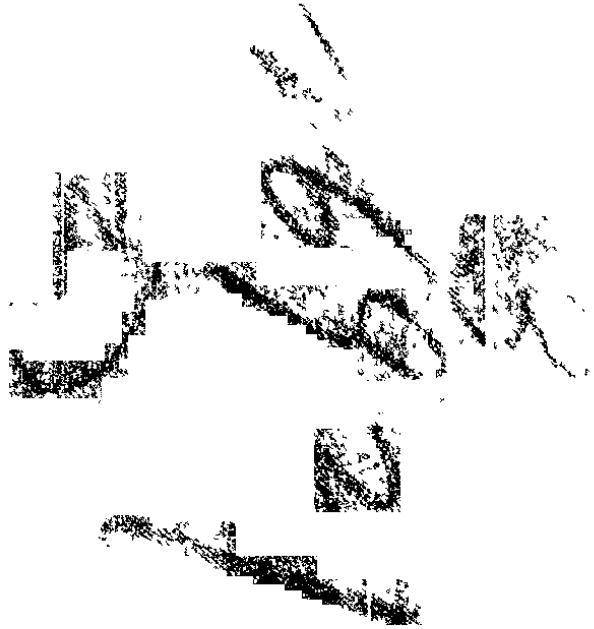

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積分問題解法

程凱丞編著

藏書

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序

微分學與積分學均以極限論爲基礎，然解決積分問題遠不如解決微分問題之易：一因求函數之微係數或微分有一定成法可按，惟積分則無之；二因積分爲微分之反運算，猶之開方之與乘方，但開方較乘方爲難，故積分亦不外是例．惟解決積分問題，亦不可不定一步驟．先將簡易函數之積分（從誘導得之）列爲一表，以爲解題時之準繩．被積函數之形狀若與表中某式相符者，則直接寫出其結果，否則須用各種方法變化之，使合於標準式之形．若被積函數爲有理分數，則分析函數爲數項．若爲無理函數，則代以新變數而使之理化．若爲對數函數，反三角函數，則實行分部．次數高者使之遞降，次數低者使之遞升．要之，各法所陳，無非化雜爲簡，避難就易，使趨於標準化而已．

本書搜集問題五百餘條，悉爲實用上常見之函數，分門別類，循序解答．每用一法，必先輸入其必要之概念及普遍之公式，此爲本書之核心，似與普通詳解一類書籍有別．凡公式貴能自導，並不以機械代替爲畢事也．閱讀解答之先，必須自動按法推演，推演不得，再閱解答，如是習一題即獲一題之益．

積分法之第二定義爲求和之極限，此定義在應用方面頗爲重要．本書之宗旨，在詳述求積分之方法，方法嫻熟，應用上當不致有所困難，故將應用問題一概略去．編者學識謭陋，尙望海內大師有以敎之．

二十五年元旦凱丞識于蘇州滄庶

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書藏

王緩之圖書

積分問題解法

第一章

緒論

1 積分學與微分學之關係. 既知某函數爲 $F(x)$ 而欲求其微係數 $F'(x)$ 或求其微分 $F'(x)dx$, 此微分學中之事也. 反之, 如已知某函數之微係數爲 $F'(x)$, 或已知其微分爲 $F'(x)dx$ 而欲求此函數, 此乃積分學中惟一之問題. 故積分學與微分學實係一種反運算之關係. 學者間有着重此種反運算之事實, 直稱積分曰反微分 (anti-differential).

2. 術語與記法. 於微分學中, 求 $F(x)$ 之微係數或求其微分之運算方法, 恆以下列二式記之:

$$(A)\quad \frac{d}{dx}F(x)=F'(x);\quad F'(x)\text{爲}F(x)\text{之微係數.}$$

$$(B)\quad dF(x)=F'(x)dx;\quad F'(x)dx\text{爲}F(x)\text{之微分.}$$

積分既爲微分之反運算, 先將 (A), (B) 兩式逆其次序而讀之曰:

(C) $F'(x)$ 之積分等於 $F(x)$.

(D) $F'(x)dx$ 之積分等於 $F(x)$.

[illegible]藏書

慣例,積分學中言函數之微分 (differential) 而不言函數之微係數 (differential coefficient). 今以 $\int F'(x)dx$ 表示 $F'(x)dx$ 之積分,於是 (L) 式之記法如下:

$$(E)\quad \int F'(x)dx = F(x).$$

式中 $F(x)$ 名曰 $F'(x)dx$ 之積分 (integral). 名此求 $F(x)$ 之方法曰積分法 (integration). 微分式前之符號 $\int$ 曰積分符號 (integral sign).

3. 積分常數與不定積分. 從上款 (E) 式,

若命 $F(x)=x^4$, 則因 $d(x^4)=4x^3dx$, $\therefore \int 4x^3dx = x^4$;

若命 $F(x)=x^4+2$, 則因 $d(x^4+2)=4x^3dx$, $\therefore \int 4x^3dx = x^4+2$;

若命 $F(x)=x^4-3$, 則因 $d(x^4-3)=4x^3dx$, $\therefore \int 4x^3dx = x^4-3$;

若命 $F(x)=x^4+c$, 則因 $d(x^4+c)=4x^3dx$, $\therefore \int 4x^3dx = x^4+c$.

因常數之微分爲零,上列諸函數,其差異之點在於常數之一項,其微分皆各相同 (卽均爲 $4x^3dx$). 然由 $4x^3dx$ 求原函數,其數實屬無窮,因每次予 c 以一定值,卽得一積分故也. 此隨意常數 c 名曰積分之常數,實際上其值得隨原始條件 (initial condition) 以定之. 又因 c 值爲不定,故稱 $F(x)+c$ 爲 $F'(x)dx$ 之不定積分 (indefinite integral). 由是得一定理如下:

[定理] 凡兩個函數之差爲一常數者,其微分恆相同. 又逆定理亦正確,卽兩個函數之微分相同者,其差爲一常數.

[逆定理之證明] 設兩函數爲 $\phi(x)$ 及 $\psi(x)$, 其公共之

微係數爲 $f(x)$, 求證 $\phi(x)$ 與 $\psi(x)$ 之差爲一常數.

[證]　命兩函數之差爲 $F(x)$;

(1)　卽　$F(x)=\phi(x)-\psi(x)$.

兩邊求微係數,得

(2)　$F'(x)=\frac{d}{dx}[\phi(x)-\psi(x)]=f(x)-f(x)=0.$　(假設)

由中值定理,

(3)　$F(x+\Delta x)-F(x)=\Delta x\,F'(x+\theta\cdot\Delta x).\ 0<\theta<1$

由 (2) 式,不論 x 之值爲何, $F(x)$ 之微係數終必爲零,

故 (3) 式中之 $F'(x+\theta\cdot\Delta x)$爲零,

卽　$F(x+\Delta x)-F(x)=0$,

(4)　故　$F(x+\Delta x)=F(x)$.

故知 x 若取一微差 Δx, (1) 式中之 $F(x)$ 並不改變其值;換言之. $\phi(x)$ 與 $\psi(x)$ 之差僅爲一常數.　Q.E.D.

將上款 (E) 式,兩邊求微分,

(F)　$d\int F'(x)dx=dF(x)=F'(x)dx,$

故知將 $F'(x)dx$ 先求其積分次求其微分,結果仍爲 $F'(x)dx$;實行求積分時,縱有積分常數之存在,終因最後之求微分以消去, 故 d 若在 $\int$ 之前,此兩記號恆可互相抵銷不論

4.　積分學中之假定.　於初等積分學中,假定每一綿續函數必有一個不定積分可求,其嚴確之證明,非本編之目的,故不贅.

5. 積分之驗證. 求得函數之積分後,欲知其結果是否正確,可將求得之積分重復求其微分,視其是否等於原來之函數即可.此種驗證適用於一般不定積分,其理由根據於積分爲微分之反運算一語,固不待言.

6. 求積分之法則. 求函數之微係數,有普通法則可守(即四步法則),惟求積分則無之.由每一求微分之結果,恆可誘導一求積分之公式,故積分法重在試探.先將簡易函數之積分列爲一表,名曰不定積分之標準式 (standard form). 運算時可將微分式與表中比較之;若發現與表中某一標準式相當者,其積分不難立即求得.若無相當者,須用各種方法展變之,使合於標準式之形.

第 二 章

不定積分之標準式

7. 標準式及其證明.

標 準 式

(1) $\int (du+dv-dw)=\int du+\int dv-\int dw.$

(2) $\int a\,dv=a\int dv.$

(3) $\int dx=x+c.$

(4) $\int v^n\,dv=\frac{v^{n+1}}{n+1}+c. \quad n\neq -1$

(5) $\int\frac{dv}{v}=\log v+c.$

(6) $\int a^v\,dv=\frac{a^v}{\log a}+c.$

(7) $\int e^v\,dv=e^v+c.$

(8) $\int \sin v\,dv=-\cos v+c.$

(9) $\int \cos v\,dv=\sin v+c.$

(10) $\int \sec^2 v\,dv=\tan v+c.$

(11) $\int \csc^2 v\,dv = -\cot v + c.$

(12) $\int \sec v \tan v\,dv = \sec v + c.$

(13) $\int \csc v \cot v\,dv = -\csc v + c.$

(14) $\int \tan v\,dv = \log \sec v + c$

(15) $\int \cot v\,dv = \log \sin v + c.$

(16) $\int \sec v\,dv = \log(\sec v + \tan v) + c.$

(17) $\int \csc v\,dv = \log(\csc v - \cot v) + c.$

(18) $\int \frac{dv}{v^2+a^2} = \frac{1}{a} \arctan \frac{v}{a} + c.$

(19) $\int \frac{dv}{v^2-a^2} = \frac{1}{2a} \log \frac{v-a}{v+a} + c.$

(20) $\int \frac{dv}{\sqrt{a^2-v^2}} = \arcsin \frac{v}{a} + c.$

(21) $\int \frac{dv}{\sqrt{v^2 \pm a^2}} = \log(v + \sqrt{v^2 \pm a^2}) + c.$

(22) $\int \frac{dv}{\sqrt{2av-v^2}} = \arccos \frac{v}{a} + c.$

(23) $\int \frac{dv}{v\sqrt{v^2-a^2}} = \frac{1}{a} \operatorname{arc\,sec} \frac{v}{a} + c.$

以上諸標準式均由微分法公式得來;其證明可將各式右邊之積分再求其微分,結果必與左邊積分記號後之微分式相等.

公式(1)之證明:

$$\because \quad d\left[\int du+\int dv-\int dw\right]=d\int du+d\int dv-d\int dw=u+v-w,$$

$$\therefore \quad \int(du+dv-dw)=\int du+\int dv-\int dw.$$

公式(2)之證明:

$$\because \quad d\left[a\int dv\right]=a\,dv,$$

$$\therefore \quad \int a\,dv=a\int dv.$$

公式(3)之證明:

$$\because \quad d(x+c)=dx,$$

$$\therefore \quad \int dx=x+c.$$

公式(4)之證明:

$$\because \quad d\left(\frac{v^{n+1}}{n+1}+c\right)=v^n\,dv,$$

$$\therefore \quad \int v^n\,dv=\frac{v^{n+1}}{n+1}+c.$$

此式除$n=-1$外,恆能成立,因若$n=-1$,則

$$\int v^{-1}\,dv=\frac{v^{-1+1}}{-1+1}+c=\frac{1}{0}+c=\infty+c,$$

故不可能.若$n=-1$時,宜歸屬於公式(5)一類中.

公式(5)之證明:

$$\because \quad d(\log v+c)=\frac{dv}{v},$$

$$\therefore \quad \int\frac{dv}{v}=\log v+c.$$

公式(6)之證明:

$$\because \quad d\left(\frac{a^v}{\log a}+c\right)=a^v\,dv,$$

$$\therefore \quad \int a^v\,dv=\frac{a^v}{\log a}+c.$$

公式(7)之證明:

$$\because \quad d(e^v+c)=e^v\,dv,$$

$$\therefore \quad \int e^v\,dv=e^v+c.$$

公式(8)之證明:

$$\because \quad d(-\cos v+c)=\sin v\,dv,$$

$$\therefore \quad \int \sin v\,dv=-\cos v+c.$$

公式(9)之證明:

$$\because \quad d(\sin v+c)=\cos v\,dv,$$

$$\therefore \quad \int \cos v\,dv=\sin v+c.$$

公式(10)之證明:

$$\because \quad d(\tan v+c)=\sec^2 v\,dv,$$

$$\therefore \quad \int \sec^2 v\,dv=\tan v+c.$$

公式(11)之證明:

$$\because \quad d(-\cot v+c)=\csc^2 v\,dv,$$

$$\therefore \quad \int \csc^2 v\,dv=-\cot v+c.$$

公式(12)之證明:

$$\because \quad d(\sec v + c) = \sec v \tan v \, dv,$$

$$\therefore \quad \int \sec v \tan v \, dv = \sec v + c.$$

公式(13)之證明:

$$\because \quad d(-\csc v + c) = \csc v \cot v \, dv,$$

$$\therefore \quad \int \csc v \cot v \, dv = -\csc v + c.$$

公式(14)之證明:

$$\because \quad d(\log \sec v + c) = \frac{d(\sec v)}{\sec v} = \frac{\sec v \tan v \, dv}{\sec v} = \tan v \, dv,$$

$$\therefore \quad \int \tan v \, dv = \log \sec v + c = -\log \cos v + c.$$

[別證]

$$\int \tan v \, dv = \int \frac{\sin v \, dv}{\cos v} = -\int \frac{-\sin v \, dv}{\cos v} = -\int \frac{d(\cos v)}{\cos v}$$

$$= -\log \cos v + c = \log \sec v + c.$$

公式(15)之證明:

$$\because \quad d(\log \sin v + c) = \frac{d(\sin v)}{\sin v} = \frac{\cos v \, dv}{\sin v} = \cot v \, dv,$$

$$\therefore \quad \int \cot v \, dv = \log \sin v + c.$$

[別證]

$$\int \cot v \, dv = \int \frac{\cos v \, dv}{\sin v} = \int \frac{d(\sin v)}{\sin v} = \log \sin v + c.$$

公式(16)之證明:

$$\because\quad d[\log(\sec v+\tan v)+c]=\frac{d(\sec v+\tan v)}{\sec v+\tan v}$$

$$=\frac{(\sec v\tan v+\sec^2 v)dv}{\sec v+\tan v}=\frac{(\sec v+\tan v)\sec v\,dv}{\sec v+\tan v}=\sec v\,dv,$$

$$\therefore\quad \int\sec v\,dv=\log(\sec v+\tan v)+c.$$

[別證]

$$\because\quad \sec v=\sec v\frac{\sec v+\tan v}{\sec v+\tan v}=\frac{\sec v\tan v+\sec^2 v}{\sec v+\tan v},$$

$$\therefore\quad \int\sec v\,dv=\int\frac{\sec v\tan v+\sec^3 v}{\sec v+\tan v}dv$$

$$=\int\frac{d(\sec v+\tan v)}{\sec v+\tan v}=\log(\sec v+\tan v)+c.$$

公式(17)之證明:

$$\because\quad d[\log(\csc v-\cot v)+c]=\frac{d(\csc v-\cot v)}{\csc v-\cot v}$$

$$=\frac{(-\csc v\cot v+\csc^2 v)dv}{\csc v-\cot v}=\frac{(\csc v-\cot v)\csc v\,dv}{\csc v-\cot v}$$

$$=\csc v\,dv,$$

$$\therefore\quad \int\csc v\,dv=\log(\csc v-\cot v)+c.$$

[別證]

$$\because\quad \csc v=\csc v\frac{\csc v-\cot v}{\csc v-\cot v}=\frac{-\csc v\cot v+\csc^2 v}{\csc v-\cot v},$$

$$\therefore\quad \int\csc v\,dv=\int\frac{-\csc v\cot v+\csc^2 v}{\csc v-\cot v}dv$$

$$=\int\frac{d(\csc v-\cot v)}{\csc v-\cot v}=\log(\csc v-\cot v)+c.$$

公式(18)之證明:

$$\because\quad d\left(\frac{1}{a}\arctan\frac{v}{a}+c\right)=\frac{1}{a}\,\frac{d\left(\frac{v}{a}\right)}{1+\left(\frac{v}{a}\right)^2}=\frac{dv}{v^2+a^2},$$

$$\therefore\quad \int\frac{dv}{v^2+a^2}=\frac{1}{a}\arctan\frac{v}{a}+c.$$

公式(19)之證明:

$$\because\quad d\left[\frac{1}{2a}\log\frac{v-a}{v+a}+c\right]=\frac{1}{2a}\,\frac{d\left(\frac{v-a}{v+a}\right)}{\frac{v-a}{v+a}}=\frac{1}{2a}\cdot\frac{2a\,dv}{v+a}\cdot\frac{v+a}{v-a}$$

$$=\frac{dv}{v^2-a^2},$$

$$\therefore\quad \int\frac{dv}{v^2-a^2}=\frac{1}{2a}\log\frac{v-a}{v+a}+c.$$

[別證]

$$\because\quad \frac{1}{v^2-a^2}=\frac{1}{2a}\left(\frac{1}{v-a}-\frac{1}{v+a}\right),$$

$$\therefore\quad \int\frac{dv}{v^2-a^2}=\frac{1}{2a}\int\left(\frac{1}{v-a}-\frac{1}{v+a}\right)dv=\frac{1}{2a}\left(\int\frac{dv}{v-a}-\int\frac{dv}{v+a}\right)$$

$$=\frac{1}{2a}[\log(v-a)-\log(v+a)]+c$$

$$=\frac{1}{2a}\log\frac{v-a}{v+a}+c.$$

公式(20)之證明:

$$\because\quad d(\arcsin\frac{v}{a}+c)=\frac{d\left(\frac{v}{a}\right)}{\sqrt{1-\left(\frac{v}{a}\right)^2}}=\frac{dv}{\sqrt{a^2-v^2}},$$

$$\therefore \quad \int \frac{dv}{\sqrt{a^2-v^2}} = \arcsin\frac{v}{a}+c.$$

公式(21)之證明:

$$\because \quad d[\log(v+\sqrt{v^2\pm a^2})+c] = \frac{d(v+\sqrt{v^2\pm a^2})}{v+\sqrt{v^2\pm a^2}}$$

$$= \frac{dv+\frac{1}{2}(v^2\pm a^2)^{-\frac{1}{2}}2v\,dv}{v+\sqrt{v^2\pm a^2}} = \frac{\left(1+\frac{v}{\sqrt{v^2\pm a^2}}\right)dv}{v+\sqrt{v^2\pm a^2}}$$

$$= \frac{(v+\sqrt{v^2\pm a^2})dv}{\sqrt{v^2\pm a^2}} \cdot \frac{1}{(v+\sqrt{v^2+a^2})} = \frac{dv}{\sqrt{v^2\pm a^2}},$$

$$\therefore \quad \int \frac{dv}{\sqrt{v^2\pm a^2}} = \log(v+\sqrt{v^2\pm a^2})+c.$$

公式(22)之證明:

$$\because \quad d(\text{arc vers}\,\frac{v}{a}+c) = \frac{d\left(\frac{v}{a}\right)}{\sqrt{2\cdot\frac{v}{a}-\left(\frac{v}{a}\right)^2}} = \frac{\frac{dv}{a}}{\frac{\sqrt{2av-v^2}}{a}}$$

$$= \frac{dv}{\sqrt{2av-v^2}},$$

$$\therefore \quad \int \frac{dv}{\sqrt{2av-v^2}} = \text{arc vers}\,\frac{v}{a}+c.$$

公式(23)之證明:

$$\because \quad d\left(\frac{1}{a}\,\text{arc sec}\,\frac{v}{a}+c\right) = \frac{1}{a}\,\frac{d\left(\frac{v}{a}\right)}{\frac{v}{a}\sqrt{\left(\frac{v}{a}\right)^2-1}} = \frac{\frac{dv}{a^2}}{\frac{v\sqrt{v^2-a^2}}{a^2}}$$

$$= \frac{dv}{v\sqrt{v^2-a^2}},$$

$$\therefore \quad \int \frac{dv}{v\sqrt{v^2-a^2}} = \frac{1}{a}\,\text{arc sec}\,\frac{v}{a}+c.$$

8. 本章問題解法. 不定積分之標準式，爲數二十有三，須一一熟記. 函數之形狀簡單者能口頭讀出其積分更佳. 運算一題，往往須引用數公式者，尤以公式(1)公式(2)爲用最廣. 又將函數與標準式比較時，應具明敏之視察，以下數款，因其簡要，以實本編.

(1) 逢二項式之整次冪者，先展開之而後實行積分.

[舉例] 求 $\int (x^2-2)^3 x^3\,dx$ 之值.

[解] $$\int (x^2-2)^3 x^3 dx = \int (x^6-6x^4+12x^2-8)x^3\,dx$$

$$= \int (x^9-6x^7+12x^5-8x^3)dx$$

$$= \int x^9dx - \int 6x^7dx + \int 12x^5dx - \int 8x^3dx \quad (\text{公式 1})$$

$$= \int x^9dx - 6\int x^7dx + 12\int x^5dx - 8\int x^3dx \quad (\text{公式 2})$$

$$= \frac{x^{9+1}}{9+1} - 6\frac{x^{7+1}}{7+1} + 12\frac{x^{5+1}}{5+1} - 8\frac{x^{3+1}}{3+1} + c \quad (\text{公式 4})$$

$$= \frac{1}{10}x^{10} - \frac{3}{4}x^8 + 2x^6 - 2x^4 + c.$$

本題有四個不定積分，應有四個隨意常數，命最後之常數 c，爲此四常數之代數和.

(2) 積分號後之函數，若不插入一常數因子，往往不能立即援用公式；其法可插入此必須之因子，惟在積分號之前，宜用此因子之倒數乘之，其值不變. 今當注意者，此插入之因子僅限於常數，若將變數亦循樣插入，必致大謬.

[舉例] 求 $\int (nx)^{\frac{1-n}{n}} dx$ 之值.

[解] 此式類似公式(4),命 $v=nx$,則 $dv=n\,dx$,若引用公式(4),積分號後尙缺少常數因子 n;於是插入此因子,而以 $\frac{1}{n}$ (卽 n 之倒數)乘於積分號之前.

$$\int (nx)^{\frac{1-n}{n}} dx=\frac{1}{n}\int (nx)^{\frac{1-n}{n}} n\,dx=\frac{1}{n}\int (nx)^{\frac{1-n}{n}} d(nx)$$

$$=\frac{1}{n}\,\frac{(nx)^{\frac{1-n}{n}+1}}{\frac{1-n}{n}+1}+c=(nx)^{\frac{1}{n}}+c.$$

(3) 有理分函數之分子,其次數較分母爲高者,先實行除法而後運用公式.

[舉例] $$\int \frac{x^3 dx}{x+1}=\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$$

$$=\int x^2\,dx-\int x\,dx+\int dx-\int\frac{1}{x+1}dx \quad \text{(公式1)}$$

$$=\frac{x^3}{3}-\frac{x^2}{2}+x+\log(x+1)+c. \quad \text{(公式4, 3, 5)}$$

(4) 分函數之分母已合於標準式,但分子爲多項式,先將此函數寫成數個分函數之和而後運用公式.

[舉例] $$\int\frac{3x-1}{x^2+9}dx=\int\left(\frac{3x}{x^2+9}-\frac{1}{x^2+9}\right)dx=\int\frac{3x\,dx}{x^2+9}-\int\frac{dx}{x^2+9}$$

$$=\frac{3}{2}\int\frac{d(x^2+9)}{x^2+9}-\int\frac{dx}{x^2+9}$$

$$=\frac{3}{2}\log(x^2+9)-\frac{1}{3}\tan^{-1}\frac{x}{3}+c.$$

(5) 分函數之分子已合於標準式,但分母爲二次三項式者,先完成平方,使成 $x^2\pm a^2$, $\sqrt{a^2-x^2}$, $\sqrt{x^2\pm a^2}$ 之形而後運用公式.

[舉例] 求 $\int\frac{dx}{x^2+2x+5}$ 之值.

[解] $\int\frac{dx}{x^2+2x+5}=\int\frac{dx}{(x^2+2x+1)+4}=\int\frac{dx}{(x+1)^2+2^2}$

$$=\frac{1}{2}\tan^{-1}\frac{x+1}{2}+c. \quad \text{(公式 18)}$$

求 $\int\frac{dv}{v^2-6v+5}$ 之值.

[解] $\int\frac{dv}{v^2-6v+5}=\int\frac{dv}{(v-3)^2-2^2}=\frac{1}{4}\log\frac{v-5}{v-1}+c.$ (公式 19)

求 $\int\frac{dx}{\sqrt{2+x-x^2}}$ 之值.

[解] $\int\frac{dx}{\sqrt{2+x-x^2}}=\int\frac{dx}{\sqrt{2-(x^2-x)}}=\int\frac{dx}{\sqrt{2-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}}$

$$=\int\frac{dx}{\sqrt{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2}}=\sin^{-1}\frac{2x-1}{3}+c. \quad \text{(公式 20)}$$

求 $\int\frac{dt}{\sqrt{t^2+t+1}}$ 之值.

[解] $\int\frac{dt}{\sqrt{t^2+t+1}}=\int\frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}}=\int\frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$

$$=\log\left(t+\frac{1}{2}+\sqrt{t^2+t+1}\right)+c. \quad \text{(公式 21)}$$

(6) 欲求積分之函數,析成二個因子,視察其一因子是否爲其他因子之微分,如適爲他一因子之微分,則可直接求出其結果.

[舉例] 求 $\int (x^3+2x)(3x^2+2)dx$ 之值.

[解] 由視察,知 $(3x^2+2)dx$ 恰爲 (x^3+2x) 之微分.

$$\therefore \int (x^3+2x)(3x^2+2)dx = \int (x^3+2x)\,d(3x^2+2x)$$

$$= \frac{(x^3+2x)^2}{2}+c. \qquad \text{(公式 4)}$$

(7) 積分方法不同,則由各種方法所得之結果,其差僅在常數一項之不同.

[舉例] 求 $\int \sin x \cos x\,dx$ 之值.

[第一解] 與公式 (4) 比較, 命 $v=\sin x$, $dv=\cos x$, $n=1$,

$$\therefore \int \sin x \cos x\,dx = \int \sin x\,d(\sin x) = \frac{\sin^2 x}{2}+c.$$

[第二解] 若命 $v=\cos x$, 則 $dv=-\sin x\,dx$, $n=1$,

$$\therefore \int \sin x \cos x\,dx = -\int \cos x(-\sin x)\,dx$$

$$= -\int \cos x\,d(\cos x) = -\frac{\cos^2 x}{2}+c'.$$

改變第二解之結果,

$$c'+\frac{-\cos^2 x}{2} = -\frac{1-\sin^2 x}{2}+c' = -\frac{1}{2}+\frac{\sin^2 x}{2}+c'.$$

故第一解與第二解之結果,僅有常數項之差異; 即 $c=-\frac{1}{2}+c'$.

(8) 分母爲二次式分子爲單項式之函數,不論其分母是否帶有平方根號,應用甚夥. 茲列一表,以爲選擇公式時之一助.

	分子爲單項一次式	分子爲零次式(卽不含變數)
分母中不含平方根號	用 $\int \frac{dv}{v}=\log v+c$	用 $\int \frac{dv}{v^2+a^2}=\frac{1}{a}\tan^{-1}\frac{v}{a}+c$ $\int \frac{dv}{v^2-a^2}=\frac{1}{2a}\log\frac{v-a}{v+a}+c$
分母中含有平方根號	用 $\int v^n dv=\frac{v^{n+1}}{n+1}+c$ $\left(n=-\frac{1}{2}\right)$	用 $\int \frac{dv}{\sqrt{a^2-v^2}}=\sin^{-1}\frac{v}{a}+c$ $\int \frac{dv}{\sqrt{v^2\pm a^2}}=\log(v+\sqrt{v^2\pm a^2})+c$

9. 公式(1)至公式(5)之例題.

1. $\int x^6\,dx=\frac{x^{6+1}}{6+1}+c=\frac{x^7}{7}+c.$

2. $\int \sqrt{x}\,dx=\int x^{\frac{1}{2}}dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{3}x^{\frac{3}{2}}+c.$

3. $\int\frac{dx}{x^3}=\int x^{-3}dx=\frac{x^{-2}}{-2}+c=-\frac{1}{2x^2}+c.$

4. $\int ax^5\,dx=a\int x^5\,dx=\frac{ax^6}{6}+c.$

5. $\int x^2\,dx=\frac{x^{2+1}}{2+1}=\frac{x^3}{3}+c.$

6. $\int x^{\frac{2}{3}}\,dx=x^{\frac{5}{3}}\Big/\frac{5}{3}+c=\frac{3}{5}x^{\frac{5}{3}}+c.$

7. $\int at^{\frac{5}{2}}\,dt=a\int t^{\frac{5}{2}}\,dt=a\cdot\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}+c=\frac{2}{7}at^{\frac{7}{2}}+c.$

8. $\int\frac{dx}{3x^2}=\frac{1}{3}\int x^{-2}\,dx=\frac{1}{3}\cdot\frac{x^{-1}}{-1}+c=-\frac{1}{3x}+c.$

9. $\int \frac{2\,dx}{ax^{\frac{1}{3}}} = \frac{2}{a}\int x^{-\frac{1}{3}}dx = \frac{2}{a}\cdot\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{5\,x^{\frac{4}{3}}}{2\,a} + c.$

10. $\int 5\,y\,dy = 5\int y\,dy = 5\cdot\frac{y^2}{2} + c = \frac{5}{2}\,y^2 + c.$

11. $\int \sqrt{2px}\,dx = \sqrt{2p}\int x^{\frac{1}{2}}\,dx = \sqrt{2p}\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x\sqrt{2px} + c.$

12. $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3}{4}\,x^{\frac{4}{3}} + c.$

13. $\int s^{-\frac{1}{2}}ds = \frac{s^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = 2\,s^{\frac{1}{2}} + c.$

14 $\int 3\,a\,\theta^2\,d\theta = 3\,a\int \theta^2\,d\theta = 3\,a\cdot\frac{\theta^3}{3} + c = a\,\theta^3 + c.$

15. $\int 5\,m^2z^6\,dz = 5\,m^2\int z^6\,dz = 5\,m^2\,\frac{z^7}{7} + c = \frac{5}{7}\,m^2z^7 + c.$

16. $\int \frac{b\,d\phi}{\sqrt[3]{\phi}} = b\int \phi^{-\frac{1}{3}}\,d\phi = b\cdot\frac{\phi^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}\,b\,\phi^{\frac{2}{3}} + c.$

17 $\int y^{-m-1}\,dy = \frac{y^{-m}}{-m} + c = \frac{1}{-my^m} + c.$

18. $\int (2x^3 - 5x^2 - 3x + 4)\,dx$

$$= \int 2x^3\,dx - \int 5x^2\,dx - \int 3x\,dx + \int 4\,dx$$

$$= 2\int x^3\,dx - 5\int x^2\,dx - 3\int x\,dx + 4\int dx$$

$$= \frac{1}{2}x^4 - \frac{5}{3}x^3 - \frac{3}{2}x^2 + 4x + c.$$

19. $\int\left(\frac{2a}{\sqrt{x}}-\frac{b}{x^2}+3c\sqrt[3]{x^2}\right)dx$

$$=\int 2ax^{-\frac{1}{2}}dx-\int bx^{-2}dx+\int 3cx^{\frac{2}{3}}dx$$

$$=2a\int x^{-\frac{1}{2}}dx-b\int x^{-2}dx+3c\int x^{\frac{2}{3}}dx$$

$$=2a\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}}-b\cdot\frac{x^{-1}}{-1}+3c\cdot\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+c$$

$$=4a\sqrt{x}+\frac{b}{x}+\frac{9}{5}cx^{\frac{5}{3}}+c.$$

20. $\int(2x^9-3x^6+12x^3-3)\,dx$

$$=\int 2x^9dx-\int 3x^6dx+\int 12x^3dx-\int 3\,dx$$

$$=\frac{2x^{10}}{10}-\frac{3x^7}{7}+\frac{12x^4}{4}-3x+c$$

$$=\frac{x^{10}}{5}-\frac{3x^7}{7}+3x^4-3x+c.$$

21. $\int\left(\sqrt[3]{x^2}-\frac{1}{\sqrt[3]{x^2}}+\frac{2}{x^5}\right)dx=\int x^{\frac{2}{3}}dx-\int x^{-\frac{2}{3}}dx+\int 2x^{-5}dx$

$$=\frac{x^{\frac{5}{3}}}{\frac{5}{3}}-\frac{x^{\frac{1}{3}}}{\frac{1}{3}}+\frac{2x^{-4}}{-4}+c$$

$$=\frac{3x^{\frac{5}{3}}}{5}-3x^{\frac{1}{3}}-\frac{1}{2}x^{-4}+c.$$

22. $\int(a^{\frac{2}{3}}-x^{\frac{2}{3}})^3dx=\int(a^2-3a^{\frac{4}{3}}x^{\frac{2}{3}}+3a^{\frac{2}{3}}x^{\frac{4}{3}}-x^2)\,dx$

$$=\int a^2dx-\int 3a^{\frac{4}{3}}x^{\frac{2}{3}}dx+\int 3a^{\frac{2}{3}}x^{\frac{4}{3}}dx-\int x^2dx$$

$$= a^2x - 3a^{\frac{4}{3}} \cdot \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + 3a^{\frac{2}{3}} \cdot \frac{x^{\frac{7}{3}}}{\frac{7}{3}} - \frac{x^3}{3} + c$$

$$= a^2x - \frac{9}{5}a^{\frac{4}{3}}x^{\frac{5}{3}} + \frac{9}{7}a^{\frac{2}{3}}x^{\frac{7}{3}} - \frac{x^3}{3} + c.$$

23. $$\int (a^2-y^2)^3 \sqrt{y}\, dy = \int (a^6 - 3a^4y^2 + 3a^2y^4 - y^6)\sqrt{y}\, dy$$

$$= \int a^6 y^{\frac{1}{2}} dy - \int 3a^4y^2y^{\frac{1}{2}} dy + \int 3a^2y^4y^{\frac{1}{2}} dy - \int y^6y^{\frac{1}{2}} dy$$

$$= \int a^6 y^{\frac{1}{2}} dy - \int 3a^4y^{\frac{5}{2}} dy + \int 3a^2y^{\frac{9}{2}} dy - \int y^{\frac{13}{2}} dy$$

$$= a^6 \cdot \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - 3a^4 \cdot \frac{y^{\frac{7}{2}}}{\frac{7}{2}} + 3a^2 \cdot \frac{y^{\frac{11}{2}}}{\frac{11}{2}} - \frac{y^{\frac{15}{2}}}{\frac{15}{2}} + c$$

$$= \frac{2}{3}a^6y^{\frac{3}{2}} - \frac{6}{7}a^4y^{\frac{7}{2}} + \frac{6}{11}a^2y^{\frac{11}{2}} - \frac{2}{15}y^{\frac{15}{2}} + c$$

$$= 2y^{\frac{3}{2}}\left(\frac{a^6}{3} - \frac{3}{7}a^4y^2 + \frac{3}{11}a^2y^4 - \frac{1}{15}y^6\right) + c.$$

24. $$\int (\sqrt{a} - \sqrt{t})^3\, dt = \int a^{\frac{3}{2}} dt - \int 3at^{\frac{1}{2}} dt + \int 3a^{\frac{1}{2}}t\, dt - \int t^{\frac{3}{2}} dt$$

$$= a^{\frac{3}{2}}t - \frac{3at^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3a^{\frac{1}{2}}t^2}{2} - \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= a^{\frac{3}{2}}t - 2at^{\frac{3}{2}} + \frac{3}{2}a^{\frac{1}{2}}t^2 - \frac{2}{5}t^{\frac{5}{2}} + c.$$

25. $$\int (a^2+b^2x^2)^{\frac{1}{2}} x\, dx = \frac{1}{2b^2}\int (a^2+b^2x^2)^{\frac{1}{2}} 2b^2x\, dx$$

$$= \frac{1}{2b^2}\int (a^2+b^2x^2)^{\frac{1}{2}} d(a^2+b^2x^2)$$

$$= \frac{1}{2b^2} \cdot \frac{(a^2+b^2x^2)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(a^2+b^2x^2)^{\frac{3}{2}}}{3b^2} + c$$

26. $\int \sqrt{a^2-x^2}\,x\,dx = \int (a^2-x^2)^{\frac{1}{2}}\,x\,dx = -\frac{1}{2}\int (a^2-x^2)^{\frac{1}{2}}\,2x\,dx$

$$= -\frac{1}{2}\int (a^2-x^2)^{\frac{1}{2}}\,d(a^2-x^2)$$

$$= -\frac{1}{2}\cdot\frac{(a^2-x^2)^{\frac{3}{2}}}{\frac{3}{2}}+c = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}+c.$$

27. $\int (3ax^2+4bx^3)^{\frac{4}{3}}(2ax+4bx^2)\,dx = ?$

命　$v = 3ax^2+4bx^3$, 則 $dv = (6ax+12bx^2)\,dx$,

∴　$\int (3ax^2+4bx^3)^{\frac{4}{3}}(2ax+4bx^2)\,dx$

$$= \frac{1}{3}\int (3ax^2+4bx^3)^{\frac{4}{3}}\,3(2ax+4bx^2)\,dx$$

$$= \frac{1}{3}\cdot\frac{(3ax^2+4bx^3)^{\frac{7}{3}}}{\frac{7}{3}}+c = \frac{1}{7}(3ax^2+4bx^3)^{\frac{7}{3}}+c.$$

28. $\int b\,(6ax^2+8bx^3)^{\frac{5}{3}}(2ax+4bx^2)\,dx = ?$

命　$v = 6ax^2+8bx^3$, 則 $dv = (12ax+24bx^2)\,dx$

$$= 6(2ax+4bx^2)\,dx,$$

∴　$\int b\,(6ax^2+8bx^3)^{\frac{5}{3}}(2ax+4bx^2)\,dx$

$$= \frac{b}{6}\int (6ax^2+8bx^3)^{\frac{5}{3}}\cdot 6(2ax+4bx^2)\,dx$$

$$= \frac{b}{6}(6ax^2+8bx^3)^{\frac{8}{3}}\Big/\frac{8}{3}+c$$

$$= \frac{b}{16}(6ax^2+8bx^3)^{\frac{8}{3}}+c.$$

29. $\int \frac{x^2\,dx}{(a^2+x^3)^{\frac{1}{2}}} = ?$

命 $v = a^2+x^3$, 則 $dv = 3x^2\,dx$,

$\therefore \int \frac{x^2 dx}{(a^2+x^3)^{\frac{1}{2}}} = \int (a^2+x^3)^{-\frac{1}{2}} x^2\,dx = \frac{1}{3}\int (a^2+x^3)^{-\frac{1}{2}} 3x^2\,dx$

$= \frac{1}{3}(a^2+x^3)^{\frac{1}{2}} 2 + c = \frac{2}{3}(a^2+x^3)^{\frac{1}{2}} + c.$

30. $\int \frac{dx}{\sqrt{1-x}} = ?$

命 $v = 1-x$, 則 $dv = -dx$,

$\therefore \int \frac{dx}{\sqrt{1-x}} = \int (1-x)^{-\frac{1}{2}}\,dx = -\int (1-x)^{-\frac{1}{2}}(-dx)$

$= -\frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c = -2(1-x)^{\frac{1}{2}} + c.$

31. $\int 2\pi y\left(\frac{y^2}{p^2}+1\right)^{\frac{1}{2}} dy = ?$

命 $v = p^2+y^2$, 則 $dv = 2y\,dy$,

$\therefore \int 2\pi y\left(\frac{y^2+p^2}{p^2}\right)^{\frac{1}{2}} dy = \int \frac{\pi}{p}(y^2+p^2)^{\frac{1}{2}} 2y\,dy$

$= \frac{\pi}{p}\int (y^2+p^2)^{\frac{1}{2}} d(y^2+p^2)$

$= \frac{\pi}{p}\,\frac{(y^2+p^2)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\pi}{3p}(y^2+p^2)^{\frac{3}{2}} + c.$

32. $\int (1+e^x)^{\frac{1}{2}} e^x\,dx = ?$

命 $v = 1+e^x$, 則 $dv = e^x\,dx$.

$$\therefore \quad \int (1+e^x)^{\frac{1}{2}} e^x dx = \frac{(1+e^x)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + c.$$

33. $$\int \sin^2 x \cos x \, dx = \int (\sin x)^2 \cos x \, dx$$

$$= \frac{(\sin x)^3}{3} + c = \frac{\sin^3 x}{3} + c.$$

$$[\because \quad v = \sin x, \, dv = \cos x \, dx, \, n = 2.]$$

34. $$\int \cos^5 x \sin x \, dx = -\int \cos^5 x(-\sin x)dx = -\frac{1}{6}\cos^6 x + c.$$

$$[\because \quad v = \cos x, \, dv = -\sin x, \, n = 5.]$$

35. $$\int \sin^3 ax \cos ax \, dx = \frac{1}{a}\int \sin^3 ax \cdot a \cos ax \, dx = \frac{1}{4a}\sin^4 ax + c.$$

$$[\because \quad v = \sin ax, \, dv = a \cos ax \, dx, \, n = 3.]$$

36. $$\int \cos^4 3x \sin 3x \, dx = -\frac{1}{3}\int \cos^4 3x(-3\sin 3x)dx$$

$$= -\frac{1}{3} \cdot \frac{1}{5}\cos^5 3x + c = -\frac{1}{15}\cos^5 3x + c.$$

$$[\because \quad v = \cos 3x, \, dv = -3\sin 3x \, dx, \, n = 4.]$$

37. $$\int \frac{x \, dx}{\sqrt{a^2 - x^2}} = \int (a^2 - x^2)^{-\frac{1}{2}} x \, dx$$

$$= -\frac{1}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x \, dx) = -(a^2 - x^2)^{\frac{1}{2}} + c.$$

$$[\because \quad v = a^2 - x^2, \, dv = -2x \, dx, \, n = -\tfrac{1}{2}.]$$

38. $$\int \frac{5a \, dt}{(b-t)^5} = \int (b-t)^{-5} 5a \, dt = -5a\int (b-t)^{-5}(-dt)$$

$$= (b-t)^{-5} a + c = \frac{a}{(b-t)^5} + c.$$

$$[\because \quad v=b-t,\ dv=-dt,\ n=-6.]$$

39. $$\int \sqrt[3]{1+x^2}\,x\,dx=\frac{1}{2}\int \sqrt[3]{1+x^2}\,2x\,dx=\frac{1}{2}\cdot\frac{3}{4}(1+x^2)^{\frac{4}{3}}+c$$

$$=\frac{3}{8}(1+x^2)^{\frac{4}{3}}+c.$$

$$[\because \quad v=1+x^2,\ dv=2x\,dx,\ n=\tfrac{1}{3}.]$$

40. $$\int \frac{s\,ds}{\sqrt[3]{1-s^2}}=-\frac{1}{2}\int (1-s^2)^{-\frac{1}{3}}d(1-s^2)$$

$$=-\frac{1}{2}\cdot\frac{(1-s^2)^{\frac{2}{3}}}{\frac{2}{3}}+c=-\frac{3}{4}(1-s^2)^{\frac{2}{3}}+c.$$

$$[\because \quad v=1-s^2,\ dv=-2s\,ds,\ n=-\tfrac{1}{3}]$$

41. $$\int \frac{u^{n-1}\,du}{(a+bu^n)^m}=\frac{1}{bn}\int (a+bu^n)^{-m}d(a+bu^n)=\frac{(a+bu^n)^{1-m}}{bn(1-m)}+c$$

$$[\because \quad v=a+bu^n,\ dv=bnu^{n-1},\ n=-m.]$$

42. $$\int \frac{2as\,ds}{(b^2-c^2s^2)^2}=-\frac{a}{c^2}\int (b^2-c^2s^2)^{-2}d(b^2-c^2s^2)$$

$$=\frac{a}{c^2(b^2-c^2s^2)}+c.$$

$$[\because \quad v=b^2-c^2s^2,\ dv=-2c^2s\,ds,\ n=-2.]$$

43. $$\int \frac{3ax\,dx}{b^2+c^2x^2}=3a\int \frac{x\,dx}{b^2+c^2x^2}=\frac{3a}{2c^2}\int \frac{2c^2x\,dx}{b^2+c^2x^2}$$

$$=\frac{3a}{2c^2}\int \frac{d(b^2+c^2x^2)}{b^2+c^2x^2}=\frac{3a}{2c^2}\log(b^2+c^2x^2)+c.$$

$$[\because \quad v=b^2+c^2x^2,\ dv=2c^2x\,dx.]$$

44. $$\int \frac{x\,dx}{x^2-1}=\frac{1}{2}\int \frac{d(x^2-1)}{x^2-1}=\frac{1}{2}\log(x^2-1)+c.$$

45. $\int \frac{(x^2-a^2)dx}{x^3-3a^2x} = \frac{1}{3}\int \frac{d(x^3-3a^2x)}{x^3-3a^2x} = \frac{1}{3}\log(x^3-3a^2x)+c.$

46. $\int \frac{5x^2dx}{10x^3+15} = \frac{1}{6}\int \frac{d(10x^3+15)}{10x^3+15} = \frac{1}{6}\log(10x^3+15)+c$

$$= \log(10x^3+15)^{\frac{1}{6}}+c.$$

47. $\int \frac{5bxdx}{8a-6bx^2} = -\frac{5}{12}\int \frac{d(8a-6bx^2)}{8a-6bx^2}$

$$= -\frac{5}{12}\log(8a-6bx^2)+c.$$

48. $\int \frac{2x-1}{2x+3}dx = \int dx - \int \frac{4}{2x+3}dx = x-\log(2x+3)^2+c.$

49. $\int \frac{x^{n-1}-1}{x^n-nx}dx = \frac{1}{n}\int \frac{d(x^n-nx)}{x^n-nx} = \frac{1}{n}\log(x^n-nx)+c.$

50. $\int \frac{(y^2-2)^3dy}{y^5} = \int\left(y-\frac{6}{y}+\frac{12}{y^3}-\frac{8}{y^5}\right)dy$

$$= \int ydy - \int\frac{6}{y}dy + \int\frac{12}{y^3}dy - \int\frac{8}{y^5}dy$$

$$= \frac{y^2}{2} - 6\int\frac{dy}{y} + \frac{12y^{-2}}{-2} - \frac{8y^{-4}}{-4}+c$$

$$= \frac{2}{y^4} - \frac{6}{y^2} + \frac{y^2}{2} - 6\log y + c.$$

51. $\int \frac{t^{n-1}dt}{a+bt^n} = \frac{1}{nb}\int \frac{d(a+bt^n)}{a+bt^n} = \frac{1}{nb}\log(a+bt^n)+c.$

52. $\int (\log a)^3\frac{da}{a} = \int (\log a)^3 d(\log a) = \frac{\log^4 a}{4}+c.$

53. $\int \frac{r^2+1}{r-1}dr = \int rdr + \int dr + \int \frac{2}{r-1}dr$

$$= \frac{r^2}{2} + r + 2\log(r-1)+c.$$

54. $\int \frac{2e^x dx}{e^x+1} = 2\int \frac{d(e^x+1)}{e^x+1} = 2\log(e^x+1)+c.$

55. $\int \frac{\sin x\,dx}{a+b\cos x} = -\frac{1}{b}\int \frac{d(a+b\cos x)}{a+b\cos x}$

$$= -\frac{1}{b}\log(a+b\cos x)+c.$$

56. $\int \frac{\sec^2\theta\,d\theta}{1+3\tan\theta} = \frac{1}{3}\int \frac{3\sec^2\theta\,d\theta}{1+3\tan\theta} = \frac{1}{3}\log(1+3\tan\theta)+c.$

57. $\int \frac{e^{3s}ds}{e^s-1} = \int\left(e^{2s}+e^s+1+\frac{1}{e^s-1}\right)ds = \int\left(e^{2s}+e^s+\frac{e^s}{e^s-1}\right)ds$

$$= \int e^{2s}ds + \int e^s ds + \int \frac{e^s ds}{e^s-1} = \frac{e^{2s}}{2}+e^s+\log(e^s-1)+c.$$

58. $\int \frac{e^r-1}{e^r+1}dr = \int\left(\frac{2e^r}{e^r+1}-1\right)dr$

$$= \int \frac{2d(e^r+1)}{e^r+1} - \int dr = \log(e^r+1)^2 - r + c.$$

10. 公式(6)至公式(7)之例題.

59. $\int b\,a^{2x}dx = b\int a^{2x}dx = \frac{b}{2}\int a^{2x}2dx$

$$= \frac{b}{2}\int a^{2x}d(2x) = \frac{b}{2}\cdot\frac{a^{2x}}{\log a}+c.$$

[$\because$ $v=2x$; 則 $dv=2dx$].

60. $\int 3e^x dx = 3\int e^x dx = 3e^x+c.$

61. $\int e^{\frac{x}{n}}dx = n\int e^{\frac{x}{n}}d\left(\frac{x}{n}\right) = ne^{\frac{x}{n}}+c.$

62. $\int e^{\sin x}\cos x\,dx = \int e^{\sin x}d(\sin x) = e^{\sin x}+c$

63. $\int e^{2\cos x}\sin x\,dx=-\frac{1}{2}\int e^{2\cos x}\,d(2\cos x)=-\frac{e^{2\cos x}}{2}+c$

64. $\int 3^{2y-1}dy=\frac{1}{2}\int 3^{2y-1}d(2y-1)=\frac{3^{2y-1}}{2\log 3}+c.$

65. $\int e^{-x}dx=-\int e^{-x}d(-x)=-e^{-x}+c.$

66. $\int e^{ax}dx=\frac{1}{a}\int e^{ax}d(ax)=\frac{1}{a}e^{ax}+c.$

67. $\int a^{2x}dx=\frac{1}{2}\int a^{2x}d(2x)=\frac{a^{2x}}{2\log a}+c.$

68. $\int 5^{x}dx=\frac{5^{x}}{\log 5}+c.$

69. $\int a^{x}e^{x}dx=\int (ae)^{x}dx=\frac{a^{x}e^{x}}{\log ae}+c$

$$=\frac{a^{x}e^{x}}{\log e+\log a}+c=\frac{a^{x}e^{x}}{1+\log a}+c.$$

70. $\int (e^{5x}+a^{5x})dx=\frac{1}{5}\int e^{5x}d(5x)+\frac{1}{5}\int a^{5x}d(5x)$

$$=\frac{1}{5}\left(e^{5x}+\frac{a^{5x}}{\log a}\right)+c.$$

71. $\int e^{x^2+4x+3}(x+2)dx=\frac{1}{2}\int e^{x^2+4x+3}(2x+4)dx$

$$=\frac{1}{2}e^{x^2+4x+3}+c.$$

72. $\int (a^{nx}-b^{mx})dx=\frac{1}{n}\int a^{nx}d(nx)-\frac{1}{m}\int b^{mx}d(mx)$

$$=\frac{a^{nx}}{n\log a}-\frac{b^{mx}}{m\log b}+c.$$

73. $\int (e^{\frac{x}{a}}+e^{-\frac{x}{a}})\,dx = a\int e^{\frac{x}{a}}\,d\left(\frac{x}{a}\right) - a\int e^{-\frac{x}{a}}\,d\left(-\frac{x}{a}\right)$

$$= a(e^{\frac{x}{a}} - e^{-\frac{x}{a}}) + c.$$

74. $\int (e^{y}+e^{-y})^2 dy = \int (e^{2y}+2+e^{-2y})\,dy = \frac{1}{2}(e^{2y}-e^{-2y}) + 2y + c.$

75. $\int \frac{(a^x-b^x)^2}{a^x b^x}\,dx = \int \left[\left(\frac{a}{b}\right)^x - 2 + \left(\frac{b}{a}\right)^x\right] dx$

$$= \int \left(\frac{a}{b}\right)^x dx - 2\int dx + \int \left(\frac{b}{a}\right)^x dx$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}} - 2x + \frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}} + c$$

$$= \frac{a^x b^{-x} - b^x a^{-x}}{\log a - \log b} - 2x + c.$$

76. $\int (e^{4x}+a^{5x}+3b^{-2x})\,dx = \frac{e^{4x}}{4} + \frac{a^{5x}}{5\log a} - \frac{3b^{-2x}}{2\log b} + c.$

11. 公式(8)至公式(17)之例題.

77. $\int \sin 2ax\,dx = \frac{1}{2a}\int \sin 2ax \cdot 2a\,dx = \frac{1}{2a}\int \sin 2ax\,d(2ax)$

$$= \frac{1}{2a}(-\cos 2ax) + c = -\frac{\cos 2ax}{2a} + c.$$

[$\because$ $v = 2ax$; 則 $dv = 2a\,dx$.]

