racks which move under pinions similar to those used with the stepped reckoner. Corresponding to each multiplier there is a tongue-plate which forms a multiplication table. For example, the "7" tongue-plate has nine pairs of tongues, the lengths of which correspond in length to so many cogs on the racks, 0, 7; 1, 4; 2, 1; 2, 8; 3, 5; etc. When 4 is multiplied by 7 the fourth rack is pushed by a short tongue on the seventh tongue-piece through two teeth, then the tongue-piece is itself displaced laterally, whilst the rack returns to its original position, and

* See Whipple, pp. 104—122.

finally a longer tongue pushes the same rack through eight teeth. Another machine of this kind is Hamann's Mercedes-Euklid Arithmometer*. This is far too elaborate for me to attempt to describe, but it is compact, strong, noiseless, and so ingenious as to render it almost impossible for an operator to make a mistake in using it without having the error shown in the windows. In this, division is effected by a series of approximations which automatically get closer and closer.

To sum up the matter we may say that the problem of performing addition and subtraction mechanically has been completely solved, but further improvements will have to be introduced before multiplication and division can be performed as expeditiously and simply.

Of late years typewriters have been introduced which are also addition machines. They can be used to type columns of figures and they have attached to them a totaliser which, when switched into use, types the total of the sums already printed. These machines can be arranged to suit any system of moneys, weights, or measures, and are now largely used in banks, insurance offices, and businesses. If a machine is in order there is no risk of error, but if it gets out of order the fact that it is regarded as absolutely reliable may cause trouble. I believe that recently at the headquarters of a large bank, where some thirty or more of these machines were used, a cog in one machine broke off without the fracture being detected. The mishap was not found until early the next morning, and meantime all the additions given by that machine were inaccurate. It was impossible to say definitely what documents had been typed on this particular machine, and some weeks elapsed before all the errors occasioned by this unlucky breakage had been tracked and corrected.

Pascal has the credit of having constructed, about 1642, the earliest addition machine of the type here described; in it the addition in each place of figures was performed separately by moving a wheel by hand, and subtraction was effected by adding the numerical complements. In 1666 Samuel Morland adapted

* See Whipple, pp. 104—122.

Pascal's machine for the addition of sums of money. Leibnitz suggested in 1671, and later in 1694 constructed, a machine which may be considered the predecessor of modern machines of the Thomas type. These machines of the seventeenth century were, however, little more than scientific toys. In the eighteenth century various technical improvements were introduced which brought the instruments into the field of practical usefulness though in fact they were as yet only rarely employed either by men of science or men of business. The machines described by Babbage during the first half of the nineteenth century were of a far more elaborate character and were designed to make and print tables of all kinds. During the closing years of that century the normal machines were greatly improved, mostly by German and American inventors, and many of the instruments now on the market are marvels of mechanical ingenuity.

CHAPTER XIII.

THREE CLASSICAL GEOMETRICAL PROBLEMS.

Among the more interesting geometrical problems of antiquity are three questions which attracted the special attention of the early Greek mathematicians. Our knowledge of geometry is derived from Greek sources, and thus these questions have attained a classical position in the history of the subject. The three questions to which I refer are (i) the duplication of a cube, that is, the determination of the side of a cube whose volume is double that of a given cube; (ii) the trisection of an angle; and (iii) the squaring of a circle, that is, the determination of a square whose area is equal to that of a given circle—each problem to be solved by a geometrical construction involving the use of straight lines and circles only, that is, by Euclidean geometry.

This limitation to the use of straight lines and circles implies that the only instruments available in Euclidean geometry are compasses and rulers. But the compasses must be capable of opening as wide as is desired, and the ruler must be of unlimited length. Further the ruler must not be graduated, for if there were two fixed marks on it we can obtain constructions equivalent to those obtained by the use of the conic sections.

With the Euclidean restriction all three problems are insoluble*. To duplicate a cube the length of whose side is a,

* See F. C. Klein, *Vorträge über ausgewählte Fragen der Elementargeometrie*, Leipzig, 1895; and F. G. Texeira, *Sur les Problèmes célèbres de la Géométrie Élémentaire non resolubles avec la Règle et le Compas*, Coimbra, 1915. It is said that the earliest rigorous proof that the problems were insoluble by Euclidean geometry was given by P. L. Wantzell in 1837.

we have to find a line of length x, such that $x^3 = 2a^3$. Again, to trisect a given angle, we may proceed to find the sine of the angle, say a, then, if x is the sine of an angle equal to one-third of the given angle, we have $4x^3 = 3x - a$. Thus the first and second problems, when considered analytically, require the solution of a cubic equation; and since a construction by means of circles (whose equations are of the form $x^2 + y^2 + ax + by + c = 0$) and straight lines (whose equations are of the form $\alpha x + \beta y + \gamma = 0$) cannot be equivalent to the solution of a cubic equation, it is inferred that the problems are insoluble if in our constructions we are restricted to the use of circles and straight lines. If the use of the conic sections is permitted, both of these questions can be solved in many ways. The third problem is different in character, but under the same restrictions it also is insoluble.

I propose to give some of the constructions which have been proposed for solving the first two of these problems. To save space I shall not draw the necessary diagrams, and in most cases I shall not add the proofs: the latter present but little difficulty. I shall conclude with some historical notes on approximate solutions of the quadrature of the circle.

The Duplication of the Cube.*

The problem of the duplication of the cube was known in ancient times as the Delian problem, in consequence of a legend that the Delians had consulted Plato on the subject. In one form of the story, which is related by Philoponus†, it is asserted that the Athenians in 430 B.C., when suffering from the plague of eruptive typhoid fever, consulted the oracle at Delos as to how they could stop it. Apollo replied that they must double the size of his altar which was in the form of a cube. To the unlearned suppliants nothing seemed more easy, and a new altar was constructed either having each of its edges

* See *Historia Problematis de Cubi Duplicatione* by N. T. Reimer, Göttingen, 1798; and *Historia Problematis Cubi Duplicandi* by C. H. Biering, Copenhagen, 1844: also *Das Delische Problem*, by A. Sturm, Linz, 1895–7. Some notes on the subject are given in my *History of Mathematics*.

† *Philoponus ad Aristotelis Analytica Posteriora*, bk. I, chap. vii.

double that of the old one (from which it followed that the volume was increased eight-fold) or by placing a similar cube altar next to the old one. Whereupon, according to the legend, the indignant god made the pestilence worse than before, and informed a fresh deputation that it was useless to trifle with him, as his new altar must be a cube and have a volume exactly double that of his old one. Suspecting a mystery the Athenians applied to Plato, who referred them to the geometricians. The insertion of Plato's name is an obvious anachronism. Eratosthenes* relates a somewhat similar story, but with Minos as the propounder of the problem.

In an Arab work, the Greek legend was distorted into the following extraordinarily impossible piece of history, which I cite as a curiosity of its kind. "Now in the days of Plato," says the writer, "a plague broke out among the children of Israel. Then came a voice from heaven to one of their prophets, saying, 'Let the size of the cubic altar be doubled, and the plague will cease'; so the people made another altar like unto the former, and laid the same by its side. Nevertheless the pestilence continued to increase. And again the voice spake unto the prophet, saying, 'They have made a second altar like unto the former, and laid it by its side, but that does not produce the duplication of the cube.' Then applied they to Plato, the Grecian sage, who spake to them, saying, 'Ye have been neglectful of the science of geometry, and therefore hath God chastised you, since geometry is the most sublime of all the sciences.' Now, the duplication of a cube depends on a rare problem in geometry, namely...." And then follows the solution of Apollonius, which is given later.

If a is the length of the side of the given cube and x that of the required cube, we have $x^3 = 2a^3$, that is, $x : a = \sqrt[3]{2} : 1$. It is probable that the Greeks were aware that the latter ratio is incommensurable, in other words, that no two integers can be found whose ratio is the same as that of $\sqrt[3]{2} : 1$, but it did not therefore follow that they could not find the ratio by

* *Archimedis Opera cum Eutocii Commentariis*, ed. Torelli, Oxford, 1792, p. 144; ed. Heiberg, Leipzig, 1880–1, vol. III, pp. 104—107.

geometry: in fact, the side and diagonal of a square are instances of lines whose numerical measures are incommensurable.

I proceed now to give some of the geometrical constructions which have been proposed for the duplication of the cube*. With one exception, I confine myself to those which can be effected by the aid of the conic sections.

Hippocrates† (circ. 420 B.C.) was perhaps the earliest mathematician who made any progress towards solving the problem. He did not give a geometrical construction, but he reduced the question to that of finding two means between one straight line (a), and another twice as long ($2a$). If these means are x and y, we have $a : x = x : y = y : 2a$, from which it follows that $x^3 = 2a^3$. It is in this form that the problem is always presented now. Formerly any process of solution by finding these means was called a mesolabum.

One of the first solutions of the problem was that given by Archytas‡ in or about the year 400 B.C. His construction is equivalent to the following. On the diameter OA of the base of a right circular cylinder describe a semicircle whose plane is perpendicular to the base of the cylinder. Let the plane containing this semicircle rotate round the generator through O, then the surface traced out by the semicircle will cut the cylinder in a tortuous curve. This curve will itself be cut by a right cone, whose axis is OA and semi-vertical angle is (say) 60°, in a point P, such that the projection of OP on the base of the cylinder will be to the radius of the cylinder in the ratio of the side of the required cube to that of the given cube. Of course the proof given by Archytas is geometrical; and it is interesting to note that in it he shows himself familiar with the results of the propositions Euc. III, 18, III, 35, and XI, 19. To

* On the application to this problem of the traditional Greek methods of analysis by Hero and Philo (leading to the solution by the use of Apollonius's circle), by Nicomedes (leading to the solution by the use of the conchoid), and by Pappus (leading to the solution by the use of the cissoid), see *Geometrical Analysis* by J. Leslie, Edinburgh, second edition, 1811, pp. 247—250, 453.

† Proclus, ed. Friedlein, pp. 212—213.

‡ *Archimedis Opera*, ed. Torelli, p. 143; ed. Heiberg, vol. III, pp. 98—103.

show analytically that the construction is correct, take OA as the axis of x, and the generator of the cylinder drawn through O as axis of z, then with the usual notation, in polar co-ordinates, if a is the radius of the cylinder, we have for the equation of the surface described by the semicircle $r = 2a \sin \theta$; for that of the cylinder $r \sin \theta = 2a \cos \phi$; and for that of the cone $\sin \theta \cos \phi = \frac{1}{2}$. These three surfaces cut in a point such that $\sin^3 \theta = \frac{1}{2}$, and therefore $(r \sin \theta)^3 = 2a^3$. Hence the volume of the cube whose side is $r \sin \theta$ is twice that of the cube whose side is a.

The construction attributed to Plato* (circ. 360 B.C.) depends on the theorem that, if CAB and DAB are two right-angled triangles, having one side, AB, common, their other sides, AD and BC, parallel, and their hypothenuses, AC and BD, at right angles, then if these hypothenuses cut in P, we have $PC : PB = PB : PA = PA : PD$. Hence, if such a figure can be constructed having $PD = 2PC$, the problem will be solved. It is easy to make an instrument by which the figure can be drawn.

The next writer whose name is connected with the problem is Menaechmus†, who in or about 340 B.C. gave two solutions of it.

In the first of these he pointed out that two parabolas having a common vertex, axes at right angles, and such that the latus rectum of the one is double that of the other, will intersect in another point whose abscissa (or ordinate) will give a solution. If we use analysis this is obvious; for, if the equations of the parabolas are $y^2 = 2ax$ and $x^2 = ay$, they intersect in a point whose abscissa is given by $x^3 = 2a^3$. It is probable that this method was suggested by the form in which Hippocrates had cast the problem: namely, to find x and y so that $a : x = x : y = y : 2a$, whence we have $x^2 = ay$ and $y^2 = 2ax$.

The second solution given by Menaechmus was as follows. Describe a parabola of latus rectum l. Next describe a rectangular hyperbola, the length of whose real axis is $4l$, and

* *Archimedis Opera*, ed. Torelli, p. 135; ed. Heiberg, vol. III, pp. 66—71.

† *Ibid.*, ed. Torelli, pp. 141—143; ed. Heiberg, vol. III, pp. 92—99.

having for its asymptotes the tangent at the vertex of the parabola and the axis of the parabola. Then the ordinate and the abscissa of the point of intersection of these curves are the mean proportionals between l and $2l$. This is at once obvious by analysis. The curves are $x^2 = ly$ and $xy = 2l^2$. These cut in a point determined by $x^3 = 2l^3$ and $y^3 = 4l^3$. Hence

$$l : x = x : y = y : 2l.$$

The solution of Apollonius*, which was given about 220 B.C., was as follows. The problem is to find two mean proportionals between two given lines. Construct a rectangle $OADB$, of which the adjacent sides OA and OB are respectively equal to the two given lines. Bisect AB in C. With C as centre describe a circle cutting OA produced in a and cutting OB produced in b, so that aDb shall be a straight line. If this circle can be so described, it will follow that $OA : Bb = Bb : Aa = Aa : OB$, that is, Bb and Aa are the two mean proportionals between OA and OB. It is impossible to construct the circle by Euclidean geometry, but Apollonius gave a mechanical way of describing it.

The only other construction of antiquity to which I will refer is that given by Diocles and Sporus†. It is as follows. Take two sides of a rectangle OA, OB, equal to the two lines between which the means are sought. Suppose OA to be the greater. With centre O and radius OA describe a circle. Let OB produced cut the circumference in C and let AO produced cut it in D. Find a point E on BC so that if DE cuts AB produced in F and cuts the circumference in G, then $FE = EG$. If E can be found, then OE is the first of the means between OA and OB. Diocles invented the cissoid in order to determine E, but it can be found equally conveniently by the aid of conics.

In more modern times several other solutions have been suggested. I may allude in passing to three given by Huygens‡,

* *Archimedis Opera*, ed. Torelli, p. 137; ed. Heiberg, vol. III, pp. 76—79. The solution is given in my *History of Mathematics*, London, 1901, p. 84.

† *Ibid.*, ed. Torelli, pp. 138, 139, 141; ed. Heiberg, vol. III, pp. 78—84, 90—93.

‡ *Opera Varia*, Leyden, 1724, pp. 393—396.

but I will enunciate only those proposed respectively by Vieta, Descartes, Gregory of St Vincent, and Newton.

Vieta's construction is as follows*. Describe a circle, centre O, whose radius is equal to half the length of the larger of the two given lines. In it draw a chord AB equal to the smaller of the two given lines. Produce AB to E so that $BE = AB$. Through A draw a line AF parallel to OE. Through O draw a line $DOCFG$, cutting the circumference in D and C, cutting AF in F, and cutting BA produced in G, so that $GF = OA$. If this line can be drawn then $AB : GC = GC : GA = GA : CD$.

Descartes pointed out† that the curves

$$x^2 = ay \quad \text{and} \quad x^2 + y^2 = ay + bx$$

cut in a point (x, y) such that $a : x = x : y = y : b$. Of course this is equivalent to the first solution given by Menaechmus, but Descartes preferred to use a circle rather than a second conic.

Gregory's construction was given in the form of the following theorem‡. The hyperbola drawn through the point of intersection of two sides of a rectangle so as to have the two other sides for its asymptotes meets the circle circumscribing the rectangle in a point whose distances from the asymptotes are the mean proportionals between two adjacent sides of the rectangle. This is the geometrical expression of the proposition that the curves $xy = ab$ and $x^2 + y^2 = ay + bx$ cut in a point (x, y) such that $a : x = x : y = y : b$.

One of the constructions proposed by Newton is as follows§. Let OA be the greater of two given lines. Bisect OA in B. With centre O and radius OB describe a circle. Take a point C on the circumference so that BC is equal to the other of the two given lines. From O draw ODE cutting AC produced in D, and BC produced in E, so that the intercept $DE = OB$. Then

* *Opera Mathematica*, ed. Schooten, Leyden, 1646, prop. v, pp. 242—243.

† *Geometria*, bk. III, ed. Schooten, Amsterdam, 1659, p. 91.

‡ Gregory of St Vincent, *Opus Geometricum Quadraturae Circuli*, Antwerp, 1647, bk. VI, prop. 138, p. 602.

§ *Arithmetica Universalis*, Ralphson's (second) edition, 1728, p. 242; see also pp. 243, 245.

$BC : OD = OD : CE = CE : OA$. Hence OD and CE are two mean proportionals between any two lines BC and OA.

*The Trisection of an Angle**.

The trisection of an angle is the second of these classical problems, but tradition has not enshrined its origin in romance. The following two constructions are among the oldest and best known of those which have been suggested; they are quoted by Pappus†, but I do not know to whom they were due originally.

The first of them is as follows. Let AOB be the given angle. From any point P in OB draw PM perpendicular to OA. Through P draw PR parallel to OA. On MP take a point Q so that if OQ is produced to cut PR in R then $QR = 2 . OP$. If this construction can be made, then $AOR = \frac{1}{3} AOB$. The solution depends on determining the position of R. This was effected by a construction which may be expressed analytically thus. Let the given angle be $\tan^{-1}(b/a)$. Construct the hyperbola $xy = ab$, and the circle $(x-a)^2 + (y-b)^2 = 4(a^2+b^2)$. Of the points where they cut, let x be the abscissa which is greatest, then $PR = x - a$, and $\tan^{-1}(b/x) = \frac{1}{3}\tan^{-1}(b/a)$.

The second construction is as follows. Let AOB be the given angle. Take $OB = OA$, and with centre O and radius OA describe a circle. Produce AO indefinitely and take a point C on it external to the circle so that if CB cuts the circumference in D then CD shall be equal to OA. Draw OE parallel to CDB. Then, if this construction can be made, $AOE = \frac{1}{3} AOB$. The ancients determined the position of the point C by the aid of the conchoid: it could be also found by the use of the conic sections.

I proceed to give a few other solutions, confining myself to those effected by the aid of conics.

* On the bibliography of the subject see the supplements to *L'Intermédiaire des Mathématiciens*, Paris, May and June, 1904.

† Pappus, *Mathematicae Collectiones*, bk. IV, props. 32, 33 (ed. Commandino, Bonn, 1670, pp. 97—99). On the application to this problem of the traditional Greek methods of analysis see *Geometrical Analysis*, by J. Leslie, Edinburgh, second edition, 1811, pp. 245—247.

Among other constructions given by Pappus* I may quote the following. Describe a hyperbola whose eccentricity is two. Let its centre be C and its vertices A and A'. Produce CA' to S so that $A'S = CA'$. On AS describe a segment of a circle to contain the given angle. Let the orthogonal bisector of AS cut this segment in O. With centre O and radius OA or OS describe a circle. Let this circle cut the branch of the hyperbola through A' in P. Then $SOP = \frac{1}{3}SOA$.

In modern times one of the earliest of the solutions by a direct use of conics was suggested by Descartes, who effected it by the intersection of a circle and a parabola. His construction† is equivalent to finding the points of intersection, other than the origin, of the parabola $y^2 = \frac{1}{4}x$ and the circle $x^2 + y^2 - \frac{13}{4}x + 4ay = 0$. The ordinates of these points are given by the equation $4y^3 = 3y - a$. The smaller positive root is the sine of one-third of the angle whose sine is a. The demonstration is ingenious.

One of the solutions proposed by Newton is practically equivalent to the third one which is quoted above from Pappus. It is as follows‡. Let A be the vertex of one branch of a hyperbola whose eccentricity is two, and let S be the focus of the other branch. On AS describe the segment of a circle containing an angle equal to the supplement of the given angle. Let this circle cut the S branch of the hyperbola in P. Then PAS will be equal to one-third of the given angle.

The following elegant solution is due to Clairaut§. Let AOB be the given angle. Take $OA = OB$, and with centre O and radius OA describe a circle. Join AB, and trisect it in H, K, so that $AH = HK = KB$. Bisect the angle AOB by OC cutting AB in L. Then $AH = 2 . HL$. With focus A, vertex H, and directrix OC, describe a hyperbola. Let the branch of

* Pappus, *Mathematicae Collectiones*, bk. IV, prop. 34, pp. 99—104.

† *Geometria*, bk. III, ed. Schooten, Amsterdam, 1659, p. 91.

‡ *Arithmetica Universalis*, problem XLII, Ralphson's (second) edition, London, 1728, p. 148; see also pp. 243—245.

§ I believe that this was first given by Clairaut, but I have mislaid my reference. The construction occurs as an example in the *Geometry of Conics*, by C. Taylor, Cambridge, 1881, No. 308, p. 126.

this hyperbola which passes through H cut the circle in P. Draw PM perpendicular to OC and produce it to cut the circle in Q. Then by the focus and directrix property we have $AP : PM = AH : HL = 2 : 1$, $\therefore AP = 2 . PM = PQ$. Hence, by symmetry, $AP = PQ = QR$. $\therefore AOP = POQ = QOR$.

I may conclude by giving the solution which Chasles* regards as the most fundamental. It is equivalent to the following proposition. If OA and OB are the bounding radii of a circular arc AB, then a rectangular hyperbola having OA for a diameter and passing through the point of intersection of OB with the tangent to the circle at A will pass through one of the two points of trisection of the arc.

Several instruments have been constructed by which mechanical solutions of the problem can be obtained.

The Quadrature of the Circle†.

The object of the third of the classical problems was the determination of a side of a square whose area should be equal to that of a given circle.

The investigation, previous to the last two hundred years, of this question was fruitful in discoveries of allied theorems, but in more recent times it has been abandoned by those who are able to realize what is required. The history of this subject has been treated by competent writers in such detail that I shall content myself with a very brief allusion to it.

Archimedes showed‡ (what possibly was known before) that the problem is equivalent to finding the area of a right-angled

* *Traité des sections coniques*, Paris, 1865, art. 37, p. 36.

† See Montucla's *Histoire des Recherches sur la Quadrature du Cercle*, edited by P. L. Lacroix, Paris, 1831; also various articles by A. De Morgan, and especially his *Budget of Paradoxes*, London, 1872. A popular sketch of the subject has been compiled by H. Schubert, *Die Quadratur des Zirkels*, Hamburg, 1889; and since the publication of the earlier editions of these *Recreations* Prof. F. Rudio of Zurich has given an analysis of the arguments of Archimedes, Huygens, Lambert, and Legendre on the subject, with an introduction on the history of the problem, Leipzig, 1892.

‡ *Archimedis Opera*, Κύκλου μέτρησις, prop. I, ed. Torelli, pp. 203—205; ed. Heiberg, vol. I, pp. 258—261, vol. III, pp. 269—277.

triangle whose sides are equal respectively to the perimeter of the circle and the radius of the circle. Half the ratio of these lines is a number, usually denoted by π.

That this number is incommensurable had been long suspected, and has been now demonstrated. The earliest analytical proof of it was given by Lambert* in 1761; in 1803 Legendre† extended the proof to show that π^2 was also incommensurable; and recently Lindemann‡ has shown that π cannot be the root of a rational algebraical equation.

An earlier attempt by James Gregory to give a geometrical demonstration of this is worthy of notice. Gregory proved§ that the ratio of the area of any arbitrary sector to that of the inscribed or circumscribed polygons is not expressible by a finite number of algebraical terms. Hence he inferred that the quadrature was impossible. This was accepted by Montucla, but it is not conclusive, for it is conceivable that some particular sector might be squared, and this particular sector might be the whole circle.

In connection with Gregory's proposition above cited, I may add that Newton|| proved that in any closed oval an arbitrary sector bounded by the curve and two radii cannot be expressed in terms of the co-ordinates of the extremities of the arc by a finite number of algebraical terms. The argument is condensed and difficult to follow: the same reasoning would show that a closed oval curve cannot be represented by an algebraical equation in polar co-ordinates. From this proposition no

* *Mémoires de l'Académie de Berlin* for 1761, Berlin, 1768, pp. 265—322.

† Legendre's *Geometry*, Brewster's translation, Edinburgh, 1824, pp. 239—245.

‡ Ueber die Zahl π, *Mathematische Annalen*, Leipzig, 1882, vol. xx, pp. 213—225. The proof leads to the conclusion that, if x is a root of a rational integral algebraical equation, then e^x cannot be rational: hence, if πi was the root of such an equation, $e^{\pi i}$ could not be rational; but $e^{\pi i}$ is equal to -1, and therefore is rational; hence πi cannot be the root of such an algebraical equation, and therefore neither can π.

§ *Vera Circuli et Hyperbolae Quadratura*, Padua, 1668: this is reprinted in Huygens's *Opera Varia*, Leyden, 1724, pp. 405—462.

|| *Principia*, bk. I, section VI, lemma XXVIII.

conclusion as to the quadrature of the circle is to be drawn, nor did Newton draw any. In the earlier editions of this work I expressed an opinion that the result presupposed a particular definition of the word oval, but on more careful reflection I think that the conclusion is valid without restriction.

With the aid of the quadratrix, or the conchoid, or the cissoid, the quadrature of the circle is easy, but the construction of those curves assumes a knowledge of the value of π, and thus the question is begged.

I need hardly add that, if π represented merely the ratio of the circumference of a circle to its diameter, the determination of its numerical value would have but slight interest. It is however a mere accident that π is defined usually in that way, and it really represents a certain number which would enter into analysis from whatever side the subject was approached.

I recollect a distinguished professor explaining how different would be the ordinary life of a race of beings born, as easily they might be, so that the fundamental processes of arithmetic, algebra and geometry were different to those which seem to us so evident, but, he added, it is impossible to conceive of a universe in which e and π should not exist.

I have quoted elsewhere an anecdote, which perhaps will bear repetition, that illustrates how little the usual definition of π suggests its properties. De Morgan was explaining to an actuary what was the chance that a certain proportion of some group of people would at the end of a given time be alive; and quoted the actuarial formula, involving π, which, in answer to a question, he explained stood for the ratio of the circumference of a circle to its diameter. His acquaintance, who had so far listened to the explanation with interest, interrupted him and exclaimed, "My dear friend, that must be a delusion, what can a circle have to do with the number of people alive at the end of a given time?" In reality the fact that the ratio of the length of the circumference of a circle to its diameter is the number denoted by π does not afford the best analytical definition of π, and is only one of its properties.

The use of a single symbol to denote this number 3·14159... seems to have been introduced about the beginning of the eighteenth century. William Jones* in 1706 represented it by π; a few years later† John Bernoulli denoted it by c; Euler in 1734 used p, and in 1736 used c; Christian Goldback in 1742 used π; and after the publication of Euler's *Analysis* the symbol π was generally employed.

The numerical value of π can be determined by either of two methods with as close an approximation to the truth as is desired.

The first of these methods is geometrical. It consists in calculating the perimeters of polygons inscribed in and circumscribed about a circle, and assuming that the circumference of the circle is intermediate between these perimeters‡. The approximation would be closer if the areas and not the perimeters were employed. The second and modern method rests on the determination of converging infinite series for π.

We may say that the π-calculators who used the first method regarded π as equivalent to a geometrical ratio, but those who adopted the modern method treated it as the symbol for a certain number which enters into numerous branches of mathematical analysis.

It may be interesting if I add here a list of some of the approximations to the value of π given by various writers§. This will indicate incidentally those who have studied the subject to the best advantage.

* *Synopsis Palmariorum Matheseos*, London, 1706, pp. 243, 263 *et seq.*

† See notes by G. Eneström in the *Bibliotheca Mathematica*, Stockholm, 1889, vol. III, p. 28; *Ibid.*, 1890, vol. IV, p. 22.

‡ The history of this method has been written by K. E. I. Selander, *Historik öfver Ludolphska Talet*, Upsala, 1868.

§ For the methods used in classical times and the results obtained, see the notices of their authors in M. Cantor's *Geschichte der Mathematik*, Leipzig, vol. I, 1880. For medieval and modern approximations, see the article by A. De Morgan on the Quadrature of the Circle in vol. XIX of the *Penny Cyclopaedia*, London, 1841; with the additions given by B. de Haan in the *Verhandelingen* of Amsterdam, 1858, vol. IV, p. 22: the conclusions were tabulated, corrected, and extended by Dr J. W. L. Glaisher in the *Messenger of Mathematics*, Cambridge, 1873, vol. II, pp. 119—128; and *Ibid.*, 1874, vol. III, pp. 27—46.

The ancient Egyptians* took 256/81 as the value of π, this is equal to 3·1605...; but the rougher approximation of 3 was used by the Babylonians† and by the Jews‡. It is not unlikely that these numbers were obtained empirically.

We come next to a long roll of Greek mathematicians who attacked the problem. Whether the researches of the members of the Ionian School, the Pythagoreans, Anaxagoras, Hippias, Antipho, and Bryso led to numerical approximations for the value of π is doubtful, and their investigations need not detain us. The quadrature of certain lunes by Hippocrates of Chios is ingenious and correct, but a value of π cannot be thence deduced; and it seems likely that the later members of the Athenian School concentrated their efforts on other questions.

It is probable that Euclid§, the illustrious founder of the Alexandrian School, was aware that π was greater than 3 and less than 4, but he did not state the result explicitly.

The mathematical treatment of the subject began with Archimedes, who proved that π is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$, that is, it lies between 3·1428... and 3·1408.... He established|| this by inscribing in a circle and circumscribing about it regular polygons of 96 sides, then determining by geometry the perimeters of these polygons, and finally assuming that the circumference of the circle was intermediate between these perimeters: this leads to a result from which he deduced the limits given above. This method is equivalent to using the proposition $\sin\theta < \theta < \tan\theta$, where $\theta = \pi/96$: the values of $\sin\theta$ and $\tan\theta$ were deduced by Archimedes from those of $\sin\frac{1}{3}\pi$ and $\tan\frac{1}{3}\pi$ by repeated bisections of the angle. With a polygon of n sides this

* *Ein mathematisches Handbuch der alten Aegypter* (*i.e.* the Rhind papyrus), by A. Eisenlohr, Leipzig, 1877, arts. 100—109, 117, 124.

† Oppert, *Journal Asiatique*, August, 1872, and October, 1874.

‡ 1 Kings, ch. 7, ver. 23; 2 Chronicles, ch. 4, ver. 2.

§ These results can be deduced from Euc. IV, 15, and IV, 8: see also book XII, prop. 16.

|| *Archimedis Opera*, Κύκλου μέτρησις, prop. III, ed. Torelli, Oxford, 1792, pp. 205—216; ed. Heiberg, Leipzig, 1880, vol. I, pp. 263—271.

process gives a value of π correct to at least the integral part of $(2\log n - 1{\cdot}19)$ places of decimals. The result given by Archimedes is correct to two places of decimals. His analysis leads to the conclusion that the perimeters of these polygons for a circle whose diameter is 4970 feet would lie between 15610 feet and 15620 feet—actually it is about 15613 feet 9 inches.

Apollonius discussed these results, but his criticisms have been lost.

Hero of Alexandria gave* the value 3, but he quoted† the result 22/7: possibly the former number was intended only for rough approximations.

The only other Greek approximation that I need mention is that given by Ptolemy‡, who asserted that $\pi = 3°\ 8'\ 30''$. This is equivalent to taking $\pi = 3 + \frac{8}{60} + \frac{30}{3600} = 3\frac{17}{120} = 3{\cdot}141\dot{6}$.

The Roman surveyors seem to have used 3, or sometimes 4, for rough calculations. For closer approximations they often employed $3\frac{1}{8}$ instead of $3\frac{1}{7}$, since the fractions then introduced are more convenient in duodecimal arithmetic. On the other hand Gerbert§ recommended the use of 22/7.

Before coming to the medieval and modern European mathematicians it may be convenient to note the results arrived at in India and the East.

Baudhayana|| took 49/16 as the value of π.

Arya-Bhata¶, circ. 530, gave 62832/20000, which is equal to 3·1416. He showed that, if a is the side of a regular polygon of n sides inscribed in a circle of unit diameter, and if b is the side of a regular inscribed polygon of $2n$ sides, then $b^2 = \frac{1}{2} - \frac{1}{2}(1 - a^2)^{\frac{1}{2}}$. From the side of an inscribed hexagon, he found successively the sides of polygons of 12, 24, 48, 96, 192,

* *Mensurae*, ed. Hultsch, Berlin, 1864, p. 188.

† *Geometria*, ed. Hultsch, Berlin, 1864, pp. 115, 136.

‡ *Almagest*, bk. VI, chap. 7; ed. Halma, vol. I, p. 421.

§ *Œuvres de Gerbert*, ed. Olleris, Clermont, 1867, p. 453.

|| The *Sulvasutras* by G. Thibaut, *Asiatic Society of Bengal*, 1875, arts. 26—28.

¶ *Leçons de calcul d'Aryabhata*, by L. Rodet in the *Journal Asiatique*, 1879, series 7, vol. XIII, pp. 10, 21.

and 384 sides. The perimeter of the last is given as equal to $\sqrt{9\cdot8694}$, from which his result was obtained by approximation.

Brahmagupta*, circ. 650, gave $\sqrt{10}$, which is equal to 3·1622.... He is said to have obtained this value by inscribing in a circle of unit diameter regular polygons of 12, 24, 48, and 96 sides, and calculating successively their perimeters, which he found to be $\sqrt{9\cdot65}$, $\sqrt{9\cdot81}$, $\sqrt{9\cdot86}$, $\sqrt{9\cdot87}$ respectively; and to have assumed that as the number of sides is increased indefinitely the perimeter would approximate to $\sqrt{10}$.

Bhaskara, circ. 1150, gave two approximations. One†—possibly copied from Arya-Bhata, but said to have been calculated afresh by Archimedes's method from the perimeters of regular polygons of 384 sides—is 3927/1250, which is equal to 3·1416: the other‡ is 754/240, which is equal to $3\cdot141\dot{6}$, but it is uncertain whether this was not given only as an approximate value.

Among the Arabs the values 22/7, $\sqrt{10}$, and 62832/20000 were given by Alkarismi§, circ. 830; and no doubt were derived from Indian sources. He described the first as an approximate value, the second as used by geometricians, and the third as used by astronomers.

In Chinese works the values 3, 22/7, 157/50 are said to occur: probably the last two results were copied from the Arabs. The Japanese‖ approximations were closer.

Returning to European mathematicians, we have the following successive approximations to the value of π: many of those prior to the eighteenth century having been calculated originally with the view of demonstrating the incorrectness of some alleged quadrature.

* *Algebra...from Brahmegupta and Bhascara*, trans. by H. T. Colebrooke, London, 1817, chap. xii, art. 40, p. 308.

† *Ibid.*, p. 87.

‡ *Ibid.*, p. 95.

§ *The Algebra of Mohammed ben Musa*, ed. by F. Rosen, London, 1831, pp. 71—72.

‖ On Japanese approximations and the methods used, see P. Harzer, *Transactions of the British Association* for 1905, p. 325.

Leonardo of Pisa*, in the thirteenth century, gave for π the value $1440/458\frac{1}{3}$, which is equal to 3·1418.... In the fifteenth century, Purbach† gave or quoted the value 62832/20000, which is equal to 3·1416; Cusa believed that the accurate value was $\frac{3}{4}(\sqrt{3}+\sqrt{6})$, which is equal to 3·1423...; and, in 1464, Regiomontanus‡ is said to have given a value equal to 3·14243.

Vieta§, in 1579, showed that π was greater than $31415926535/10^{10}$, and less than $31415926537/10^{10}$. This was deduced from the perimeters of the inscribed and circumscribed polygons of 6×2^{16} sides, obtained by repeated use of the formula $2\sin^2 \frac{1}{2}\theta = 1 - \cos\theta$. He also gave‖ a result equivalent to the formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{(2+\sqrt{2})}}{2} \frac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2} \dots.$$

The father of Adrian Metius¶, in 1585, gave 355/113, which is equal to 3·14159292..., and is correct to six places of decimals. This was a curious and lucky guess, for all that he proved was that π was intermediate between 377/120 and 333/106, whereon he jumped to the conclusion that he would obtain the true fractional value by taking the mean of the numerators and the mean of the denominators of these fractions.

In 1593 Adrian Romanus** calculated the perimeter of the inscribed regular polygon of 1073,741824 (*i.e.* 2^{30}) sides, from which he determined the value of π correct to 15 places of decimals.

* Boncompagni's *Scritti di Leonardo*, vol. II (*Practica Geometriae*), Rome, 1862, p. 90.

† Appendix to the *De Triangulis* of Regiomontanus, Basle, 1541, p. 131.

‡ In his correspondence with Cardinal Nicholas de Cusa, *De Quadratura Circuli*, Nuremberg, 1533, wherein he proved that the cardinal's result was wrong. I cannot quote the exact reference, but the figures are given by competent writers and I have no doubt are correct.

§ *Canon Mathematicus seu ad Triangula*, Paris, 1579, pp. 56, 66: probably this work was printed for private circulation only, it is very rare.

‖ *Vietae Opera*, ed. Schooten, Leyden, 1646, p. 400.

¶ *Arithmeticae libri duo et Geometriae*, by A. Metius, Leyden, 1626, pp. 88—89. [Probably issued originally in 1611.]

** *Ideae Mathematicae*, Antwerp, 1593: a rare work, which I have never been able to consult.

L. van Ceulen devoted no inconsiderable part of his life to the subject. In 1596* he gave the result to 20 places of decimals: this was calculated by finding the perimeters of the inscribed and circumscribed regular polygons of 60×2^{33} sides, obtained by the repeated use of a theorem of his discovery equivalent to the formula $1 - \cos A = 2 \sin^2 \frac{1}{2} A$. I possess a finely executed engraving of him of this date, with the result printed round a circle which is below his portrait. He died in 1610, and by his directions the result to 35 places of decimals (which was as far as he had calculated it) was engraved on his tombstone† in St Peter's Church, Leyden. His posthumous arithmetic‡ contains the result to 32 places; this was obtained by calculating the perimeter of a polygon, the number of whose sides is 2^{62}, *i.e.* 4,611686,018427,387904. Van Ceulen also compiled a table of the perimeters of various regular polygons.

Willebrord Snell§, in 1621, obtained from a polygon of 2^{30} sides an approximation to 34 places of decimals. This is less than the numbers given by van Ceulen, but Snell's method was so superior that he obtained his 34 places by the use of a polygon from which van Ceulen had obtained only 14 (or perhaps 16) places. Similarly, Snell obtained from a hexagon an approximation as correct as that for which Archimedes had required a polygon of 96 sides, while from a polygon of 96 sides he determined the value of π correct to seven decimal places instead of the two places obtained by Archimedes. The reason is that Archimedes, having calculated the lengths of the sides of inscribed and circumscribed regular polygons of n sides, assumed that the length of $1/n$th of the perimeter of the circle was intermediate between them; whereas Snell constructed

* *Vanden Circkel*, Delf, 1596, fol. 14, p. 1; or *De Circulo*, Leyden, 1619, p. 3.

† The inscription is quoted by Prof. de Haan in the *Messenger of Mathematics*, 1874, vol. III, p. 25.

‡ *De Arithmetische en Geometrische Fondamenten*, Leyden, 1615, p. 163; or p. 144 of the Latin translation by W. Snell, published at Leyden in 1615 under the title *Fundamenta Arithmetica et Geometrica*. This was reissued, together with a Latin translation of the *Vanden Circkel*, in 1619, under the title *De Circulo*; in which see pp. 3, 29—32, 92.

§ *Cyclometricus*, Leyden, 1621, p. 55.

from the sides of these polygons two other lines which gave closer limits for the corresponding arc. His method depends on the theorem $3\sin\theta/(2+\cos\theta) < \theta < (2\sin\frac{1}{3}\theta + \tan\frac{1}{3}\theta)$, by the aid of which a polygon of n sides gives a value of π correct to at least the integral part of $(4\log n - \cdot 2305)$ places of decimals, which is more than twice the number given by the older rule. Snell's proof of his theorem is incorrect, though the result is true.

Snell also added a table* of the perimeters of all regular inscribed and circumscribed polygons, the number of whose sides is 10×2^n where n is not greater than 19 and not less than 3. Most of these were quoted from van Ceulen, but some were recalculated. This list has proved useful in refuting circle-squares. A similar list was given by James Gregory†.

In 1630 Grienberger‡, by the aid of Snell's theorem, carried the approximation to 39 places of decimals. He was the last mathematician who adopted the classical method of finding the perimeters of inscribed and circumscribed polygons. Closer approximations serve no useful purpose. Proofs of the theorems used by Snell and other calculators in applying this method were given by Huygens in a work§ which may be taken as closing the history of this method.

In 1656 Wallis|| proved that

$$\frac{\pi}{2} = \frac{2.2.4.4.6.6\ldots}{1.3.3.5.5.7.7\ldots},$$

and quoted a proposition given a few years earlier by Viscount Brouncker to the effect that

$$\frac{\pi}{4} = 1 + \frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\ldots,$$

* It is quoted by Montucla, ed. 1831, p. 70.

† *Vera Circuli et Hyperbolae Quadratura*, prop. 29, quoted by Huygens, *Opera Varia*, Leyden, 1724, p. 447.

‡ *Elementa Trigonometrica*, Rome, 1630, end of preface.

§ *De Circula Magnitudine Inventa*, 1654; *Opera Varia*, pp. 351—387. The proofs are given in G. Pirie's *Geometrical Methods of Approximating to the Value of π*, London, 1877, pp. 21—23.

|| *Arithmetica Infinitorum*, Oxford, 1656, prop. 191. An analysis of the investigation by Wallis was given by Cayley, *Quarterly Journal of Mathematics*, 1889, vol. XXIII, pp. 165—169.

but neither of these theorems was used to any large extent for calculation.

Subsequent calculators have relied on converging infinite series, a method that was hardly practicable prior to the invention of the calculus, though Descartes* had indicated a geometrical process which was equivalent to the use of such a series. The employment of infinite series was proposed by James Gregory†, who established the theorem that

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots,$$

the result being true only if θ lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$.

The first mathematician to make use of Gregory's series for obtaining an approximation to the value of π was Abraham Sharp‡, who, in 1699, on the suggestion of Halley, determined it to 72 places of decimals (71 correct). He obtained this value by putting $\theta = \frac{1}{6}\pi$ in Gregory's series.

Machin§, earlier than 1706, gave the result to 100 places (all correct). He calculated it by the formula

$$\tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}.$$

De Lagny‖, in 1719, gave the result to 127 places of decimals (112 correct), calculating it by putting $\theta = \frac{1}{6}\pi$ in Gregory's series.

Hutton¶, in 1776, and Euler**, in 1779, suggested the use of

* See Euler's paper in the *Novi Commentarii Academiae Scientiarum*, Petrograd, 1763, vol. VIII, pp. 157—168.

† See the letter to Collins, dated Feb. 15, 1671, printed in the *Commercium Epistolicum*, London, 1712, p. 25, and in the Macclesfield Collection, *Correspondence of Scientific Men of the Seventeenth Century*, Oxford, 1841, vol. II, p. 216.

‡ See *Life of A. Sharp* by W. Cudworth, London, 1889, p. 170. Sharp's work is given in one of the preliminary discourses (p. 53 *et seq.*) prefixed to H. Sherwin's *Mathematical Tables*. The tables were issued at London in 1705: probably the discourses were issued at the same time, though the earliest copies I have seen were printed in 1717.

§ W. Jones's *Synopsis Palmariorum*, London, 1706, p. 243; and Maseres, *Scriptores Logarithmici*, London, 1796, vol. III, pp. vii—ix, 155—164.

‖ *Histoire de l'Académie* for 1719, Paris, 1721, p. 144.

¶ *Philosophical Transactions*, 1776, vol. LXVI, pp. 476—492.

** *Nova Acta Academiae Scientiarum Petropolitanae* for 1793, Petrograd, 1798, vol. XI, pp. 133—149: the memoir was read in 1779.

the formula $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ or $\frac{1}{4}\pi = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}$, but neither carried the approximation as far as had been done previously.

Vega, in 1789*, gave the value of π to 143 places of decimals (126 correct); and, in 1794†, to 140 places (136 correct).

Towards the end of the eighteenth century F. X. von Zach saw in the Radcliffe Library, Oxford, a manuscript by an unknown author which gives the value of π to 154 places of decimals (152 correct).

In 1837, the result of a calculation of π to 154 places of decimals (152 correct) was published‡.

In 1841 Rutherford§ calculated it to 208 places of decimals (152 correct), using the formula $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.

In 1844 Dase‖ calculated it to 205 places of decimals (200 correct), using the formula $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

In 1847 Clausen¶ carried the approximation to 250 places of decimals (248 correct), calculating it independently by the formulae $\frac{1}{4}\pi = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$ and $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

In 1853 Rutherford** carried his former approximation to 440 places of decimals (all correct), and William Shanks prolonged the approximation to 530 places. In the same year Shanks published an approximation to 607 places††: and in 1873 he carried the approximation to 707 places of decimals‡‡. These were calculated from Machin's formula.

In 1853 Richter, presumably in ignorance of what had been

* *Nova Acta Academiae Scientiarum Petropolitanae* for 1790, Petrograd, 1795, vol. IX, p. 41.

† *Thesaurus Logarithmorum* (*logarithmisch-trigonometrischer Tafeln*), Leipzig, 1794, p. 633.

‡ J. F. Callet's *Tables, etc., Précis Élémentaire*, Paris, tirage, 1837. Tirage, 1894, p. 96.

§ *Philosophical Transactions*, 1841, p. 283.

‖ *Crelle's Journal*, 1844, vol. XXVII, p. 198.

¶ Schumacher, *Astronomische Nachrichten*, vol. XXV, col. 207.

** *Proceedings of the Royal Society*, Jan. 20, 1853, vol. VI, pp. 273—275.

†† *Contributions to Mathematics*, W. Shanks, London, 1853, pp. 86—87.

‡‡ *Proceedings of the Royal Society*, 1872–3, vol. XXI, p. 318; 1873–4, vol. XXII, p. 45.

done in England, found the value of π to 333 places* of decimals (330 correct); in 1854 he carried the approximation to 400 places†; and in 1855 carried it to 500 places‡.

Of the series and formulae by which these approximations have been calculated, those used by Machin and Dase are perhaps the easiest to employ. Other series which converge rapidly are the following:

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3\,.\,2^3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5\,.\,2^5}+\ldots,$$

and

$$\frac{\pi}{4}=22\tan^{-1}\frac{1}{28}+2\tan^{-1}\frac{1}{443}-5\tan^{-1}\frac{1}{1393}-10\tan^{-1}\frac{1}{11018};$$

the latter of these is due to Mr Escott§.

As to those writers who believe that they have squared the circle their number is legion and, in most cases, their ignorance profound, but their attempts are not worth discussing here. "Only prove to me that it is impossible," said one of them, "and I will set about it immediately"; and doubtless the statement that the problem is insoluble has attracted much attention to it.

Among the geometrical ways of approximating to the truth the following is one of the simplest. Inscribe in the given circle a square, and to three times the diameter of the circle add a fifth of a side of the square, the result will differ from the circumference of the circle by less than one-seventeen-thousandth part of it.

An approximate value of π has been obtained experimentally by the theory of probability. On a plane a number of equidistant parallel straight lines, distance apart a, are ruled; and a stick of length l, which is less than a, is dropped on the plane. The probability that it will fall so as to lie across one of the lines is $2l/\pi a$. If the experiment is repeated many hundreds

* *Grünert's Archiv*, vol. XXI, p. 119.

† *Ibid.*, vol. XXIII, p. 476: the approximation given in vol. XXII, p. 473, is correct only to 330 places.

‡ *Ibid.*, vol. XXV, p. 472; and *Elbinger Anzeigen*, No. 85.

§ *L'Intermédiaire des Mathématiciens*, Paris, Dec. 1896, vol. III, p. 276.

of times, the ratio of the number of favourable cases to the whole number of experiments will be very nearly equal to this fraction: hence the value of π can be found. In 1855 Mr A. Smith* of Aberdeen made 3204 trials, and deduced $\pi = 3{\cdot}1553$. A pupil of Prof. De Morgan*, from 600 trials, deduced $\pi = 3{\cdot}137$. In 1864 Captain Fox† made 1120 trials with some additional precautions, and obtained as the mean value $\pi = 3{\cdot}1419$.

Other similar methods of approximating to the value of π have been indicated. For instance, it is known that if two numbers are written down at random, the probability that they will be prime to each other is $6/\pi^2$. Thus, in one case‡ where each of 50 students wrote down 5 pairs of numbers at random, 154 of the pairs were found to consist of numbers prime to each other. This gives $6/\pi^2 = 154/250$, from which we get $\pi = 3{\cdot}12$.

* A. De Morgan, *Budget of Paradoxes*, London, 1872, pp. 171, 172 [quoted from an article by De Morgan published in 1861].

† *Messenger of Mathematics*, Cambridge, 1873, vol. II, pp. 113, 114.

‡ Note on π by R. Chartres, *Philosophical Magazine*, London, series 6, vol. XXXIX, March, 1904, p. 315.

CHAPTER XIV.

THE PARALLEL POSTULATE.

In the last chapter I considered three classical problems. Another geometrical question, perhaps of greater interest, is concerned with whether the sum of the angles of a plane triangle is exactly equal to two right angles. This is a proposition which in ordinary textbooks on elementary geometry is enunciated—and properly so—as if it were undoubtedly true. In one sense this theorem, like the problems discussed in the last chapter, or like the algebraic solution of the general equation of the fifth degree, is insoluble; but the efforts to prove it afford materials for an interesting chapter in the history of mathematics, since many of the demonstrations formerly proposed are fallacious. The fact however that in the reasoning there are pitfalls, logical as well as mathematical, adds to the interest of the discussion, and the treacherous nature of the path makes its safe passage the more interesting*.

