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Other eds

A

COURSE OF

MATHEMATICS.

CONTAINING

THE PRINCIPLES OF

PLANE TRIGONOMETRY,

MENSURATION,

NAVIGATION, AND SURVEYING.

ADAPTED TO THE METHOD OF INSTRUCTION IN THE AMERICAN COLLEGES.

BY JEREMIAH DAY, D. D. LL.D.
PRESIDENT OF YALE COLLEGE.

NEW HAVEN:
PUBLISHED BY DURRIE AND PECK.
NEW YORK—ROBERT B. COLLINS,
254 PEARL STREET.

1853.

A

TREATISE

OF

PLANE TRIGONOMETRY.

TO WHICH IS PREFIXED

A SUMMARY VIEW OF THE NATURE AND USE OF

LOGARITHMS;

BEING

THE SECOND PART

OF

A COURSE OF MATHEMATICS,

ADAPTED TO THE METHOD OF INSTRUCTION IN THE AMERICAN COLLEGES.

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The plan upon which this work was originally commenced, is continued in this second part of the course. As the single object is to provide for *a class in college*, such matter as is not embraced by this design is excluded. The mode of treating the subjects, for the reasons mentioned in the preface to Algebra, is, in a considerable degree, diffuse. It was thought better to err on this extreme, than on the other, especially in the early part of the course.

The section on right angled triangles will probably be considered as needlessly minute. The solutions might, in all cases, be effected by the theorems which are given for oblique angled triangles. But the applications of rectangular trigonometry are so numerous, in navigation, surveying, astronomy, &c., that it was deemed important, to render familiar the various methods of stating the relations of the sides and angles; and especially to bring distinctly into view the principle on which most trigonometrical calculations are founded, the proportion between the parts of the given triangle, and a similar one formed from the sines, tangents, &c., in the tables.

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LOGARITHMS.

TRIGONOMETRY.

LOGARITHMS.

SECTION I.

NATURE OF LOGARITHMS.*

Art. 1. The operations of Multiplication and Division, when they are to be often repeated, become so laborious, that it is an object of importance to substitute, in their stead, more simple methods of calculation, such as Addition and Subtraction. If these can be made to perform, in an expeditious manner, the office of multiplication and division, a great portion of the time and labor which the latter processes require, may be saved.

Now it has been shown, (Algebra, 233, 237,) that *powers* may be multiplied, by adding their *exponents*, and divided, by subtracting their exponents. In the same manner, *roots* may be multiplied and divided, by adding and subtracting their fractional exponents. (Alg. 280, 286.) When these exponents are arranged in tables, and applied to the general purposes of calculation, they are called *Logarithms*.

2. LOGARITHMS, THEN, ARE THE EXPONENTS OF A SERIES OF POWERS AND ROOTS.†

In forming a system of logarithms, some particular number is fixed upon, as the *base*, *radix*, or first power, whose logarithm is always 1. From this, a series of powers is raised, and the exponents of these are arranged in tables for use. To explain this, let the number which is chosen for the first

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* Maskelyne's Preface to Taylor's Logarithms. Introduction to Hutton's Tables. Keil on Logarithms. Maseres Scriptores Logarithmici. Briggs' Logarithms. Dodson's Anti-logarithmic Canon. Euler's Algebra.

† See note A.

power, be represented by a. Then taking a series of powers, both direct and reciprocal, as in Alg. 207;

$$a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ \&c.$$

The logarithm of a^3 is 3, and the logarithm of a^{-1} is -1,
of a^1 is 1, of a^{-2} is -2,
of a^0 is 0, of a^{-3} is -3, &c.
Universally, the logarithm of a^x is x.

3. In the system of logarithms in common use, called *Briggs'* logarithms, the number which is taken for the radix or base is 10. The above series then, by substituting 10 for a, becomes

$$10^4,\ 10^3,\ 10^2,\ 10^1,\ 10^0,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ \&c.$$

Or 10000, 1000, 100, 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c.

Whose logarithms are

4, 3, 2, 1, 0, —1, —2, —3, &c.

4. The fractional exponents of *roots*, and of powers of roots, are converted into *decimals*, before they are inserted in the logarithmic tables. See Alg. 255.

The logarithm of $a^{\frac{1}{3}}$, or $a^{0.3333}$, is 0.3333,

of $a^{\frac{2}{3}}$, or $a^{0.6666}$, is 0.6666,

of $a^{\frac{3}{7}}$, or $a^{0.4285}$, is 0.4285,

of $a^{\frac{11}{3}}$, or $a^{3.6666}$, is 3.6666, &c.

These decimals are carried to a greater or less number of places, according to the degree of accuracy required.

5. In forming a system of logarithms, it is necessary to obtain the logarithm of each of the numbers in the natural series 1, 2, 3, 4, 5, &c.; so that the logarithm of any number may be found in the tables. For this purpose, the *radix* of the system must first be determined upon; and then every other number may be considered as some power or root of this. If the radix is 10, as in the common system, every other number is to be considered as some power of 10.

That a power or root of 10 may be found, which shall be equal to any other number whatever, or, at least, a very near approximation to it, is evident from this, that the *exponent* may be endlessly varied; and if this be increased or diminished, the *power* will be increased or diminished.

If the exponent is a fraction, and the *numerator* be increased, the power will be increased; but if the *denominator* be increased, the power will be diminished.

6. To obtain then the logarithm of any number, according to Briggs' system, we have to find a power or root of 10 which shall be equal to the proposed number. The *exponent* of that power or root is the logarithm required. Thus

$$\left.\begin{array}{r} 7=10^{0.8451} \\ 20=10^{1.3010} \\ 30=10^{1.4771} \\ 400=10^{2.6020} \end{array}\right\} \text{therefore the logarithm} \left\{\begin{array}{l} \text{of } 7 \text{ is } 0.8451 \\ \text{of } 20 \text{ is } 1.3010 \\ \text{of } 30 \text{ is } 1.4771 \\ \text{of } 400 \text{ is } 2.6020, \text{\&c.} \end{array}\right.$$

7. A logarithm generally consists of two parts, an *integer* and a *decimal.* Thus, the logarithm 2.60206, or, as it is sometimes written, 2+.60206, consists of the integer 2, and the decimal .60206. The integral part is called the *characteristic* or *index** of the logarithm; and is frequently omitted, in the common tables, because it can be easily supplied, whenever the logarithm is to be used in calculation.

By art. 3d, the logarithms of

	10000,	1000,	100,	10,	1,	.1,	.01,	.001,	&c.
are	4,	3,	2,	1,	0,	—1,	—2,	—3,	&c.

As the logarithms of 1 and of 10 are 0 and 1, it is evident, that, if any given number be between 1 and 10, its logarithm will be between 0 and 1, that is, it will be greater than 0, but less than 1. It will therefore have 0 for its index, with a decimal annexed.

Thus, the logarithm of 5 is 0.69897.

For the same reason, if the given number be between

$$\left.\begin{array}{r} 10 \text{ and } 100, \\ 100 \text{ and } 1000, \\ 1000 \text{ and } 10000, \end{array}\right\} \begin{array}{c} \text{the log.} \\ \text{will be} \\ \text{between} \end{array} \left\{\begin{array}{ll} 1 \text{ and } 2, \text{ i. e.} & 1+\text{the dec. part.} \\ 2 \text{ and } 3, & 2+\text{the dec. part.} \\ 3 \text{ and } 4, & 3+\text{the dec. part.} \end{array}\right.$$

We have, therefore, when the logarithm of an integer or mixed number is to be found, this general rule:

* The term *index*, as it is used here, may possibly lead to some confusion in the mind of the learner. For the logarithm itself is the index or exponent of a power. The characteristic, therefore, is the index of an index.

8. *The index of the logarithm is always one less, than the number of integral figures, in the natural number whose logarithm is sought:* or, the index shows how far the first figure of the natural number is removed from the place of units.

Thus, the logarithm of 37 is 1.56820.

Here, the number of figures being *two*, the index of the logarithm is 1.

The logarithm of 253 is 2.40312.

Here, the proposed number 253 consists of *three* figures, the first of which is in the second place from the unit figure. The index of the logarithm is therefore 2.

The logarithm of 62.8 is 1.79796.

Here it is evident that the mixed number 62.8 is between 10 and 100. The index of its logarithm must, therefore, be 1.

9. As the logarithm of 1 is 0, the logarithm of a number *less* than 1, that is, of any proper *fraction*, must be *negative.*

Thus, by art. 3d,
The logarithm of $\frac{1}{10}$ or .1 is —1,
of $\frac{1}{100}$ or .01 is —2,
of $\frac{1}{1000}$ or .001 is —3, &c.

10. If the proposed number is *between* $\frac{1}{100}$ and $\frac{1}{1000}$, its logarithm must be between —2 and —3. To obtain the logarithm, therefore, we must either *subtract* a certain fractional part from —2, or *add* a fractional part to —3; that is, we must either annex a *negative decimal* to —2, or a *positive one* to —3.

Thus, the logarithm
of .008 is either —2 —.09691, or —3+.90309.*

The latter is generally most convenient in practice, and is more commonly written $\bar{3}.90309$. The line over the index

* That these two expressions are of the same value will be evident, if we subtract the same quantity, +.90309 from each. The remainders will be equal, and therefore the quantities from which the subtraction is made must be equal. See note B.

denotes, that *that* is negative, while the *decimal* part of the logarithm is positive.

$$\text{The logarithm}\begin{cases}\text{of } 0.3, & \text{is } \bar{1}.47712,\\ \text{of } 0.06, & \text{is } \bar{2}.77815,\\ \text{of } 0.009, & \text{is } \bar{3}.95424,\end{cases}$$

And universally,

11. ***The negative index of a logarithm shows how far the first significant figure of the natural number, is removed from the place of units, on the right;*** in the same manner as a *positive* index shows how far the first figure of the natural number is removed from the place of units, on the left. (Art. 8.) Thus, in the examples in the last article,

The decimal 3 is in the *first* place from that of units,
6 is in the *second* place,
9 is in the *third* place;

And the indices of the logarithms are $\bar{1}$, $\bar{2}$, and $\bar{3}$.

12. It is often more convenient, however, to make the *index* of the logarithm positive, as well as the decimal part. This is done by adding 10 to the index.

Thus, for -1, 9 is written; for -2, 8, &c.
Because $-1+10=9$, $-2+10=8$, &c.

With this alteration,

$$\text{The logarithm}\begin{Bmatrix}\bar{1}.90309\\ \bar{2}.90309\\ \bar{3}.90309\end{Bmatrix}\text{becomes}\begin{cases}9.90309,\\ 8.90309,\\ 7.90309, \text{ \&c.}\end{cases}$$

This is making the index of the logarithm 10 too great. But with proper caution, it will lead to no error in practice.

13. The *sum* of the logarithms of two numbers, is the logarithm of the *product* of those numbers; and the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of one of the numbers divided by the other. (Art. 2.) In Briggs' system, the logarithm of 10 is 1. (Art. 3.) If therefore any number be multiplied or divided by 10, its logarithm will be increased or diminished by 1: and as this is an integer, it will only change the *index* of the logarithm, without affecting the decimal part.

Thus, the logarithm of	4730 is	3.67486
And the logarithm of	10 is	1.
The logarithm of the product	47300 is	4.67486
And the logarithm of the quotient	473 is	2.67486

Here the *index* only is altered, while the decimal part remains the same. We have then this important property,

14. *The* DECIMAL PART *of the logarithm of any number* ***is the same; as that of the number multiplied or divided by*** 10, 100, 1000, &c.

Thus the log. of 45670,	is 4.65963,	
4567,	3.65963,	
456.7,	2.65963,	
45.67,	1.65963,	
4.567,	0.65963,	
.4567,	$\bar{1}$.65963,	or 9.65963,
.04567,	$\bar{2}$.65963,	8.65963,
.004567,	$\bar{3}$.65963,	7.65963.

This property, which is peculiar to Briggs' system, is of great use in abridging the logarithmic tables. For when we have the logarithm of any number, we have only to change the index, to obtain the logarithm of every other number, whether integral, fractional, or mixed, consisting of the same significant figures. The decimal part of the logarithm of a fraction found in this way, is always *positive*. For it is the same as the decimal part of the logarithm of a whole number.

15. In a series of fractions *continually decreasing*, the negative indices of the logarithms *continually increase.* Thus,

In the series	1,	.1,	.01,	.001,	.0001,	.00001,	&c.
The logarithms are	0,	—1,	—2,	—3,	—4,	—5,	&c.

If the progression be continued, till the fraction is reduced to 0, the negative logarithm will become greater than any assignable quantity. The logarithm of 0, therefore, is *infinite and negative.* (Alg. 447.)

16. It is evident also, that all *negative* logarithms belong to fractions which are between 1 and 0; while *positive* loga-

rithms belong to natural numbers which are greater than 1. As the whole range of numbers, both positive and negative, is thus exhausted in supplying the logarithms of integral and fractional positive quantities; there can be no other numbers to furnish logarithms for *negative* quantities. On this account the logarithm of a negative quantity is, by some writers, considered as *impossible.* But as there is no difference in the multiplication, division, involution, &c. of positive and negative quantities, except in applying the *signs;* they may be considered as all positive, while these operations are performing by means of logarithms; and the proper signs may be *afterwards* affixed.

17. *If a series of numbers be in* GEOMETRICAL *progression, their logarithms will be in* ARITHMETICAL *progression.* For, in a geometrical series ascending, the quantities increase by a common *multiplier;* (Alg. 436.) that is, each succeeding term is the *product* of the preceding term into the ratio. But the *logarithm* of this product is the *sum* of the logarithms of the preceding term and the ratio; that is, the logarithms increase by a common *addition*, and are, therefore, in arithmetical progression. (Alg. 422.) In a geometrical progression *descending*, the terms decrease by a common *divisor*, and their logarithms, by a common *difference.**

Thus, the numbers 1, 10, 100, 1000, 10000, &c. are in geometrical progression.

And their logarithms 0, 1, 2, 3, 4, &c. are in arithmetical progression.

Universally, if in any geometrical series,

$a=$ the least term, $r=$ the ratio,

$L=$ its logarithm, $l=$ its logarithm;

Then the logarithm of ar is $L+l$, (Art. 1.)

of ar^2 is $L+2l$,

of ar^3 is $L+3l$, &c.

Here, the quantities a, ar^2, ar^3, ar^4, &c., are in geometrical progression. (Alg. 436.)

And their logarithms L, $L+l$, $L+2l$, $L+3l$, &c., are in arithmetical progression. (Alg. 423.)

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* See note C.

THE LOGARITHMIC CURVE.

19. The relations of logarithms, and their corresponding numbers, may be represented by the abscissas and ordinates of a curve. Let the line AC (Fig. 1.) be taken for unity. Let AF be divided into portions, each equal to A C, by the points 1, 2, 3, &c. Let the line a represent the *radix* of a given system of logarithms, suppose it to be 1.3; and let a^2, a^3, &c. correspond, in length, with the different powers of a. Then the distances from A to 1, 2, 3, &c., will represent the *logarithms* of a, a^2, a^3, &c. (Art. 2.) The line CH is called the *logarithmic curve*, because its *abscissas* are proportioned to the logarithms of numbers represented by its *ordinates*. (Alg. 527.)

20. As the abscissas are the distances from AC, on the line AF, it is evident, that the abscissa of the point C is 0, which is the logarithm of $1 = AC$. (Art. 2.) The distance from A to 1 is the logarithm of the ordinate a, which is the *radix* of the system. For Briggs' logarithms, this ought to be ten times AC. The distances from A to 2 is the logarithm of the ordinate a^2; from A to 3 is the logarithm of a^3, &c.

21. The logarithms of numbers less than a unit are *negative*. (Art. 9.) These may be represented by portions of the line AN, on the *opposite side* of AC. (Alg. 507.) The ordinates a^{-1}, a^{-2}, a^{-3}, &c., are less than AC, which is taken for unity; and the abscissas, which are the distances from A to —1, —2, —3, &c., are negative.

22. If the curve be continued ever so far, it will never meet the axis AN. For, as the ordinates are in geometrical progression decreasing, each is a certain portion of the preceding one. They will be diminished more and more, the farther they are carried, but can never be reduced absolutely to nothing. The axis AN is, therefore, an *asymptote* of the curve. (Alg. 545.) As the ordinate decreases, the abscissa increases; so that, when one becomes infinitely small, the other becomes infinitely great. This corresponds with what has been stated, (Art. 15.) that the logarithm of 0 is *infinite* and *negative*.

23. To find the *equation* of this curve,

Let a = the *radix* of the system,
x = any one of the abscissas,
y = the corresponding ordinate.

Then, by the nature of the curve, (Art. 19.) the *ordinate* to any point, is that power of a whose exponent is equal to the *abscissa* of the same point; that is, (Alg. 528.)

$$y = a^x.\text{*}$$

* For other properties of the logarithmic curve, see Fluxions.

SECTION II.

DIRECTIONS FOR TAKING LOGARITHMS AND THEIR NUMBERS FROM THE TABLES.*

Art. 24. The purpose which logarithms are intended to answer, is to enable us to perform arithmetical operations with *greater expedition*, than by the common methods. Before any one can avail himself of this advantage, he must become so familiar with the tables, that he can readily find the logarithm of any number; and, on the other hand, the number to which any logarithm belongs.

In the common tables, the *indices* to the logarithms of the first 100 numbers, are inserted. But, for all other numbers, the *decimal* part only of the logarithm is given; while the index is left to be supplied, according to the principles in arts. 8 and 11.

25. *To find the logarithm of any number between* 1 *and* 100:

Look for the proposed number, on the left; and against it, in the next column, will be the logarithm, with its index. Thus,

The log. of 18 is 1.25527. The log. of 73 is 1.86332.

26. *To find the logarithm of any number between* 100 *and* 1000; or of any number consisting of not more than three significant figures, with ciphers annexed.

In the smaller tables, the three first figures of each number, are generally placed in the left hand column; and the fourth figure is placed at the head of the other columns.

Any number, therefore, between 100 and 1000, may be found on the left hand; and directly opposite, in the next column, is the decimal part of its logarithm. To this the *index* must be prefixed, according to the rule in art. 8.

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* The best English Tables are Hutton's in 8vo. and Taylor's in 4to. In these, the logarithms are carried to seven places of decimals, and proportional parts are placed in the margin. The smaller tables are numerous; and, when accurately printed, are sufficient for common calculations.

The log. of 458 is 2.66087, The log. of 935 is 2.97081,
of 796 2.90091, of 386 2.58659.

If there are *ciphers* annexed to the significant figures the logarithm may be found in a similar manner. For, by art. 14, the *decimal* part of the logarithm of any number is the same, as that of the number multiplied into 10, 100, &c. All the difference will be in the *index;* and this may be supplied by the same general rule.

The log. of 4580 is 3.66087, The log. of 326000 is 5.51322,
of 79600 4.90091, of 8010000 6.90363.

27. *To find the logarithm of any number consisting of* FOUR *figures, either with, or without, ciphers annexed.*

Look for the three first figures, on the left hand, and for the fourth figure, at the head of one of the columns. The logarithm will be found, opposite the three first figures, and in the column which, at the head, is marked with the fourth figure.*

The log. of 6234 is 3.79477, The log. of 783400 is 5.89398,
of 5231 3.71858, of 6281000 6.79803.

28. *To find the logarithm of a number containing* MORE *than* FOUR *significant figures.*

By turning to the tables, it will be seen, that if the *differences* between several numbers be small, in comparison with the numbers themselves; the differences of the *logarithms* will be nearly proportioned to the differences of the *numbers.* Thus,

The log. of 1000 is 3.00000,
of 1001 3.00043,
of 1002 3.00087,
of 1003 3.00130,
of 1004 3.00173, &c.

Here the differences in the numbers are, 1, 2, 3, 4, &c. and the corresponding differences in the logarithms, are 43, 87, 130, 173, &c.

Now 43 is nearly half of 87, one third of 130, one fourth of 173, &c.

Upon this principle, we may find the logarithm of a number which is between two other numbers whose logarithms

* In Taylor's, Hutton's, and other tables, *four* figures are placed in the left hand column, and the *fifth* at the top of the page.

are given by the tables. Thus, the logarithm of 21716 is not to be found in those tables which give the numbers to four places of figures only.

But by the table, the log. of 21720 is 4.33686
and the log. of 21710 is 4.33666

The difference of the two numbers is 10; and that of the logarithms 20.

Also, the difference between 21710, and the proposed number 21716, is 6.

If, then, a difference of 10 in the numbers
make a difference of 20 in the logarithms:
A difference of 6 in the numbers, will
make a difference of 12 in the logarithms.

That is, 10 : 20 : : 6 : 12.

If, therefore, 12 be added to 4.33666, the log. of 21710;
The sum will be 4.33678, the log. of 21716.
We have, then, this

RULE.

To find the logarithm of a number consisting of more than four figures:

Take out the logarithm of two numbers, one greater, and the other less, than the number proposed: Find the difference of the two numbers, and the difference of their logarithms: Take also the difference between the least of the two numbers, and the proposed number. Then say,

As the difference of the two numbers,
To the difference of their logarithms;
So is the difference between the least of the two numbers, and the proposed number,
To the proportional part to be added to the least of the two logarithms.

It will generally be expedient to make the *four first figures*, in the least of the two numbers, the same as in the proposed number, substituting ciphers, for the remaining figures; and to make the greater number the same as the less, with the addition of a unit to the last significant figure. Thus,

For 36843, take	36840, and	36850,
For 792674	792600,	792700,
For 6537825,	6537000,	6538000, &c.

The first term of the proportion will then be 10, or 100, or 1000, &c.

Ex. 1. Required the logarithm of 362572.

The logarithm of 362600 is 5.55943
of 362500 5.55931

The differences are 100, and 12.

Then 100 : 12 : : 72 : 8.64, or 9 nearly.

And the log. 5.55931+9=5.55940, the log. required.

Ex. 2. The log. of 78264 is 4.89356.
3. The log. of 143542 is 5.15698
4. The log. of 1129535 is 6.05290.

By a little practice, such a facility, in abridging these calculations, may be acquired, that the logarithms may be taken out, in a very short time. When great accuracy is not required, it will be easy to make an allowance sufficiently near, without formally stating a proportion. In the larger tables, the proportional parts which are to be added to the logarithms, are already prepared, and placed in the margin.

29. *To find the logarithm of a* DECIMAL FRACTION.

The logarithm of a decimal is the same as that of a whole number, excepting the *index*. (Art. 14.) To find then the logarithm of a decimal, take out that of a whole number consisting of the same figures; *observing to make the negative index equal to the distance of the first significant figure of the fraction from the place of units.* (Art. 11.)

The log. of 0.07643, is $\bar{2}.88326$, or 8.88326, (Art. 12.)
of 0.00259, $\bar{3}.41330$, or 7.41330,
of 0.0006278, $\bar{4}.79782$, or 6.79782.

30. *To find the logarithm of a* MIXED *decimal number.*

Find the logarithm, in the same manner as if *all* the figures were integers; and then prefix the index which belongs to the *integral* part, according to art. 8.

The logarithm of 26.34 is 1.42062.

The index here is 1, because 1 is the index of the logarithm of every number greater than 10, and less than 100. (Art. 7.)

The log. of 2.36 is	0.37291,		The log. of 364.2 is	2.56134,
of 27.8	1.44404,		of 69.42	1.84148.

31. *To find the logarithm of a* VULGAR FRACTION.

From the nature of a vulgar fraction, the numerator may be considered as a *dividend*, and the denominator as a *divisor*; in other words, the value of the fraction is equal to the quotient of the numerator divided by the denominator. (Alg. 135.) But in logarithms, division is performed by *subtraction*; that is, the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of those numbers. (Art. 1.) To find then the logarithm of a vulgar fraction, *subtract the logarithm of the denominator* from *that of the numerator*. The difference will be the logarithm of the fraction. Or the logarithm may be found, by first reducing the vulgar fraction to a *decimal*. If the numerator is less than the denominator, the index of the logarithm must be *negative*, because the value of the fraction is less than a unit. (Art. 9.)

Required the logarithm of $\frac{34}{87}$.

The log. of the numerator is 1.53148
of the denominator 1.93952

of the fraction $\bar{1}.59196$, or 9.59196.

The logarithm of $\frac{362}{7854}$ is $\bar{2}.66362$, or 8.66362.
of $\frac{7}{6329}$ $\bar{3}.04376$, or 7.04376.

32. If the logarithm of a *mixed number* is required, reduce it to an improper fraction, and then proceed as before.

The logarithm of $3\frac{7}{9}=\frac{34}{9}$ is 0.57724.

---

33. *To find the* NATURAL NUMBER *belonging to any logarithm.*

In computing by logarithms, it is necessary, in the first place, to take from the tables the logarithms of the numbers which enter into the calculation; and, on the other hand, at the close of the operation, to find the number belonging to the logarithm obtained in the result. This is evidently done, by *reversing* the methods in the preceding articles.

Where great accuracy is not required, look in the tables for the logarithm which is *nearest* to the given one; and directly opposite, on the left hand, will be found the *three first* figures, and at the top, over the logarithm, the *fourth* figure, of the number required. This number, by pointing off decimals, or by adding ciphers, if necessary, must be made to correspond with the *index* of the given logarithm, according to arts. 8 and 11.

The natural number belonging

to 3.86493 is	7327,	to 1.62572 is	42.24,
to 2.90141	796.9,	to $\bar{2}$.89115	0.07783.

In the last example, the index requires that the first significant figure should be in the *second* place from units, and therefore a cipher must be prefixed. In other instances, it is necessary to annex ciphers on *the right*, so as to make the number of figures exceed the index by 1.

The natural number belonging

to 6.71567 is	5196000,	to $\bar{3}$.65677 is	0.004537,
to 4.67062	46840,	to $\bar{4}$.59802	0.0003963.

34. When great accuracy is required, and the given logarithm is not exactly, or very nearly, found in the tables, it will be necessary to reverse the rule in art. 28.

Take from the tables two logarithms, one the next greater, the other the next less than the given logarithm. Find the difference of the two logarithms, and the difference of their natural numbers; also the difference between the least of the two logarithms, and the given logarithm. Then say,

As the difference of the two logarithms,
To the difference of their numbers;
So is the difference between the given
logarithm and the least of the other two,
To the proportional part to be added to
the least of the two numbers.

Required the number belonging to the logarithm 2.67325.

Next great. log.	2.67330.	Its numb.	471.3.	Given log.	2.67325
Next less	2.67321.	Its numb.	471.2.	Next less	2.67321.
Differences	9		0.1		4

Then, 9 : 0.1 : : 4 : 0.044, which is to be added to the number 471.2

The number required is 471.244.

The natural number belonging
to 4.37627 is 23783.45, to 1.73698 is 54.57357,
to 3.69479 4952.08, to $\bar{1}.09214$ 0.123635.

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35. *Correction of the Tables.*—The tables of logarithms have been so carefully and so repeatedly calculated, by the ablest computers, that there is no room left to question their general correctness. They are not, however, exempt from the common imperfections of the press. But an error of this kind is easily corrected, by comparing the logarithm with any two others to whose *sum* or *difference* it ought to be equal. (Art. 1.)

Thus, $48 = 24 \times 2 = 16 \times 3 = 12 \times 4 = 8 \times 6$. Therefore, the logarithm of 48 is equal to the *sum* of the logarithms of 24 and 2, of 16 and 3, &c.

And, $3 = \frac{6}{2} = \frac{12}{4} = \frac{15}{5} = \frac{18}{6} = \frac{21}{7}$, &c. Therefore, the logarithm of 3 is equal to the *difference* of the logarithms of 6 and 2, of 12 and 4, &c.

SECTION III.

METHODS OF CALCULATING BY LOGARITHMS.

Art. 36. The arithmetical operations for which logarithms were originally contrived, and on which their great utility depends, are chiefly multiplication, division, involution, evolution, and finding the term required in single and compound proportion. The principle on which all these calculations are conducted, is this :

If the logarithms of two numbers be added, the SUM *will be the logarithm of the* PRODUCT *of the numbers ;* and,

If the logarithm of one number be subtracted from that of another, the DIFFERENCE *will be the logarithm of the* QUOTIENT *of one of the numbers divided by the other.*

In proof of this, we have only to call to mind, that logarithms are the EXPONENTS *of a series of powers and roots.* (Arts. 2, 5.) And it has been shown, that powers and roots are *multiplied* by *adding* their exponents ; and *divided*, by *subtracting* their exponents. (Alg. 233, 237, 280, 286.)

MULTIPLICATION BY LOGARITHMS.

37. ADD THE LOGARITHMS OF THE FACTORS: THE SUM WILL BE THE LOGARITHM OF THE PRODUCT.

In making the addition, 1 is to be carried, for every 10, from the decimal part of the logarithm, to the index. (Art. 7.)

	Numbers.		Logarithms.		Numbers.	Logarithms.
Mult.	36.2	(Art. 30.)	1.55871	Mult.	640	2.80618
Into	7.84		0.89432	Into	2.316	0.36474
Prod.	283.8		2.45303	Prod.	1482	3.17092

The logarithms of the two factors are taken from the tables. The product is obtained, by finding, in the tables, the natural number belonging to the sum. (Art. 33.)

Mult.	89.24	1.95056	Mult.	134.	2.12710
Into	3.687	0.56667	Into	25.6	1.40824
Prod.	329.	2.51723	Prod.	3430	3.53534

38. When any or all of the indices of the logarithms are *negative*, they are to be added according to the rules for the addition of positive and negative quantities in algebra. But it must be kept in mind, that the decimal part of the logarithm is *positive*. (Art. 10.) Therefore, that which is carried from the decimal part to the index, must be considered positive also.

Mult.	62.84	1.79824	Mult.	0.0294	$\bar{2}.46835$
Into	0.682	$\bar{1}.83378$	Into	0.8372	$\bar{1}.92283$
Prod.	42.86	1.63202	Prod.	0.0246	$\bar{2}.39118$

In each of these examples, +1 is to be carried from the decimal part of the logarithm. This, added to —1, the lower index, makes it 0; so that there is nothing to be added to the upper index.

If any perplexity is occasioned, by the addition of positive and negative quantities, it may be avoided, by borrowing 10 to the index. (Art. 12.)

Mult.	62.84	1.79824	Mult.	0.0294	8.46835
Into	0.682	9.83378	Into	0.8372	9.92283
Prod.	42.86	1.63202	Prod.	0.0246	8.39118

Here 10 is added to the negative indices, and afterwards rejected from the index of the sum of the logarithms.

Multiply	26.83	1.42862		1.42862
Into	0.00069	$\bar{4}.83885$	or	6.83885
Product	0.01851	$\bar{2}.26747$		8.26747

Here +1 carried to —4 makes it —3, which added to the upper index +1, gives —2 for the index of the sum.

Multiply	.00845	$\bar{3}.92686$ or	7.92686
Into	1068.	3.02857	3.02857
Product	9.0246	0.95543	0.95543

The product of 0.0352 into 25.38 is 0.9188
of 0.00467 into 348.1 is 1.626
of 0.0861 into 0.00843 is 0.0007258

39. *Any number of factors* may be multiplied together, by adding their logarithms. If there are several *positive*, and several *negative* indices, these are to be reduced to one, as in algebra, by taking the difference between the sum of those which are negative, and the sum of those which are positive, increased by what is carried from the decimal part of the logarithms. (Alg. 78.)

Multiply	6832	3.83455	3.83455
Into	0.00863	$\bar{3}.93601$ or	7.93601
And	0.651	$\bar{1}.81358$	9.81358
And	0.0231	$\bar{2}.36361$ or	8.36361
And	62.87	1.79844	1.79844
Prod.	55.74	1.74619	1.74619

Ex. 2. The prod. of 36.4×7.82×68.91×0.3846 is 7544.

3. The prod. of 0.00629×2.647×0.082×278.8×0.00063 is 0.0002398.

40. *Negative* quantities are multiplied, by means of logarithms, in the same manner as those which are positive. (Art. 16.) But, after the operation is ended, the proper sign must be applied to the natural number expressing the product, according to the rules for the multiplication of positive and negative quantities in algebra. The negative index of a *logarithm*, must not be confounded with the sign which denotes that the *natural number* is negative. That which the index of the logarithm is intended to show, is not whether the natural number is *positive or negative*, but whether it is *greater or less than a unit.* (Art. 16.)

Mult.	+36.42	1.56134	Mult.	—2.681	0.42830
Into	—67.31	1.82808	Into	+37.24	1.57101
Prod.	—2451	3.38942	Prod.	—99.84	1.99931

In these examples, the logarithms are taken from the tables, and added, in the same manner, as if both factors were positive. But after the product is found, the negative sign is prefixed to it, because + is multiplied into —. (Alg. 105.)

Mult.	0.263	$\bar{1}.41996$	Mult.	0.065	$\bar{2}.81291$
Into	0.00894	$\bar{3}.95134$	Into	0.693	$\bar{1}.84073$
Prod.	0.002351	$\bar{3}.37130$	Prod.	0.04504	$\bar{2}.65364$

Here, the indices of the logarithms are negative, but the product is positive, because the factors are both positive.

Mult.	—62.59	1.79650	Mult.	—68.3	1.83442
Into	—0.00863	$\bar{3}.93601$	Into	—0.0096	$\bar{3}.98227$
Prod.	+0.5402	$\bar{1}.73251$	Prod.	+0.6557	$\bar{1}.81669$

DIVISION BY LOGARITHMS.

41. From the logarithm of the DIVIDEND, SUBTRACT the logarithm of the DIVISOR; the DIFFERENCE will be the logarithm of the QUOTIENT. (Art. 36.)

	Numbers.	Logarithms.		Numbers.	Logarithms.
Divide	6238	3.79505	Divide	896.3	2.95245
By	2982	3.47451	By	9.847	0.99330
Quot.	2.092	0.32054	Quot.	91.02	1.95915

42. The *decimal* part of the logarithm may be subtracted as in common arithmetic. But for the *indices*, when either of them is negative, or the lower one is greater than the upper one, it will be necessary to make use of the general rule for subtraction in algebra; that is, to change the signs of the subtrahend, and then proceed as in addition. (Alg. 82.) When 1 is carried from the decimal part, this is to be considered affirmative, and applied to the index, before the sign is changed.

Divide	0.8697	$\bar{1}.93937$	or	9.93937
By	98.65	1.99410		1.99410
Quot.	0.008816	$\bar{3}.94527$		7.94527

In this example, the upper logarithm being less than the lower one, it is necessary to borrow 10, as in other cases of subtraction; and therefore to carry 1 to the lower index, which then becomes +2. This changed to —2, and added to —1 above it, makes the index of the difference of the logarithms —3.

Divide	29.76	1.47363		1.47363
By	6254	3.79616		3.79616
Quot.	0.00476	$\bar{3}.67747$	or	7.67747

Here, 1 carried to the lower index, makes it +4. This changed to —4, and added to 1 above it, gives —3 for the index of the difference of the logarithms.

Divide	6.832	0.83455	Divide	0.00634	$\bar{3}.80209$
By	.0362	$\bar{2}.55871$	By	62.18	1.79365
Quot.	188.73	2.27584	Quot.	0.000102	$\bar{4}.00844$

The quotient of 0.0985 divided by 0.007241, is 13.6
The quotient of 0.0621 divided by 3.68, is 0.01687

43. To divide *negative* quantities, proceed in the same manner as if they were positive, (Art. 40.) and prefix to the quotient, the sign which is required by the rules for division in algebra.

Divide	+3642	3.56134	Divide	—0.657	$\bar{1}.81757$
By	—23.68	1.37438	By	+0.0793	$\bar{2}.89927$
Quot.	—153.8	2.18696	Quot.	—8.285	0.91830

In these examples, the sign of the divisor being different from that of the dividend, the sign of the quotient must be negative. (Alg. 123.)

Divide	—0.364	$\bar{1}.56110$	Divide	—68.5	1.83569
By	—2.56	0.40824	By	+0.094	$\bar{2}.97313$
Quot.	+0.1422	$\bar{1}.15286$	Quot.	—728.7	2.86256

INVOLUTION BY LOGARITHMS.

44. Involving a quantity is multiplying it into itself. By means of logarithms, multiplication is performed by addition. If, then, the logarithm of any quantity be *added to itself*, the

logarithm of a *power* of that quantity will be obtained. But adding a logarithm, or any other quantity, to itself, is *multiplication*. The involution of quantities, by means of logarithms, is therefore performed, by multiplying the logarithms.

Thus the logarithm
of 100 is 2
of 100×100, that is, of $\overline{100^2}$ is 2+2 = 2×2.
of 100×100×100, $\overline{100^3}$ is 2+2+2 = 2×3.
of 100×100×100×100, $\overline{100^4}$ is 2+2+2+2 = 2×4.

On the same principle, the logarithm of $\overline{100^n}$ is $2\times n$.
And the logarithm of x^n, is $(\log. x)\times n$. Hence,

45. To involve a quantity by logarithms. MULTIPLY THE LOGARITHM OF THE QUANTITY, BY THE INDEX OF THE POWER REQUIRED.

The reason of the rule is also evident, from the consideration, that logarithms are the exponents of powers and roots, and a power or root is involved, by *multiplying* its index into the index of the power required. (Alg. 220, 288.)

Ex. 1. What is the cube of 6.296?

Root 6.296,	its log.	0.79906
	Index of the power	3
Power 249.6		2.39718

2. Required the 4th power of 21.32

Root 21.32	log.	1.32879
	Index	4
Power 206614		5.31516

3. Required the 6th power of 1.689

Root 1.689	log.	0.22763
	Index	6
Power 23.215		1.36578

4. Required the 144th power of			1.003
Root	1.003	log.	0.00130
		Index	144
Power	1.539		0.18720

46. It must be observed, as in the case of multiplication, (Art. 38.) that what is carried from the *decimal* part of the logarithm is *positive*, whether the index itself is positive or negative. Or, if 10 be added to a negative index, to render it positive, (Art. 12.) this will be multiplied, as well as the other figures, so that the logarithm of the square, will be 20 too great; of the cube, 30 too great, &c.

Ex. 1. Required the cube of			0.0649		
Root	0.0649	log.	$\bar{2}.81224$	or	8.81224
		Index	3		3
Power	0.0002733		$\bar{4}.43672$		6.43672

2. Required the 4th power of			0.1234		
Root	0.1234	log.	$\bar{1}.09132$	or	9.09132
		Index	4		4
Power	0.0002319		$\bar{4}.36528$		6.36528

3. Required the 6th power of			0.9977		
Root	0.9977	log.	$\bar{1}.99900$	or	9.99900
		Index	6		6
Power	0.9863		$\bar{1}.99400$		9.99400

4. Required the cube of			0.08762		
Root	0.08762	log.	$\bar{2}.94260$	or	8.94260
		Index	3		3
Power	0.0006727		$\bar{4}.82780$		6.82780

5. The 7th power of 0.9061 is 0.5015.
6. The 5th power of 0.9344 is 0.7123.

EVOLUTION BY LOGARITHMS.

47. Evolution is the opposite of involution. Therefore, as quantities are involved, by the *multiplication* of logarithms, roots are extracted by the *division* of logarithms; that is,

To extract the root of a quantity by logarithms, DIVIDE THE LOGARITHM OF THE QUANTITY, BY THE NUMBER EXPRESSING THE ROOT REQUIRED.

The reason of the rule is evident also, from the fact, that logarithms are the exponents of powers and roots, and evolution is performed, by dividing the exponent, by the number expressing the root required. (Alg. 257.)

1. Required the square root of 648.3

	Numbers.	Logarithms.
Power	648.3	2)2.81178
Root	25.46	1.40589

2. Required the cube root of 897.1

Power	897.1	3)2.95284
Root	9.645	0.98428

In the first of these examples, the logarithm of the given number is divided by 2; in the other, by 3.

3. Required the 10th root of 6948.

Power	6948	10)3.84186
Root	2.422	0.38418

4. Required the 100th root of 983.

Power	983	100)2.99255
Root	1.071	0.02992

The division is performed here, as in other cases of [decimals], by removing the decimal point to the left.

5. What is the ten thousandth root of 49680000?

Power	49680000	10000)7.69618
Root	1.00179	0.00077

We have, here, an example of the great rapidity with which arithmetical operations are performed by logarithms.

48. If the index of the logarithm is *negative*, and is *not divisible* by the given divisor, without a remainder, a difficulty will occur, unless the index be altered.

Suppose the cube root of 0.0000892 is required. The logarithm of this is $\overline{5}.95036$. If we divide the index by 3, the quotient will be —1, with —2 remainder. This remainder, if it were positive, might, as in other cases of division, be prefixed to the next figure. But the remainder is *negative*, while the decimal part of the logarithm is positive; so that, when the former is prefixed to the latter, it will make neither +2.9 nor —2.9, but —2+.9. This embarrassing intermixture of positives and negatives may be avoided, by adding to the index another negative number, to make it exactly divisible by the divisor. Thus, if to the index —5 there be added —1, the sum —6 will be divisible by 3. But this addition of a negative number must be *compensated*, by the addition of an equal positive number, which may be prefixed to the decimal part of the logarithm. The division may then be continued, without difficulty, through the whole.

Thus, if the logarithm $\overline{5}.95036$ be altered to $\overline{6}+1.95036$ it may be divided by 3, and the quotient will be $\overline{2}.65012$. We have then this rule,

49. *Add to the index*, if necessary, *such a negative number as will make it exactly divisible by the divisor, and prefix an equal positive number to the decimal part of the logarithm.*

1. Required the 5th root of 0.009642.

Power	0.009642	log.	$\overline{3}.98417$
		or	$\overline{5}+2.98417$
Root	0.3952		$\overline{1}.59683$

2. Required the 7th root of 0.0004935.

Power	0.0004935	log.	$\overline{4}.69329$
		or	7)$\overline{7}+3.69329$
Root	0.337		$\overline{1}.52761$

50. If, for the sake of performing the division conveniently, the negative index be rendered *positive*, it will be expedient to borrow as many tens, as there are units in the number denoting the root.

What is the fourth root of 0.03698 ?

Power	0.03698	$4)\bar{2}.56797$	or $4)38.56797$
Root	0.4385	$\bar{1}.64199$	9.64199

Here the index, by borrowing, is made 40 too great, that is, +38 instead of —2. When, therefore, it is divided by 4, it is still 10 too great, +9 instead of —1.

What is the 5th root of 0.008926 ?

Power	0.008926	$5)\bar{3}.95066$	or $5)47.95066$
Root	0.38916	$\bar{1}.59013$	9.59013

51. A *power of a root* may be found by first *multiplying* the logarithm of the given quantity into the index of the power, (Art. 45.) and then *dividing* the product by the number expressing the root. (Art. 47.)

1. What is the value of $(53)^{\frac{6}{7}}$, that is, the 6th power of the 7th root of 53 ?

Given number 53	log.	1.72428
Multiplying by		6
Dividing by		7)10.34568
Power required 30.06		1.47795

2. What is the 8th power of the 9th root of 654 ?

PROPORTION BY LOGARITHMS.

52. In a proportion, when three terms are given, the fourth is found, in common arithmetic, by multiplying together the second and third, and dividing by the first. But when logarithms are used, *addition* takes the place of multiplication, and *subtraction*, of division.

To find, then, by logarithms, the fourth term in a proportion, ADD THE LOGARITHMS OF THE SECOND AND THIRD TERMS, AND *from the sum* SUBTRACT THE LOGARITHM

OF THE FIRST TERM. The remainder will be the logarithm of the term required.

Ex. 1. Find a fourth proportional to 7964, 378, and 27960.

	Numbers.	Logarithms.
Second term	378	2.57749
Third term	27960	4.44654
		7.02403
First term	7964	3.90113
Fourth term	1327	3.12290

2. Find a 4th proportional to 768, 381, and 9780.

Second term	381	2.58092
Third term	9780	3.99034
		6.57126
First term	768	2.88536
Fourth term	4852	3.68590

ARITHMETICAL COMPLEMENT.

53. When one number is to be subtracted from another, it is often convenient, first to subtract it from 10, then *to add the difference* to the other number, and afterwards to reject the 10.

Thus, instead of $a-b$, we may put $10-b+a-10$.

In the first of these expressions, b is subtracted from a. In the other, b is subtracted from 10, the difference is added to a, and 10 is afterwards taken from the sum. The two expressions are equivalent, because they consist of the same terms, with the addition, in one of them, of $10-10=0$. The alteration is, in fact, nothing more than borrowing 10, for the sake of convenience, and then rejecting it in the result.

Instead of 10, we may borrow, as occasion requires, 100, 1000, &c.

Thus, $a-b=100-b+a-100=1000-b+a-1000$, &c.

54. The DIFFERENCE *between a given number and* 10, *or* 100, *or* 1000, *&c., is called the* ARITHMETICAL COMPLEMENT *of that number.*

The arithmetical complement of a number consisting of ***one*** **integral figure, either with or** without decimals, is found, **by subtracting the number from** 10. If there are *two* integral **figures, they are subtracted** from 100; if *three*, from **1000, &c.**

Thus, the arithmetical compl't of 3.46 is 10—3.46=6.54
of 34.6 is 100—34.6=65.4
of 346. is 1000—346.=654. &c.

According to the rule for subtraction in arithmetic, any number is subtracted from 10, 100, 1000, &c. by beginning on the right hand, and taking each figure from 10, after *increasing* all except the first, by *carrying* 1.

Thus, if from	10.00000
We subtract	7.63125

The difference, or arith'l compl't is 2.36875, which is obtained by taking 5 from 10, 3 from 10, 2 from 10, 4 from 10, 7 from 10, and 8 from 10. But, instead of taking each figure, *increased by* 1, from 10; we may take it *without being increased*, from 9.

Thus, 2 from 9 is the same as 3 from 10,
3 from 9, the same as 4 from 10, &c. Hence,

55. *To obtain the* ARITHMETICAL COMPLEMENT *of a number, subtract the right hand significant figure from* 10, *and each of the other figures from* 9. If, however, there are *ciphers* on the right hand of all the significant figures, they are to be set down without alteration.

In taking the arithmetical complement of a logarithm, if the index is *negative*, it must be *added* to 9; for adding a negative quantity is the same as subtracting a positive one. (Alg. 81.) The difference between —3 and +9, is not 6, but 12.

The arithmetical complement

of 6.24897 is	**3.75103**	**of $\overline{2}$.70649 is**	**11.29351**
of 2.98643	**7.01357**	**of 3.64200**	**6.35800**
of 0.62430	**9.37570**	**of 9.35001**	**0.64999**

56. The principal use of the arithmetical complement, is in working proportions by logarithms; where some of the terms are to be *added*, and one or more to be *subtracted*. In the Rule of Three or simple proportion, two terms are to be added, and from the sum, the first term is to be subtracted. But if, instead of the logarithm of the first term, we substitute its arithmetical complement, this may be *added* to the sum of the other two, or more simply, all three may be added together, by one operation. After the index is diminished by 10, the result will be the same as by the common method. For subtracting a number is the same, as adding its arithmetical complement, and then rejecting 10, 100, or 1000, from the sum. (Art. 53.)

It will generally be expedient, to place the terms in the same order, in which they are arranged in the statement of the proportion.

1. As	6273 *a. c.*	6.20252		2. As	253 *a. c.*	7.59688
Is to	769.4	2.88615		Is to	672.5	2.82769
So is	37.61	1.57530		So is	497	2.69636
To	4.613	0.66397		To	1321.1	3.12093
3. As	46.34 *a. c.*	8.33404		4. As	9.85 *a. c.*	9.00656
Is to	892.1	2.95041		Is to	643	2.80821
So is	7.638	0.88298		So is	76.3	1.88252
To	147	2.16743		To	4981	3.69729

COMPOUND PROPORTION.

57. In compound, as in single proportion, the term required may be found by logarithms, if we substitute addition for multiplication, and subtraction for division.

Ex. 1. If the interest of $365, for 3 years and 9 months, be $82.13; what will be the interest of $8940, for 2 years and 6 months?

In common arithmetic, the statement of the question is made in this manner.

$\left.\begin{matrix}365 \text{ dollars} \\ 3.75 \text{ years}\end{matrix}\right\}$: 82.13 dollars : : $\left\{\begin{matrix}8940 \text{ dollars} \\ 2.5 \text{ years}\end{matrix}\right\}$:

And the method of calculation is, to *divide* the *product* of the third, fourth, and fifth terms, by the *product* of the two first.* This, if logarithms are used, will be to *subtract* the *sum* of the logarithms of the two first terms, from the *sum* of the logarithms of the other three.

Two first terms	{ 365 log.	2.56229
	{ 3.75	0.57403
Sum of the logarithms		3.13632
Third term	82.13	1.91450
Fourth and fifth terms	{ 8940	3.95134
	{ 2.5	0.39794
Sum of the logs. of the 3d, 4th, and 5th,		6.26378
Do.	1st and 2d,	3.13632
Term required	1341	3.12746

58. The calculation will be more simple, if, instead of *subtracting* the logarithms of the two first terms, we *add* their *arithmetical complements*. But it must be observed, that *each* arithmetical complement increases the index of the logarithm by 10. If the arithmetical complement be introduced into *two* of the terms, the index of the sum of the logarithms will be 20 too great; if it be in *three* terms, the index will be 30 too great, &c.

Two first terms	{ 365 *a. c.*	7.43771
	{ 3.75 *a. c.*	9.42597
Third term	82.13	1.91450
Fourth and fifth terms	{ 8940	3.95134
	{ 2.5	0.39794
Term required	1341	23.12746

The result is the same as before, except that the index of the logarithm is 20 too great.

* See Arithmetic.

Ex. 2. If the wages of 53 men for 42 days be 2200 dollars; what will be the wages of 87 men for 34 days?

53 men, 42 days : 2200 : : 87 men, 34 days :

Two first terms	53	*a. c.*	8.27572
	42	*a. c.*	8.37675
Third term	2200		3.34242
Fourth and fifth terms	87		1.93952
	34		1.53148
Term required	2923.5		3.46589

59. In the same manner, if the product of *any number* of quantities, is to be divided, by the product of several others; we may add together the logarithms of the quantities to be divided, and the arithmetical complements of the logarithms of the divisors.

Ex. If 29.67×346.2 be divided by 69.24×7.862×497; what will be the quotient?

Numbers to be divided	29.67		1.47232
	346.2		2.53933
Divisors	69.24	*a. c.*	8.15964
	7.862	*a. c.*	9.10447
	497	*a. c.*	7.30364
Quotient	0.03797		8.5794

In this way, the calculations in *Conjoined Proportion* may be expeditiously performed.

COMPOUND INTEREST.

60. In calculating compound interest, the amount for the first year, is made the principal for the second year; the amount for the second year, the principal for the third year, &c. Now the amount at the end of each year, must be proportioned to the principal at the beginning of the year. If

the principal for the first year be 1 dollar, and if the amount of 1 dollar for 1 year $= a$; then, (Alg. 377.)

$$1 : a :: \begin{cases} a : a^2 = \text{the amount for the 2d year, or the principal for the 3d;} \\ a^2 : a^3 = \text{the amount for the third year, or the principal for the 4th;} \\ a^3 : a^4 = \text{the amount for the 4th year, or the principal for the 5th.} \end{cases}$$

That is, the amount of 1 dollar for any number of years is obtained, by finding the amount for 1 year, and involving this to a power whose index is equal to the number of years. And the amount of any other principal, for the given time, is found, by multiplying the amount of 1 dollar, into the number of dollars, or the fractional part of a dollar.

If logarithms are used, the *multiplication* required here may be performed by *addition;* and the *involution*, by *multiplication.* (Art. 45.) Hence,

61. To calculate Compound Interest, *Find the amount of 1 dollar for 1 year; multiply its logarithm by the number of years; and to the product, add the logarithm of the principal.* The sum will be the logarithm of the *amount* for the given time. From the amount subtract the principal, and the remainder will be the *interest.*

If the interest becomes due *half yearly* or *quarterly;* find the amount of one dollar, for the half year or quarter, and multiply the logarithm, by the number of half years or quarters in the given time.

If $P =$ the principal,
$a =$ the amount of 1 dollar for 1 year,
$n =$ any number of years, and
$A =$ the amount of the given principal for n years; then,

$$A = a^n \times P.$$

Taking the logarithms of both sides of the equation, and reducing it, so as to give the value of each of the four quantities, in terms of the others, we have

1. $\text{Log. } A = n \times \text{log. } a + \text{log. } P.$
2. $\text{Log. } P = \text{log. } A - n \times \text{log. } a.$
3. $\text{Log. } a = \frac{\text{log. } A - \text{log. } P.}{n}$
4. $n = \frac{\text{log. } A - \text{log. } P.}{\text{log. } a.}$

Any three of these quantities being given, the fourth may be found.

Ex. 1. What is the amount of 20 dollars, at 6 per cent. compound interest, for 100 years?

Amount of 1 dollar for 1 year	1.06	log. 0.0253059
Multiplying by		100
		2.53059
Given principal	20	1.30103
Amount required	$6786	3.83162

2. What is the amount of 1 cent, at 6 per cent. compound interest, in 500 years?

Amount of 1 dollar for 1 year	1.06	log. 0.0253059
Multiplying by		500
		12.65295
Given principal	0.01	-2.00000
Amount	$44,973,000,000	10.65295

More exact answers may be obtained, by using logarithms of a greater number of decimal places.

3. What is the amount of 1000 dollars, at 6 per cent. compound interest, for 10 years? Ans. 1790.80.

4. What principal, at 4 per cent. interest, will amount to 1643 dollars in 21 years? Ans. 721.

5. What principal, at 6 per cent., will amount to 202 dollars in 4 years? Ans. 160.

6. At what rate of interest, will 400 dollars amount to 569$\frac{1}{3}$, in 9 years? Ans. 4 per cent.

7. In how many years will 500 dollars amount to 900, at 5 per cent. compound interest? Ans. 12 years.

8. In what time will 10,000 dollars amount to 16,288, at 5 per cent. compound interest? Ans. 10 years.

9. At what rate of interest, will 11,106 dollars amount to 20.000 in 15 years? Ans. 4 per cent.

10. What principal, at 6 per cent. compound interest, will amount to 3188 dollars in 8 years? Ans. $2000.

11. What will be the amount of 1200 dollars, at 6 per cent. compound interest, in 10 years, if the interest is converted into principal every *half year*? Ans. 2167.3 dollars.

12. In what time will a sum of money double, at 6 per cent. compound interest? Ans. 11.9 years.

13. What is the amount of 5000 dollars, at 6 per cent. compound interest, for 28$\frac{1}{4}$ years? Ans. 25.942 dollars.

INCREASE OF POPULATION.

