PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

(A.)

FOR THE YEAR MDCCCXC.

VOL. 181.

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VOL. 181.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind : the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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„ „ p „ „ Proceedings only.

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  B. K.K. Zoologisch-Botanische Gesellschaft.
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  Birmingham.
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    AB. Mason College.
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  Cambridge.
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    p.   Union Society.
  Cooper's Hill.
    AB. Royal Indian Engineering College.
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    p.   Dudley and Midland Geological and Scientific Society.
  Essex.
    p.   Essex Field Club.
  Greenwich.
    A.   Royal Observatory.
  Kew.
    B.   Royal Gardens.
  Leeds.
    p.   Philosophical Society.

England and Wales (continued).
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    AB. Free Public Library.
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  London.
    AB. Admiralty.
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    AB. Chemical Society.
    p.   "Electrician," Editor of the.
    B.   Entomological Society.
    AB. Geological Society.
    AB. Geological Survey of Great Britain.
    p.   Geologists' Association.
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    p.   London Library.
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    p.   Physical Society.
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    B.   Royal College of Surgeons.
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    p.   Royal Statistical Society.
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- AB.   The War Office.
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- *p.*   Victoria Institute.
- B.   Zoological Society.

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- AB.   Free Library.
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- B.   Marine Biological Association.
- *p.*   Plymouth Institution.

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- A.   " Kew " Observatory.

Salford.
- *p.*   Royal Museum and Library.

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- *p.*   The College.

Swansea.
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Woolwich.
- AB.   Royal Artillery Library.

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Helsingfors.
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Bordeaux.
- *p.*   Académie des Sciences.
- *p.*   Faculté des Sciences.
- *p.*   Société de Médecine et de Chirurgie.
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Dijon.
- *p.*   Académie des Sciences.

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- *p.*   Faculté des Sciences.

Lyons.
- AB.   Académie des Sciences, Belles-Lettres et Arts.

Marseilles.
- *p.*   Faculté des Sciences.

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- AB.   Académie des Sciences et Lettres.
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Paris.
- AB.   Académie des Sciences de l'Institut.
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- *p.*   Bureau des Longitudes.
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- *p.*   Commission des Annales des Ponts et Chaussées.
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    p.   Naturhistorisches Museum.
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    AB. Medicinisch-Naturwissenschaftliche Gesellschaft.

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    A. Grossherzogliche Sternwarte.

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    A. Sternwarte.
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    p.   Annalen der Physik und Chemie.
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    p.   Roorkee College.

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  Madrid.
    p. Comisión del Mapa Geológico de España.
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  Gottenburg.
    AB. Kongl. Vetenskaps och Vitterhets Samhälle.
  Lund.
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  Stockholm.
    A. Acta Mathematica.
    AB. Kongliga Svenska Vetenskaps-Akademie.
    AB. Sveriges Geologiska Undersökning.
  Upsala.
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  Brooklyn.
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    p. Elliott Society of Science and Art of South
          Carolina.
  Chicago.
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United States (continued).
  Philadelphia.
    AB. Academy of Natural Sciences.
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          Fisheries.
    AB. United States Geological Survey.
    AB. United States Naval Observatory.
  West Point (N Y.)
    AB. United States Military Academy.

Adjudication of the Medals of the Royal Society for the year 1890, by the President and Council.

The Copley Medal to Simon Newcomb, For.Mem.R.S., for his Contributions to the Progress of Gravitational Astronomy.

The Rumford Medal to Heinrich Hertz, for his work in Electro-magnetic Radiation.

A Royal Medal to David Ferrier, F.R.S., for his Researches on the Localisation of Cerebral Functions.

A Royal Medal to John Hopkinson, F.R.S., for his Researches in Magnetism and Electricity.

The Davy Medal to Emil Fischer, for his Discoveries in Organic Chemistry, and especially for his Researches on the Carbo-hydrates.

The Darwin Medal to Alfred Russel Wallace, for his Independent Origination of the Theory of the Origin of Species by Natural Selection.

The Bakerian Lecture, " The Discharge of Electricity through Gases (Preliminary Communication)," was delivered by Professor A. Schuster, F.R.S.

The Croonian Lecture, " On some Relations between Host and Parasite in certain Epidemic Diseases of Plants," was delivered by Professor H. M. Ward, F.R.S.

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(A.)

VOL. 181.

LIST OF ILLUSTRATIONS.

ERRATA.

Bakerian Lecture.

Page 260, *for* $N = H_c \cos \delta_c - H \cos \delta$ *read* $N = H \cos \delta - H_c \cos \delta_c$.

A similar correction should be made in the two following equations.

Page 290, in fig. 21 the direction of the horizontal disturbing force at St. Leonards should be as in Plate 13.

Plate 13.   The angle made by the horizontal disturbing force at Campbelton with the geographical meridian is $+ 136°$ not $- 136°$.

G. H. BRYAN *on a Rotating Liquid Spheroid.*

Pages 214, 216, equations (96), (103),

$$\textit{for}\quad K_n(\zeta) - \frac{4q_2(\zeta)}{n(n+1)}\quad \textit{read}\quad K_n(\zeta) + \frac{4q_2(\zeta)}{n(n+1)},$$

and in equation (97) remove the sign $-$ on the right hand.

I. *On the Effect of Temperature on the Specific Inductive Capacity of a Dielectric.*

By W. Cassie, *M.A., Trinity College, Cambridge, Examiner in the Universities of Aberdeen and Durham.*

Communicated by J. J. Thomson, *M.A., F.R.S., Cavendish Professor of Experimental Physics in the University of Cambridge.*

The object of the experiments described in this paper was to learn how the specific inductive capacity of a dielectric is affected by change of temperature.

Cavendish observed an increase in the capacity of a glass condenser when it was heated, but gave no measure of the effect. Dr. Hopkinson observed in light flint glass an increase of $2\frac{1}{2}$ per cent. in the capacity between $12°$ and $83°$ C. And Messrs. Gibson and Barclay showed that there is no appreciable change in the case of paraffin between temperatures $-12°$ and $24°$ C. Except these, no measurements of the effect appear to have been hitherto published.

The present investigation shows an increase of specific inductive capacity with rise of temperature in all the solids* examined, and a decrease in all the liquids except one.

As paraffin, which is a substance comparatively near its melting point at ordinary temperatures, shows no change, these results seem to indicate, as far as they go, that the specific inductive capacity of a substance has a maximum value about the melting point. But it may be questioned whether the data are sufficient as yet to warrant so general an induction.

The relation which Clerk Maxwell's electromagnetic theory of light indicates between specific inductive capacity and refractive index makes it interesting to compare the effects of temperature on these two quantities. Four of the liquids

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* The results given here for mica, ebonite, and the first specimen of glass are all less than those mentioned by Professor J. J. Thomson, in his treatise on "Applications of Dynamics to Physics and Chemistry"; because the results quoted there were obtained from an earlier set of observations, in which the precautions for insulation were inferior to those described here.

investigated here are amongst those for which DALE and GLADSTONE have observed the refractive indices at several temperatures. For one of these four the temperature effects show no similarity whatever, for another the relation is fairly close to that indicated by the theory, and for the other two, though not agreeing exactly with the theory, the relation does not differ from it very greatly, not more, perhaps, than might be explained by differences in the specimens used.

The experiments were done in the Cavendish Laboratory, Cambridge, and I am glad of this opportunity of thanking Professor J. J. THOMSON, F.R.S., at whose suggestion they were undertaken, for placing the resources of the Laboratory at my disposal, and for valuable advice during the course of the work.

I. SOLIDS.

The method adopted in the case of solids was to make a condenser with the substance to be experimented upon and observe its capacity at different temperatures. The condenser consisted of a pile of alternate thin sheets of the dielectric and discs of lead foil, with flat plates of iron above and below, and a weight on the top to keep them together. The condenser could not be fixed together in any more permanent way, because the unequal expansion with heat of the different materials would have altered the degree of compression of the pile, and so produced a change of capacity greater than the effect under investigation.

Special precautions were required to secure that the insulation between the two sides of the condenser be as far as possible independent of the temperature. To secure this the condenser was suspended in a stirrup from a bracket about four feet above the air bath in which it was heated. The suspending wires, which formed the connection for the side of the condenser in contact with the stirrup, passed without touching through holes in the top of the air bath, and over a short glass tube varnished with shellac, which was fixed across the bracket above. The connection for the other side of the condenser consisted of a stiff wire, which also passed without touching through a hole in the top of the air bath, and was carried through the glass tube on the bracket. Thus the condenser had no contact with anything about the air bath, and was supported solely by the glass tube on the bracket, which was always cold and perfectly insulating. The perfect freedom of the condenser and its supports was easily tested before each observation by tapping the air bath and seeing that no motion was produced in the suspending wires.

The condenser was heated by raising the bath quickly to a temperature considerably above that aimed at, and then leaving under it a small flame adjusted by experience to give the right temperature. The temperature was read by a thermometer passing through a cork in the top of the bath. About three hours was the shortest time required to ensure that the condenser was uniformly heated throughout; it was usually left a good deal longer.

The condensers needed several heatings before settling down to a steady capacity in the cold state. The temperatures in these preliminary heatings were always above the highest used in the actual observations.

The capacities were measured by the method used by Professor J. J. THOMSON in his determination of the ratio of the electrostatic and electromagnetic units ('Phil. Trans.,' 1883).   It is thus described in his paper :—

" In a Wheatstone's bridge, A B C D, with the galvanometer at G, and the battery between A and B, the circuit B D is not closed, but the points B and D are connected with the two poles, R and S, of a commutator, between which a travelling piece, P, moves backwards and forwards ; P is connected with one plate of a condenser, the other plate of which is connected with D.   Thus when P is in contact with S, the condenser will be charged, and until it is fully charged, electricity will flow into it from the battery ; this will produce a momentary current through the various arms of the bridge.   When the moving piece P is in contact with R, the two plates of the condenser are connected, and the condenser will discharge itself through D R, and as the resistance of D R is infinitesimal in comparison with the resistance of

Fig. 1.
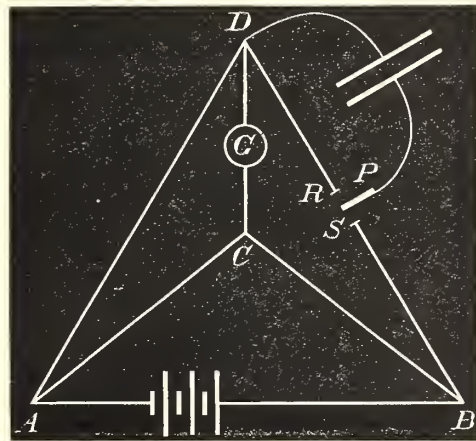
any other circuit, the discharge of the condenser will not send an appreciable amount of electricity through the galvanometer.   Thus, if we make the moving piece P oscillate quickly from R to S, there will, owing to the flow of electricity to the condenser, be a succession of momentary currents through the galvanometer.   The resistances are so adjusted that the deflection of the galvanometer produced by these momentary currents is balanced by the deflection due to the steady current through the galvanometer, and the resultant deflection is zero.   When this is the case there is a relation between the capacity of the condenser, the number of times the condenser is charged and discharged per second, and the resistances in the various arms of the bridge."

This relation, which is worked out in the paper, is expressed by taking a, b, c, d, g to represent the resistances of A C, A B, A D, B C, D C respectively.   The resistances

of D R and S B are so small as to be negligible.   Then if the condenser has a capacity C, and is charged and discharged n times per second

$$nC = \frac{a\left\{1 - \dfrac{a^2}{(a+c+g)(a+b+d)}\right\}}{cd\left\{1 + \dfrac{ab}{c(a+b+d)}\right\}\left\{1 + \dfrac{ag}{d(a+c+g)}\right\}}.$$

The resistances used were about

$$a = 10,$$
$$b = 8,$$
$$c \text{ between } 3000 \text{ and } 6000,$$
$$d \quad ,, \quad\quad 6000 \quad ,, \quad 2000,$$
$$g = 4060 \text{ ohms.}$$

With these values

$$nC = \frac{a}{cd}$$

is correct to within ·2 per cent., and is the formula used here.

The commutator P R S (fig. 2) was that used by Professor THOMSON and is thus described in his paper :—

Fig. 2.
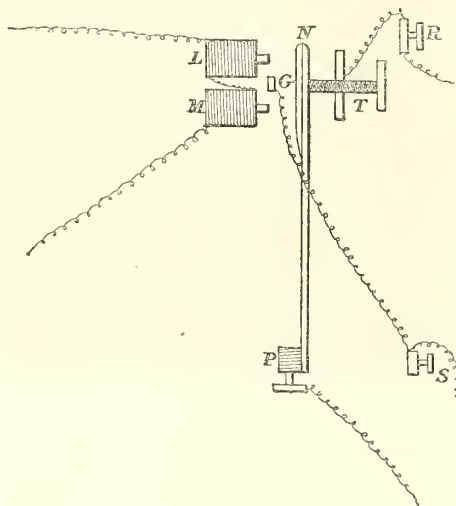
" The current from two GROVE'S cells passes first through a tuning-fork interruptor, and then through the coils L M of an electromagnet.   P N is a strip of brass with a piece of iron attached to it.   When there is no current passing through the electro-magnet, the elasticity of the rod P N makes it press against a screw T, which is electrically connected with a binding screw R : when the current passes through the electromagnet, the magnet attracts the iron attached to the rod P N and brings it into contact with the stop G, which is electrically connected with the binding screw S. The letters P, R, S indicate the same points in this figure as in fig. 1.   All the places where contact is made by the vibrating piece P N are covered with platinum, and the

whole arrangement is fastened down to an ebonite board. As the current passes intermittently through the coils L M of the electromagnet, the vibrating piece strikes alternately against the parts G and T; when it strikes against G the opposite plates of the condenser are connected with the two poles of the battery; when it strikes against T the condenser is discharged (see fig. 1)."

To be able to allow for conduction or absorption in the dielectric by means of observations with forks of different rates, we must know the comparative times during which the vibrating piece P is in contact with G, in contact with T, and travelling from one to the other. I therefore fixed a stiff arm to the end of P and took tracings with it on smoked paper, moved across while the interruptor was vibrating. These tracings showed that when the interruptor is going steadily (this is easily known by the sound) the vibrator is in contact with G and T for equal times. and the time occupied in passing from one to the other is negligible compared to the time of contact.

The battery used was six LECLANCHÉ cells, and the variable resistances were taken from a box by ELLIOTT. Tuning forks were used, making 99, 64, and 49 complete vibrations per second. The forks were worked by a current from the storage cells in the Laboratory.

The coefficients of expansion with temperature of lead, ebonite, and glass are ·000029, ·000077, and ·000009 respectively. That of mica does not seem to have been determined; but it is probably less than that of lead. So that the combined effects of expansion with rise of temperature do not affect the capacity before the fifth decimal place; and, as the results do not profess to be correct to more than four decimal places, the expansion of the materials may be neglected.

Some observations were also taken with ebonite by an electrostatic method based upon that described in MAXWELL's "Electricity and Magnetism," § 229 (second edition); the only difference being that there was no guard ring. The condenser was compared with a variable condenser by dividing a charge between them, separating them, and connecting opposite poles together and to the electrometer; then, if they were unequal, the difference deflected the electrometer. For this method a special key was required to make the connections quickly and to keep the electrometer to earth until everything except the charge to be measured had been discharged. If the electrometer were not kept to earth it would be deflected in spite of ordinary screens by induction inside the key on account of the high potentials required.

When neither of the condensers being compared has any absorption, there is no difficulty with this method, and any ordinary quadrant electrometer may be used. But when this is not the case the residual discharge will begin to come out as soon as the condensers are connected, and this makes the needle always go ultimately to the same side. The only way to get a balance is to make the variable condenser too large, and gradually diminish it until the initial motion of the needle due to it disappears. This requires a needle with a short time of swing, because with a slow

needle the residual charge comes out and overpowers the first effect before the needle has time to show it. With a small ebonite condenser it was just possible to make an ELLIOTT's electrometer sufficient by making the connection as short as possible; but with a large condenser it was useless. Consequently, this experiment would require an electrometer with a needle of very short time of swing. One with small cylindrical needle and quadrants seems most suitable.

Mica.

The mica condenser was made of twenty-two sheets of brown Muscovite with equilateral triangular markings. The sheets were about $3\frac{1}{2}$ inches diameter, and the thicknesses varied from about a quarter to about a tenth of a millimetre. Some difficulty was found in getting rid of surface conduction on the mica. No cleaning would make it insulate, but in the end perfect insulation was secured by putting a border of shellac varnish round the edges of the sheets. The shellac did not come between the plates of the condenser, and so did not affect the capacity. The highest temperatures in the preliminary heating were less than 140° C.

The insulation was constantly tested by an electrometer, and was found perfect throughout for temperatures below 110° C. Above this it began to give way.

The results are given in the following table and diagram (fig. 3).

Temperature.	Variable resistance.	Change of capacity.	Rate of increase per degree.
Fork making 99 complete vibrations per second.			
11° C.	4890		
37	4850	·0082	·00032
46	4830	·0122	·00035
56	4810	·0163	·00035
70	4790	·0204	·00035
79	4770	·0245	·00036
89	4750	·0286	·00035
107	4700	·0387	·00040
112	4600	·0591	·00059
Fork making 64 complete vibrations per second.			
14	5530		
48	5460	·0127	·00037
60	5440	·0163	·00035
70	5425	·0190	·00034
78	5390	·0253	·00039
89	5340	·0343	·00045
110	5270	·0470	·00049

The fact that both periods give the same value shows that, as was to be expected from its crystalline structure, there is no absorption in mica for those short times of charging.

Fig. 3.
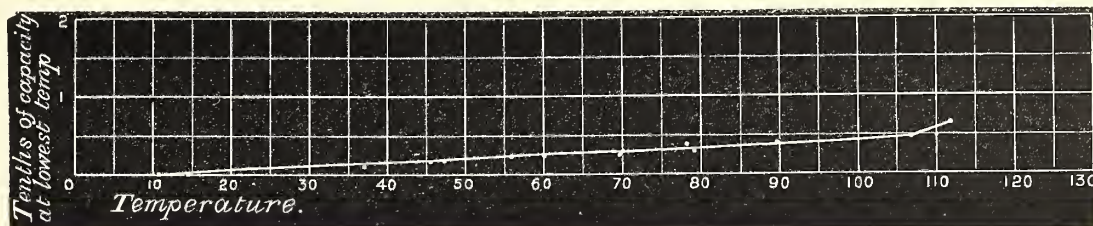
Ebonite.

The ebonite condenser was made of twenty discs of ebonite, about a third of a millimetre thick, and three and a quarter inches in diameter. The ebonite sheets were carefully cleaned with paraffin dissolved in benzol, and this gave perfect insulation without any shellac border. At temperatures above 70° C. the capacity began to increase rapidly owing to softening. The highest temperature reached in the preliminary heating was a little under 80° C.

The insulation was constantly tested as in the case of mica, and was found perfect up to about 70° C.

The results are given in the following table and diagram (fig. 4).

As the insulation was perfect, the greater capacity shown by the longer time of charging must be due to absorption—an effect which the structure of ebonite would lead us to expect. The column of values for instantaneous charge have been calculated on the assumption that the absorption goes on at the same rate from the beginning of the charge, as during the interval observed.

Temperature.	Variable resist-ance.	Change of capacity.	Rate of increase per degree.
Fork making 99 complete vibrations per second.			
13° C.	3520		
44	3460	·017	·00055
57	3415	·030	·00068
63	3355	·044	·00088
Fork making 64 complete vibrations per second.			
13	5250		
44	5143	·020	·00065
57	5060	·036	·00081
63	4960	·055	·00110
75	4800	·085	·00170

VALUES of rate of change corrected for instantaneous charge.

44° C.	·00037
57	·00043
63	·00048

Fig. 4.
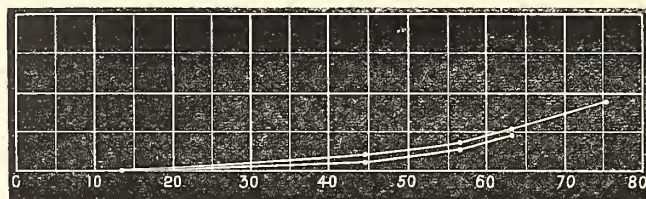
Electrostatic Observations.—In the observations on ebonite by the electrostatic method the condenser consisted of two sheets of ebonite between the iron discs, and with a copper disc between them. The condenser rested upon a glass tripod inside the air bath, and the leading wires entered the air bath through glass tubes. The capacity of this was adjusted so as to be within the range of a fine sliding condenser in the Laboratory against which it was balanced; and the readings were taken on an Elliott electrometer. It required great care and patience to make so slow an electrometer suffice, for the reasons already stated. The insulation of all parts of the apparatus also required much attention.

The cold temperature, 8°, was obtained by running iced water round the bath containing the condenser. Subsequent observations showed that this had affected the insulation and so increased the apparent diminution of capacity.

The greater rate of change shown by the electrostatic results is probably due to the time of charging by the key worked by hand being greater than with the tuning fork.

Any leak would produce an apparent decrease of capacity by this method; so that, although the electrostatic results are of less weight than the others, the agreement of the results of two methods in which the main source of error tends in opposite directions is a confirmation of their accuracy.

The electrostatic results are given in the following table :—

Temperature.	Rate of change of capacity per degree.
17° C.	
40	·0008
48	·0010
50	·0016
57	·0015
(8	·0021)

Glass.

The glass condenser was made with seventeen sheets of thin microscope-slide cover-glass, about three inches diameter. They consisted of a soda glass of high conductivity; so that, although the discs were carefully cleaned and bordered with shellac as in the case of mica, the insulation of the condenser, as tested by the electrometer, was never perfect. Since it is only the charging of the condenser that affects the galvanometer, the discharge passing sensibly all through D R (fig. 1), the defective insulation introduces simply a steady current during each time of charging. Each time the condenser is charged, the quantity of electricity passing D and B consists of (1) the charge of the condenser, and (2) the current through the condenser, which lasts during the time the vibrator P is in contact with S. For our present purpose, the current may be considered as immediately established at full strength when the circuit is closed. So that, neglecting absorption, the apparent capacity exceeds the true capacity by the conductivity of the condenser multiplied by the duration of the contact between P and S. And from observations on the apparent capacity with two forks of known speed, the true capacity can easily be found.

The highest temperature in the preliminary heating was less than 110° C.

The results are given in the following table and diagram (fig. 5):—

Temperature.	Variable resistance.	Change of capacity.	Rate of change per degree.
Fork making 99 complete vibrations per second.			
17° C.	4795		
38	4630	·034	·0016
49	4540	·053	·0017
57	4425	·077	·0019
62	4320	·099	·0022
Fork making 64 complete vibrations per second.			
17	6070		
38	5830	·039	·0018
49	5670	·066	·0021
57	5470	·098	·0024
62	5360	·117	·0026

C

RATE of change corrected for instantaneous charge.

17° to 38°	·0013
49	·0010
57	·0010
62	·0015

Fig. 5.
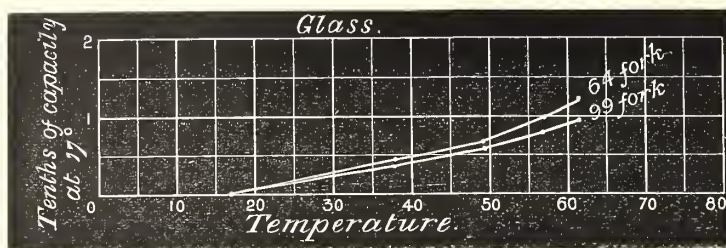
The next three tables and diagram give the results of experiments upon another condenser of microscope-slide cover-glass. This specimen of the glass had less conductivity than the previous, but the precautions for general insulation were not equal to those in the other experiments described. The condenser rested on a glass tripod inside the air bath, and the leading wires passed into the bath through glass tubes. The condenser consisted of twenty-two discs of glass cleaned and bordered with shellac in the usual way, with discs of lead foil between them.

The curves are almost exactly parallel up to between 40° and 50°, and diverge at higher temperatures in consequence of conduction; so that we may take the temperature change of specific inductive capacity for this specimen of glass to be about ·2 per cent. up to 50°.

An attempt was made to observe this temperature effect for shellac. The condenser was made by dipping the lead discs in shellac varnish, and carefully and thoroughly evaporating the alcohol. But after the condenser was made it was found that at a temperature below 50° the shellac softened, so that the plates were pressed together by the weights on the top of the condenser. To diminish the weight would have been useless; because, in any case, it would have been impossible to say how far a change of capacity was due to softening, and without a weight at all the results are quite unreliable.

Temperature.	Variable resistance.	Change of capacity in terms of capacity at lowest temperature.	Rate of change per degree.
		With 99 fork.	
18° C.	4010		
48	3750	·0645	·00215
59	3630	·0941	·00230
81	3070	·2311	·00369
101	2890	·2541	·00251
		With " 64 fork " (between 64 and 65).	
14	6110		
44	5700	·0671	·00224
50	5560	·0900	·00250
60	5270	·1351	·00293
71·5	4850	·2062	·00362
94	3840	·3551	·00444
117	3230	·4712	·00506
		With 49 fork.	
13	8100		
43	7480	·0765	·00255
64·5	7020	·1538	·00300
75	5850	·3201	·00516
115	4250	·5484	·00537

Fig. 6.
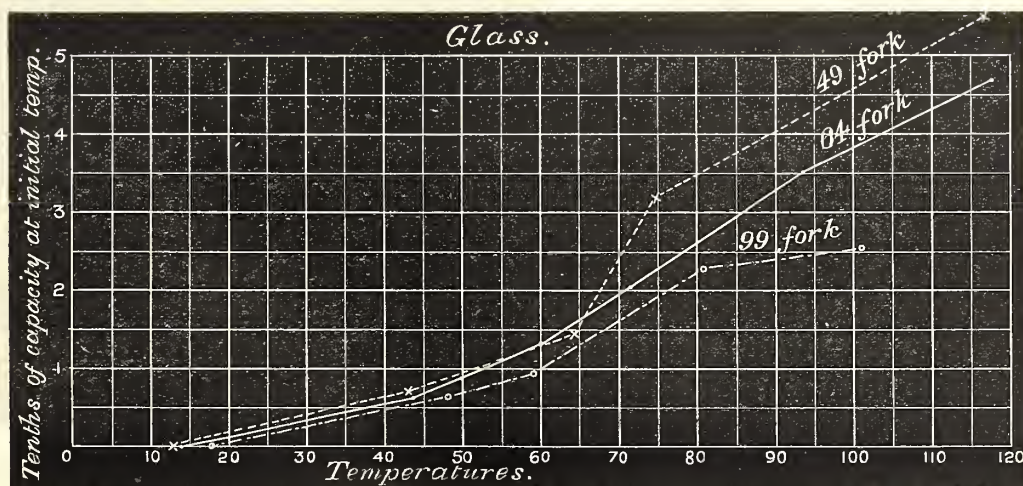
II. Liquids.

The apparatus for liquids consisted essentially of a quadrant electrometer immersed in the liquid.

The heating was done by a water bath; and on account of the highly inflammable character of some of the liquids experimented upon, the water was heated in a separate vessel at a distance, from the top and bottom of which pipes went to a and d in fig. 7. The hot water was pumped through in the direction $d\,c\,b\,a$.

Fig. 7.
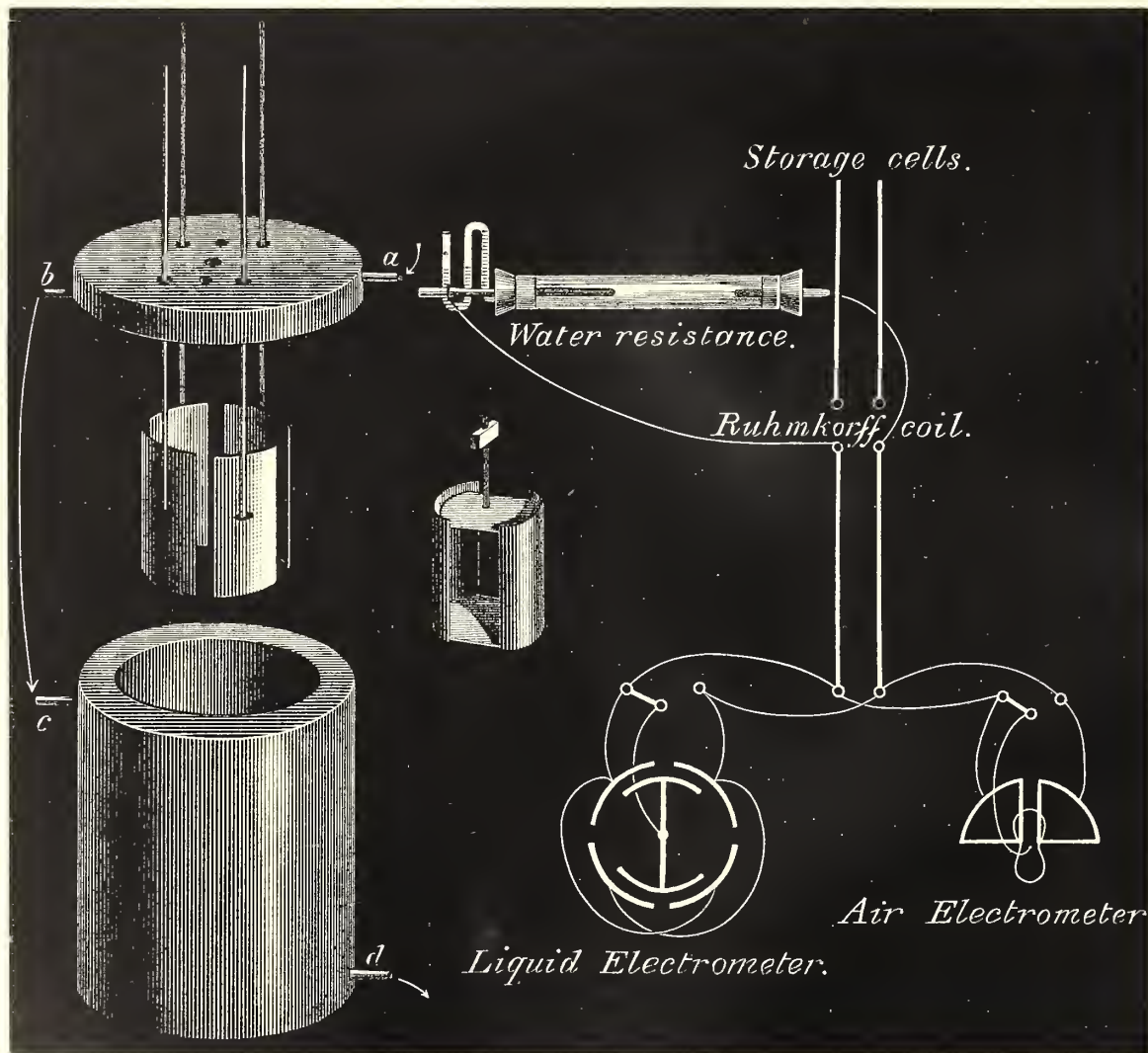
The provision for the insulation was on a similar principle to that adopted in the case of solids. Each quadrant was supported by a long stiff rod, the upper part of which was fixed in a varnished glass tube. These four glass tubes were separately

clamped to a support about 18 inches above the surface of the liquid; so that the quadrants touched nothing except the liquid and this perfectly insulating support. The needle was suspended by a fine wire from the same height so as to oscillate under torsion.

The hollow top of the water bath had seven holes through it, as shown in the figure. Through the outside four passed, without touching, the rods supporting the quadrants, through the centre one hung the needle, and through the two next the centre thermometers could be inserted. This top was permanently fixed to the support of the needle and quadrants; and the lower part of the bath containing the liquid was moved up on a smooth sliding arrangement into contact with the top so as to immerse the needle and quadrants. Above the fixed top of the bath was a box with a window on one side, through which the movements of the needle and mirror were read by a scale and telescope. The deflections of the needle were observed when connected first to one pair of quadrants and then to the other pair.

As the needle and quadrants were parts of cylinders between three and four inches diameter and about a quarter of an inch apart, a large electromotive force was required to produce a deflection. The electromotive force had also to be rapidly reversed in direction to avoid as far as possible polarisation, convection, &c., in the liquid. The electromotive force was obtained from a Ruhmkorff coil without a condenser, and with a high resistance between the terminals to prevent sparking. This high resistance consisted of a wide glass tube, about 6 inches long, filled with distilled water, and having a thick copper wire sliding through a cork at each end. By altering the distance of the ends of the copper wires in the water the resistance could be adjusted and the deflection controlled as desired. The coil was worked by a current from the storage cells in the Laboratory.

As the electromotive force given by this arrangement was variable, and also the loss of electromotive force by conduction was different for each liquid and for each temperature, it was necessary to be independent of such changes. Accordingly a second electrometer was placed between the terminals just outside the liquid one, which being always in the same state gave the comparative values of the electromotive force. This second electrometer was an Elliott's lecture-room pattern with two quadrants removed and the needle connected to one of the remaining quadrants. The advantage of this form for the present purpose was that by moving the needle up from the quadrants the deflection could be diminished to any desired extent. Then the quotient of the deflection of the liquid electrometer by the deflection of the second electrometer was proportional to the specific inductive capacity of the liquid. For every deflection of the liquid electrometer a pair of deflections of the second electrometer were taken with the needle connected first to one pair of quadrants and then to the other. The general arrangement of the apparatus is shown in the figure.

The liquids experimented upon were turpentine, carbon bisulphide, glycerine, benzene, benzylene, olive oil, and paraffin oil. Methylated spirit was also tried; but

its conductivity was so great that no deflection could be obtained. All except paraffin oil showed a decrease of specific inductive capacity with rise of temperature. The paraffin oil was that used in the lamps in the Laboratory, and its exceptional behaviour may have been due to some secondary action arising from impurity.

The results are given in the following tables and diagram. And to show the way in which the observations were made, the readings are given for turpentine at two temperatures.

	Temperature.	Ratio of specific inductive capacity to that at the lowest temperature.	Rate of decrease.
Turpentine:			
	20° C.	1·0000	
	36	·9800	·0012
	49	·9700	·0011
	62	·9600	·0009
Carbon bisulphide :			
	15	1·0000	
	35	·9130	·0040
	43	·8940	·0040
Glycerine :			
	18	1·0000	
	41	·8500	·0060
	61	·7760	·0053
Benzylene :			
	19	1·0000	
	41	·9870	·0006
	52	·9640	·0011
	63	·9350	·0015
Benzene :			
	15	1·0000	
	39	·9665	·0014
	58·5	·9507	·0012
Olive oil :			
	17	1·0000	
	38	·9530	·0021
	51	·9140	·0025
	59	·9130	·0021
	68	·8670	·0026

	Temperature.	Ratio of specific inductive capacity to that at the lowest temperature.	Rate of increase.
Paraffin oil :	18° C.	1·000	
	36·5	1·040	·0022
	49·5	1·080	·0025
	54	1·081	·0022

READINGS for Turpentine.

Temperature.	Liquid Electrometer.	Air electrometer.	Ratio of electrometer readings.
20°			
Zero	16·5		
Needle connected to one pair .	6·2	85	·098
„ „ .	6·2	— 20	
„ other „ .	26·6	— 21	·102
„ „ .	26·0	75	
Zero	16·8		
			Mean ·100
36°			
Zero	17·3		
Needle connected to one pair .	26·3	83	·098
„ „ .	26·3	— 15	
„ other „ .	7·4	— 17	·098
„ „ .	7·4	79	
			Mean ·098

Fig. 8.
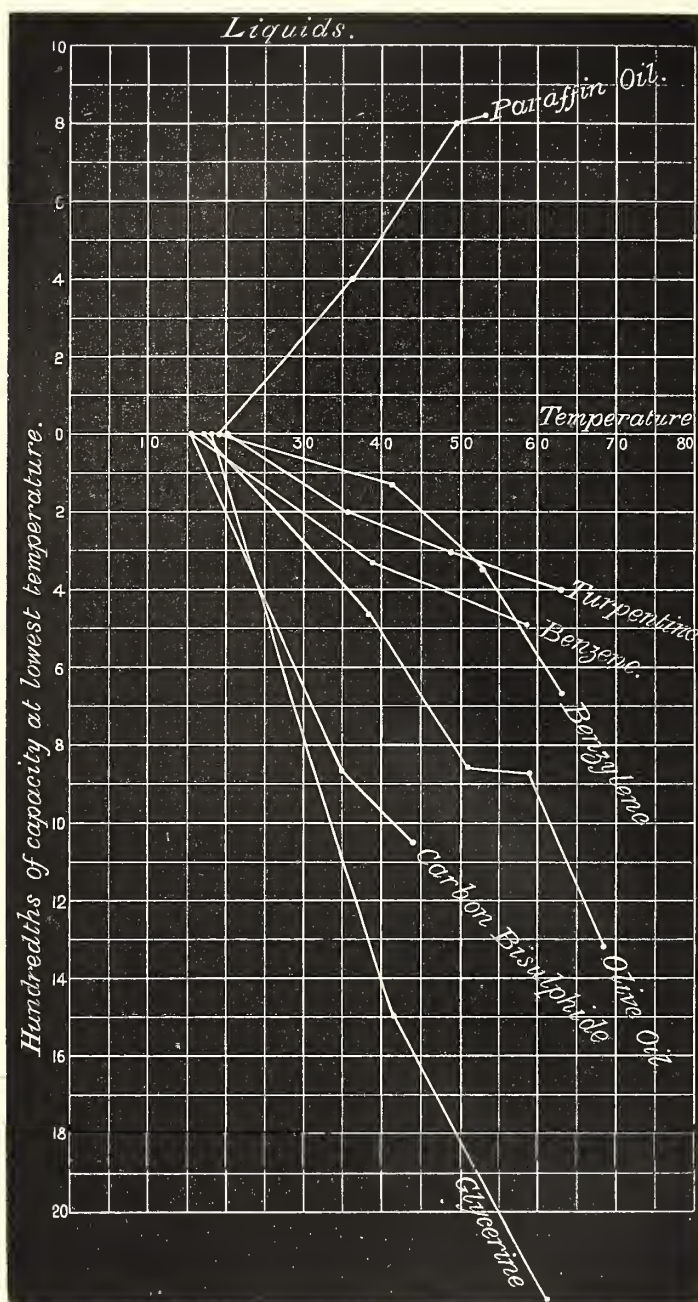
If the relation indicated by CLERK MAXWELL's electromagnetic theory of light held good, and the specific inductive capacity were equal to the square of the refractive index, then the rate of change of specific inductive capacity with temperature ought to be twice that of the refractive index. Amongst the liquids, for which DALE and GLADSTONE have observed the refractive indices at different temperatures, are four of those dealt with here. The mean rates of change, with temperature of the refractive index for the A line in the spectrum, deduced from their observations, are

Turpentine	·00035	for temperature range from 10° to 17°				
Benzene	·00040	,,	,,	,,	10	,, 39
Benzylene	·00037	,,	,,	,,	25	,, 39
Glycerine	·00018	,,	,,	,,	20	,, 48

Thus, it appears that although the two rates of change for Glycerine present no similarity whatever, those of the rest of the four are in a ratio not very far from 1 to 2, the approach being nearest in the case of Benzylene.

APPENDIX.

(Received October 18, 1889).

A suggestion having been made that, as the opposite effects of rise of temperature upon solids and liquids were observed by different methods, it would be well that both should be tested by the same method, a qualitative experiment was made on a solid by the method used for liquids, to see whether the result would agree with that already obtained.

A cylinder of glass was placed between the quadrants and needle of the liquid electrometer, leaving the needle free to oscillate; and observations were taken at different temperatures, exactly as already described for the case of a liquid dielectric. The result was always an increase of specific inductive capacity with rise of temperature. All the precautions for insulation, &c., were observed as in the experiments already described. But the heating was not maintained long enough to secure that the glass had acquired the full temperature of the air in the bath; so that the results obtained are only qualitative, the change being less than that corresponding to the temperature indicated by the air inside the bath. In one case, where the heating had lasted several hours, the rate of change rose as high as ·0024 per degree for a range of 50° C., a result sufficiently close to that obtained for glass by the other method.

II. *On the Interchange of the Variables in Certain Linear Differential Operators.*

By E. B. ELLIOTT, M.A., *Fellow of Queen's College, Oxford.*

Communicated by Professor SYLVESTER, F.R.S.

Received June 5,—Read June 20, 1889.

CONTENTS.

1. The operators to be considered, include or involve all those which have presented themselves as annihilators and generators in recent theories of functional differential invariants, reciprocants, cyclicants, &c. The general form of the binary operators, operators whose arguments are the derivatives of one dependent with regard to one independent variable, which I propose first to consider, is adopted in accordance with that used in two remarkable papers by Major MACMAHON.* They are his operators in four elements. The analogous ternary operators to which I subsequently devote attention, are distinct from his operators of six elements. Their arguments are the partial derivatives of one of three variables, supposed connected by a single relation, with regard to the two others.

* "The Theory of a Multilinear Partial Differential Operator with applications to the Theories of Invariants and Reciprocants," 'London Math. Soc. Proc.,' vol. 18, 1887, pp. 61–88. "The Algebra of Multilinear Partial Differential Operators," 'London Math. Soc. Proc.,' vol. 19, pp. 112–128.

14.3.90

The only previous contribution, of which I am aware, to the subject of the reversion of MacMahon operators, is a paper by Professor L. J. Rogers,* in which he obtains the operator reciprocal to $\{\mu, \nu ; 1, 1\}$, and alludes to the self reciprocal property of V which is expressed with more precision in (38) below.

I. *Binary Operators.*

2. Let x and y be two variables connected by any relation. Let y_r denote $\dfrac{1}{r!}\dfrac{d^r y}{dx^r}$, and x_r denote $\dfrac{1}{r!}\dfrac{d^r x}{dy^r}$.

Let ξ and η be corresponding increments of x and y, so that by Taylor's theorem

$$\eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \cdots \qquad \cdots \cdots \cdots \quad (1)$$

and

$$\xi = x_1\eta + x_2\eta^2 + x_3\eta^3 + \cdots, \qquad \cdots \cdots \cdots \quad (2)$$

the one expansion being a reversion of the other.

Let $Y_s^{(m)}$ denote the coefficient of ξ^s in the expansion of η^m, *i.e.*, of $(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$ in ascending integral powers of ξ; and $X_s^{(m)}$ the coefficient of η^s in the expansion of ξ^m, *i.e.*, of $(x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots)^m$ in ascending integral powers of η. It is supposed that m is not fractional. It need not, however, be positive. Nor is it necessary to exclude the value zero, which, though somewhat special, will be seen to be of importance later.

Let n be a positive or negative integer or zero, and let μ and ν be any numerical quantities.

Denote

$$\frac{1}{m}\Sigma\left\{(\mu + \nu s)\, X_s^{(m)}\frac{d}{dx_{n+s}}\right\} \text{ by } \{\mu, \nu ; m, n\}_x, \quad \cdots \cdots \cdots \quad (3)$$

and

$$\frac{1}{m}\Sigma\left\{(\mu + \nu s)\, Y_s^{(m)}\frac{d}{dy_{n+s}}\right\} \text{ by } \{\mu, \nu ; m, n\}_y, \quad \cdots \cdots \cdots \quad (4)$$

the summations being, with regard to s, which assumes in turn all integral values not less than the greater of the two m and $-n + 1$, so that, if $m + n > 1$, only symbols of differentiation with regard to all derivatives from y_{m+n} onwards may occur, while, if $m + n \lessgtr 1$, symbols of differentiation with regard to all derivatives may be present.

It is the operators $\{\mu, \nu ; m, n\}_x$ and $\{\mu, \nu ; m, n\}_y$ of which I propose to speak as MacMahon operators in x and y, respectively, dependent. It will be seen upon reference to the first paper referred to above that they are the results of substitution in Major MacMahon's operator $(\mu, \nu ; m, n)$ for

* "Note on Conjugate Annihilators of Homogeneous and Isobaric Differential Equations," 'Messenger of Mathematics,' vol. 18, pp. 153–158.

$$a_0, \; a_1, \; a_2, \; a_3, \; \ldots$$

of

$$0, \; x_1, \; x_2, \; x_3, \; \ldots \text{ in the one case,}$$

and of

$$0, \; y_1, \; y_2, \; y_3, \; \ldots \text{ in the other.}$$

MacMahon himself generally takes them as meaning

$$y_2, \; y_3, \; y_4, \; y_5 \ldots \ldots,$$

a fact which must not be forgotten in connecting his results with those to be here obtained.

The essential difference between the cases of $m + n \not< 1$ and $m + n < 1$ should be noticed at the outset. In the former case, the complete set of coefficients $X_s^{(m)}$ appears in the operator $\{\mu, \; \nu; \; m, \; n\}_x$. In the latter, one or more of those coefficients (a number of them equal to the excess of $-n + 1$ over m) is wanting at the beginning.

3. The aim in view is to express any MacMahon operator $\{\mu, \; \nu; \; m, \; n\}_x$ in x dependent as an operator or sum of operators of like form $\{\mu', \; \nu'; \; m', \; n'\}_y$ in y dependent. We need the linear expressions in $d/dy_1, \; d/dy_2, \; d/dy_3, \; \ldots$ which, when operating on any function of $y_1, \; y_2, \; y_3, \; \ldots$ are equivalent to $d/dx_1, \; d/dx_2, \; d/dx_3, \; \ldots$ operating on the equal function of $x_1, \; x_2, \; x_3, \; \ldots$ The expressions in question I have obtained in the second[*] of a series of papers on Cyclicants, &c. The best form for present purposes is hardly there given to the conclusions. It will therefore result in a gain of clearness and no loss of brevity if in the present article the proof is given rather than the result quoted. The same remark will apply to Article 17 below.

We may look upon $x_1, \; x_2, \; x_3, \; \ldots$ as a number of independent quantities, upon $y_1, \; y_2, \; y_3, \; \ldots$ as determinate functions of these quantities, and upon ξ and η as two other quantities connected with one another and with $x_1, \; x_2, \; x_3, \; \ldots$ by (1) or its equivalent (2).

Give x_r alone of all the quantites $x_1, \; x_2, \; x_3, \; \ldots$ an infinitesimal variation. Keep η constant. In virtue of (2) or its equivalent (1) ξ will vary in consequence of the variation of x_r. Also, as some or all of $y_1, \; y_2, \; y_3, \; \ldots$ are functions of x_r, some or all of those quantities will vary. Thus, from (2) we shall obtain

$$\delta \xi = \eta^r \, \delta x_r,$$

and from (1)

$$0 = \{y_1 + 2y_2 \xi + 3y_3 \xi^2 + \ldots\} \delta \xi + \left\{ \frac{dy_1}{dx_r} \xi + \frac{dy_2}{dx_r} \xi^2 + \frac{dy_3}{dx_r} \xi^3 + \ldots \right\} \delta x_r = 0.$$

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[*] "On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants," 'London Math. Soc. Proc.,' vol. 18, 1887, pp. 142–164.

Accordingly it follows that

$$\frac{dy_1}{dx_r}\xi + \frac{dy_2}{dx_r}\xi^2 + \frac{dy_3}{dx_r}\xi^3 + \ldots = -\eta^r\{y_1 + 2y_2\xi + 3y_3\xi^2 + \ldots\}; \quad . \quad (5)$$

and consequently, this being true for all values of ξ, that if by aid of (1) this right hand member be expanded in ascending powers of ξ, the coefficients of ξ, ξ^2, ξ^3, ... are exactly the expressions for dy_1/dx_r, dy_2/dx_r, dy_3/dx_r, ...

Now

$$\frac{d}{dx_r} = \frac{dy_1}{dx_r}\cdot\frac{d}{dy_1} + \frac{dy_2}{dx_r}\cdot\frac{d}{dy_2} + \frac{dy_3}{dx_r}\cdot\frac{d}{dy_3} + \ldots,$$

and is therefore the result of replacing each power ξ^s of ξ on the left, and therefore on the right, of (5) by the corresponding symbol d/dy_s. It follows that the expression on the right of (5) may be taken as a symbolical representation of the equivalent operator to d/dx_r, i.e., that

$$\frac{d}{dx_r} = -\left(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots\right)^r (y_1 + 2y_2\xi + 3y_3\xi^2 + \ldots)$$

$$= -\eta^r\frac{d\eta}{d\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the meaning of the symbolisation on the right is that η is to be replaced by its equivalent in terms of ξ from (1), that the differentiation with regard to ξ is partial, that the product on the right is to be expanded as a sum of multiples of powers of ξ, and that then each power ξ^s is to be replaced by the corresponding symbol of differentiation d/dy_s.

4. The proof of (6) is the same for all positive integral values (including unity) of r. Thus the means of transforming any differential operator whatever is obtained.

The rule according to which any linear operator is transformed may be very simply expressed.

Exactly companion to the symbolical notation ξ^s for d/dy_s in an operator linear in d/dy_1, d/dy_2, d/dy_3, ... is the notation η^s for d/dx_s in an operator linear in d/dx_1, d/dx_2, d/dx_3, ... Now, writing a linear operator

$$A\frac{d}{dx_\alpha} + B\frac{d}{dx_\beta} + C\frac{d}{dx_\gamma} + \ldots$$

in the symbolical form

$$A\eta^\alpha + B\eta^\beta + C\eta^\gamma + \ldots,$$

we learn by (6) that its equivalent in d/dy_1, d/dy_2, d/y_3, ... is obtained by multiplication by $-d\eta/d\xi$, expansion in terms of ξ by (1), and substitution for each power ξ^s in the expanded result, of the corresponding d/dy_s.

