deduced from one in the paper of Professor Boole on the Theory of Development:

$$f_{n+1}(x)=\frac{x}{n+1}f'_n(x)-\frac{n}{n+1}f_n(x).$$

The method of the present paper is of course of far more general application; but I have said enough in it to explain the principle on which such expansions must be conducted.

## IV. "On the Summation of Series." By W. H. L. Russell, Esq., A.B. Communicated by Professor Stokes, Sec. R.S. Received May 13, 1865.

In a Memoir published in the Philosophical Transactions for the year 1855, I applied the Theory of Definite Integrals to the summation of many intricate series. I have thought my researches on this subject might well be terminated by the following paper, in which I have pointed out methods for the summation of series of a far more complicated nature.

I commence with some remarks intended to give clear conceptions of the general method of calculation.

In any series,

$$u_0+\alpha u_1+\alpha^2u_2+\alpha^3u_3+\&c. +\alpha^x u_x+\&c.$$

Where $\alpha$ is less than unity, it is evident that we can sum the series by a definite integral when $u_x=\int du\, U_1 U^x$, $U_1$ and $U$ being functions of $u$, and the integral being taken between certain assigned limits. For it is manifest that the quantity under the integral sign then becomes a geometrical progression.

Again, for a similar reason we can express by a definite integral the sum of the series

$$u_0v_0w_0\ldots+\alpha u_1v_1w_1\ldots+\alpha^2u_2v_2w_2\ldots+\&c.$$
$$+\alpha^x u_xv_xw_x\ldots+\&c.,$$

where

$$u_x=\int du\, U_1U^x,\quad v_x=\int dv V_1V^x,$$

$$w_x=\int dw W_1W^x,\ \&c.$$

Lastly, we can sum the series

$$u_0v_0w_0\ldots+\alpha\, u_1v_1w_1\ldots+\alpha^2u_2v_2w_2\ldots+\&c.$$
$$+\alpha^x u_xv_xw_x\ldots+\&c.$$

by a definite integral when

$$u_x=\int du\,U_1U^x+\int du'\,U'_1U'^x+\int du''\,U''_1U''^x+\,..$$

$$v_x=\int dv\,V_1V^x+\int dv'\,V'_1V'^x+\int dv''\,V''_1V''^x+\,..$$

$$w_x=\int dw\,W_1W^x+\int dw'\,W'_1W'^x+\int dw''\,W''_1W''^x+\,..$$

&c. = &c.,

the number of each set of quantities $u_1$, $u'$, &c., $v_1$, $v'$, &c., $w_1$, $w'$, &c. being of course finite.

I shall now consider the series

$$\phi(0)^{\psi(0)}+\alpha\phi(1)^{\psi(1)}+\alpha^2\phi(2)^{\psi(2)}+\text{ \&c.}$$
$$+\alpha^x\phi(x)^{\psi(x)}+\text{ \&c.},$$

where $\phi(x)$ and $\psi(x)$ are rational functions of $(x)$.

Let

$$\phi(x)=\frac{(\alpha+\beta x)(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 x)\ldots}{(a+bx)(a_1+b_1x)(a_2+b_2x)\ldots},$$

$$\psi(x)=\frac{1}{m+nx}+\frac{1}{m_1+n_1x}+\frac{1}{m_2+n_2x}+\text{ \&c.}$$

Hence, by what has been said, the problem is reduced to finding a definite integral of the form $\int du\,U_1U_x$ equivalent to $\left\{\frac{\alpha+\beta x}{a+bx}\right\}^{\frac{1}{m+nx}}$.

