Groupoids and Cayley Graphs

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Abstract

Groupoids are mathematical structures that have proved to be useful in many areas, ranging from category theory and differential topology and geometry to functional analysis and operator algebras. In particular, one can associate a C^*-algebra to any locally compact groupoid. Cayley graphs of groups are used in the construction of expander graphs, which are of great interest in computer science. This paper establishes a bridge between groupoid theory and Cayley graph theory. The end goal is to use both theories as tools for problems in both areas.

1 Introduction

We start with some definitions. The graphs we will be using are directed graphs, where an edge from vertex v to vertex w is represented by the ordered pair (v, w). We allow loops and are not prejudiced against multiple edges either. With this convention, we define our Cayley graphs.

Definition 1.1. *Let* Γ *be a group and let* $A \subseteq \Gamma$*, not necessarily closed under the group operation. The* Cayley graph of Γ under alphabet A*, denoted* $Cay(\Gamma, A)$*, is a graph with vertex set* $V = G$ *and directed edge set* $E = \{(\gamma, a\gamma) : \gamma \in \Gamma, a \in A\}$.

We now define groupoids.

Definition 1.2. *A* groupoid *is a set* $\mathcal{G}$ *of "arrows" or "morphisms" connecting elements in an object set* $\mathcal{G}^{(0)}$*, with the following structure:*

1. *There are maps* **s**,**t**: $\mathcal{G} \to \mathcal{G}^{(0)}$ *which specify the* **s***ource and* **t***arget objects of each morphism.*
2. *There is a partially defined multiplication operation on the morphisms. The multiplication operates only on the set of composable pairs*

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} : \boldsymbol{s}(g) = \boldsymbol{t}(h)\}.$$

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This operation is required to be associative.

3. *There is an identity map $\epsilon : X \to \mathcal{G}$ that associates a loop morphism to every element of the object set.*
   That is, $\boldsymbol{s}(\epsilon(x)) = \boldsymbol{t}(\epsilon(x)) = x$ for all $x \in X$.

4. *Every morphism $g \in \mathcal{G}$ has an inverse morphism $g^{-1} \in \mathcal{G}$ such that $\boldsymbol{s}(g) = \boldsymbol{t}(g^{-1})$ and $\boldsymbol{t}(g) = \boldsymbol{s}(g^{-1})$.*

While there are many ways to form groupoids, one way is by the transformation groupoid construction.

Definition 1.3. *Let Γ be a group acting on a set X via a left action $\cdot$. Then the* transformation groupoid $\mathcal{T}(\Gamma, X)$ *(or $\mathcal{T}(\Gamma, X, \cdot)$ when the action is ambiguous) has object set X and morphism set $\Gamma \times X$, where a morphism (γ, x) has source $\boldsymbol{s}(\gamma, x) = x$ and target $\boldsymbol{t}(\gamma, x) = \gamma \cdot x$. The multiplication acts on the set of composable pairs $\mathcal{G}^{(2)} = \{((\gamma_1, \gamma_2 \cdot x), (\gamma_2, x))\}$ via the product $(\gamma_1, \gamma_2 \cdot x)(\gamma_2, x) = (\gamma_1\gamma_2, x)$. We denote by $\mathcal{T}(\Gamma)$ the specific transformation groupoid formed by Γ acting on itself under the action of left multiplication.*

With the idea that a groupoid is composed of morphisms that act as connected edges, it seems natural to produce a graph from a groupoid.

Definition 1.4. *Let $\mathcal{A}$ be a subset of $\mathcal{G}$. We call $\mathcal{A}$ a* groupoid alphabet. *We define the* underlying graph of $\mathcal{G}$ with groupoid alphabet $\mathcal{A}$, *denoted $U(\mathcal{G}, \mathcal{A})$, to be the graph with vertex set $V = \mathcal{G}^{(0)}$, the object set of $\mathcal{G}$, and directed edge set $E = \{(\boldsymbol{s}(\alpha), \boldsymbol{t}(\alpha)) : \alpha \in \mathcal{A}\}$.*

2 Results

The Cayley graph is a way to form a graph from a group. The transformation groupoid is a way to form a groupoid from a group. The underlying graph is a way to form a graph from a groupoid. The following theorem ties these ideas together.