61. *b.* The natural increase of population in a country, may be calculated in the same manner as compound interest; on the supposition, that the yearly rate of increase is regularly proportioned to the actual number of inhabitants. From the population at the beginning of the year, the *rate* of increase being given, may be computed the whole increase during the year. This, *added* to the number at the beginning, will give the amount, on which the increase of the *second* year is to be calculated, in the same manner as the first year's interest on a sum of money, added to the sum

itself, gives the amount on which the interest for the second year is to be calculated.

If P = the population at the beginning of the year,
a = 1 + the fraction which expresses the rate of increase,
n = any number of years ; and
A = the amount of the population at the end of n years;
then, as in the preceding article,

$$A = a^n \times P, \text{ and}$$

1. $\text{Log. } A = n \times \text{log. } a + \text{log. } P.$
2. $\text{Log. } P = \text{log. } A - n \times \text{log. } a.$
3. $\text{Log. } a = \dfrac{\text{log. } A - \text{log. } P.}{n}$
4. $n = \dfrac{\text{log. } A - \text{log. } P.}{\text{log. } a.}$

Ex. 1. The population of the United States in 1820 was 9,625,000. Supposing the yearly rate of increase to be $\frac{1}{34}$th part of the whole, what will be the population in 1830?

Here $P = 9{,}625{,}000$. $n = 10$. $a = 1 + \frac{1}{34} = \frac{35}{34}$.

And $\text{log. } A = 10 \times \text{log. } \frac{35}{34} + \text{log. } (9{,}625{,}000,)$
Therefore, A = 12,860,000, the population in 1830.

2. If the number of inhabitants in a country be five millions, at the beginning of a century; and if the yearly rate of increase be $\frac{1}{30}$; what will be the number, at the end of the century? Ans. 132,730,000.

3. If the population of a country, at the end of a century, is found to be 45,860,000; and if the yearly rate of increase has been $\frac{1}{120}$; what was the population, at the commencement of the century? Ans. 20 millions.

4. The population of the United States in 1810 was 7,240,000; in 1820, 9,625,000. What was the annual rate of increase between these two periods, supposing the increase each year to be proportioned to the population at the beginning of the year?

$$\text{Here log. } a = \frac{\text{log. } 9{,}625{,}000 - \text{log. } 7{,}240{,}000}{10}$$

Therefore, $a = 1.029$; and $\frac{29}{1000}$, or 2.9 per cent. is the rate of increase.

5. In how many years, will the population of a country advance from two millions to five millions; supposing the yearly rate of increase to be $\frac{7}{360}$? Ans. $47\frac{1}{2}$ years.

6. If the population of a country, at a given time, be seven millions; and if the yearly rate of increase be $\frac{1}{25}$th; what will be the population at the end of 35 years?

7. The population of the United States in 1800 was 5,306,000. What was it in 1780, supposing the yearly rate of increase to be $\frac{1}{28}$?

8. In what time will the population of a country advance from four millions to seven millions, if the ratio of increase be $\frac{3}{100}$?

9. What must be the rate of increase, that the population of a place may change from nine thousand to fifteen thousand, in 12 years?

If the population of a country is not affected by immigration or emigration, the rate of increase will be equal to the difference between the ratio of the *births*, and the ratio of the *deaths*, when compared with the whole population.

Ex. 10. If the population of a country, at any given time, be ten millions; and the ratio of the annual number of births to the whole population be $\frac{1}{24}$, and the ratio of deaths $\frac{1}{45}$, what will be the number of inhabitants, at the end of 60 years?

Here the yearly rate of increase $= \frac{1}{24} - \frac{1}{45} = \frac{7}{360}$.
And the population, at the end of 60 years $= 31{,}750{,}000$.

The rate of increase or decrease from *immigration* or *emigration*, will be equal to the difference between the ratio of immigration and the ratio of emigration; and if this difference be added to, or subtracted from, the difference between the ratio of the births and that of the deaths, the whole rate of increase will be obtained.

Ex. 11. If in a country, the ratio of births be $\frac{1}{30}$,
the ratio of deaths $\frac{1}{40}$,
the ratio of immigration $\frac{1}{50}$,
the ratio of emigration $\frac{1}{60}$,
and if the population this year be 10 millions, what will it be 20 years hence?

The rate of the natural increase $=\frac{1}{30}-\frac{1}{40}=\frac{1}{120}$;
That of increase from immigration $=\frac{1}{50}-\frac{1}{60}=\frac{1}{300}$;
The sum of the two is $\frac{7}{600}$;
And the population at the end of 20 years, is 12,611,000.

12. If the ratio of the births be $\frac{1}{20}$,
of the deaths $\frac{1}{30}$,
of immigration $\frac{1}{40}$,
of emigration $\frac{1}{50}$,
in what time will three millions increase to four and a half millions?

If the period in which the population will *double* be given; the numbers for several successive periods, will evidently be in a geometrical progression, of which the ratio is 2; and as the number of periods will be one less than the number of terms;

If $P=$ the first term,
$A=$ the last term,
$n=$ the number of periods;
Then will $A=P\times 2^n$, (Alg. 439.)
Or $\log. A=\log. P+n\times\log. 2$.

Ex. 1. If the descendants of a single pair double once in 25 years, what will be their number, at the end of one thousand years?

The number of periods here is 40.
And $A=2\times 2^{40}=2,199,200,000,000$.

2. If the descendants of Noah, beginning with his three sons and their wives, doubled once in 20 years for 300 years, what was their number, at the end of this time?

Ans. 196,608.

3. The population of the United States in 1820 being 9,625,000; what must it be in the year 2020, supposing it to double once in 25 years? Ans. 2,464,000,000.

4. Supposing the descendants of the first human pair to double once in 50 years, for 1650 years, to the time of the deluge, what was the population of the world, at that time?

EXPONENTIAL EQUATIONS.

62. An EXPONENTIAL equation is one in which the letter expressing the unknown quantity is an *exponent*.

Thus, $a^x=b$, and $x^x=bc$, are exponential equations. These are most easily solved by logarithms. As the two members of an equation are equal, their logarithms must also be equal. If the logarithm of each side be taken, the equation may then be reduced, by the rules given in algebra.

Ex. What is the value of x in the equation $3^x=243$?

Taking the logarithms of both sides, log. $3^x=$ log. 243.

But the logarithm of a *power* is equal to the logarithm of the root, multiplied into the index of the power. (Art. 45.)

Therefore (log. 3)$\times x=$log. 243; and dividing by log. 3.

$$x=\frac{\text{log. }243}{\text{log. }3}=\frac{2.38561}{0.47712}=5. \quad \text{So that } 3^5=243.$$

63. The preceding is an exponential equation of the simplest form. Other cases, after the logarithm of each side is taken, may be solved by *Trial and Error*, in the same manner as affected equations. (Alg. 503.) For this purpose, make two suppositions of the value of the unknown quantity, and find their errors; then say,

As the difference of the errors, to the difference of the assumed numbers;
So is the least error, to the correction required in the corresponding assumed number.

Ex. 1. Find the value of x in the equation $x^x=256$.
Taking the logarithms of both sides $(\log. x)\times x=\log. 256$.
Let x be supposed equal to 3.5, or 3.6.

By the first supposition.		By the second supposition.	
$x=3.5$, and $\log. x=$	0.54407	$x=3.6$, and $\log. x=$	0.55630
Multiplying by	3.5	Multiplying by	3.6
$(\log. x)\times x=$	1.90424	$(\log. x)\times x=$	2.00268
$\log. 256=$	2.40824	$\log. 256=$	2.40824
Error	—0.50400	Error	—0.40556
Difference of the errors		0.09844	

Then, 0.09844 : 0.1 : : 0.40556 : 0.4119, the correction. This added to 3.6, the second assumed number, makes the value of $x=4.0119$.

To correct this farther, suppose $x=4.011$, or 4.012.

By the first supposition.		By the second supposition.	
$x=4.011$, and $\log. x=$	0.60325	$x=4.012$, and $\log. x=$	0.60336
Multiplying by	4.011	Multiplying by	4.012
$(\log. x)\times x=$	2.41963	$(\log. x)\times x=$	2.42068
$\log. 256=$	2.40824	$\log. 256=$	2.40824
Error	+0.01139	Error	+0.01244
Difference of the errors		0.00105	

Then, 0.00105 : 0.001 : : 0.01139 : 0.011 very nearly.

Subtracting this correction from the first assumed number 4.011, we have the value of $x=4$, which satisfies the conditions of the proposed equation; for $4^4=256$.

2. Reduce the equation $4x^x=100x^3$. Ans. $x=5$.

3. Reduce the equation $x^x=9x$.

64. The exponent of a power may be itself a power, as in the equation

$$a^{m^x}=b;$$

where x is the exponent of the power m^x, which is the exponent of the power a^{m^x}.

Ex. 4. Find the value of x, in the equation $9^{3^x}=1000$.

$3^x\times(\text{log. }9)=\text{log. }1000.$ Therefore, $3^x=\dfrac{\text{log. }1000}{\text{log. }9}=3.14.$

Then, as $3^x=3.14.$ $x\,(\text{log. }3)=\text{log. }3.14.$

Therefore, $x=\dfrac{\text{log. }3.14}{\text{log. }3}=\frac{4969296}{4771213}=1.04.$

In cases like this, where the factors, divisors, &c., are logarithms, the calculation may be facilitated, by taking the *logarithms of the logarithms*. Thus, the value of the fraction $\frac{4969296}{4771213}$ is most easily found, by subtracting the logarithm of the logarithm which constitutes the denominator, from the logarithm of that which forms the numerator.

5. Find the value of x, in the equation $\dfrac{ba^x+d}{c}=m.$

Ans. $x=\dfrac{\text{log. }(cm-d)-\text{log. }b}{\text{log. }a.}$

SECTION IV.

DIFFERENT SYSTEMS OF LOGARITHMS, AND COMPUTATION OF THE TABLES.

65. For the common purposes of numerical computation, Briggs' system of logarithms has a decided advantage over every other. But the theory of logarithms is an important instrument of investigation, in the higher departments of mathematical science. In its numerous applications, there is frequent occasion to compare the common system with others; especially with that which was adopted by the celebrated inventor of logarithms, Lord Napier. In conducting these investigations, it is often expedient to express the logarithm of a number, in the form of a *series*.

If $a^x=N$, then x is the logarithm of N. (Art. 2.)

To find the value of x, in a series, let the quantities a and N be put into the form of a binomial, by making $a=1+b$, and $N=1+n$. Then $(1+b)^x=1+n$, and extracting the root y of both sides, we have

$$(1+b)^{\frac{x}{y}}=(1+n)^{\frac{1}{y}}$$

By the binomial theorem,

$$(1+b)^{\frac{x}{y}}=1+\frac{x}{y}(b)+\frac{x}{y}\left(\frac{x}{y}-1\right)\left(\frac{b^2}{2}\right)+\frac{x}{y}\left(\frac{x}{y}-1\right)\left(\frac{x}{y}-2\right)\left(\frac{b^3}{2.3}\right)+\&c.$$

$$(1+n)^{\frac{1}{y}}=1+\frac{1}{y}(n)+\frac{1}{y}\left(\frac{1}{y}-1\right)\left(\frac{n^2}{2}\right)+\frac{1}{y}\left(\frac{1}{y}-1\right)\left(\frac{1}{y}-2\right)\left(\frac{n^3}{2.3}\right)+\&c.$$

As these expressions will be the same, whatever be the value of y, let y be taken indefinitely great; then $\frac{x}{y}$ and $\frac{1}{y}$ being indefinitely small, in comparison with the numbers -1, -2, &c., with which they are connected, may be cancelled from the factors $\left(\frac{x}{y}-1\right)$, $\left(\frac{x}{y}-2\right)$, &c. $\left(\frac{1}{y}-1\right)$, $\left(\frac{1}{y}-2\right)$, &c. (Alg. 456.) leaving $1+\frac{x}{y}b-\frac{x}{y}\left(\frac{b^2}{2}\right)+\frac{x}{y}\left(\frac{b^3}{3}\right)-\frac{x}{y}\left(\frac{b^4}{4}\right)$, &c.

$$=1+\frac{1}{y}n-\frac{1}{y}\left(\frac{n^2}{2}\right)+\frac{1}{y}\left(\frac{n^3}{3}\right)-\frac{1}{y}\left(\frac{n^4}{4}\right), \&c.$$

Rejecting 1 from each side of the equation, multiplying by y, (Alg. 159.) and dividing by the compound factor into which x is multiplied, we have

$$x = \text{Log. N} = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \text{\&c.}}{b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \frac{1}{4}b^4 + \text{\&c.}} \quad \text{A}$$

Or, as $n = \text{N} - 1$, and $b = a - 1$,

$$\text{Log. N} = \frac{(\text{N}-1) - \frac{1}{2}(\text{N}-1)^2 + \frac{1}{3}(\text{N}-1)^3 - \frac{1}{4}(\text{N}-1)^4 + \text{\&c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{\&c.}}$$

Which is a general expression, for the logarithm of any number N, in any system in which the base is a. The numerator is expressed in terms of N only; and the denominator in terms of a only: So that, whatever be the number, the denominator will remain the same, unless the base is changed. The reciprocal of this constant denominator, viz.

$$\frac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{\&c.}}$$

is called the *Modulus* of the system of which a is the base. If this be denoted by M, then

$$\text{Log. N} = \text{M} \times \Big((\text{N}-1) - \tfrac{1}{2}(\text{N}-1)^2 + \tfrac{1}{3}(\text{N}-1)^3 - \tfrac{1}{4}(\text{N}-1)^4 + \text{\&c.}\Big)$$

66. The foundation of Napier's system of Logarithms is laid, by making the modulus equal to *unity*. From this condition the *base* is determined. Taking the equation above marked A. and making the denominator equal to 1, we have

$$x = n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5 - \text{\&c.}$$

By reverting this equation *

$$n = x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \text{\&c.}$$

Or, as by the notation, $n + 1 = \text{N} = a^x$,

$$a^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \text{\&c.}$$

If then x be taken equal to 1, we have

$$a = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \text{\&c.}$$

Adding the first fifteen terms, we have

2.7182818284

Which is the base of Napier's system, correct to ten places of decimals.

* See note D.

Napier's logarithms are also called *hyperbolic* logarithms, from certain relations which they have to the spaces between the asymptotes and the curve of an hyperbola; although these relations are not, in fact, peculiar to Napier's system.

67. The logarithms of *different* systems are compared with each other, by means of the modulus. As in the series

$$\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\frac{1}{4}(N-1)^4+\&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\&c.}$$

which expresses the logarithm of N, the *denominator* only is affected by a change of the base a; and as the value of fractions, whose numerators are given, are reciprocally as their denominators: (Alg. 360. cor. 2.)

The logarithm of a given number, in one system,
Is to the logarithm of the same number in another system;
As the modulus of one system,
To the modulus of the other.

So that, if the modulus of each of the systems be given, and the logarithm of any number be calculated in one of the systems; the logarithm of the same number in the other system may be calculated by a simple proportion. Thus, if M be the modulus in Briggs' system, and M′ the modulus in Napier's; l the logarithm of a number in the former, and l' the logarithm of the same number in the latter; then,

$$M : M' : : l : l',$$
Or, as $M'=1$,
$$M : 1 : : l : l'.$$

Therefore, $l=l'\times M$; that is, the common logarithm of a number, is equal to Napier's logarithm of the same, multiplied into the modulus of the common system.

To find this modulus, let a be the base of Briggs' system, and e the base of Napier's; and let $l.a$ denote the common logarithm of a, and $l'.a$ denote Napier's logarithm of a.

Then, $M : 1 : : l.a : l'.a$. Therefore, $M=\frac{l.a}{l'.a}$

But in the common system, $a=10$, and $l.a=1$.

So that, $M=\frac{1}{l'.10}$, that is the modulus of Briggs' system, is equal to 1 divided by Napier's logarithm of 10.

Again, $M : 1 : : l.e : l'.e$

But as e denotes Napier's base, $l'.e = 1$.

So that $M = l.e$, that is, the modulus of the common system, is equal to the common logarithm of Napier's base.

Therefore, either of the expressions, $l.e$, or $\frac{1}{l'.a}$ may be used, to convert the logarithms of one of the systems into those of the other.

The ratio of the logarithms of two numbers to each other, is the same in one system as in another. If N and n be the two numbers;

$$\begin{array}{ll} \text{Then,} & l.N : l'.N : : M : M' \\ & l.n : l'.n : : M : M' \\ \text{Therefore,} & l.N : l.n : : l'.N : l'.n. \end{array}$$

COMPUTATION OF LOGARITHMS.

68. The logarithms of most numbers can be calculated by approximation only, by finding the sum of a sufficient number of terms, in the series which expresses the value of the logarithms. According to art. 65.

$$\text{Log. } N = M \times ((N-1) - \tfrac{1}{2}(N-1)^2 + \tfrac{1}{3}(N-1)^3, \text{\&c.})$$

Or, putting as before, $n = N - 1$,

$$\text{Log. } (1+n) = M\,(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5 - \text{\&c.})$$

But this series will not converge, when n is a whole number, greater than unity. To convert it into another which will converge, let $(1-n)$ be expanded in the same manner as $(1+n)$, (Art. 65.) The formula will be the same, except that the odd powers of n will be negative instead of positive.

We shall then have,

$$\text{Log. } (1+n) = M\,(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5 - \text{\&c.})$$
$$\text{Log. } (1-n) = M(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 - \tfrac{1}{5}n^5 - \text{\&c.})$$

Subtracting the one from the other, the even powers of n disappear, and we have

$$M(2n + \tfrac{2}{3}n^3 + \tfrac{2}{5}n^5 + \tfrac{2}{7}n^7 + \text{\&c.})$$

or

$$2M\,(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \tfrac{1}{7}n^7 + \text{\&c.})$$

But this, which is the *difference* of the logarithms of $(1+n)$ and $(1-n)$ is the logarithm of the *quotient* of the one divided by the other. (Art. 36.)

That is, $\text{Log.}\ \frac{1+n}{1-n}=2M(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\frac{1}{7}n^7+\&c.$

Now put $n=\frac{1}{z-1}$

Then, $\frac{1+n}{1-n}=\frac{1+\frac{1}{z-1}}{1-\frac{1}{z-1}}=\frac{\frac{z}{z-1}}{\frac{z-2}{z-1}}=\frac{z}{z-2}$

Therefore, substituting $\frac{z}{z-2}$ for $\frac{1+n}{1-n}$, and $\frac{1}{z-1}$ for n, we have

$$\text{Log.}\ \frac{z}{z-2}=2M\left(\frac{1}{(z-1)}+\frac{1}{3(z-1)^3}+\frac{1}{5(z-1)^5}+\&c.\right)$$

Or, (Art. 36.)

$$\text{Log.}\ z-\log.(z-2)=2M\left(\frac{1}{(z-1)}+\frac{1}{3(z-1)^3}+\frac{1}{5(z-1)^5}+\&c.\right)$$

Therefore,

$$\text{Log.}\ z=\log.(z-2)+2M\left(\frac{1}{(z-1)}+\frac{1}{3(z-1)^3}+\frac{1}{5(z-1)^5}+\&c.\right)$$

This series may be applied to the computation of any number greater than 2.

To find the logarithm of 2, let $z=4$,

Then, $(z-1)=3$, and the preceding series, after transposing log. $(z-2)$ becomes

$$\text{Log.}\ 4-\log.\ 2=2M\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7},\ \&c.\right)$$

But as 4 is the square of 2; log. 4 = 2 log. 2. (Alg. 44.) So that log. 4 — log. 2 = log. 2. We have then

$$\text{Log. } 2 = 2M\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\frac{1}{9.3^9}+, \&c.\right)$$

When the logarithms of the *prime* numbers are computed, the logarithms of all other numbers may be found, by simply adding the logarithms of the factors of which the numbers are composed. (Art. 36.)

69. In Napier's system, where M=1, the logarithms may be computed, as in the following table.

NAPIER'S OR HYPERBOLIC LOGARITHMS.

$\text{Log. } 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}, \&c.\right)$	=0.693147
$\text{Log. } 3=2\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\frac{1}{7.2^7}, \&c.\right)$	=1.098612
Log. 4=2 log. 2.	=1.386294
$\text{Log. } 5=\text{log. } 3+2\left(\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\frac{1}{7.4^7}, \&c.\right)$	=1.609438
Log. 6=log. 3+log. 2.	=1.791759
$\text{Log. } 7=\text{log. } 5+2\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7}, \&c.\right)$	=1.955900
Log. 8=log. 4+log. 2.	=2.079441
Log. 9=2 log. 3.	=2.197224
Log. 10=log. 5+log. 2.	=2.302585
&c. &c.	&c.

70. To compute the logarithms of the common system, it will be necessary to find the value of the *modulus*. This is equal to 1 divided by Napier's logarithm of 10, (Art. 67.) that is,

$$\frac{1}{2.302585}=.43429448.$$

This number substituted for M, or twice the number, viz. .86858896 substituted for 2 M, in the series in art. 68. will enable us to calculate the common logarithm of any number.

COMMON OR BRIGGS' LOGARITHMS.

$$\text{Log. } 2 = .86858896\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7},\ \&c.\right) = 0.301030$$

$$\text{Log. } 3 = .86858896\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\frac{1}{7.2^7},\ \&c.\right) = 0.477121$$

$$\text{Log. } 4 = 2 \text{ log. } 2. = 0.602060$$

$$\text{Log. } 5 = \text{log. } 10 - \text{log. } 2 = 1 - \text{log. } 2. = 0.698970$$

$$\text{Log. } 6 = \text{log. } 3 + \text{log. } 2. = 0.778151$$

$$\text{Log. } 7 = .86858896\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7},\ \&c.\right) + \text{log. } 5. = 0.845098$$

$$\text{Log. } 8 = 3 \text{ log. } 2. = 0.903090$$

$$\text{Log. } 9 = 2 \text{ log. } 3. = 0.954243$$

$$\text{Log. } 10 = 1.000000$$

&c. &c.

TRIGONOMETRY.

SECTION I.

SINES, TANGENTS, SECANTS, &c.

ART. 71. TRIGONOMETRY *treats of the relations of the sides and angles of* TRIANGLES. Its first object is, to determine the length of the sides, and the quantity of the angles. In addition to this, from its principles are derived many interesting methods of investigation in the higher branches of analysis, particularly in physical astronomy. Scarcely any department of mathematics is more important, or more extensive in its applications. By trigonometry, the mariner traces his path on the ocean; the geographer determines the latitude and longitude of places, the dimensions and positions of countries, the altitude of mountains, the courses of rivers, &c., and the astronomer calculates the distances and magnitudes of the heavenly bodies, predicts the eclipses of the sun and moon, and measures the progress of light from the stars.

72. Trigonometry is either *plane* or *spherical*. The former treats of triangles bounded by *right lines;* the latter, of triangles bounded by *arcs of circles.*

Divisions of the Circle.

73. In a triangle there are two classes of quantities which are the subjects of inquiry, the *sides* and the *angles*. For the purpose of measuring the latter, a *circle* is introduced.

The periphery of every circle, whether great or small, is supposed to be divided into 360 equal parts called *degrees*, each degree into 60 *minutes*, each minute into 60 *seconds*, each second into 60 *thirds*, &c., marked with the characters °, ′, ″, ‴, &c. Thus, 32° 24′ 13″ 22‴ is 32 degrees, 24 minutes, 13 seconds, 22 thirds.*

A degree, then, is not a magnitude of a given *length;* but a certain *portion* of the whole circumference of any circle. It is evident, that the 360th part of a large circle is greater than the same part of a small one. On the other hand, the *number* of degrees in a small circle, is the same as in a large one.

The fourth part of a circle is called a *quadrant*, and contains 90 degrees.

74. To *measure* an angle, a circle is so described that its center shall be the angular point, and its periphery shall cut the two lines which include the angle. The *arc* between the two lines is considered a *measure of the angle*, because, by Euc. 33. 6, angles at the center of a given circle, have the same ratio to each other, as the arcs on which they stand. Thus the arc AB, (Fig. 2.) is a measure of the angle ACB.

It is immaterial what is the size of the circle, provided it cuts the lines which include the angle. Thus, the angle ACD (Fig. 4.) is measured by either of the arcs AG, *ag*. For ACD is to ACH, as AG to AH, or as *ag* to *ah*. (Euc. 33. 6.)

75. In the circle ADGH, (Fig. 2.) let the two diameters AG and DH be perpendicular to each other. The angles ACD, DCG, GCH, and HCA, will be right angles; and the periphery of the circle will be divided into four equal parts, each containing 90 degrees. As a right angle is subtended by an arc of 90°, the angle itself is said to contain 90°. Hence, in two right angles, there are 180°; in four right angles, 360°; and in any other angle, as many degrees, as in the arc by which it is subtended.

76. The sum of the three angles of any triangle being equal to two right angles, (Euc. 32. 1.) is equal to 180°. Hence, there can never be more than one obtuse angle in a triangle. For the sum of two obtuse angles is more than 180°.

77. *The* COMPLEMENT *of an arc or an angle, is the difference between the arc or angle and* 90 *degrees.*

The complement of the arc AB (Fig. 2.) is DB; and the complement of the angle ACB is DCB. The complement of the arc BDG is also DB.

* See note E.

The complement	of 10° is 80°,	of 60° is 30°,
	of 20° is 70°,	of 120° is 30°,
	of 50° is 40°,	of 170° is 80°, &c.

Hence, an acute angle and its complement are always equal to 90°. The angles ACB and DCB are together equal to a right angle. The two acute angles of a right angled triangle are equal to 90°: therefore each is the complement of the other.

78. *The* SUPPLEMENT *of an arc or an angle is the difference between the arc or angle and* 180 *degrees.*

The supplement of the arc BDG (Fig. 2.) is AB; and the supplement of the angle BCG is BCA.

The supplement	of 10° is 170°,	of 120° is 60°,
	of 80° is 100°,	of 150° is 30°, &c.

Hence an angle and its supplement are always equal to 180°. The angles BCA and BCG are together equal to two right angles.

79. Cor. As the three angles of a plane triangle are equal to two right angles, that is, to 180° (Euc. 32. 1.) the sum of any two of them is the supplement of the other. So that the third angle may be found, by subtracting the sum of the other two from 180°. Or the sum of any two may be found, by subtracting the third from 180°.

80. A straight line drawn from the centre of a circle to any part of the periphery, is called a *radius* of the circle. In many calculations, it is convenient to consider the radius, whatever be its length, as *a unit.* (Alg. 510.) To this must be referred the numbers expressing the lengths of other lines. Thus, 20 will be twenty times the radius, and 0.75, three fourths of the radius.

Definitions of Sines, Tangents, Secants, &c.

81. To facilitate the calculations in trigonometry, there are drawn, within and about the circle, a number of straight lines, called *Sines, Tangents, Secants, &c.* With these the learner should make himself perfectly familiar.

82. *The* SINE *of an arc is a straight line drawn from one end of the arc, perpendicular to a diameter which passes through the other end.*

Thus, BG (Fig. 3.) is the sine of the arc AG. For BG is a line drawn from the end G of the arc, perpendicular to the diameter AM which passes through the other end A of the arc.

Cor. The sine is *half the chord* of *double the arc*. The sine BG is half PG, which is the chord of the arc PAG, double the arc AG.

83. *The* VERSED SINE *of an arc is that part of the diameter which is between the sine and the arc.*

Thus, BA is the versed sine of the arc AG.

84. *The* TANGENT *of an arc, is a straight line drawn perpendicular from the extremity of the diameter which passes through one end of the arc, and extended till it meets a line drawn from the center through the other end.*

Thus, AD (Fig. 3.) is the tangent of the arc AG.

85. *The* SECANT *of an arc, is a straight line drawn from the center, through one end of the arc, and extended to the tangent which is drawn from the other end.*

Thus, CD (Fig. 3.) is the secant of the arc AG.

86. In Trigonometry, the terms *tangent* and *secant* have a more limited meaning, than in Geometry. In both, indeed, the tangent *touches* the circle, and the secant *cuts* it. But in Geometry, these lines are of no determinate length; whereas, in Trigonometry, they extend from the diameter to the point in which they intersect each other.

87. The lines just defined are sines, tangents, and secants of *arcs*. BG (Fig. 3.) is the sine of the arc AG. But this arc subtends the *angle* GCA. BG is then the sine of the arc which subtends the *angle* GCA. This is more concisely expressed, by saying that BG is the sine of the angle GCA. And universally, the sine, tangent, and secant of an *arc*, are said to be the sine, tangent, and secant of the *angle* which stands at the center of the circle, and is subtended by the arc. Whenever, therefore, the sine, tangent, or secant of an angle is spoken of; we are to suppose a circle to be drawn whose center is the angular point; and that the lines mentioned belong to that arc of the periphery which subtends the angle.

88. The *sine* and *tangent* of an acute angle, are *opposite* to the angle. But the *secant* is one of the lines which *include* the angle. Thus, the sine BG, and the tangent AD, (Fig. 3.) are opposite to the angle DCA. But the secant CD is one of the lines which include the angle.

89. *The sine complement or* COSINE *of an angle, is the sine of the* COMPLEMENT *of that angle.* Thus, if the diameter HO (Fig. 3.) be perpendicular to MA, the angle HCG is the complement of ACG; (Art. 77.) and LG, or its equal CB, is the sine of HCG. (Art. 82.) It is, therefore, the *cosine* of GCA. On the other hand, GB is the sine of GCA, and the cosine of GCH.

So also the *cotangent* of an angle is the tangent of the *complement* of the angle. Thus, HF is the cotangent of GCA. And the *cosecant* of an angle is the secant of the *complement* of the angle. Thus, CF is the cosecant of GCA.

Hence, as in a right angled triangle, one of the acute angles is the complement of the other; (Art. 77.) the sine, tangent, and secant of one of these angles, are the cosine, co-tangent, and cosecant of the other.

90. The sine, tangent, and secant of the *supplement* of an angle, are each equal to the sine, tangent, and secant of the angle itself. It will be seen, by applying the definition (Art. 82.) to the figure, that the sine of the obtuse angle GCM is BG, which is also the sine of the acute angle GCA. It should be observed, however, that the sine of an acute angle is *opposite* to it; while the sine of an obtuse angle *falls without* the angle, and is opposite to its supplement. Thus BG, the sine of the angle MCG, is not opposite to MCG, but to its supplement ACG.

The *tangent* of the obtuse angle MCG is MT, or its equal AD, which is also the tangent of ACG. And the *secant* of MCG is CD, which is also the secant of ACG.

91. But the *versed sine* of an angle *is not the same* as that of its *supplement.* The versed sine of an *acute* angle is equal to the *difference* between the cosine and radius. But the versed sine of an *obtuse* angle is equal to the *sum* of the cosine and radius. Thus, the versed sine of ACG is AB=AC—BC. (Art. 83.) But the versed sine of MCG is MB=MC+BC.

Relations of Sines, Tangents, Secants, &c., to each other.

92. The relations of the sine, tangent, secant, cosine, &c., to each other, are easily derived from the proportions of the sides of similar triangles. (Euc. 4. 6.) In the quadrant ACH, (Fig. 3.) these lines form three similar triangles, viz. ACD, BCG or LCG, and HCF. For, in each of these, there is one

right angle, because the sines and tangents are, by definition, perpendicular to AC; as the cosine and cotangent are to CH. The lines CH, BG, and AD, are parallel, because CA makes a right angle with each. (Euc. 27. 1.) For the same reason, CA, LG, and HF, are parallel. The alternate angles GCL, BGC, and the opposite angle CDA, are equal; (Euc. 29. 1.) as are also the angles GCB, LGC, and HFC. The triangles ACD, BCG, and HCF, are therefore similar.

It should also be observed, that the line BC, between the sine and the center of the circle, is parallel and equal to the cosine; and that LC, between the cosine and center, is parallel and equal to the sine; (Euc. 34. 1.) so that one may be taken for the other, in any calculation.

93. From these similar triangles, are derived the following proportions; in which R is put for radius,

sin for sine,	*cos* for cosine,
tan for tangent,	*cot* for cotangent,
sec for secant,	*cosec* for cosecant.

By comparing the triangles CBG and CAD,

1. AC : BC : : AD : BG, that is, R : cos : : tan : sin.
2. CG : CD : : BG : AD R : sec : : sin : tan.
3. CB : CA : : CG : CD cos : R : : R : sec.

Therefore $R^2 = \cos \times \sec$.

By comparing the triangles CLG and CHF,

4. CH : CL : : HF : LG, that is, R : sin : : cot : cos.
5. CG : CF : : LG : HF R : cosec : : cos : cot.
6. CL : CH : : CG : CF sin : R : : R : cosec.

Therefore $R^2 = \sin \times \text{cosec}$.

By comparing the triangles CAD and CHF,

7. CH : AD : : CF : CD, that is, R : tan : : cosec : sec.
8. CA : HF : : CD : CF R : cot : : sec : cosec.
9. AD : AC : : CH : HF tan : R : : R : cot.

Therefore $R^2 = \tan \times \cot$.

It will not be necessary for the learner to commit these proportions to memory. But he ought to make himself so familiar with the manner of stating them from the figure, as to be able to explain them, whenever they are referred to.

94. Other relations of the sine, tangent, &c., may be derived from the proposition, that the square of the hypothenuse is equal to the sum of the squares of the perpendicular sides. (Euc. 47. 1.)

In the right angled triangles CBG, CAD, and CHF, (Fig. 3.)

1. $\overline{CG}^2=\overline{CB}^2+\overline{BG}^2$, that is, $R^2=\cos^2+\sin^2$,*
2. $\overline{CD}^2=\overline{CA}^2+\overline{AD}^2$ $\quad\sec^2=R^2+\tan^2$,
3. $\overline{CF}^2=\overline{CH}^2+\overline{HF}^2$ $\quad\text{cosec}^2=R^2+\cot^2$,

And, extracting the root of both sides, (Alg. 296.)

$$R=\sqrt{\cos^2+\sin^2}=\sqrt{\sec^2-\tan^2}=\sqrt{\text{cosec}^2-\cot^2}$$

Hence, if R=1, (Alg. 510.)

$$\text{Sin}=\sqrt{1-\cos^2} \qquad \text{Sec}=\sqrt{1+\tan^2}$$

$$\text{Cos}=\sqrt{1-\sin^2} \qquad \text{Cosec}=\sqrt{1+\cot^2}$$

95. *The sine of* 90°, *The chord of* 60°, *And the tangent of* 45°, *are*, in any circle, *each equal to the radius*, and therefore equal to each other.

Demonstration.

1. In the quadrant ACH, (Fig. 5.) the arc AH is 90°. The sine of this, according to the definition, (Art. 82.) is CH, the radius of the circle.

2. Let AS be an arc of 60°. Then the angle ACS, being measured by this arc, will also contain 60°; (Art. 75.) and the triangle ACS will be equilateral. For the sum of the three angles is equal to 180°. (Art. 76.) From this, taking the angle ACS, which is 60°, the sum of the remaining two is 120°. But these two are *equal*, because they are subtended by the equal sides, CA and CS, both radii of the circle. Each, therefore, is equal to *half* 120°, that is, to 60°. All the

* Sin^2 is here put for the square of the sine, $\cos^2$ for the square of the cosine, &c.

angles being equal, the sides are equal, and therefore AS, the chord of 60°, is equal to CS, the radius.

3. Let AR be an arc of 45°. AD will be its tangent, and the angle ACD subtended by the arc, will contain 45°. The angle CAD is a right angle, because the tangent is, by definition, perpendicular to the radius AC. (Art. 84.) Subtracting ACD, which is 45°, from 90°, (Art. 77.) the other acute angle ADC will be 45° also. Therefore the two legs of the triangle ACD are equal, because they are subtended by equal angles; (Euc. 6. 1.) that is, AD the tangent of 45°, is equal to AC the radius.

Cor. The *cotangent* of 45° is also equal to radius. For the complement of 45° is itself 45°. Thus, HD, the cotangent of ACD, (Fig. 5.) is equal to AC the radius.

96. The sine of 30° is equal to *half radius*. For the sine of 30° is equal to half the chord of 60°. (Art. 82. cor.) But by the preceding article, the chord of 60° is equal to radius. Its half, therefore, which is the sine of 30°, is equal to half radius.

Cor. 1. The *cosine* of 60° is equal to half radius. For the cosine of 60° is the sine of 30°. (Art. 89.)

Cor. 2. The cosine of $30^\circ = \frac{1}{2}\surd 3$. For

$$\text{Cos}^2 30^\circ = R^2 - \sin^2 30^\circ = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

Therefore,

$$\text{Cos } 30^\circ = \surd\tfrac{3}{4} = \tfrac{1}{2}\surd 3.$$

96. *b*. The sine of $45^\circ = \frac{1}{\surd 2}$. For

$$R^2 = 1 = \sin^2 45^\circ + \cos^2 45 = 2\sin^2 45^\circ$$

Therefore, $\text{Sin } 45^\circ = \surd\frac{1}{2} = \frac{1}{\surd 2}$.

97. The *chord* of any arc is a *mean proportional*, between the *diameter* of the circle, and the *versed sine* of the arc.

Let ADB, (Fig. 6.) be an arc, of which AB is the chord, BF the sine, and AF the versed sine. The angle ABH is a right angle, (Euc. 31. 3.) and the triangles ABH, and ABF, are similar. (Euc. 8. 6.) Therefore,

$$AH : AB : : AB : AF.$$

That is, the diameter is to the chord, as the chord to the versed sine.

In Fig. 6th, let the arc AD$=a$, and ADB$=2a$. Draw BF perpendicular to AH. This will divide the right angled triangle ABH into two similar triangles. (Euc. 8. 6.) The angles ACD and AHB are equal. (Euc. 20. 3.) Therefore the four triangles ACG, AHB, FHB, and FAB, are similar; and the line BH is twice CG, because BH : CG : : HA : CA.

The sides of the four triangles are,

AG$=\sin a$,	CG$=\cos a$,	HF$=$vers. sup. $2a$,
AB$=2\sin a$,	BH$=2\cos a$,	AC$=$the radius,
BF$=\sin 2a$,	AF$=$vers $2a$,	AH$=$the diameter.

A variety of proportions may be stated, between the homologous sides of these triangles: For instance,

By comparing the triangles ACG and ABF,

AC : AG : : AB : AF, that is, $R : \sin a : : 2\sin a : \text{vers } 2a$
AC : CG : : AB : BF, $R : \cos a : : 2\sin a : \sin 2a$
AG : CG : : AF : BF, $\text{Sin } a : \cos a : : \text{vers } 2a : \sin 2a$

Therefore,

$$R \times \text{vers } 2a = 2\sin^2 a$$
$$R \times \sin 2a = 2\sin a \times \cos a$$
$$\text{Sin } a \times \sin 2a = \text{vers } 2a \times \cos a$$

By comparing the triangles ACG and BFH,

AC : CG : : BH : HF, that is, $R : \cos a : : 2\cos a : \text{vers. sup. } 2a$
AG : CG : : BF : HF, $\text{Sin } a : \cos a : : \sin 2a : \text{vers. sup. } 2a$.

Therefore,

$$R \times \text{vers. sup. } 2a = 2\cos^2 a$$
$$\text{Sin } a \times \text{vers. sup. } 2a = \cos a \times \sin 2a$$
&c. &c.

That is, the product of radius into the versed sine of the supplement of twice a given arc, is equal to twice the square of the cosine of the arc.

And the product of the sine of an arc, into the versed sine of the supplement of twice the arc, is equal to the product of the cosine of the arc, into the sine of twice the arc, &c. &c.

8

SECTION II.

THE TRIGONOMETRICAL TABLES.

ART. 98. To facilitate the operations in trigonometry, the sine, tangent, secant, &c., have been calculated for every degree and minute, and in some instances, for every second, of a quadrant, and arranged in tables. These constitute what is called the *Trigonometrical Canon.** It is not necessary to extend these tables beyond 90° ; because the sines, tangents, and secants, are of the same magnitude, in one of the quadrants of a circle, as in the others. Thus the sine of 30° is equal to that of 150°. (Art. 90.)

99. And in any instance, if we have occasion for the sine, tangent, or secant of an *obtuse angle*, we may obtain it, by looking for its equal, the sine, tangent, or secant of the *supplementary acute angle.*

100. The tables are calculated for a circle whose radius is supposed to be a *unit.* It may be an inch, a yard, a mile, or any other denomination of length. But the *sines*, *tangents*, *&c.*, must always be understood to be of the same denomination as the radius.

101. All the *sines*, except that of 90°, are *less than radius*, (Art. 82, and Fig. 3.) and are expressed in the tables by decimals.

Thus the sine of 20° is 0.34202, of 60° is 0.86603,
of 40° is 0.64279, of 89° is 0.99985, &c.

When the tables are intended to be very exact, the decimal is carried to a greater number of places.

The *tangents* of all angles less than 45° are also less than radius. (Art. 95.) But the tangents of angles greater than 45°, are *greater* than radius, and are expressed by a whole number and a decimal. It is evident that all the *secants* also

* For the *construction* of the Canon, see Section VIII.

must be greater than radius, as they extend from the center, to a point without the circle.

102. The numbers in the table here spoken of, are called *natural* sines, tangents, &c. They express the lengths of the several lines which have been defined in arts. 82, 83, &c. By means of them, the angles and sides of triangles may be accurately determined. But the calculations must be made by the tedious processes of multiplication and division. To avoid this inconvenience, another set of tables has been provided, in which are inserted the *logarithms* of the natural sines, tangents, &c. By the use of these, addition and subtraction are made to perform the office of multiplication and division. On this account, the tables of logarithmic, or as they are sometimes called, *artificial* sines, tangents, &c., are much more valuable, for practical purpose, than the *natural* sines, &c. Still it must be remembered, that the former are derived from the latter. The artificial sine of an angle, is the logarithm of the natural sine of that angle. The artificial tangent is the logarithm of the natural tangent, &c.

103. One circumstance, however, is to be attended to, in comparing the two sets of tables. The radius to which the *natural* sines, &c., are calculated, is *unity*. (Art. 100.) The secants, and a part of the tangents are, therefore, *greater* than a unit; while the sines, and another part of the tangents, are *less* than a unit. When the logarithms of these are taken, some of the indices will be *positive*, and others *negative;* (Art. 9.) and the throwing of them together in the same table, if it does not lead to error, will at least be attended with inconvenience. To remedy this, 10 is added to each of the indices. (Art. 12.) They are then all positive. Thus the natural sine of 20° is 0.34202. The logarithm of this is $\bar{1}.53405$. But the index, by the addition of 10, becomes $10-1=9$. The logarithmic sine in the tables is therefore 9.53405.*

Directions for taking Sines, Cosines, &c., from the tables.

104. The *cosine*, *cotangent*, and *cosecant* of an angle, are the sine, tangent, and secant of the *complement* of the angle. (Art. 89.) As the complement of an angle is the difference between the angle and 90°, and as 45 is the half of 90; if any given angle within the quadrant is greater than 45°, its

* Or the tables may be supposed to be calculated to the radius 10000000000, whose logarithm is 10.

complement is less; and, on the other hand, if the angle is *less* than 45°, its complement is greater. Hence, every cosine, cotangent, and cosecant of an angle greater than 45°, has its equal, among the sines, tangents, and secants of angles less than 45°, and *v. v.*

Now, to bring the trigonometrical tables within a small compass, the same column is made to answer for the *sines* of a number of angles *above* 45°, and for the *cosines* of an equal number *below* 45°.

Thus 9.23967	is the log. *sine* of 10°,	and the *cosine* of 80°,
9.53405	the *sine* of 20°,	and the *cosine* of 70°, &c.

The tangents and secants are arranged in a similar manner. Hence,

105. *To find the Sine, Cosine, Tangent, &c., of any number of degrees and minutes.*

If the given angle is *less* than 45°, look for the degrees at the *top* of the table, and the minutes on the *left*; then, opposite to the minutes, and under the word sine at the head of the column, will be found the sine; under the word tangent, will be found the tangent, &c.

The log. sin	of 43° 25′	is 9.83715	The tan	of 17° 20′	is 9.49430
	of 17° 20′	9.47411		of 8° 46′	9.18812
The cos	of 17° 20′	9.97982	The cot	of 17° 20′	10.50570
	of 8° 46′	9.99490		of 8° 46′	10.81188

The first figure is the index; and the other figures are the decimal part of the logarithm.

106. If the given angle is between 45° and 90°; look for the degrees at the *bottom* of the table, and the minutes on the *right*; then, opposite to the minutes, and *over* the word sine at the foot of the column, will be found the sine; over the word tangent, will be found the tangent, &c.

Particular care must be taken, when the angle is less than 45°, to look for the title of the column, at the *top*, and for the minutes on the *left*; but when the angle is between 45° and 90°, to look for the title of the column at the *bottom*, and for the minutes, on the *right*.

The log. sine	of 81° 21′	is 9.99503
The cosine	of 72° 10′	9.48607
The tangent	of 54° 40′	10.14941
The cotangent	of 63° 22′	9.70026

107. If the given angle is *greater* than 90°, look for the sine, tangent, &c., of its *supplement.* (Art. 98, 99.)

The log. sine of	96° 44′ is	9.99699
The cosine of	171° 16′	9.99494
The tangent of	130° 26′	10.06952
The cotangent of	156° 22′	10.35894

108. *To find the sine, cosine, tangent, &c., of any number of degrees, minutes, and* SECONDS.

In the common tables, the sine, tangent, &c., are given only to every *minute* of a degree.* But they may be found to *seconds*, by taking *proportional parts* of the difference of the numbers as they stand in the tables. For, within a single minute, the variations in the sine, tangent, &c., are nearly proportional to the variations in the angle. Hence,

To find the sine, tangent, &c., to seconds: Take out the number corresponding to the given degree and minute; and also that corresponding to the next greater minute, and find their difference. Then state this proportion;

As 60, to the given number of seconds;

So is the difference found, to the correction for the seconds.

This correction, in the case of sines, tangents, and secants, is to be *added* to the number answering to the given degree and minute; but for cosines, cotangents, and cosecants, the correction is to be *subtracted;*

For, as the sines *increase*, the cosines *decrease.*

Ex. 1. What is the logarithmic sine of 14° 43′ 10″?

The sine of	14° 43′ is	9.40490
of	14° 44′	9.40538
Difference		48

Here it is evident, that the sine of the required angle is greater than that of 14° 43′, but less than that of 14° 44′. And as the difference corresponding to a whole minute or

* In the very valuable tables of Michael Taylor, the sines and tangents are given to *every second.*

60″ is 48; the difference for 10″ must be a proportional part of 48. That is,

60″ : 10″ : : 48 : 8

the correction to be *added* to the sine of 14° 43′.

Therefore the sine of 14° 43′ 10″ is 9.40498.

2. What is the logarithmic cosine of 32° 16′ 45″?

The cosine of	32° 16′ is	9.92715
of	32° 17′	9.92707
Difference		8

Then, 60″ : 45″ : : 8 : 6 the correction to be *subtracted* from the cosine of 32° 16′.

Therefore the cosine of 32° 16′ 45″ is 9.92709.

The tangent of	24° 15′ 18″	is	9.65376
The cotangent of	31° 50′ 5″	is	10.20700
The sine of	58° 14′ 32″	is	9.92956
The cosine of	55° 10′ 26″	is	9.75670

If the given number of seconds be any even part of 60, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., the correction may be found, by taking a like part of the difference of the numbers in the tables, without stating a proportion in form.

109. *To find the degrees and minutes belonging to any given sine, tangent, &c.*

This is reversing the method of finding the sine, tangent, &c. (Art. 105, 6, 7.)

Look in the column of the same name, for the sine, tangent, &c., which is *nearest* to the given one; and if the title be at the *head* of the column, take the degrees at the *top* of the table, and the minutes on the *left*; but if the title be at the *foot* of the column, take the degrees at the *bottom*, and the minutes on the *right*.

Ex. 1. What is th number of degrees and minutes belonging to the logarithmic sine 9.62863?

The nearest sine in the tables is 9.62865. The title of sine is at the head of the column in which these numbers are found. The degrees at the top of the page are 25, and the minutes on the left are 10. The angle required is, therefore 25° 10′.

The angle belonging to

the sine	9.87993 is	49° 20′		the cos	9.97351 is	19° 48′
the tan	9.97955	43° 39′		the cotan	9.75791	60° 12′
the sec	10.65396	77° 11′		the cosec	10.49066	18° 51′

110. *To find the degrees, minutes, and* SECONDS, *belonging to any given sine, tangent, &c.*

This is reversing the method of finding the sine, tangent, &c., to seconds. (Art. 108.)

First find the difference between the sine, tangent, &c., next greater than the given one, and that which is next less; then the difference between this less number and the given one; then

As the difference first found, is to the other difference;

So are 60 seconds, to the number of seconds, which, in the case of sines, tangents, and secants, are to be *added* to the degrees and minutes belonging to the least of the two numbers taken from the tables; but for cosines, cotangents, and cosecants, are to be *subtracted.*

Ex. 1. What are the degrees, minutes, and seconds, belonging to the logarithmic sine 9.40498?

Sine next greater	14° 44′	9.40538	Given sine	9.40498
Next less	14° 43′	9.40490	Next less	9.40490
Difference		48	Difference	8

Then, 48 : 8 : : 60″ : 10″, which added to 14° 43′, gives 14° 43′ 10″ for the answer.

2. What is the angle belonging to the cosine 9.09773?

Cosine next greater	82° 48′	9.09807	Given cosine	9.09773
Next less	82° 49′	9.09707	Next less	9.09707
Difference		100	Difference	66

Then, 100 : 66 : : 60″ : 40″, which subtracted from 82° 49′, gives 82° 48′ 20″ for the answer.

It must be observed here, as in all other cases, that of the two angles, the less has the greater cosine.

The angle belonging to

the sin 9.20621 is	9° 15′ 6″	the tan 10.43434 is	69° 48′ 16″
the cos 9.98157	16° 34′ 30″	the cot 10.33554	24° 47′ 16″

Method of Supplying the Secants and Cosecants.

111. In some trigonometrical tables, the secants and cosecants are not inserted. But they may be easily obtained from the sines and cosines. For, by art. 93, proportion 3d,

$$\cos \times \sec = R^2.$$

That is, the product of the cosine and secant, is equal to the square of radius. But, in logarithms, addition takes the place of multiplication; and, in the tables of logarithmic sines, tangents, &c., the radius is 10. (Art. 103.) Therefore, in these tables,

$$\cos + \sec = 20. \quad \text{Or } \sec = 20 - \cos.$$

Again, by art. 93, proportion 6,

$$\sin \times \operatorname{cosec} = R^2.$$

Therefore, in the tables,

$$\sin + \operatorname{cosec} = 20. \quad \text{Or, } \operatorname{cosec} = 20 - \sin. \quad \text{Hence,}$$

112. To obtain the *secant*, subtract the cosine from 20; and to obtain the *cosecant*, subtract the sine from 20.

These subtractions are most easily performed, by taking the right hand figure from 10, and the others from 9, as in finding the arithmetical complement of a logarithm; (Art. 55.) observing however, to add 10 to the index of the secant or cosecant. In fact, the secant is the arithmetical complement of the cosine, with 10 added to the index.

For the secant $= 20 - \cos.$
And the arith. comp. of cos $= 10 - \cos.$ (Art. 54.)

So also the cosecant is the arithmetical complement of the sine, with 10 added to the index. The tables of secants and cosecants are, therefore, of use, in furnishing the arithmetical complement of the sine and cosine, in the following simple manner:

113. For the arithmetical complement of the *sine*, subtract 10 from the index of the cosecant; and for the arithmetical complement of the *cosine*, subtract 10 from the index of the secant.

By this, we may save the trouble of taking each of the figures from 9.

SECTION III.

SOLUTIONS OF RIGHT ANGLED TRIANGLES.

ART. 114. IN a triangle there are *six parts*, three sides, and three angles. In every trigonometrical calculation, it is necessary that some of these should be known, to enable us to find the others. *The number of parts which must be given, is* THREE, *one of which must be a* SIDE.

If only two parts be given, they will be either two sides, a side and an angle, or two angles; neither of which will limit the triangle to a particular form and size.

If *two sides* only be given, they may make any angle with each other; and may, therefore, be the sides of a thousand different triangles. Thus, the two lines *a* and *b* (Fig. 7.) may belong either to the triangle ABC, or ABC′, or ABC″. So that it will be impossible, from knowing two of the sides of a triangle, to determine the other parts.

Or, if a *side and an angle* only be given, the triangle will be indeterminate. Thus, if the side AB (Fig. 8.) and the angle at A be given; they may be parts either of the triangle ABC, or ABC′, or ABC″.

Lastly, if two *angles*, or even if *all* the angles be given, they will not determine the length of the sides. For the triangles ABC, A′B′C′, A″B″C″, (Fig. 9.) and a hundred others which might be drawn, with sides parallel to these, will all have the same angles. So that one of the parts given must always be a side. If this and any other two parts, either sides or angles, be known, the other three may be found, as will be shown, in this and the following section.

115. Triangles are either *right angled* or *oblique angled.* The calculations of the former are the most simple, and those which we have the most frequent occasion to make. A great portion of the problems in the mensuration of heights and distances, in surveying, navigation, and astronomy, are solved by rectangular trigonometry. Any triangle whatever may be divided into two right angled triangles, by drawing a perpendicular from one of the angles to the opposite side.

116. One of the six parts in a right angled triangle, is always given, viz. the right angle. This is a *constant* quantity; while the other angles and the sides are variable. It is also to be observed, that, if one of the *acute* angles is given, the other is known of course. For one is the complement of the other. (Art. 76, 77.) So that, *in a right angled triangle, subtracting one of the acute angles from* 90° *gives the other.* There remain, then, only *four* parts, one of the acute angles, and the three sides, to be sought by calculation. If any *two* of these be given, with the right angle, the others may be found.

117. To illustrate the method of calculation, let a case be supposed in which a right angled triangle CAD, (Fig. 10.) has one of its sides equal to the radius to which the trigonometrical tables are adapted.

In the first place, let the *base* of the triangle be equal to the tabular radius. Then, if a circle be described, with this radius, about the angle C as a center, DA will be the *tangent*, and DC the *secant* of that angle. (Art. 84, 85.) So that the radius, the tangent, and the secant of the angle at C, constitute the three sides of the triangle. The *tangent*, taken from the tables of natural sines, tangents, &c., will be the length of the *perpendicular;* and the *secant* will be the length of the *hypothenuse.* If the tables used be logarithmic, they will give the *logarithms* of the lengths of the two sides.

In the same manner, *any* right angled triangle whatever, whose base is equal to the radius of the tables, will have its other two sides found among the tangents and secants. Thus, if the quadrant AH, (Fig. 11.) be divided into portions of 15° each; then, in the triangle

CAD, AD will be the tan, and CD the sec of 15°,
In CAD′, AD′ will be the tan, and CD′ the sec of 30°,
In CAD″, AD″ will be the tan, and CD″ the sec of 45°, &c.