* The subject has been discussed by numerous writers. A good account of it to the end of the 18th century is given in the notes to J. Playfair's *Elements of Geometry*, Edinburgh, 1st edition, 1813, and in the Appendix to *Geometry without Axioms* by T. P. Thompson, London, 4th edition, 1833. For other and more recent researches, see H. Schotten, *Planimetrischen Unterrichts*, vol. II, Leipzig, 1893; F. Engel and P. Stäckel, *Die Theorie der Parallellinien*, Leipzig, 1895, 1899; J. Richard, *La Philosophie des Mathématiques*, Paris, 1903; J. W. Withers, *Euclid's Parallel Postulate*, Chicago, 1905; M. Simon, *Ueber die Entwicklung der Elementar-geometrie*, Leipzig, 1906. Some of the solutions offered have no interest, and are evidently fallacious. Hence I make no attempt to treat the subject exhaustively, but I mention the more plausible efforts.

Earliest Proof. Thales. We know from Geminus that this proposition was one of the first general results discovered by the Greeks*. From the extant notices the following has been suggested, with considerable probability, as indicating the manner in which it was proved; at any rate this demonstration involves nothing with which Thales, the traditional founder of the science of abstract geometry, was not acquainted, and it has been conjectured that it is in fact due to him. According to this view, it was, in the first place, stated (or more likely assumed) that in a rectangle the angles were right angles and the opposite sides equal. Hence the sum of the four angles is equal to four right angles. Next, by drawing a diagonal of a rectangle, it will be seen that any right-angled triangle can be placed in juxtaposition with an equal and similar triangle in such a way as to make up a rectangle: this step in the argument may have been suggested by the tiles used in paving floors. Hence the sum of the angles of a right-angled triangle is equal to two right angles. Lastly, any triangle ABC can be divided into two right-angled triangles by drawing a perpendicular AD from the biggest angle A to the opposite side BC. The sum of the angles of the triangle ABD is equal to two right angles. Hence the sum of the angles B and BAD is equal to one right angle. Similarly the sum of the angles C and CAD is equal to one right angle. Hence the sum of the angles B, C, BAD, and CAD, that is, the sum of the angles of the original triangle ABC, is equal to two right angles.

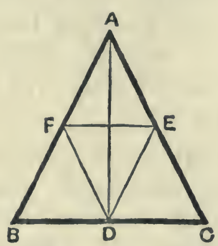
The only criticism I would make on this proof is that it rests frankly on the assumption that we can construct a rectangle, and that the opposite sides of the rectangle are equal. This, unless we refer to direct observation or measurement, involves an assumption about parallel lines, which is equivalent to that made in Euclid's postulate.

Pascal's Proof. Another proof, also resting immediately on experiment, to which I may here refer, was discovered by Pascal in the seventeenth century. It is interesting

* G. J. Allman, *Greek Geometry from Thales to Euclid*, Dublin, 1889, chap. i.

from its history. Pascal was a delicate and precocious boy, and in order to ensure his not being over-worked his father directed that his education should at first be only linguistic and literary, and should not include any mathematics. Naturally this excited the boy's curiosity, and one day, being about twelve years old, he asked in what geometry consisted. His tutor replied that it was the science of constructing exact figures and determining the relations between their parts. Pascal, stimulated no doubt by the injunction against reading it, gave up his playtime to the new amusement, and in a few weeks had discovered for himself several properties of rectilinear figures, and in particular the proposition in question.

His proof is said* to have consisted in taking a triangular piece of paper and turning over the angular points to meet at the foot of the perpendicular drawn from the biggest angle to the opposite side. The conclusion is obvious from a figure, for



if the paper be creased so that A is turned over to D, as also B and C, we get $B = FDB$, $A = FDE$, and $C = EDC$; hence $A + B + C = EDF + FDB + EDC = \pi$. But we can only prove these relations on the assumption that when the paper is folded over BF and AF will lie along DF, and thus that $BF = FA = FD$, and similarly that $CE = EA = ED$; this assumption involves properties of parallel lines. A similar proof can be obtained by turning over the angular points to meet at the centre of the inscribed circle, and according to some accounts this was the method used by Pascal. I may add in passing that his father, struck by this evidence of Pascal's geometrical ability,

* I believe that this rests merely on tradition.

gave him a copy of Euclid's *Elements*, and allowed him to take up the subject for which evidently he had a natural aptitude.

Pythagorean and Euclidean Proof. Leaving the above demonstrations which rest on observation and experiment, I proceed to the classical proof given by Euclid*. This was taken from the Pythagoreans†, and is generally attributed to Pythagoras himself. The proof rests on properties of parallel lines, and the Pythagoreans must have prefaced it by some statement of those properties, but there is now no record as to how they treated parallels.

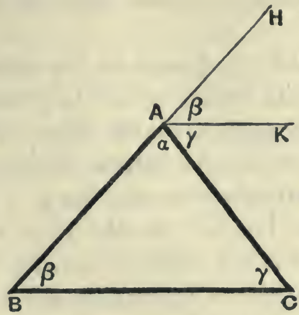
Euclid's treatment of parallels is well known. There is no doubt that he put, at the beginning of his Geometry, certain definitions, axioms, and postulates; but in the earliest manuscripts, according to Peyrard, the assumption about parallels was not stated there, but was placed in the demonstration of his proposition 29 as a fact conformable to experience, which had to be assumed for the validity of the argument. If this be so, this exceptional treatment seems to indicate that, in Euclid's opinion, the assumption was of a different character to the other postulates, and the difficulty was faced frankly without any attempt to conceal it under a vague phraseology. Unluckily the postulate is often printed in modern school books as an axiom or a self-evident statement. This misplacement may have been due in the first instance to Theon of Alexandria who, about 370 A.D., lectured on Euclid's Geometry. Our modern texts of Euclid are mainly based on Theon's lectures, and it is only comparatively recently that the commentaries on Euclid's teaching have been subjected to critical discussion.

At any rate Euclid, either at the commencement of his work or more likely in the course of his demonstration, boldly assumed that if a straight line meets two other straight lines so as to make the sum of the two interior angles on one side of it less than two right angles, then these straight lines if continually produced will meet upon that side on which these

* Euclid's *Elements*, book I, prop. 32.

† Eudemus is our authority for this: see Proclus, ed. G. Friedlein, Leipzig, 1873, p. 379.

angles are situated. Accepting this or some similar assumption, the demonstration is rigorous, and was given by him as follows.



Take any triangle ABC. Produce the side BA to any distance AH, and through A draw a line AK parallel to BC. On the assumption that his postulate is true, Euclid showed (*Euc.* I. 29) that the angle ABC must be equal to the angle HAK, and the angle ACB to the angle KAC. Hence the sum of the three angles of the triangle ABC must be equal to the sum of the angles HAK, KAC, and CAB, that is, to two right angles.

Euclid's postulate and this theorem mutually involve the one the other: if we can prove his postulate this theorem is true, if otherwise we can prove this theorem, then his postulate is true*. Hence the question with which I commenced the chapter (namely whether the sum of the angles of a triangle is, and can be shown to be, equal to two right angles) comes in effect to asking whether Euclid's postulate is true and can be proved to be true.

Features of the Problem. The postulate, as enunciated by Euclid, has the semblance of a proposition. For many centuries mathematicians believed that it could be directly deduced from the fundamental principles of geometry, and they devoted much labour to trying to prove it. The more notable of these attempts I propose to describe, but I may anticipate matters by saying that about a hundred years ago it was shown

* The demonstration is given by J. Richard, *La Philosophie des Mathématiques*, Paris, 1903, pp. 81—84.

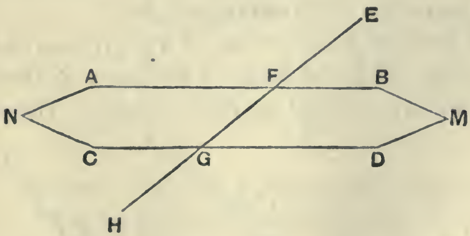
that this postulate, or any of its equivalent forms, cannot be proved. Thus in every one of the proposed demonstrations there is either a fallacy, or some assumption similar to that made by Euclid.

In order to be able to appreciate the criticisms on some of these attempts it will be convenient to preface the discussion by saying that the postulate and its conclusions do in fact involve considerations of the nature of the space considered. For example, we say that we can draw a line parallel to a given line, and that however far the lines are produced they will not meet. This is not at variance with what we observe, but we have never got to infinity to see what does happen there. Hence, though it is conformable to our experience we cannot say that it is actually true. In fact, it is not certain that the statement is absolutely true of the space we know. An example will show this. If small intelligent beings lived on a strictly circumscribed portion of the surface of a sphere, and evolved a geometry of figures drawn on that surface, they might form a body of propositions similar to those given by Euclid, and resting on the same axioms and assumptions. All their assumptions, except this postulate and the axiom about the impossibility of two straight lines enclosing a space, would be correct. But if the sphere were large enough, and they were confined to a comparatively small part of its surface, they would not be able to find out that this postulate about parallels was incorrect. Accordingly it would be not unreasonable that they should believe it to be true, though in fact it would be false.

What is here said of the surface of a sphere is by way of illustration, but it indicates the possibility of the existence of surfaces such that consistent systems of geometries, closely resembling the Euclidean geometry, might be constructed dealing with figures drawn thereon. This question is mentioned again later. The above remarks suffice, however, to show that the postulate involves properties of the space in which the figures are constructed. It follows that the best way of stating the postulate will be that which is directly characteristic of what we may call plane space, as opposed to other kinds of space. Euclid's statement answers this purpose, and it is remarkable that he should have thus gone to the root of the matter. By implication he admitted that the statement could not be demonstrated, and he frankly met the difficulty by telling his hearers that though he could not prove it, they must grant him the postulate as a foundation for his reasoning.

Attempted Demonstrations of the Postulate. I proceed now to describe a few of these attempts to prove the postulate or the proposition.

Ptolemy's Proof of the Postulate. One of the earliest of these efforts to prove the postulate was due to Ptolemy, the astronomer, in the second century after Christ. It is as follows*. Let the straight line $EFGH$ meet the two straight lines AB and CD so as to make the sum of the angles BFG and FGD equal to two right angles. It is required to prove that AB and CD are parallel. If possible let them not be parallel, then they will meet when produced say at M (or N). But the angle AFG is the supplement of BFG and is therefore equal to FGD. Similarly the angle FGC is equal to BFG. Hence the sum of the angles AFG and FGC is equal to two right angles, and therefore the lines BA and DC, if produced, will meet at M (or N). But two straight lines cannot enclose a space, therefore AB and CD cannot meet when produced, that is they are parallel.



Conversely, if AB and CD be parallel, then AF and CG are not less parallel than FB and GD; and therefore whatever be the sum of the angles AFG and FGC, such also must be the

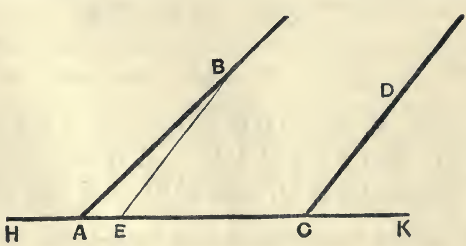
* Proclus, ed. G. Friedlein, Leipzig, 1873, pp. 362—368.

sum of the angles *FGD* and *BFG*. But the sum of the four angles is equal to four right angles, and therefore the sum of the angles *BFG* and *FGD* must be equal to two right angles.

This proof is not valid. Apart from all considerations about the nature of space, no reason is given why the sums of the angles on either side of the secant should be assumed to be equal. The whole question turns on whether the straight lines would not meet, even though the sum of the angles on one side is a little more than two right angles, and on the other a little less. It is conceivable that parallels might open out as they are prolonged, and thus that a straight line inclined at a small angle to one of them should never overtake the other, but chase it unsuccessfully through infinite space, just as a curve pursues its asymptote and never catches it.

Proclus's Proof of the Postulate. Proclus, after criticising Ptolemy's demonstration, gave a proof of his own, but in the course of it he assumed that if two intersecting straight lines be produced far enough the distance between a point on one of them and the other line can be made greater than any assigned finite length, and that if two parallel straight lines be produced indefinitely the perpendicular from a point on one of them to the other remains finite. On these assumptions the postulate can be proved. But just as we cannot assume that two converging lines (*ex. gr.* a curve and its asymptote) will ultimately meet, so we must not assume that the distance between two diverging lines will be ultimately infinite.

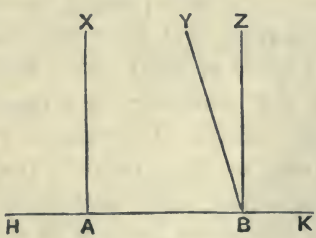
Wallis's Proof of the Postulate. I will give next a demonstration offered by J. Wallis, Savilian Professor of Geometry, in a lecture delivered at Oxford on July 11, 1663. The substance of



his argument may be put thus*. It is desired to prove that if two lines AB, CD meet a transversal $HACK$, so that the sum of the angles BAC, ACD is less than two right angles, then AB and CD must (if produced) meet. One of the angles BAC, ACD must be acute; suppose it is BAC. He first showed that, in this case, from any point B in AB we can draw a line BE which will cut AC in E, so that the sum of the angles BEC, ECD is equal to two right angles; hence the angle BEA is equal to the angle DCA. Then if we take the triangle BAE (drawn on AE as base and with B as vertex) and construct a similar triangle on AC as base, he proved that its vertex must be at a finite distance from AC, must lie on AB produced, and must lie on CD (produced if necessary). Hence AB and CD when produced must meet.

The proof is ingenious, but it rests on the assumption that it is possible to construct a triangle on any specified scale similar to a given triangle. This cannot be considered axiomatic and in fact is not true of spherical triangles. The assumption, however, is made explicitly, and it can be used instead of Euclid's postulate if it be thought desirable.

Bertrand's Proof of the Postulate. The following is another interesting demonstration. It was originally given by Bertrand



of Geneva†. Suppose AX and BY are two lines which meet a third line $HABK$ so that $XAB + YBA < \pi$. It is required to show that AX and BY must cut. For simplicity, I draw the figure so that $XAB = \pi/2$ and therefore $YBA < \pi/2$,

* J. Wallis, *Opera*, Oxford, 1693, vol. II, pp. 674—678.

† I do not know where or when it was first published. It was given by S. F. Lacroix in his *Éléments dè Géometrie*, Paris, 1802, p. 23.

but this does not affect the argument. Produce AB indefinitely in both directions to H and K. Draw BZ perpendicular to AB, and denote the angle YBZ by α. Then the area between BY and BZ is the fraction α/π of the space round B above HK, that is, α/π of the area of the plane above HK. Also the area between AX and BZ is the fraction AB/HK of the plane above HK. Now, however small α may be, α/π is a definite finite fraction, but AB/HK is indefinitely small. Hence the area between BY and BZ is greater than the area between AX and BZ. But as long as BY does not cut AX the area between BY and BZ is less than that between AX and BZ. Hence BY must cut AX.

The objection to this demonstration is that it depends upon a comparison of infinite areas. But we have no test by which we can compare such areas, and to consider the order of infinities involves questions outside the region of elementary geometry. There is also a more fundamental difficulty: the argument assumes that space is infinite, but it is possible that it may be boundless and finite, as, for instance, is the surface of a sphere.

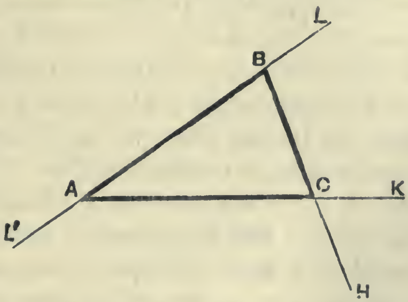
Playfair's Earlier Proof of the Postulate. I will mention next an attempt to prove this postulate by assuming that two lines which cut cannot be both parallel to another line, that is, that through a point one and only one straight line can be drawn parallel to a given straight line. It has been said that this assumption is not axiomatic, for a reason similar to that given above in my criticism of Ptolemy's Proof, but to most readers it seems simpler than Euclid's postulate, and as its meaning is easily grasped, some mathematicians prefer it to Euclid's postulate. Like the latter it is characteristic of the space considered. If this assumption is made, it is easy to show* that a transversal meeting two parallel straight lines makes the alternate angles equal, from which the other conclusions of Euclid follow.

* *Elements of Geometry*, by J. Playfair, Edinburgh, 1st edition, 1813, book I. prop 29. The book is in the same form as Euclid's *Elements* except for the substitution of this postulate for that given by Euclid.

Personally I agree with those who consider Playfair's assertion as axiomatic, that is, as being a part of our conception of plane space as derived from experience. This is not inconsistent with admitting that mathematicians can conceive a more general view of space (*i.e.* non-Euclidean space) and that to them the space of our experience is only, to the highest degree of approximation, Euclidean space*. But many writers do not accept Playfair's assertion as axiomatic. On such an issue no argument is possible.

Attempted Direct Demonstrations of the Proposition. The difficulties connected with the subject of parallelism led to various attempts to prove the proposition directly and thence to deduce some property of parallelism equivalent to Euclid's postulate. Substantially this was the method used by Thales and Pascal. I will mention one or two of these attempts.

Playfair's Rotational Proof of the Proposition. First I will describe an attempt, given by Playfair † in 1813. His argument is as follows. An angle is measured by the amount of turning



of a vector. Let ABC be any triangle. Suppose we have a rod AL placed along AB with one end at A. If we rotate it clockwise round A as a pivot through the angle BAC it will move from AL to AK. It will make no difference if we now slide the rod along AK so that the end moves from A

* See A. Cayley, *British Association Report*, London, 1883, p. 9.

† See the notes appended to J. Playfair's *Elements of Geometry*, p. 432 in the fifth edition. Playfair finds the sum of the exterior angles and thence deduces the sum of the interior angles, but the method is the same as that given above.

to C. If we now turn the rod, in the same direction as before, round C as a pivot through the angle ACB it will move from CK to CH. It will make no difference if we now slide it back along CB so that the end moves from C to B. If we now turn the rod, again in the same direction, round B as a pivot through the angle CBA it will move from BH to BA. We can then again slide the rod along BA so that the end B moves to A, when the rod will lie along AL'. Thus the rotation, always in the same direction, successively through the three angles of the triangle produces exactly the same effect as a rotation through two right angles.

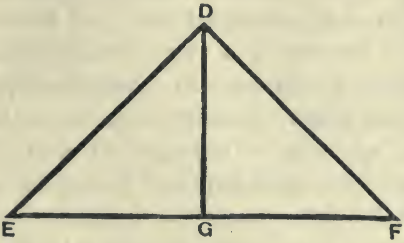
The demonstration is incorrect. In fact it is assumed that if the angle ABC in the figure above on page 311 is equal to HAK, then BC and AK will be parallel. The fallacy can be seen at once by applying the argument to the case of a spherical triangle or to one whose sides are circular arcs all convex—or all concave—to its median point.

Legendre's First Proof of the Proposition. Legendre devoted special attention to the problem and offered various demonstrations of it. I give three of them. In one, which appears in the earlier editions of his Geometry, he tried to show that the sum of the angles of a triangle could not be greater than two right angles and could not be less than two right angles, and that therefore it must be equal to two right angles. His demonstration assumes that if any number of equal triangles are placed in juxtaposition along a line it is possible to draw a triangle enclosing them all: the same assumption was made by T. P. Thompson. But unless we assume that space is infinite this is not justified: the insufficiency of the argument is clearly brought out by applying it to spherical triangles.

Legendre's Analytical Proof of the Proposition. I think however that the following is the most ingenious of the proofs given by Legendre*. A triangle ABC is completely determined by one side and two angles, say, a, B, C. Given these, the triangle can be constructed, and therefore the angle A determined. Now if the unit of length be changed the measure

* *Éléments de Géometrie*, Paris, 12th edition, p. 281.

of a will be changed but the triangle, and therefore A, will not be altered. Hence A cannot depend on the value of a; accordingly it must depend only on B and C.



Now take a right-angled triangle DEF, of which D is the right angle. Draw DG perpendicular to EF. The angle EDG in the triangle EDG is calculated from the other two angles of that triangle, namely E and a right angle, in the same way as the angle F in the triangle DEF is calculated from the other two angles of that triangle, namely E and a right angle. Hence F is equal to EDG. Similarly E is equal to GDF. Therefore the sum of the angles F and E is equal to the sum of EDG and GDF, and therefore is a right angle. Hence the sum of the angles F, E, and D of the triangle DEF is equal to two right angles. Thus the result is proved for a right-angled triangle, and it will follow for any other triangle in the same way as in Thales's proof.

J. Leslie criticised this proof on the ground that in the corresponding theorem of Spherical Trigonometry, we know that the expression for the value of the angle A involves a/R, where R is the radius of the sphere, and it is conceivable that in plane geometry there might be a length R' (the reciprocal of the space constant) which entered in a similar way in the problem: hence A might involve a and yet not change with the unit of measurement. To this it was replied that the point of Legendre's argument was that the discussion related only to plane geometry: this might, no doubt, be considered as the special case of spherical geometry in which R was infinite; if so, any term in the expression for A which involved a/R

disappeared, and thus his reasoning was valid; and to introduce an unknown quantity R' was contrary to all canons of reasoning.

Legendre's Latest Proof of the Proposition. At the end of his life, 1833, Legendre showed* that if we could construct one triangle the sum of whose angles was equal to two right angles, then the sum of the angles of every triangle would be equal to two right angles. All attempts to obtain direct proofs that such a triangle existed failed. He showed, however, that if the sum of the angles of a triangle is not equal to two right angles then linear magnitudes can be determined by angular measurements. Assuming that this latter result is impossible, the proposition is proved. The second part of the argument is in effect a translation into geometry of the analytical proof given above.

Lagrange's Memoir. Legendre's great contemporary, Lagrange, believed at one time that he had found a solution of the problem. It rested on establishing plane geometry by a generalization from geometry on a spherical surface. He commenced to read to the Institute a paper on the subject, but had hardly begun when he stopped abruptly, put his memoir in his pocket and saying "Gentlemen, I must think further about this," left the room. We do not know what his argument was, but doubtless some flaw in it flashed on him as he commenced his paper†.

Other Parallel Postulates. Euclid's postulate is in accordance with experience, and like the axioms and other postulates it rests ultimately on the results of observation, but his statement of the property in question is not easy, and it requires some thought before the point is grasped. For this reason many attempts have been made to put it in other forms which are more likely to be readily granted by an ordinary reader. I enumerate two or three of these.

It will be noticed that the demonstration offered by Wallis and the earlier one given by Playfair rest on alternative postulates about parallels. That assumed by Wallis is sufficient,

* *Mémoires de l'Institut de France*, Paris, 1833, vol. XII, pp. 367—410. The paper also contains an account of Legendre's earlier investigations.

† A. de Morgan, *Budget of Paradoxes*, London, 1872, p. 173.

but is not axiomatic, as may be seen by its incorrectness when applied to spherical triangles. It was adopted by Carnot, Laplace, and J. Delboeuf. Playfair's axiom answers the purpose as well as Euclid's: this form was also used by Ludlam. Thales's assumption that a rectangle exists also suffices. This was assumed by Clairaut.

It has been suggested that Euclid's postulate might be replaced by assuming that, if at a point A in a given line AB a line AX be drawn perpendicular to it, and at another point B in AB a line BY be drawn, making with it an acute angle, then AX and BY will cut. But essentially this is only Euclid's form expressed diagrammatically.

Another alternative form which has been suggested is to the effect that through every point within an angle a line can be drawn intersecting both sides (substantially the view of Lorentz and Legendre). This also is sufficient, but the application is less easy than that of Euclid's postulate.

It has also been proposed that we may reasonably assume that the distance between two parallel lines is always the same (Dürer, T. Simpson, R. Simpson), or that a line which is everywhere equidistant from a given straight line in the same plane is itself straight (Clavius). Neither of these forms is satisfactory. The conception of distance involves measurement, and this in turn involves a theory of incommensurable magnitudes. Thus before we can rest the theory on such a postulate, other assumptions have to be made, and the resulting discussion is neither simple nor clear.

Legendre suggested that it was sufficient to assume that the lesser of two homogeneous magnitudes if multiplied by a sufficiently large number would exceed the greater of them. But to make use of this he had to introduce the assumptions and principles of the infinitesimal calculus, and this can hardly be regarded as permissible in elementary geometry.

A different kind of postulate was suggested by Dodgson, who proposed* to replace Euclid's postulate by assuming that 2^n times the area of an equilateral four-sided figure inscribed

* C. L. Dodgson, *Curiosa Mathematica*, London, 1890, p. 35.

in a circle is greater than the area of any one of the segments of the circle which lies outside it, where n is any positive integer. Granting that we can inscribe such a figure in a circle, this assumption seems obviously true. But a comparison by the eye of the area of a rectilineal figure with an incommensurable area bounded by a curve and a straight line is contrary to all the traditions of classical geometry and to what is usually regarded as permissible in elementary geometry.

Definitions of Parallels. Other writers have tried to turn the difficulty by altering the definitions of parallel lines*. One of the best known suggestions made with this object defines parallel lines as lines which have the same direction, by which is meant lines which make the same angle with a line cutting them (Varignon, Bézout, Lacroix). The phrasings of the proposed definition vary slightly. There is no objection to this if the cutting line is fixed, but then it does not avoid the necessity of our having to assume some postulate. If, however, as is usual, the definition is taken to mean that parallel lines make equal angles with every secant, it involves an unwarrantable assumption. In fact it would seem that the term direction cannot be defined without predicating a theory or properties of parallels†.

Another suggested definition which has met with some favour is to the effect that parallel lines are lines which neither recede from nor approach each other, that is, lines whose distance apart is always the same (Wolf, Boscovich, Bonnycastle). This definition is really equivalent to the postulates laid down by Dürer and Clavius, and to give a definition which involves a disputed assumption is worse than frankly postulating the assumption. The definition, however, agrees with the popular view, but if we take a straight line, and erect at every point a perpendicular of given length, we have no right to assume that the locus of the extremities of these transverses will be a straight line, and even less right to assume that it is a

* See J. Playfair, *Elements of Geometry*, Edinburgh, 1813; notes to book I, prop. 29.

† W. Killing, *Grundlagen der Geometrie*, Paderborn, 1898.

straight line perpendicular to them; and, unless we assume that the distance between the parallel lines is measured by a transverse perpendicular to both of them, we cannot use the definition to much purpose. D'Alembert avoided these difficulties by saying that one line is parallel to another if it contains two points on the same side of and equidistant from the other; but this by itself is of no use, unless we assume Euclid's postulate or some similar property of parallels.

These definitions, if they are to be useful, involve assumptions. They slur over the real difficulty, and are less satisfactory than a frank statement of what is assumed.

*Non-Euclidean Systems**. It had long been well known that the postulate and proposition were not true in the corresponding geometry on a spherical surface—in fact the sum of the angles of a spherical triangle always exceeds two right angles—and since there were such difficulties in establishing the Euclidean postulate in plane space, mathematicians began, rather more than a century ago, to consider whether that postulate was true either necessarily or in fact. It required courage, even genius, to make such a conjecture, for though on the one hand the postulate could not be proved, there was on the other no reason to doubt its correctness, and no conclusion inconsistent with observation had been deduced from it, while at first sight nothing seemed to justify the assumption that it was not true.

Saccheri and Lambert raised this question in the eighteenth century, but their investigations, though intelligent, were incomplete and attracted little attention. Gauss went further, as appears from his correspondence in 1829 and 1831, but even before then he had shown that the proposition and postulate could be proved to be true if it were admitted that a triangle could be drawn with an area greater than a given area; this, however, he rightly regarded as non-axiomatic. Later he discussed some of the properties of hyperbolic geometry. He did not publish his results, and they did not affect the treatment of the problem by other writers.

* See R. Bonola, *La Geometria Non-Euclidea*, Bologna, 1906; and D. M. Y. Sommerville, *Bibliography of Non-Euclidean Geometry*, St Andrews, 1911.

The credit of first showing that the postulate is not necessarily true is due to Lobatschewsky and the Bolyais. They boldly assumed that the postulate was not true and that through a point a number of straight lines can be drawn parallel to a given straight line. On this assumption, they deduced a consistent body of propositions, which is termed *hyperbolic geometry*.

These investigations attracted but slight notice. The writers were almost unknown. N. I. Lobatschewsky, 1793—1856, was professor at Kasan, and his works were written in Russian. Wolfgang Bolyai, 1776—1856, was an eccentric, simple, rough-clad teacher in Transylvania. He now lies buried at his request under an apple tree, commemorating the three apples which, he said, had so profoundly affected the history of the human race—those of Eve and Paris, which had made earth a hell, and that of Newton which had raised earth again into the company of the heavenly bodies. His son John, 1802—1860, had excellent mathematical abilities, and worked out the principles of the new geometry, but he spent his life soldiering, and to him mathematics was only a recreation. Probably he valued his reputation as a musician far above his mathematical abilities. He was noted for his fiery temper; in one of his quarrels he accepted the challenge of thirteen officers of a regiment on condition that after each duel he might play to each of them a piece on his violin. He is said to have vanquished them all, and been, in consequence, retired from the army.

The subject, however, was "in the air," and attracted the attention of G. F. B. Riemann. Riemann was one of the most brilliant German mathematicians of the nineteenth century and, though short-lived, his writings have profoundly affected the development of the subject. His paper on the hypotheses on which geometry is founded was read in 1854. He showed that a consistent system of geometry of two dimensions can be constructed in which all straight lines are of a finite length. This science, now known as *elliptic geometry*, is characterised by the fact that through a point no straight line can be drawn which if produced far enough will not meet every other line. The resulting geometry may be compared with the geometry of figures drawn on the surface of a sphere; in it, space, though boundless, is finite. The discussion of Riemann's paper led to the discovery of the earlier researches of Lobatschewsky and the Bolyais. The subject has since been studied by several mathematicians of repute, notably by E. Beltrami and F. C. Klein.

Here then we have three geometries—Elliptic, Euclidean (or Parabolic or Homaloidal) and Hyperbolic—each consistent on its own hypotheses, distinguished from one another according as no straight line, or only one straight line, or a pencil of straight lines can be drawn through a point parallel to a given straight line.

In the parabolic and hyperbolic systems straight lines are infinitely long: in the elliptic they are finite. In the hyperbolic system there are no similar figures of unequal size; the area of a triangle can be deduced from the sum of its angles, which is always less than two right angles; there is a finite maximum to the area of a triangle; and its angles can be made as small as we like by making its sides sufficiently long. In the elliptic system all straight lines, if produced, are of the same finite length; any two lines intersect; and the sum of the angles of a triangle is always greater than two right angles. In the elliptic system it is possible to get from one point to a point on the other side of a plane without passing through the plane; thus a watch-dial moving face upwards continuously forward in a plane in a straight line in the direction from the mark VI to the mark XII will finally appear to a stationary observer with its face downwards; and if originally the mark III was to the right of the observer it will finally be on his left-hand.

In spite of these and other peculiarities of hyperbolic and elliptic geometries, it is impossible to prove by observation that one of them is not true of the space in which we live. For in measurements in each of these geometries we must have a unit of distance; and if we live in a space whose properties are those of either of these geometries, and such that the greatest distances with which we are acquainted (*e.g.* the distances of the fixed stars) are immensely smaller than any unit natural to the system, then it may be impossible for our observations to detect the discrepancies between these three geometries. It might indeed be possible for us by observations of the parallaxes of stars to prove that the parabolic system and either the hyperbolic or the elliptic system were false, but never can it be proved by measurements that the Euclidean geometry is true. Similar difficulties might arise in connection with excessively minute quantities. In short, though the results of Euclidean geometry are more exact than present experiments can verify for finite things, such as those with which we have to deal, yet for much larger things or much smaller things, or for parts of space at present inaccessible to us, they may not be true. Even, however, if our space is only approximately Euclidean, the propositions of ordinary geometry are none the less true of Euclidean space, though that may not be the space of our experience.

I mention later, in Chapter XX, some other problems connected with different kinds of space.

CHAPTER XV.

INSOLUBILITY OF THE ALGEBRAIC QUINTIC.

Another of the famous problems in the history of mathematics, which long proved an *ignis fatuus* to mathematicians, is the solution of the general algebraic equation of the fifth degree. By a solution of an algebraic equation we mean the expression, by a finite number of radicals and rational functions, of a root of it in terms of its coefficients. The solution of an algebraic quadratic equation presents but little difficulty. In the sixteenth century, solutions of the general cubic and quartic equations were obtained. It did not seem unnatural to suppose that by similar analytic methods the solution of quintic equations, as also those of a higher order, might be effected. This is now known to be impossible, and I propose to give a brief sketch of the reason why it is so. The proof rests on the fact that the equations $x^3=1$ and $x^5=1$ have no common complex root.

Quadratic, cubic and quartic equations can be solved by various methods; but, in effect, all of them reduce the solution of the particular equation to the solution of one, called a *resolvant*, of a lower order. These methods fail when applied to equations of an order higher than four. Lagrange was the earliest writer to ask whether this was necessarily the case. In 1770 and 1771 he published a critical examination of the known solutions of quadratic, cubic and quartic equations, and showed that such solutions were possible only because a function of the roots of such equations could be formed which had a smaller number of possible values than the order of the equation, and this function was such that its value could be determined.

For instance, take the quadratic equation $x^2 + ax + b = 0$. It has two roots x_1 and x_2. We can reduce the solution to that of a simple equation because we can form a function of x_1 and x_2 which has only one value, and this value can be determined. Such a value is $y = (x_1 - x_2)^2$, since $(x_1 - x_2)^2 = (x_2 - x_1)^2$. Moreover we have $y = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = a^2 - 4b$. Thus y is known. Hence the roots of the quadratic are given by

$$x_1 + x_2 = -a,$$
$$x_1 - x_2 = \sqrt{y}.$$

Similarly, in the cubic equation $x^3 + ax^2 + bx + c = 0$, if $y = (x_1 + \omega x_2 + \omega^2 x_3)^3$, where ω is one of the complex cube roots of unity, then y is a function of the roots which, when $x_1\, x_2\, x_3$ are interchanged, has only two values, namely $(x_1 + \omega x_2 + \omega^2 x_3)^3$ which may be also written in the form $(x_2 + \omega x_3 + \omega^2 x_1)^3$ or $(x_3 + \omega x_1 + \omega^2 x_2)^3$, and $(x_1 + \omega x_3 + \omega^2 x_2)^3$ which may be also written in the form $(x_3 + \omega x_2 + \omega^2 x_1)^3$ or $(x_2 + \omega x_1 + \omega^2 x_3)^3$. Let y and z denote these two values, that is, $y = (x_1 + \omega x_2 + \omega^2 x_3)^3$, and $z = (x_1 + \omega x_3 + \omega^2 x_2)^3$. Then $y + z = -2a^2 + 9ab - 27c = A$, say, and $yz = (a^2 - 3b)^3 = B$, say. Thus y and z are the roots of $t^2 - At + B = 0$, an equation which can be solved. And the roots of the cubic are determined by

$$x_1 + x_2 + x_3 = -a,$$
$$x_1 + \omega x_2 + \omega^2 x_3 = \sqrt[3]{y},$$
$$x_1 + \omega^2 x_2 + \omega x_3 = \sqrt[3]{z}.$$

Again, in the quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$, if $y = (x_1 - x_2 + x_3 - x_4)^2$, then y is a function of the roots which has only three values, namely, $(x_1 - x_2 + x_3 - x_4)^2$ or $(x_2 - x_1 + x_4 - x_3)^2$, $(x_1 - x_3 + x_4 - x_2)^2$ or $(x_3 - x_1 + x_2 - x_4)^2$, and $(x_1 - x_4 + x_2 - x_3)^2$ or $(x_4 - x_1 + x_3 - x_2)^2$. If these values be denoted by y, z, and u, we have $y + z + u = 3a^2 - 8b = A$, say, $yz + zu + uy = 3a^4 - 16a^2b + 16b^2 + 16bc - 64d = B$, say, and $yzu = (a^3 - 4ab + 8b)^2 = C$, say. Hence y, z, u are the roots of the equation $t^3 - At^2 + Bt + C = 0$. And the roots of the quartic are determined by

$$x_1 + x_2 + x_3 + x_4 = -a,$$
$$x_1 - x_2 + x_3 - x_4 = \sqrt{y},$$
$$x_1 - x_2 - x_3 + x_4 = \sqrt{z},$$
$$x_1 + x_2 - x_3 - x_4 = \sqrt{u}.$$

Lagrange showed that for an equation of the nth degree, an analogous function y of the roots could be formed which had only $n-1$ values, and which led to a resolvant of the degree $n-1$, but that the coefficients of this latter equation could not be obtained without the previous solution of an equation of the degree $(n-2)!$. Hence the form assumed for y did not provide a solution of a quintic or of an equation of a higher degree. But though he suspected that the general quintic and higher equations could not be solved algebraically, he failed to prove it. The subject was next taken up by P. Ruffini, 1798—1806, but his analysis lacked precision.

The earliest rigorous demonstration that quintic and higher equations cannot be solved in general terms was given by N. H. Abel in 1824, and published in *Crelle's Journal* in 1826. The result was interesting, not only in itself, but as definitely limiting a field of investigation which had attracted many workers. Abel's proof was simplified by E. Galois* in 1831, and is now accessible in various text-books. Essentially the argument is as follows.

Let $x_1, x_2, x_3, \ldots x_n$ be the roots of the equation

$$f(x) = x^n + ax^{n-1} + \ldots + k = 0.$$

Any one of these roots, x_1, is a function (which will generally involve radicals) of the coefficients $a, b, \ldots$. These coefficients are symmetrical functions of the roots, namely, $a = \Sigma x_1$, $b = \Sigma x_1 x_2, \ldots$. If these values of $a, b, \ldots$ be substituted in the expression for x_1 we get, on simplification, an identity. If we interchange x_1 and x_2, or any pair of roots, this will remain an identity. Similarly it will remain an identity, if we permute cyclically an odd number of roots, since such a permutation is equivalent to an interchange of a number of pairs of roots. For instance, if x_1 and x_2 are the roots of the quadratic equation $x^2 + ax + b = 0$, we have

$$\begin{aligned} x_1 &= \tfrac{1}{2}[-a + \sqrt{\{a^2 - 4b\}}] \\ &= \tfrac{1}{2}[(x_1 + x_2) + \sqrt{\{(x_1 + x_2)^2 - 4x_1 x_2\}}] \\ &= \tfrac{1}{2}[(x_1 + x_2) + (x_1 - x_2)]. \end{aligned}$$

* See *Liouville's Journal*, Paris, 1846, vol. XI, pp. 417—433.

This is an identity, and if we interchange x_1 and x_2 it will remain an identity.

Now suppose that we have a solution of the equation, that is, an expression for x_1 in terms of the coefficients, involving only a finite number of radicals and rational functions. This expression may be a sum of a number of quantities. To fix our ideas let us suppose that in one of these quantities we come first, in the order of operations, to a radical, say, the pth root of H, where we may without loss of generality take p as prime. Of course H is rational: it involves $a, b, c, \ldots$, and therefore is a symmetrical function of $x_1, x_2, \ldots$. The pth root of H also will be rational, but it will not be symmetrical: let us denote it by $\phi(x_1, x_2, \ldots x_n)$. For instance, in the quadratic equation $H = a^2 - 4b = (x_1 + x_2)^2 - 4x_1x_2$, which is a rational and symmetrical function of x_1 and x_2. But $\sqrt{H} = \phi = x_1 - x_2$, which is not symmetrical, though it is rational.

In the general case ϕ is rational but not symmetrical. It involves the two roots, x_1 and x_2, and therefore it must change in value if they are interchanged. Further, since the values of ϕ are determined by $\phi^p = H$, and H does not vary when the roots are interchanged, one of the values of ϕ must be deducible from the other by multiplying it by ω, where ω is a pth root (other than unity) of unity, that is,

$$\phi(x_2, x_1, x_3 \ldots) = \omega\phi(x_1, x_2, x_3 \ldots).$$

For instance, in the quadratic equation $x^2 + ax + b = 0$, we observe that, if in $\phi(x_1, x_2)$, that is, in $x_1 - x_2$, we interchange x_1 and x_2, we necessarily get $\phi(x_2, x_1) = -\phi(x_1, x_2)$ since the two values of ϕ are roots of $\phi^2 = H$, where H is invariable. Hence if one is $\pm\sqrt{H}$, the other must be $\mp\sqrt{H}$

The relation thus reached,

$$\phi(x_2, x_1, x_3, \ldots) = \omega\phi(x_1, x_2, x_3, \ldots),$$

is an identity; hence if we interchange x_1 and x_2, we have

$$\phi(x_1, x_2, x_3, \ldots) = \omega\phi(x_2, x_1, x_3, \ldots).$$

Hence $\omega^2 = 1$. And as $\omega \neq 1$, we have $\omega = -1$. Since $\omega^2 = 1$, we have $p = 2$: this shows that in the expression for a root of an equation the first radical which occurs in the order of operations must be of the second degree.

In the case of a quadratic equation this concludes the discussion, for there are only two roots which can be interchanged. We may note that if it were possible to take the value $\omega = 1$, it would at once give $x_2 = x_1$, which, in the general case, is clearly impossible.

We proceed to the case of a cubic or higher equation. We will first suppose that in the expression for x_1 we substitute the above value of ϕ, and combine it and similar square roots with the various rational functions of the coefficients, $a, b, \ldots$. As long as we only introduce such square roots we obtain a function of the roots, say K, susceptible of taking only two values, and therefore invariable when three (or any odd number) of the roots are permuted cyclically. This cannot lead to the determination of three or more roots. Hence we must, in this process of reduction and simplification, arrive, in the expression for x_1, at a radical, say the qth root of K, of a higher order than a square root. In this expression K will be invariable when three (or any odd number) of the roots are permuted cyclically.

We can express the qth root of K as a rational function of the roots $\psi(x_1, x_2, \ldots x_n)$, and, from the nature of the case, ψ takes different values when three roots are permuted cyclically. The values of ψ are roots of $\psi^q = K$. Accordingly, following the same argument as that given above, we have

$$\psi(x_2, x_3, x_1, \ldots x_n) = \omega\psi(x_1, x_2, x_3, \ldots x_n),$$

where ω is a qth root (other than unity) of unity. If we permute x_1, x_2, x_3 cyclically, we have

$$\psi(x_3, x_1, x_2, x_4, \ldots) = \omega\psi(x_2, x_3, x_1, x_4, \ldots),$$

and $$\psi(x_1, x_2, x_3, x_4, \ldots) = \omega\psi(x_3, x_1, x_2, x_4, \ldots).$$

Hence $\omega^3 = 1$. We have previously shown that the first radical involved in the general expression for a root must be a square root, and now we see that the next radical must be a cube root. We observe that the solution usually given of a cubic confirms our analysis. The solution of the cubic $x^3+bx+c=0$ is generally written as

$$x=\{-c/2+\sqrt{u}\}^{\frac{1}{3}}+\{-c/2-\sqrt{u}\}^{\frac{1}{3}},$$

where $u=c^2/4+b^3/27$. In each term of the expression for x the first surd which occurs in the order of operations is a square root and the next surd a cube root.

In the case of a cubic equation there are only three roots, and we cannot continue the process further. So also we cannot proceed further in the case of a quartic equation, for as we want to permute an odd number of roots cyclically we cannot permute more than three.

We proceed to the case of an equation of the fifth or higher degree. We have already shown that in this case if, when we substitute in the expression for x_1 the value of ϕ and similar square roots, we arrive at a radical $\sqrt[q]{K}$ for which the equivalent rational function ψ takes different values when an odd number of the roots are permuted cyclically, it follows that if we permute three roots cyclically we get $\omega^3=1$, where ω is a root (other than unity) of $\omega^q=1$. Hence $q=3$. Also if we permute five roots cyclically, we obtain by a similar argument $\omega^5=1$. Thus $q=5$. These equations for ω and values of q are inconsistent. In fact the argument shows that the first surd which has to be calculated in the general expression for x_1 is a square root, and the next surd is at the same time a complex cube root of unity and a complex fifth root of unity. This is impossible. Hence an equation of the fifth or higher degree cannot be solved by a finite number of radicals and rational functions of the coefficients.

It may be added that just as we can express the root of a cubic equation in terms of trigonometrical functions, so we can express the root of a quintic or sextic equation in terms of elliptic or hyperelliptic functions. But such functions lie outside the field of algebra.

CHAPTER XVI.

MERSENNE'S NUMBERS.

One of the unsolved riddles of higher arithmetic, to which I alluded in the second chapter, is the discovery of the method by which Mersenne or his contemporaries determined values of p which make a number of the form 2^p-1 a prime. It is convenient to describe such primes as *Mersenne's Numbers*, a name which I believe I introduced. In this chapter, for shortness, I use N to denote a number of the form 2^p-1. In a memoir in the *Messenger of Mathematics* in 1891 I gave a brief sketch of the history of the problem. I here repeat the facts in somewhat more detail, and add some notes on methods used in attacking the problem.

Mersenne's enunciation of the results associated with his name is in the preface to his *Cogitata* *. The passage is as follows:

Vbi fuerit operae pretium aduertere XXVIII numeros a Petro Bungo pro perfectis exhibitos, capite XXVIII, libri de Numeris, non esse omnes Perfectos, quippe 20 sunt imperfecti, adeovt [adeunt?] solos octo perfectos habeat......qui sunt è regione tabulae Bungi, 1, 2, 3, 4, 8, 10, 12, et 29: quique soli perfecti sunt, vt qui Bungum habuerint, errori medicinam faciant.

Porrò numeri perfecti adeo rari sunt, vt vndecim dumtaxat potuerint hactenus inueniri: hoc est, alii tres a Bongianis differentes: neque enim vllus est alius perfectus ab illis octo, nisi superes exponentem numerum 62, progressionis duplae ab 1 incipientis. Nonus enim perfectus est potestas exponentis 68 minus 1. Decimus, potestas exponentis 128, minus 1. Vndecimus denique, potestas 258, minus 1, hoc est potestas 257, vnitate decurtata, multiplicata per potestatem 256.

* *Cogitata Physico-Mathematica*, Paris, 1644, Praefatio Generalis, article 19.

Qui vndecim alios repererit, nouerit se analysim omnem, quae fuerit hactenus, superasse: memineritque interea nullum esse perfectum à 17000 potestate ad 32000; & nullum potestatum interuallum tantum assignari posse, quin detur illud absque perfectis. Verbi gratia, si fuerit exponens 1050000, nullus erit numerus progressionis duplae vsque ad 2090000, qui perfectis numeris seruiat, hoc est qui minor vnitate, primus existat.

Vnde clarum est quàm rari sint perfecti numeri, & quàm merito viris perfectis comparentur; esseque vnam ex maximis totius Matheseos difficultatibus, praescriptam numerorum perfectorum multitudinum exhibere; quemadmodum & agnoscere num dati numeri 15, aut 20 caracteribus constantes, sint primi necne, cùm nequidem saeculum integrum huic examini, quocumque modo hactenus cognito, sufficiat.

It is evident that, if p is not a prime, then N is composite, and two or more of its factors can be written down by inspection. Hence we may confine ourselves to prime values of p. Mersenne, in effect, asserted that the only values of p, not greater than 257, which make N a prime, are 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257: to these numbers 89 and 107 must be added. I gave reasons, some years ago, for thinking that 67 here is a misprint for 61, and I assume this is so. With these corrections we have no reason to doubt the truth of the statement, but it has not been definitely established.

There are 56 primes not greater than 257. The determination of the prime or composite character of N for the 9 cases when p is less than 20 presents no difficulty: in only one of them is N composite. For 2 of the remaining 47 cases (namely, when $p=23$ and 37) the decomposition of N had been given by Fermat. For 9 of them (namely, when $p=29$, 43, 73, 83, 131, 179, 191, 239, 251) the factors of N were given by Euler. He also proved that N was prime when $p=31$. Reuschle gave the factors of N when $p=47$, and Plana the factors when $p=41$. Landry and Le Lasseur discovered the factors in 9 cases, namely, when $p=53, 59, 79, 97, 113, 151, 211$, 223, and 233. Seelhoff showed that N was prime when $p=61$, Cunningham gave the factors when $p=71, 163, 173$, and 197, Cole the factors when $p=67$, Woodall the factors when $p=181$, and Powers proved that N was prime when $p=89$ and 107. It has been asserted that the prime character of N when $p=127$ has been established, but the proof has not been published or verified.

Thus there are 15 values of p for which Mersenne's statement still awaits verification. These are 101, 103, 109, 127, 137, 139, 149, 157, 167, 193, 199, 227, 229, 241, 257. For these values N is (according to Mersenne) prime when $p = 127$ and 257, and is composite for the other values. If we admit that the character of N is known when $p = 127$, the number of cases yet to be verified is reduced to 14.

To put the matter in another way. According to Mersenne's statement (corrected by the substitution of 61 for 67 and with the addition of 89 and 107 to his list) 42 of the 56 primes less than 258 make N composite and the remaining 14 primes make N prime. In 29 out of the 42 cases in which N is said to be composite we know its factors, and in 14 cases the statement is still unverified. In 12 out of the 14 cases in which it is said that N is prime the statement has been verified, and in 2 cases it is still unverified.