5. Now the symbolical form of any MacMahon operator for which $m + n \not< 1$ is very simple.   By (4) that of,

$$\{\mu, \nu \ ; \ m, n\}_y$$

is

$$\frac{1}{m} \Sigma_{s \not< m} (\mu + \nu s) \, \mathrm{Y}_s^{(m)} \, \xi^{u+s}$$

i.e.

$$\frac{\mu}{m} \xi^u (y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m + \frac{\nu}{m} \xi^{u+1} \frac{d}{d\xi} (y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$$

i.e.

$$\frac{\mu}{m} \xi^u \eta^m + \nu \xi^{u+1} \eta^{m-1} \frac{d\eta}{d\xi}. \qquad \cdots \qquad (7)$$

Thus in particular

$$\{1, 0 \ ; \ m, n,\}_y = \frac{1}{m} \xi^u \eta^m, \qquad \cdots \qquad (8)$$

and

$$\{0, 1 \ ; \ m, n\}_y = \xi^{u+1} \eta^{m-1} \frac{d\eta}{d\xi}, \qquad \cdots \qquad (9)$$

the right hand members being supposed to be expanded in terms of ξ by aid of (1), and then to have each power ξ^s of ξ which occurs replaced by the corresponding d/dy_s.

We may of course write (7)

$$\{\mu, \nu \ ; \ m, n\}_y = \mu\{1, 0 \ ; \ m, n\}_y + \nu\{0, 1 \ ; \ m, n\}_y, \qquad \cdots \qquad (10)$$

so that in (8) and (9) we have involved all MacMahon operators in y for which $m + n$ is not less than unity.   A reservation must for the moment be made of the case $m = 0$.

Exactly corresponding to (8) and (9) we have the symbolical forms

$$\{1, 0 \ ; \ m, n\}_x = \frac{1}{m} \eta^u \xi^m, \qquad \cdots \qquad (11)$$

$$\{0, 1 \ ; \ m, n\}_x = \eta^{u+1} \xi^{m-1} \frac{d\xi}{d\eta}, \qquad \cdots \qquad (12)$$

where the expansions on the right are to be in ascending powers of η by (2), and where in an expansion each η^s is to be replaced by the corresponding d/dx_s.

6. The transformation of $\{\mu, \nu \ ; \ m, n\}_x$ for the cases at present under consideration of $m + n$ not less than unity is now immediate.   By Art. 4 the transformed form of the expansion in terms of η of

$$\eta^u \xi^m \ ;$$

considered as the symbolical form of an operator in x dependent, is the expansion in terms of ξ of

$$- \xi^m \eta^n \frac{d\eta}{d\xi},$$

considered as an operator in y dependent.

In other words, by (11) and (9),

$$m\{1, 0 ; m, n\}_x = - \{0, 1 ; n + 1, m - 1\}_y \quad . \quad . \quad . \quad . \quad (13)$$

Again, the transformed form of the symbolical expansion in powers of η of

$$\eta^{n+1} \xi^{m-1} \frac{d\xi}{d\eta}$$

is, by Art. 4, the symbolical expansion in powers of ξ of

$$- \xi^{m-1} \eta^{n+1} \frac{d\xi}{d\eta} \cdot \frac{d\eta}{d\xi};$$

i.e., of

$$- \xi^{m-1} \eta^{n+1},$$

since in $d\xi/d\eta$ and $d\eta/d\xi$ the derivatives $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are not regarded as variables. In other words, by (12) and (8),

$$\{0, 1 ; m, n\}_x = - (n + 1)\{1, 0 ; n + 1, m - 1\}_y \quad . \quad . \quad . \quad (14)$$

It is to be remarked that (13) and (14) are entirely in accord. Either of them is produced from the other by the interchange of x and y and of m and $n + 1$.

From (13) and (14) by aid of (10) we produce the more general equality of operators

$$\{\mu, \nu ; m, n\}_x = - \left\{\nu(n + 1), \frac{\mu}{m} ; n + 1, m - 1\right\}_y \quad . \quad . \quad . \quad (15)$$

which may be given the more symmetrical form

$$\{m\mu, \mu' ; m, m' - 1\}_x = - \{m'\mu', \mu ; m', m - 1\}_y, \quad . \quad . \quad . \quad (16)$$

in which $m + m'$ has to be positive.

In (16) are included two interesting classes of particular cases, viz.:—

$$\{- m, 1 ; m, m - 1\}_x = \{- m, 1 ; m, m - 1\}_y, \quad . \quad . \quad . \quad (17)$$

and

$$\{m, 1 ; m, m - 1\}_x = - \{m, 1 ; m, m - 1\}_y \quad . \quad . \quad . \quad . \quad (18)$$

Corresponding to each positive degree m there are then two self-reciprocal operators.* The first is of positive character, being entirely unaltered in form by

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* Self-reciprocal operators, of course, generate from absolute reciprocants other absolute reciprocants.

interchange of x and y; and the second of negative character, persisting in form but for a change of sign. (A complex self reciprocal operator can of course be found by taking the sum or difference of any two correlative operators; e.g.,

$$m\{1, 0; m, n\}_x \mp \{0, 1; n+1, m-1\}_x).$$

At greater length (17) and (18) are

$$X_{m+1}^{(m)} \frac{d}{dx_{2m}} + 2X_{m+2}^{(m)} \frac{d}{dx_{2m+1}} + 3X_{m+3}^{(m)} \frac{d}{dx_{2m+2}} + \dots$$

$$= Y_{m+1}^{(m)} \frac{d}{dy_{2m}} + 2Y_{m+2}^{(m)} \frac{d}{dy_{m+1}} + 3Y_{m+3}^{(m)} \frac{d}{dy_{m+2}} + \dots, \quad \dots \quad (17\text{A})$$

and

$$2m \, X_m^{(m)} \frac{d}{dx_{2m-1}} + (2m+1) X_{m+1}^{(m)} \frac{d}{dx_{2m}} + (2m+2) X_{m+2}^{(m)} \frac{d}{dx_{2m+1}} + \dots$$

$$= -\left\{ 2m \, Y_m^{(m)} \frac{d}{dy_{2m-1}} + (2m+1) Y_{m+1}^{(m)} \frac{d}{dy_{2m}} + (2m+2) Y_{m+2}^{(m)} \frac{d}{dy_{2m+1}} + \dots \right\} \quad (18\text{A})$$

In particular for $m=1$ we have

$$x_2 \frac{d}{dx_2} + 2x_3 \frac{d}{dx_3} + 3x_4 \frac{d}{dx_4} + \dots = y_2 \frac{d}{dy_2} + 2y_3 \frac{d}{dy_3} + 3y_4 \frac{d}{dy_4} + \dots \quad (19)$$

and

$$2x_1 \frac{d}{dx_1} + 3x_2 \frac{d}{dx_2} + 4x_3 \frac{d}{dx_3} + \dots = -\left\{ 2y_1 \frac{d}{dy_1} + 3y_2 \frac{d}{dy_2} + 4y_3 \frac{d}{dy_3} + \dots \right\} \quad (20)$$

Again $m=2$ gives us that

$$2x_1 x_2 \frac{d}{dx_4} + 2\left(2x_1 x_3 + x_2^2\right) \frac{d}{dx_5} + 3\left(2x_1 x_4 + 2x_2 x_3\right) \frac{d}{dx_6}$$

$$+ 4\left(2x_1 x_5 + 2x_2 x_4 + x_3^2\right) \frac{d}{dx_7} + \dots, \quad \dots \quad (21)$$

and

$$4x_1^2 \frac{d}{dx_3} + 5 \cdot 2x_1 x_2 \frac{d}{dx_4} + 6\left(2x_1 x_3 + x_2^2\right) \frac{d}{dx_5} + 7\left(2x_1 x_4 + 2x_2 x_3\right) \frac{d}{dx_6} + \dots, \quad (22)$$

are self reciprocal operators of positive and negative characters respectively.

7. As other examples of the important formulæ of transformation (13) and (14), let us write down cases corresponding to $m=1$, $n \not< 0$.

For $m=1$, $n=0$ we obtain

$$x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} + x_3 \frac{d}{dx_3} + \dots = -\left\{ y_1 \frac{d}{dy_1} + 2y_2 \frac{d}{dy_2} + 3y_3 \frac{d}{dy_3} + \dots \right\}, \quad (23)$$

$$x_1 \frac{d}{dx_1} + 2x_2 \frac{d}{dx_2} + 3x_3 \frac{d}{dx_3} + \dots = -\left\{ y_1 \frac{d}{dy_1} + y_2 \frac{d}{dy_2} + y_3 \frac{d}{dy_3} + \dots \right\}, \quad (24)$$

which together are equivalent to (19) and (20) together. From (23) it follows that a homogeneous function of $x_1\, x_2\, x_3,\, \ldots$ transforms into an isobaric function of $y_1\, y_2\, y_3,$ $\ldots$, and that, i and w meaning degree and weight respectively,

$$i_x = -\, w_y,$$

while from (24) follows the equivalent fact that an isobaric function transforms into a homogeneous one, and that

$$w_x = -\, i_y.$$

From (19) follows the especially interesting fact that, if a function of $x_1,\, x_2\, x_3,\, \ldots$ is isobaric in $x_2,\, x_3,\, x_4,\, \ldots$ upon considering the weight of x_r to be $r-1$, so also is the transformed function of $y_1,\, y_2,\, y_3,\, \ldots$ isobaric in the same sense and of the same weight in $y_2,\, y_3,\, y_4,\, \ldots$

Again the substitution $m=1,\ n=1$ in (13) and (14) produces for us

$$x_1 \frac{d}{dx_2} + x_2 \frac{d}{dx_3} + x_3 \frac{d}{dx_4} + \ldots = -\tfrac{1}{2}\left\{ 2\mathrm{Y}_2^{(2)}\frac{d}{dy_2} + 3\mathrm{Y}_3^{(2)}\frac{d}{dy_3} + 4\mathrm{Y}_4^{(2)}\frac{d}{dy_4} + \ldots \right\}, \quad (25)$$

and

$$x_1 \frac{d}{dx_2} + 2x_2 \frac{d}{dx_3} + 3x_3 \frac{d}{dx_4} + \ldots = -\left\{ \mathrm{Y}_2^{(2)}\frac{d}{dy_2} + \mathrm{Y}_3^{(2)}\frac{d}{dy_3} + \mathrm{Y}_4^{(2)}\frac{d}{dy_4} + \ldots \right\}, \quad (26)$$

where

$$\mathrm{Y}_2^{(2)} = y_1^{\,2},\ \ \mathrm{Y}_3^{(2)} = 2y_1 y_2,\ \ \mathrm{Y}_4^{(2)} = 2y_1 y_3 + y_2^{\,2},\ \ \mathrm{Y}_5^{(2)} = 2y_1 y_4 + 2y_2 y_3,\ \ \ldots$$

These two transformations have been obtained by Professor ROGERS (see note to Art. 1). The second tells us that what he calls primary invariants in $x_1,\, x_2,\, x_3,\, \ldots$ have for their transforms what he calls secondary invariants in $y_1,\, y_2,\, y_3,\, \ldots$

We might now consider the results of putting $m=1$ and $n=2, 3, \ldots$ in (13) and (14). By this means the transformation of lineo-linear operators of two, three, &c., steps is effected. For the general case $m=1,\ n=n$ the results are

$$x_1 \frac{d}{dx_{n+1}} + x_2 \frac{d}{dx_{n+2}} + x_3 \frac{d}{dx_{n+3}} + \ldots$$
$$= -\frac{1}{n+1}\left\{ (n+1)\,\mathrm{Y}_{(n+1)}^{(n+1)}\frac{d}{dy_{n+1}} + (n+2)\,\mathrm{Y}_{(n+2)}^{(n+1)}\frac{d}{dy_{n+2}} + \ldots \right\} . \quad (27)$$

$$x_1 \frac{d}{dx_{n+1}} + 2x_2 \frac{d}{dx_{n+2}} + 3x_3 \frac{d}{dx_{n+3}} + \ldots = -\left\{ \mathrm{Y}_{(n+1)}^{(n+1)}\frac{d}{dy_{n+1}} + \mathrm{Y}_{(n+2)}^{(n+1)}\frac{d}{dy_{n+2}} + \ldots \right\} \quad (28)$$

Perhaps the most interesting fact to be deduced from (25) and (26) is the transformation of $\{-1, 1\,;\, 1, 1\}$, the second annihilator Ω of projective reciprocants. By subtraction of (25) from (26), or directly from (15)

$$\{-1, 1\,;\, 1, 1\}_x = \{-2, 1\,;\, 2, 0\}_y,$$

i.e.,

$$x_2 \frac{d}{dx_3} + 2x_3 \frac{d}{dx_4} + 3x_4 \frac{d}{dx_5} + \ldots$$

$$= \tfrac{1}{2} \left\{ Y_3^{(2)} \frac{d}{dy_3} + 2Y_4^{(2)} \frac{d}{dy_4} + 3Y_5^{(2)} \frac{d}{dy_5} + \ldots \right\}$$

$$= \tfrac{1}{2} \left\{ 2y_1 y_2 \frac{d}{dy_3} + 2\left(2y_1 y_3 + y_2^2\right) \frac{d}{dy_4} + 3\left(2y_1 y_4 + 2y_2 y_3\right) \frac{d}{dy_5} + \ldots \right\}$$

$$= y_1 \left\{ y_2 \frac{d}{dy_3} + 2y_3 \frac{d}{dy_4} + 3y_4 \frac{d}{dy_5} + \ldots \right\} + 2\frac{y_2^2}{2} \frac{d}{dy_4} + 3y_2 y_3 \frac{d}{dy_5}$$

$$+ 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_6} + \ldots,$$

or

$$\Omega(x, y) = y_1 \, \Omega(y, x) + 2\frac{y_2^2}{2} \frac{d}{dy_4} + 3y_2 y_3 \frac{d}{dy_5} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_6} + \ldots \quad (29)$$

Since $y_1 x_1 = 1$ we infer from this conclusion and its correlative that

$$x_1^{-\frac{1}{2}} \left\{ 2\frac{x_2^2}{2} \frac{d}{dx_4} + 3x_2 x_3 \frac{d}{dx_5} + 4\left(x_2 x_4 + \frac{x_3^2}{2}\right) \frac{d}{dx_6} + \ldots \right\} = y_1^{\frac{1}{2}} \, \Omega(y, x) - x_1^{\frac{1}{2}} \, \Omega(x, y)$$

$$= - y_1^{-\frac{1}{2}} \left\{ 2\frac{y_2^2}{2} \frac{d}{dy_3} + 3y_2 y_3 \frac{d}{dy_4} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_5} + \ldots \right\} \quad . \quad (30)$$

is a self reciprocal operator of negative character. The operator is one of considerable interest in connection with the theories of invariants and reciprocants. (See MacMahon, 'London Math. Soc. Proc.,' vol. 18, p. 75).

It also follows that the sum of $2x_1^{\frac{1}{2}} \, \Omega(x, y)$ and the operator on the left of (30) is a self reciprocal operator of positive character.

8. To complete our theory of the reversion of MacMahon operators, for which $m + n$ is not less than unity, we must consider the somewhat special and exactly correlative cases $m = 0$, $n \not< 1$, and $n = -1$, $m \not< 2$.

The operator $0\{1, 0 \,;\, 0, n\}_x$ is d/dx_n in accordance with the general definition of $m\{\mu, \nu \,;\, m, n\}_x$ in (3). Thus the identity (6) may be written, by aid of (9),

$$0\{1, 0 \,;\, 0, r\}_x = -\{0, 1 \,;\, r+1, -1\}_y, \quad \ldots \ldots \quad (31)$$

for any positive integral value, not excluding unity, of r: which is strictly in agreement with the general formula of transformation (13).

On the other hand the general definition (3) gives to $0\{0, 1 \,;\, 0, n\}_x$ no other meaning than zero. So far then the operator $\{0, 1 \,;\, 0, n\}_x$ is indeterminate in form. An interpretation of it is now sought which shall make the case not exceptional to the general formula of transformation (14).

To discover this interpretation let us reverse the order of investigation and seek the operator in x, which is equivalent to the operator in y obtained by putting $m = 0$ in the right hand member of (14).

By (8) the symbolical form of

$$- (n + 1) \{1, 0 ; n + 1, - 1\}_y$$

is

$$- \xi^{-1} \eta^{n+1}.$$

The equivalent operator in x dependent has then for its symbolical form, as in Article 4,

$$\eta^{n+1} \xi^{-1} \frac{d\xi}{d\eta},$$

i.e.,

$$\eta^n + \eta^{n+1} \frac{d}{d\eta} \log \frac{\xi}{\eta}.$$

Now

$$\frac{d}{d\eta} \log \frac{\xi}{\eta} = \frac{d}{d\eta} \log (x_1 + x_2 \eta + x_3 \eta^2 + \ldots)$$

$$= \frac{d}{d\eta} e^{\,\eta\,(x_2\, d/dx_1 + 2x_3\, d/dx_2 + 3x_4\, d/dx_3 + \ldots)} \cdot \log x_1.$$

$$= e^{\,\eta\,(x_2\, d/dx_1 + 2x_3\, d/dx_2 + 3x_4\, d/dx_3 + \ldots)} \cdot \frac{x_2}{x_1},$$

$$= \frac{x_2}{x_1} + \frac{2x_1 x_3 - x_2^2}{x_1^2} \eta + \frac{3x_1^2 x_4 - 3x_1 x_2 x_3 + x_2^3}{x_1^3} \eta^2 + \ldots$$

$$= \frac{x_2}{x_1} + \frac{G x_2}{x_1^2} \eta + \frac{G^2 x_2}{x_1^3} \cdot \frac{\eta^2}{1.2} + \frac{G^3 x_2}{x_1^4} \cdot \frac{\eta^3}{1.2.3} + \ldots, \quad \ldots \quad (32)$$

where $G = x_1 \left(x_2 \dfrac{d}{dx_1} + 2x_3 \dfrac{d}{dx_2} + 3x_4 \dfrac{d}{dx_3} + \ldots \right) - x_2 \left(x_1 \dfrac{d}{dx_1} + x_2 \dfrac{d}{dx_2} + x_3 \dfrac{d}{dx_3} + \ldots \right)$

$$= (2x_1 x_3 - x_2^2) \frac{d}{dx_2} + (3x_1 x_4 - x_2 x_3) \frac{d}{dx_3} + (4x_1 x_5 - x_2 x_4) \frac{d}{dx_4} + \ldots, \quad (32\text{A})$$

so that the numerators $G x_2$, $G^2 x_2$, $G^3 x_2$, $\ldots$, are a set of seminvariant protomorphs in x_1, x_2, $2! \, x_3$, $3! \, x_4$, $\ldots$

Consequently the transformation in x dependent of $- (n + 1) \{1, 0 ; n + 1, - 1\}_y$ is

$$\frac{d}{dx_n} + \frac{x_2}{x_1} \frac{d}{dx_{n+1}} + \frac{G x_2}{x_1^2} \frac{d}{dx_{n+2}} + \frac{G^2 x_2}{2! \, x_1^3} \frac{d}{dx_{n+3}} + \frac{G^3 x_2}{3! \, x_1^4} \frac{d}{dx_{n+4}} + \ldots ;$$

and it is accordingly this operator which has to be defined as

$$\{0, 1 ; 0, n\}_x^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

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* *Cf.* Hammond, ' London Math. Soc. Proc.,' vol. 18, p. 64, note.

that (14) may be regarded as holding for the value $m = 0$ as well as for non-vanishing values of m.

It affords an instructive verification to conduct the investigation of the same transformation in the order of Article 6.

9. For the case $n = 1$ the two formulæ of transformation,

$$0 \{1, 0 \, ; \, 0, n\}_x = - \{0, 1 \, ; \, n+1, -1\}_y,$$
$$\{0, 1 \, ; \, 0, n\}_x = - (n+1) \{1, 0 \, ; \, n+1, -1\}_y,$$

of the last article become respectively

$$\frac{d}{dx_1} = - \tfrac{1}{2} \left\{ 2Y_2^{(2)} \frac{d}{dy_1} + 3Y_3^{(2)} \frac{d}{dy_2} + 4Y_4^{(2)} \frac{d}{dy_3} + \cdots \right\}$$
$$= - \tfrac{1}{2} \left\{ 2y_1^2 \frac{d}{dy_1} + 3 \cdot 2y_1 y_2 \frac{d}{dy_2} + 4 \left(2y_1 y_3 + y_2^2\right) \frac{d}{dy_3} + \cdots \right\}, \quad (34)$$

and

$$\frac{d}{dx_1} + \frac{x_2}{x_1} \frac{d}{dx_2} + \frac{2x_1 x_3 - x_2^2}{x_1^2} \frac{d}{dx_3} + \frac{3x_1^2 x_4 - 3x_1 x_2 x_3 + x_2^3}{x_1^3} \frac{d}{dx_4} + \cdots$$
$$= - \left\{ y_1^2 \frac{d}{dy_1} + 2y_1 y_2 \frac{d}{dy_2} + \left(2y_1 y_3 + y_2^2\right) \frac{d}{dy_3} + \cdots \right\} \quad . \quad (35)$$

By combination of these we have the equivalence, free from d/dx_1 and d/dy_1,

$$\frac{x_2}{x_1} \frac{d}{dx_2} + \frac{2x_1 x_3 - x_2^2}{x_1^2} \frac{d}{dx_3} + \frac{3x_1^2 x_4 - 3x_1 x_2 x_3 + x_2^3}{x_1^3} \frac{d}{dx_4} + \cdots$$
$$= y_1 y_2 \frac{d}{dy_2} + 2 \left(y_1 y_3 + \frac{y_2^2}{2}\right) \frac{d}{dy_3} + 3 \left(y_1 y_4 + y_2 y_3\right) \frac{d}{dy_4} + \cdots$$
$$= y_1 \left(y_2 \frac{d}{dy_2} + 2y_3 \frac{d}{dy_3} + 3y_4 \frac{d}{dy_4} + \cdots\right) + 2 \frac{y_2^2}{2} \frac{d}{dy_3} + 3y_2 y_3 \frac{d}{dy_4}$$
$$+ 4 \left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_5} + \cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In like manner, for any positive integral value of n,

$$\{0, 1 \, ; \, 0, n\}_x - 0\{1, 0 \, ; \, 0, n\}_x = - (n+1) \{1, 0 \, ; \, n+1, -1\}_y$$
$$- \{0, 1 \, ; \, n+1, -1\}_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

is an equivalence of operators which do not involve any lower symbols of operation than d/dx_{n+1} and d/dy_{n+1} respectively.

10. From (34) is easily derived in exact form the known fact that the annihilator V

of pure reciprocants is, when affected with a simple multiplier, self reciprocal. We may write (34)

$$\frac{d}{dx_1} = y_1{}^2 \frac{d}{dy_1} - y_1 \left\{ 2y_1 \frac{d}{dy_1} + 3y_2 \frac{d}{dy_2} + 4y_3 \frac{d}{dy_3} + \ldots \right\}$$

$$- \left\{ 4\frac{y_2{}^2}{2}\frac{d}{dy_3} + 5y_2 y_3 \frac{d}{dy_4} + 6\left(y_2 y_4 + \frac{y_3{}^2}{2}\right)\frac{d}{dy_5} + \ldots \right\}$$

$$= y_1{}^2 \frac{d}{dy_1} - y_1 \{1,\,1\,;\,1,\,0\}_y - \mathrm{V}\,(y,\,x)\,.$$

$$\therefore\ x_1 \frac{d}{dx_1} - y_1 \frac{d}{dy_1} = -\,\{1,\,1\,;\,1,\,0\}_y - y_1{}^{-1}\mathrm{V}\,(y,\,x)\,.$$

So too, correlatively,

$$y_1 \frac{d}{dy_1} - x_1 \frac{d}{dx_1} = -\,\{1,\,1\,;\,1,\,0\}_x - x_1{}^{-1}\mathrm{V}\,(x,\,y)\,.$$

But

$$\{1,\,1\,;\,1,\,0\}_x = -\,\{1,\,1\,;\,1,\,0\}_y \quad \text{by (18) or (20)}$$

$$\therefore\ x_1{}^{-1}\mathrm{V}\,(x,\,y) = -\,y_1{}^{-1}\mathrm{V}\,(y,\,x)\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

so that, to use a familiar notation, $t^{-1}\mathrm{V}$ is a self reciprocal operator of negative character.

11. It remains to consider operators $\{\mu,\,\nu\,;\,m,\,n\}$ in cases when $m + n < 1$. In such cases the formulæ of Arts. 5 and 6 have to be replaced by others. The essential difference between them and the cases already considered lies in the fact that the lower limit of s in (3) and (4), and, therefore, in what replaces (7), is now $-n + 1$ instead of m, $i.e.$, is greater than m, so that the coefficients in $\{\mu,\,\nu\,;\,m,\,n\}_y$ are no longer multiples of the complete set of coefficients in the expansion of $(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$, but of those coefficients with the exception of one or more at the beginning.

In the present article attention is confined to the case of $m + n = 0$, $i.e.$, $n = -m$.

Proceeding to write down the symbolical form of $\{\mu,\,\nu\,;\,m,\,-m\}_y$ as in Art. (5) we see that the whole expansion from which we there started is present except the first term,

$$\frac{1}{m}(\mu + \nu m)\,\mathrm{Y}_m^{(m)}\,\xi^{n+m}.$$

Thus the symbolical form of

$$\{1,\,0\,;\,m,\,-m\}_y \quad \text{is} \quad \frac{1}{m}\,\xi^{-m}(\eta^m - y_1{}^m \xi^m), \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and that of

$$\{0,\,1\,;\,m,\,-m\}_y \quad \text{is} \quad \xi^{-m+1}\left(\eta^{m-1}\frac{d\eta}{d\xi} - y_1{}^m \xi^{m-1}\right), \quad . \quad . \quad . \quad . \quad (40)$$

the right-hand members standing for their expansions in powers of ξ.

or, replacing νm by ν,

$$m\left\{\mu, \frac{\nu}{m}; m, -m\right\}_x = -(1-m)\left\{\nu, \frac{\mu}{1-m}; 1-m, m-1\right\}_y$$
$$+ (\mu + \nu)y_1^{-m}\{0, 1; 1, -1\}_y \quad . \quad (45)$$

In particular,

$$m\left\{\mu, -\frac{\mu}{m}; m, -m\right\}_x = (1-m)\left\{\mu, -\frac{\mu}{1-m}; 1-m, m-1\right\}_y. \quad . \quad (46)$$

12. The value zero of m is somewhat special in these cases of $n = -m$, just as in the more general cases already discussed. So too, of course, is the conjugate value $m = 1$.

For (43) and (44) to hold for these special values of m we must have,

$$\left.\begin{array}{ll}
0\{1, 0; 0, 0\}_x & = - \quad \{0, 1; 1, -1\}_y + \{0, 1; 1, -1\}_y, \\
\{1, 0; 1, -1\}_x = - \quad \{0, 1; 0, 0\}_y \quad + y_1^{-1}\{0, 1; 1, -1\}_y, \\
\{0, 1; 0, 0\}_x & = - \quad \{1, 0; 1, -1\}_y + \{0, 1; 1, -1\}_y, \\
\{0, 1; 1, -1\}_x = - 0\{1, 0; 0, 0\}_y \quad + y_1^{-1}\{0, 1; 1, -1\}_y.
\end{array}\right\} \quad . \quad (47)$$

Of these four equalities the first is a mere identity of two zero operators. In fact to $0\{1, 0; 0, 0\}$ no other meaning than zero could be attached consistently with the general definition. Thus the form of $\{1, 0; 0, 0\}$ is left indeterminate.

The fourth of (47) becomes

$$\{0, 1; 1, -1\}_x = y_1^{-1}\{0, 1; 1, -1\}_y,$$

i.e.,

$$x_1^{-\frac{1}{2}}\left\{2x_2\frac{d}{dx_1} + 3x_3\frac{d}{dx_2} + 4x_4\frac{d}{dx_3} + \ldots\right\} = y_1^{-\frac{1}{2}}\left\{2y_2\frac{d}{dy_1} + 3y_3\frac{d}{dy_2}\right.$$
$$\left. + 4y_4\frac{d}{dy_3} + \ldots\right\}, \quad . \quad (48)$$

of which the left-hand member is merely $x_1^{-\frac{1}{2}}\frac{d}{dy}$, and the right $y_1^{-\frac{1}{2}}\frac{d}{dx}$, the symbols of differentiation being total.

The remaining equalities, the second and third of (47), now become the same but for an interchange of x and y. Consequently there will be complete consistency if we define the at present undetermined operator $\{0, 1; 0, 0\}$ as that which obeys the equation of transformation

$$\{0, 1; 0, 0\}_x = - \{1, 0; 1, -1\}_y + \{0, 1; 1, -1\}_y$$
$$= \{-1, 1; 1, -1\}_y$$
$$= y_2\frac{d}{dy_1} + 2y_3\frac{d}{dy_2} + 3y_4\frac{d}{dy_3} + \ldots \quad . \quad . \quad (49)$$

Now, proceeding exactly as in Art. 8, it is seen that the symbolical form of an x operator equal to this is $\eta\, d/d\eta \log (\xi/\eta)$, and that its expanded form is obtained by omitting the first term, and then putting $n = 0$ in the general value (33). Thus, the operator which has to be defined as $\{0, 1\, ;\, 0, 0\}_x$ is

$$\frac{x_2}{x_1}\frac{d}{dx_1} + \frac{Gx_2}{x_1^2}\frac{d}{dx_2} + \frac{G^2x_2}{2!\,x_1^3}\frac{d}{dx_3} + \cdots, \qquad \cdots \quad \cdots \quad (50)$$

where G is the generator defined in (32A).

13. Examples of important operators which occur among those transformed in the last two articles are before us in (48) and in the equality of (49) and (50). It is unnecessary to multiply particular instances as they can be deduced without number by giving m particular integral values in (43) . . . (46). It is to be noticed that, excluding the special cases of $m = 0$ and $m = 1$, one or other of the two equivalent operators will involve as coefficients those in a multinomial expansion of negative index. Thus, for instance,

$$2\{1, 0\, ;\, 2, -2\}_x = -\{0, 1\, ;\, -1, 1\}_y + y_1^{-2}\{0, 1\, ;\, 1, -1\}_y,$$

a particular case of (43), is at more length

$$2x_1x_2\frac{d}{dx_1} + (2x_1x_3 + x_2^2)\frac{d}{dx_2} + (2x_1x_4 + 2x_2x_3)\frac{d}{dx_3} + \cdots$$

$$= -\left\{\frac{y_2^2 - y_1y_3}{y_1^3}\frac{d}{dy_2} - 2\frac{(y_3^3 - 2y_1y_2y_3 + y_1^2y_4)}{y_1^4}\frac{d}{dy_3} + \cdots\right\}$$

$$+ \frac{1}{y_1^2}\left\{2y_2\frac{d}{dy_1} + 3y_3\frac{d}{dy_2} + 4y_4\frac{d}{dy_3} + \cdots\right\}. \qquad \cdots \quad \cdots \quad (51)$$

The transformation of the operator G of (32A) is an application of (49). Thus

$$G\,(x, y) = x_1\left(x_2\frac{d}{dx_1} + 2x_3\frac{d}{dx_2} + 3x_4\frac{d}{dx_3} + \cdots\right) - x_2\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \cdots\right)$$

$$= x_1\{-1, 1\, ;\, 1, -1\}_x - x_2\{1, 0\, ;\, 1, 0\}_x$$

$$= y_1^{-1}\{0, 1\, ;\, 0, 0\}_y - y_1^{-3}y_2\{0, 1\, ;\, 1, 0\}_y,$$

by (49) and (23),

$$= y_1^{-1}\left\{\frac{y_2}{y_1}\frac{d}{dy_1} + \frac{Gy_2}{y_1^2}\frac{d}{dy_2} + \frac{G^2y_2}{2!\,y_1^3}\frac{d}{dy_3} + \cdots\right\}$$

$$- y_1^{-3}y_2\left\{y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + 3y_3\frac{d}{dy_3} + \cdots\right\}$$

$$= \frac{2y_1y_3 - 3y_2^2}{y_1^3}\frac{d}{dy_2} + \frac{3y_1^2y_4 - 6y_1y_2y_3 + y_2^3}{y_1^4}\frac{d}{dy_3} + \cdots \qquad \cdots \quad \cdots \quad (52)$$

14. Operators for which $m + n$ is negative still remain to be considered. In particular, those of the type $\{0, 1; 0, -n'\}$ have still to be defined. About the right definition of them there can, however, after articles 8 and 12, be no doubt.

The general principle by means of which if $\{\mu, \nu; m, n\}_y$ is known, the form of $\{\mu, \nu; m, n - r\}_y$ is deduced is expressed by the rule—"Write $\{\mu, \nu, m, n\}_y$ symbolically, by putting ξ^p for each d/dy_p, divide through by ξ^r, reject all terms, if any now occur, which do not contain a positive power of ξ as factor, and then for each ξ^p write d/dy_p."

Thus to accord with (33) and (50) the right operator to be defined as

$$\{0, 1; 0, -n'\}_y,$$

where $-n'$ is a negative integer, is

$$\frac{G^{n'}y_2}{n'!\, y_1^{n'+1}} \cdot \frac{d}{dy_1} + \frac{G^{n'+1}y_2}{(n'+1)!\, y_1^{n'+2}} \cdot \frac{d}{dy_2} + \frac{G^{n'+2}y_2}{(n'+2)!\, y_1^{n'+3}} \cdot \frac{d}{dy_3} + \cdots \qquad (53)$$

We now proceed to the transformation of $\{\mu, \nu; m, -m-r\}_x$ where r is a positive integer.

The symbolical form of $m\{1, 0; m, -m-r\}_x$ is as in Arts. 6 and 11

$$\eta^{-m-r}\{\xi^m - X_m^{(m)}\eta^m - X_{m+1}^{(m)}\eta^{m+1} - \cdots - X_{m+r}^{(m)}\eta^{m+r}\}. \qquad \cdots \cdots (54)$$

The symbolical form of its transformation is, therefore,

$$-\xi^m \eta^{-m-r}\frac{d\eta}{d\xi} + X_m^{(m)}\eta^{-r}\frac{d\eta}{d\xi} + X_{m+1}^{(m)}\eta^{-r+1}\frac{d\eta}{d\xi} + \cdots + X_{m+r}^{(m)}\frac{d\eta}{d\xi}. \qquad \cdot (55)$$

This when expanded in terms of ξ can involve no zero or negative powers. For it is a sum of multiples of $\eta\, d\eta/d\xi$, $\eta^2\, d\eta/d\xi$, ... only, since $\{1, 0; m, -m-r\}_x$ is a sum of multiples of η, η^2, ... only, and these when expressed in terms of ξ are all free from ξ^0, ξ^{-1} ξ^{-2} ... Thus the coefficients of ξ^{-r}, ξ^{-r+1}, ... 1, which would appear to occur in the above symbolical form of the transformation of $\{1, 0; m, -m-r\}$ are in reality absent; and, consequently,

$$m\{1, 0; m, -m-r\}_x = -\{0, 1; 1-m-r, m-1\}_y + X_m^{(m)}\{0, 1; 1-r, -1\}_y$$

$$+ X_{m+1}^{(m)}\{0, 1; 2-r, -1\}_y + \cdots + X_{m+r-1}^{(m)}\{0, 1; 0, -1\}_y + X_{m+r}^{(m)}\{0, 1; 1, -1\}_y, \quad (56)$$

the various terms on the right consisting of the parts with positive indices of ξ from the corresponding terms of (55).

In like manner $m\{0, 1 ; m, - m - r\}_x$ whose symbolical form is

$$\eta^{1-m-r}\frac{d}{d\eta}\{\xi^m - X_m^{(m)}\eta^m - X_{m+1}^{(m)}\eta^{m+1} - \ldots - X_{m+r}^{(m)}\eta^{m+r}\}, \quad . \quad . \quad (57)$$

a form proceeding by positive integral powers of η, and therefore of ξ, transforms into

$$- m\xi^{m-1}\eta^{1-m-r} + mX_m^{(m)}\eta^{-r}\frac{d\eta}{d\xi} + (m + 1) X_{m+1}^{(m)}\eta^{1-r}\frac{d\eta}{d\xi} + \ldots$$

$$+ (m + r - 1) X_{m+r-1}^{(m)}\eta^{-1}\frac{d\eta}{d\xi} + (m + r) X_{m+r}^{(m)}\frac{d\eta}{d\xi}, \quad . \quad (58)$$

of which the terms in zero and negative powers of ξ must, as before, disappear, leaving as the result of transformation

$$m\{0, 1 ; m, - m - r\}_x = - m(1 - m - r)\{1, 0 ; 1 - m - r, m - 1\}_y$$

$$+ mX_m^{(m)}\{0, 1 ; 1 - r, - 1\}_y + (m + 1)X_{m+1}^{(m)}\{0, 1 ; 2 - r, - 1\}_y + \ldots$$

$$+ (m + r - 1)X_{m+r-1}^{(m)}\{0, 1 ; 0, - 1\}_y + (m + r)X_{m+r}^{(m)}\{0, 1 ; 1, - 1\}_y. \quad (59)$$

By addition of μ times (56) to ν times (59) the transformation of the more general $\{\mu, \nu ; m, - m - r\}_x$ is at once deduced.

15. The forms taken by (56) and (59) for the case $r = 1$, i.e., $m + n = - 1$, since

$$X_m^{(m)} = x_1^m = y_1^{-m}$$

and

$$X_{m+1}^{(m)} = mx_1^{m-1}x_2 = - my_1^{-m-2}y_2,$$

may be written

$$m\{1, 0 ; m, - m - 1\}_x = - \{0, 1 ; - m, m - 1\}_y + y_1^{-m}\{0, 1 ; 0, - 1\}_y$$

$$- my_1^{-m-2}y_2\{0, 1 ; 1, - 1\}_y. \quad (60)$$

and

$$\{0, 1 ; m, - m - 1\}_x = m\{1, 0 ; - m, m - 1\}_y + y_1^{-m}\{0, 1 ; 0, -1\}_y$$

$$- (m + 1)y_1^{-m-2}y_2\{0, 1 ; 1, - 1\}_y. \quad (61)$$

One or two particular cases of these formulæ deserve mention. The value zero of m makes (60) an identity. In (61) the substitution of the same value produces

$$\{0, 1 ; 0, -1\}_x = \{0, 1 ; 0, -1\}_y - y_1^{-2}y_2\{0, 1 ; 1, - 1\}_y,$$

in verification of which we may notice that it is unaltered by interchange of x and y, in virtue of (48).   Another way of writing the result is to say that

$$2\{0,\,1;\,0,\,-1\}_x - x_1^{-2}x_2\{0,\,1;\,1,\,-1\}_x$$
$$= 2\{0,\,1;\,0,\,-1\}_y - y_1^{-2}y_2\{0,\,1;\,1,\,-1\}_y \quad . \quad (62)$$

is a self reciprocal operator of positive character.

We are now enabled to write (61)

$$m\{0,\,1;\,m,\,-m-1\}_x$$
$$= m\{1,\,0;\,-m,\,m-1\}_y + y_1^{-m}\{0,\,1;\,0,\,-1\}_x - my_1^{-m-2}y_2\{0,\,1;\,1,\,-1\}_y,$$

which becomes (60) upon interchanging x and y, replacing m by $-m$, and using (48) and the values for x_1 and x_2, in terms of y_1 and y_2.   Thus we have another verification of the consistency of our results.

II. *Ternary Operators.*

16. Let x, y, z be three variables connected by a relation of any form known or unknown.   Let x_{rs}, y_{rs}, z_{rs} denote respectively

$$\frac{1}{r!\,s!}\frac{d^{r+s}x}{dy^r\,dz^s}, \qquad \frac{1}{r!\,s!}\frac{d^{r+s}y}{dz^r\,dx^s}, \qquad \frac{1}{r!\,s!}\frac{d^{r+s}z}{dx^r\,dy^s}.$$

Let ξ, η, ζ be any set of corresponding increments of x, y, z.   They are connected by a single relation, which may be written in either of the forms

$$\xi = (x_{10}\,\eta + x_{01}\,\zeta) + (x_{20}\,\eta^2 + x_{11}\,\eta\zeta + x_{02}\,\zeta^2)$$
$$+ (x_{30}\,\eta^3 + x_{21}\,\eta^2\zeta + x_{12}\,\eta\zeta^2 + x_{03}\,\zeta^3) + \ldots, \quad . \quad . \quad (63)$$

$$\eta = (y_{10}\,\zeta + y_{01}\,\xi) + (y_{20}\,\zeta^2 + y_{11}\,\zeta\xi + y_{02}\,\xi^2)$$
$$+ (y_{30}\,\zeta^3 + y_{21}\,\zeta^2\xi + y_{12}\,\zeta\xi^2 + y_{03}\,\xi^3) + \ldots, \quad . \quad . \quad (64)$$

$$\zeta = (z_{10}\,\xi + z_{01}\,\eta) + (z_{20}\,\xi^2 + z_{11}\,\xi\eta + z_{02}\,\eta^2)$$
$$+ (z_{30}\,\xi^3 + z_{21}\,\xi^2\eta + z_{12}\,\xi\eta^2 + z_{03}\,\eta^3) + \ldots . \quad . \quad . \quad (65)$$

Let m be a positive integer, and let $\mathrm{X}_{rs}^{(m)}$ denote the coefficient of $\eta^r\,\zeta^s$ in ξ^m when expanded in ascending products of positive integral powers of η and ζ, so that

$$\xi^m = \{\Sigma x_{pq}\, \eta^p\, \zeta^q\}^m = \underset{r+s\,\nleq\,m}{\Sigma}\, X^{(m)}_{rs}\, \eta^r\, \zeta^s;\qquad \cdots \cdots \quad (66)$$

and in like manner write

$$\eta^m = \{\Sigma y_{pq}\, \zeta^p\, \xi^q\}^m = \underset{r+s\,\nleq\,m}{\Sigma}\, Y^{(m)}_{rs}\, \zeta^r\, \xi^s,\qquad \cdots \cdots \quad (67)$$

and

$$\zeta^m = \{\Sigma z_{pq}\, \xi^p\, \eta^q\}^m = \underset{r+s\,\nleq\,m}{\Sigma}\, Z^{(m)}_{rs}\, \xi^r\, \eta^s.\qquad \cdots \cdots \quad (68)$$

We may include, if we please, the value zero of m; but the expansions of ξ^o, η^o, ζ^o consist only of the single terms $\eta^o\,\zeta^o$, $\zeta^o\,\xi^o$, $\xi^o\,\eta^o$.

The operators to be considered and transformed are the following :—

$$m'\{\mu,\, \nu,\, \nu';\, m,\, n,\, n'\}_x = \Sigma\,(\mu+\nu r+\nu's)\, X^{(m)}_{rs}\, \frac{d}{dx_{n+r,\,n'+s}},\qquad \cdots \quad (69)$$

$$m\{\mu,\, \nu,\, \nu';\, m,\, n,\, n'\}_y = \Sigma\,(\mu+\nu r+\nu's)\, Y^{(m)}_{rs}\, \frac{d}{dy_{n+r,\,n'+s}},\qquad \cdots \quad (70)$$

$$m\{\mu,\, \nu,\, \nu';\, m,\, n,\, n'\}_z = \Sigma\,(\mu+\nu r+\nu's)\, Z^{(m)}_{rs}\, \frac{d}{dz_{n+r,\,n'+s}};\qquad \cdots \quad (71)$$

where μ, ν, ν' are any numerical quantities,

    m a positive integer,

    n, n' positive integers or one or both zero, and

    r, s quantities which take in succession all zero and positive integral values subject to $r+s \nleq m$.

Cases of m negative, and of n, n' either or both less than -1, which have been dealt with in the analogous theory of binary operators, will not be here considered.

The cases of m zero, and of n or n' equal to -1, will not be entirely excluded, but will be only dealt with as far as their accordance with the results for m positive and n, n' not negative needs no elaboration to make it clear.

Thus our field of investigation *is narrower* than in that of the analogous theory hitherto considered. Were negative values of n and n' admitted, the lower limit of r in the operators (69) . . . (71) would be $-n+1$ instead of zero, and that of s would be in like manner $-n'+1$. Thus when we admit the value -1 of n we must exclude the value 0 of r, and when the value -1 of n' we must exclude the value 0 of s.

Let us now express (69), (70), (71) symbolically as follows :—

$$m\{\mu,\, \nu,\, \nu';\, m,\, n,\, n'\}_x = \Sigma\,(\mu+\nu r+\nu's)\, X^{(m)}_{rs}\, \eta^{n+r}\, \zeta^{n'+s},\qquad \cdots \quad (72)$$

$$m\{\mu,\, \nu,\, \nu';\, m,\, n,\, n'\}_y = \Sigma\,(\mu+\nu r+\nu's)\, Y^{(m)}_{rs}\, \zeta^{n+r}\, \xi^{n'+s},\qquad \cdots \quad (73)$$

$$m\{\mu,\, \nu,\, \nu';\, m,\, n,\, n'\}_z = \Sigma\,(\mu+\nu r+\nu's)\, Z^{(m)}_{rs}\, \xi^{n+r}\, \eta^{n'+s};\qquad \cdots \quad (74)$$

i.e., let us in any x-operator symbolize d/dx_{pq} by $\eta^p \zeta^q$, in any y-operator d/dy_{pq} by $\zeta^p \xi^q$, and in any z-operator d/dz_{pq} by $\xi^p \eta^q$.

We may in this way write (71) or (74)

$$m \{\mu, \nu, \nu'; m, n, n'\}_z = \mu \xi^n \eta^{n'} \{z_{10} \xi + z_{01} \eta + z_{20} \xi^2 + z_{11} \xi\eta + z_{02} \eta^2 + \ldots\}^m$$

$$+ \nu \zeta^{n+1} \eta^{n'} \frac{d}{d\xi} \{z_{10} \xi + z_{01} \eta + z_{20} \xi^2 + z_{11} \xi\eta + z_{02} \eta^2 + \ldots\}^m$$

$$+ \nu' \xi^n \eta^{n'+1} \frac{d}{d\eta} \{z_{10} \xi + z_{01} \eta + z_{20} \xi^2 + z_{11} \xi\eta + z_{02} \eta^2 + \ldots\}^m$$

$$= \mu \xi^n \eta^{n'} \zeta^m + \nu \xi^{n+1} \eta^{n'} \frac{d}{d\xi}(\zeta^m) + \nu' \xi^n \eta^{n'+1} \frac{d}{d\eta}(\zeta^m), \quad . \quad . \quad (75)$$

where ζ means the expansion in terms of ξ and η given in (65), and where the symbolization denotes that the right-hand member is to be expanded in terms of ξ and η, and to have each product $\xi^p \eta^q$ in its expansion replaced by the corresponding d/dz_{pq}, in order to produce the operator in z dependent which is represented by the notation on the left.

Thus in particular, assigning to different pairs in succession of the three parameters, μ, ν, ν', zero values,

$$m \{1, 0, 0; m, n, n'\}_z = \xi^n \eta^{n'} \zeta^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

$$m \{0, 1, 0; m, n, n'\}_z = \xi^{n+1} \eta^{n'} \frac{d}{d\xi}(\zeta^m) = m\xi^{n+1} \eta^{n'} \zeta^{m-1} \frac{d\zeta}{d\xi}, \quad . \quad . \quad (77)$$

$$m \{0, 0, 1; m, n, n'\}_z = \xi^n \eta^{n'+1} \frac{d}{d\eta}(\zeta^m) = m\xi^n \eta^{n'+1} \zeta^{m-1} \frac{d\zeta}{d\eta}, \quad . \quad . \quad (78)$$

while

$$\{\mu, \nu, \nu'; m, n, n'\}_z = \mu \{1, 0, 0; m, n, n'\}_z + \nu \{0, 1, 0; m, n, n'\}_z$$
$$+ \nu' \{0, 0, 1; m, n, n'\}_z. \quad . \quad (79)$$

Precisely similar symbolical expressions to (75) . . . (78) are, of course, assigned to the corresponding operators in x and in y dependent. We have only cyclically to interchange ξ, η, ζ once and twice respectively, and to regard the expressions on the right thus obtained as short ways of writing their expansions by aid of (63) and (64) in terms of η, ζ and ζ, ξ respectively.

17. In the present article the expression of each operative symbol d/dx_{rs}, on a function of the derivatives of x with regard to y and z, in terms of the operative symbols d/dz_{pq} on the equivalent function of the derivatives of z with regard to x and y, is investigated.

If, as in the earlier part of the last article, ξ, η, ζ are simultaneous increments of x, y, z, we may look upon

$$x_{10},\ x_{01},\ x_{20},\ x_{11},\ x_{02},\ \ldots$$

as a number of independent quantities; upon

$$y_{10},\ y_{01},\ y_{20},\ y_{11},\ y_{02},\ \ldots$$

and

$$z_{10},\ z_{01},\ z_{20},\ z_{11},\ z_{02},\ \ldots$$

as determinate functions of these quantities; upon ξ, η, ζ as three quantities connected with one another, and with x_{10}, x_{01}, x_{20}, $\ldots$ by a relation of which (63), (64), and (65) are equivalent forms.