118. In the next place, let the *hypothenuse* of a right angled triangle CBF, (Fig. 12.) be equal to the radius of the tables. Then, if a circle be described, with the given radius, and about the angle C as a center; BF will be the *sine*, and BC the *cosine* of that angle. (Art. 82. 89.) Therefore the sine of the angle at C, taken from the tables, will be the length of the *perpendicular*, and the cosine will be the length of the *base.*

And any right angled triangle whatever, whose hypothenuse is equal to the tabular radius, will have its other two sides found among the sines and cosines. Thus, if the quadrant AH, (Fig. 13.) be divided into portions of 15° each, in the points F, F′, F″, &c.; then, in the triangle,

CBF, FB will be the sin, and CB the cos, of 15°,
In CB′F′, F′B′ will be the sin, and CB′ the cos, of 30°,
In CB″F″, F″B″ will be the sin, and CB″ the cos, of 45°, &c.

119. By merely *turning to the tables*, then, we may find the parts of any right angled triangle which has one of its sides equal to the radius of the tables. But for determining the parts of triangles which have *not* any of their sides equal to the tabular radius, the following proportion is used:

As the radius of one circle,
To the radius of any other;
So is a sine, tangent, or secant, in one,
To the sine, tangent, or secant, of the same number of degrees, in the other.

In the two concentric circles AHM, *ahm*, (Fig. 4.) the arcs AG and *ag*, contain the same number of degrees. (Art. 74.) The sines of these arcs are BG and *bg*, the tangents AD and *ad*, and the secants CD and C*d*. The four triangles, CAD, CBG, C*ad*, and C*bg*, are similar. For each of them, from the nature of sines and tangents, contains one right angle; the angle at C is common to them all; and the other acute angle in each is the complement of that at C. (Art. 77.) We have, then, the following proportions. (Euc. 4. 6.)

1. CG : C*g* : : BG : *bg*.

That is, one radius is to the other, as one *sine* to the other.

2. CA : C*a* : : DA : *da*.

That is, one radius is to the other, as one *tangent* to the other.

3. CA : C*a* : : CD : C*d*.

That is, one radius is to the other, as one *secant* to the other

Cor. BG : *bg* : : DA : *da* : : CD : C*d*.

That is, as the sine in one circle, to the sine in the other; so is the tangent in one, to the tangent in the other; and so is the secant in one, to the secant in the other.

This is a general principle, which may be applied to most trigonometrical calculations. If one of the sides of the proposed triangle be made radius, each of the other sides will be the sine, tangent, or secant, of an arc described by this radius. Proportions are then stated, between these lines, and the *tabular* radius, sine, tangent, &c.

120. A line is said to be *made radius*, when a circle is described, or supposed to be described, whose semi-diameter is equal to the line, and whose center is at one end of it.

121. In any right angled triangle, *if the* HYPOTHENUSE *be made radius, one of the legs will be a* SINE *of its opposite angle, and the other leg a* COSINE *of the same angle.*

Thus, if to the triangle ABC (Fig. 14.) a circle be applied, whose radius is AC, and whose center is A, then BC will be the *sine*, and BA the *cosine*, of the angle at A. (Art. 82, 89.)

If, while the same line is radius, the other end C be made the center, then BA will be the *sine*, and BC the *cosine*, of the angle at C.

122. *If either of the* LEGS *be made radius, the other leg will be a* TANGENT *of its opposite angle, and the hypothenuse will be a* SECANT *of the same angle;* that is, of the angle between the secant and the radius.

Thus, if the *base* AB (Fig. 15.) be made radius, the center being at A, BC will be the *tangent*, and AC the *secant*, of the angle at A. (Art. 84, 85.)

But, if the *perpendicular* BC, (Fig. 16.) be made radius, with the center at C, then AB will be the *tangent*, and AC the *secant*, of the angle at C.

123. As the side which is the sine, tangent, or secant of one of the acute angles, is the cosine, cotangent, or cosecant of the other; (Art. 89.) the *perpendicular* BC (Fig. 14.) is the *sine* of the angle A, and the *cosine* of the angle C; while the *base* AB, is the *sine* of the angle C, and the *cosine* of the angle A.

If the base is made radius, as in Fig. 15, the *perpendicular* BC is the *tangent* of the angle A, and the *cotangent* of the angle C; while the *hypothenuse* is the *secant* of the angle A, and the *cosecant* of the angle C.

If the perpendicular is made radius, as in Fig. 16, the *base* AB is the *tangent* of the angle C, and the *cotangent* of the angle A; while the *hypothenuse* is the *secant* of the angle C, and the *cosecant* of the angle A.

124. Whenever a right angled triangle is proposed, whose sides or angles are required; a *similar* triangle may be formed, from the sines, tangents, &c., of the *tables*. (Art. 117, 118.) The parts required are then found, by stating proportions between the similar sides of the two triangles. If the triangle proposed be ABC, (Fig. 17.) another, *abc*, may be formed, having the same angles with the first, but differing from it in the length of its sides, so as to correspond with the numbers in the tables. If similar sides be made radius in both, the remaining similar sides will be lines of *the same name;* that is, if the perpendicular in one of the triangles be a *sine*, the perpendicular in the other will be a sine; if the base in one be a *cosine*, the base in the other will be a cosine, &c.

If the *hypothenuse* in each triangle be made radius, as in Fig. 14, the perpendicular *bc*, will be the *tabular sine* of the angle at *a;* and the perpendicular BC, will be a sine of the equal angle A, in a circle of which AC is radius.

If the *base* in each triangle be made radius, as in Fig. 15, then the perpendicular *bc*, will be the *tabular tangent* of the angle at *a;* and BC will be a tangent of the equal angle A, in a circle of which AB, is radius, &c.

125. From the relations of the similar sides of these triangles, are derived the two following *theorems*, which are sufficient for calculating the parts of any right angled triangle whatever, when the requisite data are furnished. One is used, when a *side* is to be found; the other, when an *angle* is to be found.

Theorem I.

126. When a *side* is required;

AS THE TABULAR SINE, TANGENT, &C., OF THE SAME NAME WITH THE GIVEN SIDE,
TO THE GIVEN SIDE;
SO IS THE TABULAR SINE, TANGENT, &C., OF THE SAME NAME WITH THE REQUIRED SIDE,
TO THE REQUIRED SIDE.

It will be readily seen, that this is nothing more than a statement, in general terms, of the proportions between the similar sides of two triangles, one proposed for solution, and the other formed from the numbers in the tables.

Thus, if the hypothenuse be *given*, and the base or perpendicular be *required;* then, in Fig. 14, where *ac* is the tabular radius, *bc* the tabular sine of *a*, or its equal A, and *ab* the tabular sine of C; (Art. 124.)

ac : AC : : *bc* : BC, that is, R : AC : : sin A : BC.
ac : AC : : *ab* : AB, R : AC : : sin C : AB.

In Fig. 15, where *ab* is the tabular radius, *ac* the tabular secant of A, and *bc* the tabular tangent of A;

ac : AC : : *bc* : BC, that is, sec A : AC : : tan A : BC.
ac : AC : : *ab* : AB, sec A : AC : : R : AB.

In Fig. 16, where *bc* is the tabular radius, *ac* the tabular secant of C, and *ab* the tabular tangent of C;

ac : AC : : *bc* : BC, that is, sec C : AC : : R : BC.
ac : AC : : *ab* : AB, sec C : AC : : tan C : AB.

THEOREM II.

127. When an *angle* is required;

AS THE GIVEN SIDE MADE RADIUS,
TO THE TABULAR RADIUS;
SO IS ANOTHER GIVEN SIDE,
TO THE TABULAR SINE, TANGENT, &C., OF THE SAME NAME.

Thus, if the side made radius, and one other side be given, then, in Fig. 14,

AC : *ac* : : BC : *bc*, that is, AC : R : : BC : sin A.
AC : *ac* : : AB : *ab*, AC : R : : AB : sin C.

In Fig. 15,

AB : *ab* : : BC : *bc*, that is AB : R : : BC : tan A.
AB : *ab* : : AC : *ac* AB : R : : AC : sec A.

In Fig. 16,

BC : *bc* : : AB : *ab*, that is, BC : R : : AB : tan C.
BC : *bc* : : AC : *ac*, BC : R : : AC : sec C.

It will be observed, that in these theorems, *angles* are not introduced, though they are among the quantities which are either given or required, in the calculation of triangles. But the tabular sines, tangents, &c., may be considered the *representatives* of angles, as one may be found from the other, by merely turning to the tables.

128. In the theorem for finding a *side*, the first term of the proportion is a *tabular number*. But, in the theorem for finding an *angle*, the first term is a side. Hence, in applying the proportions to particular cases, this rule is to be observed;

To find a SIDE, *begin with a tabular number,*
To find an ANGLE, *begin with a side.*

Radius is to be reckoned among the tabular numbers.

129. In the theorem for finding an *angle*, the first term is a *side made radius.* As in every proportion, the three first terms must be given, to enable us to find the fourth, it is evident, that where this theorem is applied, the side made radius must be a *given* one. But, in the theorem for finding a *side*, it is not necessary that either of the terms should be radius. Hence,

130. *To find a* SIDE, ANY *side may be made radius.*
To find an ANGLE, *a* GIVEN *side must be made radius.*

It will generally be expedient, in both cases, to make radius one of the terms in the proportion; because, in the tables of natural sines, tangents, &c., radius is 1, and in the logarithmic tables it is 10. (Art. 103.)

131. The proportions in Trigonometry are of the same nature as other simple proportions. The fourth term is found, therefore, as in the Rule of Three in Arithmetic, by *multiplying together the second and third terms, and dividing their product by the first term.* This is the mode of calculation, when the tables of *natural* sines, tangents, &c., are used. But the operation by logarithms is so much more expeditious, that it has almost entirely superseded the other method. In logarithmic calculations, addition takes the place of multiplication; and subtraction the place of division.

The logarithms expressing the lengths of the *sides* of a triangle, are to be taken from the tables of common logarithms. The logarithms of the *sines, tangents, &c.*, are found in the tables of artificial sines, &c. The calculation is then made *by adding the second and third terms, and subtracting the first.* (Art. 52.)

132. The logarithmic radius 10, or, as it is written in the tables, 10.00000, is so easily added and subtracted, that the three terms of which it is one, may be considered as, in effect, reduced to two. Thus, if the tabular radius is in the *first* term, we have only to add the other two terms, and then take 10 from the index; for this is subtracting the first term. If radius occurs in the *second* term, the first is to be subtracted from the third, after its index is increased by 10. In the same manner, if radius is in the *third* term, the first is to be subtracted from the second.

133. Every species of right angled triangles may be solved upon the principle, that the sides of similar triangles are proportional, according to the two theorems mentioned above. There will be some advantages, however, in giving the examples in distinct classes.

There must be given, in a right angled triangle, *two* of the parts, besides the right angle. (Art. 116.) These may be;

1. The hypothenuse and an angle; or
2. The hypothenuse and a leg; or
3. A leg and an angle; or
4. The two legs.

Case I.

134. Given { The hypothenuse, And an angle, } to find { The base and Perpendicular.

Ex. 1. If the hypothenuse AC, (Fig. 17.*) be 45 miles, and the angle at A 32° 20′, what is the length of the base AB, and the perpendicular BC?

In this case, as *sides* only are required, *any* side may be made radius. (Art. 130.)

If the hypothenuse be made radius, as in Fig. 14, BC will be the sine of A, and AB the sine of C, or the cosine of A. (Art. 121.) And if *abc* be a similar triangle, whose hypothenuse is equal to the *tabular* radius, *bc* will be the tabular sine of A, and *ab* the tabular sine of C. (Art. 124.)

* The parts which are *given* are distinguished by a mark across the line, or at the opening of the angle, and the parts *required*, by a cipher.

10

To find the *perpendicular*, then, by Theorem I, we have this proportion;

$ac : AC :: bc : BC.$
Or $R : AC :: \text{Sin } A : BC.$

Whenever the terms Radius, Sine, Tangent, &c., occur in a proportion like this, the *tabular* Radius, &c., is to be understood, as in Arts. 126, 127.

The numerical calculation, to find the length of BC, may be made, either by *natural* sines, or by *logarithms*. See Art. 131.

By natural Sines.

$1 : 45 :: 0.53484 : 24.068 = BC.$

Computation by Logarithms.

As radius		10.00000
To the hypothenuse	45	1.65321
So is the Sine of A	32° 20′	9.72823
To the perpendicular	24.068	1.38144

Here the logarithms of the second and third terms are added, and from the sum, the first term 10 is subtracted. (Art. 132.) The remainder is the logarithm of 24.068=BC.

Subtracting the angle at A from 90°, we have the angle at C=57° 40′. (Art. 116.) Then to find the *base* AB;

$ac : AC :: ab : AB$
Or $R : AC :: \text{Sin } C : AB = 38.023.$

Both the sides required are now found, by making the hypothenuse radius. The results here obtained may be verified, by making either of the other sides radius.

If the *base* be made radius, as in Fig. 15, the perpendicular will be the *tangent*, and the hypothenuse the *secant* of the angle at A. (Art. 122.) Then,

$\text{Sec } A : AC :: R : AB$
$R : AB :: \text{Tan } A : BC$

By making the arithmetical calculations, in these two proportions, the values of AB and BC, will be found the same as before.

If the *perpendicular* be made radius, as in Fig. 16, AB will be the *tangent*, and AC the *secant* of the angle at C. Then,

Sec C : AC : : R : BC
R : BC : : Tan C : AB

Ex. 2. If the hypothenuse of a right angled triangle be 250 rods, and the angle at the base 46° 30′; what is the length of the base and perpendicular?

Ans. The base is 172.1 rods, and the perpendic. 181.35.

CASE II.

135. Given { The hypothenuse, And one leg } to find { The angles and The other leg.

Ex. 1. If the hypothenuse (Fig. 18.) be 35 leagues, and the base 26; what is the length of the perpendicular, and the quantity of each of the acute angles?

To find the angles it is necessary that one of the *given* sides be made radius. (Art. 130.)

If the *hypothenuse* be radius, the base and perpendicular will be sines of their opposite angles. Then,

AC : R : : AB : Sin C=47° 58½′

And to find the *perpendicular* by theorem I;

R : AC : : Sin A : BC=23.43

If the *base* be radius, the perpendicular will be *tangent*, and the hypothenuse *secant* of the angle at A. Then,

AB : R : : AC : Sec A
R : AB : : Tan A : BC

In this example, where the hypothenuse and base are given, the angles can not be found by making the *perpendicular* radius. For to find an angle, a *given* side must be made radius. (Art. 130.)

136. Ex. 2. If the hypothenuse (Fig. 19.) be 54 miles, and the perpendicular 48 miles, what are the angles, and the base?

Making the *hypothenuse* radius.

AC : R : : BC : Sin A
R : AC : : Sin C : AB

The numerical calculation will give A=62° 44′ and AB =24.74.

Making the *perpendicular* radius.

BC : R : : AC : Sec C
R : BC : : Tan C : AB

The angles cannot be found by making the *base* radius, when its length is not given.

Case III.

137. Given { The angles, And one leg } to find { The hypothenuse, And the other leg.

Ex. 1. If the base (Fig. 20.) be 60, and the angle at the base 47° 12′, what is the length of the hypothenuse and the perpendicular?

In this case, as *sides* only are required, *any* side may be radius.

Making the *hypothenuse* radius.

Sin C : AB : : R : AC=88.31
R : AC : : Sin A : BC=64.8

Making the *base* radius.

R : AB : : Sec A : AC
R : AB : : Tan A : BC

Making the *perpendicular* radius.

Tan C : AB : : R : BC
R : BC : : Sec C : AC

138. Ex. 2. If the perpendicular (Fig. 21.) be 74, and the angle C 61° 27′, what is the length of the base and the hypothenuse?

Making the *hypothenuse* radius.

Sin A : BC : : R : AC
R : AC : : sin C : AB

Making the *base* radius.

Tan A : BC : : R : AB
R : AB : : sec A : AC

Making the *perpendicular* radius.

R : BC : : sec C : AC
R : BC : : tan C : AB

The hypothenuse is 154.83 and the base 136.

Case IV.

139. Given { The base, and Perpendicular } to find { The hypothenuse, And the angles.

Ex. 1. If the base (Fig. 22.) be ~~284~~ 285. and the perpendicular 193 ~~192~~, what are the angles, and the hypothenuse?

In this case, one of the legs must be made radius, to find an angle; because the hypothenuse is not given.

Making the *base* radius.

AB : R : : BC : tan A=34° 4′
R : AB : : sec A : AC=342.84

Making the *perpendicular* radius.

BC : R : : AB : tan C
R : BC : : sec C : AC

Ex. 2. If the base be 640, and the perpendicular 480, what are the angles and hypothenuse?

Ans. The hypothenuse is 800, and the angle at the base 36° 52′ 12″.

Examples for Practice.

1. Given the hypothenuse 68, and the angle at the base 39° 17′; to find the base and perpendicular.
2. Given the hypothenuse 850, and the base 594, to find the angles, and the perpendicular.
3. Given the hypothenuse 78, and perpendicular 57, to find the base, and the angles.
4. Given the base 723, and the angle at the base 64° 18′, to find the hypothenuse and perpendicular.
5. Given the perpendicular 632, and the angle at the base 81° 36′, to find the hypothenuse and the base.
6. Given the base 32, and the perpendicular 24, to find the hypothenuse, and the angles.

140. The preceding solutions are all effected, by means of the tabular sines, tangents, and secants. But, when any *two sides* of a right angled triangle are given, the third side may be found, without the aid of the trigonometrical tables, by the proposition, that *the square of the hypothenuse is equal to the sum of the squares of the two perpendicular sides.* (Euc. 47. 1.)

If the legs be given, extracting the square root of the *sum* of their squares, will give the hypothenuse. Or, if the hypothenuse and one leg be given, extracting the square root of the *difference* of the squares, will give the other leg.

Let h = the hypothenuse, p = the perpendicular, b = the base, of a right angled triangle.

Then	$h^2=b^2+p^2$, or (Alg. 296.)	$h=\sqrt{b^2+p^2}$
By trans.	$b^2=h^2-p^2$, or	$b=\sqrt{h^2-p^2}$
And	$p^2=h^2-b^2$, or	$p=\sqrt{h^2-b^2}$

Ex. 1. If the base is 32, and the perpendicular 24, what is the hypothenuse? Ans. 40.

2. If the hypothenuse is 100, and the base 80, what is the perpendicular? Ans. 60.

3. If the hypothenuse is 300, and the perpendicular 220, what is the base?

Ans. $300^2-\overline{220}^2=4160$, the root of which is 204 nearly.

141. It is generally most convenient to find the difference of the squares by *logarithms*. But this is not to be done by *subtraction*. For subtraction, in logarithms, performs the office of *division*. (Art. 41.) If we subtract the logarithm of b^2 from the logarithm of h^2, we shall have the logarithm, not of the *difference* of the squares, but of their *quotient*. There is, however, an indirect, though very simple method, by which the difference of the squares may be obtained by logarithms. It depends on the principle, that the *difference of the squares of two quantities is equal to the product of the sum and difference of the quantities.* (Alg. 235.) Thus,

$$h^2 - b^2 = (h+b) \times (h-b)$$

as will be seen at once, by performing the multiplication. The two factors may be multiplied by *adding* their logarithms. Hence,

142. *To obtain the difference of the squares of two quantities, add the logarithm of the sum of the quantities, to the logarithm of their difference.* After the logarithm of the difference of the squares is found; the *square root* of this difference is obtained, by dividing the logarithm by 2. (Art. 47.)

Ex. 1. If the hypothenuse be 75 inches, and the base 45, what is the length of the perpendicular?

Sum of the given sides	120	log.	2.07918
Difference of do.	30		1.47712
	Dividing by		2)3.55630
Side required	60		1.77815

2. If the hypothenuse is 135, and the perpendicular 108, what is the length of the base? Ans. 81.

SECTION IV.

SOLUTIONS OF OBLIQUE ANGLED TRIANGLES.

ART. 143. THE sides and angles of oblique angled triangles may be calculated by the following theorems.

THEOREM I.

In any plane triangle, THE SINES OF THE ANGLES ARE AS THEIR OPPOSITE SIDES.

Let the angles be denoted by the letters A, B, C, and their opposite sides by a, b, c, as in Fig. 23 and 24. From one of the angles, let the line p be drawn perpendicular to the opposite side. This will fall either within or without the triangle.

1. Let it fall *within* as in Fig. 23. Then, in the right angled triangles ACD, and BCD, according to art. 126,

$$R : b :: \sin A : p$$
$$R : a :: \sin B : p$$

Here, the two *extremes* are the same in both proportions. The other four terms are, therefore, *reciprocally* proportional: (Alg. 387.*) that is,

$$a : b :: \sin A : \sin B.$$

2. Let the perpendicular p fall *without* the triangle, as in Fig. 24. Then, in the right angled triangles ACD and BCD;

$$R : b :: \sin A : p$$
$$R : a :: \sin B : p$$

Therefore as before,

$$a : b :: \sin A : \sin B.$$

* Euclid 23 5.

Sin A is here put both for the sine of DAC, and for that of BAC. For, as one of these angles is the *supplement* of the other, they have the same sine. (Art. 90.)

The sines which are mentioned here, and which are used in calculation, are *tabular* sines. But the proportion will be the same, if the sines be adapted to any other radius. (Art. 119.)

THEOREM II.

144. In a plane triangle,

AS THE SUM OF ANY TWO OF THE SIDES,
TO THEIR DIFFERENCE;
SO IS THE TANGENT OF HALF THE SUM OF THE OPPOSITE ANGLES;
TO THE TANGENT OF HALF THEIR DIFFERENCE.

Thus, the sum of AB and AC, (Fig. 25.) is to their difference; as the tangent of half the sum of the angles ACB and ABC, to the tangent of half their difference.

Demonstration.

Extend CA to G, making AG equal to AB; then CG is *the sum of the two sides* AB and AC. On AB, set off AD, equal to AC; then BD *is the difference of the sides* AB and AC.

The sum of the two angles ACB and ABC, is equal to the sum of ACD and ADC; because each of these sums is the supplement of CAD. (Art. 79.) But as AC=AD by construction, the angle ADC=ACD. (Euc. 5. 1.) Therefore ACD is *half the sum* of ACB and ABC. As AB=AG, the angle AGB=ABG, or DBE. Also, GCE, or ACD=ADC=BDE. (Euc. 15. 1.) Therefore, in the triangles GCE, and DBE, the two remaining angles DEB, and CEG, are equal; (Art. 79.) So that CE is perpendicular to BG. (Euc. Def. 10. 1.) If then CE is made radius, GE is the tangent of GCE, (Art. 84.) that is, *the tangent of half the sum of the angles opposite to* AB and AC.

If from the greater of the two angles ACB and ABC, there be taken ACD their half sum; the remaining angle ECB will be their half difference. (Alg. 341.) The tangent of this angle, CE being radius, is EB, that is, *the tangent of half the difference of the angles opposite to* AB and AC. We have then,

CG=the sum of the sides AB and AC;
DB=their difference;
GE=the tangent of half the sum of the opposite angles;
EB=the tangent of half their difference.

But by similar triangles,

CG : DB : GE : EB. Q. E. D.

Theorem III.

145. If upon the longest side of a triangle, a perpendicular be drawn from the opposite angle;

As the longest side,
To the sum of the two others;
So is the difference of the latter,
To the difference of the segments made by the perpendicular.

In the triangle ABC, (Fig. 26.) if a perpendicular be drawn from C upon AB;

AB : CB+CA : : CB—CA : BP—PA.*

Demonstration.

Describe a circle on the center C, and with the radius BC. Through A and C, draw the diameter LD, and extend BA to H. Then by Euc. 35. 3.

AB×AH=AL×AD

Therefore,

AB : AD : : AL : AH

But AD=CD+CA=CB+CA
And AL=CL—CA=CB—CA
And AH=HP—PA=BP—PA (Euc. 3. 3.)

If, then, for the three last terms in the proportion, we substitute their equals, we have,

AB : CB+CA : : CB—CA : PB—PA.

146. It is to be observed, that the greater segment is next the greater side. If BC is greater than AC, (Fig. 26.) PB is

* See note F.

greater than AP. With the radius AC, describe the arc AN. The segment NP=AP. (Euc. 3. 3.) But BP is greater than NP.

147. The two segments are to each other, as the tangents of the opposite angles, or the cotangents of the adjacent angles. For, in the right angled triangles ACP, and BCP, (Fig. 26.) if CP be made radius, (Art. 126.)

R : PC : : Tan ACP : AP
R : PC : : Tan BCP : BP

Therefore, by equality of ratios, (Alg. 384.*)

Tan ACP : AP : : Tan BCP : BP

That is, the segments are as the tangents of the opposite angles. And the tangents of these are the *cotangents* of the adjacent angles A and B. (Art. 89.)

Cor. The greater segment is oppósite to the greater angle. And of the angles at the base, the less is next the greater side. If BP is greater than AP, the angle BCP is greater than ACP; and B is less than A. (Art. 77.)

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148. To enable us to find the sides and angles of an oblique angled triangle, *three* of them must be *given*. (Art. 114.)

These may be, either

1. Two angles and a side, or
2. Two sides and an angle *opposite* one of them, or
3. Two sides and the *included* angle, or
4. The three sides.

The two first of these cases are solved by theorem I, (Art. 143.) the third by theorem II, (Art. 144.) and the fourth by theorem III, (Art. 145.)

149. In making the calculations, it must be kept in mind, that the greater side is always opposite to the greater angle, (Euc. 18, 19. 1.) that there can be only one *obtuse* angle in a triangle, (Art. 76.) and, therefore, that the angles opposite to the two least sides must be *acute*.

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* Euc. 11. 5.

Case I.

9
150. Given,

Two angles, and A side,	to find	The remaining angle, and The other two sides.

The third angle is found by merely subtracting the sum of the two which are given from 180°. (Art. 79.)

The sides are found, by stating, according to theorem I, the following proportion;

As the sine of the angle opposite the ***given*** side,
To the length of the given side;
So is the sine of the angle opposite the ***required*** side,
To the length of the required side.

As a ***side*** is to be found, it is necessary to begin with a ***tabular number.***

Ex. 1. In the triangle ABC, (Fig. 27.) the side b is given 32 rods, the angle A 56° 20′, and the angle C 49° 10′, to find the angle B, and the sides a and c.

The sum of the two given angles 56° 20′+49° 10′=105° 30′; which subtracted from 180°, leaves 74° 30′ the angle B.

Then,

$$\text{Sin B} : b : : \begin{cases} \text{Sin A} : a \\ \text{Sin C} : c \end{cases}$$

Calculation by logarithms.

As the sine of B	74° 30′	*a. c.*	0.01609
To the side b	32		1.50515
So is the sine of A	56° 20′		9.92027
To the side a	27.64		1.44151

As the sine of B	74° 30′	*a. c.*	0.01609
To the side b	32		1.50515
So is the sine of C	49° 10′		9.87887
To the side c	25.13		1.40011

The *arithmetical complement* used in the first term here, may be found, in the usual way, or by taking out the *cosecant* of the given angle, and rejecting 10 from the index. (Art. 113.)

Ex. 2. Given the side b 71, the angle A 107° 6′, and the angle C 27° 40′; to find the angle B, and the sides a and c. The angle B is 45° 14′. Then,

$$\text{Sin B} : b :: \begin{cases} \text{Sin A} : a = 95.58 \\ \text{Sin C} : c = 46.43 \end{cases}$$

When one of the given angles is *obtuse*, as in this example, the sine of its *supplement* is to be taken from the tables. (Art. 99.)

Case II.

151. Given,

Two sides, and An opposite angle,	to find	The remaining side, and The other two angles.

One of the required angles is found, by beginning with a side, and, according to Theorem I, stating the proportion,

> As the side opposite the given angle,
> To the sine of that angle;
> So is the side opposite the required angle,
> To the sine of that angle.

The third angle is found, by subtracting the sum of the other two from 180°; and the remaining side is found, by the proportion in the preceding article.

152. In this second case, if the side opposite to the given angle be shorter than the other given side the solution will be *ambiguous*. Two different triangles may be formed, each of which will satisfy the conditions of the problem.

Let the side b, (Fig. 28.) the angle A, and the length of the side opposite this angle, be given. With the latter for radius, (if it be shorter than b,) describe an arc, cutting the line AH in the points B and B′. The lines BC and B′C, will be equal. So that, with the same data, there may be formed two different triangles, ABC and AB′C.

There will be the same ambiguity in the numerical calculation. The answer found by the proportion will be the *sine* of an angle. But this may be the sine, either of the *acute* angle AB′C, or of the *obtuse* angle ABC. For, BC being equal to B′C, the angle CB′B is equal to CBB′. Therefore ABC, which is the supplement of CBB′, is also the supplement of CB′B. But the sine of an angle is the same, as the sine of its supplement. (Art. 90.) The result of the calculation will, therefore, be ambiguous. In practice, however, there will generally be some circumstances which will determine whether the angle required is acute or obtuse.

If the side opposite the given angle be *longer* than the other given side, the angle which is subtended by the latter, will necessarily be acute. For there can be but one obtuse angle in a triangle, and this is always subtended by the longest side. (Art. 149.)

If the *given* angle be obtuse, the other two will, of course, be acute. There can, therefore, be no ambiguity in the solution.

Ex. 1. Given the angle A, (Fig. 28.) 35° 20′, the opposite side *a* 50, and the side *b* 70; to find the remaining side, and the other two angles.

To find the angle opposite to *b*, (Art. 151.)

$$a : \sin A : : b : \sin B$$

The calculation here gives the acute angle AB′C 54° 3′ 50″, and the obtuse angle ABC 125° 56′ 10″. If the latter be added to the angle at A 35° 20′, the sum will be 161° 16′ 10″, the supplement of which, 18° 43′ 50″, is the angle ACB. Then in the triangle ABC, to find the side c=AB,

$$\text{Sin } A : a : : \sin C : c = 27.76$$

If the *acute* angle AB′C 54° 3′ 50″ be added to the angle at A 35° 20′, the sum will be 89° 23′ 50″, the supplement of which, 90° 36′ 10″, is the angle ACB′. Then, in the triangle AB′C,

$$\text{Sin } A : CB' : : \sin C : AB' = 86.45.$$

Ex. 2. Given the angle at A, 63° 35′, (Fig. 29.) the side *b* 64, and the side *a* 72; to find the side *c*, and the angles B and C.

a : sin A : : b : sin B=52° 45′ 25″
Sin A : a : : sin C : c=72.05

The sum of the angles A and B, is 116° 20′ 25″, the supplement of which, 63° 39′ 35″, is the angle C.

In this example the solution is *not ambiguous*, because the side opposite the given angle is longer than the other given side.

Ex. 3. In a triangle of which the angles are A, B, and C, and the opposite sides a, b, and c, as before; if the angle A be 121° 40′, the opposite side a 68 rods, and the side b 47 rods; what are the angles B and C, and what is the length of the side c? Ans. B is 36° 2′ 4″, C 22° 17′ 56″, and c 30.3.

In this example also, the solution is not ambiguous, because the *given* angle is obtuse.

Case III.

153. Given,

Two sides, and } to find { The remaining side, and
The included angle, } { The other two angles.

In this case, the angles are found by theorem II. (Art. 144.) The required side may be found by theorem I.

In making the solutions, it will be necessary to observe, that by subtracting the given angle from 180°, the *sum* of the other two angles is found; (Art. 79.) and, that *adding half the difference of two quantities to their half sum gives the greater quantity, and subtracting the half difference from the half sum gives the less.* (Alg. 341.) The latter proposition may be geometrically demonstrated thus;

Let AE, (Fig. 32.) be the greater of two magnitudes, and BE the less. Bisect AB in D, and make AC equal to BE. Then,

AB is the *sum* of the two magnitudes;
CE their *difference*;
DA or DB *half* their *sum*;
DE or DC *half* their *difference*;
But DA+DE=AE the *greater* magnitude,
And DE--DE=BE the *less*.

Ex. 1. In the triangle ABC, (Fig. 30.) the angle A is given

26° 14′, the side *b* 39, and the side *c* 53; to find the angles B and C, and the side *a*.

The *sum* of the sides *b* and *c* is	53+39=92
And their *difference*	53—39=14
The sum of the angles B and C=180°—26° 14′=	153° 46′
And *half the sum* of B and C is	76° 53′

Then, by theorem II,

$(b+c):(b-c)::\tan\frac{1}{2}(B+C):\tan\frac{1}{2}(B-C)$

To and from the half sum	76°	53′	
Adding and subtracting the half difference	33	8	50
We have the greater angle	110	1	50
And the less angle	43	44	10

As the greater of the two given sides is *c*, the greater angle is C, and the less angle B. (Art. 149.)

To find the side *a*, by theorem I.

Sin B : *b* : : sin A : *a*=24.94.

Ex. 2. Given the angle A 101° 30′, the side *b* 76, and the side *c* 109; to find the angles B and C, and the side *a*.

B is 30° 57½′, C 47° 32½′, and *a* 144.8.

Case IV.

154. Given the three sides, to find the angles.

In this case, the solutions may be made, by drawing a perpendicular to the longest side, from the opposite angle. This will divide the given triangle into two *right angled* triangles. The two segments may be found by theorem III. (Art. 145.) There will then be given, in each of the right angled triangles, the hypothenuse and one of the legs, from which the angles may be determined, by rectangular trigonometry. (Art. 135.)

Ex. 1. In the triangle ABC, (Fig. 31.) the side AB is 39, AC 35, and BC 27. What are the angles?

Let a perpendicular be drawn from C, dividing the longest side AB into the two segments AP and BP. Then, by theorem III,

$$AB : AC+BC :: AC-BC : AP-BP$$

As the longest side	39 *a. c.*	8.40894
To the sum of the two others	62	1.79239
So is the difference of the latter	8	0.90309
To the difference of the segments	12.72	1.10442

The greater of the two segments is AP, because it is next the side AC, which is greater than BC. (Art. 146.)

To and from half the sum of the segments	19.5
Adding and subtracting half their difference, (Art. 153.)	6.36
We have the greater segment AP	25.86
And the less BP	13.14

Then, in each of the right angled triangles APC and BPC, we have given the hypothenuse and base; and by art. 135.

$$AC : R :: AP : \cos A = 42° 21' 57''$$
$$BC : R :: BP : \cos B = 60° 52' 42''$$

And subtracting the sum of the angles A and B from 180°, we have the remaining angle ACB=76° 45′ 21″.

Ex. 2. If the three sides of a triangle are 78, 96, and 104; what are the angles?

Ans. 45° 41′ 48″, 61° 43′ 27″, and 72° 34′ 45″.

Examples for Practice.

1. Given the angle A 54° 30′, the angle B 63° 10′, and the side *a* 164 rods; to find the angle C, and the sides *b* and *c*.
2. Given the angle A 45° 6′, the opposite side *a* 93, and the side *b* 108; to find the angles B and C, and the side *c*.
3. Given the angle A 67° 24′, the opposite side *a* 62, and the side *b* 46; to find the angles B and C, and the side *c*.
4. Given the angle A 127° 42′, the opposite side *a* 381, and the side *b* 184; to find the angles B and C, and the side *c*.

5. Given the side *b* 58, the side *c* 67, and the included angle A=36°; to find the angles B and C, and the side *a*.
6. Given the three sides, 631, 268, and 546; to find the angles.

155. The three theorems demonstrated in this section, have been here applied to *oblique angled* triangles only. But they are equally applicable to *right angled* triangles.

Thus, in the triangle ABC, (Fig. 17.) according to theorem I, (Art. 143.)

Sin B : AC : : sin A : BC

This is the same proportion as one stated in art. 134, except that, in the first term here, the *sine of* B is substituted for *radius*. But, as B is a right angle, its sine is *equal* to *radius*. (Art. 95.)

Again, in the triangle ABC, (Fig. 21.) by the same theorem;

Sin A : BC : : sin C : AB

This is also one of the proportions in rectangular trigonometry, when the hypothenuse is made radius.

The other two theorems might be applied to the solution of right angled triangles. But, when one of the angles is *known* to be a right angle, the methods explained in the preceding section, are much more simple in practice.*

* For the application of Trigonometry to the Mensuration of Heights and Distances, see Navigation and Surveying.

SECTION V.

GEOMETRICAL CONSTRUCTION OF TRIANGLES, BY THE PLANE SCALE.

ART. 156. To facilitate the construction of geometrical figures, a number of graduated lines are put upon the common two feet scale; one side of which is called the *Plane Scale*, and the other side, *Gunter's Scale*. The most important of these are the scales of *equal parts*, and the line of *chords*. In forming a given triangle, or any other right lined figure, the parts which must be made to agree with the conditions proposed, are the *lines*, and the *angles*. For the former, a scale of equal parts is used; for the latter, a line of chords.

157. The line on the upper side of the plane scale, is divided into *inches* and *tenths* of an inch. Beneath this, on the left hand, are two *diagonal* scales of equal parts,* divided into inches and half inches, by perpendicular lines. On the larger scale, one of the inches is divided into tenths, by lines which pass *obliquely* across, so as to intersect the parallel lines which run from right to left. The use of the oblique lines is to measure *hundredths* of an inch, by inclining more and more to the right, as they cross each of the parallels.

To take off, for instance, an extent of 3 inches, 4 tenths, and 6 hundredths;

Place one foot of the compasses at the intersection of the perpendicular line marked 3 with the parallel line marked 6, and the other foot at the intersection of the latter with the oblique line marked 4.

The other diagonal scale is of the same nature. The divisions are smaller, and are numbered from left to right.

158. In geometrical constructions, what is often required, is to make a figure, not *equal* to a given one, but only *similar*. Now figures are similar which have equal angles, and the

* These lines are not represented in the plate, as the learner is supposed to have the scale before him

sides about the equal angles *proportional.* (Euc. Def. 1. 6.) Thus a land surveyor, in plotting a field, makes the several lines in his plan to have the same proportion to each other, as the sides of the field. For this purpose a scale of equal parts may be used, of any dimensions whatever. If the sides of the field are 2, 5, 7, and 10 *rods*, and the lines in the plan are 2, 5, 7, and 10 *inches*, and if the angles are the same in each, the figures are similar. One is a copy of the other, upon a smaller scale.

So any two right lined figures are similar, if the angles are the same in both, and if the number of smaller parts in each side of one, is equal to the number of larger parts in the corresponding sides of the other. The several divisions on the scale of equal parts may, therefore, be considered as representing any measures of length, as feet, rods, miles, &c. All that is necessary is, that the scale be not changed, in the construction of the same figure; and that the several divisions and subdivisions be properly proportioned to each other. If the larger divisions, on the diagonal scale, are units, the smaller ones are tenths and hundredths. If the larger are tens, the smaller are units and tenths.

159. In laying down an *angle*, of a given number of degrees, it is necessary to *measure* it. Now the proper measure of an angle is an arc of a circle. (Art. 74.) And the measure of an arc, where the radius is given, is its *chord.* For the chord is the distance, in a straight line, from one end of the arc to the other. Thus the chord AB, (Fig. 33.) is a measure of the arc ADB, and of the angle ACB.

To form the *line of chords*, a circle is described, and the lengths of its chords determined for every degree of the quadrant. These measures are put on the plane scale, on the line marked CHO.

160. The chord of 60° is equal to *radius.* (Art. 95.) In laying down or measuring an angle, therefore, an arc must be drawn, with a radius which is equal to the extent from 0 to 60 on the line of chords. There are generally on the scale, two lines of chords. Either of these may be used; but the angle must be measured by the same line from which the radius is taken.

161. To *make an angle*, then, of a given number of degrees; from one end of a straight line as a center, and with a radius equal to the chord of 60° on the line of chords, describe an arc of a circle cutting the straight line. From the

point of intersection, extend the chord of the given number of degrees, applying the other extremity to the arc; and through the place of meeting, draw the other line from the angular point.

If the given angle is *obtuse*, take from the scale the chord of *half* the number of degrees, and apply it *twice* to the arc. Or make use of the chords of any two arcs whose *sum* is equal to the given number of degrees.

A *right angle* may be constructed, by drawing a perpendicular without using the line of chords.

Ex. 1. To make an angle of 32 degrees. (Fig. 33.) With the point C, in the line CH, for a center, and with the chord of 60° for radius, describe the arc ADF. Extend the chord of 32° from A to B; and through B, draw the line BC. Then is ACB an angle of 32 degrees.

2. To make an angle of 140 degrees. (Fig. 34.) On the line CH, with the chord of 60°, describe the arc ADF; and extend the chord of 70° from A to D, and from D to B. The arc ADB=70°×2=140°.

On the other hand:

162. To *measure an angle;* On the angular point as a center, and with the chord of 60° for radius, describe an arc to cut the two lines which include the angle. The distance between the points of intersection, applied to the line of chords, will give the measure of the angle in degrees. If the angle be *obtuse*, divide the arc into two parts.

Ex. 1. To measure the angle ACB. (Fig. 33.) Describe the arc ADF, cutting the lines CH and CB. The distance AB, will extend 32° on the line of chords.

2. To measure the angle ACB. (Fig. 34.) Divide the arc ADB into two parts, either equal or unequal, and measure each part, by applying its chord to the scale. The sum of the two will be 140°.

163. Besides the lines of chords, and of equal parts, on the plane scale; there are also lines of natural *sines*, *tangents*, and *secants*, marked Sin., Tan. and Sec.; of *semitangents*, marked S. T.; of *longitude*, marked Lon. or M. L.; of *rhumbs*, marked Rhu. or Rum., &c. These are not necessary in trigonometrical construction. Some of them are used in Navigation; and some of them, in the projections of the Sphere.

164. In Navigation, the quadrant, instead of being graduated in the usual manner, is divided into *eight* portions, called *Rhumbs*. The *Rhumb line*, on the scale, is a line of chords, divided into rhumbs and quarter-rhumbs, instead of degrees.

165. The line of *Longitude* is intended to show the number of geographical miles in a degree of longitude, at different distances from the equator. It is placed over the line of chords, with the numbers in an inverted order: so that the figure above shows the length of a degree of longitude, in any latitude denoted by the figure below.* Thus, at the equator, where the latitude is 0, a degree of longitude is 60 geographical miles. In latitude 40, it is 46 miles; in latitude 60, 30 miles, &c.

166. The graduation on the line of *secants* begins where the line of sines ends. For the greatest sine is only equal to radius; but the secant of the least arc is greater than radius.

167. The *semitangents* are the tangents of *half* the given arcs. Thus, the semitangent of 20° is the tangent of 10°. The line of semitangents is used in one of the projections of the sphere.

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168. In the construction of *triangles*, the sides and angles which are *given*, are laid down according to the directions in Arts. 158, 161. The parts *required* are then measured, according to Arts. 158, 162. The following problems correspond with the four cases of oblique angled triangles; (Art. 148.) but are equally adapted to right angled triangles.

169. PROB. I. *The angles and one side* of a triangle being given; to find, by construction, the other two sides.

Draw the given side. From the ends of it, lay off two of the given angles. Extend the other sides till they intersect; and then measure their lengths on a scale of equal parts.

Ex. 1. Given the side *b* 32 rods, (Fig. 27.) the angle A 56° 20′, and the angle C 49° 10′; to construct the triangle, and find the lengths of the sides *a* and *c*.

Their lengths will be 25 and $27\frac{1}{2}$.

2. In a right angled triangle, (Fig. 17.) given the hypothenuse 90, and the angle A 32° 20′, to find the base and perpendicular.

The length of AB will be 76, and of BC 48.

* Sometimes the line of longitude is placed *under* the line of chords.

3. Given the side AC 68, the angle A 124°, and the angle C 37°: to construct the triangle.

170. PROB. II. *Two sides and an opposite angle* being given, to find the remaining side, and the other two angles.

Draw one of the given sides; from one end of it, lay off the given angle; and extend a line indefinitely for the required side. From the other end of the first side, with the remaining given side for radius, describe an arc cutting the indefinite line. The point of intersection will be the end of the required side.

If the side opposite the given angle be less than the other given side, the case will be *ambiguous*. (Art. 152.)

Ex. 1. Given the angle A 63° 35′, (Fig. 29.) the side *b* 32, and the side *a* 36.

The side AB will be 36 nearly, the angle B 52° 45½′, and C 63° 39½′.

2. Given the angle A (Fig. 28.) 35° 20′, the opposite side *a* 25, and the side *b* 35.

Draw the side *b* 35, make the angle A 35° 20′, and extend AH indefinitely. From C with radius 25, describe an arc cutting AH in B and B′. Draw CB and CB′, and two triangles will be formed, ABC and AB′C, each corresponding with the conditions of the problem.

3. Given the angle A 116°, the opposite side *a* 38, and the side *b* 26; to construct the triangle.

171. PROB. III. *Two sides and the included angle* being given; to find the other side and angles.

Draw one of the given sides. From one end of it lay off the given angle, and draw the other given side. Then connect the extremities of this and the first line.

Ex. 1. Given the angle A (Fig. 30.) 26° 14′, the side *b* 78, and the side *c* 106; to find B, C, and *a*.

The side *a* will be 50, the angle B 43° 44′, and C 110° 2′.

2. Given A 86°, *b* 65, and *c* 83; to find B, C, and *a*.

172. PROB. IV. The *three sides* being given; to find the angles.

Draw one of the sides, and from one end of it, with an extent equal to the second side, describe an arc. From the other end, with an extent equal to the third side, describe a second arc cutting the first; and from the point of intersection draw the two sides. (Euc. 22. 1.)

Ex. 1. Given AB (Fig. 31.) 78, AC 70, and BC 54, to find the angles.

The angles will be A 42° 22′, B 60° 52$\frac{2}{3}$′, and C 76° 45$\frac{1}{3}$′.

2. Given the three sides 58, 39, and 46; to find the angles.

173. Any right lined figure whatever, whose sides and angles are given, may be constructed, by laying down the sides from a scale of equal parts, and the angles from a line of chords.

Ex. Given the sides AB (Fig. 35.)=20, BC=22, CD=30, DE=12; and the angles B=102°, C=130°, D=108°, to construct the figure.

Draw the side AB=20, make the angle B=102°, draw BC=22, make C=130°, draw CD=30, make D=108°, draw DE=12, and connect E and A.

The last line, EA, may be measured on the scale of equal parts; and the angles E and A, by a line of chords.

SECTION VI.

DESCRIPTION AND USE OF GUNTER'S SCALE.

ART. 174. AN expeditious method of solving the problems in trigonometry, and making other logarithmic calculations, in a mechanical way, has been contrived by Mr. Edmund Gunter. The logarithms of numbers, of sines, tangents, &c., are represented by *lines*. By means of these, multiplication, division, the rule of three, involution, evolution, &c., may be performed much more rapidly, than in the usual method by figures.

The logarithmic lines are generally placed on one side only of the scale in common use. They are,

A line of artificial *Sines* divided into *Rhumbs*, and marked,		S. R.
A line of artificial *Tangents*,	do.	T. R.
A line of the logarithms of *Numbers*,		Num.
A line of artificial *Sines*, to every *degree*,		SIN.
A line of artificial *Tangents*,	do.	TAN.
A line of *Versed Sines*,		V. S.

To these are added a line of *equal parts*, and a line of *Meridional Parts*, which are not logarithmic. The latter is used in Navigation.

The Line of Numbers.

175. Portions of the line of *Numbers*, are intended to represent the *logarithms* of the natural series of numbers 2, 3, 4, 5, &c.

The logarithms of 10, 100, 1000, &c., are 1, 2, 3, &c. (Art. 3.)

If, then, the log. of 10 be represented by a line of 1 foot;
the log. of 100 will be repres'd by one of 2 feet;
the log. of 1000 by one of 3 feet;
the lengths of the several lines being proportional to the corresponding logarithms in the tables. *Portions* of a foot will represent the logarithms of numbers between 1 and 10; and

portions of a line 2 feet long, the logarithms of numbers between 1 and 100.

On Gunter's scale, the line of the logarithms of numbers begins at a brass pin on the left, and the divisions are numbered 1, 2, 3, &c., to another pin near the middle. From this the numbers are repeated, 2, 3, 4, &c., which may be read 20, 30, 40, &c. The logarithms of numbers between 1 and 10, are represented by portions of the first half of the line; and the logarithms of numbers between 10 and 100, by portions greater than half the line, and less than the whole.

176. The logarithm of 1, which is 0, is denoted, not by any extent of line, but by a *point* under 1, at the commencement of the scale. The distances from this point to different parts of the line, represent other logarithms, of which the *figures* placed over the several divisions are the *natural numbers*. For the intervening logarithms, the intervals between the figures, are divided into tenths, and sometimes into smaller portions. On the right hand half of the scale, as the divisions which are numbered are *tens*, the subdivisions are units.

Ex. 1. To take from the scale the logarithm of 3.6; set one foot of the compasses under 1 at the beginning of the scale, and extend the other to the 6th division after the first figure 3.

2. For the logarithm of 47; extend from 1 at the beginning, to the 7th subdivision after the second figure 4.*

177. It will be observed, that the divisions and subdivisions *decrease*, from left to right; as in the tables of *logarithms*, the differences decrease. The difference between the logarithms of 10 and 100 is no greater, than the difference between the logarithms of 1 and 10.

178. The line of numbers, as it has been here explained, furnishes the logarithms of all numbers between 1 and 100.

And if the indices of the logarithms be neglected, the same scale may answer for all numbers whatever. For the *decimal* part of the logarithm of any number is the same, as that of the number multiplied or divided by 10, 100, &c. (Art. 14.) In logarithmic calculations, the use of the indices is to determine the distance of the several figures of the natural numbers from the place of units. (Art. 11.) But in those cases in which the logarithmic line is commonly used, it will

* If the compasses will not reach the distance required; first open them so as to take off *half*, or any part of the distance, and then the remaining part.

not generally be difficult to determine the local value of the figures in the result.

179. We may, therefore, consider the *point* under 1 at the left hand, as representing the logarithm of 1, or 10, or 100; or $\frac{1}{10}$, or $\frac{1}{100}$, &c., for the decimal part of the logarithm of each of these is 0. But if the first 1 is reckoned 10, all the succeeding numbers must also be increased in a tenfold ratio; so as to read, on the first half of the line, 20, 30, 40, &c., and on the other half, 200, 300, &c.

The whole extent of the logarithmic line,

is from 1 to 100,	or from 0.1 to 10,
or from 10 to 1000,	or from 0.01 to 1,
or from 100 to 10000, &c.	or from 0.001 to 0.1, &c.

Different values may, on different occasions, be assigned to the several numbers and subdivisions marked on this line. But for any one calculation, the value must remain the same.

Ex. Take from the scale 365.

As this number is between 10 and 1000, let the 1 at the beginning of the scale, be reckoned 10. Then, from this point to the second 3 is 300; to the 6th dividing stroke is 60; and half way from this to the next stroke is 5.

180. Multiplication, division, &c., are performed by the line of numbers, on the same principle, as by common logarithms. Thus,

To *multiply* by this line, *add* the logarithms of the two factors; (Art. 37.) that is, take off, with the compasses, that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line representing the logarithm of the product.

Ex. Multiply 9 into 8. The extent from 1 to 8, added to that from 1 to 9, will be equal to the extent from 1 to 72, the product.

181. To *divide* by the logarithmic line, *subtract* the logarithm of the divisor from that of the dividend; (Art. 41.) that is, take off the logarithm of the divisor, and this extent set back from the end of the logarithm of the dividend, will reach to the logarithm of the quotient.

Ex. Divide 42 by 7. The extent from 1 to 7, set back from 42, will reach to 6, the quotient.

182. *Involution* is performed in logarithms, by multiplying

the logarithm of the quantity into the index of the power; (Art. 45.) that is, by *repeating* the logarithms as many times as there are units in the index. To involve a quantity on the scale, then, take in the compasses the linear logarithm, and *double it*, *treble it*, *&c.*, according to the index of the proposed power.

Ex. 1. Required the square of 9. Extend the compasses from 1 to 9. *Twice* this extent will reach to 81, the square.

2. Required the cube of 4. The extent from 1 to 4 repeated *three times*, will reach to 64 the cube of 4.

183. On the other hand, to perform *evolution* on the scale; take *half*, *one third*, *&c.*, of the logarithm of the quantity, according to the index of the proposed root.

Ex. 1. Required the square root of 49. *Half* the extent from 1 to 49, will reach from 1 to 7, the root.

2. Required the cube root of 27. *One third* the distance from 1 to 27, will extend from 1 to 3, the root.

184. The *Rule of Three* may be performed on the scale, in the same manner as in logarithms, by adding the two middle terms, and from the sum, subtracting the first term. (Art. 52.) But it is more convenient in practice to *begin* by subtracting the first term from one of the others. If four quantities are proportional, the quotient of the first divided by the second, is equal to the quotient of the third divided by the fourth. (Alg. 364.)

Thus, if $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{a}{c}=\frac{b}{d}$. (Alg. 380.)

But in logarithms, *subtraction* takes the place of division; so that,

$\log. a - \log. b = \log. c - \log. d$. Or, $\log. a - \log. c = \log. b - \log. d$.

Hence,

185. On the scale, *the difference between the first and second terms of a proportion, is equal to the difference between the third and fourth.* Or, the difference between the first and third terms, is equal to the difference between the second and fourth.

The difference between the two terms is taken, by extending the compasses from one to the other. If the second term be greater than the first; the fourth must be greater than the third; if less, less. (Alg. 395.*) Therefore if the compasses

* Euc. 14. 5.

extend *forward* from *left to right*, that is, from a less number to a greater, from the first term to the second; they must also extend forward from the third to the fourth. But if they extend *backward*, from the first term to the second; they must extend the same way, from the third to the fourth.

Ex. 1. In the proportion 3 : 8 : : 12 : 32, the extent from 3 to 8, will reach from 12 to 32; Or, the extent from 3 to 12, will reach from 8 to 32.

2. If 54 yards of cloth cost 48 dollars, what will 18 yards cost?

54 : 48 : : 18 : 16

The extent from 54 to 48, will reach *backwards* from 18 to 16.

3. If 63 gallons of wine cost 81 dollars, what will 35 gallons cost?

63 : 81 : : 35 : 45

The extent from 63 to 81, will reach from 35 to 45.

The Line of Sines.

186. The line on Gunter's scale marked SIN. is a line of logarithmic sines, made to correspond with the line of numbers. The whole extent of the line of numbers, (Art. 179.)

is from 1 to 100, whose logs. are 0.00000 and 2.00000,
or from 10 to 1000, whose logs. are 1.00000 and 3.00000,
or from 100 to 10000, whose logs. are 2.00000 4.00000,

the *difference of the indices* of the two extreme logarithms being in each case 2.