78. $\int \cos mx\,dx = \frac{1}{m}\int \cos mx\,d(mx) = \frac{1}{m}\sin mx + c.$

79. $\int \tan bx\,dx = \frac{1}{b}\int \tan bx\,d(bx) = \frac{1}{b}\log(\sec bx) + c.$

80. $\int \sec ax\,dx = \frac{1}{a}\int \sec ax\,d(ax) = \frac{1}{a}\log(\sec ax + \tan ax) + c.$

81. $\int \csc \frac{x}{a}dx = a\int \csc \frac{x}{a}\,d\left(\frac{x}{a}\right) = a\log\left(\csc \frac{x}{a} - \cot \frac{x}{a}\right) + c.$

82. $\int \sec 3t \tan 3t\,dt = \frac{1}{3}\int \sec 3t \tan 3t\,d(3t) = \frac{1}{3}\sec 3t + c.$

83. $\int \csc ay \cot ay\,dy = \frac{1}{a}\int \csc ay \cot ay\,d(ay) = \frac{-1}{a}\csc ay + c.$

84. $\int \csc^2 3x\,dx = \frac{1}{3}\int \csc^2 3x\,d(3x) = -\frac{1}{3}\cot 3x + c.$

85. $\int \cot \frac{x}{2}dx = 2\int \cot \frac{x}{2}\,d\left(\frac{x}{2}\right) = 2\log\left(\sin \frac{x}{2}\right) + c.$

86. $\int \sec^2 x^3 \cdot x^2\,dx = \frac{1}{3}\int \sec^2 x^3 \cdot d(x^3) = \frac{1}{3}\tan x^3 + c.$

87. $\int \frac{dx}{\sin^2 x} = \int \csc^2 x\,dx = -\cot x + c.$

88. $\int \frac{ds}{\cos^2 s} = \int \sec^2 s\,ds = \tan s + c.$

89. $$\begin{aligned}\int (\tan\theta + \cot\theta)^2 d\theta &= \int (\tan^2\theta + 2 + \cot^2\theta)\,d\theta \\ &= \int (\sec^2\theta + \csc^2\theta)\,d\theta \\ &= \int \sec^2\theta\,d\theta + \int \csc^2\theta\,d\theta \\ &= \tan\theta - \cot\theta + c.\end{aligned}$$

90. $$\begin{aligned}\int (\sec\theta - \tan\theta)^2 d\theta &= \int (\sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta)\,d\theta \\ &= \int (2\sec^2\theta - 2\sec\theta\tan\theta - 1)\,d\theta \\ &= 2(\tan\theta - \sec\theta) - \theta + c.\end{aligned}$$

91. $\int(\tan 2s-1)^2 ds=\int(\tan^2 2s-2\tan 2s+1)ds$

$$=\int(\sec^2 2s-2\tan 2s)ds$$

$$=\frac{1}{2}\int\sec^2 2s\,d(2s)-\int\tan 2s\,d(2s)$$

$$=\frac{1}{2}\tan 2s-\log(\sec 2s)+c.$$

92. $\int(\cos\frac{\theta}{3}-\sin 3\theta)\,d\theta=3\sin\frac{\theta}{3}+\frac{1}{3}\cos 3\theta+c.$

93. $\int\left(\sin ax+\sin\frac{x}{a}\right)dx=-\frac{1}{a}\cos ax-a\cos\frac{x}{a}+c.$

94. $\int k\cos(a+by)\,dy=\frac{k}{b}\int\cos(a+by)\,d(a+by)$

$$=\frac{k}{b}\sin(a+by)+c.$$

95. $\int\csc^2 x^3\cdot x^2\,dx=\frac{1}{3}\int\csc^2 x^3\,d(x^3)=-\frac{1}{3}\cot x^3+c.$

96. $\int\cos(\log x)\frac{dx}{x}=\int\cos(\log x)\,d(\log x)=\sin(\log x)+c.$

97. $\int\frac{dx}{1+\cos x}=\int\frac{dx}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}=\int\frac{1-\cos x}{1-\cos^2 x}dx$

$$=\int\frac{1-\cos x}{\sin^2 x}dx=\int(\csc^2 x-\csc x\cot x)\,dx$$

$$=\int\csc^2 x\,dx-\int\csc x\cot x\,dx=-\cot x+\csc x+c$$

$$=\frac{1}{\sin x}-\frac{\cos x}{\sin x}+c=\frac{1-\cos x}{\sin x}+c$$

$$=\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}+c=\tan\frac{x}{2}+c.$$

98. $$\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{\cos^2 x}\,dx = \int (\sec^2 x - \sec x \tan x)\,dx$$
$$= \int \sec^2 x\,dx - \int \sec x \tan x\,dx = \tan x - \sec x + c.$$

99. $$\int \sin \frac{2}{3} x\,dx = \frac{3}{2}\int \sin \frac{2}{3} x\,d\left(\frac{2}{3} x\right) = -\frac{3}{2}\cos \frac{2}{3} x + c.$$

100. $$\int \cot e^x \cdot e^x\,dx = \int \cot e^x\,d(e^x) = \log (\sin e^x) + c.$$

101. $$\int \sec \frac{\theta}{2} \tan \frac{\theta}{2}\,d\theta = 2\int \sec \frac{\theta}{2} \tan \frac{\theta}{2}\,d\left(\frac{\theta}{2}\right) = 2 \sec \frac{\theta}{2} + c.$$

102. $$\int \csc \frac{a}{b}\phi \cot \frac{a}{b}\phi\,d\phi = \frac{b}{a}\int \csc \frac{a}{b}\phi \cot \frac{a}{b}\phi\,d\left(\frac{a}{b}\phi\right)$$
$$= -\frac{b}{a}\csc \frac{a}{b}\phi + c.$$

103. $$\int \cos (b+ax)\,dx = \frac{1}{a}\sin (b-ax) + c.$$

104. $$\int \sec^2 2ax\,dx = \frac{1}{2a}\int \sec^2 2ax\,d(2ax) = \frac{1}{2a}\tan 2ax + c.$$

105. $$\int \frac{dx}{\cos^2 3x} = \int \sec^2 3x\,dx = \frac{1}{3}\int \sec^2 3x\,d(3x) = \frac{1}{3}\tan 3x + c.$$

106. $$\int \frac{dt}{\tan 5t} = \int \cot 5t\,dt = \frac{1}{5}\int \cot 5t\,d(5t) = \frac{1}{5}\log (\sin 5t) + c.$$

107. $$\int \tan \frac{x}{3}\,dx = 3\int \tan \frac{x}{3}\,d\left(\frac{x}{3}\right) = 3\log \left(\sec \frac{x}{3}\right) + c.$$

108. $$\int \csc^2 (a-bx)\,dx = \frac{-1}{b}\int \csc^2 (a-bx)\,d(a-bx)$$
$$= \frac{1}{b}\cot (a-bx) + c.$$

109. $$\int \frac{d\theta}{\sin^2 4\theta} = \int \csc^2 4\theta\, d\theta = \frac{1}{4}\int \csc^2 4\theta\, d(4\theta)$$

$$= -\frac{1}{4}\cot 4\theta + c.$$

110. $$\int \frac{dy}{\cot 7y} = \int \tan 7y\, dy = \frac{1}{7}\int \tan 7y\, d(7y)$$

$$= \frac{1}{7}\log(\sec 7y) + c.$$

111. $$\int \left(\sec 2\theta - \csc\frac{\theta}{2}\right) d\theta = \int \sec 2\theta\, d\theta - \int \csc\frac{\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int \sec 2\theta\, d(2\theta) - 2\int \csc\frac{\theta}{2}\, d\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\log(\sec 2\theta + \tan 2\theta)$$

$$-2\log\left(\csc\frac{\theta}{2} - \cot\frac{\theta}{2}\right) + c.$$

112. $$\int (\tan\theta + \sec\theta)^2 d\theta = \int (\tan^2\theta + 2\sec\theta\tan\theta + \sec^2\theta)\, d\theta$$

$$= \int (2\sec^2\theta + 2\sec\theta\tan\theta - 1)\, d\theta$$

$$= \int 2\sec^2\theta\, d\theta + \int 2\sec\theta\tan\theta\, d\theta - \int d\theta$$

$$= 2\tan\theta + 2\sec\theta - \theta + c.$$

113. $$\int \left(\tan 4s - \cot\frac{s}{4}\right) ds = \int \tan 4s\, ds - \int \cot\frac{s}{4}\, ds$$

$$= \frac{1}{4}\int \tan 4s\, d(4s) - 4\int \cot\frac{s}{4}\, d\left(\frac{s}{4}\right)$$

$$= \frac{1}{4}\log(\sec 4s) - 4\log\left(\sin\frac{s}{4}\right) + c.$$

114. $$\int(\cot x-1)^2\,dx=\int(\cot^2 x-2\cot x+1)\,dx$$
$$=\int(\csc^2 x-2\cot x)\,dx$$
$$=\int\csc^2 x\,dx-2\int\cot x\,dx$$
$$=-\cot x-2\log(\sin x)+c.$$

115. $$\int(\sec t-1)^2\,dt=\int(\sec^2 t-2\sec t+1)\,dt$$
$$=\int\sec^2 t\,dt-2\int\sec t\,dt+\int dt$$
$$=\tan t-2\log(\sec t+\tan t)+t+c.$$

116. $$\int(1-\csc y)^2\,dy=\int(1-2\csc y+\csc^2 y)\,dy$$
$$=\int dy-2\int\csc y\,dy+\int\csc^2 y\,dy$$
$$=y-2\log(\csc y-\cot y)-\cot y+c.$$

117. $$\int\frac{dx}{1-\cos x}=\int\frac{(1+\cos x)\,dx}{(1+\cos x)(1-\cos x)}=\int\frac{(1+\cos x)\,dx}{1-\cos^2 x}$$
$$=\int\frac{(1+\cos x)\,dx}{\sin^2 x}=\int\frac{dx}{\sin^2 x}+\int\frac{\cos x}{\sin^2 x}\,dx$$
$$=\int\csc^2 x\,dx+\int\cot x\csc x\,dx=-\cot x-\csc x+c.$$

118. $$\int\frac{dx}{1-\sin x}=\int\frac{(1+\sin x)\,dx}{(1+\sin x)(1-\sin x)}=\int\frac{(1+\sin x)\,dx}{\cos^2 x}$$
$$=\int\frac{1}{\cos^2 x}\,dx+\int\frac{\sin x}{\cos^2 x}\,dx=\int\sec^2 x\,dx$$
$$+\int\sec x\tan x\,dx=\tan x+\sec x+c.$$

119. $\int \frac{2a\,dt}{\sin bt} = \frac{2a}{b}\int \csc bt\, d(bt) = \frac{2a}{b}\log(\csc bt - \cot bt) + c.$

120. $\int \frac{5b\,d\theta}{\cos 8\theta} = 5b\int \frac{d\theta}{\cos 8\theta} = 5b\int \sec 8\theta\, d\theta$

$$= \frac{5b}{8}\int \sec 8\theta\, d(8\theta) = \frac{5b}{8}\log(\sec 8\theta + \tan 8\theta) + c.$$

12. 公式(18)至公式(23)之例題.

121. $\int \frac{dx}{4x^2+9} = \frac{1}{2}\int \frac{2\,dx}{(2x)^2+3^2} = \frac{1}{2}\int \frac{d(2x)}{(2x)^2+3^2} = \frac{1}{6}\tan^{-1}\frac{2x}{3} + c.$

[與公式(18)比較,命 $v^2 = 4x^2$, $a^2 = 9$; 則 $v = 2x$, $dv = 2x$, $a = 3$.]

122. $\int \frac{dx}{9x^2-4} = \frac{1}{3}\int \frac{d(3x)}{(3x)^2-2^2} = \frac{1}{3}\cdot\frac{1}{4}\log\frac{3x-2}{3x+2} + c$

$$= \frac{1}{12}\log\frac{3x-2}{3x+2} + c.$$

[與公式(19)比較,命 $v^2 = 9x^2$, $a^2 = 4$; 則 $v = 3x$, $dv = 3\,dx$, $a = 2$.]

123. $\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int \frac{d(3x)}{\sqrt{4^2-(3x)^2}} = \frac{1}{3}\sin^{-1}\frac{3x}{4} + c.$

[與公式(20)比較, $a^2 = 16$, $v^2 = 9x^2$; 則 $a = 4$, $v = 3x$, $dv = 3\,dx$.]

124. $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c.$

125. $\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}} = \log(x + \sqrt{x^2-9}) + c.$

[與公式(21)比較, 命 $v^2 = x^2$, $a^2 = 9$; 則 $v = x$, $dv = dx$, $a = 3$.]

126. $\int \frac{dx}{\sqrt{x^2+9}} = \int \frac{dx}{\sqrt{x^2+3^2}} = \log(x + \sqrt{x^2+9}) + c.$

127. $\int \frac{5\,dx}{x^2+9} = 5\int \frac{dx}{x^2+3^2} = 5\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + c = \frac{5}{3}\tan^{-1}\frac{x}{3} + c.$

128. $\int \frac{b\,dx}{a^2x^2-c^2}=\frac{b}{a}\int \frac{d(ax)}{(ax)^2-c^2}=\frac{b}{a}\cdot\frac{1}{2c}\log\frac{ax-c}{ax+c}+c$

$$=\frac{b}{2ac}\log\frac{ax-c}{ax+c}+c.$$

129. $\int \frac{7x^2dx}{5-x^6}=7\int \frac{x^2dx}{(\sqrt{5})^2-(x^3)^2}=\frac{7}{3}\int \frac{d(x^3)}{(\sqrt{5})^2-(x^3)^2}$

$$=\frac{7}{6\sqrt{5}}\log\frac{x^3+\sqrt{5}}{x^3-\sqrt{5}}+c.$$

130. $\int \frac{5x\,dx}{\sqrt{1-x^4}}=5\int \frac{x\,dx}{\sqrt{1-(x^2)^2}}=\frac{5}{2}\int \frac{d(x^2)}{\sqrt{1-(x^2)^2}}=\frac{5}{2}\sin^{-1}x^2+c.$

131. $\int \frac{dx}{x\sqrt{4x^2-9}}=\int \frac{2\,dx}{2x\sqrt{(2x)^2-3^2}}=\frac{1}{3}\sec^{-1}\frac{2x}{3}+c.$

[與公式(23)比較, $v^2=(2x)^2$, $a^2=9$; 則 $v=2x$, $dv=2\,dx$, $a=3$.]

132. $\int \frac{ax\,dx}{x^4+e^4}=\int \frac{ax\,dx}{(x^2)^2+(e^2)^2}=\frac{a}{2}\int \frac{d(x^2)}{(x^2)^2+(e^2)^2}$

$$=\frac{a}{2e^2}\tan^{-1}\frac{x^2}{e^2}+c.$$

133. $\int \frac{dx}{\sqrt{6x-x^2}}=\int \frac{dv}{\sqrt{2\times 3x-x^2}}=\operatorname{vers}^{-1}\frac{x}{3}+c.$

[與公式(22)比較, $2av=6x$, $v^2=x^2$; 則 $v=x$, $dv=dx$, $a=3$.]

134. $\int \frac{e\,dt}{a^2-b^2t^2}=\frac{e}{b}\int \frac{d(bt)}{a^2-(bt)^2}=\frac{e}{2ab}\log\frac{bt+a}{bt-a}+c.$

135. $\int \frac{e^t\,dt}{\sqrt{1-e^{2t}}}=\int \frac{d(e^t)}{\sqrt{1-(e^t)^2}}=\sin^{-1}e^t+c.$

136. $\int \frac{7\,ds}{\sqrt{3-5s^2}}=\int \frac{7\,ds}{(\sqrt{3})^2-(\sqrt{5}s)^2}=\frac{7}{\sqrt{5}}\int \frac{d(\sqrt{5}s)}{(\sqrt{3})^2-(\sqrt{5}s)^2}$

$$=\frac{7}{\sqrt{5}}\sin^{-1}\frac{\sqrt{5}s}{\sqrt{3}}+c.$$

137. $$\int \frac{dv}{\sqrt{a}v^2-b} = \frac{-1}{\sqrt{a}} \int \frac{d(\sqrt{a}v)}{\sqrt{(\sqrt{a}v)^2-(\sqrt{b})^2}}$$

$$= \frac{1}{\sqrt{a}} \log (\sqrt{a}v + \sqrt{a\,v^2-b}) + c.$$

138. $$\int \frac{\cos\theta d\theta}{a^2+\sin^2\theta} = \int \frac{d(\sin\theta)}{a^2+\sin^2\theta} = \frac{1}{a} \tan^{-1}\left(\frac{\sin\theta}{a}\right) + c.$$

139. $$\int \frac{dx}{x\sqrt{1-\log^2 x}} = \int \frac{1}{\sqrt{1-\log^2 x}} \cdot \frac{dx}{x} = \int \frac{d(\log x)}{\sqrt{1-\log^2 x}}$$

$$= \sin^{-1}(\log x) + c.$$

140. $$\int \frac{dx}{\sqrt{b^2-e^2x^2}} = \frac{1}{e} \int \frac{d(ex)}{\sqrt{b^2-(ex)^2}}$$

$$= \frac{1}{e} \log (ex + \sqrt{b^2+e^2x^2}) + c.$$

141. $$\int \frac{dy}{\sqrt{b^2y^2-a^2}} = \frac{1}{b} \int \frac{d(by)}{\sqrt{(by)^2-a^2}}$$

$$= \frac{1}{b} \log (by + \sqrt{b^2y^2-a^2}) + c.$$

142. $$\int \frac{du}{\sqrt{a^2-(u+b)^2}} = \int \frac{d(u+b)}{\sqrt{a^2-(u+b)^2}} = \sin^{-1}\frac{u+b}{a} + c.$$

143. $$\int \frac{a\,dz}{(z-e)^2+b^2} = a \int \frac{d(z-e)}{(z-e)^2+b^2} = \frac{a}{b} \tan^{-1}\frac{z-e}{b} + c.$$

144. $$\int \frac{dx}{1+x+x^2} = \int \frac{4\,dx}{3+(1+2x)} = 2\int \frac{2\,dx}{3+(1+2x)^2}$$

$$= 2\int \frac{d(1+2x)}{(1+2x)^2+(\sqrt{3})^2} = \frac{2}{\sqrt{3}} \tan^{-1}\frac{2x+1}{\sqrt{3}} + c.$$

145. $$\int\frac{dx}{3x^2-2x+4}=\frac{1}{3}\int\frac{dx}{x^2-\frac{2}{3}x+\frac{4}{3}}=\frac{1}{3}\int\frac{dx}{x^2-\frac{2}{3}x+(\frac{1}{3})^2+\frac{4}{3}-\frac{1}{9}}$$

$$=\frac{1}{3}\int\frac{dx}{(x-\frac{1}{3})^2-\frac{11}{9}}=\frac{1}{3}\int\frac{d(x-\frac{1}{3})}{\left(x-\frac{1}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}$$

$$=\frac{1}{\sqrt{11}}\tan^{-1}\frac{3x-1}{\sqrt{11}}+c.$$

146. $$\int\frac{dx}{\sqrt{2-3x-4x^2}}=\frac{1}{2}\int\frac{dx}{\sqrt{\frac{1}{2}-\frac{3}{4}x-x^2}}$$

$$=\frac{1}{2}\int\frac{dx}{\sqrt{\frac{1}{2}-(x^2+\frac{3}{4}x+\frac{9}{64})+\frac{9}{64}}}$$

$$=\frac{1}{2}\int\frac{d(x+\frac{3}{8})}{\sqrt{\left(\frac{\sqrt{41}}{8}\right)^2-\left(x+\frac{3}{8}\right)^2}}$$

$$=\frac{1}{2}\sin^{-1}\frac{8x+3}{\sqrt{41}}+c.$$

147. $$\int\frac{dx}{\sqrt{3x-x^2-2}}=\int\frac{dx}{\sqrt{\frac{1}{4}-(x-\frac{3}{2})^2}}=\int\frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{3}{2})^2}}$$

$$=\sin^{-1}\frac{x-\frac{3}{2}}{\frac{1}{2}}=\sin^{-1}(2x+3)+c.$$

148. $$\int\frac{dv}{v^2-6v+5}=\int\frac{dv}{v^2-6v+9-4}=\int\frac{dv}{(v-3)^2-2^2}$$

$$=\frac{1}{4}\log\frac{v-3-2}{v-3+2}+c=\frac{1}{4}\log\frac{v-5}{v-1}+c.$$

149. $$\int\frac{dy}{y^2+3y+1}=\int\frac{dy}{\left(y+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$=\frac{1}{\sqrt{5}}\log\frac{2y+3-\sqrt{5}}{2y+3+\sqrt{5}}+c.$$

150. $$\int\frac{dz}{2z^2-2z+1}=\int\frac{2\,dz}{(2z-1)^2+1}=\int\frac{d(2z-1)}{(2z-1)^2+1}$$

$$=\tan^{-1}(2z-1)+c.$$

151. $$\int\frac{ds}{\sqrt{2as+s^2}}=\int\frac{ds}{\sqrt{s^2+2as+a^2-a^2}}=\int\frac{ds}{\sqrt{(s+a)^2-a^2}}$$

$$=\log(s+a+\sqrt{s^2+2as})+c.$$

152. $$\int\frac{dx}{x\sqrt{c^2x^2-a^2b^2}}=\int\frac{d(cx)}{cx\sqrt{(cx)^2-(ab)^2}}=\frac{1}{ab}\sec^{-1}\frac{cx}{ab}+c.$$

153. $$\int\frac{3x^2dx}{\sqrt{x^3-9x^6}}=\int\frac{x^2dx}{\sqrt{\frac{x^3}{9}-x^6}}=\frac{1}{3}\int\frac{d(x^3)}{\sqrt{\frac{x^3}{9}-x^6}}$$

$$=\frac{1}{3}\operatorname{vers}^{-1}\frac{x^3}{\frac{1}{18}}+c.$$

[與公式 (22) 比較, 命 $2av=\frac{x^3}{9}$, $v^2=x^6$; 則 $v=x^3, dv=3x^2dx, a=\frac{1}{18}$.]

154. $$\int\frac{(b+ex)\,dx}{a^2+x^2}=\int\frac{b\,dx}{a^2+x^2}+\int\frac{ex\,dx}{a^2+x^2}=b\int\frac{dx}{a^2+x^2}+e\int\frac{x\,dx}{a^2+x^2}$$

$$=\frac{b}{a}\tan^{-1}\frac{x}{a}+\frac{e}{2}\int\frac{d(x^2+a^2)}{a^2+x^2}$$

$$=\frac{b}{a}\tan^{-1}\frac{x}{a}+\frac{e}{2}\log(x^2+a^2)+c.$$

155. $$\int\frac{(3x-1)\,dx}{x^2+9}=\int\frac{3x\,dx}{x^2+9}-\int\frac{dx}{x^2+9}=3\int\frac{x\,dx}{x^2+3^2}-\int\frac{dx}{x^2+3^2}$$

$$=\frac{3}{2}\int\frac{d(x^2+9)}{x^2+9}-\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$=\frac{3}{2}\log(x^2+9)-\frac{1}{3}\tan^{-1}\frac{x}{3}+c.$$

156. $\int \frac{x+3}{\sqrt{x^2+4}}dx = \int \frac{x\,dx}{\sqrt{x^2+2^2}} + \int \frac{3\,dx}{\sqrt{x^2+2^2}}$

$$= \frac{1}{2}\int (x^2+4)^{-\frac{1}{2}}\,d(x^2+4) + 3\int \frac{dx}{\sqrt{x^2+2^2}}$$

$$= (x^2+4)^{\frac{1}{2}} + 3\log(x+\sqrt{x^2+4}) + c.$$

157. $\int \frac{2x-5}{3x^2-2}dx = \int \frac{2x}{3x^2-2}dx - 5\int \frac{dx}{3x^2-2}$

$$= \frac{1}{3}\int \frac{d(3x^2-2)}{3x^2-2} - \frac{5}{3}\int \frac{dx}{x^2-\frac{2}{3}}$$

$$= \frac{1}{3}\log(3x^2-2) - \frac{5\sqrt{3}}{6\sqrt{2}}\log\frac{x-\sqrt{\frac{2}{3}}}{x+\sqrt{\frac{2}{3}}} + c.$$

158. $\int \frac{3s-2}{\sqrt{9-s^2}}ds = \int \frac{3s\,ds}{\sqrt{9-s^2}} - 2\int \frac{ds}{\sqrt{9-s^2}}$

$$= \frac{-3}{2}\int (9-s^2)^{-\frac{1}{2}}\,d(9-s^2) - 2\int \frac{ds}{\sqrt{9-s^2}}$$

$$= -\frac{2}{2}\cdot 2\sqrt{9-s^2} - 2\sin^{-1}\frac{s}{3} + c.$$

159. $\int \frac{5t-1}{\sqrt{3t^2-9}}dt = \int \frac{5t\,dt}{\sqrt{3t^2-9}} - \int \frac{dt}{\sqrt{3t^2-9}}$

$$= \frac{5}{6}\int \frac{d(3t^2-9)}{\sqrt{3t^2-9}} - \frac{1}{\sqrt{3}}\int \frac{d(\sqrt{3}t)}{\sqrt{(\sqrt{3}t)^2-3^2}}$$

$$= \frac{5}{3}\sqrt{3t^2-9} - \frac{1}{\sqrt{3}}\log(\sqrt{3}t+\sqrt{3t^2-9}) + c$$

160. $\int \frac{dx}{\sqrt{4-25x^2}} = \int \frac{dx}{\sqrt{2^2-(5x)^2}} = \frac{1}{5}\int \frac{d(5x)}{\sqrt{2^2-(5x)^2}}$

$$= \frac{1}{5}\sin\frac{5x}{2} + c.$$

161. $$\int \frac{a\,dx}{3-12x^2} = \frac{a}{3}\int \frac{dx}{1-4x^2} = \frac{a}{3\cdot 2}\int \frac{d(2x)}{1-(2x)^2}$$

$$= \frac{-a}{6}\int \frac{d(2x)}{(2x)^2-1} = -\frac{a}{6}\cdot\frac{1}{2}\log\frac{2x-1}{2x+1}+c.$$

162. $$\int \frac{2\,dt}{3t^2-5t+2} = \frac{2}{3}\int \frac{dt}{t^2-\frac{5}{3}t+\frac{2}{3}} = \frac{2}{3}\int \frac{dt}{(t-\frac{5}{6})^2-(\frac{1}{6})^2}$$

$$= \frac{2}{3}\cdot\frac{1}{2\times\frac{1}{6}}\log\frac{3t-3}{3t-2}+c = 2\log\frac{3t-3}{3t-2}+c.$$

163. $$\int \frac{dx}{x\sqrt{9x^2-4}} = \int \frac{3\,dx}{3x\sqrt{(3x)^2-2^2}} = \frac{1}{2}\sec^{-1}\frac{3x}{2}+c.$$

164. $$\int \frac{\sin\theta\,d\theta}{\sqrt{9-4\cos^2\theta}} = -\frac{1}{2}\int \frac{d(2\cos\theta)}{\sqrt{3^2-(2\cos\theta)^2}}$$

$$= -\frac{1}{2}\sin\frac{2\cos\theta}{3}+c.$$

165. $$\int \frac{(2x-3)\,dx}{x^2+4} = \int\frac{2x\,dx}{x^2+4} - \int\frac{3\,dx}{x^2+4} = \int\frac{d(x^2+4)}{x^2+4} - 3\int\frac{dx}{x^2+4}$$

$$= \log(x^2+4) - 3\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

$$= \log(x^2+4) - \frac{3}{2}\tan^{-1}\frac{x}{2}+c.$$

166. $$\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+2^2} = \frac{1}{2}\tan^{-1}\frac{x+3}{2}+c.$$

167. $$\int \frac{y+3}{\sqrt{9y^2+1}}\,dy = \int \frac{y\,dy}{\sqrt{9y^2+1}} + 3\int \frac{dy}{\sqrt{9y^2+1}}$$

$$= \frac{1}{18}\int \frac{d(9y^2+1)}{\sqrt{9y^2+1}} + \int \frac{d(3y)}{\sqrt{(3y)^2+1}}$$

$$= \frac{1}{9}(9y^2+1)^{\frac{1}{2}} + \log(3y+\sqrt{9y^2+1})+c.$$

168. $\int \frac{2\,dx}{\sqrt{25x^2-4}} = \frac{2}{5}\int \frac{d(5x)}{\sqrt{(5x)^2-2^2}} = \frac{2}{5}\log(5x+\sqrt{25x^2+4})+c.$

169. $\int \frac{b\,dy}{12y^2+3} = \frac{b}{3}\int \frac{dy}{4y^2+1} = \frac{b}{3}\tan 2y+c.$

170. $\int \frac{d\theta}{\sqrt{3\theta^2-2}} = \frac{1}{\sqrt{3}}\int \frac{d(\sqrt{3}\theta)}{\sqrt{(\sqrt{3}\theta)^2-2}}$

$$= \frac{1}{\sqrt{3}}\log(\sqrt{3}\theta+\sqrt{3\theta^2-2})+c.$$

171. $\int \frac{dx}{x\sqrt{9x^2-16}} = \int \frac{3\,dx}{3x\sqrt{(3x)^2-4^2}} = \frac{1}{4}\sec^{-1}\frac{3x}{4}+c.$

172. $\int \frac{dz}{z\sqrt{4-(\log z)^2}} = \int \frac{1}{\sqrt{4-(\log z)^2}}\cdot\frac{dz}{z}$

$$= \int \frac{d(\log z)}{\sqrt{2^2-(\log z)^2}} = \sin^{-1}\frac{\log z}{2}+c.$$

173. $\int \frac{(t+2)dt}{4t^2-3} = \int \frac{t\,dt}{4t^2-3} + 2\int \frac{dt}{4t^2-3}$

$$= \frac{1}{8}\int \frac{d(4t^2-3)}{4t^2-3} + \int \frac{d(2t)}{(2t)^2-(\sqrt{3})^2}$$

$$= \frac{1}{8}\log(4t^2-3) + \frac{1}{2\sqrt{3}}\log\frac{2t-\sqrt{3}}{2t+\sqrt{3}}+c.$$

174. $\int \frac{(ax-b)dx}{\sqrt{1+9x^2}} = \int \frac{ax\,dx}{\sqrt{1+9x^2}} - \int \frac{b\,dx}{\sqrt{1+9x^2}}$

$$= \frac{a}{18}\int (1+9x^2)^{-\frac{1}{2}}\,d(9x^2+1) - \frac{b}{3}\int \frac{d(3x)}{\sqrt{(3x)^2+1}}$$

$$= \frac{a}{9}(1+9x^2)^{\frac{1}{2}} - \frac{b}{3}\log(3x+\sqrt{9x^2+1})+c.$$

175. $\int \frac{dx}{\sqrt{8+4x-4x^2}} = \int \frac{dx}{\sqrt{9-(4x^2-4x+1)}}$

$$= \frac{1}{2}\int \frac{d(2x-1)}{\sqrt{3^2-(2x-1)^2}} = \frac{1}{2}\sin^{-1}\frac{2x-1}{3}+c.$$

176. $$\int \frac{3\,dx}{\sqrt{5}\,x^2+1} = \frac{3}{\sqrt{5}} \int \frac{d(\sqrt{5}\,x)}{\sqrt{(\sqrt{5}\,x)^2+1}}$$
$$= \frac{3}{\sqrt{5}} \log (\sqrt{5}x + \sqrt{5\,x^2+1}) + c.$$

177. $$\frac{dw}{12\,w^2-3} = \frac{1}{6} \int \frac{d\,(2\,w)}{(2\,w)^2-1} = \frac{1}{6} \cdot \frac{1}{2} \log \frac{2\,w-1}{2\,w+1} + c$$
$$= \frac{1}{12} \log \frac{2\,w-1}{2\,w+1} + c.$$

178. $$\int \frac{3\,d\theta}{\sqrt{2-3\,\theta^2}} = 3 \int \frac{d\theta}{\sqrt{(\sqrt{2})^2-(\sqrt{3}\,\theta)^2}}$$
$$= \frac{3}{\sqrt{3}} \int \frac{d\,(\sqrt{3}\,\theta)}{\sqrt{(\sqrt{2})^2-(\sqrt{3}\,\theta)^2}}$$
$$= \sqrt{3} \sin^{-1} \sqrt{\frac{3}{2}}\,\theta + c.$$

179. $$\int \frac{dt}{\sqrt{7\,t-4\,t^2+5}} = \frac{1}{2} \int \frac{2\,dt}{\sqrt{\frac{129}{16} - \left(2\,t-\frac{7}{4}\right)^2}}$$
$$= \frac{1}{2} \sin^{-1} \frac{8\,t-7}{\sqrt{129}} + c.$$

180. $$\int \frac{e^{ax}\,dx}{e^{2ax}+1} = \frac{1}{a} \int \frac{d\,(e^{ax})}{(e^{ax})^2+1} = \frac{1}{a} \tan^{-1} e^{ax} + c.$$

181. $$\int \frac{(3\,s-5)\,ds}{\sqrt{1-9\,s^2}} = \int \frac{3s\,ds}{\sqrt{1-9\,s^2}} - \int \frac{5\,ds}{\sqrt{1-9\,s^2}}$$
$$= -\frac{1}{6} \int \frac{d(1-9\,s^2)}{\sqrt{1-9\,s^2}} - \frac{5}{3} \int \frac{d\,(3s)}{\sqrt{1-9\,s^2}}$$
$$= \frac{-1}{3} \sqrt{1-9\,s^2} - \frac{5}{3} \sin^{-1} 3s + c.$$

182. $$\int \frac{dt}{\sqrt{t^2-4\,t+2}} = \int \frac{dt}{\sqrt{t^2-4\,t+4-2}} = \int \frac{dt}{\sqrt{(t-2)^2-(\sqrt{2})^2}}$$
$$= \log (t-2+\sqrt{t^2-4\,t+2}) + c.$$

第 三 章

三角函數之積分法

13. 三角微分 (trigonometric differentials). 三角微分之求積分公式,已略述於第二章標準式中,惟其形爲最簡易者.茲將數種常見之三角微分,簡化之,使趨於標準式之形,然後其積分隨卽可得.

14. 求$\int \sin^m x \cos^n x\, dx$.

式中m或n之任一爲正整數,且爲奇數.

若m爲奇數,則由三角公式$\sin^2\theta = 1-\cos^2\theta$之關係,上式終可化爲

(*A*) $\int$(僅含$\cos x$諸項)$\sin x\, dx$.

若n爲奇數,則由三角公式$\cos^2\theta = 1-\sin^2\theta$之關係,上式終可化爲

(*B*) $\int$(僅含$\sin x$諸項)$\cos x\, dx$.

於是(*A*),(*B*)兩式均與標準式(4)相符.若m與n均爲偶數,則另用第18款方法解之.

[例1]

$$\int \cos^4 x \sin^3 x\,dx = \int \cos^4 x \sin^2 x \sin x\,dx = \int \cos^4 x(1-\cos^2 x)\sin x\,dx$$

$$= -\int (\cos^4 x - \cos^6 x)d(\cos x)$$

$$= -\int \cos^4 x\,d(\cos x) + \int \cos^6 x\,d(\cos x)$$

$$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c.$$

[例 2]

$$\int \sin^{\frac{3}{7}}\theta \cos^3\theta\ d\theta = \int \sin^{\frac{3}{7}}\theta(1-\sin^2\theta)\cos\theta\,d\theta$$

$$= \int (\sin^{\frac{3}{7}}\theta - \sin^{\frac{17}{7}}\theta)d(\sin\theta)$$

$$= \int \sin^{\frac{3}{7}}\theta\,d(\sin\theta) - \int \sin^{\frac{17}{7}}\theta\,d(\sin\theta)$$

$$= \frac{(\sin\theta)^{\frac{3}{7}+1}}{\frac{3}{7}+1} - \frac{(\sin\theta)^{\frac{17}{7}+1}}{\frac{17}{7}+1} + c$$

$$= \frac{7}{10}\sin^{\frac{10}{7}}\theta - \frac{7}{24}\sin^{\frac{24}{7}}\theta + c.$$

15. 求 $\int \tan^n x\,dx$, 或 $\int \cot^n x\,dx$.

式中 n 爲一整數.

先將一部分因子,用三角公式 $\tan^2\theta = \sec^2\theta - 1$, $\cot^2\theta = \csc^2\theta - 1$ 變化之,而後引用標準公式.

[例 1]

$$\int \tan^3 x\,dx = \int \tan x(\sec^2 x - 1)dx = \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$= \int \tan x\, d(\tan x) - \int \tan x\, dx = \frac{\tan^2 x}{2} + \log(\cos x) + c.$$

[例 2]

$$\int \cot^3 x\, dx = \int \cot x(\csc^2 x - 1)dx = \int \cot x \csc^2 x\, dx - \int \cot x\, dx$$

$$= -\int \cot x\, d(\cot x) - \int \cot x\, dx = -\frac{\cot^2 x}{2} - \log(\sin x) + c.$$

16. 求 $\int \sec^n x\, dx$, 或 $\int \csc^n x\, dx$.

若 n 爲正整數且爲偶數,則用三角公式 $\sec^2\theta = 1 + \tan^2\theta$, $\csc^2\theta = 1 + \cot^2\theta$ 變化之,然後引用標準公式(如下二例).

若 n 爲大於 1 之奇數,則先化爲 $\cos x$ 及 $\sin x$, 而後用第 19 款公式解之.

[例 1]

$$\int \sec^4 x\, dx = \int \sec^2 x(1 + \tan^2 x)dx = \int (1 + \tan^2 x)d(\tan x)$$

$$= \int d(\tan x) + \int \tan^2 x\, d(\tan x) = \tan x + \frac{\tan^3 x}{3} + c.$$

[例 2]

$$\int \csc^4 x\, dx = \int \csc^2 x(1 + \cot^2 x)dx = \int \csc^2 x\, dx + \int \cot^2 x \csc^2 x\, dx$$

$$= -\cot x - \frac{\cot^3 x}{3} + c.$$

17. 求 $\int \tan^m x \sec^n x\, dx$, 或 $\int \cot^m x \csc^n x\, dx$.

若 n 爲正整偶數,可依 16 款解之. 例如

$$\int \tan^6 x \sec^4 x\,dx = \int \tan^6 x(\tan^2 x+1)\sec^2 x\,dx$$

$$= \int \tan^8 x \sec^2 x + \int \tan^6 x \sec^2 x\,dx$$

$$= \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + c.$$

若 m 爲奇數,則依下例解之:

$$\int \tan^5 x \sec^3 x\,dx = \int \tan^4 x \sec^2 x \tan x \sec x\,dx$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \tan x \sec x\,dx$$

$$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x)\sec x \tan x\,dx$$

$$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + c.$$

18. 用倍角三角函數求 $\int \sin^m x \cos^n x\,dx$.

若 m 或 n 之任一爲正整奇數時,其最簡捷之法,已詳於14款.若 m 與 n 俱爲偶數,則改變上式之 $\sin x$ 與 $\cos x$ 爲倍角三角函數.普通用下列三式:

(1) $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$;

(2) $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$;

(3) $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

[例1]

$$\int \sin^2 x \cos^2 x\,dx = \int (\sin x \cos x)^2 dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x\,dx = \frac{x}{8} - \frac{1}{32}\sin 4x + c.$$

[例 2]

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 dx$$

$$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx$$

$$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\right) dx$$

$$= \int \frac{3}{8} dx + \frac{1}{2}\int \cos 2x\,dx + \frac{1}{8}\int \cos 4x\,dx$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$$

19. 求 $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$ 及

$$\int \cos mx \cos nx\,dx.$$ (但 $m \neq n$).

由三角公式
$$\begin{cases} (1)\ \sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B), \\ (2)\ \cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B), \\ (3)\ \cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B). \end{cases}$$

$$\text{即}\begin{cases}(4)\ \sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)=\frac{1}{2}\sin A+\frac{1}{2}\sin B,\\(5)\ \sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)=-\frac{1}{2}\cos A+\frac{1}{2}\cos B,\\(6)\ \cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)=\frac{1}{2}\cos A+\frac{1}{2}\cos B.\end{cases}$$

命 $\frac{1}{2}(A+B)=mx$, $\frac{1}{2}(A-B)=nx$;

即 $A+B=2mx$, $A-B=2nx$.

解之, $\therefore\ A=(m+n)x$, $B=(m-n)x$.

代入(4), (5), (6)三式,

$$\text{得}\begin{cases}(7)\ \sin mx\cos nx=\frac{1}{2}\sin(m+n)x+\frac{1}{2}\sin(m-n)x,\\(8)\ \sin mx\sin nx=-\frac{1}{2}\cos(m+n)x+\frac{1}{2}\cos(m-n)x,\\(9)\ \cos mx\cos nx=\frac{1}{2}\cos(m+n)x+\frac{1}{2}\cos(m-n)x.\end{cases}$$

將(7), (8), (9)各式兩邊求積分,即得下列三公式:

$$(A)\quad\int\sin mx\cos nx\,dx=-\frac{\cos(m+n)x}{2(m+n)}-\frac{\cos(m-n)x}{2(m-n)}+c.$$

$$(B)\quad\int\sin mx\sin nx\,dx=-\frac{\sin(m+n)x}{2(m+n)}+\frac{\sin(m-n)x}{2(m-n)}+c.$$

$$(C)\quad\int\cos mx\cos nx\,dx=\frac{\sin(m+n)x}{2(m+n)}+\frac{\sin(m-n)x}{2(m-n)}+c.$$

20. 本章例題.

第十四款例題

183. $\int\sin^3 x\,dx=\int(1-\cos^2 x)\sin x\,dx$

$$= \int \sin x\,dx - \int \cos^2 x \sin x\,dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c.$$

184. $$\int \sin^2 x \cos x\,dx = \int \sin^2 x\,d(\sin x) = \frac{\sin^3 x}{3} + c.$$

185. $$\int \sin x \cos x\,dx = \int \sin x\,d(\sin x) = \frac{\sin^2 x}{2} + c.$$

186. $$\int \cos^4 \theta \sin \theta\,d\theta = -\int \cos^2 \theta(-\sin \theta\,d\theta)$$

$$= -\int \cos^2\theta\,d(\cos \theta) = -\frac{\cos^3 \theta}{3} + c.$$

187. $$\int \sin^3 6\theta \cos 6\theta\,d\theta = \frac{1}{6}\int \sin^3 6\theta\,d(\sin 6\theta)$$

$$= \frac{1}{6} \cdot \frac{\sin^4 6\theta}{4} + c = \frac{1}{24}\sin^4 6\theta + c.$$

188. $$\int \cos^3 2\theta \sin 2\theta\,d\theta = -\int \cos^3 2\theta(-\sin 2\theta)d\theta$$

$$= -\frac{1}{2}\int \cos^3 2\theta\,d(\cos 2\theta)$$

$$= \frac{-\cos^4 2\theta}{8} + c.$$

189. $$\int \frac{\cos^3 x\,dx}{\sin^4 x} = \int \cot^3 x \csc x\,dx = \int (\csc^2 x - 1)\cot x \csc x\,dx$$

$$= \int \csc^3 x \cot x\,dx - \int \cot x \csc x\,dx$$

$$= -\int \csc^2 x(-\csc x \cot x)dx - \int \cot x \csc x\,dx$$

$$= -\int \csc^2 x\, d(\csc x) - \int \cot x \csc x\, dx$$

$$= -\frac{\csc^3 x}{3} + \csc x + c.$$

190. $$\int \frac{\sin^3\theta\, d\theta}{\cos^2\theta} = \int (\cos\theta)^{-2} \sin^3\theta\, d\theta$$

$$= \int (\cos\theta)^{-2}(1-\cos^2\theta)\sin\theta\, d\theta$$

$$= \int (\cos\theta)^{-2} \sin\theta\, d\theta - \int \sin\theta\, d\theta$$

$$= -\int (\cos\theta)^{-2}\, d(\cos\theta) - \int \sin\theta\, d\theta$$

$$= -\frac{(\cos\theta)^{-1}}{-1} - (-\cos\theta) + c = \sec\theta + \cos\theta + c.$$

191. $$\int \sin^5 x\, dx = \int (1-\cos^2 x)^2 \sin x\, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\, dx$$

$$= \int \sin x\, dx - 2\int \cos^2 x \sin x\, dx$$

$$+ \int \cos^4 x \sin x\, dx$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c.$$

192. $$\int \cos^5 x\, dx = \int (1-\sin^2 x)^2 \cos x\, dx$$

$$= \int (1 - 2\sin^2 x + \sin^4 x)\cos x\, dx$$

$$= \int \cos x\, dx - 2\int \sin^2 x \cos x\, dx$$

$$+\int \sin^4 x \cos x\, dx$$

$$=\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c.$$

193. $\int \sin^{\frac{2}{3}}\theta \cos^5\theta\, d\theta=\int \sin^{\frac{2}{3}}\theta(1-\sin^2\theta)^2\cos\theta\, d\theta$

$$=\int \sin^{\frac{2}{3}}\theta(1-2\sin^2\theta+\sin^4\theta)\cos\theta\, d\theta$$

$$=\int (\sin^{\frac{2}{3}}\theta-\sin^{\frac{8}{3}}\theta+\sin^{\frac{14}{3}}\theta)\,d(\sin\theta)$$

$$=\frac{3}{5}\sin^{\frac{5}{3}}\theta-\frac{6}{11}\sin^{\frac{11}{3}}\theta+\frac{3}{17}\sin^{\frac{17}{3}}\theta+c.$$

194. $\int \frac{\sin^5 y}{\sqrt{\cos y}}dy=\int (\cos^{-\frac{1}{2}}y-2\cos^{\frac{3}{2}}y+\cos^{\frac{7}{2}}y)\sin y\, dy$

$$=-2\sqrt{\cos y}+\frac{4}{5}\cos^{\frac{5}{2}}y-\frac{2}{9}\cos^{\frac{9}{2}}y+c.$$

195. $\int \frac{\cos^5 t\, dt}{\sqrt[3]{\sin t}}=\int \sin^{-\frac{1}{3}}t(1-\sin^2 t)^2\cos t\, dt$

$$=\int \sin^{-\frac{1}{3}}t(1-2\sin^2 t+\sin^4 t)\cos t\, dt$$

$$=\int (\sin^{-\frac{1}{3}}t-2\sin^{\frac{5}{3}}t+\sin^{\frac{11}{3}}t)\,d(\sin t)$$

$$=\frac{3}{2}\sin^{\frac{2}{3}}t-\frac{3}{4}\sin^{\frac{8}{3}}t+\frac{3}{14}\sin^{\frac{14}{3}}t+c.$$

196. $\int \sin^3 2\theta\, d\theta=\int (1-\cos^2 2\theta)\sin 2\theta\, d\theta$

$$=\int \sin 2\theta\, d\theta-\int \cos^2 2\theta \sin 2\theta\, d\theta$$

$$=-\frac{1}{2}\cos 2\theta+\frac{1}{6}\cos^3 2\theta+c.$$

197. $$\int \cos^3\frac{\theta}{2}\,d\theta = \int\left(1-\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\,d\theta$$

$$= 2\int\left(1-\sin^2\frac{\theta}{2}\right)d\left(\sin\frac{\theta}{2}\right)$$

$$= 2\sin\frac{\theta}{2}-\frac{2}{3}\sin^3\frac{\theta}{2}+c.$$

198. $$\int \sin 2\theta\cos 2\theta\,d\theta = \frac{1}{2}\int \sin 2\theta\, d(\sin 2\theta)$$

$$= \frac{1}{4}\sin^2 2\theta + c.$$

199. $$\int \sin^3 t\cos^3 t\,dt = \int \sin^3 t\,(1-\sin^2 t)\cos t\,dt$$

$$= \int (\sin^3 t - \sin^5 t)\,d(\sin t)$$

$$= \int \sin^3 t\, d(\sin t) - \int \sin^5 t\, d(\sin t)$$

$$= \frac{1}{4}\sin^4 t - \frac{1}{6}\sin^6 t + c.$$

200. $$\int \cos\frac{x}{a}\sin\frac{x}{a}dx = a\int \sin\frac{x}{a}\,d\left(\sin\frac{x}{a}\right) = a\cdot\frac{\left(\sin\frac{x}{a}\right)^2}{2}+c$$

$$= \frac{a}{2}\sin^2\frac{x}{a}+c.$$

201. $$\int \cos^5 ax\sin ax\,dx = -\frac{1}{a}\int \cos^5 ax\, d(\cos ax)$$

$$= -\frac{1}{6a}\cos^6 ax + c.$$

202. $$\int \sin^2\frac{2x}{3}\cos\frac{2x}{3}\,dx = \frac{3}{2}\int \sin^2\frac{2x}{3}\,d\left(\sin\frac{2x}{3}\right)$$

$$= \frac{3}{2} \cdot \frac{\sin^3 \frac{2x}{3}}{3} + c = \frac{1}{2} \sin^3 \frac{2x}{3} + c.$$

203. $$\int \cos^2 3x \sin 3x \, dx = \frac{-1}{3} \int \cos^2 3x \, d(\cos 3x)$$

$$= -\frac{1}{3} \frac{\cos^3 3x}{3} + c = -\frac{\cos^3 3x}{9} + c.$$

204. $$\int \cos^3 \frac{\phi}{2} \sin^2 \frac{\phi}{2} \, d\phi = \int \sin^2 \frac{\phi}{2} \left(1 - \sin^2 \frac{\phi}{2}\right) \cos \frac{\phi}{2} \, d\phi$$

$$= \int \left(\sin^2 \frac{\phi}{2} - \sin^4 \frac{\phi}{2}\right) \cos \frac{\phi}{2} \, d\phi$$

$$= \frac{2}{3} \sin^3 \frac{\phi}{2} - \frac{2}{5} \sin^5 \frac{\phi}{2} + c.$$

205. $$\int \sin^3 mt \cos^2 mt \, dt = \int \cos^2 mt \, (1 - \cos^2 mt) \sin mt \, dt$$

$$= \int \cos^2 mt \sin mt \, dt$$

$$- \int \cos^4 mt \sin mt \, dt$$

$$= -\frac{1}{m} \int \cos^2 mt \, d(\cos mt)$$

$$+ \frac{1}{m} \int \cos^4 mt \, d(\cos mt)$$

$$= -\frac{\cos^3 mt}{3m} + \frac{\cos^5 mt}{5m} + c.$$

206. $$\int \sin^5 as \cos as \, ds = \frac{1}{a} \int \sin^5 as \, d(\sin as)$$

$$= \frac{1}{a} \cdot \frac{\sin^6 as}{6} + c = \frac{1}{6a} \sin^6 as + c.$$

207. $\int \sin^5 nt\,dt = \int (1-\cos^2 nt)^2 \sin nt\,dt$

$$= \int (1-2\cos^2 nt+\cos^4 nt)\sin nt\,dt$$

$$= -\frac{\cos nt}{n}+\frac{2\cos^3 nt}{3n}-\frac{\cos^5 nt}{5n}+c.$$

208. $\int \sin^4 x\cos x\,dx = \int \sin^4 x\,d(\sin x) = \frac{1}{5}\sin^5 x+c.$

209. $\int \cos^4 y\sin y\,dy = -\int \cos^4 y\,d(\cos y) = -\frac{1}{5}\cos^5 y+c.$

210. $\int \cos^3(a+bt)\,dt = \int [1-\sin^2(a+bt)]\cos(a+bt)dt$

$$= \frac{1}{b}\int d\sin(a+bt)$$

$$-\frac{1}{b}\int \sin^2(a+bt)\,d\sin(a+bt)$$

$$= \frac{\sin(a+bt)}{b}-\frac{\sin^3(a+bt)}{3b}+c.$$

第十五,十六,十七款例題

211. $\int \tan^4 x\,dx = \int \tan^2 x(\sec^2 x-1)dx$

$$= \int \tan^2 x\sec^2 x\,dx-\int \tan^2 x\,dx$$

$$= \int \tan^2 x\,d(\tan)x-\int (\sec^2 x-1)dx$$

$$= \frac{\tan^3 x}{3}-\tan x+x+c.$$

212. $\int \tan^2 2x\,dx = \int (\sec^2 2x-1)\,dx = \int \sec^2 2x\,dx-\int dx$

$$=\frac{\tan 2x}{2}-x+c.$$

213. $$\int \sec^6 x\,dx=\int(\tan^2 x+1)^2\sec^2 x\,dx$$

$$=\int\tan^4 x\sec^2 x\,dx+2\int\tan^2 x\sec^2 x\,dx$$

$$+\int\sec^2 x\,dx$$

$$=\frac{\tan^5 x}{5}+2\frac{\tan^3 x}{3}+\tan x+c.$$

214. $$\int\cot^2 x\,dx=\int\csc^2 x\,dx-\int dx=-\cot x-x+c.$$

215. $$\int\cot^4\frac{x}{3}dx=\int\cot^2\frac{x}{3}(\csc^2\frac{x}{3}-1)dx$$

$$=\int\cot^2\frac{x}{3}\csc^2\frac{x}{3}dx-\int\cot^2\frac{x}{3}dx$$

$$=-3\int\cot^2\frac{x}{3}\,d\left(\cot\frac{x}{3}\right)-\int\csc^2\frac{x}{3}dx+\int dx$$

$$=-\cot^3\frac{x}{3}+3\cot\frac{x}{3}+x+c.$$

216. $$\int\cot^5\theta\,d\theta=\int\cot^3\theta(\csc^2\theta-1)d\theta$$

$$=\int\cot^3\theta\csc^2\theta\,d\theta-\int\cot^3\theta\,d\theta$$

$$=-\int\cot^3\theta\,d(\cot\theta)-\int\cot\theta(\csc^2\theta-1)d\theta$$

$$=-\int\cot^3\theta\,d(\cot\theta)-\int\cot\theta\csc^2\theta\,d\theta$$

$$+\int\cot\theta\,d\theta$$

$$= -\frac{\cot^4\theta}{4}+\frac{\cot^2\theta}{2}+\log(\sin\theta)+c.$$

217. $$\int \sec^6 x\,dx = \int (\tan^2 x+1)^2 \sec^2 x\,dx$$

$$= \int (\tan^6 x+3\tan^4 x+3\tan^2 x+1)\,d(\tan x)$$

$$= \frac{\tan^7 x}{7}+\frac{3\tan^5 x}{5}+\tan^3 x+\tan x+c.$$

218. $$\int \csc^6 x\,dx = \int (\cot^2 x+1)^2 \csc^2 x\,dx$$

$$= \int (\cot^4 x+2\cot^2 x+1)\csc^2 x\,dx$$

$$= \int \cot^4 x \csc^2 x\,dx+2\int \cot^2 x \csc^2 x\,dx$$

$$+\int \csc^2 x\,dx$$

$$= -\frac{\cot^5 x}{5}-\frac{2}{3}\cot^3 x-\cot x+c.$$

219. $$\int \tan^4\theta \sec^4\theta\,d\theta = \int \tan^4\theta(\tan^2\theta+1)\sec^2\theta\,d\theta$$

$$= \int \tan^6\theta \sec^2\theta\,d\theta+\int \tan^4\theta \sec^2\theta\,d\theta$$

$$= \frac{1}{7}\tan^7\theta+\frac{1}{5}\tan^5\theta+c.$$

220 $$\int \tan^3\theta \sec^5\theta\,d\theta = \int \tan^2\theta \sec^4\theta\cdot\tan\theta\sec\theta\,d\theta$$

$$= \int (\sec^2\theta-1)\sec^4\theta\cdot\tan\theta\sec\theta\,d\theta$$

$$= \int \sec^6\theta\, d(\sec\theta) - \int \sec^4\theta\, d(\sec\theta)$$

$$= \frac{1}{7}\sec^7\theta - \frac{1}{5}\sec^5\theta + c.$$

221. $$\int \cot^5 x \csc^4 x\, dx = \int \cot^5 x(\cot^2 x + 1)\csc^2 x\, dx$$

$$= \int \cot^7 x \csc^2 x\, dx + \int \cot^5 x \csc^2 x\, dx$$

$$= -\int \cot^7 x\, d(\cot x) - \int \cot^5 x\, d(\cot x)$$

$$= -\frac{1}{8}\cot^8 x - \frac{1}{6}\cot^6 x + c.$$

222. $$\int \tan^{\frac{3}{2}} x \sec^4 x\, dx = \int (\tan^{\frac{3}{2}} x + \tan^{\frac{7}{2}} x)\, d(\tan x)$$

$$= \frac{2}{5}\tan^{\frac{5}{2}} x + \frac{2}{9}\tan^{\frac{9}{2}} x + c.$$

223. $$\int \tan^5 y \sec^{\frac{5}{2}} y\, dy = \int (\sec^2 y - 1)^2 \sec^{\frac{1}{2}} y \cdot \tan y \sec y\, dy$$

$$= \int (\sec^4 y - 2\sec^2 y + 1)\sec^{\frac{1}{2}} y\, d(\sec y)$$

$$= \int (\sec^{\frac{9}{2}} y - 2\sec^{\frac{5}{2}} y + \sec^{\frac{1}{2}} y)\, d(\sec y)$$

$$= \frac{2}{11}\sec^{\frac{11}{2}} y - \frac{4}{7}\sec^{\frac{7}{2}} y + \frac{2}{3}\sec^{\frac{3}{2}} y + c$$

$$= 2\sec^{\frac{3}{2}} y\left(\frac{\sec^4 y}{11} - \frac{2\sec^2 y}{7} + \frac{1}{3}\right) + c.$$

221. $$\int \frac{\sec^5\theta\, d\theta}{\tan^4\theta} = \int \tan^{-4}\theta\,(\tan^2\theta + 1)^2 \sec^2\theta\, d\theta$$

$$= \int (\tan^{-4}\theta - 2\tan^{-2}\theta + 1)\, d(\tan\theta)$$

$$= \tan\theta - 2\cot\theta - \frac{1}{3}\cot^3\theta + c.$$

225. $$\int (\tan^2 z + \tan^4 z)\, dz = \int (\tan^2 z \sec^2 z)\, dz$$

$$= \int \tan^2 z\, d(\tan z) = \frac{\tan^3 z}{3} + c.$$

226. $$\int (\tan t + \cot t)^3 dt$$

$$= \int (\tan^3 t + 3\tan^2 t \cot t + 3\tan t \cot^2 t + \cot^3 t)\, dt$$

$$= \int \tan t \sec^2 t\, dt + 2\int \tan t\, dt + \int \cot t \csc^2 t\, dt$$

$$+ 2\int \cot t\, dt = \frac{\tan^2 t}{2} + 2\log(\sec t) - \frac{\cot^2 t}{2} + 2\log(\sin t) + c$$

$$= \frac{1}{2}(\tan^2 t - \cot^2 t) + \log\tan^2 t + c.$$

227. $$\int \tan^2 2t\, dt = \int (\sec^2 2t - 1)\, dt = \int \sec^2 2t\, dt - \int dt$$

$$= \frac{1}{2}\tan 2t - t + c.$$

228. $$\int \cot^2\frac{t}{2}\, dt = \int (\csc^2\frac{t}{2} - 1)\, dt = 2\int \csc^2\frac{t}{2}\, d\left(\frac{t}{2}\right) - \int dt$$

$$= -2\cot\frac{t}{2} - t + c.$$

229. $$\int \tan^3 ax\, dx = \int \tan ax(\sec^2 ax - 1)\, dx$$

$$= \int \tan ax \sec^2 ax\, dx - \int \tan ax\, dx$$

$$= \frac{1}{a}\int \tan ax\, d(\tan ax) - \frac{1}{a}\int \tan ax\, d(ax)$$

$$= \frac{1}{a}\cdot\frac{\tan^2 ax}{2} - \frac{1}{a}\cdot \log(\sec ax) + c$$

$$= \frac{\tan^2 ax}{2a} + \frac{\log(\cos ax)}{a} + c.$$

230. $$\int \cot^3 \frac{x}{a}\, dx = \int \cot \frac{x}{a}\left(\csc^2 \frac{x}{a} - 1\right) dx$$

$$= \int \cot \frac{x}{a} \csc^2 \frac{x}{a}\, dx - \int \cot \frac{x}{a}\, dx$$

$$= -a\int \cot \frac{x}{a}\, d\left(\cot \frac{x}{a}\right) - a\int \cot \frac{x}{a}\, d\left(\frac{x}{a}\right)$$

$$= -\frac{a}{2}\cot^2 \frac{x}{a} - a\log\left(\sin \frac{x}{a}\right) + c.$$

231. $$\int \frac{2\,dt}{\tan^4 t} = \int 2\cot^4 t\, dt = 2\int \cot^2 t(\csc^2 t - 1)\, dt$$

$$= 2\int \cot^2 t \csc^2 t\, dt - 2\int \cot^2 t\, dt$$

$$= -2\int \cot^2 t\, d(\cot t) - 2\int (\csc^2 t - 1)\, dt$$

$$= -\frac{2}{3}\cot^3 t + 2\cot t + 2t + c.$$

232. $$\int \frac{3\,d\theta}{\cot^2 4\theta} = \int 3\tan^2 4\theta\, d\theta = 3\int (\sec^2 4\theta - 1)\, d\theta$$

$$= \frac{3}{4}\int \sec^2 4\theta\, d(4\theta) - 3\int d\theta$$

$$= \frac{3}{4}\tan 4\theta - 3\theta + c.$$

233. $\int \sec^2\theta \tan^2\theta\, d\theta = \int \tan^2\theta\, d(\tan\theta) = \frac{1}{3}\tan^3\theta + c.$