Theorem 2.1. *Let Γ be a group and let $A \subseteq \Gamma$ be an alphabet. Let $\mathcal{A}$ be the groupoid alphabet $\mathcal{A} = \{(a, \gamma) \in \mathcal{T}(\Gamma) : a \in A\}$. Then $U(\mathcal{T}(\Gamma), \mathcal{A}) \cong Cay(\Gamma, A)$.*

Proof. Since $\mathcal{T}(\Gamma)^{(0)} = \Gamma$ and $Cay(\Gamma, A)$ has vertex set Γ, both graphs have the same vertex set. To show they have the same edge set, we first let (x, ax) be a edge in $Cay(\Gamma, A)$ for some $x \in \Gamma$, $a \in A$. Hence, the edge has source x and target ax. Thus, for the edge to be in $U(\mathcal{T}(\Gamma), \mathcal{A})$, there must be an $\alpha \in A$ such that $\mathbf{s}(\alpha) = x$ and $\mathbf{t}(\alpha) = ax$. But α is an element (γ_1, γ_2) such that $\mathbf{s}(\gamma_1, \gamma_2) = \gamma_2$, and $\mathbf{t}(\gamma_1, \gamma_2) = \gamma_1 \cdot \gamma_2 = \gamma_1\gamma_2$. Hence, if $\gamma_1 = a$ and $\gamma_2 = x$, then the edge $(\mathbf{s}(\alpha), \mathbf{t}(\alpha))$ is the edge (x, ax), so (x, ax) is an edge in $U(\mathcal{T}(\Gamma), \mathcal{A})$. Similarly, if $\alpha \in \mathcal{A}$ is of the form (a, γ), then $\mathbf{s}(\alpha) = \gamma$ and $\mathbf{t}(\alpha) = a\gamma$, so the edge in $U(\mathcal{T}(\Gamma), \mathcal{A})$ is $(\gamma, a\gamma)$, which is an edge in $Cay(\Gamma, A)$. Therefore, the graphs have the same edge set, so they are isomorphic. $\square$

Through this result, we get a correspondence between groupoids and groups that we can expand upon. One natural way to expand upon this correspondence is by building larger groupoids and groups. We have a way of building larger groups through the semi-direct product of two groups. A way to build larger groupoids is through a naturally defined direct product of two groupoids.

Definition 2.2. *Let $\mathcal{G}$ and $\mathcal{H}$ be two groupoids. Then the direct product $\mathcal{G}\times\mathcal{H}$ is a groupoid with object set $(\mathcal{G}\times\mathcal{H})^{(0)} = \mathcal{G}^{(0)}\times\mathcal{H}^{(0)}$, source map $\boldsymbol{s}(g,h) = (\boldsymbol{s}(g),\boldsymbol{s}(h))$, target map $\boldsymbol{t}(g,h) = (\boldsymbol{t}(g),\boldsymbol{t}(h))$, and composable pair set*

$$(\mathcal{G}\times\mathcal{H})^{(2)} = \{((g_1,h_1),(g_2,h_2)) : (g_1,g_2)\in\mathcal{G}^{(2)},\ (h_1,h_2)\in\mathcal{H}^{(2)}\}$$

with the product $(g_1,h_1)(g_2,h_2) = (g_1g_2, h_1h_2)$.

We also recall the definition of a semi-direct product group [4].

Definition 2.3. *Let Γ and Λ be groups and let ϕ be a homomorphism from Λ into $Aut(\Gamma)$. Let $\cdot$ denote the left action of Λ on Γ determined by ϕ. Then $\Gamma\rtimes_\phi\Lambda$ is the set $\Gamma\times\Lambda$ with the following multiplication:*

$$(\gamma_1,\lambda_1)(\gamma_2,\lambda_2) = (\gamma_1(\lambda_1\cdot\gamma_2), \lambda_1\lambda_2).$$

$\Gamma\rtimes_\phi\Lambda$ is a group under this multiplication, and is called the semi-direct product of Γ and Λ with respect to ϕ.

With a properly defined groupoid alphabet, we get the correspondence we want between the semi-direct product group and the direct product groupoid.

Theorem 2.4. *Let $\Gamma\rtimes_\phi\Lambda$ be the semi-direct product group of groups Γ and Λ with respect to the homomorphism ϕ. Let $A = \{(a_\Gamma, a_\Lambda)\}\subseteq\Gamma\times\Lambda$ be an alphabet for $\Gamma\rtimes_\phi\Lambda$. We define the groupoid alphabet $\mathcal{A}_\phi\subseteq\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda)$ by*

$$\mathcal{A}_\phi = \{((a_\Gamma(a_\Lambda\cdot\gamma)\gamma^{-1},\gamma),(a_\Lambda,\lambda)) : \gamma\in\Gamma, \lambda\in\Lambda, (a_\Gamma,a_\Lambda)\in A\}.$$

Let $U(\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda),\mathcal{A}_\phi)$ be the underlying graph of $\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda)$ with groupoid alphabet $\mathcal{A}_\phi$. Then $U(\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda),\mathcal{A}_\phi)\cong Cay(\Gamma\rtimes_\phi\Lambda, A)$.