Now the logarithmic sine of 0° 34′ 22″ 41‴ is 8.00000
And the sine of 90° (Art. 95.) is 10.00000

Here also the difference of the indices is 2. If then the point directly beneath one extremity of the line of numbers, be marked for the sine of 0° 34′ 22″ 41‴; and the point beneath the other extremity, for the sine of 90°; the interval may furnish the intermediate sine; the divisions on it being made to correspond with the decimal part of the logarithmic sines in the tables.*

* To represent the sines *less* than 34′ 22″ 41‴, the scale must be extended on the left indefinitely. For, as the sine of an arc approaches to 0, its logarithm, which is negative, increases without limit. (Art. 15.)

The first dividing stroke in the line of Sines is generally at 0° 40′, a little farther to the right than the beginning of the line of numbers. The next division is at 0° 50′; then begins the numbering of the degrees, 1, 2, 3, 4, &c., from left to right.

The Line of Tangents.

187. The first 45 degrees on this line are numbered from left to right, nearly in the same manner as on the line of Sines.

The logarithmic tangent of 0° 34′ 22″ 35‴ is 8.00000
And the tangent of 45°, (Art. 95.) 10.00000

The difference of the indices being 2, 45 degrees will reach to the end of the line. For those above 45° the scale ought to be continued much farther to the right. But as this would be inconvenient, the numbering of the degrees, after reaching 45, is *carried back* from right to left. The same dividing stroke answers for an arc and its *complement*, one above and the other below 45°. For, (Art. 93. Propor. 9.)

$$\tan : R : : R : \cot.$$

In logarithms, therefore, (Art. 184.)

$$\tan - R = R - \cot.$$

That is, the *difference* between the tangent and radius, is equal to the difference between radius and the cotangent: in other words, one is as much *greater* than the tangent of 45°, as the other is *less*. In taking, then, the tangent of an arc greater than 45°, we are to suppose the distance between 45 and the division marked with a given number of degrees, to be added to the whole line, in the same manner as if the line were continued out. In working proportions, extending the compasses *back*, from a less number to a greater, must be considered the same as carrying them *forward* in other cases. See art. 185.

Trigonometrical Proportions on the Scale.

188. In working proportions in trigonometry by the scale; *the extent from the first term to the middle term of the same*

name, will reach from the other middle term to the fourth term. (Art. 185.)

In a trigonometrical proportion, two of the terms are the lengths of sides of the given triangle; and the other two are tabular sines, tangents, &c. The former are to be taken from the line of numbers; the latter, from the lines of logarithmic sines and tangents. If one of the terms is a *secant*, the calculation cannot be made on the scale, which has commonly no line of secants. It must be kept in mind that *radius* is equal to the sine of 90°, or to the tangent of 45°. (Art. 95.) Therefore, whenever radius is a term in the proportion, one foot of the compasses must be set on the end of the line of sines or of tangents.

189. The following examples are taken from the proportions which have already been solved by numerical calculation.

Ex. 1. In Case I, of right angled triangles, (Art. 134. ex. 1.)

$$R : 45 :: \sin 32° 20' : 24$$

Here the third term is a *sine;* the first term radius is, therefore, to be considered as the sine of 90°. Then the extent from 90° to 32° 20′ on the line of sines, will reach from 45 to 24 on the line of numbers. As the compasses are set *back* from 90° to 32° 20′; they must also be set back from 45. (Art. 185.)

2. In the same case, if the base be made radius, (page 60.)

$$R : 38 :: \tan 32° 20' : 24$$

Here, as the third term is a *tangent*, the first term radius is to be considered the tangent of 45°. Then the extent from 45° to 32° 20′ on the line of tangents, will reach from 38 to 24 on the line of numbers.

3. If the perpendicular be made radius, (page 60.)

$$R : 24 :: \tan 57° 40' : 38$$

The extent from 45° to 57° 40′ on the line of tangents, will reach from 24 to 38 on the line of numbers. For the tangent of 57° 40′ on the scale, look for its *complement* 32° 20′. (Art. 187.) In this example, although the compasses extend *back* from 45° to 57° 40′; yet, as this is from a *less* number to a *greater*, they must extend *forward* on the line of numbers. (Arts. 185, 187.)

4. In art. 135, 35 : R : : 26 : sin 48°

The extent from 35 to 26 will reach from 90° to 48°.

5. In art. 136, R : 48 : : tan 27$\frac{1}{4}$° : 24$\frac{3}{4}$

The extent from 45° to 27$\frac{1}{4}$°, will reach from 48 to 24$\frac{3}{4}$.

6. In art. 150, ex. 1. Sin 74° 30′ : 32 : : sin 56° 20′ : 27$\frac{1}{2}$.

For other examples, see the several cases in Sections III. and IV.

190. Though the solutions in trigonometry may be effected by the logarithmic scale, or by geometrical construction, as well as by arithmetical computation; yet the latter method is by far the most accurate. The first is valuable principally for the *expedition* with which the calculations are made by it. The second is of use, in presenting the *form* of the triangle to the eye. But the accuracy which attends arithmetical operations, is not to be expected, in taking lines from a scale with a pair of compasses.*

* See note G.

SECTION VII.*

THE FIRST PRINCIPLES OF TRIGONOMETRICAL ANALYSIS.

ART. 191. IN the preceding sections, sines, tangents, and secants have been employed in calculating the sides and angles of triangles. But the use of these lines is not confined to this object. Important assistance is derived from them, in conducting many of the investigations in the higher branches of analysis, particularly in physical astronomy. It does not belong to an elementary treatise of trigonometry, to prosecute these inquiries to any considerable extent. But this is the proper place for *preparing the formulæ,* the applications of which are to be made elsewhere.

Positive and Negative SIGNS *in Trigonometry.*

192. Before entering on a particular consideration of the algebraic expressions which are produced by combinations of the several trigonometrical lines, it will be necessary to attend to the positive and negative *signs* in the different quarters of the circle. The sines, tangents, &c., in the tables, are calculated for a single quadrant only. But these are made to answer for the whole circle. For they are of the same length in each of the four quadrants. (Art. 90.) Some of them, however, are *positive;* while others are *negative.* In algebraic processes, this distinction must not be neglected.

193. For the purpose of tracing the changes of the signs, in different parts of the circle, let it be supposed that a straight line CT (Fig. 36.) is fixed at one end C, while the other end is carried round, like a rod moving on a pivot; so that the point S shall describe the circle ABDH. If the two diameters AD and BH, be perpendicular to each other, they will divide the circle into quadrants.

* Euler's Analysis of Infinites, Hutton's Mathematics, Lacroix's Differential Calculus, Mansfield's Essays, Legendre's, Lacroix's, Playfair's, Cagnoli's, and Woodhouse's Trigonometry.

194. In the *first quadrant* AB, the sine, cosine, tangent, &c., are considered *all positive.* In the *second quadrant* BD, the sine P′S′ continues *positive ;* because it is still on the *upper* side of the diameter AD, from which it is measured. But the *cosine*, which is measured from BH, becomes *negative*, as soon as it changes from the *right* to the *left* of this line. (Alg. 507.) In the *third quadrant* the *sine* becomes negative, by changing from the upper side to the under side of DA. The *cosine* continues negative, being still on the left of BH. In the *fourth quadrant*, the *sine* continues negative. But the *cosine* becomes positive, by passing to the right of BH.

195. The signs of the *tangents* and *secants* may be derived from those of the sines and cosines. The relations of these several lines to each other must be such, that a uniform method of calculation may extend through the different quadrants.

In the first quadrant, (Art. 93. Propor. 1.)

$$\text{R} : \cos :: \tan : \sin, \text{ that is, Tan} = \frac{\text{R} \times \sin}{\cos}.$$

The sign of the quotient is determined from the signs of the divisor and dividend. (Alg. 123.) The radius is considered as always positive. If then the sine and cosine be both positive or both negative, the tangent will be positive. But if one of these be positive, while the other is negative, the tangent will be negative.

Now by the preceding article,

In the 2d quadrant, the sine is positive, and the cosine negative.

The tangent must therefore be *negative.*

In the 3d quadrant, the sine and cosine are both negative.

The tangent must therefore be *positive.*

In the 4th quadrant, the sine is negative, and the cosine positive.

The tangent must therefore be *negative.*

196. By the 9th, 3d, and 6th proportions in Art. 93.

$$1. \text{ Tan} : \text{R} :: \text{R} : \cot, \text{ that is Cot} = \frac{\text{R}^2}{\tan}.$$

Therefore, as radius is uniformly positive, the *cotangent* must have the same sign as the tangent.

2. Cos : R : : R : sec, that is, $\text{Sec}=\frac{R^2}{\cos}$.

The *secant*, therefore, must have the same sign as the cosine.

3. Sin : R : : R : cosec, that is, $\text{Cosec}=\frac{R^2}{\sin}$.

The *cosecant*, therefore, must have the same sign as the sine.

The *versed sine*, as it is measured from A, in one direction only, is invariably positive.

197. The *tangent* AT (Fig. 36.) increases, as the arc extends from A towards B. See also Fig. 11. Near B the increase is very rapid; and when the difference between the arc and 90°, is *less* than any assignable quantity, the tangent is *greater* than any assignable quantity, and is said to be *infinite*. (Alg. 447.) If the arc is *exactly* 90 degrees, it has, strictly speaking, *no* tangent. For a tangent is a line drawn perpendicular to the diameter which passes through one end of the arc, and extended till it *meets* a line proceeding from the center through the other end. (Art. 84.) But if the arc is 90 degrees, as AB, (Fig. 36.) the angle ACB is a right angle, and therefore AT is *parallel* to CB; so that, if these lines be extended ever so far, they never can meet. Still, as an arc infinitely near to 90° has a tangent infinitely great, it is frequently said, in concise terms, that the tangent of 90° is infinite.

In the second quadrant, the tangent is, at first, infinitely great, and gradually diminishes, till at D it is reduced to nothing. In the third quadrant, it increases again, becomes infinite near H, and is reduced to nothing at A.

The *cotangent* is inversely as the tangent. It is therefore nothing at B and H, (Fig. 36.) and infinite near A and D.

198. The *secant* increases with the tangent, through the first quadrant, and becomes infinite near B; it then diminishes, in the second quadrant, till at D it is equal to the radius CD. In the third quadrant it increases again, becomes infinite near H, after which it diminishes, till it becomes equal to radius.

The *cosecant* decreases, as the secant increases, and *v. v.* It is therefore equal to radius at B and H, and infinite near A and D.

199. The *sine* increases through the first quadrant, till at B (Fig. 36.) it is equal to radius. See also Fig. 13. It then diminishes, and is reduced to nothing at D. In the third quadrant, it increases again, becomes equal to radius at H, and is reduced to nothing at A.

The *cosine* decreases through the first quadrant, and is reduced to nothing at B. In the second quadrant, it increases, till it becomes equal to radius at D. It then diminishes again, is reduced to nothing at H, and afterwards increases till it becomes equal to radius at A.

In all these cases, the arc is supposed to *begin* at A, and to extend round in the direction of BDH.

200. The *sine* and *cosine* vary from nothing to radius, which they never exceed. The *secant* and *cosecant* are never less than radius, but may be greater than any given length. The *tangent* and *cotangent* have every value from nothing to infinity. Each of these lines, after reaching its *greatest* limit, begins to *decrease;* and as soon as it arrives at its *least* limit, begins to *increase.* Thus, the sine begins to decrease, after becoming equal to radius, which is its greatest limit. But the secant begins to increase after becoming equal to radius, which is its least limit.

201. The substance of several of the preceding articles is comprised in the following tables. The first shows the *signs* of the trigonometrical lines, in each of the quadrants of the circle. The other gives the *values* of these lines, at the extremity of each quadrant.

Quadrant	1st	2d	3d	4th
Sine and cosecant	+	+	−	−
Cosine and secant	+	−	−	+
Tangent and cotangent	+	−	+	−

	0°	90°	180°	270°	360°
Sine	0	*r*	0	*r*	0
Cosine	*r*	0	*r*	0	*r*
Tangent	0	∝	0	∝	0
Cotangent	∝	0	∝	0	∝
Secant	*r*	∝	*r*	∝	*r*
Cosecant	∝	*r*	∝	*r*	∝

Here *r* is put for radius, and ∝ for infinite.

202. By comparing these two tables, it will be seen, that each of the trigonometrical lines changes from positive to negative, or from negative to positive, in that part of the circle

in which the line is either *nothing* or *infinite*. Thus, the tangent changes from positive to negative, in passing from the first quadrant to the second, through the place where it is infinite. It becomes positive again, in passing from the second quadrant to the third, through the point in which it is nothing.

203. There can be no more than 360 degrees in any circle. But a body may have a number of successive revolutions in the same circle; as the earth moves round the sun, nearly in the same orbit, year after year. In astronomical calculations, it is frequently necessary to add together parts of different revolutions. The sum may be more than 360°. But a body which has made more than a complete revolution in a circle, is only brought back to a point which it had passed over before. So the sine, tangent, &c., of an arc greater than 360°, is the same as the sine, tangent, &c., of some arc less than 360°. If an entire circumference, or a number of circumferences, be added to any arc, it will terminate in the same point as before. So that, if C be put for a whole circumference, or 360°, and x be any arc whatever;

$$\sin x = \sin (C+x) = \sin (2C+x) = \sin (3C+x), \text{ \&c.}$$
$$\tan x = \tan (C+x) = \tan (2C+x) = \tan (3C+x), \text{ \&c.}$$

204. It is evident also, that, in a number of successive revolutions, in the same circle;

The first quadrant must coincide with the	5th, 9th, 13th, 17th,
The second, with the	6th, 10th, 14th, 18th, &c.
The third, with the	7th, 11th, 15th, 19th, &c.
The fourth, with the	8th, 12th, 16th, 20th, &c.

205. If an arc extending in a certain direction from a given point, be considered *positive*; an arc extending from the same point, in an *opposite* direction, is to be considered *negative*. (Alg. 507.) Thus, if the arc extending from A to S, (Fig. 36.) be positive; an arc extending from A to S''' will be negative. The latter will not terminate in the *same quadrant* as the other; and the signs of the tabular lines must be accommodated to this circumstance. Thus, the sine of AS will be positive, while that of AS''' will be negative. (Art. 194.) When a greater arc is subtracted from a less, if the latter be positive, the *remainder* must be negative. (Alg. 58, 9.)

TRIGONOMETRICAL FORMULÆ.

206. From the view which has here been taken of the changes in the trigonometrical lines, it will be easy to see, in

what parts of the circle each of them increases or decreases. But this does not determine their exact values, except at the extremities of the several quadrants. In the analytical investigations which are carried on by means of these lines, it is necessary to calculate the changes produced in them, by a given increase or diminution of the arcs to which they belong. In this there would be no difficulty, if the sines, tangents, &c., were *proportioned* to their arcs. But this is far from being the case. If an arc is doubled, its sine is *not* exactly doubled. Neither is its tangent or secant. We have to inquire, then, in what manner the sine, tangent, &c., of one arc may be obtained, from those of other arcs already known.

The problem on which almost the whole of this branch of analysis depends, consists in deriving, from the sines and cosines of two given arcs, expressions for the sine and cosine of their *sum* and *difference*. For, by addition and subtraction, a few arcs may be so combined and varied, as to produce others of almost every dimension. And the expressions for the tangents and secants may be deduced from those of the sines and cosines.

Expressions for the SINE *and* COSINE *of the* SUM *and* DIFFERENCE *of arcs.*

207. Let a=AH, the greater of the given arcs,
And b=HL=HD, the less. (Fig. 37.)

Then $a+b$=AH+HL=AL, the *sum* of the two arcs,
And $a-b$=AH—HD=AD, their *difference.*

Draw the chord DL, and the radius CH, which may be represented by R. As DH is, by construction, equal to HL; DQ is equal to QL, and therefore DL is perpendicular to CH. (Euc. 3. 3.) Draw DO, HN, QP, and LM, each perpendicular to AC; and DS and QB parallel to AC.

From the definitions of the sine and cosine, (Arts. 82, 9.) it is evident, that

$$\text{The sine}\begin{cases}\text{of AH, that is, } \sin a=\text{HN},\\ \text{of HL,} \qquad \sin b=\text{QL},\\ \text{of AL,} \qquad \sin(a+b)=\text{LM},\\ \text{of AD,} \qquad \sin(a-b)=\text{DO},\end{cases}$$

The cosine of AH, that is, $\cos a = CN$,
of HL, $\cos b = CQ$,
of AL, $\cos(a+b) = CM$,
of AD, $\cos(a-b) = CO$.

The triangle CHN is obviously similar to CQP; and it is also similar to BLQ, because the sides of the one are perpendicular to those of the other, each to each. We have, then,

1. CH : CQ : : HN : QP, that is, $R : \cos b :: \sin a : QP$,
2. CH : QL : : CN : BL, $R : \sin b :: \cos a : BL$,
3. CH : CQ : : CN : CP, $R : \cos b :: \cos a : CP$,
4. CH : QL : : HN : QB, $R : \sin b :: \sin a : QB$.

Converting each of these proportions into an equation;

1. $QP = \frac{\sin a \cos b^{*}}{R}$ 3. $CP = \frac{\cos a \cos b}{R}$

2. $BL = \frac{\sin b \cos a}{R}$ 4. $QB = \frac{\sin a \sin b}{R}$

Then adding the first and second,

$$QP + BL = \frac{\sin a \cos b + \sin b \cos a}{R}$$

Subtracting the second from the first,

$$QP - BL = \frac{\sin a \cos b - \sin b \cos a}{R}$$

Subtracting the fourth from the third,

$$CP - QB = \frac{\cos a \cos b - \sin a \sin b}{R}$$

Adding the third and fourth,

$$CP + QB = \frac{\cos a \cos b + \sin a \sin b}{R}$$

* In these formulæ, the sign of multiplication is omitted; $\sin a \cos b$ being put for $\sin a \times \cos b$, that is, the product of the sine of a into the cosine of b.

But it will be seen, from the figure, that

$$QP+BL=BM+BL=LM=\sin(a+b)$$
$$QP-BL=QP-QS=DO=\sin(a-b)$$
$$CP-QB=CP-PM=CM=\cos(a+b)$$
$$CP+QB=CP+SD=CO=\cos(a-b)$$

208. If then, for the first member of each of the four equations above, we substitute its value, we shall have,

$$\text{I. } \sin(a+b)=\frac{\sin a\cos b+\sin b\cos a}{R}$$

$$\text{II. } \sin(a-b)=\frac{\sin a\cos b-\sin b\cos a}{R}$$

$$\text{III. } \cos(a+b)=\frac{\cos a\cos b-\sin a\sin b}{R}$$

$$\text{IV. } \cos(a-b)=\frac{\cos a\cos b+\sin a\sin b}{R}$$

Or, multiplying both sides by R,

$$R\sin(a+b)=\sin a\cos b+\sin b\cos a$$
$$R\sin(a-b)=\sin a\cos b-\sin b\cos a$$
$$R\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$R\cos(a-b)=\cos a\cos b+\sin a\sin b$$

That is, the product of radius and the *sine* of the *sum* of two arcs, is equal to the product of the sine of the first arc into the cosine of the second + the product of the sine of the second into the cosine of the first.

The product of radius and the *sine* of the *difference* of two arcs, is equal to the product of the sine of the first arc into the cosine of the second — the product of the sine of the second into the cosine of the first.

The product of radius and the *cosine* of the *sum* of two arcs, is equal to the product of the cosines of the arcs — the product of their sines.

The product of radius and the *cosine* of the *difference* of two arcs, is equal to the product of the cosines of the arcs + the product of their sines.

These four equations may be considered as fundamental propositions, in what is called the *Arithmetic* of *Sines and Cosines*, or *Trigonometrical Analysis*.

Expressions for the sine and cosine of a DOUBLE *arc.*

209. When the sine and cosine of any arc are given, it is easy to derive from the equations in the preceding article, expressions for the sine and cosine of *double* that arc. As the two arcs a and b may be of any dimensions, they may be supposed to be *equal.* Substituting, then, a for its equal b, the first and the third of the four preceding equations will become,

$$\text{R}\sin(a+a)=\sin a\cos a+\sin a\cos a$$
$$\text{R}\cos(a+a)=\cos a\cos a-\sin a\sin a$$

That is, writing $\sin^2 a$ for the square of the sine of a, and $\cos^2 a$ for the square of the cosine of a,

$$\text{I. R}\sin 2a=2\sin a\cos a$$
$$\text{II. R}\cos 2a=\cos^2 a-\sin^2 a.$$

Expressions for the sine and cosine of HALF *a given arc.*

210. The arc in the preceding equations, not being necessarily limited to any particular value, may be *half* a, as well as a. Substituting then $\frac{1}{2}a$ for a, we have,

$$\text{R}\sin a=2\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a$$
$$\text{R}\cos a=\cos^2\tfrac{1}{2}a-\sin^2\tfrac{1}{2}a$$

Putting the sum of the squares of the sine and cosine equal to the square of radius, (Art. 94.) and inverting the members of the last equation,

$$\cos^2\tfrac{1}{2}a+\sin^2\tfrac{1}{2}a=\text{R}^2$$
$$\cos^2\tfrac{1}{2}a-\sin^2\tfrac{1}{2}a=\text{R}\cos a$$

If we *subtract* one of these from the other, the terms containing $\cos^2\frac{1}{2}a$ will disappear; and if we *add* them, the terms containing $\sin^2\frac{1}{2}a$ will disappear: therefore,

$$2\sin^2\tfrac{1}{2}a=\text{R}^2-\text{R}\cos a$$
$$2\cos^2\tfrac{1}{2}a=\text{R}^2+\text{R}\cos a$$

Dividing by 2, and extracting the root of both sides,

$$\text{I. } \sin \tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2-\tfrac{1}{2}R\times\cos a}$$
$$\text{II. } \cos \tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2+\tfrac{1}{2}R\times\cos a}$$

Expressions for the sines and cosines of MULTIPLE *arcs.*

211. In the same manner, as expressions for the sine and cosine of a *double* arc, are derived from the equations in art. 208; expressions for the sines and cosines of other multiple arcs may be obtained, by substituting successively $2a$, $3a$, &c., for b, or for b and a both. Thus,

$$\text{I.}\left\{\begin{array}{l} R\sin 3a=R\sin(a+2a)=\sin a\cos 2a+\sin 2a\cos a\\ R\sin 4a=R\sin(a+3a)=\sin a\cos 3a+\sin 3a\cos a\\ R\sin 5a=R\sin(a+4a)=\sin a\cos 4a+\sin 4a\cos a\end{array}\right.$$
&c.

$$\text{II.}\left\{\begin{array}{l} R\cos 3a=R\cos(a+2a)=\cos a\cos 2a-\sin a\sin 2a\\ R\cos 4a=R\cos(a+3a)=\cos a\cos 3a-\sin a\sin 3a\\ R\cos 5a=R\cos(a+4a)=\cos a\cos 4a-\sin a\sin 4a\end{array}\right.$$
&c.

Expressions for the PRODUCTS *of sines and cosines.*

212. Expressions for the products of sines and cosines may be obtained, by adding and subtracting the four equations in art. 208, viz.

$$R\sin(a+b)=\sin a\cos b+\sin b\cos a$$
$$R\sin(a-b)=\sin a\cos b-\sin b\cos a$$
$$R\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$R\cos(a-b)=\cos a\cos b+\sin a\sin b$$

Adding the first and second,

$$R\sin(a+b)+R\sin(a-b)=2\sin a\cos b$$

Subtracting the second from the first,

$$R\sin(a+b)-R\sin(a-b)=2\sin b\cos a$$

Adding the third and fourth,

$$R\cos(a-b)+R\cos(a+b)=2\cos a\cos b$$

Subtracting the third from the fourth,

$$R\cos(a-b)-R\cos(a+b)=2\sin a\sin b$$

Inverting the members of each of these equations, and dividing by 2, we have,

$$\text{I. } \sin a \cos b = \tfrac{1}{2}\text{R} \sin(a+b) + \tfrac{1}{2}\text{R} \sin(a-b)$$
$$\text{II. } \sin b \cos a = \tfrac{1}{2}\text{R} \sin(a+b) - \tfrac{1}{2}\text{R} \sin(a-b)$$
$$\text{III. } \cos a \cos b = \tfrac{1}{2}\text{R} \cos(a-b) + \tfrac{1}{2}\text{R} \cos(a+b)$$
$$\text{IV. } \sin a \sin b = \tfrac{1}{2}\text{R} \cos(a-b) - \tfrac{1}{2}\text{R} \cos(a+b)$$

213. If b be taken equal to a, then $a+b=2a$, and $a-b=0$, the sine of which is 0, (Art. 201.); and the term in which this is a *factor*, is reduced to 0. (Alg. 112.) But the *cosine* of 0 is equal to radius, so that $\text{R}\times\cos 0=\text{R}^2$. Reducing, then, the preceding equations,

The first becomes	$\sin a \cos a = \frac{1}{2}\text{R} \sin 2a$
The third,	$\cos^2 a = \frac{1}{2}\text{R}^2 + \frac{1}{2}\text{R} \cos 2a$
The fourth,	$\sin^2 a = \frac{1}{2}\text{R}^2 - \frac{1}{2}\text{R} \cos 2a$

214. If s be the *sum*, and d the *difference* of two arcs, $\frac{1}{2}(s+d)$ will be equal to the greater, and $\frac{1}{2}(s-d)$ to the less. (Art. 153.) Substituting then, in the four equations in art. 212,

s for $a+b$,	$\frac{1}{2}(s+d)$ for a
d for $a-b$,	$\frac{1}{2}(s-d)$ for b, we have,

$$\text{I. } \sin \tfrac{1}{2}(s+d) \cos \tfrac{1}{2}(s-d) = \tfrac{1}{2}\text{R} (\sin s + \sin d)$$
$$\text{II. } \sin \tfrac{1}{2}(s-d) \cos \tfrac{1}{2}(s+d) = \tfrac{1}{2}\text{R} (\sin s - \sin d)$$
$$\text{III. } \cos \tfrac{1}{2}(s+d) \cos \tfrac{1}{2}(s-d) = \tfrac{1}{2}\text{R} (\cos d + \cos s)$$
$$\text{IV. } \sin \tfrac{1}{2}(s+d) \sin \tfrac{1}{2}(s-d) = \tfrac{1}{2}\text{R} (\cos d - \cos s)$$

Or, making $\text{R}=1$,

$$\text{I. } \sin(a+b) + \sin(a-b) = 2 \sin a \cos b$$
$$\text{II. } \sin(a+b) - \sin(a-b) = 2 \sin b \cos a$$
$$\text{III. } \cos(a-b) + \cos(a+b) = 2 \cos a \cos b$$
$$\text{IV. } \cos(a-b) - \cos(a+b) = 2 \sin a \sin b$$

215. If radius be taken equal to 1, the two first equations in art. 208, are,

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$

Multiplying these into each other,

$$\sin(a+b) \times \sin(a-b) = \sin^2 a \cos^2 b - \sin^2 b \cos^2 a$$

But by art. 94, if radius is 1,

$$\cos^2 b = 1 - \sin^2 b, \text{ and } \cos^2 a = 1 - \sin^2 a$$

Substituting, then, for $\cos^2 b$ and $\cos^2 a$, their values, multiplying the factors, and reducing the terms, we have,

$$\sin(a+b) \times \sin(a-b) = \sin^2 a - \sin^2 b$$

Or, because the difference of the squares of two quantities is equal to the product of their sum and difference, (Alg. 235.)

$$\sin(a+b) \times \sin(a-b) = (\sin a + \sin b) \times (\sin a - \sin b)$$

That is, the product of the sine of the sum of two arcs, into the sine of their difference, is equal to the product of the sum of their sines, into the difference of their sines.

Expressions for the TANGENTS *of arcs.*

216. Expressions for the *tangents* of arcs may be derived from those already obtained for the sines and cosines. By art. 93, proportion 1st,

$$R : \tan :: \cos : \sin$$

That is, $\dfrac{R}{\tan} = \dfrac{\cos}{\sin}$, and $\dfrac{\tan}{R} = \dfrac{\sin}{\cos}$, and $\tan = \dfrac{R \times \sin}{\cos}$,

$$\text{Thus, } \tan(a+b) = \frac{R \sin(a+b)}{\cos(a+b)}.$$

If, for $\sin(a+b)$ and $\cos(a+b)$ we substitute their values, as given in Art. 208, we shall have,

$$\tan(a+b) = \frac{R(\sin a \cos b + \sin b \cos a)}{\cos a \cos b - \sin a \sin b}.$$

217. Here, the value of the tangent of the sum of two arcs is expressed, in terms of the *sines* and *cosines* of the arcs. To exchange these for terms of the *tangents*, let the numerator and denominator of the second member of the equation be both divided by $\cos a \cos b$. This will not alter the value of the fraction. (Alg. 140.)

The *numerator*, divided by $\cos a \cos b$, is

$$\frac{R(\sin a \cos b + \sin b \cos a)}{\cos a \cos b} = R\left(\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}\right) = \tan a + \tan b$$

And the *denominator*, divided by cos *a* cos *b*, is

$$\frac{\cos a\cos b-\sin a\sin b}{\cos a\cos b}=1-\frac{\sin a}{\cos a}\times\frac{\sin b}{\cos b}=1-\frac{\tan a}{R}\times\frac{\tan b}{R}.$$

$$\text{Therefore, } \tan(a+b)=\frac{\tan a+\tan b}{1-\dfrac{\tan a\tan b}{R^2}}$$

The denominator of the fraction may be cleared of the divisor R^2, by multiplying both the numerator and denominator into R^2. And if we proceed in a similar manner to find the tangent of $a-b$, we shall have,

218. I. $\tan(a+b)=\dfrac{R^2(\tan a+\tan b)}{R^2-\tan a\tan b}$

II. $\tan(a+b)=\dfrac{R^2(\tan a-\tan b)}{R^2+\tan a\tan b}$

If the arcs *a* and *b* are *equal*, then substituting $\frac{1}{2}a$, a, $2a$, $3a$, &c., as in Art. 210, 211.

$$\tan a=\tan(\tfrac{1}{2}a+\tfrac{1}{2}a)=\frac{R^2(2\tan\frac{1}{2}a)}{R^2-\tan^2\frac{1}{2}a}$$

$$\tan 2a=\tan(a+a)=\frac{R^2(2\tan a)}{R^2-\tan^2 a}$$

$$\tan 3a=\tan(a+2a)=\frac{R^2(\tan a+\tan 2a)}{R^2-\tan a\tan 2a}, \text{ \&c.}$$

219. If we divide the first of the equations in Art. 214, by the second; we shall have, after rejecting $\frac{1}{2}R^2$ from the numerator and denominator, (Alg. 140.)

$$\frac{\sin\frac{1}{2}(s+d)\cos\frac{1}{2}(s-d)}{\sin\frac{1}{2}(s-d)\cos\frac{1}{2}(s+d)}=\frac{\sin s+\sin d}{\sin s-\sin d}$$

But the first member of this equation, (Alg. 155.) is equal to

$$\frac{\sin\frac{1}{2}(s+d)}{\cos\frac{1}{2}(s+d)}\times\frac{\cos\frac{1}{2}(s-d)}{\sin\frac{1}{2}(s-d)}=\frac{\tan\frac{1}{2}(s+d)}{R}\times\frac{R}{\tan\frac{1}{2}(s-d)}. \text{ (Art. 216.)}$$

Therefore,

$$\frac{\sin s+\sin d}{\sin s-\sin d}=\frac{\tan\frac{1}{2}(s+d)}{\tan\frac{1}{2}(s-d)}$$

220. According to the notation in Art. 214, s stands for the *sum* of two arcs, and d for their *difference*. But it is evident that arcs may be taken, whose sum shall be equal to *any* arc a, and whose difference shall be equal to any arc b, provided that a be *greater* than b. Substituting, then, in the preceding equation a for s, and b for d,

$$\frac{\sin a+\sin b}{\sin a-\sin b}=\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}. \quad \text{Or,}$$

$$\sin a+\sin b : \sin a-\sin b :: \tan \tfrac{1}{2}(a+b) : \tan \tfrac{1}{2}(a-b.)$$

That is, *The sum of the sines of two arcs or angles, is to the difference of those sines; as the tangent of half the sum of the arcs or angles, to the tangent of half their difference.*

By Art. 143, the *sides of triangles* are as the sines of their opposite angles. It follows, therefore, from the preceding proposition, (Alg. 389.) that the sum of any two sides of a triangle, is to their difference; as the tangent of half the sum of the opposite angles, to the tangent of half their difference.

This is the second theorem applied to the solution of oblique angled triangles, which was *geometrically* demonstrated in Art. 144.

Expressions for the *cotangents* may be obtained by putting

$$\cot=\frac{R^2}{\tan} \text{ (Art. 93.)}$$

$$\text{Thus, } \cot(a+b)=\frac{R^2}{\tan(a+b)}=\frac{R^2-\tan a \tan b}{\tan a+\tan b} \text{ (Art. 218.)}$$

Substituting $\frac{R^2}{\cot a}$ for $\tan a$, and $\frac{R^2}{\cot b}$ for $\tan b$,

$$\cot(a+b)=\frac{R^2-\frac{R^2}{\cot a}\times\frac{R^2}{\cot b}}{\frac{R^2}{\cot a}+\frac{R^2}{\cot b}}$$

Multiplying both the numerator and denominator by $\cot a \cot b$, dividing by R^2, and proceeding in the same manner, for $\cot(a-b)$ we have,

$$\text{I. } \cot(a+b)=\frac{\cot a \cot b-R^2}{\cot b+\cot a}$$

$$\text{II. } \cot(a-b)=\frac{\cot a \cot b+R^2}{\cot b-\cot a}$$

220. *b*. By comparing the expressions for the sines, and cosines, with those for the tangents and cotangents, a great variety of formulæ may be obtained. Thus, the tangent of the sum or the difference of two arcs, may be expressed in terms of the cotangent.

Putting radius $=1$, we have (Arts. 93, 220.)

$$\text{I. } \tan(a+b)=\frac{1}{\cot(a+b)}=\frac{\cot b+\cot a}{\cot a \cot b-1}$$

$$\text{II. } \tan(a-b)=\frac{1}{\cot(a-b)}=\frac{\cot b-\cot a}{\cot a \cot b+1}$$

By Art. 208,

$$\frac{\sin(a+b)}{\sin(a-b)}=\frac{\sin a \cos b+\sin b \cos a}{\sin a \cos b-\sin b \cos a}$$

Dividing the last member of the equation, in the first place by $\cos a \cos b$, as in Art. 217, and then by $\sin a \sin b$, we have

$$\frac{\sin(a+b)}{\sin(a-b)}=\frac{\tan a+\tan b}{\tan a-\tan b}=\frac{\cot b+\cot a}{\cot b-\cot a}$$

In a similar manner, dividing the expressions for the cosines, in the first place by $\sin b \cos a$, and then by $\sin a \cos b$, we obtain

$$\frac{\cos(a+b)}{\cos(a-b)}=\frac{\cot b-\tan a}{\cot b+\tan a}=\frac{\cot a-\tan b}{\cot a+\tan b}$$

Dividing the numerator and denominator of the expression for the tangent of a, (Art. 218.) by $\tan \frac{1}{2}a$, we have

$$\tan a=\frac{2}{\cot \frac{1}{2}a-\tan \frac{1}{2}a}$$

These formulæ may be multiplied almost indefinitely, by combining the expressions for the sines, tangents, &c. The following are put down without demonstrations, for the exercise of the student.

$$\tan \tfrac{1}{2}a = \cot \tfrac{1}{2}a - 2 \cot a. \qquad \tan \tfrac{1}{2}a = \frac{1-\cos a}{\sin a}$$

$$\tan \tfrac{1}{2}a = \frac{\sin a}{1+\cos a} \qquad \tan^2 \tfrac{1}{2}a = \frac{1-\cos a}{1+\cos a}$$

$$\sin a = \frac{2 \tan \tfrac{1}{2}a}{1+\tan^2 \tfrac{1}{2}a} \qquad \cos a = \frac{1-\tan^2 \tfrac{1}{2}a}{1+\tan^2 \tfrac{1}{2}a}$$

$$\cos a = \frac{\cot \tfrac{1}{2}a - \tan \tfrac{1}{2}a}{\cot \tfrac{1}{2}a + \tan \tfrac{1}{2}a} \qquad \sin a = \frac{2}{\cot \tfrac{1}{2}a + \tan \tfrac{1}{2}a}$$

$$\sin a = \frac{1}{\cot \tfrac{1}{2}a - \cot a} \qquad \sin a = \frac{1}{\cot a + \tan \tfrac{1}{2}a}$$

Expression for the *area* of a triangle, in terms of the sides.

221. Let the sides of the triangle ABC (Fig. 23.) be expressed by a, b, and c, the perpendicular CD by p, the segment AD by d, and the area by S.

Then $a^2 = b^2 + c^2 - 2cd$, (Euc. 13. 2.)

Transposing and dividing by $2c$;

$$d = \frac{b^2+c^2-a^2}{2c}. \text{ Therefore } d^2 = \frac{(b^2+c^2-a^2)^2}{4c^2}. \quad \text{(Alg. 223.)}$$

By Euc. 47, 1, $p^2 = b^2 - d^2 = b^2 - \dfrac{(b^2+c^2-a^2)^2}{4c^2}$

Reducing the fraction, (Alg. 150.) and extracting the root of both sides,

$$p=\frac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}.*$$

This gives the length of the *perpendicular*, in terms of the sides of the triangle. But the *area* is equal to the product of the base into half the perpendicular height. (Alg. 518.) that is,

$$S=\tfrac{1}{2}cp=\tfrac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}$$

Here we have an expression for the area, in terms of the sides. But this may be reduced to a form much better adapted to arithmetical computation. It will be seen, that the quantities $4b^2c^2$, and $(b^2+c^2-a^2)^2$ are both *squares*; and that the whole expression under the radical sign is the *difference* of these squares. But the difference of two squares is equal to the product of the sum and difference of their roots. (Alg. 235.) Therefore, $4b^2c^2-(b^2+c^2-a^2)^2$ may be resolved into the two factors,

$$\begin{cases} 2bc+(b^2+c^2-a^2) \text{ which is equal to } (b+c)^2-a^2 \\ 2bc-(b^2+c^2-a^2) \text{ which is equal to } a^2-(b-c)^2 \end{cases}$$

Each of these also, as will be seen in the expressions on the right, is the difference of two squares; and may, on the same principle, be resolved into factors, so that,

$$\begin{cases} (b+c)^2-a^2=(b+c+a)\times(b+c-a) \\ a^2-(b-c)^2=(a+b-c)\times(a-b+c) \end{cases}$$

Substituting, then, these four factors, in the place of the quantity which has been resolved into them, we have,

$$S=\tfrac{1}{4}\sqrt{(b+c+a)\times(b+c-a)\times(a+b-c)\times(a-b+c)}$$

* The expression for the perpendicular is the same, when one of the angles is *obtuse*, as in Fig. 24. Let $AD=d$.

Then $a^2=b^2+c^2+2cd$. (Euc. 12, 2.) And $d=\frac{-b^2-c^2+a^2}{2c}$

Therefore, $d^2=\frac{(-b^2-c^2+a^2)^2}{4c^2}=\frac{(b^2+c^2-a^2)^2}{4c^2}$ (Alg. 219.)

And $p=\frac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}$ as above.

Here it will be observed, that all the three sides, a, b, and c, are in each of these factors.

Let $h=\frac{1}{2}(a+b+c)$ *half the sum* of the sides. Then

$$S=\sqrt{h\times(h-a)\times(h-b)\times(h-c)}$$

222. For finding the area of a triangle, then, when the three sides are given, we have this general rule;

From half the sum of the sides, subtract each side severally; multiply together the half sum and the three remainders; and extract the square root of the product.

SECTION VIII.

COMPUTATION OF THE CANON.

Art. 223. The trigonometrical canon is a set of tables containing the sines, cosines, tangents, &c., to every degree and minute of the quadrant. In the computation of these tables, it is common to find, in the first place, the sine and cosine of *one minute;* and then, by successive additions and multiplications, the sines, cosines, &c., of the larger arcs. For this purpose, it will be proper to begin with an arc, whose sign or cosine is a known portion of the radius. The cosine of 60° is equal to *half radius.* (Art. 96. Cor.) A formula has been given, (Art. 210,) by which, when the cosine of an arc is known, the cosine of *half* that arc may be obtained.

By successive bisections of 60°, we have the arcs

30°	0° 28′ 7″ 30‴
15°	0 14 3 45
7° 30′	0 7 1 52 30
3° 45′	0 3 30 56 15
1° 52′ 30″	0 1 45 28 7 30
0° 56′ 15″	0 0′ 52″ 44‴ 3⁗ 45′′′′′

By formula II, art. 210,

$$\cos \tfrac{1}{2}a = \sqrt{\tfrac{1}{2}R^2 + \tfrac{1}{2}R \times \cos a}$$

If the radius be 1, and if $a=60°$, $b=30°$, $c=15°$, &c.; then

$$\cos b = \cos \tfrac{1}{2}a = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} \times \tfrac{1}{2}} = 0.8660254$$

$$\cos c = \cos \tfrac{1}{2}b = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos b} = 0.9659258$$

$$\cos d = \cos \tfrac{1}{2}c = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos c} = 0.9914449$$

$$\cos e = \cos \tfrac{1}{2}d = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos d} = 0.9978589$$

Proceeding in this manner, by repeated extractions of the square root, we shall find the cosine of

$$0^\circ\ 0'\ 52''\ 44'''\ 3''''\ 45''''' \text{ to be } 0.99999996732$$

$$\text{And the sine (Art. 94.)} = \sqrt{1 - \cos^2} = 0.00025566346$$

This, however, does not give the sine of *one minute* exactly. The arc is a little *less* than a minute. But the ratio of very small arcs to each other, is so nearly equal to the ratio of their sines, that one may be taken for the other, without sensible error. Now the circumference of a circle is divided into 21600 parts, for the arc of 1′; and into 24576, for the arc of $0^\circ\ 0'\ 52''\ 44'''\ 3''''\ 45'''''$

Therefore,

$$21600 : 24576 :: 0.00025566346 : 0.0002908882,$$

which is the sine of 1 minute very nearly.*

$$\text{And the cosine} = \sqrt{1 - \sin^2} = 0.9999999577.$$

224. Having computed the sine and cosine of one minute, we may proceed, in a contrary order, to find the sines and cosines of *larger* arcs.

Making radius $=1$, and adding the two first equations in art. 208, we have

$$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$$

Adding the third and fourth,

$$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b$$

Transposing $\sin(a-b)$ and $\cos(a-b)$

$$\text{I. } \sin(a+b) = 2 \sin a \cos b - \sin(a-b)$$
$$\text{II. } \cos(a+b) = 2 \cos a \cos b - \cos(a-b)$$

If we put $b=1'$, and $a=1'$ $2'$, $3'$, &c. successively, we shall have expressions for the sines and cosines of a series of arcs increasing regularly by one minute. Thus,

* See note H.

$$\sin(1'+1')=2\sin 1'\times\cos 1'-\sin 0=0.0005817764,$$
$$\sin(2'+1')=2\sin 2'\times\cos 1'-\sin 1'=0.0008726645,$$
$$\sin(3'+1')=2\sin 3'\times\cos 1'-\sin 2'=0.0011635526,$$
&c. &c.

$$\cos(1'+1')=2\cos 1'\times\cos 1'-\cos 0=0.9999998308,$$
$$\cos(2'+1')=2\cos 2'\times\cos 1'-\cos 1'=0.9999996192,$$
$$\cos(3'+1')=2\cos 3'\times\cos 1'-\cos 2'=0.9999993230,$$
&c. &c.

The constant multiplier here, cos 1′ is 0.9999999577, which is equal to 1—0.0000000423.

225. Calculating, in this manner, the sines and cosines from 1 minute up to 30 degrees, we shall have also the sines and cosines from 60° to 90°. For the sines of arcs between 0° and 30°, are the *cosines* of arcs between 60° and 90°. And the cosines of arcs between 0° and 30°, are the *sines* of arcs between 60° and 90°. (Art. 104.)

226. For the interval between 30° and 60°, the sines and cosines may be obtained by subtraction merely. As twice the sine of 30° is equal to radius (Art. 96,) by making $a=30°$, the equation marked I, in Article 224, will become

$$\sin(30°+b)=\cos b-\sin(30°-b.)$$

And putting $b=1', 2', 3'$, &c., successively,

$$\sin(30°\ 1')=\cos 1'-\sin(29°\ 59')$$
$$(30°\ 2')=\cos 2'-\sin(29°\ 58')$$
$$(30°\ 3')=\cos 3'-\sin(29°\ 57')$$
&c. &c.

If the *sines* be calculated from 30° to 60°, the *cosines* will also be obtained. For the sines of arcs between 30° and 45°, are the cosines of arcs between 45° and 60°. And the sines of arcs between 45° and 60°, are the cosines of arcs between 30° and 45°.* (Art. 96.)

227. By the methods which have here been explained, the *natural* sines and cosines are found.

The *logarithms* of these, 10 being in each instance added to the index, will be the *artificial* sines and cosines by which trigonometrical calculations are commonly made. (Arts. 102, 3.)

228. The *tangents*, *cotangents*, *secants*, and *cosecants*, are easily derived from the sines and cosines. By Art. 93,

* See note I.

$R : \cos :: \tan : \sin \qquad \cos : R :: R : \sec$

$R : \sin :: \cot : \cos \qquad \sin : R :: R : \operatorname{cosec}$

Therefore,

The tangent $= \frac{R \times \sin}{\cos}$ The secant $= \frac{R^2}{\cos}$

The cotangent $= \frac{R \times \cos}{\sin}$ The cosecant $= \frac{R^2}{\sin}$

Or if the computations are made by *logarithms*,

The tangent $= 10 + \sin - \cos$, The secant $= 20 - \cos$,
The cotangent $= 10 + \cos - \sin$, The cosecant $= 20 - \sin$.

SECTION IX.

PARTICULAR SOLUTIONS OF TRIANGLES.*

ART. 231. ANY triangle whatever may be solved, by the theorems in Sections III. IV. But there are other methods, by which, in certain circumstances, the calculations are rendered more expeditious, or more accurate results are obtained.

The differences in the *sines* of angles near 90°, and in the *cosines* of angles near 0°, are so small as to leave an uncertainty of several seconds in the result. The solutions should be varied, so as to avoid finding a very small angle by its cosine, or one near 90° by its sine.

The differences in the logarithmic *tangents* and *cotangents* are least at 45°, and increase towards each extremity of the quadrant. In no part of it, however, are they very small. In the tables which are carried to 7 places of decimals, the least difference for one second is 42. Any angle may be found within one second, by its tangent, if tables are used which are calculated to seconds.

But the differences in the logarithmic sines and tangents, within a few minutes of the beginning of the quadrant, and in cosines and tangents within a few minutes of 90°, though they are very large, are too *unequal* to allow of an exact determination of their corresponding angles, by taking *proportional parts* of the differences. Very small angles may be accurately found, from their sines and tangents, by the rules given in a note at the end.†

232. The following formulæ may be applied to *right angled* triangles, to obtain accurate results, by finding the sine or tangent of *half* an arc, instead of the whole.

In the triangle ABC (Fig. 20, Pl. II.) making AC radius,

$$AC : AB :: 1 : \text{Cos } A.$$

By conversion, (Alg. 389, 5.)

$$AC : AC - AB :: 1 : 1 - \text{Cos } A.$$

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* Simpson's, Woodhouse's, and Cagnoli's Trigonometry. † See note K.

Therefore,

$$\frac{AC-AB}{AC}=1-\cos A=2\sin^2\tfrac{1}{2}A.\ \text{(Art. 210.)}$$

Or,

$$\text{Sin}\ \tfrac{1}{2}A=\sqrt{\left(\frac{AC-AB}{2AC}\right)}$$

Again, from the first proportion, adding and subtracting terms, (Alg. 389, 7.)

$$AC+AB : AC-AB :: 1+\cos A : 1-\cos A.$$

Therefore,

$$\frac{AC-AB}{AC+AB}=\frac{1-\cos A}{1+\cos A}=\tan^2\tfrac{1}{2}A.\ \text{(Page 120.)}$$

Or,

$$\text{Tan}\ \tfrac{1}{2}A=\sqrt{\left(\frac{AC-AB}{AC+AB}\right)}$$

233. Sometimes, instead of having two parts of a right angled triangle given, in addition to the right angle; we have only one of the parts, and the *sum* or *difference* of two others. In such cases, solutions may be obtained by the following proportions:

By the preceding formulæ, and Arts. 140, 141,

1. $\text{Tan}^2\ \tfrac{1}{2}A=\dfrac{AC-AB}{AC+AB}$
2. $BC^2=(AC-AB)\,(AC+AB)$

Multiplying these together, and extracting the root, we have,

$$\text{Tan}\ \tfrac{1}{2}A\times BC=AC-AB$$

Therefore,

$$\text{I. Tan}\ \tfrac{1}{2}A : 1 :: AC-AB : BC$$

That is, the tangent of half of one of the acute angles, is to 1, as the difference between the hypothenuse and the side at the angle, to the other side.

If, instead of multiplying, we *divide* the first equation above by the second, we have

$$\frac{\text{Tan}\ \tfrac{1}{2}A}{BC}=\frac{1}{AC+AB}$$

Therefore,

II. $1 : \tan \frac{1}{2}A :: AC+AB : BC$

Again, in the triangle ABC, Fig. 20,

$AB : BC : 1 :: \tan A$

Therefore,

$AB+BC : AB-BC :: 1+\tan A : 1-\tan A$

Or,

$$AB+BC : AB-BC :: 1 : \frac{1-\tan A}{1+\tan A}$$

By art. 218, one of the arcs being A, and the other 45°, the tangent of which is equal to radius, we have,

$$\text{Tan}\,(45°-A) = \frac{1-\tan A}{1+\tan A}$$

Therefore,

III. $1 : \tan (45°-A) :: AB+BC : AB-BC.$

That is, unity is to the tangent of the difference between 45° and one of the acute angles: as the sum of the perpendicular sides is to their difference.

Ex. 1. In a right angled triangle, if the difference of the hypothenuse and base be 64 feet, and the angle at the base $33\frac{3}{4}°$, what is the length of the perpendicular?

Ans. 211.

2. If the sum of the hypothenuse and base be 185,3 and the angle at the base 37°, what is the perpendicular?

Ans. 620.

3. Given the sum of the base and perpendicular 128.4, and the angle at the base $41\frac{1}{4}°$, to find the sides.

$1 : \tan (45°-41\frac{1}{4}°) :: 128.4 : 8.4,$

the difference of the base and perpendicular. Half the difference added to, and subtracted from, the half sum, gives the base 68.4, and the perpendicular 60.

4. Given the sum of the hypothenuse and perpendicular 83, and the angle at the perpendicular 40°, to find the base.

5. Given the difference of the hypothenuse and perpendicular 16.5, and the angle at the perpendicular $37\frac{1}{4}°$, to find the base.

6. Given the difference of the base and perpendicular 35, and the angle at the perpendicular $27\frac{1}{4}°$, to find the sides.

234. The following solutions may be applied to the *third* and *fourth* cases of *oblique* angled triangles; in one of which, two sides and the included angle are given, and in the other, the three sides. See pages 87 and 88.

CASE III.

In astronomical calculations, it is frequently the case, that two sides of a triangle are given by their *logarithms*. By the following proposition, the necessity of finding the corresponding natural numbers is avoided.

THEOREM A. *In any plane triangle, of the two sides which include a given angle, the less is to the greater; as radius to the tangent of an angle greater than* 45°:

And radius is to the tangent of the excess of this angle above 45°; *as the tangent of half the sum of the opposite angles to the tangent of half their difference.*

In the triangle ABC, (Fig. 39.) let the sides AC and AB, and the angle A, be given. Through A draw DH perpendicular to AC. Make AD and AF each equal to AC, and AH equal to AB. And let HG be perpendicular to a line drawn from C through F.

Then AC : AB : : R : tan ACH.

And R : tan (ACH—45°) : : tan ½ (ACB+B) : tan ½ (ACB—B)

Demonstration.

In the right angled triangle ACD, as the acute angles are subtended by the equal sides AC and AD, each is 45°. For the same reason, the acute angles in the triangle CAF are each 45°. Therefore, the angle DCF is a right angle, the angles GFH and GHF are each 45°, and the line GH is equal to GF and parallel to DC.

In the triangle ACH, if AC be radius, AH, which is equal to AB, will be the tangent of ACH. Therefore,

AC : AB : : R : tan ACH.

In the triangle CGH, if CG be radius, GH, which is equal to FG, will be the tangent of HCG. Therefore,

R : tan (ACH—45°) : : CG : FG

And, as GH and DC are parallel, (Euc. 2. 6.)

$$CG : FG :: DH : FH.$$

But DH is, by construction, equal to the *sum*, and FH to the *difference* of AC and AB. And by theorem II, (Art. 144.) the sum of the sides is to their difference; as the tangent of half the sum of the opposite angles, to the tangent of half their difference. Therefore,

$$R : \tan(ACH-45°) :: \tan \tfrac{1}{2}(ACB+B) : \tan \tfrac{1}{2}(ACB\backsim B)$$

Ex. In the triangle ABC, (Fig. 30.) given the angle A = 26° 14′, the side AC = 39, and the side AB = 53.

AC	39	1.5910646	R	10.
AB	53	1.7242759	Tan 8° 39′ 9″	9.1823381
R		10.	Tan ½(B+C) 76° 53′	10.6326181
Tan 53° 39′ 9″		10.1332113	Tan ½(B∼C) 33° 8′ 50″	9.8149562

The same result is obtained here, as by theorem II, p. 75.

To find the required *side* in this third case, by the theorems in section IV, it is necessary to find, in the first place, an *angle* opposite one of the given sides. But the required side may be obtained, in a different way, by the following proposition.

THEOREM B. *In a plane triangle, twice the product of any two sides, is to the difference between the sum of the squares of those sides, and the square of the third side, as radius to the cosine of the angle included between the two sides.*

In the triangle ABC, (Fig. 23.) whose sides are a, b, and c,

$$2bc : b^2+c^2-a^2 :: R : \cos A$$

For in the right angled triangle ACD, $b : d :: R : \cos A$

Multiplying by $2c$, $2bc : 2dc :: R : \cos A$

But, by Euclid 13. 2, $2dc = b^2+c^2-a^2$

Therefore, $2bc : b^2+c^2-a^2 :: R : \cos A.$

The demonstration is the same, when the angle A is *obtuse*, as in the triangle ABC, (Fig. 24.) except that a^2 is *greater* than b^2+c^2; (Euc. 12. 2.) so that the cosine of A is *negative*. See art. 194.