From the wording of the last clause in the above quotation it has been conjectured that the result had been communicated to Mersenne, and that he published it without being aware of how it was proved. In itself this seems probable. He was a good mathematician, but not an exceptional genius. It would be strange if he established a proposition which has baffled Euler, Lagrange, Legendre, Gauss, Jacobi, and other mathematicians of the first rank; but if the proposition is due to Fermat, with whom Mersenne was in constant correspondence, the case is altered, and not only is the absence of a demonstration explained, but we cannot be sure that we have attacked the problem on the best lines.

The known results as to the prime or composite character of N, and in the latter case its smallest factor, are given in the table on the next page. The cases that remain as yet unverified are marked with an asterisk.

Before describing the methods used for attacking the problem it will be convenient to state in more detail when and by whom these results were established.

The factors (if any) of such values of N as are less than a million can be verified easily: they have been known for a long time, and I need not allude to them in detail.

TABLE OF MERSENNE'S NUMBERS.

p	*Value of* $N=2^p-1$	*Character*	*Discoverer*
1	1	prime	
2	3	prime	
3	7	prime	
5	31	prime	
7	127	prime	
11	2047 = 23 × 89	composite	
13	8191	prime	
17	131071	prime	
19	524287	prime	
23	8388607 = 47 × 178481	composite	Fermat
29	536870911 = 233 × 1103 × 2089	composite	Euler
31	2147483647	prime	Euler
37	137438953471 = 223 × 616318177	composite	Fermat
41	2199023255551 = 13367 × 164511353	composite	Plana
43	8796093022207 = 431 × 9719 × 2099863	composite	Euler
47	2351 × 4513 × 13264529	composite	Reuschle
53	6361 × 69431 × 20394401	composite	Landry
59	179951 × 3203431780337	composite	Landry
61	2305843009213693951	prime	Seelhoff
67	≡ 0 (193707721)	composite	Cole
71	≡ 0 (228479)	composite	Cunningham
73	≡ 0 (439)	composite	Euler
79	≡ 0 (2687)	composite	Le Lasseur
83	≡ 0 (167)	composite	Euler
89	618970019642690137449562111	prime	Powers
97	≡ 0 (11447)	composite	Le Lasseur
101	2535301200456458802993406410751	*	*
103	10141204801825835211973625643007	*	*
107	162259276829213363391578010288127	prime	Powers
109	649037107316853453566312041152511	*	*
113	≡ 0 (3391)	composite	Le Lasseur
127	170141183460469231731687303715884105727	*	*
131	≡ 0 (263)	composite	Euler
137	....	*	*
139	....	*	*
149	....	*	*
151	≡ 0 (18121)	composite	Le Lasseur
157	....	*	*
163	≡ 0 (150287)	composite	Cunningham
167	....	*	*
173	≡ 0 (730753)	composite	Cunningham
179	≡ 0 (359)	composite	Euler
181	≡ 0 (43441)	composite	Woodall
191	≡ 0 (383)	composite	Euler
193	....	*	*
197	≡ 0 (7487)	composite	Cunningham
199	....	*	*
211	≡ 0 (15193)	composite	Le Lasseur
223	≡ 0 (18287)	composite	Le Lasseur
227	....	*	*
229	....	*	*
233	≡ 0 (1399)	composite	Le Lasseur
239	≡ 0 (479)	composite	Euler
241	....	*	*
251	≡ 0 (503)	composite	Euler
257	....	*	*

The factors of N when $p = 11$, 23, and 37 had been indicated by Fermat *, some four years prior to the publication of Mersenne's work, in a letter dated October 18, 1640. The passage is as follows:

En la progression double, si d'un nombre quarré, généralement parlant, vous ôtez 2 ou 8 ou 32 &c., les nombres premiers moindres de l'unité qu'un multiple du quaternaire, qui mesureront le reste, feront l'effet requis. Comme de 25, qui est un quarré, ôtez 2; le reste 23 mesurera la 11e puissance -1; ôtez 2 de 49, le reste 47 mesurera la 23e puissance -1. Ôtez 2 de 225, le reste 223 mesurera la 37e puissance -1, &c.

Factors of N when $p = 29$, 43, and 73 were given by Euler† in 1732 or 1733. The fact that N is composite for the values $p = 83$, 131, 179, 191, 239, and 251 follows from a proposition enunciated, at the same time, by Euler to the effect that, if $4n + 3$ and $8n + 7$ are primes, then $2^{4n+3} - 1 \equiv 0 \pmod{8n + 7}$. This was proved by Lagrange ‡ in his classical memoir of 1775. The proposition also covers the cases of $p = 11$ and $p = 23$. This is the only general theorem on the subject which has been yet established.

The fact that N is prime when $p = 31$ was proved by Euler§ in 1771. Fermat had asserted, in the letter mentioned above, that the only possible prime factors of $2^p \pm 1$, when p was prime, were of the form $np + 1$, where n is an integer. This was proved by Euler‖ in 1748, who added that, since $2^p \pm 1$ is odd, every factor of it must be odd, and therefore if p is odd n must be even. But if p is a given number we can define n much more closely, and Euler showed that, if $p = 31$, the prime factors (if any) of N were necessarily primes of the form $248n + 1$ or $248n + 63$; also they must be less than $\sqrt{N}$, that is, than

* *Oeuvres de Fermat*, Paris, vol. II, 1894, p. 210; or *Opera Mathematica*, Toulouse, 1679, p. 164; or Brassinne's *Précis*, Paris, 1853, p. 144.

† *Commentarii Academiae Scientiarum Petropolitanae*, 1738, vol. VI, p. 105; or *Commentationes Arithmeticae Collectae*, vol. I, p. 2.

‡ *Nouveaux Mémoires de l'Académie des Sciences de Berlin*, 1775, pp. 323—356.

§ *Histoire de l'Académie des Sciences* for 1772, Berlin, 1774, p. 36. See also Legendre, *Théorie des Nombres*, third edition, Paris, 1830, vol. I, pp. 222—229.

‖ *Novi Commentarii Academiae Scientiarum Petropolitanae*, vol. I, p. 20; or *Commentationes Arithmeticae Collectae*, Petrograd, 1849, vol. I, pp. 55, 56.

46339. Hence it is necessary to try only forty divisors to see if N is prime or composite.

The factors when $p=47$; the factor 1433 when $p=179$, and the factor 1913 when $p=239$, were given by Reuschle in 1856*.

The factors of N when $p=41$ were given by Plana † in 1859. He showed that the prime factors (if any) are primes of the form $328n+1$ or $328n+247$, and lie between 1231 and $\sqrt{N}$, that is, 1048573. Hence it is necessary to try only 513 divisors to see if N is composite: the seventeenth of these divisors gives the required factors. This is the same method of attacking the problem which was used by Euler in 1771, but it would be laborious to employ it for values of p greater than 41. Plana ‡ added the forms of the prime divisors of N, if p is not greater than 101.

That N is prime when $p=127$ seems to have been verified by Lucas § in 1876 and 1877. The demonstration has not been published.

The discovery of factors of N for the values $p=53$ and 59 is due apparently to F. Landry, who established theorems on the factors (if any) of numbers of certain forms. He seems to have communicated his results to Lucas, who quoted them in the memoir cited below ||.

Factors of N when $p=79$ and 113 were given first by Le Lasseur, and were quoted by Lucas in the same memoir ||.

A factor of N when $p=233$ was discovered later by Le Lasseur, and was quoted by Lucas in 1882 ¶.

* C. G. Reuschle, *Neue Zahlentheoretische Tabellen*, Stuttgart, 1856, pp. 21, 22, 42—53.

† G. A. A. Plana, *Memorie della Reale Accademia delle Scienze di Torino*, Series 2, vol. XX, 1863, p. 130.

‡ *Ibid.*, p. 137.

§ *Sur la Théorie des Nombres Premiers*, Turin, 1876, p. 11; and *Recherches sur les Ouvrages de Léonard de Pise*, Rome, 1877, p. 26, quoted by A. J. C. Cunningham, *Proceedings of the London Mathematical Society*, Nov. 14, 1895, vol. XXVII, p. 54.

|| *American Journal of Mathematics*, 1878, vol. I, pp. 234—238.

¶ *Récréations*, 1882-3, vol. I, p. 241.

Factors of N when $p=97$, 151, 211, and 223 were determined subsequently by Le Lasseur, and were quoted by Lucas* in 1883.

That N is prime when $p=61$ had been conjectured by Landry and in 1886 a demonstration was offered by Seelhoff†. His demonstration is open to criticism, but the fact has been verified by others‡, and may be accepted as proved.

Cunningham showed in 1895§ that 7487 is a factor of N when $p=197$: in 1908|| that 150287 is a factor of N when $p=163$; in 1909¶ that 228479 is a factor of N when $p=71$; and in 1912** that 730753 is a factor of N when $p=173$. The factors of N when $p=71$ were discussed independently by Ramesam in 1912††.

That N is not prime when $p=67$ seems to have been verified by Lucas‡‡ in 1876 and 1877. The composite nature of N, when $p=67$, was confirmed by E. Fauquembergue§§, and was also implied by Lucas in 1891. The factors were given by Cole|||| in 1903.

A factor of N when $p=181$ was discovered by Woodall in 1911¶¶.

* *Récréations*, 1882–3, vol. II, p. 230.

† P. H. H. Seelhoff, *Zeitschrift für Mathematik und Physik*, 1886, vol. XXXI, p. 178.

‡ See Weber-Wellstein, *Encyclopaedie der Elementar-Mathematik*, p. 48; and F. N. Cole, *Bulletin of the American Mathematical Society*, December, 1903, p. 136.

§ A. J. C. Cunningham, *Proceedings of the London Mathematical Society*, March 14, 1895, vol. XXVI, p. 261.

|| *Ibid.*, April 30, 1908, vol. VI (2nd series), p. xxii.

¶ *L'Intermédiaire des Mathématiciens*, Paris, 1909, vol. XVI, p. 252.

** *Proceedings of the London Mathematical Society*, April 11, 1912, vol. XI, p. xxiv.

†† *Nature*, London, March 28, 1912, vol. LXXXIX, p. 87.

‡‡ *Sur la Théorie des Nombres Premiers*, Turin, 1876, p. 11, quoted by A. J. C. Cunningham, *Proceedings of the London Mathematical Society*, Nov. 14, 1895, vol. XXVII, p. 54, and *Recherches sur les Ouvrages de Léonard de Pise*, Rome, 1877, p. 26.

§§ *L'Intermédiaire des Mathématiciens*, Paris, Sept. 1894, vol. I, p. 148.

|||| F. N. Cole, "On the Factoring of Large Numbers," *Bulletin of the American Mathematical Society*, December, 1903, pp. 134—137.

¶¶ H. J. Woodall, *Nature*, London, July 20, 1911, vol. LXXXVII, p. 78.

That N is prime when $p = 89$ and $p = 107$ was proved by Powers in 1911 and 1914 respectively*.

Bickmore in the memoir† cited below showed that 5737 is another factor of N if $p = 239$. Cunningham has also shown that 55871 is another factor of N if $p = 151$, and that 54217 is another factor of N if $p = 251$.

I turn next to consider the methods by which these results can be obtained. It is impossible to believe that the statement made by Mersenne rested on an empirical conjecture, but the puzzle as to how it was discovered is still, after more than 250 years, unsolved.

I cannot offer any solution of the riddle. But it may be interesting to indicate some ways which have been used in attacking the problem. The object is to find a prime divisor q (other than N and 1) of a number N when N is of the form $2^p - 1$ and p is a prime.

I may observe that Lucas‡ showed that if we find the residue (mod N) of each term of the series 4, 14, 194, ... u_p, constructed according to the law $u_{n+1} = u_n^2 - 2$, then N is prime if the first residue which is zero lies between the $(p-1)/2$th and the pth residues. If an earlier residue is zero the theorem does not help us, but if none of the p residues is zero, N is composite. The application of the theorem to high numbers is so laborious that for the cases still unverified we are driven to seek other methods.

It can be easily shown that the prime divisor q must be of the form $2pt + 1$. Also q must be of one of the forms $8i \pm 1$: for N is of the form $2A^2 - B^2$, where A is even and B odd, hence§ any factor of it must be of the form $2a^2 - b^2$, that is, of the form $8i \pm 1$, and 2 must be a quadratic residue of q. The theory of residues is, however, of but little use in finding

* R. E. Powers, *American Mathematical Monthly*, November, 1911, vol. XVIII, pp. 195—197. *Proceedings of the London Mathematical Society*, 11 June 1919, series 2, vol. XIII, p. xxxix.

† C. E. Bickmore, *Messenger of Mathematics*, Cambridge, 1895, vol. xxv, p. 19.

‡ *American Journal of Mathematics*, 1878, vol. I, p. 316.

§ Legendre, *Théorie des Nombres*, third edition, Paris, 1830, vol. I, § 143. In the case of Mersenne's numbers, $B = b = 1$.

factors of the cases that still await solution, though the possibility some day of finding a complete series of solutions by properties of residues must not be neglected*. Our present knowledge of the means of factorizing N has been summed up in the statement† that a prime factor of the form $2pt+1$ can be found directly by rules due to Legendre, Gauss, and Jacobi, when $t=1, 3, 4, 8$, or 12; and that a factor of the form $2ptt'+1$ can be found indirectly by a method due to Bickmore when $t=1, 3, 4, 8$, or 12, and t' is an odd integer greater than 3. But this only indicates how little has yet been done towards finding a general solution of the problem.

First. There is the simple but crude method of trying all possible prime divisors q which are of the form $2pt+1$ as well as of one of the forms $8i \pm 1$.

The chief known results for the smaller factors may be summarized by saying that a prime of this form, when t is odd, will divide N when $t=1$, if $p=11, 23, 83, 131, 179, 191, 239$, or 251; when $t=3$, if $p=37, 73$, or 233; when $t=5$, if $p=43$; when $t=15$, if $p=113$; when $t=17$, if $p=79$; when $t=19$, if $p=29$, or 197; when $t=25$, if $p=47$; when $t=41$, if $p=223$; when $t=59$, if $p=97$; when $t=163$, if $p=41$; when $t=461$, if $p=163$; when $t=1525$, if $p=59$; when $t=1609$, if $p=71$. Similarly for even values of t, a prime of this form will divide N when $t=4$, if $p=11, 29, 179$, or 239; when $t=8$, if $p=11$; when $t=12$, if $p=239$; when $t=36$, if $p=29$, or 211; when $t=60$, if $p=53$, or 151; when $t=120$, if $p=181$; when $t=2112$, if $p=173$; and when $t=1445580$, if $p=67$.

Of the 29 cases in which we know that the statement of the composite character of N is correct all save 7 can be easily verified by trial in this way. For, neglecting all values of t not

* For methods of finding the residue indices of 2 see Bickmore, *Messenger of Mathematics*, Cambridge, 1895, vol. xxv, pp. 15—21; see also A. J. C. Cunningham on 2 as a 16-ic residue, *Proceedings of the London Mathematical Society*, 1895-6, vol. xxvii, pp. 85—122; and on Haupt-exponents of 2, *Quarterly Journal of Pure and Applied Mathematics*, Cambridge, 1906, vol. xxxvii, pp. 122—145.

† *Transactions of the British Association for the Advancement of Science* (Ipswich Meeting), 1895, p. 614.

exceeding, say, 60 which make q either composite or not of one of the forms $8i \pm 1$, we have in each case to try only a comparatively small number of divisors. Of the 7 other cases in which Mersenne's statement of the composite character of N has been verified, one verification ($p = 41$) is due to Plana, who frankly confessed that the result was reached "par un heureux hasard"; and a second is due to Landry ($p = 59$), who did not explain how he obtained the factors. The third is due to Cole ($p = 67$), who established it by the use of quadratic residues of N, three others ($p = 71$, 163, and 173) are due to Cunningham, and one is due to Woodall. The last five verifications involved laborious numerical work, and it is possible that the results would have been obtained as easily by trial of prime divisors of the form $2pt + 1$.

Of the 12 cases in which we know that the statement of the prime character of N is correct all save three (namely, when $p=61$, $p=89$ and $p=107$) may be verified by trial in this way, for the number of possible factors is not large.

Thus practically we may say that simple empirical trials would at once lead us to all the conclusions known except in the cases of $p=41$ due to Plana, of $p=59$ due to Landry, of $p=61$ due to Seelhoff, of $p=67$ due to Cole, of $p=71$, 163, and 173 due to Cunningham, of $p=181$ due to Woodall, and of $p=89$ and 107 due to Powers. In fact, save for these ten results the conclusions of all mathematicians to date could be obtained by anyone by a few hours' arithmetical work.

As p increases the number of factors to be tried increases so fast that, if p is large, it would be practically impossible to apply the test to obtain large factors. This is an important point, for Colonel Cunningham has stated that in the cases still awaiting verification there are no factors less than 1,000,000. Hence, we may take it as reasonably certain that this cannot have been the method by which the result was originally obtained; nor, as here enunciated, is it likely to give many factors not yet known. Of course it is possible there may be ways by which the number of possible values of t might be further limited, and if we could find them we might thus diminish the number of possible factors to be tried, but it will be observed that the values of N which still await verification are very large, for instance, when $p=257$, N contains no less than 78 digits.

It is hardly necessary to add that if q is known and is not very large we can determine whether or not it is a factor of N without the labour of performing the division.

For instance, if we want to verify that $q=13367$ is a factor of N when $p=41$, we proceed thus. Take the power of 2 nearest to q or to its square-root. We have, to the modulus q,

$$
\begin{aligned}
& 2^{14} &&= 16384 \equiv 3017 \equiv 7 \times 431,\\
\therefore\ & 2^{28} &&\equiv 49(-1377) \equiv -638,\\
\therefore\ & 2^{27} &&\equiv -319,\\
\therefore\ & 2^{14+27} &&\equiv (3017)(-319) \equiv 1,\\
\therefore\ & 2^{41} &&\equiv 1.
\end{aligned}
$$

Second. We can proceed by reducing the problem to the solution of an indeterminate equation.

It is clear that we can obtain a factor of N if we can express it as the difference of the squares, or more generally of the nth powers, of two integers u and v. Further, if we can express a multiple of N, say mN, in this form, we can find a factor of mN and (with certain obvious limitations as to the value of m) this will lead to a factor of N. It may be also added that if m can be found so that N/m is expressible as a continued fraction of a certain form, a certain continuant* defined by the form of the continued fraction is a factor of N.

Since N can always be expressed as the difference of two squares, this method seems a natural one by which to attack the problem. If we put

$$N = n^2 + a = (n+b)^2 - (b^2 + 2bn - a),$$

we can make use of the known forms of u and v, and thus

* See J. G. Birch in the *Messenger of Mathematics*, Cambridge, 1902, vol. XXII, pp. 52—55.

obtain an indeterminate equation between two variables x and y of the form

$$x^2 = (2py + H)^2 - 4(K - y),$$

where H and K are numbers which can be easily calculated. Integral values of x and y where $y < K$ will determine values of u and v, and thus give factors of N.

We can also attack the problem by indeterminate equations in another way. For the factors must be of the form $2pt + 1$ and $8ps + 1$, hence

$$\begin{aligned}(2pt+1)(8ps+1) &= N \\ &= 2^p - 1 \\ &= 2(2^{p-1} - 1) + 1, \\ \therefore \quad 4s + t + 8pst &= (2^{p-1} - 1)/p \\ &= (\text{say})\ \alpha + 8p\beta.\end{aligned}$$

Hence $\qquad 4s + t = \alpha + 8px,$ and $st = \beta - x,$

where $x \ngtr \beta$ and t is odd. These results again lead to an indeterminate equation.

But, in both cases, unless p is small, the resulting equations are intractable.

Third. A not uncommon method of attacking problems such as this, dealing with the factorization of large numbers, is through the theory of quadratic forms*. At best this is a difficult method to use, and we have no reason to think that it would have been employed by a mathematician of the seventeenth century I here content myself with alluding to it.

Fourth. There is yet another way in which the problem might be attacked. The problem will be solved if we can find an odd prime q so that to it as modulus $2^{p+y} \equiv z$, and $2^y \equiv z$, where y and z may have any values we like to choose. If such values of q, y, and z can be found, we have $2^y(2^p - 1) \equiv 0$. Therefore $2^p \equiv 1$, that is, q is a divisor of N.

* For a sketch of this see G. B. Mathews, *Theory of Numbers*, part 1, Cambridge, 1891, pp. 261—271. See also F. N. Cole's paper, "On the Factoring of Large Numbers," *Bulletin of the American Mathematical Society*, December, 1903, pp. 134—137; and *Quadratic Partitions* by A. J. C. Cunningham, London, 1904.

For example, to the modulus 23, we have

$$2^8 \equiv 3,$$

$$2^{16} \equiv 3^2.$$

Also $$2^5 \equiv 3^2.$$

Therefore $$2^{16} - 2^5 \equiv 0,$$

$$\therefore\ 2^{11} - 1 \equiv 0.$$

Without going further into the matter we may say that the *à priori* determination of the values of q, y, and z introduces us to an almost untrodden field. It is just possible (though I should suppose unlikely) that the key to the riddle is to be found on methods of finding q, y, z, to satisfy the above conditions. For instance, if we could say what was the remainder when 2^x was divided by a prime q of the form $2pt+1$, and if the remainders were the same when $x=u$ and $x=v$, then to the modulus q we should have $2^u \equiv 2^v$, and therefore $2^{u-v} \equiv 1$.

It should however be noted that Jacobi's *Canon Arithmeticus* and the similar canon drawn up by Cunningham would, if carried far enough, enable us to solve the problem by this method. Cunningham's *Canon* gives the solution of the congruence $2^x \equiv R$ for all prime moduli less than 1000, but it is of no use in determining factors of N larger than 1000. It is however possible that if R or q have certain forms an extended canon of this kind might be constructed, and thus lead to a solution of the problem

Fifth. It is noteworthy that the odd values of p specified by Mersenne are primes of one of the four forms $2^q \pm 1$ or $2^q \pm 3$, but it is not the case that all primes of these forms make N a prime, for instance, N is composite if $p = 2^3 + 3 = 11$ or if $p = 2^5 - 3 = 29$.

This fact has suggested to more than one mathematician the possibility that some test as to the prime or composite character of N when p is of one of these forms may be discoverable. Of course this is merely a conjecture. There is however this to say for it, that we know that Fermat* had paid attention to numbers of this form.

* For instance, see above, pp. 39, 40.

Sixth. The number N when expressed in the binary scale consists of 1 repeated p times. This has suggested whether the work connected with the determination of factors of N might not with advantage be expressed in the binary scale. A method based on the use of properties of this scale has been indicated by G. de Longchamps*, but as there given it would be unlikely to lead to the discovery of large divisors. I am, however, inclined to think that greater advantages would be gained by working in a scale whose radix was $4p$ or may-be $8p$—the resulting numbers being then expressed by a reasonably small number of digits. In fact when expressed in the latter scale in only one out of the cases in which the factors of N are known does the smallest factor contain more than two digits.

Seventh. I have reserved to the last the description of the method which seems to me to be the most hopeful.

We know by Fermat's Theorem that if $x+1$ is a prime then 2^x-1 is divisible by $x+1$. Hence if $2pt+1$ is a prime we have, to the modulus $2pt+1$,

$$2^{2pt}-1\equiv 0,$$

$$\therefore\ (2^p-1)(1+2^p+2^{2p}+\ldots+2^{(2t-1)p})\equiv 0.$$

Hence, a divisor of 2^p-1 will be known, if we can find a value of t such that $2pt+1$ is prime and the second factor is prime to it.

This method may be used to establish Euler's Theorem of 1732. For if we put $t=1$, and if $2p+1$ is a prime, we have, to the modulus $2p+1$,

$$(2^p-1)(2^p+1)\equiv 0.$$

Hence $2^p\equiv 1$ if 2^p+1 is prime to $2p+1$. This is the case if $p=4m+3$. Hence $2p+1$ is a factor of N if $p=11$, 23, 83, 131, 179, 191, 239, and 251, for in these cases $2p+1$ is prime.

The problem of Mersenne's Numbers is a particular case of the determination of the factors of a^n-1. This has been

* *Comptes Rendus de l'Académie des Sciences*, Paris, 1877, vol. LXXXV, pp. 950—952.

the subject of investigations by many mathematicians: an outline of their conclusions has been given by Bickmore*. I ought also to add a reference to the general method suggested by Lawrence† for the factorization of any high number: it is possible that Fermat used some method analogous to this.

Finally, I should add that machines‡ have been devised for investigating whether a number is prime, but I do not know that any have been constructed suitable for numbers as large as those involved in the numbers in question.

* *Messenger of Mathematics*, Cambridge, 1895-6, vol. xxv, pp. 1—44; also 1896-7, vol. xxvi, pp. 1—38; see also a note by Mr E. B. Escott in the *Messenger*, 1903-4, vol. xxxiii, p. 49.

† F. W. Lawrence, *Messenger of Mathematics*, Cambridge, 1894-5, vol. xxiv, pp. 100—109; *Quarterly Journal of Mathematics*, Cambridge, 1896, vol. xxviii, pp. 285—311; and *Transactions of the London Mathematical Society*, May 13, 1897, vol. xxviii, pp. 465—475.

‡ F. W. Lawrence, *Quarterly Journal of Mathematics*, Cambridge, 1896, already quoted, pp. 310—311; see also C. A. Laisant, *Comptes Rendus*, *Association Français pour l'Avancement des Sciences*, 1891 (Marseilles), vol. xx, pp. 165—168.

CHAPTER XVII.

STRING FIGURES.

An amusement of considerable antiquity consists in the production of figures, known as *Cat's-Cradles*, by twisting or weaving on the hands an endless loop of string, say, from six to seven feet long. The formation of these figures is a fascinating recreation with an interesting history. It cannot, with accuracy, be described as mathematical, but as I deliberately gave this book a title which might allow me a free hand to write on what I liked, I propose to devote a chapter to an essay on certain string figures*. The subject is extensive. I propose however merely to describe the production of a few of the more common forms, and do not concern myself with their ethnographical aspects. Should, as I hope, some of my readers find the results interesting, they may serve as an introduction to innumerable other forms which, with a little ingenuity, can be constructed on similar lines.

First we must note that there are two main types of the string figures known as Cat's-Cradles. In one, termed the European or Asiatic Variety, common in England and parts of Europe and Asia, there are two players one of whom, at each move, takes the string from the other. In this, the more

* For my knowledge of the subject I am mainly indebted to Dr A. C. Haddon, of Cambridge; to *String Figures* by C. F. Jayne, New York, 1906; and to articles by W. I. Pocock and others in *Folk-Lore*, and the *Journal of the Anthropological Society*. Since writing this chapter I have come across another book on the subject by K. Haddon, London, 1911: it contains descriptions of fifty Figures and a dozen String Tricks.

usual forms produced are supposed to suggest the creations of civilized man, such as cradles, trays, dishes, candles, &c. In the other, termed the Oceanic Variety, common among the aborigines of Oceania, Africa, Australasia and America, there is generally (but not always) only one player. In this, the more usual forms are supposed to represent, or be connected with, natural objects, such as the sun and moon, lightning, clouds, animals, &c., and on the whole these are more varied and interesting than those of the European type. There are a large number of known forms of each species, and it is easy to produce additional forms as yet undescribed. We can pass from a figure of the European type to one of the Oceanic type and vice versa, but it is believed that this transformation is an invention of recent date and has no place in the history of the game.

To describe the construction of these figures we need an accurate terminology. The following terms, introduced by Rivers and Haddon*, are now commonly used. The part of a string which lies across the palm of the hand is described as *palmar*, the part lying across the back of the hand as *dorsal*. The part of a string passed over a thumb, finger or fingers is a *loop*. A loop is described or distinguished by the projection—such as the thumb, index, middle-finger, ring-finger, or little-finger of either the right or the left hand—over which it passes. The part of the loop on the thumb side of a loop is termed *radial*, the part on the little-finger side is called *ulnar*; thus each loop is composed of a radial string and an ulnar string. If, as is not uncommon, the figure is held by some one, with his hands held apart, palm facing palm and the fingers pointing upwards, then the radial string of any loop is that nearest him, and the ulnar string is that farthest from him; in this case we may use the terms *far* and *near* instead of ulnar and radial. If there are two or more loops on one finger (or other object), the one nearest the root of the finger is termed *proximal*, the one nearest the tip or

* *Torres Straits String Figures*, by W. H. R. Rivers and A. C. Haddon, *Man*, London, 1902, pp. 147, 148.

free end of the finger is termed *distal*. If the position of the hands is unambiguous, it is often as clear to speak of taking up a string from above or below it as to say that we take it from the distal or proximal side.

The following descriptions are I believe sufficient to enable anyone to construct the figures, and I do not attempt to make them more precise. They are long, but this is only because of the difficulty of explaining the movements in print, and the figures are produced much more easily than might be inferred from the elaborate descriptions. Unless the contrary is stated all of them are made with a piece of string, some seven feet long, whose ends are tied together. In the descriptions I assume that after every movement the hands are separated so as to draw the strings tight. In the diagrams the string is represented by two parallel lines: this enables us to indicate whether one string appears in front of or behind another. Unless otherwise stated the diagrams show the figures as seen by the operator.

I recommend any one desirous of making the Figures and not already acquainted with the subject to commence with the Oceanic Varieties described on pp. 357 to 371, where only one operator is required.

Cat's-Cradles, European Varieties. I begin by discussing the form of Cat's-Cradle known in so many English nurseries. It is typical of the European Variety. It is played by two persons, P and Q, each of whom in turn takes the string off the fingers of the other. In the following descriptions, the terms near and far, or radial and ulnar, refer to the player from whom the string is being taken. In most of the forms two or more parallel horizontal strings are stretched between the hands of the player who is holding the string, and the other strings cross one another forming an even number of crosses. When there are more than two such crosses, two will usually be at the ends of the figure near the hands, and two at the sides nearer the middle of the figure. Parallel strings, if taken up, are usually taken up by hooking each little-finger in one of them. A cross, if taken up, is usually taken by inserting the thumb and index-finger of one hand in opposite angles of the cross, holding the cross with the thumb and finger, and then turning the plane containing the thumb and finger through one or two right angles. When one player takes the strings from the other, it is assumed that he draws his hands apart so as to keep the string stretched.

The initial figure is termed the *Cradle*; from this we can produce *Snuffer-Trays.* From Snuffer-Trays we can obtain forms known as a *Pound-of-Candles, Cat's-Eye,* and *Trellis-Bridge.* From each of these forms again we can proceed in various ways. I will describe first the figures produced when the string is taken off so as to lead successively from the Cradle to Snuffer-Trays, Cat's-Eye, and Fish-in-a-Dish. This is the normal sequence.

The *Cradle*—see figure i—is formed by six loops, three on each hand: there are two horizontal strings, one near and the other far, and two pairs of strings which cross one another, each cross being over one of the horizontal strings.

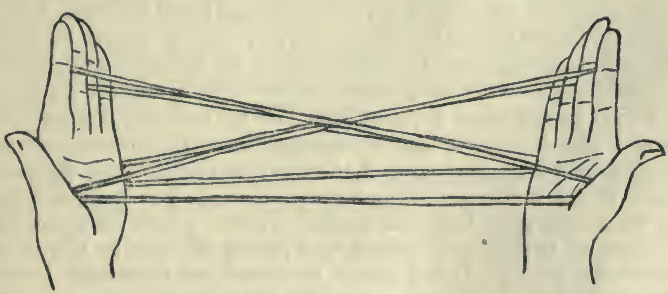


Figure i. *The Cradle.*

The *Cradle* is produced thus. *First.* One of the operators, *P*, loops the string over the four fingers of each hand, which are held upright, palm towards palm, the near or radial string lying between the thumbs and index-fingers, and the far or ulnar string beyond the little fingers. *Second.* *P* puts a second loop on the right hand by bending it over outside the radial string and up into the loop. There are now two dorsal strings and one palmar string on the right hand. *Third.* *P* puts a similar loop on the left hand by bending it over outside the radial string and up into the space between the hands. *Lastly.* *P* with the back of the right middle-finger takes up from the proximal side (*i.e.* from below) the left palmar string. And then similarly with the back of the left middle-finger takes up from the proximal side that part of the right palmar string which lies across the base of the right middle-finger. The hands are now drawn apart so as to make the strings taut.

The figure known as *Snuffer-Trays*—see figure ii—is formed by six loops, three on each hand. Four of the strings cross diamond-wise in the middle so as to form two side crosses and two end crosses, the other two strings are straight, one being near and the other far. All the strings lie in a horizontal plane. This figure is also known as *Soldier's Bed,* the *Church Window,* and the *Fish-Pond.*

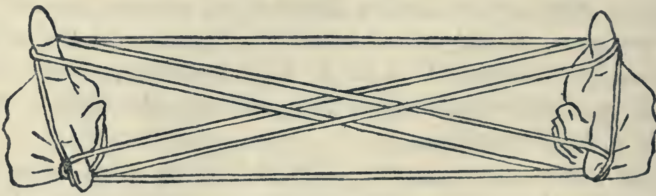


Figure ii. *Snuffer-Trays.*

Snuffer-Trays can be got from the Cradle thus. I suppose that the Cradle is held by one operator, *P*, and that the other operator, *Q*, faces *P*. *First.* *Q* inserts the thumb and index-finger of his left hand from his far side in those angles of the cross of the Cradle farthest from him which lie towards *P*'s hands. Similarly *Q* inserts the thumb and index-finger of his right hand from his near side in those angles of the cross nearest to him which lie towards *P*'s hands, both thumbs being towards *P*'s right hand, and his index-fingers towards *P*'s left hand. *Second.* *Q* takes hold of each cross by the tips of the thumb and finger, then pulls each cross outwards (*i.e.* away from the centre of the figure) over and beyond the corresponding horizontal string, then down, and then round the corresponding horizontal strings. *Third.* *Q* turns the thumbs and fingers upwards through a right angle, thus passing the cross up between the two horizontal strings. By this motion the thumb and index-finger of each hand (still holding the crossed strings) are brought against the horizontal strings. *Lastly.* *Q*, having pushed his fingers up, releases the crosses by separating his index-fingers from his thumbs, and drawing his hands apart removes the string from *P*'s hands. The diagram represents the figure as seen by *P*.

Cat's-Eye—see figure iii—also is formed by six loops, three on each hand. There are two near (or radial) thumb strings, two far (or ulnar) index strings, while the radial thumb and ulnar index strings form a diamond-shaped lozenge in the middle of the figure, giving rise to two side crosses and two end crosses. All the strings lie in a horizontal plane.

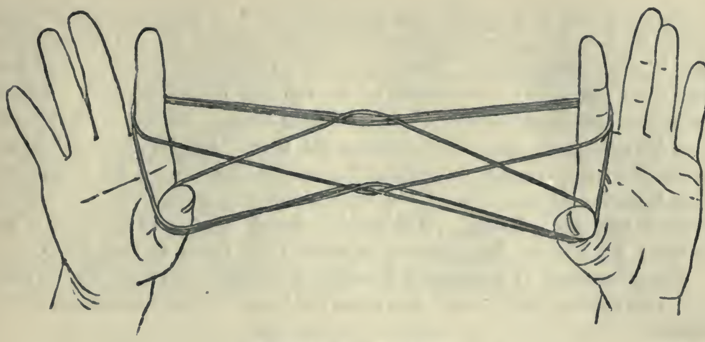


Figure iii. *Cat's-Eye.*

Cat's-Eye can be got from Snuffer-Trays thus. I suppose that Snuffer-Trays is held by *Q*. *First.* *P* inserts the thumb and index-finger of each hand from below into those angles of the two side crosses of Snuffer-Trays which lie towards *Q*'s hands, both thumbs being towards *Q*'s left hand. *Second.* *P* takes hold of each cross with the thumbs and fingers and pulls it downwards, then he separates his hands thus bringing each cross below its corresponding horizontal string, and continuing the motion he passes the cross outside, round, and then above the horizontal string. *Third.* *P* turns each thumb and finger inwards (*i.e.* towards the centre of the figure) between the two horizontal strings through two right angles taking the horizontal strings with them. *Lastly.* *P* pushes his fingers down, separates the index-fingers from the thumbs, and then, drawing his hands apart, removes the figure from *Q*'s hands.

The figure now is in a horizontal plane; but for clearness I have drawn it as seen by *Q*, when *P* lifts his hands up.

Fish-in-a-Dish—see figure iv—is composed of a central lozenge (the dish) on which rest lengthwise two strings (the fish). There are two loops on each hand.

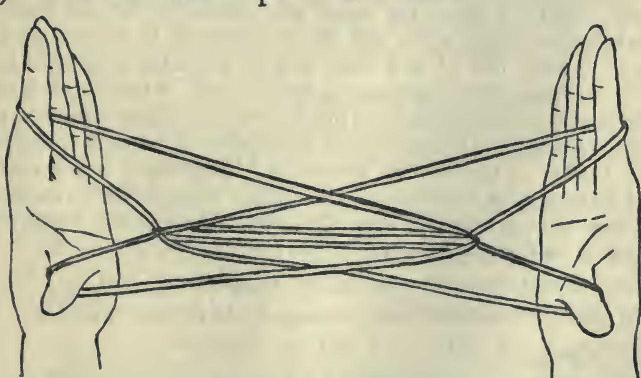


Figure iv. *Fish-in-a-Dish.*

Fish-in-a-Dish is produced from Cat's-Eye thus. I suppose that Cat's-Eye is held by *P* in the normal position, his fingers pointing down, and the figure lying in a horizontal plane. *First.* *Q* puts the thumb and index-finger of each hand from above into those angles of the two side crosses which lie towards *P*'s hands, both thumbs being towards *P*'s right hand. *Second.* *Q* turns each hand inwards (*i.e.* towards the centre of the figure) through two right angles, and as he does so, catches the sides of the central diamond on the thumb and index-fingers. At the end of the motion the thumbs and fingers will be pointing upwards. *Lastly.* *Q* draws his hands apart and thus takes the figure off *P*'s hands. The diagram represents the figure as seen by *Q*.

The same result is produced if *Q* inserts his thumbs and index-fingers from below into the two side crosses, and turns his hands inwards through two right angles.

There are a few other standard forms which may be mentioned. In the first place instead of proceeding directly from Snuffer-Trays to Cat's-Eye, we can obtain the latter figure through three intermediate forms known as a *Pound-of-Candles*, the *Hammock*, and *Lattice-Work*. The figure last named is the same as Snuffer-Trays, though it is held somewhat differently, and from it we can obtain Cat's-Eye by similar movements to those described above when forming it from Snuffer-Trays, though it will be held somewhat differently.

A *Pound-of-Candles* is formed by six parallel strings held in a horizontal plane. It is, in fact, only an extension sideways of the figure obtained in the course of the formation of the Cradle before the palmar strings are taken up; and, if desired, it can be obtained in that way.

A *Pound-of-Candles* is produced from Snuffer-Trays (assumed to be held by *Q*) thus. *First.* *P* inserts the thumb and index-finger of each hand from above into those angles of the two side crosses which lie towards *Q*'s hands. *Second.* *P* takes hold of each cross, lifts it, and takes it above, and then round the corresponding horizontal string. *Third.* *P* turns each hand upwards between the horizontal string through two right angles; this brings each horizontal string on to the thumbs and index-finger of the hand corresponding to it. *Lastly.* *P* separates his thumbs and index-fingers, and drawing his hands apart he takes the figure off *Q*'s hands. This corresponds exactly with the process above described by which Cat's-Eye is obtained from Snuffer-Trays, except that the side crosses are taken from above instead of from below.

The *Hammock* is an inverted Cradle and can be obtained directly from it without any intermediate forms. The Hammock is also known as *Baby's-Cot* and the *Manger*.

The *Hammock* can be obtained from a Pound-of-Candles (assumed to be held by *P*) thus. *First.* *Q*, holding his right-hand palm upwards, hooks with his right little-finger the ulnar thumb string from above, and then pulls it up and over the three index strings. *Second.* Q passes his left hand, held palm upwards, from above through the loop on the right little-finger, and with the little-finger of the left hand he hooks the radial index string, and pulls it up and over the pair of thumb strings. *Third.* Q places the thumb and index-finger of the right hand from the far side of P below the pair of far index strings, and similarly he places the thumb and index of the left hand from the near side of P below the pair of near thumb strings. *Fourth.* Q turns his hands inwards and upwards through two right angles; by this motion the thumbs and indices will take up these two pairs of strings. *Lastly.* Q separates his index-fingers from his thumbs, and drawing his hands apart removes the figure from P's hands.

Lattice-Work. The form known as Lattice-Work is the same as Snuffer-Trays, except that it is held with the fingers pointing downwards.

Lattice-Work can be produced from the Hammock (assumed to be held by Q) by the same process as that described above for the production of Snuffer-Trays from the Cradle, namely by P inserting the thumb and index-finger of each hand from the outside in the crosses; then pulling each cross outwards, then up and over the corresponding horizontal string; and finally turning each hand downwards through a right angle. Also, since it is the same as Snuffer-Trays, though held differently, it can be produced from Snuffer-Trays by P putting his thumbs and fingers pointing downwards in the place of those of Q which are pointing upwards.

Other standard forms are *Trellis-Bridge*, *Double-Crowns*, *Suspension-Bridge*, *Tridents*, and *See-Saw.* For the formation of these I give only a brief outline of the necessary steps.

Trellis-Bridge can be produced from Snuffer-Trays by the second player hooking his little-fingers from above in the two parallel strings, one in one string and one in the other, pulling the loops so formed over and down to opposite sides, and taking up the side crosses from above and turning the hands inwards through two right angles.

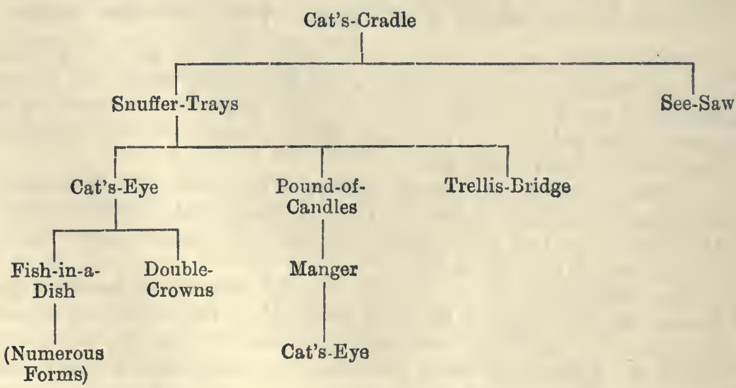
Double-Crowns can be got from Cat's-Eye by the second player putting the thumb and index-finger of each hand through the far and near angles of the end crosses, either from above or below, and making the usual turning movement inwards through two right angles.

Suspension-Bridge can be got from Fish-in-a-Dish by the second player hooking his little-fingers in the two parallel strings that form the dish, one in one string and one in the other, pulling the loops so formed up, then taking up the side crosses from above, and turning the hands inwards towards the centre of the figure through two right angles. Suspension-Bridge can be also obtained from Double-Crowns.

Tridents is produced from Suspension-Bridge by the second player releasing the left-hand thumb and index-finger, and then drawing his hands apart.

See-Saw is an arrangement of the string which each player in turn draws out. It is said that in such figures children draw the string backwards and forwards to the chant of a doggerel line *See Saw Johnnie Maw*, *See San Johnnie Man* (Jayne, p. xiii). One form of it can be obtained from the Cradle, which I will suppose is held by *P*, thus. *Q* takes one of the straight strings with the index-finger of one hand and the other straight string with the index-finger of the other hand; *P* then slips his hands out of the loops round them but retains the middle-fingers in the loops on them. If now *P* separates his hands the loops held by *Q* diminish, and vice versa. Another variety is obtained by *P* taking the ulnar base string from below in his mouth and then withdrawing his hands from the loops while retaining his middle-fingers in their respective loops: in this way we obtain a sawing figure of three loops.

The forms above described may be arranged in sequence in tabular form, as shown below. It will be noticed that parallel strings when taken up may be pulled up or down, and they may be crossed or not. So too in the horizontal figures there are often a pair of side crosses and a pair of end crosses, and either pair may be taken from above or from below, and in many cases may be crossed or not as is desired. Frequently also one player *Q* can insert two fingers in a pair of crosses and if *P* leaves go *Q* may, by drawing his hands apart, produce a new figure. By combining these motions we can obtain various forms, and can secure sequences



of them. The movements and forms described above will however illustrate the process sufficiently.

Cat's-Cradles, Oceanic Varieties. I will next enumerate a few forms of the Oceanic Variety. One or two specimens of this type are known in England, but they may be recent importations, perhaps by sailors, and not indigenous. There are so many examples of this kind, that it is difficult to select specimens, but those I have chosen will serve to illustrate the methods. In many cases the figures can be formed in more ways than one.

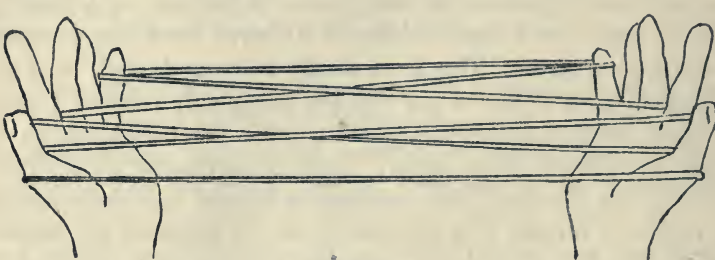
I again emphasize the fact that the figures are produced much more easily than might be inferred from the lengthy descriptions given. This is so partly because I have tried to mention every detail of the process, and partly because I constantly describe similar movements with the two hands as if they were made consecutively, whereas in practice to produce the figures effectively the movements should be simultaneous. I may add further that frequently two or more of the movements mentioned can be combined in one, and when practicable this is desirable. Also usually the more rapidly the movements are made the better is the presentation. By rotating the wrists, considerable play is given to the figure, and the movements are facilitated. Where two loops are on one finger, it is generally advisable to place one on the tip and the other on the base of the finger so as to keep them distinct.

Once a figure has been constructed or the rule given for making it understood, the brief description of the method (which in many cases I insert after the exposition of the rule) will suffice for the reproduction of the figure.

Openings A and B. The greater number of Oceanic figures begin with the same initial steps known as Opening A or Opening B. These steps I proceed to describe.

Opening A. In Opening A, the operator commences by placing the string on the left hand as follows. The hands are held with the fingers pointing upwards, and palm facing palm. The tips of the left thumb and little-finger are put together, and a loop of the string put over them. On opening the hand it will be found that the string from the far (or ulnar) side passes round the back of the little-finger, then between the ring and little-finger, then across the palm, then between the index and thumb, and then round the back of the thumb to the near (or radial) side of the hand. The string is then taken up similarly by the right hand. The hands are now drawn apart. This is called the "first position."

Next with the back of the index of the right hand take up from the proximal side (*i.e.* from below) the left palmar string, and return: in these descriptions the word *return* is used to mean a return to the position occupied at the beginning of the movement by the finger or fingers concerned. Then, similarly, with the back of the index of the left hand take up from the proximal side that part of the right palmar string which lies across the base of the right index, and return. The figure now consists of six loops on the thumb, index, and little-finger of each hand. The resulting figure, in a horizontal plane, is shown in the diagram, looking down at it from above.



Opening A.

Opening B. Opening B is obtained as above, save that, in the second part of the Opening, the right palmar string is taken up by the left index before the left palmar string is taken up by the right index. In most of the figures described below it is immaterial whether we begin with Opening A or Opening B.

Movement T. There is also another movement which is made in the construction of many of the figures and which may be described once for all.

This movement is when we have on a finger two loops, one proximal and the other distal, and the proximal loop is pulled up over the distal loop, then over the tip of the finger, and then dropped on the palmar side. I term this the *Movement T.*

A *Door.* The first example I will give is a Door—see figure v—which comes from the Apache Red Indians. It affords a good introduction to the Oceanic Varieties, for it is one of the easiest figures to construct, as the movements are simple and involve no skill in manipulation. The rubbing the hands together in the final movement has nothing to do with the formation of the figure, though it adds to its effective presentation. The diagram represents the final figure held in a horizontal plane.

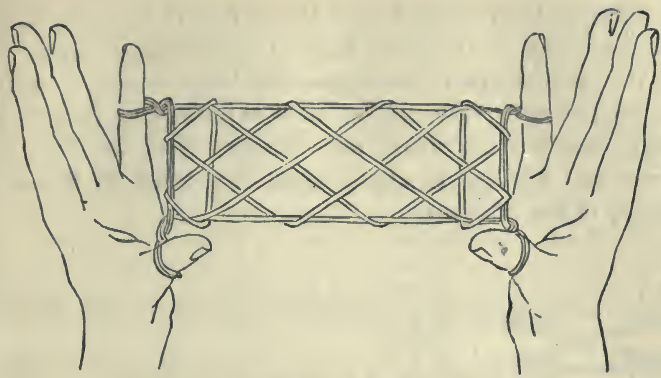


Figure v. *A Door.*

The *Door* is produced thus (Jayne, pp. 12—15). *First.* Take up the string in the form of Opening A. *Second.* With the right thumb and index lift the left index loop off that finger, put it over the left hand, and drop it on the left wrist. Make a similar movement with the other hand. *Third.* With the dorsal tip of the right thumb take up the near right little-finger string, and return. Make a similar movement with the other hand. *Fourth.* With the dorsal tip of the right little-finger take up the far right thumb string, and return. Make a similar movement with the other hand. *Fifth.* Keeping all the loops in position on both hands, with the left hand grasp tightly all the strings where they cross in the centre of the figure, and pass this bunch of strings from left to right between the right thumb and index (*i.e.* from the palmar side to the back of the hand), and let them lie on the back of the hand between the thumb and finger. Next with the left thumb and index take hold of the two loops already on the right thumb, and draw them over the tip of the right thumb. Then, continuing to hold these two loops, let the bunch of strings still lying between the right index and thumb slip over the right thumb to the palmar side, and after this replace these loops on the right thumb. Make a similar set of movements with the other hand. *Sixth.* With the right thumb and index lift the left wrist loop from the back of the left wrist up over the tips of the left thumb and fingers, and let it fall on the palmar side. Make a similar movement with the other hand. *Finally.* Rub the palms of the hands together, separate the hands, and the Door will appear.