Of the quantities x_{10}, x_{01}, x_{20}, $\ldots$ let one x_{rs} alone receive an infinitesimal variation: also of ξ, η, ζ, let η and ζ be kept constant so that ξ receives a consequent variation. Some or all of y_{10}, y_{01}, y_{20}, $\ldots$ and some or all of z_{10}, z_{01}, z_{20}, $\ldots$ will also receive consequent variations. From (63) we thus obtain

$$\delta\xi = \eta^r \zeta^s\, \delta x_{rs};$$

from (64)

$$0 = \{y_{01} + y_{11}\xi + 2y_{02}\eta + y_{21}\xi^2 + 2y_{12}\xi\eta + 3y_{03}\eta^2 + \ldots\}\,\delta\xi$$
$$+ \left\{\frac{dy_{10}}{dx_{rs}}\zeta + \frac{dy_{01}}{dx_{rs}}\xi + \frac{dy_{20}}{dx_{rs}}\zeta^2 + \frac{dy_{11}}{dx_{rs}}\zeta\xi + \frac{dy_{02}}{dx_{rs}}\xi^2 + \ldots\right\}\,\delta x_{rs};$$

and from (65)

$$0 = \{z_{10} + 2z_{20}\xi + z_{11}\eta + 3z_{30}\xi^2 + 2z_{21}\xi\eta + z_{12}\eta^2 \ldots\}\,\delta\xi$$
$$+ \left\{\frac{dz_{10}}{dx_{rs}}\xi + \frac{dz_{01}}{dx_{rs}}\eta + \frac{dz_{20}}{dx_{rs}}\xi^2 + \frac{dz_{11}}{dx_{rs}}\xi\eta + \frac{dz_{02}}{dx_{rs}}\eta^2 + \ldots\right\}\,\delta x_{rs}.$$

The three relations are identical. Let us study the identity of the first and third. We obtain from them that

$$\frac{dz_{10}}{dx_{rs}}\xi + \frac{dz_{01}}{dx_{rs}}\eta + \frac{dz_{20}}{dx_{rs}}\xi^2 + \frac{dz_{11}}{dx_{rs}}\xi\eta + \frac{dz_{02}}{dx_{rs}}\eta^2 + \ldots$$
$$= -\eta^r\zeta^s\{z_{10} + 2z_{20}\xi + z_{11}\eta + 3z_{30}\xi^2 + 2z_{21}\xi\eta + z_{12}\eta^2 + \ldots\}, \quad . \quad . \quad (80)$$

for all values of ξ and η; and, consequently, that if by aid of (65) the right hand member be like the left, expanded in powers and products of powers of ξ and η, the coefficients of corresponding terms on the two sides will be equal. In other words, each dz_{pq}/dx_{rs} is the coefficient of the corresponding $\xi^p\eta^q$.

Now, in the equivalence of operators,

$$\frac{d}{dx_{rs}} = \frac{dz_{10}}{dx_{rs}} \cdot \frac{d}{dz_{10}} + \frac{dz_{01}}{dx_{rs}} \cdot \frac{d}{dx_{01}} + \frac{dz_{20}}{dx_{rs}} \cdot \frac{d}{dz_{20}} + \frac{dz_{11}}{dx_{rs}} \cdot \frac{d}{dz_{11}} + \frac{dz_{02}}{dx_{rs}} \cdot \frac{d}{dz_{02}} + \cdots$$

each dz_{pq}/dx_{rs} is the coefficient of d/dz_{pq}.

It follows that in the expansion in terms of ξ and η of the right hand member of (80) the substitution for each $\xi^p \eta^q$ of the corresponding d/dz_{pq} exactly produces the expression for d/dx_{rs}. In other words, for each r and s, the

$$z \text{ transform of } \frac{d}{dx_{rs}} = -\eta^r \zeta^s \frac{d\zeta}{d\xi}, \qquad \qquad \cdots \quad (81)$$

where ζ and its partial differential coefficient are to be replaced by their equivalents in terms of ξ and η by (65), where the product is to be expanded in terms of ξ and η, and where in the expanded result each product $\xi^p \eta^q$ is to be replaced by the corresponding operative symbol d/dz_{pq}.

By (77) we see that d/dx_{rs} is thus replaced by a linear z-operator of the form under consideration; in fact that

$$\frac{d}{dx_{rs}} = -\{0, 1, 0 ; s + 1, -1, r\}_z. \qquad \cdots \quad (82)$$

Since the μ and the ν' of this operator are zero, the fact that n is -1 gives no difficulty as to the presence or absence of coefficients on the right like $Z_{0q}^{(s+1)}$.

In precisely the same way, by giving variations to ξ and x_{rs} in (63) and (64) instead of (63) and (65), we might have obtained

$$\frac{d}{dx_{rs}} = -\{0, 0, 1 ; r + 1, s, -1\}_y, \qquad \cdots \quad (83)$$

of which y-operator the symbolical form is

$$-\zeta^s \eta^r \frac{d\eta}{d\xi}.$$

18. The rules for transforming any linear x-operator to its equivalent forms in y and in z dependent, are now very simply expressed just as was the analogous rule in in Art. 4. Since the x-operator

$$\frac{d}{dx_{rs}} \quad \text{ or } \quad \eta^r \zeta^s$$

has for its equivalent y-operator

$$-\eta^r \zeta^s \frac{d\eta}{d\xi},$$

and for its equivalent z-operator

$$-\eta^r \zeta^s \frac{d\zeta}{d\xi},$$

these rules are merely—To find the equivalent y-operator to a given linear x-operator, multiply its symbolical form by $-\,d\eta/d\xi$; and to find its equivalent z-operator, multiply that symbolical form by $-\,d\zeta/d\xi$. The y-operator thus obtained has of course to be expanded in terms of ζ and ξ by (64), and the z-operator in terms of ξ and η by (65), before being intelligible except by means of (76) to (78).

In verification let it be noticed that, since by first principles of the theory of partial differentiation the three sets of ratios

$$\frac{d\xi}{d\eta} \;:\; \frac{d\xi}{d\zeta} \;:\; -1\,,$$

$$-1 \;:\; \frac{d\eta}{d\zeta} \;:\; \frac{d\eta}{d\xi}\,,$$

$$\frac{d\zeta}{d\eta} \;:\; -1 \;:\; \frac{d\zeta}{d\xi}\,,$$

are equal, precisely the same results are obtained by cyclical interchange of x, y, z and ξ, η, ζ.

19. Now as in (76)

$$m\{1, 0, 0\,;\, m, n, n'\}_x = \eta^n \zeta^{n'} \xi^m.$$

Its forms in y and z respectively are then

$$-\,\zeta^{n'} \xi^m \eta^n \frac{d\eta}{d\xi}\,, \quad \text{and} \quad -\,\xi^m \eta^n \zeta^{n'} \frac{d\zeta}{d\xi}\,.$$

Of these by two cyclic interchanges in (78), and by (77) itself, respectively, the expressions are

$$-\,\{0, 0, 1\,;\, n+1, n', m-1\}_y, \quad \text{and} \quad -\,\{0, 1, 0\,;\, n'+1, m-1, n\}_z$$

consequently

$$m\{1, 0, 0\,;\, m, n, n'\}_x = -\,\{0, 0, 1\,;\, n+1, n', m-1\}_y$$
$$= -\,\{0, 1, 0\,;\, n'+1, m-1, n\}_z. \quad (84)$$

In the same way the y and z transforms of

$$\{0, 1, 0\,;\, m, n, n'\}_x, \quad i.e., \quad \eta^{n+1} \zeta^{n'} \xi^{m-1} \frac{d\xi}{d\eta}\,,$$

are

$$-\,\zeta^{n'} \xi^{m-1} \eta^{n+1} \frac{d\xi}{d\eta}\cdot\frac{d\eta}{d\xi}\,, \quad \text{and} \quad -\,\xi^{m-1} \eta^{n+1} \zeta^{n'} \frac{d\xi}{d\eta}\cdot\frac{d\zeta}{d\xi}\,,$$

i.e.,

$$-\,\zeta^{n'} \xi^{m-1} \eta^{n+1}\,, \quad \text{and} \quad +\,\xi^{m-1} \eta^{n+1} \zeta^{n'} \frac{d\zeta}{d\eta}\,,$$

by the equalities of ratios at the end of the last article.   Accordingly

$$\{0, 1, 0 ; m, n, n'\}_x = - (n + 1)\{1, 0, 0 ; n + 1, n', m - 1\}_y$$
$$= \{0, 0, 1 ; n' + 1, m - 1, n\}_z. \quad . \quad (85)$$

And once more, precisely in the same way,

$$\eta^n \zeta^{n'+1} \xi^{m-1} \frac{d\xi}{d\zeta}, \quad \zeta^{n'+1} \xi^{m-1} \eta^n \frac{d\eta}{d\zeta}, \quad \text{and} \quad - \xi^{m-1} \eta^n \zeta^{n'+1}$$

are equivalent operators in x, y, z respectively dependent, so that also

$$\{0, 0, 1 ; m, n, n'\}_x = \{0, 1, 0 ; n + 1, n', m - 1\}_y$$
$$= - (n' + 1) \{1, 0, 0 ; n' + 1, m - 1, n\}_z. \quad . \quad (86)$$

Of these sets of equalities (85) and (86) may in reality be deduced from (84) by cyclical interchanges of the variables and alteration of parameters.   The independent investigation above is justified by the verification it affords.

The general formulæ of transformation, including (84), (85), (86), follow from them by (79), and are

$$\{\mu, \nu, \nu' ; m, n, n'\}_x = \{- \nu (n + 1), \nu', - \frac{\mu}{m} ; n + 1, n', m - 1\}_y$$

$$= \{- \nu' (n' + 1), - \frac{\mu}{m}, \nu ; n' + 1, m - 1, n\}_z. \quad . \quad (87)$$

Included, it is interesting to notice that we have three distinct classes of self reproductive or cyclically persistent operators, of characters corresponding each to one of the cube roots of unity, viz.:

$$\{- m, 1, 1 ; m, m - 1, m - 1\}_x = \{- m, 1, 1 ; m, m - 1, m - 1\}_y$$
$$= \{- m, 1, 1 ; m, m - 1, m - 1\}_z \quad . \quad . \quad (88)$$

$$\{- m, \omega, \omega^2 ; m, m - 1, m - 1\}_x = \omega \{- m, \omega, \omega^2 ; m, m - 1, m - 1\}_y$$
$$= \omega^2 \{- m, \omega, \omega^2 ; m, m - 1, m - 1\}_z . \quad (89)$$

$$\{- m, \omega^2, \omega ; m, m - 1, m - 1\}_x = \omega^2 \{- m, \omega^2, \omega ; m, m - 1, m - 1\}_y$$
$$= \omega \{- m, \omega^2, \omega ; m, m - 1, m - 1\}_z . \quad (90)$$

20. Some of the simplest, and most important so far as actual experience goes, examples of the formulæ now proved will be considered in what follows.

The only lineo-linear operators, of the classes with which we are dealing, both of whose cyclical transformations are also lineo-linear, are found by putting m, $n + 1$ and $n' + 1$ all equal to unity in the results of the last article. Thus

$$\{1, 0, 0 ; 1, 0, 0\}_x = - \{0, 0, 1 ; 1, 0, 0\}_y = - \{0, 1, 0 ; 1, 0, 0\}_z, \qquad (91)$$

with the correlative equalities obtained by writing y, z, x and z, x, y respectively for x, y, z, involve the aggregate of all such operators. At length the equalities (91) are

$$x_{10} \frac{d}{dx_{10}} + x_{01} \frac{d}{dx_{01}} + x_{20} \frac{d}{dx_{20}} + x_{11} \frac{d}{dx_{11}} + x_{02} \frac{d}{dx_{02}} + x_{30} \frac{d}{dx_{30}} + x_{21} \frac{d}{dx_{21}}$$

$$+ x_{12} \frac{d}{dx_{12}} + x_{03} \frac{d}{dx_{03}} + \dots$$

$$= - \left\{ y_{01} \frac{d}{dy_{01}} + y_{11} \frac{d}{dy_{11}} + 2y_{02} \frac{d}{dy_{02}} + y_{21} \frac{d}{dy_{21}} + 2y_{12} \frac{d}{dy_{12}} + 3y_{03} \frac{d}{dy_{03}} + \dots \right\}$$

$$= - \left\{ z_{10} \frac{d}{dz_{10}} + 2z_{20} \frac{d}{dz_{20}} + z_{11} \frac{d}{dz_{11}} + 3z_{30} \frac{d}{dz_{30}} + 2z_{21} \frac{d}{dz_{21}} + z_{12} \frac{d}{dz_{12}} + \dots \right\} . \qquad (92)$$

We thus learn that, if a function of the derivatives of x with regard to y and z is homogeneous, the equivalent function of the derivatives of y with regard to z and x is isobaric in second suffixes, while the equivalent function of the derivatives of z with regard to x and y is isobaric in first suffixes ; and that

$$i (x, yz) = - w_2 (y, zx) = - w_1 (z, xy), \qquad \dots \dots \qquad (93)$$

where the notation explains itself. The correlative facts are

$$- w_1 (x, yz) = i (y, zx) = - w_2 (z, xy), \qquad \dots \dots \qquad (94)$$

and

$$- w_2 (x, yz) = - w_1 (y, zx) = i (z, xy). \qquad \dots \dots \qquad (95)$$

The same aggregate as is involved in (91) and its correlatives is also expressed by the facts that

$$\{ - 1, 1, 1 ; 1, 0, 0\}, \qquad \dots \dots \dots \dots \qquad (96)$$

$$\{ - 1, \omega, \omega^2 ; 1, 0, 0\}, \qquad \dots \dots \dots \dots \qquad (97)$$

$$\{ - 1, \omega^2, \omega ; 1, 0, 0\}, \qquad \dots \dots \dots \dots \qquad (98)$$

obtained by giving m the value 1 in (88) to (90), are cyclically persistent lineo-linear operators of characters 1, ω, ω^2 respectively.

If the operation be on a homogeneous and doubly isobaric function we are thus told that

$$- i + w_1 + w_2 \qquad - i + \omega w_1 + \omega^2 w_2, \qquad - i + \omega^2 w_1 + \omega w_2 \quad . \quad (99)$$

are characteristics which persist after one cyclical transformation but for the multipliers 1, ω, ω^2 respectively, and after a second but for 1, ω^2, ω.

21. The quadro-linear operators (linear operators with coefficients quadratic in the derivatives) both whose cyclic transformations are also quadro-linear, are obtained by giving to every one of m, $n + 1$, $n' + 1$ the value 2 in the formulæ of Art. 19. Their aggregate is involved in

$$2 \{1, 0, 0 ; 2, 1, 1\}_x = - \{0, 0, 1 ; 2, 1, 1\}_y = - \{0, 1, 0 ; 2, 1, 1\}_z, \quad . \quad (100)$$

and its two correlatives in y, z, x and z, x, y.

The same system is expressed by the three cyclically persistent quadro-linear operators

$$\{- 2, 1, 1 ; 2, 1, 1\}, \text{ of character } 1, \quad . \quad . \quad . \quad . \quad . \quad (101)$$

$$\{- 2, \omega, \omega^2 ; 2, 1, 1\}, \qquad ,, \qquad \omega, \quad . \quad . \quad . \quad . \quad . \quad (102)$$

$$\{- 2, \omega^2, \omega ; 2, 1, 1\}, \qquad ,, \qquad \omega^2. \quad . \quad . \quad . \quad . \quad . \quad (103)$$

Of these the first expanded to a few terms is

$$X_{30}^{(2)} \frac{d}{dx_{41}} + X_{21}^{(2)} \frac{d}{dx_{32}} + X_{12}^{(2)} \frac{d}{dx_{23}} + X_{03}^{(2)} \frac{d}{dx_{14}} + 2 \left(X_{40}^{(2)} \frac{d}{dx_{51}} + \cdots \right)$$
$$+ 3 \left(X_{50}^{(2)} \frac{d}{dx_{61}} + \cdots \right) + \cdots, \quad . \quad . \quad (104)$$

where

$$X_{20}^{(2)} = x_{10}{}^2, \; X_{11}^{(2)} = 2x_{10}x_{01}, \; X_{02}^{(2)} = x_{01}{}^2,$$

$$X_{30}^{(2)} = 2x_{10}x_{20}, \; X_{21}^{(2)} = 2x_{10}x_{11} + 2x_{01}x_{20}, \; X_{12}^{(2)} = 2x_{10}x_{02} + 2x_{01}x_{11}, \; X_{03}^{(2)} = 2x_{01}x_{02},$$

$$X_{40}^{(2)} = 2x_{10}x_{30} + x_{20}{}^2, \; X_{31}^{(2)} = 2x_{10}x_{21} + 2x_{01}x_{12} + 2x_{20}x_{11}, \; X_{22}^{(2)} = 2x_{10}x_{12} + 2x_{01}x_{21} + x_{11}{}^2$$
$$+ 2x_{20}x_{02}, \; X_{13}^{(2)} = 2x_{10}x_{03} + 2x_{01}x_{12} + 2x_{11}x_{02}, \; X_{04}^{(2)} = 2x_{01}x_{03} + x_{02}{}^2,$$

$$X_{50}^{(2)} = 2x_{10}x_{40} + 2x_{20}x_{30}, \; \dots,$$

and generally

$$X_{mn}^{(2)} = \sum_{r+s \,\not\lessdot\, 1}^{r+s \,\not\gtrdot\, m+n-1} \left(x_{rs} x_{m-r,\, n-s} \right).$$

The two imaginary cyclically persistent quadro-linear operators (102) and (103) are easily written out in like manner. They commence with terms in d/dx_{31}, d/dx_{22}, d/dx_{13}, which it is to be observed are wanting from the above.

Once more by giving to each of m, $n + 1$, $n' + 1$ the value 3 in Art. 19, an aggregate is obtained of linear operators with coefficients of the third degree, whose transforms have both of them coefficients of the third degree also. The aggregate may, as before, be considered involved in three cyclically persistent operators of the type, one of each character. Similarly as to operators with coefficients of any higher degree.

22. Some of the most important linear operators which have been used in recent theories of functional invariants, cyclicants, &c., have the property of persistence of degree in the derivatives after one cyclical transformation, but not after a second.

Such operators occur among those obtained by putting $n + 1 = m$ in (87), viz.,

$$\{\mu,\, \nu,\, \nu';\; m,\, m-1,\, n'\}_x = \left\{-\nu m,\, \nu',\, -\frac{\mu}{m};\; m,\, n',\, m-1\right\}_y$$
$$= \left\{-\nu'(n'+1),\, -\frac{\mu}{m'},\, \nu;\; n'+1,\, m-1,\, m-1\right\}_z \quad . \quad . \quad (105)$$

In particular, there are three classes of operators which have a property closely akin to that of persisting in form after a first cyclical transformation, being, in fact, only altered by the interchange of first and second suffixes : they are

$$\{-m,\, 1,\, 1;\; m,\, m-1,\, n'\}_x = \{-m,\, 1,\, 1;\; m,\, n',\, m-1\}_y$$
$$= \{-(n'+1),\, 1,\, 1;\; n'+1,\, m-1,\, m-1\}_z \quad . \quad . \quad . \quad (106)$$

$$\{-m,\, \omega,\, \omega^2;\; m,\, m-1,\, n'\}_x = \omega \{-m,\, \omega,\, \omega^2;\; m,\, n',\, m-1\}_y$$
$$= \omega^2 \{-(n'+1),\, \omega,\, \omega^2;\; n'+1,\, m-1,\, m-1\}_z \quad . \quad . \quad (107)$$

$$\{-m,\, \omega^2,\, \omega;\; m,\, m-1,\, n'\}_x = \omega^2 \{-m,\, \omega^2,\, \omega;\; m,\, n',\, m-1\}_y$$
$$= \omega \{-(n'+1),\, \omega^2,\, \omega;\; n'+1,\, m-1,\, m-1\}_z \quad . \quad . \quad (108)$$

It is to be noticed, in the case of the first of these, that the second cyclical transformation, which is of different degree from the first, is quite symmetrical in first and second suffixes.

Among the operators comprised in (106) occur the two, which I have called ω_1 and ω_2,[*] two of the six form annihilators of projective cyclicants, viz.,

$$\omega_1(x,\, yz) = \underset{r+s \nleqq 1}{\Sigma} \left\{(r+s-1)\, x_{rs}\, \frac{d}{dx_{r,\, s+1}}\right\},$$

$$\omega_2(x,\, yz) = \underset{r+s \nleqq 1}{\Sigma} \left\{(r+s-1)\, x_{rs}\, \frac{d}{dx_{r+1,\, s}}\right\},$$

or in present notation $\{-1, 1, 1 \,;\, 1, 0, 1\}$ and $\{-1, 1, 1 \,;\, 1, 1, 0\}$.   For the transformation of these we have, by putting 1 for each of m and n' in (106),

$$\omega_1(x, yz) = \omega_2(y, zx) = \{-2, 1, 1 \,;\, 2, 0, 0\}_z, \quad . \quad . \quad . \quad . \quad (109)$$

of which right hand operator the expansion is

$$Z^{(2)}_{30}\frac{d}{dz_{30}} + Z^{(2)}_{21}\frac{d}{dz_{21}} + Z^{(2)}_{12}\frac{d}{dz_{12}} + Z^{(2)}_{03}\frac{d}{dz_{03}} + 2\left(Z^{(2)}_{40}\frac{d}{dz_{40}} + \ldots\right) + 3\left(Z^{(2)}_{50}\frac{d}{dz_{50}} + \ldots\right) + \ldots$$

where the coefficients have meanings, as in Art. 21.

Closely resembling, but distinct from ω_1 and ω_2, are Mr. FORSYTH's Δ_2 and Δ_1,[*] i.e.,

$$\Delta_2(x, yz) = \underset{r+s \,\not<\, 1}{\Sigma}\left\{(r+s)x_{rs}\frac{d}{dx_{r,s+1}}\right\} = \{0, 1, 1 \,;\, 1, 0, 1\}_x,$$

$$\Delta_1(x, yz) = \underset{r+s \,\not<\, 1}{\Sigma}\left\{(r+s)x_{rs}\frac{d}{dx_{r+1,s}}\right\} = \{0, 1, 1 \,;\, 1, 1, 0\}_x.$$

These are also transformed by means of the present article, but have not the property of companionship belonging to ω_1 and ω_2.   In fact, by (105),

$$\Delta_2(x, yz) = \{-1, 1, 0 \,;\, 1, 1, 0\}_y = \{-2, 0, 1 \,;\, 2, 0, 0\}_z, \quad . \quad . \quad (110)$$

and, by (105), with z, x, y put for x, y, z,

$$\Delta_1(x, yz) = \{-2, 1, 0 \,;\, 2, 0, 0\}_y = \{-1, 0, 1 \,;\, 1, 0, 1\}_z. \quad . \quad . \quad (111)$$

23. The special importance of many operators in which the first derivatives do not occur is well known.   The form of such operators (in z dependent) is symbolically

$$\xi^p\eta^q\left(a + b\xi\frac{d}{d\xi} + c\eta\frac{d}{d\eta}\right)(\zeta - z_{10}\xi - z_{01}\eta)^m.$$

As every such operator is a sum of multiples of complete operators $\{\mu, \nu, \nu' \,;\, m, n, n'\}_z$ so that their theory is implicitly involved in that above discussed, no attempt will be made here to develope it independently.   In the present article, however, an interesting class of cyclically persistent operators will be obtained, and a method of procedure in a much wider class of cases will be thus exemplified.

It is required to prove that the result of replacing each first derivative by zero in

$$\{-m, 1, 1 \,;\, m, 0, 0\}$$

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* See his Memoir " A Class of Functional Invariants," ' Phil. Trans.,' A., vol. 180 (1889), pp. 71–118.

is, but for a first derivative factor, an operator which persists in form after one and two cyclical interchanges of the variables.

Symbolically we have, *if square brackets indicate that in an operator first derivatives are thus omitted,*

$$[-m, 1, 1 \; ; \; m, 0, 0]_x = \frac{1}{m}\left(-m + \eta\frac{d}{d\eta} + \zeta\frac{d}{d\zeta}\right)(\xi - x_{10}\eta - x_{01}\zeta)^m,$$

$$= (\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{-\xi + x_{10}\eta + x_{01}\zeta + \eta\frac{d\xi}{d\eta}\right.$$

$$\left. - x_{10}\eta + \zeta\frac{d\xi}{d\zeta} - x_{01}\zeta\right\}$$

$$= (\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{\eta\frac{d\xi}{d\eta} + \zeta\frac{d\xi}{d\zeta} - \xi\right\}.$$

The y transform of this operator is therefore, by Art. 18,

$$-(\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{\eta\frac{d\xi}{d\eta} + \zeta\frac{d\xi}{d\zeta} - \xi\right\}\frac{d\eta}{d\xi},$$

which, since

$$x_{10} : x_{01} : -1 = -1 : y_{10} : y_{01} = z_{01} : -1 : z_{10},$$

and

$$\frac{d\xi}{d\eta} : \frac{d\xi}{d\zeta} : -1 = -1 : \frac{d\eta}{d\zeta} : \frac{d\eta}{d\xi} = \frac{d\zeta}{d\eta} : -1 : \frac{d\zeta}{d\xi},$$

may be written

$$(-1)^{m-1}x_{10}{}^{m-1}(\eta - y_{10}\zeta - y_{01}\xi)^{m-1}\left\{\zeta\frac{d\eta}{d\zeta} + \xi\frac{d\eta}{d\xi} - \eta\right\},$$

and is consequently

$$(-1)^{m-1}x_{10}{}^{m-1}[-m, 1, 1 \; ; \; m, 0, 0]_y.$$

In exactly the same way the z transform of the same operator is

$$(-1)^{m-1}x_{01}{}^{m-1}[-m, 1, 1 \; ; \; m, 0, 0]_z.$$

Thus we have the formula of transformation

$$[-m, 1, 1 \; ; \; m, 0, 0]_x = \left(-\frac{1}{y_{01}}\right)^{m-1}[-m, 1, 1 \; ; \; m, 0, 0]_y = \left(-\frac{1}{z_{10}}\right)^{m-1}[-m, 1, 1 \; ; \; m, 0, 0]_z,$$

which may be written in a form even more clearly expressive of the cyclically persistent property, viz.,

$$\left(\frac{1}{x_{10}x_{01}}\right)^{(m-1)/3}[-m, 1, 1 \; ; \; m, 0, 0]_x = \left(\frac{1}{y_{10}y_{01}}\right)^{(m-1)/3}[-m, 1, 1 \; ; \; m, 0, 0]_y$$

$$= \left(\frac{1}{z_{10}z_{01}}\right)^{(m-1)/3}[-m, 1, 1 \; ; \; m, 0, 0]_z. \quad . \quad (112)$$

A first case of this formula occurs in (96). A second is that the operator on the right of (109) is the sum of terms involving z_{10}, z_{01}, and of an operator free from those derivatives which, when affected with the factor $(1/z_{10}z_{01})^{\frac{1}{3}}$, has the persistent property. The fact is interesting in its bearing on the theory of the transformation of ω_1 and ω_2. In fact (109) may be written

$$\omega_1(x, yz) = \omega_2(y, zx) = z_{10}\omega_2(z, xy) + z_{01}\omega_1(z, xy) + [-2, 1, 1 ; 2, 0, 0]_z.$$

From this and its two correlatives a verification of the formula (112) for the case of $m = 2$ is readily obtained. The first few terms of this cyclically persistent operator are

$$\left(\frac{1}{x_{10}x_{01}}\right)^{\frac{1}{3}}\left\{2\frac{x_{20}^2}{2}\frac{d}{dx_{40}} + 2x_{20}x_{11}\frac{d}{dx_{31}} + 2\left(x_{20}x_{02} + \frac{x_{11}^2}{2}\right)\frac{d}{dx_{22}} + 2x_{11}x_{02}\frac{d}{dx_{13}} + 2\frac{x_{02}^2}{2}\frac{d}{dx_{04}}\right.$$

$$+ 3x_{20}x_{30}\frac{d}{dx_{50}} + 3\left(x_{20}x_{21} + x_{11}x_{30}\right)\frac{d}{dx_{41}} + 3\left(x_{20}x_{12} + x_{11}x_{21} + x_{02}x_{30}\right)\frac{d}{dx_{32}}$$

$$+ 3\left(x_{20}x_{03} + x_{11}x_{12} + x_{02}x_{21}\right)\frac{d}{dx_{23}} + 3\left(x_{11}x_{03} + x_{02}x_{12}\right)\frac{d}{dx_{14}} + 3\,x_{02}x_{03}\frac{d}{dx_{05}}$$

$$\left. + \dots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

With it compare the analogous operator (30) of the binary theory. That, like this, is, of course, one of a whole class of persistent operators.

24. The transformation of the four form annihilators Ω_1, Ω_2, V_1, V_2 of pure cyclicants[*] will be now considered. They are all operators free from first derivatives, and might as such be treated by the methods of the last article. It seems preferable, however, to take them in connection with the formulæ of transformation

$$\frac{d}{dx_{10}} = -\{0, 0, 1 ; 2, 0, -1\}_y = -\{0, 1, 0 ; 1, -1, 1\}_z, \quad . \quad . \quad (114)$$

$$\frac{d}{dx_{01}} = -\{0, 0, 1 ; 1, 1, -1\}_y = -\{0, 1, 0 ; 2, -1, 0\}_z, \quad . \quad . (114\text{A})$$

which are particular cases of (82) and (83), or again of (87) with $m = 0$.

The forms of the four annihilators are

$$\Omega_1(z, xy) = \underset{m + n \not< 2}{\Sigma}\left\{mz_{mn}\frac{d}{dz_{m-1, n+1}}\right\}$$

$$= \eta\frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \eta\frac{d\zeta}{d\xi} - z_{10}\eta$$

$$= \{0, 1, 0 ; 1, -1, 1\}_z - z_{10}\frac{d}{dz_{01}}, \quad . \quad . \quad . \quad . \quad . \quad (115)$$

* See my papers in the 'London Math. Soc. Proc.,' vol. 18, pp. 142, &c.; vol. 19, pp. 6, &c., and pp. 337, &c.; vol. 20, pp. 131, &c.

$$\Omega_2(z, xy) = \underset{m+n \nleq 2}{\Sigma} \left\{ n z_{mn} \frac{d}{dz_{m+1,\, n-1}} \right\}$$

$$= \xi \frac{d}{d\eta}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \{0, 0, 1 ; 1, 1, -1\}_z - z_{01}\frac{d}{dz_{10}} ; \quad . \quad . \quad . \quad . \quad . \quad (116)$$

$$V_1(z, xy) = \underset{m+n \nleq 4}{\Sigma} \left\{ \overset{r+s \ngtr m+n-2}{\underset{r+s \nleq 2}{\Sigma}} (r z_{rs} z_{m-r,\, n-s}) \frac{d}{dz_{m-1,\, n}} \right\}$$

$$= \tfrac{1}{2}\frac{d}{d\xi}\{(\zeta - z_{10}\xi - z_{01}\eta)^2\}$$

$$= \zeta\frac{d\zeta}{d\xi} - z_{10}\xi\frac{d\zeta}{d\xi} - z_{01}\eta\frac{d\zeta}{d\xi} - z_{10}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \{0, 1, 0 ; 2, -1, 0\}_z - z_{10}\{0, 1, 0 ; 1, 0, 0\}_z - z_{01}\{0, 1, 0 ; 1, -1, 1\}_z$$

$$- z_{10}\left\{ \{1, 0, 0 ; 1, 0, 0\}_z - z_{10}\frac{d}{dz_{10}} - z_{01}\frac{d}{dz_{01}} \right\}$$

$$= \{0, 1, 0 ; 2, -1, 0\}_z - z_{10}\{1, 1, 0 ; 1, 0, 0\}_z$$

$$- z_{01}\Omega_1(z, xy) + z_{10}^2\frac{d}{dz_{10}}, \quad . \quad . \quad . \quad (117)$$

$$V_2(z, xy) = \underset{m+n \nleq 4}{\Sigma} \left\{ \overset{r+s \ngtr m+n-2}{\underset{r+s \nleq 2}{\Sigma}} (s z_{rs} z_{m-r,\, n-s}) \frac{d}{dz_{m,\, n-1}} \right\}$$

$$= \tfrac{1}{2}\frac{d}{d\eta}\{(\zeta - z_{10}\xi - z_{01}\eta)^2$$

$$= \{0, 0, 1 ; 2, 0, -1\}_z - z_{01}\{1, 0, 1 ; 1, 0, 0\}_z - z_{10}\Omega_2(z, xy) + z_{01}^2\frac{d}{dz_{01}}. \quad (118)$$

Thus (114A) may be written

$$\frac{d}{dx_{01}} = -\Omega_2(y, zx) - y_{01}\frac{d}{dy_{10}} = -\{0, 1, 0 ; 2, -1, 0\}_z,$$

and the result of one cyclical interchange in (114) may be written

$$-\Omega_1(x, yz) - x_{10}\frac{d}{dx_{01}} = \frac{d}{dy_{10}} = -\{0, 0, 1 ; 2, 0, -1\}_z ;$$

from which two sets of identities, by aid of the facts that

$$x_{10} : x_{01} : -1 = -1 : y_{10} : y_{01} = z_{01} : -1 : z_{10},$$

it follows at once that

$$\frac{1}{x_{01}} \Omega_1(x,\, yz) = -\frac{1}{y_{10}} \Omega_2(y,\, zx)$$

$$= -z_{01}\{0,\, 1,\, 0;\; 2,\, -1,\, 0\}_z + z_{10}\{0,\, 0,\, 1;\; 2,\, 0,\, -1\}_z,. \qquad (119)$$

which, and its correlatives, obtained by one and two cyclical transformations effect the transformation of Ω_1 and Ω_2.

Again, we may write (114A)

$$\frac{d}{dx_{01}} = -\Omega_2(y,\, zx) - y_{01}\frac{d}{dy_{10}}$$

$$= -V_1(z,\, xy) - z_{01}\Omega_1(z,\, xy) - z_{10}\{1,\, 1,\, 0;\; 1,\, 0,\, 0\}_z + z_{10}{}^2 \frac{d}{dz_{10}},.$$

and (114) in like manner

$$\frac{d}{dx_{10}} = -V_2(y,\, zx) - y_{10}\Omega_2(y,\, zx) - y_{01}\{1,\, 0,\, 1;\; 1,\, 0,\, 0\}_y + y_{01}{}^2 \frac{d}{dy_{01}}$$

$$= -\Omega_1(z,\, xy) - z_{10}\frac{d}{dz_{01}}.$$

From these it follows at once that

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} = y_{10}\frac{d}{dy_{10}} + y_{01}\frac{d}{dy_{01}} - \frac{1}{y_{01}}V_2(y,\, zx) - \{1,\, 0,\, 1;\; 1,\, 0,\, 0\}_y$$

$$= z_{10}\frac{d}{dz_{10}} + z_{01}\frac{d}{dz_{01}} - \frac{1}{z_{10}}V_1(z,\, xy) - \{1,\, 1,\, 0;\; 1,\, 0,\, 0\}_z.$$

By a cyclical interchange of the variables we have, also,

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} - \frac{1}{x_{10}}V_1(x,\, yz) - \{1,\, 1,\, 0;\; 1,\, 0,\, 0\}_x = y_{10}\frac{d}{dy_{10}} + y_{01}\frac{d}{dy_{01}}$$

$$= z_{10}\frac{d}{dz_{10}} + z_{01}\frac{d}{dz_{01}} - \frac{1}{z_{01}}V_2(z,\, xy) - \{1,\, 0,\, 1;\; 1,\, 0,\, 0\}_z;$$

and by a second cyclical interchange a third set of such equalities is obtained. From the two sets that have been written out, upon subtraction we obtain

$$\frac{1}{x_{10}}V_1(x,\, yz) + \{1,\, 1,\, 0;\; 1,\, 0,\, 0\}_x = -\frac{1}{y_{01}}V_2(y,\, zx) - \{1,\, 0,\, 1;\; 1,\, 0,\, 0\}_y$$

$$= -\frac{1}{z_{10}}V_1(z,\, xy) + \frac{1}{z_{01}}V_2(z,\, xy) + \{0,\, -1,\, 1;\; 1,\, 0,\, 0\}_z$$

Now, from (91) and its correlative obtained by a cyclical interchange, the second parts of these three equal operators are themselves equal. Consequently

$$\frac{1}{x_{10}} V_1(x, yz) = - \frac{1}{y_{01}} V_2(y, zx) = - \frac{1}{z_{10}} V_1(z, xy) + \frac{1}{z_{01}} V_2(z, xy), \quad . \quad (120)$$

which, and its correlatives, are the formulæ for the transformation of V_1 and V_2.

It is easy and very instructive to prove (120) directly from the symbolical expressions in (117) and (118) by the method of Art. 19 or 23.

Of other important operators Mr. FORSYTH's Δ_4 and Δ_3 ('Phil. Trans.,' A., vol. 180, p. 74) should have their formulæ of transformation noted. They are the complete operators $\{0, 1, 0 ; 1, -1, 1\}$ and $\{0, 0, 1 ; 1, 1, -1\}$ of which Ω_1 and Ω_2 are all but the first terms. Thus their formulæ of transformation are merely (114) and (114A) themselves, *i.e.*, cyclically interchanging the variables once,

$$\Delta_4(x, yz) = - \frac{d}{dy_{10}} = \{0, 0, 1 ; 2, 0, -1\}_z, \quad . \quad . \quad . \quad . \quad (121)$$

$$\Delta_3(x, yz) = \{0, 1, 0 ; 2, -1, 0\}_y = - \frac{d}{dz_{01}}. \quad . \quad . \quad . \quad . \quad (122)$$

III. The Bakerian Lecture.—*A Magnetic Survey of the British Isles for the Epoch January* 1, 1886.

By A. W. Rücker, *M.A., F.R.S., and* T. E. Thorpe, *B.Sc., Ph.D., F.R.S.*

Lecture delivered April 11,—MS. received June 6, 1889.

[Plates 1–14.]

Contents.

Two Magnetic Surveys of the British Isles have been made previous to that of which an account is given in this paper. The observations necessary for these were taken between the years 1834–38 and 1857–62, and the results were reduced to the epoch 1842–5 by Sir E. SABINE, in a paper published in the 'Philosophical Transactions' for 1870 ('Phil. Trans.,' vol. 160, 1870, p. 265). As a full account of both surveys is given in that paper, it is unnecessary to describe them in detail here. The first was made by five observers, viz., Sir E. SABINE, Captain J. C. Ross, Mr. R. W. Fox, and Professors LLOYD and JOHN PHILLIPS. In the second survey (1857–58), Mr. WELSH, Superintendent of the Kew Observatory, made an admirable series of observations in Scotland, though, unfortunately, the exposure to which he was subjected brought on an illness which terminated in his death. Sir EDWARD SABINE made observations on the Force and Dip at 24 stations in England, and some declinations determined by several naval officers between the years 1855 and 1861, were utilized. Altogether observations were made at 243 stations. ('Phil. Trans.,' vol. 162, 1872, p. 319.)

It has, we believe, for some time been thought by those interested in terrestrial magnetism, that another survey of the United Kingdom should be undertaken, and we ourselves drew attention to the matter in a paper " On the Irregularities in Magnetic Inclination on the West Coast of Scotland " ('Roy. Soc. Proc.,' vol. 36, 1884, p. 10). Not only was this desirable in order that the secular changes in the direction of the lines of equal Inclination, Force, and Declination might be re-determined, but also because the earlier surveys left much to be desired with regard to the distribution of the stations and the number of the Declination observations. Thus the Declination was determined about the epochs 1836 and 1857 at 84 stations only, of which 11 were common to the two surveys. The maps given by Sir E. SABINE in the paper already referred to show that different districts have received very different degrees of attention. Stations where the Dip has been determined cluster thickly about the coast of Scotland to the south of Oban, about the English lakes, and the south coast of England, and are thinly distributed in the North of Scotland, in the eastern counties of England, and in the centre of Ireland. In like manner, while (owing chiefly to the labours of Mr. WELSH) the Declination had been measured at 40 places in Scotland, it had been observed with adequate instruments at only 28 stations in England, and 16 in Ireland. We have therefore undertaken, and, in the course of the five years 1884–88, both inclusive, have completed a new magnetic survey of the British Isles.

In our opinion, it would probably have been better if a larger number of observers had been engaged in the task, so that it could have been finished in a shorter time. In fact, we originally made a proposal in this sense which was, we believe, brought before the Kew Committee of the Royal Society. It was considered that the objections to the employment of many instruments and observers were sufficient to prevent the acceptance of such a scheme. With proper organisation, we believe that the errors thus introduced would have been much less serious than those due to the uncertainty as to the true secular corrections at any particular station. As, however, it was more important that the survey should be made than that it should be made under the best possible conditions, we have in undertaking the task ourselves devoted most of our vacations and spare time to it, so that there should be as little delay as possible, and have collected all the facts which throw light on the value of the secular corrections.

Although the re-determination of the lines of equal magnetic Declination, Force, and Dip has been the main object of our investigation, a number of other questions have naturally come under consideration. We have made some alterations in the Kew magnetometer which have been described to the Physical Society ('Phil. Mag.,' August, 1888, p. 122); the method of presenting the results of the experiments has been modified so as to afford a greater test of their accuracy; the validity of the method of correcting for diurnal variation and disturbance, especially at places far distant from the base station, has been reconsidered, and finally, and perhaps chiefly, we have given more attention than our predecessors have done to the distribution and causes of "local magnetic attraction." In view of the difficulties caused by such disturbances, we have taken special pains to indicate the position of our stations as accurately as possible. This has been done, not merely by verbal description, but by taking out the latitudes and longitudes from the inch Ordnance maps or the Admiralty charts, with far greater accuracy than is necessary for the calculations in which these quantities are afterwards employed. For similar reasons we have selected, when possible, public parks, open commons, or other situations which are likely to be still available when the survey is repeated. It must, however, be remarked that, at some places which were not included in our original programme, observations have been made mainly because a favourable opportunity presented itself, and that, in such cases, it was not always possible to exercise the same care in the selection of a site.

Epoch.

The epoch of the survey is January 1, 1886, to which date all the observations have been reduced.

Instruments.

The survey of Scotland was mainly made with a set of instruments which belong to Professor RÜCKER. They are a Kew Magnetometer by ELLIOTT BROS., No. 60, and a

Dip Circle by Dover, No. 74.   These instruments were also used by Professor RÜCKER in his portion of the surveys of England, Wales, and Ireland.   The instruments employed by Dr. THORPE in these countries are the property of the Science and Art Department.   They are of the same patterns and by the same makers ; the Magnetometer and Dip Circle are numbered 61 and 83 respectively.

Arrangements have been made for placing this latter set of instruments in the Collection of Scientific Apparatus at South Kensington.   They will be kept for surveying only, and will·be available for comparison with instruments which may be used by future observers.

Base Station.

Our base station is the Kew Observatory.   Both sets of instruments were tested there before they came into our possession, and we have also made numerous comparisons with the Kew instruments during the progress of the survey.   We take this opportunity of expressing the great obligation we are under to Mr. G. M. WHIPPLE, the Superintendent, and to Mr. T. BAKER, the First Assistant in the Observatory, who, with the approval of the Kew Committee, rendered us every assistance in their power.   Our frequent visits and requests for information as to diurnal variation and disturbance have imposed much additional labour on these gentlemen.   The help we required has, however, invariably been given with a readiness and heartiness which merit our grateful acknowledgments.

We tested our instruments at Kew in 1884, 1886, and 1887.

A very large magnetic disturbance set in on March 30, 1886, the 31st was also much disturbed and the effects were still to be distinguished on April 1.   We have, therefore, neglected the observations made on that day, but, with this exception, the following tables contain all the determinations made by us in the Magnetic House.

The first three columns give the date, instrument, and observer, and that headed S the value of the element obtained.

The numbers given are not corrected either for diurnal variation or disturbance.

In the column headed K is the corresponding value read off from the curves of the continuously recording instruments at Kew, standardised by means of the monthly observations which are taken there.

The differences are given in the next column.   They are very nearly constant, but are rather larger than we should *a priori* have expected.   Our own instruments are evidently in accord.

Subtracting the mean difference from the Kew values, we should, if both sets of observations were perfect, reproduce our own numbers, so that a comparison between columns S and K $-$ β gives the error of experiment and comparison which is tabulated in the last column.

OBSERVATIONS in the Kew Magnetic House.

Declination.

Date.	Instru-ment.	Observer.	S (Survey Instrument.)	K (Kew Instrument.)	K − S = β.	K − β = K′.	S − K′.
1884.			° ′	° ′	′	° ′	′
July 17 . .	60	T	18 26·3	18 28·6	2·3	18 26·1	+ 0 2
,, 18 . .	60	R	18 27·4	18 29·9	2·5	18 27·4	0·0
1886.							
April 2 . .	61	T	18 12·1	18 14·4	2·3	18 11·9	+ 0·2
1887.							
Sept. 30 . .	60	R	18 8·8	18 13·1	4·3	18 10·6	− 1·8
			18 5·5	18 9·4	3·9	18 6·9	− 1·4
Oct. 11 . .	61	T	18 6·0	18 8·7	2·7	18 6·2	− 0·2
			18 6·9	18 10·4	3·5	18 7·9	− 1·0
,, 12 . .	60	R	18 9·5	18 11·9	2·4	18 9·4	+ 0·1
			18 11·1	18 13·4	2·3	18 10·9	+ 0·2
,, 13 . .	61	T	18 5·8	18 6·7	0·9	18 4·2	+ 1·6
			18 8·6	18 8·8	0·2	18 6·3	+ 2·3
,, 18 . .	61	T	18 6·3	18 8·4	2·1	18 5·9	+ 0·4
			18 6·0	18 10·4	4·4	18 7·9	− 1·9
,, 19 . .	60	R	18 8·1	18 10·3	2·2	18 7·8	+ 0·3
			18 6·9	18 8·7	1·8	18 6·2	+ 0·7
					β = 2·5	Mean .	= ± 0·82

Horizontal Force.

Date.	Instru-ment.	Observer.	S (Survey Instrument.)	K (Kew Instrument.)	K − S = β.	K − β = K′.	S − K′ = 0·000.
1884.							
July 17 . .	60	T	1·8080	1·8055	− 0·0025	1·8084	− 4
1886.							
April 2 . .	61	T	1·8091	1·8070	− 0·0021	1·8099	− 8
	60	R	1·8107	1·8072	− 0·0035	1·8101	+ 6
,, 22 . .	60	R	1·8110	1·8082	− 0·0028	1·8111	− 1
1887.							
Sept. 30 . .	60	R	1·8117	1·8087	− 0·0030	1·8116	+ 1
Oct. 11 . .	61	T	1·8114	1·8084	− 0·0030	1·8113	+ 1
,, 12 . .	60	R	1·8096	1·8070	− 0·0026	1·8099	− 3
,, 13 . .	61	T	1·8092	1·8062	− 0·0030	1·8091	+ 1
,, 18 . .	61	T	1·8114	1·8088	− 0·0026	1·8117	− 3
			1·8117	1·8088	− 0·0029	1·8117	0
,, 19 . .	60	R	1·8113	1·8074	− 0·0039	1·8103	+ 10
					β = − 0·0029	Mean .	± 3·5

To obviate, as far as possible, the use of decimals, the Forces are throughout this paper expressed in terms of metric units (metre, gram, second), and all numerical values of Forces can be reduced to C.G.S. units by dividing by 10.

In the case of the Dips, of which no continuous record is kept, the following plan was adopted. We give, under the column S, our results corrected for diurnal variation and disturbance as hereafter described, each being the mean of the two needles. In column K are the values obtained at Kew at the nearest date to the time of our observations taken from the published record. The dates in question are July 28 and 31, 1884, Sept. 22, 24, and October 25, 26, 1887. The mean of the last two is taken as applying to the whole period of the 1887 observations.

INCLINATION.

Date.	Instrument.	Observer.	S (Survey Instrument).	K (Kew Instrument).	K − S = β.	K − β = K′.	S − K′.
July 17, 1884 .	74	R	67° 36′·0	67° 38′·8	2′·8	67° 36′·1	− 0′·1
„ 18, „ .	„	T	67 35·8	..	3·0	..	− 0·3
„ 19, „ .	„	R	67 36·4	..	2·4	..	+ 0·3
Sept. 30, 1887 .	„	„	67 34·9	67 37·5	2·6	67 34·8	+ 0·1
Oct. 11, „ .	83	T	67 34·8	..	2·7	..	0·0
„ 13, „ .	„	„	67 35·0	..	2·5	..	+ 0·2
„ 18, „ .	„	„	67 34·2	..	3·3	..	− 0·6
„ 19, „ .	74	R	67 34·9	..	2·6	..	+ 0·1
					β = 2·7	Mean . .	± 0·2

The mirrors used to form an image of the Sun which can be viewed through the telescope were tested independently. We were, however, unfortunate in that on most of the days when we observed at Kew the Sun was invisible. The meridian mark also, which is used in the Kew measurements, cannot be seen from the lawn immediately in front of the Magnetic House, and we were, therefore, obliged to use a rod which was fixed at some distance, as nearly as might be in the meridian line in which the instrument was also placed. On one occasion (October 11, 1887) the latter adjustment was inaccurate, and a correction was applied deduced from the angle subtended at the pillar in the magnetic house by the centre of the tripod and the rod, and from the distances of the tripod and pillar from the rod. At Parsonstown the instrument was placed in the meridian line used for the Observatory of the Earl of Rosse, and a check on the declinometer was thus obtained. From what has been said it will have been seen that the determinations of the geographical meridian at Kew were not made under the most favourable conditions, but the following Table is sufficient to show that no important error attached to the indications of the instruments.

Azimuth of Kew Meridian Mark from Sun Observations.

Date.	Instrument.	Observer.	Azimuth.
			° ′
April 22, 1886	60	R	2 48·4
Oct. 11, 1887	61	T	2 49·9
„ 12, „	60	R	2 48·0
„ 12, „	„	„	2 50·1
„ 19, „	„	„	2 48·2
Mean	..	..	2 48·9
Value employed at Kew	..	..	2 48·7

At Parsonstown the difference between the reading for the fixed mark and the meridian, as calculated from a sun observation, was 1′·1.