234. $\int \csc^2\theta \cot^2\theta\, d\theta = -\int \cot^2\theta\, d(\cot\theta) = -\frac{1}{3}\cot^3\theta + c.$

235. $\int \frac{a\,dx}{\tan^2 4x} = \int a\cot^2 4x\,dx = a\int (\csc^2 4x - 1)\,dx$

$$= a\int \csc^2 4x\,dx - a\int dx = -\frac{a}{4}\cot 4x - ax + c.$$

236. $\int \tan^3 t \sec^2 t\,dt = \int \tan^3 t\, d(\tan t) = \frac{1}{4}\tan^4 t + c.$

237. $\int \cot^5 y \csc^2 y\,dy = -\int \cot^5 y\, d(\cot y) = -\frac{1}{6}\cot^6 y + c.$

238. $\int \frac{b\,d\theta}{\cot^3\theta} = \int b\tan^3\theta\,d\theta = b\int \tan\theta\,(\sec^2\theta - 1)\,d\theta$

$$= b\int \tan\theta \sec^2\theta\,d\theta - b\int \tan\theta\,d\theta$$

$$= \frac{b\tan^2\theta}{2} - b\log(\sec\theta) + c.$$

239. $\frac{\sqrt{\tan x}}{\cos^2 x}dx = \int \tan^{\frac{1}{2}} x \sec^3 x\,dx$

$$= \int \tan^{\frac{1}{2}} x\, d(\tan x) = \frac{2}{3}\tan^{\frac{3}{2}} x + c.$$

240. $\int \frac{\cot^{\frac{2}{3}} x}{\sin^2 x}dx = \int \cot^{\frac{2}{3}} x \csc^2 x\,dx$

$$= -\int \cot^{\frac{2}{3}} x\, d(\cot x) = -\frac{3}{5}\cot^{\frac{5}{3}} x + c.$$

241. $\int \tan x \sec^2 x\, dx = \int \tan x\, d(\tan x) = \frac{\tan^2 x}{2} + c.$

242. $\int \cot x \csc^2 x\, dx = -\int \cot x\, d(\cot x) = -\frac{1}{2}\cot^2 x + c.$

第十八,十九款例題

243. $$\int \cos^2 x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \int \frac{1}{2}\, dx + \int \frac{1}{2}\cos 2x\, dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c.$$

244. $$\int \sin^4 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$$

245. $$\int \sin^6 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^3 dx$$

$$= \int \left(\frac{1}{8} - \frac{3}{8}\cos 2x + \frac{3}{8}\cos^2 2x - \frac{1}{8}\cos^3 2x\right) dx$$

$$= \int \left(\frac{1}{8} - \frac{3}{8}\cos 2x + \frac{3}{16} + \frac{3}{16}\cos 4x\right.$$

$$\left. - \frac{1}{8}\cos 2x + \frac{1}{8}\cos 2x \sin^2 2x\right) dx$$

$$= \int \frac{5}{16} dx - \frac{1}{2} \int \cos 2x \, dx + \frac{3}{16} \int \cos 4x \, dx$$

$$+ \frac{1}{16} \int \sin^2 2x \, d(\sin 2x)$$

$$= \frac{5}{16} x - \frac{1}{4} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{48} \sin^3 2x + c.$$

246. $\int \cos^6 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right)^3 dx$

$$= \int \left(\frac{1}{8} + \frac{3}{8} \cos 2x + \frac{3}{8} \cos^2 2x + \frac{1}{8} \cos^3 2x\right) dx$$

$$= \int \left(\frac{1}{8} + \frac{3}{8} \cos 2x + \frac{3}{16} + \frac{3}{16} \cos 4x\right.$$

$$\left. + \frac{1}{8} \cos 2x - \frac{1}{8} \sin^2 2x \cos 2x\right) dx$$

$$= \frac{5}{16} \int dx + \frac{1}{2} \int \cos 2x \, dx$$

$$+ \frac{3}{16} \int \cos 4x \, dx - \frac{1}{8} \int \sin^2 2x \cos 2x \, dx$$

$$= \frac{5}{16} x + \frac{1}{4} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{16} \cdot \frac{1}{3} \sin^3 2x + c$$

$$= \frac{1}{16} (5x + 4 \sin 2x + \frac{3}{4} \sin 4x - \frac{1}{3} \sin^3 2x) + c.$$

247. $\int \sin^4 \theta \cos^2 \theta \, d\theta = \frac{1}{4} \int \sin^2 2\theta \sin^2 \theta \, d\theta$

$$= \frac{1}{4} \int \sin^2 2\theta \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) d\theta$$

$$= \frac{1}{8} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4\theta - \sin^2 2\theta \cos 2\theta\right) d\theta$$

$$=\frac{\theta}{16}-\frac{1}{64}\sin 4\theta-\frac{1}{48}\sin^3 2\theta+c.$$

248. $$\int \sin^4 t\cos^4 t\,dt=\int(\sin t\cos t)^4 dt=\int\left(\frac{1}{2}\sin 2t\right)^4 dt$$

$$=\frac{1}{16}\int(\sin^2 2t)^2\,dt=\frac{1}{16}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4t\right)^2 dt$$

$$=\frac{1}{16}\int\left(\frac{1}{4}-\frac{1}{2}\cos 4t+\frac{1}{4}\cos^2 4t\right)dt$$

$$=\frac{1}{16}\int\left(\frac{1}{4}-\frac{1}{2}\cos 4t+\frac{1}{8}+\frac{1}{8}\cos 8t\right)dt$$

$$=\frac{1}{16}\int\frac{3}{8}dt-\frac{1}{16}\int\frac{1}{2}\cos 4t\,dt$$

$$+\frac{1}{16}\int\cos 8t\,dt$$

$$=\frac{3}{128}t-\frac{1}{128}\sin 4t+\frac{1}{128}\sin 8t+c$$

$$=\frac{1}{128}(t-\sin 4t+\sin 8t)+c.$$

249. $$\int\cos^6\theta\sin^2\theta\,d\theta=\int\frac{1}{4}\sin^2 2\theta\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)^2 d\theta$$

$$=\frac{1}{16}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4\theta\right)d\theta$$

$$+\frac{1}{16}\int\sin^2 2\theta\,d(\sin 2\theta)$$

$$+\frac{1}{64}\int\left(\frac{1}{2}-\frac{1}{2}\cos 8\theta\right)d\theta$$

$$=\frac{5}{128}\int d\theta-\frac{1}{128}\int\cos 4\theta\,d(4\theta)$$

$$+\frac{1}{16}\int \sin^2 2\theta\, d(\sin 2\theta)$$

$$-\frac{1}{128}\int \frac{1}{8}\cos 8\theta\, d(8\theta)$$

$$=\frac{1}{128}\left(5\theta-\sin 4\theta+\frac{8\sin^3 2\theta}{3}-\frac{\sin 8\theta}{7}\right)+c.$$

250. $$\int \cos 3y \sin 5y\, dy=-\frac{\cos(5+3)y}{2(5+3)}-\frac{\cos(5-3)y}{2(5-3)}+c$$

$$=-\frac{\cos 8y}{16}-\frac{\cos 2y}{4}+c.$$

[與19款公式(*A*)比較, $m=5, n=3$.]

251. $$\int \sin 6z \sin 5z\, dz=-\frac{\sin(6+5)z}{2(6+5)}+\frac{\sin(6-5)z}{2(6-5)}+c$$

$$=-\frac{\sin 11z}{22}+\frac{\sin z}{2}+c.$$

[應用19款公式(*B*), 命 $m=6, n=5$.]

252. $$\int \cos 7s \cos 4s\, ds=\frac{\sin(7+4)s}{2(7+4)}+\frac{\sin(7-4)s}{2(7-4)}+c$$

$$=\frac{\sin 11s}{22}+\frac{\sin 3s}{6}+c.$$

[應用19款公式(*C*), 命 $m=7, n=4$.]

第 四 章

偏分積分法

21. 有理分數. 若分數式之分子分母均爲有理整函數時,(卽式中之變數旣不含分指數,又不含負指數);則名此分數式爲有理分數 (rational fraction). 普通假定有理分數之分子,其次數必較分母爲低;若較分母爲高,先實行除法;卽

$$\frac{f(x)}{\phi(x)}=Q(x)+\frac{R(x)}{\phi(x)},$$

式中 $Q(x)$ 爲商, $R(x)$ 爲剩餘, $R(x)$ 之次數當然必低於 $\phi(x)$.

有理分數之分母,若能析成數個實數質因子,則此分數恆能分成數個偏分數,此事已證明於代數學中. 若微分式中含有此種分數而欲求其積分,先將此有理分數化成數個偏分數之和後,再引用標準公式,卽能求得其積分. 此法名曰偏分積分法 (integration by partial fractions).

22. 分母之因子均爲一次式且無重複者. 若分母有一因子 $x-a$, 卽可假定與此因子相對應之偏分數爲 $\frac{A}{x-a}$. 於是求 $\frac{A}{x-a}dx$ 之積分如下:

(A) $\int \frac{A}{x-a}dx = A\int \frac{dx}{x-a} = A\log(x-a)+c.$

[應用標準式(2),(5).]

[舉例] 求 $\int \frac{x^4\,dx}{(x^2-1)(x+2)}$.

[解] 因分子之次數較分母爲高,先行除法.

(1) $\frac{x^4}{(x^2-1)(x+2)} = x-2+\frac{5x^2-4}{(x^2-1)(x+2)}$.

(1)式右邊$x,-2$兩項之積分,已有標準式可用,當不成問題.茲將最末一項化成偏分數如下:

(2) 命 $\frac{5x^2-4}{(x+1)(x-1)(x+2)} = \frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{x-1}$.

式中A,B,C爲所求之常數.

將(2)式簡之,得

(3) $5x^2-4 = A(x+1)(x-1)+B(x-1)(x+2)+C(x+1)(x+2)$.

集項,得

(4) $5x^2-4 = (A+B+C)x^2+(B+3C)x-A-2B+2C$.

比較兩邊係數,得聯立方程如次:

$$\begin{cases} A+B+C=5, \\ B+3C=0, \\ -A-2B+2C=-4. \end{cases}$$

解之,

$$\therefore\quad A=\frac{16}{3};\quad B=-\frac{1}{2};\quad C=\frac{1}{6}.$$

代入(2)式,

$$\therefore \quad \frac{5x^2-4}{(x+1)(x-1)(x+2)}=\frac{16}{3(x+2)}-\frac{1}{2(x+1)}+\frac{1}{6(x-1)}.$$

$$\therefore \quad \int\frac{x^4\,dx}{(x^2-1)(x+2)}=\int\left(x-2+\frac{16}{3(x+2)}-\frac{1}{2(x+1)}+\frac{1}{6(x-1)}\right)dx$$

$$=\int x\,dx-2\int dx+\frac{16}{3}\int\frac{dx}{x+2}-\frac{1}{2}\int\frac{dx}{x+1}+\frac{1}{6}\int\frac{dx}{x-1}$$

$$=\frac{x^2}{2}-2x+\frac{16}{3}\log(x+2)$$

$$-\frac{1}{2}\log(x+1)+\frac{1}{6}(\log x-1)+c$$

$$=\frac{x^2}{2}-2x+\frac{1}{6}\log\frac{x-1}{(x+1)^3}+\frac{16}{3}\log(x+2)+c.$$

由(3)式求A, B, C之值,以下法爲較捷:

因(3)式爲恆等式,不論予x以何值均能成立.

命 $x=-2$, $\quad 16=3A$, $\quad\therefore\quad A=\frac{16}{3}$.

命 $x=1$, $\quad 1=6C$, $\quad\therefore\quad C=\frac{1}{6}$.

命 $x=-1$, $\quad 1=-2B$, $\quad\therefore\quad B=-\frac{1}{2}$.

23. 分母之因子均爲一次式但有重複者. 分母中有因子爲$(x-a)^n$,卽可假定與此因子相對應之n個偏分數爲

$$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}+\cdots\cdots+\frac{L}{(x-a)^n}.$$

式中第一項之積分,由22款方法求之.其餘悉用標準公式(4).

例如

(B) $$\int \frac{L\,dx}{(x-a)^n} = A\int (x-a)^{-n}\,dx = \frac{L}{(1-n)(x-a)^{n-1}} + c.$$

[舉例] 求 $\int \frac{x^3+1}{x(x-1)^3}\,dx.$

[解] 命 $$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

即 $$x^3+1 = A(x-1)^3 + Bx + Cx(x-1) + Dx(x-1)^2,$$

$$x^3+1 = (A+D)x^3 + (-3A+C-2D)x^2 + (3A+B-C+D)x - A.$$

比較 x 同次冪之係數，得

$$\begin{cases} A+D=1, \\ -3A+C-2D=0, \\ 3A+B-C+D=0, \\ -A=1. \end{cases}$$

解之， $A=-1,\quad B=2,\quad C=1,\quad D=2.$

$$\therefore\quad \frac{x^3+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{2}{x-1}.$$

求積分，

$$\therefore\quad \int \frac{x^3+1}{x(x-1)^3}\,dx = \int \left(-\frac{1}{x} + \frac{2}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{2}{x-1}\right)dx$$

$$= -\log x - \frac{1}{(x-1)^2} - \frac{1}{x-1} + 2\log(x-1) + c$$

$$= -\frac{x}{(x-1)^2} + \log\frac{(x-1)^2}{x} + c.$$

[別解] 命 $$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{\phi(x)}{(x-1)^3}.$$

即　$x^3+1=A(x-1)^3+x\cdot\phi(x)$.

命　$x=0$,　$\therefore\quad A=\frac{1}{-1}=-1$.

$\because\quad x^3+1=-(x-1)^3+x\cdot\phi(x)$,

$\therefore\quad \phi(x)=\frac{x^3+1+(x-1)^3}{x}=2x^2-3x+3$.

$$\therefore\quad \int\frac{x^3+1}{x(x-1)^3}dx=-\int\frac{dx}{x}+\int\frac{2x^2-3x+3}{(x-1)^3}dx$$

$$=-\int\frac{dx}{x}+2\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}+2\int\frac{dx}{(x-1)^3}$$

$$=-\log x+2\log(x-1)-\frac{1}{x-1}-\frac{1}{(x-1)^2}+c.$$

24. 分母有二次質因子且無重複者. 分母如有二次質因子x^2+px+q, 即可假定與此因子相對應之偏分數爲$\frac{Ax+B}{x^2+px+q}$. 茲求$\frac{Ax+B}{x^2+px+q}dx$之積分如下:

$$(C)\quad \int\frac{(Ax+B)dx}{x^2+px+q}=\int\frac{\left(Ax+\frac{Ap}{2}-\frac{Ap}{2}+B\right)dx}{x^2+px+q}$$

[分子加減同一常數$\frac{Ap}{2}$]

$$=\int\frac{\left(Ax+\frac{Ap}{2}\right)dx}{x^2+px+q}+\int\frac{\left(-\frac{Ap}{2}+B\right)dx}{x^2+px+q}$$

[由標準公式(1)]

$$=\frac{A}{2}\int\frac{(2x+p)dx}{x^2+px+q}+\left(\frac{2B-Ap}{2}\right)\int\frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)}$$

[由標準公式(2),並將第二積分之分母完成平方]

$$=\frac{A}{2}\log(x^2+px+q)+\frac{2B-Ap}{\sqrt{4q-p^2}}\tan^{-1}\frac{2x+p}{\sqrt{4q-p^2}}+c.$$

[由標準公式(5)及(18)]

[舉例] 求$\int\frac{4}{x^3+4x}dx$.

[解] 命 $\frac{4}{x^3+4x}=\frac{4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$.

即 $4=A(x^2+4)+x(Bx+C)=(A+B)x^2+Cx+4A$.

比較x之同次冪,

$$A+B=0,$$

$$C=0,$$

$$4A=4.$$

解之, $A=1,\quad B=-1,\quad C=0.$

$$\therefore\quad \frac{4}{x^3+4x}=\frac{1}{x}-\frac{x}{x^2+4}.$$

$$\therefore\quad \int\frac{4\,dx}{x^3+4x}=\int\frac{dx}{x}-\int\frac{x\,dx}{x^2+4}$$

$$=\log x-\frac{1}{2}\log(x^2+4)+c$$

$$=\log\frac{x}{\sqrt{x^2+4}}+c.$$

[別解] 命 $\frac{4}{x^3+4x}=\frac{4}{x(x^2+4)}=\frac{A}{x}+\frac{\phi(x)}{x^2+4}$.

即 $4=A(x^2+4)+x\cdot\phi(x)$.

命　$x=0$,　$4=4A$,　$\therefore$　$A=\frac{4}{4}=1$.

$\because$　$4=(x^2+4)+x\cdot\phi(x)$,

$\therefore$　$\phi(x)=\frac{-x^2}{x}=-x$.

$$\therefore\quad \int\frac{4\,dx}{x^3+4x}=\int\frac{1}{x}dx+\int\frac{-x\,dx}{x^2+4}=\int\frac{dx}{x}-\frac{1}{2}\int\frac{d(x^2+4)}{x^2+4}$$

$$=\log x-\frac{1}{2}\log(x^2+4)+c=\log\frac{x}{\sqrt{x^2+4}}+c.$$

25. 分母有二次質因子且係重複者. 分母如有因子如$(x^2+px+q)^n$者,可假定與此因子相對應之n個偏分數爲

$$(1)\quad \frac{Ax+B}{(x^2+px+q)^n}+\frac{Cx+D}{(x^2+px+q)^{n-1}}+\cdots\cdots+\frac{Lx+M}{x^2+px+q}.$$

最末一項之求積分公式,已詳述於24款中.茲述第一項之求法於下:

$$\int\frac{(Ax+B)dx}{(x^2+px+q)^n}=\int\frac{\left(Ax+\frac{Ap}{2}-\frac{Ap}{2}\right)dx}{(x^2+px+q)^n}$$

$$\left[\text{分子同時加減以常數}\ \frac{Ap}{2}\right]$$

$$=\int\frac{\left(Ax+\frac{Ap}{2}\right)dx}{(x^2+px+q)^n}+\int\frac{\left(-\frac{Ap}{2}+B\right)dx}{(x^2+px+q)^n}$$

[標準公式(1)]

$$=\frac{A}{2}\int(x^2+px+q)^{-n}(2x+p)\,dx+\left(\frac{2B-Ap}{2}\right)\int\frac{dx}{(x^2+px+q)^n}.$$

右端第一積分顯然與標準公式(4)相符,惟第二積分尙待解

決.暫書其結果如次:

$$(2)\quad \int\frac{(Ax+B)dx}{(x^2+px+q)^n}=\frac{A}{2(1-n)(x^2+px+q)^{n-1}}+\left(\frac{2B-Ap}{2}\right)\int\frac{dx}{(x^2+px+q)^n}.$$

茲將函數 $\frac{\left(x+\frac{p}{2}\right)}{(x^2+px+q)^{n-1}}$ 求微分,得

$$\frac{d}{dx}\left[\frac{x+\frac{p}{2}}{(x^2+px+q)^{n-1}}\right]=\frac{1}{(x^2+px+q)^{n-1}}-\frac{2(n-1)\left(x+\frac{p}{2}\right)^2}{(x^2+px+q)^n},$$

或

$$d\left[\frac{x+\frac{p}{2}}{(x^2+px+q)^{n-1}}\right]=\left[\frac{1}{(x^2+px+q)^{n-1}}-\frac{2(n-1)\left(x+\frac{p}{2}\right)^2}{(x^2+px+q)^n}\right]dx.$$

又因 $x^2+px+q=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)$,

$$\therefore\quad \left(x+\frac{p}{2}\right)^2=x^2+px+q-\left(q-\frac{p^2}{4}\right).$$

代入上式,得

$$(3)\quad d\left[\frac{x+\frac{p}{2}}{(x^2+px+q)^{n-1}}\right]=\left[\frac{-(2n-3)}{(x^2+px+q)^{n-1}}+\frac{2(n-1)\left(q-\frac{p^2}{4}\right)}{(x^2+px+q)^n}\right]dx.$$

將(3)式兩端各求積分,

$$\frac{x+\frac{p}{2}}{(x^2+px+q)^{n-1}}=-(2n-3)\int\frac{dx}{(x^2+px+q)^{n-1}}+2(n-1)\left(q-\frac{p^2}{4}\right)\int\frac{dx}{(x^2+px+q)^n}.$$

依 $\int\frac{dx}{(x^2+px+q)^n}$ 解之,

$$(4)\quad \therefore \int \frac{dx}{(x^2+px+q)^n} = \frac{x+\frac{p}{2}}{2(n-1)\left(q-\frac{p^2}{4}\right)(x^2+px+q)^{n-1}}$$

$$+\frac{2n-3}{2(n-1)\left(q-\frac{p^2}{4}\right)}\int \frac{dx}{(x^2+px+q)^{n-1}}.$$

將(4)式之結果代入(2)式並簡化之,故得公式

$$(D)\quad \int \frac{(Ax+B)dx}{(x^2+px+q)^n} = \frac{A(p^2-4q)+(2B-Ap)(2x+p)}{2(n-1)(4q-p^2)(x^2+px+q)^{n-1}}$$

$$+\frac{(2B-Ap)(2n-3)}{(n-1)(4q-p^2)}\int \frac{dx}{(x^2+px+q)^{n-1}}.$$

由(D)式,可知原微分式之分母已由n次降爲$(n-1)$次,如將(D)式遞次用之,最後之積分終可降爲

$$(5)\qquad \int \frac{dx}{x^2+px+q}$$

之形,而(5)之積分,可完成平方法求之.於是(1)式中其他各項(末項除外)悉可依此法求之而告解決.

[舉例]　求$\int \frac{2x^3+x+3}{(x^2+1)^2}dx$.

[解]　由視察,分母中二次因子重複二遍.

命　$\frac{2x^3+x+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)^2}+\frac{Cx+D}{x^2+1}$.

即　$2x^3+x+3=(Ax+B)+(Cx+D)(x^2+1)$,

或　$2x^3+x+3=Cx^3+Dx^2+(A+C)x+B+D$.

比較x同次冪之係數,故得

$$C=2,\ D=0,\ A+C=1,\ B+D=3.$$

解之, $\therefore\ A=-1,\ B=3,\ C=2,\ D=0.$

$$\therefore\ \int\frac{2x^3+x+3}{(x^2+1)^2}dx=\int\frac{-x+3}{(x^2+1)^2}dx+\int\frac{2x\,dx}{x^2+1}$$

$$=\log(x^2+1)+\int\frac{-x+3}{(x^2+1)^2}dx.$$

上式右端最末之一積分須用25款公式(D)求之.

$$\because\ A=-1,\ B=3,\ p=0,\ q=1,\ n=2.$$

代入25款公式(D),

$$\int\frac{-x+3}{(x^2+1)^2}dx=\frac{1+3x}{2(x^2+1)}+\frac{3}{2}\int\frac{dx}{x^2+1}=\frac{1+3x}{2(x^2+1)}+\frac{3}{2}\tan^{-1}x.$$

$$\therefore\ \int\frac{2x^3+x+3}{(x^2+1)^2}dx=\log(x^2+1)+\frac{1+3x}{2(x^2+1)}+\frac{3}{2}\tan^{-1}x+c.$$

26. 本章例題.

第22款之例題

253. 求$\int\frac{(2x+3)dx}{x^3+x^2-2x}$.

[解] 命 $\frac{2x+3}{x^3+x^2-2x}=\frac{2x+3}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}.$

即 $2x+3=A(x-1)(x+2)+B(x+2)x+C(x-1)x.$

命 $x=0$; 則 $3=-2A,\ \therefore\ A=-\frac{3}{2}.$

命 $x=1$; 則 $5=3B,\ \therefore\ B=\frac{5}{3}.$

命 $x=-2$; 則 $-1=6C,\ \therefore\ C=-\frac{1}{6}.$

$$\therefore \quad \frac{2x+3}{x(x-1)(x+2)}=-\frac{3}{2x}+\frac{5}{3(x-1)}-\frac{1}{6(x+2)}.$$

$$\therefore \quad \int\frac{2x+3}{x(x-1)(x+2)}dx=-\frac{3}{2}\int\frac{dx}{x}+\frac{5}{3}\int\frac{dx}{x-1}-\frac{1}{6}\int\frac{dx}{x+2}$$

$$=-\frac{3}{2}\log x+\frac{5}{3}\log(x-1)-\frac{1}{6}\log(x+2)+c$$

$$=\log\frac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}(x+2)^{\frac{1}{6}}}+c.$$

254. 求 $\int\frac{x\,dx}{(x+1)(x+3)(x+5)}$.

[解] 命 $\frac{x}{(x+1)(x+3)(x+5)}=\frac{A}{x+1}+\frac{B}{x+3}+\frac{C}{x+5}$.

即 $x=A(x+3)(x+5)+B(x+1)(x+5)+C(x+1)(x+3)$.

命 $x=-1$; 則 $-1=2\cdot4A$, $\therefore\ A=-\frac{1}{8}$.

命 $x=-3$; 則 $-3=-2\cdot2B=-4B$, $\therefore\ B=\frac{3}{4}$.

命 $x=-5$; 則 $-5=(-4)(-2)C=8C$, $\therefore\ C=-\frac{5}{8}$.

$$\therefore \quad \int\frac{x\,dx}{(x+1)(x+3)(x+5)}=-\int\frac{dx}{8(x+1)}+\int\frac{3\,dx}{4(x+3)}-\int\frac{5\,dx}{8(x+5)}$$

$$=-\frac{1}{8}\log(x+1)+\frac{6}{8}\log(x+3)$$

$$-\frac{5}{8}\log(x+5)+c$$

$$=\frac{1}{8}\log\frac{(x+3)^6}{(x+5)^5(x+1)}+c.$$

255. 求 $\int \frac{(3x-1)dx}{x^2+x-6}$.

[解] 命 $\frac{3x-1}{x^2+x-6}=\frac{3x-1}{(x-2)(x+3)}=\frac{A}{x+3}+\frac{B}{x-2}$.

即 $3x-1=A(x-2)+B(x+3)$.

命 $x=-3$; 則 $-10=-5A$, $\therefore$ $A=2$.

命 $x=2$; 則 $5=5B$, $\therefore$ $B=1$.

$\therefore$ $\int \frac{(3x-1)dx}{(x+3)(x-2)}=\int \frac{2dx}{x+3}+\int \frac{dx}{x-2}=2\log(x+3)+\log(x-2)+c$

$=\log(x+3)^2(x-2)+c$.

256. 求 $\int \frac{(2x^3+1)dx}{x^2+3x+2}$.

[解] 因分子之次數較分母爲高,先行除法.

$$\frac{2x^3+1}{(x+1)(x+2)}=2x-6+\frac{14x+13}{(x+1)(x+2)}.$$

次將末項析爲偏分數:

命 $\frac{14x+13}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$.

即 $14x+13=A(x+2)+B(x+1)$.

若命 $x=-2$; 則 $-15=-B$, $\therefore$ $B=15$.

若命 $x=-1$; 則 $-1=A$, $\therefore$ $A=-1$.

$\therefore$ $\frac{2x^3+1}{(x+1)(x+2)}=2x-6-\frac{1}{x+1}+\frac{15}{x+2}$.

$\therefore$ $\int \frac{2x^3+1}{(x+1)(x+2)}dx=\int 2xdx-\int 6dx-\int \frac{dx}{x+1}+\int \frac{15}{x+2}dx$

$$=x^2-6x-\log(x+1)+15\log(x+2)+c$$

$$=x^2-6x+\log\frac{(x+2)^{15}}{x+1}+c.$$

257. 求 $\int\frac{(a-b)\,y\,dy}{y^2-(a+b)\,y+ab}$.

〔解〕 命 $\frac{(a-b)y}{y^2-ay-by+ab}=\frac{(a-b)y}{y(y-a)-b(y-a)}$

$$=\frac{(a-b)y}{(y-b)(y-a)}=\frac{A}{y-a}+\frac{B}{y-b}.$$

即 $(a-b)y=A(y-b)+B(y-a)$.

命 $y=a$; 則 $(a-b)a=(a-b)A$, $\therefore$ $A=a$.

命 $y=b$; 則 $(a-b)b=(b-a)B$, $\therefore$ $B=-b$.

由是 $\frac{(a-b)y}{(y-b)(y-a)}=\frac{a}{y-a}-\frac{b}{y-b}$.

求積分:

$$\int\frac{(a-b)y}{y^2-ay-by+ab}dy=\int\frac{a\,dy}{y-a}-\int\frac{b\,dy}{y-b}$$

$$=a\log(y-a)-b\log(y-b)+c$$

$$=\log\frac{(y-a)^a}{(y-b)^b}+c.$$

258. 求 $\int\frac{dx}{x^2+4x}$.

〔解〕 命 $\frac{1}{x^2+4x}=\frac{1}{x(x+4)}=\frac{A}{x}+\frac{B}{x+4}$.

即 $1=A(x+4)+Bx$.

命 $x=0$; 則 $1=4A$, $\therefore$ $A=\frac{1}{4}$

命 $x=-4$; 則 $1=-4B$, $\therefore$ $B=-\frac{1}{4}$.

$\therefore$ $\frac{1}{x(x+4)}=\frac{1}{4x}-\frac{1}{4(x+4)}$.

求積分:

$$\int\frac{dx}{x(x+4)}=\int\frac{dx}{4x}-\int\frac{dx}{4(x+4)}=\frac{1}{4}\int\frac{dx}{x}-\frac{1}{4}\int\frac{dx}{x+4}$$

$$=\frac{1}{4}\log x-\frac{1}{4}\log(x+4)+c=\frac{1}{4}\log\frac{x}{x+4}+c.$$

259. 求$\int\frac{dx}{2x^2+3x+1}$.

[解] 命 $\frac{1}{2x^2+3x+1}=\frac{1}{(2x+1)(x+1)}=\frac{A}{2x+1}+\frac{B}{x+1}$.

即 $1=A(x+1)+B(2x+1)$.

命 $x=-1$; 則 $1=-B$, $\therefore$ $B=-1$.

命 $x=-\frac{1}{2}$; 則 $1=\frac{1}{2}A$, $\therefore$ $A=2$.

$\therefore$ $\frac{1}{2x^2+3x+1}=\frac{2}{2x+1}-\frac{1}{x+1}$.

求積分:

$$\int\frac{dx}{2x^2+3x+1}=\int\frac{2\,dx}{2x+1}-\int\frac{dx}{x+1}$$

$$=\log(2x+1)-\log(x+1)+c$$

$$=\log\frac{2x+1}{x+1}+c.$$

260. 求$\int\frac{(x^3+6x-8)dx}{x^3-4x}$.

[解] 命 $\frac{x^2+6x-8}{x^3-4x}=\frac{x^2+6x-8}{x(x+2)(x-2)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}$.

即 $x^2+6x-8=A(x+2)(x-2)+Bx(x-2)+Cx(x+2)$.

命 $x=2$; 則 $8=8C$, $\therefore$ $C=1$.

命 $x=0$; 則 $-8=-4A$, $\therefore$ $A=2$.

命 $x=-2$; 則 $-16=8B$, $\therefore$ $B=-2$.

$\therefore$ $\frac{x^2+6x-8}{x^3-4x}=\frac{2}{x}-\frac{2}{x+2}+\frac{1}{x-2}$.

求積分:

$$\int\frac{(x^2+6x-8)dx}{x^3-4x}=2\int\frac{dx}{x}-2\int\frac{dx}{x+2}+\int\frac{dx}{x-2}$$

$$=2\log x-2\log(x+2)+\log(x-2)+c$$

$$=\log\frac{x^2(x-2)}{(x+2)^2}+c.$$

261. 求 $\int\frac{(2x-1)dx}{(x-1)(x-2)}$.

[解] 命 $\frac{2x-1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$.

即 $2x-1=A(x-2)+B(x-1)$.

命 $x=1$; 則 $1=-A$, $\therefore$ $A=-1$.

命 $x=2$; 則 $3=B$, $\therefore$ $B=3$.

$\therefore$ $\frac{2x-1}{(x-1)(x-2)}=-\frac{1}{x-1}+\frac{3}{x-2}$.

求積分:

$$\int\frac{(2x-1)dx}{(x-1)(x-2)}=-\int\frac{dx}{x-1}+3\int\frac{dx}{x-2}$$

$$= -\log(x-1)+3\log(x-2)+c$$

$$=\log\frac{(x-2)^3}{x-1}+c.$$

262. 求 $\int\frac{(x-1)dx}{x^2+6x+8}$.

[解] 命 $\frac{x-1}{x^2+6x+8}=\frac{x-1}{(x+2)(x+4)}=\frac{A}{x+2}+\frac{B}{x+4}$.

即 $x-1=A(x+4)+B(x+2)$.

命 $x=-2$; 則 $-3=2A$, $\therefore\ A=\frac{-3}{2}$.

命 $x=-4$; 則 $-5=-2B$, $\therefore\ B=\frac{5}{2}$.

$\therefore\ \frac{x-1}{x^2+6x+8}=-\frac{3}{2(x+2)}+\frac{5}{2(x+4)}$.

求積分:

$$\int\frac{(x-1)dx}{x^2+6x+8}=-\frac{3}{2}\int\frac{dx}{x+2}+\frac{5}{2}\int\frac{dx}{x+4}$$

$$=-\frac{3}{2}\log(x+2)+\frac{5}{2}\log(x+4)+c$$

$$=\frac{1}{2}[\log(x+4)^5-\log(x+2)^3]+c$$

$$=\frac{1}{2}\log\frac{(x+4)^5}{(x+2)^3}+c.$$

268. 求 $\int\frac{(x^2+x-1)dx}{x^3+x^2-6x}$.

[解] 命 $\frac{x^2+x-1}{x^3+x^2-6x}=\frac{x^2+x-1}{x(x^2+x-6)}=\frac{x^2+x-1}{x(x+3)(x-2)}$

$$=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-2}$$

即　$x^2+x-1=A(x+3)(x-2)+B(x-2)x+C(x+3)x.$

命　$x=0$;　則　$-1=-6A$,　$\therefore$　$A=\frac{1}{6}$.

命　$x=-3$;　則　$5=15B$,　$\therefore$　$B=\frac{1}{3}$.

命　$x=2$;　則　$5=10C$,　$\therefore$　$C=\frac{1}{2}$.

$\therefore$　$\frac{x^2+x-1}{x^3+x^2-6x}=\frac{1}{6x}+\frac{2}{6(x+3)}+\frac{3}{6(x-2)}$

求積分:

$$\int\frac{(x^2+x-1)dx}{x^3+x^2-6x}=\int\frac{dx}{6x}+\int\frac{2dx}{6(x+3)}+\int\frac{3dx}{6(x-2)}$$

$$=\frac{1}{6}\log x+\frac{2}{6}\log(x+3)+\frac{3}{6}\log(x-2)+c$$

$$=\frac{1}{6}[\log x+2\log(x+3)+3\log(x-2)]+c$$

$$=\frac{1}{6}\log(x+2)(x+3)^2(x-2)^3+c$$

$$=\log\sqrt[6]{(x+2)(x+3)^2(x-2)^3}+c.$$

264.　求$\int\frac{x^5+x^4-8}{x^3-4x}dx$.

[解]　$\frac{x^5+x^4-8}{x^3-4x}=x^2+x+4+\frac{16x-8}{x^3-4x}$.

命　$\frac{16x-8}{x^3-4x}=\frac{16x-8}{x(x^2-4)}=\frac{16x-8}{x(x+2)(x-2)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}$

即　$16x-8=A(x+2)(x-2)+B(x-2)x+C(x+2)x.$

命　$x=0$;　則　$-8=-4A$,　$\therefore$　$A=2$.

命　$x=-2$;　則　$-40=8B$,　$\therefore$　$B=-5$.

命 $x=2$; 則 $24=8C$, $\therefore\ C=3$.

$$\therefore\ \frac{x^5+x^4-8}{x^3-4x}=x^2+x+4+\frac{2}{x}-\frac{5}{x+2}+\frac{3}{x-2}.$$

$$\therefore\ \int\frac{x^5+x^4-8}{x^3-4x}dx=\int x^2dx+\int x\,dx+\int 4\,dx$$

$$+2\int\frac{dx}{x}-5\int\frac{dx}{x+2}+3\int\frac{dx}{x-2}$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+2\log x-5\log(x+2)$$

$$+3\log(x-2)+c$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+\log\frac{x^2(x-2)^3}{(x+2)^5}+c.$$

265. 求 $\int\frac{(t^2+pq)dt}{t(t-p)(t+q)}$.

[解] 命 $\frac{t^2+pq}{t(t-p)(t+q)}=\frac{A}{t}+\frac{B}{t-p}+\frac{C}{t+q}$

即 $t^2+pq=A(t-p)(t+q)+Bt(t+q)+Ct(t-p)$.

命: $t=0$, $pq=-pqA$, $\therefore\ A=-1$;

$t=p$, $p(p+q)=p(p+q)B$, $\therefore\ B=1$;

$t=-q$, $q(q+p)=q(q+p)C$, $\therefore\ C=1$.

$$\therefore\ \int\frac{(t^2+pq)dt}{t(t-p)(t+q)}=-\int\frac{dt}{t}+\int\frac{dt}{t-p}+\int\frac{dt}{t+q}$$

$$=-\log t+\log(t-p)+\log(t+q)+c$$

$$=\log\frac{(t-p)(t+q)}{t}+c.$$

第23款之例題

266. 求$\int \frac{dx}{(x-1)^2(x-2)}$.

[解] 命 $\frac{1}{(x-1)^2(x-2)}=\frac{A}{x-2}+\frac{B}{(x-1)^2}+\frac{C}{x-1}$.

即 $1=A(x-1)^2+B(x-2)+C(x-1)(x-2)$,

$1=(A+C)x^2+(B-2A-3C)x+A-2B+2C$

$$\therefore \begin{cases} A \quad +C=0, \\ -2A+B-3C=0, \\ A-2B+2C=1. \end{cases}$$

解之, $A=1,\ B=-1,\ C=-1.$

$$\therefore \frac{1}{(x-1)^2(x-2)}=\frac{1}{x-2}+\frac{-1}{(x-1)^2}+\frac{-1}{x-1}.$$

$$\therefore \int \frac{dx}{(x-1)^2(x-2)}=\int \frac{dx}{x-2}-\int \frac{dx}{(x-1)^2}-\int \frac{dx}{x-1}$$

$$=\log(x-2)-\frac{(x-1)^{-1}}{-1}-\log(x-1)+c$$

$$=\frac{1}{x-1}+\log\frac{x-2}{x-1}+c.$$

267. 求$\int \frac{x^2+1}{(x-1)^3}dx$.

[解] 命 $\frac{x^2+1}{(x-1)^3}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}$.

即 $x^2+1=A+B(x-1)+C(x-1)^2$,

$x^2+1=Cx^2+(B-2C)x+A-B+C.$

比較x同次冪之係數,

$$\begin{cases} C=1, \\ B-2C=0, \\ A-B+C=1. \end{cases}$$

解之，$C=1,\ B=2,\ A=2.$

$$\therefore\ \frac{x^2+1}{(x-1)^3}=\frac{2}{(x-1)^3}+\frac{2}{(x-1)^2}+\frac{1}{x-1}.$$

$$\therefore\ \int\frac{(x^2+1)dx}{(x-1)^3}=2\int(x-1)^{-3}dx+2\int(x-1)^{-2}dx+\int\frac{dx}{x-1}$$

$$=\frac{1}{(x-1)^2}-\frac{2}{x-1}+\log(x-1)+c.$$

268. 求$\int\frac{y^2\,dy}{y^3+5y^2+8y+4}$.

[解] 命 $\frac{y^2}{y^3+5y^2+8y+4}=\frac{y^2}{(y+1)(y+2)^2}=\frac{A}{y+1}+\frac{\phi(y)}{(y+2)^2}$.

$\therefore\ y^2=A(y+2)^2+\phi(y)(y+1).$

命 $y=-1,\ \therefore\ A=1.$

$\because\ y^2=(y+2)^2+\phi(y)(y+1),$

$\therefore\ \phi(y)=[y^2-(y+2)^2]\div(y+1)=-4.$

$$\therefore\ \int\frac{y^2\,dy}{y^3+5y^2+8y+4}=\int\frac{dy}{y+1}-4\int\frac{dy}{(y+2)^2}$$

$$=\log(y+1)+\frac{4}{y+2}+c.$$

269. 求$\int\frac{as^2\,ds}{(s+a)^3}$.

[解] 命 $\frac{as^2}{(s+a)^3}=\frac{A}{(s+a)^3}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)}$.

$\therefore\quad as^2=A+B(s+a)+C(s+a)^2.$

即　$as^2=Cs^2+(2aC+B)s+A+aB+a^2C.$

比較 s 同次冪之係數，

$$\begin{cases} C=a, \\ B+2aC=0, \\ A+aB+a^2C=0. \end{cases}$$

解之，　$C=a,\ B=-2a^2,\ A=a^3.$

$$\therefore\quad \int\frac{as^2\,ds}{(s+a)^3}=\int\frac{a^3\,ds}{(s+a)^3}-\int\frac{2a^2\,ds}{(s+a)^2}+\int\frac{a\,ds}{s+a}$$

$$=-\frac{a^3}{2(s+a)^2}+\frac{2a^2}{s+a}+a\log(s+a)+c$$

270.　求 $\int\frac{dx}{x^2(1+x)}$.

[解]　命　$\frac{1}{x^2(1+x)}=\frac{A}{1+x}+\frac{\phi(x)}{x^2}.$

$\therefore\quad 1=Ax^2+\phi(x)(1+x).$

命　$x=-1,\qquad \therefore\quad A=1.$

$\because\quad 1=x^2+\phi(x)(1+x).$

$\therefore\quad \phi(x)=\frac{1-x^2}{1+x}=1-x.$

$$\therefore\quad \int\frac{dx}{x^2(1+x)}=\int\frac{dx}{1+x}+\int\frac{(1-x)dx}{x^2}$$

$$=\int\frac{dx}{1+x}+\int x^{-2}\,dx-\int\frac{dx}{x}$$

$$=\log(1+x)-\frac{1}{x}+\log x+c=\log x(1+x)-\frac{1}{x}+c.$$

271. 求 $\int \frac{dt}{t(1+t)^2}$.

[解] 命 $\frac{1}{t(1+t)^2}=\frac{A}{t}+\frac{\phi(t)}{(1+t)^2}$.

$\therefore\ 1=A(1+t)^2+\phi(t)t.$

命 $t=0,\quad \therefore\ A=1.$

$\because\ 1=(1+t)^2+\phi(t)t.$

$\therefore\ \phi(t)=\frac{-t^2-2t}{t}=-t-2=-(t+2).$

$\therefore\ \frac{1}{t(1+t)^2}=\frac{1}{t}-\frac{t+2}{(1+t)^2}=\frac{1}{t}-\frac{1}{(1+t)^2}-\frac{1}{1+t}.$

$\therefore\ \int\frac{dt}{t(1+t)^2}=\int\frac{dt}{t}-\int\frac{dt}{(1+t)^2}-\int\frac{dt}{1+t}$

$$=\log t+\frac{1}{1+t}-\log(1+t)+c$$

$$=\frac{1}{1+t}+\log\frac{t}{1+t}+c.$$

272. 求 $\int\frac{(1+3x)dx}{x+2x^2+x^3}$.

[解] 命 $\frac{1+3x}{x+2x^2+x^3}=\frac{1+3x}{x(1+2x+x^2)}=\frac{1+3x}{x(1+x)^2}=\frac{A}{x}+\frac{\phi(x)}{(1+x)^2}.$

$\therefore\ 1+3x=A(1+x)^2+\phi(x)x.$

命 $x=0,\quad \therefore\ A=1.$

$\because\ 1+3x=(1+x)^2+\phi(x)x.$

$\therefore\ \phi(x)=\frac{1+3x-(x+1)^2}{x}=\frac{-x^2+x}{x}=1-x.$

$$\therefore\quad \frac{1+3x}{x+2x^2+x^3}=\frac{1}{x}+\frac{1-x}{(1+x)^2}=\frac{1}{x}+\frac{2}{(x+1)^2}-\frac{1}{x+1}.$$

$$\therefore\quad \int\frac{(1+3x)dx}{x+2x^2+x^3}=\int\frac{dx}{x}+2\int\frac{dx}{(x+1)^2}-\int\frac{dx}{x+1}$$

$$=\log x+\frac{-2}{x+1}-\log(x+1)+c$$

$$=-\frac{2}{x+1}+\log\frac{x}{x+1}+c.$$

273. 求$\int\frac{(x-8)dx}{x^3-4x^2+4x}$.

[解] 命 $\frac{x-8}{x^3-4x^2+4x}=\frac{x-8}{x(x^2-4x+4)}=\frac{x-8}{x(x-2)^2}=\frac{A}{x}+\frac{\phi(x)}{(x-2)^2}$.

$\therefore\quad x-8=A(x-2)^2+\phi(x)x.$

若命 $x=0,\ -8=4A,\qquad \therefore\quad A=-2.$

$$\because\qquad \phi(x)=\frac{x-8+2(x-2)^2}{x}=2x-7.$$

$$\therefore\quad \frac{x-8}{x^3-4x^2+4x}=-\frac{2}{x}+\frac{2x-7}{(x-2)^2}=\frac{-2}{x}+\frac{2}{x-2}-\frac{3}{(x-2)^2}.$$

$$\therefore\quad \int\frac{(x-8)dx}{x^3-4x^2+4x}=-2\int\frac{dx}{x}+2\int\frac{dx}{x-2}-3\int\frac{dx}{(x-2)^2}$$

$$=-2\log x+2\log(x-2)+\frac{3}{x-2}+c$$

$$=\frac{3}{x-2}+\log\frac{(x-2)^2}{x^2}+c.$$

274. 求$\int\frac{(x^5-x^3+1)dx}{x^4-x^3}$.

[解] $\frac{x^5-x^3+1}{x^4-x^3}=x+1+\frac{1}{x^4-x^3}=x+1+\frac{1}{x^3(x-1)}.$

命 $\dfrac{1}{x^3(x-1)}=\dfrac{A}{x-1}+\dfrac{\phi(x)}{x^3}$.

$\therefore\quad 1=Ax^3+\phi(x)(x-1)$.

命 $x=1,\qquad \therefore\quad A=1$.

$\because\quad \phi(x)=\dfrac{1-x^3}{x-1}=-x^2-x-1$.

$\therefore\quad \dfrac{x^5-x^3+1}{x^4-x^3}=x+1+\dfrac{1}{x-1}-\dfrac{1}{x}-\dfrac{1}{x^2}-\dfrac{1}{x^3}$.

$\therefore\quad \int\dfrac{(x^5-x^3+1)dx}{x^4-x^3}=\int x\,dx+\int dx+\int\dfrac{dx}{x-1}$

$$-\int\frac{dx}{x}-\int x^{-2}\,dx-\int x^{-3}\,dx$$

$$=\frac{x^2}{2}+x+\log(x-1)-\log x+\frac{1}{x}+\frac{1}{2x^2}+c$$

$$=\frac{x^2}{2}+x+\log\frac{x-1}{x}+\frac{1}{x}+\frac{1}{2x^2}+c.$$

275. 求 $\int\dfrac{(3x+2)dx}{x(x+1)^3}$.

[解] 命 $\dfrac{3x+2}{x(x+1)^3}=\dfrac{A}{x}+\dfrac{\phi(x)}{(x+1)^3}$.

$\therefore\quad 3x+2=A(x+1)^3+\phi(x)x$.

命 $x=0,\qquad \therefore\quad A=2$.

$\because\quad \phi(x)=\dfrac{3x+2-2(x+1)^3}{x}=-2x^2-6x-3$.

$\therefore\quad \dfrac{3x+2}{x(x+1)^3}=\dfrac{2}{x}+\dfrac{-2x^2-6x-3}{(x+1)^3}$

$$=\frac{2}{x}-\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}.$$

$$\therefore \int\frac{(3x+2)dx}{x(x+1)^3}=2\int\frac{dx}{x}-2\int\frac{dx}{x+1}$$

$$-2\int(x+1)^{-2}dx+\int(x+1)^{-3}dx$$

$$=2\log x-2\log(x+1)+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+c$$

$$=\frac{4x+3}{2(x+1)^2}+\log\frac{x^2}{(x+1)^2}+c.$$

276. 求$\int\left(\frac{m}{z+m}-\frac{nz}{(z+n)^2}\right)dz.$

[解] $$\int\left(\frac{m}{z+m}-\frac{nz}{(z+n)^2}\right)dz=\int\frac{m}{z+m}dz-\int\frac{nz}{(z+n)^2}dz.$$

右端第一個積分之計算,毫無問題;其第二個之算法如下:

命 $$\frac{nz}{(z+n)^2}=\frac{A}{(z+n)^2}+\frac{B}{z+n}.$$

$$\therefore\ nz=A+B(z+n)=Bz+A+Bn.$$

比較z之同次冪係數:

$$\begin{cases}B=n,\\ A+Bn=0.\end{cases}$$

解之, $$\therefore\ B=n,\quad A=-n^2.$$

$$\therefore \int\left(\frac{m}{z+m}-\frac{nz}{(z+n)^2}\right)dz=\int\frac{m\,dz}{z+m}-\int\frac{nz\,dz}{(z+n)^2}$$

$$=m\int\frac{dz}{z+m}-\int\left[\frac{-n^2}{(z+n)^2}+\frac{n}{z+n}\right]dz$$

$$=m\int\frac{dz}{z+m}+n^2\int\frac{dz}{(z+n)^2}-\int\frac{n\,dz}{z+n}$$

$$=m\log(z+m)+n^2\cdot\frac{-1}{z+n}-n\log(z+n)+c$$

$$=\log\frac{(z+m)^m}{(z+n)^n}-\frac{n^2}{z+n}+c.$$

第24款例題

277. 求$\int\frac{x\,dx}{(x+1)(x^2+4)}$.

[解] 命 $\frac{x}{(x+1)(x^2+4)}=\frac{A}{x+1}+\frac{\phi(x)}{x^2+4}$.

$\therefore\quad x=A(x^2+4)+\phi(x)(x+1)$.

命 $x=-1$, $\therefore\quad A=\frac{-1}{5}$.

$\because\quad x=-\frac{1}{5}(x^2+4)+\phi(x)(x+1)$.

$\therefore\quad \phi(x)=\frac{x^2+5x+4}{x+1}=\frac{1}{5}x+\frac{4}{5}=\frac{1}{5}(x+4)$.

$\therefore\quad \int\frac{x\,dx}{(x+1)(x^2+4)}=-\int\frac{dx}{5(x+1)}+\int\frac{x+4}{5(x^2+4)}dx$

$$=-\frac{1}{5}\int\frac{dx}{x+1}+\frac{1}{5}\int\frac{x\,dx}{x^2+4}+\frac{4}{5}\int\frac{dx}{x^2+2^2}$$

$$=-\frac{1}{5}\log(x+1)+\frac{1}{10}\log(x^2+4)$$

$$+\frac{4}{10}\tan^{-1}\frac{x}{2}+c$$

$$=\frac{1}{10}\log\frac{x^2+4}{(x+1)^2}+\frac{2}{5}\tan^{-1}\frac{x}{2}+c$$

278. 求$\int\frac{dx}{x(x^2+1)}$.

[解] 命 $\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{\phi(x)}{x^2+1}$.

$\therefore\quad 1=A(x^2+1)+\phi(x)x.$

命 $x=0,\qquad \therefore\quad A=1.$

$\because\quad \phi(x)=\frac{1-x^2-1}{x}=-x.$

$$\therefore\quad \int\frac{dx}{x(x^2+1)}=\int\frac{dx}{x}+\int\frac{-x}{x^2+1}dx=\int\frac{dx}{x}-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}$$

$$=\log x-\frac{1}{2}\log(x^2+1)+c=\log\frac{x}{\sqrt{x^2+1}}+c.$$

279. 求 $\int\frac{(2x^2-3x-3)dx}{(x-1)(x^2-2x+5)}$.

[解] 命 $\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}=\frac{A}{x-1}+\frac{\phi(x)}{x^2-2x+5}$.

$\therefore\quad 2x^2-3x-3=A(x^2-2x+5)+\phi(x)(x-1).$

命 $x=1,\ 2-3-3=A(1-2+5),\quad \therefore\quad A=-\frac{4}{4}=-1.$

$\because\quad \phi(x)=\frac{2x^2-3x-3+(x^2-2x+5)}{x-1}=3x-2.$

$$\therefore\quad \int\frac{(2x^2-3x-3)dx}{(x-1)(x^2-2x+5)}=\int\frac{-dx}{x-1}+\int\frac{(3x-2)dx}{x^2-2x+5}. \qquad (1)$$

上式右端第一積分用標準式(5),第二個積分求法如下:

先命 $v=x^2-2x+5$; 則 $dv=2x\,dx-2\,dx$, $(3x-2)dx=dv+x\,dx$.

$$\therefore\quad \int\frac{(3x-2)dx}{x^2-2x+5}=\int\frac{d(x^2-2x+5)}{x^2-2x+5}+\int\frac{x\,dx}{x^2-2x+5}. \qquad (2)$$

更命 $v=x^2-2x+5$; 則 $dv=2xdx-2\,dx$, $x\,dx=\frac{dv}{2}+dx$.

於是(2)式右端第二個積分又如下法求出:

$$\int\frac{x\,dx}{x^2-2x+5}=\frac{1}{2}\int\frac{d(x^2-2x+5)}{x^2-2x+5}+\int\frac{dx}{x^2-2x+5}. \qquad (3)$$

將(2),(3)之結果代入(1)中,

$$\therefore\quad \int\frac{(2x^2-3x-3)\,dx}{(x-1)(x^2-2x+5)}=-\int\frac{dx}{x-1}+\int\frac{(3x-2)\,dx}{x^2-2x+5}$$

$$=-\int\frac{dx}{x-1}+\int\frac{d(x^2-2x+5)}{x^2-2x+5}+\int\frac{x\,dx}{x^2-2x+5}$$

$$=-\int\frac{dx}{x-1}+\int\frac{d(x^2-2x+5)}{x^2-2x+5}$$

$$+\frac{1}{2}\int\frac{d(x^2-2x+5)}{x^2-2x+5}+\int\frac{dx}{x^2-2x+5}$$

$$=-\log(x-1)+\log(x^2-2x+5)$$

$$+\frac{1}{2}\log(x^2-2x+5)+\frac{1}{2}\tan^{-1}\frac{x-1}{2}+c$$

$$=\log\frac{(x^2-2x+5)^{\frac{3}{2}}}{x-1}+\frac{1}{2}\tan^{-1}\frac{x-1}{2}+c.$$

280. 求$\int\frac{x^2dx}{1-x^4}$.

[解] 命 $\frac{x^2}{1-x^4}=\frac{x^2}{(1+x^2)(1-x)(1+x)}=\frac{A}{1+x}+\frac{B}{1-x}+\frac{\phi(x)}{1+x^2}$.