More briefly thus. Opening A. Index strings over wrists. Each thumb over one and takes up one. Each little-finger takes up one. Thumb loops over groups of strings. Wrist loops over hands. Extend.

Climbing a Tree. I select this as another easy example starting from Opening A. The tree—see figure vi—consists of two straight strings, which slope slightly towards one another: if the figure here delineated is turned through a right angle, the straight lines representing the tree will be upright, and this is the way in which the figure is normally produced. Up this tree, some loops, which represent a boy, are made to ascend. The production is very simple, but it is interesting because the design, and nothing more, was obtained from the Blacks of Queensland, and this is a conjectural restoration, by Pocock, of the way it was produced.

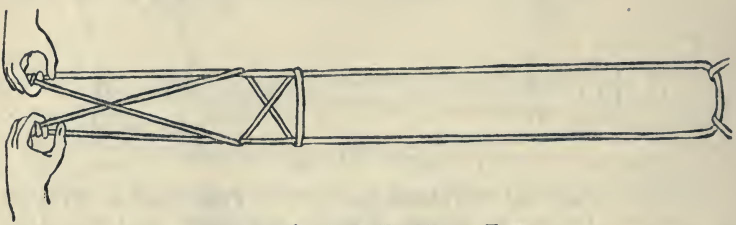


Figure vi. *Boy Climbing a Tree.*

The *Boy Climbing a Tree* may be produced thus. *First.* Take up the string in the form of Opening A. *Second.* Pass the little-fingers over four strings (namely, all the strings except the near or radial thumb string), then with the backs of the little-fingers take up the near thumb string, and return. *Third.* Make movement T on the little-finger loops. *Fourth.* Bend the right index over that part of the right palmar string which lies along the base of the finger, and press the tip of the finger on the palm. Make a similar movement with the left index. *Fifth.* Holding the strings loosely, release the thumbs and pull the index loops over the knuckles so that they hang on the cross strings. *Finally.* Put the far (or ulnar) little-finger string under one foot or on any fixed object. Then release the little-fingers, and pull steadily with the index-fingers on the strings they hold. This will cause the loop round the two straight strings to rise. The two strings starting from the foot represent the tree, and the loops, which ascend on them, as the fingers pull, represent the boy climbing up the tree. If in the fourth movement we put the middle-fingers instead of the index-fingers through the index loops, we shall slightly modify the result.

More briefly thus. Opening A. Each little-finger over 4, and take up one. T to little-finger loops. Each index on palm. Release thumbs. Index loops over knuckles. Far little-finger string under foot. Release little-fingers, and pull.

Throwing the Spear is a New Guinea figure. It is peculiar in that the design can be transferred or thrown from one hand to the other as often as is desired. The construction is very simple.

Throwing the Spear was obtained by Haddon. It is made thus (Jayne, pp. 131—132). *First.* Take up the string in the form of Opening A. *Second.*

Take the loop off the left index, put it on the right index, and pass it down the finger over the right index loop to the base of the finger. *Third.* Take off from the right index the original right index loop, release it, and extend the hands. This is *the spear*, the handle being held by the left hand.

More briefly thus. Opening A. Left index loop off finger, and then over right index loop to base of right index. Release original right index loop, and pull.

To throw the spear from one hand to the other, proceed thus. Put the left index between the left ulnar thumb string and the left radial little-finger string, then push it up from below into the right index loop. Then opening the right hand, drawing it sharply to the right, and at the same time taking the right index out of its loop, the spear is transferred to the other hand. The process can be indefinitely repeated with one hand or the other.

Diamonds. Numerous lattice-work forms have been collected in which diamonds or lozenge-shaped figures are strung in a row, or in two or more rows, between two parallel strings. I describe a few of these.

Triple Diamonds. This is an interesting figure—see figure vii—and lends itself to a catch described hereafter. It comes from the Natik natives in the Caroline Islands, but possibly is of negro origin. It is not symmetrical.

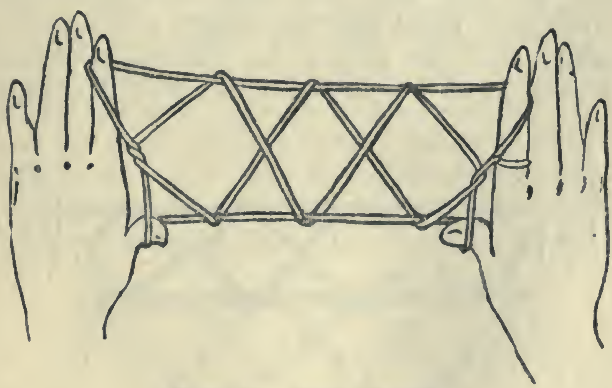


Figure vii. *Triple Diamonds.*

The formation of *Triple Diamonds* is described by Mrs Jayne (pp. 142—146), and may be effected as follows. *First.* Take up the string in the form of Opening A. *Second.* Take the right hand out of all the loops, and let the string hang straight down from the left hand, which is held upright with the fingers pointing upwards. *Third.* Put the tips of the right thumb and little-finger together, and insert them from the right side into the left index loop.

Next, separate the right thumb and little-finger, take the loop off the left index, and draw the hands apart. *Fourth.* Put the right index-finger under the left palmar string, and draw the loop out on the back of the finger. *Fifth.* Bend the right thumb over one string (viz. the near or radial right index string), take up from below on the back of the thumb the far or ulnar right index string, and return. *Sixth.* Bend the left thumb away from you over one string (viz. the far or ulnar thumb string) and take up from below on its back the near or radial little-finger string, and return. *Seventh.* With the back of the tip of the right index-finger pick up from below the near right index string, and return. *Eighth.* With the back of the tip of the left index pick up from below the far left thumb string (not the string passing across the palm), and return. These strings on the index-fingers should be kept well up at the tips by pressing the middle-fingers against them, and the radial left thumb string should now cross between the left thumb and index; if it be not in this position it should be shifted there. *Ninth.* Make movement T on the thumb loops of each hand. *Lastly.* Release the loops from the little-fingers, and extend the figure between the thumbs and the tips of the index-fingers: it is usual but not necessary, at the same time, to rotate the hands to face outwards.

More briefly thus. Opening A. Right hand out. Right little-finger and thumb through index loop. Right index takes up palmar string. Each thumb over one and takes up one. Right index picks up near right index string and left index picks up far left thumb string. T to thumb loops. Release little-fingers and extend.

Quadruple Diamonds. This design—see figure viii—was given me by a friend who was taught it when a boy in Lancashire. It is the same as one described by Mrs Jayne (pp. 24—27), which was derived by her from the Osage Red Indians.

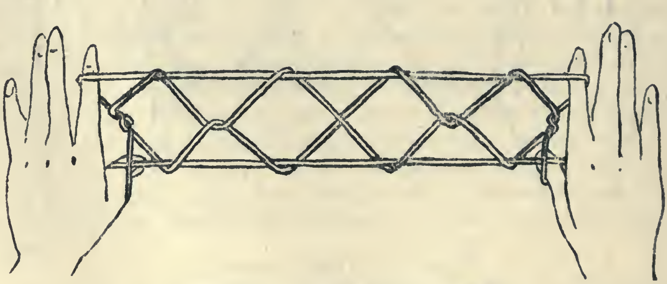


Figure viii. *Quadruple Diamonds.*

Quadruple Diamonds can be produced thus. *First.* Take up the string in the form of Opening A. *Second.* Release the thumbs. *Third.* Pass the right thumb away from you under all the strings, and take up from below with the back of the thumb the far right little-finger string, and return. Make a similar movement with the other hand. *Fourth.* Pass each thumb away from you over the near index string, and take up from below with the back of the thumb the corresponding far index string, and return. *Fifth.* Release the little-fingers. *Sixth.* Pass each little-finger towards you over one string (viz. the near index string) and take up from below on the back of the little-finger the corresponding far thumb string, and return. *Seventh.* Release the thumbs. *Eighth.* Pass each thumb away from you over the two index strings and take up from below, with the back of the thumb, the corresponding far little-finger string, and return. *Ninth.* With the right thumb and index pick up the left near index string, close to the left index and above the left palmar string, and put it over the tip of the left thumb. Next make movement T on the left thumb loops. Make a similar movement with the other hand. *Finally.* Insert each index into the small triangle near it whose sides are formed by the corresponding palmar string and its immediate continuation. Then, rotate the right hand counter-clockwise and the left hand clockwise. In making this movement the little-finger loops and the proximal index loops will fall off, and for the production of the figure it is essential they should do so. At the end of the movement the palms of the hands should be facing outwards and away from you, the thumbs lowest and pointing away from you, and the index-fingers pointing upwards. On separating the hands the *Diamonds* will appear. It is a matter of indifference whether the top string is taken on the middle-fingers or on the index-fingers. If the fourth step be taken before the third we get a form of Double Diamonds.

More briefly thus. Opening A. Let go thumbs. Each thumb under 3 and takes up one. Each thumb over one and picks up one. Release little-fingers. Each little-finger over one and picks up one. Release thumbs. Each thumb over 2 and picks up one. Near index strings on tips of thumbs. T to loops on thumbs. Index-fingers in triangle. Release little-fingers, and extend.

Multiple Diamonds. In the figures above described the diamonds are in a row. I give an instance—see figure ix—derived from the Natik natives, where the diamonds are in three rows. The reader will be easily able to introduce modifications in the process of the formation of Multiple Diamonds which will lead to other figures which he can thus invent for himself.

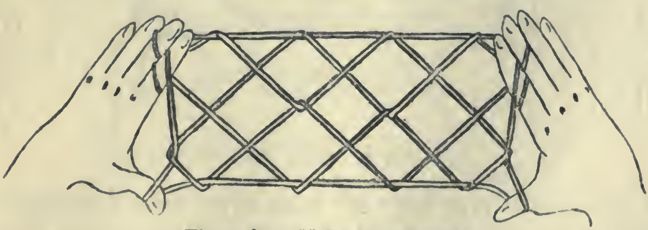


Figure ix. *Multiple Diamonds.*

Multiple Diamonds may be formed thus (Jayne, pp. 150—156). *First.* Take up the string in the form of Opening A. *Second.* With the teeth draw the far little-finger string towards you over all the strings. Then, bending the left index over the left string of the loop held by the teeth, pick up from below on the back of the index the right string of the loop, and return. Next bend the right index over to the left, and pick up from below on its back the left string of the loop, and return. Release the loop held by the teeth. *Third.* Release the thumbs. *Fourth.* Put each thumb away from you, under both index loops, and pick up on its back the near little-finger string, and return. *Fifth.* Pass each thumb up over the lower near index string, and put its tip from below into the upper index loop. *Sixth.* Make movement T on the thumb loops. *Seventh.* Withdraw each index from the loop which passes around both thumb and index. *Eighth.* Transfer the thumb loops to the index-fingers by putting each index from below into the thumb loop, and withdrawing the thumb. *Ninth.* Repeat the fourth movement, namely, put each thumb away from you under both index loops, and pick up on its back the near little-finger string, and return. *Tenth.* Repeat the fifth movement, namely, pass each thumb up over the lower near index string, and put its tip from below into the upper index loop. *Eleventh.* Make movement T on the thumb loops. *Twelfth.* Bend each middle-finger over the upper far index string, and take up from below on the back of the finger the lower near index string (*i.e.* the one passing from index to index), and return. *Lastly.* Keeping the middle and index-fingers close together, release the loops from the little-fingers, and extend the figure keeping the palms turned away from you.

Many Stars. A somewhat similar figure—see figure x—is made by the Navaho Mexican Indians, and by the Oregon Indians. The Oregon method is much the more artistic, since the movements are carried on by both hands simultaneously

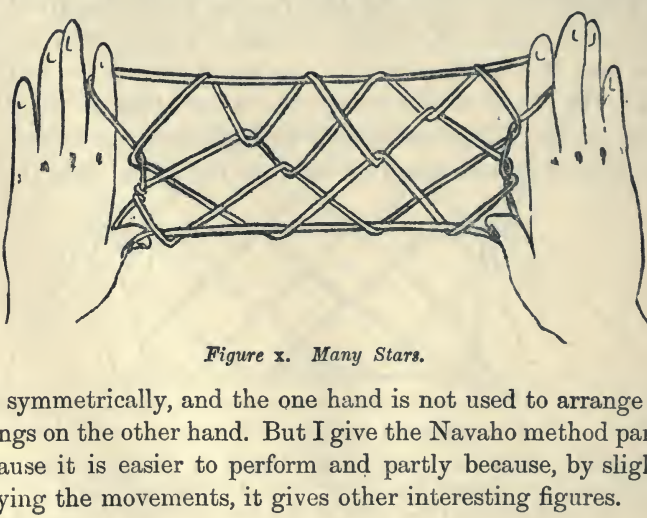


Figure x. *Many Stars.*

and symmetrically, and the one hand is not used to arrange the strings on the other hand. But I give the Navaho method partly because it is easier to perform and partly because, by slightly varying the movements, it gives other interesting figures.

Many Stars. The formation of this figure has been described by Haddon (*American Anthropologist*, vol. v, 1903, p. 222) and Mrs Jayne (pp. 48—53). It is produced thus. *First.* Take up the string in the form of Opening A. *Second.* Pass each thumb away from you over three strings (viz. the far thumb and both index strings) and pick up from below on its back the near little-finger string, and return. *Third.* Bend each middle-finger down towards you over two strings (viz. both the index strings) and take up from below on its back the far thumb string, and return. *Fourth.* Release the thumbs. *Fifth.* Pass each thumb away from you over one string (viz. the near index string), under the remaining five strings, and pick up on its back the far little-finger string, and return. *Sixth.* Release the little-fingers. *Seventh.* Take the far string of the right middle-finger loop, pass it under the near string of that loop, and then, taking it over the other strings, put it over the tips of the right thumb and index, so as to be the distal loop on them. Release the right middle-finger. Make a similar movement with the other hand. *Eighth.* Make movement T on the loops on the thumbs and index-fingers. There is now on each hand a string passing from the thumb to the index, and on each of these strings are two loops, one nearer you than the other. *Ninth.* Bend each thumb away from you over the upper string of these nearer loops. *Lastly.* Rotate the hands so that the palms face away from you, the fingers point up, and the thumbs are stretched as far from the hands as possible.

More briefly thus. Opening A. Each thumb over 3 and picks up one. Each middle-finger over 2 and picks up one. Release thumbs. Each thumb over one, under 5, and picks up one. Release little-fingers. Take up near string of each middle-finger loop, turn it over, and transfer it to tips of corresponding thumb and index-finger. T to loops on thumbs and index-fingers. Place thumbs on upper strings of near loops. Rotate, and extend.

Owls. Certain figures, called Owls, can be produced like Many Stars save for the interpolation or the alteration of one movement. Their resemblance to Owls is slight, but they

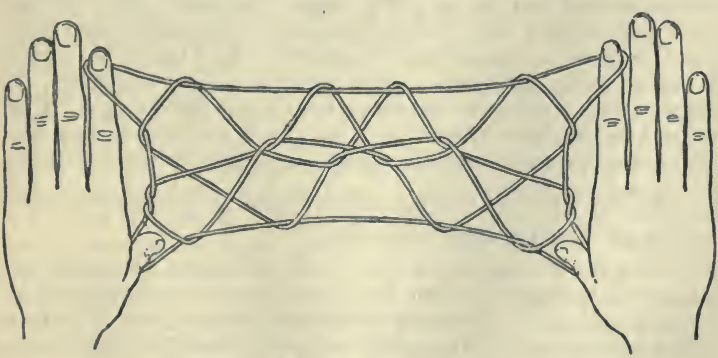


Figure xi. *An Owl.*

may be taken without much straining to represent *Bats.* One instance—see figure xi—will suffice.

The example of an *Owl* which I select is produced thus (Jayne, pp. 54—55). Immediately after taking up the string in the form of Opening A, give a twist to the index loops by bending each index down between the far index string and the near little-finger string and, keeping the loop on it, bringing it towards you up between the near index string and the far thumb string. Continue with the second and subsequent movements described in Many Stars.

In another example (Jayne, pp. 55—56) all the movements are the same as in Many Stars save that in the fifth movement the far little-finger string is drawn from above, instead of from below, through the thumb loops.

Single Stars. Other figures, which we may call Single Stars or Diamonds, can be produced like Many Stars save for the alteration or omission of one movement. One instance—see figure xii—will suffice.

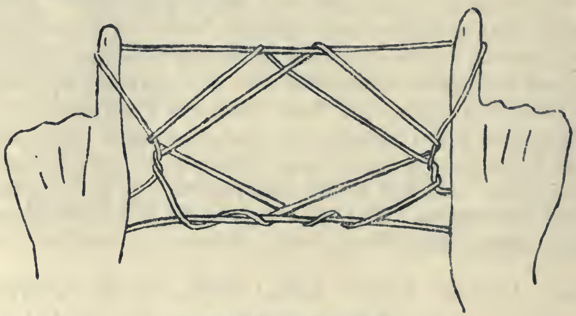


Figure xii. *North Star.*

The example of the Single Stars which I select is termed the *North Star*, and is produced thus (Jayne, p. 65). Replace the second, third, and fourth movements in Many Stars by the following. Bend each middle-finger towards you over the index loop, and take up from below on the back of the finger the far thumb string. Release the thumbs, and return the middle-fingers. The effect of this is to transfer the thumb loops to the middle-fingers. This is followed by the fifth and subsequent movements described in Many Stars. This figure may at our option be regarded as a single or double diamond.

W. W. Another elegant design, forming two interlaced W's, can be produced somewhat similarly.

This figure is made in the same way as North Star, except that after transferring the thumb loops to the middle-fingers, a twist is given to each middle-finger loop by bending each middle-finger down on the far side of the far middle-finger string and (keeping the loop on it) bringing it towards you up between the near middle-finger string and the far index string.

The reader who has followed me in my descriptions of the movements for producing Many Stars will find it easy to make other modifications which lead to other figures.

The Setting Sun is derived from the Torres Straits natives. This figure, if well done, is effective, but the process is somewhat long. The method and result illustrate another type of string figure.

The Setting Sun is produced thus (Rivers and Haddon, p. 150; Jayne, pp. 21—24). *First.* Take up the string in the form of Opening A. *Second.* Pass the little-fingers over four strings (viz. the radial or near little-finger string, the index loops, and the ulnar or far thumb string), insert them into the thumb-loops from above, take up with the backs of the little-fingers the near thumb string, and return. *Third.* Release the thumbs. *Fourth.* Pass the thumbs under the index loops, take up from below the two near strings of the little-finger loops and return, passing under the index loops. *Fifth.* Release the little-fingers. *Sixth.* Pass the little-fingers over the index loops, and take up from below the two far strings of the thumb loops and return. This arrangement is known as the *Lem Opening*.

I continue, from the Lem Opening, the movements for the production of the Setting Sun. *Seventh.* Transfer the loop on the left index to the right index; and then transfer the loop originally on the right index to the left index, by taking it over the original left index loop. *Eighth.* Pass the right middle-finger from the distal side (*i.e.* from above) through the right index loop and take up from the proximal side (*i.e.* from below) the two far thumb strings. Return the middle-finger through the index loops. Make similar movements with the other hand. *Ninth.* Release the thumbs and index-fingers. *Tenth.* Pass the thumbs from below into the middle-finger loops, and then transfer the middle-finger loops to the thumbs. Extend the figure with the thumbs towards the body. There will now be in the middle of the figure a triangle whose apex is towards your body, whose base is formed by the two far little-finger strings, and whose sides are formed by the mid-parts of the two near thumb strings. On either side of this triangle is a small four-sided figure. *Eleventh.* Insert the index-fingers from above into these quadrilaterals, and with the backs of the index-fingers take up the strings forming the sides of the triangle, and return. *Twelfth.* Put each middle-finger from above through the index loop, and take up from below the two far thumb strings, and return through the index loop. *Thirteenth.* Release the thumbs and index-fingers, put the index-fingers into the middle-finger loops to make them wider; and with the thumbs manipulate the figure so as to make an approximate semicircle (the sun on the horizon) with four diverging loops (the rays). *Finally.* Release the loops on the index and middle-fingers, separate the hands, and the semicircle will slowly disappear representing the setting of the sun.

The Head Hunters is another and more difficult example, in which the Lem Opening is used. It too is derived from the Torres Straits, and is interesting because it is a graphical illustration of a story.

The Head Hunters are produced thus (Rivers and Haddon, p. 150; Jayne, pp. 16—20). *First.* Make the Lem Opening, which involves six movements. Continue thus. *Seventh.* Insert the index-fingers from below into the central triangle and take up on their backs the near thumb strings. *Eighth.* Loop the lowest or proximal index string of each hand over the two upper or distal strings and over the tip of the index on to its palmar aspect. *Ninth.* Release the thumbs. *Tenth.* Take the index loops off the right hand, twist them tightly three or four times, and let the twist drop. Similarly form a twist out of the loops on the left index.

This is the figure. If now the little-finger loops are drawn slowly apart, the two index loops will approach each other and become entangled. One represents a Murray man, and the other a Dauar man. They "fight, fight, fight," and, if worked skilfully, one loop, the victor, eventually remains, while a kink in the string represents all that is left of the other loop. The victorious loop can now be drawn to one hand along the two strings, sweeping the kink in front of it: it represents the victor carrying off the head of his opponent. Sometimes, if the two index loops are twisted exactly alike, they both break up, representing a duel fatal to both parties. In the hands of the Murray man who showed the figure to Dr Haddon, the result of the fight always led to the defeat of the Dauar warrior.

It is not easy to make the figure so as to secure a good fight. For the benefit of any who wish to predict the result I may add that if, in the first position, there be a knot in the right palmar string the left loop will be usually victorious over the right loop, and vice versa.

The Parrot Cage. As another instance I give the Parrot Cage. This is a string figure made by Negroes on the Gold Coast in West Africa. The construction and design are not interesting in themselves, but the method used is somewhat different to those employed in the foregoing examples, and for this reason I insert it.

The Parrot Cage is made thus (Yõruba Figures by J. Parkinson, *Journal of the Anthropological Institute*, London, 1906, vol. xxxvi, p. 136). *First.* Take up the string in the form of Opening A. *Second.* Transfer the little-finger loops of each hand to the ring-fingers of that hand, the index loops to the middle-fingers, and the thumb loops to the index-fingers. By an obvious modification of Opening A, the string can be taken up initially in this form. *Third.* Lace the dorsal loops thus. Turn the back of the right hand upwards, pull the dorsal string of the ring-finger loop a little way out, and through it pass the dorsal string of the middle-finger loop. Similarly pass the dorsal string of the index loop through the middle-finger loop, and then put the index loop on the thumb. Pull all the strings taut. Repeat the same movements with the other hand. *Fourth.* Lace the palmar loops thus. Holding the right hand with the fingers pointing upwards, take up with the dorsal tip of the right thumb the nearest string which passes from hand to hand, pass the string already on the thumb over the string so taken up and then over the tip of the thumb on to the far and palmar side of the thumb. Repeat the same process successively with the far index string, with the near middle-finger string, with the far proximal middle-finger string, with the remaining far middle-finger string, and with the far ring-finger string. Repeat all these movements with the left hand. *Finally.* Transfer the thumb loops to the little-fingers, and extend the figure. This is the cage.

See-Saw. I described above a couple of See-Saw figures made from the European Opening. Such figures can be also produced from Opening A.

A *See-Saw* arrangement can be made by two players, *P* and *Q*, thus (Haddon quoted by Jayne, p. xiii). *First.* The string is taken up by *P*, with his hands pointing upwards and palm facing palm, in the form of Opening A. *Second.* *Q* hooks his right index from above in the straight thumb string and pulls it away from *P*. *Third.* *Q*, passing his hand from *P* over the other strings, hooks his left index in the straight little-finger string, and pulls it towards *P*. *Finally.* *P* releases all but the index loops. The sawing movement can then be made.

Lightning. I proceed next to give two examples of figures which do not start from Opening A: both are easy to produce. The first instance I select, known as *Lightning*—see figure xiii—is derived from the North American Red Indians. The final movement should be performed sharply, so that the zig-zag lightning may flash out suddenly.

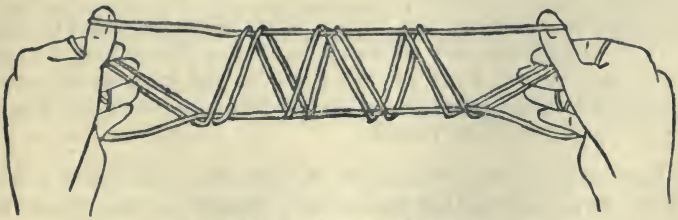


Figure xiii. *Zig-zag Lightning.*

Lightning is produced thus (Jayne, pp. 216—219). *First.* Hold the string in one place between the tips of the thumb and index-finger of the right hand and in another place between the tips of the thumb and index-finger of the left hand, so that a piece passes between the hands and the rest hangs down in a loop. With the piece between the hands make a ring, hanging down, by putting the right-hand string away from you over the left-hand string. Next, insert the index-fingers towards you in the ring and put the thumbs away from you into the long hanging loop. Separate the hands, and turn the index-fingers upward and outward with the palms of the hands facing away from you. Then, turn the hands so that the palms are almost facing you, and the thumbs and the palms come toward you and point upward. You now have a long crossed loop on each thumb and a single cross in the centre of the figure. This is the Navaho Opening.

I continue from the Navaho Opening the movements for the production of Lightning. *Second.* Pass each thumb away from you over the radial or near index string and take up from below with the back of the thumb the far index string, and return the thumb to its former position. *Third.* Pass each middle-finger toward you over the near index string, and take up from below on the back of the finger the far thumb string and return the middle-finger to its original position. *Fourth.* Bend each ring-finger toward you over the far middle-finger string, take up from below with the back of the finger the near index string, and return the ring-finger to its position. *Fifth.* Pass each little-finger over the far ring-finger string, take up from below on the back of the finger the far middle-finger string, and return the little-finger to its position. You now have two twisted strings passing between the two little-fingers, two strings laced round the other fingers, and two loose strings (which may represent thunder clouds) passing over the thumbs. *Sixth.* Holding the hands with the fingers pointing upwards, put the tips of the thumbs from below (or if it is easier, from above) into the small spaces between the little-fingers and the twisted strings on them. *Finally.* With the thumbs raise (or depress, as the case may be) the near ring-finger string, and separate the hands so as to make the little-finger strings taut, turn the hands outwards so that the palms are away from you, and at the same time throw or jerk the thumb loops off the thumbs so that they hang away from you over the tightly drawn strings between the little-fingers. These movements will cause the strings between the little-fingers to untwist, making the lightning spring into view. The diagram represents the figure when the thumb in the sixth movement is inserted from below in the loop.

More briefly thus. Form Navaho Ring. Each thumb over 2 and takes up one. Each middle-finger over one and takes up one. Each ring-finger over one and takes up one. Each little-finger over one and takes up one. Each thumb released, placed under (or over) the near little-finger string, and pressed up (or down) while hands are rotated.

A *Butterfly.* As yet another of figures of this kind I will give the formation of that known as a *Butterfly*—see figure xiv. It is also derived from the North American Red Indians.

The *Butterfly* is produced as follows (Jayne, pp. 219—221). Begin with the Navaho Opening, that is, make the first movements as when forming *Lightning*. *Second.* Twist each index loop by rotating each index down toward you and up again. In all make (say) three or four such twists on each index loop. *Third.* Put the right thumb from below into the right index loop, and, without removing the index, separate the thumb from the index. Make movement T on the thumb loops. Repeat the movement with the other hand. *Fourth.* Bring the hands close together with the index and thumb of the one hand pointing toward the index and thumb of the other hand; then hang the right index loop β on the tip of the left index, and the right thumb loop δ on the tip of the left thumb. Thus

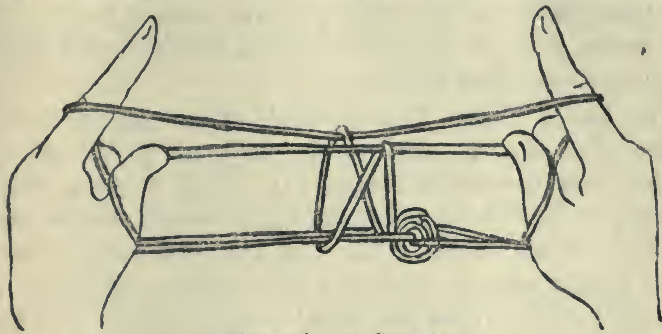


Figure xiv. *A Butterfly.*

on the left index there are two loops, α and β; on the left thumb there are two loops, γ and δ; and the right hand is free. *Fifth.* Put the tips of the right index and thumb against the left thumb knuckle. Take up with the right index from the proximal side the loop δ and take up with the right thumb from the distal side the loop γ. With the right thumb and index grasp the loops α and β where they lie on the top of the left index. Remove the left hand. Then, holding the right hand up, from the left, put the left index into the loop β, and the left thumb into the loop α. *Finally.* Placing the hands with the thumbs up and the fingers pointing away from you, draw the hands slowly apart, and when the strings have partially rolled up in the middle of the figure pull down with the middle, the ring, and the little-fingers of each hand the far index string and the near thumb string. The butterfly will now appear; its wings being held up by the strings extended between the widely separated thumbs and index-fingers, and its proboscis appearing on the strings held down by the other fingers.

More briefly thus. Form Navaho Ring. Twist index loops. Thumbs into index loops and T to thumb loops. Take up figure afresh with thumbs and index-fingers, and pull.

There are numerous other figures which can be formed from an endless loop of string. For examples I refer the reader to Mrs Jayne's fascinating volume.

String Tricks. Of string tricks there are two classes. One comprises tricks in which the hand of one player is unexpectedly caught by the other player pulling one string or certain strings in the figure, whereas it is left free if other strings are pulled.

The other comprises tricks where the string is released from or is made to take some position which *prima facie* is impossible. Aborigines are usually very proud of their ability to perform such feats, and Mrs Jayne acutely remarks that it is "delightful to witness their pleasure when they are successful, and their gratification at the observer's astonishment, which it will amply pay him to make very evident."

The *Lizard Twist*. I give a couple of examples of Catch-Tricks. The first I select is one brought by Rivers and Haddon from the Torres Straits. If a loop of string hangs from (say) the left hand and if the right hand is twisted once round one of the strings of the loop, the right hand or wrist will be caught when the string is pulled by the left hand. If the right hand is twisted first round one string of the loop and then round the other string, it will in general be caught more firmly. The Lizard movement is a way of taking the twist on the second string so as to undo the effect of the first twist.

There is no difficulty in taking up the string so that the hand is caught. To take up the string so that the hand is not caught, proceed thus (Rivers and Haddon, *Man*, 1902, p. 152; Jayne, pp. 337—339). *First*. Hold the string by the left hand, held rather high, the string hanging down on the right and left sides in a loop. *Second*. Put the right hand away from you through the loop. Turn the right hand round the right pendant string clockwise; this will be done by pointing the fingers to the right, then towards you, and then upwards. *Third*. Keeping the fingers pointing upwards, move the right hand to the left between your body and the pendant strings, then clockwise beyond the left pendant string, then away from you, then to the right, and finally towards you through the loop. *Lastly*. Draw the hand down and to the right, and it will come free from the noose round the wrist.

The *Caroline Catch*. The other example of a Catch-Trick which I propose to describe is derived from natives in the Caroline Isles. First, form the figure described above (p. 361) as Triple Diamonds. Next a second person puts his hand through the middle-lozenge shaped space in this figure. Then his wrist will be caught in a loop if the strings be dropped from the right hand and the left-hand strings pulled to the left. On the other hand his wrist will not be caught if the strings be dropped from the left hand, and the right-hand strings pulled to the right.

Of the second class of string tricks there are numerous examples familiar to English schoolboys—presumably well-known over large parts of the world, and I conjecture of considerable antiquity. In many old journals and books written for boys these are described, but such descriptions are often vague. Recently W. I. Pocock and Mrs Jayne have described some of them in accurate language. I give as typical examples: Threading the Needle; the Yam Thief, otherwise known as the Mouse Trick; the Halter; the Fly on the Nose; the Hand-Cuff; and the Elusive Loop. Of these the first, second, and fifth, are well-known. I owe my knowledge of the others to Mrs Jayne's book*.

Threading the Needle. This is a familiar trick. The effect is that a small loop (representing the eye of the needle) held by the left thumb and index is threaded by a string held by the right hand throughout the motion. The trick is best worked with a single piece of string, but a doubled string will answer the purpose.

Threading the Needle is performed thus (Jayne, pp. 354—355). *First.* Take hold of a piece of string, *ABC*, some three or four feet long, at a point *B* about 8 inches from one end *A*. Hold this piece *AB* in the left palm and hold the left hand so that the thumb points to the right. *Second.* Holding the other end, *C*, of the string with the right hand, wind the rest of the string *BC*, beginning with the *B* end, round the left thumb, above the thumb when moving towards the body and under the thumb when moving away from the body. Leave about 6 inches at the end *C* unwound. *Third.* Out of this 6 inches, make a small loop by carrying the end of *C* held by the right hand to the far side of the rest of the loop. With the tips of the left thumb and index-finger hold this loop where the strings cross so that it stands erect. *Fourth.* Pick up with the right thumb and index-finger the end *A* of the piece of string *AB* which is in the left palm and open the left palm. Make this piece of string *AB* taut, and carry the end of it *A* to the right close under and between the left thumb and index. If this is properly done the piece *AB* will pass near *B* between the far and near portions of the cross of the loops, and then will be caught by the left thumb and index: thus between the thumb and index there are now three pieces of string. *Fifth.* Make passes with the end *A* as if you

* The second is described by Rivers and Haddon in *Man*, London, October, 1902, pp. 141, 153. All, except the Hand-Cuff, are described in C. F. Jayne's *String Figures*, New York, 1906, chapter viii. See, also, W. I. Pocock, *Folk-Lore*, London, 1906, pp. 349—373.

were trying to thread it through the erect loop held by the left hand. *Lastly.* Pass the right hand sharply to the left over and beyond the left hand. This will carry the piece *AB* beyond the two strings of the loop. Hence the loop which is still held up by the left thumb and index appears to have been threaded by the right-hand string, but in reality the part of the string which hangs from the right hand is drawn between the left thumb and index up into the loop.

The *Yam Thief* or *Mouse Trick.* This is also a familiar trick, and is interesting as having stories connected with its performance.

The *Yam Thief* or *Mouse Trick* is effected thus (Jayne, pp. 340—343). *First.* Hold the left hand with the palm facing you, the thumb upright, and the fingers pointing to the right. With the right hand, loop the string over the left thumb, cross the strings, and let one string hang down over the palm and the other over the back of the left hand. *Second.* Pass the right index from below under (*i.e.* on the proximal side of) the pendant palmar string and then between the left thumb and index, and with the palmar tip of the right index loop up a piece of the string hanging on the back of the left hand. Pull this loop back between the left thumb and index and on the upper (or distal) side of the left palmar string. Then with the right index give the loop one twist clockwise, and put it over the left index. Pull the two pendant strings in order to hold tight the loops on the thumb and index. *Third.* In the same way pass the right index from below under the pendant palmar string and then between the left index and middle-fingers, and with the palmar tip of the right index loop up a piece of the pendant dorsal string. Pull it back between the left index and the middle-fingers and on the upper side of the left palmar string. With the right index give the loop one twist clockwise, and put it over the left middle-finger. *Fourth.* In the same way pick up a loop of the pendant dorsal string, and put it on the left ring-finger. *Fifth.* In the same way pick up a loop of the pendant dorsal string, and put it on the left little-finger. *Sixth.* Take off the left thumb loop, or slip it to the tip of the thumb and hold it between the left thumb and index. The pendant dorsal string on the left hand can be pulled to show that the loops are still on the fingers. *Finally.* Pull the left pendant palmar string and all the string will come away.

In one version of the story the thumb loop represents the owner of a yam or cabbage patch. He is supposed to be asleep. The loops successively taken up from the dorsal string represent the yams or cabbages dug up by a thief from the patch, and secured by him in bundles on the fingers. The loop coming off the thumb represents the owner waking, and going out to see what is the matter. He walks down the dorsal side, sees the yams, pulls at the dorsal string, is satisfied that his yams are still on the ground, and returns to catch the thief. Meanwhile the thief walks down the palmar side of the hand, and as the owner returns from the dorsal side, the left palmar string is pulled and the thief disappears with the stolen yams.

Cheating the Halter. This is a trick of the Philippinoes.

A halter is put round the neck, but by a movement, which in effect reverses the turn on the neck, the string, when pulled, comes off.

The *Halter Trick* is performed thus (Jayne, pp. 339—340). *First.* Put your head through a loop of string, and let the rest of the loop hang down in front of you. *Second.* Pass the right string round the neck from the left side, draw the loop tight, and let it hang down in front of you. *Third.* Put the hanging loop on the hands, and form Opening A. *Fourth.* Pass the index loops over the head, and remove the hands and fingers from the other loops; a loop now hangs down in front of you. *Lastly.* If this loop, or either string of it, be pulled all the strings will come off the neck.

We can vary the presentation by twisting the left string round the neck from the right side. In this case we must use Opening B.

The *Fly on the Nose.* In this trick the string seems to come away although looped on to the nose.

The *Fly on the Nose* is performed thus (Jayne, pp. 348—349). *First.* Hold the string at some point with the thumb and index-finger of one hand; and take hold of the string at a place, some 9 or 10 inches off, with the thumb and index of the other hand. *Second.* Make a small ring hanging down by passing the right hand to the left and on the near side of the string held by the left hand. There is now a long loop and a small ring, both hanging down, the right string of the ring being the left string of the loop. *Third.* Hold the place where the strings cross between the teeth, the string originally held by the right hand being on the lower side. Hold the strings or one string of the long loop with the left hand. *Fourth.* Place the right index from the far side through the ring, taking the ring string up to the root of the fingers. Close the right fist, and carry the fist (holding the ring) round in a circle, first back to the far side, then to the right, and so on round the right string of the loop, to its original position. *Fifth.* Open the index, keeping the rest of the fist shut, and put the tip of the index on the tip of the nose. *Finally.* Let go with the mouth and pull the left string of the loop. The whole string will then come away.

Of course the same result can be obtained if the ring is made by passing the right hand to the left on the far side of the string held by the left hand, so that when the cross is held in the mouth the string originally held by the right hand is uppermost. But in this case the ring should be taken by the left index, which is carried round the left string of the loop.

The *Hand-Cuff Trick.* This is a very ancient string trick. Two players, *P* and *Q*, are connected thus. One end of a piece of string is tied round *P*'s right wrist, the other is tied round his left wrist. Another piece of string is passed through the space bounded by the string tied to *P*'s wrist, his body, and his arms. The ends of this piece are then tied one on *Q*'s right wrist, and the other on *Q*'s left wrist. The players desire to free themselves from the entanglement.

The *Hand-Cuff Trick*. Either player can free himself thus. *First*. *P* takes up a small loop *L* near the middle of the string tied to his wrists, and pulling the loop to one of *Q*'s wrists on the palmar side of it, passes it, from the elbow to the finger side, between *Q*'s wrist and the loop on that wrist. Next, *P* draws this loop *L* sufficiently far through until he is able to pass it, first over *Q*'s hand, and then under the wrist loop on the outer side of *Q*'s wrist, passing this time from the finger to the elbow side of *Q*. When in this position *P* can pull his own string clear of *Q* on the outer side of *Q*'s arm.

The *Elusive Loop*. This is a trick in which a loop is offered to some one, and then unexpectedly disappears. Almost any of the forms in which the string is so arranged that if pulled it runs off the fingers—and there are many examples of this kind, *ex. gr.* the Yam Thief—lends itself to this presentation. I give an instance derived from the Torres Straits where the loop is supposed to represent a Yam.

The offer of the *Elusive Yam* is performed thus (Jayne, pp. 352—354). *First*. Take up the string in the first position. *Second*. Pass each index away from you over the little-finger string and to the far side of it, draw the string towards you in the bend of the index, and then turn the index up towards you in its usual position, thus twisting the string round the tip of the finger. *Third*. Pass each thumb away from you under the far index string, pick up from below on the back of the thumb the near index string which crosses the palm obliquely, and return the thumb under the near thumb string to its original position. *Fourth*. Pass each little-finger towards you over the far index string, and pick up from below on the back of the little-finger the near string which passes directly from hand to hand, and return the little-finger. *Fifth*. Pass each thumb away from you, and pick up from below the near string of the figure, and return the thumb. *Lastly*. Release the loop from the left index, and hold it erect between the left index and thumb. This loop represents a Yam.

One boy, who is supposed to be hungry, says *Have you any food for me*? Thereupon another boy, who has made the figure, offers the loop or Yam to the first boy saying, *Take it if you can*. On this the first boy grabs at the Yam, while the second boy pulls the right-hand strings. If the former is quick enough he gets the Yam, but if not, it disappears, and all the strings with it. The same trick can be repeated with the right hand.

More briefly thus. First position. Index-fingers, take up twist on far string. Thumbs, under one, pick up palmar near index string, and return under two. Little fingers, pick up near string. Thumbs, pick up near string. Release loop from left index.

There also are some string tricks which must be familiar to most of my readers. I give as well-known examples the Button-Hole Trick and the Loop Trick. These require no skill and present no difficulty. It is with some hesitation that I describe them, but age gives them a certain claim. I have no idea when or by whom they were invented. The devices used are so obvious that the tricks will hardly bear repetition.

The *Button-Hole Trick.* In this a loop of string is passed through a button-hole or ring, and on each side of the hole a finger or thumb is put into the loop on that side. The object is to free the loop from that hole. It may be done in several ways which however do not differ in principle. I give two methods.

The *Button-Hole Trick* may be done thus. *First.* Pass a loop of string through a button-hole or key-ring. *Second.* Hold the thumbs upright, and insert them from below in the two loops one on each side of the button-hole. Move the hands a little out from the body and towards each other. *Third.* Hook the right little-finger into the right-hand string of the left thumb loop, and pull it across to the right hand. *Fourth.* Pass the left hand above the right little-finger loop, hook the left little-finger into the left hand string of the right thumb loop and pull it across to the left hand. *Fifth.* Drop the right thumb loop and put the right thumb into the right little-finger loop. *Finally.* Withdraw both little-fingers, and separate the hands; the string will then come off the button-hole. In fact the effect of the movements described is to get both thumbs in one loop.

This trick may be also performed by two people, *P* and *Q*, by the following movements, the description of which I take from Pocock's paper in *Folk-Lore*, 1906, p. 355. *First.* Pass a loop of the string through a ring. *Second.* *P*, holding his thumbs upright, inserts them from below in the two loops, one on each side of the ring. The string nearest him will be the radial, that farthest from him the ulnar string. *Third.* *Q*, who is facing *P*, puts his left index from below into the loop on *P*'s right thumb. *Q* then takes up on the back of his left index-finger the ulnar string at some point, say *H*, of it between the ring and *P*'s right thumb. At the same time *Q*, with the inside of his left little-finger, hooks the same string at some point between *H* and *P*'s right thumb and draws it towards himself. Thus there is now one loop on the back of *Q*'s left index-finger and another on the inside of his left little-finger. *Fourth.* *Q* places the tip of his left index-finger on the tip of *P*'s right thumb and transfers the loop on that finger to *P*'s thumb. At the same time *Q* draws the little-finger loop well away towards *P*'s left side. *Fifth.* *Q* pushes his left index from above into the loop near *P*'s left thumb but on the other side of the ring, and with the back of the finger he picks up the radial string. *Sixth.* *Q* transfers this index loop to *P*'s right thumb, by touching the tips as before. *Seventh.* *Q* grasps the ring with his right hand, and at the same time drops the little-finger loop. *Finally.* *P* separates his hands and the ring comes away from the string.

The *Loop Trick* is merely an illustration of how easily an unobservant person can be deceived. It requires two persons, *P* and *Q*, and is performed thus.

The *Loop Trick*. A string is looped on the index-fingers held upright of a player *P*: thus there are two parallel strings, a near and a far one. The loop is taken up by another player, *Q*, thus. *First.* *Q* presses his left index on both strings about half-way between *P*'s hands and holds them firmly down. *Second.* *P* moves his right index over the strings until the tip meets the tip of his left index, and if he likes shifts the right index loop from the right finger to the base of the left finger. *Lastly.* *Q* slips his left index off the distal string, and at the same time pulls the left index, which now rests on only one string, and of course the loop comes away.

This is the common presentation of the trick, but W. I. Pocock remarks that it is somewhat less easy to detect the method used if the strings be struck sharply down with the right hand at the instant when the left index is pulled.

The *Waistcoat Puzzle*. There is one other trick of this type which was, I believe, published for the first time in this book in 1888. It is applicable to a man wearing a coat and waistcoat of the usual pattern. The problem is to take off the waistcoat, which may be unbuttoned, without pulling it over the head and without taking off the coat.

Those of my readers who are conversant with the history of white magic will recollect that Pinetti, the celebrated conjurer of the eighteenth century, in his performance at Versailles before the French court in 1782, relieved some of the courtiers of their shirts without disturbing the rest of their attire. It was his most striking trick, and not a very seemly one. In fact he threw a shawl over the person operated on, and then pulled the back of the shirt over his head; an obvious method which I have barred in the waistcoat trick. The victim and the spectators were uncritical, and Pinetti's technical skill was sufficient to conceal the *modus operandi* from them.

The *Waistcoat Puzzle* can be done thus. *First.* Take the left corner (or lappel) of the coat and push it through the left armhole of the waistcoat, from outside to inside. *Second.* Put the left hand and arm through the same armhole. The effect of this is to leave the left armhole of the waistcoat at the back of the neck. *Third.* Take the right lappel of the coat and put it through the left armhole of the waistcoat. *Fourth.* Put the right hand and arm through the same armhole. *Finally.* Pass the waistcoat down the right sleeve of the coat.

I should have liked to add another section to this chapter on knots and lashings. Some references to the mathematics of the subject will be found in papers by Listing, Tait, Böddicker *, but its presentation in a popular form is far from easy, and this chapter has already run to dimensions which forbid any extension of it.

* J. B. Listing, *Vorstudien zur Topologie*, Die Studien, Göttingen, 1847, Part x; O. Böddicker, *Erweiterung der Gauss'schen Theorie der Verschlingungen*, Stuttgart, 1876; and P. G. Tait, *Collected Scientific Papers*, Cambridge, vol. I, 1898, pp. 273—347. The more common forms of knots and lashings and their special uses are described in most naval, engineering, and scouting manuals.

CHAPTER XVIII.

ASTROLOGY.

Astrologers professed to be able to foretell the future, and within certain limits to control it. I propose to give in this chapter a concise account of the rules they used for this purpose*.

I have not attempted to discuss the astrology of periods earlier than the middle ages, for the technical laws of the ancient astrology are not known with accuracy. At the same time there is no doubt that, as far back as we have any definite historical information, the art was practised in the East; that thence it was transplanted to Egypt, Greece, and Rome; and that the medieval astrology was founded on it. It is probable that the rules did not differ materially from those described in this chapter†, and it may be added that the more intelligent thinkers of the old world recognized that the art had no valid pretences to accuracy. I may note also that the history of the development of the art ceases with the general acceptance of the Copernican theory, after which the practice of astrology rapidly became a mere cloak for imposture.

* I have relied mainly on the *Manual of Astrology* by Raphael—whose real name was R. C. Smith—London, 1828, to which the references to Raphael hereafter given apply; and on Cardan's writings, especially his commentary on Ptolemy's work and his *Geniturarum Exempla.* I am indebted also for various references and gossip to Whewell's *History of the Inductive Sciences*; to various works by Raphael, published in London between 1825 and 1832; and to a pamphlet by M. Uhlemann, entitled *Grundzüge der Astronomie und Astrologie*, Leipzig, 1857.

† On the influences attributed to the planets, see *The Dialogue of Bardesan on Fate*, translated by W. Cureton in the *Spicilegium Syriacum*, London, 1855.

All the rules of the medieval astrology—to which I confine myself—are based on the Ptolemaic astronomy, and originate in the *Tetrabiblos** which is said, it may be falsely, to have been written by Ptolemy himself. The art was developed by numerous subsequent writers, especially by Albohazen†, and Firmicus. The last of these collected the works of most of his predecessors in a volume‡, which remained a standard authority until the close of the sixteenth century.

I may begin by reminding the reader that though there was a fairly general agreement as to the methods of procedure and interpretation—which alone I attempt to describe—yet there was no such thing as a fixed code of rules or a standard text-book. It is therefore difficult to reduce the rules to any precise and definite form, and almost impossible, within the limits of a chapter, to give detailed references. At the same time the practice of the elements of the art was tolerably well established and uniform, and I feel no doubt that my account, so far as it goes, is substantially correct.