We also thought that it would be desirable to compare the instruments under the normal conditions of use in the field. We therefore made three sets of observations at Ard Point, Stornoway; at Bunnahabhain, in the Sound of Islay; and at Stranraer. At the first station the results of the individual experiments made with each instrument were in good accord, but, as the declinations differed from each other by an amount very much greater than any possible error of experiment, viz., 18′, it was obvious that the station was affected by local magnetism. At Bunnahabhain we took the precaution to exchange positions; these observations were made on a patch of limestone, and were in agreement. At Stranraer the sky became overcast, and it was impossible to move the instruments after the first determination of the geographical meridian. The results show that the station was a good one, and we therefore give here the declinations obtained at Bunnahabhain and Stranraer. As we shall have to refer to them again, we reproduce only the data which bear upon the point under discussion. At Bunnahabhain the thread of instrument 61 was broken, and had to be replaced in the field, which accounts for the fact that the first observation made with it is not in very good accord with the others. The Greenwich mean time is reckoned throughout from midnight.

Bunnahabhain.—Aug. 25, 1888.

G. M. T.	Station.	Instrument.	Observer.	Declination.
h. m.				° ′
10 55	1	61	T.	22 44·0
11 36	2	„	„	22 46·0
10 29	2	60	R.	22 46·0
12 25	1	„	„	22 47·3
14 30	1	„	„	22 47·9
15 9	1	„	„	22 47·3

STRANRAER.—Aug. 28, 1888.

G. M. T.	Station.	Instrument.	Observer.	Declination.
h. m. 11 35	1	61	T.	o ′ 21 13·7
10 54	2	60	R.	21 12·5
13 6	2	,,	,,	21 13·0

During the survey we employed various chronometers, some of which were hired from Messrs. DENT, and others were lent to us by the late Professor BALFOUR STEWART, F.R.S., and by Captain WHARTON, R.N., F.R.S., Chief Hydrographer to the Admiralty, to whom our thanks are due.

We enjoyed an important advantage over our predecessors in that we were able to determine the rates of our chronometers frequently by comparison with Greenwich, by means of the 10 A.M. and 1 P.M. telegraphic signals, of which the former is sent to all post-offices in the kingdom. We have to thank Mr. PREECE, F.R.S., Chief Electrician to the General Post-office, and Mr. J. C. LAMB, the Head of the Telegraph Department, for their kindness in giving or obtaining for us permission to receive the signals. Many of the local post-office officials not only afforded us every facility for correcting our chronometers in accordance with their instructions, but gave us additional help in the selection of suitable stations. When possible, we received the signal every day, and rarely omitted more than two consecutive days. At places where the signal had undergone one or more re-transmissions, an error on this account was inevitable. By the kindness of the Earl of ROSSE, we were able to determine its magnitude on the occasion of our visit to Parsonstown. The time signal was observed both by Professor RÜCKER and Herr BÖDDEKKER, the Superintendent of Lord ROSSE's Observatory, who agreed exactly as to the apparent error of the chronometer. The value thus obtained differed by 4 sec. from that given by the observatory clock, the error of which was known from star observations. We are inclined to think that this amount was rarely exceeded, or the rates would have varied more widely than was actually the case. No special precautions against error had been taken at Parsonstown, whereas in many districts we had, when practicable, made arrangements at the central office that particular care should be taken on those mornings when we informed the authorities that we intended to receive the signal. We have on several occasions received at the same post-office signals sent by two different routes, involving re-transmission, and have never detected any appreciable difference.

When visiting outlying stations at sea we kept one chronometer on board in a fixed position. This served as the standard. Another instrument, which was frequently compared with it, was used for making the comparisons with Greenwich and for the work of the actual observations. At these periods of our survey longer intervals necessarily elapsed between successive receptions of the time signal, during which we

depended on the standard. With the exception of two or three cases, which are referred to in the detailed account of the observations at the stations affected, we do not think that any error exceeding a minute of arc can have been introduced into the declinations from uncertainty as to the true time.

Selection of Stations.

In selecting stations we have aimed at uniformity of distribution over the whole area under investigation, and, subject to this condition, have chosen places at which observations have previously been made. We have also avoided situations where disturbance by so-called local attraction was known to be great, except in special cases to be discussed hereafter. On many tours we have carried geological maps which have aided us in choosing sites.

In all we have made observations at 54 stations in Scotland, 102 in England, Wales,* and the Channel Isles, and 44 in Ireland. At many of these places which are counted as a single station our instruments have been set up in different localities in order to study or eliminate the local effects of magnetic rocks. These sub-stations were in some instances only a few yards and in others several miles apart. The more important of them are indicated by letters in the lists given hereafter, and if these are added they bring the number of stations up to 213. In addition to these we observed on the island of Canna at 23 places. If we took minor changes of position into account this total would be considerably increased, and as we have been able to use the observations made at Greenwich and Stonyhurst, and the data for Cherbourg and Berck-sur-Mer, furnished by the survey of France, recently completed by M. MOUREAUX (" Détermination des Éléments Magnétiques en France," par M. TH. MOUREAUX. Paris: GAUTHIER-VILLARS, 1886), our conclusions are based upon observations made at about 250 different places.

The average distance apart in normal districts is about 30 miles. At most of the principal stations we made two Dip observations, one with each needle, and a complete set of measurements for the determination of the Declination and Horizontal Force. At many places we determined the Declination twice, making two independent sets of observations on the Sun and the needle. In some cases only the geographical meridian or only the direction of the magnetic axis of the needle was re-determined. When time was short or the weather unfavourable the deflection experiment was omitted. The following Table shows the total number of observations made. By magnetic meridian we mean determination of the direction of the magnetic axis, which, when combined with the Sun observations or geographical meridian, gives the Declination. We do not include 23 Declinations taken on Canna by means of an azimuth compass and chart.

* Exclusive of five supplementary stations along the valley of the Wye, at which Dip observations only were made in 1889 (*vide* p. 84).

	Stations.	Dips.	Deflections.	Vibrations.	Geographical meridian.	Magnetic meridian.
Scotland	54	120	57	66	89	93
England and Wales	102	213	94	135	171	185
Ireland	44	89	35	58	84	84
Total . . .	200	422	186	259	344	362

Most of these observations were taken by one or other of us. We have, however, to thank Mr. A. P. LAURIE, now Fellow of King's College, Cambridge, for observations of Dip at 8 stations in Scotland. The stations on the West Coast of Scotland were, for the most part, visited in Dr. THORPE's yacht " Coventina." During the first year (1884) we generally observed together. Afterwards, in order to save time, we travelled separately.

Method of Taking the Observations.

The conditions and time of the observations necessarily varied, but the greatest number were made as follows :—Shortly before 10 A.M. we visited the post-office to correct our chronometers by the time signal from Greenwich. We then drove to the station which had been selected the night before or earlier in the morning. The first observation taken was the solar azimuth, which was finished about an hour before noon. This was followed by the declination, vibration, and deflection in this order. The dip was then determined, and, if time and weather allowed, we often repeated the azimuth and declination. The Sun observations were hardly ever taken within an hour of noon.

In a variable climate like that of the United Kingdom the weather often presents a serious difficulty. We carried with us a small tent, in which the dip and vibration observations could be made, and we were also provided with waterproof covers for the Magnetometer and Dip Circle. We were thus able to make the Dip observations, for example, during showers, while the Magnetometer, though outside the tent, was protected by its covering. The case containing the dip needles was carried in a box filled with soda-lime to prevent the axles being injured by rust.

During sunshine we shaded the deflecting magnet in the deflection experiment by a cardboard case, or by throwing a light piece of cloth over it. At 42 principal and subsidiary stations the force was determined by means of the vibration experiment only, the deflection being omitted and the moment of the magnet deduced from the values obtained at neighbouring stations.

The only part of the observations which requires special comment is the use of the mirror employed in the sun observations. The adjustment can be effected by means of the reflected image of the cross wires in the telescope. We found, however, that

in practice in the field this method is troublesome. It is difficult to see the image unless the mirror is very bright, and there is fairly strong sunshine. On days when the weather and sunshine are uncertain it is very annoying to have to spend time when the sun is visible in making preliminary adjustments. From and after our visit to Stornoway in 1884, *i.e.*, after observations had been made at about a dozen stations, we adopted a different plan. The mirror was frequently adjusted either indoors in accordance with the directions of the Admiralty ' Manual,' or by means of some elevated object in the field. In the latter case the error of parallelism was first got rid of by observing the object when the observer's back was turned towards it, and the mirror was nearly vertical. The mirorr was adjusted until the image appeared in the same position as before when it was reversed in its bearings. The object was then viewed by reflection when the instrument was turned through 180°, *i.e.*, when the mirror was nearly horizontal and the axis of the mirror was made perpendicular to the axis of the telescope. By repeating these processes twice or thrice a perfect adjustment was obtained, and it was found that under ordinary circumstances the instrument could be carried about from place to place for some time without this adjustment being seriously affected. To make certain, however, that all was right, we always (when possible) took observations of the sun both in the "front" and "back" positions, *i.e.*, with the mirror approximately horizontal and vertical and reversed the mirror in its bearings in both positions. By the latter precaution the error of parallelism (if any existed) was eliminated, and the observation thus afforded data for the calculation of the error of collimation, and for the corrections which in consequence of this error must be applied to the "front" observations.

By this method the actual adjustment of the mirror could be made at any time or place. If the time at our disposal was short, or the weather uncertain, the instrument was regarded as ready for use as soon as it was set up in the field, and thus no time was lost, but the observations were so taken that any error of adjustment could be calculated and allowed for. Lastly the agreement of the measurements made in the front and back positions gives a valuable test both of the adjustments and observations.

In the case of the observations made in 1888 with Magnetometer No. 60, the method of correcting the mirror by the image of the cross wires was again resorted to. Some modifications of the instrument which we have introduced (*loc. cit.*) now make this method practicable, and it was found to work well under the new conditions.

Diurnal Variation and Disturbance.

In the survey of 1857 corrections for diurnal variation and disturbance were applied to the declination observations only. These two quantities are intimately connected. The total deflections of the continuously recording instruments at any instant are the sums of the normal diurnal variations and the disturbances. If the latter are assumed to occur simultaneously and in equal intensity over an area such as that of the United

Kingdom, and if the G.M.T. at which the observations were made is known they may be determined from the curves obtained at Kew. On the other hand the magnitude of the diurnal variation at a given instant depends on the local time, and in the case of stations so far apart as Kew and Valentia, the diurnal variation of the declination would, between 10 and 11 a.m. G.M.T., differ by several minutes of arc.

We have therefore taken out from the Kew curves the deviations from the mean at the G.M.T. of our observations, and subtracting the corresponding value of the diurnal variations at Kew, have called these differences the temporary disturbances at the time of the observations.

We have assumed further, that the Kew mean curves of diurnal variations may be taken as applying to the whole of the United Kingdom, but we have, in all cases, used the correction corresponding to the local time. Thus the total correction is the algebraic sum of the diurnal variation at the local time, and of the disturbance registered at Kew at the G.M.T. at which the observations were taken.

As proof of the validity of this method, we have picked out all the stations in Ireland at which two or more observations were made at times when the diurnal variations differed by more than four minutes.

As full particulars are given later on, it is only necessary here to tabulate the corrections for diurnal variation and disturbance in addition to the final results. It may, however, be remarked that in the majority of cases the determinations of the geographical, as well as of the magnetic meridian, are quite independent, and that at Athlone, Gort, and Westport, the results given were obtained on different days. The letters v and Δ are used throughout this memoir to indicate the departure from the mean value of the element at the time of experiment, due to diurnal variation and to temporary disturbance respectively.

Station.	v.	Δ.	Corrected Declination.
Athlone	+ 6·3	+ 1·5	22 16·5
	− 3·9	0·0	22 16·8
Cavan	− 3·7	+ 1·5	22 26·1
	+ 6·2	− 4·0	22 26·7
	+ 3·5	− 5·0	22 27·4
Charleville	+ 4·9	+ 1·0	22 19·4
	+ 0·7	+ 2·5	22 19·1
Cork	0·0	0·0	22 3·8
	+ 5·8	− 1·0	22 1·7
Gort	+ 3·9	− 1·0	22 38·2
	− 4·2	+ 1·0	22 39·1
Lismore	− 1·0	0·0	21 54·1
	+ 6·4	+ 3·5	21 54·0
Westport	+ 0·7	0·0	23 5·5
	+ 1·8	− 0·5	23 4·2
	+ 6·3	+ 2·0	23 5·4

The following are the best stations for the application of a similar test to Scotland. The two sets of observations at Scarnish and Portree were taken on two different days. Those at Stornoway were on the same day, but at different stations. The observations at Bunnahabhain and Stranraer, which have been already discussed, are here arranged in order of time without reference to the station or instrument :—

Station.	$v.$	$\Delta.$	Corrected Declination.
Scarnish	$+ 2\cdot4$	..	24 49·8
	$- 3\cdot6$	$+ 3\cdot0$	24 51·8
Portree (2)	$+ 3\cdot3$	$+ 2\cdot0$	22 21·6
	$+ 3\cdot7$	$- 2\cdot0$	22 22·7
Stornoway, Ard Point (1)	$+ 2\cdot0$	$- 2\cdot0$	23 50·7
	$+ 6\cdot3$	$- 1\cdot0$	23 48·4
	$+ 4\cdot4$	..	23 50·7
Ard Point (2) . . .	$+ 2\cdot4$	$- 3\cdot0$	24 8·7
	$+ 6\cdot4$	$- 1\cdot0$	24 7·9
	$+ 4\cdot4$	..	24 8·3
Bunnahabhain . . .	$- 0\cdot1$	..	22 46·0
	$+ 1\cdot7$	..	22 44·0
	$+ 4\cdot3$	$- 2\cdot0$	22 46·0
	$+ 6\cdot3$	$- 1\cdot0$	22 47·3
	$+ 5\cdot7$	..	22 47·9
	$+ 4\cdot6$	..	22 47·3
Stranraer	$+ 1\cdot9$	$- 2\cdot0$	21 12·5
	$+ 4\cdot6$	$- 3\cdot0$	21 13·7
	$+ 6\cdot4$	$- 1\cdot0$	21 13·0

In some cases where there is an appearance of a regular change, such as would be caused if the diurnal variation was not properly estimated, it can be otherwise accounted for. Thus the first four observations at Bunnahabhain were independent in every particular, but the last two depend upon the same Sun observation as the fourth. This probably accounts for the somewhat closer agreement in these cases. The magnitude of the errors which would be introduced if the disturbance corrections were omitted can be estimated by noting that at Cavan the first and last observations when uncorrected give 22° 27′·6 and 22° 22′·4, those at Portree 22° 23′·6 and 22° 20′·7, and those at Scarnish 24° 49′·8 and 24° 54′·8 respectively, instead of the much more accordant numbers entered in the Table.

These results when corrected are quite as good as those we obtained when observing at Kew, and furnish a complete *a posteriori* justification of the method of correction.

The values of the diurnal variation employed for the year 1884–5 were furnished to us by Mr. WHIPPLE. For 1885–6 and 1886–7 we used the Tables given in the reports of the Kew Committee for those years ('Roy. Soc. Proc.,' vol. 41, p. 415, and vol. 43, p. 226).

These differed rather markedly as to the magnitude of the maximum variation from the curve employed for 1884–5, the difference amounting to about 1′·5. As the report

of the Kew Committee for 1887–8 was not published when the observations made in 1888 were reduced, we used the same Table as for the 1887 observations.

The question as to whether we should apply similar corrections to the other elements has been carefully considered, and we have on the whole decided to do so.

Though we have worked up the observations on the Horizontal Force so that each experiment gives two nearly independent results, the times at which these have been taken are practically identical, and they do not serve as a test of the advantage of applying corrections. In 18 cases we have repeated either the deflection or the vibration experiment (generally the latter) in the field on the same day. The differences between the results when uncorrected and corrected for diurnal variations and disturbance are as follows. They are expressed in terms of 0·0001 metric or 0·00001 C.G.S. unit.

TABLE I.

Station.	Uncorrected.	Corrected.
Loch Aylort	− 5	0
Ayr	− 4	− 4
Stornoway	− 3	− 5
	+ 3	+ 2
Cambridge	+ 1	+ 1
Northampton	+18	− 7
Stoke	− 5	− 6
Swansea	+ 3	− 1
Taunton	+17	+16
Thirsk	0	+ 1
Tunbridge Wells	+17	− 3
Worthing	+ 8	− 2
Ballywilliam	0	− 1
Bangor	+ 4	− 6
Coleraine	− 7	−11
Drogheda	+ 9	+ 3
Kildare	−10	+ 5
Londonderry	+ 6	+ 5

Probable difference { Uncorrected . . 5·9
Corrected . . . 4·0

The values of H obtained by us at Kew in 1887 also afford a test of the method of correction. The diurnal variation and disturbance are expressed in the following Table in terms of 0·0001 metric unit :—

Date.	Deflection.		Vibration.		H.	
	v_1.	Δ_1.	v_2.	Δ_2.	Observed.	Corrected.
September 30	+ 7	0	+ 3	0	1·8117	1·8112
October 11	2	0	− 2	0	1·8114	1·8114
,, 12	3	− 20	− 1	− 20	1·8096	1·8115
,, 13	− 5	− 20	− 10	− 10	1·8092	1·8114
,, 18	− 5	0	− 13	0	1·8114	1·8123
, 18	− 5	0	− 11	0	1·8117	1·8125
,, 19	− 4	− 10	− 10	0	1·8113	1·8125

As then the diurnal variation between the hours at which our observations were generally taken amounts to 0·0025 metric unit, and as the probable difference between the corrected semi-independent results at the stations referred to in Table I., p. 67, is ·0004 (the probable error being half that quantity), and as the probable error of the seven Kew observations is less than ·0004, there seems no doubt that the correction ought to be applied.

In the case of the Dip the diurnal range (which is more than 2′) exceeds the error of experiment, but the application of a correction for disturbance is more difficult and uncertain. It has to be deduced from the Vertical and Horizontal Force magnetograms and thus the chances of error are increased. The quantities to be dealt with are generally small, and the uncertainty of the readings taken from the curves unquestionably affects the results. We have not many data which furnish any *a posteriori* arguments of importance, as in most cases the times at which the two Dip observations were taken were separated by too short an interval for the diurnal variation to have altered much. The observations made at Kew in 1887 are, however, decidedly improved by correction.

Date.	Instrument.	Needle.	Dip observed.	v.	Δ.	Dip corrected.
September 30	74	1	67° 35·4′	− 0·7	+ 0·7	67° 35·4′
	74	2	34·7	− 0·8	+ 1·3	34·2
October 11	83	1	67 33·8	− 0·4	0	67 34·2
	83	2	34·9	− 0·5	0	35·4
,, 13	83	1	67 36·0	− 0·4	+ 1·4	67 35·0
	83	2	36·1	− 0·5	+ 1·6	35·0
,, 18	83	1	67 34·2	− 0·3	+ 0·3	67 34·2
	83	2	34·2	− 0·4	+ 0·3	34·3
,, 19	74	1	67 35·0	− 0·3	0	67 35·3
	74	2	33·5	− 0·3	− 0·7	34·5

The greatest differences in the corrected and uncorrected results are 1′·2 and 2′·6 respectively.

On the whole, then, we think the application of the correction for diurnal range is advantageous, but the correction for disturbance appears to be somewhat uncertain in its effect as applied to the field observations, and we have retained it chiefly for the sake of uniformity with our treatment of the other elements.

The formulæ used in the application of the corrections to the Horizontal Force and Dip were as follows : —

If H′, m' and H, m be the values of the Horizontal Force and the moment of the magnet deduced from the uncorrected and corrected observations respectively, and if ϕ and ψ be the total increments of the Horizontal Force due to diurnal variation and disturbance at the times when the deflection and vibration experiments are made,

$$\text{H} = \text{H}' - (\phi + \psi)/2 ;$$
$$m = m' \left\{ 1 + (\phi - \psi)/2\text{H}' \right\}.$$

As the disturbances of the Dips have to be deduced from those of the Horizontal and Vertical Forces, the calculations involved are rather troublesome.

If $d\theta$ be the change, expressed in minutes of arc, produced by increments dV and dH in the Vertical and Horizontal Forces respectively,

$$d\theta = a\,d\text{V} - b\,d\text{H},$$

where a and b are quantities which depend on the Dip and Horizontal Force at the station. Tables were prepared in which the values of these were entered for each complete degree between 66° and 72°, and for each tenth of a unit of Horizontal Force between 1·4 and 1·8. As they vary slowly, the value corresponding to any given Dip or Force could be readily determined, and the value of $d\theta$ was thus deduced.

Method of Tabulating the Observations.

We have ventured to modify considerably the methods ordinarily adopted of presenting the results of a survey such as our own. It is often very difficult for any one who studies the records of magnetic observations to learn anything as to whether the instruments were in good or bad adjustment, or whether the actual observations were careful or careless. Sometimes superfluous data are given which supply no information on these points. Thus, the quantities log mX and log m/X, or mX and m/X, which are sometimes tabulated in records of Force observations, add but little to their value. In an observatory, indeed, they ought to be nearly constant from month to month ; but as, in both of them, the changes in the value of the moment of the magnet and those in the Force are mixed up with the error of experiment, it is not easy to draw any definite conclusion from them. To test the observations, we should compare quantities the difference between which ought theoretically to vanish. In the case of a survey, log mX and log m/X must vary largely from station to station, and they are, therefore, quite worthless as tests of the observations.

There are now five British magnetic observatories working with instruments of substantially the same patterns, and we think it would be very desirable if an agreement could be come to as to the form in which the results of the observations are published. In our own case we have been anxious to check the numbers given by experiment in every way, as it is only thus that we are able to determine what reliance may be placed on observations at stations where time or weather prevented the repetition of one or more of the measurements.

The following is an example of the method of tabulation adopted in the case of the Declinations :—

Σ is the interval between the southing of the Sun and the mean of the times at which the "front" observations were made on the Sun's azimuth. It is taken as positive if the observations were made after noon.

Alt. signifies the altitude of the Sun above the horizon at the same time.

μ is the correction made by means of the "back" observations to the geographical meridian determined from the front observations alone. This quantity serves to indicate the order of the error that may have been introduced by the omission of the back readings on some occasions.

G.M. is the reading on the horizontal circle which corresponds to the geographical meridian. In most cases two determinations of the geographical and magnetic meridians were made at an interval of some hours. In this time the tripod may have shifted slightly, owing to unequal sinking of the ground, warping of the legs, &c. Hence, if no fixed mark was read, each geographical meridian must be regarded as corresponding only to the magnetic meridian which was observed nearly at the same time as itself. As it is often difficult to select a suitable fixed mark, we have, whether a fixed mark was observed or not, regarded each reading for the geographical meridian as corresponding to that for the magnetic meridian which was nearest to it in point of time.

In the case of the magnetic meridian (the reading for which is indicated by M.M.) we tabulate the following data :

(1) The G.M.T. at which the observation was made reckoned from midnight.

(2) Half the difference between the readings when the magnet is erect and inverted is the angle between the magnetic and geometrical axes of the magnet. If then from this quantity (ξ) we subtract its mean value as determined from all the observations made in the same year (ξ_0), we obtain a measure of the error due to inaccuracy of reading or adjustment, and to imperfect compensation of the diurnal variation or of any disturbance which may have been occurring at the time.

(3) ω is the error which would have been caused had the torsion of the thread not been allowed for.

(4) v and Δ are the deviations of the element from its mean value due to diurnal variation and disturbance respectively. The first, as has been already explained, is calculated for local time from the Kew curves of diurnal variation, the latter is the

disturbance registered by the Kew curves at the same G.M.T. as that at which the observations were taken.

(5) δ (obs.) is the value of the Declination obtained by subtracting the reading for the magnetic from that for the corresponding geographical meridian.

(6) σ is the secular correction to Jan. 1, 1886, obtained as is hereafter explained.

(7) δ (red.) is the mean value of the Declination reduced to the epoch of the survey.

As an example we take Athlone, where observations were made on two different days. On May 8, no back observations were taken, so that μ could not be calculated.

DECLINATION.   Athlone.   Observer, T. E. T.   Magnetometer 61.

Geographical Meridian.

Date, 1887.	Σ.	Alt.	μ.	G.M.
	h. m. s.	° ′	′	° ′
May 8	+1 40 59	48 42	..	338 38·1
„ 9	−3 43 58	33 51	− 0·2	210 31·2

Magnetic Meridian.

Date.	G.M.T.	ξ − ξ₀.	w.	v.	Δ.	M.M.
	h. m.	′	′	′	′	° ′
May 8	12 39	+0·2	−0·1	+6·3	+1·5	316 21·6
„ 9	9 9	−0·4	0·0	−3·9	0·0	188 14·4

Declination.

δ (obs.).	Mean.	σ.	δ (red.).
° ′	° ′	′	° ′
22 16·5	22 16·6	+10·1	22 26·7
22 16·8			

In the case of the Horizontal Force observations the deflections were always taken when the distances between the centres of the magnets were 0·3 and 0·4 metre. If then l_1 and l_2 are the common logarithms of the values of m/H obtained at the two distances, the factor P used in the corrections may be calculated by the formula

$$P = 0·4737 (l_1 − l_2) − 1·947 (l_1 − l_2)^2$$

of which the second term is very small. ('Nature,' Aug. 13, 1887, p. 366; Sept. 29, p. 508; Dec. 8, p. 127; and Jan. 19, 1888, p. 272).

The following table gives the mean values of P obtained for each year during the survey.

	Magnetometer.	
	60	61
1884	0·000817	..
1885	0·000866	..
1886	0·000828	0·000753
1887	0·000809	0·000692
1888	0·000800	0·000706

The ordinary method of determining the Inclination with two needles gives two independent values, the agreement of which furnishes a test of the accuracy of the observations. It is, however, usual to give only one value of the Horizontal Force deduced from the means of the values of m/H and mH calculated from the deflection and vibration experiments. The constancy or regular change of the value of m affords a means of determining whether the observations are sufficiently good, but it seems to us that nothing is for this purpose so satisfactory as the agreement or disagreement between two independent experiments. The results of the ordinary deflection and vibration observations can readily be presented in such a form as to satisfy this condition with only an insignificant addition to the labour of reduction.

The vibration experiment furnishes twelve independent determinations of the time of 100 vibrations, six of which are taken when the scale of the vibrating magnet is apparently moving to the right, and six when it appears to be moving to the left. If we take the mean of the six observations furnished by the first and last three of each of these groups, we have two independent determinations of the vibration period, based in each case on six observations, of which three were taken when the movement appeared to be to the right, and three when it appeared to the left.

If we agree always to combine the first and second of these groups with the values of m/H obtained when the distance between the magnets is 0·3 and 0·4 metre respectively, we obtain from each experiment two values of m and H. Except in so far as errors may arise in the determination of the temperature, or in the adjustment of the instrument (levelling, &c.), these are absolutely independent, and thus furnish a satisfactory check on the accuracy of the observations. It would, however, be hardly worth while for the sake of this advantage to go through the labour of repeating the reduction of the vibrations. As the two vibration periods are very nearly the same, this is unnecessary. Let T be the time deduced from all the observations, and $T \pm dT$ the times given by the first and second groups of observations selected as above described. Then if D be the value of m/H given by the deflection experiment,

$$H = \sqrt{\frac{\pi^2 K}{T^2} \div D},$$

$$\frac{dH}{H} = -\frac{dT}{T},$$

and in like manner

$$\frac{dm}{m} = -\frac{dT}{T}.$$

Thus, if we use the mean of the times for deducing two values of H and m from the two deflection experiments (as is done in the ordinary method of reduction) we may obtain the independent values of H and m by altering the means in the ratio of the independent to the mean times.

Thus, at Horsham, the mean time of vibration was $4^s \cdot 1650$, and using this the deflections at the two distances gave

$$H = 1 \cdot 8365 \text{ and } 1 \cdot 8366$$

and

$$m = 0 \cdot 0091956 \text{ and } 0 \cdot 0091952$$

respectively.

The actual periods of vibration for 100 oscillations observed were—

Apparently—	
Moving to right.	Moving to left.
m. s.	m. s.
6 56·6	6 56·6
56·5	56·6
56·5	56·5
56·5	56·6
56·3	56·4
56·4	56·5

Taking the means of the observations recorded above and below the line we get for the time of one oscillation

$$4^s \cdot 1655 \quad \text{and} \quad 4^s \cdot 1645.$$

These independent times differ from the mean by $0 \cdot 012$ and $- 0 \cdot 012$ per cent. respectively.

Hence, altering the values of H and m in the same proportion, but in the opposite direction to that indicated by the signs of these quantities, we get

$$H = 1 \cdot 8363 \quad \text{and} \quad 1 \cdot 8368,$$

and

$$m = 0 \cdot 0091945 \quad \text{and} \quad 0 \cdot 0091963.$$

These quantities are quite independent, except as regards the temperature corrections and instrumental adjustments, and their difference is a measure of the errors of observation and imperfect compensation of the diurnal variation and disturbance. In tabulating the Force observations we give the G.M.T. at which the deflection and vibration observations were made, and the extreme temperatures during the progress of the experiments, the observed deflections and the independent times calculated as above described.

The agreement of the times of vibration serves to test the vibration experiments. As a measure of the accuracy of the deflections we take the ratio of the two values of m/H calculated by combining the mean time of vibration with each of the two deflections.

The logarithms of these two quantities occur in the calculations, and all that is necessary is to subtract them and look out the corresponding natural number. As this quantity is very nearly unity we multiply by 100 and give the fractional part only. The symbol ϵ is thus the percentage difference between the results given by the two deflections when combined with the mean time of vibration.

The following form, which is that which has been used, explains itself.

The results of the deflections differed by 0·008 per cent., the times of vibration by 0^s·001. There was no disturbance, and the Horizontal Force, owing to diurnal variation, was 0·0007 and 0·0003 metric unit above the average, at the times which correspond to the middle of the deflection and vibration experiments respectively.

The corrected values are obtained from the mean values by applying the corrections for diurnal variation and disturbance.

HORIZONTAL FORCE.   Horsham.   Observer, A. W. R.   Magnetometer 60.

Deflection.

Date, 1888.	G.M.T.	t.	Observed deflection.	ϵ.	v_1.	Δ_1.
April 21	h. m. 14 44	° 11·8 to 10·7	° ′ ″ 21 51 55 9 0 8	+ 0·008	+ 0·0007	0

Vibration.

Date.	G.M.T.	t.	T.	v_2.	Δ_2.
April 21	h. m. 13 34	° 16·5 to 17·4	s. 4·1655 4·1645	+ 0·0003	0

Values of m.

Observed.	Mean.	Corrected.
0·0091945 0·0091963	0·0091954	0·0091965

Values of H.

Observed.	Mean.	Corrected.	σ.	Reduced.
1·8363 1·8368	1·8365	1·8360	− 0·0051	1·8309

At places where the deflection experiment was not made, the moment of the magnet was calculated from the other moments observed during the same tour. It was assumed that the strength of the magnet diminished regularly, and a linear expression was determined by equations of condition which gave the rate of decrease. A careful analysis of the Scotch observations, which were the first reduced, proved that this method was satisfactory.

The greatest difference between observation and calculation occurred at Kirkwall, and amounted to 0·234 per cent. It is certain that in this case the difference is not due to errors of experiment, but to a real change in the moment, as the observations obtained at the next station (Lerwick) confirmed it. We are also able to assign a probable cause for the alteration, as the "Coventina" was caught in a "roost" off Stromness, and experienced rather rough treatment, during which the magnet may have been jarred.

The mean difference (irrespective of sign) at all the stations was 0·057 per cent.; and if we put aside about one-fifth of the whole number of stations at which, owing to bad meteorological or magnetic weather, or some similar cause, the conditions under which the observations were taken were not very favourable, the mean difference at the remainder was 0·028 per cent.

It must, of course, be remembered that this quantity is not a measure of the error of experiment, but of the algebraical sum of that error, and of the deviation of the rate of decrease of the moment of the magnet from perfect uniformity. The conclusion we came to from the discussion of the Scotch observations was that if the observed moment differed from that calculated by the linear formula by 0·1 per cent., either a real change had taken place in the magnet or the observation had been affected by some disturbing cause, which it was in general easy to specify. In the later tours the moments of the magnets changed very slowly, and it was sufficient to take the mean

of the moments obtained at one or two stations immediately preceding and following that at which the deflection experiment had been omitted. The total number of stations at which vibrations only were observed was 42.

In the case of the Dip observations the correction for diurnal variation was deduced from the Kew curves for the diurnal variation of the Horizontal and Vertical Forces, which were supplied to us by Mr. WHIPPLE.

The disturbance corrections were calculated as previously described.

The following form was used in working up the results :—

INCLINATION.   Bunnahabhain.   Observers: Needle 1, A. W. R. ;
Needle 2, T. E. T.   Dip Circle, DOVER, 83.

Date 1888.	Needle.	G. M. T.	Observed.	v.	Δ.	Corrected.
Aug. 25 . .	1	h. m. 12 46	$\overset{\circ}{70}$ $\overset{'}{40}$·2	+ 0·1	+ 0·7	$\overset{\circ}{70}$ 39·4
	2	14 38	70 39·4	− 0·4	. .	70 39·8

Mean.	σ.	Reduced.
$\overset{\circ}{70}$ 39·6	+ $\overset{'}{3}$·4	$\overset{\circ}{70}$ 43·0

All the observations were tabulated in these forms, and we propose to deposit copies of them with the Royal Society. In this paper we give fewer data. It will, however, be more convenient to describe these under the heading "Results of the Observations," p. 93, and we now proceed to use the tabulated results in a discussion of the accuracy of our work.

Errors of Experiment.

The completion of so large a number of observations as those involved in our survey, carried out for the most part by two observers with different sets of instruments, affords a good opportunity of testing the accuracy of experiment in the field, especially as we have followed a regular practice of determining both the geographical and magnetic meridians by means of independent experiments made at an interval of several hours. Again, our method of tabulating the results of the Force observations furnishes a good test of the various parts of the experiments, and, as will be seen hereafter, we are able, in the case of the Dip observations, to arrive at conclusions as to the small errors due to the imperfections even of the excellent needles supplied by Mr. DOVER.

In the first place then, we take as the error of a Declination, the difference from the mean of the values obtained at the same station at an interval so small that the secular correction is not involved (*i.e.*, a few hours or days), when both the geographical and magnetic meridians are determined independently.

The error of a Force observation is taken as half the difference between the two *independent values,* calculated as above described.

Lastly, the error of a Dip observation is half the difference between the results given with needles 1 and 2.   In the case of the last two, it is evident that both quantities may be affected with small errors which are not thus detected, such as that due to uncertainty as to temperature, in the case of the Forces, and errors of setting in the magnetic meridian, in that of the Dips.

Taking, however, these quantities and treating them as though all the results in each group were measures of the same quantity, we get the following values of the probable error :—

	Number of observations.	Probable error.
Declination	97·5	0'·699
Horizontal Force . . .	196	0·00028 (M.U.)
Dip	190	0'·15

In this Table, we count each double observation as one.   The fraction, in the case of the Declination, is due to the fact that, at some stations, an odd number of observations was made.

It is useful to analyse the observations still further.   In so doing, we have generally treated Scotland as a whole.   During our earlier tours we observed together, and sometimes one observer would take the geographical and the other the magnetic meridian, one would observe the deflection and the other the time of vibration.   It is, therefore, difficult to separate the results.   In England and Ireland we always observed apart; and, with the exception of a short tour in 1886, when Dr. THORPE used Dip Circle No. 74, we always used different instruments.   The mean values of the quantities are taken *irrespective* of *sign,* so that they indicate the average value whether positive or negative.

The definitions of the quantities used as tests, are given on p. 70.   The symbol dS indicates the mean error of the element.

DECLINATION Observations.

	Observer.	Instrument.	Mean values of			
			μ.	$\xi - \xi_0$.	ω.	dS.
Scotland	R. and T.	60 and 61	$\pm 0\cdot31$	$\pm 0\cdot39$	$\pm 0\cdot75$	$\pm 0\cdot77$
England	T.	61	$\pm 0\cdot39$	$\pm 0\cdot32$	$\pm 0\cdot28$	$\pm 0\cdot71$
	R.	60	$\pm 0\cdot45$	$\pm 0\cdot35$	$\pm 0\cdot39$	$\pm 0\cdot85$
Ireland	T.	61	$\pm 0\cdot48$	$\pm 0\cdot24$	$\pm 0\cdot37$	$\pm 0\cdot51$
	R.	60	$\pm 0\cdot39$	$\pm 0\cdot32$	$\pm 0\cdot42$	$\pm 0\cdot63$

Roughly then, the average errors due to imperfect adjustment of the mirror, to inaccuracy in determining the magnetic axis of the magnet, and to torsion, ought not, in each case, to exceed about 0·4 of a minute of arc, assuming that no correction was attempted, as would be the case, for instance, with regard to μ, if no back observations of the sun were taken. As in the great majority of cases, all three were corrected by front and back observations on the sun, by observing the magnet erect and inverted, and by determining the torsion, we may be sure that the actual errors were much below these amounts. It seems, therefore, probable that the observations are affected with small errors which are not so capable of correction, for the mean and probable values of dS appear to prove that it is an even chance that each of two independent declinations differ from their mean by more than 0′·7.

In the following Table we give a similar analysis of the Force observations. The quantity $\epsilon/2$ is the percentage difference of either deflection experiment from their mean.

dT is the average difference in seconds between each "independent" determination of the time of a single oscillation (as defined on p. 73) and their mean.

dH is the average difference between each independent determination of the Horizontal Forces and the mean of the two involved in each complete experiment. In all cases the averages are, of course, taken without reference to sign.

HORIZONTAL Force.

	Observer.	Instrument.	Mean values of		
			$\epsilon/2$.	dT.	dH.
			Per cent.	s.	M.U.
Scotland . .	R. and T.	60 and 61	$\pm 0\cdot041$	$\pm 0\cdot00043$	$\pm 0\cdot00033$
England . .	T.	61	$0\cdot031$	$0\cdot00040$	$0\cdot00030$
	R.	60	$0\cdot023$	$0\cdot00035$	$0\cdot00025$
Ireland . .	T.	61	$0\cdot033$	$0\cdot00032$	$0\cdot00028$
	R.	60	$0\cdot017$	$0\cdot00040$	$0\cdot00022$

We have analysed the dips in a somewhat different way.

If one needle tends to give a slightly higher or lower result than the other, the mean value of the difference of the results, always taken in the same order—viz., dip given by Needle 1, minus dip given by Needle 2—ought to give the bias or measure of the difference due to imperfections in their construction. If this quantity be subtracted from the difference of the results in each observation, we get the error of experiment, which is the difference due to imperfection in observing and correcting. The mean of the last quantities, taken without reference to sign, is the mean error of experiment.

Thus, if we take the difference as positive when Needle 1 is the higher, out of 23 stations in Ireland at which observations were made with the same instrument (74), it was positive at 15 and negative at 8. The mean of all the differences showed that, on the average, Needle 1 gave results 0′·31 higher, and Needle 2 results 0′·31 lower, than the mean of the two needles; so that there is evidence of a *bias* which would make Needle 1 give results 0′·6 higher than Needle 2. That this number is worthy of credit is shown by the close agreement of the values obtained with the same instrument in England, and by the practical identity of all the numbers obtained for Dip Circle 83 in England, Scotland, and Ireland.

The axles of the needles of Dip Circle No. 74 appear to have undergone some slight alteration between the end of 1885 and the beginning of 1886, but there seems no doubt that since that date Needles No. 1 in the two instruments have read about 0′·3 and 0′·2 respectively above the mean of the two needles.

The following Table gives a summary of the results :—

INSTRUMENT 74.

	Observers.	Number of stations, with difference.		Bias. Upper sign, Needle 1. Lower sign, Needle 2.
		Positive.	Negative.	
Scotland . .	T., R., and L.	18	31	∓ 0·25
England . .	T.	8	2	± 0·25
	R.	24	12	± 0·33
Ireland . .	R.	15	8	± 0·31

INSTRUMENT 83.

	Observers.	Number of stations, with difference.		Bias. Upper sign, Needle 1. Lower sign, Needle 2.
		Positive.	Negative.	
Scotland . .	T. and R.	6	3	± 0·19
England . .	T.	25	16	± 0·20
Ireland . . .	T.	11	7	± 0·20

An analysis of the errors and corrections such as we have carried out appears to constitute a very delicate test of the observer and instrument combined. Thus it is curious to note that (although the differences are small) one of us has on the whole been more successful with the Declinations and the other with the Force observations. As has already been remarked, the observations in Scotland were often divided between us, so that we do not attempt to analyse them, but both in England and Ireland Dr. THORPE's Declinations agree the better by about 0'·26, and Professor RÜCKER's Forces by about 0·00011 metric unit. The facts that in Ireland Professor RÜCKER's mirror appears to have been generally in the better adjustment, and that in England Dr. THORPE's observations of the period of oscillation of the needle were the better, indicate that the causes of the differences in the results were probably a more perfect management of the torsion of the thread in the Declination observations, and a slight superiority in the deflection observations in the Force measurements.

It will, we think, be granted that in all cases the observations are as good as the occasion requires, and we discuss these minute differences not because we think they produce any appreciable effect on the results of our survey, but as an illustration of the fact that our method of tabulating the observations, conveys information as to the personal equations of the observers which is not afforded by the results as ordinarily presented.

We think it possible that if we had been able to carry out the work in a more leisurely manner, to observe azimuths only when the Sun was low in the heavens, and to wait for fine weather, the results might have been in closer accord, but on the other hand we do not think that any improvement so obtained would have been of practical importance. The uncertainty of the value of the secular correction at any given station, and the changes produced in disturbed districts by a slight alteration in the position of the instruments, are far more important than any residual errors by which our observations may possibly be affected.

Secular Corrections.

The secular correction is often appreciably different at neighbouring stations. Thus the annual change in the Dip in the years 1883–4 and 1884–5 was 1′·8 and 1′·7 at Greenwich, and 1′·4 and 1′·2 at Kew.

The differences in the Declination change are still more marked, as the following Table shows.

Year.	Declination.		Annual variation.	
	Greenwich.	Kew.	Greenwich.	Kew.
1880	18° 32·6′	18° 59·0′	5·5′	8·5′
1881	27·1	50·5	4·8	5·7
1882	22·3	44·8	7·3	4·8
1883	15·0	40·0	7·4	8·0
1884	7·6	32·0	5·9	7·3
1885	1·7	24·7	. .	. .

It will be noted that the annual variations differed in 1880 by 3′·0, and that the average difference irrespective of sign is 1′·4.

It does not therefore appear advisable to reduce the observations to one epoch by means of the annual variations as determined at a single observatory.

On the whole a better result will probably be attained if we collect all the evidence at our disposal and deduce from it an average secular change.

The comparison of our own observations with those of Mr. WELSH affords the means of determining this quantity for Scotland, and proves that the rate of decrease of the Dip is less in the northern than in the southern stations, a result which is in accord with previous observations.

The mean decrease in Scotland between 1837 and 1857 was 1′·94. ('Brit. Assoc. Rep.,' 1859, p. 169.) In the nearly corresponding interval between 1837 and 1860 it was 2′·05 on the northern border of England and 2′·68 on the south coast. ('Brit. Assoc. Rep.,' 1861, p. 260.)

Unfortunately very few Dip observations were made in England and none were made in Ireland in the 1857 survey, but there are a number of stations common to the surveys of 1837 and 1886. We can thus determine the secular decrease during the last 50 years. In the following Table the stations are grouped according to their geographical distribution, the districts being North and South Scotland, North, Central, and South England and Wales, and North and South Ireland.

The data for the 1837 survey are taken from the paper by the late Sir EDWARD SABINE, already cited ('Phil. Trans.,' 1870, Vol. 160, p. 271).

TABLE II.—Mean Annual Decrease of Inclination between 1837 and 1886.

Station.	Mean Annual Decrease.	Station.	Mean Annual Decrease.
Lerwick	1·22	Malvern	2·33
Aberdeen	1·58	Brecon	2·24
Kirkwall	1·41	Aberystwith	2·27
Wick	1·47	Harwich	2·02
Golspie	1·48	Clifton	2·25
Inverness	1·57	Swansea	2·22
Fort Augustus	1·44		
		Kew	2·04
Edinburgh	1·43	Ilfracombe	2·04
Glasgow	1·53	Salisbury	2·28
Helensburgh	1·38	Dover	2·10
Campbelton	1·70	Exeter	2·06
Cumbrae	1·23	Ryde	2·26
Berwick	1·77	Weymouth	2·32
		Plymouth	2·26
Alnwick	1·67	Falmouth	2·46
Newcastle	1·75		
Carlisle	1·96	Londonderry	2·04
Whitehaven	1·77	Strabane	1·87
Thirsk	1·88	Bangor	1·98
Scarborough	1·80	Armagh	2·09
Manchester	2·10	Enniskillen	2·04
Birkenhead	1·98	Ballina	2·06
Holyhead	2·02	Westport	2·12
		Clifden	2·13
Cromer	1·81		
King's Lynn	1·96	Galway	1·99
Nottingham	2·06	Dublin	2·06
Pwllheli	2·04	Limerick	2·21
Shrewsbury	1·70	Waterford	2·31
Birmingham	2·10	Killarney	2·45
Cambridge	2·02	Cork	2·32
Lowestoft	1·84	Valentia	2·50

There is a steady increase as we pass from north to south, and from east to west. This is brought out in the next table, in which the mean decrease is given for each district, and also for the stations in the easterly and westerly parts of it.

District.	Number of Stations.	Mean annual decrease of inclination between 1837–1886 for		
		Whole district.	Western half.	Eastern half.
North of Scotland	7	1·45	1·49	1·42
South of Scotland	6	1·51	..	..
North of England and North Wales	9	1·88	1·97	1·77
Midlands and Wales	14	2·05	2·10	2·00
South of England	9	2·20	2·24	2·17
North of Ireland	8	2·04	2·07	1·98
South of Ireland	7	2·26	2·37	2·15

In the case of twelve Scotch and six English stations we are able to compare the rate of change in the two periods 1837–1857 and 1857–1886.   The data are taken from the sources above described.

Station.	Mean annual decrease of inclination.		Ratio.
	1837–1857.	1857–1888.	
Lerwick	1·65	0·96	1·72
Aberdeen	2·02	1·33	1·52
Kirkwall	1·97	1·07	1·84
Wick	2·02	1·11	1·82
Golspie.	1·68	1·38	1·22
Inverness	1·92	1·37	1·40
Fort Augustus	1·80	1·21	1·49
Edinburgh	1·82	1·14	1·60
Glasgow	1·80	1·42	1·27
Helensburgh.	1·48	1·32	1·12
Campbelton	1·99	1·39	1·43
Cumbrae	1·53	0·94	1·63
Berwick	2·48	1·33	1·87
Scarborough	2·02	1·63	1·24
Stonyhurst	..	1·75	..
Cromer	2·22	1·44	1·54
Cambridge	2·50	1·58	1·58
Lowestoft	2·17	1·54	1·41
Kew	2·65	1·62	1·63
St. Leonards	..	1·82	..
Plymouth	2·73	1·89	1·44

The mean ratio for North Scotland is 1·57, for South Scotland 1·49, and for England 1·47.

These figures prove that the ratio of the mean annual decrease, in the intervals 1837–57 and 1857–87 has been nearly constant all over Great Britain, but that the rate of change has been rather more rapid in the north.

If we assume that the ratio has been 1·5 for England and Ireland, we may deduce x, the mean annual decrease between 1857–87, from the corresponding quantity given in Table II., p. 82, for the interval 1837–87, by the formula

$$1\text{·}5 \times 20x + 30x = 50d,$$
$$x = 5d/6.$$

The same quantity is determined for stations in Scotland by direct experiment by the comparison of our own observations with those of Mr. WELSH.   The results are given in the following Table.

Mean Annual Decrease of Inclination in Scotland between 1857 and 1884–8.

Station.	Mean annual decrease.	Station.	Mean annual decrease.
Lerwick.	0·96	Fort Augustus	1·21
Kirkwall	1·07	Aberdeen	1·33
Stromness	1·27	Pitlochrie	1·40
Thurso	1·15	Oban.	1·27
Wick	1·11	Port Askaig	1·38
Loch Inver.	1·32	Lochgoilhead	1·12
Stornoway	0·90	Edinburgh.	1·14
Callernish	0·99	Glasgow	1·42
Golspie	1·38	Berwick.	1·33
Banff.	1·33	Cumbrae	0·94
Elgin	1·30	Campbelton	1·39
Inverness	1·37	Ayr	1·56
Kyle Akin	1·12	Dumfries	1·45
Dalwhinnie	1·41	Stranraer	1·54

Using these figures and the numbers calculated as above described from Table II., p. 82, we get the mean annual decrease between 1857 and 1886 in different parts of the United Kingdom, as follows :—

District.	Mean annual decrease between 1857–8 and 1884–7.
N. of Scotland	1·20
S. of Scotland	1·36
N. of England and N. Wales	1·57
Midlands and Wales	1·71
S. of England	1·84
N. of Ireland	1·68
S. of Ireland	1·90

Although we give these numbers as showing the mean results for districts of considerable size, the figures obtained at the individual stations appear worthy of careful study.

The observed values of the annual decrease for 1857–87 for Scotland, and those calculated for the same period for England and Ireland from the 1837–87 surveys are inserted in the accompanying map (fig. 1).

The general increase from east to west is evident, but in Great Britain there are some stations the annual decrease at which is less than at those which are nearest to them on the eastern side.

To test whether the differences were accidental a number of supplementary stations were taken on the Wye.

At Malvern and Brecon the calculated secular rates between 1857 and 1886 are — 1′·94 and — 1′·87, and at Clifton and Cardiff — 1′·87 and — 1′·85. In each case

the smaller value occurs at the western station.   Between them lie Hereford, Ross, and Chepstow, and the rates obtained from observations at these in April, 1889, are − 1′·85, − 1′·83, and − 1′·84, which are in all cases less than those at Malvern and Clifton.   These observations then support the view that there is a real maximum followed by a minimum rate of change of Dip in this neighbourhood.