From this theorem are derived expressions, both for the *sides* of a triangle, and for the cosines of the *angles*. Converting the last proportion into an equation, and proceeding in the same manner with the other sides and angles, we have the following expressions:

For the angles. For the sides.

$$\begin{cases} \text{Cos A} = R \times \dfrac{b^2+c^2-a^2}{2bc} \\ \text{Cos B} = R \times \dfrac{a^2+c^2-b^2}{2ac} \\ \text{Cos C} = R \times \dfrac{a^2+b^2-c^2}{2ab} \end{cases} \qquad \begin{cases} a = \sqrt{\left(b^2+c^2-\dfrac{2bc \cos A}{R}\right)} \\ b = \sqrt{\left(a^2+c^2-\dfrac{2ac \cos B}{R}\right)} \\ c = \sqrt{\left(a^2+b^2-\dfrac{2ab \cos C}{R}\right)} \end{cases}$$

These formulæ are useful, in many trigonometrical investigations; but are not well adapted to logarithmic computation.

CASE IV.

When the *three sides* of a triangle are given, the angles may be found, by either of the following theorems; in which a, b, and c, are the sides, A, B, and C, the opposite angles, and h=half the sum of the sides.

$$\text{THEOREM C.} \quad \begin{cases} \text{Sin A} = \dfrac{2R}{bc}\sqrt{h\,(h-a)\,(h-b)\,(h-c)} \\ \text{Sin B} = \dfrac{2R}{ac}\sqrt{h\,(h-a)\,(h-b)\,(h-c)} \\ \text{Sin C} = \dfrac{2R}{ab}\sqrt{h\,(h-a)\,(h-b)\,(h-c)} \end{cases}$$

The quantities under the radical sign are the same in all the equations.

In the triangle ACD, (Fig. 23.)

$R : b :: \sin A : p$. Therefore, $\sin A \times b = R \times p$.

But $p = \dfrac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}$. (Art. 221, p. 121.)

This, by the reductions in page 122, becomes

$$p=\frac{\sqrt{2h\times2(h-a)\times2(h-b)\times2(h-c)}}{2c}$$

Substituting this value of p, and reducing,

$$\text{Sin A}=\frac{2\text{R}}{bc}\sqrt{h(h-a)(h-b)(h-c)}$$

The arithmetical calculations may be made, by adding the logarithms of the factors under the radical sign, dividing the sum by 2, and to the quotient, adding the logarithms of radius and 2, and the arithmetical complements of the logarithms of b and c. (Arts. 39, 47, 59.)

Ex. Given $a=134$, $b=108$, and $c=80$, to find A, B, and C.

For the angle A.

h	161	log.	2.2068259
$h-a$	27	log.	1.4313638
$h-b$	53	log.	1.7242759
$h-c$	81	log.	1.9084850
			2)7.2709506
			* 3.6354753
R×2		log.	10.3010300
			13.9365053
b	108	a. c.	7.9665762
c	80	a. c.	8.0969100
Sin A.			9.9999915
A = 89° 28′ 31″			

For the angle B.

			13.9365053
a	134	a. c.	7.8728952
c	80	a. c.	8.0969100
Sin B.			9.9063105
B = 53° 42′ 9″			

For the angle C.

			13.9365053
a	134	a. c.	7.8728952
b	108	a. c.	7.9665762
Sin C.			9.7759767
C = 36° 39′ 20″			

THEOREM D.

$$\begin{cases}\text{Sin }\tfrac{1}{2}\text{A}=\text{R}\sqrt{\dfrac{(h-b)(h-c)}{bc}}\\[2ex]\text{Sin }\tfrac{1}{2}\text{B}=\text{R}\sqrt{\dfrac{(h-a)(h-c)}{ac}}\\[2ex]\text{Sin }\tfrac{1}{2}\text{C}=\text{R}\sqrt{\dfrac{(h-a)(h-b)}{ab}}\end{cases}$$

By Art. 210, $2\,\text{Sin}^2\tfrac{1}{2}\text{A}=\text{R}^2-\text{R}\times\cos\text{A}$.

Substituting for cos A, its value, as given in page 132,

$$2\,\text{Sin}^2\tfrac{1}{2}\text{A}=\text{R}^2-\text{R}^2\times\frac{b^2+c^2-a^2}{2bc}$$

* This is the logarithm of the *area* of the triangle. (Art. 222.)

But $R^2=R^2\times\frac{2bc}{2bc}$. And $-R^2\times\frac{b^2+c^2-a^2}{2bc}=R^2\times\frac{a^2-b^2-c^2}{2bc}$

$$\text{Therefore, } 2\,\text{Sin}^2\tfrac{1}{2}A=R^2\times\frac{2bc+a^2-b^2-c^2}{2bc}$$

But $2bc+a^2-b^2-c^2=a^2-(b-c)^2=(a+b-c)\,(a-b+c)$ (Alg. 235.)

Putting then $h=\frac{1}{2}(a+b+c)$, reducing, and extracting;

$$\text{Sin }\tfrac{1}{2}A=R\sqrt{\frac{(h-b)\,(h-c)}{bc}}$$

Ex. Given a, b, and c, as before, to find A and B.

For the angle A.

$h-b$	53		1.7242759
$h-c$	81		1.9084850
b	108	a. c.	7.9665762
c	80	a. c.	8.0969100
			2)19.6962471
Sin $\frac{1}{2}$A			9.8481235
A = 69° 38′ 31″			

For the angle B.

$h-a$	27		1.4313638
$h-c$	81		1.9084850
a	134	a. c.	7.8728952
c	80	a. c.	8.0969100
			2)19.3096540
Sin $\frac{1}{2}$B			9.6548270
B = 53° 42′ 9″			

THEOREM E.
$$\begin{cases}\text{Cos }\tfrac{1}{2}A=R\sqrt{\dfrac{h\,(h-a)}{bc}}\\[2ex]\text{Cos }\tfrac{1}{2}B=R\sqrt{\dfrac{h\,(h-b)}{ac}}\\[2ex]\text{Cos }\tfrac{1}{2}C=R\sqrt{\dfrac{h\,(h-c)}{ab}}\end{cases}$$

By Art. 210, $2\,\text{Cos}^2\tfrac{1}{2}A=R^2+R\times\cos A$.

Substituting and reducing, as in the demonstration of the last theorem,

$$2\,\text{Cos}^2\tfrac{1}{2}A=R^2\times\frac{2bc+b^2+c^2-a^2}{2bc}=R^2\times\frac{(b+c+a)\,(b+c-a)}{2bc}$$

Putting $h=\frac{1}{2}(a+b+c)$, reducing and extracting,

$$\text{Cos }\tfrac{1}{2}A=R\sqrt{\frac{h\,(h-a)}{bc}}$$

Ex. Given the sides 134, 108, 80; to find B and C.

For the angle B.

h	161		2.2068259
$h-b$	53		1.7242759
a	134	a. c.	7.8728952
c	80	a. c.	8.0969100
			2)19.9009070
Cos ½B			9.9504535

B = 53° 42′ 9″

For the angle C

h	161		2.2068259
$h-c$	81		1.9084850
a	134	a. c.	7.8728952
b	108	a. c.	7.9665762
			2)19.9547823
Cos ½C			9.9773911

C = 36° 39′ 20″

THEOREM F.

$$\begin{cases} \text{Tan } \tfrac{1}{2}A = R\sqrt{\dfrac{(h-b)(h-c)}{h(h-a)}} \\ \text{Tan } \tfrac{1}{2}B = R\sqrt{\dfrac{(h-a)(h-c)}{h(h-b)}} \\ \text{Tan } \tfrac{1}{2}C = R\sqrt{\dfrac{(h-a)(h-b)}{h(h-c)}} \end{cases}$$

The tangent is equal to the product of radius and the sine, divided by the cosine. (Art. 216.) By the last two theorems, then,

$$\text{Tan } \tfrac{1}{2}A = \frac{R \sin \tfrac{1}{2}A}{\cos \tfrac{1}{2}a} = R^2\sqrt{\frac{(h-b)(h-c)}{bc}} \div R\sqrt{\frac{h(h-a)}{bc}}$$

$$\text{That is, } \tan \tfrac{1}{2}A = R\sqrt{\frac{(h-b)(h-c)}{h(h-a)}}$$

Ex. Given the sides as before, to find A and C.

For the angle A.

$h-b$	53		1.7242759
$h-c$	81		1.9084850
$h-a$	27	a. c.	8.5686362
h	161	a. c.	7.7931741
			2)19.9945712
Tan ½A			9.9972856

A = 89° 38′ 31″

For the angle C.

$h-a$	27		1.4313638
$h-b$	53		1.7242759
$h-c$	81	a. c.	8.0915150
h	161	a. c.	7.7931741
			2)19.0403288
Tan ½C			9.5201644

C = 36° 39′ 20″

The three last theorems give the angle required, ***without ambiguity***. For the *half* of any angle must be less than 90°.

Of these different methods of solution, each has its advantages in particular cases. It is expedient to find an angle, sometimes by its sine, sometimes by its cosine, and sometimes by its tangent.

By the first of the four preceding theorems, marked C, D, E, and F, the calculation is made for the *sine* of the *whole* angle; by the others, for the *sine*, *cosine*, or *tangent*, of *half* the angle. For finding an angle near 90°, each of the three last theorems is preferable to the first. In the example above, A would have been uncertain to several seconds, by theorem C, if the other two angles had not been determined also.

But for a very *small* angle, the first method has an advantage over the others. The third, by which the calculation is made for the *cosine* of half the required angle, is in this case the most defective of the four. The second will not answer well for an angle which is almost 180°. For the *half* of this is almost 90°; and near 90°, the differences of the sines are very small.

NOTES.

NOTE A. Page 1.

THE *name* Logarithm is from λόγος, *ratio*, and ἀριθμὸς, *number*. Considering the ratio of a to 1 as a *simple* ratio, that of a^2 to 1 is a *duplicate* ratio, of a^3 to 1 a *triplicate* ratio, &c. (Alg. 354.) Here the *exponents* or *logarithms* 2, 3, 4, &c., show how many times the simple ratio is *repeated as a factor*, to form the compound ratio. Thus, the ratio of 100 to 1, is the *square* of the ratio of 10 to 1; the ratio of 1000 to 1, is the *cube* of the ratio of 10 to 1, &c. On this account, logarithms are called the *measures* of ratios; that is, of the ratios which different numbers bear to unity. See the Introduction to Hutton's Tables, and Mercator's Logarithmo-Technia, in Maseres' Scriptores Logarithmici.

NOTE B. p. 4.

If 1 be added to $-.09691$, it becomes $1-.09691$, which is equal to $+.90309$. The decimal is here rendered positive, by *subtracting* the figures from 1. But it is made 1 too great. This is compensated, by adding -1 to the *integral* part of the logarithm. So that $-2-.09691=-3+.90309$.

In the same manner, the decimal part of any logarithm which is wholly negative, may be rendered positive, by subtracting it from 1, and adding -1 to the index. The subtraction is most easily performed, by taking the right hand significant figure from 10, and each of the other figures from 9. (Art. 55.)

On the other hand, if the index of a logarithm be negative, while the decimal part is positive; the whole may be rendered negative, by subtracting the decimal part from 1, and taking -1 from the index.

Note C. p. 7.

It is common to *define* logarithms to be a series of numbers in arithmetical progression, corresponding with another series in geometrical progression. This is calculated to perplex the learner, when, upon opening the tables, he finds that the natural numbers, as they stand there, instead of being in *geometrical*, are in *arithmetical* progression; and that the logarithms are *not* in arithmetical progression.

It is true, that a geometrical series may be obtained, by taking out, here and there, a few of the natural numbers; and that the logarithms of these will form an arithmetical series. But the definition is not applicable to the whole of the numbers and logarithms, as they stand in the tables.

The supposition that positive and negative numbers have the same series of logarithms, (p. 7.) is attended with some theoretical difficulties. But these do not affect the practical rules for calculating by logarithms.

Note D. p. 43.

To revert a series, of the form

$$x = an + bn^2 + cn^3 + dn^4 + en^5 +, \text{ \&c.}$$

that is, to find the value of n, in terms of x, *assume* a series, with indeterminate co-efficients, (Alg. 490. *b*.)

$$\text{Let } n = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 +, \text{ \&c.}$$

Finding the powers of this value of n, by multiplying the series into itself, and arranging the several terms according to the powers of x; we have

$$\begin{array}{llll}
n^2 = A^2x^2 + 2ABx^3 & \left.\begin{matrix}+2AC \\ +\ B^2\end{matrix}\right\} x^4 & \left.\begin{matrix}+2BC \\ +2AD\end{matrix}\right\} x^5 +, \text{ \&c.} \\
n^3 = \quad A^3x^3 & +3A^2Bx^4 & \left.\begin{matrix}+3A^2C \\ +3AB^2\end{matrix}\right\} x^5 +, \text{ \&c.} \\
n^4 = & \quad A^4x^4 & +4A^3Bx^5 +, \text{ \&c.} \\
n^5 = & & \quad A^5x^5 +, \text{ \&c.}
\end{array}$$

Substituting these values, for n and its powers, in the first series above, we have

$$x=\left\{\begin{matrix} aAx+aB \\ +bA^2 \end{matrix}\right\} x^2 \left.\begin{matrix} +aC \\ +2bAB \\ +\ cA^3 \end{matrix}\right\} x^3 \left.\begin{matrix} +\ aD \\ +\ 2bAC \\ +\ bB^2 \\ +3cA^2B \\ +\ dA^4 \end{matrix}\right\} x^4 \left.\begin{matrix} +\ aE \\ +\ 2bBC \\ +\ 2bAD \\ +3cA^2C \\ +3cAB^2 \\ +4dA^3B \\ +\ eA^5 \end{matrix}\right\} x^5$$

Transposing x, and making the co-efficients of the several powers of x each equal to 0, we have

$aA-1=0,$
$aB+bA^2=0,$
$aC+2bAB+cA^3=0,$
$aD+2bAC+bB^2+3cA^2B+dA^4=0,$
$aE+2bBC+2bAD+3cA^2C+3cAB^2+4dA^3B+eA^5=0.$

And reducing the equations,

$$A=\frac{1}{a}$$

$$B=-\frac{b}{a^3}$$

$$C=\frac{2b^2-ac}{a^5}$$

$$D=-\frac{5b^3-5abc+a^2d}{a^7}$$

$$E=\frac{14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e}{a^9}$$

These are the values of the co-efficients A, B, C, &c., in the assumed series

$$n=Ax+Bx^2+Cx^3+Dx^4+Ex^5+, \&c.$$

Applying these results to the logarithmic series; (Art. 66. p. 43.)

$$x=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-, \&c.$$

in which

$$a=1,\ b=-\tfrac{1}{2},\ c=\tfrac{1}{3},\ d=-\tfrac{1}{4},\ e=\tfrac{1}{5},$$

we have, in the inverted series,

$$n=Ax+Bx^2+Cx^3+Dx^4+Ex^5+, \&c.$$

$$A=\frac{1}{a}=1 \qquad D=\frac{1}{2.3.4}$$

$$B=-b=\tfrac{1}{2}$$

$$C=2b^2-ac=\frac{1}{2.3} \qquad E=\frac{1}{2.3.4.5}$$

Therefore,

$$n=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+, \&c.$$

Note E, p. 50.

According to the scheme lately introduced into France, of dividing the denominations of weights, measures, &c., into tenths, hundredths, &c., the fourth part of a circle is divided into 100 degrees, a degree into 100 minutes, a minute into 100 seconds, &c. The whole circle contains 400 of these degrees; a plane triangle 200. If a right angle be taken for the measuring *unit;* degrees, minutes, and seconds, may be written as decimal fractions. Thus, 36° 5′ 49″ is 0.360549.

According to the French division $\left\{\begin{array}{c} 10^\circ=9^\circ \\ 100'=54' \\ 1000''=324'' \end{array}\right\}$ English.

Note F, p. 82.

If the perpendicular be drawn from the angle opposite the longest side, it will always fall *within* the triangle; because the other two angles must, of course, be acute. But if one of the angles at the base be *obtuse,* the perpendicular will fall *without* the triangle, as CP, (Fig. 38.)

In this case, the side on which the perpendicular falls, is to the sum of the other two; as the difference of the latter, to the *sum* of the segments made by the perpendicular.

The demonstration is the same, as in the other case, except that AH=BP+PA, instead of BP—PA.

Thus, in the circle BDHL, (Fig. 38.) of which C is the center,

$$AB\times AH=AL\times AD\text{; therefore } AB : AD :: AL :: AH.$$

$$\begin{aligned}\text{But } AD&=CD+CA=CB+CA\\ \text{And } AL&=CL-CA=CB-CA\\ \text{And } AH&=HP+PA=BP+PA.\end{aligned}$$

Therefore,

$$AB : CB+CA :: CB-CA : BP+PA$$

When the three sides are given, it may be known whether one of the angles is obtuse. For any angle of a triangle is obtuse or acute, according as the square of the side subtending the angle is *greater*, or *less*, than the sum of the squares of the sides containing the angle. (Euc. 12, 13. 2.)

Note G. p. 104.

Gunter's *Sliding Rule*, is constructed upon the same principle as his scale, with the addition of a slider, which is so contrived as to answer the purpose of a pair of compasses, in working proportions, multiplying, dividing, &c. The lines on the *fixed part* are the same as on the scale. The *slider* contains two lines of numbers, a line of logarithmic sines, and a line of logarithmic tangents.

To *multiply* by this, bring 1 on the slider, against one of the factors on the fixed part; and against the other factor on the slider, will be the product on the fixed part. To divide, bring the divisor on the slider, against the dividend on the fixed part; and against 1 on the slider, will be the quotient on the fixed part. To work a *proportion*, bring the first term on the slider, against one of the middle terms on the fixed part; and against the other middle term on the slider, will be the fourth term on the fixed part. Or the first term may be taken on the fixed part; and then the fourth term will be found on the slider.

Another instrument frequently used in trigonometrical constructions, is

THE SECTOR.

This consists of two equal scales movable about a point as a center. The lines which are drawn on it are of two kinds; some being parallel to the sides of the instrument, and others diverging from the central point, like the radii of a circle. The latter are called the *double* lines, as each is repeated upon the two scales. The *single* lines are of the same nature, and have the same use, as those which are put upon the common scale; as the lines of equal parts, of chords, of latitude, &c., on one face; and the logarithmic lines of numbers, of sines, and of tangents, on the other.

The *double* lines are

A line of *Lines*, or equal parts, marked	Lin. or L.
A line of *Chords*,	Cho. or C.
A line of natural *Sines*,	Sin. or S.
A line of natural *Tangents* to 45°,	Tan. or T.
A line of tangents *above* 45°,	Tan. or T.
A line of natural *Secants*,	Sec. or S·
A line of *Polygons*,	Pol. or P.

The double lines of *chords*, of *sines*, and of *tangents* to 45°, are all of the same radius; beginning at the central point, and terminating near the other extremity of each scale; the chords at 60°, the sines at 90°, and the tangents at 45°. (See Art. 95.) The line of *lines* is also of the same length, containing ten equal parts which are numbered, and which are again subdivided. The radius of the lines of secants, and of tangents above 45°, is about one fourth of the length of the other lines. From the end of the radius, which for the secants is at 0, and for the tangents at 45°, these lines extend to between 70° and 80°. The line of polygons is numbered 4, 5, 6, &c., from the extremity of each scale, towards the center.

The simple principle on which the utility of these several pairs of lines depends is this, that *the sides of similar triangles are proportional.* (Euc. 4. 6.) So that sines, tangents, &c., are furnished *to any radius*, within the extent of the opening of the two scales. Let AC and AC′ (Fig. 40.) be any pair of lines on the sector, and AB and AB′ equal portions of these lines. As AC and AC′ are equal, the triangle ACC′ is isosceles, and similar to ABB′. Therefore,

$$AB : AC :: BB' : CC'.$$

Distances measured from the center on either scale, as AB and AC, are called *lateral distances.* And the distances between corresponding points of the two scales, as BB′ and CC′, are called *transverse distances.*

Let AC and CC′ be radii of two circles. Then, if AB be the chord, sine, tangent, or secant, of any number of degrees in one; BB′ will be the chord, sine, tangent, or secant, of the same number of degrees in the other. (Art. 119.) Thus, to find the *chord* of 30°, to a radius of four inches, open the sector so as to make the transverse distance from 60 to 60, on the lines of chords, four inches; and the distance from 30 to 30, on the same lines, will be the chord required. To find the *sine* of 28°, make the distance from 90 to 90, on the lines of sines, equal to radius; and the distance from 28 to 28 will be the sine. To find the *tangent* of 37°, make the distance from 45 to 45, on the lines of tangents, equal to radius; and the distance from 37 to 37 will be the tangent. In finding *secants*, the distance from 0 to 0 must be made radius. (Art. 201.)

To lay down an *angle* of 34°, describe a circle, of any convenient radius, open the sector, so that the distance from 60 to 60 on the lines of chords shall be equal to this radius, and to the circle apply a chord equal to the distance from 34 to 34. (Art. 161.) For an angle above 60°, the chord of *half* the number of degrees may be taken, and applied *twice* on the arc, as in Art. 161.

The line of *polygons* contains the chords of arcs of a circle which is divided into equal portions. Thus, the distances from the center of the sector to 4, 5, 6, and 7, are the chords of $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ of a circle. The distance 6 is the radius. (Art. 95.) This line is used to make a regular polygon, or to inscribe one in a given circle. Thus, to make a *pentagon* with the transverse distance from 6 to 6 for radius, describe a circle, and the distance from 5 to 5 will be the length of one of the sides of a pentagon inscribed in that circle.

The line of *lines* is used to divide a line into equal or proportional parts, to find fourth proportionals, &c. Thus, to divide a line into 7 equal parts, make the length of the given line the transverse distance from 7 to 7, and the distance from 1 to 1 will be one of the parts. To find $\frac{3}{5}$ of a line, make the transverse distance from 5 to 5 equal to the given line; and the distance from 3 to 3 will be $\frac{3}{5}$ of it.

In working the *proportions in trigonometry* on the sector, the lengths of the sides of triangles are taken from the line of lines, and the degrees and minutes from the lines of sines, tangents, or secants. Thus, in Art. 135, ex. 1,

$$35 : R :: 26 : \sin 48^{\circ}.$$

To find the fourth term of this proportion by the sector, make the lateral distance 35 on the line of lines, a transverse distance from 90 to 90 on the lines of sines; then the lateral distance 26 on the line of lines, will be the transverse distance from 48 to 48 on the lines of sines.

For a more particular account of the construction and uses of the Sector, see Stone's edition of Bion on Mathematical Instruments, Hutton's Dictionary, and Robertson's Treatise on Mathematical Instruments.

Note H. p. 124.

The error in supposing that arcs less than 1 minute are proportional to their sines, cannot affect the first ten places of decimals. Let AB and AB′ (Fig. 41.) each equal 1 minute. The tangents of these arcs BT and B′T are equal, as are also the sines BS and B′S. The arc BAB′ is greater than BS + B′S, but less than BT + B′T. Therefore BA is greater than BS, but less than BT: that is, *the difference between the sine and the arc is less than the difference between the sine and the tangent.*

Now the sine of 1 minute is	0.000290888216
And the tangent of 1 minute is	0.000290888204
The difference is	0.0000000000012

The difference between the sine and the arc of 1 minute is less than this; and the error in supposing that the sines of $1'$, and of $0'\ 52''\ 44'''\ 3''''\ 45'''''$ are proportional to their arcs, as in Art. 223, is still less.

Note I. p. 125.

There are various ways in which sines and cosines may be more *expeditiously* calculated, than by the method which is given here. But as we are already supplied with accurate trigonometrical tables, the computation of the canon is, to the great body of our students, a subject of speculation, rather than of practical utility. Those who wish to enter into a minute examination of it, will of course consult the treatises in which it is particularly considered.

There are also numerous formulæ of *verification*, which are used to detect the errors with which any part of the calculation is liable to be affected. For these, see Legendre's and Woodhouse's Trigonometry, Lacroix's Differential Calculus, and particularly Euler's Analysis of Infinites.

Note K, p. 127.

The following rules for finding the sine or tangent of a very small arc, and, on the other hand, for finding the arc from its sine or tangent, are taken from Dr. Maskelyne's Introduction to Taylor's Logarithms.

To find the logarithmic SINE *of a very small arc.*

From the sum of the constant quantity 4.6855749, and the logarithm of the given arc reduced to seconds and decimals, subtract one third of the arithmetical complement of the logarithmic cosine.

To find the logarithmic TANGENT *of a very small arc.*

To the sum of the constant quantity 4.6855749, and the logarithm of the given arc reduced to seconds and decimals, add two thirds of the arithmetical complement of the logarithmic cosine.

To find a small arc from its logarithmic SINE.

To the sum of the constant quantity 5.3144251, and the given logarithmic sine, add one third of the arithmetical complement of the logarithmic cosine. The remainder diminished by 10, will be the logarithm of the number of seconds in the arc.

To find a small arc from its logarithmic TANGENT.

From the sum of the constant quantity 5.3144251, and the given logarithmic tangent, subtract two thirds of the arithmetical complement of the logarithmic cosine. The remainder, diminished by 10, will be the logarithm of the number of seconds in the arc.

For the demonstration of these rules, see Woodhouse's Trigonometry, p. 189

A

TABLE

OF

NATURAL SINES AND TANGENTS;

TO EVERY TEN MINUTES OF A DEGREE.

If the given angle is less than 45°, look for the title of the column, at the *top* of the page; and for the degrees and minutes, on the *left*. But if the angle is between 45° and 90°, look for the title of the column, at the *bottom;* and for the degrees and minutes, on the *right*.

D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
0° 0′	0.0000000	0.0000000	Infinite.	1.0000000	90° 0′
10	0029089	0029089	343.77371	0.9999958	50
20	0058177	0058178	171.88540	9999831	40
30	0087265	0087269	114.58865	9999619	30
40	0116353	0116361	85.939791	9999323	20
0° 50′	0145439	0145454	68.750087	9998942	89° 10′
1° 0′	0.0174524	0.0174551	57.289962	0.9998477	89° 0′
10	0203608	0203650	49.103881	9997927	50
20	0232690	0232753	42.964077	9997292	40
30	0261769	0261859	38.188459	9996573	30
40	0290847	0290970	34.367 71	9995770	20
1° 50′	0319922	0320086	31.241577	9994881	88° 10′
2° 0′	0.0348995	0.0349208	28.636253	0.9993908	88° 0′
10	0378065	0378335	26.431600	9992851	50
20	0407131	0407469	24.541758	9991709	40
30	0436194	0436609	22.903766	9990482	30
40	0465253	0465757	21.470401	9989171	20
2° 50′	0494308	0494913	20.205553	9987775	87° 10′
3° 0′	0.0523360	0.0524078	19.081137	0.9986295	87° 0′
10	0552406	0553251	18.074977	9984731	50
20	0581448	0582434	17.169337	9983082	40
30	0610485	0611626	16.349855	9981348	30
40	0639517	0640829	15.604784	9979530	20
3° 50′	0668544	0670043	14.924417	9977627	86° 10′
4° 0′	0.0697565	0.0699268	14.300666	0.9975641	86° 0′
10	0726580	0728505	13.726738	9973569	50
20	0755589	0757755	13.196883	9971413	40
30	0784591	0787017	12.706205	9969173	30
40	0813587	0816293	12.250505	9966849	20
4° 50′	0842576	0845583	11.826167	9964440	85° 10′
5° 0′	0.0871557	0.0874887	11.430052	0.9961947	85° 0′
10	0900532	0904206	11.059431	9959370	50
20	0929499	0933540	10.711913	9956708	40
30	0958458	0962890	10.385397	9953962	30
40	0987408	0992257	10.078031	9951132	20
5° 50′	1016351	1021641	9.7881732	9948217	84° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
6° 0′	0.1045285	0.1054042	9.5143645	0.9945219	84° 0′
10	1074210	1080462	9.2553035	9942136	50
20	1103126	1109899	9.0098261	9938969	40
30	1132032	1139356	8.7768874	9935719	30
40	1160929	1168832	8.5555468	9932384	20
6° 50′	1189816	1198329	8.3449558	9928065	83° 10′
7° 0′	0.1218693	0.1227846	8.1443464	0.9925462	83° 0′
10	1247560	1257384	7.9530224	9921874	50
20	1276416	1286943	7.7703506	9918204	40
30	1305262	1316525	7.5957541	9914449	30
40	1334096	1346129	7.4287064	9910610	20
7° 50′	1362919	1375757	7.2687255	9906687	82° 10′
8° 0′	0.1391731	0.1404085	7.1153697	0.9902681	82° 0′
10	1420531	1435084	6.9682335	9898590	50
20	1449319	1464784	6.8269437	9894416	40
30	1478094	1494510	6.6911562	9890159	30
40	1506857	1524262	6.5605538	9885817	20
8° 50′	1535607	1554040	6.4348428	9881392	81° 10′
9° 0′	0.1564345	0.1583844	6.3137515	0.9876883	81° 0′
10	1593069	1613677	6.1970279	9872291	50
20	1621779	1643537	6.0844381	9867615	40
30	1650476	1673426	5.9757644	9862856	30
40	1679159	1703344	5.8708042	9858013	20
9° 50′	1707828	1733292	5.7693688	9853087	80° 10′
10° 0′	0.1736482	0.1763270	5.6712818	0.9848078	80° 0′
10	1765121	1793279	5.5763786	9842985	50
20	1793746	1823319	5.4845052	9837808	40
30	1822355	1853390	5.3955172	9832549	30
40	1850949	1883495	5.3092793	9827206	20
10° 50′	1879528	1913632	5.2256647	9821781	79° 10′
11° 0′	0.1908090	0.1943803	5.1445540	0.9816272	79° 0′
10	1936636	1974008	5.0658352	9810680	50
20	1965166	2004248	4.9894027	9805005	40
30	1993679	2034523	4.9151570	9799247	30
40	2022176	2064834	4.8430045	9793406	20
11° 50′	2050655	2095181	4.7728568	9787483	78° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
12° 0′	0.2079117	0.2125566	4.7046301	0.9781476	78° 0′
10	2107561	2155988	4.6382457	9775387	50
20	2135088	2186448	4.5736287	9769215	40
30	2164396	2216947	4.5107085	9762960	30
40	2192786	2247485	4.4494181	9756623	20
12° 50′	2221158	2278063	4.3896940	9750203	77° 10′
13° 0′	0.2249511	0.2308682	4.3314759	0.9743701	77° 0′
10	2277844	2339342	4.2747066	9737116	50
20	2306159	2370044	4.2193318	9730449	40
30	2334454	2400788	4.1652998	9723699	30
40	2362729	2431575	4.1125614	9716867	20
13° 50′	2390984	2462405	4.0610700	9709953	76° 10′
14° 0′	0.2419219	0.2493280	4.0107809	0.9702957	76° 0′
10	2447433	2524200	3.9616518	9695879	50
20	2475627	2555165	3.9136420	9688719	40
30	2503800	2586176	3.8667131	9681476	30
40	2531952	2617234	3.8208281	9674152	20
14° 50′	2560082	2648339	3.7759519	9666746	75° 10′
15° 0′	0.2588190	0.2679492	3.7320508	0.9659258	75° 0′
10	2616277	2710694	3.6890927	9651689	50
20	2644342	2741945	3.6470467	9644037	40
30	2672384	2773245	3.6058835	9636305	30
40	2700403	2804597	3.5655749	9628490	20
15° 50′	2728400	2835999	3.5260938	9620594	74° 10′
16° 0′	0.2756374	0.2867454	3.4874144	0.9612617	74° 0′
10	2784324	2898961	3.4495120	9604558	50
20	2812251	2930521	3.4123626	9596418	40
30	2840153	2962135	3.3759434	9588197	30
40	2868032	2993803	3.3402326	9579895	20
16° 50′	2895887	3025527	3.3052091	9571512	73° 10′
17° 0′	0.2923717	0.3057307	3.2708526	0.9563048	73° 0′
10	2951522	3089143	3.2371438	9554502	50
20	2979303	3121036	3.2040638	9545876	40
30	3007058	3152988	3.1715948	9537170	30
40	3034788	3184998	3.1397194	9528382	20
17° 50′	3062492	3217067	3.1084210	9519514	72° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
18° 0′	0.3090170	0.3219197	3.0776835	0.9510565	72° 0′
10	3117822	3281387	3.0474915	9501536	50
20	3145448	3313639	3.0178301	9492426	40
30	3173047	3345953	2.9886850	9483237	30
40	3200619	3378330	2.9600422	9473966	20
18° 50′	3228164	3410771	2.9318885	9464616	71° 10′
19° 0′	0.3255682	0.3443276	2.9042109	0.9455186	71° 0′
10	3283172	3475846	2.8769970	9445675	50
20	3310634	3508483	2.8502349	9436085	40
30	3338069	3541186	2.8239129	9425415	30
40	3365475	3573956	2.7980198	9416665	20
19° 50′	3392852	3606795	2.7725448	9406835	70° 10′
20° 0′	0.3420201	0.3639702	2.7474774	0.9396926	70° 0′
10	3447521	3672680	2.7228076	9386938	50
20	3474812	3705728	2.6985254	9376869	40
30	3502074	3738847	2.6746215	9366722	30
40	3529306	3772038	2.6510867	9356495	20
20° 50′	3556508	3805302	2.6279121	9346189	69° 10′
21° 0′	0.3583679	0.3838640	2.6050891	0.9335804	69° 0′
10	3610821	3872053	2.5826094	9325340	50
20	3637932	3905541	2.5604649	9314797	40
30	3665012	3939105	2.5386479	9304176	30
40	3692061	3972746	2.5171507	9293475	20
21° 50′	3719079	4006465	2.4959661	9282696	68° 10′
22° 0′	0.3746066	0.4040262	2.4750869	0.9271839	68° 0′
10	3773021	4074139	2.4545061	9260902	50
20	3799944	4108097	2.4342172	9249888	40
30	3826834	4142136	2.4142136	9238795	30
40	3853693	4176257	2.3944889	9227624	20
22° 50′	3880518	4210460	2.3750372	9216375	67° 10′
23° 0′	0.3907311	0.4244748	2.3558524	0.9205049	67° 0′
10	3934071	4279121	2.3369287	9193644	50
20	3960798	4313579	2.3182606	9182161	40
30	3987491	4348124	2.2998425	9170601	30
40	4014150	4382756	2.2816693	9158963	20
23° 50′	4040775	4417477	2.2637357	9147247	66° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

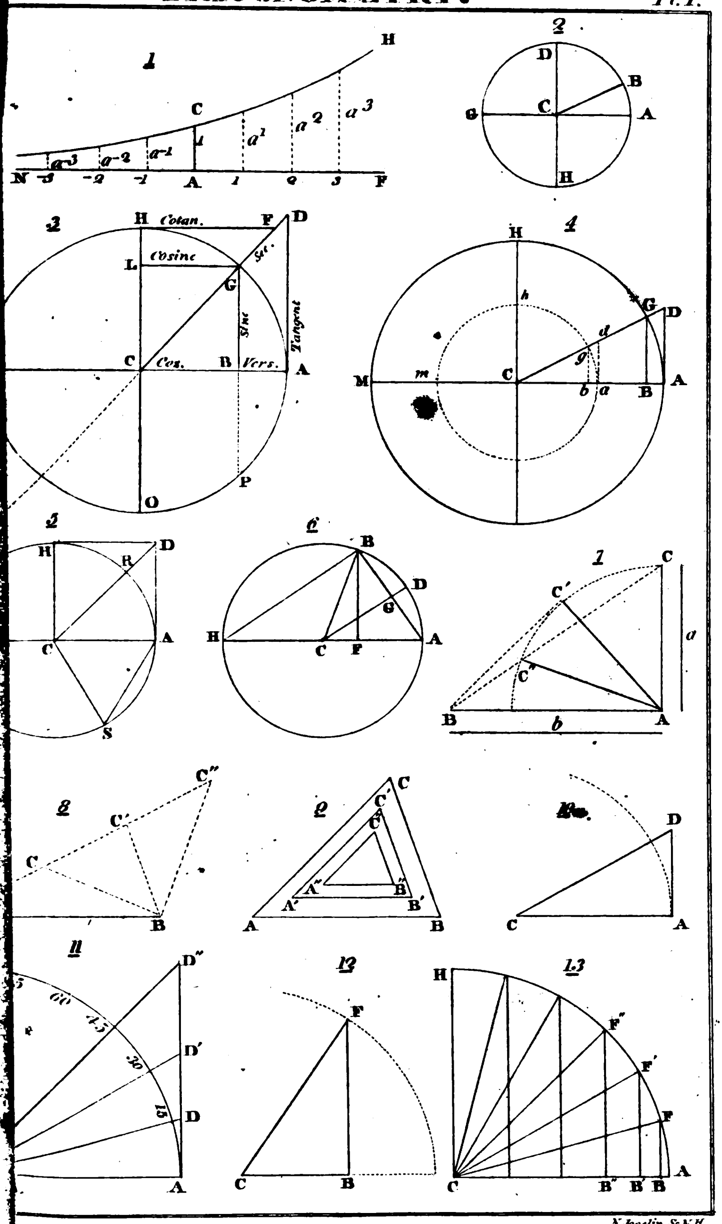
D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
24° 0′	0.4067366	0.4452287	2.2460368	0.9135455	66° 0′
10	4093923	4487187	2.2285676	9123584	50
20	4120445	4522179	2.2113234	9111637	40
30	4146932	4557263	2.1942997	9099613	30
40	4173385	4593439	2.1774920	9087511	20
24° 50′	4199801	4627710	2.1608958	9075333	65° 10′
25° 0′	0.4226183	0.4663077	2.1445069	0.9063078	65° 0′
10	4252528	4698539	2.1283213	9050746	50
20	4278838	4734098	2.1123348	9038338	40
30	4305111	4769755	2.0965436	9025853	30
40	4331348	4805512	2.0809438	9013292	20
25° 50′	4357548	4841368	2.0655318	9000654	64° 10′
26° 0′	0.4383711	0.4877326	2.0503038	0.8987940	64° 0′
10	4409838	4913386	2.0352565	8975151	50
20	4435927	4949549	2.0203862	8962285	40
30	4461978	4985816	2.0056897	8949344	30
40	4487992	5022189	1.9911637	8936326	20
26° 50′	4513967	5058668	1.9768050	8923234	63° 10′
27° 0′	0.4539905	0.5095254	1.9626105	0.8910065	63° 0′
10	4565804	5131950	1.9485772	8896822	50
20	4591665	5168755	1.9347020	8883503	40
30	4617486	5205671	1.9209821	8870108	30
40	4643269	5242698	1.9074147	8856639	20
27° 50′	4669012	5279839	1.8939971	8843095	62° 10′
28° 0′	0.4694716	0.5317094	1.8807265	0.8829476	62° 0′
10	4720380	5354465	1.8676003	8815782	50
20	4746004	5391952	1.8546159	8802014	40
30	4771588	5429557	1.8417709	8788171	30
40	4797131	5467281	1.8290628	8774254	20
28° 50′	4822634	5505125	1.8164892	8760263	61° 10′
29° 0′	0.4848096	0.5543091	1.8040478	0.8746197	61° 0′
10	4873517	5581179	1.7917362	8732058	50
20	4898897	5619391	1.7795524	8717844	40
30	4924236	5657728	1.7674940	8703557	30
40	4949532	5696191	1.7555590	8689196	20
29° 50′	4974787	5734783	1.7437453	8674762	60° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

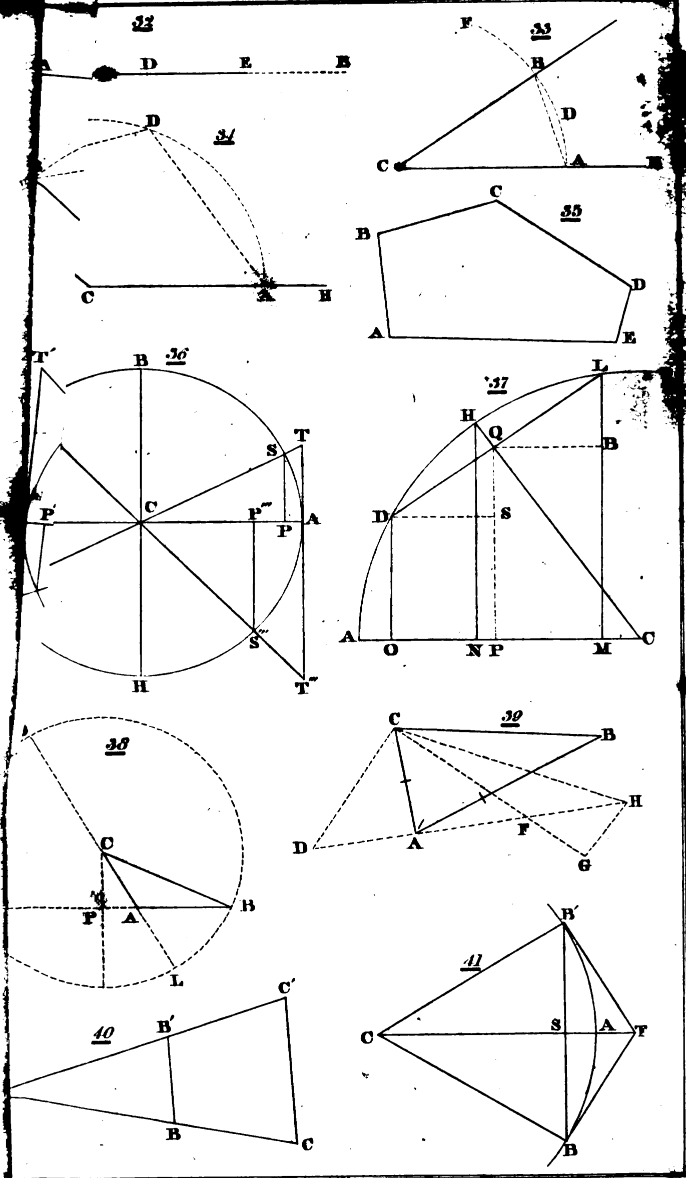
D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
30° 0′	0.5000000	0.5773503	1.7320508	0.8660254	60° 0′
10	5025170	5812353	1.7204736	8645673	50
20	5050298	5851335	1.7090116	8631019	40
30	5075384	5890450	1.6976631	8616292	30
40	5100426	5929699	1.6864261	8601491	20
30° 50′	5125425	5969084	1.6752988	8585619	59° 10′
31° 0′	0.5150381	0.6008606	1.6642795	0.8571673	59° 0′
10	5175293	6048266	1.6533663	8556655	50
20	5200161	6088067	1.6425576	8541564	40
30	5224986	6128008	1.6318517	8526402	30
40	5249766	6168092	1.6212469	8511167	20
31° 50′	5274502	6208320	1.6107417	8495860	58° 10′
32° 0′	0.5299193	0.6248694	1.6003345	0.8480481	58° 0′
10	5323839	6289214	1.5900238	8465030	50
20	5348440	6329883	1.5798079	8449508	40
30	5372996	6370703	1.5696856	8433914	30
40	5397507	6411673	1.5596552	8418249	20
32° 50′	5421971	6452797	1.5497155	8402513	57° 10′
33° 0′	0.5446390	0.6494076	1.5398650	0.8386706	57° 0′
10	5470763	6535511	1.5301023	8370827	50
20	5495090	6577103	1.5204261	8354878	40
30	5519370	6618856	1.5108352	8338858	30
40	5543603	6660769	1.5013282	8322768	20
33° 50′	5567790	6702845	1.4919039	8306607	56° 10′
34° 0′	0.5591929	0.6745085	1.4825610	0.8290376	56° 0′
10	5616021	6787492	1.4732983	8274074	50
20	5640066	6830066	1.4641147	8257703	40
30	5664062	6872810	1.4550090	8241262	30
40	5688011	6915725	1.4459801	8224751	20
34° 50′	5711912	6958813	1.4370268	8208170	55° 10′
35° 0′	0.5735764	0.7002075	1.4281480	0.8191520	55° 0′
10	5759568	7045515	1.4193427	8174801	50
20	5783323	7089133	1.4106098	8158013	40
30	5807030	7132931	1.4019483	8141155	30
40	5830687	7176911	1.3933571	8124229	20
35° 50′	5854294	7221075	1.3848353	8107234	54° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
36° 0′	0.5877853	0.7265425	1.3763819	0.8090170	54° 0′
10	5901361	7309963	1.3679959	8073038	50
20	5924819	7354691	1.3596764	8055837	40
30	5948228	7399611	1.3514224	8038569	30
40	5971586	7444724	1.3432331	8021232	20
36° 50′	5994893	7490033	1.3351075	8003827	53° 10′
37° 0′	0.6018150	0.7535541	1.3270448	0.7986355	53° 0′
10	6041356	7581248	1.3190441	7968815	50
20	6064511	7627157	1.3111046	7951208	40
30	6087614	7673270	1.3032254	7933533	30
40	6110666	7719589	1.2954057	7915792	20
37° 50′	6133666	7766118	1.2876447	7897983	52° 10′
38° 0′	0.6156615	0.7812856	1.2799416	0.7880108	52° 0′
10	6179511	7859808	1.2722957	7862165	50
20	6202355	7906975	1.2647062	7844157	40
30	6225146	7954359	1.2571723	7826082	30
40	6247885	8001963	1.2496933	7807940	20
38° 50′	6270571	8049790	1.2422685	7789733	51° 10′
39° 0′	0.6293204	0.8097840	1.2348972	0.7771460	51° 0′
10	6315784	8146118	1.2275786	7753121	50
20	6338310	8194625	1.2203121	7734716	40
30	6360782	8243364	1.2130970	7716246	30
40	6383201	8292337	1.2059327	7697710	20
39° 50′	6405566	8341547	1.1988184	7679110	50° 10′
40° 0′	0.6427876	0.8390996	1.1917536	0.7660444	50° 0′
10	6450132	8440688	1.1847376	7641714	50
20	6472334	8490624	1.1777698	7622919	40
30	6494480	8540807	1.1708496	7604060	30
40	6516572	8591240	1.1639763	7585136	20
40° 50′	6538609	8641926	1.1571495	7566148	49° 10′
41° 0′	0.6560590	0.8692867	1.1503684	0.7547096	49° 0′
10	6582516	8744067	1.1436326	7527980	50
20	6604386	8795528	1.1369414	7508800	40
30	6626200	8847253	1.1302944	7489557	30
40	6647959	8899244	1.1236909	7470251	20
41° 50′	6669661	8951506	1.1171305	7450881	48° 10′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

D. M.	Sine.	Tangent.	Cotangent.	Cosine.	D. M.
42° 0′	0.6691306	0.9004040	1.1106125	0.7431448	48° 0′
10	6712895	9056851	1.1041365	7411953	50
20	6734427	9109940	1.0977020	7392394	40
30	6755902	9163312	1.0913085	7372773	30
40	6777320	9216969	1.0849554	7353090	20
42° 50′	6798681	9270914	1.0786423	7333345	47° 10′
43° 0′	0.6819984	0.9325151	1.0723687	0.7313537	47° 0′
10	6841229	9379683	1.0661341	7293668	50
20	6862416	9434513	1.0599381	7273736	40
30	6883546	9489646	1.0537801	7253744	30
40	6904617	9545083	1.0476598	7233690	20
43° 50′	6925630	9600829	1.0415767	7213574	46° 10′
44° 0′	0.6946584	0.9656888	1.0355303	0.7193398	46° 0′
10	6967479	9713262	1.0295203	7173161	50
20	6988315	9769956	1.0235461	7152863	40
30	7009093	9826973	1.0176074	7132504	30
40	7029811	9884316	1.0117038	7112086	20
44° 50′	7050469	9941991	1.0058348	7091607	45° 10′
45° 0′	0.7071068	1.0000000	1.0000000	0.7071068	45° 0′
D. M.	Cosine.	Cotangent.	Tangent.	Sine.	D. M.

The *Secants and Cosecants*, which are not inserted in this table, may be easily supplied. If 1 be divided by the cosine of an arc, the quotient will be the secant of that arc. (Art. 228.) And if 1 be divided by the sine, the quotient will be the cosecant.


Cotan.
Cosine
Sine
Tangent
Cos.
Vers.
Sec.
N. Jocelin Sc. N.H.


32
A D E B
33
F B D C A E
34
D C A H
35
C B D A E
36
T' B S T C P' P''' P A S''' H T'''
37
L H Q B D S A O N P M C
38
C Q P A B L
39
C B H D A F G
40
B' C' B C
41
B' C S A T B
N. Jocelin Sc. N.H.

A

PRACTICAL APPLICATION

OF

THE PRINCIPLES OF GEOMETRY

TO THE

MENSURATION

OF

SUPERFICIES AND SOLIDS:

BEING

THE THIRD PART

OF

A COURSE OF MATHEMATICS,

ADAPTED TO THE METHOD OF INSTRUCTION IN THE AMERICAN COLLEGES.

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The following short Treatise contains little more than an application of the principles of Geometry, to the numerical calculation of the superficial and solid contents of such figures as are treated of in the Elements of Euclid. As the plan proposed for the work of which this number is a part, does not admit of introducing rules and propositions which are not demonstrated; the particular consideration of the areas of the Conic Sections and other curves, with the contents of solids produced by their revolution, is reserved for succeeding parts of the course. The student would be little profited by applying arithmetical calculation, in a mechanical way, to figures of which he has not yet learned even the definitions. But as this number may fall into the hands of some who will not read those which are to follow, the principal rules for conic areas and solids, and for the gauging of casks, are given without demonstrations, in the appendix. Those who wish to take a complete view of Mensuration, in all its parts, are referred to the valuable treatise of Dr. Hutton on the subject.

CONTENTS.

SECTION I.

AREAS OF FIGURES BOUNDED BY RIGHT LINES.

ART. 1. THE following definitions, which are nearly the same as in Euclid, are inserted here for the convenience of reference.

I. *Four-sided* figures have different names, according to the relative position and length of the sides. A *parallelogram* has its opposite sides equal and parallel; as ABCD. (Fig. 2.) A *rectangle*, or *right parallelogram*, has its opposite sides equal, and all its angles right angles; as AC. (Fig. 1.) A *square* has all its sides equal, and all its angles right angles; as ABGH. (Fig. 3.) A *rhombus* has all its sides equal, and its angles oblique; as ABCD. (Fig. 3.) A *rhomboid* has its opposite sides equal, and its angles oblique; as ABCD. (Fig. 2.) A *trapezoid* has only two of its sides parallel; as ABCD. (Fig. 4.) Any other four sided figure is called a *trapezium*.

II. A figure which has more than four sides is called a *polygon*. A *regular* polygon has all its sides equal, and all its angles equal.

III. The *height* of a *triangle* is the length of a perpendicular, drawn from one of the angles to the opposite side; as CP. (Fig. 5.) The *height* of a *four sided* figure is the perpendicular distance between two of its parallel sides; as CP. (Fig. 4.)

IV. The *area* or *superficial contents* of a figure is the *space* contained within the line or lines by which the figure is bounded.

2. In calculating areas, some particular portion of surface is fixed upon, as the *measuring unit*, with which the given figure is to be compared. This is commonly a *square*; as a square inch, a square foot, a square rod, &c. For this reason, determining the quantity of surface in a figure is called *squaring it*, or finding its *quadrature*; that is, finding a square or number of squares to which it is equal.

3. The *superficial* unit has generally the same name, as the *linear* unit which forms the side of the square.

The side of a square inch is a linear inch;
of a square foot, a linear foot;
of a square rod, a linear rod, &c.

There are some superficial measures, however, which have no corresponding denominations of length. The *acre*, for instance, is not a square which has a line of the same name for its side.

The following tables contain the linear measures in common use, with their corresponding square measures.

Linear Measures.				*Square Measures.*			
12	inches	=1	foot.	144	inches	=1	foot.
3	feet	=1	yard.	9	feet	=1	yard.
6	feet	=1	fathom.	36	feet	=1	fathom.
$16\frac{1}{2}$	feet	=1	rod.	$272\frac{1}{4}$	feet	=1	rod.
$5\frac{1}{2}$	yards	=1	rod.	$30\frac{1}{4}$	yards	=1	rod.
4	rods	=1	chain.	16	rods	=1	chain.
40	rods	=1	furlong.	1600	rods	=1	furlong.
320	rods	=1	mile.	102400	rods	=1	mile.

An *acre* contains 160 square rods, or 10 square chains.

By reducing the denominations of square measure, it will be seen that

1 sq. mile=640 acres=102400 rods=27878400 feet=4014489600 inches.
1 acre=10 chains=160 rods=43560 feet=6272640 inches.

The fundamental problem in the mensuration of superficies is the very simple one of determining the area of a *right parallelogram.* The contents of other figures, particularly those which are rectilinear, may be obtained by finding parallelograms which are equal to them, according to the principles laid down in Euclid.

PROBLEM I.

To find the area of a PARALLELOGRAM, *square, rhombus, or rhomboid.*

4. MULTIPLY THE LENGTH BY THE PERPENDICULAR HEIGHT OR BREADTH.

It is evident that the number of *square* inches in the parallelogram AC (Fig. 1.) is equal to the number of *linear* inches in the length AB, repeated as many times as there are inches in the breadth BC. For a more particular illustration of this see Alg. 511—514.

The oblique parallelogram or rhomboid ABCD, (Fig. 2.) is equal to the right parallelogram GHCD. (Euc. 36. 1.) The area, therefore, is equal to the length AB multiplied into the perpendicular height HC. And the rhombus ABCD, (Fig. 3.) is equal to the *parallelogram* ABGH. As the sides of a *square* are all equal, its area is found, by *multiplying one of the sides into itself.*

Ex. 1. How many square feet are there in a floor $23\frac{1}{2}$ feet long, and 18 feet broad? Ans. $23\frac{1}{2}\times18=423$.

2. What are the contents of a piece of ground which is 66 feet square? Ans. 4356 sq. feet=16 sq. rods.

3. How many square feet are there in the four sides of a room which is 22 feet long, 17 feet broad, and 11 feet high? Ans. 858.

Art. 5. If the sides and angles of a parallelogram are given, the perpendicular height may be easily found by trigonometry. Thus, CH (Fig. 2.) is the perpendicular of a right angled triangle, of which BC is the hypothenuse. Then, (Trig. 134.)

$$R : BC :: \sin B : CH.$$

The area is obtained by multiplying CH thus found, into the length AB.

Or, to reduce the two operations to one,

As radius,
To the sine of any angle of a parallelogram;
So is the product of the sides including that angle,
To the area of the parallelogram.

