HW: $f(x) = \sin x$, $x \in [0, \pi]$

Constr. Taylor series of $f(\cdot)$ around $x_0 = \frac{\pi}{4}$ for $n = 0, 1, \ldots, 10$.

Plot these series (for each n).

$R(x)$: Find (estimate the max) of $R(x)$ at $x = \frac{\pi}{2}$.

$$R(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt$$

$$= \frac{1}{n!} f^{(n+1)}(\xi) \int_{x_0}^{x} (x-t)^n dt$$

$$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

$\xi \in [x_0, x]$

$$|R(x)| \leqslant \frac{(x-x_0)^{n+1}}{(n+1)!} \underbrace{\max_{t \in [x_0, x]} |f^{(n+1)}(t)|}_{\leqslant 1}$$

Note: Motivation for pf of Taylor series expansion (similar tech will be used):

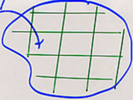
- * higher order error analysis of trap. rule (not in A.)
- Richardson extrap.
- * Clenshaw - Curtis quadrature
- * Chebyshev poly (orthogonal)
- * Recent devel. using Chebyshev poly. to solve L2-ODE-VC (Lin. 2nd order Ordin. Diff. Eq. Varying Coeff.)
- * comb. of symbolic + numeric

Note: Quadrature = num. int.

↑ Quadrilateral

Greeks : Meas. area

Quadrilateral



Cubature
cube

$$\text{Area} = \sum \text{Quad.} \qquad (1)$$

$$\text{Vol} = \sum \text{cubes} \qquad (2)$$

Num. Int. using Taylor series
cont'd p. 2-2 (1)

$$I_n = \int_0^1 f_n(x)\,dx = \int_0^1 \sum_{j=1}^{n} \frac{x^{j-1}}{j!}\,dx$$

$$= \sum_{j=1}^{n} \frac{1}{j!}\,\frac{1}{j} \qquad (3)$$

$$(4) \quad f(x) - f_n(x) = R_n(x) = \frac{(x-0)^n}{(n+1)!} \uparrow \exp[\xi]$$

$$\xi \in [0, x]$$

ξ is a func. of $x \Rightarrow \xi_x = \xi(x)$

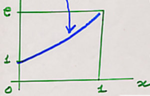
$$E_n := I - I_n$$
$$= \int_0^1 [f(x) - f_n(x)]\,dx$$
$$= \int_0^1 \underbrace{\frac{x^n}{(n+1)!}}_{W(x)} \underbrace{\exp[\xi(x)]}_{g(x)}\,dx$$

IMVT

$$\overset{\downarrow}{=} \underbrace{g(\alpha)}_{= \exp[\xi(\alpha)]} \underbrace{\int_0^1 W(x)\,dx}_{\frac{1}{(n+1)!(n+1)}}, \quad \alpha \in [0,1]$$

$\exp[\xi(\alpha)]$

$\xi \in [0,1]$



$\min_{\alpha \in [0,1]} g(\alpha) = 1$

$\max_{\alpha \in [0,1]} g(\alpha) = e$

$$\frac{1}{(n+1)!(n+1)} \leq E_n \leq \frac{e}{(n+1)!(n+1)}$$

$I_6 = 1.3178\ldots$ A. p.250

$2.83 \times 10^{-5} \leqslant E_6 = I - I_6 \leqslant 7.70 \times 10^{-5}$

HW: $I = \int_0^1 \frac{e^x - 1}{x}\, dx$

Use 3 methods to find I_n :

1) Taylor series exp. fn
2) Comp. Trap. rule
3) Comp. Simpson rule

$n = 2, 4, 8, \ldots$ until error of order 10^{-6}.