$\therefore\quad x^2=A(1-x)(1+x^2)+B(1+x)(1+x^2)+\phi(x)(1-x^2)$.

命 $x=1$, $\therefore\quad B=\frac{1}{4}$;

命 $x=-1$, $\therefore\quad A=\frac{1}{4}$.

$\therefore\quad \frac{x^2}{1-x^4}=\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{\phi(x)}{1+x^2}$.

依$\phi(x)$解之,

$$\frac{\phi(x)}{1+x}=\frac{x^2}{1-x^4}-\frac{1}{4(1+x)}-\frac{1}{4(1-x)}=\frac{x^2}{1-x^4}-\frac{1-x+1+x}{4(1+x)(1-x)}$$

$$=\frac{x^2}{1-x^4}-\frac{1}{2(1-x^2)}=\frac{2x^2-1-x^2}{2(1-x^2)(1+x^2)}$$

$$=\frac{x^2-1}{2(1-x^2)(1+x^2)}=\frac{-1}{2(1+x^2)}.$$

$\because\quad \phi(x)=(1+x^2)\cdot\frac{-1}{2(1+x^2)}=-\frac{1}{2}.$

$$\therefore\quad \int\frac{x^2\,dx}{1-x^4}=\int\frac{dx}{4(1+x)}+\int\frac{dx}{4(1-x)}+\int-\frac{dx}{2(1+x^2)}$$

$$=\frac{1}{4}\log(1+x)-\frac{1}{4}\log(1-x)-\frac{1}{2}\tan^{-1}x+c$$

$$=\frac{1}{4}\log\frac{1+x}{1-x}-\frac{1}{2}\tan^{-1}x+c.$$

281. 求$\int\frac{dx}{x^3+1}$.

[解] 命 $\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{\phi(x)}{x^2-x+1}.$

$\therefore\quad 1=A(x^2-x+1)+\phi(x)(x+1).$

命 $x=-1,\ 1=A(1+1+1),\quad \therefore\quad A=\frac{1}{3}.$

$$\because\quad \phi(x)=\left(1-\frac{x^2-x+1}{3}\right)\div(x+1)=\frac{-x^2+x+2}{3(x+1)}=-\frac{1}{3}x+\frac{2}{3}.$$

$$\therefore\quad \int\frac{dx}{x^3+1}=\int\frac{dx}{3(x+1)}-\frac{1}{3}\int\frac{x\,dx}{x^2-x+1}+\frac{2}{3}\int\frac{dx}{x^2-x+1}. \quad (1)$$

上式右端第一積分可應用標準公式(5),第三積分可完

成平方法求之;其第二積分求法如下:

命 $x^2-x+1=v$; 則 $dv=2x\,dx-dx$, $x\,dx=\frac{1}{2}(dv+dx)$.

$$\therefore\quad -\frac{1}{3}\int\frac{x\,dx}{x^2-x+1}=-\frac{1}{3}\cdot\frac{1}{2}\int\frac{d(x^2-x+1)}{x^2-x+1}-\frac{1}{3}\cdot\frac{1}{2}\int\frac{dx}{x^2-x+1}$$

代入(1)式,

$$\therefore\quad \int\frac{dx}{x^3+1}=\frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{6}\int\frac{d(x^2-x+1)}{x^2-x+1}$$

$$-\frac{1}{6}\int\frac{dx}{x^2-x+1}+\frac{2}{3}\int\frac{dx}{x^2-x+1}$$

$$=\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)$$

$$-\frac{1}{6}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{2}{3}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)$$

$$-\frac{1}{6}\times\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+\frac{2}{3}\times\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$$

$$=\frac{2}{6}\log(x+1)-\frac{1}{6}\log(x^2-x+1)$$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$$

$$=\frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c.$$

282. 求 $\int\frac{z^2\,dz}{z^4+z^2-2}$.

[解] 命 $\frac{z^2}{z^4+z^2-2}=\frac{z^2}{(z^2+2)(z^2-1)}=\frac{A}{z+1}+\frac{B}{z-1}+\frac{\phi(x)}{z^2+2}$.

$\therefore\quad z^2=A(z-1)(z^2+2)+B(z+1)(z^2+2)+\phi(x)(z+1)(z-1)$. (1)

命 $z=1,\quad 1=B\cdot 2\cdot 3=6B,\qquad \therefore\quad B=\frac{1}{6}$.

命 $z=-1,\ 1=A(-2)3=-6A,\quad \therefore\quad A=-\frac{1}{6}$.

代入(1)式依$\phi(x)$解之,

$$\phi(x)=\left[z^2+\frac{(z-1)(z^2+2)}{6}-\frac{(z+1)(z^2+2)}{6}\right]\div(z+1)(z-1)$$

$$=\frac{6z^2+(z^2+2)(z-1-z-1)}{6(z+1)(z-1)}=\frac{6z^2+(z^2+2)-2}{6(z^2-1)}$$

$$=\frac{6z^2-2z^2-4}{6(z^2-1)}=\frac{4(z^2-1)}{6(z^2-1)}=\frac{4}{6}.$$

$$\therefore\quad \frac{z^2}{z^4+z^2-2}=-\frac{1}{6(z+1)}+\frac{1}{6(z-1)}+\frac{4}{6(z^2+2)}.$$

$$\therefore\quad \int\frac{z^2\,dz}{z^4+z^2-z}=-\frac{1}{6}\int\frac{dz}{z+1}+\frac{1}{6}\int\frac{dz}{z-1}+\frac{4}{6}\int\frac{dz}{z^2+(\sqrt{2})^2}$$

$$=-\frac{1}{6}\log(z+1)+\frac{1}{6}\log(z-1)$$

$$+\frac{4}{6}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c$$

$$=\frac{1}{6}\log\frac{z-1}{z+1}+\frac{2}{3\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c.$$

283. 求$\int\frac{dx}{(x^2+1)(x^2+x)}$.

[解] 命 $\frac{1}{(x^2+1)(x^2+x)}=\frac{1}{x(x+1)(x^2+1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{\phi(x)}{x^2+1}$.

$$\therefore\quad 1=A(x+1)(x^2+1)+B(x^2+1)x+\phi(x)(x+1)x.$$

命 $x=0$, $\therefore\quad A=1.$

命 $x=-1$, $\therefore\quad B=-\frac{1}{2}.$

$$\because\quad \phi\,x=\left[1-(x+1)(x^2+1)+\frac{x(x^2+1)}{2}\right]\div x(x+1)$$

$$=\frac{-x^3-2x^2-x}{2}\times\frac{1}{x(x+1)}=\frac{-x(x+1)^2}{2}\times\frac{1}{x(x+1)}$$

$$=\frac{-(x+1)}{2}=-\frac{x}{2}-\frac{1}{2}.$$

$$\therefore\quad \int\frac{dx}{(x^2+1)(x^2+x)}=\int\frac{dx}{x}-\frac{1}{2}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{x\,dx}{x^2+1}-\frac{1}{2}\int\frac{dx}{x^2+1}$$

$$=\log x-\frac{1}{2}\log(x+1)-\frac{1}{4}\log(x^2+1)$$

$$-\frac{1}{2}\tan^{-1}x+c$$

$$=\frac{1}{4}\log\frac{x^4}{(x+1)^2(x^2+1)}-\frac{1}{2}\tan^{-1}x+c.$$

284. 求 $\int\frac{(x^3-6)dx}{x^4+6x^2+8}$.

[解] 命 $\frac{x^3-6}{x^4+6x^2+8}=\frac{x^3-6}{(x^2+4)(x^2+2)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+2}.$

$$\therefore\quad x^3-6=(Ax+B)(x^2+2)+(Cx+D)(x^2+4).$$

即 $x^3-6=(A+C)x^3+(B+D)x^2+(2A+4C)x+2B+4D.$

比較兩端 x 同次冪之係數,

$$A+C=1,$$

$$B+D=0,$$

$$2A+4C=0,$$

$$2B+4D=-6.$$

解之，　　$A=2,\ B=3,\ C=-1,\ D=-3.$

$$\therefore\quad \frac{x^3-6}{(x^2+4)(x^2+2)}=\frac{2x+3}{x^2+4}-\frac{x+3}{x^2+2}.$$

$$\therefore\quad \int\frac{(x^3-6)dx}{(x^2+4)(x^2+2)}=\int\frac{x+3}{x^2+4}dx-\int\frac{x+3}{x^2+2}dx$$

$$=\int\frac{2x\,dx}{x^2+4}+3\int\frac{dx}{x^2+4}-\int\frac{x\,dx}{x^2+2}-3\int\frac{dx}{x^2+2}$$

$$=\log(x^2+4)+\frac{3}{2}\tan^{-1}\frac{x}{2}-\frac{1}{2}\log(x^2+2)$$

$$-\frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c$$

$$=\log\frac{x^2+4}{\sqrt{x^2+2}}+\frac{3}{2}\tan^{-1}\frac{x}{2}-\frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c$$

285.　求$\int\frac{2x\,dx}{(1+x^2)(3+x^2)}$.

〔解〕　命　$\frac{2x}{(1+x^2)(3+x^2)}=\frac{Ax+B}{1+x^2}+\frac{Cx+D}{3+x^2}$.

$$\therefore\quad 2x=(Ax+B)(3+x^2)+(Cx+D)(1+x^2).$$

即　$2x=(A+C)x^3+(B+D)x^2+(3A+C)x+3B+D.$

比較兩端x同次冪之係數，

$$A+C=0,$$

$$B+D=0,$$

$$3A+C=2,$$

$$3B+D=0.$$

解之, $A=1,\ B=0,\ C=-1,\ D=0.$

$$\therefore\quad \frac{2x}{(1+x^2)(3+x^2)}=\frac{x}{1+x^2}-\frac{x}{3+x^2}.$$

$$\therefore\quad \int\frac{2x\,dx}{(1+x^2)(3+x^2)}=\int\frac{x\,dx}{1+x^2}-\int\frac{x\,dx}{3+x^2}$$

$$=\frac{1}{2}\int\frac{2x\,dx}{x^2+1}-\frac{1}{2}\int\frac{2x\,dx}{x^2+3}$$

$$=\frac{1}{2}\log(x^2+1)-\frac{1}{2}\log(x^2+3)+c$$

$$=\frac{1}{2}\log\frac{x^2+1}{x^2+3}+c.$$

286. 求 $\int\frac{(2x^2+x+3)dx}{(x+1)(x^2+1)}$.

[解] 命 $\frac{2x^2+x+3}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{\phi(x)}{x^2+1}$.

$\therefore\quad 2x^2+x+3=A(x^2+1)+\phi(x)(x+1).$

命 $x=-1,\ 4=2A,\qquad \therefore\quad A=2$

$$\because\quad \phi(x)=\frac{2x^2+x+3-2(x^2+1)}{x+1}=\frac{x+1}{x+1}=1.$$

$$\therefore\quad \int\frac{(2x^2+x+3)dx}{(x+1)(x^2+1)}=\int\frac{2\,dx}{x+1}+\int\frac{dx}{x^2+1}$$

$$=2\log(x+1)+\tan^{-1}x+c.$$

287. 求 $\int\frac{dy}{1-y^3}$.

[解] 命 $\frac{1}{1-y^3}=\frac{A}{1-y}+\frac{\phi(y)}{y^2+y+1}$.

$\therefore\quad 1=A(y^2+y+1)+\phi(y)(1-y).$

命　$y=1,\ 1=A(1+1+1),\qquad \therefore\ A=\frac{1}{3}.$

$$\because\ \phi(y)=\frac{1-\frac{1}{3}(y^2+y+1)}{1-y}=\frac{\frac{1}{3}y^2+\frac{1}{3}y-\frac{2}{3}}{y-1}=\frac{1}{3}y+\frac{2}{3}.$$

$$\therefore\ \int\frac{dy}{1-y^3}=-\frac{1}{3}\int\frac{d(1-y)}{1-y}+\frac{1}{3}\int\frac{(y+2)dy}{y^2+y+1}$$

$$=-\frac{1}{3}\int\frac{-dy}{1-y}+\frac{1}{6}\int\frac{(2y+4)dy}{y^2+y+1}$$

$$=-\frac{1}{3}\log(1-y)+\frac{1}{6}\int\frac{2y+1}{y^2+y+1}dy+\frac{1}{6}\int\frac{3\,dy}{y^2+y+1}$$

$$=-\frac{2}{6}\log(1-y)+\frac{1}{6}\log(y^2+y+1)$$

$$+\frac{1}{6}\cdot 3\int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{6}\log\frac{y^2+y+1}{y^2-2y+1}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}+c.$$

288.　求$\int\frac{(5x^2-1)dx}{(x^2+3)(x^2-2x+5)}$.

[解]　命　$\frac{5x^2-1}{(x^2+3)(x^2-2x+5)}=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2-2x+5}$.

$$\therefore\ 5x^2-1=(Ax+B)(x^2-2x+5)+(Cx+D)(x^2+3).$$

即　$5x^2-1=(A+C)x^3+(-2A+B+D)x^2$

$$+(5A-2B+3C)x+5B+3D.$$

比較兩端x之同次冪係數,得四元一次聯立方程式如次:

$$\begin{cases} A+C=0, \\ -2A+B+D=5, \\ 5A-2B+3C=0, \\ 5B+3D=-1. \end{cases}$$

解之, $A=-2,\ B=-2,\ C=2,\ D=3.$

$$\therefore\ \frac{5x^2-1}{(x^2+3)(x^2-2x+5)}=-\frac{2x+2}{x^2+3}+\frac{2x+3}{x^2-2x+5}.$$

$$\therefore\ \int\frac{(5x^2-1)dx}{(x^2+3)(x^2-2x+5)}=-\int\frac{2x+2}{x^2+3}dx+\int\frac{2x+3}{x^2-2x+5}dx$$

$$=-\int\frac{2x\,dx}{x^2+3}-2\int\frac{dx}{x^2+3}$$

$$+\int\frac{(2x-2)dx}{x^2-2x+5}+5\int\frac{dx}{x^2-2x+5}$$

$$=-\log(x^2+3)-2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

$$+\log(x^2-2x+5)+5\int\frac{dx}{(x-1)^2+2^2}$$

$$=\log\frac{x^2-2x+5}{x^2+3}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

$$+\frac{5}{2}\tan^{-1}\frac{x-1}{2}+c.$$

第25款之例題

289. 求 $\int\frac{x^3+x-1}{(x^2+2)^2}dx$

[解] 命 $\frac{x^3+x-1}{(x^2+2)^2}=\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}.$

$$\therefore\quad x^3+x-1=Ax+B+(Cx+D)(x^2+2).$$

$$\text{即}\quad x^3+x-1=Cx^3+Dx^2+(A+2C)x+B+2D.$$

比較x同次冪之係數,

$$C=1,$$

$$D=0,$$

$$A+2C=1,$$

$$B+2D=-1.$$

解之,
$$A=-1,\ B=-1,\ C=1,\ D=0.$$

$$\therefore\quad \frac{x^3+x-1}{(x^2+2)^2}=-\frac{x+1}{(x^2+2)^2}+\frac{x}{x^2+2}.$$

$$\therefore\quad \int\frac{(x^3+x-1)dx}{(x^2+2)^2}=-\int\frac{(x+1)dx}{(x^2+2)^2}+\frac{1}{2}\int\frac{d(x^2+2)}{x^2+2}$$

$$=\frac{1}{2}\log(x^2+2)-\frac{1}{2}\int(x^2+2)^{-2}\,2x\,dx$$

$$-\int\frac{dx}{(x^2+2)^2}$$

$$=\frac{1}{2}\log(x^2+2)+\frac{1}{2}(x^2+2)^{-1}-\int\frac{dx}{(x^2+2)^2}.\quad(1)$$

上式右端最後一積分用25款公式(D)求之,與(D)式比較:

$$A=0,\ B=1,\ p=0,\ q=2,\ n=2.$$

代入25款公式(D),得

$$\int\frac{dx}{(x^2+2)^2}=\frac{x}{2\times2(x^2+2)}+\frac{2}{8}\int\frac{dx}{x^2+2}$$

$$=\frac{x}{4(x^2+2)}+\frac{2}{8\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c.$$

代入(1)式,

$$\therefore \quad \int\frac{(x^3+x-1)dx}{(x^2+2)^2}=\frac{1}{2}\log(x^2+2)+\frac{1}{2}(x^2+2)$$

$$-\frac{x}{4(x^2+2)}-\frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c$$

$$=\log(x^2+2)^{\frac{1}{2}}+\frac{2-x}{4(x^2+2)}-\frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+c.$$

290. 求 $\int\frac{dx}{(x^2+1)^2}$.

[解] 與25款公式(D)比較: $A=0,\ B=1,\ p=0,\ q=1,\ n=2$;

代入(D)式,故

$$\int\frac{dx}{(x^2+1)^2}=\frac{4x}{8(x^2+1)}+\frac{2}{4}\int\frac{dx}{x^2+1}=\frac{x}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x+c.$$

291. 求 $\int\frac{2x\,dx}{(x+1)(x^2+1)^2}$.

[解] 命 $\frac{2x}{(x+1)(x^2+1)^2}=\frac{A}{x+1}+\frac{\phi(x)}{(x^2+1)^2}$.

$\therefore\ 2x=A(x^2+1)^2+\phi(x)(x+1)$.

命 $x=-1$, $\quad\therefore\ A=-\frac{2}{4}=-\frac{1}{2}$.

$$\because\quad \phi(x)=\frac{2x+\frac{1}{2}(x^2+1)^2}{x+1}=\frac{x^4+2x^2+4x+1}{2(x+1)}$$

$$=\frac{1}{2}(x^3-x^2+3x+1),$$

$$\therefore\quad \frac{2x}{(x+1)(x^2+1)^2}=\frac{-1}{2(x+1)}+\frac{x^3-x^2+3x+1}{2(x^2+1)^2}. \qquad (1)$$

再命　$\dfrac{x^3-x^2+3x+1}{(x^2+1)^2}=\dfrac{Bx+C}{(x^2+1)^2}+\dfrac{Dx+E}{x^2+1}$.

則　$x^3-x^2+3x+1=Bx+C+(Dx+E)(x^2+1)$

$=Dx^3+Ex^2+(B+D)x+C+E.$

比較兩端x同次冪之係數,

$$D=1,$$

$$E=-1$$

$$B+D=3,$$

$$C+E=1.$$

解之,得　$D=1, E=-1, B=2, C=2.$

$$\therefore\quad \frac{x^3-x^2+3x+1}{2(x^2+1)^2}=\frac{x+1}{(x^2+1)^2}+\frac{x-1}{2(x^2+1)}.$$

代入(1)式,

$$\therefore\quad \frac{2x}{(x+1)(x^2+1)^2}=-\frac{1}{2(x+1)}+\frac{x+1}{(x^2+1)^2}+\frac{x-1}{2(x^2+1)}.$$

$$\therefore\quad \int\frac{2x\,dx}{(x+1)(x^2+1)^2}=-\frac{1}{2}\int\frac{dx}{x+1}+\int\frac{x+1}{(x^2+1)^2}dx+\int\frac{(x-1)dx}{2(x^2+1)}$$

$$=-\frac{1}{2}\int\frac{dx}{x+1}+\int\frac{x\,dx}{(x^2+1)^2}+\int\frac{dx}{(x^2+1)^2}$$

$$+\int\frac{x\,dx}{2(x^2+1)}-\int\frac{dx}{2(x^2+1)}$$

$$=-\frac{1}{2}\log(x+1)-\frac{1}{2(x^2+1)}+\int\frac{dx}{(x^2+1)^2}$$

$$+\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x.$$

由例題290之結果,上式右端第三個積分之值爲

$$\int \frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + c.$$

代入上式,

$$\therefore \int \frac{2x\,dx}{(x+1)(x^2+1)^2} = -\frac{1}{2}\log(x+1) - \frac{1}{2(x^2+1)}$$

$$+ \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x$$

$$+ \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$$

$$= \frac{1}{4}\log\frac{x^2+1}{(x+1)^2} + \frac{x-1}{2(x^2+1)} + c.$$

292. 求 $\int\left(\frac{x^2-a^2}{x^2+a^2}\right)^2 dx.$

[解] $\left(\frac{x^2-a^2}{x^2+a^2}\right)^2 = \frac{x^4-2a^2x^2+a^4}{x^4+2a^2x^2+a^4} = 1 - \frac{4a^2x^2}{(x^2+a^2)^2}.$ (1)

命 $\frac{4a^2x^2}{(x^2+a^2)^2} = \frac{Ax+B}{(x^2+a^2)^2} + \frac{Cx+D}{x^2+a^2},$

$\therefore\ 4a^2x^2 = Ax+B+(Cx+D)(x^2+a^2)$

$= Cx^3+Dx^2+(A+a^2C)x+B+a^2D.$

比較左右兩端 x 同次冪之係數,

$$C=0,$$

$$D=4a^2,$$

$$A+a^2C=0,$$

$$B+a^2D=0.$$

解之，　　$A=0,\ B=-4a^4,\ C=0,\ D=4a^2.$

$$\therefore\quad \frac{4a^2x^2}{(x^2+a^2)^2}=-\frac{4a^4}{(x^2+a^2)^2}+\frac{4a^2}{x^2+a^2}.$$

代入(1)式，

$$\therefore\quad \left[\frac{x^2-a^2}{x^2+a^2}\right]^2=1+\frac{4a^4}{(x^2+a^2)^2}-\frac{4a^2}{x^2+a^2}.$$

$$\therefore\quad \int\left[\frac{x^2-a^2}{x^2+a^2}\right]^2dx=\int dx+4a^4\int\frac{dx}{(x^2+a^2)^2}-4a^2\int\frac{dx}{x^2+a^2}$$

$$=x-\frac{4a^2}{a}\tan^{-1}\frac{x}{a}+4a^4\int\frac{dx}{(x^2+a^2)^2}.\quad(2)$$

將(2)式右端最後一積分與25款公式(*D*)比較：

$$A=0,\ B=1,\ p=0,\ q=a^2,\ n=2.$$

代入25款公式(*D*)，得

$$\int\frac{dx}{(x^2+a^2)^2}=\frac{4x}{8a^2(x^2+a^2)}+\frac{2}{4a^2}\int\frac{dx}{x^2+a^2}$$

$$=\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^2}\cdot\frac{1}{a}\tan^{-1}\frac{x}{a}.\quad(3)$$

將(3)式之結果代入(2)式，

$$\therefore\quad \int\left[\frac{x^2-a^2}{x^2+a^2}\right]^2dx=x-4a\tan^{-1}\frac{x}{a}-\frac{4a^4x}{2a^2(x^2+a^2)}$$

$$+\frac{4a^4}{2a^3}\tan^{-1}\frac{x}{a}+c$$

$$=x-2a\tan^{-1}\frac{x}{a}-\frac{2a^2x}{x^2+a^2}+c.$$

293.　求$\int\frac{(4x+3)dx}{(4x^2+3)^3}$.

[解] $\frac{4x+3}{(4x^2+3)^3}=\frac{1}{64}\left[\frac{4x+3}{(x^2+\frac{3}{4})^3}\right]$. (1)

因分母$x^2+\frac{3}{4}$之指數爲3,故25款之公式 (D) 須應用二次,與公式(D)比較:

$$A=4,\ B=3,\ p=0,\ q=\frac{3}{4},\ n=3.$$

$$\therefore\ \int\frac{1}{64}\left[\frac{4x+3}{(x^2+\frac{3}{4})^3}\right]dx=\frac{1}{64}\left[\frac{-12+12x}{12(x^2+\frac{3}{4})^2}+3\int\frac{dx}{(x^2+\frac{3}{4})^2}\right]$$

$$=\frac{1}{64}\left[\frac{x-1}{(x^2+\frac{3}{4})^2}+3\int\frac{dx}{(x^2+\frac{3}{4})^2}\right]. \quad (2)$$

更將(2)式右端最後之一積分與25款公式(D)比較:

$$A=0,\ B=1,\ p=0,\ q=\frac{3}{4},\ n=2.$$

$$\therefore\ 3\int\frac{dx}{(x^2+\frac{3}{4})^2}=3\cdot\frac{2x}{3(x^2+\frac{3}{4})}+3\cdot\frac{2}{3}\int\frac{dx}{x^2+\frac{3}{4}}$$

$$=\frac{2x}{x^2+\frac{3}{4}}+\frac{2\times2}{\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}. \quad (3)$$

將(2),(3)之結果代入(1)式,

$$\therefore\ \int\frac{(4x+3)dx}{(4x^2+3)^3}=\int\frac{1}{64}\left[\frac{4x+3}{(x^2+\frac{3}{4})^3}\right]dx$$

$$=\frac{1}{64}\left[\frac{x-1}{(x^2+\frac{3}{4})^2}+3\int\frac{dx}{(x^2+\frac{3}{4})^2}\right]$$

$$=\frac{1}{64}\left[\frac{x-1}{(x^2+\frac{3}{4})^2}+\frac{2x}{x^2+\frac{3}{4}}+\frac{4}{\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}\right]+c$$

$$=\frac{1}{64}\left[\frac{16x-16+8x^3+24x}{(4x^2+3)^2}\right]$$

$$+\frac{1}{16\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}+c$$

$$=\frac{x^3+5x-2}{8(4x^3+3)^2}+\frac{1}{16\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}+c.$$

294. 求$\int\frac{(3x+2)dx}{(x^2-3x+3)^2}$.

[解] 與25款公式(D)比較, $A=3$, $B=2$, $p=-3$ $q=3$, $n=2$.

$$\therefore\ \int\frac{(3x+2)dx}{(x^2-3x+3)^2}=\frac{3(9-12)+(4+9)(2x-3)}{2(12-9)(x^2-3x+3)}$$

$$+\frac{4+9}{3}\int\frac{dx}{x^2-3x+3}$$

$$=\frac{-9+13(2x-3)}{6(x^2-3x+3)}+\frac{13}{3}\int\frac{dx}{\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{26x-39-9}{6(x^2-3x+3)}+\frac{13}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-3}{\sqrt{3}}+c$$

$$=\frac{13x-24}{3(x^2-3x+3)}+\frac{26}{3\sqrt{3}}\tan^{-1}\frac{2x-3}{\sqrt{3}}+c.$$

第 五 章

代換積分法

27. 代換積分法. 凡有理分數,其分母倘能析成一次或二次實數因子,其積分終必可求;且得以初等函數(即代數函數,對數函數,反三角函數等)表示之.凡有理函數恆可化爲有理分數之形,故有理分數之可積分性,亦即有理函數之可積分性也.有理分數之積分法,第四章中言之詳矣.至於無理代數函數(即式中變數含有根號)之積分,能求者實屬寥寥;然若將無理函數中之變數用一適當之新變數代換之,則此函數往往趨向二途:[第一]代換後已改爲有理函數;[第二]代換後顯然與標準式相符合.有此二端,無理函數之求積分始無問題.名此方法曰代換積分法 (integration by substitution). 又因無理函數代以適當之新變數後,即改爲有理函數,故此法亦名理化積分法(integration by rationalization).

28 微分式含 $\sqrt{a^2-x^2}$ 或 $\sqrt{x^2\pm a^2}$ 之代換公式. 微分式中若含有 $\sqrt{a^2-x^2}$ 或 $\sqrt{x^2\pm a^2}$,而欲求其積分,其最簡捷之法,恆用三角函數代換之:

(*A*)　如遇 $\sqrt{a^2-x^2}$, 命 $x=a\sin z$, 或 $x=a\cos z$;

(*B*)　如遇 $\sqrt{x^2+a^2}$, 命 $x=a\tan z$, 或 $x=a\cot z$;

(*C*)　如遇 $\sqrt{x^2-a^2}$, 命 $x=a\sec z$, 或 $x=a\csc z$.

至於欲問何以必如此代換法?其理由甚爲簡單,因由三角恆等式,有 $\sin^2\theta+\cos^2\theta=1$, $1+\tan^2\theta=\sec^2\theta$; $1+\cot^2\theta=\csc^2\theta$ 之關係,如以上式分別一一代換之,其結果亦均一一改爲有理式. 總之,代換之目的在求與原式同值之有理函數或合於標準公式之形狀以便從事求此函數之積分而已.

[舉例]　試用本款之方法,證明第二章標準公式(20)及標準公式(21).

(1)　求證 $\int\frac{dv}{\sqrt{a^2-v^2}}=\sin^{-1}\frac{v}{a}+c.$

證:　命 $v=a\sin z$;

則 $dv=a\cos z\,dz$, $v^2=a^2\sin^2 z$.

$$\therefore\quad\int\frac{dv}{\sqrt{a^2-v^2}}=\int\frac{a\cos z\,dz}{\sqrt{a^2-a^2\sin^2 z}}=\int\frac{a\cos z\,dz}{a\sqrt{1-\sin^2 z}}$$

$$=\int\frac{\cos z\,dz}{\cos z}=\int dz=z+c=\sin^{-1}\frac{v}{a}+c.$$

$$\left[\because\quad\text{由假設}\sin z=\frac{v}{a},\ \therefore\quad z=\sin^{-1}\frac{v}{a}.\right]$$

(2)　求證 $\int\frac{dv}{\sqrt{v^2+a^2}}=\log(v+\sqrt{v^2+a^2})+c.$

證:　命 $v=a\tan z$;

則 $dv=a\sec^2 z\,dz$, $v^2=a^2\tan^2 z$.

$$\therefore \int \frac{dv}{\sqrt{v^2+a^2}} = \int \frac{a\sec^2 z\,dz}{\sqrt{a^2\tan^2 z+a^2}} = \int \frac{\sec^2 z\,dz}{\sqrt{\tan^2 z+1}}$$

$$= \int \frac{\sec^2 z\,dz}{\sec z} = \int \sec z\,dz$$

$$= \log(\sec z+\tan z)+c$$

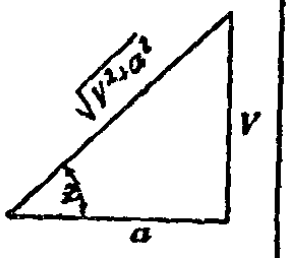
$$= \log\left(\frac{\sqrt{v^2+a^2}}{a}+\frac{v}{a}\right)+c$$

$$= \log(v+\sqrt{v^2+a^2})-\log a+c.$$

命 $c-\log a=C$.

$$\therefore \int \frac{dv}{\sqrt{v^2+a^2}} = \log(v+\sqrt{v^2+a^2})+C.$$

[由假設 $\tan z=\frac{v}{a}$, a 爲直角三角形之鄰邊, v 爲對邊, 故斜邊爲 $\sqrt{v^2+a^2}$, $\therefore \sec z=\frac{\sqrt{v^2+a^2}}{a}$.



]

(3) 求證 $\int \frac{dv}{\sqrt{v^2-a^2}} = \log(v+\sqrt{v^2-a^2})+c.$

證: 命 $v=a\sec z$;

則 $dv=a\sec z\tan z\,dz$, $v^2=a^2\sec^2 z$.

$$\therefore \int \frac{dv}{\sqrt{v^2-a^2}} = \int \frac{a\sec z\tan z\,dz}{\sqrt{a^2\sec^2 z-a^2}} = \int \frac{a\sec z\tan z\,dz}{a\tan z}$$

$$= \int \sec z\,dz = \log(\sec z+\tan z)+c$$

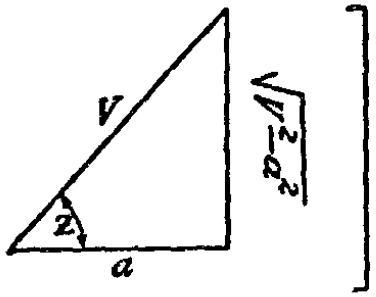
$$=\log\left(\frac{v}{a}+\frac{\sqrt{v^2-a^2}}{a}\right)+c$$

$$=\log(v+\sqrt{v^2-a^2})-\log a+C.$$

命 $c-\log a=C.$

$$\therefore\ \int\frac{dv}{\sqrt{v^2-a^2}}=\log(v+\sqrt{v^2-a^2})+C.$$

由假設 $v=a\sec z$, 即 $\sec z=\frac{v}{a}$, v 爲直角三角形之斜邊, a 爲鄰邊, 故對邊爲 $\sqrt{v^2-a^2}$, $\therefore\ \tan z=\frac{\sqrt{v^2-a^2}}{a}$.



29　**微分式僅含 x 之分指數之代換公式.**　微分式中若僅含 x 之分指數, 用

$$(D)\qquad x=z^n$$

代之, 則原式即可改成爲有理函數. 式中 z 爲代入之新變數, n 爲各分指數之最小公分母.

[舉例]　求 $\int\frac{x^{\frac{2}{3}}-x^{\frac{1}{4}}}{x^{\frac{1}{2}}}dx$

[解]　因 x 之分指數爲 $\frac{2}{3}$, $\frac{1}{4}$, $\frac{1}{2}$; 其最小公分母爲 12.

命 $x=z^{12}$;

則 $dx=12z^{11}dz$, $x^{\frac{1}{2}}=z^6$, $x^{\frac{2}{3}}=z^8$, $x^{\frac{1}{4}}=z^3$.

$$\therefore \int \frac{x^{\frac{2}{3}}-x^{\frac{1}{4}}}{x^{\frac{1}{2}}}dx=\int \frac{(z^8-z^3)\,12\,z^{11}\,dz}{z^6}=12\int(z^{13}-z^8)\,dz$$

$$=\frac{6}{7}z^{14}-\frac{4}{3}z^9+c=\frac{6}{7}x^{\frac{7}{6}}-\frac{4}{3}x^{\frac{3}{4}}+c.$$

30. 微分式僅含 $a+bx$ 之分指數之代換公式. 微分式中若僅含 $a+bx$ 之分指數, 用

$$(E) \qquad a+bx=z^n$$

代之, 則原式即可改爲有理式. 式中 z 爲新變數, n 爲各分指數之最小公分母.

[舉例] 求 $\int \frac{dx}{(x+1)^{\frac{2}{3}}-(x+1)^{\frac{1}{2}}}$.

[解] 因 $1+x$ 之分指數爲 $\frac{2}{3}$, $\frac{1}{2}$. 其 L.C.D. $=6$.

命 $x+1=z^6$; 則 $dx=6z^5\,dz$, $(x+1)^{\frac{2}{3}}=z^4$, $(x+1)^{\frac{1}{2}}=z^3$.

$$\therefore \int \frac{dx}{(x+1)^{\frac{2}{3}}-(x+1)^{\frac{1}{2}}}=\int \frac{6z^5dz}{z^4-z^3}=\int \frac{6z^2dz}{z-1}$$

$$=\int\left(6z+6+\frac{6}{z-1}\right)dz$$

$$=6\int z\,dz+6\int dz+6\int \frac{dz}{z-1}$$

$$=3z^2+6z+6\log(z-1)+c$$

$$=3(x+1)^{\frac{1}{3}}+6(x+1)^{\frac{1}{6}}+6\log[(x+1)^{\frac{1}{6}}-1]+c$$

$$=3\{(x+1)^{\frac{1}{3}}+2(x+1)^{\frac{1}{6}}+2\log[(x+1)^{\frac{1}{6}}-1]\}+c.$$

31. 微分式僅含$\sqrt{a+bx+x^2}$之代換公式. 微分式中若含有$a+bx+x^2$之平方根號,別無其他根號者,用

$$(F)\qquad \sqrt{a+bx+x^2}=z-x.$$

代之,則原式即可改爲有理式.因若將(F)平方後且依x解之,則得

$$x=\frac{z^2-a}{b+2z};\quad dx=\frac{2(z^2+bz+a)dz}{(b+2z)^2};$$

$$\sqrt{a+bx+x^2}=z-x=z-\frac{z^2-a}{b+2z}=\frac{z^2+bz+a}{b+2z}.$$

由是$x, dx, \sqrt{a+bx+x^2}$均用新變數z表之,且爲有理式.又若遇無理式如$\sqrt{C+Bx+Ax^2}$者,其中$A>0$,可改變其形如次:

$$\sqrt{C+Bx+Ax^2}=\sqrt{A}\sqrt{\frac{C}{A}+\frac{B}{A}x+x^2},$$

與代換公式(F)比較:$a=\frac{C}{A}$;$b=\frac{B}{A}$.故此類之代換公式亦併屬於本款中

[舉例] 求 $\int\frac{dx}{x\sqrt{x^2+2x-1}}$.

[解] 命$\sqrt{x^2+2x-1}=z-x$.

平方之,　　$x^2+2x-1=z^2-2zx+x^2$.

集項,且依x解之,　　$x=\frac{z^2+1}{2(z+1)}$.

$$\therefore\quad dx=\frac{1}{2}d\left(\frac{z^2+1}{z+1}\right)=\frac{1}{2}\frac{(z+1)2z\,dz-(z^2+1)dz}{(z+1)^2}$$

$$=\frac{(z^2+2z-1)dz}{2(z+1)^2}$$

而 $\sqrt{x^2+2x-1}=z-x=z-\dfrac{z^2+1}{2(z+1)}=\dfrac{2z(z+1)-z^2-1}{2(z+1)}=\dfrac{z^2+2z-1}{2(z+1)}.$

$$\therefore\ \int\frac{dx}{x\sqrt{x^2+2x-1}}=\int\frac{(z^2+2z-1)dz}{2(z+1)^2}\cdot\frac{2(z+1)}{z^2+1}\cdot\frac{2(z+1)}{z^2+2z-1}$$

$$=\int\frac{2dz}{z^2+1}=2\int\frac{dz}{z^2+1}=2\tan^{-1}z+c$$

$$=2\tan^{-1}(x+\sqrt{x^2+2x-1})+c.$$

[$\because\ \sqrt{x^2+2x-1}=z-x,\quad\therefore\quad z=x+\sqrt{x^2+2x-1}.$]

32. 微分式僅含 $\sqrt{a+bx-x^2}$ 之代換公式. 微分式中若含有 $a+bx-x^2$ 之平方根號，別無其他根號者，用

(G) $\sqrt{a+bx-x^2}=\sqrt{(x-m)(n-x)}=(x-m)z$ [或 $=(n-x)z$ 亦可.] 但 $x-m$ 及 $n-x$ 皆爲 $a+bx-x^2$ 之實數因子. 因若將 (G) 平方之，則得

$$(x-m)(n-x)=(x-m)^2z^2.$$

兩端各消去 $x-m$，然後依 x 解之，

$$x=\frac{mz^2+n}{z^2+1};\quad dx=\frac{2(m-n)z\,dz}{(z^2+1)^2};$$

$$\sqrt{a+bx-x^2}=(x-m)z=\left(\frac{mz^2+n}{z^2+1}-m\right)z=\frac{(n-m)z}{z^2+1}.$$

由是 x, dx, $\sqrt{a+bx-x^2}$ 均用新變數 z 表之，且均爲有理式. 又若遇無理式之形如 $\sqrt{C+Bx-Ax^2}$ 者，其中 $A>0$，可改變其形如次：

$$\sqrt{C+Bx-Ax^2}=\sqrt{A}\sqrt{\frac{C}{A}+\frac{B}{A}x-x^2},$$

與代換公式(G)比較：$a=\frac{C}{A}$, $b=\frac{B}{A}$, 故此類之代換公式亦併屬於本款中.

[舉例]　求 $\int\frac{dx}{x\sqrt{2+x-x^2}}$.

[解]　因 $2+x-x^2=-(x^2-x-2)=-(x-2)(x+1)=(2-x)(x+1)$.

故命 $\sqrt{(2-x)(x+1)}=(2-x)z$. [若命 $\sqrt{(2-x)(x+1)}=(x+1)z$ 亦無不可.]

平方之，　　$(2-x)(x+1)=(2-x)^2z^2$.

兩端各消去 $2-x$，　　$x+1=(2-x)z^2$.

故 x 解之，　　$x=\frac{2z^2+1}{z^2+1}$.

$$\therefore\quad dx=d\left(\frac{2z^2+1}{z^2+1}\right)=\frac{(z^2+1)4z\,dz-(2z^2-1)2z\,dz}{(z^2+1)^2}=\frac{6z\,dz}{(z^2+1)^2}.$$

而 $$\sqrt{(2-x)(x+1)}=(2-x)z=\left(2-\frac{2z^2+1}{z^2+1}\right)z=\frac{3z}{z^2+1}.$$

$$\therefore\quad \int\frac{dx}{x\sqrt{2+x-x^2}}=\int\frac{6z(z^2+1)^2dz}{3z(z^2+1)^2(2z^2-1)}=\int\frac{2\,dz}{2z^2-1}$$

$$=2\int\frac{dz}{(\sqrt{2}z)^2-1}=\frac{2}{\sqrt{2}}\int\frac{d(\sqrt{2}z)}{(\sqrt{2}z)^2-1}$$

$$=\frac{2}{\sqrt{2}}\cdot\frac{1}{2}\log\frac{\sqrt{2}z-1}{\sqrt{2}z+1}+c$$

$$=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2x+2}-\sqrt{2-x}}{\sqrt{2x+2}+\sqrt{2-x}}+c.$$

33. 二項微分. 假設有微分式爲

(1)　　$x^m(a+bx^n)^p\,dx$.

其中a,b爲隨意常數；m,n,p爲有理數,則名此微分式爲二項微分 (binomial differential).

在討論二項微分之可積分性之先,必須略變二項微分之形狀,其要點有二:[第一]括號內x之指數n須改爲正數;[第二]x之指數m與n必設法使變爲整數.若n本爲整正數,m本爲整數,以上二事均可告免.否則宜遵從以下說法:

[第一] 若n爲負數,則$-n$必爲正數,即

$$x^m(a+bx^n)^p\,dx=x^{m+np}(ax^{-n}+b)^p\,dx.$$

故不論原微分式內n之符號爲正爲負,終可設法使n爲正

[第二] 若m與n均爲分數,或有一爲分數時;則實行通分後,m與n必有一整數最小公分母,設此整數最小公分母爲a,於是將原微分式以新變數代換之如下:

命 $x=z^a$; 則 $dx=az^{a-1}\,dz$,

$$x^m(a+bx^n)^p\,dx=az^{ma+a-1}(a+bz^{na})^p\,dz.$$

上式右端之形狀與左端無異(仍爲二項微分式).因a爲m,n之最小公分母,且爲整數,故ma與na亦爲整數,由是右端新變數之指數均各爲整數.

二項微分x之指數n既改爲整正數,m亦已改爲整數.然後就二項式之指數p討論之:若p爲整數,則展開之,逐項求其積分,可無問題.普遍言之,假定p爲分數,以$\frac{r}{s}$表p之值,r與s爲整數.

綜上結論，凡二項微分必可化成

$$x^m(a+bx^n)^{\frac{r}{s}}dx$$

之形，其中 m, n, r, s 均爲整數，且 n 爲正.

34. 二項微分之可積分性之條件. 二項微分既由上款改變其指數後，茲再討論其可積分性之條件於下：假定二項微分式爲

$$(1)\qquad x^m(a+bx^n)^{\frac{r}{s}}dx.$$

式中 m, n, r, s 均爲整數，且 n 爲正.

[討論一] 若命 $a+bx^n=z^s$.

則
$$(a+bx^n)^{\frac{1}{s}}=z,\quad (a+bx^n)^{\frac{r}{s}}=z^r;$$

$$x=\left[\frac{z^s-a}{b}\right]^{\frac{1}{n}},\quad x^m=\left[\frac{z^s-a}{b}\right]^{\frac{m}{n}};$$

$$dx=\frac{s}{bn}z^{s-1}\left(\frac{z^s-a}{b}\right)^{\frac{1}{n}-1}dz.$$

代入(1)式，則得

$$x^m(a+bx^n)^{\frac{r}{s}}dx=\frac{s}{bn}z^{r+s-1}\left(\frac{z^s-a}{b}\right)^{\frac{m+1}{n}-1}.$$

上式右端 $\frac{m+1}{n}$ 若爲一整數或等於零則爲一有理式.

[討論二] 若命 $a+bx^n=z^sx^n$.

則
$$x^n=\frac{a}{z^s-b},\quad a+bx^n=z^sx^n=\frac{az^s}{z^s-b};$$

$$(a+bx^n)^{\frac{r}{s}}=a^{\frac{r}{s}}(z^s-b)^{-\frac{r}{s}}z^r;$$

$$x=a^{\frac{1}{n}}(z^s-b)^{-\frac{1}{n}},\quad x^m=a^{\frac{m}{n}}(z^s-b)^{-\frac{m}{n}};$$

$$dx=-\frac{s}{n}a^{\frac{1}{n}}z^{s-1}(z^s-b)^{-\frac{1}{n}-1}dz.$$

代入(1)式,則得

$$x^m(a+bx^n)^{\frac{r}{s}}dx=-\frac{s}{n}a^{\frac{m+1}{n}+\frac{r}{s}}(z^s-b)^{-\left(\frac{m+1}{n}+\frac{r}{s}+1\right)}z^{r+s-1}dz.$$

上式右端$\frac{m+1}{n}+\frac{r}{s}$若爲一整數或等於零則爲一有理式.

基上討論,二項徵分式爲

$$x^m(a+bx^n)^{\frac{r}{s}}dx.$$

(一)若$\frac{m+1}{n}$=整數或等於零時,用公式$a+bx^n=z^s$代換之.

(二)若$\frac{m+1}{n}+\frac{r}{s}$=整數或等於零時,用公式$a+bx^n=z^sx^n$代換之.

[舉例一] 求 $\int\frac{dy}{y(a^2+y^2)^{\frac{1}{2}}}$.

[解] $\because\ \frac{1}{y(a^2+y^2)^{\frac{1}{2}}}=y^{-1}(a^2+y^2)^{-\frac{1}{2}}.$

$$m=-1,\ n=2,\ r=-1,\ s=2;\ \frac{m+1}{n}=\frac{-1+1}{2}=\frac{0}{2}=0.$$

由34款討論一,命$a^2+y^2=z^2$,

則 $y^2=z^2-a^2,\ y=(z^2-a^2)^{\frac{1}{2}},\ dy=(z^2-a^2)^{-\frac{1}{2}}z\,dz.$

$$\therefore\ \int\frac{dy}{y(a^2+y^2)^{\frac{1}{2}}}=\int\frac{z\,dz}{(z^2-a^2)z}=\int\frac{dz}{z^2-a^2}=\frac{1}{2a}\log\frac{z-a}{z+a}+c$$

$$=\frac{1}{2a}\log\frac{\sqrt{y^2+a^2}-a}{\sqrt{y^2+a^2}+a}+c.$$

[舉例二] 求 $\int\frac{dx}{x^2(a+x^3)^{\frac{5}{3}}}$.

[解]　∵ $\dfrac{1}{x^2(a+x^3)^{\frac{5}{3}}}=x^{-2}(a+x^3)^{-\frac{5}{3}}$.

$m=-2, n=3, r=-5, s=3$; $\dfrac{m+1}{n}+\dfrac{r}{s}=\dfrac{-2+1}{3}-\dfrac{5}{3}=-2$.

由34款討論二, 命 $a+x^3=z^3x^3$,

則 $x^3=\dfrac{a}{z^3-1}$, $x=\left(\dfrac{a}{z^3-1}\right)^{\frac{1}{3}}$, $dx=-\dfrac{a^{\frac{1}{3}}z^2\,dz}{(z^3-1)^{\frac{4}{3}}}$;

$x^2=\left(\dfrac{a}{z^3-1}\right)^{\frac{2}{3}}$; $a+x^3=z^3\left(\dfrac{a}{z^3-1}\right)=\dfrac{az^3}{z^3-1}$.

$$\therefore \int\frac{dx}{x^2(a+x^3)^{\frac{5}{3}}}=\int-\frac{a^{\frac{1}{3}}z^2\,dz}{(z^3-1)^{\frac{4}{3}}\left(\frac{a}{z^3-1}\right)^{\frac{2}{3}}\left(\frac{az^3}{z^3-1}\right)^{\frac{5}{3}}}$$

$$=-\int\frac{z^3-1}{a^3z^3}\,dz=-\frac{1}{a^3}\int dz+\frac{1}{a^3}\int z^{-3}\,dz$$

$$=-\frac{1}{a^3}z-\frac{1}{2a^3z^2}+c=-\frac{1}{a^3}\left(z+\frac{1}{2z^2}\right)+c$$

$$=-\frac{1}{a^3}\left(\frac{2z^3+1}{2z^2}\right)+c=-\frac{1}{a^3}\left[\frac{2\left(\frac{x+x^3}{x^3}\right)+1}{\left(\frac{a+x^3}{x^3}\right)^{\frac{2}{3}}}\right]+c$$

$$=-\frac{1}{a^3}\left[\frac{2a+2x^3+x^3}{(a+x^3)^{\frac{2}{3}}}\right]+c$$

$$=-\frac{3x^3+2a}{2a^2x(a+x^3)^{\frac{2}{3}}}+c.$$

35. 三角微分之代換公式. 微分式若含有 $\sin\theta, \cos\theta$, 且爲有理式. 則 $\sin\theta, \cos\theta, d\theta$ 之代換公式求之如下:

由三角公式:

(1) $$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

(2) $$\cos\theta=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}.$$

因 $$\sin\frac{\theta}{2}=\frac{1}{\csc\frac{\theta}{2}}=\frac{1}{\sqrt{\cot^2\frac{\theta}{2}+1}}=\frac{\tan\frac{\theta}{2}}{\sqrt{1+\tan^2\frac{\theta}{2}}},$$

又因 $$\cos\frac{\theta}{2}=\frac{1}{\sec\frac{\theta}{2}}=\frac{1}{\sqrt{1+\tan^2\frac{\theta}{2}}}.$$

命 $$\tan\frac{\theta}{2}=z,\quad 或\quad \theta=2\tan^{-1}z;$$

故得 $$\sin\frac{\theta}{2}=\frac{z}{\sqrt{1+z^2}},\ \cos\frac{\theta}{2}=\frac{1}{\sqrt{1+z^2}}.$$

代入(1),(2)兩式,

(*H*) $$\therefore\quad \sin\theta=\frac{2z}{1+z^2},\ \cos\theta=\frac{1-z^2}{1+z^2},\ d\theta=\frac{2dz}{1+z^2}.$$

故 $\sin\theta$, $\cos\theta$, $d\theta$ 用上列三公式代換後,仍爲含新變數之有理式. 又因 $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$ 均可用 $\sin\theta$ 或 $\cos\theta$, 或二者表之,故任意有理三角微分均可用上列三公式代換之. 由是可知任意有理三角微分皆可求其積分.

[舉例] 求 $\int\frac{dx}{1-\sin x}$.

[解] 因微分式爲含 $\sin x$ 之有理式,故用本款公式(*H*)代換之,卽

$$\sin x=\frac{2z}{1+z^2},\quad dx=\frac{2dz}{1+z^2}.$$

$$\therefore\quad \int\frac{dx}{1-\sin x}=\int\frac{\frac{2dz}{1+z^2}}{1-\frac{2z}{1+z^2}}=\int\frac{2}{1+z^2}\cdot\frac{1+z^2}{z^2-2z+1}dz$$

$$=\int\frac{2dz}{z^2-2z+1}=2\int\frac{dz}{(z-1)^2}=-\frac{2}{z-1}+c$$

$$=\frac{2}{1-z}+c=\frac{2}{1-\tan\frac{x}{2}}+c.$$

36. 代換雜例. 代換之目的,無非改變微分式爲有理式,以便求其積分.但亦有代換後而終不克有理化者.究應如何代換方能變爲有理式,殊無成法可言.惟賴多解問題,以增經驗而已.

[雜例一]　求 $\int\frac{\sqrt{1+\log x}}{x}dx.$

[解]　命 $1+\log x=z$; 則 $x=e^{z-1}, dx=e^{z-1}dz.$

$$\therefore\quad \int\frac{\sqrt{1+\log x}}{x}dx=\int\frac{z^{\frac{1}{2}}}{e^{z-1}}\cdot e^{z-1}dz=\int z^{\frac{1}{2}}dz$$

$$=\frac{2}{3}z^{\frac{3}{2}}+c=\frac{2}{3}(1+\log x)^{\frac{3}{2}}+c.$$

[雜例二]　求 $\int\frac{dx}{x(a^3+x^3)}.$

[解]　命 $x^3=z$; 則 $x=z^{\frac{1}{3}}, dx=\frac{1}{3}z^{-\frac{2}{3}}dz, a^3+x^3=a^3+z.$

$$\therefore \int\frac{dx}{x(x^3+a^3)}=\int\frac{dz}{3z^{\frac{2}{3}}\cdot z^{\frac{1}{3}}(a^3+z)}=\int\frac{dz}{3a^3z+3z^2}$$

$$=\frac{1}{3}\int\frac{dz}{z^2+a^3z}=\frac{1}{3}\int\frac{dz}{\left(z+\frac{a^3}{2}\right)-\left(\frac{a^3}{2}\right)^2}$$

$$=\frac{1}{3}\cdot\frac{1}{a^3}\log\frac{2z+a^3-a^3}{2z+a^3+a^3}+c$$

$$=\frac{1}{3a^3}\log\frac{z}{z+a^3}+c=\frac{1}{3a^3}\log\frac{x^3}{x^3+a^3}+c.$$

[雜例三] 求 $\int\frac{x^2-x}{(x-2)^3}dx.$

[解] 命 $x-2=z$; 則 $x=z+2, x^2=z^2+4z+4, dx=dz.$

$$\therefore \int\frac{x^2-x}{(x-2)^3}dx=\int\frac{z^2+4z+4-z-2}{z^3}dz=\int\frac{z^2+3z+2}{z^3}dz$$

$$=\int\frac{dz}{z}+3\int z^{-2}dz+2\int z^{-3}dz=\log z-\frac{3}{z}-\frac{1}{z^2}+c$$

$$=\log(x-2)-\frac{3}{x-2}-\frac{1}{(x-2)^2}+c$$

$$=\log(x-2)-\frac{3x-5}{(x-2)^2}+c.$$

[雜例四] 求 $\int\frac{dx}{(a^2+x^2)^{\frac{3}{2}}}.$

[解] 命 $x=\frac{1}{z}$; 則 $x^2=\frac{1}{z^2}$ $dx=-\frac{dz}{z^2}.$

$$\therefore \int\frac{dx}{(a^2+x^2)^{\frac{3}{2}}}=-\int\frac{dz}{z^2\left(a^2+\frac{1}{z^2}\right)^{\frac{3}{2}}}=-\int\frac{z\,dz}{(a^2z^2+1)^{\frac{3}{2}}}$$

$$= -\frac{1}{2a^2}\int (a^2z^2+1)^{-\frac{3}{2}}\,d(a^2z^2+1)$$

$$= \frac{1}{a^2(a^2z^2+1)^{\frac{1}{2}}}+c = \frac{1}{a^2\left(\frac{a^2}{x^2}+1\right)^{\frac{1}{2}}}+c$$

$$= \frac{x}{a^2\sqrt{a^2+x^2}}+c.$$

[雜例五]　求 $\int \frac{dx}{x\sqrt{1+x+x^2}}$.

[解]　命 $x=\frac{1}{z}$; 則 $x^2=\frac{1}{z^2}$, $dx=-\frac{dz}{z^2}$.

$$\therefore \quad \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{-dz}{\sqrt{1+z+z^2}} = -\int \frac{dz}{\sqrt{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\log\left(z+\frac{1}{2}+\sqrt{1+z+z^2}\right)+c$$

$$= -\log\left(\frac{1}{x}+\frac{1}{2}+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\right)+c$$

$$= -\log\left(\frac{2+x+2\sqrt{x^2+x+1}}{2x}\right)+c$$

$$= \log\frac{2x}{2+x+2\sqrt{x^2+x+1}}+c.$$

[雜例六]　求 $\int \frac{dx}{e^{2x}-2e^x}$.

[解]　命 $e^x=z$, 則 $e^{2x}=z^2$; $dz=e^xdx$, $\therefore$ $dx=\frac{dz}{e^x}=\frac{dz}{z}$.

$$\therefore \quad \int \frac{dx}{e^{2x}-2e^x} = \int \frac{\frac{dz}{z}}{z^2-2z} = \int \frac{dz}{z^2(z-2)}.$$

命 $\dfrac{1}{z^2(z-2)}=\dfrac{A}{z-2}+\dfrac{\phi(z)}{z}$,

故 $1=Az^2+\phi(z)(z-2)$.