For *the area*$=AB\times CH$, (Fig. 2.) But $CH=\dfrac{BC\times\sin B}{R}$.

Therefore,

The area$=\dfrac{AB\times BC\times\sin B}{R}$. Or, $R : \sin B :: AB\times BC :$ *the area.*

Ex. If the side AB be 58 rods, BC 42 rods, and the angle B 63°, what is the area of the parallelogram?

60 40 60

As radius		10.00000
To the sine of B	63°	9.94988
So is the product of AB	58	1.76343
Into BC (Trig. 39.)	42	1.62325
To the area	2170.5 sq. rods	3.33656

2. If the side of a rhombus is 67 feet, and one of the angles 73°, what is the area? Ans. 4292.7 feet.

6. When the dimensions are given in feet and inches, the multiplication may be conveniently performed by the arithmetical rule of *Duodecimals;* in which each inferior denomination is one twelfth of the next higher. Considering a foot as the measuring *unit*, a prime is the twelfth part of a foot; a second, the twelfth part of a prime, &c. It is to be observed, that, in measures of *length*, *inches* are *primes;* but in *superficial* measure they are *seconds*. In both, a prime is $\frac{1}{12}$ of a foot. But $\frac{1}{12}$ of a *square* foot is a parallelogram, a foot long and an inch broad. The twelfth part of this is a square inch, which is $\frac{1}{144}$ of a square foot.

Ex. 1. What is the surface of a board 9 feet 5 inches, by 2 feet 7 inches.

F	′	″
9	5′	
2	7	
18	10	
5	5	11
24	3	11″, or 24 feet 47 inches.

2. How many feet of glass are there in a window 4 feet 11 inches high, and 3 feet 5 inches broad?

Ans. 16 F. 9′ 7″, or 16 feet 115 inches.

7. If the area and one side of a parallelogram be given, the other side may be found by *dividing the area by the given side*. And if the area of a *square* be given, the side may be found by *extracting the square root of the area*. This is merely reversing the rule in art. 4. See Alg. 520, 521.

Ex. 1. What is the breadth of a piece of cloth which is 36 yds. long, and which contains 63 square yds.

Ans. $1\frac{3}{4}$ yds.

2. What is the side of a square piece of land containing 289 square rods?

3. How many yards of carpeting $1\frac{1}{4}$ yard wide, will cover a floor 30 feet long and $22\frac{1}{2}$ broad?

Ans. $30\times22\frac{1}{2}$ feet$=10\times7\frac{1}{2}=75$ yds. And $75\div1\frac{1}{4}=60$.

4. What is the side of a square which is equal to a parallelogram 936 feet long and 104 broad?

5. How many panes of 8 by 10 glass are there, in a window 5 feet high, and 2 feet 8 inches broad?

PROBLEM II.

To find the area of a TRIANGLE.

8. RULE I. MULTIPLY ONE SIDE BY HALF THE PERPENDICULAR FROM THE OPPOSITE ANGLE. Or, multiply half the side by the perpendicular. Or, multiply the whole side by the perpendicular, and take half the product.

The area of the triangle ABC, (Fig. 5.) is equal to $\frac{1}{2}$ PC$\times$AB, because a parallelogram of the same base and height is equal to PC$\times$AB, (Art. 4.) and by Euc. 41, 1, the triangle is half the parallelogram.

Ex. 1. If AB (Fig. 5.) be 65 feet, and PC 31.2, what is the area of the triangle? Ans. 1014 square feet.

2. What is the surface of a triangular board, whose base is 3 feet 2 inches, and perpendicular height 2 feet 9 inches?

Ans. 4 F. 4′ 3″, or 4 feet 51 inches.

9. If two sides of a triangle and the included angle, are given, the perpendicular on one of these sides may be easily found by rectangular trigonometry. And the area may be calculated in the same manner as the area of a parallelogram in art. 5. In the triangle ABC, (Fig. 2.)

$$R : BC :: \sin B : CH$$

And because the triangle is half the parallelogram of the same base and height,

As radius,
To the sine of any angle of a triangle;
So is the product of the sides including that angle,
To twice the area of the triangle. (Art. 5.)

Ex. If AC (Fig. 5.) be 39 feet, AB 65 feet, and the angle at A 53° 7′ 48″, what is the area of the triangle?

Ans. 1014 square feet.

9. *b*. If *one side* and the *angles* are given; then

As the product of radius and the sine of the angle opposite the given side,

To the product of the sines of the two other angles;
So is the square of the given side,
To twice the area of the triangle.

If PC (Fig. 5.) be perpendicular to AB.

$$R : \sin B :: BC : CP$$
$$\sin ACB : \sin A :: AB : BC$$

Therefore, (Alg. 390, 382.)

$R \times \sin ACB : \sin A \times \sin B :: AB \times BC : CP \times BC :: \overline{AB}^2 : AB \times CP =$ twice the area of the triangle.

Ex. If one side of a triangle be 57 feet, and the angles at the ends of this side 50° and 60°, what is the area?

Ans. 1147 sq. feet.

10. If the *sides* only of a triangle are given, an angle may be found, by oblique trigonometry, Case IV, and then the perpendicular and the area may be calculated. But the area may be more directly obtained, by the following method.

Rule II. When the three sides are given, *from half their sum subtract each side severally, multiply together the half sum and the three remainders, and extract the square root of the product.*

If the sides of the triangle are a, b, and c, and if h=half their sum, then

$$\textit{The area} = \sqrt{h \times (h-a) \times (h-b) \times (h-c)}$$

For the demonstration of this rule, see Trigonometry, Art. 221.

If the calculation be made by *logarithms*, add the logarithms of the several factors, and half their sum will be the logarithm of the area. (Trig. 39, 47.)

Ex. 1. In the triangle ABC, (Fig. 5.) given the sides a 52 feet, b 39, and c 65; to find the side of a square which has the same area as the triangle.

$\frac{1}{2}(a+b+c)=h=78$ $\quad h-b=39$
$h-a=26$ $\quad h-c=13$

Then the area $=\sqrt{78 \times 26 \times 39 \times 13} = 1014$ square feet.

By logarithms.

The half sum	=78	1.89209
First remainder	=26	1.41497
Second do.	=39	1.59106
Third do.	=13	1.11394
		2)6.01206
The area required	=1014	2)3.00603
Side of the square	=31.843 (Trig. 47.)	1.50301

2. If the sides of a triangle are 134, 108, and 80 rods, what is the area? Ans. 4319.

3. What is the area of a triangle whose sides [redacted], 264, and [redacted] feet?

11. In an *equilateral* triangle, one of whose sides is a, the expression for the area becomes

$$\sqrt{h\times(h-a)\times(h-a)\times(h-a)}$$

But as $h=\frac{3}{2}a$, and $h-a=\frac{3}{2}a-a=\frac{1}{2}a$, the area is

$$\sqrt{\tfrac{3}{2}a\times\tfrac{1}{2}a\times\tfrac{1}{2}a\times\tfrac{1}{2}a}=\sqrt{\tfrac{3}{16}a^4}=\tfrac{1}{4}a^2\sqrt{3}\ \text{(Alg. 271.)}$$

That is, the area of an equilateral triangle is equal to $\frac{1}{4}$ the square of one of its sides, multiplied into the square root of 3, which is 1.732.

Ex. 1. What is the area of a triangle whose sides are each 34 feet? Ans. $500\frac{1}{2}$ feet.

2. If the sides of a triangular field are each [redacted] rods, how many acres does it contain?

PROBLEM III.

To find the area of a TRAPEZOID.

21. MULTIPLY HALF THE SUM OF THE PARALLEL SIDES INTO THEIR PERPENDICULAR DISTANCE.

The area of the trapezoid ABCD, (Fig. 4.) is equal to half the sum of the sides AB and CD, multiplied into the perpendicular distance PC or AH. For the whole figure is made up of the two triangles ABC and ADC; the area of the first of which is equal to the product of half the base AB into the perpendicular PC, (Art. 8.) and the area of the other is equal to the product of half the base DC into the perpendicular AH or PC.

Ex. If AB (Fig. 4.) be 46 feet, BC 31, DC 38, and the angle B 70°, what is the area of the trapezoid?

R : BC : : sin B : PC=29.13. And 42×29.13=1223¼.

2. What are the contents of a field which has two parallel sides [redacted] and [redacted] rods, distant from each other [redacted] rods?

PROBLEM IV.

To find the area of a TRAPEZIUM, *or of an irregular* POLYGON.

13. DIVIDE THE WHOLE FIGURE INTO TRIANGLES, BY DRAWING DIAGONALS, AND FIND THE SUM OF THE AREAS OF THESE TRIANGLES. (Alg. 519.)

If the perpendiculars in two triangles fall upon the *same diagonal*, the area of the trapezium formed of the two triangles, is equal to half the product of the diagonal into the sum of the perpendiculars.

Thus the area of the trapezium ABCH, (Fig. 6.) is

$$\tfrac{1}{2}BH\times AL+\tfrac{1}{2}BH\times CM=\tfrac{1}{2}BH\times(AL+CM.)$$

Ex. In the irregular polygon ABCDH, (Fig. 6.)

if the diagonals $\left\{\begin{array}{l}BH=36,\\ CH=32,\end{array}\right.$ and the perpendiculars $\left\{\begin{array}{l}AL=5.3\\ CM=9.3\\ DN=7.3\end{array}\right.$

The area=18×14.6+16×7.3=379.6.

14. If the diagonals of a *trapezium* are given, the area may be found, nearly in the same manner as the area of a parallelogram in Art. 5, and the area of a triangle in Art. 9.

In the trapezium ABCD, (Fig. 8.) the sines of the four angles at N, the point of intersection of the diagonals, are all equal. For the two acute angles are *supplements* of the other two, and therefore have the same sine. (Trig. 90.) Putting, then, sin N for the sine of each of these angles, the areas of the four triangles of which the trapezium is composed, are given by the following proportions; (Art. 9.)

$$R : \sin N : : \left\{\begin{array}{l}BN\times AN : 2\,\textit{area}\ ABN\\ BN\times CN : 2\,\textit{area}\ BCN\\ DN\times CN : 2\,\textit{area}\ CDN\\ DN\times AN : 2\,\textit{area}\ ADN\end{array}\right.$$

And by addition, (Alg. 388, Cor. 1.*)

R : sin N : : BN×AN+BN×CN+DN×CN+DN×AN : 2 *area* ABCD.

The 3d term=(AN+CN)×(BN+DN)=AC×BD, by the figure.

Therefore, R : sin N : : AC×BD : 2 *area* ABCD. That is,

> As Radius,
> To the sine of the angle at the intersection of the diagonals of a trapezium;
> So is the product of the diagonals,
> To twice the area of the trapezium.

It is evident that this rule is applicable to a parallelogram, as well as to a trapezium.

If the diagonals intersect at *right angles*, the sine of N is equal to radius; (Trig. 95.) and therefore the product of the diagonals is equal to twice the area. (Alg. 395.†)

Ex. 1. If the two diagonals of a trapezium are 37 and 62, and if they intersect at an angle of 54°, what is the area of the trapezium? Ans. 928.

2. If the diagonals are [redacted] and [redacted], and the angle of intersection [redacted], what is the area of the trapezium?

14. *b.* When a trapezium can be *inscribed in a circle*, the area may be found by either of the following rules.

I. *Multiply together any two adjacent sides, and also the two other sides; then multiply half the sum of these products by the sine of the angle included by either of the pairs of sides multiplied together.*

Or,

II. ***From half the sum of all the sides, subtract each side severally, multiply together the four remainders, and extract the square root of the product.***

If the sides are a, b, c, and d; and if h=half their sum;

$$\text{The area}=\sqrt{(h-a)\times(h-b)\times(h-c)\times(h-d)}$$

* Euclid 2, 5. Cor. † Euclid 14. 5.

If the trapezium ABCD, (Fig. 33.) can be inscribed in a circle, the sum of the opposite angles BAD and BCD is 180° (Euc. 22. 3.) Therefore, the *sine* of BAD is equal to that of BCD or P′CD.

If s = the sine of either of these angles, radius being 1, and if

$$AB=a,\quad BC=b,\quad CD=c.\quad AD=d;$$

The triangle $BAD=\frac{1}{2}ad\times s$, And $BCD=\frac{1}{2}bc\times s$; (Art. 9.)

Therefore,

1. *The area of* $ABCD=\frac{1}{2}(ad+bc)\times s$.

To obtain the value of s, in terms of the sides of the trapezium, draw DP and DP′ perpendicular to BA and BC.

Then, Rad. : s : : AD : DP : : CD : DP′.

Also, $AP^2=AD^2-DP^2$, and $CP'^2=CD^2-DP'^2$.

So that $\begin{cases} DP=AD\times s=ds \\ DP'=CD\times s=cs \end{cases}$ And $\begin{cases} AP=\sqrt{d^2-d^2s^2}=d\sqrt{1-s^2} \\ CP'=\sqrt{c^2-c^2s^2}=c\sqrt{1-s^2} \end{cases}$

But by the figure $\begin{cases} BP=AB-AP=a-d\sqrt{1-s^2} \\ BP'=BC+CP'=b+c\sqrt{1-s^2} \end{cases}$

And $\overline{BP}^2+\overline{DP}^2=\overline{DB}^2=\overline{BP'}^2+\overline{DP'}^2$

That is $a^2-2ad\sqrt{1-s^2}+d^2=b^2+2bc\sqrt{1-s^2}+c^2$

Reducing the equation, we have

$$s^2=1-\frac{(b^2+c^2-a^2-d^2)^2}{(2ad+2bc)^2}, \text{ and}$$

$$s=\frac{\sqrt{(2ad+2bc)^2-(b^2+c^2-a^2-d^2)^2}}{2ad+2bc}$$

Substituting for s in the first rule, the value here found, we have the area of the trapezium, equal to

$$\tfrac{1}{4}\sqrt{(2ad+2bc)^2-(b^2+c^2-a^2-d^2)^2}$$

The expression under the radical sign is the difference of two squares, and may be resolved, as in Trig. 221, into the factors

$$(\overline{b+c}^2-\overline{a-d}^2)\times(\overline{a+d}^2-\overline{b-c}^2)$$

and these again into

$$(a+b+c-d)(b+c+d-a)(a+b+d-c)(a+d+c-b)$$

If then h = half the sum of the sides of the trapezium,

II. *The area*$=\sqrt{(h-a)\times(h-b)\times(h-c)\times(h-d)}$

If one of the sides, as d, is supposed to be diminished, till it is reduced to nothing; the figure becomes a *triangle*, and the expression for the area is the same as in art. 10. See Hutton's Mensuration.

PROBLEM V.

To find the area of a REGULAR POLYGON.

15. MULTIPLY ONE OF ITS SIDES INTO HALF ITS PERPENDICULAR DISTANCE FROM THE CENTER, AND THIS PRODUCT INTO THE NUMBER OF SIDES.

A regular polygon contains as many equal triangles as the figure has sides. Thus, the hexagon ABDFGH (Fig. 7.) contains six triangles, each equal to ABC. The area of one of them is equal to the product of the side AB, into half the perpendicular CP. (Art. 8.) The area of the whole, therefore, is equal to this product multiplied into the *number* of sides.

Ex. 1. What is the area of a regular octagon, in which the length of a side is ~~60~~, and the perpendicular from the center ~~72.426~~? 70 Ans. 17382.

2. What is the area of a regular decagon whose sides are 46 each, and the perpendicular 70.7867?

16. If only the length and number of sides of a regular polygon be given, the *perpendicular* from the center may be easily found by trigonometry. The periphery of the circle in which the polygon is inscribed, is divided into as many equal parts as the polygon has sides. (Euc. 16. 4. Schol.) The arc, of which one of the sides is a chord, is therefore known; and of course, the angle at the center subtended by this arc.

Let AB (Fig. 7.) be one side of a regular polygon inscribed in the circle ABDG. The perpendicular CP bisects the line AB, and the angle ACB. (Euc. 3. 3.) Therefore, BCP is the same part of 360°, which BP is of the perimeter of the polygon. Then, in the right angled triangle BCP, if BP be radius, (Trig. 122.)

R : BP : : cot BCP : CP. That is,

As Radius,
To half of one of the sides of the polygon;
So is the cotangent of the opposite angle,
To the perpendicular from the center.

Ex. 1. If the side of a regular hexagon (Fig. 7.) be 38 inches, what is the area?

$$\text{The angle BCP} = \tfrac{1}{12} \text{ of } 360^\circ = 30^\circ. \quad \text{Then,}$$

$$R : 19 : : \cot 30^\circ : 32.909 = CP, \text{ the perpendicular,}$$

$$\text{And the area} = 19 \times 32.909 \times 6 = 3751.6$$

2. What is the area of a regular decagon whose sides are each 62 feet? Ans. 29576.

17. From the proportion in the preceding article, a *table* of perpendiculars and areas may be easily formed, for a series of polygons, of which each side is a unit. Putting R=1, (Trig. 100.) and n=the number of sides, the proportion becomes

$$1 : \tfrac{1}{2} : : \cot \frac{360}{2n} : \textit{the perpendicular}$$

$$\text{So that, } \textit{the perp.} = \tfrac{1}{2} \cot \frac{360}{2n}$$

And the *area* is equal to half the product of the perpendicular into the number of sides. (Art. 15.)

Thus, in the trigon, or equilateral triangle, the perpendicular $= \tfrac{1}{2} \cot \frac{360^\circ}{6} = \tfrac{1}{2} \cot 60^\circ = 0.2886752$.

And the area $= 0.4330127$.

In the tetragon, or square, the perpendicular $= \tfrac{1}{2} \cot \frac{360^\circ}{8} = \tfrac{1}{2} \cot 45^\circ = 0.5$. And the area $= 1$.

In this manner, the following table is formed, in which the side of each polygon is supposed to be a unit.

A TABLE OF REGULAR POLYGONS.

Names.	Sides.	Angles.	Perpendiculars.	Areas.
Trigon,	3	60°	0.2886752	0.4330127
Tetragon,	4	45°	0.5000000	1.0000000
Pentagon,	5	36°	0.6881910	1.7204774
Hexagon,	6	30°	0.8660254	2.5980762
Heptagon,	7	$25\frac{5}{7}$	1.0382601	3.6339124
Octagon,	8	$22\frac{1}{2}$	1.2071069	4.8284271
Nonagon,	9	20°	1.3737385	6.1818242
Decagon,	10	18°	1.5388418	7.6942088
Undecagon,	11	$16\frac{4}{11}$	1.7028439	9.3656399
Dodecagon,	12	15°	1.8660252	11.1961524

By this table may be calculated the area of any other regular polygon, of the same number of sides with one of these. For the areas of similar polygons are as the *squares* of their homologous sides. (Euc. 20, 6.)

To find, then, the area of a regular polygon, *multiply the square of one of its sides by the area of a similar polygon of which the side is a unit.*

Ex. 1. What is the area of a regular decagon whose sides are each ~~102~~ rods? 110 Ans. 80050.5 rods.

2. What is the area of a regular dodecagon whose sides are each 87 feet?

SECTION II.*

THE QUADRATURE OF THE CIRCLE AND ITS PARTS.

ART. 18. *Definition* I. A CIRCLE is a plane bounded by a line which is equally distant in all its parts from a point within called the center. The bounding line is called the *circumference* or periphery. An *arc* is any portion of the circumference. A semi-circle is half, and a quadrant one fourth, of a circle.

II. A *Diameter* of a circle is a straight line drawn through the center, and terminated both ways by the circumference. A *Radius* is a straight line extending from the center to the circumference. A *Chord* is a straight line which joins the two extremities of an arc.

III. A Circular *Sector* is a space contained between an arc and the two radii drawn from the extremities of the arc. It may be *less* than a semi-circle, as ACBO, (Fig. 9.) or *greater*, as ACBD.

IV. A Circular *Segment* is the space contained between an arc and its chord, as ABO or ABD. (Fig. 9.) The chord is sometimes called the *base* of the segment. The *height* of a segment is the perpendicular from the middle of the base to the arc, as PO. (Fig. 9.)

V. A Circular *Zone* is the space between two parallel chords, as AGHB. (Fig. 15.) It is called the *middle* zone, when the two chords are equal.

VI. A Circular *Ring* is the space between the peripheries of two concentric circles, as AA′, BB′. (Fig. 13.)

VII. A *Lune* or Crescent is the space between two circular arcs which intersect each other, as ACBD. (Fig. 14.)

19. The *Squaring of the Circle* is a problem which has exercised the ingenuity of distinguished mathematicians for

* Wallis's Algebra, Legendre's Geometry, Book IV, and Note IV. Hutton's Mensuration, Horseley's Trigonometry, Book I, Sec. 3; Introduction to Euler's Analysis of Infinites, London Phil. Trans. Vol. VI, No. 75, LXVI, p. 476, LXXXIV, p. 217, and Hutton's abridgment of do. Vol. II, p. 547.

many centuries. The result of their efforts has been only an *approximation* to the value of the area. This can be carried to a degree of exactness far beyond what is necessary for practical purposes.

20. If the *circumference* of a circle of given diameter were known, its area could be easily found. For the area is equal to the product of half the circumference into half the diameter. (Sup. Euc. 5, 1.*) But the circumference of a circle has never been exactly determined. The method of approximating to it is by inscribing and circumscribing *polygons*, or by some process of calculation which is, in principle, the same. The perimeters of the polygons can be easily and exactly determined. That which is circumscribed is *greater*, and that which is inscribed is *less*, than the periphery of the circle; and by increasing the number of sides, the difference of the two polygons may be made less than any given quantity. (Sup. Euc. 4, 1.)

21. The side of a *hexagon* inscribed in a circle, as AB, (Fig. 7.) is the chord of an arc of 60°, and therefore equal to the radius. (Trig. 95.) The chord of *half* this arc, as BO, is the side of a polygon of 12 equal sides. By repeatedly bisecting the arc, and finding the chord, we may obtain the side of a polygon of an immense number of sides. Or we may calculate the *sine*, which will be half the chord of double the arc, (Trig. 82, cor.); and the *tangent*, which will be half the side of a similar *circumscribed* polygon. Thus the sine AP, (Fig. 7,) is half of AB, a side of the inscribed hexagon; and the tangent NO is half of NT, a side of the circumscribed hexagon. The difference between the sine and the arc AO is less than the difference between the sine and the tangent. In the section on the computation of the canon, (Trig. 223.) by 12 successive bisections, beginning with 60 degrees, an arc is obtained which is the $\frac{1}{24576}$ of the whole circumference.

The *cosine* of this, if radius be 1, is found to be .99999996732
The *sine* is .00025566346

And the tangent$=\dfrac{\text{sine}}{\text{cosine}}$ (Trig. 228.) $=$.00025566347

The diff. between the sine and tangent is only .00000000001
And the difference between the sine and the *arc* is still less.

* In this manner, the *Supplement* to *Playfair's Euclid* is referred to in this work.

Yale

Taking then .000255663465 for the length of the arc, multiplying by 24576, and retaining 8 places of decimals, we have 6.28318531 for the whole circumference, the radius being 1. Half of this,

3.14159265

is the circumference of a circle whose radius is $\frac{1}{2}$, and *diameter* 1.

22. If this be multiplied by 7, the product is 21.99+ or 22 nearly. So that,

Diam : Circum : : 7 : 22, nearly.

If 3.14159265 be multiplied by 113, the product is 354.9999+, or 355, very nearly. So that,

Diam : Circum : : 113 : 355, very nearly.

The first of these ratios was demonstrated by Archimedes.

There are various methods, principally by infinite series and fluxions, by which the labor of carrying on the approximation to the periphery of a circle may be very much abridged. The calculation has been extended to nearly 150 places of decimals.* But four or five places are sufficient for most practical purposes.

After determining the ratio between the diameter and the circumference of a circle, the following problems are easily solved.

PROBLEM.

To find the CIRCUMFERENCE *of a circle from its diameter.*

23. MULTIPLY THE DIAMETER BY 3.14159.†

Or,

Multiply the diameter by 22 *and divide the product by* 7. Or, multiply the diameter by 355, and divide the product by 113. (Art. 22.)

Ex. 1. If the diameter of the earth be 7930 miles, what is the circumference? Ans. 249128 miles.

2. How many miles does the earth move, in revolving round the sun; supposing the orbit to be a circle whose diameter is 190 million miles? Ans. 596,902,100.

* See note A. † In many cases, 3.1416 will be sufficiently accurate.

3. What is the circumference of a circle whose diameter is 769843 rods?

PROBLEM II.

To find the DIAMETER *of a circle from its circumference.*

24. DIVIDE THE CIRCUMFERENCE BY 3.14159.

Or,

Multiply the circumference by 7, and divide the product by 22. Or, multiply the circumference by 113, and divide the product by 355. (Art. 22.)

Ex. 1. If the circumference of the sun be 2,800,000 miles, what is his diameter? Ans. 891,267.

2. What is the diameter of a tree which is 5½ feet round?

25. As multiplication is more easily performed than division, there will be an advantage in exchanging the *divisor* 3.14159 for a *multiplier* which will give the same result. In the proportion

$$3.14159 : 1 :: \text{Circum} : \text{Diam.}$$

to find the fourth term, we may divide the second by the first, and multiply the quotient into the third. Now, $1 \div 3.14159 = 0.31831$. If, then, the circumference of a circle be multiplied by .31831, the product will be the diameter.*

Ex. 1. If the circumference of the moon be 6850 miles, what is her diameter? Ans. 2180.

2. If the whole extent of the orbit of Saturn be 5650 million miles, how far is he from the sun?

3. If the periphery of a wheel be 4 feet 7 inches, what is its diameter?

PROBLEM III.

To find the length of an ARC *of a circle.*

26. *As* 360°, *to the number of degrees in the arc;*
So is the circumference of the circle, to the length of the arc.

The circumference of a circle being divided into 360°, (Trig. 73.) it is evident that the length of an arc of any less number of degrees must be a proportional part of the whole.

* See note B.

Ex. What is the length of an arc of 16°, in a circle whose radius is 50 feet?

The circumference of the circle is 314.159 feet. (Art. 23.)

Then 360 : 16 : : 314.159 : 13.96 feet.

2. If we are 95 millions of miles from the sun, and if the earth revolves round it in $365\frac{1}{4}$ days, how far are we carried in 24 hours? Ans. 1 million 634 thousand miles.

27. The length of an arc may also be found, by multiplying the diameter into the number of degrees in the arc, and this product into .0087266, which is the length of *one* degree, in a circle whose diameter is 1. For 3.14159 ÷ 360=0.0087266. And in different circles, the circumferences, and of course the degrees, are as the diameters. (Sup. Euc. 8, 1.)

Ex. 1. What is the length of an arc of 10° 15′ in a circle whose radius is 68 rods? Ans. 12.165 rods.

2. If the circumference of the earth be 24913 miles, what is the length of a degree at the equator?

28. The length of an arc is frequently required, when the *number of degrees* is not given. But if the radius of the circle, and either the *chord* or the *height* of the arc, be known; the number of degrees may be easily found.

Let AB (Fig. 9.) be the chord, and PO the height, of the arc AOB. As the angles at P are right angles, and AP is equal to BP; (Art. 18. Def. 4.) AO is equal to BO. (Euc. 4, 1.) Then,

BP is the *sine*, and CP the *cosine*,
OP the *versed sine*, and BO the *chord*; } of *half* the arc AOB.

And in the right angled triangle CBP,

CB : R : : { BP : sin BCP or BO
{ CP : cos BCP or BO

Ex. 1. If the radius CO (Fig. 9.)=25, and the chord AB =43.3; what is the length of the arc AOB?

CB : R : : BP : sin BCP or BO=60° very nearly.

The circumference of the circle =3.14159×50=157.08.

And 360° : 60° : : 157.08 : 26.18= OB. Therefore, AOB=52.36.

2. What is the length of an arc whose chord is $216\frac{1}{2}$, in a circle whose radius is 125? Ans. 261.8.

29. If only the *chord* and the *height* of an arc be given, the radius of the circle may be found, and then the length of the arc.

If BA (Fig. 9.) be the chord, and PO the height of the arc AOB, then (Euc. 35. 3.)

$$DP=\frac{\overline{BP}^2}{OP}. \quad \text{And } DO=OP+DP=OP+\frac{\overline{BP}^2}{OP}.$$

That is, the *diameter* is equal to the height of the arc, + the square of half the chord divided by the height.

The diameter being found, the length of the arc may be calculated by the two preceding articles.

Ex. 1. If the chord of an arc be 173.2, and the height 50, what is the length of the arc?

$$\text{The diameter}=50+\frac{\overline{86.6}^2}{50}=200.$$ The arc contains 120°, (Art. 28.) and its length is 209.44. (Art. 26.)

2. What is the length of an arc whose chord is ~~120~~ 150, and height ~~45~~? 50 Ans. 160.8.*

PROBLEM IV.

To find the AREA *of a* CIRCLE.

30. MULTIPLY THE SQUARE OF THE DIAMETER BY THE DECIMALS .7854.

Or,

MULTIPLY HALF THE DIAMETER INTO HALF THE CIRCUMFERENCE. Or, multiply the whole diameter into the whole circumference, and take $\frac{1}{4}$ of the product.

The area of a circle is equal to the product of half the diameter into half the circumference; (Sup. Euc. 5, 1.) or which is the same thing, $\frac{1}{4}$ the product of the diameter and circumference. If the diameter be 1, the circumference is 3.14159; (Art. 23.) one fourth of which is 0.7854 nearly. But the areas of different circles are to each other, *as the squares of their diameters*. (Sup. Euc. 7, 1.)† The area of any circle, therefore, is equal to the product of the square of

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* See note C. † Euclid 2. 12.

its diameter into 0.7854, which is the area of a circle whose diameter is 1.

Ex. 1. What is the area of a circle whose diameter is 623 feet? Ans. 304836 square feet.

2. How many acres are there in a circular island whose diameter is 124 rods? Ans. 75 acres, and 76 rods.

3. If the diameter of a circle be 113, and the circumference 355, what is the area? Ans. 10029.

4. How many square yards are there in a circle whose diameter is 7 feet?

31. If the *circumference* of a circle be given, the area may be obtained, by first finding the diameter; or, without finding the diameter, by multiplying the square of the circumference by .07958.

For, if the circumference of a circle be 1, the diameter $= 1 \div 3.14159 = 0.31831$; and $\frac{1}{4}$ the product of this into the circumference is .07958 the area. But the areas of different circles, being as the squares of their diameters, are also as the squares of their *circumferences*. (Sup. Euc. 8, 1.)

Ex. 1. If the circumference of a circle be 136 feet, what is the area? Ans. 1472 feet.

2. What is the surface of a circular fish-pond, which is 10 rods in circumference?

32. If the area of a circle be *given*, the diameter may be found, by dividing the area by .7854, and extracting the square root of the quotient.

This is reversing the rule in art. 30.

Ex. 1. What is the diameter of a circle whose area is 380.1336 feet?

Ans. $380.1336 \div .7854 = 484$. And $\sqrt{484} = 22$.

2. What is the diameter of a circle whose area is 19.635?

33. The area of a circle, is to the area of the *circumscribed square*; as .7854 to 1; and to that of the *inscribed* square as .7854 to $\frac{1}{2}$.

Let ABDF (Fig. 10.) be the inscribed square and LMNO, the circumscribed square, of the circle ABDF. The area of the circle is equal to $\overline{AD}^2 \times .7854$. (Art. 30.) But the area of the circumscribed square (Art. 4.) is equal to $ON^2 = AD^2$.

And the smaller square is half of the larger one. For the latter contains 8 equal triangles, of which the former contains only 4.

Ex. What is the area of a square inscribed in a circle whose area is 159? Ans. .7854 : $\frac{1}{2}$: : 159 : 101.22.

PROBLEM V.

To find the area of a SECTOR *of a circle.*

34. MULTIPLY THE RADIUS INTO HALF THE LENGTH OF THE ARC. Or,

As 360, TO THE NUMBER OF DEGREES IN THE ARC;
So IS THE AREA OF THE CIRCLE, TO THE AREA OF THE SECTOR.

It is evident, that the area of the sector has the same ratio to the area of the circle, which the length of the arc has to the length of the whole circumference; or which the number of *degrees* in the arc has to the number of degrees in the circumference.

Ex. 1. If the arc AOB (Fig. 9.) be 120°, and the diameter of the circle 226; what is the area of the sector AOBC?

The area of the whole circle is 40115. (Art. 30.)
And 360° : 120° : : 40115 : $13371\frac{2}{3}$, the area of the sector.

2. What is the area of a quadrant whose radius is 621?

3. What is the area of a semi-circle whose diameter is 328?

4. What is the area of a sector which is less than a semicircle, if the radius be 15, and the chord of its arc 12?

Half the chord is the sine of 23° $34\frac{3}{4}'$ nearly. (Art. 28.)
The whole arc, then, is 47° $9\frac{1}{2}'$
The area of the circle is 706.86
And 360° : 47° $9\frac{1}{2}'$: : 706.86 : 92.6 the area of the sector

5. If the arc ADB (Fig. 9.) be 240 degrees, and the radius of the circle 113, what is the area of the sector ADBC?

PROBLEM VI.

To find the area of a SEGMENT *of a circle.*

35. FIND THE AREA OF THE SECTOR WHICH HAS THE SAME ARC, AND ALSO THE AREA OF THE TRIANGLE FORMED BY THE CHORD OF THE SEGMENT AND THE RADII OF THE SECTOR.

THEN, IF THE SEGMENT BE LESS THAN A SEMI-CIRCLE, SUBTRACT THE AREA OF THE TRIANGLE FROM THE AREA OF THE SECTOR. BUT, IF THE SEGMENT BE GREATER THAN A SEMI-CIRCLE, ADD THE AREA OF THE TRIANGLE TO THE AREA OF THE SECTOR.

If the triangle ABC, (Fig. 9.) be taken from the sector AOBC, it is evident the difference will be the segment AOBP, less than a semi-circle. And if the same triangle be added to the sector ADBC, the sum will be the segment ADBP, greater than a semi-circle.

The area of the triangle (Art. 8.) is equal to the product of half the chord AB into CP, which is the difference between the radius and PO the height of the segment. Or CP is the *cosine* of half the arc BOA. If this cosine, and the chord of the segment are not given, they may be found from the arc and the radius.

Ex. 1. If the arc AOB (Fig. 9.) be 120°, and the radius of the circle be 113 feet, what is the area of the segment AOBP?

In the right angled triangle BCP,

R : BC : : sin BCO : BP=97.86, half the chord. (Art. 28.)

The cosine PC=$\frac{1}{2}$CO (Trig. 96, Cor.)	=56.5
The area of the sector AOBC (Art. 34.)	=13371.67
The area of the triangle ABC=BP×PC	= 5528.97
The area of the segment, therefore,	= 7842.7

2. If the base of a segment, less than a semi-circle, be 10 feet, and the radius of the circle 12 feet, what is the area of the segment?

The arc of the segment contains	49$\frac{1}{4}$ degrees.	(Art. 28.)
The area of the sector	=61.89	(Art. 34.)
The area of the triangle	=54.54	
And the area of the segment	= 7.35 square feet.	

3. What is the area of a circular segment, whose height is 19.2 and base 70? Ans. 947.86.

4. What is the area of the segment ADBP, (Fig. 9.) if the base AB be 195.7, and the height PD 169.5?

Ans. 32272.*

36. The area of any figure which is bounded *partly* by arcs of circles, and partly by right lines, may be calculated, by finding the areas of the segments under the arcs, and then the area of the rectilinear space between the chords of the arcs and the other right lines.

Thus, the Gothic arch ACB, (Fig. 11.) contains the two segments ACH, BCD, and the plane triangle ABC.

Ex. If AB (Fig. 11.) be 110, each of the lines AC and BC 100, and the height of each of the segments ACH, BCD 10.435; what is the area of the whole figure?

The areas of the two segments are	1404
The area of the triangle ABC is	4593.4
And the whole figure is	5997.4

PROBLEM VII.

To find the area of a circular ZONE.

37. FROM THE AREA OF THE WHOLE CIRCLE, SUBTRACT THE TWO SEGMENTS ON THE SIDES OF THE ZONE.

If from the whole circle (Fig. 12.) there be taken the two segments ABC and DFH, there will remain the zone ACDH.

Or, the area of the zone may be found, by subtracting the segment ABC from the segment HBD: Or, by adding the two small segments GAH and VDC, to the trapezoid ACDH. See art. 36.

The latter method is rather the most expeditious in practice, as the two segments at the end of the zone are *equal.*

Ex. 1. What is the area of the zone ACDH, (Fig. 12.) if AC is 7.75, DH 6.93, and the diamater of the circle 8?

* For the method of finding the areas of segments by a *table*, see note D.

The area	of the whole circle is		50.26
	of the segment ABC	17.32	
	of the segment DFH	9.82	
	of the zone ACDH		23.12

2. What is the area of a zone, one side of which is 23.25, and the other side 20.8, in a circle whose diameter is 24?

Ans. 208.

38. If the *diameter* of the circle is not given, it may be found from the sides and the breadth of the zone.

Let the center of the circle be at O. (Fig. 12.) Draw ON perpendicular to AH, NM perpendicular to LR, and HP perpendicular to AL. Then,

$AN=\frac{1}{2}AH$, (Euc. 3. 3.) $MN=\frac{1}{2}(LA+RH)$
$LM=\frac{1}{2}LR$, (Euc. 2. 6.) $PA=LA-RH$.

The triangles APH and OMN are similar, because the sides of one are perpendicular to those of the other, each to each. Therefore,

$$PH : PA :: MN : MO$$

MO being found, we have $ML-MO=OL$.

And the *radius* $CO=\sqrt{OL^2+CL^2}$. (Euc. 47. 1.)

Ex. If the breadth of the zone ACDH (Fig. 12.) be 6.4, and the sides 6.8 and 6; what is the radius of the circle?

$PA=3.4-3=0.4$. And, $MN=\frac{1}{2}(3.4+3)=3.2$.

Then, $6.4 : 0.4 :: 3.2 : 0.2=MO$. And, $3.2-0.2=3=OL$

And the radius $CO=\sqrt{3^2+(3.4)^2}=4.534$.

PROBLEM VIII.

To find the area of a LUNE *or crescent.*

39. FIND THE DIFFERENCE OF THE TWO SEGMENTS WHICH ARE BETWEEN THE ARCS OF THE CRESCENT AND ITS CHORD.

If the segment ABC, (Fig. 14.) be taken from the segment ABD; there will remain the lune or crescent ACBD.

Ex. If the chord AB be 88, the height CH 20, and the height DH 40; what is the area of the crescent ACBD?

The area of the segment ABD is	2698
of the segment ABC	1220
of the crescent ACBD	1478

PROBLEM IX.

To find the area of a RING, included between the peripheries of two concentric circles.

40. FIND THE DIFFERENCE OF THE AREAS OF THE TWO CIRCLES.

Or,

Multiply the product of the sum and difference of the two diameters by .7854.

The area of the ring (Fig. 13.) is evidently equal to the difference between the areas of the two circles AB and A′B′.

But the area of each circle is equal to the square of its diameter multiplied into .7854. (Art. 30.) And the *difference* of these squares is equal to the product of the sum and difference of the diameters. (Alg. 235.) Therefore the area of the ring is equal to the product of the sum and difference of the two diameters multiplied by .7854.

Ex. 1. If AB (Fig. 13.) be 221. and A′B′ 106, what is the area of the ring?

Ans. $(221^2 \times .7854) - (106^2 \times .7854) = 29535$.

2. If the diameters of Saturn's larger ring be 205,000 and 190,000 miles, how many square miles are there on one side of the ring?

Ans. $395000 \times 15000 \times .7854 = 4,653,495,000$.

PROMISCUOUS EXAMPLES OF AREAS.

Ex. 1. What is the expense of paving a street 20 rods long, and 2 rods wide, at 5 cents for a square foot?

Ans. 544½ dollars.

2. If an equilateral triangle contains ~~as many square feet~~ as there are inches in one of its sides; what is the area of the triangle?

Let x=the number of square feet in the area.

Then $\frac{x}{12}$=the number of linear feet in one of the sides.

And, (Art. 11.) $x=\frac{1}{4}\left(\frac{x}{12}\right)^2\times\sqrt{3}=\frac{x^2}{576}\times\sqrt{3}$.

Reducing the equation, $x=\frac{576}{\sqrt{3}}$=332.55 the area.

3. What is the side of a square whose area is equal to that of a circle 452 feet in diameter?

Ans. $\sqrt{(452)^2\times.7854}$=400.574. (Art. 30 and 7.)

4. What is the diameter of a circle which is equal to a square whose side is 36 feet?

Ans. $\sqrt{(36)^2\div0.7854}$=40.6217. (Art. 4. and 32.)

5. What is the area of a square inscribed in a circle whose diameter is 132 feet?

Ans. 8712 square feet. (Art. 33.)

6. How much carpeting, a yard wide, will be necessary to cover the floor of a room which is a regular octagon, the sides being ~~8~~ feet each? Ans. $34\frac{1}{3}$ yards.

7. If the diagonal of a square be ~~16~~ feet, what is the area? Ans. 128 feet. (Art. 14.)

8. If a carriage wheel ~~four feet~~ in diameter revolve ~~300~~ times, in going round a circular green; what is the area of the green?

Ans. $4154\frac{1}{3}$ sq. rods, or 25 acres, 3 qrs. and $34\frac{1}{3}$ rods.

9. What will be the expense of papering the sides of a room, at 10 cents a square yard; if the room be 21 feet long, 18 feet broad, and 12 feet high; and if there be deducted 3 windows, each 5 feet by 3, two doors 8 feet by $4\frac{1}{2}$, and one fire-place 6 feet by $4\frac{1}{2}$? Ans. 8 dollars 80 cents.

10. If a circular pond of water ~~10~~ rods in diameter be surrounded by a gravelled walk ~~$8\frac{1}{4}$~~ feet wide; what is the area of the walk? Ans. $16\frac{1}{2}$ sq. rods. (Art. 40.)

11. If CD (Fig. 17.) the base of the isosceles triangle VCD, be 60 feet, and the area 1200 feet; and if there be cut off, by the line LG parallel to CD, the triangle VLG, whose area is 432 feet; what are the sides of the latter triangle?

Ans. 30, 30, and 36 feet.

12. What is the area of an equilateral triangle inscribed in a circle whose diameter is 52 feet?

Ans. 878.15 sq. feet.

13. If a circular piece of land is enclosed by a fence, in which 10 rails make a rod in length; and if the field contains as many square rods, as there are rails in the fence; what is the value of the land at 120 dollars an acre?

Ans. 942.48 dollars.

14. If the area of the equilateral triangle ABD (Fig. 9.) be 219.5375 feet; what is the area of the circle OBDA, in which the triangle is inscribed?

The sides of the triangle are each 22.5167. (Art. 11.)
And the area of the circle is 530.93.

15. If 6 concentric circles are so drawn, that the space between the least or 1st, and the 2d is 21.2058,
between the 2d and 3d 35.343,
between the 3d and 4th 49.4802,
between the 4th and 5th 63.6174,
between the 5th and 6th 77.7546;
what are the several diameters, supposing the longest to be equal to 6 times the shortest?

Ans. 3, 6, 9, 12, 15, and 18.

16. If the area between two concentric circles be 1202.64 square inches, and the diameter of the lesser circle be 19 inches, what is the diameter of the other?

17. What is the area of a circular segment, whose height is 9, and base 24?

SECTION III.

SOLIDS BOUNDED BY PLANE SURFACES.

ART. 41. DEFINITION I. A *prism* is a solid bounded by plane figures or faces, two of which are parallel, similar, and equal; and the others are parallelograms.

II. The parallel planes are sometimes called the *bases* or *ends;* and the other figures, the *sides* of the prism. The latter taken together constitute the *lateral surface.*

III. A prism is *right* or *oblique*, according as the sides are perpendicular or oblique to the bases.

IV. The *height* of a prism is the perpendicular distance between the planes of the bases. In a right prism, therefore, the height is equal to the length of one of the sides.

V. A *Parallelopiped* is a prism whose bases are parallelograms.

VI. A *Cube* is a solid bounded by six equal squares. It is a right prism whose sides and bases are all equal.

VII. A *Pyramid* is a solid bounded by a plane figure called the base, and several triangular planes, proceeding from the sides of the base, and all terminating in a single point. These triangles taken together constitute the *lateral surface.*

VIII. A pyramid is *regular*, if its base is a regular polygon, and if a line from the center of the base to the vertex of the pyramid is *perpendicular* to the base. This line is called the *axis* of the pyramid.

IX. The *height* of a pyramid is the perpendicular distance from the summit to the plane of the base. In a *regular* pyramid, it is the length of the *axis.*

X. The *slant-height* of a regular pyramid, is the distance from the summit to the middle of one of the sides of the base.

XI. A *frustum* or *trunk* of a pyramid is a portion of the solid next the base, cut off by a plane parallel to the base. The *height* of the frustum is the perpendicular distance of the two parallel planes. The *slant-height* of a frustum of a *regular* pyramid, is the distance from the middle of one of the sides of the base, to the middle of the corresponding side in the plane above. It is a line passing on the surface of the frustum, through the middle of one of its sides.

XII. A *Wedge* is a solid of five sides, viz. a rectangular base, two rhomboidal sides meeting in an edge, and two triangular ends; as ABHG. (Fig. 20.) The base is ABCD, the sides are ABHG and DCHG, meeting in the edge GH, and the ends are BCH and ADG. The *height* of the wedge is a perpendicular drawn from any point in the edge, to the plane of the base, as GP.

XIII. A *Prismoid* is a solid whose ends or bases are parallel, but not similar, and whose sides are quadrilateral. It differs from a prism or a frustum of a pyramid, in having its ends dissimilar. It is a *rectangular* prismoid, when its ends are right parallelograms.

XIV. A *linear side* or *edge* of a solid is the line of intersection of two of the planes which form the surface.

42. The common *measuring unit* of solids is a *cube*, whose sides are squares of the same name. The sides of a cubic inch are square inches; of a cubic foot, square feet, &c. Finding the *capacity*, *solidity*,* or *solid contents* of a body, is finding the number of cubic measures, of some given denomination contained in the body.

In solid measure.

1728	cubic inches	=1	cubic foot,
27	cubic feet	=1	cubic yard,
$4492\frac{1}{8}$	cubic feet	=1	cubic rod,
32768000	cubic rods	=1	cubic mile,
282	cubic inches	=1	ale gallon,
231	cubic inches	=1	wine gallon,
2150.42	cubic inches	=1	bushel,
1	cubic foot of pure water weighs 1000 avoirdupois ounces, or $62\frac{1}{2}$ pounds.		

* See note E.

PROBLEM I.

To find the SOLIDITY *of a* PRISM.

43. **MULTIPLY THE AREA OF THE BASE BY THE HEIGHT.**

This is a general rule, applicable to parallelopipeds whether right or oblique, cubes, triangular prisms, &c.

As *surfaces* are measured, by comparing them with a right *parallelogram* (Art. 3.); so *solids* are measured, by comparing them with a right *parallelopiped.*

If ABCD (Fig. 1.) be the base of a right parallelopiped, as a stick of timber standing erect, it is evident that the number of *cubic* feet contained in *one* foot of the height, is equal to the number of *square* feet in the area of the base. And if the solid be of any other height, instead of one foot, the contents must have the same ratio. For parallelopipeds of the same base are to each other as their heights. (Sup. Euc. 9. 3.) The solidity of a right parallelopiped, therefore, is equal to the *product of its length, breadth, and thickness.* See Alg. 523.

And an *oblique* parallelopiped being equal to a right one of the same base and altitude, (Sup. Euc. 7. 3.) is equal to the area of the base multiplied into the perpendicular height. This is true also of *prisms,* whatever be the form of their bases. (Sup. Euc. 2. Cor. to 8, 3.)

44. As the sides of a *cube* are all *equal,* the solidity is found by *cubing one of its edges.* On the other hand, if the solid contents be given, the length of the edges may be found, by *extracting the cube root.*

45. When solid measure is cast by *Duodecimals,* it is to be observed that *inches* are not *primes* of feet, but *thirds.* If the unit is a cubic foot, a solid which is an inch thick and a foot square is a prime; a parallelopiped a foot long, an inch broad, and an inch thick is a second, or the twelfth part of a prime; and a cubic inch is a third, or a twelfth part of a second. A linear inch is $\frac{1}{12}$ of a foot, a square inch $\frac{1}{144}$ of a foot, and a cubic inch $\frac{1}{1728}$ of a foot.

Ex. 1. What are the solid contents of a stick of timber which is 31 feet long, 1 foot 3 inches broad, and 9 inches thick? Ans. 29 feet 9′′, or 29 feet 108 inches.

2. What is the solidity of a wall which is 22 feet long, 12 feet high, and 2 feet 6 inches thick?

Ans. 660 cubic feet.

3. What is the capacity of a cubical vessel which is 2 feet 3 inches deep?

Ans. 11 F. 4′ 8″ 3‴, or 11 feet 675 inches.

4. If the base of a prism be 108 square inches, and the height 36 feet, what are the solid contents?

Ans. 27 cubic feet.

5. If the height of a square prism be $2\frac{1}{4}$ feet, and each side of the base $10\frac{1}{3}$ feet what is the solidity?

The area of the base $=10\frac{1}{3}\times10\frac{1}{3}=106\frac{7}{9}$ sq. feet.
And the solid contents$=106\frac{7}{9}\times2\frac{1}{4}=240\frac{1}{4}$ cubic feet.

6. If the height of a prism be 23 feet, and its base a regular pentagon, whose perimeter is 18 feet, what is the solidity?

Ans. 512.84 cubic feet.

46. The number of *gallons* or *bushels* which a vessel will contain may be found, by calculating the capacity in *inches*, and then dividing by the number of inches in 1 gallon or bushel.

The *weight of water* in a vessel of given dimensions is easily calculated; as it is found by experiment, that a cubic foot of pure water weighs 1000 ounces avoirdupois. For the weight in ounces, then, multiply the cubic feet by 1000; or for the weight in pounds, multiply by $62\frac{1}{2}$.

Ex. 1. How many ale gallons are there in a cistern which is 11 feet 9 inches deep, and whose base is 4 feet 2 inches square?

The cistern contains 352500 cubic inches;
And $352500\div282=1250$.

2. How many wine gallons will fill a ditch 3 feet 11 incnes wide, 3 feet deep, and 462 feet long? Ans. 40608.

3. What weight of watei can be put into a cubical vessel 4 feet deep? Ans. 4000 lbs.

PROBLEM II.

To find the LATERAL SURFACE *of a* RIGHT PRISM.

47. MULTIPLY THE LENGTH INTO THE PERIMETER OF THE BASE.

Each of the sides of the prism is a right parallelogram, whose area is the product of its length and breadth. But the breadth is one side of the base; and therefore, the sum of the breadths is equal to the perimeter of the base.

Ex. 1. If the base of a right prism be a regular hexagon whose sides are each 2 feet 3 inches, and if the height be 16 feet, what is the lateral surface? Ans. 216 square feet.

If the areas of the two ends be added to the lateral surface, the sum will be the whole surface of the prism. And the superficies of any solid bounded by planes, is evidently equal to the areas of all its sides.

Ex. 2. If the base of a prism be an equilateral triangle whose perimeter is 6 feet, and if the height be 17 feet, what is the surface?

The area of the triangle is 1.732. (Art. 11.)
And the whole surface is 105.464.

PROBLEM III.

To find the SOLIDITY *of a* PYRAMID.

48. MULTIPLY THE AREA OF THE BASE INTO $\frac{1}{3}$ OF THE HEIGHT.

The solidity of a *prism* is equal to the product of the area of the base into the height. (Art. 43.) And a pyramid is $\frac{1}{3}$ of a prism of the same base and altitude. (Sup. Euc. 15, 3. Cor. 1.) Therefore the solidity of a pyramid whether right or oblique, is equal to the product of the base into $\frac{1}{3}$ of the perpendicular height.

Ex. 1. What is the solidity of a triangular pyramid, whose height is 60, and each side of whose base is 4?

The area of the base is 6.928
And the solidity is 138.56.

2. Let ABC (Fig. 16.) be one side of an oblique pyramid whose base is 6 feet square; let BC be 20 feet, and make an angle of 70 degrees with the plane of the base; and let CP be perpendicular to this plane. What is the solidity of the pyramid?

In the right angled triangle BCP, (Trig. 134.)

R : BC : : sin B : : PC=18.79.

And the solidity of the pyramid is 225.48 feet.

3. What is the solidity of a pyramid whose perpendicular height is 72, and the sides of whose base are 67, 54, and 40?

Ans. 25920.

PROBLEM IV.

To find the LATERAL SURFACE *of a* REGULAR PYRAMID.

49. MULTIPLY HALF THE SLANT-HEIGHT INTO THE PERIMETER OF THE BASE.

Let the triangle ABC (Fig. 18.) be one of the sides of a regular pyramid. As the sides AC and BC are equal, the angles A and B are equal. Therefore a line drawn from the vertex C to the middle of AB is *perpendicular* to AB. The area of the triangle is equal to the product of half this perpendicular into AB. (Art. 8.) The perimeter of the base is the sum of its sides, each of which is equal to AB. And the areas of all the equal triangles which constitute the lateral surface of the pyramid, are together equal to the product of the perimeter into half the slant-height CP.

The *slant-height* is the hypothenuse of a right angled triangle, whose legs are the axis of the pyramid, and the distance from the center of the base to the middle of one of the sides. See Def. 10.

Ex. 1. What is the lateral surface of a regular hexagonal pyramid, whose axis is 20 feet, and the sides of whose base are each 8 feet?

The square of the distance from the center of the base to one of the sides. (Art. 16.) =48.

The slant-height (Euc. 47. 1.) $=\sqrt{48+(20)^2}=21.16$.

And the lateral surface $=21.16\times4\times6=507.84$ sq. feet.

2. What is the whole surface of a regular triangular pyramid whose axis is 8, and the sides of whose base are each 20.78?

The lateral surface is	312
The area of the base is	187
And the whole surface is	499

3. What is the lateral surface of a regular pyramid whose axis is 12 feet, and whose base is 18 feet square?

Ans. 540 square feet.

The lateral surface of an *oblique* pyramid may be found, by taking the sum of the areas of the unequal triangles which form its sides.

PROBLEM V.

To find the SOLIDITY *of a* FRUSTUM *of a pyramid.*

50. ADD TOGETHER THE AREAS OF THE TWO ENDS, AND THE SQUARE ROOT OF THE PRODUCT OF THESE AREAS; AND MULTIPLY THE SUM BY $\frac{1}{3}$ OF THE PERPENDICULAR HEIGHT OF THE SOLID.

Let CDGL (Fig. 17.) be a vertical section, through the middle of a frustum of a right pyramid CDV whose base is a *square.*

Let $CD=a$, $LG=b$, $RN=h$.