Proof. $Cay(\Gamma\rtimes_\phi\Lambda, A)$ has vertex set $\Gamma\times\Lambda$, and $U(\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda),\mathcal{A}_\phi)$ has vertex set $(\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda))^{(0)} = \mathcal{T}(\Gamma)^{(0)}\times\mathcal{T}(\Lambda)^{(0)} = \Gamma\times\Lambda$. Thus, the graphs have the same vertex set. To show they have the same edge set, first we let (x, ax) be an edge in $Cay(\Gamma\rtimes_\phi\Lambda)$. That is, $x = (\gamma,\lambda)$ and $a = (a_\Gamma, a_\Lambda)$, with $ax = (a_\Gamma(a_\Lambda\cdot\gamma), a_\Lambda\lambda)$. But, the element $\alpha = ((a_\Gamma(a_\Lambda\cdot\gamma)\gamma^{-1},\gamma),(a_\Lambda,\lambda))\in\mathcal{A}_\phi$ has source $\mathbf{s}(\alpha) = (\mathbf{s}(a_\Gamma(a_\Lambda\cdot\gamma)\gamma^{-1}), \mathbf{s}(a_\Lambda,\lambda)) = (\gamma,\lambda) = x$ and target $\mathbf{t}(\alpha) = (\mathbf{t}(a_\Gamma(a_\Lambda\cdot\gamma)\gamma^{-1}), \mathbf{t}(a_\Lambda,\lambda)) = (a_\Gamma(a_\Lambda\cdot\gamma)\gamma^{-1}\gamma, a_\Lambda\lambda) = (a_\Gamma(a_\Lambda\cdot\gamma), a_\Lambda\lambda) = ax$. Hence, the edge (x, ax) is in $U(\mathcal{T}(\Gamma)\times\mathcal{T}(\Lambda),\mathcal{A}_\phi)$.
Similarly, if $\alpha\in\mathcal{A}_\phi$ is of the form $((a_\Gamma(a_\Lambda\cdot\gamma)\gamma^{-1},\gamma),(a_\Lambda,\lambda))$, then $\mathbf{s}(\alpha) = (\gamma,\lambda)$ and $\mathbf{t}(\alpha) = (a_\Gamma(a_\Lambda\cdot\gamma), a_\Lambda\lambda)$, so the edge is in $Cay(\Gamma\rtimes_\phi\Lambda)$. Thus, the graphs have the same edge sets, and are therefore isomorphic. □

3 Applications

3.1 Expander Graphs

Now that we have established that we can create a groupoid alphabet that fits the semi-direct product of any two groups, we can relate groupoids to expander graph applications of Cayley graphs. In *Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders and Extractors* by Reingold, Vadhan, and Wigderson [7], it was shown that a new type of graph product, known as the zig-zag graph product, has the special property that the zig-zag product of two expander graphs is always an expander graph. Later, the paper by Alon, Lubotzky, and Wigderson [1] related this idea to Cayley graphs by showing that under some special circumstances, the zig-zag product of two Cayley graphs of groups is isomorphic to the Cayley graph of the semi-direct product of the groups. This is important because it revealed that for groups with generators known to yield good expander graphs, a proper choice of alphabet on the semi-direct product group would yield a new expander graph. With the results established in this paper, we have essentially the same property with groupoids. If we have groups yielding expander graphs, we can choose the proper group alphabet for the semi-direct product and construct a groupoid containing this information so that the underlying graph with the correct groupoid alphabet is an expander graph.

For example, suppose Γ and Λ are groups with alphabets A_Γ and A_Λ that yield Cayley graphs that are good expanders. Under the alphabet A detailed in [1], we know that $Cay(\Gamma \rtimes \Lambda)$ is an expander graph. Using A, we can construct the groupoid alphabet $\mathcal{A}$ such that $U(\mathcal{T}(\Gamma) \times \mathcal{T}(\Lambda), \mathcal{A})$ is an expander graph. This raises the question of whether it may be possible to use groupoids to produce expander graphs in new ways.

3.2 Operator Algebras

On the other hand, it seems feasible that this connection could be used to attack problems on the groupoid theory side as well. It is known that given a groupoid, together with a suitable topological structure, one can construct a C^*-algebra [8]. As Paterson [6] states, operator algebras are generally quite complex, so deriving an algebra from an object such as a group or groupoid can be extremely useful. Motivated by the "noncommutative geometry" program initiated by Connes [3], some authors have been studying the elements of the groupoid C^*-algebra as quantum observables of one or many-particle systems. Since we now have established a correlation between groupoids and Cayley graphs, it seems that we now also consider possible relationships between Cayley graphs and certain operator algebras, or even some particle systems on the graphs. Through this relationship, we may discover a way to construct an operator algebra directly from a Cayley graph. It will be interesting to see whether this program is related to the types of constructions being done by operator algebraists, as in [5].

4 Summary

In this paper, we have shown that there is a type of groupoid that encapsulates all of the same information as a Cayley graph. With this result, we can use group theory or graph theory to approach problems typically associated with groupoids, and we can use groupoid theory to address problems for which Cayley graphs tend to be used.

References

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