Fig. 1.
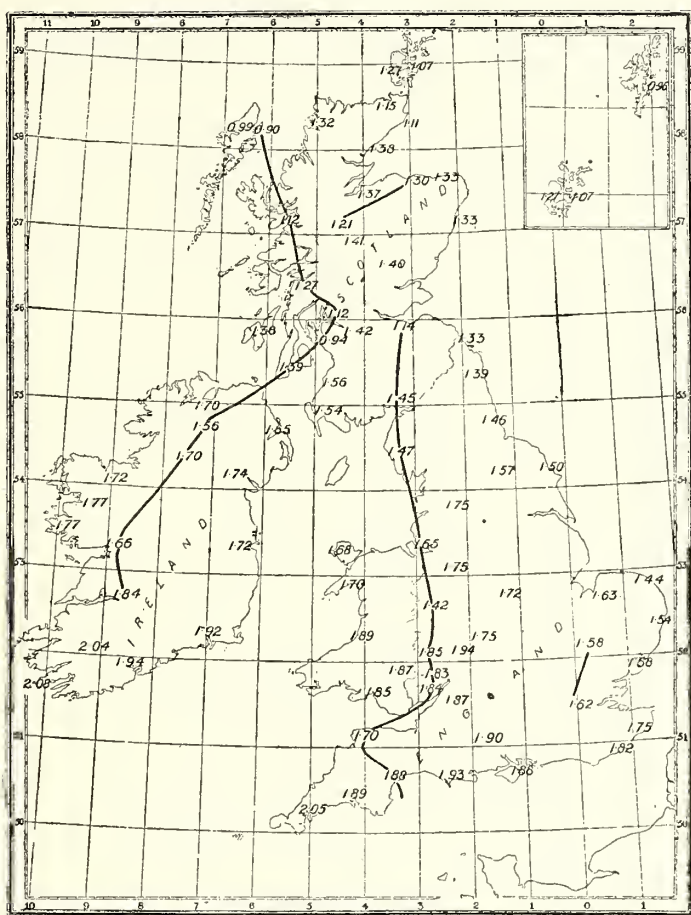
Secular change of Dip, 1858–86.

This induces us to show in fig. 1 that if all stations at which there is a maximum rate of change are joined, we get two long lines running from Edinburgh to Exeter, and from Stornoway to Limerick.   The stations are not sufficiently numerous to enable us to draw any certain conclusions about them, but the point ought not to be lost sight of when the survey is repeated.

For the present we note only that of the three pairs of neighbouring abnormal stations at which observations were made in all three surveys, all three are abnormal (in the sense that the rate of change is greater at the easterly station) whether we take the interval 1837–87 or 1857–87.   In both cases the rate at Inverness is greater than at Fort Augustus, at Glasgow than at Cumbrae, and at Berwick than at Edinburgh.

The distribution of these stations affords, at all events, *prima facie* evidence that the differences between them are not accidental. That they cannot be due to errors of measurement is evident when we remember that an error of 0'·1 in the annual rate of change determined from experiments separated by an interval of fifty years would require an error of 5' in one or other of the observations, or of 2'·5 in both. We therefore decided to regard the figures in fig. 1 as giving the true rate of secular change of Dip between 1857 and 1887 at the stations indicated.

The determination of the mean rate of decrease of Dip between the years 1857 and 1887 does not, however, give the value of that quantity during the period of our survey. As the rate is diminishing it would be less in recent than in earlier years.

The evidence we are able to collect on this head is confined to data from Greenwich, Kew, and Stonyhurst, and to a few stations at which we observed both near the beginning and near the end of our survey. In the following Table we have entered such facts as were known to us at the time when the progress of the reductions made it necessary to come to a decision on this point.

Our own observations and those taken at Stonyhurst are corrected for diurnal variation and disturbance. The Table illustrates the importance of these corrections when conclusions are to be drawn from observations between which the interval of time is small.

TABLE III.

Station.	Date.	Dip.	Date.	Dip.	Interval in years.	Secular rate of change.	Assumed rate.
Loch Aylort .	Sept., 1884	71 24·0	Aug., 1888	71 22·3	4	− 0·4	− 1·0
Kerrera . . .	Aug., 1884	70 51·1	Aug., 1885	70 48·7	1	− 2·4	− 1·2
Stranraer . .	Sept., 1884	70 14·1	Aug., 1888	70 11·1	4	− 0·8	− 1·4
Stonyhurst .	1884	69 16·2	1887	69 9·0	3	− 2·4	− 1·6
Kew	1884	67 39·5	1888	67 36·4	4	− 0·8	− 1·5
Greenwich . .	1884	67 30·2	1886	67 27·0	2	− 1·6	− 1·5
Leeds . . .	Sept., 1886	69 9·7	Dec., 1888	69 6·9	2·2	− 1·3	− 1·5
					Mean .	− 1·4	− 1·4

Note.—Stornoway is omitted because the observations were made in 1884 in the Castle grounds and in 1888 on Ard Point.

These figures illustrate the fact that there may be a considerable discrepancy at any particular station between the actual secular change at a given time and that deduced from the mean taken over a long series of years. On the whole, we think it probable that the present rate of change of Dip is somewhat less than its mean value between 1857–87, and we have, therefore, reduced the numbers for that period by one tenth, neglecting fractions of a tenth of a minute. This *assumed rate* gives a correct mean rate for the stations referred to above in Table III., though it is considerably too high for some and too low for others. It is probable that in the case of Kerrera the observed rate of change is not quite correct. The site is subject to local

attraction, and a single pair of observations made at so short an interval as a year is hardly sufficient to rely on. The results from Stonyhurst, however, prove that the assumed and actual rates of change may differ by 0'·8, and that, thus, Dips reduced to January 1, 1886, from the end of 1888, may be about 2'·5 wrong. To prevent, as far as possible, any effect on the general run of the isoclinal lines from this uncertainty, we put down on a map all the assumed rates at places for which they could be determined, and took the rate at any other place to be that given by the nearest of these stations.

The rates of decrease thus obtained for the south of England are considerably less than those assumed by M. MOUREAUX for the north of France, which do not differ much from — 2'·7. These values are obtained for the interval 1858–1885 by comparing M. MOUREAUX's observations with LAMONT's map. We think, however, that the observations of the Rev. S. J. PERRY, F.R.S., reduced to the epoch, January 1, 1869, prove that in France as in England the rate of change of Dip has been diminishing.

Twelve stations were common to Father PERRY and M. MOUREAUX; at one of these, Marseilles, the station of the earlier observer was no doubt seriously affected by local disturbance. The rate of change for the epoch 1858–69 is much larger than for any other station, and for 1869–85 it is much too small. Rejecting this observation we think the following table shows that M. MOUREAUX's numbers are too large.

SECULAR Change of Dip in France.

Station.	Jan. 1, 1869.	Jan. 1, 1885.	Secular variation.	
			1869–85.	1859–85.
Amiens.	66 40·3	66 8·0	— 2·0	— 2·7
Bordeaux	63 23·0	62 41·8	— 2·6	— 3·1
Clermont	63 36·4	62 52·1	— 2·8	— 2·8
Dijon	64 24·5	63 53·4	— 1·9	— 2·7
Grenoble	62 54·2	62 6·9	— 2·9	— 2·6
Monaco.	61 22·1	60 41·2	— 2·5	— 2·8
Moulins	64 4·9	63 30·1	— 2·2	— 2·7
Paris	65 52·5	65 17·3	— 2·2	— 2·8
Périgueux. . . .	63 23·9	62 44·9	— 2·4	— 3·1
Poitiers	64 28·1	63 55·6	— 2·0	— 2·8
Toulouse	62 1·1	61 23·9	— 2·3	— 3·2
Mean	. .	. .	— 2·35	— 2·85

We turn next to the secular change of the Declination.

A number of stations are common to our own survey and those of 1837 and 1857, and we have also observed at several places where the Declination was determined in 1872. The mean annual decrease calculated for the observations is exhibited in the following table :—

TABLE IV.—Table of Mean Annual Decrease of Declination.

Station.	1837 to 1886.	1857 to 1886.	1872 to 1886.
Lerwick	8·4	10·5	8·2
Kirkwall	7·9	10·5	..
Thurso	..	10·6	9·9
Loch Eriboll	..	10·1	9·4
Wick	8·1	10·5	9·3
Stornoway	..	{8·2	..
Callernish	..	9·8	..
Loch Inver	..	11·8	..
Golspie	8·1	10·4	..
Elgin	..	8·8	..
Banff	..	8·9	..
Loch Maddy	..	9·8	..
Kyle Akin	..	8·8	8·9
Inverness	7·9	9·0	..
Aberdeen	7·7	8·9	8·9
Loch Boisdale	..	..	12·0
Fort Augustus	..	9·1	..
Dalwhinnie	..	8·8	..
Pitlochrie	..	8·9	..
Oban	..	8·1	12·9
Lochgoilhead	..	..	..
Edinburgh	..	9·1	8·4
Glasgow	..	9·0	..
Cumbrae	..	..	..
Ayr	..	8·7	..
Campbelton	..	9·0	..
Dumfries	..	8·4	..
Stranraer	..	8·5	..
Liverpool	..	8·2	..
Cromer	7·3	..	7·0
King's Lynn	..	..	6·6
Lowestoft	7·1	..	..
Harwich	7·3	..	7·3
Kew	..	7·8	..
Greenwich	7·3	7·6	7·3
Milford	..	..	7·7
Plymouth	..	8·3	..
Falmouth	..	8·4	..
Jersey (St. Helier's)	..	7·6	..
Londonderry	7·7	..	..
Westport	7·6	..	..
Dublin	7·5	..	..
Limerick	6·9	..	..
Wexford	..	..	9·1
Waterford	6·7	..	..
Killarney	6·6	..	..
Valentia	6·9	..	8·4
Cork	6·2	..	..

These results indicate—(1) that the secular change is greater in the northern than in the southern parts of the United Kingdom; (2) that it was much larger between the years 1857–86 than during the interval 1837–57 ; and, lastly, (3) that, in the north at all events, the rate of annual decrease is again diminishing.

The first of these conclusions differs from that arrived at by Sir E. SABINE when collating the 1837 and 1857 surveys.   He assumed ('Phil. Trans.,' vol. 160, 1870, p. 268) "an annual decrease of West Declination of approximately 5'·6 in Scotland and the north of England, increasing to 6'·2 in the middle and southern parts of England and to 6' in Ireland."

Our conclusion that, at the present time, the rate is greater in the north is borne out by the values deduced by M. MOUREAUX from his own observations and the maps of LAMONT (*loc. cit.*, p. 162).   The values he gives for stations on the Mediterranean and on the English Channel are about −6'·5 and −7'·7 respectively, which indicates an increase of about 0'·2 on the rate for each degree of latitude as we go north.

The truth of the second conclusion is rendered more apparent by calculating the annual rate for the epoch 1837–57 instead of 1857–86, as shown in the following Table for the Scotch stations, for which the requisite data are available :—

MEAN Annual Decrease of Declination.

Station.	1837 to 1857.	1857 to 1886.	Station.	1837 to 1857.	1857 to 1886.
Lerwick . . .	− 5'·5	− 10'·5	Inverness . . .	− 6'·3	− 9'·0
Kirkwall . . .	− 4·5	− 10·5	Aberdeen . . .	− 6·8	− 8·9
Wick	− 4·8	− 10·5	Loch Inver . .	− 2·9 (?)	− 11·8
Golspie	− 5·0	− 10·4			

The third conclusion is supported by a comparison of M. MOUREAUX'S rates for the epoch 1859–1885 with those obtained by collating as before his results and those of the Rev. S. J. PERRY (*loc. cit.*).

Station.	Declination.		Secular rate of change.	
	Jan. 1, 1869.	Jan. 1, 1885.	1869–85.	1859–85.
Amiens	18 19·0	16 34·7	− 6·5	− 7·4
Bordeaux	18 12·5	16 45·7	− 5·4	− 7·0
Clermont	16 27·6	15 25·0	− 3·9	− 6·9
Dijon	16 36·7	14 45·2	− 7·0	− 7·2
Grenoble	15 49·3	14 11·0	− 6·1	− 6·8
Marseilles	15 41·5	14 0·0	− 6·3	− 6·7
Monaco	14 31·4	13 10·5	− 5·0	− 6·5
Moulins	16 29·2	15 25·6	− 4·0	− 6·9
Paris	17 50·5	16 10·2	− 6·3	− 7·4
Périgueux	17 40·9	16 8·6	−· 5·8	− 7·0
Poitiers	18 18·4	16 40·8	− 6·1	− 7·1
Toulouse	17 7·3	15 41·4	− 5·4	− 6·8
Mean . . .	..	..	− 5·6	− 7·0

This result—that the rate of decrease is diminishing—is in accord with the conclusions drawn by M. MOUREAUX himself from a comparison of his own observations with those on which a magnetic map of France for the epoch 1875 was based by M. MARIÉ-DAVY (*loc. cit.*, p. 166). He did not, however, make use of this fact, as the number of observations employed by MARIÉ-DAVY was small, and they were taken at stations irregularly distributed over France. He does not cite the observations of Father PERRY, which, as the above Table shows, strongly support the same view.

In the next Table we give such observations as were made at English observatories or by ourselves during the progress of the survey which bear on the question under discussion. No very definite conclusions can be drawn from them.

Two single observations at the interval of a year, as at Kerrera and Glasgow, are hardly sufficient for the purpose of deducing a secular rate. Loch Aylort and Stornoway are both disturbed stations and the results are thus less trustworthy. The general evidence that the secular change is greater in higher latitudes is opposed by the fact that Stonyhurst gives a less value than Kew and Greenwich. If the increase with latitude which obtains in France were maintained in England, the value at Paris being taken at − 6′·3, we could deduce − 6′·8 for London ; − 7′·7 for Edinburgh ; and − 8′·2 for Wick ; of which the two latter are less than the values for the epoch 1872–86 given in Table IV., p. 88.

Station.	Date.	Declination.	Date.	Declination.	Interval in years.	Secular rate of change.
		° ′		° ′		′
Stornoway . . (Ard Point)	} Sept., 1884	24 20·6 {	Aug., 1885	24 12·4	0·96	− 8·5
			Aug., 1888	23 49·9	4	− 7·7
Loch Aylort .	Sept., 1884	23 40·2	Aug., 1888	22 41·8	4	− 14·6
Kerrera . . .	Aug., 1884	22 25·4	Aug., 1885	22 15·2	1	− 10·2
Glasgow . .	Aug., 1884	21 21·3	Aug., 1885	21 18·2	1	− 3·1
Stranraer . .	Sept., 1884	21 46·2	Aug., 1888	21 13·1	4	− 8·2
Stonyhurst . .	1884	19 52·8	1887	19 35·2	3	− 5·9
Reading . .	April 30, 1886	18 13·1	May 30, 1888	17 53·9	2	− 9·6
Kew	1884	18 33·9	1888	18 9·3	4	− 6·1
Greenwich . .	1884	18 7·6	1888	{ 17 40·0* (approx.)	} 4	− 6·9

From Table IV. we see that the rates at Greenwich for the three intervals, 1837–86, 1857–86, and 1872–86 are nearly the same, the differences between them and the mean rate during the period of our survey (6′·9) being 0′·4, 0′·7, and 0′·4 respectively. Assuming that if we subtract these numbers from the rates calculated for the same epochs for the other English stations in Table IV. we obtain the present rates at the stations, we find as a mean rate 7′·0, which agrees closely with that of Greenwich.

On the whole we decided to take a secular rate of − 7′·0 in England south of a line joining Redcar and Barrow, and also in Wales, and in Ireland south of a line joining Dublin and Donegal.

For the remainder of England and Ireland we have assumed the rate to be − 8′·0 per annum. At the time when most of the Scotch observations were reduced we had not M. MOUREAUX's results before us, and we hardly felt justified in departing from the rates given by the comparison of our observations with Mr. WELSH's, on the evidence of the small number of stations at which the Declination was determined in 1872. We, therefore, employed rates varying from − 10′·4 in the north of Scotland to − 8′·8 in the south, the change on the east coast being somewhat greater than on the west. These may be a little too large but the error certainly does not exceed the variation which occurs between neighbouring stations. For the later Scotch observations we used a rate of − 9′·0.

The data for the determination of the secular change of the Horizontal Force are more scanty than in the case of the other elements.

At the time of the 1837 survey the measures were comparative only, and though Sir E. SABINE afterwards reduced them to absolute values by means of the known absolute value and secular rate at Kew, we do not think any useful end would be attained by discussing them.

M. MOUREAUX found that for the interval 1848–85, the secular variation in France attains its maximum + 0·0027 (M.U.) at Bordeaux and decreases slowly towards the

* From information kindly supplied by the Astronomer-Royal.

east, being ·0025 at Paris and + 0·0023 at Nice and Mezières (*loc. cit.*, p. 167). The mean value obtained by the Rev. S. J. PERRY in 1869 by comparison with LAMONT was 0·0023 ('Phil. Trans.,' Vol. 160, 1870, p. 48), which was increasing by about 0·000008 per annum and would correspond to 0·0025 at the present time.

By comparison of our own observations with those of WELSH we obtain 0·0018 as a mean for Scotland between 1857 and 1886.

The following Tables give these values and those which were obtained at the English observatories or by ourselves during our survey.

Secular Change in Horizontal Force.

Station.	WELSH.	R. and T.	Difference.	Interval.	Secular rate of change.
	Uncorrected for v and Δ.				
				Years.	
Dumfries	1·6006	1 6522	·0516	27	·0019
Stranraer	1·5853	1·6433	·0580	27	·0021
Ayr	1·5756	1 6317	·0561	$27\frac{3}{4}$	·0020
Oban	1·5451	1·6083	·0632	31	·0020
Fort Augustus	1·5134	1·5643	·0509	28	·0018
Inverness	1·5093	1·5653	·0560	28	·0020
Aberdeen	1·5166	1·5735	·0569	$27\frac{1}{2}$	·0021
Pitlochrie	1·5341	1·5926	·0585	$27\frac{3}{4}$	·0021
Dalwhinnie	1·5377	1·5926	·0549	$27\frac{3}{4}$	·0020
Edinburgh	1·5665	1·6162	·0497	$27\frac{1}{2}$	·0018
	1·5697	. .	·0465	$26\frac{3}{4}$	·0017
Kyle Akin	1·5009	1·5407	·0398	. .	·0015
Stornoway	$\left\{\begin{matrix}1·4773\\1·4783\\1·4794\end{matrix}\right\}$1·4783	1·5122	·0339	$26\frac{1}{12}$	·0013
Callernish	1·4731	1·5236	·0505	27	·0019
Loch Inver	1·4499	1·4956	·0457	$26\frac{1}{12}$	·0018
Thurso	1·4763	1·5209	·0446	$26\frac{3}{4}$	·0017
Lerwick	1·4313	1·4718	·0405	27	·0015
Kirkwall	$\left\{\begin{matrix}1·4669\\1·4725\\1·4753\end{matrix}\right\}$1·4716	1·5111	·0395	27	·0015
Wick	1·4706	1·5117	·0411	$26\frac{3}{4}$	·0015
Golspie	1·4893	1·5390	·0497	$26\frac{3}{4}$	·0019
Mean . . .	. .	. .	. .	. . . .	·0018

Station.	Date.	H.	Date.	H.	Interval in years.	Secular rate of change.
Stornoway . .	Aug. 19, 1885	1·5200	Aug. 14, 1888	1 5220	3	+ ·0007
Loch Aylort . .	Sept. 12, 1884	1·5577	Aug. 2, 1888	1·5774	4	+ ·0049
Stranraer . .	Sept. 18, 1884	1·6423	Aug. 28, 1888	1·6470	4	+ ·0012
Stonyhurst . .	1884	1·6954	1887	1·7024	3	+ ·0023
Kew	1884	1·8056	1887	1·8091	3	+ ·0012
	1883	1·8023	1887	1·8091	4	+ ·0017
Greenwich . .	1883	1·8100	1885	1·8156	2	+ ·0028
	1883	1·8100	1886	1·8157	3	+ ·0019

The results at Kew and Greenwich show that the secular variation varies so much from year to year that it is practically impossible to draw any conclusions from so short a period as 2 or 3 years, and our own observations on Scotland show that if to this difficulty, that due to a slight variation in the position in a disturbed district is added, the results are still less trustworthy.   At the time when the reductions were made the Greenwich results for 1886 and the Stonyhurst results for 1887 had not been published.   We therefore took the mean of the values for Stonyhurst 1883–1886, Kew 1883–87, and Greenwich 1883–1885, or $+0\cdot0022$ the annual increase for England.   This value is in fair accord with M. Moureaux's results.   In Scotland we took the number given by the comparison of our own and Welsh's observations, viz., $+0\cdot0018$, and for Ireland $+0\cdot0020$.

Results of the Observations.

Having thus described the observations and the methods of correction and reduction to epoch, we now proceed to give a more detailed account of the results at each station.   In doing this we have attempted to distinguish between facts which it is necessary to give for the information of most of those who may read our paper, and details which ought to be preserved, but which will nevertheless only be of interest to observers who may for any cause wish to undertake a detailed examination of our results.   We have therefore decided to publish in this paper only a description of each station, the hours at which the observations were taken, the results corrected for diurnal variation and disturbance, and the mean reduced to epoch.   As Magnetic Observatories and provincial Colleges are now rapidly multiplying, it is not too much to hope that special studies of the districts in their immediate neighbourhood may from time to time be made by those connected with such institutions.   It is, therefore, we think, important that the descriptions of the stations should be readily accessible.   In adding to these only the times at which the observations were taken, and the actual and reduced results, we are publishing far less than has been usual.   Thus, in the case of the Rev. S. J. Perry's survey of the east of France, he gave for the Force observations the date, hour, and temperature for both vibrations and deflections, the time of one vibration, log mX, the distances of the magnets in the deflection experiment, the observed deflections, and log m/X.   M. Moureaux has not given quite so many details of the Force observations, but in the case of the Declination he gives the individual readings for the determination of the geographical meridian, &c., so that the description of his work at each station occupies about two quarto pages. For our own part we have no fault to find with the publication of these details; on the contrary, we have found them to be useful; but the large number of stations included in our survey would, we fear, make this paper inordinately long if we adopted a similar plan.   We therefore purpose to place in the hands of the Royal Society bound copies of the details of the observations and calculations, and also of

the forms described on pp. 71-74, in which the results are analysed and the corrections for diurnal and secular variation and disturbance are applied. It will thus be possible for those who may desire to do so to inspect these, and the data used in the preparation of the Tables on pp. 77-80 will be on record, while this paper will, we hope, contain sufficient to enable future observers who are not specially interested in the details of the calculations and reductions to find the positions where we observed, to know when we observed there, and to judge from the final results of the general accuracy of the observations. We have followed the plan adopted by M. MOUREAUX of giving all the facts with respect to each station together, which we think the most convenient. Tables are also given on pp. 251-258 in which the final results are entered in tabular form for comparison with the values obtained by calculation from formulæ to be hereafter discussed.

The stations are arranged in the following order:—Three groups are formed, comprising Scotland, England and Wales, and Ireland, this being the chronological order of the bulk of the observations in each of these countries. In each group the stations are arranged in alphabetical order, and they are numbered continuously throughout. These numbers are affixed to the positions of the stations as given on Plate I., which serves as an Index map.

The Scotch stations, from Aberdeen to Wick, are numbered from 1 to 54, and it should be mentioned that the name of a Loch is regarded as determining the initial letter. Thus East Loch Tarbert is found under T.

The English and Welsh stations, including the Isle of Man and the Channel Isles, from Aberystwith to Worthing, are numbered from 55 to 156.

The Irish stations, from Armagh to Wicklow, are numbered from 157 to 200.

Thus anyone desirous of looking up the observations at a particular place, can easily do so from a knowledge of its name, while the stations in any particular district can be found in Plate I., and then referred to by means of the corresponding numbers.

The data given in each case, are as follows :—

(1.) The number and name of the station.
(2.) Date of the observations.
(3.) Initials of the observer and numbers of the instruments.
(4.) Latitude and longitude of the station.
(5.) Verbal description of the station.

For the Declination we give :—

(1.) The time from the southing of the sun (Σ), at which the geographical meridian was determined by sun observations, a positive sign indicating the afternoon.
(2.) The G.M.T. of the determination of the magnetic meridian.
(3.) The observed Declinations with all corrections applied (δ).
(4.) The mean observed Declination reduced to the epoch, January 1, 1886 (δ_0).

For the Inclination we give :—

    (1.) The number of the needle.

    (2.) The G.M.T. at which the observation was made.

    (3.) The observed Dip, with all corrections applied (θ).

    (4.) The mean observed Dip reduced to epoch (θ_0).

For the Horizontal Force we give : —

    (1.) The G.M.T. at which the deflection (D) and vibration (V) were observed.

    (2.) The corrected independent forces found as described on p. 73 (H), but corrected for diurnal variation and disturbance.

    (3.) The mean observed Horizontal Force reduced to epoch.

Longitudes are to be taken as west of Greenwich, unless the contrary is expressly stated.

DESCRIPTIONS OF SCOTCH STATIONS.

1. ABERDEEN.   April 6 and 7, 1885 ; A. W. R. and T. E. T. (60, 74).   Lat. 57° 9′ 50″ ; Long. 2° 6′ 5″.   In a field behind Professor MILLIGAN's house, immediately opposite to King's College.   Tower of King's College E.S.E. ; hermitage in Miss LESLIE's park W. by S., and the centre of the gate of Miss LESLIE's lodge S.E. by S.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
April 6 „ 7	h. m. .. −2 42	h. m. 14 49 9 53	° ′ 20 24·3 20 21·6	° ′ 20 16·3

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
April 6	1 2	h. m. 14 46 15 28	° ′ 71 12·4 71 14·1	° ′ 71 12·3

Horizontal Force.

Date.	G.M.T.		H.	H_0.
April 6	D V	h. m. 15 56 14 10	1·5724 1·5719	1·5734

2. ARINAGOWER (Coll).   August 11, 1885; A. W. R. (60).   Lat. 56° 37′ 5″; Long. 6° 31′ 12″.   Near the landing-place on the west side of the bay.   This observation was taken during an unexpected detention of the steamer.   The declination was the only element determined.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −0 52	h. m. 12 1	° ′ 23 44·2	° ′ 23 40·4

3. LOCH AYLORT (Gobbar Island).   September 12, 1884; A. W. R. and T. E. T. (60, 74). August 2, 1888; T. E. T. (61, 83).   Lat. 56° 51′ 5″; Long. 5° 47′ 0″.   On the east side of the island.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Sept. 12, 1884 Aug. 2, 1888	h. m. + 2 48 + 1 15	h. m. 15 42 { 12 16 { 13 24	° ′ 23 40·2 22 40·5 22 43·1	° ′ 23 27·2 23 5·9

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Sept. 12, 1884 Aug. 2, 1888	1 2 1 2	h. m. 14 44 15 46 12 38 13 27	° ′ 71 24·8 71 23·2 71 22·0 71 22·7	° ′ 71 22·6 71 25·4

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 12, 1884 Aug. 2, 1888	D V D V	h. m. 16 39 17 14 15 9 14 28 13 0	1·5571 } 1·5583 } 1·5576 } 1·5578 } 1·5772 1·5776	1·5600 1·5600 1·5727

4. AYR.   May 25, 1885 ; A. W. R. (60, 74).   Lat. 55° 27′ 30″ ; Long. 4° 37′ 10″.
On the low green about 150 yards from the Castle.   Identical with Mr. WELSH's
station.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −3 55 −1 29	h. m. 8 55	° ′ 21 23·0 21 23·4	° ′ 21 17·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 32 13 15	° ′ 70 21·6 70 23·1	° ′ 70 21·4

Horizontal Force.

G.M.T.		H.	H_0.
D V D V	h. m. 11 21 9 25 .. 10 28	1·6337 1·6336 1·6331 1·6334	1·6345

5. BALLATER.   April 7 and 8, 1885 ; A. W. R. and T. E. T. (60, 74).   Lat. 57° 2′ 53″ ;
Long. 3° 2′ 6″.   Fifty yards from the Invercauld Arms Hotel ; near the river.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
April 7 „ 8	h. m. + 3 30 − 2 51	h. m. 16 26 ..	° ′ 20 36·9 20 36·1	° ′ 20 29·5

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
April 7	1	h. m. 15 39	$\overset{\circ}{71}\ 14\overset{\prime}{\cdot}9$	$\overset{\circ}{71}\ 15\overset{\prime}{\cdot}4$
	2	16 22	71 17·8	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
April 7	D	h. m. 15 11	1·5703	1·5714
	V	17 2	1·5699	

6. BANAVIE.   August 4, 1885.   A. W. R. and A. P. L. (60, 74).   Lat. 56° 51′ 0″;
Long. 5° 5′ 40″.   Field close to and on the North side of the Hotel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 4 16	h. m. 17 39	$\overset{\circ}{22}\ 10\overset{\prime}{\cdot}6$	$\overset{\circ}{22}\ 6\overset{\prime}{\cdot}7$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 17 11	$\overset{\circ}{71}\ 11\overset{\prime}{\cdot}7$	$\overset{\circ}{71}\ 11\overset{\prime}{\cdot}4$
2	18 5	71 12·1	

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 18 23	1·5928	1·5940
V	17 10	1·5938	

7. BANFF.   July 9, 1885 ; A. W. R. (60, 74).   Lat. 57° 39′ 57″ ; Long. 2° 31′ 17″. In the grounds of the old Castle ; on the lawn in front of the house formerly occupied by Dr. BREMNER, now by Sheriff SCOTT-MONCRIEFF.   Same station as that at which Mr. WELSH observed.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −2 40	h. m. 10 28	° ′ 21 8·8	° ′ 21 4·5

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 58 13 39	° ′ 71 19·7 71 19·6	° ′ 71 19·0

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 11 48 11 7	1·5669 1·5681	1·5684

8. BERWICK.   April 2, 1885 ; A. W. R. and T. E. T. (60, 74).   Lat. 55° 46′ 4″ ; Long. 1° 59′ 52″.   On a bastion.   Powder-magazine distant about 150 yards N. Church, N.W., and works on sandspit at mouth of river S.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −3 5	h. m. 9 50	° ′ 19 42·9	° ′ 19 36·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	10 57	70 15·8	
2	11 34	70 17·8	70 15·9

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	11 35	1·6474	
V	10 28	1·6467	1·6483

9. BOAT OF GARTEN. July 31, 1885 ; A. W. R. and A. P. L. (60, 74). Lat. 57° 15′ 0″; Long. 3° 45′ 13″. On the Green to the North of the Station, about 200 yards from the Hotel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+ 3 37	15 31	22 11·7	22 7·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	15 51	71 15·0	
2	18 0	71 19·3	71 16·3
1	18 28	71 16·0	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	16 43	1·5776	
V	15 6	1·5781	1·5786

10. LOCH BOISDALE (S. Uist).   August 31, 1884 ; A. W. R. and T. E. T. (60, 74).
Lat. 57° 8′ 55″; Long. 7° 18′ 0″.   In the Bay on the Island of Kisgay.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m. − 1 51 + 4 56	h. m. 11 19 16 45	° ′ 23 7·1 23 7·1	° ′ 22 53·3

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
1 2	h. m. 11 29 12 30	° ′ 71 41·0 71 39·9	° ′ 71 39·3

Horizontal Force.

G.M.T.		H.	H$_0$.
D V	h. m. 12 49 12 2	1·5284 1·5288	1·5310

11. BUNNAHABHAIN (Islay).   August 25, 1888 ; A. W. R. and T. E. T. (60, 61, 83).
Lat. 55° 53′ 0″; Long. 6° 8′ 0″.   At the bottom of the road, 150 yards N. of
the Distillery and about 10 yards from the beach.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m. −2 32 −1 12	h. m. 10 55 11 36	° ′ 22 44·0 22 46·0	° ′
−2 29	10 29 12 25	22 46·0 22 47·3	23 10·3
−1 16	14 30 15 9	22 47·9 22 47·3	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	12 46	70 39·4	70 43·0
2	14 38	70 39·8	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	15 39	1·6292	1·6243
V	14 55	1·6291	

12. CALLERNISH (Lewis).   August 20, 1885 ; T. E. T. (60, 74).   Lat. 58° 11′ 0″ ; Long. 6° 42′ 0″.   Fifty yards S.S.W. (magnetic) from front of Garynahine inn.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+ 5 2	18 3	23 44·1	23 40·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	7 26	72 7·5	72 7·1
2	8 1	72 7·4	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	18 21	1·5224	1·5231

13. CAMPBELTON.   August 22, 1884 ; A. W. R. and T. E. T. (60, 74).   Lat. 55° 25′ 30″;
    Long. 5° 36′ 5″.   200 yards N.E. from Lime Crags.   400 yards E.S.E. from old
    Parish Church.   Same station as that at which Mr. WELSH observed.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m.	h. m.	° ′	° ′
− 1 20		22 21·1	
− 1 13	14 36	22 21·0	22 8·1
+ 1 43		22 19·8	

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
	h. m.	° ′	° ′
1	12 56	70 35·2	70 34·2
2	13 27	70 36·9	

Horizontal Force.

G.M.T.		H.	H$_0$.
	h. m.		
D	12 58	1·6236	1·6244
V	11 49	1·6204	

14. CANNA.   August 30, 1884 ; A. W. R. and T. E. T. (60, 74).   Lat. 57° 3′ 30″
    Long. 6° 29′ 20″.   On the high land N.E. of the Harbour, and S.W. of Compass
    Hill.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m.	h. m.	° ′	° ′
− 4 35	8 12	21 21·8	21 8·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
2	h. m. 8 38	o ʹ 72 46·3	o ʹ 72 45·0

Horizontal Force.

G.M.T.		H.	H_0
D V	h. m. 9 10 8 30	1·5065 1·5072	1·5092

15. CARSTAIRS.   May 25 and 26, 1885 ; A. W. R. (60, 74).   Lat. 55° 41′ 10″; Long. 3° 40′ 11″.   In Mr. MONTEITH's grounds, about 400 yards from the station, and about 50 yards from the road leading from the station to the house.   Nearly the same position as that at which Mr. WELSH observed.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
May 25	h. m. + 5 55	h. m. 18 35	o ʹ 20 57·6	o ʹ 20 52·2

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
May 26	1 2	h. m. 9 49 10 23	o ʹ 70 16·4 70 16·4	o ʹ 70 15·7

Horizontal Force.

Date.	G.M.T.		H.	H_0.
May 25	D V	h. m. 19 42 19 5	1·6434 1·6440	1·6448

16. CRIANLARICH. September 17, 1884; A. W. R. (60, 74). Lat. 56° 23′ 25″;
Long. 4° 37′ 6″. Near the inn, about 80 yards from the roads to Dalmally and
Loch Lomond.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. − 1 47 + 2 49	h. m. 11 10	° ′ 22 2·2 22 3·8	° ′ 21 50·6

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 8 31 9 8	° ′ 70 54·5 70 53·1	° ′ 70 52·5

Horizontal Force.

G.M.T.		H.	H₀.
D V	h. m. 12 52 11 32	1·5943 1·5971	1·5980

17. CRIEFF. July 28, 1885; A. W. R. and A. P. L. (60, 74). Lat. 56° 22′ 27″;
Long. 3° 50′ 22″. In the grounds of MORRISON's Academy, about half-way
between the school and the gate.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. + 1 46	h. m. 14 48	° ′ 21 37·6	° 21 33·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 36 14 49	$\overset{\circ}{70}$ $\overset{\prime}{53 \cdot 4}$ 70 54·8	$\overset{\circ}{70}$ $\overset{\prime}{53 \cdot 6}$

Horizontal Force.

G.M.T.		H.	H_0
D V	h. m. 15 55 15 19	1·6071 1·6072	1·6079

18. CUMBRAE.   July 24, 1888; T. E. T. (61, 83).   Lat. 55° 47′ 45″; Long. 4° 53′ 40″.   Eight yards to the N. of the Monument to the " Shearwater's " midshipmen, at the N.E. end of the Island.   Mr. WELSH's Station.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 4 19	h. m. 17 0	$\overset{\circ}{21}$ $\overset{\prime}{13 \cdot 2}$	$\overset{\circ}{21}$ $\overset{\prime}{37 \cdot 2}$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 41 18 0	$\overset{\circ}{71}$ $\overset{\prime}{1 \cdot 0}$ 70 59·7	$\overset{\circ}{71}$ $\overset{\prime}{2 \cdot 3}$

Horizontal Force.

G.M.T.		H.	H_0.
V.	h. m. 17 16	1·5957	1·5911

19. DALWHINNIE. July 30, 1885. A. W. R. and A. P. L. (60, 74). Lat. 56° 55′ 52″; Long. 4° 14′ 12″. About 150 yards from the Hotel; on the opposite side of the river.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 54	h. m. 16 10	21 49·2	21 45·5

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 9 17 35	70 59·7 71 1·6	71 0·1

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 14 42 15 49	1·5901 1·5902	1·5909

20. DUMFRIES. September 20, 1884 and July 11, 1885. A. W. R. (60, 74). Lat. 55° 2′ 10″; Long. 3° 35′ 30″. On Mr. STOTT's farm at Lower Netherwood. In a field about 100 yards W. of the farm-house; 39 and 50 paces from the N. and E. walls of the field respectively. Nearly the same station as that of Mr. WELSH.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
July 11.	h. m. + 4 51	h. m. 16 33	20 51·5	20 47·4

P 2

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Sept. 20.	1 2	h. m. 12 6 12 46	70° 4·2′ 70 4·5	70° 2·6′

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 20.	D V	h. m. 13 38 14 19	1·6524 1·6514	1·6542

21. DUNDEE. April 9, 1885 ; A. W. R. and T. E. T. (60, 74). Lat. 56° 28′ 17″ ; Long. 2° 56′ 58″. In the Baxter Park. The tower of Morgan's Hospital N. by W. ; tower of lodge S.W. (100 yards) ; Park Pavilion N.N.E. (100 yards) ; Ogilvie Church N.W.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +1 33	h. m. 12 45	20° 51·2′	20° 44·5′

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h m. 11 56 12 38	70° 51·7′ 70 54·6	70° 52·2′

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 11 31 12 16	1·5990 1·5989	1·6002

22. EDINBURGH.   April 3, 1885 ; A. W. R. and T. E. T. (60, 74).   Lat. 55° 57′ 52″. Long. 3° 12′ 28″.   In the Arboretum of the Botanic Gardens.   Inverleith House N.N.W. 100 yards.   Cathedral S.W. by S.   Melville College S.S.E.   Donaldson's Hospital W.S.W.   The small magnetic house in the gardens in which Mr. WELSH probably observed had been removed.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +2 14	h. m. 13 56	° ′ 20 53·8	° ′ 20 47·2

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2 2	h. m. 11 57 12 36 14 21	° ′ 70 37·2 70 39·5 70 40·8	° ′ 70 38·5

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 22 12 51	1·6175 1·6165	1·6183

23. ELGIN.   July 8, 1885 ; A. W. R. (60, 74).   Lat. 57° 38′ 40″; Long. 3° 19′ 0″. In the grounds of North College, the residence of G. SMITH, Esq., 100 yards N.N.E. of the Cathedral Tower and 80 yards E.S.E. of the house.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 33	h m. 12 29	° ′ 21 1·8	° ′ 20 57·5

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	14	26	71	32·5	71	32·0
2	15	2	71	32·7		

Horizontal Force.

G.M.T.			H.	H_0.
	h.	m.		
D	13	8	1·5566	1·5577
V	12	11	1·5570	

24. LOCH ERIBOLL (Camas Bay). August 23, 1885; T. E. T. (60, 74). Lat. 58° 29′ 15″; Long. 4° 39′ 20″. Near the stream running into Camas Bay, ¼ mile N.E. (magnetic) from the ruin in the bight.

Declination.

Σ.		G.M.T.		δ.		δ_0.	
h.	m.	h.	m.	°	′	°	′
+ 3	31	14	52	22	21·7	22	18·1
+ 4	34						

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	11	40	72	9·4	72	9·4
2	12	13	72	10·3		

Horizontal Force.

G.M.T.			H.	H_0.
	h.	m.		
D	13	7	1·5184	1·5198
V	14	29	1·5201	

25. FAIRLIE.   August 14, 1884.   A. W. R. and T. E. T. (60, 74).   Lat. 55° 45′ 30″;
Long. 4° 51′ 5″.   In a field on the high ground to the rear of the village.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 4 16 8	70 43·7 70 44·2	70 42·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 50 13 56	1·6150 1·6143	1·6172

26. FORT AUGUSTUS.   August 3, 1885.   A. W. R. and A. P. L. (60, 74).   Lat.
57° 8′ 30″; Long. 4° 40′ 32″.   In a field on the south side of the Abbey, and
about 150 yards from the building.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 37	h. m. 14 38	21 49·4	21 45·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 47 15 51	71 29·4 71 27·1	71 27·7

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 15 47	1·5634	1·5641

27. GAIRLOCH.   September 9, 1884 ; A. W. R. and T. E. T. (60, 74).   Lat. 57° 42′ 40″;
Long. 5° 40′ 55.″   On the rising ground behind the pier in Flowerdale Bay.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 3 23	h. m. 16 33	° ′ 22 28·0	° ′ 22 14·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 39 17 36	° ′ 71 45·9 71 45·3	° ′ 71 44·3

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 17 37 17 2	1·5323 1·5336	1·5353

28. GLASGOW.   August 13, 1884, and July 27, 1885 ; A. W. R. (60, 74).
Lat. 55° 52′ 43″ ; Long. 4° 17′ 39″.   In a field to the West of the Observatory ;
48 paces from the building.

Declination.

Date.		G.M.T.	δ.	δ_0.
Aug. 13, 1884 July 27, 1885	h. m. + 3 47 + 2 43	h. m. 16 53 15 54	° ′ 21 21·3 21 18·2	° ′ 21 11·5

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Aug. 13, 1884	1	h. m. 12 12	70 44·6	70 44·7
	2	12 18	70 48·4	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Aug. 13, 1884	D	h. m. 14 17	1·6038	1·6064
	V	15 5	1·6038	

29. GOLSPIE. July 4, 1885; A. W. R. (60, 74). Lat. 57° 58′ 20″; Long. 3° 58′ 15″. The Dips were taken in a field by the road in front of the Sutherland Arms Hotel and about 50 yards to the West of it. The other observations in a field behind the Hotel about 80 yards S.W. of some cottages and 50 yards N.N.W. of the Bank.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 9 + 2 39	h. m. 15 33	21 34·9 21 35·6	21 30·2

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 12 20	71 46·8	71 46·7
2	12 55	71 47·8	

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 16 37	1·5372	1·5382
V	16 1	1·5374	

30. HAWICK.   March 31 and April 1, 1885 ; A. W. R. and T. E. T. (60, 74).
    Lat. 55° 25′ 58″; Long. 2° 47′ 58″.   In the Park of Sillerbit Hall (T. LAIDLAW,
    Esq.) to the N. of the town.   The Hall bears 400 yards E.N.E. and the Lodge
    120 yards S.S.E.

Declination.

Date.	Σ.	G.M.T.	δ.	δ₀.
	h. m.	h. m.	° ′	° ′
March 31 . .	. .	12 43	20 21·5	20 16·0
April 1 . . .	+ 1 50	13 34	20 23·5	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ₀.
		h. m.	° ′	° ′
March 31 . .	1	11 15	70 7·5	70 7·3
	2	11 54	70 9·0	

Horizontal Force.

Date.	G.M.T.		H.	H₀.
		h. m.		
March 31 . .	D	11 14	1·6473	1·6487
	V	12 1	1·6476	

31. LOCH INVER.   September 6, 1884 ; A. W. R. and T. E. T. (60, 74).   Lat.
    58° 9′ 30″ ; Long. 5° 14′ 40″.   Near the Coast-guard Station, on the bank of
    the River Inver.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m.	h. m.	° ′	° ′
+ 2 53	14 2	22 21·8	22 7·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 48 13 46	° ′ 72 2·6 72 1·0	° ′ 72 0·2

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 13 14 13 25	1·4968 1·4965	1·4990

32. INVERNESS. August 1, 1885 ; A. W. R. and A. P. L. (60, 74). Lat. 57° 28′ 30″ ; Long. 4° 13′ 20″. In the garden of Mr. MITCHELL's house, near the Castle. Mr WELSH's station.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +2 32	h. m. 16 18	° ′ 21 47·2	° ′ 21 43·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 44 16 45	° ′ 71 30·0 71 33·2	° ′ 71 31·1

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 16 56 15 50	1·5638 1·5631	1·5642

33. IONA.   September 15 and 16, 1884 ; A. W. R. (60, 74).   Lat. 56° 20′ 0″ ;
Long. 6° 23′ 40″.   In a field behind the Inn, and about 50 yards from it.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Sept. 15 . .	h. m. + 4 6	h. m. 17 4	$\overset{\circ}{23}$ 41′·0	$\overset{\circ}{23}$ 28′·6

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Sept. 16 . .	1 2	h. m. 9 35 10 10	$\overset{\circ}{70}$ 57′·1 70 57·5	$\overset{\circ}{70}$ 55·8

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 15 . .	V	h. m. 17 47	1·6162	1·6185

34. KIRKWALL.   August 27, 1885 ; T. E. T. (60, 74).   Lat. 58° 59′ 12″ ; Long.
2° 57′ 15″.   At Battery Point and close to the road.   Cathedral tower bearing
S.W. by W   Cairn on Wideford Hill, bearing W.N.W.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 46 + 4 31	h. m. 12 19 . .	$\overset{\circ}{21}$ 32′·9 21 32·7	$\overset{\circ}{21}$ 29·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 31 14 59	72 12·4 72 13·8	72 12·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 15 55 11 51	1·5104 1·5100	1·5108

35. KYLE AKIN.   September 11, 1884; A. W. R. and T. E. T. (60, 74).   Lat. 57° 16′ 35″; Long. 5° 44′ 0″.   Near the shore; between the Inn and Kyle House.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 2 49	h. m. 10 29	23 23·4	23 10·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 10 15 11 1	71 40·9 71 38·8	71 38·5

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 8 56 11 21	1·5432 1·5452	1·5465

36. LAIRG. July 6, 1885 ; A. W. R. (60, 74). Lat. 58° 1′ 30″ ; Long. 4° 24′ 0″.
40 yards S.W. of Church, and about 200 yards N.E. of the Hotel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 44	h. m. 15 56	° ′ 21 55·2	° ′ 21 50·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 13 18 5	° ′ 71 51·0 71 50·8	° ′ 71 50·3

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 17 7 16 23	1·5347	1·5356

37. LERWICK. August 30, 1885; T. E. T. (60, 74). Lat. 60° 8′ 53″ ; Long. 1° 7′ 47″.
Towards the South Ness, ½ mile due S. (mag.) from Fort Charlotte. Mr.
WELSH'S station.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 53	h. m. 12 38	° ′ 20 33·1	° ′ 20 29·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	15 36	72 46·7	72 47·1
2	16 3	72 48·1	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	14 39	1·4712	1·4710
V	13 34	1·4696	

38. LOCHGOILHEAD. July 21, 1888; T. E. T. (61, 83). Lat. 56° 10′ 20″; Long. 4° 54′ 0″. In a field about 80 yards N.E. of the Church.   Mr. WELSH's station.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{\delta}_0$.
h. m.	h. m.	° ′	° ′
+ 4 40	15 48	21 29·2	21 54·2
	18 6	21 30·8	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	17 32	70 42·4	70 46·1
2	17 48	70 43·6	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	16 29	1·6069	1·6021
V	16 4	1·6065	

39. LOCH MADDY (N. Uist); September 1 and 2, 1884; A. W. R. and T. E. T. (60, 74). Lat. 57° 35′ 50″; Long. 7° 9′ 0″. On the most westerly of the Reelee Islands.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Sept. 1 „ 2	h. m. . . −3 14	h. m. 18 34 10 29	° ′ 23 31·3 23 32·3	° ′ 23 18·0

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Sept. 1	2	h. m. 18 55	° ′ 71 53·3	° ′ 71 52·1

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 2 „ 1	D V	h. m. 12 26 19 13	1·5346 1·5336	1·5365

40A. OBAN. August 13, 1885, and July 30, 1888; T. E. T. (60, 61, 83). Lat. 56° 25′ 9″; Long. 5° 28′ 30″. Same station as that of Mr. WELSH.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Aug. 13, 1885 July 30, 1888	h. m. +3 38 −1 41	h. m. 16 37 11 20 12 59	° ′ 22 12·0 21 47·3 21 45·2	° ′ 22 8·7 22 10·2

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
July 30	1	h. m. 12 43	° ′ 70 50·7	° ′ 70 53·2
	2	13 7	70 49·5	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
July 30	D	h. m. 12 24	1·6090	1·6044
	V	11 36	1·6091	

40B.—OBAN (Kerrera).   August 26, 1884, and August 6, 1885.   A. W. R. and T. E. T. (60, 74).   Lat. 56° 25′ 20″; Long. 5° 30′ 0″.   Near the shore of Ardentraive Bay.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Aug. 26, 1884	h. m. + 2 5	h. m. 15 32	° ′ 22 25·4	° ′ 22 12·4
	+ 3 55	..	22 25·5	22 11·4
„ 6, 1885	− 1 25	12 32	22 15·2	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Aug. 26, 1884	1	h. m. 15 18	° ′ 70 50·5	° ′ 70 49·5
	2	15 47	70 51·8	
„ 6, 1885	1	13 14	70 49·1	70 48·2
	2	14 11	70 48·3	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Aug. 26, 1884	D	h. m. 17 30	1·6066	1·6092
	V	16 44	1·6071	
„ 6, 1885	D	13 30	1·6110	1·6114
	V	12 13	1·6105	

41. PITLOCHRIE. July 29, 1885 ; A. W. R and A. P. L. (60, 74). Lat. 56° 42' 7'' ;
   Long. 3° 43' 26''. In the grounds of the Hydropathic Establishment, about
   100 yards S.S.W. of the building ; near the lower road through the grounds.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m. + 3 9	h. m. 16 35	° ' 21 12·2	° ' 21 8·3

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
1 2	h. m. 17 55 16 58	° ' 70 58·7 70 57·1	° ' 70 57·4

Horizontal Force.