By similar triangles, $LG : CD :: RV : NV$.

Subtracting the antecedents, (Alg. 389.)

$LG : CD-LG :: RV : NV-RV=RN.$

$$\text{Therefore } RV=\frac{RN\times LG}{CD-LG}=\frac{hb}{a-b}$$

The square of CD is the base of the pyramid CDV;
And the square of LG is the base of the small pyramid LGV.
Therefore, the solidity of the larger pyramid (Art. 48.) is

$$\overline{CD}^2\times\tfrac{1}{3}(RN+RV)=a^2\times\tfrac{1}{3}\left(h+\frac{hb}{a-b}\right)=\frac{ha^3}{3a-3b}$$

And the solidity of the smaller pyramid is equal to

$$\overline{LG}^2\times\tfrac{1}{3}RV=b^2\times\frac{hb}{3a-3b}=\frac{hb^3}{3a-3b}.$$

If the smaller pyramid be taken from the larger, there will remain the frustum CDLG, whose solidity is equal to

$$\frac{ha^3-hb^3}{3a-3b}=\tfrac{1}{3}h\times\frac{a^3-b^3}{a-b}=\tfrac{1}{3}h\times(a^2+ab+b^2) \quad \text{(Alg. 466.)}$$

Or, because $\sqrt{a^2b^2}=ab$, (Alg. 259.)

$$\tfrac{1}{3}h\times(a^2+b^2+\sqrt{a^2b^2})$$

Here h, the height of the frustum, is multiplied into a^2 and b^2, the areas of the two ends, and into $\sqrt{a^2 b^2}$ the square root of the products of these areas.

In this demonstration, the pyramid is supposed to be *square.* But the rule is equally applicable to a pyramid of any other form. For the solid contents of pyramids are equal, when they have equal heights and bases, whatever be the *figure* of their bases. (Sup. Euc. 14. 3.) And the sections parallel to the bases, and at equal distances, are equal to one another. (Sup. Euc. 12. 3. Cor. 2.)*

Ex. 1. If one end of the frustum of a pyramid be 9 feet square, the other end 6 feet square, and the height 36 feet, what is the solidity?

The areas of the two ends are 81 and 36.
The square root of their product is 54.
And the solidity of the frustum $=(81+36+54)\times 12=2052$.

2. If the height of a frustum of a pyramid be 24, and the areas of the two ends 441 and 121; what is the solidity?
Ans. 6344.

3. If the height of a frustum of a hexagonal pyramid be 48, each side of one end 26, and each side of the other end 16; what is the solidity?
Ans. 56034.

PROBLEM VI.

To find the LATERAL SURFACE *of a* FRUSTUM *of a regular pyramid.*

51. MULTIPLY HALF THE SLANT-HEIGHT BY THE SUM OF THE PERIMETERS OF THE TWO ENDS.

Each side of a frustum of a regular pyramid is a *trapezoid*, as ABCD. (Fig. 19.) The slant-height HP, (Def. 11.) though it is oblique to the base of the solid, is perpendicular to the line AB. The area of the trapezoid is equal to the product of half this perpendicular into the sum of the parallel sides AB and DC. (Art. 12.) Therefore the area of all the equal trapezoids which form the lateral surface of the frustum, is

* See note F.

equal to the product of half the slant-height into the sum of the perimeters of the ends.

Ex. If the slant-height of a frustum of a regular octagonal pyramid be 42 feet, the sides of one end 5 feet each, and the sides of the other end 3 feet each; what is the lateral surface?

Ans. 1344 square feet.

52. If the slant-height be not given, it may be obtained from the perpendicular height, and the dimensions of the two ends. Let GD (Fig. 17.) be the slant-height of the frustum CDGL, RN or GP the perpendicular height, ND and RG the radii of the circles inscribed in the perimeters of the two ends. Then, PD is the difference of the two radii:

And the slant-height $GD = \sqrt{(GP^2 + PD^2)}$.

Ex. If the perpendicular height of a frustum of a regular hexagonal pyramid be 24, the sides of one end 13 each, and the sides of the other end 8 each; what is the whole surface?

$\sqrt{(BC^2 - BP^2)} = CP$, (Fig. 7.) that is, $\sqrt{(13^2 - 6.5^2)} = 11.258$

And $\sqrt{8^2 - 4^2} = 6.928$

The difference of the two radii is, therefore, 4.33

The slant-height $= \sqrt{(24^2 + 4.33^2)} = 24.3875$
The lateral surface is 1536.4
And the whole surface, 2141.75

53. The height of the *whole pyramid* may be calculated from the dimensions of the frustum. Let VN (Fig. 17.) be the height of the pyramid, RN or GP the height of the frustum, ND and RG the radii of the circles inscribed in the perimeters of the ends of the frustum.

Then, in the similar triangles GPD and VND,

DP : GP : : DN : VN.

The height of the frustum subtracted from VN, gives VR the height of the small pyramid VLG. The *solidity* and *lateral surface* of the frustum may then be found, by subtracting from the whole pyramid, the part which is above the cutting plane. This method may serve to verify the calculations which are made by the rules in arts. 50 and 51.

Ex. If one end of the frustum CDGL (Fig. 17.) be 90 feet square, the other end 60 feet square, and the height RN 36 feet; what is the height of the whole pyramid VCD? and what are the solidity and lateral surface of the frustum?

DP=DN—GR=45—30=15. And, GP=RN=36.

Then, 15 : 36 : : 45 : 108=VN, the height of the whole pyramid.

And, 108—36=72=VR, the height of the part VLG.

The solidity	of the large pyramid is	291600	(Art. 48.)
	of the small pyramid	86400	
	of the frustum CDGL	205200	
The lateral surface	of the large pyramid is	21060	(Art. 49.)
	of the small pyramid	9360	
	of the frustum	11700	

PROBLEM VII.

To find the SOLIDITY *of a* WEDGE.

54. ADD THE LENGTH OF THE EDGE TO TWICE THE LENGTH OF THE BASE, AND MULTIPLY THE SUM BY $\frac{1}{6}$ OF THE PRODUCT OF THE HEIGHT OF THE WEDGE AND THE BREADTH OF THE BASE.

Let L=AB the length of the base. (Fig. 20.)
l=GH the length of the edge.
b=BC the breadth of the base.
h=PG the height of the wedge.
Then, L—l=AB—GH=AM.

If the length of the base and the edge be *equal*, as BM and GH, (Fig. 20.) the wedge MBHG is half a parallelopiped of the same base and height. And the solidity (Art. 43.) is equal to half the product of the height, into the length and breadth of the base; that is to $\frac{1}{2}bhl$.

If the length of the base be *greater* than that of the edge, as ABGH; let a section be made by the plane GMN, parallel to HBC. This will divide the whole wedge into two parts MBHG and AMG. The latter is a pyramid, whose solidity (Art. 48.) is $\frac{1}{3}bh\times(L-l)$

The solidity of the parts together, is, therefore,

$$\tfrac{1}{2}bhl+\tfrac{1}{3}bh\times(L-l)=\tfrac{1}{6}bh3l+\tfrac{1}{6}bh2L-\tfrac{1}{6}bh2l=\tfrac{1}{6}bh\times(2L+l)$$

If the length of the base be *less* than that of the edge, it is evident that the pyramid is to be *subtracted* from half the parallelopiped, which is equal in height and breadth to the wedge, and equal in length to the edge.

The solidity of the wedge is, therefore,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh\times(l-L)=\tfrac{1}{6}bh3l-\tfrac{1}{6}bh2l+\tfrac{1}{6}bh2L=\tfrac{1}{6}bh\times(2L+l)$$

Ex. 1. If the base of a wedge be 35 by 15, the edge 55, and the perpendicular height 12.4; what is the solidity?

Ans. $(70+55)\times\frac{15\times12.4}{6}=3875.$

2. If the base of a wedge be 27 by 8, the edge 36, and the perpendicular height 42; what is the solidity?

Ans. 5040.

PROBLEM VIII.

To find the SOLIDITY *of a rectangular* PRISMOID.

55. TO THE AREAS OF THE TWO ENDS, ADD FOUR TIMES THE AREA OF A PARALLEL SECTION EQUALLY DISTANT FROM THE ENDS, AND MULTIPLY THE SUM BY $\frac{1}{6}$ OF THE HEIGHT.

Let L and B (Fig. 21.) be the length and breadth of one end,
l and b the length and breadth of the other end,
M and m the length and breadth of the section in the middle.
and h the height of the prismoid.

The solid may be divided into two wedges, whose bases are the ends of the prismoid, and whose edges are L and l. The solidity of the whole, by the preceding article, is

$$\tfrac{1}{6}Bh\times(2L+l)+\tfrac{1}{6}bh\times(2l+L)=\tfrac{1}{6}h(2BL+Bl+2bl+bL)$$

As M is equally distant from L and l,

$$2M=L+l, 2m=B+b, \text{ and } 4Mm=(L+l)(B+b)=BL+Bl+bL+lb.$$

Substituting $4Mm$ for its value, in the preceding expression for the solidity, we have

$$\tfrac{1}{6}h(BL+bl+4Mm)$$

That is, the solidity of the prismoid is equal to $\frac{1}{6}$ of the height, multiplied into the areas of the two ends, and 4 times the area of the section in the middle.

This rule may be applied to prismoids of other forms. For, whatever be the figure of the two ends, there may be drawn in each, such a number of small rectangles, that the sum of them shall differ less, than by any given quantity, from the figure in which they are contained. And the solids between these rectangles will be rectangular prismoids.

Ex. 1. If one end of a rectangular prismoid be 44 feet by 23, the other end 36 by 21, and the perpendicular height 72; what is the solidity?

The area of the larger end	$=44\times23=1012$
of the smaller end	$=36\times21=\ \ 756$
of the middle section	$=40\times22=\ \ 880$

And the solidity $=(1012+756+4\times880)\times12=63456$ feet.

2. What is the solidity of a stick of hewn timber, whose ends are 30 inches by 27, and 24 by 18, and whose length is 48 feet? Ans. 204 feet.

Other solids not treated of in this section, if they be bounded by plane surfaces, may be measured by supposing them to be divided into prisms, pyramids, and wedges. And, indeed, every such solid may be considered as made up of triangular pyramids.

THE FIVE REGULAR SOLIDS.

56. **A SOLID IS SAID TO BE REGULAR, WHEN ALL ITS SOLID ANGLES ARE EQUAL, AND ALL ITS SIDES ARE EQUAL AND REGULAR POLYGONS.**

The following figures are of this description;

1. The *Tetraedron*,		four triangles;
2. The *Hexaedron or cube*,		six squares;
3. The *Octaedron*,	whose sides are	eight triangles;
4. The *Dodecaedron*,		twelve pentagons;
5. The *Icosaedron*,		twenty triangles.*

Besides these five, there can be no other regular solids. The only plane figures which can form such solids, are triangles, squares, and pentagons. For the plane angles which contain any solid angle, are together less than four right angles or 360°. (Sup. Euc. 21, 2.) And the least number which can form a solid angle is three. (Sup. Euc. Def. 8, 2.) If they are angles of equilateral *triangles*, each is 60°. The sum of *three* of them is 180°, of *four* 240°, of *five* 300°, and of *six* 360°. The latter number is too great for a solid angle.

The angles of *squares* are 90° each. The sum of *three* of these is 270°, of four 360°, and of any other greater number, still more.

The angles of regular *pentagons* are 108° each. The sum of *three* of them is 324°; of four, or any other greater number, more than 360°. The angles of all other regular polygons are still greater.

In a regular solid, then, each solid angle must be contained by three, four, or five equilateral triangles, by three squares, or by three regular pentagons.

57. As the sides of a regular solid are similar and equal, and the angles are also alike; it is evident that the sides are all equally distant from a central point in the solid. If then, planes be supposed to proceed from the several edges to the center, they will divide the solid into as many equal *pyramids*, as it has sides. The base of each pyramid will be one of the sides; their common vertex will be the central point; and their height will be a perpendicular from the center to one of the sides.

* For the geometrical construction of these solids, see Legendre's Geometry; Appendix to Books VI and VII.

PROBLEM IX.

To find the SURFACE *of a* REGULAR SOLID.

58. MULTIPLY THE AREA OF ONE OF THE SIDES BY THE NUMBER OF SIDES.

Or,

MULTIPLY THE SQUARE OF ONE OF THE EDGES, BY THE SURFACE OF A SIMILAR SOLID WHOSE EDGES ARE 1.

As all the sides are *equal*, it is evident that the area of one of them, multiplied by the number of sides, will give the area of the whole.

Or, if a *table* is prepared, containing the surfaces of the several regular solids whose linear edges are *unity;* this may be used for other regular solids, upon the principle, that the areas of similar polygons are as the squares of their homologous sides. (Euc. 20. 6.) Such a table is easily formed, by multiplying the area of one of the sides, as given in art. 17 by the number of sides. Thus, the area of an equilateral triangle whose side is 1, is 0.4330127. Therefore, the surface

Of a regular tetraedron $=.4330127\times4 =1.7320508$.
Of a regular octaedron $=.4330127\times8 =3.4641016$.
Of a regular icosaedron $=.4330127\times20=8.6602540$.

See the table in the following article.

Ex. 1. What is the surface of a regular dodecaedron whose edges are each 25 inches?

The area of one of the sides is 1075.3.

And the surface of the whole solid $=1075.3\times12=12903.6$.

2. What is the surface of a regular icosaedron whose edges are each 102? Ans. 90101.3.

PROBLEM X.

To find the SOLIDITY *of a* REGULAR SOLID.

59. MULTIPLY THE SURFACE BY $\frac{1}{3}$ OF THE PERPENDICULAR DISTANCE FROM THE CENTER TO ONE OF THE SIDES.

Or,

MULTIPLY THE CUBE OF ONE OF THE EDGES, BY THE SOLIDITY OF A SIMILAR SOLID WHOSE EDGES ARE 1.

As the solid is made up of a number of equal pyramids, whose bases are the sides, and whose height is the perpendicular distance of the sides from the center (Art. 57.); the solidity of the whole must be equal to the areas of all the sides multiplied into $\frac{1}{3}$ of this perpendicular. (Art. 48.)

If the contents of the several regular solids whose edges are 1, be inserted in a *table*, this may be used to measure other similar solids. For two similar regular solids contain the same number of similar pyramids; and these are to each other as the *cubes* of their linear sides or edges. (Sup. Euc. 15. 3. Cor. 3.)

A TABLE OF REGULAR SOLIDS WHOSE EDGES ARE 1.

Names.	No. of sides.	Surfaces.	Solidities.
Tetraedron	4	1.7320508	0.1178513
Hexaedron	6	6.0000000	1.0000000
Octaedron	8	3.4641016	0.4714045
Dodecaedron	12	20.6457288	7.6631189
Icosaedron	20	8.6602540	2.1816950

For the method of calculating the last column of this table, see Hutton's Mensuration, Part III. Sec. 2.

Ex. What is the solidity of a regular octaedron whose edges are each 32 inches? Ans. 15447 inches.

SECTION IV.*

THE CYLINDER, CONE, AND SPHERE.

ART. 61. DEFINITION I. A *right cylinder* is a solid described by the revolution of a rectangle about one of its sides. The *ends or bases* are evidently equal and parallel circles. And the *axis*, which is a line passing through the middle of the cylinder, is perpendicular to the bases.

The ends of an *oblique* cylinder are also equal and parallel circles; but they are not perpendicular to the axis. The *height* of a cylinder is the perpendicular distance from one base to the plane of the other. In a right cylinder, it is the length of the axis.

II. A *right cone* is a solid described by the revolution of a right angled triangle about one of the sides which contain the right angle. The *base* is a circle, and is perpendicular to the *axis*, which proceeds from the middle of the base to the vertex.

The base of an *oblique* cone is also a circle, but is not perpendicular to the axis. The *height* of a cone is the perpendicular distance from the vertex to the plane of the base. In a right cone, it is the length of the axis. The *slant-height* of a right cone is the distance from the vertex to the circumference of the base.

III. A *frustum* of a cone is a portion cut off by a plane parallel to the base. The *height* of the frustum is the perpendicular distance of the two ends. The *slant-height* of a frustum of a right cone, is the distance between the peripheries of the two ends, measured on the outside of the solid; as AD. (Fig. 23.)

IV. A *sphere* or *globe* is a solid which has a center equally distant from every part of the surface. It may be described by the revolution of a semicircle about a diameter. A *radius* of the sphere is a line drawn from the center to any part of

* Hutton's Mensuration, West's Mathematics, Legendre's, Clairaut's, and Camus's Geometry.

the surface. A *diameter* is a line passing through the center, and terminated at both ends by the surface. The *circumference* is the same as the circumference of a circle whose plane passes through the center of the sphere. Such a circle is called a *great circle.*

V. A *segment* of a sphere is a part cut off by any plane. The *height* of the segment is a perpendicular from the middle of the base to the convex surface, as LB. (Fig. 12.)

VI. A *spherical zone* or frustum is a part of the sphere included between two parallel planes. It is called the *middle zone*, if the planes are equally distant from the center. The *height* of a zone is the distance of the two planes, as LR. (Fig. 12.*)

VII. A *spherical sector* is a solid produced by a *circular* sector, revolving in the same manner as the semicircle which describes the whole sphere. Thus, a spherical sector is described by the circular sector ACP (Fig. 15.) or GCE revolving on the axis CP.

VIII. A solid described by the revolution of any figure about a fixed axis, is called a *solid of revolution.*

PROBLEM I.

To find the CONVEX SURFACE *of a* RIGHT CYLINDER.

62. MULTIPLY THE LENGTH INTO THE CIRCUMFERENCE OF THE BASE.

If a right cylinder be covered with a thin substance like paper, which can be spread out into a plane; it is evident that the plane will be a *parallelogram*, whose length and breadth will be equal to the length and circumference of the cylinder. The area must, therefore, be equal to the length multiplied into the circumference. (Art. 4.)

Ex. 1. What is the convex surface of a right cylinder which is 42 feet long, and 15 inches in diameter?

Ans. $42 \times 1.25 \times 3.14159 = 164.933$ sq. feet.

* According to some writers, a spherical *segment* is either a solid which is cut off from a sphere by a single plane, or one which is included between two planes: and a *zone* is the *surface* of either of these. In this sense, the term zone is commonly used in geography.

2. What is the whole surface of a right cylinder, which is 2 feet in diameter and 36 feet long?

The convex surface is	226.1945
The area of the two ends (Art. 30.) is	6.2832
The whole surface is	232.4777

3. What is the whole surface of a right cylinder whose axis is 82, and circumference 74? Ans. 6624.32.

63. It will be observed that the rules for the *prism* and *pyramid* in the preceding section, are substantially the same, as the rules for the *cylinder* and *cone* in this. There may be some advantage, however, in considering the latter by themselves.

In the base of a *cylinder*, there may be inscribed a polygon, which shall differ from it less than by any given space. (Sup. Euc. 6. 1. Cor.) If the polygon be the base of a *prism*, of the same height as the cylinder, the two solids may differ less than by any given quantity. In the same manner, the base of a *pyramid* may be a polygon of so many sides, as to differ less than by any given quantity, from the base of a *cone* in which it is inscribed. A cylinder is therefore considered, by many writers, as a prism of an infinite number of sides; and a cone, as a pyramid of an infinite number of sides. For the meaning of the term "infinite," when used in the mathematical sense, see Alg. Sec. xv.

PROBLEM II.

To find the SOLIDITY *of a* CYLINDER.

64. MULTIPLY THE AREA OF THE BASE BY THE HEIGHT

The solidity of a *parallelopiped* is equal to the product of the base into the perpendicular altitude. (Art. 43.) And a parallelopiped and a cylinder which have equal bases and altitudes are equal to each other. (Sup. Euc. 17. 3.)

Ex. 1. What is the solidity of a cylinder, whose height is 121, and diameter 45.2?

Ans. $\overline{45.2}^2 \times .7854 \times 121 = 194156.6$.

2. What is the solidity of a cylinder, whose height is 424, and circumference 213? Ans. 1530837.

3. If the side AC of an oblique cylinder (Fig. 22.) be 27, and the area of the base 32.61, and if the side make an angle of 62° 44′ with the base, what is the solidity?

R : AC : : sin A : BC=24 the perpendicular height.
And the solidity is 782.64.

4. The Winchester bushel is a hollow cylinder, $18\frac{1}{2}$ inches in diameter, and 8 inches deep. What is its capacity?

The area of the base $=(18.5)^2 \times .7853982 = 268.8025$.
And the capacity is 2150.42 cubic inches. See the table in Art. 42.

PROBLEM III.

To find the CONVEX SURFACE *of a* RIGHT CONE.

65. MULTIPLY HALF THE SLANT-HEIGHT INTO THE CIRCUMFERENCE OF THE BASE.

If the convex surface of a right cone be spread out into a plane, it will evidently form a *sector* of a circle whose radius is equal to the slant-height of the cone. But the area of the sector is equal to the product of half the radius into the length of the arc. (Art. 34.) Or if the cone be considered as a pyramid of an infinite number of sides, its lateral surface is equal to the product of half the slant-height into the perimeter of the base. (Art. 49.)

Ex. 1. If the slant-height of a right cone be 82 feet, and the diameter of the base 24, what is the convex surface?

Ans. $41 \times 24 \times 3.14159 = 3091.3$ square feet.

2. If the axis of a right cone be 48, and the diameter of the base 72, what is the whole surface?

The slant-height $=\sqrt{(36^2+48^2)}=60$. (Euc. 47. 1.)	
The convex surface is	6786
The area of the base	4071.6
And the whole surface	10857.6

3. If the axis of a right cone be 16, and the circumference of the base 75.4; what is the whole surface?

Ans. 1206.4.

PROBLEM IV.

To find the SOLIDITY *of a* CONE.

66. MULTIPLY THE AREA OF THE BASE INTO $\frac{1}{3}$ OF THE HEIGHT.

The solidity of a *cylinder* is equal to the product of the base into the perpendicular height. (Art. 64.) And if a cone and a cylinder have the same base and altitude, the cone is $\frac{1}{3}$ of the cylinder. (Sup. Euc. 18. 3.) Or if a cone be considered as a pyramid of an infinite number of sides, the solidity is equal to the product of the base into $\frac{1}{3}$ of the height, by art. 48.

Ex. 1. What is the solidity of a right cone whose height is 663, and the diameter of whose base is 101?

Ans. $\overline{101}^2 \times .7854 \times 221 = 1770622$.

2. If the axis of an oblique cone be 738, and make an angle of 30° with the plane of the base; and if the circumference of the base be 355, what is the solidity?

Ans. 1233536.

PROBLEM V.

To find the CONVEX SURFACE *of a* FRUSTUM *of a right cone.*

67. MULTIPLY HALF THE SLANT-HEIGHT BY THE SUM OF THE PERIPHERIES OF THE TWO ENDS.

This is the rule for a frustum of a *pyramid*; (Art. 51.) and is equally applicable to a frustum of a *cone*, if a cone be considered as a pyramid of an infinite number of sides. (Art. 63.)

Or thus,

Let the sector ABV (Fig. 23.) represent the convex surface of a right cone, (Art. 65.) and DCV the surface of a portion of the cone, cut off by a plane parallel to the base. Then will ABCD be the surface of the frustum.

Let $AB=a$, $DC=b$, $VD=d$, $AD=h$.

Then the area $ABV=\frac{1}{2}a\times(h+d)=\frac{1}{2}ah+\frac{1}{2}ad$. (Art. 34.)

And the area $DCV=\frac{1}{2}bd$.

Subtracting the one from the other,

The area $ABDC=\frac{1}{2}ah+\frac{1}{2}ad-\frac{1}{2}bd$.

But $d:d+h::b:a$. (Sup. Euc. 8. 1.) Therefore $\frac{1}{2}ad-\frac{1}{2}bd=\frac{1}{2}bh$.

The surface of the frustum, then, is equal to

$\frac{1}{2}ah+\frac{1}{2}bh$. or $\frac{1}{2}h\times(a+b)$

Cor. The surface of the frustum is equal to the product of the slant-height into the circumference of a circle which is *equally distant* from the two ends. Thus, the surface ABCD (Fig. 23.) is equal to the product of AD into MN. For MN is equal to half the sum of AB and DC.

Ex. 1. What is the convex surface of a frustum of a right cone, if the diameters of the two ends be 44 and 33, and the slant-height 84? Ans. 10159.8.

2. If the perpendicular height of a frustum of a right cone be 24, and the diameters of the two ends 80 and 44, what is the whole surface?

Half the difference of the diameters is 18.

And $\sqrt{18^2+24^2}=30$, the slant-height, (Art. 52.)	
The convex surface of the frustum is	5843
The sum of the areas of the two ends is	6547
And the whole surface is	12390

PROBLEM VI.

To find the SOLIDITY *of a* FRUSTUM *of a cone.*

68. ADD TOGETHER THE AREAS OF THE TWO ENDS, AND THE SQUARE ROOT OF THE PRODUCT OF THESE AREAS; AND MULTIPLY THE SUM BY $\frac{1}{3}$ OF THE PERPENDICULAR HEIGHT.

This rule, which was given for the frustum of a *pyramid*, (Art. 50.) is equally applicable to the frustum of a cone; because a cone and a pyramid which have equal bases and altitudes are equal to each other.

Ex. 1. What is the solidity of a mast which is 72 feet long, 2 feet in diameter at one end, and 18 inches at the other?

Ans. 174.36 cubic feet.

2. What is the capacity of a conical cistern which is 9 feet deep, 4 feet in diameter at the bottom, and 3 feet at the top?

Ans. 87.18 cubic feet=652.15 wine gallons.

3. How many gallons of ale can be put into a vat in the form of a conic frustum, if the larger diameter be 7 feet, the smaller diameter 6 feet, and the depth 8 feet?

PROBLEM VII.

To find the SURFACE *of a* SPHERE.

69. MULTIPLY THE DIAMETER BY THE CIRCUMFERENCE.

Let a hemisphere be described by the quadrant CPD, (Fig. 25.) revolving on the line CD. Let AB be a side of a regular polygon inscribed in the circle of which DBP is an arc. Draw AO and BN perpendicular to CD, and BH perpendicular to AO. Extend AB till it meets CD continued. The triangle AOV, revolving on OV as an axis, will describe a right *cone*. (Defin. 2.) AB will be the slant-height of a *frustum* of this cone extending from AO to BN. From G the middle of AB, draw GM parallel to AO. The surface of the frustum described by AB, (Art. 67. Cor.) is equal to

$$AB \times circ\ GM.^*$$

From the center C draw CG, which will be perpendicular to AB, (Euc. 3. 3.) and the radius of a circle inscribed in the polygon. The triangles ABH and CGM are similar, because the sides are perpendicular, each to each. Therefore,

$$HB \text{ or } ON : AB :: GM : GC :: circ\ GM : circ\ GC.$$

So that $ON \times circ\ GC = AB \times circ\ GM$, that is, the surface of the frustum is equal to the product of ON the perpendicular height, into *circ* GC, the perpendicular distance from the center of the polygon to one of the sides.

* By *circ* GM is meant the circumference of a circle the radius of which is GM.

In the same manner it may be proved, that the surfaces produced by the revolution of the lines BD and AP about the axis DC, are equal to

$$ND \times circ\ GC, \qquad \text{and } CO \times circ\ GC.$$

The surface of the whole solid, therefore, (Euc. 1.2.) is equal to

$$CD \times circ\ GC.$$

The demonstration is applicable to a solid produced by the revolution of a polygon of *any* number of sides. But a polygon may be supposed which shall differ less than by any given quantity from the circle in which it is inscribed; (Sup. Euc. 4. 1.) and in which the perpendicular GC shall differ less than by any given quantity from the radius of the circle. Therefore, the surface of a *hemisphere* is equal to the product of its radius into the circumference of its base; and *the surface of a sphere is equal to the product of its diameter into its circumference.*

Cor. 1. From this demonstration it follows, that the surface of any *segment* or *zone* of a sphere is equal to the product of the height of the segment or zone into the circumference of the sphere. The surface of the zone produced by the revolution of the arc AB about ON, is equal to $ON \times circ$ CP. And the surface of the segment produced by the revolution of BD about DN is equal to $DN \times circ$ CP.

Cor. 2. The surface of a sphere is equal to four times the area of a circle of the same diameter; and therefore, the convex surface of a hemisphere is equal to twice the area of its base. For the area of a circle is equal to the product of half the diameter into half the circumference; (Art. 30.) that is, to $\frac{1}{4}$ the product of the diameter and circumference.

Cor. 3. The surface of a sphere, or the convex surface of any spherical segment or zone, is equal to that of the circumscribing cylinder. A hemisphere described by the revolution of the arc DBP, is circumscribed by a cylinder produced by the revolution of the parallelogram D*d*CP. The convex surface of the cylinder is equal to its height multiplied by its circumference. (Art. 62.) And this is also the surface of the hemisphere.

So the surface produced by the revolution of AB is equal to that produced by the revolution of *ab*. And the surface produced by BD is equal to that produced by *bd*.

Ex. 1. Considering the earth as a sphere 7930 miles in diameter, how many square miles are there on its surface?
Ans. 197,558,500.

2. If the circumference of the sun be 2,800,000 miles, what is his surface? Ans. 2,495,547,600,000 sq. miles.

3. How many square feet of lead will it require, to cover a hemispherical dome whose base is 13 feet across?
Ans. $265\frac{1}{2}$.

PROBLEM VIII.

To find the SOLIDITY *of a* SPHERE.

70. 1. MULTIPLY THE CUBE OF THE DIAMETER BY .5236.

Or,

2. MULTIPLY THE SQUARE OF THE DIAMETER BY $\frac{1}{6}$ OF THE CIRCUMFERENCE.

Or,

3. MULTIPLY THE SURFACE BY $\frac{1}{6}$ OF THE DIAMETER.

1. A sphere is *two thirds* of its circumscribing cylinder. (Sup. Euc. 21. 3.) The height and diameter of the cylinder are each equal to the diameter of the sphere. The solidity of the cylinder is equal to its height multiplied into the area of its base, (Art. 64.) that is putting D for the diameter,

$$D\times D^2\times.7854 \quad \text{or} \quad D^3\times.7854.$$

And the solidity of the *sphere*, being $\frac{2}{3}$ of this, is

$$D^3\times.5236.$$

2. The base of the circumscribing cylinder is equal to half the circumference multiplied into half the diameter; (Art. 30.) that is, if C be put for the circumference,

$$\tfrac{1}{4}C\times D\text{; and the solidity is } \tfrac{1}{4}C\times D^2.$$

Therefore, the solidity of the sphere is

$$\tfrac{2}{3} \text{ of } \tfrac{1}{4}C\times D^2=D^2\times\tfrac{1}{6}C.$$

3. In the last expression, which is the same as $C \times D \times \frac{1}{6}D$, we may substitute S, the surface, for $C \times D$. (Art. 69.) We then have the solidity of the sphere equal to

$$S \times \tfrac{1}{6}D.$$

Or, the sphere may be supposed to be filled with small *pyramids*, standing on the surface of the sphere, and having their common vertex in the center. The number of these may be such, that the difference between their sum and the sphere shall be less than any given quantity. The solidity of each pyramid is equal to the product of its base into $\frac{1}{3}$ of its height. (Art. 48.) The solidity of the whole, therefore, is equal to the product of the surface of the sphere into $\frac{1}{3}$ of its radius, or $\frac{1}{6}$ of its diameter.

71. The numbers 3.14159, .7854, .5236, should be made perfectly familiar. The first expresses the ratio of the *circumference* of a circle to the *diameter*; (Art. 23.) the second, the ratio of the *area* of a circle to the square of the diameter (Art. 30.); and the third, the ratio of the *solidity* of a sphere to the *cube* of the diameter. The second is $\frac{1}{4}$ of the first, and the third is $\frac{1}{6}$ of the first.

As these numbers are frequently occurring in mathematical investigations, it is common to represent the first of them by the Greek letter π. According to this notation,

$$\pi = 3.14159, \qquad \tfrac{1}{4}\pi = .7854, \qquad \tfrac{1}{6}\pi = .5236.$$

If D = the *diameter*, and R = the *radius* of any circle or sphere;

$$\text{Then,} \quad D = 2R \quad D^2 = 4R^2 \quad D^3 = 8R^3.$$

$$\left.\begin{matrix}\text{And } \pi D \\ \text{Or, } 2\pi R\end{matrix}\right\} = \text{the } periph. \left.\begin{matrix}\tfrac{1}{4}\pi D^2 \\ \text{or } \pi R^2\end{matrix}\right\} = \begin{matrix}\text{the } area \text{ of} \\ \text{the circ. or}\end{matrix} \left.\begin{matrix}\tfrac{1}{6}\pi D^3 \\ \tfrac{4}{3}\pi R^3\end{matrix}\right\} = \text{the}$$

solidity of the sphere.

Ex. 1. What is the solidity of the earth, if it be a sphere 7930 miles in diameter?

Ans. 261,107,000,000 cubic miles.

2. How many wine gallons will fill a hollow sphere 4 feet in diameter?

Ans. The capacity is 33.5104 feet = $250\frac{2}{3}$ gallons.

3. If the diameter of the moon be 2180 miles, what is its solidity? Ans. 5,424,600,000 miles.

72. **If the solidity of a sphere be *given*, the diameter may be found by reversing the first rule in the preceding article; that is, *dividing by* .5236 and *extracting the cube root of the quotient.***

Ex. 1. What is the diameter of a sphere whose solidity is 65.45 cubic feet? Ans. 5 feet.

2. What must be the diameter of a globe to contain 16755 pounds of water? Ans. 8 feet.

PROBLEM IX.

To find the CONVEX SURFACE *of a* SEGMENT *or* ZONE *of a sphere.*

73. MULTIPLY THE HEIGHT OF THE SEGMENT OR ZONE INTO THE CIRCUMFERENCE OF THE SPHERE.

For the demonstration of this rule, see art. 69.

Ex. 1. If the earth be considered a perfect sphere 7930 miles in diameter, and if the polar circle be $23° 28'$ from the pole, how many square miles are there in one of the frigid zones?

If PQOE (Fig. 15.) be a meridian on the earth, ADB one of the polar circles, and P the pole; then the frigid zone is a spherical segment described by the revolution of the arc APB about PD. The angle ACD subtended by the arc AP is $23°$ $28'$. And in the right angled triangle ACD,

$$R : AC :: \cos ACD : CD = 3637.$$

Then, $CP - CD = 3965 - 3637 = 328 = PD$ the height of the segment.

$$\text{And } 328 \times 7930 \times 3.14159 = 8171400 \text{ the surface.}$$

2. If the diameter of the earth be 7930 miles, what is the surface of the torrid zone, extending $23° 28'$ on each side of the equator?

If EQ (Fig. 15.) be the equator, and GH one of the tropics, then the angle ECG is $23° 28'$. And in the right angled triangle GCM, R : CG : : sin ECG : GM=CN=1578.9 the height of half the zone.

The surface of the whole zone is 78669700.

3. What is the surface of each of the temperate zones?

The height DN=CP—CN—PD=2058.1

And the surface of the zone is 51273000.

The surface of the two temperate zones is	102,546,000
of the two frigid zones	16,342,800
of the torrid zone	78,669,700
of the whole globe	197,558,500

PROBLEM X.

To find the SOLIDITY *of a spherical* SECTOR.

74. MULTIPLY THE SPHERICAL SURFACE BY $\frac{1}{3}$ OF THE RADIUS OF THE SPHERE.

The spherical sector, (Fig. 24.) produced by the revolution of ACBD about CD, may be supposed to be filled with *small pyramids*, standing on the spherical surface ADB, and terminating in the point C. Their number may be so great, that the height of each shall differ less than by any given length from the radius CD, and the sum of their bases shall differ less than by any given quantity from the surface ABD. The solidity of each is equal to the product of its base into $\frac{1}{3}$ of the radius CD. (Art. 48.) Therefore, the solidity of all of them, that is, of the sector ADBC, is equal to the product of the spherical surface into $\frac{1}{3}$ of the radius.

Ex. Supposing the earth to be a sphere 7930 miles in diameter, and the polar circle ADB (Fig. 15.) to be 23° 28′ from the pole; what is the solidity of the spherical sector ACBP? Ans. 10,799,867,000 miles.

PROBLEM XI.

To find the SOLIDITY *of a spherical* SEGMENT.

75. MULTIPLY HALF THE HEIGHT OF THE SEGMENT INTO THE AREA OF THE BASE, AND THE CUBE OF THE HEIGHT INTO .5236; AND ADD THE TWO PRODUCTS.

As the *circular* sector AOBC (Fig. 9.) consists of two parts, the segment AOBP and the triangle ABC; (Art. 35.) so the *spherical* sector produced by the revolution of AOC about OC consists of two parts, the *segment* produced by the revolution of AOP, and the *cone* produced by the revolution of ACP. If then the cone be subtracted from the sector, the remainder will be the segment.

Let $CO=R$, the radius of the sphere,
$PB=r$, the radius of the base of the segment,
$PO=h$, the height of the segment,
Then $PC=R-h$, the axis of the cone.

The sector$=2\pi R\times h\times\frac{1}{3}R$ (Arts. 71, 73, 74.)$=\frac{2}{3}\pi hR^2$.
The cone$=\pi r^2\times\frac{1}{3}(R-h)$ (Arts. 71, 66.)$=\frac{1}{3}\pi r^2R-\frac{1}{3}\pi hr^2$.

Subtracting the one from the other,

The segment $=\frac{2}{3}\pi hR^2-\frac{1}{3}\pi r^2R+\frac{1}{3}\pi hr^2$.

But $DO\times PO=\overline{BO}^2$ (Trig. 97.*)$=\overline{PO}^2+\overline{PB}^2$ (Euc. 47. 1.)

That is, $2Rh=h^2+r^2$. So that, $R=\dfrac{h^2+r^2}{2h}$

$$\text{And } R^2=\left(\frac{h^2+r^2}{2h}\right)^2=\frac{h^4+2h^2r^2+r^4}{4h^2}$$

Substituting then, for R and R^2, their values, and multiplying the factors,

$$\text{The segment}=\tfrac{1}{6}\pi h^3+\tfrac{1}{3}\pi hr^2+\tfrac{1}{6}\frac{\pi r^4}{h}-\tfrac{1}{6}\pi hr^2-\tfrac{1}{6}\frac{\pi r^4}{h}+\tfrac{1}{3}\pi hr^2$$

which, by uniting the terms, becomes

$$\tfrac{1}{2}\pi hr^2+\tfrac{1}{6}\pi h^3.$$

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* Euclid 31, 3, and 8, 6. Cor.

The first term here is $\frac{1}{2}h\times\pi r^2$, half the height of the segment multiplied into the area of the base; (Art. 71.) and the other $h^3\times\frac{1}{6}\pi$, the cube of the height multiplied into .5236.

If the segment be *greater* than a hemisphere, as ABD; (Fig. 9.) the cone ABC must be *added* to the sector ACBD.

Let PD$=h$ the height of the segment,
Then, PC$=h$—R the axis of the cone.

$$\text{The sector ACBD}=\tfrac{2}{3}\pi h R^2$$

$$\text{The cone}=\pi r^2\times\tfrac{1}{3}(h-R)=\tfrac{1}{3}\pi hr^2-\tfrac{1}{3}\pi r^2 R$$

Adding them together, we have as before,

$$\text{The segment}=\tfrac{2}{3}\pi hR^2-\tfrac{1}{3}\pi r^2R+\tfrac{1}{3}\pi hr^2.$$

Cor. The solidity of a spherical segment is equal to half a cylinder of the same base and height + a sphere whose diameter is the height of the segment. For a cylinder is equal to its height multiplied into the area of its base; and a sphere is equal to the cube of its diameter multiplied by .5236.

Thus, if Oy (Fig. 15.) be half Ox, the spherical segment produced by the revolution, of Oxt is equal to the cylinder produced by $tvyx$ + the sphere produced by $Oyxz$; supposing each to revolve on the line Ox.

Ex. 1. If the height of a spherical segment be 8 feet, and the diameter of its base 25 feet; what is the solidity?

Ans. $(25)^2\times.7854\times4+8^3\times.5236=2231.58$ feet.

2. If the earth be a sphere 7930 miles in diameter, and the polar circle 23° 28′ from the pole, what is the solidity of one of the frigid zones? Ans. 1,303,000,000 miles.

PROBLEM XII.

To find the SOLIDITY *of a spherical* ZONE *or frustum.*

76. FROM THE SOLIDITY OF THE WHOLE SPHERE, SUBTRACT THE TWO SEGMENTS ON THE SIDES OF THE ZONE.

Or,

ADD TOGETHER THE SQUARES OF THE RADII OF THE TWO ENDS, AND $\frac{1}{3}$ THE SQUARE OF THEIR DISTANCE; AND MULTIPLY THE SUM BY THREE TIMES THIS DISTANCE, AND THE PRODUCT BY .5236.

If from the whole sphere, (Fig. 15.) there be taken the two segments ABP and GHO, there will remain the zone or frustum ABGH.

Or, the zone ABGH is equal to the difference between the segments GHP and ABP.

Let $NP=H$, $DP=h$, the *heights* of the two segments.

$GN=R$, $AD=r$, the *radii* of their bases.

$DN=d=H-h$ the *distance* of the two bases, or the height of the zone.

Then the larger segment $=\frac{1}{2}\pi HR^2+\frac{1}{6}\pi H^3$
And the smaller segment $=\frac{1}{2}\pi hr^2+\frac{1}{6}\pi h^3$ (Art. 75.)

Therefore the zone $ABGH=\frac{1}{6}\pi(3HR^2+H^3-3hr^2-h^3)$

By the properties of the circle, (Euc. 35, 3.)

$$ON\times H=R^2. \quad \text{Therefore, } (ON+H)\times H=R^2+H^2$$

$$\text{Or, } OP=\frac{R^2+H^2}{H}$$

$$\text{In the same manner, } OP=\frac{r^2+h^2}{h}$$

$$\text{Therefore, } 3H\times(r^2+h^2)=3h\times(R^2+H^2.)$$

$$\text{Or, } 3Hr^2+3Hh^2-3hR^2-3hH^2=0. \quad \text{(Alg. 178.)}$$

To reduce the expression for the solidity of the zone to the required form, without altering its value, let these terms be added to it: and it will become

$$\tfrac{1}{6}\pi(3HR^2+3Hr^2-3hR^2-3hr^2+H^3-3H^2h+3Hh^2-h^3)$$

Which is equal to

$$\tfrac{1}{6}\pi\times3(H-h)\times(R^2+r^2+\tfrac{1}{3}(H-h)^2)$$

Or, as $\frac{1}{6}\pi$ equals .5236 (Art. 71.) and $H-h$ equals d,

$$\text{The zone} = .5236\times3d\times(R^2+r^2+\tfrac{1}{3}d^2)$$

Ex. 1. If the diameter of one end of a spherical zone is 24 feet, the diameter of the other end 20 feet, and the distance of the two ends, or the height of the zone 4 feet; what is the solidity? Ans. 1566.6 feet.

2. If the earth be a sphere 7930 miles in diameter, and the obliquity of the ecliptic 23° 28′; what is the solidity of one of the temperate zones? Ans. 55,390,500,000 miles.

3. What is the solidity of the torrid zone?
Ans. 147,720,000,000 miles.

The solidity	of the two temperate zones is	110,781,000,000
	of the two frigid zones	2,606,000,000
	of the torrid zone	147,720,000,000
	of the whole globe	261,107,000,000

4. What is the convex surface of a spherical zone, whose breadth is 4 feet, on a sphere of 25 feet diameter?

5. What is the solidity of a spherical segment, whose height is 18 feet, and the diameter of its base 40 feet?

PROMISCUOUS EXAMPLES OF SOLIDS.

Ex. 1. How much water can be put into a cubical vessel three feet deep, which has been previously filled with cannon balls of the same size, 2, 4, 6, or 9 inches in diameter, regularly arranged in tiers, one directly above another?

Ans. $96\frac{1}{5}$ wine gallons.

2. If a cone or pyramid, whose height is three feet, be divided into three equal portions, by sections parallel to the base; what will be the heights of the several parts?

Ans. 24.961, 6.488, and 4.551 inches.

3. What is the solidity of the greatest square prism which can be cut from a cylindrical stick of timber, 2 feet 6 inches in diameter and 56 feet long?*

Ans. 175 cubic feet.

4. How many such globes as the earth are equal in bulk to the sun; if the former is 7930 miles in diameter, and the latter 890,000?

Ans. 1,413,678.

5. How many cubic feet of wall are there in a conical tower 66 feet high, if the diameter of the base be 20 feet from outside to outside, and the diameter of the top 8 feet; the thickness of the wall being 4 feet at the bottom, and decreasing regularly, so as to be only 2 feet at the top?

Ans. 7188.

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* The common rule for measuring *round timber* is to multiply the square of the *quarter-girt* by the length. The quarter-girt is one fourth of the circumference. This method does not give the whole solidity. It makes an allowance of about one fifth, for waste in hewing, bark, &c. The solidity of a cylinder is equal to the product of the length into the area of the base.

If C = the circumference, and $\pi = 3.14159$, then (Art. 31.)

$$\text{The area of the base} = \frac{C^2}{4\pi} = \left(\frac{C}{\sqrt{4\pi}}\right)^2 = \left(\frac{C}{3.545}\right)^2$$

If then the circumference were divided by 3.545 instead of 4, and the quotient squared, the area of the base would be correctly found. See note G.

6. If a metallic globe filled with wine, which cost as much at 5 dollars a gallon, as the globe itself at 20 cents for every square inch of its surface; what is the diameter of the globe?

Ans. 55.44 inches.

7. If the circumference of the earth be 25,000 miles, what must be the diameter of a metallic globe, which, when drawn into a wire $\frac{1}{20}$ of an inch in diameter, would reach round the earth? Ans. 15 feet and 1 inch.

8. If a conical cistern be 3 feet deep, $7\frac{1}{2}$ feet in diameter at the bottom, and 5 feet at the top; what will be the depth of a fluid occupying half its capacity?

Ans. 14.535 inches.

9. If a globe 20 inches in diameter be perforated by a cylinder 16 inches in diameter, the axis of the latter passing through the center of the former; what part of the solidity, and the surface of the globe will be cut away by the cylinder?

Ans. 3284 inches of the solidity, and 502,655 of the surface.

10. What is the solidity of the greatest cube which can be cut from a sphere three feet in diameter?

Ans. $5\frac{1}{5}$ feet.

11. What is the solidity of a conic frustum, the altitude of which is 36 feet, the greater diameter 16, and the lesser diameter 8?

12. What is the solidity of a spherica. segment 4 feet high, cut from a sphere 16 feet in diameter?

SECTION V.

ISOPERIMETRY.*

Art. 77. It is often necessary to compare a number of different figures or solids, for the purpose of ascertaining which has the *greatest area*, within a given perimeter, or the *greatest capacity* under a given surface. We may have occasion to determine, for instance, what must be the form of a fort, to contain a given number of troops, with the least extent of wall; or what the shape of a metallic pipe to convey a given portion of water, or of a cistern to hold a given quantity of liquor, with the least expense of materials.

78. Figures which have equal perimeters are called *Isoperimeters.* When a quantity is *greater* than any other of the same class, it is called a *maximum.* A multitude of straight lines, of different lengths, may be drawn within a circle. But among them all, the *diameter* is a *maximum.* Of all *sines* of angles, which can be drawn in a circle, the sine of 90° is a *maximum.*

When a quantity is *less* than any other of the same class, it is called a *minimum.* Thus, of all straight lines drawn from a given point to a given straight line, that which is *perpendicular* to the given line is a *minimum.* Of all straight lines drawn from a given point in a circle, to the circumference, the *maximum* and *minimum* are the two parts of the diameter which pass through that point. (Euc. 7, 3.)

In isoperimetry, the object is to determine, on the one hand, in what cases the area is a *maximum*, within a given perimeter; or the capacity a *maximum*, within a given surface: and on the other hand, in what cases the perimeter is a *minimum* for a given area, or the surface a *minimum*, for a given capacity.

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* Emerson's, Simpson's, and Legendre's Geometry, Lhuillier, Fontenelle, Hutton's Mathematics, and Lond. Phil. Trans. Vol. 75.

PROPOSITION I.

79. *An* Isosceles Triangle *has a greater area than any scalene triangle, of equal base and perimeter.*

If ABC (Fig. 26.) be an isosceles triangle whose equal sides are AC and BC; and if ABC′ be a scalene triangle on the same base AB, and having AC′+BC′=AC+BC; then the area of ABC is greater than that of ABC′.

Let perpendiculars be raised from each end of the base, extend AC to D, make C′D′ equal to AC′, join BD, and draw CH and C′H′ parallel to AB.

As the angle CAB=ABC, (Euc. 5, 1.) and ABD is a right angle, ABC+CBD=CAB+CDB=ABC+CDB. Therefore CBD=CDB, so that CD=CB; and by construction, C′D′=AC′. The perpendiculars of the equal right angled triangles CHD and CHB are equal; therefore, BH=½BD. In the same manner, AH′=½AD′. The line AD=AC+BC=AC′+BC′=D′C′+BC′. But D′C′+BC′>BD′. (Euc. 20, 1.) Therefore, AD>BD′; BD>AD′, (Euc. 47, 1.) and ½BD>½AD′. But ½BD, or BH, is the height of the isosceles triangle; (Art. 1.) and ½AD′ or AH′, the height of the scalene triangle; and the areas of two triangles which have the same base are as their heights. (Art. 8.) Therefore the area of ABC is greater than that of ABC′. Among all triangles, then, of a given perimeter, and upon a given base, the isosceles triangle is a *maximum.*

Cor. The isosceles triangle has a *less perimeter* than any scalene triangle of the same base and area. The triangle ABC′ being less than ABC, it is evident the perimeter of the former must be enlarged, to make its area equal to the area of the latter.

PROPOSITION II.

80. *A triangle in which two given sides make a* right angle, *has a greater area than any triangle in which the same sides make an oblique angle.*

If BC, BC′, and BC″ (Fig. 27.) be equal, and if BC be perpendicular to AB; then the right angled triangle ABC, has a greater area than the acute angled triangle ABC′, or the oblique angled triangle ABC″.

Let P′C′ and PC″ be perpendicular to AP. Then, as the three triangles have the same base AB, their areas are as their heights; that is, as the perpendiculars BC, P′C′, and PC″. But BC is equal to BC′, and therefore greater than P′C′. (Euc. 47. 1.) BC is also equal to BC″, and therefore greater than PC″.

PROPOSITION III.

81. *If all the sides* EXCEPT ONE *of a polygon be given, the area will be the greatest, when the given sides are so disposed, that the figure may be* INSCRIBED IN A SEMICIRCLE, *of which the undetermined side is the diameter.*

If the sides AB, BC, CD, DE, (Fig. 28.) be given, and if their position be such that the area, included between these and another side whose length is not determined, is a *maximum;* the figure may be inscribed in a semicircle, of which the undetermined side AE is the diameter.

Draw the lines AD, AC, EB, EC. By varying the angle at D, the triangle ADE may be enlarged or diminished, without affecting the area of the other parts of the figure. The whole area, therefore, cannot be a *maximum*, unless this triangle be a *maximum*, while the sides AD and ED are given. But if the triangle ADE be a *maximum*, under these conditions, the angle ADE is a right angle; (Art. 80.) and therefore the point D is in the circumference of a circle, of which AE is the diameter. (Euc. 31, 3.) In the same manner it may be proved, that the angles ACE and ABE are right angles, and therefore that the points C and B are in the circumference of the same circle.

The term *polygon* is used in this section to include *triangles*, and *four-sided* figures, as well as other right-lined figures.

82. The area of a polygon, inscribed in a semicircle, in the manner stated above, will not be altered by varying the *order* of the given sides.

The sides AB, BC, CD, DE, (Fig. 28.) are the *chords* of so many arcs. The sum of these arcs, in whatever order they are arranged, will evidently be equal to the semicircumference. And the *segments* between the given sides and the arcs will be the same, in whatever part of the circle they are situated. But the area of the polygon is equal to the area of the semicircle, diminished by the sum of these segments.

83, If a polygon, of which all the sides except one are given, be inscribed in a semicircle whose diameter is the undetermined side; a polygon having the same given sides, cannot be inscribed in any *other* semicircle which is either greater or less than this, and whose diameter is the undetermined side.

The given sides AB, BC, CD, DE, (Fig. 28.) are the chords of arcs whose sum is 180 degrees. But in a larger circle, each would be the chord of a less number of degrees, and therefore the sum of the arcs would be less than 180°: and in a smaller circle, each would be the chord of a greater number of degrees, and the sum of the arcs would be greater than 180°.

PROPOSITION IV.

84. *A polygon* INSCRIBED IN A CIRCLE *has a greater area, than any polygon of equal perimeter, and the same number of sides, which cannot be inscribed in a circle.*

If in the circle ACHF, (Fig. 30.) there be inscribed a polygon ABCDEFG; and if another polygon *abcdefg* (Fig. 31.) be formed of sides which are the same in number and length, but which are so disposed, that the figure cannot be inscribed in a circle; the area of the former polygon is greater than that of the latter.

Draw the diameter AH, and the chords DH and EH. Upon *de* make the triangle *deh* equal and similar to DEH, and join *ah*. The line *ah* divides the figure *abcdhefg* into two parts, of which *one at least* cannot, by supposition, be inscribed in a semicircle of which the diameter is AH, nor in any other semicircle of which the diameter is the undetermined side. (Art. 83.) It is therefore less than the corresponding part of the figure ABCDHEFG. (Art. 81.) And the other part of *abcdhefg* is not greater than the corresponding part of ABCDHEFG. Therefore, the whole figure ABCDHEFG is greater than the whole figure *abcdhefg*. If from these there be taken the equal triangles DEH and *deh*, there will remain the polygon ABCDEFG greater than the polygon *abcdefg*.

85. A polygon of which all the sides are given in number and length, can not be inscribed in circles of different diameters. (Art. 83.) And the area of the polygon will not be altered, by changing the *order* of the sides. (Art. 82.)

PROPOSITION V.

86. *When a polygon has a greater area than any other, of the same number of sides, and of equal perimeter, the sides are* EQUAL.

The polygon ABCDF (Fig. 29.) cannot be a *maximum*, among all polygons of the same number of sides, and of equal perimeters, unless it be equilateral. For if any two of the sides, as CD and FD, are unequal, let CH and FH be equal, and their sum the same as the sum of CD and FD. The isosceles triangle CHF is greater than the scalene triangle CDF (Art. 79.); and therefore the polygon ABCHF is greater than the polygon ABCDF; so that the latter is not a *maximum.*

PROPOSITION VI.