命 $z=2$, $\therefore$ $A=\dfrac{1}{4}$.

$$\phi(z=\left[1-\frac{z^2}{4}\right]\div(z-2)=\frac{4-z^2}{4(z-2)}=-\frac{(z-2)(z+2)}{4(z-2)}=-\frac{z}{4}-\frac{1}{2}.$$

$$\therefore\quad \int\frac{dx}{e^{2x}-2e^x}=\frac{1}{4}\int\frac{dz}{z-2}-\frac{1}{4}\int\frac{z\,dz}{z^2}-\frac{1}{2}\int\frac{dz}{z^2}$$

$$=\frac{1}{4}\log(z-2)-\frac{1}{4}\log z+\frac{1}{2z}+c$$

$$=\frac{1}{4}\log(e^x-2)-\frac{1}{4}\log e^x+\frac{1}{2e^x}+c$$

$$=\frac{1}{4}\log(e^x-2)-\frac{x}{4}+\frac{1}{2e^x}+c.$$

37. 本章例題.

第28款例題

295. 求 $\displaystyle\int\frac{\sqrt{x^2-a^2}}{x}dx$.

[解] 命 $x=a\sec z$; 則 $dx=a\sec z\tan z\,dz$, $\sqrt{x^2-a^2}=a\tan z$.

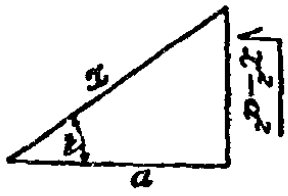
$$\therefore\quad \int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{a\tan z}{a\sec z}a\sec z\tan z\,dz=\int a\tan^2 z\,dz$$

$$=a\int(\sec^2 z-1)\,dz=a\tan z-a\,z+c$$

$$=\sqrt{x^2-a^2}-a\sec^{-1}\frac{x}{a}+c.$$

由假設 $\sec z=\frac{x}{a}$，　$\therefore$　$z=\sec^{-1}\frac{x}{a}$；

$a\tan z=\sqrt{x^2-a^2}$.



296. 求 $\int\frac{dx}{x^4\sqrt{x^2-1}}$.

[解]　命　$x=\sec z$；[28款代換公式C.]

則　$dx=\sec z\tan z\,dz$, $x^4=\sec^4 z$, $\sqrt{x^2-1}=\tan z$.

$$\therefore\quad\int\frac{dx}{x^4\sqrt{x^2-1}}=\int\frac{\sec z\tan z\,dz}{\sec^4 z\tan z}=\int\frac{dz}{\sec^3 z}=\int\cos^3 z\,dz$$

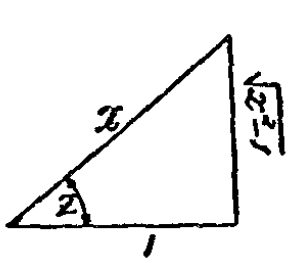
$$=\int(1-\sin^2 z)\,d(\sin z)=\sin z-\frac{\sin^3 z}{3}+c$$

$$=\frac{\sqrt{x^2-1}}{x}-\frac{1}{3}\cdot\frac{(\sqrt{x^2-1})^3}{x^3}+c$$

$$=\frac{(2x^2+1)\sqrt{x^2-1}}{3x^3}+c.$$

由假設 $\sec z=\frac{x}{1}$,

$\therefore$　$\sin z=\frac{\sqrt{x^2-1}}{x}$.



297. 求 $\int\sqrt{a^2-x^2}\,dx$.

[解]　由28款代換公式(A),

命　$x=a\sin z$；則 $dx=a\cos z\,dz$, $\sqrt{a^2-x^2}=a\cos z$.

$$\therefore \quad \int \sqrt{a^2-x^2}\,dx = \int a\cos z \cdot a\cos z\,dz = a^2\int \cos^2 z\,dz$$

$$= a^2\int\left(\frac{1}{2}+\frac{1}{2}\cos 2z\right)dz = \frac{a^2}{2}z+\frac{a^2}{4}\sin 2z+c$$

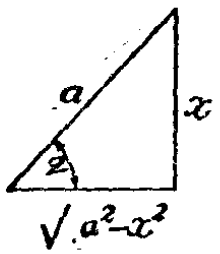
$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{4}\cdot 2\cos z\sin z+c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{a^2}{4}\cdot 2\cdot\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a}+c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c.$$

[由假設 $\sin z=\frac{x}{a}$;

$\therefore \quad z=\sin^{-1}\frac{x}{a}$, $\cos z=\frac{\sqrt{a^2-x^2}}{a}$.



]

208. 求 $\int\frac{\sqrt{x^2+a^2}}{x^2}dx$.

[解] 由28款代換公式(*B*),

命 $x=a\tan z$; 則 $dx=a\sec^2 z\,dz$, $\sqrt{x^2+a^2}=a\sec z$.

$$\therefore \quad \int\frac{\sqrt{x^2+a^2}}{x^2}dx = \int\frac{a\sec z}{a^2\tan^2 z}\,a\sec^2 z\,dz = \int\sec z\csc^2 z\,dz$$

$$= \int\sec z(1+\cot^2 z)dz = \int(\sec z+\sec z\cot^2 z)dz$$

$$= \int(\sec z+\csc z\cot z)dz$$

$$= \int\sec z\,dz+\int\csc z\cot z\,dz$$

$$= \log(\sec z + \tan z) - \csc z + c$$

$$= \log\left(\frac{\sqrt{x^2+a^2}+x}{a}\right) - \frac{\sqrt{x^2+a^2}}{x} + c$$

$$= \log(\sqrt{x^2+a^2}+x) - \log a - \frac{\sqrt{x^2+a^2}}{x} + c$$

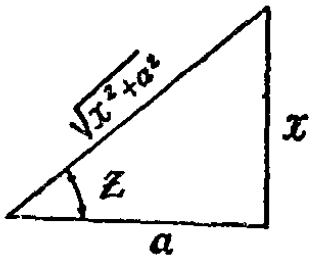
$$= \log(\sqrt{x^2+a^2}+x) - \frac{\sqrt{x^2+a^2}}{x} + c'.$$

由假設 $\tan z = \frac{x}{a}$,

$\therefore \sec z = \frac{\sqrt{x^2+a^2}}{x}$,

$\tan z = \frac{x}{a}$;

又命 $c - \log a = c'$.



299. 求 $\int \frac{x^2 dx}{\sqrt{1-x^2}}$.

[解] 命 $x = \sin z$; 則 $dx = \cos z\, dz$, $\sqrt{1-x^2} = \cos z$.

$$\therefore \int \frac{x^2 dx}{\sqrt{1-x^2}} = \int \frac{\sin^2 z \cos z dz}{\cos z} = \int \sin^2 z\, dz$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2z\right) dz$$

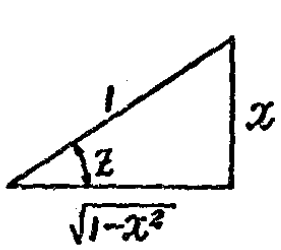
$$= \frac{1}{2}\int dz - \frac{1}{2}\int \cos 2z\, dz = \frac{z}{2} - \frac{1}{4}\sin 2z + c$$

$$= \frac{1}{2}\sin^{-1} x - \frac{1}{4} \cdot 2 \sin z \cos z + c$$

$$= -\frac{1}{2}\sin^{-1} x - \frac{1}{2} x\sqrt{1-x^2} + c.$$

由假設 $\sin z=\frac{x}{1}$, $\therefore\ z=\sin^{-1}x$;

$\cos z=\frac{\sqrt{1-x^2}}{1}=\sqrt{1-x^2}$.



300. 求 $\int\frac{dx}{x^4\sqrt{x^2+1}}$.

[解] 命 $x=\tan z$; 則 $dx=\sec^2 z\,dz$, $\sqrt{x^2+1}=\sec z$.

$$\therefore\ \int\frac{dx}{x^4\sqrt{x^2+1}}=\int\frac{\sec^2 z\,dz}{\tan^4 z\sec z}=\int\frac{\cos^3 z}{\sin^4 z}dz=\int\frac{(1-\sin^2 z)\cos z\,dz}{\sin^4 z}$$

$$=\int\frac{1-\sin^2 z}{\sin^4 z}d(\sin z)=\int\sin^{-4}d(\sin z)$$

$$-\int\sin^{-2}z\,d(\sin z)=-\frac{1}{3\sin^3 z}+\frac{1}{\sin z}+c$$

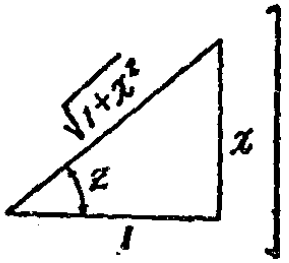
$$=-\frac{1}{3}\left(\frac{\sqrt{1+x^2}}{x}\right)^3+\frac{\sqrt{1+x^2}}{x}+c$$

$$=-\frac{(1+x^2)^{\frac{3}{2}}-3x^2\sqrt{1+x^2}}{3x^3}+c$$

$$=\frac{(2x^2-1)\sqrt{1+x^2}}{3x^3}+c.$$

由假設 $\tan z=\frac{x}{1}$,

$\therefore\ \sin z=\frac{x}{\sqrt{1+x^2}}$.



301. 求 $\int\frac{dx}{x^2\sqrt{x^2+a^2}}$.

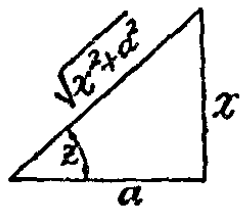
[解]　命　$x=a\tan z$；則 $dx=a\sec^2 z\,dz$, $\sqrt{x^2+a^2}=a\sec z$.

$$\therefore\quad \int\frac{dx}{x^2\sqrt{x^2+a^2}}=\int\frac{a\sec^2 z\,dz}{a^2\tan^2 z\cdot a\sec z}=\int\frac{\sec z\,dz}{a^2\tan^2 z}$$

$$=\frac{1}{a^2}\int\frac{\cos z\,dz}{\sin^2 z}=\frac{1}{a^2}\int(\sin z)^{-2}\,d(\sin z)$$

$$=-\frac{\csc z}{a^2}+c=-\frac{\sqrt{x^2+a^2}}{a^2 x}+c.$$

$$\left[\begin{array}{ll}\because & \tan z=\dfrac{x}{a},\\ \therefore & \csc z=\dfrac{\sqrt{x^2+a^2}}{x}.\end{array}\right.$$



$$\left.\vphantom{\begin{array}{ll}\because & \tan z=\dfrac{x}{a},\\ \therefore & \csc z=\dfrac{\sqrt{x^2+a^2}}{x}.\end{array}}\right]$$

302.　求 $\displaystyle\int\frac{\sqrt{x^2-a^2}}{x^4}dx.$

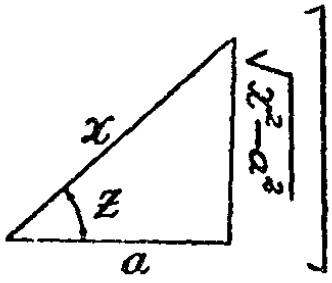
[解]　命　$x=a\sec z$；則 $dx=a\sec z\tan z\,dz$, $\sqrt{x^2-a^2}=a\tan z$.

$$\therefore\quad \int\frac{\sqrt{x^2-a^2}}{x^4}dx=\int\frac{a\tan z}{a^4\sec^4 z}\cdot a\sec z\tan z\,dz=\frac{1}{a^2}\int\frac{\tan^2 z\,dz}{\sec^3 z}$$

$$=\frac{1}{a^2}\int\sin^2 z\cos z\,dz=\frac{1}{3a^2}\sin^3 z+c$$

$$=\frac{1}{3a^2}\left(\frac{\sqrt{x^2-a^2}}{x}\right)^3+c=\frac{(x^2-a^2)^{\frac{3}{2}}}{3a^2x^3}+c.$$

$$\left[\begin{array}{ll}\because & \sec z=\dfrac{x}{a},\\ \therefore & \sin z=\dfrac{\sqrt{x^2-a^2}}{x}.\end{array}\right.$$



$$\left.\vphantom{\begin{array}{ll}\because & \sec z=\dfrac{x}{a},\\ \therefore & \sin z=\dfrac{\sqrt{x^2-a^2}}{x}.\end{array}}\right]$$

303. 求 $\int x^3(1+x^2)^{\frac{1}{2}}\,dx$.

[解] 命 $x=\tan z$, 則 $dx=\sec^2 z\,dz$, $(1+x^2)^{\frac{1}{2}}=\sec z$.

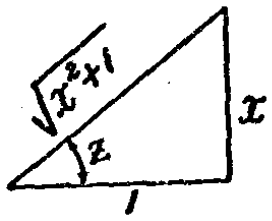
$$\therefore\quad \int x^3(1+x^2)^{\frac{1}{2}}\,dx=\int \tan^3 z\sec z\cdot\sec^2 z\,dz$$

$$=\int(\sec^2 z-1)\sec^2 z\,d(\sec z)$$

$$=\int \sec^4 z\,d(\sec z)-\int \sec^2 z\,d(\sec z)$$

$$=\frac{\sec^5 z}{5}-\frac{\sec^3 z}{3}+c=\frac{(x^2+1)^{\frac{5}{2}}}{5}-\frac{(x^2+1)^{\frac{3}{2}}}{3}+c$$

$$\left[\quad \because\quad \tan z=x=\frac{x}{1},\qquad \therefore\quad \sec z=\sqrt{x^2+1}.\right.$$



$$\left.\quad\right]$$

304. 求 $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$.

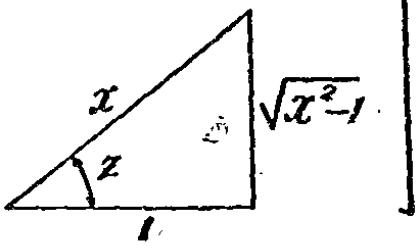
[解] 命 $x=\sec z$; 則 $dx=\sec z\tan z\,dz$, $\sqrt{x^2-1}=\tan z$.

$$\therefore\quad \int \frac{1}{x^3\sqrt{x^2-1}}\,dx=\int \frac{\sec z\tan z\,dz}{\sec^3 z\tan z}=\int \frac{dz}{\sec^2 z}=\int \cos^2 z\,dz$$

$$=\int\left(\frac{1}{2}+\frac{1}{2}\cos 2z\right)dz=\frac{z}{2}+\frac{1}{4}\sin 2z+c$$

$$=\frac{1}{2}\sec^{-1}x+\frac{\sqrt{x^2+1}}{2x^2}+c.$$

$$\left[\begin{array}{l} \because \quad \sec z=\frac{x}{1}, \quad \therefore \quad z=\sec^{-1}x; \\ \\ \sin z=\frac{\sqrt{x^2-1}}{x}; \quad \cos z=\frac{1}{x}. \end{array}\right.$$



第29款第30款之例題

305. 求 $\int \frac{3\sqrt{x}\,dx}{2\sqrt{x}-\sqrt[3]{x^2}}$.

[解] 式中x之分指數爲$\frac{1}{2}$, $\frac{1}{2}$, $\frac{2}{3}$; L.C.M.$=6$.

命 $x=z^6, dx=6z^5\,dz, \sqrt{x}=z^3, \sqrt[3]{x^2}=z^4$.

$$\therefore \int \frac{3\sqrt{x}\,dx}{2\sqrt{x}-\sqrt[3]{x^2}}=\int \frac{3z^3\cdot 6z^5\,dz}{2z^3-z^4}=\int \frac{18z^5\,dz}{z^3(2-z)}=-18\int \frac{z^3\,dz}{z-2}$$

$$=-18\int\left(z^4+2z^3+4z^2+8z+16+\frac{32}{z-2}\right)dz$$

$$=-18\left[\frac{z^5}{5}+\frac{2z^4}{4}+\frac{4z^3}{3}+\frac{8z^2}{2}\right.$$

$$\left.+16z+32\log(z-2)\right]+c$$

$$=-18\left[\frac{x^{\frac{5}{6}}}{5}+\frac{x^{\frac{2}{3}}}{2}+\frac{4x^{\frac{1}{2}}}{3}+4x^{\frac{1}{3}}+16x^{\frac{1}{6}}\right.$$

$$\left.+32\log(x^{\frac{1}{6}}-2)\right]+c.$$

306. 求 $\int \frac{x^{\frac{2}{3}}-x^{\frac{1}{3}}}{6x^{\frac{1}{4}}}dx$.

[解] 式中 x 之分指數爲 $\frac{3}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; L.C.M. = 12.

命 $x = z^{12}$; 則 $dx = 12\,z^{11}\,dz$, $x^{\frac{3}{2}} = z^{18}$, $x^{\frac{1}{3}} = z^4$, $x^{\frac{1}{4}} = z^3$.

$$\therefore \int \frac{x^{\frac{3}{2}} - x^{\frac{1}{3}}}{6\,x^{\frac{1}{4}}}\,dx = \int \frac{z^{18} - z^4}{6\,z^3} \cdot 12\,z^{11}\,dz = 2\int z^{26}\,dz - 2\int z^{12}\,dz$$

$$= \frac{2\,z^{27}}{27} - \frac{2\,z^{13}}{13} + c = \frac{2\,x^{\frac{27}{12}}}{27} - \frac{2\,x^{\frac{13}{12}}}{13} + c.$$

307. 求 $\int \frac{dx}{1+\sqrt{x}}$.

[解] 命 $x = z^2$; 則 $dx = 2\,z\,dz$, $\sqrt{x} = z$.

$$\therefore \int \frac{dx}{1+\sqrt{x}} = \int \frac{2\,z\,dz}{1+z} = 2\int \frac{z\,dz}{1+z} = 2\int dz - 2\int \frac{dz}{1+z}$$

$$= 2\,z - 2\log(1+z) + c = 2\sqrt{x} - 2\log(1+\sqrt{x}) + c.$$

308. 求 $\int \frac{x^{\frac{3}{2}}dx}{1+x}$.

[解] 命 $x = z^2$; 則 $dx = 2\,z\,dz$, $x^{\frac{3}{2}} = (z^2)^{\frac{3}{2}} = z^3$.

$$\therefore \int \frac{x^{\frac{3}{2}}dx}{1+x} = \int \frac{z^3}{1+z^2} \cdot 2\,z\,dz = 2\int \frac{z^4\,dz}{1+z^2} = 2\int \left(z^2 - 1 + \frac{1}{1+z^2}\right)dz$$

$$= 2 \cdot \frac{z^3}{3} - 2\,z + 2\tan^{-1} z + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} - 2\sqrt{x} + 2\tan^{-1}\sqrt{x} + c.$$

309. 求 $\int \frac{x^{\frac{1}{2}}dx}{x^{\frac{3}{4}} + 1}$.

[解]　命　$x=z^8$. (8 爲 $\frac{1}{2}$, $\frac{3}{4}$ 之 L.C.D.)

則　$dx=8z^7dz$, $x^{\frac{1}{2}}=(z^8)^{\frac{1}{2}}=z^4$, $x^{\frac{3}{4}}=(z^8)^{\frac{3}{4}}=z^6$.

$$\therefore \int\frac{z^4}{z^6+1}\cdot 8z^7dz=8\int\frac{z^{11}dz}{z^6+1}=8\int\left(z^5-\frac{z^5}{z^6+1}\right)dz$$

$$=8\cdot\frac{z^6}{6}-8\cdot\frac{1}{6}\log(z^6+1)+c$$

$$=\frac{4}{3}x^{\frac{6}{8}}-\frac{4}{3}\log(x^{\frac{6}{8}}+1)+c$$

$$=\frac{4}{3}x^{\frac{3}{4}}-\frac{4}{3}\log(x^{\frac{3}{4}}+1)+c.$$

310.　求　$\int\frac{dx}{x^{\frac{5}{8}}-x^{\frac{1}{8}}}$.

[解]　命　$x=z^8$, $dx=8z^7dz$, $x^{\frac{5}{8}}=z^5$, $x^{\frac{1}{8}}=z$.

$$\therefore \int\frac{dx}{x^{\frac{5}{8}}-x^{\frac{1}{8}}}=\int\frac{8z^7dz}{z^5-z}=8\int\frac{z^7dz}{z(z^4-1)}=8\int\frac{z^6dz}{z^4-1}$$

$$=8\int\left(z^2-\frac{z^2}{z^4-1}\right)dz=8\int z^2dz-8\int\frac{z^2}{z^4-1}dz$$

$$=\frac{8}{3}z^3-8\int\frac{z^2}{z^4-1}dz. \qquad (1)$$

(1) 式右端第二積分用偏分積分法求之如下:

命　$\frac{z^2}{z^4-1}=\frac{z^2}{(z-1)(z+\ )(z^2+1)}=\frac{A}{z-1}+\frac{B}{z+1}+\frac{\phi(z)}{z^2+1}$.

$\therefore$　$z^2=A(z^2+1)(z+1)+B(z^2+1)(z-1)+\phi(z)(z-1)(z+1)$

命　$z=1$,　$\therefore$　$A=\frac{1}{4}$.

命 $z=-1$, $\therefore\ B=-\frac{1}{4}$.

$$\because\ \phi(z)=\frac{4z^2-(z^2+1)(z+1)+(z^2+1)(z-1)}{4(z-1)(z+1)}=\frac{4z^2-2z^2-2}{4(z^2-1)}$$

$$=\frac{2(z^2-1)}{4(z^2-1)}=\frac{1}{2}.$$

$$\therefore\ \int\frac{z^2}{z^4-1}dz=\frac{1}{4}\int\frac{dz}{z-1}-\frac{1}{4}\int\frac{dz}{z+1}+\frac{1}{2}\int\frac{dz}{z+1}$$

$$=\frac{1}{4}\log(z-1)-\frac{1}{4}\log(z+1)+\frac{1}{2}\tan^{-1}z. \qquad (2)$$

將(2)式之結果代入(1)式,

$$\therefore\ \int\frac{dx}{x^{\frac{6}{8}}-x^{\frac{4}{8}}}=\frac{8}{3}z^3-2\log(z-1)+2\log(z+1)-4\tan^{-1}z+c$$

$$=\frac{8}{3}z^3+2\log\frac{z+1}{z-1}-4\tan^{-1}z+c$$

$$=\frac{8}{3}x^{\frac{3}{8}}+2\log\frac{x^{\frac{1}{8}}+1}{x^{\frac{1}{8}}-1}-4\tan^{-1}x^{\frac{1}{8}}+c.$$

311. 求 $\int\frac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}$.

[解] 因 $1+x$之分指數爲$\frac{3}{2}$, $\frac{1}{2}$; L.C.D.=2.

命 $1+x=z^2$; 則 $dx=2z\,dz$, $(1+x)^{\frac{3}{2}}=z^3$, $(1+x)^{\frac{1}{2}}=z$.

$$\therefore\ \int\frac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}=\int\frac{2z\,dz}{z^3+z}=2\int\frac{dz}{z^2+1}$$

$$=2\tan^{-1}z+c=2\tan^{-1}(1+x)^{\frac{1}{2}}+c.$$

312. 求 $\int \frac{(x-2)^{\frac{2}{3}}dx}{(x-2)^{\frac{2}{3}}+3}$.

[解] 命 $x-2=z^3$; 則 $dx=3z^2dz$, $(x-2)^{\frac{2}{3}}=z^2$.

$$\therefore \int \frac{(x-2)^{\frac{2}{3}}dx}{(x-2)^{\frac{2}{3}}+3}=\int \frac{z^2\cdot 3z^2dz}{z^2+3}=\int \frac{3z^4dz}{z^2+3}$$

$$=3\int\left(z^2-3+\frac{9}{z^2+3}\right)dz$$

$$=3\left[\frac{z^3}{3}-3z+9\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}\right]+c$$

$$=(x-2)-9(x-2)^{\frac{1}{3}}$$

$$+9\sqrt{3}\tan^{-1}\left(\frac{\sqrt[3]{x-2}}{\sqrt{3}}\right)+c.$$

313. 求 $\int \frac{ydy}{\sqrt{2+4y}}$.

[解] 命 $2+4y=z^2$; $dy=\frac{1}{4}2zdz=\frac{1}{2}zdz$, $y=\frac{z^2-2}{4}$.

$$\therefore \int \frac{ydy}{\sqrt{2+4y}}=\int \frac{\frac{z^2-2}{4}\cdot\frac{1}{2}zdz}{z}=\int \frac{z^2-2}{8}dz$$

$$=\int \frac{z^2}{8}dz-2\int \frac{dz}{8}=\frac{z^3}{24}-\frac{z}{4}+c$$

$$=\frac{(2+4y)^{\frac{3}{2}}}{24}-\frac{(2+4y)^{\frac{1}{2}}}{4}+c.$$

314. 求 $\int \frac{x\,dx}{(2x+3)^{\frac{4}{3}}}$.

[解] 命 $2x+3=z^3$, 則 $x=\frac{z^3-3}{2}$, $dx=\frac{1}{2}\cdot 3z^2\,dz$.

$$\therefore \int \frac{x\,dx}{(2x+3)^{\frac{4}{3}}}=\int \frac{(z^3-3)3z^2\,dz}{4z^4}=\frac{3}{4}\int z\,dz-\frac{3}{4}\int 3z^2\,dz$$

$$=\frac{3z^2}{8}+\frac{9}{4z}+c=\frac{3(2x+3)^{\frac{2}{3}}}{8}+\frac{9}{4(2x+3)^{\frac{1}{3}}}+c$$

$$=\frac{3(2x+3)+18}{8(2x+3)^{\frac{1}{3}}}+c=\frac{6x+27}{8(2x+3)^{\frac{1}{3}}}+c.$$

315. 求 $\int \frac{dx}{x(x+1)^{\frac{1}{3}}}$.

[解] 命 $1+x=z^3$; 則 $x=z^3-1$, $dx=2z\,dz$.

$$\therefore \int \frac{dx}{x(x+1)^{\frac{1}{3}}}=\int \frac{2z\,dz}{z(z^3-1)}=2\int \frac{dz}{z^2-1}=2\cdot\frac{1}{2}\log\frac{z-1}{z+1}+c$$

$$=\log\frac{(x+1)^{\frac{1}{3}}-1}{(x+1)^{\frac{1}{3}}+1}+c.$$

316. 求 $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx$.

[解] 命 $x+1=z^2$; 則 $dx=2z\,dz$.

$$\therefore \int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx=\int \frac{z+1}{z-1}\,2z\,dz=\int \frac{2z^2\,dz}{z-1}+\int \frac{2z\,dz}{z-1}$$

$$=\int 2z\,dz+\int 2\,dz+2\int \frac{2dz}{z-1}+\int 2dz+\int \frac{2dz}{z-1}$$

$$=z^2+2z+2\log(z-1)+2z+2\log(z-1)+c$$

$$=z^2+4z+4\log(z-1)+c$$

$$=x+1+4\sqrt{x+1}+4\log(\sqrt{x+1}-1)+c.$$

317. 求 $\int\frac{dx}{(x+2)\sqrt{x+1}}$.

[解] 命 $x+1=z^2$; 則 $dx=2z\,dz$, $\sqrt{x+1}=z$, $2+x=1+z^2$.

$$\therefore\ \int\frac{dx}{(x+2)\sqrt{x+1}}=\int\frac{2z\,dz}{(z^2+1)z}=2\int\frac{dz}{z^2+1}=2\tan^{-1}z+c$$

$$=2\tan^{-1}(x+1)^{\frac{1}{2}}+c.$$

318. 求 $\int\frac{x+1}{x\sqrt{x-2}}dx$.

[解] 命 $x-2=z^2$; 則 $dx=2z\,dz$, $x=z^2+2$.

$$\therefore\ \int\frac{x+1}{x\sqrt{x-2}}dx=\int\frac{(z^2+3)2z\,dz}{(z^2+2)z}=\int\frac{2z^2dz}{z^2+2}+6\int\frac{dz}{z^2+2}$$

$$=\int 2z\,dz-\int\frac{4\,dz}{z^2+2}+6\int\frac{dz}{z^2+2}$$

$$=2z^2+\frac{2}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c$$

$$=2\sqrt{x-2}+\frac{2}{\sqrt{2}}\tan^{-1}\sqrt{\frac{x-2}{2}}+c.$$

319. 求 $\int\frac{dt}{2t^{\frac{1}{2}}+t^{\frac{1}{3}}}$.

[解] 命 $t=z^6$ (6爲分指數之 L.C.D.);

則 $dt=6z^5\,dz$, $t^{\frac{1}{2}}=z^3$, $t^{\frac{1}{3}}=z^2$.

$$\therefore \int \frac{dt}{2t^{\frac{1}{3}}+t^{\frac{1}{2}}} = \int \frac{6z^5\,dz}{2z^3+z^2} = \int \frac{6z^3\,dz}{2z+1}$$

$$= \int \left(3z^2 - \frac{3}{2}z + \frac{3}{4} - \frac{3}{4(2z+1)}\right) dz$$

$$= z^3 - \frac{3}{4}z^2 + \frac{3}{4}z - \frac{3}{4}\log(2z+1) + c$$

$$= t^{\frac{1}{2}} - \frac{3}{4}t^{\frac{1}{3}} + \frac{3}{4}t^{\frac{1}{6}} - \frac{3}{4}\log(2t^{\frac{1}{6}}+1) + c.$$

320. 求 $\int \frac{dx}{1-\sqrt{x}}$.

[解] 命 $x=z^2$; 則 $\sqrt{x}=z$, $dx=2z\,dz$.

$$\therefore \int \frac{dx}{1-\sqrt{x}} = \int \frac{2z\,dz}{1-z} = \int -2\,dz + \int \frac{2\,dz}{1-z}$$

$$= -2z - 2\log(z-1) + c$$

$$= -2\sqrt{x} - 2\log(\sqrt{x}-1) + c$$

$$= -2[\sqrt{x} + \log(\sqrt{x}-1)] + c.$$

321. 求 $\int \frac{y^{\frac{1}{7}}+y^{\frac{1}{2}}}{y^{\frac{8}{7}}+y^{\frac{15}{14}}} dy$.

[解] 命 $y=z^{14}$; 則 $dy=14z^{13}\,dz$, $y^{\frac{1}{7}}=z^2$, $y^{\frac{1}{2}}=z^7$, $y^{\frac{8}{7}}=z^{16}$, $y^{\frac{15}{14}}=z^{15}$.

$$\therefore \int \frac{y^{\frac{1}{7}}+y^{\frac{1}{2}}}{y^{\frac{8}{7}}+y^{\frac{15}{14}}} dy = \int \frac{z^2+z^7}{z^{16}+z^{15}} \cdot 14z^{13}\,dz = \int \frac{z^2(1+z^5)}{z^{15}(1+z)} \cdot 14z^{13} dz$$

$$= 14\int \frac{z^5+1}{z+1} dz = 14\int \frac{(z+1)(z^4-z^3+z^2-z+1)dz}{z+1}$$

$$= 14\int (z^4-z^3+z^2-z+1)dz$$

$$=14\left(\frac{z^5}{5}-\frac{z^4}{4}+\frac{z^3}{3}-\frac{z^2}{2}+z\right)+c$$

$$=14\left(\frac{y^{\frac{5}{14}}}{5}-\frac{y^{\frac{2}{7}}}{4}+\frac{y^{\frac{3}{14}}}{3}-\frac{y^{\frac{1}{7}}}{2}+y^{\frac{1}{14}}\right)+c.$$

322. 求 $\int\frac{x\,dx}{(a+b\,x)^{\frac{3}{2}}}$.

[解] 命 $a+b\,x=z^2$; 則 $x=\frac{z^2-a}{b}$, $dx=\frac{1}{b}2\,z\,dz$.

$$\therefore\quad\int\frac{x\,dx}{(a+b\,x)^{\frac{3}{2}}}=\int\frac{(z^2-a)2\,z\,dz}{b^2z^3}=\frac{2}{b^2}\int\frac{z^2-a}{z^2}\,dz$$

$$=\frac{2}{b^2}\int dz-\frac{2\,a}{b^2}\int\frac{dz}{z^2}=\frac{2}{b^2}z+\frac{2\,a}{b^2}\cdot\frac{1}{z}+c$$

$$=\frac{2\,z^2-2\,a}{b^2\,z}+c=\frac{2\,(2\,a+b\,x)}{b^2\,(a+b\,x)^{\frac{1}{2}}}+c.$$

323. 求 $\int y(a+y)^{\frac{1}{3}}\,dy$.

[解] 命 $a+y=z^3$; 則 $y=z^3-a$, $dy=3\,z^2\,dz$.

$$\therefore\quad\int y(a+y)^{\frac{1}{3}}dy=\int(z^3-a)z\cdot3\,z^2\,dz=\int(3\,z^6-3\,a\,z^3)\,dz$$

$$=3\int z^6\,dz-3\,a\int z^3\,dz=\frac{3}{7}z^7-\frac{3}{4}a\,z^4+c$$

$$=\frac{3}{7}(a+y)^{\frac{7}{3}}-\frac{3}{4}a(a+y)^{\frac{4}{3}}+c$$

$$=\frac{3}{28}(a+y)^{\frac{4}{3}}(4\,y-3\,a)+c.$$

324 求 $\int\frac{dx}{1+\sqrt[3]{x+1}}$.

[解] 命 $x+1=z^3$; 則 $dx=3z^2dz$.

$$\therefore \int\frac{dx}{1+\sqrt[3]{x+1}}=\int\frac{3z^2dz}{z+1}=\int\left(3z-3+\frac{3}{z+1}\right)dz$$

$$=\frac{3}{2}z^2-3z+3\log(z+1)+c$$

$$=\frac{3}{2}(x+1)^{\frac{2}{3}}-3(x+1)^{\frac{1}{3}}+3\log(\sqrt[3]{x+1}+1)+c.$$

第31款第32款之例題

325. 求 $\int\frac{dx}{x\sqrt{x^2-x+2}}$.

[解] 由31款代換公式(F),

命 $\sqrt{x^2-x+2}=z-x$.

平方之, $x^2-x+2=z^2-2xz+x^2$, $\therefore\ x=\frac{z^2-2}{2z-1}$.

$$dx=\frac{2(z^2-2)-2z(2z-1)}{(2z-1)^2}dz=\frac{2z(2z-1)-2(z^2-2)}{(2z-1)^2}dz.$$

$$\sqrt{x^2-x+2}=z-x=z-\frac{z^2-2}{2z-1}=\frac{z^2-z+2}{2z-1}.$$

$$\therefore \int\frac{dx}{x\sqrt{x^2-x+2}}=\int\frac{2z(2z-1)-2(z^2-2)}{(2z-1)^2}\cdot\frac{2z-1}{z^2-2}\cdot\frac{2z-1}{z^2-z+2}dz$$

$$=\int\frac{2(z^2-z+2)dz}{(z^2-2)(z^2-z+2)}=\int\frac{2dz}{z^2-2}$$

$$=\frac{1}{\sqrt{2}}\log\frac{z-\sqrt{2}}{z+\sqrt{2}}+c$$

$$=\frac{1}{\sqrt{2}}\log\frac{\sqrt{x^2-x+2}+x-\sqrt{2}}{\sqrt{x^2-x+2}+x+\sqrt{2}}+c$$

326. 求 $\int \frac{dx}{x\sqrt{x^2+4x-4}}$.

[解] 由31款代換公式(F),

命 $\sqrt{x^2+4x-4}=z-x$.

平方之, $x^2+4x-4=z^2-2xz+x^2$, $\therefore\ x=\frac{z^2+4}{2(2+z)}$.

$$dx=\frac{2z(z+2)-(z^2+4)}{2(2+z)^2}dz=\frac{4z+2z^2-z^2+4}{2(2+z)^2}dz$$

$$=\frac{z^2+4z-4}{2(z+2)^2}dz=\frac{(z+2)^2}{2(z+2)^2}dz=\frac{dz}{2}.$$

$$\sqrt{x^2+4x-4}=z-x=z-\frac{z^2+4}{2(2+z)}=\frac{2z(2+z)-z^2-4}{2(2+z)}$$

$$=\frac{4z+2z^2-z^2+4}{2(2+z)}=\frac{(2+z)^2}{2(2+z)}=\frac{2+z}{2}.$$

$$\therefore\ \int\frac{dx}{x\sqrt{x^2+4x-4}}=\int\frac{2(2+z)2}{(z^2+4)(z+2)}\cdot\frac{dz}{2}=\int\frac{2\,dz}{z^2+4}$$

$$=\frac{2}{2}\tan^{-1}\frac{z}{2}+c=\tan^{-1}\frac{\sqrt{x^2+4x-4}}{2}+c.$$

327. 求 $\int\frac{dx}{\sqrt{1+x+x^2}}$.

[解] 由31款代換公式(F),

命 $\sqrt{1+x+x^2}=z-x$.

平方之且依x解之,

$$x=\frac{z^2-1}{2z+1};\ dx=\frac{2(z^2+z+1)dz}{(2z+1)^2},$$

$$\sqrt{1+x+x^2}=z-x=z-\frac{z^2-1}{2z+1}=\frac{z^2+z+1}{2z+1}.$$

$$\therefore \int \frac{dx}{\sqrt{1+x+x^2}} = \int \frac{(2z+1)\,2(z^2+z+1)\,dz}{(z^2+z+1)(2z+1)^2}$$

$$= \int \frac{2\,dz}{2z+1} = \log(2z+1)+c$$

$$= \log(2x+1+2\sqrt{1+x+x^2})+c.$$

328. 求 $\int \frac{(2x+x^2)^{\frac{1}{2}}}{x^2}\,dx.$

[解] 由31款代換公式(F),

命 $\sqrt{2x+x^2} = z-x.$

平方之且依 x 解之, $x = \frac{z^2}{2(1+z)}.$ $dx = \frac{z(z+2)}{2(z+1)^2}\,dz.$

$$\sqrt{2x+x^2} = z - \frac{z^2}{2(z+1)} = \frac{2z+2z^2-z^2}{2(z+1)} = \frac{z(z+2)}{2(z+1)}.$$

$$\therefore \int \frac{(2x+x^2)^{\frac{1}{2}}}{x^2}\,dx = \int \frac{z(z+2)\,z(z+2)\,4(z+1)^2}{2(z+1)\,2(z+1)^2\,z^4}\,dz$$

$$= \int \frac{(z+2)^2\,dz}{(z+1)\,z^2} = \int \frac{z^2+4z+4}{(z+1)\,z^3}\,dz. \qquad (1)$$

由偏分積分法,

命 $\frac{z^2+4z+4}{z^2(z+1)} = \frac{A}{z+1} + \frac{\phi(z)}{z^2}.$

$\therefore$ $z^2+4z+4 = Az + \phi(z)(z+1).$

命 $z=-1$, $A = 1-4+4 = 1.$

$\because$ $\phi(z) = \frac{z^2+4z+4-z^2}{z+1} = \frac{4(z+1)}{z+1} = 4.$

$\therefore$ $\frac{z^2+4z+4}{z^2(z+1)} = \frac{1}{z+1} + \frac{4}{z^2}.$ (2)

將(2)式之結果代入(1)式,

$$\therefore \quad \int \frac{(2x+x^2)^{\frac{1}{2}}}{x^2}dx = \int \frac{z^2+4z+4}{(z+1)z^2}dz = \int \frac{dz}{z+1} + 4\int z^{-2}dz$$

$$= \log(z+1) - \frac{4}{z} + c$$

$$= \log(x+1+\sqrt{2x+x^2}) - \frac{4}{x+\sqrt{2x+x^2}} + c.$$

329. 求 $\int \frac{\sqrt{x^2+4x}}{x^2}dx.$

[解] 由31款代換公式(F),

命 $\sqrt{x^2+4x} = z-x.$

平方之, $x^2+4x = z^2-2xz+x^2.$

依x解之, $4x+2xz = z^2$, $\therefore \quad x = \frac{z^2}{2(2+z)}.$

$$\therefore \quad dx = \frac{1}{2}d\left(\frac{z^2}{2+z}\right) = \frac{2z(z+2)-z^2}{2(z+2)^2} = \frac{z^2+4z}{2(z+2)^2} = \frac{z(z+4)}{2(z+2)^2}.$$

$$\sqrt{x^2+4x} = z-x = z - \frac{z^2}{2(2+z)} = \frac{z^2+4z}{2(2+z)} = \frac{z(z+4)}{2(2+z)}.$$

$$\therefore \quad \int \frac{\sqrt{x^2+4x}}{x^2}dx = \int \frac{z(z+4)z(z+4)4(2+z)^2}{2(2+z)(z+2)^2 \cdot z^4 \cdot 2}dz$$

$$= \int \frac{(z+4)^2dz}{(z+2)z^2} = \int \frac{z^2+8z+16}{(z+2)z^2}dz. \qquad (1)$$

由偏分積分法,

命 $\frac{z^2+8z+16}{(z+2)z^2} = \frac{A}{z+2} + \frac{\phi(z)}{z^2}.$

$\therefore \quad z^2+8z+16 = Az^2+\phi(z)(z+2).$

命 $z=-2$; 則 $4-16+16=4A$, $\therefore\ A=1$.

$\because\ \phi(z)=\dfrac{z^2+8z+16-z^2}{z+2}=\dfrac{8z+16}{z+2}=\dfrac{8(z+2)}{z+2}=8.$

$$\therefore\ \int\frac{z^2+8z+16}{(z+2)z^2}dz=\int\frac{dz}{z+2}+8\int z^{-2}dz$$

$$=\log(z+2)-\frac{8}{z}+c.$$

代入(1)式,

$$\therefore\ \int\frac{\sqrt{x^2+4x}}{x^2}dx=\int\frac{z^2+8z+16}{(z+2)z^2}dz=\log(z+2)-\frac{8}{z}+c$$

$$=\log(x+\sqrt{x^2+4x}+2)-\frac{8}{x+\sqrt{x^2+4x}}+c.$$

330. 求 $\displaystyle\int\frac{dx}{x\sqrt{x^2+x+1}}$.

[解] 由31款代換公式(F),

命 $\sqrt{x^2+x+1}=z-x.$

平方之, $x^2+x+1=z^2-2xz+x^2.$

依 x 解之, $\therefore\ x=\dfrac{z^2-1}{1+2z}.$

$$dx=d\left(\frac{z^2-1}{1+2z}\right)=\frac{(1+2z)2z-(z^2-1)2}{(1+2z)^2}dz=\frac{2(z^2+z+1)dz}{(1+2z)^2}.$$

$$\sqrt{x^2+x+1}=z-x=z-\frac{z^2-1}{1+2z}=\frac{z+2z^2-z^2+1}{1+2z}=\frac{z^2+z+1}{1+2z}.$$

$$\therefore\ \int\frac{dx}{x\sqrt{x^2+x+1}}=\int\frac{2(z^2+z+1)(1+2z)(1+2z)dz}{(1+2z)^2(z^2-1)(z^2+z+1)}$$

$$=\int 2\cdot\frac{dz}{z^2-1}=2\cdot\frac{1}{2}\log\frac{z-1}{z+1}+c$$

$$=\log\frac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}+c.$$

331. 求 $\int\frac{dx}{\sqrt{2+x-x^2}}$.

[解] 因 $2+x-x^2=-(x^2-x-2)=-(x+1)(x-2)=(x+1)(2-x)$.

故由32款代換公式(G),

命 $\sqrt{(x+1)(2-x)}=(x+1)z$.

平方之,消去$x+1$,且依x解之;

$$\therefore\quad x=\frac{2-z^2}{z^2+1},\ dx=\frac{-6z\,dz}{(z^2+1)^2},\ \sqrt{2+x-x^2}=(x+1)z=\frac{3z}{z^2+1}.$$

$$\therefore\quad\int\frac{dx}{\sqrt{2+x-x^2}}=\int\frac{-6z\,dz(z^2+1)}{(z^2+1)^2 3z}=-2\int\frac{dz}{z^2+1}$$

$$=-2\tan^{-1}z+c=-2\tan^{-1}\sqrt{\frac{2-x}{x+1}}+c.$$

332. 求 $\int\frac{dx}{(2ax-x^2)^{\frac{3}{2}}}$.

[解] 因 $2ax-x^2=x(2a-x)$. 故由32款代換公式(G),

命 $\sqrt{2ax-x^2}=xz$.

平方之, $x(2a-x)=x^2z^2$.

消去x, $2a-x=xz^2$, $\therefore\quad x=\frac{2a}{z^2+1}$.

$$dx=d\left(\frac{2a}{z^2+1}\right)=\frac{-4az}{(z^2+1)^2}dz,$$

$$\sqrt{2ax-x^2}=xz=\frac{2az}{z^2+1},\ (2ax-x^2)^{\frac{3}{2}}=\frac{8a^3z^3}{(z^2+1)^3}.$$

$$\therefore \int \frac{dx}{(2ax-x^2)^{\frac{3}{2}}} = \int \frac{-4a\,z(z^2+1)^3}{(z^2+1)^2\,8a^3z^3}dz = -\int \frac{(z^2+1)\,dz}{2a^2z^2}$$

$$= -\frac{1}{2a^2}\int (1+z^{-2})\,dz = -\frac{1}{2a^2}\left[z-\frac{1}{z}\right]+c$$

$$= -\frac{1}{2a^2}\left[\frac{\sqrt{x(2a-x)}}{x}-\frac{x}{\sqrt{x(2a-x)}}\right]+c$$

$$= -\frac{1}{2a^2}\left[\frac{x(2a-x)-x^2}{x\sqrt{x(2a-x)}}\right]+c$$

$$= -\frac{1}{2a^2}\left[\frac{2(a-x)}{\sqrt{x(2a-x)}}\right]+c$$

$$= \frac{1}{a^2}\left[\frac{x-a}{\sqrt{x(2a-x)}}\right]+c = \frac{x-a}{a^2\sqrt{2ax-x^2}}+c.$$

333. 求 $\int \frac{dx}{x\sqrt{5x-6-x^2}}$.

[解] 因 $5x-6-x^2 = -(x^2-5x+6) = -(x-3)(x-2) = (3-x)(x-2)$.

由32款代換公式(G),

命 $\sqrt{(3-x)(x-2)} = (3-x)z$.

平方之, $(3-x)(x-2) = (3-x)^2z^2$,

消去 $3-x$, $x-2 = (3-x)z^2$.

依 x 解之, $x = \frac{3z^2+2}{z^2+1}$.

$$dx = d\left(\frac{3z^2+2}{z^2+1}\right) = \frac{(z^2+1)6z-(3z^2+2)2z}{(z^2+1)^2}dz = \frac{2z\,dz}{(z^2+1)^2}.$$

$$(5x-6-x^2)^{\frac{1}{2}} = (3-x)z = \left(3-\frac{3z^2+2}{z^2+1}\right)z = \frac{z}{z^2+1}.$$

$$\therefore \int \frac{dx}{x\sqrt{5x-6-x^2}} = \int \frac{(z^2+1)(z^2+1)2z}{(3z^2+2)z(z^2+1)^2}dz = 2\int \frac{dz}{3z^2+2}$$

$$= \frac{2}{\sqrt{3}}\int \frac{d(\sqrt{3}z)}{(\sqrt{3}z)^2+(\sqrt{2})^2}$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\frac{\sqrt{3}}{\sqrt{2}}z + c$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \tan^{-1}\sqrt{\frac{3(x-2)}{2(3-x)}} + c.$$

334. 求 $\int \frac{x\,dx}{(2+3x-2x^2)^{\frac{3}{2}}}$.

[解] 因 $2+3x-2x^2=(2-x)(2x+1)$. 故由32款代換公式(G),

命 $\sqrt{(2-x)(2x+1)}=(2-x)z$.

平方之, $(2-x)(2x+1)=(2-x)^2z^2$.

即 $2x+1=(2-x)z^2$, $x=\frac{2z^2-1}{z^2+2}$.

$$dx=d\left(\frac{2z^2-1}{z^2+2}\right)=\frac{(z^2+2)4z-(2z^2-1)2z}{(z^2+2)^2}dz=\frac{10z\,dz}{(z^2+2)^2}.$$

$$(2+3x-2x^2)^{\frac{1}{2}}=(2-x)z=\left(2-\frac{2z^2-1}{z^2+2}\right)z=\frac{5z}{z^2+2}.$$

$$(2+3x-2x^2)^{\frac{3}{2}}=\left(\frac{5z}{z^2+2}\right)^3=\frac{125z^3}{(z^2+2)^3}.$$

$$\therefore \int \frac{x\,dx}{(2+3x-2x^2)^{\frac{3}{2}}} = \int \frac{(2z^2-1)(z^2+2)^3 10z\,dz}{(z^2+2)(z^2+2)^2 125z^3}$$

$$= \int \frac{2(2z^2-1)dz}{25z^2} = \frac{4}{25}\int dz - \frac{2}{25}\int \frac{dz}{z^2}$$

$$=\frac{4z}{25}+\frac{2}{25z}+c=\frac{4z^2+2}{25z}+c$$

$$=\frac{4(2-x)(2x+1)+2(2-x)^2}{(2-x)^2}\cdot\frac{2-x}{25\sqrt{(2-x)(2x+1)}}+c$$

$$=\frac{2(2-x)[2(2x+1)+2-x]}{(2-x)^2}\cdot\frac{2-x}{25\sqrt{(2-x)(2x+1)}}+c$$

$$=\frac{2(4+3x)}{25\sqrt{(2-x)(2x+1)}}+c=\frac{8+6x}{25\sqrt{2+3x-2x^2}}+c.$$

第34款例題(二項微分可積分性之條件)

335. 求 $\int x^3(1+x^2)^{\frac{1}{2}}dx.$

[解] $m=3, n=2, r=1, s=2$; $\therefore$ $\frac{m+1}{n}=\frac{4}{2}=2$ 爲一整數.

由34款討論一, 命 $1+x^2=z^2$ (z 爲代入之新變數).

則 $x^3=(z^2-1)^{\frac{3}{2}}, x=(z^2-1)^{\frac{1}{2}}, dx=\frac{1}{2}(z^2-1)^{-\frac{1}{2}}2z\,dz=\frac{z\,dz}{(z^2-1)^{\frac{1}{2}}}.$

$$\therefore\quad \int x^3(1+x^2)^{\frac{1}{2}}dx=\int(z^2-1)^{\frac{3}{2}}z\cdot\frac{z\,dz}{(z^2-1)^{\frac{1}{2}}}=\int z^2(z^2-1)\,dz$$

$$=\int(z^4-z^2)dz=\frac{z^5}{5}-\frac{z^3}{3}+c$$

$$=\frac{1}{5}(1+x^2)^{\frac{5}{2}}-\frac{1}{3}(1+x^2)^{\frac{3}{2}}+c.$$

336. 求 $\int\theta^5(1+\theta^2)^{\frac{3}{2}}d\theta.$

[解] $m=5, n=2, r=2, s=3$; $\therefore$ $\frac{m+1}{n}=\frac{5+1}{2}=3$ 爲一整數.

由34款討論一，命 $1+\theta^2=z^3$, $z=(1+\theta^2)^{\frac{1}{3}}$, $\theta=(z^3-1)^{\frac{1}{2}}$,

$$d\theta=\frac{1}{2}(z^3-1)^{-\frac{1}{2}}\,3z^2\,dz,\ \theta^5=(z^3-1)^{\frac{5}{2}},\ (1+\theta^2)^{\frac{2}{3}}=z^2.$$

$$\therefore\quad \int\theta^5(1+\theta^2)^{\frac{2}{3}}\,d\theta=\int(z^3-1)^{\frac{5}{2}}z^2-\frac{1}{2}(z^3-1)^{-\frac{1}{2}}\,3z^2\,dz$$

$$=\int\frac{3}{2}z^4(z^3-1)^2\,dz=\frac{3}{2}\int z^4(z^6-2z^3+1)\,dz$$

$$=\frac{3}{2}\int z^{10}\,dz-\frac{3}{2}\int z^7\,dz+\frac{3}{2}\int z^4\,dz$$

$$=\frac{3}{2}\cdot\frac{1}{11}z^{11}-\frac{3}{2}\cdot\frac{1}{8}z^8+\frac{3}{2}\cdot\frac{1}{5}z^5+c$$

$$=\frac{3}{22}z^{11}-\frac{3}{16}z^8+\frac{3}{10}z^5+c$$

$$=\frac{3}{22}(1+\theta^2)^{\frac{11}{3}}-\frac{3}{16}(1+\theta^2)^{\frac{8}{3}}+\frac{3}{10}(1+\theta^2)^{\frac{5}{3}}+c.$$

337　求 $\displaystyle\int\frac{x^3\,dx}{(a+bx^2)^{\frac{5}{2}}}$.

[解]　$m=3, n=2, r=3, s=2$; $\dfrac{m+1}{n}=\dfrac{3+1}{2}=2$ 爲一整數.

由34款討論一，命 $a+bx^2=z^2$.

則　$x=\left(\dfrac{z^2-a}{b}\right)^{\frac{1}{2}}$, $dx=\dfrac{z\,dz}{b^{\frac{1}{2}}(z^2-a)^{\frac{1}{2}}}$, $(a+bx^2)^{\frac{5}{2}}=z^5$.

$$\therefore\quad \int\frac{x^3\,dx}{(a+bx^2)^{\frac{5}{2}}}=\int\left(\frac{z^2-a}{b}\right)^{\frac{3}{2}}\cdot\frac{z\,dz}{b^{\frac{1}{2}}(z^2-a)^{\frac{1}{2}}}\cdot\frac{1}{z^5}$$

$$=\frac{1}{b^2}\int(1-az^{-2})\,dz=\frac{1}{b^2}(z+az^{-1})+c$$

$$=\frac{2a+bx^2}{b^2\sqrt{a+bx^2}}+c.$$

338. 求 $\int\frac{dx}{x^4\sqrt{1+x^2}}$

[解] $m=-4, n=2, \frac{r}{s}=-\frac{1}{2}$; $\frac{m+1}{n}+\frac{r}{s}=\frac{-4+1}{2}-\frac{1}{2}=-2$ 爲一整數.

由34款討論二, 命 $1+x^2=z^2x^2$.

則 $z=\frac{\sqrt{1+x^2}}{x}$; $x^2=\frac{1}{z^2-1}$, $1+x^2=\frac{z^2}{z^2-1}$, $\sqrt{1+x^2}=\frac{z}{(z^2-1)^{\frac{1}{2}}}$;

$$x=\frac{1}{(z^2-1)^{\frac{1}{2}}},\ x^4=\frac{1}{(z^2-1)^2};\ dx=-\frac{z\,dz}{(z^2-1)^{\frac{3}{2}}}.$$

$$\therefore\ \int\frac{dx}{x^4\sqrt{1+x^2}}=-\int\frac{z\,dz(z^2-1)^2(z-1)^{\frac{1}{2}}}{(z^2-1)^{\frac{3}{2}}\cdot z}=-\int(z^2-1)\,dz$$

$$=z-\frac{z^3}{3}+c=\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}+c.$$

339. 求 $\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}$.

[解] $m=0, n=2, \frac{r}{s}=-\frac{3}{2}$; $\frac{m+1}{n}+\frac{r}{s}=\frac{1}{2}-\frac{3}{2}=-1$ 爲一整數.

由34款討論二, 命 $1+x^2=z^2x^2$.

則 $x=(z^2-1)^{-\frac{1}{2}}$, $dx=-\frac{1}{2}(z^2-1)^{-\frac{3}{2}}2z\,dz=\frac{-z\,dz}{(z^2-1)^{\frac{3}{2}}}$;

$$x^2=(z^2-1)^{-1}=\frac{1}{z^2-1},\quad 1+x^2=1+\frac{1}{z^2-1}=\frac{z^2}{z^2-1},$$

$$(1+x^2)^{\frac{3}{2}}=\left(\frac{z^2}{z^2-1}\right)^{\frac{3}{2}}=\frac{z^3}{(z^2-1)^{\frac{3}{2}}}.$$

$$\therefore\quad \int\frac{dx}{(1+x^2)^{\frac{3}{2}}}=\int\frac{-z\,dz(z^2-1)^{\frac{3}{2}}}{(z^2-1)^{\frac{3}{2}}\cdot z^3}=-\int\frac{dz}{z^2}=\frac{1}{z}+c$$

$$=\frac{1}{\frac{\sqrt{1+x^2}}{x}}+c=\frac{x}{\sqrt{1+x^2}}+c.$$

340. 求 $\int\frac{x^3\,dx}{\sqrt{1+x^2}}$.

[解] $m=3, n=2, \frac{r}{s}=-\frac{1}{2}$; $\frac{m+1}{n}=\frac{3+1}{2}=2$ 爲一整數.

由 34 款討論一, 命 $1+x^2=z^2$.