G.M.T.		H.	H$_0$.
D V	h. m. 17 10 16 11	1·5891 1·5891	1·5899

42. PORT ASKAIG (Islay). August 25, 1884 ; A. W. R. and T. E. T. (60, 74).
   Lat. 55° 50' 40'' ; Long. 6° 6' 55''. Between the Inn and the Quay.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m. − 2 44	h. m. 10 18	° ' 23 13·7	° ' 23 0·7

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
1 2	h. m. 12 23 12 22	° ' 70 37·9 70 38·1	° ' 70 36·2

Horizontal Force.

G.M.T.		H.	H$_0$.
	h. m.		
D	12 2	1·6313	
V	10 38	1 6319	1·6340

43. PORTREE.   T. E. T. (61, 83).

(*a*) August 9, 1888 ; Lat. 57° 24′ 35″, Long. 6° 11′ 40″.   On the N. shore of the bay.   Boat-house 20° W.

(*b*) August 9 and 10, 1888.   Lat. 57° 24′ 10″, Long. 6° 11′ 5″.   On the S. side of the bay, on a rock close to the shore.   Portree Landing Stage 12° W. of N. (mag.).   School-house 35° W. of N.   Station (*a*) 22° E. of N.

(*c*) August 10, 1888.   Lat. 57° 24′ 15″, Long. 6° 11′ 50″.   On the edge of the bay ; S. of the town, and within a dozen yards of the shore.   Station (*b*) bearing 100° E. of N. ; School-house 2° E. of N. ; Station (*a*) bearing 57° E. of N. (approx.) ; St. Columba's Church steeple 24° E. of N.

Declination.

Date.	Σ.	G.M.T.	δ.	δ$_0$.
	h. m.	h. m.	° ′	° ′
Aug. 9 (*a*)	−2 30	10 26	24 37·6	
,, 9 (*b*)	+4 24	15 50	22 21·6	
,, 10 (*b*)	−2 42	11 25	22 22·7	22 42·3
,, 10 (*c*)	+1 12	13 24	19 50·8	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ$_0$.
		h. m.	° ′	° ′
Aug. 9 (*a*)	1	13 2	72 16·3	
	2	13 21	72 14·7	
,, 10 (*b*)	1	10 34	71 5·9	
	2	10 51	71 5·5	72 1·2
,, 10 (*c*)	1	14 31	72 33·9	
	2	14 56	72 33·6	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Aug. 9.	D	(a) 12 9	1·5223	
	V	(a) 11 10	1·5226	1·5177
„ 10.	D	(c) 12 38	1·5265	
	V	(c) 13 10	1·5252	1·5211
„ 9.	V	(b) 15 36	1·5906	1·5859
„ 10.	V	(b) 11 16	1·5941	1·5894

NOTE.—The ground at Portree was known to be extremely bad as there is much basaltic rock in the neighbourhood. The observations were made with a view of gaining information as to the magnitude of the disturbing forces.

44. Row (Gairloch). July 23, 1888; T. E. T. (61, 83). Lat. 56° 1′ 0″; Long. 4° 46′ 50″. Near the shore of the loch; Pier end bears due W. (mag.); Roseneath House 37° W. of S.; and Roseneath Point 17° W. of S. (mag.).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 2 23	10 24	21 24·4	
	12 0	21 22·6	21 47·7
+ 1 16	13 13	21 24·0	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	12 28	70 48·4	
2	12 52	70 47·5	70 51·0

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	11 31	1·6025	
V	10 53	1·6023	1·5978

45. SCARNISH (Tiree). August 8 and 10, 1885 ; A. W. R. and A. P. L. (60, 74). Lat. 56° 30′ 12″; Long. 6° 47′ 20″. Dips about 15 yards in front of Hotel. Declinations: Station I. 30 yards W.S.W. of Inn. Station II. Inn bears S.W. by S. (200 yards); Harbour Mouth bears S. Station III. Inn bears S.W. by S. (250 yards). On returning to Oban, when a comparison could be made with Greenwich time, the rate of the chronometer was found to have altered suddenly during the visit to Tiree. The difference between the values of the declinations obtained at Stations I. and II. on the 8th and 10th is probably due to this fact.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
Aug. 8 (1)	− 1 43	11 36	24 49·8 }	24 46·9
„ 10 (1)	− 2 49	9 24	24 51·8 }	
„ 10 (3)	+ 5 57	17 0	24 52·4 }	24 47·3
	..	18 51	24 50·0 }	
„ 8 (2)	+ 3 27	16 34	23 52·1 }	23 49·6
„ 10 (2)	+ 4 13	16 30	23 54·9 }	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
Aug. 8	1	11 43	71 20·0	71 19·4
	2	12 50	71 19·9	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Aug. 8	D	13 6	1·5898	1·5909
	V	12 9	1·5907	

46. SOA (Skye).   August 29, 1884 ; A. W. R. and T. E. T. (60, 74).   Lat. 57° 9′ 45″; Long. 6° 10′ 12″. On the N.E. point; about 300 yards E. of the anchorage, and near the Ru Mhoil Dearg.

Declination.

Σ.	G.M.T.	δ.	$\hat{\iota}_0$.
h. m.	h. m.	° ′	° ′
− 4 41	8 38	23 28·3	23 14·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 8 48 9 31	° ′ 72 0·4 72 1·5	° ′ 71 59·6

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 10 16 9 21	1·5051 1·5045	1·5072

47. STIRLING.   July 10, 1885;   A. W. R. (60, 74).   Lat. 56° 7′ 2″; Long. 3° 56′ 55″.   In the King's Park; in the centre of the plain to the S. of the Castle.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 42	h. m. 14 53	° ′ 21 33·0	° ′ 21 28·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 54 16 26	° ′ 70 53·7 70 54·1	° ′ 70 53·3

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 17 11 14 32	1·5938 1·5935	1·5945

48A. STORNOWAY (Ard Point).   September 4 and 5, 1884, and August 19, 1885, A. W. R. and T. E. T. (60, 74); and August 14, 1888, T. E. T. (61, 83), A. W. R. (60, 74).   Lat. 58° 12′ 10″; Long. 6° 23′ 40″.   On the top of the hillock on the Point.   The Declinations taken on this spot agreed well with that obtained in the Castle Grounds, and were therefore regarded as normal.   In 1888, simultaneous observations were made by both of us, which agree among themselves but differ considerably from each other.   Dr. THORPE occupied the old station near the top of the hillock, Professor RÜCKER was about 50 yards distant on lower ground and nearer the mainland.   There can be little doubt that the station is disturbed.

Declination.

Date.	Σ.	G.M.T.	δ.	δ₀.
	h. m.	h. m.	° ′	° ′
Sept. 4, 1884	− 2 22	10 54	24 21·0	
„ 5, „	− 1 42	11 3	24 20·2	24 9·8
Aug. 19, 1885	+ 3 11	14 7	24 12·4	24 9·5
„ 14, 1888	− 1 52	11 2	T ⎧ 23 50·7	
		12 28	T ⎨ 23 48·4	24 13·9
	+ 3 24	15 15	⎩ 23 50·7	
	− 1 45	11 8	R ⎧ 24 8·7	
		12 37	R ⎨ 24 7·9	24 32.3
	+ 3 23	15 15	⎩ 24 8·3	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ₀.
		h. m.	° ′	° ′
Aug. 14, 1888	1	13 3	72 8·4	72 10·5
	2	13 31	72 8·5	

Horizontal Force.

Date.	G.M.T.		H.	H₀.
		h. m.		
Aug. 19, 1885	D	16 44	1·5197	
	V	15 13	1·5210	1·5210
	V	17 42	1·5191	
Aug. 14, 1888	D	12 1	1·5205	1·5205
	V	11 25	1·5242 ⎫ R	
	D	12 0	1·5243 ⎭	1·5195
	V	11 20	1·5222 ⎫ T	
			1·5218 ⎭	1·5173

48B. STORNOWAY. September 4, 1884, A. W. R. and T. E. T. (60, 74). Lat. 58° 12′ 40″; Long. 6° 23′ 35″. In the Castle Grounds.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 32	h. m. 14 20	24 18.4	24 7.6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 21 12 54	72 9.4 72 11.0	72 9.1

Horizontal Force.

	G.M.T.	H.	H_0.
D	h. m. 13 5	1.5119 1.5124	1.5145
V	13 44 13 59	1.5122 1.5125	1.5147

49. STRACHUR. August 16 and 17, 1884; A. W. R. and T. E. T. (60, 74). Lat. 56° 10′ 20″; Long. 5° 4′ 40″. On the lawn in front of Strachur House.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Aug. 17	h. m. − 4 4	h. m. 8 59	22 1.5	21 48.9

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Aug. 16	1 2	h. m. 12 17 12 19	70 44.4 70 44.9	70 42.9

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Aug. 16	D V	h. m. 15 38 16 19	1·6068 1·6073	1·6095

50. STRANRAER.   September 18, 1884; A. W. R. (60, 74) ; and August 28, 1888 ; A. W. R. (60, 83), and T. E. T. (61, 83).   Lat. 54° 54′ 25″ ; Long. 5° 2′ 10″. In a field 300 yards N.W. by W. of Schuchan Church.   The same position was occupied on both occasions.   Near Mr. WELSH's station.

Declination.

Date.	Σ.	G.M.T.	$\hat{\delta}$.	$\hat{\delta}_0$.
Sept. 18, 1884 Aug. 28, 1888	h. m. + 2 7 − 1 30 − 2 10	h. m. 16 14 11 35 10 54 13 6	° ′ 21 46·2 21 13·7 21 12·5 21 13·0	° ′ 21 35·0 21 37·6 T 21 36·6 R

Inclination.

Date.	Needle.	G.M.T.	θ	θ_0
Sept. 18, 1884 Aug. 28, 1888	1 2 1 2	h. m. 15 1 15 37 11 16 12 43	° ′ 70 14·6 70 13·7 70 11·7 70 10·5	° ′ 70 12·3 70 14·8

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 18, 1884 Aug. 28, 1888	D V D V D V	h. m. 17 25 16 39 11 50 11 21 12 37 12 3	1·6421 1·6426 1·6466 } R 1·6471 } 1·6473 } T 1·6473 }	1·6446 1·6420 1·6425

51. STROMNESS (Orkneys). August 25, 1885; T. E. T. (60, 74). Lat. 58° 57′ 30″;
Long. 3° 17′ 12″. 60 yards S.E. from the door of the house on Holm of Stromness.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 9	h. m. 12 29	° ′ 21 31·5	° ′ 21 27·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 1 14 28	° ′ 72 12·3 72 11·9	° ′ 72 11·7

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 12 8	1·5143	1·5149

52. E. LOCH TARBERT (Loch Fyne). August 19 and 20, 1884; A. W. R. and T. E. T.
(60, 74). Lat. 55° 51′ 56″; Long. 5° 24′ 25″. At the back of the town, near the
Castle.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Aug. 19 „ 20	h. m. + 5 43 − 3 20	h. m. 18 50 9 46	° ′ 22 14·4 22 19·3	° ′ 22 4·3

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
Aug. 20	1	10 2	70 47·8	70 46·8
	2	10 26	70 49·5	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Aug. 20	D	11 33	1·6027	1·6053
	V	10 22	1·6031	

53. THURSO.   July 3, 1885 ; A. W. R. (60, 74).   Lat. 58° 35′ 30″ ; Long. 3° 31′ 15″ ;
On the green by the right bank of the river, about 30 yards above the bridge.
The station occupied by Mr. WELSH had been built over.

Declination.

Σ.	G.M.T.	δ.	δ.
h. m.	h. m.	° ′	° ′
− 1 44	11 11	21 43·5	21 38·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	13 7	72 2·9	72 1·1
2	13 43	72 0·6	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	14 38	1·5208	1·5217
V	11 54		

54. WICK.   July 2, 1885 ; A. W. R. (60, 74).   Lat. 58° 26′ 20″ ; Long. 3° 5′ 45″
On the lawn in front of Rosebank.   Same station as that of Mr. WELSH.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 2 52	h. m. 11 6	°　　　′ 21 20·4	°　　′ 21 15·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 25 14 0	°　　′ 72 10·1 72 10·7	°　　′ 72 9·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 11 56 10 44	1·5139 1·5131	1·5144

DESCRIPTIONS OF ENGLISH STATIONS.

55. ABERYSTWITH.   May 18, 1886 ; A. W. R. (60, 74).   Lat. 52° 23′ 51″; Long. 4° 5′ 13″.   About a mile to the South of the town, between the River Ystwith and the sea, near the point marked Pen-y-ro on the Ordnance map.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 1 19	11 19	19 54·6	19 56·5
	13 42	19 51·7	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	18 2	68 34·4	68 34·7
2	13 4	68 33·9	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	12 16	1·7508	1·7500
V	12 34	1·7485	1·7477

56. ALDERNEY.   April 3 and 4, 1888 ; T. E. T. (61, 83).   Lat. 49° 43′ 10″; Long. 2° 11′ 0″.   In the N.E. corner of the garden of Scott's Hotel; about 70 yards from the house.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
April 3	+ 1 54	12 56	17 46·1	
	..	15 9	17 46·5	18 2·8
	+ 4 27	16 48	17 45·1	
,, 4	− 4 2	7 54	17 49·3	

Inclination.

Date.	Needle.	G.M.T.	$\theta.$	$\theta_0.$
		h. m.	° ′	° ′
April 3	1	16 1	66 35·2	66 38·9
	2	16 23	66 33·7	

Horizontal Force.

Date.	G.M.T.		H.	$H_0.$
		h. m.		
April 3	D	14 45	1·8755	1·8705
	V	13 41	1·8756	
			1·8719	1·8677
	V	15 21	1·8735	

57. ALNWICK.   September 16, 1886.   A. W. R.   Lat. 55° 25′ 19″; Long. 1° 43′ 53″. In the Deer Park; on the N. side of the drive, and about one-third of a mile from the reservoir.   Alnwick Church S.E. by E.   Castle E. by S.

Declination.

$\Sigma.$	G.M.T.	$\delta.$	$\delta_0.$
h. m.	h. m.		
− 1 7	11 26	19 39·4	19 45·0

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
	h. m.	° ′	° ′
1	13 1	70 2·1	70 3·6
2	13 32	70 3·3	

Horizontal Force.

G.M.T.		H.	$H_0.$
	h. m.		
P	14 20	1·6529	1·6511
V	11 50	1·6526	

58. ALRESFORD.   April 28, 1888 ; A. W. R. (60, 74).   Lat. 51° 4′ 46″ ; Long. 1° 9′ 56″.
On Tichborne Down.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 39 3 45	h. m. 13 16 16 19·5	° ′ 17 52·8 17 53·2	° ′ 18 9·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 26 16 2	° ′ 67 17·8 67 19·0	° ′ 67 22·3

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 14 42 14 2	1·8292 1·8293	1·8241

59. APPLEBY.   September 15, 1886 ; A. W. R. (60, 74).   Lat. 54° 34′ 10″ ; Long.
2° 29′ 3″.   On the Holme, a common close to the Eden, to the S. of the town.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −1 1 +2 7	h. m. 11 47 13 51	° ′ 20 1·5 19 59·0	° ′ 20 5·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 46 13 15	° ′ 69 43·7 69 44·1	° ′ 69 44·9

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	12 2	1·6706	1·6690

60. BARROW. August 25, 1886; A. W. R. (60, 74). Lat. 54° 7′ 24″; Long. 3° 13′ 0″. In a field about 200 yards W. from the road which leads to Barrow from Furness Abbey. About $1\frac{1}{2}$ miles from the Abbey on the brow of the hill above Barrow.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+2 15	14 52	20 4·6	20 9·3
+5 18	17 5	20 2·9	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	16 3	69 29·5	69 30·6
2	16 32	69 29·7	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	17 59	1·6886	1·6875
V	15 13	1·6892	

61. BEDFORD. April 21, 1888 ; T. E. T. (61, 83). Lat. 52° 8′ 3″; Long. 0° 26′ 51″. On a road on the Bower estate, half-a-mile to the E. of the town-bridge, and about 80 yards from the river.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
−1 24	9 51	18 10·9	18 27·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ $'$	$\circ$ $'$
1	11 13	68 4·4	68 7·3
2	11 30	68 3·3	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	10 8	1·7756	1·7705

62. BIRKENHEAD.   August 23, 1886 ; A. W. R. (60, 74).   Lat. 53° 24′ 4″; Long. 3° 4′ 18″.   1st station, 130 yards S. of Observatory ; 2nd station, 100 yards S. of Observatory.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	$\circ$ $'$	$\circ$ $'$
− 1 1	(1) 12 23	19 54·8	19 58·3
+ 5 2	(2) 17 32	19 52·1	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ $'$	$\circ$ $'$
1	15 37	69 4·0	69 4·3
2	16 4	69 2·6	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	13 29	1·7188	1·7176
V	12 50	1·7192	

63. BIRMINGHAM.   May 7, 1886; A. W. R. (60, 74).   Lat. 52° 27′ 37″; Long. 1° 53′ 40″.
In Calthorpe Park, at the S. end.   About 50 yards from the East, and 200 yards
from the South railings.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. − 1 14 + 2 44	h. m. 11 15 14 29	° ′ 18 42·1 18 39·7	° ′ 18 44·0

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 13 24 13 55	° ′ 68 21·3 68 20·1	° ′ 68 21·3

Horizontal Force.

G.M.T.		H.	H₀.
D V	h. m. 12 22 11 37	1·7672 1·7683	1·7669

64. BRAINTREE.   September 22, 1888; T. E. T. (61, 83).   Lat. 51° 52′ 41″; Long.
0° 32′ 40″ E.   In a field to the W. of the Church, distant 400 yards ; 400 yards
N. of the Railway to Dunmow.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. + 3 34	h. m. 15 51	° ′ 17 36·4	° ′ 17 55·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 16 17 32	$\overset{\circ}{67}$ $\overset{'}{41 \cdot 8}$ 67 40·8	$\overset{\circ}{67}$ $\overset{'}{45 \cdot 4}$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 16 40 16 15	1·8005 1·7999	1·7942

65. BRECON.   May 26, 1886 ; A. W. R. (60, 74).   Lat. 51° 56′ 56″ ; Long. 3° 24′ 42″. In the fields (Newton Port) to the W. of the town, and close to the river. Church in centre of town S.E. by E.   About $\frac{3}{4}$ of a mile E. by N. of Castle Hotel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 4 42	h. m. 17 30	$\overset{\circ}{19}$ $\overset{'}{35 \cdot 1}$	$\overset{\circ}{19}$ $\overset{'}{38 \cdot 6}$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
2	h. m. 19 12	$\overset{\circ}{68}$ $\overset{'}{15 \cdot 1}$	$\overset{\circ}{68}$ $\overset{'}{15 \cdot 8}$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 18 28 17 51	1·7710 1·7710	1·7701

66. BUDE HAVEN.   T. E. T. (61, 83).

(*a*) April 11 and 15, 1887 ; Lat. 50° 49′ 34″ ; Long. 4° 32′ 37″.   On the upper walk of the kitchen garden of the Falcon Hotel, and on the W. front of the house.

(*b*) April 12, 1887 ; Lat. 50° 49′ 42″; Long. 4° 32′ 54″.   On Efford Down near the Compass Tower, which bore 75° W. of N. (mag.).   Belfry of Efford House 10° E. of S. (mag.).   Entrance lock to Bude Canal 68° E. of N. (mag.).

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
April 11 . .	− 1 46	11 2	19 50·1	
	+ 3 7	15 2	19 52·2	20 0·1
,, 22 . .	− 1 32	11 13	19 43·9	
	+ 2 44	14 41	19 43·5	19 52·9
,, 15 . .	− 0 20	12 26	19 44·2	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
April 11 . .	1	15 58	67 44·8	
	2	14 31	67 40·2	
,, 12 . .	1	13 46	67 41·8	67 44·2
	2	14 16	67 41·1	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
April 11 . .	D	12 14	1·8126	
	V	11 47	1·8100	1·8085
12 . .	D	12 2	1·8103	
	V	11 36	1·8104	1·8075

67. CAMBRIDGE.   T. E. T. (61, 83).

May 18, 1886; Lat. 52° 11′ 40″; Long. 0° 7′ 17″ E.   In the lane off the Trumpington Road, past the Botanic Gardens, and on the right hand side.

May 19 and 20, 1886; Lat. 52° 11′ 23″; Long. 0° 7′ 16″ E.   In the Leys School Grounds.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
May 18.	+2 28			
	+2 37			
„ 19.	..	11 47	18 2·1	
		15 35	18 0·5	18 5·0
„ 20.	−1 51	10 34	18 2·1	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
May 18.	1	16 39	68 1·8	
	2	17 11	68 1·9	68 2·4

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
May 19.	D	12 37	1·7799	
	V	11 11	1·7786	1·7784
„ 20.	V	10 58	1·7797	
			1·7790	1·7785

68. CARDIFF.   May 25, 1886; A. W. R. (60, 74).   Lat. 51° 29′ 36″; Long. 3° 10′ 33″. In a field W. of the town, on Crwys Farm.   House E.N.E., about 80 yards distant; gate of barracks, 200 yards N. by W.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+3 0	16 1	19 16·6	
+4 33	17 5	19 15·9	19 19·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 38 18 3	67 52·8 67 50·5	67 52·3

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 16 22	1·7953	1·7944

69. CARDIGAN.   May 29, 1886 ; A. W. R. (60, 74).   Lat. 52° 5′ 20″; Long. 4° 40′ 9″. In a field about half a mile to the west of the town.

Declination.

Σ.	G.M.T.	δ.	$\hat{\delta}_0$.
h. m. + 1 12	h. m. 12 15	20 22·2	20 25·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 6 . 11 43	68 30·3 68 31·0	68 31·3

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 13 47	1·7544	1·7535

70. CARLISLE. August 28, 1886; A. W. R. (60, 74). Lat. 54° 53′ 55″; Long. 2° 55′ 40″. On the Swifts, about 100 yards from the footpath on the S. side, and 250 yards E. by S. of the Grand Stand. Cathedral bore W.S.W. ½ S.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 0 4J + 1 30	h. m. 12 1 14 41	° ′ 20 20·5 20 19·8	° ′ 20 25·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 17 14 14	° ′ 69 52·1 69 54·0	° ′ 69 54·0

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 12 25	1·6640	1·6625

71. CHESTERFIELD. September 14 and 15, 1887; A. W. R. (60, 74). Lat. 53° 14′ 3″; Long. 1° 24′ 37″. In a field on Mr. PENISTONE's farm to the N. of the main road. Church bearing W.N.W. about a mile distant. Cemetery S.W. by W. about half a mile.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Sept. 15	h. m. − 2 36 − 2 10	h. m. 10 19 ··	° ′ 19 0·3 18 59·8	° ′ 19 11·9

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Sept. 14	1 2	h. m. 11 40 12 10	68 47·1 68 44·5	68 48·5

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 14	D V	h. m. 14 16 13 26	1·7383 1·7393	1·7351

72. CHICHESTER.   September 18, 1888; T. E. T. (61, 83). Lat. 50° 50′ 0″; Long. 0° 47′ 2″.   In the meadows between the Cathedral and the Railway Station; close to the ditch.   Railway Station bearing S.E.   Cathedral tower N.E., distant a quarter of a mile.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 0 52	h. m. 11 32	17 46·6	18 5·5

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 27 13 43	67 7·6 67 7·0	67 11·6

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 36 11 54	1·8456 1·8455	1·8395

73. Clenchwarton.   August 2, 1888 ; A. W. R. (60, 74).   Lat. 52° 45′ 20″; Long. 0° 21′ 20″ E.   In the grounds of a house on the road from King's Lynn to Sutton Bridge.   Not quite 3 miles from King's Lynn.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m. −0 29	h. m. 11 58	° ′ 17 51·4	° ′ 18 10·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 13 45	° ′ 68 14·3	° ′ 68 17·9

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 49 12 16	1·7722 1·7717	1·7662

74. Clifton.   April 22, 1886 ; T. E. T. (61, 74).   Lat. 51° 27′ 15″; Long. 2° 37′ 4″. On the Down ; on the N. side of the Stoke road, about 60 yards from it, and about ¼ mile from the Clifton Cricket Club Pavilion.   Steeple of Christ Church Congregational Chapel bearing E.   Pumping station S.E. about 150 yards.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m. +0 15	h. m. 11 56	° ′ 19 8·6	° ′ 19 10·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 13 12	° ′ 67 48·9	° ′ 67 48·7
2	13 47	67 47·6	

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 14 49	1·8001	1·7996
V	11 30	1·8006	

75. CLOVELLY.   April 16, 1887 ; T. E. T. (61, 83).   Lat. 50° 59′ 48″;   Long. 4° 23′ 50″.   To the W. of the village; between it and Clovelly Court.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 0 33	h. m. 12 9	° ′ 19 44·2	° ′ 19 53·8
+ 3 4	15 45	19 45·5	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 14 23	° ′ 67 47·8	° ′ 67 49·9
2	14 57	67 47·6	

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 11 27	1·8027	1·8000
V	12 44	1·8030	

76. COALVILLE.　April 30, 1888; T. E. T. (61, 83).　Lat. 52° 43′ 41″; Long. 1° 21′ 18″.　In a field about 100 yards E. of Coalville Station (L. and N.W. Railway), and E. of town.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 26	h. m. 14 28	18 24·6	18 41·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 11 58 12 17	68 20·3 68 20·6	68 24·1

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 13 1 14 15	1·7583 1·7587	1·7534

77. COLCHESTER.　September 24, 1888; T. E. T. (61, 83).　Lat. 51° 53′ 30″; Long. 0° 54′ 0″ E.　In the Castle grounds; to the S. of the main tower, distant 20 yards.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 13 + 1 18	h. m. 10 52 12 52	17 36·2 17 36·2	17 55·2

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 11 21 11 37	67 31·3 67 31·1	67 35·3

Horizontal Force.

	G.M.T.	H.	H$_0$.
	h. m.		
D	12 28	1·8078	1·8012
V	12 2	1·8066	

78. CROMER. May 10, 1886; T. E. T. (61, 83).   Lat. 52° 55' 20"; Long. 1° 18' 24" E. In a field S.S.W. of the Lighthouse; Railway Station about three-quarters of a mile away bearing W.; town bearing N.W. by N.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° '	° '
+ 1 0	13 21	17 32·7	17 35·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° '	° '
1	14 26	68 19·6	68 20·0
2	14 53	68 19·2	

Horizontal Force.

	G.M.T.	H.	H$_0$.
	h. m.		
V.	13 40	1·7611	1·7603

79. DOVER. September 28, 1887; T. E. T. (61, 83).   Lat. 51° 6' 53"; Long. 1° 17' 52" E.   In the Public Recreation Ground, near Archcliff Fort; between the town and Shakespeare Cliff.   End of Dover Breakwater 63° E. of S. (mag.); Flagstaff on Shakespeare Cliff 65° W. of S. (mag.).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° '	° '
− 1 58	10 20	16 44·1	16 57·2
+ 0 56	12 26	16 46·2	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\overset{\circ}{67}$ $\overset{\prime}{5\cdot8}$	$\overset{\circ}{}$ $\overset{\prime}{}$
1	11 15	67 5·8	67 8·0
2	11 31	67 4·7	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	11 59	1·8377	1·8336
V	10 34	1·8372	

80. FALMOUTH.   April 8, 1887 ; T. E. T. (61, 83).   Lat. 50° 8′ 47″ ; Long. 5° 4′ 21″.
At the Observatory.

Declination.

Σ.	G.M.T.	$\hat{\delta}$.	$\hat{\delta}_0$.
h. m.	h. m.	$\overset{\circ}{19}$ $\overset{\prime}{43\cdot3}$	$\overset{\circ}{}$ $\overset{\prime}{}$
− 1 51	11 10	19 43·3	19 53·4
+ 2 39	15 42	19 45·8	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\overset{\circ}{67}$ $\overset{\prime}{12\cdot3}$	$\overset{\circ}{}$ $\overset{\prime}{}$
1	14 31	67 12·3	67 15·0
2	13 45	67 13·1	

Horizontal Force.

G.M.T.		H.	$\bar{H}_0$.
	h. m.		
D	11 59	1·8354	1·8323
V	11 35	1·8348	

81. GAINSBOROUGH. September 19, 1887 ; A. W. R. (60, 74). Lat. 53° 23′ 23″ ; Long. 0° 44′ 51″. In a field to the W. of the Upton Road, and about 1 mile S. (mag.) of the town.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +3 56	h. m. 16 16	° ′ 18 24·5	° ′ 18 36·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 53 18 12	° ′ 68 46·4 68 46·7	° ′ 68 49·3

Horizontal Force.

	G.M.T.	H.	H_0.
D V	h. m. 17 14 16 31	1·7362 1·7356	1·7321

82. GIGGLESWICK. September 14, 1886 ; A. W. R. (60, 74). Lat. 54° 4′ 18″ ; Long. 2° 17′ 48″. In the cricket field of Giggleswick School, 150 yards N.W. of the Pavilion, and near the W. end of the field.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +1 33 +3 16	h. m. 12 2 15 8	° ′ 19 31·2 19 29·8	° ′ 19 35·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ $\prime$	$\circ$ $\prime$
1	10 59	69 21·4	69 22·3
2	11 19	69 21·0	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	13 16	1·6976	1·6962
V	12 37	1·6979	

83. GLOUCESTER. May 24, 1886; A. W. R. (60, 74). Lat. 51° 52′ 12″; Long. 2° 14′ 50″. In the fields to the E. of the town, about 300 yards E. by N. of Alexandra Terrace ; Cathedral bearing W.

Declination.

Σ.	G.M.T.	$\hat{\delta}$.	δ_0.
h. m.	h. m.	$\circ$ $\prime$	$\circ$ $\prime$
+1 12	..	19 9·4	19 12·9
+3 26	16 4	19 9·7	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ $\prime$	$\circ$ $\prime$
1	13 46	68 4·5	68 4·3
2	14 19	68 2·8	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	15 1	1·7823	1·7811
V	12 46	1·7817	

84. GRANTHAM. May 22, 1886 ; T. E. T. (61, 83). Lat. 52° 54' 54" ; Long. 0° 37' 46" W. In a field to the E. of the town, and across the river. The Church bore about S.W. ; distant 400 yards.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +1 36	h. m. 13 13	$18°$ $25·6'$	$18°$ $29·0'$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 9 14 30	$68°$ $26·6'$ 68 28·3	$68°$ $28·0'$

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 12 56	1·7536	1·7527

85 and 86. GUERNSEY. T. E. T. (61, 83).

85. L'ERÉE. Lat. 49° 27' 50" ; Long. 2° 35' 40". Fifty yards due S. of L'Erée Hotel, and about 15 yards from the road to Roquaine Castle.

86. PETER PORT. Lat. 49° 27' 45" ; Long. 2° 31' 45". In the Gardens behind the Royal Hotel, and about 60 yards from the house.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
April 2 . . .	h. m. + 2 32	h. m. 14 22	18 $16·6'$	$18°$ $32·7'$
„ 4 . . .	+ 2 43	14 19	18 2·3	18 18·4

Inclination.

Date.	Needle.	G M.T.	θ.	θ_0.
		h. m.	° ′	° ′
April 2 . . .	1	13 2	66 28·9	66 34·1
	2	13 38	66 30·3	
,, 4 . . .	1	16 2	66 29·0	66 32·4
	2	16 17	66 26·8	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
April 2 . . .	D	15 20	1·8798	1·8746
	V	14 10	1·8795	
,, 4 . . .	D	15 28	1·8843	1·8796
	V	14 37	1·8849	

87. HARWICH. May 8, 1886; T. E. T. (61, 83). Lat. 51° 56′ 48″; Long. 1° 17′ 5″ E. In a field to the W. of the town; Harwich Church bearing E. by N.; Great Eastern Hotel, N.E., and Railway Station, E.S.E.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 1 10	11 38	17 15·5	17 18·8
	13 41	17 16·0	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	12 30	67 37·0	67 38·4
2	13 6	67 38·7	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	14 17	1·8040	1·8031
V	11 13	1·8039	

88. HARPENDEN.   October 5 and 6, 1887, and May 1, 1888 ; T. E. T. (61, 83).
    Lat. 51° 47′ 27″; Long. 0° 21′ 15″.   1st Station. At the N. end of the Common ;
    200 yards S. of the Railway Hotel, and W. of the Railway Station.   2nd
    Station. In Dr. GILBERT's garden.   To the W. of the lower Common.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
May 1, 1888	h. m. + 5 13	h. m. 14 33	° ′ 17 59·6	° ′ 18 16·5

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Oct. 6, 1887·	1	h. m. 16 13	° ′ 67 53·4	° ′
	2	16 32	67 48·4	67 53·5
May 1, 1888	1	18 7	67 48·2	
	2	17 48	67 47·6	67 51·4

Horizontal Force.

Date.		G M.T.	H.	H_0.
Oct. 5, 1887	D	h. m. 15 36	1·7961 ⎱ 1·7970 ⎰	1·7926
	V	15 13		
May 1, 1888	V	14 50	1·7971	1·7920
	V	15 6	1·7960	1·7909

89. HASLEMERE. September 30, 1887; T. E. T. (61, 83). Lat. 51° 6′ 4″; Long. 0° 44′ 40″.
    On the tennis-lawn of Professor WILLIAMSON's house, High Pitfold, and about
    50 yards S. of it.   About 600 feet above sea level.

Declination.

Σ	G.M.T.	δ.	δ_0.
h. m. — 1 6	h. m. 11 12	° ′ 17 55·6	° ′ 18 7·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 12 57	$\overset{\circ}{67}$ $\overset{\prime}{17 \cdot 8}$	$\overset{\circ}{67}$ $\overset{\prime}{20 \cdot 6}$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 17 11 46	1·8321 1·8319	1·8282

90. HOLYHEAD.   May 4, 1887 ; T. E. T. (61, 83).   Lat. 53° 17′ 53″; Long. 4° 38′ 22″.
On the road to Porth Dafarch, 20 yards from the roadside, and about a mile from
the Railway Station which bore 50° E. of N. ; Monument on Black Bridge, 55°
E. of N. ; Spire of New Church, 30° E. of N. ; Windmill, 80° E. of N. ;
Summit of Holyhead Mountain, 30° W. of N.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 51 + 1 42 + 2 0	h. m. 10 50 13 42	$\overset{\circ}{20}$ $\overset{\prime}{40 \cdot 3}$ 20 41·8	$\overset{\circ}{20}$ $51 \cdot 1$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 46 13 10	$\overset{\circ}{69}$ $22 \cdot 2$ 69 20·1	$\overset{\circ}{69}$ $23 \cdot 1$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 11 50 11 11	1·6988 1·6986	1·6958

91. HORSHAM. April 21, 1888 ; A. W. R. (60, 74). Lat. 51° 4′ 16″ ; Long. 0° 21′ 54″.
In a field on the N.W. side of Barber's Green, 1½ miles W. of Horsham.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+ 0 44	13 14	17 47·0	
+ 3 16	15 37	17 47·3	18 3·3
+ 5 4	17 26	17 46·1	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	16 17	67 11·6	67 15·2
2	16 39	67 11·4	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	14 44	1·8358	1·8309
V	13 34	1·8363	

92. HULL. Sept. 16 and 17, 1887 ; A. W. R. (60, 74). Lat. 53° 44′ 40″ ; Long.
0° 22′ 5″. In the Botanic Gardens.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
Sept. 16 . .	+ 5 38	18 1	18 45·7	18 57·8
„ 17 . .	− 3 3	9 20	18 46·2	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	$\circ$ $\prime$	$\circ$ $\prime$
Sept. 16 . .	1	16 1	69 1·8	69 3·9
	2	16 44	69 1·3	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Sept. 16 . .	V	17 18	1·7163	1·7125

93. ILFRACOMBE.   April 25, 1886 ; T. E. T. (61, 83).   Lat. 51° 12′ 38″ ; Long. 4° 7′ 36″. In a field near the Torr's Walk, between the town and the sea.   Parish Church, bearing S. by E. (½ mile).   Ilfracombe Hotel, E. by N.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	h. m.	$\circ$ $\prime$
− 1 32	11 12	19 45·5	19 46·5
+ 1 51	13 48	19 43·3	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ $\prime$	$\circ$ $\prime$
1	13 15	67 53·3	67 53·8

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	14 48	1·7962	1·7952
V	11 33	1·7957	

94, 95, 96. JERSEY.   T. E. T. (61, 83).

    94. Grouville.   Lat. 49° 12′ 5″; Long. 2° 1′ 20″.   In the eastern ditch of the Fort.   Railway Station bore about W.

    95. S. Louis.   Lat. 49° 12′ 0″; Long. 2° 5′ 40″.

    96. S. Owen.   Lat. 49° 13′ 30″; Long. 2° 12′ 5″.   Within 300 yards of the shore, on the flat ground near the Marée.   Corbière Lighthouse, 40° W. of S. (mag.).

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
Grouville March 30	+1 19	13 16	18 19·9	
	+2 20	14 24	18 19·9	18 36·0
S. Louis „ 31	+1 20	12 12	17 34·4	
	..	13 56	17 32·9	17 49·7
S. Owen April 1	+2 21	12 37	18 7·6	
	..	15 1	18 9·7	18 24·7

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
Grouville March 31	1	9 49	66 3·4	
	2	9 9	66 5·1	66 8·7
S. Louis „ 31	1	14 32	66 7·8	
	2	15 0	66 9·2	66 13·0
S. Owen April 1	1	14 4	66 9·8	
	2	14 20	66 12·2	66 15·5

Horizontal Force.

Date.		G.M.T.	H.	H_0.
		h. m.		
Grouville March 30	D	15 13	1·9099 ⎫	
	V	14 7	1·9106 ⎬	1·9053
S. Louis „ 31	D	13 6	1·8938 ⎫	
	V	12 28	1·8934 ⎬	1·8887
S. Owen April 1	D	13 28	1·8952 ⎫	
	V	12 51	1·8955 ⎬	1·8904

97. KENILWORTH.   April 16, 1888 ; T. E. T. (61, 83).   Lat. 52° 20′ 51″ ; Long. 1° 34′ 43″.   In a field near the Abbey Hotel, 60 yards from the road ; Castle bearing 65° W. of N., and S. Nicholas' Church steeple 24° W. of N. (all magnetic).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −1 23 +1 16	h. m. 11 9 13 42	° ′ 18 45·2 18 44·8	° ′ 19 1·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 47 13 5	° ′ 68 24·1 68 26·2	° ′ 68 28·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 4 11 28	1·7628 1·7624	1·7576

98. KETTERING.   May 1, 1888 ; T. E. T. (61, 83).   Lat. 52° 23′ 44″ ; Long. 0° 44′ 10″.   In the second field past the Railway Arch, W. of the Church, and 70 yards from the Northampton Road.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 2 37	h. m. 9 40	° ′ 18 19·1	° ′ 18 36·0

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
	h. m.	° ′	° ′
1	10 56	68 7·7	68 10·7
2	11 12	68 6·3	

Horizontal Force.

G.M.T.		H.	$H_0.$
	h. m.		
V	10 0	1·7707	1·7656

99. KEW. Lat. 51° 28′ 6″; Long. 0° 18′ 45″. In the Magnetic House at the Observatory.

Declination.

Date.	Observer.	Instrument.	G.M.T.	$\delta.$	$\delta_0.$
			h. m.	° ′	° ′
July 17, 1884	T.	60	16 9	18 23·7	18 15·8
„ 18, „	R.	60	11 59	18 21·9	18 14·0
April 2, 1886	T.	61	17 14	18 11·0	18 12·5
September 30, 1887 . .	R.	60	12 33	18 3·4	18 14·0
			15 38	18 5·1	18 15·7
October 11, 1887 . . .	T.	61	12 11	18 3·9	18 13·8
			15 33	18 4·0	18 13·9
„ 12, „ . . .	R.	60	12 14	18 4·8	18 14·7
			14 21	18 4·3	18 14·2
„ 13, „ . . .	T.	61	10 33	18 5·3	18 15·2
			15 30	18 4·6	18 14·5
„ 18, „ . . .	T.	61	10 47	18 4·1	18 14·0
			14 56	18 2·6	18 12·5
„ 19, „ . . .	R.	60	11 9	18 4·8	18 14·7
			16 1	18 6·7	18 16·6
				Mean . .	18 14·4

It will be observed that the differences between these results are not exactly in accord with those which would be deduced from the table on p. 58.

This is due to the fact that in the above table the observations have been treated as though Kew were an ordinary station, whereas, on p. 58, the corrections have been applied in the most accurate way.

In the first place, no attempt is made on p. 58 to reduce the observations to epoch,

whereas in the above table they are reduced by the mean coefficient of secular change at Kew (6'·1) during the period of the survey, together with a mean correction for monthly variation.

In the next place in the reduction on p. 58, no distinction was made between the diurnal variation and disturbance, the total divergence from the mean being read off directly. The corrections applied in the above table were obtained by reading off the disturbances only and afterwards adding the value of the diurnal variation.

The differences are a little larger than we should have expected, but this is probably due to the fact that, as the corrections were wanted quickly, the constants of the curves were only determined provisionally.

That the accuracy attained was practically sufficient is proved by a comparison of the numbers in the above table with those on p. 58.

The mean difference between our results and those given by curves treated as accurately as possible is $\pm\ 0'·81$, while the mean difference between the numbers given above and their mean is $\pm\ 0'·83$.

The Kew value for January, 1886, is $18°\ 16'·3$, and ours, from the above table when reduced to the Kew instrument by adding the mean difference, is $18°\ 14'·4 + 2'·5 = 18°\ 16'·9$.

If from each of our uncorrected observations given on p. 58, we subtract the difference between the corresponding Kew value and $18°\ 16'·3$, we reduce to January, 1886, by one operation, in which no distinction is made between disturbance and the diurnal, monthly, and secular variations. The largest of the numbers so obtained is $18°\ 16'·1$, and the smallest $18°\ 11'·9$, while the largest and smallest in the above table are $18°\ 16'·6$ and $18°\ 12'·5$. Thus, not only are the mean results obtained by the two methods in close accord, but the range of variation of the results is in each case practically identical; in other words, they agree to within the limits of the error of experiment.

Horizontal Force.

Date.	Observer.	Instrument.	H.	H_0.
July 17, 1884	R.	60	1·8075	1·8100
April 2, 1886	T.	61	1·8082	1·8078
„ 2, „	R.	60	1·8098	1·8094
„ 22, „	R.	60	1·8101	1·8096
September 30, 1887 . . .	R.	60	1·8112	1·8082
October 11, 1887	T.	61	1·8114	1·8084
„ 12, „	R.	60	1·8115	1·8085
„ 13, „	T.	61	1·8114	1·8084
„ 18, „	T.	61	1·8123	1·8092
„ 18, „	T.	61	1·8125	1·8094
„ 19, „	R.	60	1·8125	1·8094
			Mean . . .	1·8089

Inclination.

Date.	Needle.	Observer.	Dip Circle.	G.M.T.	θ.	θ_0.
				h. m.	° ′	° ′
July 17, 1884 . .	1	R.	74	15 38	67 36·0	67 34·8
„ 18, „ . .	1	T.	74	12 9	67 35·7	67 34·5
	2	T.	74	12 38	67 36·0	67 34·8
„ 19, „ . .	1	R.	74	12 12	67 36·7	67 35·5
	2	R.	74	12 47	67 36·1	67 34·9
September 30, 1887	1	R.	74	16 43	67 35·4	67 36·8
	2	R.	74	17 10	67 34·2	67 35·6
October 11, 1887 .	1	T.	83	14 33	67 34·2	67 35·6
	2	T.	83	15 7	67 35·4	67 36·8
„ 13, „ .	1	T.	83	14 27	67 35·0	67 36·4
	2	T.	83	15 0	67 35·0	67 36·4
„ 18, „ .	1	T.	83	13 52	67 24·2	67 35·6
	2	T.	83	14 28	67 34·3	67 35·7
„ 19, „ .	1	R.	74	13 38	67 35·3	67 36·7
	2	R.	74	14 2	67 34·5	67 35·9
					Mean . .	67 35·8

In the following Table the values for January 1, 1886, deduced above from the survey instruments, are given in Column I. In Column II. are the mean differences between the survey and Kew instruments (see p. 58), and hence in Column III. the Kew (calculated) values are deduced. In Column IV. are the published elements for that epoch ('Roy. Soc. Proc.,' vol. 41, 1887, p. 416), the Declination being corrected for diurnal variation.

I.	II.	III.	IV.
18° 14′·4	+ 2′·5	18° 16′·9	18° 16′·3
1·8089	− 0·0029	1·8060	1·8093
67° 35′·8	+ 2′·7	67° 38′·5	67° 37′·4

The agreement between Columns III. and IV. is satisfactory in the cases of the Declination and Dip. In that of the Horizontal Force the actual value in January, 1886, seems to have been rather high. As a good deal will hereafter be said about the relative values of the elements at Kew and Greenwich, and as we have not compared our instruments with those at Greenwich, it has been thought better to treat both these observatories in the same way, and therefore to use the numbers in Column IV. as the Kew values.

100. KING'S LYNN.

(*a*) May 20, 1886 ; T. E. T. (61, 83).   Lat. 52° 45′ 43″; Long. 0° 24′ 30″ E.
In a field to the E. of the town ; S. Nicholas' Church half-a-mile away,
bearing W.S.W. ; Waterworks, S.W., quarter of a mile away.

(*b*) August 2, 1888 ; A. W. R. (60, 74).   Lat. 52° 45′ 20″; Long. 0° 26′ 0″ E.
In a field on the S. side of the road from Gaywood to Gayton Road
Station.   About half a mile from the village of Gaywood.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
May 20, 1886	+5 12	16 14	17 54·5	17 57·9
(Gaywood) Aug. 2, 1888	+3 26	15 47	17 42·8	18 1·7

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
May 20, 1886	2	17 15	68 17·3	68 17·8
Aug. 2, 1888	1	16 39	68 13·2	68 16·8

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
May 20, 1886	V	15 51	1·7664	1·7656
Aug. 2, 1888	V	16 5	1·7703	1·7646

101. KING'S SUTTON.   April 14, 1888; T. E. T. (61, 83).   Lat. 52° 1′ 9″; Long. 1° 16′ 19″.
In a field 25 yards E. of the road to Aynho, and parallel to the Great Western
line ; King's Sutton Church bearing 35° W. of N., and distant 300 yards.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+1 27	13 45	18 35·4	18 51·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 35 16 1	68 2·6 68 2·4	68 6·4

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 14 55 14 23	1·7825 1·7832	1·7778

102. LAMPETER. May 31, 1886 ; A. W. R. (60, 74). Lat. 52° 6′ 38″ ; Long. 4° 4′ 36″.
In a field to the S. of the town. Church bearing a little to the W. of N.
Distance from the Church, about two-thirds of that between it and the river ;
250 yards from the nearest point of a bend in the river.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. — 1 35	h. m. 11 2	° ′ 19 51·7	° ′ 19 55·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 34 12 59	68 24·6 68 25·1	68 25·5

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 13 38 11 26	1·7569 1·7567	1·7559

103. LEEDS.   September 22 and 24, 1886; A. W. R. (60, 74).   Lat. 53° 50′ 58″; Long. 1° 35′ 3″.   In the centre of a field behind St. Helen's, the residence of O. Eddison, Esq., at Adel.   The Dip observations were repeated in the same position on December 31, 1888.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
Sept. 22, 1886	+ 4 2	16 38	19 3·9	19 8·9

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
Sept. 24, 1886	1	9 58	69 10·7	69 10·8
	2	10 21	69 8·8	
Dec. 31, 1888	1	10 57	69 7·5	69 11·4
	2	11 30	69 6·3	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Sept. 22, 1886	V	16 51	1·7098	1·7082

104. LEICESTER.   June 19, 1886; T. E. T. (61, 83).   Lat. 52° 37′ 38″; Long. 1° 6′ 41″. Twenty yards E. of Evington Road; Evington, 1½ mile away, bearing S.E. Railway Station (Midland) ¾ of a mile away, bearing N.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+ 1 57	13 4	18 18·0	18 23·6
	16 42	18 21·0	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	o ′	o ′
1	15 48	68 23·0	68 24·0
2	16 16	68 23·5	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	14 46	1·7548	1·7538
V	13 42		

105. LINCOLN. April 26, 1888 ; T. E. T. (61, 83). Lat. 53° 12′ 27″; Long. 0° 31′ 14″. On the South Common, S. of the Cathedral, and about 60 yards from the road.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	o ′	o ′
+3 39	15 35	18 2·2	18 18·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	o ′	o ′
1	16 43	68 40·3	68 43
2	16 57	68 38·3	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	16 0	1·7446	1·7395

106. LLANDUDNO. May 14, 1886; A. W. R. (60, 74). Lat. 53° 19′ 5″; Long. 3° 50′ 23″. Near the Beach to the W. of the Town, steeple bearing E. by N. In a little Bay about 300 yards S. of the road round Great Orme's Head.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 2 45 + 1 37	h. m. 9 1 14 2	° ′ 20 49·5 20 46·9	° ′ 20 51·5

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 36 13 8	° ′ 69 12·5 69 10·4	° ′ 69 12·0

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 11 15 10 29	1·7087 1·7098	1·7084

107. LLANGOLLEN. May 10, 1886; A. W. R. (60, 74). Lat. 52° 58′ 23″; Long. 3° 10′ 13″. On a hillock in a field on the N. of the town, near the road which runs parallel to the canal, and through which the footpath to Castle Dinas Bran passes. Llangollen Church bearing S. Bridge over Canal S. by E. (80 yards distant). Castle Dinas Bran N.E. by N.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 30	h. m. 11 15	° ′ 20 5·3	° ′ 20 8·4

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 39 14 9	68 50·5 68 47·3	° ′ 68 49·4

Horizontal Force.