There were two distinct problems with which astrologers concerned themselves. One was the determination in general outline of the life and fortunes of an enquirer: this was known as *natal astrology*, and was effected by the erection of a *scheme of nativity*. The other was the means of answering any specific question about the individual: this was known as *horary astrology*. Both depended on the casting or erecting of a *horoscope*. The person for whom it was erected was known as the *native*.

A horoscope was *cast* according to the following rules§. The space between two concentric and similarly situated squares was divided into twelve spaces, as shown in the annexed diagram. These twelve spaces were known technically as *houses*; they were numbered consecutively 1, 2, ..., 12 (see figure); and

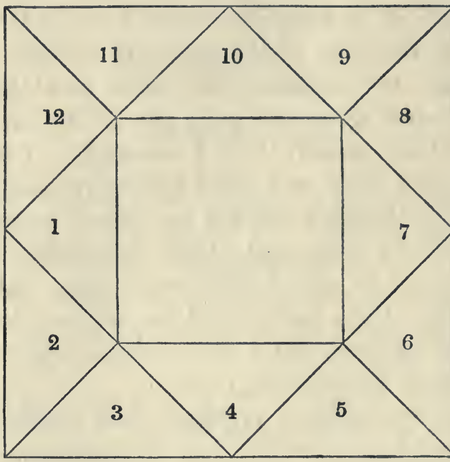
* There is an English translation by J. Wilson, London [*n.d.*]; and a French translation is given in Halma's edition of Ptolemy's works.

† *De Judiciis Astrorum*, ed. Liechtenstein, Basle, 1571.

‡ *Astronomicorum*, eight books, Venice, 1499.

§ Raphael, pp. 91—109.

were described as the first house, the second house, and so on. The dividing lines were termed *cusps*: the line between the houses 12 and 1 was called the cusp of the first house, the line between the houses 1 and 2 was called the cusp of the second house, and so on, finally the line between the houses 11 and 12



was called the cusp of the twelfth house. Each house had also a name of its own—thus the first house was called the ascendant house, the eighth house was called the house of death, and so on—but as these names are immaterial for my purpose I shall not define them.

Next, the positions which the various astrological signs and planets had occupied at some definite time and place (for instance, the time and place of birth of the native, if his nativity was being cast) were marked on the celestial sphere. This sphere was divided into twelve equal spaces by great circles drawn through the zenith, the angle between any two consecutive circles being 30°. The first circle was drawn through the East point, and the space between it and the next circle towards the North corresponded to the first house, and sometimes was called the first house. The next space, proceeding from East to North, corresponded to the second house, and so on. Thus each of the twelve spaces between these circles corresponded to one of the twelve houses, and each of the circles to one of the cusps.

In delineating* a horoscope, it was usual to begin by inserting the zodiacal signs. A zodiacal sign extends over 30°, and was marked on the cusp which passed through it: by its side was written a number indicating the distance to which its influence extended in the earlier of the two houses divided by the cusp. Next the positions of the planets in these signs were calculated, and each planet was marked in its proper house and near the cusp belonging to the zodiacal sign in which the planet was then situated: it was followed by a number indicating its right ascension measured from the beginning of the sign. The name of the native and the date for which the horoscope was cast were inserted usually in the central square. The diagram near the end of this chapter is a facsimile of the horoscope of Edward VI as cast by Cardan and will serve as an illustration of the above remarks.

We are now in a position to explain how a horoscope was *read* or interpreted. Each house was associated with certain definite questions and subjects, and the presence or absence in that house of the various signs and planets gave the answer to these questions or information on these subjects.

These questions cover nearly every point on which information would be likely to be sought. They may be classified roughly as follows. For the answer, so far as it concerns the native, to all questions connected with his life and health, look in house 1; for questions connected with his wealth, refer to house 2; for his kindred and communications to him, refer to 3; for his parents and inheritances, refer to 4; for his children and amusements, refer to 5; for his servants and illnesses, refer to 6; for his marriage and amours, refer to 7; for his death, refer to 8; for his learning, religion, and travels, refer to 9; for his trade and reputation, refer to 10; for his friends, refer to 11; and finally for questions connected with his enemies, refer to house 12.

* Raphael, pp. 118—131.

I proceed to describe briefly the influences of the planets, and shall then mention those of the zodiacal signs; I should note however that in practice the signs were in many respects considered to be more influential than the planets.

The astrological "planets" were seven in number, and included the Sun and the Moon. They were Saturn or the Great Infortune, Jupiter or the Great Fortune, Mars or the Lesser Infortune, the Sun, Venus or the Lesser Fortune, Mercury, and the Moon: the above order being that of their apparent times of rotation round the earth.

Each of them had a double signification. In the first place it impressed certain characteristics, such as good fortune, feebleness, &c., on the dealings of the native with the subjects connected with the house in which it was located; and in the second place it imported certain objects into the house which would affect the dealings of the native with the subjects of that house.

To describe the exact influence of each planet in each house would involve a long explanation, but the general effect of their presence may be indicated roughly as follows*. The presence of Saturn is malignant: that of Jupiter is propitious: that of Mars is on the whole injurious: that of the Sun indicates respectability and moderate success: that of Venus is rather favourable: that of Mercury implies rapid practical action: and lastly the presence of the Moon merely faintly reflects the influence of the planet nearest her, and suggests rapid changes and fickleness. Besides the planets, the Moon's nodes and some of the more prominent fixed stars† also had certain influences.

These vague terms may be illustrated by taking a few simple cases.

For example, in casting a nativity, the life, health, and general career of the native were determined by the first or ascendant house, whence comes the expression that a man's fortune is in the ascendant. Now the most favourable planet

* Raphael, pp. 70—90, pp. 204—209.

† Raphael, pp. 129—131.

was Jupiter. Therefore, if at the instant of birth Jupiter was in the first house, the native might expect a long, happy, healthy life; and being born under Jupiter he would have a "jovial" disposition. On the other hand, Saturn was the most unlucky of all the planets, and was as potent as malignant. If at the instant of birth he was in the first house, his potency might give the native a long life, but it would be associated with an angry and unhappy temper, a spirit covetous, revengeful, stern, and unloveable, though constant in friendship no less than in hate, which was what astrologers meant by a "saturnine" character. Similarly a native born under Mercury, that is, with Mercury in the first house, would be of a mercurial nature, while anyone born under Mars would have a martial bent.

Moreover it was the prevalent opinion that a jovial person would have his horoscope affected by Jupiter, even if that planet had not been in the ascendant at the time of birth. Thus the horoscope of an adult depended to some extent on his character and previous life. It is hardly necessary to point out how easily this doctrine enabled an astrologer to make the prediction of the heavens agree with facts that were known or probable.

In the same way the other houses are affected. For instance, no astrologer, who believed in the art, would have wished to start on a long journey when Saturn was in the ninth house or house of travels; and if, at the instant of birth, Saturn was in that house, the native would always incur considerable risk on his journeys.

Moreover every planet was affected to some extent by its aspect (conjunction, opposition, or quadrature) to every other planet according to elaborate rules* which depended on their positions and directions of motion: in particular the angular distance between the Sun and the Moon—sometimes known as the "part of fortune"—was regarded as specially important, and this distance affected the whole horoscope. In general, conjunction was favourable, quadrature unfavourable, and opposition ambiguous.

* Raphael, pp. 132—170.

Each planet not only influenced the subjects in the house in which it was situated, but also imported certain objects into the house. Thus Saturn was associated with grandparents, paupers, beggars, labourers, sextons, and gravediggers. If, for example, he was present in the fourth house, the native might look for a legacy from some such person; if he was present in the twelfth house, the native must be careful of the consequences of the enmity of any such person; and so on.

Similarly Jupiter was associated generally with lawyers, priests, scholars, and clothiers; but, if he was conjoined with a malignant planet, he represented knaves, cheats, and drunkards. Mars indicated soldiers (or, if in a watery sign, sailors on ships of war), masons, doctors, smiths, carpenters, cooks, and tailors; but, if afflicted with Mercury or the Moon, he denoted the presence of thieves. The Sun implied the action of kings, goldsmiths, and coiners; but, if afflicted by a malignant planet, he denoted false pretenders. Venus imported musicians, embroiderers, and purveyors of all luxuries; but, if afflicted, prostitutes and bullies. Mercury imported astrologers, philosophers, mathematicians, statesmen, merchants, travellers, men of intellect, and cultured workmen; but, if afflicted, he signified the presence of pettifoggers, attorneys, thieves, messengers, footmen, and servants. Lastly, the presence of the Moon introduced sailors and those engaged in inferior offices.

I come now to the influence and position of the zodiacal signs. So far as the first house was concerned, the sign of the zodiac which was there present was even more important than the planet or planets, for it was one of the most important indications of the duration of life.

Each sign was connected with certain parts of the body—*ex. gr.* Aries influenced the head, neck and shoulders—and that part of the body was affected according to the house in which the sign was. Further each sign was associated with certain countries and connected the subjects of the house in which the sign was situated with those countries: *ex. gr.* Aries was associated especially with events in England, France, Syria, Verona, Naples, &c.

The sign in the first house determined also the character and appearance of the native*. Thus the character of a native born under Aries (*m*) was passionate; under Taurus (*f*) was dull and cruel; under Gemini (*m*) was active and ingenious; under Cancer (*f*) was weak and yielding; under Leo (*m*) was generous, resolute, and ambitious; under Virgo (*f*) was sordid and mean; under Libra (*m*) was amorous and pleasant; under Scorpio (*f*) was cold and reserved; under Sagittarius (*m*) was generous, active, and jolly; under Capricorn (*f*) was weak and narrow; under Aquarius (*m*) was honest and steady; and under Pisces (*f*) was phlegmatic and effeminate.

Moreover the signs were regarded as alternately masculine and feminine, as indicated above by the letters *m* or *f* placed after each sign. A masculine sign is fortunate, and all planets situated in the same house have their good influence rendered thereby more potent and their unfavourable influence mitigated. But all feminine signs are unfortunate, their direct effect is evil, and they tend to nullify all the good influence of any planet which they afflict (*i.e.* with which they are connected), and to increase all its evil influences, while they also import an element of fickleness into the house and often turn good influences into malignant ones. The precise effect of each sign was different on every planet.

I think the above account is sufficient to enable the reader to form a general idea of the manner in which a horoscope was cast and interpreted, and I do not propose to enter into further details. This is the less necessary as the rules—especially as to the relative importance to be assigned to various planets when their influence was conflicting—were so vague that astrologers had little difficulty in finding in the horoscope of a client any fact about his life of which they had information or any trait of character which they expected him to possess.

That this vagueness was utilized by quacks is notorious, but no doubt many an astrologer in all honesty availed himself of it, whether consciously or unconsciously. It must be remembered also that the rules were laid down at a time when

* Raphael, pp. 61—69.

men were unacquainted with exact sciences, with the possible exception of mathematics, and further that, if astrology had been reduced to a series of inelastic rules applicable to all horoscopes, the number of failures to predict the future correctly would have rapidly led to a recognition of the folly of the art. As it was, the failures were frequent and conspicuous enough to shake the faith of most thoughtful men. Moreover it was a matter of common remark that astrologers showed no greater foresight in meeting the difficulties of life than their neighbours, while they were neither richer, wiser, nor happier for their supposed knowledge. But though such observations were justified by reason they were often forgotten in times of difficulty and danger. A prediction of the future and the promise of definite advice as to the best course of action, revealed by the heavenly bodies themselves, appealed to the strongest desires of all men, and it was with reluctance that the futility of the advice was gradually recognized.

The objections to the scheme had been stated clearly by several classical writers. Cicero* pointed out that not one of the futures foretold for Pompey, Crassus, and Caesar had been verified by their subsequent lives, and added that the planets, being almost infinitely distant, cannot be supposed to affect us. He also alluded to the fact, which was especially pressed by Pliny†, that the horoscopes of twins are practically identical though their careers are often very different, or as Pliny put it, every hour in every part of the world are born lords and slaves, kings and beggars.

In answer to the latter obvious criticism astrologers replied by quoting the anecdote of Publius Nigidius Figulus, a celebrated Roman astrologer of the time of Julius Caesar. It is said that when an opponent of the art urged as an objection the different fates of persons born in two successive instants, Nigidius bade him make two contiguous marks on a potter's wheel, which was revolving rapidly near them. On stopping the wheel, the two marks were found to be far removed from

* Cicero, *De Divinatione*, II, 42.

† Pliny, *Historia Naturalis*, VII, 49; XXIX, 1.

each other. Nigidius received the name of Figulus, the potter, in remembrance of this story, but his argument, says St Augustine*, who gives us the narrative, was as fragile as the ware which the wheel manufactured.

On the other hand Seneca and Tacitus may be cited as being on the whole favourable to the claims of astrology, though both recognized that it was mixed up with knavery and fraud. An instance of successful prediction which is given by the latter of these writers† may be used more correctly as an illustration of how the ordinary professors of the art varied their predictions to suit their clients and themselves. The story deals with the first introduction of the astrologer Thrasyllus to the emperor Tiberius. Those who were brought to Tiberius on any important matter were admitted to an interview in an apartment situated on a lofty cliff in the island of Capreae. They reached this place by a narrow path overhanging the sea, accompanied by a single freedman of great bodily strength; and on their return, if the emperor had conceived any doubts of their trustworthiness, a single blow buried the secret and its victim in the ocean. After Thrasyllus had, in this retreat, stated the results of his art as they concerned the emperor, the latter asked the astrologer whether he had calculated how long he himself had to live. The astrologer examined the aspect of the stars, and while he did this showed, as the narrative states, hesitation, alarm, increasing terror, and at last declared that the present hour was for him critical, perhaps fatal. Tiberius embraced him, and told him he was right in supposing he had been in danger but that he should escape it; and made him thenceforth a confidential counsellor. But Thrasyllus would have been but a sorry astrologer had he not foreseen such a question and prepared an answer which he thought fitted to the character of his patron.

A somewhat similar story is told‡ of Louis XI of France.

* St Augustine, *De Civitate Dei*, bk. v, chap. iii; *Opera Omnia*, ed. Migne, vol. VII, p. 143.

† *Annales*, VI, 22: quoted by Whewell, *History of the Inductive Sciences*, vol. I, p. 313.

‡ *Personal Characteristics from French History*, by Baron F. Rothschild,

He sent for a famous astrologer whose death he was meditating and asked him to show his skill by foretelling his own future. The astrologer replied that his fate was uncertain, but it was so inseparably interwoven with that of his questioner that the latter would survive him but by a few hours, whereon the superstitious monarch not only dismissed him uninjured, but took steps to secure his subsequent safety. The same anecdote is also related of a Scotch student who, being captured by Algerian pirates, predicted to the Sultan that their fates were so involved that he should predecease the Sultan by only a few weeks. This may have been good enough for a barbarian, but with most civilized monarchs probably it would be less effectual, as certainly it is less artistic, than the answer of Thrasyllus.

I may conclude by mentioning a few notable cases of horoscopy.

Among the most successful instances of horoscopy enumerated by Raphael* is one by W. Lilly, given in his *Monarchy or No Monarchy*, published in 1651, in which he predicted a plague in London so terrible that the number of deaths should exceed the number of coffins and graves, to be followed by "an exorbitant fire." The prediction was amply verified in 1665 and 1666. In fact Lilly's success was embarrassing, for the Committee of the House of Commons, which sat to investigate the causes of the fire and ultimately attributed it to the papists, thought that he must have known more about it than he chose to declare, and on October 25, 1666, summoned him before them: Lilly proved himself a match for his questioners.

An even more curious instance of a lucky hit is told of Flamsteed†, the first astronomer royal. It is said that an old lady who had lost some property wearied Flamsteed by her perpetual requests that he would use his observatory to discover

London, 1896, p. 10. The story was introduced by Sir Walter Scott in *Quentin Durward*, chap. xv.

* *Manual of Astrology*, p. 37.

† The story, though in a slightly different setting, is given in *The London Chronicle*, Dec. 3, 1771, and it is there stated that Flamsteed attributed the result to the direct action of the devil.

her property for her. At last, tired out with her importunities, he determined to show her the folly of her demand by making a prediction, and, after she had found it false, to explain again to her that nothing else could be expected. Accordingly he drew circles and squares round a point that represented her house and filled them with all sorts of mystical symbols. Suddenly striking his stick into the ground he said, "Dig there and you will find it." The old lady dug in the spot thus indicated, and found her property; and it may be conjectured that she believed in astrology for the rest of her life.

Perhaps the belief that the royal observatory was built for such purposes may still be held, for De Morgan, writing in 1850, says that "persons still send to Greenwich to have their fortunes told, and in one case a young gentleman wrote to know who his wife was to be, and what fee he was to remit."

It is easier to give instances of success in horoscopy than of failure. Not only are all ambiguous predictions esteemed to be successful, but it is notorious that prophecies which have been verified by the subsequent course of events are remembered and quoted, while the far more numerous instances in which the prophecies have been falsified are forgotten or passed over in silence.

As exceptionally well-authenticated instances of failures I may mention the twelve cases collected by Cardan in his *Geniturarum Exempla*. These are good examples because Cardan was not only the most eminent astrologer of his time, but was a man of science, and perhaps it is not too much to say was accustomed to accurate habits of thought; moreover, I believe he was honest in his belief in astrology. To English readers the most interesting of these is the horoscope of Edward VI of England, the more so as Cardan has left a full account of the affair, and has entered into the reasons of his failure to predict Edward's death.

To show how Cardan came to be mixed up in the transaction I should explain that in 1552 Cardan went to Scotland to prescribe for John Hamilton, the archbishop of St Andrews, who was ill with asthma and dropsy and about whose treatment the physicians had disagreed*. On his return through London, Cardan stopped with Sir John Cheke, the Professor of Greek at Cambridge, who was tutor to the young king. Six months previously, Edward had been attacked by measles and small-pox which had made his health even weaker than before. The king's guardians were especially anxious to know how long he would live, and they asked Cardan to cast Edward's nativity with particular reference to that point.

The Italian was granted an audience in October, of which he wrote a full account in his diary, quoted in the *Geniturarum Exempla*. The king, says he†, was "of a stature somewhat below the middle height, pale faced, with grey eyes, a grave aspect, decorous, and handsome. He was rather of a bad habit of body than a sufferer from fixed diseases, and had a somewhat projecting shoulder-blade." But, he continues, he was a boy of most extraordinary wit and promise. He was then but fifteen years old and he was already skilled in music and dialectics, a master of Latin, English, French, and fairly proficient in Greek, Italian, and Spanish. He "filled with the highest expectation every good and learned man, on account of his ingenuity and suavity of manners....When a royal gravity was called for, you would think that it was an old man you saw, but he was bland and companionable as became his years. He played upon the lyre, took concern for public affairs, was liberal of mind, and in these respects emulated his father, who, while he studied to be [too] good, managed to seem bad." And in another place‡ he

* Luckily they left voluminous reports on the case and the proper treatment for it. The only point on which there was a general agreement was that the phlegm, instead of being expectorated, collected in his Grace's brains, and that thereby the operations of the intellect were impeded. Cardan was celebrated for his success in lung diseases, and his remedies were fairly successful in curing the asthma. His fee was 500 crowns for travelling expenses from Pavia, 10 crowns a day, and the right to see other patients; the archbishop actually gave him 2300 crowns in money and numerous presents in kind; his fees from other persons during the same time must have amounted to about an equal sum (see Cardan's *De Libris Propriis*, ed. 1557, pp. 159—175; *Consilia Medica*, *Opera*, vol. IX, pp. 124—148; *De Vita Propria*, ed. 1557, pp. 138, 193 *et seq.*).

† I quote from Morley's translation, vol. II, p. 135 *et seq.*

‡ *De Rerum Varietate*, p. 285.

describes him as "that boy of wondrous hopes." At the close of the interview Cardan begged leave to dedicate to Edward a work on which he was then engaged. Asked the subject of the work, Cardan replied that he began by showing the cause of comets. The subsequent conversation, if it is reported correctly, shows good sense on the part of the young king.

I have reproduced above a facsimile of Cardan's original drawing of Edward's horoscope. The horoscope was cast and read with unusual care. I need not quote the minute details given about Edward's character and subsequent career, but obviously the predictions were founded on the impressions derived from the above-mentioned interview. The conclusion about his length of life was that he would certainly live past middle age, though after the age of 55 years 3 months and 17 days various diseases would fall to his lot*.

In the following July the king died, and Cardan felt it necessary for his reputation to explain the cause of his error. The title of his dissertation is *Quae post consideravi de eodem*†. In effect his explanation is that a weak nativity can never be

* *Geniturarum Exempla*, p. 19.
† *Ibid.*, p. 23.

predicted from a single horoscope, and that to have ensured success he must have cast the nativity of every one with whom Edward had come intimately into contact; and, failing the necessary information to do so, the horoscope could be regarded only as a probable prediction.

This was the argument usually offered to account for non-success. A better defence would have been the one urged by Raphael* and by Southey† that there might be other planets unknown to the astrologer which had influenced the horoscope, but I do not think that medieval astrologers assigned this reason for failure.

I have not alluded to the various adjuncts of the art, but astrologers so frequently claimed the power to be able to raise spirits that perhaps I may be pardoned for remarking that I believe some of the more important and elaborate of these deceptions were effected not infrequently by means of mirrors and lenses or perhaps by the use of a magic lantern, the pictures being sometimes thrown on to a fixed surface or a mirror and at other times on to a cloud of smoke which caused the images to move and finally disappear in a fantastic way capable of many explanations‡.

I would conclude by repeating again that though the practice of astrology was so often connected with impudent quackery, yet one ought not to forget that most physicians and men of science in medieval Europe were astrologers or believers in the art. These observers did not consider that its rules were definitely established, and they laboriously collected much of the astronomical evidence that was to crush it. Thus, though there never was a time when astrology was not practised by knaves, there was a period of intellectual development when it was honestly accepted as a difficult but a real science.

* *The Familiar Astrologer*, London, 1832, p. 248.

† *The Doctor*, chap. xcii.

‡ See *ex. gr.* the life of Cellini, chap. xiii, Roscoe's translation, pp. 144—146. See also Sir David Brewster's *Letters on Natural Magic*.

CHAPTER XIX.

CRYPTOGRAPHS AND CIPHERS.

The art of constructing cryptographs or ciphers—intelligible to those who know the key and unintelligible to others—has been studied for centuries. Their usefulness on certain occasions, especially in time of war, is obvious, while their right interpretation may be a matter of great importance to those from whom the key is concealed. But the romance connected with the subject, the not uncommon desire to discover a secret, and the implied challenge to the ingenuity of all from whom it is hidden, have attracted to the subject the attention of many to whom its utility is a matter of indifference.

Among the best known of the older authorities on the subject are J. Tritheim of Spanheim, G. Porta of Naples, G. Cardan, J. F. Niceron, J. Wilkins, and E. A. Poe. More modern writers are J. E. Bailey in the *Encyclopaedia Britannica*, E. B. von Wostrowitz of Vienna, 1881, F. Delastelle of Paris, 1902, and J. L. Klüber of Tübingen, 1809. My knowledge, however, is largely the result of casual reading, and I prefer to discuss the subject as it has presented itself to me, with no attempt to make it historically complete.

Most writers use the words cryptograph and cipher as synonymous. I employ them, however, with different meanings, which I proceed to define.

A cryptograph may be defined as a manner of writing in which the letters or symbols employed are used in their normal sense, but are so arranged that the communication is intelligible only to those possessing the key: the word is also sometimes used to denote the communication made. A simple example is a communication in which every word is spelt backwards. Thus:

ymene deveileb ot eb gniriter troper noitisop no ssorc daor.

A cipher may be defined as a manner of writing by characters arbitrarily invented or by an arbitrary use of letters, words, or characters in other than their ordinary sense, intelligible only to those possessing the key: the word is also sometimes used to denote the communication made. A simple example is when each letter is replaced by the one that immediately follows it in the natural order of the alphabet, *a* being replaced by *b*, *b* by *c*, and so on, and finally *z* by *a*. In this cipher the above message would read:

fofnz cfmjfwfe up cf sfujsjoh sfqpsu qptjujpo po dsptt spbe.

In both cryptographs and ciphers the essential feature is that the communication may be freely given to all the world though it is unintelligible save to those who possess the key. The key must not be accessible to anyone, and if possible it should be known only to those using the cryptograph or cipher. The art of constructing a cryptograph lies in the concealment of the proper order of the essential letters or words: the art of constructing a cipher lies in concealing what letters or words are represented by the symbols used.

In an actual communication cipher symbols may be arranged cryptographically, and thus further hinder a reading of the message. Thus the message given above might be put in a cryptographic cipher as

znfof efwfjmfc pu fc hojsjufs uspqfs opjujtpq op ttpsd ebps.

If the message were written in a foreign language it would further diminish the chance of it being read by a stranger through whose hands it passed. But I may confine myself to messages in English, and for the present to simple cryptographs and ciphers.

A communication in cryptograph or cipher must be in writing or in some permanent form. Thus to make small muscular movements—such, *ex. gr.*, as talking on the fingers, or breathing long and short in the Morse dot and dash system, or making use of pre-arranged signs by a fan or stick, or flashing signals by light—do not here concern us.

The mere fact that the message is concealed or secretly conveyed does not make it a cryptograph or cipher. The majority of stories dealing with secret communications are concerned with the artfulness with which the message is concealed or conveyed and have nothing to do with cryptographs or ciphers. Many of the ancient instances of secret communication are of this type. Illustrations are to be found in messages conveyed by pigeons, or wrapped round arrows shot over the head of a foe, or written on the paper wrapping of a cigarette, or by the use of ink which becomes visible only when the recipient treats the paper on which it is written by some chemical or physical process.

Again, a communication in a foreign language or in any recognized notation like shorthand is not an instance of a cipher. A letter in Chinese or Polish or Russian might be often used for conveying a secret message from one part of England to another, but it fails to fulfil our test that if published to all the world it would be concealed, unless submitted to some special investigation. On the other hand, in practice, foreign languages or systems of shorthand which are but little known may serve to conceal a communication better than an easy cipher, for in the last case the key may be found with but little trouble, while in the other cases, though the key may be accessible, it is probable that there are only a few who know where to look for it.

Cryptographs. I proceed to enumerate some of the better known types of cryptographs. There are at least three distinct types. The types are not exclusive, and any particular cryptograph may comprise the distinctive feature of two or all the types.

First Type of Cryptographs (*Transposition Type*). A cryptograph of the first type is one in which the successive letters or words of the message are re-arranged in some pre-determined manner.

One of the most obvious cryptographs of this type is to write each word or the message itself backwards. Here is an instance in which the whole message is written backwards:

tsop yb tnes tnemeerga fo seniltuo smret ruo tpecca yeht.

It is unnecessary to indicate the division into words by leaving spaces between them, and we might introduce capitals or make a pretence of other words, as thus:

Ts opybtne stne meer gafos eniltu Osmret ruot peccaye ht.

A recipient who was thus mis-led would be very careless. Preferably, according to modern practice, we should write the message in groups of five letters each: the advantage of such a division being that the number of such groups can be also communicated, and the casual omission of letters thus detected.

Systems of this kind which depend on altering the places of letters or lines in some pre-arranged manner have always been common. One example is where the letters which make up the communication are written vertically up or down. Thus the message: *The pestilence continues to increase,* might be written in 8 columns as shown in the margin, and then sent in five letter groups as *eiotn litsn,* etc. If before reading off the message the 8 columns were interchanged according to some prearranged scheme the cryptograph would be greatly improved, and this is said to be a method used in the German Army. This cryptograph might be further obscured by writing the 32 letters according to the Route Method described below.

e i o t n l i t
s n t i o e t h
a c s n c n s e
e r e u e c e p.

Another method is to write successively the 1st, 18th, 35th letters of the original message, and then the 2nd, 19th, 36th letters, and so on. If, however, we know the clue number, say c, it is easy enough to read the communication. For if it divides into the number of letters n times with a remainder r it suffices to re-write the message in lines putting $n+1$ letters in each of the first r lines, and n letters in each of the last $c-r$ lines, and then the communication can be read by reading the columns downwards. For instance, if the following communication, containing 270 letters, were received: *Ahtze ipqhg esoae ouazs esewa eqtmu sfdtb enzce sjteo ttqiz yczht zjioa rhqet tjrfe sftnz mroom ohyea rziaq neorn breot lennk aerwi zesju asjod ezwjz zszjb rritt jnfjl weuzr oqyfo htqay eizsl eopji dihal oalhp epkrh eanaz srvli imosi adygt pekij scerq vvjqj qajqn yjint kaehs bhsnb goaot qetqe uuesa yqurn tpebq stzam ztqrj*, and the clue number were *17* we should put *16* letters in each of the first *15* lines and *15* letters in each of the last *2* lines. The communication could then be discovered by reading the columns downwards: the letters *j*, *q* and *z* marking the ends of words.

A better cryptograph of this kind may be made by arranging the letters cyclically, and agreeing that the communication is to be made by selected letters, as, for instance, every seventh, second, seventh, second, and so on. Thus if the communication were *Ammunition too low to allow of a sortie*, which consists of 32 letters, the successive significant letters would come in the order 7, 9, 16, 18, 25, 27, 2, 4, 13, 15, 24, 28, 5, 8, 20, 22, 1, 6, 21, 26, 11, 14, 32, 10, 31, 12, 17, 23, 3, 29, 30, 19—the numbers being selected as in the decimation problem given above at the end of chapter I, and being struck out from the 32 cycle as soon as they are determined. The above communication would then read *Ttrio oalmo laoon msueo awotn lioti fw*. This is a good method, but it is troublesome to use, and for that reason is not to be recommended.

In another cryptograph of this type, known as the *Route Method*, the words are left unaltered, but are re-arranged in a

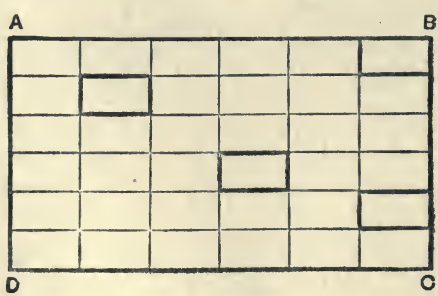
11	8	13	2	15	4	*w*
x	1	10	17	6	*y*	*z*
9	12	7	14	3	16	5

pre-determined manner. Thus, to take a very simple example, the words might be written in tabular form in the order shown in the diagram, certain spaces being filled with dummy words *x*, *y*, *z*, ..., and the message being sent in the order 11, 8, 13, 2, 15, 4, *w*, *x*, 1, ... This method was used successfully by the Federals in the American Civil War, 1861—1865, equivalents for proper names being used. It is easy to work, but the key would soon be discovered by modern experts.

A double cryptograph is said to have been used by the Nihilists in Russia from 1890—1900. Such double transpositions are always awkward, and mistakes, which would make the message unintelligible, may be easily introduced, but if time is of little importance, and the message is unlikely to fall into the hands of any but ordinary officials, the concealment is fairly effective, though a trained specialist who had several messages in it could work out the key.

Second Type of Cryptographs. A cryptograph of the second type is one in which the message is expressed in ordinary writing, but in it are introduced a number of dummies or non-significant letters or digits thus concealing which of the letters are relevant.

One way of picking out those letters which are relevant is by the use of a perforated card of the shape of (say) a sheet of note-paper, which when put over such a sheet permits only such letters as are on certain portions of it to be visible. Such a card is known as a *grille.* An example of a grille with four openings is figured below. A communication made in this



way may be easily concealed from anyone who does not possess a card of the same pattern. If the recipient possesses such a card he has only to apply it in order to read the message. This method was used by Richelieu.

The use of the grille may be rendered less easy to detect if it be used successively in different positions, for instance, with the edges *AB* and *CD* successively put along the top of the paper containing the message. Below, for instance, is a message which, with the aid of the grille figured above, is at once intelligible. On applying the grille to it with the line *AB* along the top *HK* we get the first half of the communication, namely, 1000 *rifles se.* On applying the grille with the

H *K*

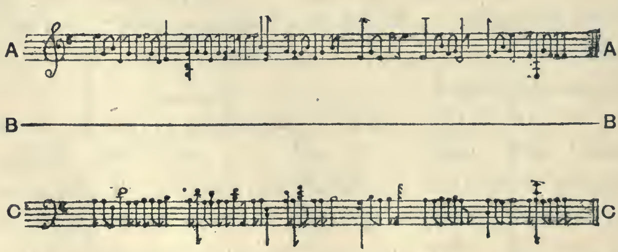
981	264	070	523	479	100
NTT	*ORI*	*SON*	*SON*	*AHY*	*DTC*
BFS	*PUM*	*OLT*	*KFE*	*LJO*	*EGX*
AEU	*QJT*	*EGO*	*FLE*	*HVE*	*WLA*
FML	*AES*	*REM*	*REM*	*ODA*	*SSE*
YZZ	*EPD*	*QJC*	*EKS*	*TIM*	*OEF*

line *CD* along the top *HK* we get the rest of the message, namely, *nt to L to-day.* The other spaces in the paper are filled with non-significant letters or numerals in any way we please. Of course any one using such a grille would not divide the sheet of paper on which the communication was written into cells, but in the figure I have done so in order to render the illustration clearer.

We can avoid the awkward expedient of having to use a perforated card, which may fall into undesired hands, by introducing a certain pre-arranged number of dummies or non-significant letters or symbols between those which make up the message. For instance, we might arrange that (say) only every alternate second and third letter shall be relevant. Thus the first, third, sixth, eighth, eleventh, &c., letters are those that make up the message. Such a communication would be two and a half times as long as the message, and this might be a great disadvantage if time in sending the message was of importance.

Another method, essentially the same as the grille method, is to arrange that every *n*th word shall give the message, the other words being non-significant, though of course inserted as far as possible so as to make the complete communication run as a whole. But the difficulty of composing a document of this kind and its great length render it unsuitable for any purpose except an occasional communication composed at leisure and sent in writing. This method is said to have been used by the Earl of Argyle when plotting against James II.

Third Type of Cryptographs. A kind of secret writing which may perhaps be considered to constitute a third type of cryptograph is a communication on paper which is legible only when the paper is folded in a particular way. An example is a message written across the edges of a strip of paper wrapped spiral-wise round a stick called a *scytale.* When the paper is unwound and taken off the stick the letters appear broken, and may seem to consist of arbitrary signs, but by wrapping the paper round a similar stick the message can be again read. This system is said to have been used by the Lacedemonians. The concealment can never have been effectual against an intelligent reader who got possession of the paper. As another illustration take the appended communication which



is said to have been given to the Young Pretender during his wanderings after Culloden. If it be creased along the lines *BB* and *CC* (*CC* being along the second line of the second score), and then folded over, with *B* inside, so that the crease *C* lies over the line *A* (which is the second line of the first score) thus leaving only the top and bottom of the piece of paper visible, it will be found to read *Conceal yourself, your foes look for you.* I have seen what purports to be the original, but of the truth of the anecdote I know nothing, and the desirability of concealing himself must have been so patent that it was hardly necessary to communicate it by a cryptograph.

Ciphers. I proceed next to some of the more common types of ciphers. It is immaterial whether we employ special characters to denote the various letters; or whether we use the letters in a non-natural sense, such as the letter *z* for *a*, the letter *y* for *b*, and so on. In the former case it is desirable to use symbols, for instance, musical notes, which are not likely to attract special notice. Geometrical figures have also been used for the same purpose. It is not even necessary to employ written signs. Natural objects have often been used, as in a necklace of beads, or a bouquet of flowers, where the different shaped or coloured beads or different flowers stand for different letters or words. An even more subtle form of disguising the cipher is to make the different distances between consecutive knots or beads indicate the different letters. Of all such systems we may say that a careful scrutiny shows that different symbols are being used, and as soon as the various symbols are distinguished one from the other no additional complication is introduced, while for practical purposes they give more trouble to the sender and the recipient than those written in symbols in current use. Accordingly I confine myself to ciphers written by the use of the current letters and numerals. There are four types of ciphers.

First Type of Ciphers. Simple Substitution Alphabets. A cipher of the first type is one in which the same letter or word is always represented by the same symbol, and this symbol always represents the same letter or word.

Perhaps the simplest illustration of a cipher of this type is to employ one language, written as far as practical in the alphabet of another language. It is said that during the Indian Mutiny messages in English, but written in Greek characters, were used freely, and successfully baffled the ingenuity of the enemy, into whose hands they fell.

A common cipher of this type is made by using the actual letters of the alphabet, but in a non-natural sense as indicating other letters. Thus we may use each letter to represent the one immediately following it in the natural order of the alphabet—the letters being supposed to be cyclically arranged—*a* standing for *b* wherever it occurs, *b* standing for *c*, and so on, and finally *z* standing for *a*: this scheme is said to have been used by the Carthaginians and Romans.

More generally we may write the letters of the alphabet in a line, and under them re-write the letters in any order we like. For instance

a b c d e f g h i j k l m n o p q r s t u v w x y z
o l k m a z s q x e u f y r t h c w b v n i d g j p

In such a scheme, we must in our communication replace *a* by *o*, *b* by *l*, etc. The recipient will prepare a key by re-arranging the letters in the second line in their natural order and placing under them the corresponding letter in the first line. Then whenever *a* comes in the message he receives he will replace it by an *e*; similarly he will replace *b* by *s*, and so on.

A cipher of this kind is not uncommonly used in military signalling, the order of the letters being given by the use of a key word. Ciphers of this class were employed by the British forces in the Sudan and South African campaigns. If, for instance, *Pretoria* is chosen as the key word, we write the letters in this order, striking out any which occur more than once, and continue with the unused letters of the alphabet in their natural order, writing the whole in two lines thus:

p r e t o i a b c d f g h
z y x w v u s q n m l k j

Then in using the cipher *p* is replaced by *z* and vice versa, *r* by *y*, and so on. A long message in such a cipher would be easily discoverable, but it is rapidly composed by the sender and read by the receiver, and for some purposes may be useful, especially if the discovery of the purport of the message is, after a few hours, immaterial.

The key to ciphers of this type may usually be found by using tables of the normal frequency with which letters may be expected to occur. Such tables, and other characteristic features of the English, French, German, Italian, Dutch, Latin, and Greek languages, were given by D. A. Conrad in 1742*. His results have since been revised, and extended to Russian, Spanish, and other tongues. In English the percentage scale of frequency of the letters is approximately as follows:—*e*, 12·0; *t*, 9·4; *a*, 7·8; *o*, 7·5; *i*, 7·4; *n*, 7·3; *s*, 6·8; *r*, 5·9; *h*, 5·7; *d*, 3·9; *l*, 3·6; *u*, 3·0; *c*, 2·8; *m*, 2·7; *f*, 2·5; *p*, 1·9; *g*, 1·8; *y*, 1·8; *b*, 1·7; *w*, 1·7; *v*, 1·1; *k*, 0·6; *j*, 0·3; *q*, 0·3; *x*, 0·3; *z*, 0·2. The order of frequency for combinations of two letters is *th, he, in, an, on, re, ti, er, it, nt, es, to, st*; of three letters is *the, ion*, &c., &c.; of four letters is *tion, that*, &c., &c.; and of double letters is *tt, ss*, &c., &c. Other peculiarities, such as that *h, l, m, n, v*, and *y*, when at the beginning of a word, must be followed by a vowel, that *q* must be followed by *u* and another vowel, have been classified and are important. I need not go here into further details. Unless, however, the message runs to 400 words or more, we cannot reasonably expect to find the scale of frequency the same as in Conrad's Table.

In ciphers of this class it is especially important to avoid showing the division into words, for a long word may easily betray the secret. For instance, if the decipherer has reason to suspect that the message related to something connected with Birmingham, and he found that a particular word of ten letters had its second and fifth letters alike, as also its fourth and tenth letters, he would naturally see how the key would work if the word represented Birmingham, and on this hypothesis would at once know the letters represented by eight symbols. With reasonable luck this should suffice to enable him to tell if the hypothesis was tenable. To avoid this risk it

* *Gentleman's Magazine*, 1742, vol. XII, pp. 133—135, 185—186, 241—242, 473—475. See also the *Collected Works of E. A. Poe* in 4 volumes, vol. I, p. 30 *et seq.*

is usual to send the cipher in groups of five letters, and, before putting it into cipher, to separate the words in the message by letters like *j*, *q*, *x*, *z*.

Ciphers of this type suggest themselves naturally to those approaching the subject for the first time, and are commonly made by merely shifting the letters a certain number of places forward. If this is done we may decrease the risk of detection by altering the amount of shifting at short (and preferably irregular) intervals. Thus it may be agreed that if initially we shift every letter one place forward then whenever we come to the letter (say) *n* we shall shift every letter one more place forward. In this way the cipher changes continually, and is essentially changed to one of the third class; but even with this improvement it is probable that an expert would decode a fairly long message without much difficulty.

We can have ciphers for numerals as well as for letters: such ciphers are common in many shops. Any word or sentence containing ten different letters will answer the purpose. Thus, an old tradesman of my acquaintance used the excellent precept *Be just O Man*—the first letter representing 1, the second 2, and so on. In this cipher the price 10/6 would be marked *bn/t*. This is an instance of a cipher of the first type.

Second Type of Ciphers. A cipher of the second type is one in which the same letter or word is, in some or all cases, represented by more than one symbol, and this symbol always represents the same letter or word. Such ciphers were uncommon before the Renaissance, but the fact that to those who held the key they were not more difficult to write or read than ciphers of the first type, while the key was not so easily discovered, led to their common adoption in the seventeenth century.

A simple instance of such a cipher is given by the use of numerals to denote the letters of the alphabet. Thus *a* may be represented by 11 or by 37 or by 63, *b* by 12 or by 38 or by 64, and so on, and finally *z* by 36 or by 62 or by 88, while we can use 89 or 90 to signify the end of a word and the numbers 91 to 99 to denote words or sentences which constantly occur. Of course in practice no one would employ the numbers in an order like this, which suggests their meaning, but it will serve to illustrate the principle.

The cipher can be improved by introducing after every (say) eleventh digit a non-significant digit. If this is done the recipient of the message must erase every twelfth digit before he begins to read the message. With this addition the difficulty of discovering the key is considerably increased.

The same principle is sometimes applied with letters instead of numbers. For instance, if we take a word (say) of n letters, preferably all different, and construct a table as shown below of n^2 cells, each cell is defined by two letters of the key word. Thus, if we choose the word *smoking-cap* we shall have 100

	S	*M*	*O*	*K*	*I*	*N*	*G*	*C*	*A*	*P*
S	a	b	c	d	e	f	g	h	i	j
M	k	l	m	n	o	p	q	r	s	t
O	u	v	w	x	y	z	a	b	c	d
K	e	f	g	h	i	j	k	l	m	n
I	o	p	q	r	s	t	u	v	w	x
N	y	z	a	b	c	d	e	f	g	h
G	i	j	k	l	m	n	o	p	q	r
C	s	t	u	v	w	x	y	z		
A										
P										

cells, and each cell is determined uniquely by the two letters denoting its row and column. If we fill these cells in order with the letters of the alphabet we shall have a system similar to that explained above, where *a* will be denoted by *ss* or *og* or *no*, and so for the other letters. The last 22 cells may be used to denote the first 22 letters of the alphabet, or better, three or four of them may be used after the end of a word to show that it is ended, and the rest may be used to denote words or sentences which are likely to occur frequently. The statement in cipher however is twice as long as when it is in clear.

Like the similar cipher with numbers this can be improved by introducing after every *m*th letter any single letter which it is agreed shall be non-significant. To decipher a communication so written it is necessary to know the clue word and the clue number.

Here for instance is a communication written in the above cipher with the clue word *smoking-cap*, and with 7 as the clue number: *ngmks igrio icpss amcks cakqi gnass nxmig poasu iamno cmpam inscn ogcpn cisyi kskam sssgn nncae kknoo mkhsc pcmsc bgpng siaws sgigg ndiic a*. In this sentence the letters denoting the 79th, 80th, 81st, and 82nd cells have been used to denote the end of a word, and no use has been made of the last 18 cells.

Another cipher of this type is made as follows. The sender and recipient of the message furnish themselves with identical copies of some book. In the cipher only numerals are used, and these numerals indicate the locality of the letters in the book. For example, the first letter in the communication might be indicated by 79–8–5, meaning that it is the 5th letter in the 8th line of the 79th page. But though secrecy might be secured, it would be very tedious to prepare or decode a message, and the method is not as safe as some of those described below.

Another cipher of this type is obtained by the sender and receiver agreeing on some common book of reference and further on a number which, if desired, may be communicated as part of the message. To employ this method the page of the book indicated by the given number must be used. The first letter in it is taken to signify *a*, the next *b*, and so on—any letter which occurs a second time or more frequently being neglected. It may be also arranged that after *n* letters of the message have been ciphered, the next *n* letters shall be written in a similar cipher taken from the *p*th following page of the book, and so on. Thus the possession of the code-book would be of little use to anyone who did not also know the numbers employed. It is so easy to conceal the clue number that with ordinary prudence it would be almost impossible for an unauthorized person to discover a message sent in this cipher.

The clue number may be communicated indirectly in many ways. For instance, it may be arranged that the number to be used shall be the number sent, plus (say) q, or that the number to be used shall be an agreed multiple of the number actually sent.

Third Type of Ciphers. Complex Shifting Alphabets. A cipher of the third type is one in which the same symbol represents sometimes one letter or word and sometimes another.

A simple example, known as *Gronfeld's Method*, is the employment of pre-arranged numbers in shifting forward the letters that make the communication. For instance, if we agree on the key number 6814, then the first letter in the communication is replaced by the sixth letter which follows it in the natural order of the alphabet: for instance, if it were an *a* it would be replaced by *g*. The next letter is replaced by the eighth letter which follows it in the natural order of the alphabet: for instance, if it were an *a* it would be replaced by *i*. The next letter is replaced by the first after it; the next by the fourth after it; the next by the sixth; and so on to the end of the message. Of course to read the message the recipient would reverse the process. If the letters of the alphabet are written at uniform intervals along a ruler, and another ruler similarly marked with the digits is made to slide along it, the letter corresponding to the shifting of any given number of places can be read at once. Here is such a message:—*Cisvg vumya vijnp vgzsi ybpjp woiy.* Such ciphers are easy to make and read by those who have the key. But in recent years their construction has been subjected to critical analysis, and experts now can generally obtain the key number if the message contains 80 or 100 words; an example of the way by which this is done is given below. It would be undesirable to allow the division into words to appear in the message, and either the words must be run on continuously, or preferably the less common letters *j*, *q*, *z* may be used to mark the division of words and the message then written in five letter groups.

It is most important to conceal the number of digits in the key number. The difficulty of discovering the key number is increased if after every (say) *m*th letter (or word) a non-significant letter is inserted. I suggest this as an improvement in the cipher.

Here for instance is a communication written in this cipher with the clue numbers 4276 and 7: *atpzn hvaxu xhiep xafwg hzniy prpsi kbdkz yygkq prgez uytlk obldi febzm xlpog quyit cmgxk ckuex vsqka ziagg sigay tnvvs styvu aslyw gjuzm csfct qbpwj vaepf xhibw pxiul txlav vtqzo xwkvt uvvfh cqbxn pvism phzmq tuwxj ykeev ltif.* The recipient would begin by striking out every eighth letter. He would then shift back every letter 4, 2, 7, 6, 4, 2, &c., places respectively, and in reading it, would leave out the letters *j*, *q*, and *z* as only marking the ends of words.

With these modifications, this is an excellent cipher, and it has the additional merit of not materially lengthening the message. It can be rendered still more difficult by arranging that either or both the clue numbers shall be changed according to some definite scheme, and it may be further agreed that they shall change automatically every day or week.

A similar system, now known as the *St Cyr Method*, was proposed by Wilkins*. He took a key word, such as *prudentia*, and constructed as many alphabets as there were letters in it, each alphabet being arranged cyclically and beginning respectively with the letters *p*, *r*, *u*, *d*, *e*, *n*, *t*, *i*, and *a*. He thus got a table like the following, giving nine possible letters which might stand for any letter of the alphabet. Using this we may vary the cipher in successive words or letters of the communication. Thus the message *The prisoners have mutinied and seized the railway station* would, according as the cipher changes in successive words or letters, read as *Hwt fhziedvhi bupy pxwmqmhg erh ervmrq max zirteig station* or as *Hyy svvlwnthm lehx uuhzgmiq tvd gvcciq mqe frcoanr atpkcrr.*

The name by which the method is known is derived from the fact that it was taught at St Cyr under Napoleon. This system is said to have been widely employed by both armies in the Franco-German war in 1870—1871. The construction of military ciphers must be so simple that messages can be rapidly

* *Mercury*, by J. Wilkins, London, 1641, pp. 59, 60.

enciphered and decoded by non-experts: the St Cyr code fulfils this requirement.

a	b	c	d	e	f	g	h	i	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	k	l	m	n	o
r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	k	l	m	n	o	p	q
u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	k	l	m	n	o	p	q	r	s	t
d	e	f	g	h	i	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c
e	f	g	h	i	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d
n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	k	l	m
t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	k	l	m	n	o	p	q	r	s
i	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h
a	b	c	d	e	f	g	h	i	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

The St Cyr scheme is essentially the same as Gronfeld's. For instance, in the St Cyr system the key word *gibe* leads to the same result as the key number 6814 by Gronfeld's method. One advantage of the St Cyr system over that advocated by Gronfeld is that key words are more easily recollected than key numbers. Another advantage comes from the fact that the employment of words is equivalent to using 26 digits instead of 10: thus the key word *kings* is equivalent to a number whose digits, from left to right, are 11, 9, 14, 7, 19. Messages in this cipher of any considerable length can be read by the same rules as are used to discover the key in Gronfeld's code. To hamper a decipherer I recommend the introduction, as in Gronfeld's method, of a non-significant letter after every *m*th letter.