則 $(1+x^2)^{\frac{1}{2}}=z$; $x^2=z^2-1$, $x=(z^2-1)^{\frac{1}{2}}$, $dx=\frac{z\,dz}{(z^2-1)^{\frac{1}{2}}}$.

$$\therefore\quad \int\frac{x^3\,dx}{\sqrt{1+x^2}}=\int\frac{(z^2-1)^{\frac{3}{2}}z\,dz}{(z^2-1)^{\frac{1}{2}}z}=\int(z^2-1)\,dz=\int z^2\,dz-\int dz$$

$$=\frac{z^3}{3}-z+c=\frac{(1+x^2)^{\frac{3}{2}}}{3}-(1+x^2)^{\frac{1}{2}}+c$$

$$=\frac{1}{3}(x-2)(1+x^2)^{\frac{1}{2}}+c.$$

341. 求 $\int\frac{dx}{x\sqrt{a^2-x^2}}$.

[解] $m=-1, n=2, \frac{r}{s}=-\frac{1}{2}, \frac{m+1}{n}=\frac{-1+1}{2}=0$; 故

由34款討論一, 命 $a^2-x^2=z^2$.

則 $x^2=a^2-z^2$, $x=\sqrt{a^2-z^2}$, $dx=\frac{1}{2}(a^2-z^2)^{-\frac{1}{2}}(-2z)dz$

$$=\frac{-z\,dz}{(a^2-z^2)^{\frac{1}{2}}},\ (a^2-x^2)^{\frac{1}{2}}=(z^2)^{\frac{1}{2}}=z.$$

$$\therefore \int\frac{dx}{x\sqrt{a^2-x^2}}=\int\frac{-z\,dz}{(a^2-z^2)^{\frac{1}{2}}(a^2-z^2)^{\frac{1}{2}}z}=\int\frac{-dz}{a^2-z^2}=\int\frac{dz}{z^2-a^2}$$

$$=\frac{1}{2a}\log\frac{z-a}{z+a}+c=\frac{1}{2a}\log\frac{\sqrt{a^2-x^2}-a}{\sqrt{a^2-x^2}+a}+c.$$

342. 求 $\int\frac{a\,dx}{x^2(1+x^2)^{\frac{3}{2}}}$.

[解] $m=-2,\ n=2,\ \frac{r}{s}=-\frac{3}{2}$; $\frac{m+1}{n}+\frac{r}{s}=\frac{-2+1}{2}-\frac{3}{2}=-2$

爲一整數.故由34款討論二,

命 $1+x^2=x^2z^2$.

則 $x^2-x^2z^2=1, x^2=\frac{1}{1-z^2}, x=(1-z^2)^{-\frac{1}{2}}, dx=\frac{z\,dz}{(1-z^2)^{\frac{3}{2}}}$;

$$1+x^2=x^2z^2=\left(\frac{1}{1-z^2}\right)z^2=\frac{z^2}{1-z^2},\ (1+x^2)^{\frac{3}{2}}=\frac{z^3}{(1-z^2)^{\frac{3}{2}}}.$$

$$\therefore \int\frac{a\,dx}{x^2(1+x^2)^{\frac{3}{2}}}=\int\frac{a(1-z^2)(1-z^2)^{\frac{3}{2}}z\,dz}{z^3(1-z^2)^{\frac{3}{2}}}=a\int\frac{(1-z^2)dz}{z^2}$$

$$=a\int z^{-2}dz-a\int dz=-\frac{a}{z}-a\,z+c$$

$$=-a\left(\frac{1}{z}+z\right)+c=-a\left(\frac{x}{\sqrt{1+x^2}}+\frac{\sqrt{1+x^2}}{x}\right)+c$$

$$=-a\left(\frac{x^2+1+x^2}{x\sqrt{1+x^2}}\right)+c=-a\cdot\frac{1}{\sqrt{1+x^2}}\left(\frac{2x^2+1}{x}\right)+c$$

$$=-a(1+x^2)^{-\frac{1}{2}}\left(2x+\frac{1}{x}\right)+c.$$

343. 求 $\int\frac{x^5\,dx}{\sqrt{a^2-x^2}}=\int x^5(a^2-x^2)^{-\frac{1}{2}}\,dx.$

[解] $m=5,\ n=2,\ \frac{r}{s}=-\frac{1}{2};\ \frac{m+1}{n}=\frac{5+1}{2}=3$ 爲一整數

由34款討論一，命 $a^2-x^2=z^2.$

則 $x^2=a^2-z^2,\ x=(a^2-z^2)^{\frac{1}{2}},\ x^5=(a^2-z^2)^{\frac{5}{2}};$

$$dx=d(a^2-z^2)^{\frac{1}{2}}=\frac{1}{2}(a^2-z^2)^{-\frac{1}{2}}(-2z)dz=\frac{-z\,dz}{(a^2-z^2)^{\frac{1}{2}}},$$

$$(a^2-x^2)^{\frac{1}{2}}=(z^2)^{\frac{1}{2}}=z$$

$$\therefore\quad\int\frac{x^5\,dx}{\sqrt{a^2-x^2}}=\int\frac{-(a^2-z^2)^{\frac{5}{2}}z\,dz}{(a^2-z^2)^{\frac{1}{2}}z}=\int-(a^2-z^2)^2\,dz$$

$$=\int(-a^4+2a^2z^2-z^4)\,dz$$

$$=\int-a^4\,dz+\int2a^2z^2-\int z^4\,dz$$

$$=-a^4z+\frac{2a^2z^3}{3}-\frac{z^5}{5}+c$$

$$= -a^4(a^2-x^2)^{\frac{1}{2}} + \frac{2a^2}{3}(a^2-x^2)^{\frac{3}{2}} - \frac{1}{5}(a^2-x^2)^{\frac{5}{2}} + c$$

$$= -\frac{(a^2-x^2)^{\frac{1}{2}}}{15}[15a^4 - 10a^2(a^2-x^2)$$

$$+ 3(a^2-x^2)^2] + c$$

$$= -\frac{\sqrt{a^2-x^2}}{15}(15a^4 - 10a^4 + 10a^2x^2 + 3a^4$$

$$- 6a^2x^2 + 3x^4) + c$$

$$= -\frac{\sqrt{a^2-x^2}}{15}(8a^4 + 4a^2x^2 + 3x^4) + c.$$

344. 求 $\int \sqrt{a^2-x^2}\,dx = \int (a^2-x^2)^{\frac{1}{2}}\,dx.$

[解] $m=0, n=2, \frac{r}{s}=\frac{1}{2}; \frac{m+1}{n}+\frac{r}{s}=\frac{1}{2}+\frac{1}{2}=1$ 爲一整數.

由34款討論二, 命 $a^2-x^2=z^2x^2.$

則 $a^2=z^2x^2+x^2,\ x^2=\frac{a^2}{z^2+1},\ x=\frac{a}{\sqrt{z^2+1}},\ dx=\frac{-az\,dz}{(z^2+1)^{\frac{3}{2}}};$

$$a^2-x^2=z^2x^2=z^2\left(\frac{a^2}{z^2+1}\right)=\frac{a^2z^2}{z^2+1},\quad \sqrt{a^2-x^2}=\frac{az}{(z^2+1)^{\frac{1}{2}}}.$$

$$\therefore\ \int \sqrt{a^2-x^2}\,dx = \int \frac{-az\cdot az\,dz}{(z^2+1)^{\frac{1}{2}}(z^2+1)^{\frac{3}{2}}} = \int \frac{-a^2z^2\,dz}{(z^2+1)^2}. \tag{1}$$

上式右端之積分已化爲有理式,須用偏分積分法求之:

命 $$\frac{-a^2z^2}{(z^2+1)^2} = \frac{Az+B}{z^2+1} + \frac{Cz+D}{(z^2+1)^2}. \tag{2}$$

$$\therefore \quad -a^2z^2=(Az+B)(z^2+1)+Cz+D.$$

即
$$-a^2z^2=Az^3+Bz^2+(A+C)z+B+D.$$

比較左右兩端z之同次冪係數，得聯立方程如下：

$$A=0,$$

$$B=-a^2,$$

$$A+C=0,$$

$$B+D=0.$$

解之，　$A=0,\ B=-a^2,\ C=0,\ D=a^2$

$$\therefore \quad \frac{-a^2z^2}{(z^2+1)^2}=\frac{-a^2}{z^2+1}+\frac{a^2}{(z^2+1)^2}.$$

$$\therefore \quad \int\frac{-az^2}{(z^2+1)^2}=-a^2\int\frac{dz}{z^2+1}+\int\frac{a^2dz}{(z^2+1)^2}$$

$$=-a^2\tan^{-1}z+\int\frac{a^2dz}{(z^2+1)^2}. \tag{3}$$

上式右端最末一積分用第四章25款偏分積分法公式(D)求之，與公式D比較：

$$A=0,\ B=a^2,\ p=0,\ q=1,\ n=2.$$

$$\int\frac{a^2dz}{(z^2+1)^2}=\frac{2a^2(2z)}{2\cdot4(z^2+1)}+\frac{2a^2}{4}\int\frac{dz}{z^2+1}=\frac{a^2z}{2(z^2+1)}+\frac{a^2}{2}\tan^{-1}z. \tag{4}$$

將(3)，(4)之結果代入(1)式，

$$\therefore \quad \int\sqrt{a^2-x^2}\,dx=\int\frac{-a^2z^2dz}{(z^2+1)^2}=-a^2\int\frac{dz}{z^2+1}+\int\frac{a^2dz}{(z^2+1)^2}$$

$$=-a^2\tan^{-1}z+\int\frac{a^2dz}{(z^2+1)^2}$$

$$= -a^2 \tan^{-1} z + \frac{a^2 z}{2(z^2+1)} + \frac{a^2}{2} \tan^{-1} z + c$$

$$= \frac{a^2 z}{2(z^2+1)} - \frac{a^2}{2} \tan^{-1} z + c$$

$$= \frac{a^2 \sqrt{a^2-x^2}}{x} \cdot \frac{x^2}{2a^2} - \frac{a^2}{2} \tan^{-1} \frac{\sqrt{a^2-x^2}}{x} + c$$

$$= \frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2} \tan^{-1} \frac{\sqrt{a^2-x^2}}{x} + c.$$

345. 求 $\int u(1+u)^{\frac{3}{2}} du.$

[解] $m=1,\ n=1,\ \frac{r}{s}=\frac{3}{2};\ \frac{m+1}{n}=\frac{1+1}{1}=2$ 爲一整數.

由34款討論一, 命 $1+u=z^2.$

則 $u=z^2-1,\ du=2z\,dz,\ (1+u)^{\frac{3}{2}}=(z^2)^{\frac{3}{2}}=z^3.$

$$\therefore \int u(1+u)^{\frac{3}{2}} du = \int (z^2-1)z^3 \cdot 2z\,dz = \int (z^5-z^3)2z\,dz$$

$$= 2\int (z^6-z^4)dz = 2\int z^6\,dz - 2\int z^4\,dz$$

$$= \frac{2}{7}z^7 - \frac{2}{5}z^5 + c = \frac{2}{7}(1+u)^{\frac{7}{2}} - \frac{2}{5}(1+u)^{\frac{5}{2}} + c$$

$$= \frac{2}{35}(1+u)^{\frac{5}{2}}[5(1+u)-7] + c$$

$$= \frac{2}{35}(1+u)^{\frac{5}{2}}(5u-2) + c.$$

346. 求 $\int t^3(1+2t^2)^{\frac{3}{2}} dt$

[解]　$m=3,\ n=2,\ \frac{r}{s}=\frac{3}{2};\ \frac{m+1}{n}=\frac{3+1}{2}=2$ 爲一整數.

由34款討論一，命　$1+2t^2=z^2$.

則　$2t^2=z^2-1,\ t^2=\frac{z^2-1}{2},\ t=\frac{(z^2-1)^{\frac{1}{2}}}{\sqrt{2}},\ t^3=\frac{(z^2-1)^{\frac{3}{2}}}{\sqrt{2^3}};$

$$(1+2t^2)^{\frac{3}{2}}=(z^2)^{\frac{3}{2}}=z^3;\ dt=\frac{1}{\sqrt{2}}\cdot\frac{1}{2}(z^2-1)^{-\frac{1}{2}}2z\,dz$$

$$=\frac{z\,dz}{\sqrt{2}(z^2-1)^{\frac{1}{2}}}.$$

$$\therefore\quad \int t^3(1+2t^2)^{\frac{3}{2}}dt=\int\frac{(z^2-1)^{\frac{3}{2}}\cdot z^3\cdot z\,dz}{\sqrt{2^3}\sqrt{2}(z^2-1)^{\frac{1}{2}}}=\int\frac{(z^2-1)z^4\,dz}{4}$$

$$=\frac{1}{4}\int(z^5-z^4)dz=\frac{1}{4}\int z^5\,dz-\frac{1}{4}\int z^4\,dz$$

$$=\frac{1}{4}\left(\frac{z^7}{7}-\frac{z^5}{5}\right)+c=\frac{1}{4}\left[\frac{1}{7}(1+2t^2)^{\frac{7}{2}}\right.$$

$$\left.-\frac{1}{5}(1+2t^2)^{\frac{5}{2}}\right]+c$$

$$=\frac{1}{4}(1+2t^2)^{\frac{5}{2}}\left(\frac{1+2t^2}{7}-\frac{1}{5}\right)+c$$

$$=\frac{(1+2t^2)^{\frac{5}{2}}}{4}\cdot\frac{2(5t^2-1)}{35}+c$$

$$=\frac{1}{70}(1+2t^2)^{\frac{5}{2}}(5t^2-1)+c.$$

第35款例題

347. 求 $\int \frac{dx}{1+\sin x}$.

[解] 由35款代換公式(*H*),

命 $\sin x = \frac{2z}{1+z^2}$, $dx = \frac{2\,dz}{1+z^2}$.

$$\therefore \int \frac{dx}{1+\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{z^2+2z+1}{1+z^2}} = \int \frac{2\,dz}{z^2+2z+1}$$

$$= \int \frac{2\,dz}{(z+1)^2} = 2\int (z+1)^{-2}\,dz = -\frac{2}{1+z} + c$$

$$= -\frac{2}{1+\tan\frac{x}{2}} + c.$$

$\left[\text{由35款假設 } z = \tan\frac{x}{2}.\right]$

348. 求 $\int \frac{dx}{1-\cos x}$.

[解] 由35款代換公式(*H*),

命 $\cos x = \frac{1-z^2}{1+z^2}$, $dx = \frac{2\,dz}{1+z^2}$.

$$\therefore \int \frac{dx}{1-\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{1+z^2-1+z^2}{1+z^2}}$$

$$= \int \frac{2\,dz}{1+z^2} \cdot \frac{1+z^2}{2z^2} = \int \frac{dz}{z^2} = \int z^{-2}\,dz$$

$$= -\frac{1}{z} + c = -\frac{1}{\tan\frac{x}{2}} + c.$$

$$\left[\text{由假設 } z=\tan\frac{x}{2}.\right]$$

349. 求 $\int\frac{dy}{3+2\cos y}$.

[解] 由35款代換公式(H),

命 $\cos y=\frac{1-z^2}{1+z^2}$, $dy=\frac{2dz}{1+z^2}$.

$$\therefore\int\frac{dy}{3+2\cos y}=\int\frac{\frac{2dz}{1+z^2}}{3+2\cdot\frac{1-z^2}{1+z^2}}=\int\frac{2dz}{5+z^2}=2\int\frac{dz}{z^2+(\sqrt{5})^2}$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\frac{z}{\sqrt{5}}+c=\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan\frac{y}{2}}{\sqrt{5}}\right)+c.$$

$$\left[\text{由35款假設 } z=\tan\frac{y}{2}.\right]$$

350. 求 $\int\frac{d\theta}{2+\cos\theta}$.

[解] 由35款代換公式(H),

命 $\cos\theta=\frac{1-z^2}{1+z^2}$, $d\theta=\frac{2dz}{1+z^2}$.

$$\therefore\int\frac{d\theta}{2+\cos\theta}=\int\frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}}=\int\frac{2dz}{3+z^2}=2\int\frac{dz}{z^2+(\sqrt{3})^2}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right)+c.$$

351. 求 $\int\frac{(1+\sin x)dx}{\sin x(1+\cos x)}$.

[解] 由35款代換公式(*H*),

命 $\sin x=\frac{2z}{1+z^2}$, $\cos x=\frac{1-z^2}{1+z^2}$, $dx=\frac{2dz}{1+z^2}$

$$\therefore \int\frac{(1+\sin x)dx}{\sin x(1+\cos x)}=\int\frac{\left(1+\frac{2z}{1+z^2}\right)\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2}\left(1+\frac{1-z^2}{1+z^2}\right)}=\int\frac{(1+z^2+2z)dz}{z(1+z^2+1-z^2)}$$

$$=\int\frac{(z^2+2z+1)dz}{2z}=\frac{1}{2}\int\left(z+2+\frac{1}{z}\right)dz$$

$$=\frac{1}{2}\left(\frac{z^2}{2}+2z+\log z\right)+c$$

$$=\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\log\left(\tan\frac{x}{2}\right)+c.$$

352. 求 $\int\frac{dx}{1+\cos x}$.

[解] 由35款代換公式(*H*),

命 $\cos x=\frac{1-z^2}{1+z^2}$, $dx=\frac{2dz}{1+z^2}$.

$$\therefore \int\frac{dx}{1+\cos x}=\int\frac{\frac{2dz}{1+z^2}}{1+\frac{1-z^2}{1+z^2}}=\int\frac{2dz}{2}=\int dz=z+c=\tan\frac{x}{2}+c.$$

353. 求 $\int\frac{dx}{5+3\cos x}$.

[解] 由35款代換公式(*H*),

命 $\cos x=\frac{1-z^2}{1+z^2}$, $dx=\frac{2dz}{1+z^2}$.

$$\therefore \int \frac{dx}{5+3\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{5+3\cdot\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{2(z^2+4)}{1+z^2}} = \int \frac{2\,dz}{2(z^2+4)}$$

$$= \int \frac{dz}{z^2+2^2} = \frac{1}{2}\tan^{-1}\frac{z}{2}+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)+c.$$

354. 求 $\int \frac{dx}{5-3\cos x}$.

[解] 由35款代換公式(H),

命 $\cos x = \frac{1-z^2}{1+z^2}$, $dx = \frac{2\,dz}{1+z^2}$.

$$\therefore \int \frac{dx}{5-3\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{5-\frac{3(1-z^2)}{1+z^2}} = \int \frac{2\,dz}{5+5z^2-3+3z^2}$$

$$= \int \frac{dz}{4z^2+1} = \frac{1}{2}\int \frac{d(2z)}{(2z)^2+1} = \frac{1}{2}\tan^{-1}2z+c$$

$$= \frac{1}{2}\tan^{-1}\left(2\tan\frac{x}{2}\right)+c.$$

355. 求 $\int \frac{dx}{5-4\cos 2x}$.

[解] 命 $\tan x = z$, $x = \tan^{-1}z$, $dx = d(\tan^{-1}z) = \frac{dz}{1+z^2}$;

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1$$

$$= \frac{2}{\sec^2 x} - 1 = \frac{2}{1+\tan^2 x} - 1 = \frac{2}{1+z^2} - 1 = \frac{1-z^2}{1+z^2}.$$

$$\therefore \int \frac{dx}{5-4\cos 2x} = \int \frac{\frac{dz}{1+z^2}}{5-\frac{4(1-z^2)}{1+z^2}} = \int \frac{\frac{dz}{1+z^2}}{\frac{9z^2+1}{1+z^2}} = \int \frac{dz}{(3z)^2+1}$$

$$= \frac{1}{3}\int \frac{d(3z)}{(3z)^2+1} = \frac{1}{3}\tan^{-1} 3z + c$$

$$= \frac{1}{3}\tan^{-1}(3\tan x) + c.$$

356. 求 $\int \frac{dt}{2-\cos t}$.

[解] 由35款代換公式(H),

命 $\cos t = \frac{1-z^2}{1+z^2}$, $dt = \frac{2dz}{1+z^2}$.

$$\therefore \int \frac{dt}{2-\cos t} = \int \frac{\frac{2dz}{1+z^2}}{2-\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{2+2z^2-1+z^2}{1+z^2}} = \int \frac{2dz}{3z^2+1}$$

$$= 2\int \frac{dz}{(\sqrt{3}z)^2+1} = \frac{2}{\sqrt{3}}\int \frac{d(\sqrt{3}z)}{(\sqrt{3}z)^2+1}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}(\sqrt{3}z) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{t}{2}\right) + c.$$

357. 求 $\int \frac{dx}{5+4\sin 2x}$.

[解] 命 $\tan x = z$; 則 $x = \tan^{-1} z$, $dx = d(\tan^{-1} z) = \frac{dz}{1+z^2}$;

$$\sin 2x = 2\cos x \sin x = \frac{2}{\sec x \cdot \csc x} = \frac{2}{\sqrt{(1+\tan^2 x)(1+\cot^2 x)}}$$

$$= \frac{2}{\sqrt{(1+z^2)\left(1+\frac{1}{z^2}\right)}} = \frac{2z}{1+z^2}.$$

$$\therefore \int \frac{dx}{5+4\sin 2x} = \int \frac{\frac{dz}{1+z^2}}{5+4\cdot\frac{2z}{1+z^2}} = \int \frac{dz}{5z^2+8z+5}$$

$$= \frac{1}{5}\int \frac{dz}{z^2+\frac{8}{5}z+1}$$

$$= \frac{1}{5}\int \frac{dz}{z^2+\frac{8}{5}z+\left(\frac{4}{5}\right)^2+1-\frac{16}{25}}$$

$$= \frac{1}{5}\int \frac{dz}{\left(z+\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}$$

$$= \frac{1}{5}\cdot\frac{5}{3}\tan^{-1}\left(\frac{5z+4}{3}\right)+c$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{5\tan x+4}{3}\right)+c.$$

358. 求 $\int \frac{\cos x\,dx}{1+\cos x}$.

[解] 由35款代換公式(*H*),

命 $\cos x = \frac{1-z^2}{1+z^2}$, $dz = \frac{2dz}{1+z^2}$.

$$\therefore \int \frac{\cos x\,dx}{1+\cos x} = \int \frac{\frac{1-z^2}{1+z^2}\cdot\frac{2dz}{1+z^2}}{1+\frac{1-z^2}{1+z^2}} = \int \frac{2(1-z^2)dz}{(1+z^2)^2}\cdot\frac{1+z^2}{2}$$

$$= \int \frac{(1-z^2)dz}{1+z^2} = \int\left(-1+\frac{2}{1+z^2}\right)dz$$

$$= -\int dz+2\int \frac{dz}{z^2+1} = -z+2\tan^{-1}z+c$$

$$= -\tan\frac{x}{2}+2\tan^{-1}\left(\tan\frac{x}{2}\right)+c.$$

359. 求 $\int \frac{\sin x\,dx}{1+\sin x}$.

[解] 由35款代換公式(H),

令 $\sin x=\frac{2z}{1+z^2}$, $dx=\frac{2\,dz}{1+z^2}$.

$$\therefore \int \frac{\sin x\,dx}{1+\sin x}=\int \frac{\frac{2z}{1+z^2}\cdot\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}}=\int \frac{\frac{4z\,dz}{(1+z^2)^2}}{\frac{z^2+2z+1}{1+z^2}}$$

$$=\int \frac{4z\,dz}{(z^2+1)(z+1)^2}=\int \frac{2dz}{z^2+1}-\int 2(z+1)^{-2}\,dz$$

$$=2\tan^{-1}z+\frac{2}{z+1}+c$$

[用偏分積分法]

$$=2\tan^{-1}\left(\tan\frac{x}{2}\right)+\frac{2}{1+\tan\frac{x}{2}}+c$$

$$=x+\frac{2}{1+\tan\frac{x}{2}}+c.$$

360. 試用35款代換公式(H),求證第二章標準公式(16)及(17).

(一) 求證 $\int \sec v\,dv=\log(\sec v+\tan v)+c$.

[證] $\int \sec v\,dv=\int \frac{dv}{\cos v}=\int \frac{2dz}{1+z^2}\cdot\frac{1+z^2}{1-z^2}=2\int \frac{dz}{1-z^2}$

$$=\int \frac{dz}{1+z}+\int \frac{dz}{1-z}=\log(1+z)-\log(1-z)+c$$

[用偏分積分法]

$$=\log\frac{1+z}{1-z}+c=\log\frac{1+\tan\frac{v}{2}}{1-\tan\frac{v}{2}}+c$$

$$=\log\frac{1+\frac{\sqrt{1-\cos v}}{\sqrt{1+\cos v}}}{1-\frac{\sqrt{1-\cos v}}{\sqrt{1+\cos v}}}+c$$

$$=\log\frac{\sqrt{1+\cos v}+\sqrt{1-\cos v}}{\sqrt{1+\cos v}-\sqrt{1-\cos v}}+c$$

$$=\log\frac{1+\cos v+1-\cos v+2\sqrt{1-\cos^2 v}}{1+\cos v-1+\cos v}+c$$

$$=\log\frac{2+2\sin v}{2\cos v}+c=\log\frac{1+\sin v}{\cos v}+c$$

$$=\log(\sec v+\tan v)+c.$$

(二)　求證　$\int \csc v\,dv=\log(\csc v-\cot v)+c.$

[證]　$$\int \csc v\,dv=\int\frac{dv}{\sin v}=\int\frac{\frac{2\,dz}{1+z^2}}{\frac{2z}{1+z^2}}=\int\frac{dz}{z}=\log z+c$$

$$=\log\left(\tan\frac{v}{2}\right)+c=\log\left(\frac{\sqrt{1-\cos v}}{\sqrt{1+\cos v}}\right)+c$$

$$=\log\left(\frac{1-\cos v}{\sin v}\right)+c=\log(\csc v-\cot v)+c.$$

第 六 章

分部積分法

38. 分部積分法之公式. 設 u, v 爲單自變數(例如 x)之函數; 則求 uv 之微係數公式爲

$$(1)\quad \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}.$$

求 uv 之微分公式爲

$$(2)\quad d(uv)=u\,dv+v\,du.$$

遷項 $$u\,dv=d(uv)-v\,du,$$

兩端各求積分,

$$(A)\qquad \int u\,dv=uv-\int v\,du.$$

故 $u\,dv$ 之積分隨右端 $uv-\int v\,du$ 而決定. 若左端 $u\,dv$ 之積分不能直接求得, 則由 (A) 式右端間接求其積分. 此法名曰分部積分法 (integration by parts). (A) 爲分部積分之公式; 其重要用途有三:

(1) 求兩函數乘積之積分;

(2) 求對數函數之積分;

(3) 求反三角函數之積分.

應用公式(A)時,當注意者亦有三:

(1) 須將微分式分爲u及dv兩部分;且自變數之微分dx恆爲公式中dv之一部分.

(2) 公式(A)右端之v爲左端dv之積分;故應用(A)式時,dv之積分定必可求.

(3) 若微分式之形狀爲兩函數之乘積,其形狀較複雜之一部分能求得其積分者,可假定此部分爲dv.

[舉例一]　求 $\int \log x\,dx$.

[解]　命 $u=\log x,\ dv=dx$;

則 $du=d(\log x)=\dfrac{dx}{x},\ v=\int dx=x$.

代入本款公式(A),

$$\therefore \int \underbrace{\log x}_{u}\underbrace{dx}_{dv}=\underbrace{\log x}_{u}\cdot\underbrace{x}_{v}-\int \underbrace{x}_{v}\cdot\underbrace{\frac{dx}{x}}_{du}=\log x\cdot x-x+c$$

$$=x(\log x-1)+c.$$

[舉例二]　求 $\int x e^x\,dx$.

[解]　命 $u=e^x,\ dv=x\,dx$;

則 $du=e^x\,dx,\ v=\int x\,dx=\dfrac{x^2}{2}$.

代入本款公式(A),

$$\therefore \quad \int x e^x dx = e^x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} e^x dx.$$

上式右端之微分式 $\frac{x^2}{2} e^x dx$ 較左端 $x e^x dx$ 更爲複雜,顯係選擇 u 與 dv 時之不適當.茲變更如次:

命 $u = x,\ dv = e^x dx;$

則 $du = dx,\ v = \int e^x dx = e^x.$

代入公式(A),

$$\therefore \quad \int x e^x dx = x \cdot e^x - \int e^x dx = x \cdot e^x - e^x + c = e^x(x-1) + c.$$

[舉例三] 求 $\int x^2 e^{ax} dx.$

[解] 命 $u = x^2,\ dv = e^{ax} dx;$

則 $du = 2x\,dx,\ v = \int e^{ax} dx = \frac{e^{ax}}{a}.$

代入公式(A),

$$\therefore \quad \int x^2 e^{ax} dx = x^2 \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} 2x\,dx$$

$$= x^2 \cdot \frac{e^{ax}}{a} - \frac{2}{a} \int x e^{ax} dx. \qquad (1)$$

上式右端之積分已較左端略爲簡單;於是再應用公式(A)於右端最末之一積分,或可求得其積分:

命 $u = x,\ dv = e^{ax} dx;$

則 $du = dx,\ v = \int e^{ax} dx = \frac{e^{ax}}{a}.$

代入公式(A),

$$\int x\,e^{ax}\,dx = x\cdot\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}\,dx = \frac{x\,e^{ax}}{a} - \frac{e^{ax}}{a^2}. \qquad (2)$$

將(2)式之結果代入(1)式,

$$\therefore\ \int x^2\,e^{ax}\,dx = \frac{x^2e^{ax}}{a} - \frac{2}{a}\left(\frac{x\,e^{ax}}{a} - \frac{e^{ax}}{a^2}\right) + c$$

$$= \frac{x^2e^{ax}}{a} - \frac{2\,x\,e^{ax}}{a^2} + \frac{2\,e^{ax}}{a^3} + c$$

$$= \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c.$$

39. 本章例題.

361. 求 $\int \theta \sin\theta\,d\theta$.

[解] 命 $u=\theta,\quad dv=\sin\theta\,d\theta$;

則 $du=d\theta,\quad v=\int \sin\theta\,d\theta = -\cos\theta.$

$$\therefore\ \int \theta\sin\theta\,d\theta = -\theta\cos\theta - \int(-\cos\theta)\,d\theta$$

$$= -\theta\cos\theta + \sin\theta + c.$$

362. 求 $\int x\log x\,dx$.

[解] 命 $u=\log x,\quad dx=x\,dx$;

則 $du=d(\log x)=\frac{dx}{x},\quad v=\int x\,dx=\frac{x^2}{2}.$

$$\therefore\ \int x\log x\,dx = \log x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{dx}{x} = \frac{x^2}{2}\log x - \frac{x^2}{4} + c.$$

363. 求 $\int \sin^{-1} x\,dx$.

[解] 命 $u=\sin^{-1} x$, $dv=dx$;

則 $du=d(\sin^{-1} x)=\dfrac{dx}{\sqrt{1-x^2}}$, $v=\int dx=x$.

$\therefore$ $\int \sin^{-1} x\,dx=x\cdot\sin^{-1} x-\int \dfrac{x\,dx}{\sqrt{1-x^2}}$

$=x\sin^{-1} x+\sqrt{1-x^2}+c$.

364. 求 $\int x^2\log x\,dx$.

[解] 命 $u=\log x$, $dv=x^2\,dx$;

則 $du=d(\log x)=\dfrac{dx}{x}$, $v=\int x^2\,dx=\dfrac{x^3}{3}$.

$\therefore$ $\int x^2\log x\,dx=\log x\cdot\dfrac{x^3}{3}-\int \dfrac{x^3}{3}\cdot\dfrac{dx}{x}=\dfrac{x^3}{3}\log x-\dfrac{x^3}{9}+c$

$=\dfrac{x^3}{3}\left(\log x-\dfrac{1}{3}\right)+c$.

365. 求 $\int \tan^{-1} x\,dx$.

[解] 命 $u=\tan^{-1} x$, $dv=dx$;

則 $du=d(\tan^{-1} x)=\dfrac{dx}{1+x^2}$, $v=\int dx=x$.

$\therefore$ $\int \tan^{-1} x\,dx=x\cdot\tan^{-1} x-\int \dfrac{x\,dx}{1+x^2}$

$=x\cdot\tan^{-1} x-\dfrac{1}{2}\log(1+x^2)+c$.

366. 求 $\int x^n\log x\,dx$.

[解]　命　$u=\log x,\quad dv=x^n\,dx;$

則　$du=d(\log x)=\dfrac{dx}{x},\ v=\int x^n dx=\dfrac{x^{n+1}}{n+1}.$

$$\therefore\quad \int x^n\log x\,dx=\frac{x^{n+1}}{n+1}\cdot\log x-\int\frac{x^{n+1}}{n+1}\cdot\frac{dx}{x}$$

$$=\frac{x^{n+1}}{n+1}\left(\log x-\frac{1}{n+1}\right)+c.$$

867. 求　$\int x\tan^{-1}x\,dx.$

[解]　命　$u=\tan^{-1}x,\quad dv=x\,dx;$

則　$du=d(\tan^{-1}x)=\dfrac{dx}{1+x^2},\ v=\int x\,dx=\dfrac{x^2}{2}.$

$$\therefore\quad \int x\tan^{-1}x\,dx=\frac{x^2}{2}\cdot\tan^{-1}x-\int\frac{x^2}{2}\cdot\frac{dx}{1+x^2}$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1+\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c$$

$$=\frac{x^2+1}{2}\tan^{-1}x-\frac{1}{2}x+c.$$

868. 求　$\int\cot^{-1}y\,dy.$

[解]　命　$u=\cot^{-1}y,\quad dv=dy;$

則　$du=d(\cot^{-1}y)=-\dfrac{dy}{1+y^2},\ v=\int dy=y.$

$$\therefore\quad \int\cot^{-1}y\,dy=y\cot^{-1}y+\int\frac{y\,dy}{1+y^2}$$

$$=y\cot^{-1}y+\frac{1}{2}\log(1+y^2)+c.$$

369. 求 $\int x\cdot a^x\,dx$.

[解] 命 $u=x,\quad dv=a^x\,dx$;

則 $du=dx,\ v=\int a^x\,dx=\dfrac{a^x}{\log a}$.

$$\therefore\quad \int x\cdot a^x\,dx=x\cdot\frac{a^x}{\log a}-\int\frac{a^x}{\log a}\,dx=\frac{x\cdot a^x}{\log a}-\frac{a^x}{(\log a)^2}+c$$

$$=a^x\left[\frac{x}{\log a}-\frac{1}{(\log a)^2}\right]+c.$$

370. 求 $\int t^2 a^t\,dt$.

[解] 命 $t^2=u,\quad dv=a^t\,dt$;

則 $du=2t\,dt,\quad v=\int a^t\,dt=\dfrac{a^t}{\log a}$.

$$\therefore\quad \int t^2 a^t\,dt=t^2\cdot\frac{a^t}{\log a}-\int\frac{a^t}{\log a}\cdot 2t\,dt$$

$$=\frac{t^2a^t}{\log a}-\frac{2}{\log a}\int t\,a^t\,dt$$

$$=\frac{t^2a^t}{\log a}-\frac{2}{\log a}\left(\frac{t\,a^t}{\log a}-\frac{a^t}{\log^2 a}\right)+c$$

$$=a^t\left(\frac{t^2}{\log a}-\frac{2t}{\log^2 a}+\frac{2}{\log^3 a}\right)+c.$$

371. 求 $\int\cos\theta\log\sin\theta\,d\theta$.

[解] 命 $u=\log\sin\theta,\quad dv=\cos\theta\,d\theta$;

則 $du=d(\log\sin\theta)=\dfrac{d(\sin\theta)}{\sin\theta}=\dfrac{\cos\theta}{\sin\theta}d\theta,$

$$v=\int\cos\theta\,d\theta=\sin\theta.$$

代入公式(A),

$$\therefore \int \cos\theta \log\sin\theta\, d\theta = \sin\theta \log\sin\theta - \int \sin\theta \frac{\cos\theta}{\sin\theta} d\theta$$

$$= \sin\theta \log\sin\theta - \sin\theta + c$$

$$= \sin\theta (\log\sin\theta - 1) + c.$$

372. 求 $\int x^2 e^x dx$.

[解] 命 $u = x^2$, $dv = e^x dx$;

則 $du = 2x\,dx$, $v = \int e^x dx = e^x$.

$$\therefore \int x^2 e^x dx = x^2 e^x - \int e^x 2x\, dx = x^2 e^x - 2(x e^x - \int e^x dx)$$

$$= x^2 e^x - 2x e^x + 2e^x + c = e^x(x^2 - 2x + 2) + c.$$

373. 求 $\int x \sin x \cos x\, dx$.

[解] 命 $u = x$, $dv = \sin x \cos x\, dx$;

則 $du = dx$, $v = \int \sin x \cos x\, dx = \int \sin x\, d(\sin x) = \frac{\sin^2 x}{2}$.

$$\therefore \int x \sin x \cos x\, dx = x \cdot \frac{\sin^2 x}{2} - \frac{1}{2}\int \sin^2 x\, dx$$

$$= \frac{x \sin^2 x}{2} - \frac{1}{2}\int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{x \sin^2 x}{2} - \frac{1}{2}\left(\frac{x}{2} - \frac{1}{4}\sin 2x\right) + c$$

$$= \frac{x \sin^2 x}{2} - \frac{x}{4} + \frac{\sin 2x}{8} + c$$

$$= \frac{2x \sin^2 x - x}{4} + \frac{\sin 2x}{8} + c$$

$$=\frac{x(2\sin^2 x-1)}{4}+\frac{\sin 2x}{8}+c$$

$$=\frac{x\cos 2x}{4}+\frac{\sin 2x}{8}+c$$

$$=\frac{1}{4}(x\cos 2x+\frac{1}{2}\sin 2x)+c.$$

374. 求 $\int x^2 e^{-x}\,dx$.

[解] 命 $u=x^2$, $dv=e^{-x}\,dx$;

則 $du=2x\,dx$, $v=\int e^{-x}\,dx=-e^{-x}$.

$$\therefore \int x^2 e^{-x}\,dx=-x^2 e^{-x}-\int -e^{-x}\,2x\,dx=-x^2 e^{-x}+2\int e^{-x}x\,dx$$

$$=-x^2 e^{-x}+2(-x e^{-x}+\int e^{-x}\,dx)+c$$

$$=-x^2 e^{-x}-2x e^{-x}-2e^{-x}+c$$

$$=-e^{-x}(x^2+2x+2)+c.$$

375. 求 $\int \tan^{-1}\sqrt{x}\,dx$.

[解] 命 $u=\tan^{-1}\sqrt{x}$, $dv=dx$;

則 $du=d(\tan^{-1}\sqrt{x})=\frac{d(\sqrt{x})}{1+x}$, $v=\int dx=x$.

$$\therefore \int \tan^{-1}\sqrt{x}\,dx=x\tan^{-1}\sqrt{x}-\int\frac{x}{1+x}d\sqrt{x}$$

$$=x\tan^{-1}\sqrt{x}-\int d\sqrt{x}+\int\frac{d\sqrt{x}}{1+x}+c$$

$$=x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}\sqrt{x}+c.$$

376. 求 $\int x\log x\,dx$.

[解] 命 $u=\log x,\quad dv=x\,dx$;

則 $du=d(\log x)=\dfrac{dx}{x},\ v=\int x\,dx=\dfrac{x^2}{2}$.

$\therefore\ \int x\log x\,dx=\dfrac{x^2}{2}\log x-\int\dfrac{x^2}{2}\cdot\dfrac{dx}{x}=\dfrac{x^2}{2}\log x-\dfrac{x^2}{4}+c$

$=\dfrac{x^2}{2}\left(\log x-\dfrac{1}{2}\right)+c.$

377. 求 $\int\theta^2\sin\theta\,d\theta$.

[解] 命 $u=\theta^2\quad dv=\sin\theta\,d\theta$;

則 $du=2\theta\,d\theta,\quad v=\int\sin\theta\,d\theta=-\cos\theta$.

$\therefore\ \int\theta^2\sin\theta\,d\theta=-\theta^2\cos\theta-\int-\cos\theta\,2\theta\,d\theta$

$=-\theta^2\cos\theta+2\theta\sin\theta-2\int\sin\theta\,d\theta$

$=-\theta^2\cos\theta+2\theta\sin\theta+2\cos\theta+c.$

378. 求 $\int x^4\log x\,dx$.

[解] 命 $u=\log x,\quad dv=x^4\,dx$;

則 $du=d(\log x)=\dfrac{dx}{x},\ v=\int x^4\,dx=\dfrac{x^5}{5}$.

$\therefore\ \int x^4\log x\,dx=\dfrac{x^5}{5}\log x-\int\dfrac{x^5}{5}\cdot\dfrac{dx}{x}=\dfrac{x^5}{5}\log x-\dfrac{x^5}{25}+c$

$=\dfrac{x^5}{5}\left(\log x-\dfrac{1}{5}\right)+c.$

379. 求 $\int s^2 \log s \, ds$.

[解] 命 $u=\log s$, $dv=s^2\,ds$;

則 $du=d(\log s)=\dfrac{ds}{s}$, $v=\int s^2\,ds=\dfrac{s^3}{3}$.

$\therefore$ $\int s^2 \log s \, ds=\dfrac{s^3}{3}\log s-\int \dfrac{s^3}{3}\cdot\dfrac{ds}{s}=\dfrac{s^3}{3}\log s-\dfrac{s^3}{9}+c$

$=\dfrac{s^3}{3}\left(\log s-\dfrac{1}{3}\right)+c.$

380. 求 $\int x^3 e^{ax}\,dx$.

[解] 命 $u=x^3$, $dv=e^{ax}\,dx$;

則 $du=3x^2\,dx$, $v=\int e^{ax}\,dx=\dfrac{e^{ax}}{a}$.

$\therefore$ $\int x^3 e^{ax}\,dx=x^3\cdot\dfrac{e^{ax}}{a}-\int\dfrac{e^{ax}}{a}\cdot 3x^2\,dx=x^3\cdot\dfrac{e^{ax}}{a}-\dfrac{3}{a}\int e^{ax}x^2\,dx$

$=\dfrac{e^{ax}}{a}\cdot x^3-\dfrac{3}{a^2}\cdot e^{ax}(a^2x^2-2ax+2)+c$

$=\dfrac{e^{ax}}{a}\cdot x^3-\dfrac{3}{a^2}\cdot e^{ax}x^2+\dfrac{6}{a^3}\cdot e^{ax}x-\dfrac{6}{a^4}e^{ax}+c$

$=\dfrac{e^{ax}}{a}\left(x^3-\dfrac{3x^2}{a}+\dfrac{6x}{a^2}-\dfrac{6}{a^3}\right)+c.$

381. 求 $\int \phi^2 \sin\phi\,d\phi$.

[解] 命 $u=\phi^2$, $dv=\sin\phi\,d\phi$;

則 $du=2\phi\,d\phi$, $v=\int\sin\phi\,d\phi=-\cos\phi$.

$$\therefore \int \phi^2 \sin\phi\, d\phi = -\phi^2\cos\phi - \int -\cos\phi\, 2\phi\, d\phi$$

$$= -\phi^2\cos\phi + \int \cos\phi\, 2\phi\, d\phi$$

$$= -\phi^2\cos\phi + 2(\phi\sin\phi - \int \sin\phi)\, d\phi$$

$$= -\phi^2\cos\phi + 2(\phi\sin\phi + \cos\phi) + c$$

$$= 2\cos\phi + 2\phi\sin\phi - \phi^2\cos\phi + c.$$

382. 求 $\int (\log x)^2\, dx$.

[解] 命 $u = (\log x)^2$, $dv = dx$;

則 $du = d(\log^2 x) = 2\log x \frac{dx}{x}$, $v = \int dx = x$.

$$\therefore \int (\log x)^2\, dx = x(\log x)^2 - \int x \cdot 2\log x \cdot \frac{dx}{x}$$

$$= x\log^2 x - 2\int \log x\, dx$$

$$= x\log^2 x - 2x(\log x - 1) + c$$

$$= x(\log^2 x - 2\log x + 2) + c.$$

383. 求 $\int \frac{\log x\, dx}{(x+1)^2}$.

[解] 命 $u = \log x$, $dv = \frac{dx}{(x+1)^2}$;

則 $du = \frac{dx}{x}$, $v = \int \frac{dx}{(x+1)^2} = \int (x+1)^{-2}\, dx = -\frac{1}{x+1}$.

$$\therefore \int \frac{\log x\, dx}{(x+1)^2} = -\frac{\log x}{x+1} - \int -\frac{1}{x+1}\cdot\frac{dx}{x} = -\frac{\log x}{x+1} + \int \frac{dx}{(x+1)x}$$

$$= -\frac{\log x}{x+1} + \int \frac{dx}{x} - \int \frac{dx}{x+1}$$

$$= -\frac{\log x}{x+1} + \log x - \log(x+1) + c$$

$$= \frac{x}{x+1}\log x - \log(x+1) + c.$$

384. 求 $\int x\tan^2 x\,dx$.

[解] 命 $u = x$, $dv = \tan^2 x\,dx$;

則 $du = dx$, $v = \int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x$.

$\therefore$ $\int x\tan^2 x\,dx = x(\tan x - x) - \int (\tan x - x)\,dx$

$$= x\tan x - x^2 - \log\sec x + \frac{x^2}{2} + c$$

$$= x\tan x - \frac{x^2}{2} + \log\cos x + c.$$

385. 求 $\int x^2 \sin^{-1} x\,dx$.

[解] 命 $u = \sin^{-1} x$, $dv = x^2 dx$;

則 $du = d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}$, $v = \int x^2 dx = \frac{x^3}{3}$.

$\therefore$ $\int x^2 \sin^{-1} x\,dx = \frac{x^3}{3}\sin^{-1} x - \int \frac{x^3}{3}\cdot\frac{dx}{\sqrt{1-x^2}}$

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

[用代換積分法]

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{3}\left(-\frac{x^2\sqrt{1-x^2}}{3}\right.$$

$$\left. - \frac{2}{3}\sqrt{1-x^2}\right) + c$$

$$=\frac{x^3}{3}\sin^{-1}x+\frac{x^2}{9}\sqrt{1-x^2}+\frac{2}{9}\sqrt{1-x^2}+c$$

$$=\frac{x^3}{3}\sin^{-1}x+\frac{x^2+2}{9}\sqrt{1-x^2}+c.$$

386. 求 $\int \sec^2\theta \log\tan\theta\, d\theta$.

[解] 命 $u=\log\tan\theta$, $dv=\sec^2\theta\, d\theta$;

則 $du=d(\log\tan\theta)=\dfrac{d(\tan\theta)}{\tan\theta}=\dfrac{\sec^2\theta\, d\theta}{\tan\theta}$,

$$v=\int \sec^2\theta\, d\theta=\tan\theta.$$

$$\therefore \int \sec^2\theta \log\tan\theta\, d\theta=\tan\theta\cdot\log\tan\theta-\int\tan\theta\cdot\frac{\sec^2\theta\, d\theta}{\tan\theta}$$

$$=\tan\theta\cdot\log\tan\theta-\int\sec^2\theta\, d\theta$$

$$=\tan\theta\cdot\log\tan\theta-\tan\theta+c$$

$$=\tan\theta(\log\tan\theta-1)+c.$$

387. 求 $\int \log(\log x)\dfrac{dx}{x}$.

[解] 命 $u=\log(\log x)$, $dv=\dfrac{dx}{x}$;

則 $du=\dfrac{dx}{x}\cdot\dfrac{1}{\log x}$, $v=\int\dfrac{dx}{x}=\log x$.

$$\therefore \int \log(\log x)\frac{dx}{x}=\log x\cdot\log(\log x)-\int\frac{dx}{x}\cdot\frac{1}{\log x}\cdot\log x$$

$$=\log x\cdot\log(\log x)-\log x+c.$$

388. 求 $\int\dfrac{\log(x+1)\,dx}{\sqrt{x+1}}$.

[解] 命 $u=\log(x+1)$, $dv=\dfrac{dx}{\sqrt{x+1}}$;

則 $du=\dfrac{dx}{x+1}$, $v=\int\dfrac{dx}{\sqrt{x+1}}=2\sqrt{x+1}$.

$$\therefore\ \int\frac{\log(x+1)\,dx}{\sqrt{x+1}}=2\sqrt{x+1}\log(x+1)-\int 2\sqrt{x+1}\cdot\frac{dx}{x+1}$$

$$=2\sqrt{x+1}\log(x+1)-4\sqrt{x+1}+c$$

$$=2\sqrt{x+1}\,[\log(x+1)-2]+c.$$

389. 求 $\int x^3(a-x^2)^{\frac{1}{2}}\,dx$.

[解] 命 $u=x^2$, $dv=(a-x^2)^{\frac{1}{2}}x\,dx$;

則 $du=2x\,dx$, $v=\int(a-x^2)^{\frac{1}{2}}x\,dx=-\dfrac{1}{3}(a-x^2)^{\frac{3}{2}}$.

$$\therefore\ \int x^3(a-x^2)^{\frac{1}{2}}dx=-\frac{x^2}{3}(a-x^2)^{\frac{3}{2}}+\int\frac{1}{3}(a-x^2)^{\frac{3}{2}}2x\,dx$$

$$=-\frac{x^2}{3}(a-x^2)^{\frac{3}{2}}-\frac{1}{3}\int(a-x^2)^{\frac{3}{2}}d(a-x^2)$$

$$=-\frac{1}{3}x^2(a-x^2)^{\frac{3}{2}}-\frac{2}{15}(a-x^2)^{\frac{5}{2}}+c.$$

390. 求 $\int\sqrt{a^2-x^2}\,dx$.

[解] 命 $u=\dfrac{\sqrt{a^2-x^2}}{x}$, $dv=x\,dx$;

則 $du=-\dfrac{a^2dx}{x^2\sqrt{a^2-x^2}}$, $v=\dfrac{x^2}{2}$.

$$\therefore\ \int\sqrt{a^2-x^2}\,dx=\frac{\sqrt{a^2-x^2}}{x}\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\left(-\frac{a^2dx}{x^2\sqrt{a^2-x^2}}\right)$$

$$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$$

391. 求 $\int\frac{x^3\,dx}{\sqrt{1-x^2}}$.

[解] 命 $u=x^2$, $dv=\frac{x\,dx}{\sqrt{1-x^2}}$;

則 $du=2x\,dx$, $v=-\sqrt{1-x^2}$.

$$\therefore\ \int\frac{x^3\,dx}{\sqrt{1-x^2}}=x^2(-\sqrt{1-x^2})-\int(-\sqrt{1-x^2})\,2x\,dx$$

$$=-x^2\sqrt{1-x^2}-\int\sqrt{1-x^2}\,d(1-x^2)$$

$$=-x^2\sqrt{1-x^2}-\frac{2}{3}(1-x^2)^{\frac{3}{2}}+c$$

$$=-\frac{1}{3}(x^2+2)\sqrt{1-x^2}+c.$$

392. 求 $\int\frac{(\log x)^2\,dx}{x^{\frac{5}{2}}}$.

[解] 命 $u=\log^2 x$, $dv=x^{-\frac{5}{2}}dx$;

則 $du=2\log x\frac{dx}{x}$, $v=\frac{-2}{3x^{\frac{3}{2}}}$.

$$\therefore\ \int\frac{(\log x)^2\,dx}{x^{\frac{5}{2}}}=\log^2 x\cdot\frac{-2}{3x^{\frac{3}{2}}}-\int-\frac{2}{3x^{\frac{3}{2}}}\cdot 2\log x\frac{dx}{x}$$

$$=-\frac{2}{3x^{\frac{3}{2}}}\log^2 x+\frac{4}{3}\int\frac{\log x\,dx}{x^{\frac{5}{2}}}$$

$$=-\frac{2}{3x^{\frac{3}{2}}}\log^2 x+\frac{4}{3}\left(-\frac{2\log x}{3x^{\frac{3}{2}}}-\int-\frac{2}{3x^{\frac{3}{2}}}\cdot\frac{dx}{x}\right)$$

$$= -\frac{2\log^2 x}{3x^{\frac{3}{2}}} - \frac{8\log x}{9x^{\frac{3}{2}}} - \frac{16}{27x^{\frac{3}{2}}} + c$$

$$= -\frac{2}{3x^{\frac{3}{2}}}\left(\log^2 x + \frac{4}{3}\log x + \frac{8}{9}\right) + c.$$

893. 求 $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$.

[解] 命 $u = x$, $dv = \frac{x\,dx}{\sqrt{a^2-x^2}}$;

則 $du = dx$, $v = \int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$.

$$\therefore \int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = x\,(-\sqrt{a^2-x^2}) - \int -\sqrt{a^2-x^2}\,dx$$

$$= -x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2}\,dx$$

$$= -x\sqrt{a^2-x^2} + \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$= -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$$

894. 求 $\int \sqrt{a^2+x^2}\,dx$.

[解] 命 $u = \frac{\sqrt{a^2+x^2}}{x}$, $dv = x\,dx$;

則 $du = -\frac{a^2\,dx}{x^2\sqrt{a^2+x^2}}$, $v = \frac{x^2}{2}$

$$\therefore \int \sqrt{a^2+x^2}\,dx = \frac{\sqrt{a^2+x^2}}{x}\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\left(-\frac{a^2}{x^2}\cdot\frac{dx}{\sqrt{a^2+x^2}}\right)$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2+x^2}}$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x + \sqrt{a^2+x^2}\right) + c.$$

第七章

遞化積分法

40. 二項微分之遞化公式. 由第五章83款之方法，凡二項微分式恆可化爲

$$x^m(a+bx^n)^p dx$$

之形；其中 p 爲有理數，m 爲整數，n 爲正整數. 且於84款已約略討論其可積分性之條件. 倘二項微分式歷遍以前之方法仍不能立卽求得其積分，則用下列四公式求之：

$$(A)\quad \int x^m(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)b} - \frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}(a+bx^n)^p dx.$$

$$(B)\quad \int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

$$(C)\quad \int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)a} - \frac{(np+n+m+1)b}{(m+1)a}\int x^{m+n}(a+bx^n)^p dx.$$

$$(D)\quad \int x^m(a+bx^n)^p dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{n(p+1)a} + \frac{np+n+m+1}{n(p+1)a}\int x^m(a+bx^n)^{p+1}dx.$$

上列諸公式左端之積分,均隨右端而決定.每公式右端各有兩項:其一不含積分號;其他另一項含有積分號,已較左端爲簡易.如遞次用之,終可使左端之二項微分式改爲標準簡易之形.名此四公式曰遞化公式.此法曰遞化積分法(integration by successive reduction).遞化公式均從分部積分法得來其詳情如下:

(一) 公式(A)之求法:

求 $\int x^m(a+bx^n)^p dx$

假設 $u = x^{m-(n-1)}$, $dv = (a+bx^n)^p x^{n-1}dx$;

則 $du = (m-n+1)x^{m-n}dx$, $v = \dfrac{(a+bx^n)^{p+1}}{nb(p+1)}$.

代入第六章38款公式(A),

$$(1)\quad \int x^m(a+bx^n)^p dx = \int \underbrace{x^{m-n+1}}_{u}\underbrace{(a+bx^n)^p x^{n-1}dx}_{dv}$$

$$= \underbrace{\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}}_{uv} - \underbrace{\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx}_{\int v\,du}.$$

但 $\int x^{m-n}(a+bx^n)^{p+1}dx = \int x^{m-n}(a+bx^n)^p(a+bx^n)dx$

$$= a\int x^{m-n}(a+bx^n)^p dx + b\int x^m(a+bx^n)^p dx.$$

代入(1)式,

$$\int x^m(a+bx^n)^p dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{(m-n+1)a}{nb(p+1)}\int x^{m-n}(a+bx^n)^p dx$$

$$-\frac{m-n+1}{n(p+1)}\int x^m(a+bx^n)^p dx.$$

將上式右端最末一項移於左端,集項,且依$\int x^m(a+bx^n)^p dx$解之,即得(A)式如次:

$$(A)\quad \int x^m(a+bx^n)^p dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}$$

$$-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p dx.$$

[討論] 公式(A)右端之分母不能爲零,即

$$np+m+1\neq 0.$$

若$np+m+1=0$時,即不能應用(A)式.但若$np+m+1=0$,即$\frac{m+1}{n}+p=0$時,用第五章34款之代換公式已能解決.

又比較(A)式左右兩端,可知左端之m在右端已降爲$m-n$,如將(A)式遞次用之,m可降至n之任何倍.