87. *A* REGULAR POLYGON *has a greater area than any other polygon of equal perimeter, and of the same number of sides.*

For, by the preceding article, the polygon which is a *maximum* among others of equal perimeters, and the same number of sides, is *equilateral*, and by art. 84, it may be *inscribed in a circle.* But if a polygon inscribed in a circle is equilateral, as ABDFGH (Fig. 7.) it is also *equiangular*. For the sides of the polygon are the bases of so many isosceles triangles, whose common vertex is the center C. The angles at these bases are all equal; and two of them, as AHC and GHC, are equal to AHG one of the angles of the polygon. The polygon, then, being equiangular, as well as equilateral, is a *regular* polygon. (Art. 1. Def. 2.)

Thus an *equilateral triangle* has a greater area, than any other triangle of equal perimeter. And a *square* has a greater area, than any other four-sided figure of equal perimeter.

Cor. A regular polygon has a *less perimeter* than any other polygon of equal area, and the same number of sides.

For if, with a given perimeter, the regular polygon is greater than one which is not regular; it is evident the perimeter of the former must be diminished, to make its area equal to that of the latter.

PROPOSITION VII.

88. *If a polygon be* DESCRIBED ABOUT A CIRCLE, *the areas of the two figures are as their perimeters.*

Let ST (Fig. 32.) be one of the sides of a polygon, either regular or not, which is described about the circle LNR. Join OS and OT, and to the point of contact M draw the radius OM, which will be perpendicular to ST. (Euc. 18, 3.) The triangle OST is equal to half the base ST multiplied into the radius OM. (Art. 8.) And if lines be drawn, in the same manner, from the center of the circle, to the extremities of the several sides of the circumscribed polygon, each of the triangles thus formed will be equal to half its base multiplied into the radius of the circle. Therefore the area of the whole polygon is equal to half its perimeter multiplied into the radius: and the area of the circle is equal to half its circumference multiplied into the radius. (Art. 30.) So that the two areas are to each other as their perimeters.

Cor. 1. If different polygons are described about the same circle, their areas are to each other as their perimeters. For the area of each is equal to half its perimeter, multiplied into the radius of the inscribed circle.

Cor. 2. The *tangent* of an arc is always greater than the arc itself. The triangle OMT (Fig. 32.) is to OMN, as MT to MN. But OMT is greater than OMN, because the former includes the latter. Therefore, the tangent MT is greater than the arc MN.

PROPOSITION VIII.

89. *A* CIRCLE *has a greater area than any polygon of equal perimeter.*

If a circle and a regular polygon have the same center, and equal perimeters; each of the sides of the polygon must fall partly *within* the circle. For the area of a *circumscribing* polygon is greater than the area of the circle, as the one includes the other: and therefore, by the preceding article, the *perimeter* of the former is greater than that of the latter.

Let AD then (Fig. 32.) be one side of a regular polygon, whose perimeter is equal to the circumference of the circle RLN. As this falls partly within the circle, the perpendicular OP is less than the radius OR. But the area of the polygon is equal to half its perimeter multiplied into this perpendicular (Art. 15.); and the area of the circle is equal to half its circumference multiplied into the radius. (Art. 30.) The circle then is greater than the given regular polygon; and therefore greater than any other polygon of equal perimeter. (Art. 87.)

Cor. 1. A circle has a *less perimeter*, than any polygon of equal area.

Cor. 2. Among regular polygons of a given perimeter, that which has the *greatest number of sides*, has also the *greatest area.* For the greater the number of sides, the more nearly does the perimeter of the polygon approach to a coincidence with the circumference of a circle.*

PROPOSITION IX.

90. *A right* PRISM *whose bases are* REGULAR POLYGONS, *has a less surface than any other right prism of the same solidity, the same altitude, and the same number of sides.*

If the altitude of a prism is given, the area of the base is as the solidity (Art. 43.); and if the number of sides is also given, the perimeter is a *minimum*, when the base is a regular

* For a rigorous demonstration of this, see Legendre's Geometry, Appendix to Book iv.

polygon. (Art. 87. Cor.) But the lateral surface is as the perimeter. (Art. 47.) Of two right prisms, then, which have the same altitude, the same solidity, and the same number of sides, that whose bases are regular polygons has the least *lateral* surface, while the areas of the ends are equal.

Cor. A right prism whose bases are regular polygons has a *greater solidity*, than any other right prism of the same surface, the same altitude, and the same number of sides.

PROPOSITION X.

91. *A right* CYLINDER *has a less surface, than any right prism of the same altitude and solidity.*

For if the prism and cylinder have the same altitude and solidity, the areas of their bases are equal. (Art. 64.) But the *perimeter* of the cylinder is less, than that of the prism (Art. 89. Cor. 1.); and therefore its lateral surface is less, while the areas of the ends are equal.

Cor. A right cylinder has a *greater solidity*, than any right prism of the same altitude and surface.

PROPOSITION XI.

92. *A* CUBE *has a less surface than any other right parallelopiped of the same solidity.*

A parallelopiped is a prism, any one of whose faces may be considered a base. (Art. 41. Def. I. and V.) If these are not all *squares*, let one which is not a square be taken for a base. The perimeter of this may be diminished, without altering its area (Art. 87. Cor.); and therefore the surface of the solid may be diminished, without altering its altitude or solidity. (Art. 43, 47.) The same may be proved of each of the other faces which are not squares. The surface is therefore a *minimum*, when *all* the faces are squares, that is, when the solid is a *cube*.

Cor. A cube has a *greater solidity* than any other right parallelopiped of the same surface.

PROPOSITION XII.

93. *A* CUBE *has a greater solidity, than any other right parallelopiped, the sum of whose length, breadth, and depth is equal to the sum of the corresponding dimensions of the cube.*

The solidity is equal to the product of the length, breadth, and depth. If the length and breadth are unequal, the solidity may be increased, without altering the sum of the three dimensions. For the product of two factors whose sum is given, is the greatest when the factors are equal. (Euc. 27. 6.) In the same manner, if the breadth and depth are unequal, the solidity may be increased, without altering the sum of the three dimensions. Therefore, the solid can not be a *maximum*, unless its length, breadth, and depth are equal.

PROPOSITION XIII.

94. *If a* PRISM BE DESCRIBED ABOUT A CYLINDER, *the capacities of the two solids are as their surfaces.*

The capacities of the solids are as the *areas* of their bases, that is, as the *perimeters* of their bases. (Art. 88.) But the lateral surfaces are also as the perimeters of the bases. Therefore the *whole* surfaces are as the solidities.

Cor. The capacities of different prisms, described about the same right cylinder, are to each other as their surfaces.

PROPOSITION XIV.

95. *A right cylinder* WHOSE HEIGHT IS EQUAL TO THE DIAMETER OF ITS BASE *has a greater solidity than any other right cylinder of equal surface.*

Let C be a right cylinder whose height is equal to the diameter of its base; and C′ another right cylinder having the same surface, but a different altitude. If a square prism P be described about the former, it will be a *cube*. But a square prism P′ described about the latter will not be a cube.

Then the surfaces of C and P are as their bases (Art. 47. and 88.); which are as the bases of C′ and P′, (Sup. Euc. 7, 1.); so that,

$surf C : surf P :: base C : base P :: base C' : base P' :: surf C' : surf P'$

But the surface of C is, by supposition, equal to the surface of C′. Therefore, (Alg. 395.) the surface of P is equal to the surface of P′. And by the preceding article,

$solid P : solid C :: surf P : surf C :: surf P' : surf C' :: solid P' : solid C'$

But the solidity of P is greater than that of P′. (Art. 92. Cor.) Therefore, the solidity of C is greater than that of C′.

Schol. A right cylinder whose height is equal to the diameter of its base, is that which *circumscribes a sphere.* It is also called *Archimedes' cylinder;* as he discovered the ratio of a sphere to its circumscribing cylinder; and these are the figures which were put upon his tomb.

Cor. Archimedes' cylinder has a *less surface*, than any other right cylinder of the same capacity.

PROPOSITION XV.

96. *If a* SPHERE BE CIRCUMSCRIBED *by a solid bounded by plane surfaces; the capacities of the two solids are as their surfaces.*

If planes be supposed to be drawn from the center of the sphere, to each of the edges of the circumscribing solid, they will divide it into as many pyramids as the solid has faces. The base of each pyramid will be one of the faces; and the height will be the radius of the sphere. The capacity of the pyramid will be equal, therefore, to its base multiplied into $\frac{1}{3}$ of the radius (Art. 48.); and the capacity of the whole circumscribing solid, must be equal to its whole surface multiplied into $\frac{1}{3}$ of the radius. But the capacity of the sphere is also equal to its surface multiplied into $\frac{1}{3}$ of its radius. (Art. 70.)

Cor. The capacities of different solids circumscribing the same sphere, are as their surfaces.

PROPOSITION XVI.

97. ***A* SPHERE *has a greater solidity than any regular polyedron of equal surface.***

If a sphere and a regular polyedron have the same center, and equal surfaces; each of the faces of the polyedron must fall partly *within* the sphere. For the solidity of a *circumscribing* solid is greater than the solidity of the sphere, as the one includes the other: and therefore, by the preceding article, the *surface* of the former is greater than that of the latter.

But if the faces of the polyedron fall partly within the sphere, their perpendicular distance from the center must be less than the radius. And therefore, if the surface of the polyedron be only equal to that of the sphere, its solidity must be less. For the solidity of the polyedron is equal to its surface multiplied into $\frac{1}{3}$ of the distance from the center. (Art. 59.) And the solidity of the sphere is equal to its surface multiplied into $\frac{1}{3}$ of the radius.

Cor. A sphere has a *less surface* than any regular polyedron of the same capacity.

For other cases of Isoperimetry, see Fluxions.

APPENDIX.—PART I.

CONTAINING RULES, WITHOUT DEMONSTRATIONS, FOR THE MENSURATION OF THE CONIC SECTIONS, AND OTHER FIGURES NOT TREATED OF IN THE ELEMENTS OF EUCLID.*

PROBLEM I.

To find the area of an ELLIPSE.

101. Multiply the product of the transverse and conjugate axes into .7854.

Ex. What is the area of an ellipse whose transverse axis is 36 feet, and conjugate 28? Ans. 791.68 feet.

PROBLEM II.

To find the area of a SEGMENT *of an ellipse, cut off by a line perpendicular to either axis.*

102. If either axis of an ellipse be made the diameter of a circle; and if a line perpendicular to this axis cut off a segment from the ellipse, and from the circle;

The diameter of the circle, is to the other axis of the ellipse; As the circular segment, to the elliptic segment.

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* For demonstrations of these rules, see Conic Sections, Spherical Trigonometry, and Fluxions, or Hutton's Mensuration.

Ex. What is the area of a segment cut off from an ellipse whose transverse axis is 415 feet, and conjugate 332; if the height of the segment is 96 feet, and its base is perpendicular to the transverse axis?

The circular segment is 23680 feet.
And the elliptic segment 18944

PROBLEM III.

To find the area of a conic PARABOLA.

103. Multiply the base by $\frac{2}{3}$ of the height.

Ex. If the base of a parabola is 26 inches, and the height 9 feet; what is the area? Ans. 13 feet.

PROBLEM IV.

To find the area of a FRUSTUM *of a parabola, cut off by a line parallel to the base.*

104. Divide the difference of the cubes of the diameters of the two ends, by the difference of their squares; and multiply the quotient by $\frac{2}{3}$ of the perpendicular height.

Ex. What is the area of a parabolic frustum, whose height is 12 feet, and the diameters of its ends 20 and 12 feet?

Ans. 196 feet.

PROBLEM V.

To find the area of a conic HYPERBOLA.

105. Multiply the base by $\frac{2}{3}$ of the height; and correct the oduct by subtracting from it the series

$$2bh\times\left(\frac{z}{1.3.5}+\frac{z^2}{3.5.7}+\frac{z^3}{5.7.9}+\frac{z^4}{7.9.11}+\&c.\right)$$

In which $\begin{cases} b = \text{the base or double ordinate,} \\ h = \text{the height or abscissa,} \\ z = \text{the height divided by the sum of the height and transverse axis.} \end{cases}$

The series converges so rapidly, that a few of the first terms will generally give the correction with sufficient exactness. This correction is the difference between the hyperbola, and a parabola of the same base and height.

Ex. If the base of a hyperbola be 24 feet, the height 10 and the transverse axis 30; what is the area?

The base $\times \frac{2}{3}$ the height is	160.
The first term of the series is	0.016666
The second	0.000592
The third	0.000049
The fourth	0.000006
Their sum	0.017313
This into $2bh$ is	8.31
And the area corrected is	151.69

PROBLEM VI.

To find the area of a SPHERICAL TRIANGLE *formed by three arcs of great circles of a sphere.*

106. As 8 right angles or 720°,
To the excess of the 3 given angles above 180°;
So is the whole surface of the sphere,
To the area of the spherical triangle.

Ex. What is the area of a spherical triangle, on a sphere whose diameter is 30 feet, if the angles are 130°, 102°, and 68°? Ans. 471.24 feet.

PROBLEM VII.

To find the area of a SPHERICAL POLYGON *formed by arcs of great circles.*

107. As 8 right angles, or 720°,
To the excess of all the given angles above the product of the number of angles—2 into 180°;
So is the whole surface of the sphere,
To the area of the spherical polygon.

Ex. What is the area of a spherical polygon of seven sides, on a sphere whose diameter is 17 inches; if the sum of all the angles is 1080°? Ans. 227 inches.

PROBLEM VIII.

To find the lunar surface included between two great circles of a sphere.

108. As 360°, to the angle made by the given circles;
So is the whole surface of the sphere, to the surface between the circles.

Or,

The lunar surface is equal to the breadth of the middle part of it, multiplied into the diameter of the sphere.

Ex. If the earth be 7930 miles in diameter, what is the surface of that part of it which is included between the 65th and 83d degree of longitude?

Ans. 9,878,000 square miles.

PROBLEM IX.

To find the solidity of a SPHEROID, *formed by the revolution of an ellipse about either axis.*

109. Multiply the product of the fixed axis and the square of the revolving axis, into .5236.

Ex. 1. What is the solidity of an oblong spheroid, whose longest and shortest diameters are 40 and 30 feet?

Ans. $40\times\overline{30}^2\times.5236=18850$ feet.

2. If the earth be an oblate spheroid, whose polar and equatorial diameters are 7930 and 7960 miles; what is its solidity? Ans. 263,000,000,000 miles.

PROBLEM X.

To find the solidity of the MIDDLE FRUSTUM *of a spheroid, included between two planes which are perpendicular to the axis, and equally distant from the center.*

110. Add together the square of the diameter of one end, and twice the square of the middle diameter; multiply the sum by $\frac{1}{3}$ of the height, and the product by .7854.

If D and d=the two diameters, and h=the height;

$$\text{The solidity}=(2D^2+d^2)\times\tfrac{1}{3}h\times.7854.$$

Ex. If the diameter of one end of a middle frustum of a spheroid be 8 inches, the middle diameter 10 and the height 30, what is the solidity?

Ans. 2073.4 inches.

Cor. *Half* the middle frustum is equal to a frustum of which one of the ends passes through the center.

If then D and d=the diameters of the two ends, and h=the height,

$$\text{The solidity}=(2D^2+d^2)\times\tfrac{1}{3}h\times.7854.$$

PROBLEM XI.

To find the solidity of a PARABOLOID.

111. Multiply the area of the base by half the height.

Ex. If the diameter of the base of a paraboloid be 12 feet, and the height 22 feet, what is the solidity?

Ans. 1243 feet.

PROBLEM XII.

To find the solidity of a FRUSTUM *of a paraboloid.*

112. Multiply the sum of the areas of the two ends by half their distance.

Ex. If the diameter of one end of a frustum of a paraboloid be 8 feet, the diameter of the other end 6 feet, and the length 24 feet; what is the solidity?

Ans. $942\frac{1}{2}$ feet.

Cor. If a cask be in the form of *two equal* frustums of a paraboloid; and
If D=the middle diam. d=the end diam. and h=the length;

$$\text{The solidity}=(D^2+d^2)\times\tfrac{1}{2}h\times.7854.$$

PROBLEM XIII.

To find the solidity of a HYPERBOLOID, *produced by the revolution of a hyperbola on its axis.*

113. Add together the square of the radius of the base, and the square of the diameter of a section which is equally distant from the base and the vertex; multiply the sum by the height, and the product by .5236.
If R=the radius of the base, D=the middle diameter, and h=the height;

$$\text{The solidity}=(R^2+D^2)\times h\times.5236.$$

Ex. If the diameter of the base of a hyperboloid be 24, the square of the middle diameter 252, and the height 10, what is the solidity? Ans. 2073.4.

PROBLEM XIV.

To find the solidity of a FRUSTUM *of a hyperboloid.*

114. Add together the squares of the radii of the two ends, and the square of the middle diameter; multiply the sum by the height, and the product by .5236.
If R and r=the two radii, D=the middle diameter, and h=the height;

$$\text{The solidity}=(R^2+r^2+D^2)\times h\times.5236.$$

Ex. If the diameter of one end of a frustum of a hyperboloid be 32, the diameter of the other end 24, the square of the middle diameter $793\frac{3}{5}$, and the length 20, what is the solidity?

Ans. 12499.3.

PROBLEM XV.

To find the solidity of a CIRCULAR SPINDLE, *produced by the revolution of a circular segment about its base or chord as an axis.*

115. From $\frac{1}{3}$ of the cube of half the axis, subtract the product of the central distance into half the revolving circular segment, and multiply the remainder by four times 3.14159.

If a=the erea of the revolving circular segment,
l=half the length or axis of the spindle,
c=the distance of this axis from the center of the circle to which the revolving segment belongs;
The solidity$=(\frac{1}{3}l^3-\frac{1}{2}ac)\times4\times3.14159$.

Ex. Let a circular spindle be produced by the revolution of the segment ABO (Fig. 9.) about AB. If the axis AB be 140, and OP half the middle diameter of the spindle be 38.4; what is the solidity?

The area of the revolving segment is	3791
The central distance PC	44.6
The solidity of the spindle	374402

PROBLEM XVI.

To find the solidity of the MIDDLE FRUSTUM *of a circular spindle.*

116. From the square of half the axis of the whole spindle, subtract $\frac{1}{3}$ of the square of half the length of the frustum; multiply the remainder by this half length; from the product subtract the product of the revolving area into the central distance; and multiply the remainder by twice 3.14159.

If L=half the length or axis of the whole spindle,
l=half the length of the middle frustum,
c=the distance of the axis from the center of the circle,
a=the area of the figure which, by revolving, produces the frustum;
The solidity$=(\overline{L^2-\frac{1}{3}l^2}\times l-ac)\times2\times3.14159$.

Ex. If the diameter of each end of a frustum of a circular spindle be 21.6, the middle diameter 60, and the length 70; what is the solidity?

The length of the whole spindle is	79.75
The central distance	11.5
The revolving area	1703.8
The solidity	136751.5

PROBLEM XVII.

To find the solidity of a PARABOLIC SPINDLE, *produced by the revolution of a parabola about a double ordinate or base.*

117. Multiply the square of the middle diameter by $\frac{8}{15}$ of the axis, and the product by .7854.

Ex. If the axis of a parabolic spindle be 30, and the middle diameter 17, what is the solidity?

Ans. 3631.7.

PROBLEM XVIII.

To find the solidity of the MIDDLE FRUSTUM *of a parabolic spindle.*

118. Add together the square of the end diameter, and twice the square of the middle diameter; from the sum subtract $\frac{2}{5}$ of the square of the difference of the diameters, and multiply the remainder by $\frac{1}{3}$ of the length, and the product by .7854.

If D and d = the two diameters, and l = the length;
The solidity $= (2D^2 + d^2 - \frac{2}{5}(D-d)^2) \times \frac{1}{3} l \times .7854$.

Ex. If the end diameters of a frustum of a parabolic spindle be each 12 inches, the middle diameter 16, and the length 30; what is the solidity? Ans. 5102 inches.

APPENDIX.—PART II.

GAUGING OF CASKS.

ART. 119. GAUGING is a practical art, which does not admit of being treated in a very scientific manner. Casks are not commonly constructed in exact conformity with any regular mathematical figure. By most writers on the subject, however, they are considered as nearly coinciding with one of the following forms;

1. 2.	The middle frustum	of a spheroid, of a parabolic spindle.
3. 4.	The equal frustums	of a paraboloid, of a cone.

The *second* of these varieties agrees more nearly than any of the others, with the forms of casks, as they are commonly made. The first is too much curved, the third too little, and the fourth not at all, from the head to the bung.

120. Rules have already been given, for finding the capacity of each of the four varieties of casks. (Arts. 68, 110, 112, 118.) As the dimensions are taken in *inches*, these rules will give the contents in cubic inches. To abridge the computation, and adapt it to the particular measures used in gauging, the factor .7854 is divided by 282 or 231; and the quotient is used instead of .7854, for finding the capacity in ale gallons or wine gallons.

$$\text{Now } \frac{.7854}{282} = .002785, \text{ or } .0028 \text{ nearly};$$

$$\text{And } \frac{.7854}{231} = .0034$$

If then .0028 and 0034 be substituted for .7854, in the rules referred to above; the contents of the cask will be given in ale gallons and wine gallons. These numbers are to each other nearly as 9 to 11.

PROBLEM I.

To calculate the contents of a cask, in the form of the middle frustum of a SPHEROID.

121. Add together the square of the head diameter, and twice the square of the bung diameter; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or by .0034 for wine gallons.

If D and d=the two diameters, and l=the length;

The capacity in inches$=(2D^2+d^2)\times\frac{1}{3}l\times.7854$. (Art. 110.)

And by substituting .0028 or .0034 for .7854, we have the capacity in ale gallons or wine gallons.

Ex. What is the capacity of a cask of the first form, whose length is 30 inches, its head diameter 18, and its bung diameter 24?

Ans. 41.3 ale gallons, or 50.2 wine gallons.

PROBLEM II.

To calculate the contents of a cask, in the form of the middle frustum of a PARABOLIC SPINDLE.

122. Add together the square of the head diameter, and twice the square of the bung diameter, and from the sum subtract $\frac{2}{5}$ of the square of the difference of the diameters; multiply the remainder by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $=(2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times$ .7854. (Art. 118.)

Ex. What is the capacity of a cask of the second form, whose length is 30 inches, its head diameter 18, and its bung diameter 24?

Ans. 40.9 ale gallons, or 49.7 wine gallons.

PROBLEM III.

To calculate the contents of a cask, in the form of two equal frustums of a PARABOLOID.

123. Add together the square of the head diameter, and the square of the bung diameter; multiply the sum by half the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $=(D^2+d^2)\times\frac{1}{2}l\times.7854$. (Art. 112. Cor.)

Ex. What is the capacity of a cask of the third form, whose dimensions are, as before, 30, 18, and 24?

Ans. 37.8 ale gallons, or 45.9 wine gallons.

PROBLEM IV.

To calculate the contents of a cask, in the form of two equal frustums of a CONE.

124. Add together the square of the head diameter, the square of the bung diameter, and the product of the two diameters; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $=(D^2+d^2+Dd)\times\frac{1}{3}l\times.7854$. (Art. 68.)

Ex. What is the capacity of a cask of the fourth form, whose length is 30, and its diameters 18 and 24?

Ans. 37.3 ale gallons, or 45.3 wine gallons.

125. The preceding rules, though correct in theory, are not very well adapted to practice, as they suppose the form of the cask to be *known*. The two following rules, taken from Hutton's Mensuration, may be used for casks of the usual forms. For the first, *three* dimensions are required, the length, the head diameter, and the bung diameter. It is evident that no allowance is made by this, for different degrees of curvature from the head to the bung. If the cask is more or less curved than usual, the following rule is to be preferred, for which *four* dimensions are required, the head

and bung diameters, and a third diameter taken in the middle between the bung and the head. For the demonstration of these rules, see Hutton's Mensuration, Part v. Sec. 2. Ch. 5. and 7.

PROBLEM V.

To calculate the contents of any common cask from THREE *dimensions.*

126. Add together

25 times the square of the head diameter,
39 times the square of the bung diameter, and
26 times the product of the two diameters;

Multiply the sum by the length, divide the product by 90, and multiply the quotient by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $=(39\,D^2+25d^2+26\,Dd)\times\frac{l}{90}\times.7854$.

Ex. What is the capacity of a cask whose length is 30 inches, the head diameter 18, and the bung diameter 24?

Ans. 39 ale gallons, or $47\frac{1}{3}$ wine gallons.

PROBLEM VI.

To calculate the contents of a cask from FOUR *dimensions, the length, the head and bung diameters, and a diameter taken in the middle between the head and the bung.*

127. Add together the squares of the head diameter, of the bung diameter, and of double the middle diameter; multiply the sum by $\frac{1}{6}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If D=the bung diameter, d=the head diameter, m=the middle diameter, and l=the length;

The capacity in inches $=(D^2+d^2+\overline{2m}^2)\times\frac{1}{6}l\times.7854$.

Ex. What is the capacity of a cask, whose length is 30 inches, the head diameter 18, the bung diameter 24, and the middle diameter $22\frac{1}{2}$?

Ans. 41 ale gallons, or $49\frac{2}{3}$ wine gallons.

128. In making the calculations in gauging, according to the preceding rules, the multiplications and divisions are frequently performed by means of a *Sliding Rule*, on which are placed a number of logarithmic lines, similar to those on Gunter's Scale. See Trigonometry, Sec. vi. and Note G. p. 141.

Another instrument commonly used in gauging is the *Diagonal Rod*. By this, the capacity of a cask is very expeditiously found, from a single dimension, the distance from the bung to the intersection of the opposite stave with the head. The measure is taken by extending the rod through the cask, from the bung to the most distant part of the head. The number of gallons corresponding to the length of the line thus found, is marked on the rod. The *logarithmic* lines on the gauging rod are to be used in the same manner, as on the sliding rule.

ULLAGE OF CASKS.

129. When a cask is *partly* filled, the whole capacity is divided, by the surface of the liquor, into two portions; the *least* of which, whether full or empty, is called the *ullage*. In finding the ullage, the cask is supposed to be in one of two positions; either *standing*, with its axis perpendicular to the horizon; or *lying*, with its axis parallel to the horizon. The rules for ullage which are *exact*, particularly those for lying casks, are too complicated for common use. The following are considered as sufficiently near approximations. See Hutton's Mensuration.

PROBLEM VII.

To calculate the ullage of a* STANDING *cask.

130. Add together the squares of the diameter at the surface of the liquor, of the diameter of the nearest end, and of double the diameter in the middle between the other two; multiply the sum by $\frac{1}{6}$ of the distance between the surface and the nearest end, and the product by .0028 for ale gallons or .0034 for wine gallons.

If D = the diameter of the surface of the liquor,
d = the diameter of the nearest end,
m = the middle diameter, and
l = the distance between the surface and the nearest end;
The ullage in inches $=(D^2+d^2+\overline{2m^2})\times\frac{1}{6}l\times.7854$.

Ex. If the diameter at the surface of the liquor, in a standing cask, be 32 inches, the diameter of the nearest end 24, the middle diameter 29, and the distance between the surface of the liquor and the nearest end 12; what is the ullage?
Ans. $27\frac{4}{5}$ ale gallons, or $33\frac{3}{4}$ wine gallons.

PROBLEM VIII.

To calculate the ullage of a LYING *cask.*

131. Divide the distance from the bung to the surface of the liquor, by the whole bung diameter, find the quotient in the column of heights or versed sines in a table of circular segments, take out the corresponding segment, and multiply it by the whole capacity of the cask, and the product by $1\frac{1}{4}$ for the part which is empty.

If the cask be not half full, divide the depth of the liquor by the whole bung diameter, take out the segment, multiply, &c., for the contents of the part which is full.

Ex. If the whole capacity of a lying cask be 41 ale gallons, or $49\frac{2}{3}$ wine gallons, the bung diameter 24 inches and the distance from the bung to the surface of the liquor 6 inches; what is the ullage?
Ans. $7\frac{3}{4}$ ale gallons, or $9\frac{1}{2}$ wine gallons.

NOTES

Note A. p. 16.

One of the earliest approximations to the ratio of the circumference of a circle to its diameter, was that of *Archimedes.* He demonstrated that the ratio of the perimeter of a regular inscribed polygon of 96 sides, to the diameter of the circle, is greater than $3\frac{10}{71} : 1$; and that the ratio of the perimeter of a circumscribed polygon of 192 sides, to the diameter, is less than $3\frac{10}{71} : 1$, that is, than 22 : 7.

Metius gave the ratio of 355 : 113, which is more accurate than any other expressed in small numbers. This was confirmed by *Vieta,* who by inscribed and circumscribed polygons of 393216 sides, carried the approximation to ten places of figures, viz.

3.141592653.

Van Ceulen of Leyden afterwards extended it, by the laborious process of repeated bisections of an arc, to 36 places. This calculation was deemed of so much consequence at the time, that the numbers are said to have been put upon his tomb.

But since the invention of *fluxions,* methods much more expeditious have been devised, for approximating to the required ratio. These principally consist in finding the sum of a series, in which the length of an arc is expressed in terms of its *tangent.*

If t = the tangent of an arc, the radius being 1,

The arc $= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} -$ &c. See Fluxions.

This series is in itself very simple. Nothing more is necessary to make it answer the purpose in practice, than that the arc be *small*, so as to render the series sufficiently converging, and that the tangent be expressed in such simple numbers, as can easily be raised to the several powers. The given series will be expressed in the most simple numbers, when the arc is 45°, whose tangent is equal to radius. If the radius be 1,

The arc of $45° = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ &c. And this multiplied by 8 gives the length of the whole circumference.

But a series in which the tangent is smaller, though it be less simple than this, is to be preferred, for the rapidity with which it converges. As the tangent of $30° = \sqrt{\frac{1}{3}}$, if the radius be 1,

$$\text{The arc of } 30° = \sqrt{\tfrac{1}{3}} \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \text{\&c.}\right)$$

And this muitiplied into 12 will give the whole circumference.

This was the series used by Dr. Halley. By this also, Mr. *Abraham Sharp* of Yorkshire computed the circumference to 72 places of figures, Mr. *John Machin*, Professor of Astronomy in Gresham College, to 100 places, and M. De Lagny to 128 places. Several expedients have been devised, by Machin, Euler, Dr. Hutton, and others, to reduce the labor of summing the terms of the series. See Euler's Analysis of Infinites, Hutton's Mensuration, Appendix to Maseres on the Negative Sign, and Lond. Phil. Trans. for 1776. For a demonstration that the diameter and the circumference of a circle are incommensurable, see Legendre's Geometry, Note iv.

The circumference of a circle whose diameter is 1, is

**3.1415926535, 8979323846, 2643383279,
5028841971, 6939937510, 5820974944,
5923078164, 0628620899, 8628034825,
3421170679, 8214808651, 3272306647,
0938446 + or 7 —.**

Note B. p. 17.

The following multipliers may frequently be useful;

The diam'r of a circle $\left\{\begin{array}{l}\times.8862=\text{the side of an equal square.}\\ \times.707\ =\text{the side of an ins'bed sq're.}\\ \times.866\ =\text{the side of an inscribed equilateral triangle.}\end{array}\right.$

The circumf. $\left\{\begin{array}{l}\times.2821=\text{the side of an equal square.}\\ \times.2251=\text{the side of an inscribed square.}\\ \times.2756=\text{the side of an ins'bed eq'lat. triang.}\end{array}\right.$

The side of a sq. $\left\{\begin{array}{l}\times1.128=\text{the diameter of an equal circle.}\\ \times3.545=\text{the circumf. of an equal circle.}\\ \times1.414=\text{the diam. of the circums. circle.}\\ \times4.443=\text{the cir. of the circumsc. circle.}\end{array}\right.$

Note C. p. 19.

The following approximating rule may be used for finding the arc of a circle.

1. The arc of a circle is nearly equal to $\frac{1}{3}$ of the difference between the chord of the whole arc, and 8 times the chord of half the arc

2. If h=the *height* of an arc, and d=the diameter of the circle;

$$\text{The arc}=2d\sqrt{\frac{3h}{3d-h}}\qquad\text{Or,}$$

3. $$\text{The arc}=2\sqrt{dh}\times\left(1+\frac{h}{2.3d}+\frac{3h^2}{2.4.5d^2}+\frac{3.5h^3}{2.4.6.7d^3}\ \&c.\right)\text{Or,}$$

4. $$\text{The arc}=\tfrac{2}{3}\left(5d\sqrt{\frac{5h}{5d-3h}}+4\sqrt{dh}\right)\text{ very nearly.}$$

5. If s=the *sine* of an arc, and r=the radius of the circle;

$$\text{The arc}=s\times\left(1+\frac{s^2}{2.3r^2}+\frac{3s^4}{5.2.4r^4}+\frac{3.5s^6}{7.2.4.6r^6}\ \&c.\right)$$

See Hutton's Mensuration.

Note D. p. 23.

To expedite the calculation of the areas of circular segments, a *table* is provided, which contains the areas of segments in a circle whose diameter is 1. See the table at the end of the book, in which the diameter is supposed to be divided into 1000 equal parts. By this may be found the areas of segments of other circles. For the heights of similar segments of different circles are as the diameters. If then the height of any given segment be divided by the diameter of the circle, the quotient will be the height of a similar segment in a circle whose diameter is 1 The area of the latter is found in the table; and from the properties of similar figures, the two segments are to each other, as the squares of the diameters of the circles. We have then the following rule:

To find the area of a circular SEGMENT *by the* TABLE.

Divide the height of the segment by the diameter of the circle; look for the quotient in the column of heights in the table; take out the corresponding number in the column of areas; and multiply it by the square of the diameter.

It is to be observed, that the figures in each of the columns in the table are *decimals.*

If accuracy is required, and the quotient of the height divided by the diameter, is *between* two numbers in the column of heights; allowance may be made for a *proportional part* of the difference of the corresponding numbers in the column of areas; in the same manner, as in taking out logarithms.

Segments *greater than a semicircle* are not contained in the table. If the area of such a segment is required, as ABD (Fig. 9.), find the area of the segment ABO, and subtract this from the area of the whole circle.

Or,

Divide the height of the given segment by the diameter, subtract the quotient from 1, find the remainder in the column of heights, subtract the corresponding area from .7854, and multiply this remainder by the square of the diameter.

Ex. 1. What is the area of a segment whose height is 16, the diameter of the circle being 48? Ans. 528.

2. What is the area of a segment whose height is 32, the diameter being 48? Ans. 1281.55.

The following rules may also be used for a circular segment.

1. To the chord of the whole arc, add $\frac{4}{3}$ of the chord of half the arc, and multiply the sum by $\frac{2}{5}$ of the height.

If C and c = the two chords, and h = the height;

$$\text{The segment} = (C + \tfrac{4}{3}c)\tfrac{2}{5}h \text{ nearly.}$$

2. If h = the height of the segment, and d = the diameter of the circle;

$$\text{The segment} = 2h\sqrt{dh} \times \left(\frac{2}{3} - \frac{h}{5d} - \frac{h^2}{28d^2} - \frac{h^3}{72d^3} \text{ \&c.}\right)$$

Note E. p. 29.

The term *solidity* is used here in the customary sense, to express the magnitude of any geometrical quantity of three dimensions, length, breadth, and thickness; whether it be a solid body, or a fluid, or even a portion of empty space. This use of the word, however, is not altogether free from objection. The same term is applied to one of the general properties of matter; and also to that peculiar quality by which certain substances are distinguished from *fluids*. There seems to be an impropriety in speaking of the *solidity* of a body of *water*, or of a vessel which is *empty*. Some writers have therefore substituted the word *volume* for solidity. But the latter term, if it be properly defined, may be retained without danger of leading to mistake.

Note F. p. 35.

The *geometrical* demonstration of the rule for finding the solidity of a frustum of a pyramid, depends on the following proposition:

A frustum of a triangular pyramid is equal to three pyramids; the greatest and least of which are equal in height to the frustum, and have the two ends of the frustum for their bases; and the third is a mean proportional between the other two.

Let ABCDFG (Fig. 34.) be a frustum of a triangular pyramid. If a plane be supposed to pass through the points AFC, it will cut off the pyramid ABCF. The height of this is evidently equal to the height of the frustum, and its base is ACB, the greater end of the frustum.

Let another plane pass through the points AFD. This will divide the remaining part of the figure into two triangular pyramids AFDG and AFDC. The height of the former is equal to the height of the frustum, and its base is DFG, the smaller end of the frustum.

To find the magnitude of the third pyramid AFDC, let F be now considered as the vertex of this, and of the second pyramid AFDG. Their bases will then be the triangles ADC and ADG. As these are in the same plane, the two pyramids have the same altitude, and are to each other as their bases. But these triangular bases, being between the same parallels, are as the lines AC and DG. Therefore, the pyramid AFDC is to the pyramid AFDG as AC to DG; and $\overline{AFDC}^2 : \overline{AFDG}^2 :: \overline{AC}^2 : \overline{DG}^2$. (Alg. 391.) But the pyramids ABCF and AFDG, having the same altitude, are as their bases ABC and DFG, that is, as $\overline{AC}^2$ and $\overline{DG}^2$. (Euc. 19, 6.) We have then

$$\left.\begin{array}{l}\overline{AFDC}^2 : \overline{AFDG}^2 :: \overline{AC}^2 : \overline{DG}^2 \\ ABCF : AFDG :: \overline{AC}^2 : \overline{DG}^2\end{array}\right\}.$$

$$\text{Therefore, } \overline{AFDC}^2 : \overline{AFDG}^2 :: ABCF : AFDG.$$

$$\text{And } \overline{AFDC}^2 = AFDG \times ABCF.$$

That is, the pyramid AFDC is a mean proportional between AFDG and ABCF.

Hence, the solidity of a frustum of a triangular pyramid is equal to $\frac{1}{3}$ of the height, multiplied into the sum of the areas of the two ends and the square root of the product of these areas. This is true also of a frustum of any other pyramid. (Sup. Euc. 12, 3. Cor. 2.)

If the smaller end of a frustum of a pyramid be enlarged, till it is made equal to the other end; the frustum will become a *prism*, which may be divided into three *equal* pyramids. (Sup. Euc. 15 3.)

NOTE G. p. 59.

The following simple rule for the solidity of round timber, or of any cylinder, is nearly exact:

Multiply the length into twice the square of $\frac{1}{5}$ of the circumference.

If C = the circumference of a cylinder;

The area of the base $=\frac{C^2}{4\pi}=\frac{C^2}{12.566}$ But $2\left(\frac{C}{5}\right)^2=\frac{C^2}{12.5}$

It is common to measure *hewn* timber, by multiplying the length into the square of the *quarter-girt*. This gives exactly the solidity of a parallelopiped, if the ends are *squares*. But if the ends are parallelograms, the area of each is *less* than the square of the quarter-girt. (Euc. 27. 6.)

Timber which is *tapering* may be exactly measured by the rule for the frustum of a pyramid or cone (Art. 50, 68.); or, if the ends are not similar figures, by the rule for a prismoid. (Art. 55.) But for common purposes, it will be sufficient to multiply the length by the area of a section *in the middle* between the two ends

A TABLE

OF THE SEGMENTS OF A CIRCLE, WHOSE DIAMETER IS 1, AND IS SUPPOSED TO BE DIVIDED INTO 1000 EQUAL PARTS.

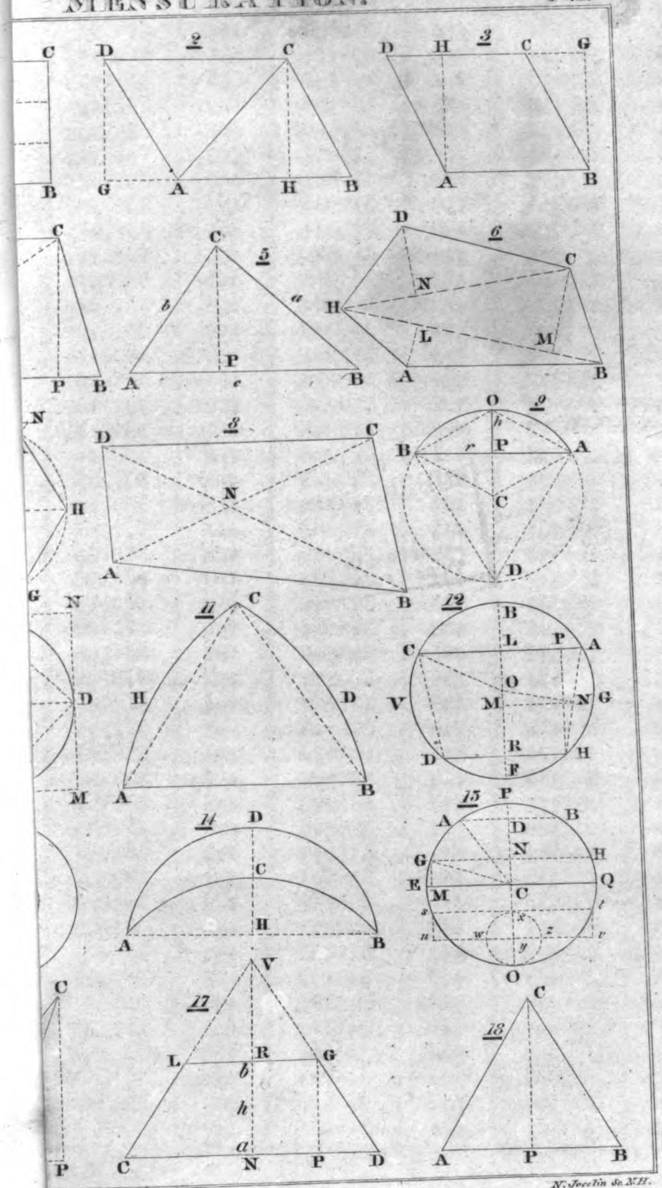
Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
.001	.000042	.034	.008273	.067	.022652
002	000119	035	008638	068	023154
003	000219	036	009008	069	023659
004	000337	037	009383	070	024168
005	000471	038	009763	071	024680
006	000618	039	010148	072	025195
007	000779	040	010537	073	025714
008	000952	041	010932	074	026236
009	001135	042	011331	075	026761
010	001329	043	011734	076	027289
011	001533	044	012142	077	027821
012	001746	045	012554	078	028356
013	001968	046	012971	079	028894
014	002199	047	013392	080	029435
015	002438	048	013818	081	029979
016	002685	049	014247	082	030126
017	002940	050	014681	083	031076
018	003202	051	015119	084	031629
019	003472	052	015561	085	032186
020	003748	053	016007	086	032745
021	004032	054	016457	087	033307
022	004322	055	016911	088	033872
023	004618	056	017369	089	034441
024	004921	057	017831	090	035011
025	005231	058	018296	091	035585
026	005546	059	018766	092	036162
027	005867	060	019239	093	036741
028	006194	061	019716	094	037323
029	006527	062	020206	095	037909
030	006865	063	020690	096	038496
031	007209	064	021178	097	039087
032	007558	065	021659	098	039680
.033	.007913	.066	.022154	.099	.040276

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
.100	.040875	.144	.069625	.188	.102334
101	041476	145	070328	189	103116
102	042080	146	071033	190	103900
103	042687	147	071741	191	104685
104	043296	148	072450	192	105472
105	043908	149	073161	193	106261
106	044522	150	073874	194	107051
107	045139	151	074589	195	107842
108	045759	152	075306	196	108636
109	046381	153	076026	197	109430
110	047005	154	076747	198	110226
111	047632	155	077469	199	111024
112	048262	156	078194	200	111823
113	048894	157	078921	201	112624
114	049528	158	079649	202	113426
115	050165	159	080380	203	114230
116	050804	160	081112	204	115035
117	051446	161	081846	205	115842
118	052090	162	082582	206	116650
119	052736	163	083320	207	117460
120	053385	164	084059	208	118271
121	054036	165	084801	209	119083
122	054689	166	085544	210	119897
123	055345	167	086289	211	120712
124	056003	168	087036	212	121529
125	056663	169	087785	213	122347
126	057326	170	088535	214	123167
127	057991	171	089287	215	123988
128	058658	172	090041	216	124810
129	059327	173	090797	217	125634
130	059999	174	091554	218	126459
131	060672	175	092313	219	127285
132	061348	176	093074	220	128113
133	062026	177	093836	221	128942
134	062707	178	094601	222	129773
135	063389	179	095366	223	130605
136	064074	180	096134	224	131438
137	064760	181	096903	225	132272
138	065449	182	097674	226	133108
139	066140	183	098447	227	133945
140	066833	184	099221	228	134784
141	067528	185	099997	229	135624
142	068225	186	100774	230	136465
.143	.068924	.187	.101553	.231	.137307

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
.232	.138150	.277	.177330	.322	.218533
233	138995	278	178225	323	219468
234	139841	279	179122	324	220404
235	140688	280	180019	325	221340
236	141537	281	180918	326	222277
237	142387	282	181817	327	223215
238	143238	283	182718	328	224154
239	144091	284	183619	329	225093
240	144944	285	184521	330	226033
241	145799	286	185425	331	226974
242	146655	287	186329	332	227915
243	147512	288	187234	333	228858
244	148371	289	188140	334	229801
245	149230	290	189047	335	230745
246	150091	291	189955	336	231689
247	150953	292	190864	337	232634
248	151816	293	191775	338	233580
249	152680	294	192684	339	234526
250	153546	295	193596	340	235473
251	154412	296	194509	341	236421
252	155280	297	195422	342	237369
253	156149	298	196337	343	238318
254	157019	299	197252	344	239268
255	157890	300	198168	345	240218
256	158762	301	199085	346	241169
257	159636	302	200003	347	242121
258	160510	303	200922	348	243074
259	161386	304	201841	349	244026
260	162263	305	202761	350	244980
261	163140	306	203683	351	245934
262	164019	307	204605	352	246889
263	164899	308	205527	353	247845
264	165780	309	206451	354	248801
265	166663	310	207376	355	249757
266	167546	311	208301	356	250715
267	168430	312	209227	357	251673
268	169315	313	210154	358	252631
269	170202	314	211082	359	253590
270	171089	315	212011	360	254550
271	171978	316	212940	361	255510
272	172867	317	213871	362	256471
273	173758	318	214802	363	257433
274	174649	319	215733	364	258395
275	175542	320	216666	365	259357
.276	.176435	.321	.217599	.366	.260320

Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
.367	.261284	.412	.305155	.457	.349752
368	262248	413	306140	458	350748
369	263213	414	307125	459	351745
370	264178	415	308110	460	352742
371	265144	416	309095	461	353739
372	266111	417	310081	462	354736
373	267078	418	311068	463	355732
374	268045	419	312054	464	356730
375	269013	420	313041	465	357727
376	269982	421	314029	466	358725
377	270951	422	315016	467	359723
378	271920	423	316004	468	360721
379	272890	424	316992	469	361719
380	273861	425	317981	470	362717
381	274832	426	318970	471	363715
382	275803	427	319959	472	364713
383	276777	428	320948	473	365712
384	277748	429	321938	474	366710
385	278721	430	322928	475	367709
386	279694	431	.323918	476	368708
387	280668	432	324909	477	369707
388	281642	433	325900	478	370706
389	282617	434	326892	479	371705
390	283592	435	327882	480	372704
391	284568	436	328874	481	373703
392	285544	437	329866	482	374702
393	286521	438	330858	483	375702
394	287498	439	331850	484	376702
395	288476	440	332843	485	377701
396	289454	441	333836	486	378701
397	290432	442	334829	487	379700
398	291411	443	335822	488	380700
399	292390	444	336816	489	381699
400	293369	445	337810	490	382699
401	294349	446	338804	491	383699
402	295330	447	339798	492	384699
403	296311	448	340793	493	385699
404	297292	449	341787	494	386699
405	298273	450	342782	495	387699
406	299255	451	343777	496	388699
407	300238	452	344772	497	389699
408	301220	453	345768	498	390699
409	302203	454	346764	499	391699
410	303187	455	347759	.500	.392699
.411	.304171	.456	.348755		

MENSURATION. Pl. I.



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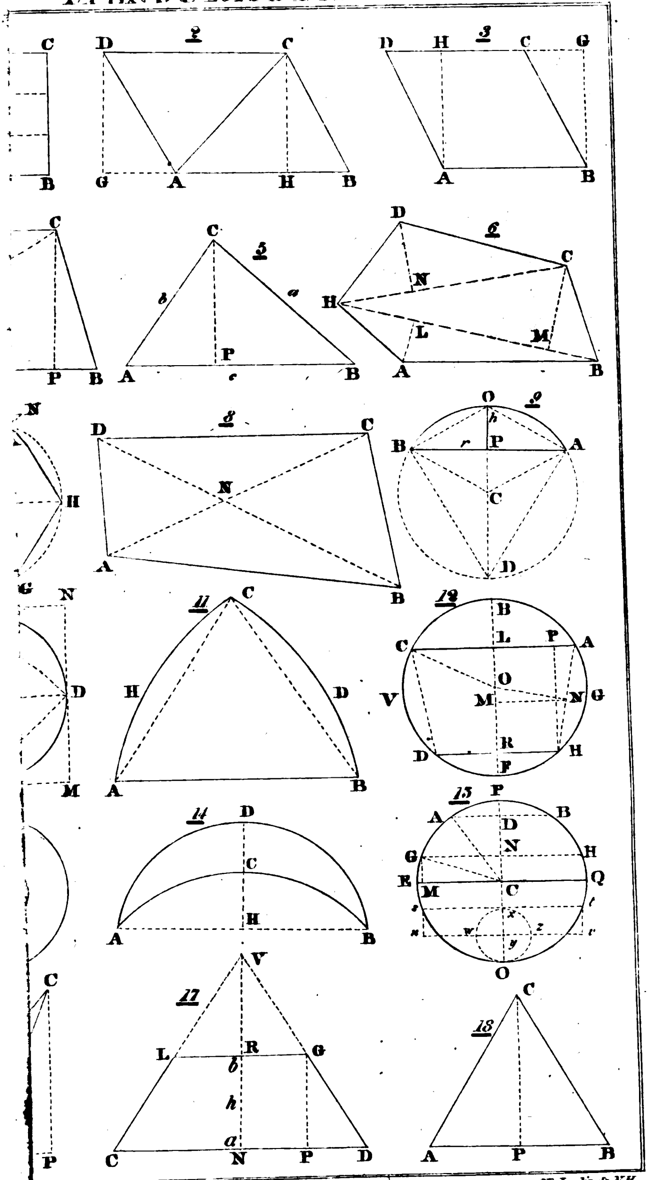
Height.	Area Seg.	Height.	Area Seg.	Height.	Area Seg.
.367	.261284	.412	.305155	.457	.349752
368	262248	413	306140	458	350748
369	263213	414	307125	459	351745
370	264178	415	308110	460	352742
371	265144	416	309095	461	353739
372	266111	417	310081	462	354736
373	267078	418	311068	463	355732
374	268045	419	312054	464	356730
375	269013	420	313041	465	357727
376	269982	421	314029	466	358725
377	270951	422	315016	467	359723
378	271920	423	316004	468	360721
379	272890	.424	316992	469	361719
380	273861	425	317981	470	362717
381	274832	426	318970	471	363715
382	275803	427	319959	472	364713
383	276777	428	320948	473	365712
384	277748	429	321938	474	366710
385	278721	430	322928	475	367709
386	279694	431	.323918	476	368708
387	280668	432	324909	477	369707
388	281642	433	325900	478	370706
389	282617	434	326892	479	371705
390	283592	435	327882	480	372704
391	284568	436	328874	481	373703
392	285544	437	329866	482	374702
393	286521	438	330858	483	375702
394	287498	439	331850	484	376702
395	288476	440	332843	485	377701
396	289454	441	333836	486	378701
397	290432	442	334829	487	379700
398	291411	443	335822	488	380700
399	292390	444	336816	489	381699
400	293369	445	337810	490	382699
401	294349	446	338804	491	383699
402	295330	447	339798	492	384699
403	296311	448	340793	493	385699
404	297292	449	341787	494	386699
405	298273	450	342782	495	387699
406	299255	451	343777	496	388699
407	300238	452	344772	497	389699
408	301220	453	345768	498	390699
409	302203	454	346764	499	391699
410	303187	455	347759	.500	.392699
.411	.304171	.456	.348755		



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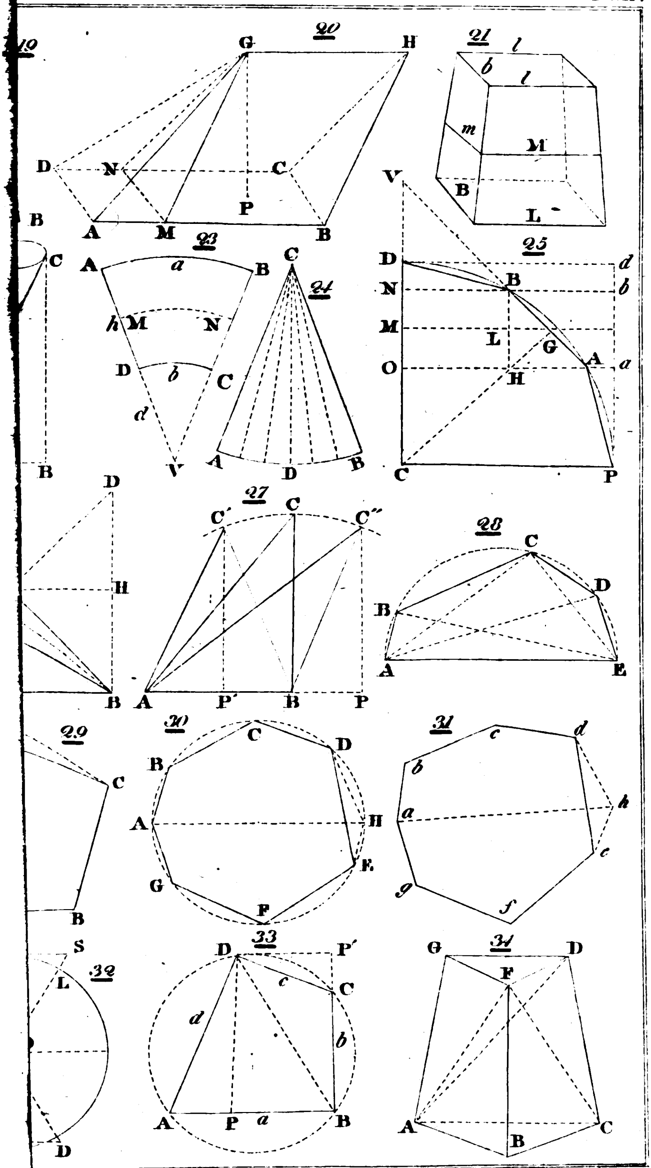

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