The *Beaufort Cipher*, introduced in the British Navy by Admiral Beaufort in 1857, is of the St Cyr type. Its inventor thought it insoluble, but French writers have shown that no special difficulties occur in the discovery of the key word in the solution, though the analysis is tedious.

A far better system of this kind is the *Playfair Cipher*. In this 25 cells arranged in the form of a square are filled by the letters of a key word such as Manchester (striking out any letter which occurs more than once) and the remaining letters of the alphabet, thus:—

m	*a*	*n*	*c*	*h*
e	*s*	*t*	*r*	*b*
d	*f*	*g*	*i*	*j*
k	*l*	*o*	*p*	*u*
v	*w*	*x*	*y*	*z*

Only 25 cells are available, so *k* has been used for both *k* and *q*. The message is then divided into pairs of consecutive letters, but to prevent any pair consisting of the same two letters, a dummy letter like *z* is, when necessary, introduced. If both letters of a pair appear in the same vertical (or horizontal) line of the square, each of them is replaced by the letter in the square immediately to its right (or below it)—the letters in every line being treated as in cyclical order. If the letters in a pair do not appear in the same line in the square they must necessarily be at opposite angles of some rectangle, and they are replaced by those at the other angles of the rectangle, each by that which is in the same horizontal line. Thus the message *will meet you at noon* would first be written *wi lz lm ez et yo ua tn oz on*; then be put in cipher as *yf uw ka bv sr xp lh gt ux pc*; and finally be sent as *yfuwk abvsr xplhg tuxpc.* It is curious that this cipher is not used more extensively, for the discovery of the key is difficult, even to specialists.

Fourth Type of Ciphers. A cipher of the fourth type is one in which each letter is always represented by the same symbol, but more than one letter may be represented by the same symbol. Such ciphers were not uncommon at the beginning of the nineteenth century, and were usually framed by means of a key sentence containing about as many letters as there are letters in the alphabet.

Thus if the key phrase is *The fox jumped over the garden gate,* we write under it the letters of the alphabet in their usual sequence as shown below:

T h e f o x j u m p e d o v e r t h e g a r d e n g a t e.
a b c d e f g h i j k l m n o p q r s t u v w x y z a b c.

Then we write the message replacing *a* by *t* or *a*, *b* by *h* or *t*, *c* by *e*, *d* by *f*, and so on. Here is such a message. *M foemho nea ge eoo jmdhohg avf teg ev ume afrmeo.* But it will be observed that in the cipher *a* may represent *a* or *u*, *d* may represent *l* or *w*, *e* may represent *c* or *k* or *o* or *s* or *x*, *g* may represent *t* or *z*, *h* may represent *b* or *r*, *o* may represent *e* or *m*, *r* may represent *p* or *v*, and *t* may represent *a* or *b* or *q*. And the recipient, in deciphering it, must judge as best he can what is the right meaning to be assigned to these letters when they appear.

An instance of a cipher of the fourth type is afforded by a note sent by the Duchesse de Berri to her adherents in Paris, in which she employed the key phrase

l e g o u v e r n e m e n t p r o v i s o i r e.
a b c d e f g h i j k l m n o p q r s t u v x y.

Hence in putting her message into cipher she replaced *a* by *l*, *b* by *e*, *c* by *g*, and so on. She forgot however to supply the key to the recipients of the message, but her friend Berryer had little difficulty in reading it by the aid of the rules I have indicated, and thence deduced the key phrase she had employed.

Desiderata in Cryptographs and Ciphers. Having mentioned various classes of cryptographs and ciphers, I may add that the shorter a message in cryptograph, the more easily it is read. On the other hand, the longer a message in cipher, the easier it is to get the key. In choosing a cipher for practical purposes, which will usually imply that it can be telegraphed or telephoned, we should seek for one in which only current letters, symbols, or words are employed; such that its use does not unduly lengthen the message; such that the key to it can be reproduced at will and need not be kept in a form which might betray the secret to an unauthorized person; such that the key to it changes or can be changed at short intervals; and such that it is not ambiguous. Many ciphers of the second and third types fulfil these conditions; in particular the Gronfeld or St Cyr Method, or the Playfair Cipher, may be noted. A cipher written cryptographically, or a cryptograph written in cipher, or a cipher again enciphered by another process, is almost insoluble even by experts, unless accidents reveal something in the construction, but it is troublesome to make, and such elaborate processes are suited only for the study, where the time spent in making them up and deciphering them is not of much consequence.

Cipher Machines. The use of instruments giving a cipher, which is or can be varied constantly and automatically, has been often recommended*. The possession of the key of the instrument as well as a knowledge of the clue word is necessary to enable anyone to read a message, but the risk of some instrument, when set, falling into unauthorized hands must be taken into account. Since equally good ciphers can be constructed without the use of mechanical devices I do not think their employment can be recommended.

On the Solution of Cryptographs and Ciphers. Much ingenuity has been shown in devising means for reading messages written in cryptograph or cipher. It is a fascinating pursuit, but I can find space for only a few remarks about it.

In such problems we must begin by deciding whether the message is a cryptograph or a cipher. If it is a combination of both, the problem is one of extreme difficulty, and is likely to baffle anyone but a specialist, but such combinations are unusual, and most secret messages belong to one class or the other.

If the scale of frequency of the letters agrees generally with Conrad's Table, presumably the message is in cryptograph,

* See, for instance, the descriptions of those devised by Sir Charles Wheatstone, given in his *Scientific Papers*, London, 1879, pp. 342—347; and by Capt. Bazeries in *Comptes Rendus*, *Association Français pour l'Avancement des Sciences*, vol. xx (Marseilles), 1891, p. 160 *et seq.*

though we must allow for the possibility that dummy letters, like *j*, *k*, *x*, and *z*, have been introduced either to separate words or deliberately to confuse those not in the secret. A short sentence of this kind may be read by an amateur, but only an expert is likely to discover the key to a long and well constructed cryptographic message.

If the message is long enough, say about 80 words, and the scale of frequency of particular letters differs markedly from Conrad's scale, there is a presumption that the message is in cipher. If the numbers of the two scales agree generally, probably a simple substitution alphabet has been used, *i.e.* it is a cipher of the first type, and generally the discovery of the key is easy. If it is not a cipher of this type, we must next try to find whether it is of any of the other recognised types. The majority of other ciphers are included in Gronfeld's number (or the St Cyr word) system, and here I will confine myself to a discussion of how such ciphers may be read.

The discovery of a key to a cipher of this kind is best illustrated by a particular case. I will apply the method to the message *cisvg vumya vijnp vgzsi ybpjp woiy*. This is an example of a Gronfeld's cipher with no additional complications introduced, but the message is short, and it so happens that the letters used are not in the normal scale of frequency; yet it can be read with ease and certainty.

The first thing is to try to find the number of digits in the key number. Now we notice that the pair of letters *vg* occurs twice, with an interval of 12. If in each case these represent the same pair of letters in the original message, the number of digits in the key number must be 12 or a divisor of 12. Again the pair of letters *ij* occurs twice, with an interval of 8, and this suggests that the number of digits in the key number is 8 or a divisor of 8. Accordingly we conjecture that the key number is one of either 2 or 4 digits: this conclusion is strengthened by noting the intervals between the recurrences of the same letters throughout the message. We may put 2 on one side till after we have tried 4, for anyone using Gronfeld's method would be unlikely to employ a key number less than 100. Accordingly we first try 4, and if that fails try 2. Had no clue of this kind been obtained from the recurrence of a pair of letters, we should have had to try successively making the key numbers comprise 2, 3, 4, 5, ... digits, but here (and in most messages) a cursory examination suggests the number of digits in the key number. We commence then by assuming provisionally that the key number has 4 digits. Accordingly we must now re-write our message in columns, each of 4 letters, giving altogether 4 lines, thus:

c g y j g y p y
i v a n z b w
s u v p s p o
v m i v i j i

If the Gronfeld method was used, the letters in each of these lines were obtained from the corresponding letters in the original message by a simple substitution alphabet. Had the message been long we could probably obtain this alphabet at once by Conrad's Table. Here, however, the message is so short that the Table is not likely to help us decisively, and we must expect to be obliged to try several shifts of the alphabet in each line.

In the first line *y* occurs three times, and *g* twice. According to Conrad's Table, the most common letters in English are *e, t, a, o, i, n, s, r, h*. Probably *y* stands for one of these and *g* for another. If *y* is made to stand successively for each of these, it is equivalent to putting every letter θ places backward, where θ is successively 20, 5, 24, 10, 16, 11, 6, 7, 17. Similarly, making *g* stand successively for *e, t, a, o, i, n, s, r, h*, we have θ equal to 2, 13, 6, 18, 24, 19, 14, 15, 25. Altogether this gives us 16 systems for the representation of the first line. We might write these out on 16 slips, and provisionally reject any slip in which many unusual letters appear, but obviously, the most probable hypothesis is that where *y* stands for *s*, and *g* for *a*, both of which changes give $\theta = 6$, or that where *y* stands for *a*, and *g* for *i*, both of which changes give $\theta = 24$: these give for the first line either *w, a, s, d, a, s, j, s*, or *e, i, a, l, i, a, r, a*.

In the second line no letter occurs more than once, so we get no clue from Conrad's Table. This could not happen if the message were of any considerable length.

In the third line *p* occurs twice, and *s* twice. Hence, as before, we must make *p* and *s* successively stand for the letters *e, t, a, o, i, n, s, r, h*. These give respectively $\theta = 11, 22, 15, 1, 7, 2, 23, 24, 8$, and $\theta = 14, 25, 18, 4, 10, 5, 0, 1, 11$. Altogether this gives us 16 systems for the representation of this line. Obviously the most probable hypothesis is that where $\theta = 11$, $p = e$, $s = h$, or that where $\theta = 1$, $p = o$, and $s = r$: these give for the third line either *h, j, k, e, h, e, d*, or *r, t, u, o, r, o, n*.

In the fourth line *i* occurs three times. As before, make *i* stand successively for *e, t, a, o, i, n, s, r, h*. Of these the first, where $\theta = 4$, is the most probable. The slip corresponding to this is *r, i, e, r, e, f, e*.

Now try combinations of these slips each in its proper line until, when we read the message in columns, we get the beginning of a word; if words appear in more than one column it is almost certain that we are right. We begin by taking the five slips which are indicated as being specially probable. The slip in the first line derived from $\theta = 6$, the slip in the third line derived from $\theta = 1$, and the slip derived from $\theta = 4$ in the fourth line give *w . rra . tis . ued . ora . res . ofj . nes*, and of course the solution is obvious. The key number was 6814, and the message is deciphered by using 6814 backwards. The corresponding St Cyr key word is *gibe*. The message was *Warrant issued for arrest of Jones*.

If the combination of the slips is troublesome we can sometimes get assistance by choosing those combinations which make the recurring pairs of letters (here *vg* and *ij*) represent pairs which occur in Conrad's Table. Also the occurrence of double letters in the cipher will often settle what combinations of slips are possible.

It may be said that this is a tedious operation. Of course it is. Deciphering is bound to be troublesome, but a great deal of the work can be done by unskilled clerks working under the direction of experts. The longer the message, the fewer the slips we have to try, and had the above message been three times as long, we could have solved the problem with half the trouble. The above example was not complicated by employing dummy letters or artificial alphabets: their use increases the difficulty of the decipherer, but if the message is a long one, the difficulties are not insuperable. Specialists, especially if working in combination, are said to select the right methods with almost uncanny quickness.

This chapter has already run to such a length that I cannot find space to describe more than one or two ciphers that appear in history.

It is said that Julius Caesar in making secret memoranda was accustomed to move every letter four places forward, writing *d* for *a*, *e* for *b*, &c. This would be a very easy instance of a cipher of the first type, but it may have been effective at that time. His nephew Augustus sometimes used a similar cipher, in which each letter was moved forward one place*.

Bacon proposed a cipher in which each letter was denoted by a group of five letters consisting of *A* and *B* only. Since there are 32 such groups, he had 6 symbols to spare, which he could use to separate words or to which he could assign special meanings. A message in this cipher would be five times as long as the original message. This may be compared with the far superior system of the five (or four) digit code-book system in use at the present time.

In the Morse code employed in telegraphy, as in the Baconian system, only two signs are used, commonly a dot or a short mark or a motion to the left, and a dash or a long mark or a motion to the right. The Morse Alphabet is as follows: *a* (· —), *b* (— · · ·), *c* (— · — ·), *d* (— · ·), *e* (·), *f* (· · — ·), *g* (— — ·), *h* (· · · ·), *i* (· ·), *j* (· — — —), *k* (— · —), *l* (· — · ·), *m* (— —), *n* (— ·), *o* (— — —), *p* (· — — ·), *q* (— — · —), *r* (· — ·), *s* (· · ·), *t* (—), *u* (· · —), *v* (· · · —), *w* (· — —), *x* (— · · —),

* Of some of Caesar's correspondence, Suetonius says (cap. 56) *si quis investigare et persequi velit, quartam elementorum literam, id est, d pro a, et perinde reliquas commutet.* And of Augustus he says (cap. 88) *quoties autem per notas scribit, b pro a, c pro b, ac deinceps eadem ratione, sequentes literas ponit; pro x autem duplex a.*

y (— · — —), *z* (— — · ·). Since there are 30 possible permutations of two signs taken not more than four together, this leaves four signals unemployed, (— — — —), (— — — ·), (· · — —), (· — · —), which might have been utilized for special signals. In telegraphy there are also recognized signs or combinations for numerals, for the ends of words and messages, and for various calls between the sender and the recipient of a message.

Charles I used ciphers freely in important correspondence —the majority being of the second type. He was foolish enough to take a cabinet, containing many confidential notes in cipher, with him to the field of Naseby, where they fell into the hands of Fairfax*. In these papers each letter was represented by a number. Clues were provided by the King who had written over the number the letter which it represented. Thus in two letters written in 1643, *a* is represented by 17 or 18, *b* by 13, *c* by 11 or 12, *d* by 5, *e* by 7 or 8 or 9 or 10, *f* by 15 or 16, *g* by 21, *h* by 31 or 32, *i* by 27 or 28, *k* by 25, *l* by 23 or 24, *m* by 42 or 44, *n* by 39 or 40 or 41, *o* by 35 or 36 or 37 or 38, *p* by 33 or 34, *r* by 50 or 51 or 52, *s* by 47 or 48, *t* by 45 or 46, *u* by 62 or 63, *w* by 58, and *y* by 74 or 77. Numbers of three digits were used to represent particular people or places. Thus 148 stood for *France*, 189 for the *King*, 260 for the *Queen*, 354 for *Prince Rupert*, and so on. Further, there were a few special symbols, thus *k*1 stood for *of*, *n*1 for *to*, and *f*1 for *is*. The numbers 2 to 4 and 65 to 72 were non-significant, and were to be struck out or neglected by the recipient of the message. Each symbol is separated from that which follows it by a full-stop.

A similar, though less elaborate, system was used by the French in the Peninsular War. An excellent illustration of the inherent defects of this method is to be found in the writings of the late Sir Charles Wheatstone. A paper in cipher, every page of which was initialed by Charles I, and countersigned by Lord Digby, was purchased some years

* *First Report of the Royal Commission on Historical Manuscripts*, 1870, pp. 2, 4.

ago by the British Museum. It was believed to be a state paper of importance. It consists of a series of numbers without any clue to their meaning, or any indication of a division between words. The task of reading it was rendered the more difficult by the supposition, which proved incorrect, that the document was in English; but notwithstanding this, Sir Charles Wheatstone discovered the key*. In this cipher *a* was represented by any of the numbers 12, 13, 14, 15, 16, or 17, *b* by 18, 19, and so on, while some 65 special words were represented by particular numbers: in all about 150 different symbols were used.

The famous diary of Samuel Pepys is commonly said to have been written in cipher, but in reality it is written in shorthand according to a system invented by T. Shelton†. It is however somewhat difficult to read, for the vowels are usually omitted, and Pepys used some arbitrary signs for terminations, particles, and certain words—so far turning it into a cipher. Further, in certain places, where the matter is such that it can hardly be expressed with decency, he changed from English to a foreign language, or inserted non-significant letters. Shelton's system had been forgotten when attention was first attracted to the diary. Accordingly we may say that, to those who first tried to read it, it was written in cipher, but Pepy's contemporaries would have properly described it as being written in shorthand, though with a few modifications of his own invention.

A system of shorthand specially invented for the purpose is a true cipher. Such a system in which the letters were represented by four strokes varying in length and position was employed by Charles I. Another such system in which each letter is represented either by a dot or by a line of constant length was used by the Earl of Glamorgan, better known by his subsequent title of Marquess of Worcester, in 1645; each of these

* The document, its translation, and the key used, are given in Wheatstone's *Scientific Papers*, London, 1879, pp. 321—341.

† *Tachy-graphy*, by T. Shelton. The earliest edition I have seen is dated 1641. A somewhat similar system by W. Cartwright was issued by J. Rich under the title *Semographie*, London, 1644.

was a cipher of the first type and had the defects inherent in almost every cipher of this kind: in fact Glamorgan's letter was deciphered, and the system was discovered by H. Dircks*. Obsolete systems of shorthand† may be thus used as ciphers.

It is always difficult to read a very short message in cipher, since necessarily the clues are few in number. When the Chevalier de Rohan was sent to the Bastille, on suspicion of treason, there was no evidence against him except what might be extracted from Monsieur Latruaumont. The latter died without making any admission. De Rohan's friends had arranged with him to communicate the result of Latruaumont's examination, and accordingly in sending him some fresh body linen they wrote on one of the shirts *Mg dulhxcclgu ghj yxuj, lm ct ulgc alj.* For twenty-four hours de Rohan pored over the message, but, failing to read it, he admitted his guilt, and was executed November 27, 1674. The cipher is a simple one of the first type, but the communication is so short that unless the key were known it would not be easy to read it. Had de Rohan suspected that the second word was *prisonnier*, it would have given him 7 out of the 12 letters used, and as the first and third words suggest the symbols used for *l* and *t*, he could hardly have failed to read the message.

Marie Antoinette used what was in effect a St Cyr cipher, consisting of 11 substitution alphabets employed in succession. The first alphabet was *n, o, p, ... z, a, b, ... l, m*; the next, *o, p, q, ... m, n*; the next *p, q, r, ... n, o*; and so on. An expert would easily read a message in this cipher.

One of the systems in use to-day is the five digit code-book cipher, to which I have already alluded. In this, a code dictionary is prepared in which every word likely to be used is printed, and the words are numbered consecutively 00000, 00001, ... up, if necessary, to 99999. Thus each word is

* *Life of the Marquis of Worcester* by H. Dircks, London, 1865. Worcester's system of shorthand was described by him in his *Century of Inventions*, London, 1663, sections 3, 4, 5.

† Various systems, including those used in classical and medieval times, are described in the *History of Shorthand* by T. Anderson, London, 1882.

represented by a number of five digits, and there are 10^5 such numbers available. The message is first written down in words. Below that it is written in numbers, each word being replaced by the number corresponding to it. To each of these numbers is added some definite pre-arranged clue number—the words in the dictionary being assumed to be arranged cyclically, so that if the resulting number exceeds 10^5 it is denoted only by the excess above 10^5. The resulting numbers are sent as a message. On receipt of the message it is divided into consecutive groups of five numbers, each group representing a word. From each number is subtracted the pre-arranged clue number, and then the message can be read off by the code dictionary. If and when such a message is published, the construction of the sentences is usually altered before publication, so that the key may not be discoverable by anyone in possession of the code-book or who has seen the cipher message. This is a rule applicable to all cryptographs and ciphers.

This is a cipher with 10^5 symbols, and as each symbol consists of five digits, a message of n words is denoted by $5n$ digits, and probably is not longer than the message when written in the ordinary way. Since however the number of words required is less than 10^5, the spare numbers may be used to represent collocations of words which constantly occur, and if so the cipher message may be slightly shortened.

If the clue number is the same all through the message it would be possible by not more than 10^5 trials to discover the message. This is not a serious risk, but, slight though it is, it can be avoided if the clue number is varied; the clue number might, for instance, be 781 for the next three words, 791 for the next five words, 801 for the next seven words, and so on. Further it may be arranged that the clue numbers shall be changed every day; thus on the seventh day of the month they might be 781, 791, &c., and on the eighth day 881, 891, &c., and so on.

This cipher can however be further improved by inserting at some step, say after each mth digit, an unmeaning digit. For example, if, in the original message written in numbers, we insert a 9 after every seven digits we shall get a collection of words (each represented by five digits), most of which would have no connection with the original message, and probably the number of digits used in the message itself would no longer be a multiple of 5. Of course the receiver has only to reverse the process in order to read the message.

It is however unnecessary to use five symbols for each word. For if we make a similar code with the twenty-six letters of the alphabet instead of the ten digits, four letters for each word or phrase would give us 26^4, that is, 456976 possible variations. Thus the message would be shorter and the power of the code increased. Further, if we like to use the ten digits and the twenty-six letters of the alphabet—all of which are easily telegraphed—we could, by only using three symbols, obtain 36^3, that is, 46656 possible words, which would be sufficient for all practical purposes.

This code, at any rate with these modifications, is undecipherable by strangers, but it has the disadvantages that those who use it must always have the code dictionary available, and that it takes a considerable time to code or decode a communication. For practical purposes its use would be confined to communications which could be deciphered at leisure in an office.

NOTE. Mr C. H. Harrison has pointed out that an objection to the Playfair system described above on pages 411—412 is that if a particular word is repeated its central letters can appear in the cipher in only two forms. If this word is a long one and it is guessed correctly the key-word can generally be found.

Mr Harrison has devised a slide-rule cipher of the complex shifting substitution type but such that the recurrence of the shifts is non-cyclical. I have not space to describe it here, but it seems safer than most of those mentioned in this chapter, and its use, in an office with intelligent clerks, would present no difficulty. I am not however convinced that it would be easier or safer to use than a simple cryptographic cipher. (See p. 396.)

CHAPTER XX.

HYPER-SPACE*.

I propose to devote the remaining pages to the consideration, from the point of view of a mathematician, of certain properties of space, time, and matter, and to a sketch of some hypotheses as to their nature. Philosophers tell us that space, time, and matter are the "categories under which physical phenomena are concerned." They cannot be defined, and such

* On the possibility of the existence of space of more than three dimensions see C. H. Hinton, *Scientific Romances*, London, 1886, a most interesting work, from which I have derived much assistance in compiling the earlier part of this chapter; his later work, *The Fourth Dimension*, London, 1904, may be also consulted. See also G. F. Rodwell, *Nature*, May 1, 1873, vol. VIII, pp. 8, 9; and E. A. Abbott, *Flatland*, London, 1884.

On Non-Euclidean geometry, see chapter XIV above. The theory is due primarily to N. I. Lobatschewsky, *Geometrische Untersuchungen zur Theorie der Parallellinien*, Berlin, 1840 (originally given in a lecture in 1826); to C. F. Gauss (*ex. gr.* letters to Schumacher, May 17, 1831, July 12, 1831, and Nov. 28, 1846, printed in Gauss's collected works); and to J. Bolyai, Appendix to the first volume of his father's *Tentamen*, Maros-Vásárkely, 1832; though the subject had been discussed by J. Saccheri as long ago as 1733: its development was mainly the work of G. F. B. Riemann, *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, written in 1854, *Göttinger Abhandlungen*, 1866–7, vol. XIII, pp. 131—152 (translated in *Nature*, May 1 and 8, 1873, vol. VIII, pp. 14—17, 36—37); H. L. F. von Helmholtz, *Göttinger Nachrichten*, June 3, 1868, pp. 193—221; and E. Beltrami, *Saggio di Interpretazione della Geometria non-Euclidea*, Naples, 1868, and the *Annali di Matematica*, series 2, vol. II, pp. 232—255: see an article by von Helmholtz in the *Academy*, Feb. 12, 1870, vol. I, pp. 128—131. In recent years the theory has been treated by several mathematicians.

On hyper-space, see V. Schlegel, *Enseignement Mathématique*, Paris, vol. II, 1900; and D. M. Y. Sommerville, *Bibliography of Non-Euclidean Geometry*, St Andrews, 1911.

explanations of them as have been offered involve difficulties of the highest order and are far from simplifying our conceptions of them. I shall not discuss the metaphysical theories that profess to account for the origin of our conceptions of them, for these theories rest on assertions which are incapable of definite proof—a foundation which does not commend itself to a scientific student. The means of measuring space, time, and mass, and the investigation of their properties fall within the domain of mathematics.

I devote this chapter to considerations connected with space, leaving the subjects of time and mass to the following two chapters.

I confine my remarks to two speculations which recently have attracted considerable attention. These are (i) the possibility of the existence of space of more than three dimensions, and (ii) the possibility of kinds of geometry, especially of two dimensions, other than those which are treated in the usual text-books: some aspects of the latter question have been already considered in chapter XIII. These problems are related. The term hyper-space was used originally of space of more than three dimensions, but now it is often employed to denote also any non-Euclidean space. I attach the wider meaning to it, and it is in that sense that this chapter is on the subject of hyper-space.

In regard to the first of these questions, the conception of a world of more than three dimensions is facilitated by the fact that there is no difficulty in imagining a world confined to only two dimensions—which we may take for simplicity to be a plane, though equally well it might be a spherical or other surface. We may picture the inhabitants of flatland as moving either on the surface of a plane or between two parallel and adjacent planes. They could move in any direction along the plane, but they could not move perpendicularly to it, and would have no consciousness that such a motion was possible. We may suppose them to have no thickness, in which case they would be mere geometrical abstractions; or, preferably, we may think of them as having a small but uniform thickness, in which case they would be realities.

Several writers have amused themselves by expounding and illustrating the conditions of life in such a world. To take a very simple instance* a knot is impossible in flatland, a simple alteration which alone would make some difference in the experience of the inhabitants as compared with our own.

If an inhabitant of flatland was able to move in three dimensions, he would be credited with supernatural powers by those who were unable so to move; for he could appear or disappear at will, could (so far as they could tell) create matter or destroy it, and would be free from so many constraints to which the other inhabitants were subject that his actions would be inexplicable to them.

We may go one step lower, and conceive of a world of one dimension—like a long tube—in which the inhabitants could move only forwards and backwards. In such a universe there would be lines of varying lengths, but there could be no geometrical figures. To those who are familiar with space of higher dimensions, life in line-land would seem somewhat dull. It is commonly said that an inhabitant could know only two other individuals; namely, his neighbours, one on each side. If the tube in which he lived was itself of only one dimension, this is true; but we can conceive an arrangement of tubes in two or three dimensions, where an occupant would be conscious of motion in only one dimension, and yet which would permit of more variety in the number of his acquaintances and conditions of existence.

Our conscious life is in three dimensions, and naturally the idea occurs whether there may not be a fourth dimension. No inhabitant of flatland could realize what life in three dimensions

* It is obvious that a knot cannot be tied in space of two dimensions. As long ago as 1876, F. C. Klein showed that knots cannot exist in space of four dimensions; see *Mathematische Annalen*, Leipzig, 1876, vol. IX, p. 478. It is not easy to give a definition of a knot in hyper-space, but, taking it in its ordinary sense, it would seem that it is only in space of three dimensions that knots can be tied in strings: see D. M. Y. Sommerville, *Messenger of Mathematics*, N.S., vol. XXXVI, 1907, pp. 139—144.

would mean, though, if he evolved an analytical geometry applicable to the world in which he lived, he might be able to extend it so as to obtain results true of that world in three dimensions which would be to him unknown and inconceivable. Similarly we cannot realize what life in four dimensions is like, though we can use analytical geometry to obtain results true of that world, or even of worlds of higher dimensions. Moreover the analogy of our position to the inhabitants of flatland enables us to form some idea of how inhabitants of space of four dimensions would regard us.

Just as the inhabitants of flatland might be conceived as being either mere geometrical abstractions, or real and of a uniform thickness in the third dimension, so, if there is a fourth dimension, we may be regarded either as having no thickness in that dimension, in which event we are mere (geometrical) abstractions—as indeed idealist philosophers have asserted to be the case—or as having a uniform thickness in that dimension, in which event we are living in four dimensions although we are not conscious of it. In the latter case it is reasonable to suppose that the thickness in the fourth dimension of bodies in our world is small and possibly constant; it has been conjectured also that it is comparable with the other dimensions of the molecules of matter, and if so it is possible that the constitution of matter and its fundamental properties may supply experimental data which will give a physical basis for proving or disproving the existence of this fourth dimension.

If we could look down on the inhabitants of flatland we could see their anatomy and what was happening inside them. Similarly an inhabitant of four-dimensional space could see inside us.

An inhabitant of flatland could get out of a room, such as a rectangle, only through some opening, but, if for a moment he could step into three dimensions, he could reappear on the other side of any boundaries placed to retain him. Similarly, if we came across persons who could move out of a closed prison-cell without going through any of the openings in it, there might be some reason for thinking that they did it by passing first in the direction of the fourth dimension and then back again into our space. This however is unknown.

Again, if a finite solid was passed slowly through flatland, the inhabitants would be conscious only of that part of it which was in their plane. Thus they would see the shape of the object gradually change and ultimately vanish. In the same way, if a body of four dimensions was passed through our space, we should be conscious of it only as a solid body, namely, the section of the body by our space, whose form and appearance gradually changed and perhaps ultimately vanished. It has been suggested that the birth, growth, life, and death of animals may be explained thus as the passage of finite four-dimensional bodies through our three-dimensional space. I believe that this idea is due to Hinton.

The same argument is applicable to all material bodies. The impenetrability and inertia of matter are necessary consequences; the conservation of energy follows, provided that the velocity with which the bodies move in the fourth dimension is properly chosen: but the indestructibility of matter rests on the assumption that the body does not pass completely through our space. I omit the details connected with change of density as the size of the section by our space varies.

We cannot prove the existence of space of four dimensions, but it is interesting to enquire whether it is probable that such space actually exists. To discuss this, first let us consider how an inhabitant of flatland might find arguments to support the view that space of three dimensions existed, and then let us see whether analogous arguments apply to our world. I commence with considerations based on geometry and then proceed to those founded on physics.

Inhabitants of flatland would find that they could have two triangles of which the elements were equal, element to element, and yet which could not be superposed. We know that the explanation of this fact is that, in order to superpose them, one of the triangles would have to be turned over so that its under-surface came on to the upper side, but of course such a movement would be to them inconceivable. Possibly however they might have suspected it by noticing that inhabitants of one-dimensional space might experience a similar difficulty in comparing the equality of two lines, ABC and $CB'A'$, each defined by a set of three points. We may suppose that the lines are equal and such that corresponding points in them could be superposed by rotation round C—a movement inconceivable to the inhabitants—but an inhabitant of such a world in moving along from A to A' would not arrive at the corresponding points in the two lines in the same relative order, and thus might hesitate to believe that they were equal. Hence inhabitants of flatland might infer by analogy that by turning one of the triangles over through three-dimensional space they could make them coincide.

We have a somewhat similar difficulty in our geometry. We can construct triangles in three dimensions—such as two spherical triangles—whose elements are equal respectively one to the other, but which cannot be superposed. Similarly we may have two helices whose elements are equal respectively, one having a right-handed twist and the other a left-handed twist, but it is impossible to make one fill exactly the same parts of space as the other does. Again, we may conceive of two solids, such as a right hand and a left hand, which are exactly similar and equal but of which one cannot be made to occupy exactly the same position in space as the other does. These are difficulties similar to those which would be experienced by the inhabitants of flatland in comparing triangles; and it may be conjectured that in the same way as such difficulties in the geometry of an inhabitant in space of one dimension are explicable by temporarily moving the figure into space of two dimensions by means of a rotation round a point, and as such difficulties in the geometry of flatland are explicable by temporarily moving the figure into space of three dimensions by means of a rotation round a line, so such difficulties in our geometry would disappear if we could temporarily move our figures into space of four dimensions by means of a rotation round a plane—a movement which of course is inconceivable to us.

Next we may enquire whether the hypothesis of our existence in a space of four dimensions affords an explanation of any difficulties or apparent inconsistencies in our physical science*. The current conception of the luminiferous ether, the explanation of gravity, and the fact that there are only a finite number of kinds of matter, all the atoms of each kind being similar, present such difficulties and inconsistencies. To see whether the hypothesis of a four-dimensional space gives any aid to their elucidation, we shall do best to consider first the analogous problems in two dimensions.

We live on a solid body, which is nearly spherical, and which moves round the sun under an attraction directed to it. To realize a corresponding life in flatland we must suppose that the inhabitants live on the rim of a (planetary) disc which rotates round another (solar) disc under an attraction directed towards it. We may suppose that the planetary world thus formed rests on a smooth plane, or other surface of constant curvature; but the pressure on this plane and even its existence would be unknown to the inhabitants, though they would be conscious of their attraction to the centre of the disc on which they lived. Of course they would be also aware of the bodies, solid, liquid, or gaseous, which were on its rim, or on such points of its interior as they could reach.

Every particle of matter in such a world would rest on this plane medium. Hence, if any particle was set vibrating, it would give up a part of its motion to the supporting plane. The vibrations thus caused in the plane would spread out in all directions, and the plane would communicate vibrations to any other particles resting on it. Thus any form of energy caused by vibrations, such as light, radiant heat, electricity, and possibly attraction, could be transmitted from one point to another without the presence of any intervening medium which the inhabitants could detect.

If the particles were supported on a uniform elastic plane film, the intensity of the disturbance at any other point would vary inversely as the distance of the point from the source of

* See a note by myself in the *Messenger of Mathematics*, Cambridge, 1891, vol. XXI, pp. 20—24, from which the above argument is extracted. The question has been treated by Hinton on similar lines.

disturbance; if on a uniform elastic solid medium, it would vary inversely as the square of that distance. But, if the supporting medium was vibrating, then, wherever a particle rested on it, some of the energy in the plane would be given up to that particle, and thus the vibrations of the intervening medium would be hindered when it was associated with matter.

If the inhabitants of this two-dimensional world were sufficiently intelligent to reason about the manner in which energy was transmitted they would be landed in a difficulty. Possibly they might be unable to explain gravitation between two particles—and therefore between the solar disc and their disc—except by supposing vibrations in a rigid medium between the two particles or discs. Again, they might be able to detect that radiant light and heat, such as the solar light and heat, were transmitted by vibrations transverse to the direction from which they came, though they could realize only such vibrations as were in their plane, and they might determine experimentally that in order to transmit such vibrations a medium of great rigidity (which we may call ether) was necessary. Yet in both the above cases they would have also distinct evidence that there was no medium capable of resisting motion in the space around them, or between their disc and the solar disc. The explanation of these conflicting results lies in the fact that their universe was supported by a plane, of which they were necessarily unconscious, and that this rigid elastic plane was the ether which transmitted the vibrations.

Now suppose that the bodies in our universe have a uniform thickness in the fourth dimension, and that in that direction our universe rests on a homogeneous elastic body whose thickness in that direction is small and constant. The transmission of force and radiant energy, without the intervention of an intervening medium, may be explained by the vibrations of the supporting space, even though the vibrations are not themselves in the fourth dimension. Also we should find, as in fact we do, that the vibrations of the luminiferous ether are hindered when it is associated with matter. I have assumed that the thickness of the supporting space is small and uniform, because then the intensity of the energy transmitted from a source to any point would vary inversely as the square of the distance, as is the case; whereas if the supporting space was a body of four dimensions, the law would be that of the inverse cube of the distance.

The application of this hypothesis to the third difficulty mentioned above—namely, to show why there are in our universe only a finite number of kinds of atoms, all the atoms of each kind having in common a number of sharply defined properties—will be given later*.

Thus the assumption of the existence of a four-dimensional homogeneous elastic body on which our three-dimensional universe rests, affords an explanation of some difficulties in our physical science.

It may be thought that it is hopeless to try to realize a figure in four dimensions. Nevertheless attempts have been made to see what the sections of such a figure would look like.

If the boundary of a solid is $\phi(x, y, z) = 0$, we can obtain some idea of its form by taking a series of plane sections by planes parallel to $z = 0$, and mentally superposing them. In four dimensions the boundary of a body would be $\phi(x, y, z, w) = 0$, and attempts have been made to realize the form of such a body by making models of a series of solids in three dimensions formed by sections parallel to $w = 0$. Again, we can represent a solid in perspective by taking sections by three co-ordinate planes. In the case of a four-dimensional body the section by each of the four co-ordinate solids will be a solid, and attempts have been made by drawing these to get an idea of the form of the body. Of course a four-dimensional body will be bounded by solids.

The possible forms of regular bodies in four dimensions, analogous to polyhedrons in space of three dimensions, have been discussed by Stringham†.

* See below, p. 475.

† *American Journal of Mathematics*, 1880, vol. III, pp. 1—14.

I now turn to the second of the two problems mentioned at the beginning of the chapter: namely, the possibility of there being kinds of geometry other than those which are treated in the usual elementary text-books. This subject is so technical that in a book of this nature I can do little more than give a sketch of the argument on which the idea is based.

The Euclidean system of geometry, with which alone most people are acquainted, rests on a number of independent axioms and postulates. Those which are necessary for Euclid's geometry have, within recent years, been investigated and scheduled. They include not only those explicitly given by him, but some others which he unconsciously used. If these are varied, or other axioms are assumed, we get a different series of propositions, and any consistent body of such propositions constitutes a system of geometry. Hence there is no limit to the number of possible non-Euclidean geometries that can be constructed.

Among Euclid's axioms and postulates is one on parallel lines, which is usually stated in the form that if a straight line meets two straight lines, so as to make the sum of the two interior angles on the same side of it less than two right angles, then these straight lines being continually produced will at length meet upon that side on which are the angles whose sum is less than two right angles. Expressed in this form the axiom is far from obvious, and from early times numerous attempts have been made to prove it. All such attempts failed, and it is now known that the axiom cannot be deduced from the other axioms assumed by Euclid. I have already discussed this question in chapter XIII, and I do not propose to add here anything more. The conclusion was that three consistent systems of geometry could be constructed, termed respectively hyperbolic, parabolic or Euclidean, and elliptic. These are distinguished from one another according as no straight line (that is, a geodetic line), or only one straight line, or a pencil of straight lines can be drawn through a point parallel to a given straight line.

To work out a body of propositions relating to figures on a surface (that is, a two-dimensional space) analogous to that given by Euclid relating to figures drawn on a plane, it is necessary that it should be possible at any point on the surface to construct a figure congruent to a given figure; this is equivalent to saying that if we take up a triangle drawn anywhere on the surface and move it to another part of the surface, it will lie flat on the surface there. This is so only if the measure of curvature at every point of the surface is constant. Such surfaces of constant curvature are spherical surfaces, where the product is positive; plane surfaces, where it is zero; pseudo-spherical surfaces, where it is negative. A tractroid, that is, a figure produced by the revolution of a tractrix about its asymptote, is an example of a pseudo-spherical surface; it is saddle-shaped at every point. Hence on spheres, planes, and tractroids we can construct these systems of geometry. And these systems are respectively examples of hyperbolic, Euclidean, and elliptic geometries.

Moreover if any surface is bent without dilation or contraction, the measure of curvature remains unaltered. Thus from these three surfaces we can form others on which congruent figures, and therefore consistent systems of geometry, can be constructed. For instance, a plane can be rolled into a cone or cylinder, and the system of geometry on a conical or cylindrical surface will be similar to that on a plane. Similarly a hemisphere can be rolled up into a sort of spindle, and the system of geometry on such a spindle will be similar to that on a sphere. In fact there are three kinds of surfaces of constant positive curvature, which are respectively spherical, spindle-shaped, and bolster-shaped, and on each of these a system of hyperbolic geometry can be constructed. So too there are three kinds of surfaces of constant negative curvature.

Throughout this discussion I have tacitly assumed that the measure of distance employed remains the same wherever it is employed. If this is not so, we may evolve in plane space non-Euclidean geometries which are not inconsistent with experience. Suppose, to take one example, that a foot-rule shrunk as it was moved away from some point of the plane—as it might do by a fall of temperature. Then a distance, which we should describe as finite, might when measured by this rule appear to be infinite, since repeated applications of the ever-shortening rule would not cover it. Thus the boundary of what we should describe as a finite area round the point would to those who were confined to the area of this foot-rule appear to be infinitely distant from the point. If the law of shrinking be properly chosen, the geometry of figures in this area would be hyperbolic. The length of the foot-rule might also alter in such a way as to lead to an elliptic geometry*.

Thus the conception of hyperbolic, Euclidean, and elliptic geometries can be reached from the theory of the measure of distances as well as from the theory of parallels and of congruent figures. This view has led to further discussion of the essential characteristics of space by F. C. Klein, S. Lie, D. Hilbert, M. L. Gérard, A. N. Whitehead, and others.

The above remarks refer only to space of two dimensions. Naturally there arises the question whether there are different kinds of non-Euclidean space of three or more dimensions. Riemann showed that there are three kinds of non-Euclidean space of three dimensions having properties analogous to the three kinds of non-Euclidean space of two dimensions already discussed. These are differentiated by the test whether at every point no geodetical surface, or one geodetical surface, or a fasciculus of geodetical surfaces can be drawn parallel to a given surface: a geodetical surface being defined as such that every geodetic line joining two points on it lies wholly on the surface. It may be added that each of the three systems of geometry of two dimensions described above may be deduced as properties of a surface in each of these three kinds of non-Euclidean space of three dimensions.

It is evident that the properties of non-Euclidean space of three dimensions are deducible only by the aid of mathematics, and cannot be illustrated materially, for in order to realize or construct surfaces in non-Euclidean space of two dimensions we think of or use models in space of three dimensions; similarly

* See A. Cayley, *Collected Mathematical Papers*, Cambridge, 1896, vol. XI, p. 435 *et seq.*

the only way in which we could construct models illustrating non-Euclidean space of three dimensions would be by utilizing space of four dimensions.

We may proceed yet further and conceive of non-Euclidean geometries of more than three dimensions, but this remains, as yet, an unworked field.

Returning to the former question of non-Euclidean geometries of two dimensions, I wish again to emphasize the fact that, if the axioms enunciated in the usual books on elementary geometry are replaced by others, it is possible to construct other consistent systems of geometry. For instance, just as one kind of non-Euclidean geometry has been constructed by assuming that Euclid's parallel postulate is not true, so D. Hilbert and M. Dehn of Göttingen have elaborated another kind, known as non-Archimedian geometry. Archimedes had assumed as axiomatic that if A and B are magnitudes of the same kind and order, it is possible to find a multiple of A which is greater than B, which implied that the geometrical magnitudes considered are continuous. If this be denied, Hilbert and Dehn showed* that it is still possible to construct consistent systems of geometry closely analogous to that given by Euclid. Assuming that in these a pencil of straight lines can be drawn through a point parallel to a given straight line, then in one form, known as the non-Legendrian system, the angle-sum of a triangle is greater than two right angles, while in another form, termed the semi-Euclidean system, the sum-angle is equal to two right angles. I do not however concern myself here further with these systems, for the methods and results appeal only to the professional mathematician. On the other hand the elliptic, parabolic, and hyperbolic systems described in chapter XIV have a special interest, from the somewhat sensational fact that they lead to no results necessarily inconsistent with the properties, as far as we can observe them, of the space in which we live; we are not at present acquainted with any other systems which are consistent with our experience. We may, however, fairly say that of these systems the Euclidean is the simplest.

* M. Dehn, *Mathematischen Annalen*, Leipzig, 1900, vol. LVII, pp. 404—439.

If we go a step further and ask what is meant by saying that a geometry is true or false we land ourselves in an interminable academic dispute. Some philosophers hold that certain axioms are necessarily true independent of all experience, or at any rate are necessarily true as far as our experience extends. Others agree with Poincaré, that the selection of a geometry is really a matter of convenience, and that that geometry is the best which enables us to state the known physical laws in the simplest form; or, more generally, that it is desirable to choose axioms and to define quantities so as to permit the expression in as simple a way as possible of all observed laws and facts in nature. But for practical purposes the conclusion is immaterial, and at any rate the discussion belongs to metaphysics rather than mathematics.

CHAPTER XXI.

TIME AND ITS MEASUREMENT.

The problems connected with time are totally different in character from those concerning space which I discussed in the last chapter. I there stated that the life of people living in space of one dimension would be uninteresting, and that probably they would find it impossible to realize life in space of higher dimensions. In questions connected with time we find ourselves in a somewhat similar position. Mentally, we can realize a past and a future—thus going backwards and forwards—actually we go only forwards. Hence time is analogous to space of one dimension. Were our time of two dimensions, the conditions of our life would be infinitely varied, but we can form no conception of what such a phrase means, and I do not think that any attempts have been made to work it out.

The idea of time, when we examine it carefully, involves many difficulties. For instance, we speak of an instant of time as if it were absolutely definite. If so we could represent it by a point on a line, and the idea of simultaneity would be simple, for two events could be regarded as simultaneous when their representative points were coincident. But in reality sensations have an appreciable duration, even though it be very small. This duration may be represented by an interval on a line, and it would seem reasonable to say that two events are simultaneous when their representative intervals have a common part; hence two events which are simultaneous with the same event are not necessarily simultaneous with one another. Here, however, I exclude these quasi-metaphysical questions, and concern myself mainly with questions concerning the measurement of time, and I shall treat these rather from a historical than from a physical point of view.

In order to measure anything we must have an unalterable unit of the same kind, and we must be able to determine how often that unit is contained in the quantity to be measured. Hence only those things can be measured which are capable of addition to things of the same kind.

Thus to measure a length we may take a foot-rule, and by applying it to the given length as often as is necessary, we shall find how many feet the length contains. But in comparing lengths we assume as the result of experience that the length of the foot-rule is constant, or rather that any alteration in it can be determined; and, if this assumption was denied, we could not prove it, though, if numerous repetitions of the experiment under varying conditions always gave the same result, probably we should feel no doubt as to the correctness of our method.

It is evident that the measurement of time is a more difficult matter. We cannot keep a unit by us in the same way as we can keep a foot-rule; nor can we repeat the measurement over and over again, for time once passed is gone for ever. Hence we cannot appeal directly to our sensations to justify our measurement. Thus, if we say that a certain duration is four hours, it is only by a process of reasoning that we can show that each of the hours is of the same duration.

The establishment of a scientific unit for measuring durations has been a long and slow affair. The process seems to have been as follows. Originally man observed that certain natural phenomena recurred after the interval of a day, say from sunrise to sunrise. Experience—for example, the amount of work that could be done in it—showed that the length of every day was about the same, and, assuming that this was accurately so, man had a unit by which he could measure durations. The present subdivision of a day into hours, minutes, and seconds is artificial, and apparently is derived from the Babylonians.

Similarly a month and a year are natural units of time though it is not easy to determine precisely their beginnings and endings.

So long as men were concerned merely with durations which were exact multiples of these units or which needed only a rough estimate, this did very well; but as soon as they tried to compare the different units or to estimate durations measured by part of a unit they found difficulties. In particular it cannot have been long before it was noticed that the duration of the same day differed in different places, and that even at the same place different days differed in duration at different times of the year, and thus that the duration of a day was not an invariable unit.

The question then arises as to whether we can find a fixed unit by which a duration can be measured, and whether we have any assurance that the seconds and minutes used to-day for that purpose are all of equal duration. To answer this we must see how a mathematician would define a unit of time. Probably he would say that experience leads us to believe that, if a rigid body is set moving in a straight line without any external force acting on it, it will go on moving in that line; and those times are taken to be equal in which it passes over equal spaces: similarly, if it is set rotating about a principal axis passing through its centre of mass, those times are taken to be equal in which it turns through equal angles. Our experiences are consistent with this statement, and that is as high an authority as a mathematician hopes to get.

The spaces and the angles can be measured, and thus durations can be compared. Now the earth may be taken roughly as a rigid body rotating about a principal axis passing through its centre of mass, and subject to no external forces affecting its rotation: hence the time it takes to turn through four right angles, *i.e.* through 360°, is always the same; this is called a sidereal day: the time to turn through one twenty-fourth part of 360°, *i.e.* through 15°, is an hour: the time to turn through one-sixtieth part of 15°, *i.e.* through 15′, is a minute: and so on.

If, by the progress of astronomical research, we find that there are external forces affecting the rotation of the earth, mathematics would have to be invoked to find what the time of rotation would be if those forces ceased to act, and this would give us a correction to be applied to the unit chosen. In the same way we may say that although an increase of temperature affects the length of a foot-rule, yet its change of length can be determined, and thus applied as a correction to the foot-rule when it is used as the unit of length. As a matter of fact there is reason to think that the earth takes about one sixty-sixth of a second longer to turn through four right angles now than it did 2500 years ago, and thus the duration of a second is just a trifle longer to-day than was the case when the Romans were laying the foundations of the power of their city.

The sidereal day can be determined only by refined astronomical observations and is not a unit suitable for ordinary purposes. The relations of civil life depend mainly on the sun, and he is our natural time-keeper. The true solar day is the time occupied by the earth in making one revolution on its axis relative to the sun; it is true noon when the sun is on the meridian. Owing to the motion of the sun relative to the earth, the true solar day is about four minutes longer than a sidereal day.