(二) 公式(B)之求法:

將二項微分析成因數,即

$$(2)\quad \int x^m(a+bx^n)^p dx=\int x^m(a+bx^n)^{p-1}(a+bx^n)dx$$

$$=a\int x^m(a+bx^n)^{p-1}dx+b\int x^{m+n}(a+bx^n)^{p-1}dx.$$

次將上式右端最末一積分應用公式(A).以$m+n$代公

式(A)中之m；以$p-1$代公式(A)中之p，則得

$$b\int x^{m+n}(a+bx^n)^{p-1}dx=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}$$

$$-\frac{a(m+1)}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

代入(2)式，集項，即得(B)式如次：

$$(B)\quad \int x^m(a+bx^n)^p dx=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}$$

$$+\frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$$

[討論] 與公式(A)討論相同.惟每用公式(B)一次,其中$a+bx^n$之指數由p改爲$p-1$.故若用n次，$a+bx^n$之指數由p改爲$p-n$.

(三) 公式(C)之求法：

將公式(A)依$\int x^{m-n}(a+bx^n)^p\,dx$解之，並以$m+n$代公式(A)中之$m$，即得公式(C)，

$$(C)\quad \int x^m(a+bx^n)^p dx=\frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)}$$

$$-\frac{b(np+n+m+1)}{a(m+1)}\int x^{m+n}(a+bx^n)^p dx.$$

[討論] 公式(C)右端之分母不能爲零，即

$$m+1\neq 0,\text{ 或 }\frac{m+1}{n}\neq 0.$$

若$m+1=0$,公式(C)即不能用.但$\frac{m+1}{n}=0$時,用第五章34

款討論之結果已能解決.

又公式(C)每用一次,左端x之指數m,在右端已改爲$m+n$.

(四)　公式(D)之求法:

將公式(B)依$\int x^m(a+bx^n)^{p-1}dx$解之,並以$p+1$代公式(B)中之p,卽得公式(D),

$$(D)\quad \int x^m(a+bx^n)^p dx = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)} + \frac{np+n+m+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}dx.$$

[討論]　公式(D)右端之分母爲零時,卽$p+1=0$時,此式卽不能應用.但若$p+1=0$,卽$p=-1$,左端已爲有理式矣.

又公式(D)如應用一次,$a+bx^n$之指數由p改爲$p+1$.

[舉例]　求$\int \frac{x^4\,dx}{(a^2-x^2)^{\frac{1}{2}}}$.

[解]　視察原式,$m=4$,$n=2$,$p=-\frac{1}{2}$,$a=a^2$,$b=-1$,$np+m+1=2\left(-\frac{1}{2}\right)+4+1\neq 0$,故若應用公式(A)一遍,其中$x^4$卽改爲$x^{4-2}$;如再引用公式(A),則$x^2$卽改爲$x^0(=1)$.由是卽有標準公式可用,其解法如下:

應用公式(A),

$$\int \frac{x^4\,dx}{(a^2-x^2)^{\frac{1}{2}}} = \int x^4(a^2-x^2)^{-\frac{1}{2}}dx$$

$$= \frac{x^3(a^2-x^2)^{\frac{1}{2}}}{-[(-1)+4+1]} + \frac{3a^2}{4}\int x^2(a^2-x^2)^{-\frac{1}{2}}dx. \quad (1)$$

將上式右端最末之一項再用公式(A)求之,即

$$\frac{3a^2}{4}\int x^2(a^2-x^2)^{-\frac{1}{2}}dx=\frac{3a^2}{4}\left(-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2-x^2}}\right)$$

$$=\frac{3a^2}{4}\left(-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)$$

$$=-\frac{3a^2x}{8}\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}. \qquad (2)$$

代入(1)式,

$$\therefore\ \int\frac{x^4\,dx}{(a^2-x^2)^{\frac{1}{2}}}=-\frac{x^3}{4}\sqrt{a^2-x^2}-\frac{3a^2x}{8}\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}+c$$

$$=\sqrt{a^2-x^2}\left(-\frac{x^3}{4}-\frac{3a^2x}{8}\right)+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}+c.$$

41. 三角微分之遞化公式. 設有三角函數之微分式爲

$$\int\sin^m x\cos^n x\,dx.$$

仍用第六章分部積分式(A),誘導三角微分遞化公式如下:

$$(E)\quad\int\sin^m x\cos^n x\,dx=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}$$

$$+\frac{m-1}{m+n}\int\sin^m x\cos^{n-2}x\,dx.$$

$$(F)\quad\int\sin^m x\cos^n x\,dx=-\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}$$

$$+\frac{m-1}{m+n}\int\sin^{m-2}x\cos^n x\,dx.$$

$$(G)\quad\int\sin^m x\cos^n x\,dx=-\frac{\sin^{m+1}x\cos^{n+1}x}{n+1}$$

$$+\frac{m+n+2}{n+1}\int\sin^m x\cos^{n+2}x\,dx.$$

$$(H)\quad \int \sin^m x\cos^n x\,dx=\frac{\sin^{m+1}x\cos^{n+1}x}{m+1}$$

$$+\frac{m+n+2}{m+1}\int \sin^{m+2}x\cos^n x\,dx.$$

公式(*E*)左端之n在右端已降爲$n-2$；但$m+n=0$時，此式不能用.

公式(*F*)左端之m在右端已降爲$m-2$；但$m+n=0$時，此式不能用.

公式(*G*)左端之n在右端已升爲$n+2$；但$n+1=0$時，此式不能用.

公式(*H*)左端之m在右端已升爲$m+2$；但$m+1=0$時，此式不能用.

(一)　公式(*E*)之求法：

由第六章38款公式(*A*)，

$$\int u\,dv=uv-\int v\,du.$$

命　$u=\cos^{n-1}x,\ dv=\sin^m x\cos x\,dx$；

則　$du=-(n-1)\cos^{n-2}x\sin x\,dx,\quad v=\frac{\sin^{m+1}x}{m+1}$.

代入公式(*A*)，則得

(1)　$$\int \sin^m x\cos^n x\,dx=\int \underbrace{\cos^{n-1}x}_{u}\ \underbrace{\sin^m x\cos x\,dx}_{dv}$$

$$=\underbrace{\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}}_{uv}+\underbrace{\frac{n-1}{m+1}\int \sin^{m+2}x\cos^{n-2}x\,dx}_{\int v\,du}.$$

但因 $\int \sin^{m+2}x\cos^{n-2}x\,dx=\int \sin^{m}x(1-\cos^{2}x)\cos^{n-2}x\,dx$

$$=\int \sin^{m}x\cos^{n-2}x\,dx-\int \sin^{m}x\cos^{n}x\,dx. \quad (2)$$

將(2)式之結果代入1)式,集項,且依$\int \sin^{m}x\cos^{n}x\,dx$解之,即得公式(*E*),

$$(E)\quad \int \sin^{m}x\cos^{n}x\,dx=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}\int \sin^{m}x\cos^{n-2}x\,dx.$$

(二) 公式(*F*)之求法:

命 $u=\sin^{m-1}x,\ dv=\cos^{n}x\sin x\,dx$;

則 $du=(m-1)\sin^{m-2}x\cos x\,dx,\quad v=-\frac{\cos^{n+1}x}{n+1}$.

代入38款公式(*A*),則得

$$(2)\quad \int \sin^{m}x\cos^{n}x\,dx=\int \underbrace{\sin^{m-1}x}_{u}\ \underbrace{\cos^{n}x\sin x\,dx}_{dv}$$

$$=\underbrace{-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}}_{uv}+\underbrace{\frac{m-1}{n+1}\int \sin^{m-2}x\cos^{n+2}x\,dx}_{\int v\,du}$$

但因 $\int \sin^{m-2}x\cos^{n+2}x\,dx=\int \sin^{m-2}x\cos^{n}x(1-\sin^{2}x)dx$

$$=\int \sin^{m-2}x\cos^{n}x-\int \sin^{m}x\cos^{n}x\,dx. \quad (3)$$

將(3)式之結果代入(2)式,集項,且依$\int \sin^{m}x\cos^{n}x\,dx$解之,即得公式(*F*),

$$(F)\quad \int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx.$$

(三)　公式(G)之求法:

將公式(E)依右端之積分解之,且以$n+2$代入公式(E)中之n,即得公式(G),

$$(G)\quad \int \sin^m x\cos^n x\,dx = -\frac{\sin^{m+1}x\cos^{n+1}x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x\cos^{n+2}x\,dx.$$

(四)　公式(H)之求法:

將公式(F)依右端之積分解之,且以$m+2$代入公式(F)中之m,即得公式(H),

$$(H)\quad \int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n+1}x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2}x\cos^n x\,dx.$$

[討論]　公式(E)及公式(F)右端之分母不能爲零,即$m+n\neq 0$;公式(G)中$n+1\neq 0$;公式(H)中$m+1\neq 0$.以上四公式右端之分母有一爲零時,則該式即不能應用.但若$m+n=0$,$n+1=0$,$m+1=0$時,由第三章之方法已可解決.

由上結論,若有三角函數微分式爲

$$\int \sin^m x\cos^n x\,dx,$$

其中 m, n 爲整數時，用上列四公式，終可逐次化爲標準簡易之形.

[舉例] 求 $\int \sin^4 x \cos^2 x\, dx$.

[解] $m+n=4+2\neq 0$，先應用公式 (E)，將 $\cos^2 x$ 消去，

$$\int \sin^4 x \cos^2 x\, dx = \frac{\sin^5 x \cos x}{6} + \frac{1}{6}\int \sin^4 x\, dx. \qquad (1)$$

(1) 式右端之積分含有 $\sin^4 x$，且 $m+n=4+0\neq 0$，故再引用公式 (F)，

$$\frac{1}{6}\int \sin^4 x\, dx = \frac{1}{6}\left(-\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\, dx\right)$$

$$= -\frac{\sin^3 x \cos x}{24} + \frac{3}{24}\int \sin^2 x\, dx. \qquad (2)$$

(2) 式右端之積分含有 $\sin^2 x$，且 $m+n=2+0\neq 0$，故須再引用公式 (F)，將 $\sin^2 x$ 消去，

$$\frac{3}{24}\int \sin^2 x\, dx = \frac{3}{24}\left(-\frac{\sin x \cos x}{2} + \frac{x}{2}\right)$$

$$= -\frac{3 \sin x \cos x}{48} + \frac{3x}{48}. \qquad (3)$$

將 (3) 式之結果代入 (2)，復將 (2) 式之結果代入 (1) 式，

$$\therefore \int \sin^4 x \cos^2 x\, dx = \frac{\sin^5 x \cos x}{6} - \frac{\sin^3 x \cos x}{24} - \frac{3 \sin x \cos x}{48} + \frac{3x}{48} + c$$

$$= \frac{1}{2}\cos x\left(\frac{1}{3}\sin^5 x - \frac{1}{12}\sin^3 x - \frac{1}{8}\sin x\right) + \frac{x}{16} + c.$$

42. 本章例題.

第40款例題

895. 求 $\int (a^2 + x^2)^{\frac{1}{2}} dx$.

[解]　$m=0, n=2, p=\frac{1}{2}, a=a^2, b=1;\ np+m+1=1+0+1\neq 0.$

應用公式(*B*),

$$\therefore\ \int (a^2+x^2)^{\frac{1}{2}}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2+x^2}}$$

$$=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log(x+\sqrt{a^2+x^2})+c.$$

396. 求 $\int\frac{dx}{x^3(x^2-1)^{\frac{1}{2}}}$.

[解]　$m=-3, n=2, p=\frac{-1}{2}, a=-1, b=1;\ m+1=-3+1=-2\neq 0.$

應用公式(*C*),

$$\therefore\ \int\frac{dx}{x^3(x^2-1)^{\frac{1}{2}}}=\frac{\sqrt{x^2-1}}{2x^2}+\frac{1}{2}\int x^{-1}(x^2-1)^{-\frac{1}{2}}dx$$

$$=\frac{(x^2-1)^{\frac{1}{2}}}{2x^2}+\frac{1}{2}\sec^{-1}x+c.$$

397. 求 $\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}$.

[解]　$m=2, n=2, p=-\frac{1}{2}, a=a^2, b=-1;\ np+m+1=-1+2+1\neq 0.$

應用公式(*A*),

$$\therefore\ \int x^2(a^2-x^2)^{-\frac{1}{2}}dx=-\frac{x(a^2-x^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\int(a^2-x^2)^{-\frac{1}{2}}dx$$

$$=-\frac{x(a^2-x^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$$

398. 求 $\int \frac{x^3\,dx}{\sqrt{a^2+x^2}}$.

[解] $m=3, n=2, p=-\frac{1}{2}, a=a^2, b=1;\ np+m+1=3\neq 0.$

應用公式(A),

$$\therefore \int \frac{x^3\,dx}{\sqrt{a^2+x^2}}=\frac{x^2(a^2+x^2)^{\frac{1}{2}}}{-1+3+1}-\frac{2a^2}{3}\int x(a^2+x^2)^{-\frac{1}{2}}dx$$

$$=\frac{x^2(a^2+x^2)^{\frac{1}{2}}}{3}-\frac{2a^2}{3}(a^2+x^2)^{\frac{1}{2}}+c$$

$$=\frac{1}{3}(a^2+x^2)^{\frac{1}{2}}(x^2-2a^2)+c.$$

399. 求 $\int \frac{x^5\,dx}{\sqrt{1-x^2}}$.

[解] $m=5, n=2, p=-\frac{1}{2}, a=1, b=-1;\ np+m+1=-1+5+1\neq 0.$

應用公式(A),

$$\therefore \int \frac{x^5\,dx}{\sqrt{1-x^2}}=\frac{x^4(1-x^2)^{\frac{1}{2}}}{-5}+\frac{4}{5}\int x^3(1-x^2)^{-\frac{1}{2}}\,dx. \qquad (1)$$

上式右端之積分, $m=3, n=2, p=-\frac{1}{2}, a=1, b=-1$, 且

$np+m+1=-1+3+1=3\neq 0.$

再應用公式(A),

$$\frac{4}{5}\int x^3(1-x^2)^{-\frac{1}{2}}dx=\frac{4}{5}\cdot\frac{x^2(1-x^2)^{\frac{1}{2}}}{-3}+\frac{2}{3}\int x(1-x^2)^{-\frac{1}{2}}dx$$

$$=-\frac{4}{15}x^2(1-x^2)^{\frac{1}{2}}-\frac{8}{15}(1-x^2)^{\frac{1}{2}}. \qquad (2)$$

將(2)式之結果代入1)式,

$$\therefore \int \frac{x^5\,dx}{\sqrt{1-x^2}} = \frac{x^4(1-x^2)^{\frac{1}{2}}}{-5} - \frac{4}{15}x^2(1-x^2)^{\frac{1}{2}} - \frac{8}{15}(1-x^2)^{\frac{1}{2}} + c$$

$$= -\left(\frac{x^4}{5} + \frac{4x^2}{15} + \frac{8}{15}\right)\sqrt{1-x^2} + c.$$

400. 求 $\int x^2\sqrt{a^2-x^2}\,dx.$

[解] $m=2, n=2, p=\frac{1}{2}, a=a^2, b=-1;\ np+m+1=1+2+1\neq 0.$

應用公式(A),

$$\therefore \int x^2\sqrt{a^2-x^2} = \frac{x(a^2-x^2)^{\frac{3}{2}}}{-4} + \frac{a^2}{4}\int (a^2-x^2)^{\frac{1}{2}}\,dx. \qquad (1)$$

(1)式最末一項之積分,其中 $m=0,\ n=2,\ p=\frac{1}{2},\ a=a^2,$ $b=-1;\ np+m+1=1+0+1=2\neq 0.$

再應用公式(B),

$$\frac{a^2}{4}\int (a^2-x^2)^{\frac{1}{2}}dx = \frac{a^2}{4}\left[\frac{x(a^2-x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2}\int (a^2-x^2)^{-\frac{1}{2}}dx\right]$$

$$= \frac{a^2x}{8}(a^2-x^2)^{\frac{1}{2}} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}. \qquad (2)$$

將(2)式之結果代入(1)式,

$$\therefore \int x^2\sqrt{a^2-x^2} = -\frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{a^2x}{8}(a^2-x^2)^{\frac{1}{2}} + \frac{a^4}{8}\sin^{-1}\frac{x}{a} + c$$

$$= \frac{x}{8}\sqrt{a^2-x^2}\,(2x^2-a^2) + \frac{a^4}{8}\sin^{-1}\frac{x}{a} + c.$$

401. 求 $\int \frac{dx}{(a^2-x^2)^2}.$

[解] $m=0, n=2, p=-2, a=a^2, b=-1;\ p+1=-2+1\neq 0$

應用公式(D),

$$\therefore \int \frac{dx}{(a^2-x^2)^2} = -\frac{x(a^2+x^2)^{-1}}{2(-2+1)a^2} + \frac{-4+2+0+1}{2(-2+1)a^2}\int (a^2+x^2)^{-1}\,dx$$

$$= \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^2}\int \frac{dx}{a^2+x^2}$$

$$= \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + c.$$

402. 求 $\int \frac{dx}{x^3\sqrt{a^2-x^2}}$.

[解] $m=-3, n=2, p=-\frac{1}{2}, a=a^2, b=-1$; $m+1=-3+1=-2$.

應用公式(C),

$$\therefore \int \frac{dx}{x^3\sqrt{a^2-x^2}} = \frac{x^{-2}(a^2-x^2)^{\frac{1}{2}}}{-2a^2}$$

$$-\frac{(-1+2-3+1)(-1)}{(-3+1)a^2}\int x^{-1}(a^2-x^2)^{-\frac{1}{2}}\,dx$$

$$= \frac{\sqrt{a^2-x^2}}{-2a^2x^2} + \frac{1}{2a^2}\int \frac{dx}{x\sqrt{a^2-x^2}} \qquad (1)$$

上式右端之積分,其中 $m=-1, n=2, p=-\frac{1}{2}, a=a^2, b=-1$; 但 $m+1=-1+1=0$, 故公式(C)不能再用. 又 $m+1=0$, 即 $\frac{m+1}{n}=0$, 故此積分顯然可用第五章34款討論一之方法求之.

命 $a^2-x^2=z^2$ (z爲代入之新變數);

則 $x=(a^2-z^2)^{\frac{1}{2}}, dx=\frac{-z\,dz}{\sqrt{a^2-z^2}}$.

$$\therefore \int \frac{dx}{x\sqrt{a^2-x^2}} = \int \frac{-z\,dz}{z(a^2-z^2)^{\frac{1}{2}}\,(a^2-z^2)^{\frac{1}{2}}} = \int \frac{-dz}{a^2-z^2}$$

$$= \int \frac{dz}{z^2-a^2} = \frac{1}{2a}\log\frac{z-a}{z+a}$$

$$= \frac{1}{2a}\log\frac{\sqrt{a^2-x^2}-a}{\sqrt{a^2-x^2}+a} = \frac{1}{2a}\log\left(\frac{x}{a+\sqrt{a^2-x^2}}\right)^2$$

$$= \frac{2}{2a}\log\frac{x}{a+\sqrt{a^2-x^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}}. \quad (2)$$

將(2)式之結果代入(1)式,

$$\therefore \int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{2a^2}\left(\frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}}\right)+c$$

$$= -\frac{\sqrt{a^2-x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{x}{a+\sqrt{a^2-x^2}}+c.$$

403. 求 $\int \frac{x^3\,dx}{(a^2+x^2)^{\frac{3}{2}}}$.

[解] $m=3, n=2, p=-\frac{3}{2}, a=a^2, b=1$; $np+m+1=-3+3+1\neq 0$.

應用公式(A),

$$\therefore \int \frac{x^3\,dx}{(a^2+x^2)^{\frac{3}{2}}} = x^2(a^2+x^2)^{-\frac{1}{2}} - 2a^2\int x(a^2+x^2)^{-\frac{3}{2}}\,dx$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} - a^2\int (a^2+x^2)^{-\frac{3}{2}}\,d(x^2+a^2)$$

$$= \frac{x^2}{\sqrt{a^2+x^2}} + \frac{2a^2}{\sqrt{a^2+x^2}} + c.$$

404. 求 $\int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}}$.

[解] $m=0, n=2, p=-\frac{5}{2}, a=a^2, b=1; p+1=-\frac{5}{2}+1\neq 0.$

應用公式(*D*),

$$\therefore \int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}}=\frac{x}{3a^2}(a^2-x^2)^{-\frac{3}{2}}+\frac{2}{3a^2}\int (a^2-x^2)^{-\frac{3}{2}}dx. \quad (1)$$

(1)式右端之積分,其中 $m=0, n=2, p=-\frac{3}{2}, a=a^2, b=-1;$

$p+1=-\frac{3}{2}+1=-\frac{1}{2}\neq 0.$

故再應用公式(*D*),

$$\frac{2}{3a^2}\int (a^2-x^2)^{-\frac{3}{2}}dx=\frac{2}{3a^2}\left[\frac{x}{a^2}(a^2-x^2)^{-\frac{1}{2}}+0\right]$$

$$=\frac{2x}{3a^4}(a^2-x^2)^{-\frac{1}{2}}. \quad (2)$$

將(2)式之結果代入(1)式,

$$\therefore \int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}}=\frac{x}{3a^2}(a^2-x^2)^{-\frac{3}{2}}+\frac{2x}{3a^4}(a^2-x^2)^{-\frac{1}{2}}+c$$

$$=\frac{(3a^2-2x^2)x}{3a^4(a^2-x^2)^{\frac{3}{2}}}+c.$$

405. 求 $\int (x^2+a^2)^{\frac{3}{2}}dx.$

[解] $m=0, n=2, p=\frac{3}{2}, a=a^2, b=1; np+m+1=3+0+1=4.$

先引用公式(*B*),

$$\therefore \int (a^2+x^2)^{\frac{3}{2}}dx=\frac{x(a^2+x^2)^{\frac{3}{2}}}{4}+\frac{3a^2}{4}\int (a^2+x^2)^{\frac{1}{2}}dx. \quad (1)$$

再引用公式(*B*)於(1)式右端之積分,

$$\frac{3a^2}{4}\int(a^2+x^2)^{\frac{1}{2}}dx=\frac{3a^2}{4}\left[\frac{x(a^2+x^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\int(a^2+x^2)^{-\frac{1}{2}}dx\right]$$

$$=\frac{3a^2x}{8}(a^2+x^2)^{\frac{1}{2}}+\frac{3a^4}{8}\log(x+\sqrt{a^2+x^2}).\quad(2)$$

將(2)式之結果代入(1)式,

$$\therefore\quad\int(a^2+x^2)^{\frac{3}{2}}dx=\frac{x}{4}(a^2+x^2)^{\frac{3}{2}}+\frac{3a^2x}{8}(a^2+x^2)^{\frac{1}{2}}$$

$$+\frac{3a^4}{8}\log(x+\sqrt{a^2+x^2})+c$$

$$=\frac{x}{8}(a^2+x^2)^{\frac{1}{2}}(5a^2+2x^2)$$

$$+\frac{3a^4}{8}\log(x+\sqrt{a^2+x^2})+c.$$

406. 求 $\int x^2(x^2+a^2)^{\frac{1}{2}}dx$.

[解] 先引用公式(*A*),次引用公式(*B*),

$$\therefore\quad\int x^2(x^2+a^2)^{\frac{1}{2}}dx=\frac{x(x^2+a^2)^{\frac{3}{2}}}{4}-\frac{a^2}{4}\int(x^2+a^2)^{\frac{1}{2}}dx$$

$$=\frac{x}{4}(x^2+a^2)^{\frac{3}{2}}-\frac{a^2}{4}\left\{\frac{x}{2}(x^2+a^2)^{\frac{1}{2}}\right.$$

$$\left.+\frac{a^2}{2}\log(x+\sqrt{x^2+a^2})\right\}+c$$

$$=\frac{x}{8}(x^2+a^2)^{\frac{1}{2}}(2x^2+a^2)$$

$$-\frac{a^4}{8}\log(x+\sqrt{x^2+a^2})+c.$$

407. 求 $\int \frac{x^2\,dx}{\sqrt{2ax-x^2}}$.

[解] 引用公式(A)二次,

$$\therefore \int \frac{x^2\,dx}{\sqrt{2ax-x^2}} = \int x^{\frac{3}{2}}(2a-x)^{-\frac{1}{2}}dx$$

$$= -\frac{x^{\frac{3}{2}}(2a-x)^{\frac{1}{2}}}{2} + \frac{6a}{4}\int x^{\frac{1}{2}}(2a-x)^{-\frac{1}{2}}dx$$

$$= -\frac{x^{\frac{3}{2}}}{2}(2a-x)^{\frac{1}{2}} + \frac{3}{2}a\left[\frac{x^{\frac{1}{2}}(2a-x)^{\frac{1}{2}}}{-1}\right.$$

$$\left. + \frac{2a}{2}\int x^{-\frac{1}{2}}(2a-x)^{-\frac{1}{2}}dx\right]$$

$$= -\frac{x^{\frac{3}{2}}}{2}(2a-x)^{\frac{1}{2}} - \frac{3ax^{\frac{1}{2}}}{2}(2a-x)^{\frac{1}{2}}$$

$$+ \frac{3a^2}{2}\int \frac{dx}{\sqrt{2ax-x^2}}$$

$$= -\frac{1}{2}\sqrt{2ax-x^2}(x+3a) + \frac{3a^2}{2}\operatorname{vers}^{-1}\frac{x}{a} + c.$$

408. 求 $\int \frac{dx}{x^2(a^2-x^2)^{\frac{1}{2}}}$.

[解] 引用公式(C),

$$\therefore \int \frac{dx}{x^2(a^2-x^2)^{\frac{1}{2}}} = \frac{x^{-1}(a^2-x^2)^{\frac{1}{2}}}{-a^2}$$

$$+ \frac{(-1+2-2+1)}{-a^2}\int (a^2-x^2)^{-\frac{1}{2}}dx$$

$$= -\frac{(a^2-x^2)^{\frac{1}{2}}}{a^2x} + 0 + c = \frac{-\sqrt{a^2-x^2}}{a^2x} + c.$$

409. 求 $\int \frac{t\,dt}{\sqrt{2at-t^2}}$.

[解] 引用公式(A),

$$\therefore \int \frac{t\,dt}{\sqrt{2at-t^2}} = \int t^{\frac{1}{2}} \cdot \frac{dt}{\sqrt{2a-t}} = -t^{\frac{1}{2}}(2a-t)^{\frac{1}{2}} + a\int \frac{dt}{\sqrt{2at-t^2}}$$

$$= -(2at-t^2)^{\frac{1}{2}} + a\,\text{vers}^{-1}\frac{t}{a} + c.$$

410. 求 $\int \frac{y^3\,dy}{\sqrt{2ry-y^2}}$.

[解] 引用公式(A)兩次,

$$\therefore \int \frac{y^3\,dy}{\sqrt{2ry-y^2}} = \int y^{\frac{5}{2}} \frac{dy}{\sqrt{2r-y}} = \frac{y^{\frac{5}{2}}(2r-y)^{\frac{1}{2}}}{-(-\frac{1}{2}+\frac{5}{2}+1)}$$

$$+\frac{5}{3}r\int y^{\frac{3}{2}}(2r-y)^{-\frac{1}{2}}dy$$

$$= \frac{-y^{\frac{5}{2}}}{3}(2r-y)^{\frac{1}{2}} + \frac{5r}{3}\Big[\frac{y^{\frac{3}{2}}(2r-y)^{\frac{1}{2}}}{-2}$$

$$+\frac{3r}{2}\int y^{\frac{1}{2}}(2r-y)^{-\frac{1}{2}}dy\Big]$$

$$= -\frac{y^{\frac{5}{2}}}{3}(2r-y)^{\frac{1}{2}} - \frac{5ry^{\frac{3}{2}}}{6}(2r-y)^{\frac{1}{2}}$$

$$+\frac{5}{2}r^2\Big[-y^{\frac{1}{2}}(2r-y)^{\frac{1}{2}} + r\int \frac{dy}{\sqrt{2ry-y^2}}\Big]$$

$$= -\frac{1}{3}y^{\frac{5}{2}}(2r-y)^{\frac{1}{2}} - \frac{5r}{6}y^{\frac{3}{2}}(2r-y)^{\frac{1}{2}}$$

$$-\frac{5r^2}{2}y^{\frac{1}{2}}(2r-y) + \frac{5r^3}{2}\,\text{vers}^{-1}\,\frac{y}{r} + c$$

$$= -\frac{1}{6}(2y^2+5ry+15r^2)\sqrt{2r-y}$$

$$+\frac{5}{2}r^3\operatorname{vers}^{-1}\frac{y}{r}+c.$$

411. 求 $\int\frac{ds}{(a^2+s^2)^3}$.

[解] 引用公式(D)兩次,

$$\therefore\ \int\frac{ds}{(a^2+s^2)^3} = -\frac{s(a^2+s^2)^{-2}}{-4a^2}+\frac{-6+2+1}{-4a^2}\int(a^2+s^2)^{-2}ds$$

$$=\frac{s}{4a^2}(a^2+s^2)^{-2}+\frac{3}{4a^2}\left[\frac{s(a^2+s^2)^{-1}}{2a^2}\right.$$

$$\left.+\frac{-4+2+1}{-2a^2}\int(a^2+s^2)^{-1}ds\right]$$

$$=\frac{s}{4a^2(a^2+s^2)^2}+\frac{3s}{8a^4(a^2+s^2)}+\frac{3}{8a^5}\tan^{-1}\frac{s}{a}+c.$$

412. 求 $\int\frac{\theta^3d\theta}{\sqrt{1-\theta^3}}$.

[解] 引用公式(A)兩次,

$$\therefore\ \int\frac{\theta^3d\theta}{\sqrt{1-\theta^3}} = \frac{\theta^6(1-\theta^3)^{\frac{1}{2}}}{-(-\frac{3}{2}+8+1)}+\frac{12}{15}\int\theta^5(1-\theta^3)^{-\frac{1}{2}}d\theta$$

$$=-\frac{2\theta^6}{15}(1-\theta^3)^{\frac{1}{2}}+\frac{4}{5}\left[\frac{\theta^3(1-\theta^3)^{\frac{1}{2}}}{-(-\frac{3}{2}+6)}\right.$$

$$\left.+\frac{6}{9}\int\theta^2(1-\theta^3)^{-\frac{1}{2}}d\theta\right]$$

$$=-\frac{2\theta^6}{15}(1-\theta^3)^{\frac{1}{2}}-\frac{8\theta^3}{45}(1-\theta^3)^{\frac{1}{2}}$$

$$+\frac{8}{15}\left[-\frac{2}{3}(1-\theta^3)^{\frac{1}{2}}\right]+c$$

$$=(1-\theta^3)^{\frac{1}{2}}\left(-\frac{2\theta^6}{15}-\frac{8\theta^3}{45}-\frac{16}{45}\right)+c.$$

413. 求 $\int\frac{x^2dx}{(a^2-x^2)^2}$.

[解] 先引用公式(A),次引用公式(D),

$$\therefore\quad\int\frac{x^2dx}{(a^2-x^2)^2}=x(a^2-x^2)^{-1}-a^2\int(a^2-x^2)^{-2}dx$$

$$=x(a^2-x^2)^{-1}-a^2\left[\frac{x(a^2-x^2)^{-1}}{2a^2}\right.$$

$$\left.+\frac{1}{2a^2}\int(a^2-x^2)^{-1}dx\right]$$

$$=x(a^2-x^2)^{-1}-\frac{1}{2}x(a^2-x^2)^{-1}$$

$$-\frac{1}{2}\cdot\frac{1}{2a}\log\frac{x+a}{x-a}+c$$

$$=\frac{1}{2}x(a^2-x^2)^{-1}-\frac{1}{4a}\log\frac{x+a}{x-a}+c.$$

414. 求 $\int\frac{dx}{x^2(1+x^2)^{\frac{3}{2}}}$.

[解] 先引用公式(C),次引用公式(D),

$$\therefore\quad\int\frac{dx}{x^2(1+x^2)^{\frac{3}{2}}}=\frac{x^{-1}(1+x^2)^{-\frac{1}{2}}}{-1}-\frac{(-3+2-2+1)}{-1}\int(1+x^2)^{-\frac{3}{2}}dx$$

$$=-x^{-1}(1+x^2)^{-\frac{1}{2}}-2\left[\frac{-x(1+x^2)^{-\frac{1}{2}}}{2(-\frac{1}{2})}\right.$$

$$\left.+\frac{-3+2+1}{2(-\frac{1}{2})}\int(1+x^2)^{-\frac{1}{2}}dx\right]$$

$$= -x^{-1}(1+x^2)^{-\frac{1}{2}} - 2x(1+x^2)^{-\frac{1}{2}} + c$$

$$= -\frac{1+2x^2}{x\sqrt{1+x^2}} + c.$$

415. 求 $\int \frac{x^5\,dx}{\sqrt{1-x^3}}$.

[解] 引用公式(A),

$$\therefore \int \frac{x^5\,dx}{\sqrt{1-x^3}} = -\frac{2}{9}x^3(1-x^3)^{\frac{1}{2}} + \frac{2}{3}\int (1-x^3)^{-\frac{1}{2}}x^3\,dx$$

$$= -\frac{2}{9}x^3(1-x^3)^{\frac{1}{2}} - \frac{2}{9}\cdot 2(1-x^3)^{\frac{1}{2}} + c$$

$$= -\frac{2}{9}(1-x^3)^{\frac{1}{2}}(x^3+2) + c.$$

416. 求 $\int t^5\sqrt{a^2+t^3}\,dt$.

[解] 引用公式(A),

$$\therefore \int t^5\sqrt{a^2+t^3}\,dt = \frac{t^3(a^2+t^3)^{\frac{3}{2}}}{\frac{15}{2}} - \frac{3a^2}{\frac{15}{2}}\int t^2(a^2+t^3)^{\frac{1}{2}}dt$$

$$= \frac{2}{15}t^3(a^2+t^3)^{\frac{3}{2}} - \frac{2a^2}{5}\cdot\frac{1}{3}\cdot\frac{2}{3}(a^2+t^3)^{\frac{3}{2}} + c$$

$$= \frac{2}{45}(a^2+t^3)^{\frac{3}{2}}(3t^3-2a^2) + c.$$

417. 求 $\int \frac{z^4\,dz}{\sqrt{z^4+9}}$.

[解] 引用公式(A),

$$\therefore \int \frac{z^5\,dz}{\sqrt{z^4+9}} = \frac{z^2(z^4+9)^{\frac{1}{2}}}{4} - \frac{18}{4}\int \frac{z\,dz}{\sqrt{(z^2)^2+9}}$$

$$= \frac{1}{4}z^2(z^4+9)^{\frac{1}{2}} - \frac{9}{2}\cdot\frac{1}{2}\log(z^2+\sqrt{z^4+9})+c$$

$$= \frac{1}{4}z^2(z^4+9)^{\frac{1}{2}} - \frac{9}{4}\log(z^2+\sqrt{z^4+9})+c.$$

418. 求 $\int \frac{s^7\,ds}{(a+bs^4)^{\frac{2}{3}}}$.

[解] 引用公式(A),

$$\therefore \int \frac{s^7\,ds}{(a+bs^4)^{\frac{2}{3}}} = \frac{3s^4}{16b}(a+bs^4)^{\frac{1}{3}} - \frac{(7-4+1)a}{(-\frac{8}{3}+7+1)b}\int s^3(a+bs^4)^{-\frac{2}{3}}ds$$

$$= \frac{3s^4}{16b}(a+bs^4)^{\frac{1}{3}} - \frac{12a}{16b}\cdot\frac{3}{4b}(a+bs^4)^{\frac{1}{3}}+c$$

$$= \frac{3s^4}{16b}(a+bs^4)^{\frac{1}{3}} - \frac{9a}{16b^2}(a+bs^4)^{\frac{1}{3}}+c$$

$$= \frac{3}{16b}(a+bs^4)^{\frac{1}{3}}\left(s^4-\frac{3a}{b}\right)+c.$$

419. 求 $\int \frac{\sqrt{4-y^2}}{y^4}dy$.

[解] 引用公式(C),

$$\int \frac{\sqrt{4-y^2}}{y^4}dy = \frac{y^{-3}(4-y^2)^{\frac{3}{2}}}{-12}+0+c = -\frac{(4-y^2)^{\frac{3}{2}}}{12y^3}+c.$$

420. 求 $\int \frac{\theta^3\,d\theta}{\sqrt{3-\theta^2}}$.

[解] 引用公式(A)兩次,

$$\therefore \int \frac{\theta^5 d\theta}{\sqrt{3-\theta^3}} = \frac{\theta^5(3-\theta^3)^{\frac{1}{2}}}{-\frac{15}{2}} + \frac{36}{15}\int \theta^5(3-\theta^3)^{-\frac{1}{2}} d\theta$$

$$= -\frac{2}{15}\theta^5(3-\theta^3)^{\frac{1}{2}} + \frac{12}{5}\left[\frac{\theta^3(3-\theta^3)^{\frac{1}{2}}}{-\frac{9}{2}}\right.$$

$$\left. + 2\int \theta^2(3-\theta^3)^{-\frac{1}{2}} d\theta \right]$$

$$= -\frac{2}{15}\theta^5(3-\theta^3)^{\frac{1}{2}} - \frac{8}{15}\theta^3(3-\theta^3)^{\frac{1}{2}}$$

$$+ \frac{24}{5}\left(-\frac{1}{3}\right)2(3-\theta^3)^{\frac{1}{2}} + c$$

$$= -\frac{2}{15}(3-\theta^3)^{\frac{1}{2}}(\theta^6+4\theta^3+24) + c.$$

421. 求 $\int \frac{\sqrt{a^2+x^2}}{x} dx$.

[解] 引用公式(*B*),

$$\therefore \int \frac{\sqrt{a^2+x^2}}{x} dx = (a^2+x^2)^{\frac{1}{2}} + a^2 \int \frac{dx}{x\sqrt{a^2+x^2}} \qquad (1)$$

(1)式右端之積分因 $m+1=-1+1=0$, 故不能用公式(*C*)求之,但 $m+1=0$, 卽 $\frac{m+1}{n}=0$ (n 爲 2). 由34款討論一之代換方法求之.

$$\int \frac{dx}{x\sqrt{a^2+x^2}} = \frac{1}{2a}\log\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}. \qquad (2)$$

將(2)之結果代入(1)式,

$$\therefore \int \frac{\sqrt{a^2+x^2}}{x} dx = (a^2+x^2)^{\frac{1}{2}} + \frac{a}{2}\log\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a} + c.$$

422. 求 $\int \frac{x^3 dx}{\sqrt{1-x^2}}$.

[解] 引用公式(A),

$$\therefore \int \frac{x^3 dx}{\sqrt{1-x^2}} = -\frac{1}{3}x^2(1-x^2)^{\frac{1}{2}} + \frac{2}{3}\int x(1-x^2)^{-\frac{1}{2}} dx$$

$$= -\frac{1}{3}x^2(1-x^2)^{\frac{1}{2}} - \frac{2}{3}(1-x^2)^{\frac{1}{2}} + c$$

$$= -\frac{1}{3}(x^2+2)(1-x^2)^{\frac{1}{2}} + c.$$

第41款例題

423. 求 $\int \frac{dx}{\sin^4 x \cos^2 x}$.

[解] 因 $n+1 = -2+1 = -1 \neq 0$, 引用公式(G),

$$\therefore \int \frac{dx}{\sin^4 x \cos^2 x} = \int (\sin x)^{-4}(\cos x)^{-2} dx$$

$$= -\frac{\sin^{-3} x \cos^{-1} x}{-2+1} + \frac{-4-2+2}{-2+1}\int \sin^{-4} x\, dx$$

$$= \frac{1}{\sin^3 x \cos x} + 4\int \sin^{-4} x\, dx. \qquad (1)$$

(1)式右端之積分,其中 $m+1 = -4+1 = -3 \neq 0$, 再引用公式(H),

$$4\int \sin^{-4} x\, dx = \frac{4\sin^{-3} x \cos x}{-3} + \frac{4 \cdot 2}{3}\int \sin^{-2} x\, dx$$

$$= -\frac{4\cos x}{3\sin^3 x} + \frac{8}{3}\int \sin^{-2} x\, dx. \qquad (2)$$

(2)式右端之積分,其中 $m+1 = -2+1 = -1 \neq 0$, 更引用公式(H),

$$\frac{8}{3}\int \sin^{-2}x\,dx=\frac{8}{3}\cdot\frac{\sin^{-1}x\cos x}{-1}+0=-\frac{8}{3}\cdot\frac{\cos x}{\sin x}. \qquad (3)$$

將(3)之結果代入(2),復將(2)之結果代入(1),

$$\therefore\quad \int\frac{dx}{\sin^4 x\cos^2 x}=\frac{1}{\sin^3 x\cos x}-\frac{4\cos x}{3\sin^3 x}-\frac{8}{3}\frac{\cos x}{\sin x}+c.$$

424. 求 $\int \sin^2 x\cos^4 x\,dx.$

[解] 因 $m+n=2+4\neq 0$,先引用公式(*F*),

$$\therefore\quad \int \sin^2 x\cos^4 x\,dx=-\frac{\sin x\cos^5 x}{6}+\frac{1}{6}\int\cos^4 x\,dx. \qquad (1)$$

(1)式右端之積分可引用公式(*E*),$(m+n=0+4\neq 0)$,

$$\frac{1}{6}\int\cos^4 x\,dx=\frac{1}{6}\left[\frac{\sin x\cos^3 x}{4}+\frac{3}{4}\int\cos^2 x\,dx\right]$$

$$=\frac{\sin x\cos^3 x}{24}+\frac{3}{24}\int\cos^2 x\,dx. \qquad (2)$$

(2)式右端之積分再引用公式(*E*),$(m+n=0+2\neq 0)$,

$$\frac{3}{24}\int\cos^2 x\,dx=\frac{3}{24}\left[\frac{\sin x\cos x}{2}+\frac{x}{2}\right]=\frac{3\sin x\cos x}{48}+\frac{3x}{48}. \qquad (3)$$

將(3)之結果代入(2),復將(2)之結果代入(1),

$$\therefore\quad \int\sin^2 x\cos^4 x\,dx=-\frac{\sin x\cos^5 x}{6}+\frac{\sin x\cos^3 x}{24}$$

$$+\frac{\sin x\cos x}{16}+\frac{x}{16}+c$$

$$=-\frac{\sin x\cos^5 x}{6}+\frac{\sin x\cos^3 x}{24}$$

$$+\frac{1}{16}(\sin x\cos x+x)+c.$$

425　求 $\int\frac{\cos^4 x\,dx}{\sin^2 x}$.

[解] 先引用公式(*H*),

$$\therefore\quad \int \sin^{-2}x\cos^4 x\,dx=\frac{\sin^{-1}x\cos^5 x}{-1}+\frac{-2+4+2}{-1}\int\cos^4 x\,dx$$

$$=-\frac{\cos^5 x}{\sin x}-4\int\cos^4 x\,dx. \tag{1}$$

再將(1)式右端之積分引用公式(*E*),

$$4\int\cos^4 x\,dx=\frac{4\sin x\cos^3 x}{4}+\frac{3\cdot 4}{4}\int\cos^2 x\,dx. \tag{2}$$

將更(2)式右端之積分引用公式(*E*),

$$3\int\cos^2 x\,dx=3\cdot\frac{\sin x\cos x}{2}+3\cdot\frac{1}{2}\int dx$$

$$=\frac{3\sin x\cos x}{2}+\frac{3x}{2}. \tag{3}$$

將(3)之結果代入(2),復將(2)之結果代入(1),

$$\therefore\quad \int\frac{\cos^4 x\,dx}{\sin^2 x}=-\frac{\cos^5 x}{\sin x}-\sin x\cos^3 x$$

$$-\frac{3}{2}\sin x\cos x-\frac{3}{2}x+c$$

$$=-\frac{1}{2}\cdot\frac{\cos x}{\sin x}(2\cos^4 x+2\sin^2 x\cos^2 x$$

$$+3\sin^2 x)-\frac{3}{2}x+c$$

$$=-\frac{\cot x}{2}(3-\cos^2 x)-\frac{3}{2}x+c.$$

426. 求 $\int \sec^3 x\,dx$.

[解] 因 $\int \sec^3 x\,dx = \int \frac{1}{\cos^3 x}dx$; 故可引用公式(G),

$$\therefore \int \sec^3 x\,dx = \int \frac{dx}{\cos^3 x} = -\frac{\sin x \cos^{-2} x}{-2} + \frac{1}{2}\int \cos^{-1} x\,dx$$

$$= \frac{1}{2}\tan x \sec x + \frac{1}{2}\log(\sec x + \tan x) + c.$$

427. 求 $\int \csc^3 x\,dx$.

[解] 因 $\int \csc^3 x\,dx = \int \frac{dx}{\sin^3 x} = \int \sin^{-3} x\,dx$; 故可引用公式(H),

$$\therefore \int \csc^3 x\,dx = \int \sin^{-3} x\,dx = \frac{\sin^{-2} x \cos x}{-2} + \frac{1}{2}\int \sin^{-1} x\,dx$$

$$= -\frac{1}{2}\cot x \csc x + \frac{1}{2}\log(\csc x - \cot x) + c.$$

428. 求 $\int \frac{\cos^4 \theta\,d\theta}{\sin^3 \theta}$

[解] 先引用公式 E) 兩次,再引用公式(H)一次,

$$\therefore \int \frac{\cos^4 \theta\,d\theta}{\sin^3 \theta} = \sin^{-2}\theta \cos^3 \theta + 3\int \sin^{-3}\theta \cos^2 \theta\,d\theta$$

$$= \frac{\cos^3 \theta}{\sin^2 \theta} + 3\left[-\frac{\cos \theta}{\sin^2 \theta} - \int \sin^{-3}\theta\,d\theta\right]$$

$$= \frac{\cos^3 \theta}{\sin^2 \theta} - \frac{3\cos \theta}{\sin^2 \theta} - 3\left[-\frac{\cos \theta}{2\sin^2 \theta} + \frac{1}{2}\int \frac{d\theta}{\sin \theta}\right]$$

$$= \frac{\cos^3 \theta}{\sin^2 \theta} - \frac{3\cos \theta}{\sin^2 \theta} + \frac{3\cos \theta}{2\sin^2 \theta} - \frac{3}{2}\log(\csc \theta - \cot \theta) + c$$

$$=\frac{\cos^3\theta}{\sin^4\theta}-\frac{3\cos\theta}{2\sin^2\theta}-\frac{3}{2}\log(\csc\theta-\cot\theta)+c$$

$$=-\frac{\cos\theta}{2\sin^2\theta}-\cos\theta-\frac{3}{2}\log\tan\frac{\theta}{2}+c.$$

429. 求 $\int\sin^6\theta\,d\theta$.

[解] 引用公式(*F*)三次,

$$\therefore\quad\int\sin^6\theta\,d\theta=-\frac{\sin^5\theta\cos\theta}{6}+\frac{5}{6}\int\sin^4\theta\,d\theta$$

$$=-\frac{\sin^5\theta\cos\theta}{6}+\frac{5}{6}\left[-\frac{\sin^3\theta\cos\theta}{4}+\frac{3}{4}\int\sin^2\theta\,d\theta\right]$$

$$=-\frac{\sin^5\theta\cos\theta}{6}-\frac{5}{24}\sin^3\theta\cos\theta+\frac{5}{8}\int\sin^2\theta\,d\theta$$

$$=-\frac{\sin^5\theta\cos\theta}{6}-\frac{5}{24}\sin^3\theta\cos\theta$$

$$+\frac{5}{8}\left[-\frac{\sin\theta\cos\theta}{2}+\frac{1}{2}\int d\theta\right]$$

$$=-\frac{\sin^5\theta\cos\theta}{6}-\frac{5}{24}\sin^3\theta\cos\theta$$

$$-\frac{5}{16}\sin\theta\cos\theta+\frac{5}{16}\theta+c.$$

$$=-\frac{1}{2}\cos\theta\left(\frac{1}{3}\sin^5\theta+\frac{5}{12}\sin^3\theta+\frac{5}{8}\sin\theta\right)+\frac{5}{16}\theta+c.$$

430. 求 $\int\csc^5\theta\,d\theta$.

[解] 因 $\int\csc^5\theta\,d\theta=\int\frac{d\theta}{\sin^5\theta}=\int\sin^{-5}\theta\,d\theta$; 故可引用公式(*H*)兩次,

$$\therefore \int \csc^5\theta\,d\theta = \int \sin^{-5}\theta\,d\theta = -\frac{\sin^{-4}\theta\cos\theta}{4} + \frac{3}{4}\int \sin^{-3}\theta\,d\theta$$

$$= -\frac{\cos\theta}{4\sin^4\theta} + \frac{3}{4}\left[\frac{\sin^{-2}\theta\cos\theta}{-2} + \frac{1}{2}\int (\sin\theta)^{-1}\,d\theta\right]$$

$$= -\frac{\cos\theta}{4\sin^4\theta} - \frac{3\cos\theta}{8\sin^2\theta} + \frac{3}{8}\log(\csc\theta - \cot\theta) + c$$

$$= -\frac{\cos\theta}{4}\left(\frac{1}{\sin^4\theta} - \frac{3}{2\sin^2\theta}\right) + \frac{3}{8}\log(\csc\theta - \cot\theta) + c.$$

431. 求 $\int \sec^7\theta\,d\theta$.

[解] 因 $\int \sec^7\theta\,d\theta = \int \frac{d\theta}{\cos^7\theta} = \int \cos^{-7}\theta\,d\theta$; 故可引用公式(G)三次,

$$\therefore \int \sec^7\theta\,d\theta = \int \cos^{-7}\theta\,d\theta = -\frac{\sin\theta\cos^{-6}\theta}{-6} + \frac{-5}{-6}\int \cos^{-5}\theta\,d\theta$$

$$= \frac{1}{6}\sin\theta\cos^{-6}\theta + \frac{5}{6}\left[-\frac{\sin\theta\cos^{-4}\theta}{-4}\right.$$

$$\left. + \frac{-3}{-4}\int \cos^{-3}\theta\,d\theta\right]$$

$$= \frac{1}{6}\sin\theta\cos^{-6}\theta + \frac{5}{24}\sin\theta\cos^{-4}\theta + \frac{5}{8}\int \cos^{-3}\theta\,d\theta$$

$$= \frac{1}{6}\sin\theta\cos^{-6}\theta + \frac{5}{24}\sin\theta\cos^{-4}\theta$$

$$+ \frac{5}{8}\left[-\frac{\sin\theta\cos^{-2}\theta}{-2} + \frac{1}{2}\int (\cos\theta)^{-1}\,d\theta\right]$$

$$= \frac{1}{6}\sin\theta\cos^{-6}\theta + \frac{5}{24}\sin\theta\cos^{-4}\theta$$

$$+ \frac{5}{16}\sin\theta\cos^{-2}\theta + \frac{5}{16}\log(\sec\theta + \tan\theta) + c$$

$$=\frac{\sin\theta}{2\cos^2\theta}\left(\frac{1}{3\cos^4\theta}+\frac{5}{12\cos^2\theta}+\frac{5}{8}\right)$$

$$+\frac{5}{16}\log(\sec\theta+\tan\theta)+c.$$

432. 求 $\int\cos^8 t\,dt$

[解] 引用公式(*E*)四次,

$$\therefore\quad\int\cos^8 t\,dt=\frac{\sin t\cos^7 t}{8}+\frac{7}{8}\left[\frac{\sin t\cos^5 t}{6}\right.$$

$$\left.+\frac{5}{6}\left\{\frac{\sin t\cos^3 t}{4}+\frac{3}{4}\left(\frac{\sin t\cos t}{2}+\frac{1}{2}\int dt\right)\right\}\right]$$

$$=\frac{\sin t\cos^7 t}{8}+\frac{7\sin t\cos^5 t}{48}+\frac{35\sin t\cos^3 t}{192}$$

$$+\frac{35\sin t\cos t}{128}+\frac{35t}{128}+c$$

$$=\frac{\sin t}{8}\left(\cos^7 t+\frac{7}{6}\cos^5 t+\frac{35}{24}\cos^3 t\right.$$

$$\left.+\frac{35}{16}\cos t\right)+\frac{35t}{128}+c.$$

433. 求 $\int\frac{dy}{\sin^4 y\cos^3 y}$.

[解] 先引用公式(*H*)兩次,再引用公式(*G*),

$$\therefore\quad\int\frac{dy}{\sin^4 y\cos^3 y}=\int(\sin y)^{-4}(\cos y)^{-3}\,dy$$

$$=-\frac{1}{3}\sin^{-3}y\cos^{-2}y+\frac{5}{3}\int\sin^{-2}y\cos^{-3}y\,dy$$

$$= -\frac{1}{3}\sin^{-3} y\cos^{-2} y$$

$$+\frac{5}{3}\left[\frac{(\sin y)^{-1}\cos^{-2} y}{-1}+3\int \cos^{-3} y\,dy\right]$$

$$= -\frac{1}{3}\sin^{-3} y\cos^{-2} y$$

$$-\frac{5}{3}(\sin y)^{-1}\cos^{-2} y+5\int \cos^{-3} y\,dy$$

$$= -\frac{1}{3}\sin^{-3} y\cos^{-2} y-\frac{5}{3}(\sin y)^{-1}\cos^{-2} y$$

$$+5\left[-\frac{\sin y\cos^{-2} y}{-2}+\frac{1}{2}\int (\cos y)^{-1}\,dy\right]$$

$$= -\frac{1}{3\sin^3 y\cos^2 y}-\frac{5}{3\sin y\cos^2 y}$$

$$+\frac{5\sin y}{2\cos^2 y}+\frac{5}{2}\log(\sec y+\tan y)+c$$

$$= -\frac{1}{\cos^2 y}\left(\frac{1}{3\sin^3 y}+\frac{5}{3\sin y}-\frac{5\sin y}{2}\right)$$

$$+\frac{5}{2}\log(\sec y+\tan y)+c.$$

434. 求 $\int \sin^6 x\cos^4 x\,dx$.

[解] 先引用公式(E)兩次,

$$\therefore \int \sin^6 x\cos^4 x\,dx = \frac{\sin^7 x\cos^3 x}{10}+\frac{3}{10}\int \sin^6 x\cos^2 x\,dx$$

$$= \frac{\sin^7 x\cos^3 x}{10}+\frac{3}{10}\left\{\frac{\sin^7 x\cos x}{8}\right.$$

$$\left.+\frac{1}{8}\int \sin^6 x\,dx\right\}$$

$$= \frac{\sin^7 x \cos^3 x}{10} + \frac{3}{80}\sin^7 x \cos x$$

$$+ \frac{3}{80}\int \sin^6 x\, dx. \qquad (1)$$

再將(1)式右端之積分應用公式(F)三次,

$$\frac{3}{80}\int \sin^6 x\, dx = \frac{3}{80}\left\{ -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\int \sin^4 x\, dx \right\}$$

$$= -\frac{1}{160}\sin^5 x \cos x + \frac{1}{32}\left\{ -\frac{\sin^3 x \cos x}{4} \right.$$

$$\left. + \frac{3}{4}\int \sin^2 x\, dx \right\}$$

$$= -\frac{\sin^5 x \cos x}{160} - \frac{\sin^3 x \cos x}{128}$$

$$+ \frac{3}{128}\left\{ -\frac{\sin x \cos x}{2} + \frac{1}{2}\int dx \right\}$$

$$= -\frac{\sin^5 x \cos x}{160} - \frac{\sin^3 x \cos x}{128}$$

$$- \frac{3 \sin x \cos x}{256} + \frac{3x}{256}. \qquad (2)$$

將(2)式之結果代入(1)式,

$$\therefore \quad \int \sin^6 x \cos^4 x\, dx = \frac{\sin x \cos x}{2}\left(\frac{\sin^6 x \cos^2 x}{5} + \frac{3 \sin^6 x}{40} \right.$$

$$\left. - \frac{\sin^4 x}{80} - \frac{\sin^2 x}{64} - \frac{3}{128} \right) + \frac{3x}{256} + c.$$

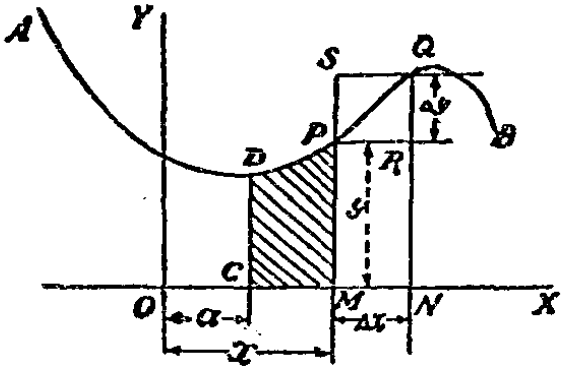
第八章

定限積分

43. 面積之微分. 設AB曲線之方程式爲

$$y=\phi(x).$$

假定$\phi(x)$爲一綿續函數.CD爲固定縱坐標.MP爲變動縱坐標.u爲面積$CMPD$之度支量.若自變數自a變至x,y亦自CD變至MP;由是面積u自零增至$CMPD$,故u亦爲x之函數.今命x之徵差爲Δx,u之徵差爲$\Delta u(=MNQP)$.