Now

$$\left\{\frac{\alpha+\beta x}{a+bx}\right\}^{\frac{1}{m+nx}}=\frac{1}{\Gamma\left\{\frac{1}{m+nx}\right\}}\int_0^1 du\,.\,u^{\frac{a+bx}{\alpha+\beta x}-1}\left(\log_\epsilon\frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$=\frac{1}{\pi}\sin\frac{\pi}{m+nx}\Gamma\left\{\frac{m+nx-1}{m+nx}\right\}\int_0^1 du\,.\,u^{\frac{a+bx}{\alpha+\beta x}-1}\left(\log_e\frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$\sin\frac{\pi}{m+nx}=\frac{1}{\sqrt{2}}\cos\left\{\frac{\pi}{4}-\frac{\pi}{m+nx}\right\}-\frac{1}{\sqrt{2}}\sin\left\{\frac{\pi}{4}-\frac{\pi}{m+nx}\right\}$$

$$=\frac{\sqrt{2}}{\pi}\int_0^1 ds\log_\epsilon{}^{-\frac{1}{2}}\frac{1}{s}\,.\,s^{\frac{1}{m+nx}-1}\int_0^\infty dz\cos(m+nx)z^2\cos 2\sqrt{\pi}\,.\,z$$

$$-\frac{\sqrt{2}}{\pi}\int_0^1 ds\log_\epsilon{}^{-\frac{1}{2}}\frac{1}{s}\,.\,s^{\frac{1}{m+nx}-1}\int_0^\infty dz\sin(m+nx)z^2\cos 2\sqrt{\pi}\,.\,z$$

$$\Gamma\left\{\frac{m+nx-1}{m+nx}\right\}=\int_0^\infty \epsilon^{-v}\,dv\,\epsilon^{-\frac{1}{m+nx}}$$

$$\int_0^1 du\,.\,u^{\frac{a+bx}{\alpha+\beta x}-1}\left(\log_\epsilon\frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$=\int_{\frac{1}{\epsilon}}^1 du\,.\,\epsilon^{\log_\epsilon u\left(\frac{a+bx}{\alpha+\beta x}-1\right)}\,\epsilon^{\log_\epsilon\log_\epsilon\frac{1}{u}\left(\frac{1}{m+nx}-1\right)}$$

(where $\log_\epsilon\log_\epsilon\frac{1}{u}$ is negative),

$$+\int_0^{\frac{1}{\epsilon}} du\,\epsilon^{\log_\epsilon u\left(\frac{a+bx}{\alpha+\beta x}-1\right)}\,\epsilon^{\log_\epsilon\log_\epsilon\frac{1}{u}\left(\frac{1}{m+nx}-1\right)}$$

(where $\log_\epsilon\log_\epsilon\frac{1}{u}$ is positive).

By means of these transformations the series is reduced to forms considered in my previous investigations; for the general term of the transformed series included under the signs of definite integration is of the form

$$\mathrm{P}\alpha^x\,\epsilon^{\frac{\phi}{m+nx}}\{\cos h(m+nx)-\sin k(m+nx)\},$$

a form which I have discussed in the memoir in the Philosophical Transactions mentioned at the beginning of this paper.

Next let us investigate the series

$$\sqrt[n]{\psi(0)+\sqrt[m]{\chi(0)}}+\alpha\sqrt[n]{\psi(1)+\sqrt[m]{\chi(1)}}+\ldots+\alpha^x\sqrt[n]{\psi(x)+\sqrt{\chi(x)}}+\ldots,$$

where $\psi(x)$ and $\chi(x)$ are identical functions of $(x)$.

We transform as follows:—

$$\sqrt[n]{\psi(x)+\sqrt[m]{\chi(x)}}=\frac{1}{\Gamma\left(\frac{1}{n}\right)}\int_0^1 du\,u^{\frac{1}{\psi(x)+\sqrt[m]{\chi(x)}}-1}\left(\log_\epsilon\frac{1}{u}\right)^{n-1}$$