The true solar day is not however always of the same duration. This is inconvenient if we measure time by clocks (as now for nearly two centuries has been usual in Western Europe) and not by sun-dials, and therefore we take the average duration of the true solar day as the measure of a day: this is called the mean solar day. Moreover to define the noon of a mean solar day we suppose a point to move uniformly round the ecliptic coinciding with the sun at each apse, and further we suppose a fictitious sun, called the mean sun, to move in the celestial equator so that its distance from the first point of Aries is the same as that of this point: it is mean noon when this mean sun is on the meridian. The mean solar day is divided into hours, minutes, and seconds; and these are the usual units of time in civil life.

The time indicated by our clocks and watches is mean solar time; that marked on ordinary sun-dials is true solar time. The difference between them is the equation of time: this may amount at some periods of the year to a little more than a quarter of an hour. In England we take the Greenwich meridian as our origin for longitudes, and instead of local mean solar time we take Greenwich mean solar time as the civil standard.

Of course mean time is a comparatively recent invention. The French were the last civilized nation to abandon the use of true time: this was in 1816.

Formerly there was no common agreement as to when the day began. In parts of ancient Greece and in Japan the interval from sunrise to sunset was divided into twelve hours, and that from sunset to sunrise into twelve hours. The Jews, Chinese, Athenians, and, for a long time, the Italians, divided their day into twenty-four hours, beginning at the hour of sunset, which of course varies every day: this method is said to have been used as late as the latter half of the nineteenth century in certain villages near Naples, except that the day began half-an-hour after sunset—the clocks being re-set once a week. Similarly the Babylonians, Assyrians, Persians, and until recently the modern Greeks and the inhabitants of the Balearic Islands counted the twenty-four hours of the day from sunrise. Until 1750, the inhabitants of Basle reckoned the twenty-four hours from our 11.0 p.m. The ancient Egyptians and Ptolemy counted the twenty-four hours from noon: this is the practice of modern astronomers. In Western Europe the day is taken to begin at midnight—as was first suggested by Hipparchus—and is divided into two equal periods of twelve hours each.

The week of seven days is an artificial unit of time. It had its origin in the East, and was introduced into the West probably during the second century by the Roman emperors, and, except during the French Revolution, has been subsequently in general use among civilized races. The names of the days are derived from the seven astrological planets, arranged, as was customary, in the order of their apparent times of rotation round the earth, namely, Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon. The twenty-four hours of the day were dedicated successively to these planets: and the day was consecrated to the planet of the first hour.

Thus if the first hour was dedicated to Saturn, the second would be dedicated to Jupiter, and so on; but the day would be Saturn's day. The twenty-fourth hour of Saturn's day would be dedicated to Mars, thus the first hour of the next day would belong to the Sun; and the day would be Sun's day. Similarly the next day would be Moon's day; the next, Mars's day; the next, Mercury's day; the next, Jupiter's day; and the next, Venus's day.

The astronomical month is a natural unit of time depending on the motion of the moon, and containing about 29½ days. The months of the calendar have been evolved gradually as convenient divisions of time, and their history is given in numerous astronomies. In the original Julian arrangement the months in a leap year contained alternately 31 and 30 days, while in other years February had 29 days. This was altered by Augustus in order that his month should not be inferior to one named after his uncle.

The solar tropical year is another natural unit of time. According to a recent determination, it contains 365·242216 days, that is, $365^{d.}\ 5^{h.}\ 48^{m.}\ 47^{s.}\cdot 4624$. Civilized races usually number the passing years consecutively from some fixed date. The Romans reckoned from the traditional date of the foundation of their city. In the sixth century of our era it was suggested that the birth of Christ was a more fitting epoch from which to reckon dates, but it was not until the ninth century that this suggestion was generally adopted.

The Egyptians knew that the year contained between 365 and 366 days, but the Romans did not profit by this information, for Numa is said to have reckoned 355 days as constituting a year—extra months being occasionally intercalated, so that the seasons might recur at about the same period of the year.

In 46 B.C. Julius Caesar decreed that thenceforth the year should contain 365 days, except that in every fourth or leap year one additional day should be introduced. He ordered this rule to come into force on January 1, 45 B.C. The change was made on the advice of Sosigenes of Alexandria.

It must be remembered that the year 1 A.D. follows immediately 1 B.C., that is, there is no year 0, and thus 45 B.C. would be a leap year. All historical dates are given now as if the Julian calendar was reckoned backwards as well as forwards from that year*. As a matter of fact, owing to a mistake in the original decree, the Romans, during the first 36 years after 45 B.C., intercalated the extra day every third year, thus producing an error of 3 days. This was remedied by Augustus, who directed that no intercalation of an extra day should be made in any of the twelve years A.U.C. 746 to 757 inclusive, but that the intercalation should be again made in the year A.U.C. 761 (that is, 8 A.D.) and every succeeding fourth year.

The Julian calendar made the year, on an average, contain 365·25 days. The actual value is, very approximately, 365·242216 days. Hence the Julian year is too long by about 11¼ minutes: this produces an error of nearly one day in 128 years. If the extra day in every thirty-second leap year had been omitted—as was suggested by some unknown Persian astronomer—the error would have been less than one day in 100,000 years. It may be added that Sosigenes was aware that his rule made the year slightly too long.

The error in the Julian calendar of rather more than eleven minutes a year gradually accumulated, until in the sixteenth century the seasons arrived some ten days earlier than they should have done. In 1582 Gregory XIII corrected this by omitting ten days from that year, which therefore contained only 355 days. At the same time he decreed that thenceforth every year which was a multiple of a century should be or not be a leap year according as the multiple was or was not divisible by four.

The fundamental idea of the reform was due to Lilius, who died before it was carried into effect. The work of framing the new calendar was entrusted to Clavius, who explained the

* Herschel, *Astronomy*, London, 11th ed. 1871, arts. 916—919.

principles and necessary rules in a prolix but accurate work* of over 700 folio pages. The plan adopted was due to a suggestion of Pitatus made in 1552 or perhaps 1537: the alternative and more accurate proposal of Stöffler, made in 1518, to omit one day in every 134 years, being rejected by Lilius and Clavius for reasons which are not known.

Clavius believed the year to contain 365·2425432 days, but he framed his calendar so that a year, on the average, contained 365·2425 days, which he thought to be wrong by one day in 3323 years: in reality it is a trifle more accurate than this, the error amounting to one day in about 3600 years.

The change was unpopular, but Riccioli† tells us that, as those miracles which take place on fixed dates—*ex. gr.* the liquefaction of the blood of S. Januarius—occurred according to the new calendar, the papal decree was presumed to have a divine sanction—Deo ipso huic correctioni Gregorianae subscribente—and was accepted as a necessary evil.

In England a bill to carry out the same reform was introduced in 1584, but was withdrawn after being read a second time; and the change was not finally effected till 1752, when eleven days were omitted from that year. In Roman Catholic countries the new style was adopted in 1582. In the German Lutheran States it was made in 1700. In England, as I have said above, it was introduced in 1752; and in Ireland it was made in 1782. It is well known that the Greek Church still adheres to the Julian calendar.

The Mohammedan year contains 12 lunar months, or $354\frac{1}{3}$ days, and thus has no connection with the seasons.

The Gregorian change in the calendar was introduced in order to keep Easter at the right time of year. The date of Easter depends on that of the vernal equinox, and as the Julian calendar made the year of an average length of 365·25 days instead of 365·242216 days, the vernal equinox came earlier and earlier in the year, and in 1582 had regreded to within about ten days of February.

* *Romani Calendarii a Greg. XIII Restituti Explicatio*, Rome, 1603.

† *Chronologia Reformata*, Bonn, 1669, vol. II, p. 206.

The rule for determining Easter is as follows*. In 325 the Nicene Council decreed that the Roman practice should be followed; and after 463 (or perhaps, 530) the Roman practice required that Easter-day should be the first Sunday after the full moon which occurs on or next following the vernal equinox—full moon being assumed to occur on the fourteenth day from the day of the preceding new moon (though as a matter of fact it occurs on an average after an interval of rather more than $14\frac{3}{4}$ days), and the vernal equinox being assumed to fall on March 21 (though as a matter of fact it sometimes falls on March 22).

This rule and these assumptions were retained by Gregory on the ground that it was inexpedient to alter a rule with which so many traditions were associated; but, in order to save disputes as to the exact instant of the occurrence of the new moon, a mean sun and a mean moon defined by Clavius were used in applying the rule. One consequence of using this mean sun and mean moon and giving an artificial definition of full moon is that it may happen, as it did in 1818 and 1845, that the actual full moon occurs on Easter Sunday. In the British Act, 24 Geo. II. cap. 23, the explanatory clause which defines full moon is omitted, but practically full moon has been interpreted to mean the Roman ecclesiastical full moon; hence the Anglican and Roman rules are the same. Until 1774 the German Lutheran States employed the actual sun and moon. Had full moon been taken to mean the fifteenth day of the moon, as is the case in the civil calendar, then the rule might be given in the form that Easter-day is the Sunday on or next after the calendar full moon which occurs next after March 21.

Assuming that the Gregorian calendar and tradition are used, there still remains one point in this definition of Easter which might lead to different nations keeping the feast at different times. This arises from the fact that local time is introduced. For instance the difference of local time between

* De Morgan, *Companion to the Almanac*, London, 1845, pp. 1—36; *Ibid.*, 1846, pp. 1—10.

Rome and London is about 50 minutes. Thus the instant of the first full moon next after the vernal equinox might occur in Rome on a Sunday morning, say at 12.30 a.m., while in England it would still be Saturday evening, 11.40 p.m., in which case our Easter would be one week earlier than at Rome. Clavius foresaw the difficulty, and the Roman Communion all over the world keep Easter on that day of the month which is determined by the use of the rule at Rome. But presumably the British Parliament intended time to be determined by the Greenwich meridian, and if so the Anglican and Roman dates for Easter might differ by a week; whether such a case has ever arisen or been discussed I do not know, and I leave to ecclesiastics to say how it should be settled.

The usual method of calculating the date on which Easter-day falls in any particular year is involved, and possibly the following simple rule* may be unknown to some of my readers.

Let m and n be numbers as defined below. (i) Divide the number of the year by 4, 7, 19; and let the remainders be a, b, c respectively. (ii) Divide $19c+m$ by 30, and let d be the remainder. (iii) Divide $2a+4b+6d+n$ by 7, and let e be the remainder. (iv) Then the Easter full moon occurs d days after March 21; and Easter-day is the $(22+d+e)$th of March or the $(d+e-9)$th day of April, except that if the calculation gives $d=29$ and $e=6$ (as happens in 1981) then Easter-day is on April 19 and not on April 26, and if the calculation gives $d=28$, $e=6$, and also $c>10$ (as happens in 1954) then Easter-day is on April 18 and not on April 25, that is, in these two cases Easter falls one week earlier than the date given by the rule. These two exceptional cases cannot occur in the Julian calendar, and in the Gregorian calendar they occur only very rarely. It remains to state the values of m and n for the particular period. In the Julian calendar we have $m=15$, $n=6$. In the Gregorian calendar we have, from 1582 to 1699 inclusive, $m=22$, $n=2$; from 1700 to 1799, $m=23$, $n=3$; from 1800 to 1899, $m=23$, $n=4$; from 1900 to 2099, $m=24$,

* It is due to Gauss; his proof is given in Zach's *Monatliche Correspondenz*, August, 1800, vol. II, pp. 221—230.

$n=5$; from 2100 to 2199, $m=24$, $n=6$; from 2200 to 2299, $m=25$, $n=0$; from 2300 to 2399, $m=26$, $n=1$; and from 2400 to 2499, $m=25$, $n=1$. Thus for the year 1908 we have $m=24$, $n=5$; hence $a=0$, $b=4$, $c=8$; $d=26$; and $e=2$: therefore Easter Sunday was on April 19. After the year 4200 the form of the rule will have to be slightly modified.

The dominical letter and the golden number of the ecclesiastical calendar can be at once determined from the values of b and c. The epact, that is, the moon's age at the beginning of the year, can be also easily calculated from the above data in any particular case; the general formula was given by Delambre, but its value is required so rarely by any but professional astronomers and almanac-makers that it is unnecessary to quote it here.

We can evade the necessity of having to recollect the values of m and n by noticing that, if N is the given year, and if $\{N/x\}$ denotes the integral part of the quotient when N is divided by x, then m is the remainder when $15+\xi$ is divided by 30, and n is the remainder when $6+\eta$ is divided by 7: where, in the Julian calendar, $\xi=0$, and $\eta=0$; and, in the Gregorian calendar, $\xi=\{N/100\}-\{N/400\}-\{N/300\}$, and $\eta=\{N/100\}-\{N/400\}-2$.

If we use these values of m and n, and if we put for a, b, c their values, namely, $a=N-4\{N/4\}$, $b=N-7\{N/7\}$, $c=N-19\{N/19\}$, the rule given on the last page takes the following form. Divide $19N-\{N/19\}+15+\xi$ by 30, and let the remainder be d. Next divide $6(N+d+1)-\{N/4\}+\eta$ by 7, and let the remainder be e. Then Easter full moon is on the dth day after March 21, and Easter-day is on the $(22+d+e)$th of March or the $(d+e-9)$th of April as the case may be; except that if the calculation gives $d=29$, and $e=6$, or if it gives $d=28$, $e=6$, and $c>10$, then Easter-day is on the $(d+e-16)$th of April.

Thus, if $N=1920$, we divide

$$19(1920)-101+15+(19-4-6) \text{ by } 30,$$

which gives $d=13$, and then we proceed to divide

$$6(1920+13+1)-480+(19-4-2) \text{ by } 7,$$

which gives $e=0$: therefore Easter-day will be on April 4.

The above rules cover all the cases worked out with so much labour by Clavius and others*.

I may add here a rule, quoted by Zeller, for determining the day of the week corresponding to any given date. Suppose that the pth day of the qth month of the year N *anno domini* is the rth day of the week, reckoned from the preceding Saturday. Then r is the remainder when

$$p+2q+\{3(q+1)/5\}+N+\{N/4\}-\eta$$

is divided by 7; provided January and February are reckoned respectively as the 13th and 14th months of the preceding year.

For instance, Columbus first landed in the New World on October 12, 1492. Here $p=12$, $q=10$, $N=1492$, $\eta=0$. If we divide $12+20+6+1492+373$ by 7 we get $r=6$; hence it was on a Friday. Again, Charles I was executed on January 30, 1649 N.S. Here $p=30$, $q=13$, $N=1648$, $\eta=0$, and we find $r=3$; hence it was on a Tuesday. As another example, the battle of Waterloo was fought on June 18, 1815. Here $p=18$, $q=6$, $N=1815$, $\eta=12$, and we find $r=1$; hence it took place on a Sunday.

Various rules have been given for obtaining these results with less arithmetical calculation, but they depend on the construction of tables which must be consulted in all cases. One rule of this kind is given in *Whitaker's Almanac*. Lightning calculators use such rules and commit the tables to memory. The same results can be also got by mechanical contrivances. The best instrument of this kind with which

* Most of the above-mentioned facts about the calendar are taken from Delambre's *Astronomie*, Paris, 1814, vol. III, chap. xxxviii; and his *Histoire de l'astronomie moderne*, Paris, 1821, vol. I, chap. i: see also A. De Morgan, *The Book of Almanacs*, London, 1851; S. Butcher, *The Ecclesiastical Calendar*, Dublin, 1877; and C. Zeller, *Acta Mathematica*, Stockholm, 1887, vol. IX, pp. 131—136: on the chronological details see J. L. Ideler, *Lehrbuch der Chronologie*, Berlin, 1831.

I am acquainted is one called the World's Calendar, invented by J. P. Wiles, and issued in London in 1906.

I proceed now to give a short account of some of the means of measuring time which were formerly in use.

Of devices for measuring time, the earliest of which we have any positive knowledge are the *styles* or *gnomons* erected in Egypt and Asia Minor. These were sticks placed vertically in a horizontal piece of ground, and surrounded by three concentric circles, such that every two hours the end of the shadow of the stick passed from one circle to another. Some of these have been found at Pompeii and Tusculum.

The *sun-dial* is not very different in principle. It consists of a rod or style fixed on a plate or dial; usually, but not necessarily, the style is placed so as to be parallel to the axis of the earth. The shadow of the style cast on the plate by the sun falls on lines engraved there which are marked with the corresponding hours.

The earliest sun-dial, of which I have read, is that made by Berosus in 540 B.C. One was erected by Meton at Athens in 433 B.C. The first sun-dial at Rome was constructed by Papirius Cursor in 306 B.C. Portable sun-dials, with a compass fixed in the face, have been long common in the East as well as in Europe. Other portable instruments of a similar kind were in use in medieval Europe, notably the sun-rings, hereafter described, and the sun-cylinders*.

I believe it is not generally known that a sun-dial can be so constructed that the shadow will, for a short time near sunrise and sunset, move backwards on the dial†. This was discovered by Nonez. The explanation is as follows. Every day the sun appears to describe a circle round the pole, and the line joining the point of the style to the sun describes a right cone whose axis points to the pole. The section of this cone by the dial is the curve described by the

* Thus Chaucer in the *Shipman's Tale*, "by my chilindre it is prime of day," and Lydgate in the *Siege of Thebes*, "by my chilyndre I gan anon to see...that it drew to nine."

† Ozanam, 1803 edition, vol. III, p. 321; 1840 edition, p. 529.

extremity of the shadow, and is a conic. In our latitude the sun is above the horizon for only part of the twenty-four hours, and therefore the extremity of the shadow of the style describes only a part of this conic. Let QQ' be the arc described by the extremity of the shadow of the style from sunrise at Q to sunset at Q', and let S be the point of the style and F the foot of the style, *i.e.* the point where the style meets the plane of the dial. Suppose that the dial is placed so that the tangents drawn from F to the conic QQ' are real, and that P and P', the points of contact of these tangents, lie on the arc QQ'. If these two conditions are fulfilled, then the shadow will regrede through the angle QFP as its extremity moves from Q to P, it will advance through the angle PFP' as its extremity moves from P to P', and it will regrede through the angle $P'FQ'$ as its extremity moves from P' to Q'.

If the sun's apparent diurnal path crosses the horizon—as always happens in temperate and tropical latitudes—and if the plane of the dial is horizontal, the arc QQ' will consist of the whole of one branch of a hyperbola, and the above conditions will be satisfied if F is within the space bounded by this branch of the hyperbola and its asymptotes. As a particular case, in a place of latitude 12° N. on a day when the sun is in the northern tropic (of Cancer) the shadow on a dial whose face is horizontal and style vertical will move backwards for about two hours between sunrise and noon.

If, in the case of a given sun-dial placed in a certain position, the conditions are not satisfied, it will be possible to satisfy them by tilting the sun-dial through an angle properly chosen. This was the rationalistic explanation, offered by the French encyclopaedists, of the miracle recorded in connection with Isaiah and Hezekiah*. Suppose, for instance, that the style is perpendicular to the face of the dial. Draw the celestial sphere. Suppose that the sun rises at M and culminates at N, and let L be a point between M and N on the sun's diurnal path. Draw a great circle to touch the sun's diurnal path MLN at L, let this great circle cut the celestial meridian in A

* 2 Kings, chap. xx, *vv.* 9—11.

and A', and of the arcs AL, $A'L$ suppose that AL is the less and therefore is less than a quadrant. If the style is pointed to A, then, while the sun is approaching L, the shadow will regrede, and after the sun passes L the shadow will advance. Thus if the dial is placed so that a style which is normal to it cuts the meridian midway between the equator and the tropic, then between sunrise and noon on the longest day the shadow will move backwards through an angle

$$\sin^{-1}(\cos\omega\sec\tfrac{1}{2}\omega)-\cot^{-1}\{\sin\omega\cos(l-\tfrac{1}{2}\omega)(\cos^2 l-\sin^2\omega)^{-\frac{1}{2}}\},$$

where l is the latitude of the place and ω is the obliquity of the ecliptic.

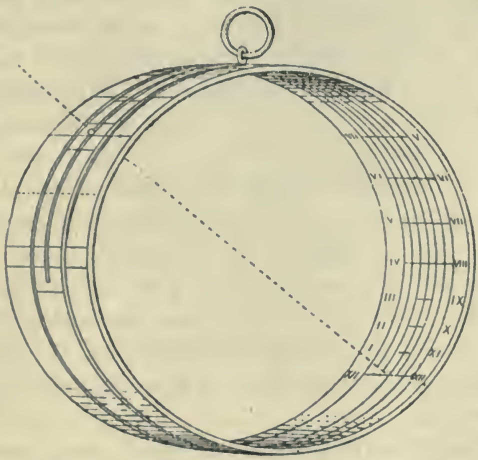
The above remarks refer to the sun-dials in ordinary use. In 1892 General Oliver brought out in London a dial with a solid style, the section of the style being a certain curve whose form was determined empirically by the value of the equation of time as compared with the sun's declination*. The shadow of the style on the dial gives the local mean time, though of course in order to set the dial correctly at any place the latitude of the place must be known: the dial may be also set so as to give the mean time at any other locality whose longitude relative to the place of observation is known.

The *sun-ring* or *ring-dial* is another instrument for measuring solar time†. One of the simplest type is figured in the diagram below. The sun-ring consists of a thin brass band, about a quarter of an inch wide, bent into the shape of a circle, which slides between two fixed circular rims—the radii of the circles being about one inch. At one point of the band there is a hole; and when the ring is suspended from a fixed point attached to the rims so that it hangs in a vertical plane containing the sun, the light from the sun shines through this hole and makes a bright speck on the opposite inner or concave surface of the ring. On this surface the hours are marked, and, if the ring is properly adjusted, the spot of light will fall on the hour which indicates the solar time. The

* An account of this sun-dial with a diagram was given in *Knowledge*, London, July 1, 1892, pp. 133, 134.

† See Ozanam, 1803 edition, vol. III, p. 317; 1840 edition, p. 526.

adjustment for the time of year is made as follows. The rims between which the band can slide are marked on their outer or convex side with the names of the months, and the band containing the hole must be moved between the rims until the hole is opposite to that month for which the ring is being used.



For determining times near noon the instrument is reliable, but for other hours in the day it is accurate only if the time of year is properly chosen, usually near one of the equinoxes. This defect may be corrected by marking the hours on a curved brass band affixed to the concave surface of the rims. I possess two specimens of rings of this kind. These rings were distributed widely. Of my two specimens, one was bought in the Austrian Tyrol and the other in London. Astrolabes and sea-rings can be used as sun-rings.

Clepsydras or water-clocks, and *hour-glasses* or sand-clocks, afford other means of measuring time. The time occupied by a given amount of some liquid or sand in running through a given orifice under the same conditions is always the same, and by noting the level of the liquid which has run through the orifice, or which remains to run through it, a measure of time can be obtained.

The burning of graduated candles gives another way of measuring time, and we have accounts of those used by Alfred the Great for the purpose. Incense sticks were used by the Chinese in a similar way.

Modern *clocks* and *watches** comprise a train of wheels turned by a weight, spring, or other motive power, and regulated by a pendulum, balance, fly-wheel, or other moving body whose motion is periodic and time of vibration constant. The direction of rotation of the hands of a clock was selected originally so as to make the hands move in the same direction as the shadow on a sun-dial whose face is horizontal—the dial being situated in our hemisphere.

The invention of clocks with wheels is attributed by tradition to Pacificus of Verona, circ. 850, and also to Gerbert, who is said to have made one at Magdeburg in 996: but there is reason to believe that these were sun-clocks. The earliest wheel-clock of which we have historical evidence was one sent by the Sultan of Egypt in 1232 to the Emperor Frederick II, though there seems to be no doubt that they had been made in Italy at least fifty years earlier.

The oldest clock in England of which we know anything was one erected in 1288 in or near Westminster Hall out of a fine imposed on a corrupt Lord Chief Justice. The bells, and possibly the clock, were staked by Henry VIII on a throw of dice and lost, but the site was marked by a sun-dial, destroyed early in the nineteenth century, and bearing the inscription *Discite justiciam moniti.* In 1292 a clock was erected in Canterbury Cathedral at a cost of £30. One erected at Glastonbury Abbey in 1325 is at present in the Kensington Museum and is in excellent condition. Another made in 1326 for St Albans Abbey showed the astronomical phenomena, and seems to have been one of the earliest clocks that did so. One put up at Dover in 1348 is still in good working order. The clocks at Peterborough and Exeter were of about the same date, and portions of them remain *in situ.* Most of these early clocks

* See *Clock and Watch Making* by Lord Grimthorpe, 7th edition, London, 1883.

were regulated by horizontal balances: pendulums being then unknown. Of the elaborate clocks of a later date, that at Strasburg made by Dasypodius in 1571, and that at Lyons constructed by Lippeus in 1598, are especially famous: the former was restored in 1842, though in a manner which destroyed most of the ancient works.

In 1370 Vick constructed a clock for Charles V with a weight as motive power and a vibrating escapement—a great improvement on the rough time-keepers of an earlier date.

The earliest clock regulated by a pendulum seems to have been made in 1621 by a clockmaker named Harris, of Covent Garden, London, but the theory of such clocks is due to Huygens*. Galileo had discovered previously the isochronism of a pendulum, but did not apply it to the regulation of the motion of clocks. Hooke made such clocks, and possibly discovered independently this use of the pendulum: he invented or re-invented the anchor pallet.

A watch may be defined as a clock which will go in any position. Watches, though of a somewhat clumsy design, were made at Nuremberg by P. Hele early in the sixteenth century —the motive power being a ribbon of steel, wound round a spindle, and connected at one end with a train of wheels which it turned as it unwound. Possibly a few similar timepieces had been made in the previous century; by the end of the sixteenth century they were not uncommon. At first they were usually made in the form of fanciful ornaments such as skulls, or as large pendants, but about 1620 the flattened oval form was introduced, rendering them more convenient to carry in a pocket or about the person. In the seventeenth century their construction was greatly improved, notably by the introduction of the spring balance by Huygens in 1674, and independently by Hooke in 1675—both mathematicians having discovered that small vibrations of a coiled spring, of which one end is fixed, are practically isochronous. The fusee had been used by R. Zech of Prague in 1525, but was re-invented by Hooke.

* *Horologium Oscillatorium*, Paris, 1673.

Clocks and watches are usually moved and regulated in the manner indicated above. Other motive powers and other devices for regulating the motion may be met with occasionally. Of these I may mention a clock in the form of a cylinder, usually attached to another weight as in Atwood's machine, which rolls down an inclined plane so slowly that it takes twelve hours to roll down, and the highest point of the face always marks the proper hour*.

A water-clock made on a somewhat similar plan is described by Ozanam† as one of the sights of Paris at the beginning of the last century. It was formed of a hollow cylinder divided into various compartments each containing some mercury, so arranged that the cylinder descended with uniform velocity between two vertical pillars on which the hours were marked at equidistant intervals.

Other ingenious ways of concealing the motive power have been described in the columns of *La Nature*‡. Of such mysterious timepieces the following are not uncommon examples, and probably are known to most readers of this book. One kind of clock consists of a glass dial suspended by two thin wires; the hands however are of metal, and the works are concealed in them or in the pivot. Another kind is made of two sheets of glass in a frame containing a spring which gives to the hinder sheet a very slight oscillatory motion—imperceptible except on the closest scrutiny—and each oscillation moves the hands through the requisite angles. Some so-called perpetual motion timepieces were described above on page 96. Lastly, I have seen in France a clock the hands of which were concealed at the back of the dial, and carried small magnets; pieces of steel in the shape of insects were placed on the dial, and, following the magnets, served to indicate the time.

The position of the sun relative to the points of the compass

* Ozanam, 1803 edition, vol. II, p. 39; 1840 edition, p. 212; or *La Nature*, Jan. 23, 1892, pp. 123, 124.

† Ozanam, 1803 edition, vol. II, p. 68; 1840 edition, p. 225.

‡ See especially the volumes issued in 1874, 1877, and 1878.

determines the solar time. Conversely, if we take the time given by a watch as being the solar time—and it will differ from it by only a few minutes at the most—and we observe the position of the sun, we can find the points of the compass*. To do this it is sufficient to point the hour-hand to the sun, and then the direction which bisects the angle between the hour and the figure XII will point due south. For instance, if it is four o'clock in the afternoon, it is sufficient to point the hour-hand (which is then at the figure IIII) to the sun, and the figure II on the watch will indicate the direction of south. Again, if it is eight o'clock in the morning, we must point the hour-hand (which is then at the figure VIII) to the sun, and the figure X on the watch gives the south point of the compass.

Between the hours of six in the morning and six in the evening the angle between the hour and XII which must be bisected is less than 180°, but at other times the angle to be bisected is greater than 180°; or perhaps it is simpler to say that at other times the rule gives the north point and not the south point.

The reason is as follows. At noon the sun is due south, and it makes one complete circuit round the points of the compass in 24 hours. The hour-hand of a watch also makes one complete circuit in 12 hours. Hence, if the watch is held in the plane of the ecliptic with its face upwards, and the figure XII on the dial is pointed to the south, both the hour-hand and the sun will be in that direction at noon. Both move round in the same direction, but the angular velocity of the hour-hand is twice as great as that of the sun. Hence the rule. The greatest error due to the neglect of the equation of time is less than 2°. Of course in practice most people, instead of holding the face of the watch in the ecliptic, would hold it horizontal, and in our latitude no serious error would be thus introduced.

* The rule is given by W. H. Richards, *Military Topography*, London, 1883, p. 31, though it is not stated quite correctly. I do not know who first enunciated it.

In the southern hemisphere where at noon the sun is due north the rule requires modification. In such places the hour-hand of a watch (held face upwards in the plane of the ecliptic) and the sun move in opposite directions. Hence, if the watch is held so that the figure XII points to the sun, then the direction which bisects the angle between the hour of the day and the figure XII will point due north.

CHAPTER XXII.

MATTER AND ETHER THEORIES.

Matter, like space and time, cannot be defined, but either the statement that matter is whatever occupies space or the statement that it is anything which can be seen, touched, or weighed, suggests its more important characteristics to anyone already familiar with it.

The means of measuring matter and some of its properties are treated in most text-books on mechanics, and I do not propose to discuss them. I confine the chapter to an account of some of the hypotheses formerly held by physicists as to the ultimate constitution of matter, but I exclude metaphysical conjectures which, from their nature, are incapable of proof and are not subject to mathematical analysis. The question is intimately associated with the explanation of the phenomena of attraction, light, chemistry, electricity, and other branches of physics.

I commence with a list of some of the more plausible of the hypotheses formerly proposed which accounted for the obvious properties of matter, and shall then discuss how far they explain or are consistent with other facts*. The interest of the list is

* For the earlier investigations I have based my account mainly on *Recent Advances in Physical Science*, by P. G. Tait, Edinburgh, 1876 (chaps. xii, xiii); and on the article *Atom* by J. Clerk Maxwell in the *Encyclopaedia Britannica* or his *Collected Works*, vol. II, pp. 445—484. For the more recent speculations see J. J. Thomson, *Electricity and Matter*, Westminster, 1904; J. Larmor, *Aether and Matter*, Cambridge, 1900; and E. T. Whittaker, *History of the Theories of Aether and Electricity*, Dublin, 1910.

largely historical, for within the last few years new views as to the constitution of matter have been propounded, which cannot be discussed satisfactorily in a book like this.

I. Hypothesis of Continuous Matter. It may be supposed that matter is homogeneous and continuous, in which case there is no limit to the infinite divisibility of bodies. This view was held by Descartes*.

This conjecture is consistent with the facts deducible by untrained observation, but there are many other phenomena for which it does not account; moreover there seems to be no way of reconciling such a structure of matter either with the facts of chemical changes or with the results of spectrum analysis. At any rate the theory must be regarded as extremely improbable.

II. Atomic Theories. If matter is not continuous we must suppose that every body is composed of aggregates of molecules. If so, it seems probable that each such molecule is built up by the association of two or more atoms, that the number of kinds of atoms is finite, and that the atoms of any particular kind are alike. As to the nature of the atoms the following hypotheses have been made.

(i) *Popular Atomic Hypothesis*. The popular view is that every atom of any particular kind is a minute indivisible article possessing definite qualities, everlasting in its form and properties, and infinitely hard.

This statement is plausible, but the difficulties to which it leads appear to be insuperable. In fact we have reason to think that the atoms which form a molecule are composite systems in incessant vibration at a rate characteristic of the molecule, and it is most probable that they are elastic.

Newton seems to have hazarded a conjecture of this kind when he suggested† that the difficulties, connected with the fact that the velocity of sound was one-ninth greater than that required by theory, might be overcome if the particles of air

* Descartes, *Principia*, vol. II, pp. 18, 23.

† Newton, *Principia*, bk. II, prop. 50.

were little rigid spheres whose distance from one another under normal conditions was nine times the diameter of any one of them. This was ingenious, but obviously the view is untenable, because, if such a structure of air existed, the air could not be compressed beyond a certain limit, namely, about 1/1021st part of its original volume, which has been often exceeded. The true explanation of the difficulty noticed by Newton was given by Laplace.

(ii) *Boscovich's Hypothesis.* In 1759 Boscovich suggested* that the facts might be explained by supposing that an atom was an infinitely small indivisible mass which was a centre of force—the law of force being attractive for sensible distances, alternately attractive and repulsive for minute distances, and repulsive for infinitely small distances. In this theory all action between bodies is action at a distance.

He explained the apparent extension of bodies by saying that two parts are consecutive (or similarly that two bodies are in contact) when the nearest pair of atoms in them are so close to one another that the repulsion at any point between them is sufficiently great to prevent any other atom coming between them. It is essential to the theory that the atom shall have a mass but shall not have dimensions.

This hypothesis is not inconsistent with any known facts, but it has been described, perhaps not unjustly, as a mere mathematical fiction, and certainly it is opposed to the apparent indications of our senses. At any rate it is artificial, though it may be a prejudice to regard that as an argument against its adoption. To some extent this view was accepted by Faraday.

Sir William Thomson, afterwards Lord Kelvin, showed† that, if we assume the existence of gravitation, then each of the above hypotheses will account for cohesion.

* *Philosophiae Naturalis Theoria Redacta ad Unicam Legem Virium*, Vienna, 1759.

† *Proceedings of the Royal Society of Edinburgh*, April 21, 1862, vol. IV, pp. 604—606.

(iii) *Hypothesis of an Elastic Solid Ether.* Some physicists tried to explain the known phenomena by properties of the medium through which our impressions are derived. By postulating that all space is filled with a medium possessed of many of the characteristics of an elastic solid, it was shown by Fresnel, Green, Cauchy, Neumann, MacCullagh, and others that a large number of the properties of light and electricity may be explained. In spite of the difficulties to which this hypothesis necessarily leads, and of its inherent improbability, it has been discussed by Stokes, Lamé, Boussinesq, Sarrau, Lorentz, Lord Rayleigh, and Kirchhoff.

This hypothesis was modified and rendered somewhat more plausible by von Helmholtz, Lommel, Ketteler*, and Voigt, who based their researches on the assumption of a mutual reaction between the ether and the material molecules located in it: on this view the problems connected with refraction and dispersion have been simplified. Finally, Sir William Thomson in his Baltimore Lectures, 1885, suggested a mechanical analogue to represent the relations between matter and this ether, by which a possible constitution of the ether can be realized. He also suggested later a form of *labile ether*, from whose properties most of the more familiar physical phenomena can be deduced, provided the arrangement can be considered stable; a labile ether is an elastic solid, and its properties in two dimensions may be compared with those of a soap-bubble film, in three dimensions.

It is, however, difficult to criticize any of these hypotheses as a theory of the constitution of matter until the arrangement of the atoms or their nature is more definitely expressed.

III. Dynamical Theories. In more recent years the suggestion was made that the so-called atoms may be forms of motion (*ex. gr.* permanent eddies) in one elementary material known as the ether; on this view all the atoms are constituted of the same matter, but the physical conditions are different for the different kinds of atoms. It has been said that there is an

* *Theoretische Optik*, Braunschweig, 1885.

initial difficulty in any such hypothesis, since the all-pervading elementary fluid must possess inertia, so that to explain matter we assume the existence of a fluid possessing one of the chief characteristics of matter. This is true as far as it goes, but it is not more unreasonable than to attribute all the fundamental properties of matter to the atoms themselves, as is done by many writers. The next paragraph contains a statement of one of the earliest attempts to formulate a dynamical atomic hypothesis.

(i) *The Vortex Ring Hypothesis.* This hypothesis assumes that each atom is a vortex ring in an incompressible frictionless homogeneous fluid.

Vortex rings—though, since friction is brought into play, of an imperfect character—can be produced in air by many smokers. Better specimens can be formed by taking a card-board box in one side of which a circular hole is cut, filling it with smoke, and hitting the opposite side sharply. The tendency of the particles forming a ring to maintain their annular connection may be illustrated by placing such a box on one side of a room in a direct line with the flame of a lighted candle on the other side. If properly aimed, the ring will travel across the room and put out the flame. If the box is filled only with air, so that the ring is not visible, the experiment is more effective.

In 1858 von Helmholtz* showed that a closed vortex filament in a perfect fluid is indestructible and retains certain characteristics always unaltered. In 1867 Sir William Thomson propounded† the idea that matter consists of vortex rings in a fluid which fills space. If the fluid is perfect we could neither create new vortex rings nor destroy those already created, and thus the permanence of the atoms is explained. Moreover the atoms would be flexible, compressible, and in incessant vibration

* *Crelle's Journal*, 1858, vol. LV, pp. 25—55; translated by Tait in the *Philosophical Magazine*, June, 1867, supplement, series 4, vol. XXXIII, pp. 485—512.

† *Proceedings of the Royal Society of Edinburgh*, Feb. 18, 1867, vol. VI, pp. 94—105.

at a definite fundamental rate. This rate is very rapid, and Sir William Thomson gave the number of vibrations per second of a sodium ring as probably being greater than 10^{14}.

By a development of this hypothesis Sir J. J. Thomson* showed, some years ago, that chemical combination may be explained. He supposed that a molecule of a compound is formed by the linking together of vortex filaments representing atoms of different elements: this arrangement may be compared with that of helices on an anchor ring. For stability not more than six filaments may be combined together, and their strengths must be equal. Another way of explaining chemical combination on the vortex atom hypothesis has been suggested by W. M. Hicks. It is known† that a spherical mass of fluid, whose interior possesses vortex motion, can move through liquid like a rigid sphere, and he has shown that one of these spherical vortices can swallow up another, thus forming a compound element.

(ii) *The Vortex Sponge Hypothesis.* Any vortex atom hypothesis labours under the difficulty of requiring that the density of the fluid ether shall be comparable with that of ordinary matter. In order to obviate this and at the same time to enable it to transmit transversal radiations Sir William Thomson suggested what has been termed, not perhaps very happily, the vortex sponge hypothesis‡: this rests on the assumption that laminar motion can be propagated through a turbulently moving inviscid liquid. The mathematical difficulties connected with such motion have prevented an adequate discussion of this hypothesis, and I confine myself to merely mentioning it.

These hypotheses, of vortex motion in a fluid, account for the indestructibility of matter and for many of its properties. But in order to explain statical electrical attraction it would

* *A Treatise on the Motion of Vortex Rings*, Cambridge, 1883.

† See a memoir by M. J. M. Hill in the *Philosophical Transactions of the Royal Society*, London, 1894, part I, pp. 213—246.

‡ *Philosophical Magazine*, London, October, 1887, series 5, vol. XXIV, pp. 342—353.

seem necessary to suppose that the ether is elastic; in other words, that an electric field must be a field of strain. If so, complete fluidity in the ether would be impossible, and hence the above theories are now regarded as untenable.

(iii) *The Ether-Squirts Hypothesis.* Karl Pearson* suggested another dynamical theory in which an atom is conceived as a point at which ether is pouring into our space from space of four dimensions.

If an observer lived in two-dimensional space filled with ether and confined by two parallel and adjacent surfaces, and if through a hole in one of these surfaces fresh ether were squirted into this space, the variations of pressure thereby produced might give the impression of a hard impenetrable body. Similarly an ether-squirt from space of four dimensions into our space might give us the impression of matter.

It seems necessary on this hypothesis to suppose that there are also ether-sinks, or atoms of negative mass; but as ether-squirts would repel ether-sinks we may suppose that the latter have moved out of the universe known to our senses.

By defining the mass of an atom as the mean rate at which ether is squirting into our space at that point, we can deduce the Newtonian law of gravitation, and by assuming certain periodic variations in the rate of squirting we can deduce some of the phenomena of cohesion, of chemical action, and of electromagnetism and light. But of course the hypothesis rests on the assumption of the existence of a world beyond our senses.

(iv) *The Electron Hypothesis.* MacCullagh, in 1837 and 1839, proposed to account for optical phenomena on the assumption of an elastic ether possessing elasticity of the type required to enable it to resist rotation. This suggestion has been recently modified and extended by Sir Joseph Larmor†, and, as now enunciated, it accounts for many of the electrical and magnetic (as well as the optical) properties of matter.

* *American Journal of Mathematics*, 1891, vol. XIII, pp. 309—362.

† *Philosophical Transactions of the Royal Society*, London, 1894, pp. 719—822; 1895, pp. 695—743.

The hypothesis is however very artificial. The assumed ether is a rotationally elastic incompressible fluid. In this fluid Larmor introduces monad electric elements or *electrons*, which are nuclei of radial rotational strain. He supposes that these electrons constitute the basis of matter. He further supposes that an electrical current consists of a procession of these electrons, and that a magnetic particle is one in which these entities are revolving in minute orbits. Dynamical considerations applied to such a system lead to an explanation of nearly all the more obvious phenomena. By further postulating that the orbital motion of electrons in the atom constitute it a fluid vortex it is possible to apply the hydrodynamical pulsatory theory of Bjerknes or Hicks and obtain an explanation of gravitation.

Thus on this view mass is explained as an electrical manifestation. Electricity in its turn is explained by the existence of electrons, that is, of nuclei of strain in the ether, which are supposed to be in incessant and rapid motion. Whilst, to render this possible, properties are attributed to the ether which are apparently inconsistent with our experience of the space it fills. Put thus, the hypothesis seems very artificial. Perhaps the utmost we can say for it is that, from some points of view, it may, so far as analysis goes, be an approximation to the true theory; in any case much work will have to be done before it can be considered established even as a working hypothesis.

(v) *Recent Developments.* Most of the above was written in 1891. Since then investigations on radio-activity have opened up new avenues of conjecture which tend to strengthen the electron theory as a working hypothesis. More than thirty years ago Clerk Maxwell had shown that light and electricity were closely connected phenomena. It was then believed that both were due to waves in the hypothetical ether, but it was supposed that the phenomena of matter on the one side and of light and electricity on the other were sharply distinguished one from the other. The differences, however, between matter and light tend to disappear as investigations proceed. In 1895

Röntgen established the existence of rays which could produce light, which had the same velocity as light, which were not affected by a magnet, and which could traverse wood and certain other opaque substances like glass. A year later Becquerel showed that uranium was constantly emitting rays which, though not affecting the eye as light, were capable of producing an image on a photographic plate. Like Röntgen rays they can go through thin sheets of metal; like heat rays they burn the skin; like electricity they generate ozone from oxygen. Passed into the air they enable it to conduct the electric current. Their speed has been measured and found to be rather more than half that of light and electricity. It was soon found that thorium possessed a similar property, but in 1903 Curie showed that radium possessed radio-activity to an extent previously unsuspected in any body, and in fact the rays were so powerful as to make the substance directly visible. Further experiments showed that numerous bodies are radio-active, but the effects are so much more marked in radium that it is convenient to use that substance for most experimental purposes.

Radium gives off no less than three kinds of rays besides a radio-active emanation. In these discharges there appears to be a gradual change from what had been supposed to be an elementary form of matter to another. This leads to the belief that of the known forms of matter some, perhaps even all, are not absolutely stable. On the other hand, it may be that only radio-active bodies are unstable, and that in their disintegration we are watching the final stage in the evolution of stable and constant forms of matter. It may, however, in any case turn out that some, or perhaps all, of the so-called elements may be capable of resolution into different combinations of electrons or electricity.

At an earlier date J. J. Thomson had concluded that the glow, seen when an electric current passes through a high vacuum tube, is due to a rush of minute particles across the tube. He calculated their weight, their velocity, and the charge of electricity transported by or represented by them, and found these to be constant. They were deflected like Becquerel rays. All space seems to contain them, and electricity, if not identical with them, is at least carried by them. This suggested that these minute particles might be electrons. If so, they might thus give the ultimate explanation of electricity as well as matter, and the atom of the chemist would be not an irreducible unit of matter, but a system comprising numerous such minute particles. These conclusions are consistent with those subsequently deduced from experiments with radium. In 1904 the hypothesis was carried one stage further. In that year J. J. Thomson investigated the conditions of stability of certain systems of revolving particles; and on the hypothesis that an atom of matter consists of a number of particles carrying negative charges of electricity revolving in orbits within a sphere of positive electrification he deduced many of the properties of the different chemical atoms corresponding to different possible stable systems of this kind. His scheme led to results agreeing closely with the results of Mendelejev's periodic hypothesis according to which some or all of the properties of an element are a periodic function of its atomic weight. An interesting consequence of this view is that Franklin's description of electricity as subtle particles pervading all bodies may turn out to be substantially correct. It is also remarkable that corpuscles somewhat analogous to those whose existence was suggested in Newton's corpuscular theory of light should be now supposed to exist in cathode and Becquerel rays.

(vi) These facts have been utilized by G. Le Bon who suggested that electricity and matter may be regarded as intermediate stages in the flux of the ether, the former being one stage in the incessant dissociation of matter which arises from ether and ultimately is resolved again into ether. The theory is ingenious but appears untenable.

(vii) *Ether as matter.* Recently another hypothesis as to the nature of the ether was put forward by D. I. Mendelejev, the distinguished chemist. In the grouping of the elements according to his periodic law, there were originally twelve series and eight groups. All the elements since discovered have fallen into place in the sequence, and possess properties in general accordance with his grouping, but helium, argon, and other similar inactive elements, constitute a ninth or zero group.

In this scheme hydrogen, with an atomic weight 1·008, is in the first series and first group. It has the lowest atomic weight of any element yet known, but there may be lighter elements, *ex. gr.* one in the first series and zero group. Mendelejev has however suggested that just as a zero group has been now discovered so there may be a zero series; and that the element (if there is one) in the zero group and zero series might be expected to possess properties closely resembling those of the hypothetical ether. It would be the lightest and simplest form of matter, of great elasticity, and with an atomic weight of perhaps about $1/10^6$ as compared with hydrogen. The velocity of its atoms would be so great as to make it all pervading, and it would appear to be capable of doing all that is required from the mysterious ether. The hypothesis is attractive and intelligible.

(viii) *The Bubble Hypothesis**. The difficulty of conceiving the motion of matter through a solid elastic medium has been met in another way, namely, by suggesting that what we call matter is a deficiency of the ether, and that this region of deficiency can move through the ether in a manner somewhat analogous to that in which a bubble can move in a liquid. To express this in technical language we may suppose the ether to consist of an arrangement of minute uniform spherical grains piled together so closely that they cannot change their neighbours, although they can move relatively one to another. Places where the number of grains is less or greater than the number necessary to render the piling normal, move through the medium, as a wave moves through water, though the grains do not move with them. Places where the ether is in excess of the normal amount would repel one another and move away

* O. Reynolds, *Submechanics of the Universe*, Cambridge, 1903.

out of our ken, but places where it is below the normal amount would attract each other according to the law of gravity, and constitute particles of matter which would be indestructible. It is alleged that the theory accounts for the known phenomena of gravity, electricity, and light, provided the size of its grains is properly chosen. Reynolds has calculated that for this purpose their diameter should be rather more than 5×10^{-18} centimetres, and that the pressure in the medium would be about 10^4 tons per square centimetre. This theory is in itself more plausible than the electron hypothesis, but its consequences have not yet been fully worked out.

Returning from these novel hypotheses to the classical theories of matter, we may now proceed a step further. Before a hypothesis on the structure of matter can be ranked as a scientific theory we may reasonably expect it to afford some explanation of three facts. These are (*a*) the Newtonian law of attraction; (*b*) the fact that there are only a finite number of ultimate kinds of matter—such as oxygen, iron, etc.—which can be arranged in a series such that the properties of the successive members are connected by a regular law; and (*c*) the main results of spectrum analysis.

In regard to the first point (*a*), we can say only that none of the above theories are inconsistent with the known laws of attraction; and as far as the ether-squirts, the electron, and the bubble hypotheses are concerned, they have been elaborated into a form from which the gravitational law of attraction can be deduced. But we may still say that as to the cause of gravity —or indeed of force—we know nothing.

Newton, in his correspondence with Bentley, while declaring his ignorance of the cause of gravity, refused to admit the possibility of force acting at a finite distance through a vacuum. "You sometimes speak of gravity," said he*, "as essential and inherent to matter: pray do not ascribe that notion to me, for the cause of gravity is what I do not pretend to know." And in another

* Letter dated Jan. 17, 1693. I quote from the original, which is in the Library of Trinity College, Cambridge; it is printed in the *Letters to Bentley*, London, 1756, p. 20.

place he wrote*, "'Tis inconceivable, that inanimate brute matter should (without the mediation of something else which is not material) operate upon and affect other matter without mutual contact; as it must if gravitation in the sense of Epicurus, be essential and inherent in it...That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance thro' a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws, but whether this agent be material or immaterial, I have left to the consideration of my readers."