G.T.M.		H.	H_0.
D V	h. m. 12 40 11 51	1·7338 1·7340	1·7331

108. LLANIDLOES.   May 19, 1886 ; A. W. R. (60, 74).   Lat. 52° 56′ 57″; Long. 3° 32′ 20″.   In the garden of the Inn.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 2 31 + 2 21	h. m. 9 57 14 53	° ′ 19 52·2 19 48·7	° ′ 19 53·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 7 3	° ′ 68 33·2	° ′ 68 33·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 17 56 15 13	1·7504 1·7515	1·7501

109. LOUGHBOROUGH.   April 30, 1888 ; T. E. T. (61, 83).   Lat. 52° 46′ 37″ ;
Long. 1° 13′ 3″.   In a field to the N. of the London and North-Western Railway
Station, distant 100 yards.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. − 2 22	h. m. 10 3	$\overset{\circ}{18}$ 2·4	$\overset{\circ}{18}$ 18·7

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1	h. m. 10 47	$\overset{\circ}{68}$ 24·0	$\overset{\circ}{68}$ 27·7

Horizontal Force.

G.M.T.		H.	H₀.
V	h. m. 10 16	1·7582	1·7531

110. LOWESTOFT.   May 9, 1886 ; T. E. T. (61, 83).   Lat. 52° 27′ 54″ ; Long.
1° 43′ 56″ E.   In a field W.S.W. of the town 1¼ mile away.   S. John's Church,
S. Lowestoft bore E.N.E.   Kirkley Old Church, E. by S. ; Cemetery, half a
mile away, S.E. by E.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. −1 13 +2 2	h. m. 11 3 13 34	$\overset{\circ}{17}$ 21·4 17 20·5	$\overset{\circ}{}$ 17 24·0

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 12 21 12 53	$\overset{\circ}{}$ 68 00·0 67 59·7	$\overset{\circ}{}$ 68 00·4

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	14 28	1·7804	1·7797
V	11 22	1·7806	

111. MABLETHORPE.   May 23, 1886 ; T. E. T. (61, 83).   Lat. 53° 20′ 20″; Long. 0° 15′ 41″ E.   In a field 60 yards from the Victoria Road, and 400 yards S. of the " Book in Hand " Hotel.   Flagstaff of coastguard station bearing N.E. (mag.). Convalescent Home bearing S.E. by S. (mag.).   Railway Station bearing N.W. (mag.).

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m.	h. m.	° ′	° ′
−1 20	11 0	18 13·5	18 16·5
+2 15	13 53	18 12·7	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	12 59	68 42·0	68 42·7
2	13 28	68 42·5	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	12 5	1·7381	1·7370
V	11 20	1·7378	

112. MALVERN.   A. W. R. (60, 74).

(a) Colwall Green, May 22, 1886; Lat. 52° 4′ 0″; Long. 2° 21′ 40″. About 250 yards S. of the bridge over the railway, in the space between two roads which bifurcate about 100 yards away.

(b) Great Malvern, May 21, 1886; Lat. 52° 6′ 22″; Long. 2° 18′ 10″. At the west end of Barnard's Green, about 20 yards S. of the south side of a triangle formed by three roads; near a pond.

(c) Malvern Wells, May 22, 1886; Lat. 52° 4″ 49″; Long. 2° 18′ 30″. In a field to the N. of the Hanley Road from Malvern Wells; and N.W. of the cross-roads which are E. of the Midland Station; about 10 yards from the road.

(d) Mathon, May 22, 1886; Lat. 52° 6′ 40″; Long. 2° 22′ 6″. In a field to the S. of the road to Mathon; a large pond on the other side of the road, 250 yards N.E. by N.; small church at West Malvern nearly due E.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
May 21 Great Malvern. . . .	+ 1 33	12 40	19 31·3 ⎱	19 33 0
	+ 4 11	16 57	19 28·0 ⎰	
„ 22 Mathon	+ 1 19	12 45	18 43·1	18 46·5
„ 22 Colwall Green. . . .	+ 2 9	14 37	19 0·3 ⎱	19 3·6
	+ 3 5		19 0·1 ⎰	
„ 22 Malvern Wells. . . .	+ 4 12	16 32	19 19·0	19 22·4

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
May 21 Great Malvern. . . .	1	12 57	68 14·8	68 14·2
	2	16 34	68 12·3	

Horizontal Force.

Date.		G.M.T.	H.	H_0.
		h. m.		
May 21 Great Malvern. . . .	D	14 28	1·7691	1·7687
	V	13 12	1·7700	
„ 22 Mathon	V	13 1	1 7664	1·7655
„ 22 Colwall Green. . . .	V	14 53	1·7636	1·7627
„ 22 Malvern Wells. . . .	V	16 47	1·7691	1·7682

113. MANCHESTER.   June 20 and 21, 1886 ; T. E. T. (61, 83).   Lat. 53° 27′ 40″;
    Long. 2° 16′ 54″.   Old Trafford.   On the lawn of the Royal Botanic Gardens,
    near the Winter House.   Second series of Dip Observations, close to the lake,
    and about 100 yards S.E. of the first station.

Declination.

Date.	Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
June 20	h. m. + 2 25	h. m. 14 13	° ′ 19 12·5	° ′ 19 16·7

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
June 20	1 2	h. m. 13 3 12 13	° ′ 69 3·2 69 3·0	° ′ 69 3·9

Horizontal Force.

Date.		G.M.T.	H.	H_0.
June 21 ,, 20 ,, 21	D V V	h. m. 16 33 13 51 17 8	1·7138 1·7133	1·7128 1·7123

114. MANTON.   April 21, 1888 ; T. E. T. (61, 83).   Lat. 52° 37′ 32″ ; Long. 0° 41′ 51″.
    On the hill about 400 yards from the Railway Station.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m. + 3 16	h. m. 15 13	° ′ 18 5·2	° ′ 18 21·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 1 16 46	68 13·8 68 13·0	68 17·1

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 16 15 15 34	1·7711 1·7714	1·7661

115. MARCH.   July 27, 1888 ; A. W. R. (60, 74).   Lat. 52° 32′ 0″; Long. 0° 4′ 0″ E. About a mile from March Church, on the south side of the road which runs west towards Barrow Moor.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +3 10	h. m. 15 36	17 43·9	18 2·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 46 18 14	68 6·8 68 6·0	68 10·3

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 16 36 15 53	1·7780 1·7772	1·7719

116. MELTON MOWBRAY.   T. E. T. (61, 83).

   (*a*) April 22, 1888 ; Lat. 52° 44′ 54″; Long. 0° 52′ 46″.   In a field about a mile
       S. (mag.) of the Midland Station.   20 yards E. of the Sandy Road.

   (*b*) April 30, 1888 ; Lat. 52° 45′ 47″; Long. 0° 53′ 20″.   In a field by the river,
       400 yards W. of the Church.   Egerton Lodge bearing about N.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
April 22	− 0 30	11 2	18 55·0	
	− 0 12	11 2	18 54·3	19 10·9
	− 0 5	13 8	18 53·7	
„ 30	+ 5 22	17 44	18 47·5	19 4·2

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0
		h. m.	° ′	° ′
April 22	1	13 43	68 29·0	68 31·5
	2	14 0	68 26·7	
„ 30	1	18 20	68 36·1	68 39·7
	2	18 32	68 36·0	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
April 22	D	12 26	1·7673 ⎱	1·7620
	V	11 17	1·7670 ⎰	
	V	12 55	1·7669 ⎱	1·7620
			1·7674 ⎰	
„ 30	V	17 55	1·7466	1·7415

117. MILFORD HAVEN.   July 29, 1887; A. W. R. (60, 74).   Lat. 51° 42′ 24″;
Long. 4° 56′ 47″.   In a field belonging to Mr. CARON.   100 yards S.W. by W.
of the main road which leads inland, and about 50 yards N.W. of a short lane
leading to Mr. CARON's house.   Barracks due S. on the other side of the inlet;
about $\frac{1}{3}$ of a mile N.N.W. of the quay.

Declination.

Σ.	G.M.T.	$\hat{c}$.	δ_0.
h. m.	h. m.	° ′	° ′
− 1 13	10 29	19 56·5	20 8·8
+ 1 51	13 58	19 56·8	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	11 46	68 7·9	68 9·9
2	12 12	68 6·6	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	12 58	1·7844	1·7808
V	13 41	1·7843	

118. NEWARK.   April 27, 1888; T. E. T. (61, 83).   Lat. 53° 4′ 35″; Long. 0° 48′ 43″.
In the middle of the Castle grounds.

Declination.

Σ.	G.M.T.	$\hat{c}$.	δ_0.
h. m.	h. m.	° ′	° ′
+ 1 29	14 18	18 29·5	18 46·2

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	12	8	68	30·5		
2	12	25	68	28·9	68	33·4

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	13 1	1·7505	
V	14 0	1·7525	1·7464

119. NEWCASTLE. September 17, 1886; A. W. R. (60, 74). Lat. 54° 59′ 25″; Long. 1° 39′ 44″. At the W. end of the moor, about 200 yards from the road. First Station about 150 yards from a Colliery; second, about 200.

Declination.

Σ.		G.M.T.		δ.		δ_0.	
h.	m.	h.	m.	°	′	°	′
− 0	59	11	37	19	25·6		
+ 1	44	13	32	19	23·9	19	30·3

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	13	10	69	48·0		
2	14	17	69	49·0	69	49·5

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	12 22	1·6681	1·6665

120. NORTHAMPTON.　October 7, 1887 ; T. E. T. (61, 83).　Lat. 52° 13′ 1″ ; Long. 0° 53′ 52″.　In a field on the W. side of the High Road, and nearly opposite Queen Eleanor's Cross, distant about 30 yards.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 1 48	10 27	18 29·7	18 41·7
+ 1 25	13 37	18 30·5	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	12 15	68 6·3	68 9·4
2	12 47	68 7·2	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	11 22	1·7708	1·7671
V	10 46	1·7713	
V	13 51	1·7701	1·7664
		1·7705	

121. NOTTINGHAM.　April 28, 1888; T. E. T. (61, 83).　Lat. 52° 57′ 0″; Long. 1° 9′ 20″. To the W. of Tunnel Road, due W. of the Castle, towards the Bowling Green.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 0 28	9 35	18 27·1	18 44·9
+ 3 20	15 9	18 25·4	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 15 52	$68° 33'\!·7$	$68° 37'\!·6$
2	16 9	68 34·1	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	11 0	1·7520	1·7471
V	9 54	1·7524	
V	10 24	1·7515 1·7524	1·7468

122. OXFORD. May 5 and 6, 1886; A. W. R. (60, 74). Lat. 51° 45′ 34″; Long. 1° 15′ 6″. In the Parks, near the Physical Laboratory. Ventilating flue in the middle of the Physical Lecture Room bears magnetic S.; Cricket Pavilion E. by N.; 29 paces W. from Plantation.

Declination.

Date.	Σ.	G.M.T.	$\hat{\delta}$.	$\hat{\delta}_0$.
	h. m.	h. m.		
May 5.	+5 15	17 51	$18° 29'\!·0$	$18° 33'\!·7$
„ 6.	−1 15	11 5	18 32·2	

Inclination.

Date.	Needle.	G.M.T.	.	θ_0.
		h. m.		
May 6.	1	14 24	$67° 57'\!·1$	$67° 57'\!·5$
	2	14 55	67 56·8	

Horizontal Force.

Date.		G.M.T.	H.	H_0.
		h. m.		
May 6.	D	12 18	1·7900	1·7890
	V	11 34	1·7894	

123. PETERBOROUGH.   May 21, 1886 ; T. E. T. (61, 83).   Lat. 52° 34′ 16″ ; Long. 0° 15′ 57″.   In the meadows to the W. of G.N. Railway Station, and about three quarters of a mile from it ; 200 yards N. of the River Nen, and 50 yards S. of the Thorpe Road.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. +1 23 +4 10	h. m. 12 30 15 40	18 18·6 18 18·5	18 21·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 50 15 18	68 14·3 68 14·1	68 14·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 13 56 12 52	1·7692 1·7708	1·7692

124.   PLYMOUTH.   April 7, 1887, T. E. T. (61, 83).   Lat. 50° 21′ 59″; Long. 4° 8′ 35″. On the West Hoe, S. of the Drake Monument.   W. end of Breakwater, 40° W. of S. ; E. end of Breakwater, 10° W. of S.   Old Eddystone Tower, 42° E. of S., 70 yards away.   Grand Hotel, 55° W. of N.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 33	h. m. 11 13 14 25	19 22·6 19 22·8	19 31·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 13 27	° ′ 67 12·9	° ′ 67 14·7
2	13 56	67 12·1	

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 12 5	1·8343	1·8309
V	11 39	1·8331	

125. PORT ERIN (Isle of Man). August 8, 1887 ; T. E. T. (61, 83). Lat. 54° 5′ 4″; Long. 4° 46′ 7″. On the S. side of the Harbour near the Breakwater.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 5	h. m. 11 46	° ′ 20 40·3	° ′ 20 55·4
+ 2 37	13 47	20 44·2	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 14 26	° ′ 69 46·2	° ′ 69 48·1
2	14 45	69 45·7	

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 15 36	1·6714	1·6678
V	12 38	1·6712	

126. PRESTON.   August 24, 1886 ; A. W. R. (60, 74).   Lat. 53° 42′ 46″ ; Long.
2° 43′ 18″.   On Farrington Moor, about 2 miles S.W. of the town, and 200 yards
from a railway crossing.   Close to cross-roads running N. and S. and E. and W.
Spire of St. James' Layland, S. by W., ½ W.   Farrington Factory, E.S.E.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+ 1 45	14 25	19 49·7	19 52·3
+ 4 48	17 24	19 45·2	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	16 29	69 13·7	69 14·7

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	15 30	1·7070	1·7053
V	14 48	1·7064	

127. PURFLEET.   April 14, 1888 ; A. W. R. (60, 74).   Lat. 51° 29′ 7″ ; Long.
0° 14′ 58″ E.   Near a footpath which leads to the woods behind the village.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+ 1 32	15 30	17 38·9	17 54·5
+ 4 44	16 55	17 37·4	

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	14	40	67	27·2	67	30·9
2	15	6	67	27·8		

Horizontal Force.

G.M.T.			H.	H_0.
	h.	m.		
D	16	21	1·8178	1·8134
V	15	48	1·8190	

128. PWLLHELI. May 15, 1886; A. W. R. (60, 74). Lat. 52° 52′ 55″; Long. 4° 24′ 35″. On the beach to the S. of the town and harbour. About 100 yards E. of the road which leads to the beach. The first declination was taken on the shore of the harbour about 100 yards off.

Declination.

Σ.		G.M.T.		δ.		δ_0.	
h.	m.	h.	m.	°	′	°	′
− 1	46	11	0	20	40·3	20	41·9
+ 2	9	14	6	20	36·9		

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	12	15	68	49·7	68	50·9
2	12	48	68	51·0		

Horizontal Force.

G.M.T.			H.	H_0.
	h.	m.		
V	13	54	1·7415	1·7407

129. RAMSEY (Isle of Man). August 3, 1887 ; T. E. T. (61, 83). Lat. 54° 19' 22";
Long. 4° 22' 48". On the N. shore of the harbour and about 60 yards W.S.W.
of the end of the pier. Ramsey Church bearing due S. ; end of North pier
bearing E.S.E.

Declination.

Σ.	G.M.T.	δ.	$δ_0$.
h. m.	h. m.	° '	° '
− 1 31	11 19	20 40·9	20 54·9
+ 1 52	13 54	20 42·8	

Inclination.

Needle.	G.M.T.	θ.	$θ_0$.
	h. m.	° '	° '
1	12 22	69 54·0	69 55·0
2	12 44	69 51·6	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	14 47	1·6653	1·6617
V	11 40	1·6651	

130. RANMORE. May 21, 1888 ; T. E. T. (61, 83). Lat. 51° 14' 38"; Long.
0° 21' 36". On Ranmore Common, half-way between the Post Office and the
Church, and 20 yards N. of the road.

Declination.

Σ.	G.M.T.	δ.	$δ_0$.
h. m.	h. m.	° '	° '
− 0 57	11 20	17 53·1	18 8·9
+ 1 18	13 6	17 49·9	

Inclination.

Needle.	G.M.T.	θ.	$θ_0$.
	h. m.	° '	° '
1	9 10	67 16·7	67 20·0
2	10 27	67 16·1	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	12 11	1·8310	1·8261
V	11 36	1·8318	

131. READING.   T. E. T. (61, 74).
  (*a*) April 20, 1886.   Lat. 51° 27′ 57″;   Long. 0° 58′ 46″.   On the river-bank opposite Caversham Church.
  (*b*) May 30, 1888.   Lat. 51° 27′ 56″;   Long. 0° 58′ 50″.   Close to the former station.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	· h. m.	h. m.	° ′	° ′
April 20, 1886	− 1 21	11 14	18 13·6	18 15·2
	+ 2 24	14 1	18 12·7	
	+ 4 12			
May 30, 1888	+ 1 4	12 1	17 53·9	18 11·6

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
April 20	1	15 47	67 41·6	67 40·7
	2	16 45	67 38·9	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
April 20	D	12 40	1·8112	1·8110
	V	13 27	1·8121	
May 30	V	12 15	1·8194	1·8141

132. REDCAR.   September 18, 1886 ; A. W. R. (60, 74).

(*a*) Lat. 54° 34′ 33″ ; Long. 1° 0′ 22″.   In a field on the spur of the hills due S. of Marske Church, and about three-quarters of a mile from it ; about 20 yards E. of the road which runs inland from Marske.

(*b*) Lat. 54° 35′ 46″ ; Long. 1° 1′ 15″.   In a straight line between Marske Church and the Sea.   About 250 yards from the Church.

Declination.

Σ.	G.M.T.	δ.	$\hat{\delta}_0$.
h. m. + 1 21 + 3 45	h. m. 11 22 15 11	° ′ 19 2·0 18 58·0	° ′ 19 5·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 14 2	° ′ 69 30·5	° ′ 69 31·5

Horizontal Force.

	G.M.T.	H.	H_0.
D V	h. m. 12 15 11 42	1·6865 1·6861	1·6847

133. RYDE.   April 28, 1886 ; T. E. T. (61, 74).   Lat. 50° 43′ 13″ ; Long. 1° 10′ 44″. In a field W. of Pellshurst, near Ryde.   Parish Church bore E. by N., Mr. Hunter's house W.N.W.   About 40 yards N. of the road.

Declination.

Σ.	G.M.T.	$\hat{\delta}$.	$\hat{\delta}_0$.
h. m. − 1 29 + 2 6	h. m. 11 4 13 50	° ′ 18 1·1 17 57·8	° ′ 18 1·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 24 12 58	$\overset{\circ}{67}$ $\overset{\prime}{6 \cdot 1}$ 67 8·5	$\overset{\circ}{67}$ $\overset{\prime}{7 \cdot 8}$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 14 39 11 22	1·8398 1·8399	1·8391

134. ST. CYRES.   April 26, 1886 ; T. E. T. (61, 74).   Lat. 50° 46′ 30″; Long. 3° 35′ 26″.   In the middle of a field to the S. of the S.W. Railway, near the road, and N. of the river Avon.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m. + 2 43	h. m. 13 25	$\overset{\circ}{19}$ $\overset{\prime}{26 \cdot 4}$	$\overset{\circ}{19}$ $\overset{\prime}{28 \cdot 6}$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 27 14 6	$\overset{\circ}{67}$ $\overset{\prime}{26 \cdot 4}$ 67 25·1	$\overset{\circ}{67}$ $\overset{\prime}{26 \cdot 2}$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 15 27 13 10	1·8269 1·8266	1·8260

135. St. Leonards. August 9, 1886 ; A. W. R. (60, 74). Lat. 50° 50′ 56″; Long. 0° 31′ 5″ E. On the Common N. of the road three-quarters of a mile W. of Bo Peep Station. About 100 yards N.W. of Coastguard Station.

Declination.

Σ.	G.M.T.	δ.	δ_0.
+ 2 41 + 5 35	h. m. 15 13	° ′ 17 21·4 17 19·0	° ′ 17 24·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 21 17 58	° ′ 66 59·1 66 56·7	° ′ 66 58·9

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 16 17 15 37	1·8452 1·8449	1·8437

136. Salisbury. April 27, 1886 ; T. E. T. (61, 74) ; Lat. 51° 5′ 3″ ; Long. 1° 48′ 6″. To the N. of the town, and S. of Old Sarum, in a field E. of the Avon ; 60 yards from the Old Castle or Stratford Road.

Declination.

Σ.	G.M.T.	δ.	δ_0
h. m. − 1 3 + 1 59	h. m. 10 46 13 46	° ′ 18 22·5 18 21·0	° ′ 18 23·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 38 13 15	67 25·9 67 24·4	67 25·6

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 14 35 11 34	1·8253 1·8246	1·8242

137. Scarborough.   September 21, 1886 ; A. W. R. (60, 74); Lat. 54° 15′ 55″; Long. 0° 23′ 24″.   In a field near the edge of the cliff on the south side of the town.   About 200 yards from the residence of Alderson Smith, Esq.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 57	h. m. 13 17	18 43·4	18 48·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 31 17 9	69 14·3 69 15·2	69 15·6

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 38 14 19	1·7032 1·7035	1·7017

138. SHREWSBURY.   May 8, 1886 ; A. W. R. (60, 74).   Lat. 52° 42′ 11″ ; Long. 2° 45′ 36″.   In the School Grounds, about 100 yards S.W. by W. of the new buildings.   On grass W. of road and E. of the Cricket Ground.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m.	h. m.	° ′	° ′
− 1 26	11 7	19 38·1	19 41·2

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m.	° ′	° ′
	12 40	68 35·9	68 36·4

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	11 27	1·7352	1·7344
V	11 53	1·7349	1·7341

139. SOUTHEND.   May 24, 1887 ; T. E. T. (61, 83) Lat. 51° 32′ 49″ ; Long. 0° 43′ 11″ E.   In a field, about 20 yards from the point where the Sutton Road bends at right angles towards Prittlewell, half a mile N. of Southend Station, and about half a mile E. of Prittlewell Church.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m.	h. m.	° ′	°
+ 2 32	14 3	17 34·0	17 44·4

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
	h. m.	° ′	° ′
1	15 12	67 28·9	
2	14 56	67 28·5	67 30·8

Horizontal Force.

G.M.T.		H.	$H_0.$
	h. m.		
D	13 12	1·8135	
V	13 46	1·8152	1·8112

140. SPALDING.   April 26, 1888 ; T. E. T. (61, 83).   Lat. 52° 47′ 5″; Long. 0° 8′ 48″.
Seventy yards W. of the church.

Declination.

Σ.	G.M.T.	$\delta.$	$\delta_0.$
h. m.	h. m.	° ′	° ′
−1 5	11 10·5	17 34·9	17 51·6

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
	h. m.	° ′	° ′
1	12 36	68 19·9	
2	12 50	68 19·4	68 23·1

Horizontal Force.

G.M.T.		H.	$H_0.$
	h. m.		
D	12 0	1·7576 ⎫	1·7512
V	11 25	1·7541 ⎬	
V	11 25	1·7487	1·7436

141. STOKE-ON-TRENT.   September 13, 1887 ; A. W. R. (60, 74).   Lat. 52° 57' 47" ;
Long. 2° 12' 20".   In Trentham Park, about 350 yards S.W. of the house.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. +2 29	h. m. 11 22 14 14	° ' 19 11·8 19 10·0	° ' 19 22·7

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 13 33 13 58	° ' 68 41·4 68 40·5	° ' 68 43·6

Horizontal Force.

	G.M.T.	H.	H₀.
D	h. m. 12 35	1·7443 1·7439	1·7404
V V	11 37 11 55	1·7435 1·7436	1·7398

142. SUTTON BRIDGE.   July 31, 1888 ; A. W. R. (60, 74).   Lat. 52° 45' 40" ;
Long. 0° 11' 50" E.   In a field on the E. bank of the River, about a quarter of
a mile from the Railway.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. +1 39 +5 34	h. m. 14 8 17 24	° ' 17 36·2 17 34·3	° ' 17 54·1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 15 30 15 54	$\overset{\circ}{68}\ \overset{\prime}{18\cdot2}$ 68 16·8	$\overset{\circ}{68}\ \ \overset{\prime}{21\cdot1}$

Horizontal Force.

G.M.T.	H.	H_0.
V	h. m. 14 29	1·7677	1·7620

143. SWANSEA.   May 28, 1886; A. W. R. (60, 74).   Lat. 51° 36′ 50″; Long. 3° 58′ 57″. In Cwmdonkin Park, one and a half mile N.W. of the town, and 150 yards N.W. of the Reservoir.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −1 4 +2 22	h. m. 11 39 13 15	$\overset{\circ}{19}\ \overset{\prime}{43\cdot2}$ 19 41·0	$\overset{\circ}{19}\ \ \overset{\prime}{45\cdot6}$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 22 16 59	$\overset{\circ}{67}\ \overset{\prime}{59\cdot5}$ 67 58·6	$\overset{\circ}{67}\ \ \overset{\prime}{59\cdot7}$

Horizontal Force.

G.M.T.	H.	H_0.
D	h. m. 14 0	$\left.\begin{array}{l}1\cdot7939 \\ 1\cdot7943\end{array}\right\}$	1·7932
V V	12 26 13 0	$\left.\begin{array}{l}1\cdot7939 \\ 1\cdot7941\end{array}\right\}$	1 7931

144. SWINDON.   April 21, 1886 ; T. E. T. (61, 74).   Lat. 51° 34′ 12″ ; Long. 1° 46′ 54″.
In a field N.N.E. (mag.) of railway station, about a quarter of a mile from railway
line, and near a farm.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 10 44 11 18	° ′ 67 49·8 67 52·1	° ′ 67 51·4

Horizontal Force.

	G.M.T.	H.	H_0.
D V	h. m. 12 5 9 48	1·7934 1·7941	 1·7930

145. TAUNTON.   April 23, 1886 ; T. E. T. (61, 74).   Lat. 51° 0′ 52″ ; Long. 3° 5′ 32″.
In a field to the S.E. of the town ; King's College, about quarter of a mile away,
bearing S.W. by W. (mag.).   Trinity Church a quarter of a mile away, N.N.W.
(mag.).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. −2 35 +1 30	h. m. 12 32	° ′ 19 8·8 19 8·3	° ′ 19 10·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 10 34 11 11	° ′ 67 32·0 67 32·5	° ′ 67 32·7

Horizontal Force.

G.M.T.		H.	H$_0$.
	h. m.		
D	13 15	1·8151 ⎱ 1·8156 ⎰	1·8146
V	11 50	1·8169 ⎱ 1·8170 ⎰	1·8162
V	12 7		

146. THETFORD.   September 20, 1887 ; A. W. R. (60, 74).   Lat. 52° 23′ 57″; Long. 0° 43′ 12″ E.   In the Hundred-acre Field, on the estate of the Maharajah Duleep Singh.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m.	h. m.	° ′	° ′
+ 3 32	13 44	17 29·8	17 41·2
+ 3 48	. .	17 28·8	

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
	h. m.	° ′	° ′
1	16 5	67 58·9	68 1·4
2	16 31	67 58·7	

Horizontal Force.

G.M.T.		H.	H$_0$.
	h. m.		
D	14 43	1·7831	1·7791
V	13 55	1·7828	

147. THIRSK.   September 20, 1886 ; A. W. R. (60, 74).   Lat. 54° 14′ 35″; Long.
1° 20′ 38″.   In the Kilvington fields, about half a mile N. of the Church.   About
50 yards W.S.W. of the high road to York, and between it and the stream.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 11 + 2 21	h. m. 11 16 15 14	° ′ 19 20·6 19 13·1	° ′ 19 21·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 51 14 44	° ′ 69 27·6 69 26·8	° ′ 69 28·3

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 12 42	1·6930 1·6926	1·6912
V V	11 35 12 1	1·6926 1·6932	

148. TILNEY.   August 1, 1888 ; A. W. R.   Lat. 52° 42′ 10″; Long. 0° 17′ 11″ E.
In a field behind a farm house on the main road from Wisbech, and about half
way between Tilney Buck and the point where the road crosses the Five-Mile
Drain.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 27	h. m. 12 10	° ′ 17 39·2	° ′ 17 58·1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 13 57	68 17·1	68 20·7

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 11 20	1·7707	1·7655
V	12 1	1·7717	

149. TUNBRIDGE WELLS. September 27, 1887; T. E. T. (61, 83). Lat. 51° 7′ 36″; Long. 0° 15′ 37″ E. On the Common, about 200 yards from the London Road; Trinity Church bearing 45° E. of N. (mag.). St. Peter's Church bearing 95° E. of N. (mag.). St. Mark's Church bearing 35° W. of S. (mag.).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 49 + 1 39	h. m. 10 31 13 52	17 29·1 17 29·3	17 41·3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 9 12 32	67 7·8 67 8·3	67 10·8

Horizontal Force.

G.M.T.		H.	H_0.
D	h. m. 11 34	1·8340 ⎱ 1·8335 ⎰	1·8299
V V	10 54 14 7	1·8333 ⎱ 1·8335 ⎰	1·8296

150. WALLINGFORD. May 29, 1888; T. E. T. (61, 83). Lat. 51° 36′ 3″; Long. 1° 7′ 21″.
Below Wallingford Bridge, 70 yards away, on the left bank, opposite the Church.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. + 0 49	h. m. 13 18	° ′ 18 3·9	° ′ 18 21·6

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 14 48 15 1	° ′ 67 44·9 67 44·2	° ′ 67 48·4

Horizontal Force.

	G.M.T.	H.	H₀.
D V	h. m. 14 7 13 31	1·8040 1·8039	1·7986

151. WEYMOUTH. April 6, 1887; T. E. T. (61, 83). Lat. 50° 36′ 16″; Long. 2° 26′ 52″.
On the S. side of the Nothe, close to the shore. End of Portland Breakwater,
28° E. of S. (mag.). Bincleaves House, 60° W. of S. (mag.). Coastguard
Station on Nothe, 55° W. of N. (mag.).

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. − 1 26 + 2 4	h. m. 11 17 14 3	° ′ 18 37·3 18 38·6	° ′ 18 46·7

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 13 25 13 46	° ′ 67 10·3 67 8·9	° ′ 67 11·7

Horizontal Force.

G.M.T.		H.	H$_0$.
	h. m.		
D	12 24	1·8357	1·8329
V	11 48	1·8358	

152. WHEELOCK.   February 18, 1887; T. E. T. (83).   Lat. 53° 7′ 50″; Long. 2° 22′ 30″. In the field behind Wettenhall Cottage.

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
	h. m.	° ′	° ′
1	13 37	68 46·8	68 49·0
2	14 42	68 47·9	

153. WHITEHAVEN.   August 27, 1886 ;   A. W. R.   (60, 74).   Lat. 54° 32′ 37″; Long. 3° 34′ 20″.   In Midgey Mount ; S.E. of the Castle, and in the grounds. About 300 yards from Cukickle Gate.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m.	h. m.	° ′	° ′
− 1 40	11 2	20 35·5	20 41·6
+ 2 22	14 21	20 36·6	

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
	h. m.	° ′	° ′
1	13 19	69 45·8	69 47·6
2	13 52	69 47·7	

Horizontal Force.

G.M.T.		H.	H$_0$.
	h. m.		
D	12 14	1·6740	1·6727
V	11 23	1·6743	

154. WINDSOR. May 31, 1888; T. E. T. (61, 83). Lat. 51° 29′ 22″; Long. 0° 37′ 25″. To the W. of the town, near the point where the river makes a bend towards the Railway Bridge.

Declination.

Σ.	G.M.T.	$\delta.$	$\delta_0.$
h. m. + 3 47	h. m. 16 10	$18°$ 12′·2	$18°$ 29′·9

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
1 2	h. m. 17 20 17 30	$67°$ 35′·5 67 34·3	$°$ ′ 67 38·8

Horizontal Force.

G.M.T.		H.	$H_0.$
D V	h. m. 16 48 16 22	1·8132 1·8142	1·8084

155. WISBECH. July 31, 1888; A. W. R. (60, 74). Lat. 52° 40′ 30″; Long. 0° 8′ 20″ E. In a field belonging to Mr. SHARP to the S. of the Leverington Road. About 50 yards E. of an octagonal pigeon house.

Declination.

Σ.	G.M.T.	$\delta.$	$\delta_0.$
h. m. − 2 26	h. m. 9 57	$17°$ 46′·7	$18°$ 5′·6

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
1 2	h. m. 17 53 18 12	$68°$ 17′·2 68 13·7	$°$ ′ 68 19·0

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	16 47	1·7707	1·7653
V	16 1	1·7713	

156. WORTHING. September 29, 1887 ; T. E. T. (61, 83). Lat. 50° 48′ 35″ ; Long. 0° 23′ 22″. In a field adjoining a new road W. of West Worthing, about 300 yards from the beach and three-quarters of a mile from the Pier ; about half-way between the railway line and the beach. Tarring Church 10° E. of N. (mag.). S. Botolph's (W. Worthing) 55° E. of N. (mag.).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
−1 10	11 6	17 46·2	17 59·0
+2 56	14 12	17 47·6	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	12 46	67 4·3	67 6·4
2	13 12	67 2·9	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	11 59	1·8443 ⎱ 1·8439 ⎰	1·8403
V	11 20	1·8445 ⎱	1·8401
V	14 25	1·8433 ⎰	

DESCRIPTIONS OF IRISH STATIONS.

157. ARMAGH Observatory.   August 15, 1887 ; T. E. T. (61, 83).   Lat. 54° 21′ 10″;
Long. 6° 38′ 53″.   In a field belonging to the Observatory about 100 yards S. of
the house, and close to the position on which LLOYD and ROSS had made
observations.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+1 14	14 13	22 3·8	22 16·5
+3 15	16 1	22 2·5	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	15 9	69 55·3	69 57·6
2	15 29	69 55·2	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	16 40	1·6659	1·6625
V	14 26	1·6656	

158. ATHLONE.   T. E. T. (61, 83).
   (*a*) May 8, 1887.   Lat. 53° 26′ 8″ ; Long. 7° 57′ 22″.   On the N. bank of the
      Shannon and about 50 yards from the river's edge.   About 1 mile from
      the town.
   (*b*) May 9, 1887.   In the middle of the kitchen garden, 150 yards behind
      the Prince of Wales' Hotel, and near the Protestant Church.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
May 8	+ 1 41	12 39	22 16·5	22 26·7
„ 9	− 3 44	9 9	22 16·8	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
May 8	1 2	h. m. 13 25 13 55	° ′ 69 38·5 69 37·6	° ′ 69 40·0

Horizontal Force.

Date.	G.M.T.		H.	H_0.
May 8	D V	h. m. 15 18 12 14	1·6874 1·6885	1·6852

159. BAGNALSTOWN.   September 8, 1887 ; A. W. R. (60, 74).   Lat. 52° 41′ 37″ ; Long. 6° 57′ 36″.   In a field to the W. of the Enniscorthy-road, about a quarter of a mile from the point where it crosses the railway.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 3 27½	h. m. 9 9	° ′ 21 43·3	° ′ 21 55·0

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 9 47	° ′ 69 2·2	° ′ 69 5·1

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 9 20	1·7242	1·7208

160. BALLINA.   September 2, 1887 ; A. W. R. (60, 74).   Lat. 54° 7′ 10″; Long. 9° 9′ 0″.   About 400 yards S. of Belleck Castle.   In a field on the west side of the road which runs north from the town ; about a quarter of a mile from the river.

Declination.

Σ.	G.M.T.	δ.	$\hat{\delta}_0$.
h. m.	h. m.	° ′	° ′
−2 1	11 6	23 15·3	23 26·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	13 20	70 22·8	70 25·8
2	13 42	70 23·8	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	12 3	1·6356	1·6323
V	11 23	1·6357	

161. BALLYWILLIAM.   September 8, 1887 ; A. W. R. (60, 74).   Lat. 52° 26′ 37″; Long. 6° 52′ 0″.   In a field W. of the road, nearly S. of the station, and about one-third of a mile from it.

Declination.

Σ.	G.M.T.	δ.	$\hat{\delta}_0$.
h. m.	h. m.	° ′	° ′
−0 49	11 56	21 25·6	21 37·3

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
1	h. m. 13 49	° ′ 69 2·7	° ′ 69 5·6

Horizontal Force.

G.M.T.		H.	$H_0.$
D	h. m. 13 7	1·7247 1·7268	1·7223
V V	12 13 12 29	1·7255 1·7258	1·7222

162. BANGOR.   August 18, 1887 ;   T. E. T. (61, 83) ;   Lat. 54° 39′ 57″ ;   Long. 5° 39′ 50″.   Station 200 yards E. (mag.) from the Harbour.

Declination.

$\Sigma.$	G.M.T.	$\delta.$	$\delta_0.$
h. m. − 1 27 + 2 5	h. m. 11 39 13 53	° ′ 21 30·6 21 31·5	° ′ 21 44·4

Inclination.

Needle.	G.M.T.	$\theta.$	$\theta_0.$
1 2	h. m. 12 48 13 15	° ′ 69 59·7 69 58·1	° ′ 70 1·3

Horizontal Force.

G.M.T.		H.	$H_0.$
D	h. m. 15 7	1·6636 1·6633	1·6601
V V	11 53 14 10	1·6624 1·6632	1·6595

163. BANTRY.   August 9, 1887 ;   A. W. R. (30, 74) ;   Lat. 51° 40′ 25″ ;   Long. 9° 28′ 53″.   In a field opposite to Ivy Cottage, about a mile and a quarter W. of the town.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 35 + 5 50	h. m. 15 53 18 13	$\overset{\circ}{22}$ 29′.7 22 27.8	$\overset{\circ}{22}$ 40.3

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 17 9 17 41	$\overset{\circ}{68}$ 43′.1 68 42.8	$\overset{\circ}{68}$ 46.0

Horizontal Force.

	G.M.T.	H.	H_0.
D V	h. m. 18 59 16 10	1.7561 1.7561	1.7529

164. CARRICK-ON-SHANNON.   May 12, 1887 ; T. E. T. (61, 83).   Lat. 53° 56′ 36″ ; Long. 8° 5′ 46″.   To the N. of the Shannon Bridge, about 70 yards from the road, and close to the Quay-side ; nearly opposite the door of the Royal Irish Constabulary Office.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m −2 49 −1 24	h. m. 10 29	$\overset{\circ}{22}$ 52′.7 22 53.1	$\overset{\circ}{23}$ 3.1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\overset{\circ}{6}9\ 5\overset{\prime}{1}\cdot7$	$\overset{\circ}{\ }\ \overset{\prime}{\ }$
1	11 55	69 51·7	69 53·3
2	11 58	69 50·9	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	10 42	1·6727	1·6700

165. CASTLEREAGH. May 9, 1887; T. E. T. (61, 83). Lat. 53° 45′ 34″; Long. 8° 29′ 7″.
About 150 yards W. of the Castlereagh Railway Station.

Declination.

Σ.	G.M.T.	$\hat{\delta}$.	δ_0.
h. m.	h. m.	$\overset{\circ}{2}3\ \ \overset{\prime}{1}\cdot0$	$\overset{\circ}{2}3\ 1\overset{\prime}{1}\cdot1$
−0 32	12 22	23 1·0	23 11·1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\overset{\circ}{\ }\ \overset{\prime}{\ }$	$\overset{\circ}{\ }\ \overset{\prime}{\ }$
1	13 16	69 53·9	69 56·3
2	13 36	69 54·7	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
V	12 35	1·6743	1·6716

166. CAVAN.   September 6, 1887; A. W. R. (60, 74).   Lat. 53° 59′ 4″; Long. 7° 21′ 43″.
In a field about 100 yards N. of the College.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 3 29 + 1 29	h. m. 9 13 14 10 15 45	° ′ 22 26·1 22 26·7 22 27·4	° ′ 22 37·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 18 12 52	° ′ 69 54·6 69 55·1	° ′ 69 57·3

Horizontal Force.

	G.M.T.	H.	H_0.
D V	h. m. 13 22 9 24	1·6660 1·6666	1·6629

167. CHARLEVILLE.   August 6, 1887; A. W. R. (60, 74).   Lat. 52° 20′ 54″; Long.
8° 40′ 22″.   In a field about halfway between the Station and the Town.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 3 56 + 4 37	h. m. 15 6 17 52	° ′ 22 19·4 22 19·1	° ′ 22 30·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 43 17 30	$\overset{\circ}{69} \quad 4{\cdot}1$ 69 0·8	$\overset{\circ}{69} \quad 5{\cdot}3$

Horizontal Force.

G.M.T.		H.	H_0.
V V	h. m. 15 36 15 20	1·7261 1·7255	1·7226

168. CLIFDEN.   August 30, 1887; A. W. R. (60, 74). Lat. 53° 29′ 35″; Long. 10° 4′ 10″. In a field on the S. side of the road which runs from Clifden to Ballymaconry. Three-quarters of a mile from the upper gate to the grounds of Clifden Castle.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 35 + 1 49	h. m. 11 30 14 44	$\overset{\circ}{24} \quad 8{\cdot}2$ 24 9·2	$\overset{\circ}{24} \quad 20{\cdot}7$

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 24 13 48	$\overset{\circ}{70} \quad 2{\cdot}3$ 70 1·9	$\overset{\circ}{70} \quad 4{\cdot}8$

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 12 20 11 40	1·6660 1·6668	1·6631

169. COLERAINE. August 21, 1887; T. E. T. (61, 83). Lat. 55° 7′ 31″; Long. 6° 40′ 24″.
On the West Side of the River and 200 yards S. of the Bridge.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 0 39	h. m. 13 32 15 23	° ′ 22 24·3 22 22·5	° ′ 22 36·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 15 50	° ′ 70 45·2	° ′ 70 47·7

Horizontal Force.

	G.M.T.	H.	H_0.
D	h. m. 14 35	1·6123 ⎱ 1·6125 ⎰	1·6091
V V	14 2 15 8	1·6111 ⎱ 1·6115 ⎰	1·6080

170. COOKSTOWN JUNCTION. August 19, 1887 ; T. E. T. (61, 83). Lat. 54° 44′ 56″ ;
Long. 6° 16′ 1″. About 300 yards W. of Cookstown Junction Station, on the
road to Randalstown.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 0 28 + 1 38	h. m. 13 26	° ′ 21 19·6 21 19·3	° ′ 21 32·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ ′	$\circ$ ′
1	14 38	69 31·8	
2	15 4	69 32·0	69 34·5

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	15 43	1·6862	
V	13 52	1·6865	1·6830

171. CORK.   August 8, 1887 ; A. W. R. (60, 74).   Lat. 51° 53′ 30″ ; Long. 8° 29′ 30″.
In the quarry behind Queen's College.   The second Declination was taken in a
field about 250 yards to the E. of the first station.   The difference between the
two results is larger than was anticipated.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	$\circ$ ′	$\circ$ ′
− 3 44	10 40	22 3·8 ⎫	
	14 36	22 1·7 ⎬	22 14·3
+ 2 49	15 49	22 10·3	22 21·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	$\circ$ ′	$\circ$ ′
1	12 52	68 44·2	
2	13 25	68 43·2	68 46·4

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	12 6	1·7537	
V	11 26	1·7540	1·7506

172. DONEGAL.   August 23, 1887 ; T. E. T. (61, 83).   Lat. 54° 39′ 5″ ; Long. 8° 6′ 57″.
On the N. bank of the Port, nearly opposite the ruins of the Abbey, which bore
S.S.W.   Village bore due E.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 0 11 + 1 53	h. m. 11 42 15 31	° ′ 23 6·9 23 6·4	° ′ 23 20·1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 46 14 32	° ′ 70 12·5 70 13·1	° ′ 70 15·3

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 15 7 12 2	1·6482 1·6483	1·6449

173. DROGHEDA.   May 15, 1887 ; T. E. T. (61, 83).   Lat. 53° 42′ 48″ ; Long. 6° 22′ 4″.
In a field S.S.W. of Rathmullan House, about 1 mile west of Drogheda Cathedral,
and on the S. side of the River Boyne.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 0 38 + 1 0	h. m. 11 29 13 4	° ′ 21 43·8 21 42·3	° ′ 21 54·7

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 13 59	° ′ 69 34·5	° ′ 69 36·3
2	14 14	69 33·8	

Horizontal Force.

	G.M.T.	H.	H_0.
D	h. m. 12 19	1·6921	1·6894
V	11 11	1·6921	
D	. .	1·6924	1·6897
V	12 50	1·6924	

174. DUBLIN (Trinity College). May 5 and 6, 1887; T. E. T. (61, 83). Lat. 53° 20′ 35″; Long. 6° 15′ 24″. 1st Station. On the path before the Magnetic House, between it and the University Buildings. About 20 yards from the former. 2nd Station. On the middle walk and almost due E. of the Provost's House; to the N.E. of the entrance of the Magnetic Observatory.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
May 5	h. m. +1 30	h. m. 13 28	° ′ 21 29·5	° ′ 21 40·8
„ 6	−1 6	11 46	21 29·4	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
May 5	1	h. m. 15 30	° ′ 69 13·9	° ′ 69 15·7
	2	15 57	69 13·3	

Horizontal Force.

Date.	G.M.T.		H.	$H_0.$
		h. m.		
May 5	D	12 5	1·7108	1·7079
	V	13 11	1·7105	
„ 6	V	12 5	1·7125	1·7095
			1·7120	

175. ENNISKILLEN.

May 14, 1887 ; T. E. T. (61, 83).   Lat. 54° 21′ 18″ ; Long. 7° 39′ 50″.   At Silverhill, at the S. end of lower Lough Erne.   Portora Old Castle, bearing due E. (mag.) ; Devenish Round Tower, 38° E. of N. (mag.) ; Royal Schools, Portora, 35° E. of S. (mag.).

September 3, 1887 ; A. W. R. (60, 74).   Lat. 54° 21′ 7″ ; Long. 7° 39′ 8″.   In a field close to the river, and about 150 yards east of the Royal Schools.

Declination.

Date.	Σ.	G.M.T.	$\delta.$	$\delta_0.$
	h. m.	h. m.	° ′	° ′
Sept. 3 (R.)	+2 39	15 44	22 53·1	
	+2 49			23 5·3
	+5 44	18 41	22 51·0	

Inclination.

Date.	Needle.	G.M.T.	$\theta.$	$\theta_0.$
		h. m.	° ′	° ′
May 14 (T.)	1	13 5	70 12·8	
	2	13 22	70 12·0	
Sept. 3 (R.)	1	17 9	70 12·3	70 14·2
	2	17 29	70 11·5	
	2	18 24	70 10·8	

Horizontal Force.

Date.	G.M.T.		H.	$H_0.$
		h. m.		
May 14 (T.)	D	11 8	1·6526	1·6489
	V	11 42	1·6507	
Sept. 3 (R.)	V	16 17	1·6496	1·6465
	V	15 58	1·6501	

176. GALWAY.   A. W. R. (60, 74).

(a) August 25, 1887 ; Lat. 53° 16′ 37″; Long. 9° 3′ 42″.   On the Lawn in front of Queen's College.

(b) August 26, 1887 ; Lat. 53° 17′ 30″ ; Long. 9° 3′ 21″.   At the lower end of Loch Cool, on the S. side, near the entrance of the river.   The first station was chosen as being near to the College, but as it is on the granite, and was found to be highly disturbed, the second position was selected on the limestone.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
Aug. 25	+ 2 15	15 18	23 55·3	24 6·3
	+ 4 28	17 49	23 53·6	
„ 26	+ 2 14	15 26	23 18·0	
	+ 2 31	..	23 18·5	23 29·8
	+ 5 14	..	23 17·3	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
Aug. 25	1	16 51	69 53·5	69 55·5
	2	17 23	69 52·6	
„ 26	1	17 1	69 40·2	
	1	17 31	69 42·3	69 44·1
	2	17 16	69 42·3	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Aug. 25 (1)	D	16 12	1·6602	1·6568
	V	15 33	1·6600	
„ 26 (1)	V	12 55	1·6631	1·6598
			1·6631	
(2)	D	16 22	1·6898	1·6867
	V	16 46	1·6903	

177. GORT.　August 24 and 25, 1887 ; A. W. R. (60, 74).　Lat. 53° 4′ 20″ ; Long. 8° 49″ 15″.　In a field to the E. of the road which runs N. from the town, due N. of the Church, and about 500 yards from it.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
Aug. 24	+ 4 28	15 41	22 38·2	22 50·5
„ 25	− 4 17	8 46	22 39·1	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
Aug. 24	1	16 47	69 29·5	69 31·7
	2	17 17	69 28·7	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
Aug. 24	V	16 6	1·7046	1·7013

178. GREENORE.　August 14, 1887 ; T. E. T. (61, 83).　Lat. 54° 1′ 35″ ; Long. 6° 7′ 47″. On the shore near the Coast Guard Station.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 0 10	12 43	22 1·3	22 14·5
+ 3 41	15 17	22 1·0	

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	14	42	69	41·7	69	42·2
2	15	1	69	37·5		

Horizontal Force.

	G.M.T.		H.	H_0.
	h.	m.		
D	15	50	1·6868	1·6833
V	13	13	1·6863	

179. KELLS.   May 16, 1887; T. E. T. (61, 83).   Lat. 53° 43′ 8″; Long. 6° 52′ 46″. In a field half a mile S. of the Kells Railway Station and about 50 yards E. of the main road.

Declination.

Σ.		G.M.T.		δ.		δ_0.	
h.	m.	h.	m.	°	′	°	′
— 0	14	12	30	21	56·7	22	7·0

Inclination.

Needle.	G.M.T.		θ.		θ_0.	
	h.	m.	°	′	°	′
1	13	51	69	36·9	69	38·7
2	14	6	69	36·2		

Horizontal Force.