$$\epsilon^{\frac{\log_\epsilon u}{\psi(x)+\sqrt[m]{\chi(x)}}}=\frac{\sqrt{\psi(x)+\sqrt[m]{\chi(x)}}}{2\sqrt{\pi}\log_\epsilon^{\frac{1}{2}}\frac{1}{u}}\int_{-\infty}^\infty d\rho\,\epsilon^{-\frac{\psi(x)+\sqrt[m]{\chi(x)}}{4\log_\epsilon\frac{1}{u}}\rho^2}\cos\rho,$$

remembering that $\log_\epsilon u$ is negative,

$$\sqrt{\psi(x)+\sqrt[m]{\chi(x)}}=\frac{\psi(x)+\sqrt[m]{\chi(x)}}{\sqrt{\pi}}\int_{-\infty}^\infty \epsilon^{-v^2(\psi(x)+\sqrt[m]{\chi(x)})}dv,$$

$$\epsilon^{-\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\sqrt[m]{\chi(x)}}=\frac{1}{\pi}\int_0^\pi\frac{(1-\sqrt[m]{\chi(x)}^2)\mathrm{F}(\theta)d\theta}{1-2\sqrt[m]{\chi(x)}\cos\theta+\sqrt[m]{\chi(x)}^2},$$

where

$$F(\theta)=\epsilon^{-\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\cos\theta}\cos\left\{\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\sin\theta\right\},$$

and $\sqrt[m]{\chi(x)}$ is supposed less than unity.

Hence also

$$\epsilon^{-\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\sqrt[m]{\chi(x)}}$$

$$=\frac{1}{\pi}\int_0^\pi\frac{(1-\sqrt[m]{\chi(x)^2})F_1(\chi(x),\epsilon^{i\theta})F_1(\chi(x),\epsilon^{-i\theta})F(\theta)d\theta}{1-2\chi(x)\cos m\theta+\chi(x)^2},$$

where we render the denominator rational by multiplication, and suppose

$$F_1(\chi(x),\epsilon^{i\theta})=\chi(x)^{\frac{m-1}{m}}+\epsilon^{i\theta}\chi(x)^{\frac{m-2}{m}}+\epsilon^{2i\theta}\chi(x)^{\frac{m-3}{m}}+\ldots,$$

$$F_1(\chi(x),\epsilon^{-i\theta})=\chi(x)^{\frac{m-1}{m}}+\epsilon^{-i\theta}\chi(x)^{\frac{m-2}{m}}+\epsilon^{-2i\theta}\chi(x)^{\frac{m-3}{m}}+\ldots,$$

$$\frac{1}{1-2\chi(x)\cos m\theta+\chi(x)^2}=\frac{\chi_1(x)^2}{(\chi_1(x)-\cos m\theta)^2+\sin^2 m\theta},$$

where

$$\chi_1(x)=\frac{1}{\chi(x)}.$$

$\chi(x)$ is less than unity, hence $\chi_1(x)$ is greater than unity, and therefore $\chi_1(x)-\cos m\theta$ is always positive; hence

$$\frac{1}{1-2\chi(x)\cos m\theta+\chi(x)^2}=\frac{\chi(x)^2}{\sin m\theta}\int_0^\infty dz\epsilon^{-(\chi_1(x)-\cos m\theta)z}\sin(z\sin m\theta).$$

The general term of the series included under the signs of definite integration is now of the form

$$P\psi(x)\epsilon^{\alpha\psi(x)}\epsilon^{-\frac{\beta}{\chi(x)}}\chi(x)^{\frac{r}{m}}\alpha^x,$$

belonging to a class which I have considered in my former memoir.