I have already alluded to conjectural explanations of gravity dependent on the ether-squirts, the electron, and the bubble hypotheses. Of other conjectures as to the cause of gravity, three, which do not involve the idea of force acting at a distance, may be here mentioned:

(1) The first of these conjectures was propounded by Newton in the *Queries* at the end of his *Opticks*, where he suggested as a possible explanation the existence of a stress in the ether surrounding a particle of matter†.

This was elaborated on a statical basis by Clerk Maxwell, who showed‡ that the stress would have to be at least 3000 times as great as that which the strongest steel would support. Sir William Thomson suggested§ a dynamical way of producing the stress by supposing that space is filled with an incompressible fluid, constantly being annihilated by each atom of matter at a rate proportional to its mass, a constant supply

* Letter dated Feb. 25, 1693; *Letters to Bentley*, London, 1756, pp. 25, 26.

† Quoted by S. P. Rigaud in his *Essay* on the *Principia*, Oxford, 1838, appendix, pp. 68—70. On other guesses by Newton see Rigaud, text, pp. 61—62, and references there given.

‡ Article *Attraction*, in *Encyclopaedia Britannica*, or *Collected Works*, vol. II, p. 489.

§ *Proceedings of the Royal Society of Edinburgh*, Feb. 7, 1870, vol. VII, pp. 60—63.

being kept up at an infinite distance. It is true that this avoids Clerk Maxwell's difficulty, but we have no right to introduce such sinks and sources of fluid unless we have other grounds for believing in their existence. The conclusion is that Newton's conjecture is very improbable unless we adopt the ether-squirts theory: on that hypothesis it is a plausible explanation.

I should add that Maclaurin implies* that though the above explanation was Newton's early opinion, yet his final view was that he could not devise any tenable hypothesis about the cause of gravitation.

(2) In 1782 Le Sage of Geneva suggested† that gravity was caused by the bombardment of streams of ultramundane corpuscles. These corpuscles are supposed to come in all directions from space and to be so small that inter-collisions are rare.

A body by itself in space would receive on an average as many blows on one side as on another, and therefore would have no tendency to move. But, if there are two bodies, each will screen the other from some of the bombarding corpuscles. Thus each body will receive more blows on the side remote from the other body than on the side turned towards it. Hence the two bodies will be impelled each towards the other.

In order to make this force between two particles vary directly as the product of their masses and inversely as the square of the distance between them, Le Sage showed that it was sufficient to suppose that the mass of a body was proportional to the area of a section at right angles to the direction in which it was attracted. This requires that the constitution of a body shall be molecular, and that the distances between consecutive molecules shall be very large compared with the sizes of the molecules. On the vortex hypothesis we may suppose that the ultramundane corpuscles are vortex rings.

* *An Account of Sir Isaac Newton's Philosophical Discoveries*, London, 1748, p. 111.

† *Mémoires de l'Académie des Sciences* for 1782, Berlin, 1784, pp. 404—432: see also the first two books of his *Traité de Physique*, Geneva, 1818.

This is ingenious, and it is possible that if the corpuscles were perfectly elastic the theory might be tenable*. But the results of Clerk Maxwell's numerical calculation show, first, that the particles must be imperfectly elastic; second, that merely to produce the effect of the attraction of the earth on a mass of one pound would require that Le Sage's corpuscles should expend energy at the rate of at least billions† of foot-pounds per second; and third, that it is probable that the effect of such a bombardment would be to raise the temperature of all bodies beyond a point consistent with our experience. Finally, it seems probable that the distance between consecutive molecules would have to be considerably greater than is compatible with the results given below.

Tait summed up the objections to these two hypotheses by saying‡, "One common defect of these attempts is...that they all demand some prime mover, working beyond the limits of the visible universe or inside each atom: creating or annihilating matter, giving additional speed to spent corpuscles, or in some other way supplying the exhaustion suffered in the production of gravitation. Another defect is that they all make gravitation a mere difference-effect, as it were; thereby implying the presence of stores of energy absolutely gigantic in comparison with anything hitherto observed, or even suspected to exist, in the universe; and therefore demanding the most delicate adjustments, not merely to maintain the conservation of energy which we observe, but to prevent the whole solar and stellar systems from being instantaneously scattered in fragments through space. In fact, the cause of gravitation remains undiscovered."

(3) There is another conjecture on the cause of gravity which I may mention§. It is possible that the attraction of one particle on another might be explained if both of them

* See a paper by Sir William Thomson in the *Proceedings of the Royal Society of Edinburgh*, Dec. 18, 1871, vol. VII, pp. 577—589.

† I use billion with the English (and not the French) meaning, that is, a billion $=10^{12}$.

‡ *Properties of Matter*, London, 1885, art. 164.

§ See an article by myself in the *Messenger of Mathematics*, Cambridge, 1891, vol. XXI, pp. 20—24.

rested on a homogeneous elastic body capable of transmitting energy. This is the case if our three-dimensional universe rests in the direction of a fourth dimension on a four-dimensional homogeneous elastic body (which we may call the ether) whose thickness in the fourth dimension is small and constant.

The results of spectrum analysis lead us to suppose that every molecule of matter in our universe is in constant vibration. On the above hypothesis these vibrations would cause a disturbance in the supporting space, *i.e.* in the ether. This disturbance would spread out uniformly in all directions; the intensity diminishing as the square of the distance from the centre of vibration, but the rate of vibration remaining unaltered. The transmission of light and radiant heat may be explained by such vibrations transversal to the direction of propagation. It is possible that gravity may be caused by vibrations in the supporting space which are wholly longitudinal or are compounded of vibrations which are partly longitudinal and partly transversal in any of the three directions at right angles to the direction of propagation. If we define the mass of a molecule as proportional to the intensity of these vibrations caused by it, then at any other point in space the intensity of the vibration there would vary as the mass of the molecule and inversely as the square of the distance from the molecule; hence, if we may assume that such vibrations of the medium spreading out from any centre would draw to that centre a particle of unit mass at any other point with a force proportional to the intensity of the vibration there, then the Newtonian law of attraction would follow. This conjecture is consistent either with Boscovich's hypothesis or with the vortex theory. It would be interesting if the results of a branch of pure mathematics so abstract as the theory of hyper-space should be found to be closely connected with one of the most fundamental problems of material science.

I should sum up the effect of this discussion on gravity on the relative probabilities of the hypotheses as to the constitution of matter enumerated above, by saying that it does not enable us to discriminate between them.

The fact that the number of kinds of matter (chemical elements) is finite and the consequences of spectrum analysis are closely related. The results of spectrum analysis show that every molecule of any species of matter, such as hydrogen, vibrates with (so far as we can tell) exactly equal sets of periods of vibration. This then is one of the characteristics of the particular kind of matter, and it is probable that any explanation of why the molecules of each kind have a definite set of periods of vibration will account also for the fact that the number of kinds of matter is finite.

Various attempts to explain why the molecules of matter are capable only of certain definite periods of vibration have been made, and it may be interesting if I give them briefly.

(1) To begin with, I may note the conjecture that it depends on properties of time. This, however, is impossible, for the continuity of certain spectra proves that in these cases there is nothing which prevents the period of vibration from taking any one of millions of different values: thus no explanation dependent on the nature of time is permissible.

(2) It has been suggested that there may have been a sorting agency, and only selected specimens of the infinite number of species formed originally have got into our universe. The objection to this is that no explanation is offered as to what has become of the excluded molecules.

(3) The finite number of species might be explained by supposing a physical connection to exist between all the molecules in the universe, just as two clocks whose rates are nearly the same tend to go at the same rate if their cases are connected.

Clerk Maxwell's objection to this is that we have no other reason for supposing that such a connection exists, but if we are living in a space of four dimensions as suggested above in chapter XIX, this connection does exist, for all the molecules rest on one and the same body. This body is capable of transmitting vibrations, hence, no matter how the molecules were set vibrating originally, they would fall into certain groups, and all the members of each group would vibrate at the same rate. It was the possibility of obtaining thus a physical connection between the various particles in our universe that first suggested to me the idea of a supporting medium in a fourth dimension.

(4) If we accept Boscovich's hypothesis or that of an elastic solid ether, and if we may lay it down as axiomatic that the mass of every sub-atom is the same, we may conceive that the number of ways of combining the sub-atoms into a permanent system is limited, and that the period of vibration depends on the form in which the sub-atoms are combined into an atom. This view is not inconsistent with any known facts. I may add that it is probable that the chemical atom is the essential vibrating system, for the sodium spectrum, to take one instance, is the same as that of all its compounds.

(5) In the same way we may suppose that the vortex rings are formed so that they can have only a definite number of stable forms produced by interlinking or kinking.

(6) Similarly we may modify the popular hypothesis by treating the atoms as indivisible aggregates of sub-atoms which are in all respects equal and similar, and can be combined in only a limited number of forms which are permanent. But most of the old difficulties connected with the atoms arise again in connection with the sub-atoms.

(7) I am not aware that Clerk Maxwell discussed any other hypotheses in connection with this point, but it has been suggested recently that, if the various forms of matter were evolved originally out of some one primitive material, then there may have been periodic disturbances in this matter when the atoms were being formed, such that they were produced only at some definite phase in the period*.

Thus, if the disturbance is represented by the swinging of a pendulum in a resisting medium, it might be supposed that the atoms were formed at the points of maximum amplitude, and we should expect that the atoms successively thrown off would form a series having the properties of its successive members connected by a regular periodic law. This conjecture,

* See *Nature*, Sept. 2, 1886, vol. xxxiv, pp. 423—432.

when worked out in some detail, led to the conclusion that some elements which ought to have appeared in the series were missing, but it was possible to predict their properties and to suggest the substances with which they were most likely to be found in combination. Guided by these theoretical conclusions a careful chemical analysis revealed the fact that such elements did exist.

That this hypothesis has led to new discoveries is something in its favour, but I do not wish to be understood to say that it is a theory which leads to results that have been verified subsequently. I should say rather that we have obtained an analogy which is sufficiently like the truth to suggest new discoveries. Such analogies are often the precursors of laws, so that it is not unreasonable to hope that ere long our knowledge of this border-land of chemistry and physics may be more definite, and thus that molecular physics may be brought within the domain of mathematics. It is however very remarkable that J. J. Thomson's conclusions on the stability of the orbital systems he devised should agree so closely with Mendelejev's periodic law.

On the whole Clerk Maxwell thought that the phenomena point to a common origin of all molecules of the same kind, that this was an event not belonging to that order of nature under which we live, but must have originated when or before the existing order was established, and that so long as the present order exists it is immutable.

This is equivalent to saying that we have arrived at a point beyond which our limited experience does not enable us to carry the explanation.

That we should be able to form an approximate idea of the size of the molecules of matter is a testimony to the extraordinary development of mathematical physics in the course of the nineteenth century.

Sir William Thomson suggested* four distinct methods of

* See *Nature*, March 31, 1870, vol. I, pp. 551—553; and Tait's *Recent Advances*, pp. 303—318. The fourth method had been proposed by Loschmidt in 1863.

attacking the problem. They lead to results which are not very different.

The first of these rests on an assertion of Cauchy that the phenomena of prismatic colours show that the distance between consecutive molecules of matter is comparable with the wave-lengths of light. Taking the most unfavourable case this would seem to indicate that in a transparent homogeneous solid or liquid medium there are not more than 64×10^{24} molecules in a cubic inch, that is, that the distance between consecutive molecules is greater than $1/(4 \times 10^{8})$th of an inch.

The second method is founded on the amount of work required to draw out a film of liquid, such as a soap-bubble, to a given thickness. This can be calculated from experiments in a capillary tube, and it is found that, if a soap-bubble could be drawn out to a thickness of $1/10^{8}$th of an inch there would be but a few molecules in its thickness. This method is not quantitative.

Thirdly, Thomson proved that the contact phenomena of electricity require that in an alloy of brass the distance between two molecules, one of zinc and one of copper, shall be greater than $1/(7 \times 10^{8})$th of an inch; hence the number of molecules in a cubic inch of zinc or copper is not greater than 35×10^{25}.

Lastly, the kinetic theory of gases leads to the conclusion that certain phenomena of temperature and viscosity depend, *inter alia*, on inter-molecular collisions, and so on the sizes and velocities of the molecules, while the average velocity with which the molecules move increases with the temperature. This leads to the conclusion that the distance between two consecutive molecules of a gas at normal pressure and temperature is greater than $1/(6 \times 10^{6})$th of an inch, and is less than $1/10^{7}$th of an inch; while the actual size of the molecule is a trifle greater than $1/(3 \times 10^{20})$th of a cubic inch; and the number of molecules in a cubic inch is about 3×10^{20}.

Thus it would seem that a cubic inch of gas at ordinary pressure and temperature contains about 3×10^{20} molecules, all similar and equal, and each molecule has a volume of about $1/(3 \times 10^{25})$th of a cubic inch; while a cubic inch of the simplest solid or liquid contains rather less than 10^{27} molecules, and perhaps each molecule has a volume of about $1/(3 \times 10^{26})$th of a cubic inch. For instance, if a pea or a drop of water whose radius is 1/16th inch was magnified to the size of the earth, then there would be about thirty molecules in every cubic foot of it, and probably the size of a molecule would be about the same as that of a fives-ball. The average size of the minute drops of water in a very light cloud can be calculated from the coloured rings produced when the sun or moon shines through it. The radius of a drop is about 1/30000th of an inch. Such a drop therefore would contain about 2×10^{13} separate molecules. In gases and vapours, the number of atoms required to make up one of these molecules can be estimated, but in liquids the number is not as yet known.

Loschmidt asserted that a cube whose side is 1/4000th of a millimetre is the smallest object which can be made visible at the present time. Such a cube of oxygen or nitrogen would contain from 60 to 100 millions of molecules of the gas. Also on an average about 50 elementary molecules of the so-called elements are required to constitute one molecule of organic matter. At least half of every living organism consists of water, and we may for the moment suppose that the remainder consists of organic matter. Hence the smallest living being which is visible under the microscope contains from 30 to 50 millions of elementary molecules which are combined in the form of water, and from 30 to 50 millions of elementary molecules which are combined so as to make not more than one million organic molecules.

Hence a very simple organism might be built up out of as few as a million similar organic molecules. Clerk Maxwell did not consider that this was sufficient to justify the current conclusions of physiologists, and said that they must not suppose that structural details of infinitely small dimensions can furnish by themselves an explanation of the variety known to exist in the properties and functions of the most minute organisms; but physiologists have replied that whether their conjectures be right or wrong Clerk Maxwell's argument is vitiated by his non-consideration of differences due to the physical (as opposed to the chemical) structure of the organism and the consequent motions of the component parts.

Throughout this chapter I have written as if the mass of a body were independent of whether it is or is not in motion relative to the hypothetical ether. This is assumed in the usual, or Newtonian, system of dynamics, but it has been recently called in question, notably by H. A. Lorentz, A. Einstein, and H. Minkowski.

The ultimate reason for this scepticism is the absence of any recognizable phenomena arising from the earth's motion relative to the ether: a question which was the subject of a series of experiments made in 1882 by A. A. Michelson and E. W. Morley. To account for this, Einstein propounded a theory of Relativity* in which he assumed certain relations between the measures of space and time employed by two observers who have a mutual relative velocity v. If the origin of coordinates be the same for both observers at the instant at which they both commence to reckon time, and if the axis of x be taken in the direction of v, he assumes that the relations between the coordinates of a point and the times T, t which have apparently elapsed at any subsequent instant are

$$X=\beta(x-vt), \quad Y=y, \quad Z=z, \quad T=\beta(t-vx/c^2),$$

where X, Y, Z, T refer to the observations of the first observer, and x, y, z, t to those of the other; c is the velocity of light; and $\beta=(1-v^2/c^2)^{-\frac{1}{2}}$. If v be negligible compared with c, these relations are the same as in the Newtonian system.

The theory leads to the result that moving bodies contract in the direction of their advance, and the greater the velocity the greater the contraction; thus, since the earth rotates from

* For an account of the theory, see N. R. Campbell, *Philosophical Magazine*, London, April, 1911, pp. 502—517

west to east, the bulk of a man walking eastwards will be somewhat smaller than his bulk when he walks westwards. Again, on this theory the mass of an electron may be taken to increase with its velocity, and it would become indefinitely great if its velocity were equal to that of light. At present the theory is beyond the range of direct verification, but it is not inconsistent with known facts, it does provide an explanation of the motion of the apse line of Mercury's orbit which had not previously been brought under the Newtonian law in gravitational astronomy, and possibly it may account for some obscure phenomena connected with the theory of atoms and ions.

INDEX.

CAMBRIDGE: PRINTED BY J. B. PEACE, M.A., AT THE UNIVERSITY PRESS.

A SHORT ACCOUNT OF THE

HISTORY OF MATHEMATICS

BY W. W. ROUSE BALL.

[*Sixth Edition,* 1914. *Pp.* xxiv + 522. *Price* 12*s.* 6*d. net.*]

MACMILLAN AND CO. LTD., LONDON AND NEW YORK.

THIS book gives an account of the lives and discoveries of those mathematicians to whom the development of the subject is mainly due. The use of technicalities has been avoided and the work is intelligible to any one acquainted with the elements of mathematics.

It commences with an account of the origin and progress of Greek mathematics, from which the Alexandrian, the Indian, and the Arab schools may be said to have arisen. Next the mathematics of medieval Europe and the renaissance are described. The latter part of the book is devoted to the history of modern mathematics, beginning with the invention of analytical geometry and the infinitesimal calculus. The history is brought down to the present time.

This excellent summary of the history of mathematics supplies a want which has long been felt in this country. The extremely difficult question, how far such a work should be technical, has been solved with great tact. . . The work contains many valuable hints, and is thoroughly readable. The biographies, which include those of most of the men who played important parts in the development of culture, are full and general enough to interest the ordinary reader as well as the specialist. Its value to the latter is much increased by the numerous references to authorities, a good table of contents, and a full and accurate index.—*The Saturday Review*.

Mr. Ball's book should meet with a hearty welcome, for though we possess other histories of special branches of mathematics, this is the first serious attempt that has been made in the English language to give a systematic account of the origin and development of the science as a whole. It is written too in an attractive style. Technicalities are not too numerous or obtrusive, and the work is interspersed with biographical sketches and

anecdotes likely to interest the general reader. Thus the tyro and the advanced mathematician alike may read it with pleasure and profit.—*The Athenæum.*

A wealth of authorities, often far from accordant with each other, renders a work such as this extremely formidable; and students of mathematics have reason to be grateful for the vast amount of information which has been condensed into this short account. . . . In a survey of so wide extent it is of course impossible to give anything but a bare sketch of the various lines of research, and this circumstance tends to render a narrative scrappy. It says much for Mr. Ball's descriptive skill that his history reads more like a continuous story than a series of merely consecutive summaries.—*The Academy.*

We can heartily recommend to our mathematical readers, and to others also, Mr. Ball's *History of Mathematics.* The history of what might be supposed a dry subject is told in the pleasantest and most readable style, and at the same time there is evidence of the most careful research.—*The Observatory.*

All the salient points of mathematical history are given, and many of the results of recent antiquarian research; but it must not be imagined that the book is at all dry. On the contrary the biographical sketches frequently contain amusing anecdotes, and many of the theorems mentioned are very clearly explained so as to bring them within the grasp of those who are only acquainted with elementary mathematics.—*Nature.*

Le style de M. Ball est clair et élégant, de nombreux aperçus rendent facile de suivre le fil de son exposition et de fréquentes citations permettent à celui qui le désire d'approfondir les recherches que l'auteur n'a pu qu'effleurer. . . . Cet ouvrage pourra devenir très utile comme manuel d'histoire des mathématiques pour les étudiants, et il ne sera pas déplacé dans les bibliothèques des savants.—*Bibliotheca Mathematica.*

The author modestly describes his work as a compilation, but it is thoroughly well digested, a due proportion is observed between the various parts, and when occasion demands he does not hesitate to give an independent judgment on a disputed point. His verdicts in such instances appear to us to be generally sound and reasonable. . . . To many readers who have not the courage or the opportunity to tackle the ponderous volumes of Montucla or the (mostly) ponderous treatises of German writers on special periods, it may be somewhat of a surprise to find what a wealth of human interest attaches to the history of so "dry" a subject as mathematics. We are brought into contact with many remarkable men, some of whom have played a great part in other fields, as the names of Gerbert, Wren, Leibnitz, Descartes, Pascal, D'Alembert, Carnot, among others may testify, and with at least one thorough blackguard (Cardan); and Mr. Ball's pages abound with quaint and amusing touches characteristic of the authors under consideration, or of the times in which they lived.—*Manchester Guardian.*

There can be no doubt that the author has done his work in a very excellent way. . . . There is no one interested in almost any part of mathematical science who will not welcome such an exposition as the present, at once popularly written and exact, embracing the entire subject. . . . Mr. Ball's work is destined to become a standard one on the subject.—*The Glasgow Herald.*

A most interesting book, not only for those who are mathematicians, but for the much larger circle of those who care to trace the course of general scientific progress. It is written in such a way that those who have only an elementary acquaintance with the subject can find on almost every page something of general interest.—*The Oxford Magazine.*

La lecture en est singulièrement attachante et instructive.—*Bulletin des sciences mathématiques.*

A PRIMER OF THE
HISTORY OF MATHEMATICS

BY W. W. ROUSE BALL.

[*Fourth Edition*, 1914. *Pp.* iv + 149. *Price* 2*s.* 6*d. net.*]

MACMILLAN AND CO. LTD., LONDON AND NEW YORK.

THIS book contains a sketch in popular language of the history of mathematics; it includes notices of the lives and surroundings of those to whom the development of the subject is mainly due as well as of their discoveries.

This Primer is written in the agreeable style with which the author has made us acquainted in his previous essays; and we are sure that all readers of it will be ready to say that Mr. Ball has succeeded in the hope he has formed, that "it may not be uninteresting" even to those who are unacquainted with the leading facts. It is just the book to give an intelligent young student, and should allure him on to the perusal of Mr. Ball's "Short Account." The present work is not a mere *réchauffé* of that, though naturally most of what is here given will be found in equivalent form in the larger work. . . . The choice of material appears to us to be such as should lend interest to the study of mathematics and increase its educational value, which has been the author's aim. The book goes well into the pocket, and is excellently printed.—*The Academy.*

We have here a new instance of Mr. Rouse Ball's skill in giving in a small space an intelligible account of a large subject. In 137 pages we have a sketch of the progress of mathematics from the earliest records up to the middle of this century, and yet it is interesting to read and by no means a mere catalogue.—*The Manchester Guardian.*

It is not often that a reviewer of mathematical works can confess that he has read one of them through from cover to cover without abatement of interest or fatigue. But that is true of Mr. Rouse Ball's wonderfully entertaining little "History of Mathematics," which we heartily recommend to even the quite rudimentary mathematician. The capable mathematical master will not fail to find a dozen interesting facts therein to season his teaching.—*The Saturday Review.*

A fascinating little volume, which should be in the hands of all who do not possess the more elaborate *History of Mathematics* by the same author. —*The Mathematical Gazette.*

This excellent sketch should be in the hands of every student, whether he is studying mathematics or no. In most cases there is an unfortunate lack of knowledge upon this subject, and we welcome anything that will help to supply the deficiency. The primer is written in a concise, lucid and easy manner, and gives the reader a general idea of the progress of mathematics that is both interesting and instructive.—*The Cambridge Review.*

Mr. Ball has not been deterred by the existence and success of his larger "History of Mathematics" from publishing a simple compendium in about a quarter of the space. . . . Of course, what he now gives is a bare outline of the subject, but it is ample for all except the most advanced proficients. There is no question that, as the author says, a knowledge of the history of a science lends interest to its study, and often increases its educational value. We can imagine no better cathartic for any mathematical student who has made some way with the calculus than a careful perusal of this little book.—*The Educational Times.*

The author has done good service to mathematicians by engaging in work in this special field. . . . The Primer gives, in a brief compass, the history of the advance of this branch of science when under Greek influence, during the Middle Ages, and at the Renaissance, and then goes on to deal with the introduction of modern analysis and its recent developments. It refers to the life and work of the leaders of mathematical thought, adds a new and enlarged value to well-known problems by treating of their inception and history, and lights up with a warm and personal interest a science which some of its detractors have dared to call dull and cold.—*The Educational Review.*

It is not too much to say that this little work should be in the possession of every mathematical teacher. . . . The Primer gives in a small compass the leading events in the development of mathematics. . . . At the same time, it is no dry chronicle of facts and theorems. The biographical sketches of the great workers, if short, are pithy, and often amusing. Well-known propositions will attain a new interest for the pupil as he traces their history long before the time of Euclid.—*The Journal of Education.*

This is a work which all who apprehend the value of "mathematics" should read and study . . . , and those who wish to learn how to think will find advantage in reading it.—*The English Mechanic.*

The subject, so far as our own language is concerned, is almost Mr. Ball's own, and those who have no leisure to read his former work will find in this Primer a highly-readable and instructive chapter in the history of education. The condensation has been skilfully done, the reader's interest being sustained by the introduction of a good deal of far from tedious detail.—*The Glasgow Herald.*

Mr. W. W. Rouse Ball is well known as the author of a very clever history of mathematics, besides useful works on kindred subjects. His latest production is *A Primer of the History of Mathematics*, a book of one hundred and forty pages, giving in non-technical language a full, concise, and readable narrative of the development of the science from the days of the Ionian Greeks until the present time. Anyone with a leaning towards algebraic or geometrical studies will be intensely interested in this account of progress from primitive usages, step by step, to our present elaborate systems. The lives of the men who by their research and discovery helped along the good work are described briefly, but graphically. . . . The Primer should become a standard text-book.—*The Literary World.*

This is a capital little sketch of a subject on which Mr. Ball is an acknowledged authority, and of which too little is generally known. Mr. Ball, moreover, writes easily and well, and has the art of saying what he has to say in an interesting style.—*The School Guardian.*

MATHEMATICAL RECREATIONS AND ESSAYS

BY W. W. ROUSE BALL.

[*Seventh Edition*, 1917. *Pp.* xvi + 492. *Price* 10*s.* 6*d. net.*]

MACMILLAN AND CO. LTD., LONDON AND NEW YORK.

THIS work is divided into two parts; the first is on mathematical recreations and puzzles, the second includes some miscellaneous essays and an account of various problems of historical interest. In both parts questions which involve advanced mathematics are excluded.

The mathematical recreations include numerous elementary questions and paradoxes, as well as problems such as the proposition that to colour a map not more than four colours are necessary, the explanation of the effect of a cut on a tennis ball, the fifteen puzzle, the eight queens problem, the fifteen school-girls, the construction of magic squares, the theory and history of mazes, and the knight's path on a chess-board.

The second part commences with an account of calculating prodigies and machines, and the history of some half-dozen celebrated problems in mathematics. These are followed by essays on String Figures, Astrology, and Ciphers. The last three chapters are devoted to an account of certain hypotheses as to the nature of Space and Mass, and the means of measuring Time.

Mr. Ball has attained a position in the front rank of writers on subjects connected with the history of mathematics, and this brochure will add another to his successes in this field. In it he has collected a mass of information bearing upon matters of more general interest, written in a style which is eminently readable, and at the same time exact. He has done his work so thoroughly that he has left few ears for other gleaners. The nature of the work is completely indicated to the mathematical student by its title. Does he want to revive his acquaintance with the *Problèmes Plaisans et Délectables* of Bachet, or the *Récréations Mathématiques et Physiques* of

Ozanam? Let him take Mr. Ball for his companion, and he will have the cream of these works put before him with a wealth of illustration quite delightful. Or, coming to more recent times, he will have full and accurate discussion of 'the fifteen puzzle,' 'Chinese rings,' 'the fifteen school-girls problem' *et id genus omne*. Sufficient space is devoted to accounts of magic squares and unicursal problems (such as mazes, the knight's path, and geometrical trees). These, and many other problems of equal interest, come under the head of 'Recreations.' The problems and speculations include an account of the Three Classical Problems; there is also a brief sketch of Astrology; and interesting outlines of the present state of our knowledge of hyper-space and of the constitution of matter. This enumeration badly indicates the matter handled, but it sufficiently states what the reader may expect to find. Moreover for the use of readers who may wish to pursue the several heads further, Mr. Ball gives detailed references to the sources from whence he has derived his information. These *Mathematical Recreations* we can commend as suited for mathematicians and equally for others who wish to while away an occasional hour.—*The Academy*.

The idea of writing some such account as that before us must have been present to Mr. Ball's mind when he was collecting the material which he has so skilfully worked up into his *History of Mathematics*. We think this because . . . many bits of ore which would not suit the earlier work find a fitting niche in this. Howsoever the case may be, we are sure that non-mathematical, as well as mathematical, readers will derive amusement, and, we venture to think, profit withal, from a perusal of it. The author has gone very exhaustively over the ground, and has left us little opportunity of adding to or correcting what he has thus reproduced from his note-books. The work before us is divided into two parts: mathematical recreations and mathematical problems and speculations. All these matters are treated lucidly, and with sufficient detail for the ordinary reader, and for others there is ample store of references. . . . Our analysis shows how great an extent of ground is covered, and the account is fully pervaded by the attractive charm Mr. Ball knows so well how to infuse into what many persons would look upon as a dry subject.—*Nature*.

A fit sequel to its author's valuable and interesting works on the history of mathematics. There is a fascination about this volume which results from a happy combination of puzzle and paradox. There is both milk for babes and strong meat for grown men. . . . A great deal of the information is hardly accessible in any English books; and Mr. Ball would deserve the gratitude of mathematicians for having merely collected the facts. But he has presented them with such lucidity and vivacity of style that there is not a dull page in the book; and he has added minute and full bibliographical references which greatly enhance the value of his work.—*The Cambridge Review*.

Mathematicians with a turn for the paradoxes and puzzles connected with number, space, and time, in which their science abounds, will delight in *Mathematical Recreations and Problems of Past and Present Times*.—*The Times*.

Mathematicians have their recreations; and Mr. Ball sets forth the humours of mathematics in a book of deepest interest to the clerical reader, and of no little attractiveness to the layman. The notes attest an enormous amount of research.—*The National Observer*.

Mr. Ball, to whom we are already indebted for two excellent Histories of Mathematics, has just produced a book which will be thoroughly appreciated by those who enjoy the setting of the wits to work. . . . He has collected a vast amount of information about mathematical quips, tricks, cranks, and puzzles—old and new; and it will be strange if even the most learned do not find something fresh in the assortment.—*The Observatory*.

Mr. Rouse Ball has the true gift of story-telling, and he writes so pleasantly that though we enjoy the fulness of his knowledge we are tempted to forget the considerable amount of labour involved in the preparation of his book. He gives us the history and the mathematics of many problems . . . and where the limits of his work prevent him from dealing fully with the points raised, like a true worker he gives us ample references to original memoirs. . . . The book is warmly to be recommended, and should find a place on the shelves of every one interested in mathematics and on those of every public library.—*The Manchester Guardian.*

A work which will interest all who delight in mathematics and mental exercises generally. The student will often take it up, as it contains many problems which puzzle even clever people.—*The English Mechanic and World of Science.*

This is a book which the general reader should find as interesting as the mathematician. At all events, an intelligent enjoyment of its contents presupposes no more knowledge of mathematics than is now-a-days possessed by almost everybody.—*The Athenæum.*

An exceedingly interesting work which, while appealing more directly to those who are somewhat mathematically inclined, it is at the same time calculated to interest the general reader. . . . Mr. Ball writes in a highly interesting manner on a fascinating subject, the result being a work which is in every respect excellent.—*The Mechanical World.*

É um livro muito interessante, consagrado a recreios mathematicos, alguns dos quaes são muito bellos, e a problemas interessantes da mesma sciencia, que não exige para ser lido grandes conhecimentos mathematicos e que tem em gráo elevado a qualidade de instruir, deleitando ao mesmo tempo.—*Journal de sciencias mathematicas, Coimbra.*

The work is a very judicious and suggestive compilation, not meant mainly for mathematicians, yet made doubly valuable to them by copious references. The style in the main is so compact and clear that what is central in a long argument or process is admirably presented in a few words. One great merit of this, or any other really good book on such a subject, is its suggestiveness; and in running through its pages, one is pretty sure to think of additional problems on the same general lines.—*Bulletin of the New York Mathematical Society.*

A book which deserves to be widely known by those who are fond of solving puzzles . . . and will be found to contain an admirable classified collection of ingenious questions capable of mathematical analysis. As the author is himself a skilful mathematician, and is careful to add an analysis of most of the propositions, it may easily be believed that there is food for study as well as amusement in his pages. . . . Is in every way worthy of praise.—*The School Guardian.*

Once more the author of a *Short History of Mathematics* and a *History of the Study of Mathematics at Cambridge* gives evidence of the width of his reading and of his skill in compilation. From the elementary arithmetical puzzles which were known in the sixteenth and seventeenth centuries to those modern ones the mathematical discussion of which has taxed the energies of the ablest investigator, very few questions have been left unrepresented. The sources of the author's information are indicated with great fulness. . . . The book is a welcome addition to English mathematical literature.—*The Oxford Magazine.*

A HISTORY OF THE STUDY OF

MATHEMATICS AT CAMBRIDGE

BY W. W. ROUSE BALL.

[*Pp.* xvi. + 264. *Price* 6s.]

THE UNIVERSITY PRESS, CAMBRIDGE.

THIS work contains an acoount of the development of the study of mathematics in the university of Cambridge from the twelfth century to the middle of the nineteenth century, and a description of the means by which proficiency in that study was tested at various times.

The first part of the book is devoted to a brief account of the more eminent of the Cambridge mathematicians, the subject matter of their works, and their methods of exposition. The second part treats of the manner in which mathematics was taught, and of the exercises and examinations required of students in past times. A sketch is given of the origin and history of the Mathematical Tripos; this includes the substance of the earlier parts of the author's work on that subject, Cambridge, 1880. To explain the relation of mathematics to other departments of study an outline of the general history of the university and the organization of education therein is added.

The present volume is very pleasant reading, and though much of it necessarily appeals only to mathematicians, there are parts—*e.g.* the chapters on Newton, on the growth of the tripos, and on the history of the university—which are full of interest for a general reader. . . . The book is well written, the style is crisp and clear, and there is a humorous appreciation of some of the curious old regulations which have been superseded by time and change of custom. Though it seems light, it must represent an extensive study and investigation on the part of the author, the essential results of which are skilfully given. We can most thoroughly commend Mr. Ball's volume to all readers who are interested in mathematics or in the growth and the position of the Cambridge school of mathematicians.—*The Manchester Guardian.*

Voici un livre dont la lecture inspire tout d'abord le regret que des travaux analogues n'aient pas été faits pour toutes les Écoles célèbres, et avec autant de soin et de clarté. . . . Toutes les parties du livre nous ont vivement intéressé.—*Bulletin des sciences mathématiques.*

A book of pleasant and useful reading for both historians and mathematicians. Mr. Ball's previous researches into this kind of history have already established his reputation, and the book is worthy of the reputation of its author. It is more than a detailed account of the rise and progress of mathematics, for it involves a very exact history of the University of Cambridge from its foundation.—*The Educational Times.*

Mr. Ball is far from confining his narrative to the particular science of which he is himself an acknowledged master, and his account of the study of mathematics becomes a series of biographical portraits of eminent professors and a record not only of the intellectual life of the *élite* but of the manners, habits, and discussions of the great body of Cambridge men from the sixteenth century to our own. . . . He has shown how the University has justified its liberal reputation, and how amply prepared it was for the larger freedom which it now enjoys.—*The Daily News.*

Mr. Ball has not only given us a detailed account of the rise and progress of the science with which the name of Cambridge is generally associated but has also written a brief but reliable and interesting history of the university itself from its foundation down to recent times. . . . The book is pleasant reading alike for the mathematician and the student of history.—*St. James's Gazette.*

A very handy and valuable book containing, as it does, a vast deal of interesting information which could not without inconceivable trouble be found elsewhere. . . . It is very far from forming merely a mathematical biographical dictionary, the growth of mathematical science being skilfully traced in connection with the successive names. There are probably very few people who will be able thoroughly to appreciate the author's laborious researches in all sorts of memoirs and transactions of learned societies in order to unearth the material which he has so agreeably condensed. . . . Along with this there is much new matter which, while of great interest to mathematicians, and more especially to men brought up at Cambridge, will be found to throw a good deal of new and important light on the history of education in general.—*The Glasgow Herald.*

Exceedingly interesting to all who care for mathematics. . . . After giving an account of the chief Cambridge Mathematicians and their works in chronological order, Mr. Rouse Ball goes on to deal with the history of tuition and examinations in the University . . . and recounts the steps by which the word "tripos" changed its meaning "from a thing of wood to a man, from a man to a speech, from a speech to two sets of verses, from verses to a sheet of coarse foolscap paper, from a paper to a list of names, and from a list of names to a system of examination."—Never did word undergo so many alterations.—*The Literary World.*

In giving an account of the development of the study of mathematics in the University of Cambridge, and the means by which mathematical proficiency was tested in successive generations, Mr. Ball has taken the novel plan of devoting the first half of his book to . . . the more eminent Cambridge mathematicians, and of reserving to the second part an account of how at various times the subject was taught, and how the result of its study was tested. . . . Very interesting information is given about the work of the students during the different periods, with specimens of problem-papers as far back as 1802. The book is very enjoyable, and gives a capital and accurate digest of many excellent authorities which are not within the reach of the ordinary reader.—*The Scots Observer.*

AN ESSAY ON

THE GENESIS, CONTENTS, AND HISTORY OF

NEWTON'S "PRINCIPIA"

By W. W. ROUSE BALL.

[*Pp.* x.+175. *Price* 6s. *net.*]

MACMILLAN AND CO. LTD., LONDON AND NEW YORK.

THIS work contains an account of the successive discoveries of Newton on gravitation, the methods he used, and the history of his researches.

It commences with a review of the extant authorities dealing with the subject. In the next two chapters the investigations made in 1666 and 1679 are discussed, some of the documents dealing therewith being here printed for the first time. The fourth chapter is devoted to the investigations made in 1684: these are illustrated by Newton's professorial lectures (of which the original manuscript is extant) of that autumn, and are summed up in the almost unknown memoir of February, 1685, which is here reproduced from Newton's holograph copy. In the two following chapters the details of the preparation from 1685 to 1687 of the *Principia* are described, and an analysis of the work is given. The seventh chapter comprises an account of the researches of Newton on gravitation subsequent to the publication of the first edition of the *Principia*, and a sketch of the history of that work.

In the last chapter, the extant letters of 1678-1679 between Hooke and Newton, and of those of 1686-1687 between Halley and Newton, are reprinted, and there are also notes on the extant correspondence concerning the production of the second and third editions of the *Principia*.

For the essay which we have before us, Mr. Ball should receive the thanks of all those to whom the name of Newton recalls the memory of a great man. The *Principia*, besides being a lasting monument of Newton's life, is also to-day the classic of our mathematical writings, and will be so for some time to come. . . . The value of the present work is also enhanced by the fact that, besides containing a few as yet unpublished letters, there are collected in its pages quotations from all documents, thus forming a complete summary of everything that is known on the subject. . . . The author is so well known a writer on anything connected with the history of mathematics, that we need make no mention of the thoroughness of the essay, while it would be superfluous for us to add that from beginning to end it is pleasantly written and delightful to read. Those well acquainted with the *Principia* will find much that will interest them, while those not so fully enlightened will learn much by reading through the account of the origin and history of Newton's greatest work.—*Nature.*

An Essay on Newton's Principia will suggest to many something solely mathematical, and therefore wholly uninteresting. No inference could be more erroneous. The book certainly deals largely in scientific technicalities which will interest experts only; but it also contains much historical information which might attract many who, from laziness or inability, would be very willing to take all its mathematics for granted. Mr. Ball carefully examines the evidence bearing on the development of Newton's great discovery, and supplies the reader with abundant quotations from contemporary authorities. Not the least interesting portion of the book is the appendix, or rather appendices, containing copies of the original documents (mostly letters) to which Mr. Ball refers in his historical criticisms. Several of these bear upon the irritating and unfounded claims of Hooke.—*The Athenæum.*

La savante monographie de M. Ball est rédigée avec beaucoup de soin, et à plusieurs égards elle peut servir de modèle pour des écrits de la même nature. —*Bibliotheca Mathematica.*

Newton's *Principia* has world-wide fame as a classic of mathematical science. But those who know thoroughly the contents and the history of the book are a select company. It was at one time the purpose of Mr. Ball to prepare a new critical edition of the work, accompanied by a prefatory history and notes, and by an analytical commentary. Mathematicians will regret to hear that there is no prospect in the immediate future of seeing this important book carried to completion by so competent a hand. They will at the same time welcome Mr. Ball's *Essay on the Principia* for the elucidations which it gives of the process by which Newton's great work originated and took form, and also as an earnest of the completed plan.—*The Scotsman.*

In this essay Mr. Ball presents us with an account highly interesting to mathematicians and natural philosophers of the origin and history of that remarkable product of a great genius *Philosophiae Naturalis Principia Mathematica*, 'The Mathematical Principles of Natural Philosophy,' better known by the short term *Principia*. . . . Mr. Ball's essay is one of extreme interest to students of physical science, and it is sure to be widely read and greatly appreciated.—*The Glasgow Herald.*

To his well-known and scholarly treatises on the *History of Mathematics* Mr. W. W. Rouse Ball has added *An Essay on Newton's Principia.* Newton's *Principia*, as Mr. Ball justly observes, is the classic of English mathematical writings; and this sound, luminous, and laborious essay ought to be the classical account of the *Principia*. The essay is the outcome of a critical edition of Newton's great work, which Mr. Ball tells us that he once contemplated. It is much to be hoped that he will carry out his intention, for no English mathematician is likely to do the work better or in a more reverent spirit. . . . It is unnecessary to say that Mr. Ball has a complete knowledge of his subject. He writes with an ease and clearness that are rare.—*The Scottish Leader.*

Le volume de M. Rouse Ball renferme tout ce que l'on peut désirer savoir sur l'histoire des *Principes*; c'est d'ailleurs l'œuvre d'un esprit clair, judicieux, et méthodique.—*Bulletin des Sciences Mathématiques.*

Mr. Ball has put into small space a very great deal of interesting matter, and his book ought to meet with a wide circulation among lovers of Newton and the *Principia.*—*The Academy.*

Admirers of Mr. W. W. Rouse Ball's *Short Account of the History of Mathematics* will be glad to receive a detailed study of the history of the *Principia* from the same hand. This book, like its predecessors, gives a very lucid account of its subject. We find in it an account of Newton's investigations in his earlier years, which are to some extent collected in the tract *de Motu* (the germ of the *Principia*) the text of which Mr. Rouse Ball gives us in full. In a later chapter there is a full analysis of the *Principia* itself, and after that an account of the preparation of the second and third editions. Probably the part of the book which will be found most interesting by the general reader is the account of the correspondence of Newton with Hooke, and with Halley, about the contents or the publication of the *Principia.* This correspondence is given in full, so far as it is recoverable. Hooke does not appear to advantage in it. He accuses Newton of stealing his ideas. His vain and envious disposition made his own merits appear great in his eyes, and be-dwarfed the work of others, so that he seems to have believed that Newton's great performance was a mere expanding and editing of the ideas of Mr. Hooke—ideas which were meritorious, but after all mere guesses at truth. This, at all events, is the most charitable view we can take of his conduct. Halley, on the contrary, appears as a man to whom we ought to feel most grateful. It almost seems as though Newton's physical insight and extraordinary mathematical powers might have been largely wasted, as was Pascal's rare genius, if it had not been for Halley's single-hearted and self-forgetful efforts to get from his friend's genius all he could for the enlightenment of men. It was probably at his suggestion that the writing of the *Principia* was undertaken. When the work was presented to the Royal Society, they undertook its publication, but, being without the necessary funds, the expense fell upon Halley. When Newton, stung by Hooke's accusations, wished to withdraw a part of the work, Halley's tact was required to avert the catastrophe. All the drudgery, worry, and expense fell to his share, and was accepted with the most generous good nature. It will be seen that both the technical student and the general reader may find much to interest him in Mr. Rouse Ball's book.—*The Manchester Guardian.*

Une histoire trés bien faite de la genèse du livre immortel de Newton. . . . Le livre de M. Ball est une monographie précieuse sur un point important de l'histoire des mathématiques. Il contribuera à accroître, si c'est possible, la gloire de Newton, en révélant à beaucoup de lecteurs, avec quelle merveilleuse rapidité l'illustre géomètre anglais a élevé à la science ce monument immortel, les *Principia.*—*Mathesis.*

THE HISTORY OF

TRINITY COLLEGE, CAMBRIDGE

By W. W. ROUSE BALL.

[*Pp.* xiv + 183. *Price* 2*s.* 6*d. net.*]

MACMILLAN AND CO. Ltd., LONDON AND NEW YORK.

This booklet gives a popular account of the History of Trinity College, Cambridge. It was written mainly for the use of the author's pupils, and contains such information and gossip about the College and life there in past times as he believed would be interesting to most undergraduates and members of the House.

This . . . little volume seems to us to do more for its subject than many of the more formal volumes . . . treating of the separate colleges of the English universities. . . . In nine short, extremely readable, and truly informing chapters it gives the reader a very vivid account at once of the origin and development of the University of Cambridge, of the rise and gradual supremacy of the colleges, . . . and the subsequent fortunes of the premier college of Cambridge. The subject is treated . . . under four great periods—namely, that during the Middle Ages, that during the Renaissance, that under the Elizabethan statutes, and that during the last half-century. No one who begins Mr. Ball's book will lay it down till he has read it from beginning to end.—*The Glasgow Herald.*

It is a sign of the times, and a very satisfactory one, when . . . a tutor . . . takes the trouble to make the history of his college known to his pupils. Considering the lack of good books about the Universities, we may thank Mr. Ball that he has been good enough to print for a larger circle. Though he modestly calls his book only "Notes," yet it is eminently readable, and there is plenty of information, as well as abundance of good stories, in its pages.—*The Oxford Magazine.*

Mr. Ball has put not only the pupils for whom he compiled these notes, but the large world of Trinity men, under a great obligation by this compendious but lucid and interesting history of the society to whose service he is devoted. The value of his contribution to our knowledge is increased by the extreme simplicity with which he tells his story, and the very suggestive details which, without much comment, he has selected, with admirable discernment, out of the wealth of materials at his disposal. His initial account of the development of the University is brief but extremely clear, presenting us with facts rather than theories, but establishing, with much distinctness, the essential difference between the hostels, out of which the more modern colleges grew, and that monastic life which poorer students were often tempted to join.—*The Guardian.*

An interesting and valuable book. . . . It is described by its author as "little more than an orderly transcript" of what, as a Fellow and Tutor of the College, he has been accustomed to tell his pupils. But while it does not pretend either to the form or to the exhaustiveness of a set history, it is scholarly enough to rank as an authority, and far more interesting and readable than most academic histories are. It gives an instructive sketch of the development of the University and of the particular history of Trinity, noting its rise and policy in the earlier centuries of its existence, until, under the misrule of Bentley, it came into a state of disorder which nearly resulted in its dissolution. The subsequent rise of the College and its position in what Mr. Ball calls the Victorian renaissance, are drawn in lines no less suggestive; and the book, as a whole, cannot fail to be welcome to every one who is closely interested in the progress of the College.—*The Scotsman.*

Mr. Ball has succeeded very well in giving in this little volume just what an intelligent undergraduate ought and probably often does desire to know about the buildings and the history of his College. . . . The debt of the "royal and religious foundation" to Henry VIII. is explained with fulness, and there is much interesting matter as to the manner of life and the expenses of students in the sixteenth century.—*The Manchester Guardian.*

TRINITY COLLEGE, CAMBRIDGE

By W. W. ROUSE BALL.

[*Pp.* xiv + 107. *Price* 2*s. net.*]

J. M. DENT AND CO., LONDON.

This booklet contains a somewhat more popular sketch of the history, external and internal, of the College, with notes on some of its famous past members. It is intended to supply such information as all those in any way interested in the matter would desire to have. It is illustrated by Mr. Edmund H. New.

Mr. Rouse Ball is a sound antiquary and an accomplished writer. He is also in close touch with the actual life of the great home of learning through which he guides us in his skilful pages. His topographical descriptions are clear and concise, his historical sketches, both of the external and the internal life of the College are interesting and lively, while the occasional light which he throws upon the habits and ways of collegians, ancient as well as modern, is extremely valuable.—*The Guardian.*

The skill with which the . . . subjects have been treated will be recognised and appreciated by all readers. Not less adequate are the author's description of the College buildings, his account of Trinity life, customs and traditions, and his references to the many eminent men who have added lustre to the great College in successive generations.—*The World.*

A charming book . . . which tells just what every Trinity man should wish to know about his College, its buildings and its famous sons.—*The Oxford Magazine.*

In his account of the College, Mr. Rouse Ball is equally at home in dealing with the history, the architecture, the collegiate life, and the personal associations which gather so closely around the College. His anecdotes and tales are chosen with judgment, and told with a vivacity and humour which add materially to the delightfulness of the book.—*The Bookseller.*

This book is pleasant, it is anecdotical, it is practical, furnishing just the details that one wants, with the relief of the agreeable and entertaining.—*The Spectator.*