作$MNRP$與$MNQS$兩矩形,由圖所示,

$$\text{面積}MNRP<\text{面積}MNQP<\text{面積}MNQS,$$

即
$$y\cdot\Delta x<\Delta u<(y+\Delta y)\Delta x.$$

以Δx除之,
$$y<\frac{\Delta u}{\Delta x}<y+\Delta y.$$

若Δx趨近於極限零時,Δy亦趨近於極限零,$y+\Delta y$趨近於極限$y(=MP)$.

$$\therefore \quad \frac{du}{dx}=y\,dx=\phi(x)dx.$$

(A)　　$\therefore \quad du=y\,dx=\phi(x)dx\ [\because\ y=\phi(x)].$

故知曲線AB,x軸,及兩縱坐標所圍面積之微分等於末端縱坐標與其對應橫坐標之乘積.

44. 定限積分.

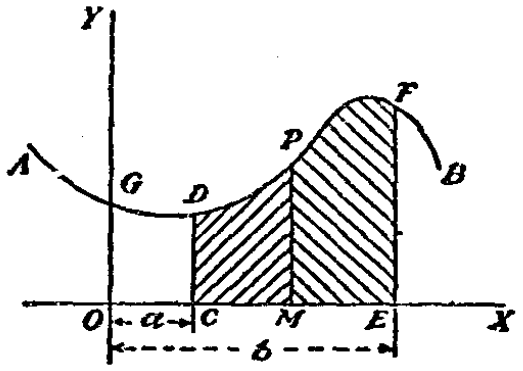
將43款(A)式求積分.

$$u=\int\phi(x)dx.$$

u爲面積,故上式右端之積分必存在

(B)　命　$u=\int\phi(x)dx=F(x)+c.$

上式c爲積分常數,今決定其值如下:爲便利起見,計算面積,先從y軸起算.



若$x=a$,則$u=$面積$OCDG$.

若$x=b$,則$u=$面積$OEFG$.

(C)　若$x=0$,則$u=0$.

將(C)代入(B)式,

$\therefore \quad 0=F(0)+c$, 卽 $c=-F(0)$.

c值既經決定,代回(B)式,

(D)　　$u=F(x)-F(0).$

(D)式乃從y軸至任一縱坐標之面積.

今求CD,EF兩縱坐標間之面積如下.

由 (D) 式, 面積 $OCDG = F(a) - F(0)$,

面積 $OEFG = F(b) - F(0)$,

但 面積 $CEFD =$ 面積 $OEFG -$ 面積 $OCDG$.

$$(E) \qquad \therefore \text{ 面積 } CEFD = F(b) - F(a).$$

慣例, (E) 式右端 $F(b) - F(a)$ 之差, 用下列記法示之:

$$(F) \qquad F(b) - F(a) = \int_a^b y\,dx = \int_a^b \phi(x)dx.$$

其意義爲面積微分 $(= y\,dx)$ 自 a 至 b 間之積分, 名 a 曰下限 (lower limit), b 爲上限 (upper limit), 名此運算爲兩限間之積分法. 因 (F) 之值恆一定, 且積分常數 $[-F(0)]$ 在運算中消去, 故曰定限積分 (definite integral).

若 x, y 僅表平面上一點之縱橫坐標, 則定限積分確表面積. 但若 y 表速度, x 表時間, 則 AB 爲一運動曲線, 定限積分表所經之距離. 故定限積分之物理解釋, 須視 x, y 所代表之量爲轉移, 其意義固不僅在面積一端也.

45. 定限積分之性質.

(A) 上下限互換, 則定限積分之符號改變.

因 $\int_a^b \phi(x)dx = F(b) - F(a)$, $\int_b^a \phi(x)dx = F(a) - F(b)$

$$= -[F(b) - F(a)];$$

故 $$\int_a^b \phi(x)dx = -\int_b^a \phi(x)\,dx.$$

(B) 定限積分可分解爲若干定限積分之和.

因 $\int_a^{x_1}\phi(x)dx=F(x_1)-F(a),$ (1)

又 $\int_{x_1}^{b}\phi(x)dx=F(b)-F(x_1).$ (2)

(1), (2) 兩式相加,

$$\int_a^{x_1}\phi(x)dx+\int_{x_1}^{b}\phi(x)dx=F(b)-F(a).$$

但 $F(b)-F(a)=\int_a^b\phi(x)dx.$

$$\therefore\ \int_a^{x_1}\phi(x)dx+\int_{x_1}^{b}\phi(x)dx=\int_a^b\phi(x)dx.$$

(C) 定限積分爲上下兩限之函數.

因 $\int_a^b\phi(x)dx=F(b)-F(a).$

故上式左端隨$\phi(x)$之形, 及 a, b 值而定. $\phi(x)$之性質與其中之文字無關, 卽

$$\phi(x)=\phi(z)=\phi(y)=\phi(u)=\cdots\cdots$$

故左端之定限積分爲其兩限之函數.

46. 定限積分之計算法.

[第一步] 用以前諸章之法,求出微分式之不定積分,略去其積分常數.

[第二步] 將不定積分中之變數,先代入以上限而計其值;次代入以下限而計其值. 由第一值減去第二值卽得所求.

若 a, b 有一爲無窮限時, 則由下列兩定義求之:

(A) b爲正無窮限時, 用公式

$$\int_a^{+\infty}\phi(x)dx=\lim_{b=+\infty}\int_a^b\phi(x)dx.$$

(B) a 爲負無窮限時，用公式

$$\int_{-\infty}^b\phi(x)dx=\lim_{a=-\infty}\int_a^b\phi(x)dx.$$

但假定二式右端之極限均屬存在.

[舉例] (1) 求 $\int_1^3 x^2\,dx$.

[解] $\int_1^3 x^2\,dx=\left|\frac{x^3}{3}\right|_1^3=\frac{3^3}{3}-\frac{1}{3}=\frac{26}{3}$.

(2) 求 $\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta$.

[解] $\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta=\left|-\cos\theta\right|_0^{\frac{\pi}{2}}=-\cos\frac{\pi}{2}+\cos 0^0=-0+1=1.$

(3) 求 $\int_1^\infty\frac{dx}{x^2}$.

[解] $\int_1^\infty\frac{dx}{x^2}=\lim_{b=+\infty}\int_1^b\frac{dx}{x^2}=\lim_{b=+\infty}\left|-\frac{1}{x}\right|_1^b=\lim_{b=+\infty}\left|-\frac{1}{b}+1\right|=1.$

(4) 求 $\int_{-\infty}^1 e^x\,dx$.

[解] $\int_{-\infty}^1 e^x\,dx=\lim_{a=-\infty}\int_a^1 e^x\,dx=\lim_{a=-\infty}\left|e^x\right|_a^1=\lim_{a=-\infty}\left|e-e^a\right|=e.$

47. 變數代換上下限亦宜代換. 由代換積分法，求得新變數之積分後，若仍欲原變數表之，則多一層代回之運算. 今求兩限間之積分，如已代換新變數後，將上下兩限亦同時代換之；則以前代回原變數之手續可省.

[舉例]　求 $\int_0^3 \frac{dx}{(2+x)\sqrt{1+x}}$.

[解]　命　$1+x=z^2$.

若　$x=3$, 則 $z=2$;

$x=0$, 則 $z=1$.

又因　$\sqrt{1+x}=z$;　$2+x=2+z^2-1=1+z^2$;　$dx=2z\,dz$.

$$\therefore \int_0^3 \frac{dx}{(2+x)\sqrt{1+x}}=\int_1^2 \frac{2z\,dz}{(z^2+1)z}=2\int_1^2 \frac{dz}{z^2+1}=2\left|\tan^{-1}z\right|_1^2$$

$$=2\tan^{-1}2-\frac{2\pi}{4}=2\tan^{-1}2-\frac{\pi}{2}.$$

48.　本章例題.

435.　$\int_2^3 6x^2\,dx=\left|2x^3\right|_2^3=2\cdot 3^3-2\cdot 2^3=54-16=38.$

436.　$\int_1^e \frac{dx}{x}=\left|\log x\right|_1^e=\log e-\log 1=1-0=1.$

437.　$\int_0^a (a^2x-x^3)dx=\left|\frac{a^2x^2}{2}-\frac{x^4}{4}\right|_0^a=\frac{a^4}{2}-\frac{a^4}{4}=\frac{a^4}{4}.$

438.　$\int_1^4 \frac{dx}{x^{\frac{3}{2}}}=\left|-\frac{2}{\sqrt{x}}\right|_1^4=-\frac{2}{2}+\frac{2}{1}=-1+2=1.$

439.　$\int_0^1 \frac{dx}{\sqrt{3-2x}}=-\frac{1}{2}\left|2\sqrt{8-2x}\right|_0^1=-1+\sqrt{3}.$

440.　$\int_0^2 \frac{x^3\,dx}{x+1}=\left|\frac{x^3}{3}-\frac{x^2}{2}+x-\log(x+1)\right|_0^2$

$$=\frac{8}{3}-\frac{4}{2}+2-\log 3=\frac{8}{3}-\log 3.$$

441. $$\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{2-3x^2}} = \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{(\sqrt{2})^2-(\sqrt{3}x)^2}}$$

$$= \frac{1}{\sqrt{3}} \int_0^{\frac{1}{\sqrt{3}}} \frac{d(\sqrt{3}x)}{\sqrt{(\sqrt{2})^2-(\sqrt{3}x)^2}}$$

$$= \frac{1}{\sqrt{3}} \left| \sin^{-1} \frac{\sqrt{3}}{\sqrt{2}} x \right|_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{\sqrt{3}} \sin^{-1} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \sin^{-1} 0 = \frac{\pi}{4\sqrt{3}}.$$

442. $$\int_2^3 \frac{3x\,dx}{2\sqrt[4]{x^2-4}} = \frac{3}{2} \int_2^3 \frac{x\,dx}{\sqrt[4]{x^2-4}} = \frac{3}{4} \int_2^3 (x^2-4)^{-\frac{1}{4}} 2x\,dx$$

$$= \frac{3}{4} \left| \frac{(x^2-4)^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} \right|_2^3 = \frac{3}{4} \left| \frac{4(x^2-4)^{\frac{3}{4}}}{3} \right|_2^3$$

$$= \left| (x^2-4)^{\frac{3}{4}} \right|_2^3 = 5^{\frac{3}{4}} = \sqrt[4]{125}.$$

443. $$\int_0^1 \frac{dy}{y^2-y+1} = \int_0^1 \frac{dy}{y^2-y+\frac{1}{4}+\frac{3}{4}} = \int_0^1 \frac{dy}{\left(y-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \left| \frac{2}{\sqrt{3}} \tan^{-1} \frac{2}{\sqrt{3}} \left(y-\frac{1}{2}\right) \right|_0^1$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \tan^{-1} \left(-\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{3\sqrt{3}}.$$

444. $$\int_2^3 \frac{t\,dt}{1+t^2} = \int_2^3 \frac{1}{2} \cdot \frac{2t\,dt}{1+t^2} = \frac{1}{2} \left| \log(1+t^2) \right|_2^3$$

$$= \frac{1}{2} \left| \log 10 - \log 5 \right| = \frac{1}{2} \log 2.$$

445. $\int_0^{\pi} \sin x\,dx = \Big| -\cos x \Big|_0^{\pi} = -(-1)-(-1) = 1+1 = 2.$

446. $\int_0^{\frac{\pi}{4}} \sec^4\theta\,d\theta = \int_0^{\frac{\pi}{4}} (1+\tan^2\theta)\sec^2\theta\,d\theta$

$$= \int_0^{\frac{\pi}{4}} \sec^2\theta\,d\theta + \int_0^{\frac{\pi}{4}} \tan^2\theta\sec^2\theta\,d\theta$$

$$= \Big| \tan\theta \Big|_0^{\frac{\pi}{4}} + \Big| \frac{\tan^3\theta}{3} \Big|_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \tan 0^\circ + \frac{1}{3}\tan^3\frac{\pi}{4} - \frac{1}{3}\tan^3 0^\circ$$

$$= 1+\frac{1}{3} = \frac{4}{3}.$$

447. $\int_0^{2r} \frac{\sqrt{2r}}{\sqrt{x}}\,dx = \sqrt{2r}\int_0^{2r} x^{-\frac{1}{2}}\,dx = \sqrt{2r}\,\Big| 2x^{\frac{1}{2}} \Big|_0^{2r}$

$$= \sqrt{2r}\cdot 2\sqrt{2r} = 4r.$$

448. $\int_0^5 \left(\frac{3}{5}\sqrt{t} - \frac{3}{25}t^2\right)dt = \frac{3}{5}\int_0^5 (\sqrt{t} - \frac{1}{5}t^2)dt$

$$= \frac{3}{5}\Big| \frac{2}{3}t^{\frac{3}{2}} - \frac{t^3}{15} \Big|_0^5 = \frac{3}{5}\Big| \frac{2}{3}\cdot 5^{\frac{3}{2}} - \frac{5^3}{15} - 0 \Big|$$

$$= \frac{3}{5}\left| \frac{10\sqrt{5}}{3} - \frac{25}{3} \right| = 2\sqrt{5} - 5.$$

449. $\int_0^r \frac{r\,dx}{\sqrt{r^2-x^2}} = r\,\Big| \sin^{-1}\frac{x}{r} \Big|_0^r = r\sin^{-1}(1) = \frac{\pi r}{2}.$

450. $\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^3\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sin^3\theta(1-\sin^2\theta)\cos\theta\,d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin^3\theta - \sin^5\theta) d(\sin\theta)$$

$$= \left| \frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{6} \right|_0^{\frac{\pi}{2}} = \frac{1}{4}\sin^4\frac{\pi}{2}$$

$$- \frac{1}{6}\sin^6\frac{\pi}{2} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

451. $$\int_0^{2r} \frac{2\sqrt{2r}\,dy}{\sqrt{2r-y}} = 2\sqrt{2r}\int_0^{2r} \frac{dy}{\sqrt{2r-y}}$$

$$= -2\sqrt{2r}\int_0^{2r} (2r-y)^{-\frac{1}{2}} d(2r-y)$$

$$= -2\sqrt{2r}\left| \frac{(2r-y)^{\frac{1}{2}}}{\frac{1}{2}} \right|_0^{2r}$$

$$= -4\sqrt{2r}\left| \sqrt{2r-y} \right|_0^{2r} = 4\sqrt{2r}\cdot\sqrt{2r} = 8r.$$

452. $$\int_{-b}^{b} \frac{\pi}{a^4}(y^2-b^2)^4\,dy$$

$$= \frac{\pi}{a^4}\int_{-b}^{b} (y^8 - 4b^2y^6 + 6b^4y^4 - 4b^6y^2 + b^8)dy$$

$$= \frac{\pi}{a^4}\left| \frac{y^9}{9} - \frac{4b^2y^7}{7} + \frac{6b^4y^5}{5} - \frac{4b^6y^3}{3} + b^8y \right|_{-b}^{b}$$

$$= \frac{2\pi}{a^4}\left[\frac{b^9}{9} - \frac{4b^9}{7} + \frac{6b^9}{5} - \frac{4b^9}{3} + b^9 \right]$$

$$= \frac{2\pi b^9}{a^4}\left(\frac{1}{9} - \frac{4}{7} + \frac{6}{5} - \frac{4}{3} + 1 \right)$$

$$= \frac{2\pi b^9}{a^4}\cdot\frac{384}{945} = \frac{256\pi b^9}{315a^4}.$$

453. $\int_0^{\pi} 2a(2+2\cos\theta)^{\frac{1}{2}}\,d\theta = 2a\int_0^{\pi}\frac{2}{\sqrt{2}}(1+\cos\theta)^{\frac{1}{2}}d\theta$

$$= 2a\int_0^{\pi} 2\cos\frac{\theta}{2}\,d\theta$$

$$= 8a\int_0^{\pi}\cos\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right)$$

$$= 8a\left|\sin\frac{\theta}{2}\right|_0^{\pi} = 8a\sin\frac{\pi}{2} = 8a.$$

454. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan\theta\,d\theta = \left|\log\sec\theta\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \log\sqrt{2}-\log\sqrt{2} = 0.$

455. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec\theta\,d\theta = \left|\log(\sec\theta+\tan\theta)\right|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$$= \log(\sqrt{2}+1)-\log\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)$$

$$= \log(1+\sqrt{2})-\log\sqrt{3} = \log\left(\frac{1+\sqrt{2}}{\sqrt{3}}\right).$$

456. $\int_0^{\frac{\pi}{2}}\frac{\cos\theta\,d\theta}{1+\sin^2\theta} = \int_0^{\frac{\pi}{2}}\frac{d(\sin\theta)}{(\sin\theta)^2+1} = \left|\tan^{-1}(\sin\theta)\right|_0^{\frac{\pi}{2}}$

$$= \tan^{-1}(1)-\tan^{-1}(0) = \frac{\pi}{4}-0 = \frac{\pi}{4}.$$

457. $\int_1^{+\infty}\frac{dx}{x^4} = \underset{b=+\infty}{\text{limit}}\left|-\frac{1}{3x^3}\right|_1^b = \underset{b=+\infty}{\text{limit}}\left|-\frac{1}{3b^3}+\frac{1}{3}\right| = \frac{1}{3}$

458. $\int_1^{+\infty}\frac{dx}{x\sqrt{2x^2-1}} = \underset{b=+\infty}{\text{limit}}\left|\text{arc sec}\sqrt{2}\,x\right|_1^b$

$$= \lim_{b=+\infty} \left| \operatorname{arc\,sec} \sqrt{2}\, b - \operatorname{arc\,sec} \sqrt{2} \right|$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

459. $$\int_0^{+\infty} \frac{dx}{x^2+a^2} = \lim_{b=+\infty} \left| \frac{1}{a} \operatorname{arc\,tan} \frac{x}{a} \right|_0^b$$

$$= \lim_{b=+\infty} \left| \frac{1}{a} \operatorname{arc\,tan} \frac{b}{a} \right| = \frac{\pi}{2a}.$$

460. $$\int_1^{+\infty} \frac{dx}{a^2+b^2x^2} = \int_0^{+\infty} \frac{1}{b} \cdot \frac{d(bx)}{(bx)^2+a^2} = \frac{1}{b} \lim_{B=+\infty} \int_0^B \frac{d(bx)}{(bx)^2+a^2}$$

$$= \frac{1}{b} \lim_{B=+\infty} \left| \frac{1}{a} \operatorname{arc\,tan} \frac{bx}{a} \right|_0^B$$

$$= \frac{1}{ab} \lim_{B=+\infty} \left| \operatorname{arc\,tan} \frac{bB}{a} \right| = \frac{\pi}{2ab}.$$

461. $$\int_0^{+\infty} e^{-x}\, dx = \lim_{b=+\infty} \int_0^b e^{-x}\, dx = -\lim_{b=+\infty} \int_0^b e^{-x}\, d(-x)$$

$$= -\lim_{b=+\infty} \left| e^{-x} \right|_0^b = -\lim_{b=+\infty} \left| e^{-b} - e^0 \right|$$

$$= -(-1) = 1.$$

462 $$\int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2} = \lim_{\substack{b=+\infty \\ a=-\infty}} \int_a^b \frac{dx}{x^2+2x+1+1}$$

$$= \lim_{\substack{b=+\infty \\ a=-\infty}} \int_a^b \frac{d(x+1)}{(x+1)^2+1}$$

$$= \lim_{\substack{b=+\infty \\ a=-\infty}} \left| \tan^{-1}(x+1) \right|_a^b$$

$$= \lim_{\substack{b=+\infty \\ a=-\infty}} \left| \operatorname{arc\,tan}(b+1) - \operatorname{arc\,tan}(a+1) \right|$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi.$$

463. $$\int_0^{+\infty} e^{-ax}\,dx = \lim_{b=+\infty}\int_0^b e^{-ax}\,dx = -\frac{1}{a}\lim_{b=+\infty}\int_0^b e^{-ax}\,d(-ax)$$

$$= -\frac{1}{a}\lim_{b=+\infty}\left| e^{-ax}\right|_0^b = -\frac{1}{a}\lim_{b=+\infty}\left| e^{-ab} - e^0\right|$$

$$= -\frac{1}{a}(-1) = \frac{1}{a}.$$

464. $$\int_a^{+\infty}\frac{dx}{(a+x)^n} = \lim_{b=+\infty}\int_a^b\frac{dx}{(a+x)^n} = \lim_{b=+\infty}\left|\frac{(a+x)^{1-n}}{1-n}\right|_a^b$$

$$= \lim_{b=+\infty}\left|\frac{(a+b)^{1-n}}{1-n} - \frac{(a+a)^{1-n}}{1-n}\right| = -\frac{(2a)^{1-n}}{1-n}$$

$$= \frac{(2a)^{1-n}}{n-1} = \frac{1}{(n-1)(2a)^{n-1}}.$$

以下數題，其不定積分由偏分積分法求之.

465. $$\int_1^2\frac{dx}{x^2+4x} = \int_1^2\frac{dx}{4x} - \int_1^2\frac{dx}{4(x+4)}$$

$$= \left[\frac{1}{4}\log x\right]_1^2 - \left[\frac{1}{4}\log(x+4)\right]_1^2$$

$$= \left[\frac{1}{4}\log 2 - \frac{1}{4}\log 1\right] - \left[\frac{1}{4}\log 6 - \frac{1}{4}\log 5\right]$$

$$= \frac{1}{4}\log 10 - \frac{1}{4}\log 6 = \frac{1}{4}\log\frac{10}{6} = \frac{1}{4}\log\frac{5}{3}.$$

466. $$\int_0^5\frac{dx}{1+3x+2x^2} = \int_0^5\frac{2\,dx}{2x+1} - \int_0^5\frac{dx}{x+1}$$

$$= \left[\log(2x+1)\right]_0^5 - \left[\log(x+1)\right]_0^5$$

$$= \log 11 - \log 6 = \log\frac{11}{6}.$$

467. $\int_1^4 \frac{1+3x}{x(x+1)^2}dx = \int_1^4 \frac{dx}{x} + \int_1^4 \frac{2\,dx}{(x+1)^2} - \int_1^4 \frac{dx}{x+1}$

$$= \left|\log x\right|_1^4 + \left|\frac{-2}{x+1}\right|_1^4 - \left|\log(x+1)\right|_1^4$$

$$= \log 4 - \frac{2}{5} + 1 - \log\frac{5}{2} = \log\frac{8}{5} + \frac{3}{5}.$$

468. $\int_0^3 \frac{2x\,dx}{(x^2+3)(x^2+1)} = \int_0^3 \frac{x\,dx}{x^2+1} - \int_0^3 \frac{x\,dx}{x^2+3}$

$$= \frac{1}{2}\int_0^3 \frac{d(x^2+1)}{x^2+1} - \frac{1}{2}\int_0^3 \frac{d(x^2+3)}{x^2+3}$$

$$= \frac{1}{2}\left|\log(x^2+1)\right|_0^3 - \frac{1}{2}\left|\log(x^2+3)\right|_0^3$$

$$= \frac{1}{2}[\log 10 - \log 1] - \frac{1}{2}[\log 12 - \log 3]$$

$$= \frac{1}{2}(\log 10 - \log 12 + \log 3) = \frac{1}{2}\log\frac{30}{12}$$

$$= \frac{1}{2}\log\frac{5}{2} = \log\sqrt{\frac{5}{2}}.$$

以下數題，其不定積分由代換積分法求之.

469. 求 $\int_0^a \sqrt{a^2-x^2}\,dx.$

[解] 令 $x = a\sin\theta$，則 $dx = a\cos\theta\,d\theta$，$\sqrt{a^2-x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$

$x = a,\ \theta = \frac{\pi}{2};\quad x = 0,\ \theta = 0.$

$$\therefore \int_0^a \sqrt{a^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta = a^2\left|\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right|_0^{\frac{\pi}{2}}$$

$$= a^2\cdot\frac{\pi}{4} - 0 = \frac{a^2\pi}{4}.$$

470. 求 $\int_0^{12} \frac{x\,dx}{(2x+3)^{\frac{1}{3}}}$.

[解] 命 $(2x+3)^{\frac{1}{3}}=z$, 即 $z^3=2x+3$; $x=\frac{z^3-3}{2}$, $dx=\frac{1}{2}3z^2dz$.

若 $x=12$, 則 $z=3$; $x=0$, 則 $z=\sqrt[3]{3}$.

$$\therefore \int_0^{12}\frac{x\,dx}{(2x+3)^{\frac{1}{3}}}=\int_{\sqrt[3]{3}}^{3}\frac{(z^3-3)3z^2dz}{4z^1}=\frac{3}{4}\int_{\sqrt[3]{3}}^{3}z\,dz-\frac{3}{4}\int_{\sqrt[3]{3}}^{3}3z^2dz$$

$$=\frac{3}{4}\left|\frac{z^2}{2}+\frac{3}{z}\right|_{\sqrt[3]{3}}^{3}=\frac{3}{4}\left(\frac{9}{2}+1-\frac{3^{\frac{3}{3}}}{2}-\frac{3}{2\sqrt[3]{3}}\right)$$

$$=\frac{3}{8}\left(11-\frac{9}{\sqrt[3]{3}}\right).$$

471. 求 $\int_0^{16}\frac{x^{\frac{1}{4}}dx}{1+x^{\frac{1}{2}}}$.

[解] 命 $x=z^4$;

則 $dx=4z^3\,dz$, $x^{\frac{1}{2}}=z^2$, $x^{\frac{1}{4}}=z$.

又 $x=16$, $z=2$; $x=0$, $z=0$.

$$\therefore \int_0^{16}\frac{x^{\frac{1}{4}}dx}{1+x^{\frac{1}{2}}}=\int_0^{2}\frac{z\cdot 4z^3\,dz}{1+z^2}=4\int_0^{2}\left(z^2-1+\frac{1}{1+z^2}\right)dz$$

$$=4\int_0^{2}z^2\,dz-4\int_0^{2}dz+4\int_0^{2}\frac{dz}{1+z^2}$$

$$=\left|\frac{4z^3}{3}-4z+4\tan^{-1}z\right|_0^{2}=\frac{8}{3}+4\tan^{-1}2.$$

472. 求 $\int_{\frac{\pi}{2}}^{\pi}\frac{dx}{1-\cos x}$.

[解] 命 $\tan\frac{x}{2}=z$;

則 $\cos x=\frac{1-z^2}{1+z^2}$, $dx=\frac{2\,dz}{1+z^2}$. (參觀35款)

若 $x=\pi$, 則 $z=\infty$; $x=\frac{\pi}{2}$, 則 $z=1$.

$$\therefore \int_{\frac{\pi}{2}}^{\pi}\frac{dx}{1-\cos x}=\int_1^{\infty}\frac{\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}}=\int_1^{\infty}\frac{dz}{z^2}=\lim_{b=\infty}\left|-\frac{1}{z}\right|_1^b$$

$$=\lim_{b=\infty}\left|-\frac{1}{b}+1\right|=-0+1=1.$$

以下數題之不定積分,用分部積分法求之.

473. 求 $\int_0^1 x\log x\,dx$.

[解] 命 $u=\log x$, $dv=x\,dx$;

則 $du=d(\log x)=\frac{dx}{x}$, $v=\frac{1}{2}x^2$.

代入38款公式(A),

$$\therefore \int_0^1 x\log x\,dx=\left[\frac{x^2}{2}\log x-\int\frac{x^2}{2}\cdot\frac{dx}{x}\right]_0^1$$

$$=\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_0^1=-\frac{1}{4}.$$

474. 求 $\int_0^1 s^2\log s\,ds$.

[解] 命 $u=\log s$, $dv=s^2\,ds$;

則 $du=d(\log s)=\frac{ds}{s}$, $v=\frac{1}{3}s^3$.

$$\therefore \quad \int_0^1 s^2 \log s \, ds = \left[\frac{s^3}{3}\left(\log s - \frac{1}{3}\right)\right]_0^1 = \frac{1}{3}\left(-\frac{1}{3}\right) = -\frac{1}{9}.$$

475. 求 $\int_0^1 \sin^{-1} z \, dz$.

[解] 命 $u = \sin^{-1} z$, $dv = dz$;

則 $du = d(\sin^{-1} z) = \dfrac{dz}{\sqrt{1-z^2}}$, $v = z$.

$$\therefore \quad \int_0^1 \sin^{-1} z \, dz = \left[z \sin^{-1} z - \int z \cdot \frac{dz}{\sqrt{1-z^2}}\right]_0^1$$

$$= \left[z \sin^{-1} z + \sqrt{1-z^2}\right]_0^1 = \frac{\pi}{2} - 1.$$

476. 求 $\int_0^1 \tan^{-1} \theta \, d\theta$.

[解] 命 $u = \tan^{-1} \theta$, $dv = d\theta$;

則 $du = d(\tan^{-1} \theta) = \dfrac{d\theta}{1+\theta^2}$, $v = \theta$.

$$\therefore \quad \int_0^1 \tan^{-1} \theta \, d\theta = \left[\theta \tan^{-1} \theta - \int \theta \cdot \frac{d\theta}{1+\theta^2}\right]_0^1$$

$$= \left[\theta \tan^{-1} \theta - \log (1+\theta^2)^{\frac{1}{2}}\right]_0^1 = \frac{\pi}{4} - \log \sqrt{2}.$$

第九章

累次積分法與偏積分法

49. 累次積分法 設x之函數爲

$$y=F(x).$$

其微係數$F'(x)$仍爲自變數x之函數;且新函數[即$F'(x)$]之微差對於自變數微差之比之極限往往仍屬存在,故新函數復可求其微係數.如此累次行之,名$\frac{d^2y}{dx^2}$爲第二微係數;$\frac{d^3y}{dx^3}$爲第三微係數;$\frac{d^ny}{dx^n}$爲第n微係數.

今命

$$\frac{dy}{dx}=F'(x),$$

$$\frac{d^2y}{dx^2}=F''(x),$$

$$\frac{d^3y}{dx^3}=F'''(x),$$

.....

$$\frac{d^ny}{dx^n}=F^{(n)}(x).$$

故累次微分式爲

$$dy = F'(x)dx,$$

$$d(dy) = d^2y = F''(x)dx^2,$$

$$d(d^2y) = d^3y = F'''(x)dx^3,$$

.....

$$d(d^{n-1}y) = d^ny = F^{(n)}(x)dx^n.$$

累次微分之逆運算曰累次積分法(successive integration). 其記法如次:

若: $d^2y = F''(x)dx^2$, 則 $y = \int\int F''(x)dx^2$. (1)

$d^3y = F'''(x)dx^3$, 則 $y = \int\int\int F'''(x)dx^3$. (2)

$d^4y = F^{IV}(x)dx^4$, 則 $y = \int\int\int\int F^{IV}(x)dx^4$. (3)

.....

稱(1)曰二重積分(double integral);(2)曰三重積分(triple integral);餘仿此.普遍言之,累次積分法之次數若爲兩次或多於兩次,其積分名曰重積分(multiple integral). 其積分號上若有上下兩限,則名此積分曰定限積分;否則曰不定積分.

[舉例] 曲線上每點之縱坐標,其第二微係數恆爲橫坐標之4倍,求此曲線之方程式.

[解] 由題意 $\frac{d^2y}{dx^2} = 4x.$

其第二微分爲 $d^2y = 4x\,dx^2.$

$$\therefore\quad y=\int\int 4x\,dx\,dx=4\int\int x\,dx^2=4\int\left(\frac{x^2}{2}+c_1\right)dx$$

$$=4\cdot\frac{1}{2}\cdot\frac{x^3}{3}+c_1x+c_2.$$

60. 偏積分法. 微分式含有二個或多於二個自變數而求積分時,先就其中任一自變數而求積分,假定其餘之自變數均爲常數;如此輪流爲之,此法曰偏積分法(partial integration). 自變數若有兩個,名此積分曰二重積分;自變數若有三個,則曰三重積分(此處命名與49款相同;本款之自變數x,y,z若均用x表之,其意義與49款無異;故49款之重積分可視爲本款之特例).

偏積分法計算之次序,其在最內層之積分號與在最內層之微分相當;故積分號之次序自右推算至左,而微分則自左推算至右.

[舉例] 求 $\int_1^2\int_0^z\int_0^{x\sqrt{3}}\frac{x}{x^2+y^2}dy\,dx\,dz.$

[解] $$\int_1^2\int_0^z\int_0^{x\sqrt{3}}\frac{x}{x^2+y^2}dy\,dx\,dz=\int_1^2\int_0^z\left[\int_0^{x\sqrt{3}}\frac{x}{x^2+y^2}dy\right]dx\,dz$$

(假定x,z爲常數)

$$=\int_1^2\int_0^z\left[x\cdot\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^{x\sqrt{3}}dx\,dz$$

$$=\int_1^2\int_0^z\tan^{-1}\sqrt{3}\,dx\,dz$$

$$=\int_1^2\int_0^z\frac{\pi}{3}dx\,dz$$

$$=\frac{\pi}{3}\int_1^2\left[\int_0^z dx\right]dz$$

(假定 z, y 爲常數)

$$=\frac{\pi}{3}\int_1^2\Big[\,x\,\Big]_0^z dz=\frac{\pi}{3}\int_1^2 z\,dz$$

(假定 y, x 爲常數)

$$=\frac{\pi}{3}\Big[\frac{z^2}{2}\Big]_1^2=\frac{\pi}{3}\Big[\frac{4}{2}-\frac{1}{2}\Big]$$

$$=\frac{\pi}{3}\cdot\frac{3}{2}=\frac{\pi}{2}.$$

51. 本章例題.

477. 已知 $\frac{d^3y}{dx^3}=0$, 求 y.

[解] $y=\int\int\int 0\,dx\,dx\,dx=\int\int c_1\,dx\,dx=\int(c_1x+c_2)dx=\frac{c_1x^2}{2}+c_2x+c_3.$

478. 已知 $\frac{d^3y}{dx^3}=ax^2$, 求 y.

[解] $y=\int\int\int ax^2\,dx^3=\int\int\Big(\frac{1}{3}ax^3+c_1\Big)dx^2=\int\Big(\frac{1}{12}ax^4+c_1x+c_2\Big)dx$

$$=\frac{ax^5}{60}+\frac{c_1x^2}{2}+c_2x+c_3$$

479. 已知 $d^3y=\frac{2\,dx^3}{x^3}$, 求 y.

[解] $y=\int\int\int 2x^{-3}\,dx^3=\int\int\Big[-2\cdot\frac{1}{2x^2}+c_1\Big]dx^2=\int\Big(\frac{1}{x}+c_1x+c_2\Big)dx$

$$=\log x+\frac{c_1x^2}{2}+c_2x+c_3$$

480. 已知 $d^3\rho=\sin\theta\,d\theta^3$, 求 ρ.

[解] $\rho=\iiint\sin\theta\,d\theta^3=\iint(-\cos\theta+c_1)\,d\theta^2$

$=\int(-\sin\theta+c_1\theta+c_2)\,d\theta=\cos\theta+\frac{c_1\theta^2}{2}+c_2\theta+c_3.$

481. 已知 $d^3s=\left(3t^3-\frac{1}{t^3}\right)dt^3$, 求 s.

[解] $s=\iiint\left(3t^3-\frac{1}{t^3}\right)dt^3=\iint\left(t^3+\frac{1}{2t^2}+c_1\right)dt^2$

$=\int\left(\frac{t^4}{4}-\frac{1}{2t}+c_1t+c_2\right)dt=\frac{t^5}{20}-\frac{1}{2}\log t+\frac{c_1t^2}{2}+c_2t+c_3.$

482. 已知 $d^2\rho=\sin\phi\cos^2\phi\,d\phi^2$, 求 ρ.

[解] $\rho=\iint\sin\phi\cos^2\phi\,d\phi^2=\int\left(-\frac{\cos^3\phi}{3}+c_1\right)d\phi$

$=\int-\frac{1}{3}(\cos\phi-\sin^2\phi\cos\phi)\,d\phi+\int c_1\,d\phi$

$=\frac{\sin^3\phi}{9}-\frac{1}{3}\sin\phi+c_1\phi+c_2.$

483. 已知 $\frac{d^4y}{dx^4}=x\cos x$, 求 y.

[解] $y=\iiint\int x\cos x\,dx^4.$

由分部積分法, 命 $u=x$, $dv=\cos x\,dx$;

則 $du=dx$, $v=\sin x$.

代入38款公式 (A),

$\therefore\quad y=\iiint\int x\cos x\,dx^4=\iiint(x\sin x-\int\sin x\,dx)\,dx^3$

$=\iiint(x\sin x+\cos x+c_1)\,dx^3$

$$=\int\int\left[(-x\cos x+\sin x)+(\sin x+c_1x+c_2)\right]dx^2$$

$$=\int\left[(-x\sin x-\cos x)-\cos x-\cos x+\frac{c_1x^2}{2}+c_2x\right]dx$$

$$=(x\cos x-\sin x)-\sin x-\sin x-\sin x+\frac{c_1x^3}{6}+\frac{c_2x^2}{2}+c_3x+c_4$$

$$=x\cos x-4\sin x+\frac{c_1x^3}{6}+\frac{c_2x^2}{2}+c_3x+c_4.$$

484. 已知 $d^3y=\sin^3x\,dx^3$, 求 y.

[解] $$y=\int\int\int\sin^3x\,dx^3=\int\int\int\sin x(1-\cos^2x)dx^3$$

$$=\int\int\int(\sin x-\sin x\cos^2x)dx^3=\int\int(-\cos x+\frac{\cos^3x}{3}+c_1)dx^2$$

$$=\int\int[-\cos x+\frac{\cos x(1-\sin^2x)}{3}+c_1]dx^2$$

$$=\int[-\sin x+\frac{\sin x}{3}-\frac{\sin^3x}{9}+c_1x+c_2]dx$$

$$=\cos x-\frac{\cos x}{3}+\frac{\cos x}{9}-\frac{1}{9}\cdot\frac{\cos^3x}{3}+\frac{c_1x^2}{2}+c_2x+c_3$$

$$=\frac{7\cos x}{9}-\frac{\cos^3x}{27}+\frac{c_1x^2}{2}+c_2x+c_3.$$

485. 求 $\int_{\frac{b}{2}}^{b}\int_{0}^{\frac{r}{b}}r\,d\theta\,dr$.

[解] $$\int_{\frac{b}{2}}^{b}\int_{0}^{\frac{r}{b}}r\,d\theta\,dr=\int_{\frac{b}{2}}^{b}\left[r\theta\right]_{0}^{\frac{r}{b}}dr=\int_{\frac{b}{2}}^{b}\frac{r^2}{b}dr=\left[\frac{r^3}{3b}\right]_{\frac{b}{2}}^{b}$$

$$=\frac{b^3}{3b}-\frac{\frac{b^3}{8}}{3b}=\frac{b^2}{3}-\frac{b^2}{24}=\frac{7b^2}{24}.$$

486. 求 $\int_0^a \int_0^b xy(x-y)\,dy\,dx.$

[解] $$\int_0^a \int_0^b xy(x-y)\,dy\,dx = \int_0^a \left[\frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^b dx = \int_0^a \left(\frac{b^2x^2}{2} - \frac{b^3x}{3}\right)dx$$

$$= \left[\frac{b^2x^3}{6} - \frac{b^3x^2}{6}\right]_0^a = \frac{b^2a^3}{6} - \frac{a^2b^2}{6} = \frac{a^2b^2}{6}(a-b).$$

487. $$\int_0^a \int_{y-a}^{2y} xy\,dx\,dy = \int_0^a \left[\frac{yx^2}{2}\right]_{y-a}^{2y} dy$$

$$= \int_0^a \left[\frac{4y^3}{2} - \frac{y}{2}(y^2 - 2ay + a^2)\right] dy$$

$$= \frac{1}{2}\int_0^a (3y^3 + 2ay^2 - a^2y)\,dy$$

$$= \frac{1}{2}\left[\frac{3}{4}y^4 + \frac{2}{3}ay^3 - \frac{a^2}{2}y^2\right]_0^a$$

$$= \frac{1}{2}\left[\frac{3}{4}a^4 + \frac{2}{3}a^4 - \frac{1}{2}a^4\right]$$

$$= \frac{1}{2} \cdot \frac{9a^4 + 8a^4 - 6a^4}{12}$$

$$= \frac{1}{2} \cdot \frac{11a^4}{12} = \frac{11a^4}{24}.$$

488. 求 $\int_0^a \int_0^x \int_0^y x^3 y^3 z\,dz\,dy\,dx.$

[解] $$\int_0^a \int_0^x \int_0^y x^3 y^3 z\,dz\,dy\,dx = \int_0^a \int_0^x \left[\frac{x^3y^3z^2}{2}\right]_0^y dy\,dx$$

$$= \int_0^a \int_0^x \frac{x^3xy^5}{2}\,dy\,dx = \int_0^a \left[\frac{x^3y^6}{12}\right]_0^x dx$$

$$= \int_0^a \left[\frac{x^9}{12}\right] dx = \left[\frac{x^{10}}{120}\right]_0^a = \frac{a^{10}}{120}.$$

489. 求 $\int_b^a \int_\beta^\alpha \gamma^2 \sin\theta \, d\theta \, dr.$

[解] $$\int_b^a \int_\beta^\alpha \gamma^2 \sin\theta \, d\theta \, d\gamma = \int_b^a \Big[-\gamma^2 \cos\theta \Big]_\beta^\alpha d\gamma$$

$$= \int_b^a \gamma^2 (\cos\beta - \cos\alpha) d\gamma$$

$$= \Big[(\cos\beta - \cos\alpha) \frac{\gamma^3}{3} \Big]_b^a$$

$$= (\cos\beta - \cos\alpha) \frac{a^3 - b^3}{3}.$$

490. 求 $\int_0^\pi \int_0^{a\cos\theta} \rho \sin\theta \, d\rho \, d\theta.$

[解] $$\int_0^\pi \int_0^{a\cos\theta} \rho \sin\theta \, d\rho \, d\theta = \int_0^\pi \Big[\sin\theta \cdot \frac{\rho^2}{2} \Big]_0^{a\cos\theta} d\theta$$

$$= \int_0^\pi \frac{a^2 \cos^2\theta \sin\theta \, d\theta}{2}$$

$$= -\frac{a^2}{2} \int_0^\pi \cos^2\theta \, d(\cos\theta)$$

$$= \Big[-\frac{a^2}{2} \cdot \frac{\cos^3\theta}{3} \Big]_0^\pi$$

$$= \Big[\frac{-a^2 \cos^3\theta}{6} \Big]_0^\pi = \Big[\frac{-a^2(-1)}{6} - \frac{-a^2(1)}{6} \Big]$$

$$= \Big[\frac{a^2}{6} + \frac{a^2}{6} \Big] = \frac{2}{6} a^2 = \frac{1}{3} a^2.$$

491. 求 $\int_0^1 \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx.$

[解] $$\int_0^1 \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx = \int_0^1 \int_0^x \Big[e^{x+y+z} \Big]_0^{x+y} dy \, dx$$

$$= \int_0^1 \int_0^x \left[e^{2(x+y)} - e^{x+y} \right] dy\, dx = \int_0^1 \left[\frac{e^{2(x+y)}}{2} - e^{x+y} \right]_0^x dx$$

$$= \int_0^1 \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx = \int_0^1 \left(\frac{e^{4x}}{2} - \frac{3\,e^{2x}}{2} + e^x \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3\,e^{2x}}{4} + e^x \right]_0^1 = \frac{e^4}{8} - \frac{3}{4} e^2 + e - \frac{1}{8} + \frac{3}{4} - 1$$

$$= \frac{e^4 - 3}{8} - \frac{3e^2}{4} + e.$$

492. 求 $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dz\, dy\, dx.$

[解] $$\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dz\, dy\, dx = \int_0^a \int_0^b \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^c dy\, dx$$

$$= \int_0^a \int_0^b \left((cx^2 + cy^2 + \frac{c^3}{3} \right) dy\, dx = \int_0^a \left[c\,x^2 y + \frac{c y^3}{3} + \frac{c^3}{3} y \right]_0^b dx$$

$$= \int_0^a \left(c\,bx^2 + \frac{c\,b^3}{3} + \frac{c^3 b}{3} \right) dx = \left[\frac{cb}{3} x^3 + \frac{cb^3}{3} x + \frac{c^3 b}{3} x \right]_0^a$$

$$= \frac{a^3\,bc}{3} + \frac{a\,b^3 c}{3} + \frac{a\,bc^3}{3} = \frac{a\,bc}{3} (a^2 + b^2 + c^2).$$

493 求 $\int_b^a \int_0^b \int_a^{2a} x^2\, y^2\, z\, dz\, dy\, dx.$

[解] $$\int_b^a \int_0^b \int_a^{2a} x^2\, y^2\, z\, dz\, dy\, dx = \int_b^a \int_0^b \left[\frac{x^2\, y^2\, z^2}{2} \right]_a^{2a} dy\, dx$$

$$= \int_b^a \int_0^b \left[\frac{4a^2 x^2 y^2}{2} - \frac{a^2\, x^2\, y^2}{2} \right] dy\, dx = \int_b^a \int_0^b \left(\frac{3}{2} a^2 x^2 y^2 \right) dy\, dx$$

$$= \int_b^a \left[\frac{3\,a^2\, x^2\, y^3}{2 \cdot 3} \right]_0^b dx = \int_b^a \left(\frac{a^2\, b^3\, x^2}{2} \right) dx = \left[\frac{a^2\, b^3\, x^3}{6} \right]_b^a$$

$$= \frac{a^5\, b^3}{6} - \frac{a^2\, b^6}{6} = \frac{a^2\, b^3}{6} (a^3 - b^3).$$

494. 求 $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta\,dr\,d\theta$

[解] $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\sin\theta\,dr\,d\theta=\int_0^{\pi}\left[\frac{\sin\theta r^3}{3}\right]_0^{a(1+\cos\theta)}d\theta$

$$=\int_0^{\pi}\left[\frac{\sin\theta\,a^3(1+\cos\theta)^3}{3}\right]d\theta$$

$$=\frac{a^3}{3}\int_0^{\pi}\sin\theta(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)d\theta$$

$$=\frac{a^3}{3}\left[-\cos\theta-\frac{3}{2}\cos^2\theta-\cos^3\theta-\frac{1}{4}\cos^4\theta\right]_0^{\pi}$$

$$=\frac{a^3}{3}(1+1+1+1-\frac{3}{2}-1+\frac{3}{2}+1)=\frac{4a^3}{3}.$$

495. 求 $\int_0^a\int_0^{\sqrt{x}} dy\,dx.$

[解] $\int_0^a\int_0^{\sqrt{x}}dy\,dx=\int_0^a\left[y\right]_0^{\sqrt{x}}dx=\int_0^a\sqrt{x}\,dx$

$$=\int_0^a x^{\frac{1}{2}}dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a=\frac{2}{3}a^{\frac{3}{2}}.$$

496. 求 $\int_0^b\int_t^{10t}\sqrt{st-t^2}\,ds\,dt.$

[解] $\int_0^b\int_t^{10t}\sqrt{st-t^2}\,ds\,dt=\int_0^b\left[\frac{2}{3t}(st-t^2)^{\frac{3}{2}}\right]_t^{10t}dt=\int_0^b\frac{2}{3t}(10t^2-t^2)^{\frac{3}{2}}dt$

$$=\int_0^b\frac{2}{3t}(9t^2)^{\frac{3}{2}}dt=\int_0^b\frac{2\cdot27t^3}{3t}dt=\int_0^b18t^2dt=\left[\frac{18t^3}{3}\right]_0^b=6b^3.$$

497. 求 $\int_0^1\int_0^{v^2}e^{\frac{u}{v}}du\,dv.$

[解] $\int_0^1\int_0^{v^2}e^{\frac{u}{v}}du\,dv=\int_0^1\left[v\,e^{\frac{u}{v}}\right]_0^{v^2}dv=\int_0^1(v\,e^v-v)dv$

$$= \int_0^1 v\,e^v\,dv - \int_0^1 v\,dv = \left[ve^v - e^v - \frac{v^2}{2}\right]_0^1 = e - e - \frac{1}{2} + 1 = \frac{1}{2}.$$

498. 求 $\int_0^a \int_{\frac{x^2}{a}}^{x} \frac{x\,dy\,dx}{x^2+y^2}$.

[解] $$\int_0^a \int_{\frac{x^2}{a}}^{x} \frac{x}{x^2+y^2}\,dy\,dx = \int_0^a \left[x \cdot \frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{\frac{x^2}{a}}^{x} dx$$

$$= \int_0^a \left[\tan^{-1}1 - \tan^{-1}\frac{x}{a}\right]dx = \int_0^a \left(\frac{\pi}{4} - \tan^{-1}\frac{x}{a}\right)dx$$

$$= \left[\frac{\pi}{4}x - x\tan^{-1}\frac{x}{a} + \frac{a}{2}\log(x^2+a^2)\right]_0^a$$

$$= \frac{\pi}{4}a - a\tan^{-1}1 + \frac{a}{2}\log 2a^2 - \frac{a}{2}\log a^2$$

$$= \frac{\pi}{4}a - \frac{\pi}{4}a + \frac{a}{2}\log\frac{2a^2}{a^2} = \frac{a}{2}\log 2.$$

499. 求 $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} (x+y)\,dx\,dy$

[解] $$\int_{-a}^{a} \int_0^{\frac{y^2}{a}} (x+y)\,dx\,dy = \int_{-a}^{a} \left[\frac{x^2}{2} + yx\right]_0^{\frac{y^2}{a}} dy = \int_{-a}^{a} \left(\frac{y^4}{2a^2} + \frac{y^3}{a}\right)dy$$

$$= \left[\frac{y^5}{10a^2} + \frac{y^4}{4a}\right]_{-a}^{a} = \frac{a^5}{10a^2} + \frac{a^4}{4a} - \left(-\frac{a^5}{10a^2}\right) - \frac{a^4}{4a} = \frac{2a^5}{10a^2} = \frac{a^3}{5}$$

500. 求 $\int_0^{\frac{\pi}{2}} \int_{a\cos\theta}^{a} \rho^4\,d\rho\,d\theta$

[解] $$\int_0^{\frac{\pi}{2}} \int_{a\cos\theta}^{a} \rho^4\,d\rho\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{\rho^5}{5}\right]_{a\cos\theta}^{a} d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{a^5}{5} - \frac{a^5\cos^5\theta}{5}\right)d\theta$$

$$= \frac{a^5}{5}\int_0^{\frac{\pi}{2}} d\theta - \frac{a^5}{5}\int_0^{\frac{\pi}{2}} \cos^5\theta\,d\theta = \frac{a^5}{5}\int_0^{\frac{\pi}{2}} d\theta$$

$$-\frac{a^5}{5}\int_0^{\frac{\pi}{2}}(1-2\sin^2\theta+\sin^4\theta)\cos\theta\,d\theta$$

$$=\frac{a^5}{5}\Big[\theta\Big]_0^{\frac{\pi}{2}}-\frac{a^5}{5}\Big[\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{a^5}{5}\Big[\theta-\sin\theta+\frac{2}{3}\sin^3\theta-\frac{1}{5}\sin^5\theta\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{a^5}{5}\Big(\frac{\pi}{2}-1+\frac{2}{3}-\frac{1}{5}\Big)=\frac{a^5}{5}\Big(\frac{\pi}{2}-\frac{8}{15}\Big)=\frac{a^5}{10}\Big(\pi-\frac{16}{15}\Big).$$

501. 求 $\int_0^b\int_y^{10y}\sqrt{xy-y^2}\,dx\,dy.$

[解] 命 $xy-y^2=z^2$;

則 $dz=\dfrac{2z\,dz}{y}$, $\sqrt{xy-y^2}=z.$

又 $x=10y$, 則 $z=3y$;

$x=y$, 則 $z=0.$

$$\therefore\quad\int_0^b\int_y^{10y}\sqrt{xy-y^2}\,dx\,dy=\int_0^b\int_0^{3y}\frac{z\,2z\,dz}{y}dy$$

$$=\int_0^b\int_0^{3y}\frac{2z^2}{y}\,dz\,dy=\int_0^b\Big[\frac{2z^3}{3y}\Big]_0^{3y}dy$$

$$=\int_0^b\frac{2\cdot27y^3}{3y}\,dy=\int_0^b18y^2\,dy=\Big[\frac{18y^3}{3}\Big]_0^b=6b^3.$$

502. 求 $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{a}}dz\,dy\,dx.$

[解] $$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{a}}dz\,dy\,dx=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\Big[z\Big]^{\frac{x^2+y^2}{a}}dy\,dx$$

$$=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\Big(\frac{x^2+y^2}{a}\Big)dy\,dx=\int_0^{2a}\Big[\frac{x^2}{a}y+\frac{1}{3a}y^3\Big]_0^{\sqrt{2ax-x^2}}dx$$

$$=\int_0^{2a}\Big[\frac{x^2\sqrt{2ax-x^2}}{a}+\frac{(2ax-x^2)^{\frac{3}{2}}}{3a}\Big]dx.$$

上式右端之積分,用代換積分法求之:

命 $x=2a\sin^2\theta$;

則 $dx=4a\sin\theta\cos\theta\,d\theta$, $\sqrt{2ax-x^2}$

$=\sqrt{4a^2\sin^2\theta-4a^2\sin^4\theta}=2a\sin\theta\cos\theta$, $(2ax-x^2)^{\frac{5}{2}}$

$=8a^5\sin^5\theta\cos^5\theta$, $x^2=4a^2\sin^4\theta$.

$$\therefore\ \int_0^{2a}\left[\frac{x^2\sqrt{2ax-x^2}}{a}+\frac{(2ax-x^2)^{\frac{5}{2}}}{3a}\right]dx$$

$$=\frac{1}{3a}\int_0^{2a}\left[3x^2(2ax-x^2)^{\frac{1}{2}}+(2ax-x^2)^{\frac{5}{2}}\right]dx$$

$$=\frac{1}{3a}\int_0^{2a}(2ax-x^2)^{\frac{1}{2}}(3x^2+2ax-x^2)dx$$

$$=\frac{1}{3a}\int_0^{2a}(2ax-x^2)^{\frac{1}{2}}(2x^2+2ax)dx$$

$$=\frac{1}{3a}\int_0^{\frac{\pi}{2}}2a\sin\theta\cos\theta(8a^2\sin^4\theta+2a\cdot2a\sin^2\theta)4a\sin\theta\cos\theta\,d\theta$$

$$=\frac{1}{3a}\int_0^{\frac{\pi}{2}}64a^4\sin^6\theta\cos^2\theta\,d\theta+\frac{1}{3a}\int_0^{\frac{\pi}{2}}32a^4\sin^4\theta\cos^2\theta\,d\theta$$

$$=\frac{64a^4}{3a}\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^2\theta\,d\theta+\frac{32a^4}{3a}\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta$$

$$=\frac{64a^3}{3}\left[\sin\theta\cos\theta\left(\frac{\sin^6\theta}{8}-\frac{\sin^4\theta}{48}-\frac{5\sin^2\theta}{192}-\frac{15}{384}\right)\right.$$

$$\left.+\frac{15}{384}\theta\right]_0^{\frac{\pi}{2}}+\frac{32a^3}{3}\left[\frac{\cos\theta}{2}\left(\frac{\sin^5\theta}{3}-\frac{\sin^3\theta}{12}-\frac{\sin\theta}{8}\right)+\frac{\theta}{16}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{64a^3}{3}\cdot\frac{15}{384}\cdot\frac{\pi}{2}+\frac{32a^3}{3}\cdot\frac{1}{16}\cdot\frac{\pi}{2}=\frac{5}{12}a^3\pi+\frac{1}{3}a^3\pi$$

$$=\frac{9}{12}a^3\pi=\frac{3}{4}a^3\pi.$$

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