Let us now consider the series

$$\sqrt[n]{\phi(0)}^{\sqrt[m]{\psi(0)}}+\alpha.\sqrt[n]{\phi(1)}^{\sqrt[m]{\psi(1)}}+\alpha^2.\sqrt[n]{\phi(2)}^{\sqrt[m]{\psi(2)}}+\ldots$$

$$+\alpha x.\sqrt[n]{\phi(x)}^{\sqrt[m]{\psi(x)}}+\&c.,$$

$\phi(x)$ and $\psi(x)$ being rational functions of $(x)$.

$$\sqrt[n]{\phi(x)}^{\sqrt[m]{\psi(x)}}=\frac{1}{\Gamma\sqrt[m]{\psi(x)}}\int_0^1 u^{\frac{1}{\sqrt[n]{\phi(x)}}-1}\left(\log_\epsilon\frac{1}{u}\right)^{\sqrt[m]{\psi(x)}-1}du,$$

where $\psi(x)$ must be supposed less than unity, in order that the following transformation may hold :—

$$\frac{1}{\Gamma\sqrt[m]{\psi(x)}}=\frac{\sin\pi\sqrt[m]{\psi(x)}}{\pi}\Gamma(1-\sqrt[m]{\psi x})$$

$$\sin\pi\sqrt[m]{\psi(x)}=\frac{1}{2\pi}\int_0^\pi\frac{d\theta(1-\sqrt[m]{\psi(x)^2})d\theta(\sin\pi\epsilon^{i\theta}+\sin\pi\epsilon^{-i\theta})}{1-2\cos\theta\sqrt[m]{\psi(x)}+\sqrt[m]{\psi(x)^2}}$$

$$=\frac{1}{2\pi}\int_0^\pi\frac{d\theta(1-\sqrt[m]{\psi(x)^2})F(\psi x\,\epsilon^{i\theta})F(\psi(x),\epsilon^{-i\theta})(\sin\pi\epsilon^{i\theta}+\sin\pi\epsilon^{-i\theta})}{1-2\cos m\theta,\ \psi(x)+\psi(x)^2},$$

where

$$F(\psi(x),\epsilon^{i\theta})\ =\psi(x)^{\frac{m-1}{m}}+\epsilon^{i\theta}\psi(x)^{\frac{m-2}{m}}+\ldots.,$$

$$F(\psi(x),\epsilon^{-i\theta})=\psi(x)^{\frac{m-1}{m}}+\epsilon^{-i\theta}\psi(x)^{\frac{m-2}{m}}+\ldots.;$$

also

$$\Gamma(1-\sqrt[m]{\psi(x)}=\int_0^\infty\epsilon^{-v}v^{-\sqrt[m]{(\psi x)}}dv.$$

The remainder of the process will be evident from the two former examples.

## V. "On a Theorem concerning Discriminants." By J. J. Sylvester, F.R.S. Received May 27, 1865.

Let $F(a,b,c,d)=a^2d^2+4a^3c+4d^3b-3a^2b^2-6abcd$, and let $a, b, c, d$ be four quantities all greater than zero, which make this function vanish.

(1) The cubic equation in $x$, $F(a,x,c,d)$ will have two positive roots $(b,b_1)$; so $F(a,b_1,x,d)$ will have two such roots $(c,c_1)$, $F(a,x,c_1,d)$ two such $(b_1,b_2)$, $F(a,b_2,x,d)$ two such $(c_1,c_2)$, and so on *ad infinitum*; we may thus generate the infinite series $b_1\,c_1\,b_2\,c_2$ . . . . . . . .

Similarly, beginning with the equation $F(a,b,x,d)$, and proceeding as above, we shall obtain a similar series, $c', b', c'', b''$ . . .; and combining the two together, and with the initial quantities $b, c$, we obtain a series proceeding to infinity in both directions . . . . . $b''\,c''\,b'\,c'\,b\,c\,b_1\,c_1\,b_2\,c_2$ . . . . .

(2) The four quantities

$$\frac{\delta F}{\delta a},\ \frac{\delta F}{\delta b},\ \frac{\delta F}{\delta c},\ \frac{\delta F}{\delta d},$$

where F represents $F(a,b,c,d)$, will present one or the other of the three following successions of sign,

| | | | |
|---|---|---|---|
| + | − | + | − |
| − | + | − | + |
| 0 | 0 | 0 | 0 |

(3) When the last is the case, *i.e.* when the differential derivatives all