	G.M.T.		H.	H_0.
	h.	m.		
D	13	18	1·6949	1·6925
V	12	42	1·6955	

180. KILDARE.   May 6 and 7, 1887; T. E. T. (61, 83).   Lat. 53° 9′ 25″;
Long. 6° 54′ 30″.   In the centre of the enclosed ground attached to the ruins of
the Castle—a plot of ground given by the Duke of LEINSTER to the town.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
	h. m.	h. m.	° ′	° ′
May 6	+6 7	18 56	21 50·2	22 0·4
„ 7	−2 38	11 7	21 50·4	

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
May 7	1	12 8	69 15·0	69 17·2
	2	12 28	69 15·1	

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
May 7	D	10 39	1·7101	1·7071
„ 6	V	19 16	1·7096	
„ 7	V	11 20	1·7105	1·7076
			1·7101	

181. KILKENNY.   September 7, 1887; A. W. R. (60, 74).   Lat. 52° 38′ 41″;
Long. 7° 15′ 30″.   In the centre of the grounds of the Catholic College.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
+1 50	14 42	21 49·0	21 58·7
+4 11	16 20	21 45·0	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	15 38	69 1·8	
2	15 57	69 2·4	69 5·0

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	13 38	1·7290	
V	14 58	1·7295	1·7258

182. KILLARNEY.   August 11, 1887 ; A. W. R. (60, 74).   Lat. 52° 3′ 50″ ; Long. 9° 32′ 20″.   In the grounds of the Victoria Hotel.   To the E. of the road leading from the Hotel to the Lake.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
− 1 ᵟ 0	11 28	22 44·1	
+ 3 25	15 23	22 44·2	22 55·8

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	13 56	68 54·2	
2	15 49	68 53·1	68 56·5

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	12 23	1·7435	
V	12 59	1·7436	1·7405

183. KILRUSH.   August 22, 1887 ; A. W. R. (60, 74).   Lat. 52° 37′ 56″ ;
Long. 9° 29′ 40″.   In a field to the W. of the road from the village to the quay,
and about half-way between them.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 4 24	h. m. 16 32	° ′ 23 0·1	° ′
	17 43	22 59·1	23 11·4
+ 6 12	18 33	22 59·3	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	15 50	69 20·5	
2	16 10	69 20·3	69 23·2

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	18 5	1·7124	
V	17 31	1·7123	1·7090

184. LEENANE.   August 31, 1887 ; A. W. R. (60, 74).   Lat. 53° 35′ 43″ ; Long.
9° 42′ 28″.   On the Green, immediately in front of the Hotel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 1 34	h. m. 14 31	° ′ 23 24·0	° ′
+ 3 10	15 3	23 24·5	23 36·2

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1	h. m. 16 2	° ′ 70 5·3	° ′ 70 8·0

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 15 50	1·6545	1·6512

185. LIMERICK.   August 20, 1887 ; A. W. R. (60, 74).   Lat. 52° 39′ 13″ ; Long. 8° 38′ 46″.   In a field at Summerville, the residence of J. BANNATYNE, Esq. The same position as in the survey of 1838.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 27 + 2 48	h. m. 11 50 12 42 13 57 15 13	° ′ 22 24·2 22 24·8 22 24·8 22 25·4	° ′ 22 36·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 32 14 53	° ′ 69 5·9 69 6·1	° ′ 69 8·8

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m. 13 25 12 27	1·7266 1·7271	1·7235

186. LISDOONVARNA.   August 23, 1887; A. W. R. (60, 74).   Lat. 53° 1' 47";
Long. 9° 17' 34".   In a field behind the Queen's Hotel.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. + 4 17 + 5 25	h. m. 16 17 18 26	° ′ 22 48·1 22 45·2	° ′ 22 58·5

Inclination.

Needle.	G.M.T.	θ.	θ₀.
1 2	h. m. 17 25 17 46	° ′ 69 30·3 69 28·4	° ′ 69 31·8

Horizontal Force.

G.M.T.		H.	H₀.
D V	h. m. 18 51 16 35	1·7078 1·7073	1·7042

187. LISMORE.   August 1, 1887; A. W. R. (60, 74).   Lat. 52° 8' 15"; Long. 7° 55' 12".
In a field, quarter of a mile E. by S. of the Cathedral.

Declination.

Σ.	G.M.T.	δ.	δ₀.
h. m. − 3 29	h. m. 10 23 13 24	° ′ 21 54·1 21 54·0	° ′ 22 5·5

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	11 52	68 46·2	68 48·6
2	12 16	68 45·6	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	12 52	1·7463	1·7427
V	10 49	1·7455	

188. LONDONDERRY.   August 20, 1887 ;   T. E. T. (61, 83).   Lat. 55° 1′ 24″ ;   Long. 7° 18′ 6″.   In a field adjoining the Moville Road, about 2 miles from Derry, and between the Road and the Lough.   Boom Hall Lodge Gate 70° E. of N. ; distant 50 yards.   Boom Hall 35° W. of S. ; distant 400 yards.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
— 0 25	12 52	22 35·5	22 50·5
+ 1 52	14 13	22 38·7	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	15 28	70 24·6	70 26·9
2	15 56	70 24·2	

Horizontal Force.

	G.M.T.	H.	H_0.
	h. m.		
D	16 21	1·6368	1·6333
		1·6365	
V	13 22	1·6376	1·6338
V	13 49	1·6367	

189. OUGHTERARD.   August 27, 1887 ; A. W. R. (60, 74).   Lat. 53° 26′ 10″ ; Long. 9° 19′ 1″.   On the shore of Lough Corrib, to the E. of the Town ; on limestone.

Declination.

Σ.	G.M.T.	δ.	δ$_0$.
h. m. + 3 11 + 3 25	h. m. 17 0 ..	° ′ 23 28·0 23 29·5	° ′ 23 40·6

Inclination.

Needle.	G.M.T.	θ.	θ$_0$.
1 2	h. m. 18 1 18 28	° ′ 69 54·1 69 54·2	° ′ 69 56·7

Horizontal Force.

G.M.T.		H.	H$_0$.
V.	h. m. 17 15	1·6795	1·6762

190. PARSONSTOWN.   August 4 and 5, 1887 ; A. W. R. (60, 74).   Lat. 53° 5′ 47″ ; Long. 7° 54′ 57″.   In the grounds of Birr Castle, near the meridian mark used for the telescope of the Earl of ROSSE.

Declination.

Date.	Σ.	G.M.T.	δ.	δ$_0$.
August 4 ,, 5	h. m. (18 20 G.M.T.) (10 40 ,,)	h. m. 18 49 11 7	° ′ 22 14·8 22 16·3	° ′ 22 27·0

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
August 5	1	13 8	69 27·8	69 30·3

Horizontal Force.

Date.	G.M.T.		H.	H_0.
		h. m.		
August 5	D	12 8	1·7022	1·6989
	V	11 27	1·7020	

191. SLIGO.   May 13, 1887 ; T. E. T. (61, 83).   Lat. 54° 16′ 34″ ; Long. 8° 28′ 36″. On a promontory in the harbour, to the W. of the town, and about a quarter of a mile distant from the bridge.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m.	h. m.	° ′	° ′
−1 46	12 24	22 55·0	23 4·6
+1 44	13 53	22 53·9	

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	11 35	70 15·4	70 17·8
2	12 2	70 16·3	

Horizontal Force.

G.M.T.		H.	H_0.
	h. m.		
D	13 23	1·6453	51·6430
V	12 39	1·6461	

192. STRABANE. August 22, 1887; T. E. T. (61, 83). Lat. 54° 49′ 48″; Long. 7° 28′ 20″. In a field midway between the Railway Station and the Bridge at Lifford. Railway Station bore 300 yards E.S.E. (mag.); Bridge, W.N.W. (mag.); and Church at Lifford, N.N.W. (mag.).

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 0 17 + 1 1	h. m. 12 33 16 21	° ′ 22 34·3 22 32·6	° ′ 22 46·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 14 44 15 20	° ′ 70 20·4 70 21·7	° ′ 70 23·5

Horizontal Force.

G.M.T.		H.	H_0.
D V	h. m 15 55 12 46	1·6391 1·6380	1·6352

193. TIPPERARY. August 3, 1887, A. W. R. (60, 74). Lat. 52° 28′ 36″; Long. 8° 9′ 12″. 1st Station. In a field on the W. side of the road to Limerick Junction, about a third of a mile from the point at which it crosses the railway. 2nd Station. In a field opposite the Hotel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 5 52	h m. 19 1	° ′ 22 11·1	° ′ 22 22·6

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 42 13 13	69 2·2 69 2·6	69 4·9

Horizontal Force.

G.M.T.		H.	H_0.
D V D V	h. m. 11 59 11 11 20 15 19 41	1·7317 1·7321 1·7292 1·7289	1·7287 1·7258

194. TRALEE.   August 19, 1887; A. W. R. (60, 74).   Lat. 52° 16′ 15″; Long. 9° 44′ 30″.   In a field about a mile to the W. of the town.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 δ 2	h. m. 13 36 14 10	69 7·7 69 5·6	69 9·4

195. VALENTIA.   August 16 and 17, 1887; A. W. R. (60, 74).   Lat. 51° 55′ 34″; Long. 10° 17′ 40″ : Lat. 51° 55′ 22″; Long. 10° 17′ 25″.   Station 1.   On the cliffs to the N. of the town.   Station 2.   In a field to the S. of the town.

Declination.

Date.	Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
Aug. 16 (1) „ 17 (2)	h. m. + 5 19 + 5 35 − 1 25 + 1 54	h. m. 18 44 .. 13 2 13 57	23 4·1 23 3·3 23 5·1 23 4·2	23 16·0

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Aug. 17 (2)	1 2	h. m. 12 8 15 11	68 52·9 68 50·4	68 54·7

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Aug. 17 (2)	V V	h. m. 13 25 13 48	1·7485 1·7478	1·7452 1·7445

196. WATERFOOT. August 25, 1887 ; T. E. T. (61, 83). Lat. 55° 3′ 11″; Long. 6° 2′ 32″. The Declination was taken on the shore, and before the Chapel; the Forces in the road to the N. of the Chapel.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. + 2 11	h. m. 14 59	22 1·5	22 15·1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 16 38 16 55	70 26·3 70 26·5	70 29·0

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 16 3	1·6326	1·6293

2 G 2

197. WATERFORD.   August 2, 1887 ; A. W. R. (60, 74).   Lat. 52° 16′ 54″ ; Long. 7° 8′ 41″.   In a field on the E. of the Kilmacow Road near the junction of the Lower Limestone and Shale.   About a third of a mile from the junction of the Limerick and Kilkenny Railways.

Declination.

Σ.	G.M.T.	δ.	δ_0.
h. m. − 1 44 + 2 17	h. m. 12 7 14 40	° ′ 21 15·1 21 17·7	° ′ 21 27·9

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 11 44 14 12	° ′ 68 49·6 68 52·5	° ′ 68 53·7

Horizontal Force.

G.M.T.		H.	H_0.
D $^\delta$ V	h. m. · · 12 31	1·7362 1·7361	1·7329

198. WESTPORT. May 9 and 10, 1887 ; T. E. T. (61, 83). Lat. 53° 48′ 8″ ; Long. 9° 31′ 17″.   In the demesne of the Marquis of SLIGO, and about 100 yards from the main entrance from the town.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
May 9 ,, 10	h. m. + 4 48 − 1 55 + 2 4	h. m. 17 54 11 10 14 13	° ′ 23 5·5 23 4·2 23 5 4	° ′ 23 15·1

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
		h. m.	° ′	° ′
May 10	1	13 25	70 16·9	70 17·9
	2	13 42	70 15·5	

Horizontal Force.

Date.		G.M.T.	H.	H_0.
		h. m.		
May 10	D	12 21	1·6536	1·6505
			1·6529	
., 9	V	18 10	1·6545	1·6514
,, 10	V	11 25	1·6537	

199. WEXFORD.   September 8, 1887 ; A. W. R. (60, 74). Lat. 52° 21′ 27″ ;
Long. 6° 27′ 18″.   On the limestone on the opposite side of the river (Ardcavan)
to the Town.   In a field on the Western side of the main road, about a quarter
of a mile N. of the point where the roads to the Wooden Bridge and Ferry
Bank unite.

Declination.

Σ.	G.M.T.	$\hat{c}$.	$\hat{c}_0$.
h. m.	h. m.	° ′	° ′
+ 4 24	17 12	21 6·4	21 18·1

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	18 22	68 55·0	68 56·2
2	18 41	68 51·6	

Horizontal Force.

G.M.T.		H.	H_0.
V	h. m. 17 49	1·7358	1·7324

200. WICKLOW.   September 9 and 10, 1887 ; A. W. R. (60, 74).   Lat. 52° 58′ 53″; Long. 6° 3′ 30″.   Half a mile W. of the town, in a field on the N. side of the road through Ballynerrin, opposite to and a little beyond the old Roman Catholic Chapel.

Declination.

Date.	Σ.	G.M.T.	δ.	δ_0.
Sept. 9 „ 10	h. m. +5 18 −1 55	h. m. 18 5 10 55	° ′ 21 5·0 21 14·4	° ′ 21 21·4

Inclination.

Date.	Needle.	G.M.T.	θ.	θ_0.
Sept. 10	1 2	h. m. 11 44 12 11	° ′ 69 7·6 69 6·9	° ′ 69 9·9

Horizontal Force.

Date.	G.M.T.		H.	H_0.
Sept. 9	D V	h. m. 18 49 18 18	1·7155 1·7165	1·7126

SUPPLEMENTARY STATIONS.

(Dips only observed. T. E. T. (83).)

201. CHEPSTOW.   April 22, 1889.   In the courtyard of the Castle.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	18 9	67 51·9	67 57·5
2	18 26	67 51·9	

202. GOODRICH CASTLE.   April 19, 1889.   On the Barbican.   To the East of the entrance to the Castle.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	16 47	68 3·3	68 9·0
2	16 52	68 3·5	

203. HEREFORD.   April 17, 1889.   On the slope of Aylstone Hill overlooking Lugg Meadow.   About a mile from Hereford, and down a short lane which leads past Lugg Vale, the residence of Mr. HAWKINS.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
	h. m.	° ′	° ′
1	16 1	68 10·2	68 15·6
2	16 20	68 9·9	

204. Ross.   April 18, 1889.   In the garden of the Royal Hotel.   South of the house and between the large yew and the wall of the Prospect.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 12 29 13 5	° ′ 68 2·2 68 0·5	° ′ 68 6·9

205. TINTERN.   April 22, 1889.   On the Monmouth Road, above the Railway Station.

Inclination.

Needle.	G.M.T.	θ.	θ_0.
1 2	h. m. 13 30 13 49	° ′ 67 56·2 67 55·2	° ′ 68 1·3

CALCULATION OF THE ISOMAGNETIC LINES.

(1.) *The Isogonal Lines.*

The isogonals, isoclinals, isodynamics and lines of equal Horizontal Force are drawn through points, at which the values of the magnetic elements to which they refer are equal. A general term which shall include them all would be useful, and we venture to suggest *isomagnetic* as the most obvious and convenient. In nothing do different magnetic surveys differ more widely than in the methods employed of drawing these lines. Some observers have calculated them by least squares; others give maps on which they are exhibited, but say nothing about the principles in accordance with which they have been drawn. But, inasmuch as the main object and result of a survey is the delineation of these isomagnetics, it seems to us that it is most important that they should be drawn with all the accuracy that the observations will allow. Before proceeding to discuss the method of doing this, it will be well to consider the exact meaning to be attached to the operation, and to attempt to give greater precision to the language in which the various curves, and the physical phenomena which they represent, are described.

If we suppose that a series of magnetic curves are drawn, in which all distortions due to local magnetism are neglected, except those which are on a scale comparable with the dimensions of the earth itself, they would be the *terrestrial isomagnetic lines ;* on the other hand, lines which showed every disturbance, however large or small, would be the *true isomagnetic lines.*

The object of a survey is to determine as nearly as possible the forms of the true lines, and to deduce from them the directions of the terrestrial lines in the district under investigation. Between these two extremes various grades of accuracy of detail intervene, and the terrestrial lines may be regarded as affected with distortions of different orders due to disturbances of various magnitudes. It is of course impossible to frame definitions which shall accurately distinguish between them, but it is nevertheless convenient to recognise three classes, into which they may be divided with respect to any particular survey.

We may regard a disturbance as being of the third, second, or first class according as its range is less than the average distance between the stations, greater than this distance, but small compared with the dimensions of the entire area under survey, or such as to involve the whole or a considerable fraction of that area.

The term *local* may be reserved for disturbances of the third class, which affect only a single station or its immediate neighbourhood, and are represented by minor bends or small loops in the isomagnetic curves.

Those of the second class may be called *regional* disturbances. They are represented by considerable distortions of the curves, but do not seriously interfere with the determination of the general direction or average distance apart of the terrestrial

lines. A disturbance of the first class is however an obstacle to any inference as to the relation between the true and the terrestrial curves. It may distort them similarly, and so lead to false conclusions as to their undisturbed directions, or it may introduce such widespread and important irregularities that any conclusion deduced from the data afforded by the survey would be manifestly uncertain.

It is to be noticed that the same cause may produce disturbances of different classes in the lines corresponding to different elements. Thus a broad band of magnetic rock at right angles to the magnetic meridian, the extremities of which lay outside the district under investigation, would produce little effect on the Declination, but might affect the Dip to a much larger extent.

If an equation can be found to a family of curves which represents, as closely as possible, the general direction of the true isomagnetics of a particular kind throughout the whole country, the curves are the nearest approach to the terrestrial lines which can be deduced from the observations. The undulations of the district lines on each side of these are evidence of regional disturbances, while the still more sinuous lines obtained by joining the points calculated as corresponding to any particular value of the element from the values in two neighbouring stations are influenced by the local disturbances also.

If the district under survey is very small, the assumption that the terrestrial lines are straight is very approximately true, but the nature of the curve indicated by a straight line on a map depends on the projection on which the map is drawn.

The English observers have generally determined the position of the station by its distance north or south of a particular line of latitude, and east or west of a particular meridian, both distances being expressed in geographical miles. That this system is open to objection is evident by considering its application to a simple ideal case in which the geographical meridians are themselves supposed to be the isogonal lines. It would evidently be better under such conditions to take the latitude and longitude as coordinates, and inasmuch as in the neighbourhood of the United Kingdom there is an approximation to such an arrangement, it is probable that the latitude and longitude are at least as convenient as any other data for the determination of the position of the stations. By this plan also, we are saved the trouble of converting the longitude east or west of Greenwich into miles east or west of the prime meridian. We have therefore taken the latitude and longitude as our coordinates.

In the next place it is, we think, important to find the general equations to the isomagnetic lines, but it would be difficult to determine their form with sufficient accuracy by a graphical process. We therefore decided to trace them by a preliminary process of calculation which was carried out as follows.

The country was divided into nine overlapping districts bounded by lines of latitude and longitude. If the stations within any district were not uniformly distributed, they were weighted, so that the weighted number of stations per unit of area should be everywhere about the same. In speaking of the mean value of any quantity in a

district, it will be understood that the values appropriate to any station are throughout properly weighted.   The means of the latitudes, longitudes, and declinations determined the central station of the district and the declination at that station (δ_0').   It was then assumed, if l' and λ' be the *district coordinates* of a station, *i.e.*, the differences between its latitude and longitude and those of the central station, that the declination is connected with these quantities by the linear equation—

$$\delta = \delta_0' + xl' + y\lambda'.$$

Two equations of condition were then formed by adding the equations thus obtained—(1), for all stations to the north of the central station ; (2), for all stations to the east of it, and dividing by the number of stations employed, each multiplied by its proper weight.

By solving these for x and y, the rates of change of the Declination per degree of latitude and longitude respectively were obtained.[*]

To test this method, it was applied in the case of the Dips to the whole of Scotland. This portion of Great Britain furnished a severe test, as from the irregularity of its form it is not particularly well adapted for the application of the method of equations of condition.   The calculation was also made by the method of least squares. Both calculations were repeated twice, viz., with and without the inclusion of Soa and Canna, at which the local disturbance is very consirdeable.   The central station was not that given by the mean latitude and longitude, but that obtained from WELSH's Dip observations, viz., lat. 56° 48′ N. ; long. 4° 19′ W.   In this particular again, the conditions of the selected example were unfavourable to the method of equations of condition.

In the following Table, u is the angle made by the lines of equal Inclination with the geographical meridian, r is the change in Inclination (expressed in minutes) per geographical mile, measured at right angles to the isoclinals, and θ_0' is the Dip at the central station :—

		θ_0'.	u.	r.
		° ′	° ′	
Including Soa and Canna . .{	Least squares.	71 9·7	67 23	0·625
	Equations of condition .	71 9·1	68 7	0·670
Excluding Soa and Canna . .{	Least squares. . . .	71 8·0	72 49	0·595
	Equations of condition .	71 7·8	72 42	0·609

[*] A plan very similar to that above described was employed by Dr. VAN RIJCKEVORSEL in working up the results of his survey of the Indian Archipelago ('Magnetische Opneming van den Indischen Archipel in de Jaren 1874–77, gedaan door Dr. VAN RIJCKEVORSEL.'   Amsterdam, J. MÜLLER, 1879).

These results prove that in this case, at all events, the differences between the results of the two methods of calculation are not greater than those produced in the numbers given by the method of least squares according as stations affected with considerable disturbances (23′ and 76′ respectively), and amounting to 4 per cent. only of the total number, are included or excluded.

We do not think, therefore, that it is in general advisable to use so cumbrous a method as that of least squares, when the addition of a station or two may modify the results to an extent far exceeding the error with which numbers obtained by the equations of condition are likely to be affected. If, however, the district under investigation is of such a shape that the effects of change of latitude and longitude respectively cannot be easily separated, it may be desirable either to modify the rule for obtaining the equations, or to employ least squares.

In the case of a district so large as Scotland there is another objection to the use of least squares, viz., that the fundamental assumption on which that method is based is almost certainly not true when applied to it. We shall show hereafter, as is indeed already known for other districts, that the errors are not irregularly distributed over the entire area, but that large fractions of the whole are affected with errors of a particular kind. We cannot, therefore, regard the employment of least squares as theoretically better, while it is certainly practically more inconvenient than the method of equations of condition. The following Table contains the boundaries of the nine districts, the latitudes and the longitudes of the central stations, the values of the Declination at the central stations (δ_0'), and of the change in Declination per degree of latitude and longitude ($d\delta'/dl$ and $d\delta'/d\lambda$) both expressed in minutes of arc.

TABLE V.

District.	Boundaries.		Central Station.		δ_0'.		$x = \dfrac{d\delta'}{dl}$.	$y = \dfrac{d\delta'}{d\lambda}$.
	Lat. N.	Long.	Lat. N.	Long. W.				
1.	All Scotland		56 48·0	4 19·0	21	38·8	14·5	40·1
II.	54 to 57	0 to 6 W.	55 27·3	3 41·6	20	55·6	16·7	36·4
III.	52 55	0 5 W.	53 26·7	2 26·0	19	39·0	15·5	33·6
IV.	50 53	2 E. 3 W.	51 47·7	0 17·5	18	6·6	17·4	28·9
V.	53 55 30	5 W. 10 W.	54 36·5	7 36·5	22	41·3	17·2	32·5
VI.	52 55	3 W. 8 W.	53 29·0	5 43·0	21	25·6	20·9	31·6
VII.	49 52	1 W. 6 W.	50 47·0	3 1·1	19	6·2	17·8	28·9
VIII.	51 54	5 W. 11 W.	52 57·1	8 13·1	22	35·0	27·3	30·1
IX.	50 53	3 W. 8 W.	51 49·5	4 47·4	20	19·7	22·4	29·2

In District I., on account of its irregular form, the method of equations of condition is not very suitable, and the method of least squares has been used. In order to compare this with the formula obtained by BALFOUR STEWART from WELSH's observation the

same coordinates have been employed as he used (the geographical-mile system), hence the values of $d\delta'/dl$ and $d\delta'/d\lambda$ can only be considered as applying to the central stations. By means of the first the Declination at any point on the meridian through that station can be calculated, and for other points on the parallel of latitude passing through such a point, the formula $y = y_0 \cos l / \cos l_0$ must be employed where y_0 and l_0 refer to the central station. In all the other districts the values of x and y are valid for the whole district.

By means of these formulæ the Declination was calculated for all points within the United Kingdom defined by whole degrees of longitude and half degrees of latitude, e.g., for lat. 50° 30', long. 2° E., 1° E., 0, 1° W., and so on. Where the districts overlapped the means of the numbers thus obtained were taken. All these values are given in the following table. The figures in brackets at the end of a row indicate the number of the district from which it was deduced. Where two or more districts overlap, the individual declinations are given in italics and the mean in ordinary type.

Throughout the central parts of the kingdom the agreement between the numbers given by the linear formulæ proper to different districts is sufficiently close to leave little doubt that the mean cannot be more than 1' or 2' wrong. Where greater differences appear it is generally easy to account for them. Thus, lat. 53° 30', long. 10° W. is in the highly disturbed region in the west of Galway. Lat. 52° 30', long. 0°, is close to a remarkable and hitherto unsuspected disturbance in the eastern counties, of which we shall have more to say hereafter. The large differences on the border of Districts I. and II. (lat. 56° 30') are, perhaps, in part due to the irregularities in the shapes of these districts owing to which the formulæ are not obtained under favourable conditions. Both the other elements, however, agree with the Declination in indicating violent local disturbance in this region, and there can be no doubt that the discrepancies are due to a physical cause.

TABLE VI.—Declinations Calculated

Lat.	Longitude.					
	10° W.	9°	8°	7°.	6°.	5°.
60 30	° ′	° ′	° ′	° ′	° ′	° ′
59 36						
58 30				(1) 23 46·2	23 7·9	22 29·6
57 30			(1) 24 13·7	23 34·4	22 55·1	22 15·8
56 30			(1) 24 2·8	23 22·4	22 42·0 / (2) 22 38·1 / 22 40·0	22 1·6 / 22 1·7 / 22 1·6
55 30			(5) 23 18·7	22 46·3	(2) 22 21·4 / 22 13·8 (5) / 22 17·6	21 45·0
54 30		(5) 23 34·0	23 1·5	22 29·0 / (6) 22 27·4 / 22 28·2	(2) 22 4·7 / 21 56·5 (5) / 21 55·8 / 21 59·0	21 28·3 / 21 24·2 / 21 26·2
53 30	(5) 23 49·2 / (8) 23 43·7 / 23 46·4	23 16·8 / 23 13·6 / 23 15·2	22 43·3 / 22 43·5 / 22 43·4	22 11·8 / (6) 22 6·5 / 22 13·4 / 22 10·6	21 39·3 (5) / 21 34·9 / 21 43·3 (8) / 21 39·2	21 3·4
52 30	(8) 23 16·4	22 46·3	22 16·2	(6) 21 45·6 / 21 46·1 / (9) 21 39·4 / 21 43·4	21 14·0 / 21 16·0 (8) / 21 10·2 / 21 13·4	20 42·4 / 20 41·0 / 20 41·7
51 30	(8) 22 49·1	22 19·0	21 48·9	21 18·8 / (9) 21 17·0 / 21 17·9	20 48·7 (8) / 20 47·8 / 20 48·2	20 18·6 / (7) 20 16·3 / 20 17·4
50 30				(9) 20 54·6	20 25·4	19 56·2 / (7) 19 58·5 / 19 57·3
49 30						(7) 19 40·7

from the District Lines.

Longitude.							Lat.
4°.	3°.	2°.	1°.	0°.	1°E.	2°.	
° ′	(1) 21 46·6	21 10·0	20 34·0	19 58·0 (1)	° ′	° ′	60 30
(1) 22 6·2	21 29·2	20 52·0	20 14·9 (1)				59 30
21 51·4	21 13·2	20 34·9	19 56·7 (1)				58 30
21 36·5	20 57·2	20 17·9	19 38·6 (1)				57 30
21 21·2 21 25·3 21 23·2	20 40·8 20 48·9 20 44·8	20 0·4 20 12·5 20 6·4	19 20·0 (1) 19 36·1 (2) 19 28·0				56 30
21 8·6	20 32·2	19 55·8	19 19·4 (2)				55 30
20 51·9 (3) 20 47·9 20 52·7 (6) 20 50·8	20 15·5 20 14·3 20 14·9	19 39·1 19 40·7 19 39·9	19 2·7 (2) 19 7·1 19 4·9	18 33·5 (3)			54 30
(3) 20 32·4 20 31·8 (6) 20 32·1	19 58·8	19 25·2	18 51·6	18 18·0 (3)			53 30
20 10·8 (6) (3) 20 16·9 20 11·8 20 13·2	19 43·3 (4) 19 37·1 19 42·6 (9) 19 41·0	19 9·7 19 8·2 19 9·0	18 36·1 18 39·3 18 37·7	18 2·5 (3) 18 10·4 18 6·4	17 41·5	17 12·6 (4)	52 30
19 49·4 19 47·4 19 48·4	19 20·2 (9) (4) 19 19·7 19 18·5 19 19·5	18 50·8 18 49·6 18 50·2	18 21·9 18 20·7 (7) 18 21·3	17 53·0	17 24·1	16 55·2 (4)	51 30
19 27 0 19 29·6 19 28·3	18 57·8 (9) (4) 19 2·3 19 0·7 19 0·3	18 33·4 18 31·8 18 32·6	18 4·5 18 2·9 (7) 18 3·7	17 35·6	17 6· 7	16 37·8 (4)	50 30
19 11·8	18 42·9	18 14·0	17 45·1 (7)				49 30

Isogonal lines were next drawn by the aid of the mean values of the Declination given in this Table, the points at which they intersect any particular line of latitude or longitude being calculated on the assumption that the rate of change of the Declination with latitude or longitude may be regarded as constant over a single degree. The curves thus obtained are shown in dotted lines in Plate II. As they were deduced from the linear district formulæ they may be called the *district curves*.

By drawing a smooth curve to coincide as nearly as possible with any one of the longer of these broken district curves, a close approximation to the corresponding isogonal could be obtained, but in order that the intervals between the curves might be properly spaced out, it was thought better to obtain a general formula by which they could be expressed.

It appears from Table V., p. 236, that y increases with the latitude, but is nearly independent of the longitude. The values of x are more irregular. After several trials it was found that the mean Declinations at the central stations of the various districts could be reproduced very accurately by means of the formula,

$$\delta = 19^\circ \ 11' + 19'\cdot1 \ (l - 49\cdot5) - 3'\cdot5 \cos \{45^\circ \ (l - 49\cdot5)\}$$
$$+ \{26'\cdot6 + 1'\cdot5 \ (l - 49\cdot5)\} \ (\lambda - 4),$$

where l and λ are the numerical values of the latitude and longitude expressed in degrees and fractions of a degree.

It must be distinctly understood that this formula has no theoretical value except in so far as it expresses satisfactorily the equation to smooth curves drawn according to a definite rule to represent the general form of the broken district curves.

In applying it to the various stations the curves given by the periodic terms were drawn, and the values corresponding to the latitude and longitude of each read off. It is possible that errors of $0'\cdot2$ or $0'\cdot3$ may have occurred in this process. The mean values of the Declinations at the central stations of each district are in the following Table compared with the results given by the formula.

District.	Declination at central station.		Difference.
	Mean of values observed in district.	Calculated.	
I.	21 38·8	21 39·4	− 0·6
II.	20 55·6	20 54·1	+ 1·5
III.	19 39·0	19 38·9	+ 0·1
IV.	18 6·6	18 4·4	+ 2·2
V.	22 41·3	22 41·5	− 0·2
VI.	21 25·6	21 26·5	− 0·9
VII.	19 6·2	19 5·6	+ 0·6
VIII.	22 35·0	22 34·4	+ 0·6
IX.	20 19·7	20 19·8	− 0·1

In District III. we included Stonyhurst ; in District IV. Greenwich, Kew, and Berck-sur-Mer, and in District VII., Cherbourg.   The data for the French stations were those given by M. MOUREAUX, reduced, of course, to January 1, 1886.

The values of the constants in the equation were not finally chosen until the Declination had been determined by means of a preliminary formula closely agreeing with that given above for all the points indicated in Table VI., pp. 238–9, and the values given by it and by the district curves compared.   Over the greater part of the country the agreement is extremely close.   It is not necessary to reproduce the numbers here, as in Plate II. we have plotted down both the broken district curves, and also the smooth curves given by the equation.   The former are dotted, the latter are continuous lines.   The agreement is all that can be desired, except, perhaps, on the coast of Norfolk, and on the west and north coasts of Scotland.

Taking, however, the English Channel first into consideration, we have calculated the points at which our 16° 53′, 17° 53′, etc., isogonals cut lat. 50°.   When secular change is thus allowed for, these correspond to the positions of the 17°, 18° isogonals on January 1, 1885, *i.e.*, with those of M. MOUREAUX.   We have also measured the corresponding points from his map.   The result is shown in the following Table :—

Isogonal Jan. 1, 1885.	Longitude from Greenwich.	
	R. and T.	MOUREAUX.
17	1 17 E.	1 9 E.
18	0 55 W.	0 57 W.
19	3 7 W.	2 55 W.
20	5 19 W.	4 58 W.

The 20° isogonal does not pass through France, and, therefore, M. MOUREAUX's map is not of special authority on its direction, but there seems no doubt that while the 18° isogonals cut latitude 50° at the same point, there is a considerable divergence between the others.

According to the general formula, 27′·35 is the change of Declination per degree of longitude on lat. 50°.   The two Channel Districts in Table V., p. 236, viz., VII. and IV. agree in giving 28′·9 for this number, and if we take this as correct and adhere to 55′ W. for the 18° isogonal, we obtain the following values :—

Isogonal.	R. and T.	MOUREAUX.
17	1 10 E.	1 9 E.
18	0 55 W.	0 57 W.
19	3 0 W.	2 55 W.
20	5 5 W.	4 58 W.

These are in much better accord than those obtained from the general formula, and the conclusions which may be drawn from them are that in latitude 50°,

(1.) M. Moureaux's lines are certainly not too far to the west;

(2.) And our 17° isogonal is not too far to the west.

Both these conclusions are important. M. Moureaux believes that there is what we should call a considerable regional disturbance in Brittany. The isogonals drawn by him sweep out to the west in the western part of France and do not resume their normal course until they have reached the English coast. In the English Channel, therefore, they are deflected to the west, and as both our district lines, and to a much more marked extent our terrestrial lines are still further west, it is evident that M. Moureaux has not in any way exaggerated the westerly tendency of the lines in the western parts of the Channel, and that, therefore, our observations tend to confirm his view. It is probable that our terrestrial curves are in this district a little too far to the west, but it must be remembered that lat. 50° is almost outside the region of our survey, and that in the Channel Isles, which with Cherbourg, are our only stations to the south of it, there are considerable disturbances.

In fig. 2 the continuous lines are our terrestrial isogonals, which on January 1, 1886, corresponded to M. Moureaux's 17°, 18°, etc., isogonals on January 1, 1885, the secular change being about 7'. M. Moureaux's curves are shown with dashes and dots, and the hypothetical connections in dots. A satisfactory comparison on the borders of the areas of two surveys cannot, however, be made unless there is a closer agreement in the methods of working up the results of the observations than there is between our own and M. Moureaux's.

Coming next to the 17° isogonal, we find that on the coast of Norfolk the calculated Declinations are generally much lower than the observed values. This would be remedied by moving the 17° isogonal further to the east, but the fact that it is already to the east of M. Moureaux's line is a strong argument against such a course.

It is noticeable that Sir Fred. Evans appears to have found some difficulty in this part of the country ('Phil. Trans.,' 1872, vol. 162, p. 350). His curves are shown in Plate II. The points at which the isogonals cut lat. 52° are given below, and it will be seen that the distance between them is a maximum in the centre of England.

Isogonal (1872).	Long.	Difference of Long.
°	°	°
25	9·54 W.	
24	7·86	1·68
23	6·16	1·70
22	4·40	1·76
21	2·40	2·00
20	0·40	2·00
19	1·47 E.	1·87

Fig. 2.
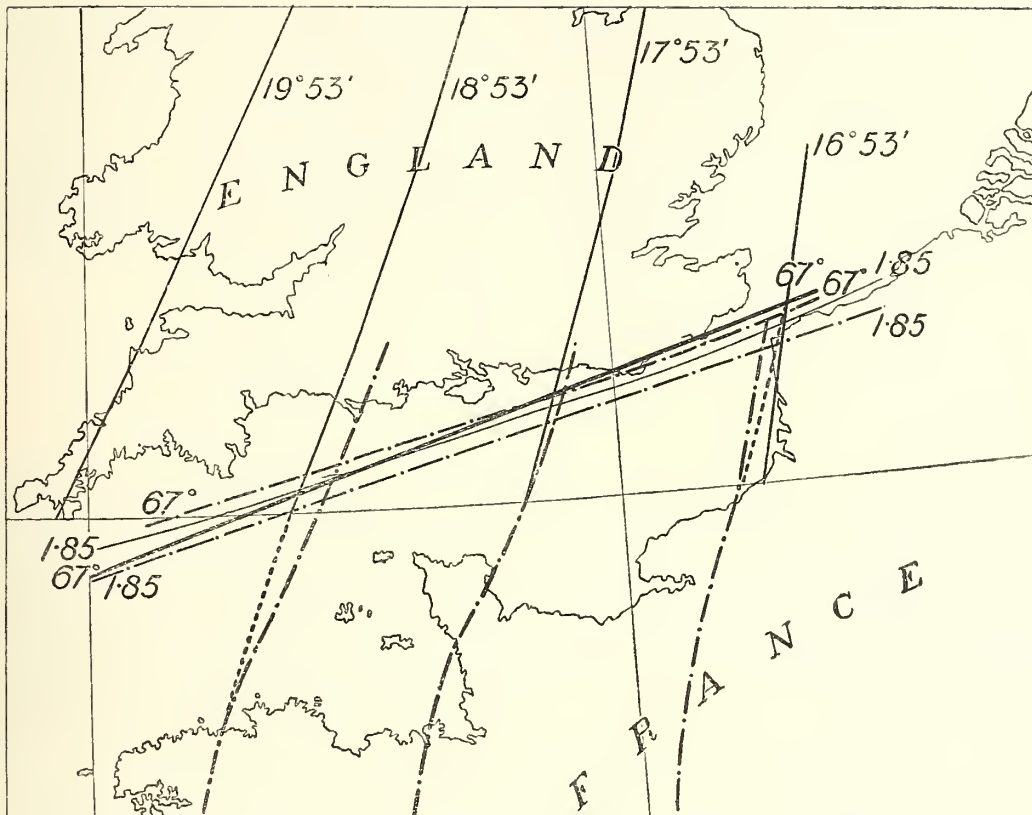
Isomagnetics in the border district of the English and French Surveys,
————— R. and T., —·—·—·— MOUREAUX, ... hypothetical connections.

We think the agreement between our calculated and observed curves is too close to leave any doubt that in this and neighbouring latitudes the law that the distance between the points of intersection with a line of latitude is constant is very approximately true.   At all events, there is no trace of a change in this distance amounting to a quarter of a degree as is shown by the 1872 lines.   As the observations on which Sir F. EVANS's map were based were comparatively few in number and were made exclusively at coast stations the accurate delineation of the isogonals was not easy. He makes the observed Declinations too small in the eastern counties.   If the distances between the isogonals had been kept constant the 19° line would have been pushed further to the west, and a closer agreement with our observations would have been attained.   Considering the circumstances under which Sir F. EVANS's map was drawn, we think the concordance between the general directions of the lines is satisfactory.· We are also of opinion that our 17° isogonal is not too far to the west, and that the fact that in the eastern counties the calculated are less than the observed Declinations is due to a real physical cause.   M. MOUREAUX's results confirm our own as to the 18° isogonal, and if we suppose that the easterly tendency of the lines which

he assumes in the western parts of the English Channel is continued to the Straits of Dover, his 17° isogonal and our own would be in practical accord.

As regards the considerable deviation between the district and calculated terrestrial lines in the North of Scotland, we have no hesitation in adhering to the calculated lines as being probably the more correct. The masses of basalt and other igneous rocks which occur in the West of Scotland and North of Ireland produce a very marked effect on the magnetic elements. The general formula gives the Declination at the outlying station at Lerwick almost as accurately as the local district equations obtained from Scotland only. The three values are :—

$$
\begin{array}{lccc}
\text{Observed Declination} & \ldots & 20° & 29'{\cdot}7 \\
\text{Calculated by} \left\{ \begin{array}{l} \text{Local formula} \ldots \\ \text{General formula} \ldots \end{array} \right. & & 20 \quad 31{\cdot}3 \\ & & 20 \quad 33{\cdot}7
\end{array}
$$

Regarding this station as giving a fixed point it is evident that in drawing the curves to it from the South of Scotland, great weight ought to be given to the general law which is found to be obeyed with accuracy in Great Britain from the English Channel to the Tay, and in the South and East of Ireland. The only way of eliminating the effect of regional disturbances of such magnitude as these which exist in Scotland, is by studying the shape of the terrestrial lines in adjacent districts. Observations to the north of Scotland would be very valuable for this purpose, but as it would be difficult to obtain them we think that the results of the general formula must be accepted as a close approximation to the truth. We shall return to this subject in the discussion of local disturbances.

(2.) *The Isoclinal Lines.*

The isoclinals were obtained in a precisely similar manner to that above described. The districts were the same, but the positions of the central stations were in some cases slightly different, as at a few places Dips had been observed without Declinations and *vice versâ*.

In District VII. the inclusion of the Channel Isles led to rates of change with latitude and longitude so widely different from those obtained elsewhere that it was thought better to omit them. When this was done the coefficients assumed normal values. We give in the next Table the latitude and longitude of the central stations, and the corresponding values of the Dips, and the rates of change of Inclination per degree of latitude and longitude.

TABLE VII.

District.	Central Station.		$D = \theta_0{}'$.	$x = \dfrac{d\theta'}{dl}$.	$y = \dfrac{d\theta'}{d\lambda}$.
	Lat. N.	Long. W.			
I.	56 48·0	4 19·0	71 8·0	34·1	5·8
II.	55 27·3	3 41·6	70 19·6	31·9	8·2
III.	53 26·7	2 26·0	69 3·0	36·1	7·1
IV.	51 47·7	0 17·4	67 45·6	40·8	7·6
V.	54 1·4	7 39·3	69 59·8	38·3	9·8
VI.	53 26·7	5 42·0	69 24·8	36·2	6·3
VII.	51 8·1	3 9·6	67 41·2	38·8	6·6
VIII.	52 57·1	8 13·1	69 24·3	38·7	8·5
IX.	51 49·5	4 47·4	68 18·2	39·1	6·8

A general formula was next found to embrace the whole country. For this purpose a Table similar to Table VI., pp. 238-9, was prepared, and the district isoclinal lines were drawn from it on curve paper, on a purely artificial system, in which all degrees of latitude and longitude were regarded as of equal length. When thus drawn the mean directions of the isoclinals were nearly straight lines and practically parallel.

The equation to the 67° isoclinal was

$$l - 49°·92 + 0·2\,(\lambda - 4) = 0,$$

where l and λ are the latitude and longitude expressed in degrees and fractions of a degree. If then s be the length of the perpendicular on this line from any point, the Dip at the station indicated by that point would (if the lines were equidistant) be given by the equation

$$\theta = 67 + As,$$

where A is a constant. This condition was not fulfilled. The distance between the lines increased approximately in the proportion of their distance from the 67° line, but a small periodic term was necessary in addition to this correction to produce the desired accuracy.

The following plan was finally adopted :—

If we write

$$p = l - 49·92 + 0·2\,(\lambda - 4)$$

and

$$q = p - 0·1 \sin (20\,p)$$

we get the Dip in degrees from the equation

$$\theta = 67° + \frac{1·0083\,q}{1·456 + 0·03\,q}.$$

The Dips given by this formula for the central stations in each district are, in the following Table compared with those given in Table VII., p. 245.

District.	Inclination at Central Station.		Difference.
	Mean of Values observed in District.	Calculated.	
I.	71 8·0	71 10·3	−2·3
II.	70 19·6	70 21·2	−1·6
III.	69 3·0	69 1·8	+1·2
IV.	67 45·6	67 44·5	+0·9
V.	69 59·8	69 59·3	+0·5
VI.	69 24·8	69 25·3	−0·5
VII.	67 41·2	67 41·2	0·0
VIII.	69 24·3	69 25·6	−1·3
IX.	68 18·2	68 19·7	−1·5

In Plate III. we show the broken curves obtained by the district lines and also the terrestrial isoclinals as represented by the formula.

The agreement is on the whole satisfactory, but it is possible that a closer approximation to the true terrestrial lines might have been obtained had we made the inclination to the geographical meridian increase rather more rapidly in the west. This would have diminished the discrepancy in the south of Ireland. On the other hand it would have considerably increased it in the north of Ireland and the central districts of Scotland, in which large regional disturbances undoubtedly exist. It does not, however, appear to be safe to depart from the rule that the terrestrial lines deduced from any survey should give, as nearly as possible, the mean directions of the true lines in the district under investigation. This end is better attained by our formulæ than if the agreement were closer in the south of Ireland.

Our 67° isoclinal agrees almost exactly with that of M. Moureaux in the more easterly parts of the English Channel (see fig. 2, p. 243). Both just cut Dungeness and Beachy Head. For the longitude of Falmouth, however, our line is 10 or 12 miles to the south of that of M. Moureaux.

As it is at this point much nearer to the English than the French coast, our result is probably the more trustworthy, and this opinion is confirmed by a study of the map of the French Survey. The isoclinals drawn by M. Moureaux upon a map on Mercator's projection are curved in the south of France, the convexity being towards the north. The curvature becomes less as the Dip increases, and the 67° line is represented as quite straight. M. Moureaux cannot have much to guide him in drawing this line, and a divergence such as that which exists between his line and ours could easily be introduced by an apparently trifling error in the estimation of the rate of disappearance of the curvature of the lines. Our lines would be slightly concave to the north if drawn upon a map on Mercator's projection.

(3.) *Lines of equal Horizontal Force.*

These lines were treated in the same way as the isogonals and isoclinals, and in the following Table the constants for the nine districts are tabulated as before. As Professor BALFOUR STEWART did not deduce the lines of equal Horizontal Force from Mr. WELSH's observations, there was not as much reason in this case as in the others for taking the whole of Scotland as District I.   We therefore included in this district only all stations north of lat. 56°.

District.	Central Station.		H_0'.	$x = \dfrac{d\mathrm{H}}{dl}$.	$y = \dfrac{d\mathrm{H}}{d\lambda}$.
	Lat. N.	Long. W.			
I.	57 26·3	4 30·0	1·5580	−0·03930	−0·00575
II.	55 27·3	3 41·6	1·6363	−0·03324	−0·00882
III.	53 26·7	2 26·0	1·7164	−0·03868	−0·00661
IV.	51 47·7	0 17·4	1·7970	−0·04374	−0·00620
V.	54 2·9	7 36·5	1·6650	−0·04380	−0·01046
VI.	53 29·0	5 43·0	1·6979	−0·03771	−0·00439
VII.	50 49·3	2 57·6	1·8212	−0·04134	−0·00598
VIII.	52 57·1	8 13·1	1·7053	−0·04237	−0·00628
IX.	51 49·5	4 47·4	1·7694	−0·04212	−0·00598

By means of these data a Table like Table VI., pp. 238–9, was prepared, and lines of equal Horizontal Force were drawn on curve paper as in the case of the isoclinals (see p. 245).   These at once showed that the lines above and below that corresponding to 1·7 units were differently disposed.   The mean direction of each of the southern lines can be accurately represented by a linear function of the latitude and longitude, their departure from parallelism is not great, and their mean distances are nearly the same. On the other hand, though the lines in District I., *i.e.*, in the extreme north of Scotland, are parallel to the 1·7 line, which runs from the neighbourhood of Miltown Malbay to that of Scarborough, the intermediate isodynamics make smaller angles with the geographical meridian, and their average distance is greater than in the south.   It is, however, difficult to decide what is the direction of the terrestrial lines.   Too much weight must not be attached to the fact that the lines in the north of Scotland agree with those in the Midlands and south of England, as they are deduced from a single district of irregular form and the seat of great local and regional disturbances.   There is also reason to suppose from what is known of the lines of equal Horizontal Force on the continent that the distance between them increases towards the east, and those crossing England are so represented on M. MOUREAUX's map.   It may well be, therefore, that the diverging lines are those which agree most closely with the terrestrial lines of equal Horizontal Force.

On the whole then, the risk of introducing fictitious disturbances by attempting to include the whole country under one simple law would be very great.

We decided, therefore, in calculating the lines of equal Horizontal Force, to employ different formulæ for districts to the north and south of the 1·7 line.

Taking the southern district first, the lines may be regarded as straight, but, inasmuch as they run across the whole breadth of the kingdom, a very small error in the calculated slope is important. A rather complicated formula is therefore unfortunately necessary.

If c be the mean distance, expressed in degrees of latitude and measured along longitude 5° W., between any line and that which corresponds to 1·85 units, we have a relation of the form

$$1·85 - H = ac,$$

where a is a constant.

The 1·85 line cuts longitude 5° W. in latitude 49°·83, and thus the equation to the isodynamic through c is

$$l - 49·83 = -(\lambda - 5)m + c,$$

where m measures the slope of the line.

Neither a nor m are quite constant, but both are functions of c, so that

$$1/a = 24·47 \left(1 + \frac{c}{1000}\right);$$

$$m = ·157 - ·0019c - ·0155 \sin(50c).$$

Thus, if we wish to find where the line corresponding to H cuts longitude λ, we find c from the equation

$$c = 24·47(1·85 - H) + 0·001c^2,$$

where the value of c, used in the small term in c^2, is the approximate value obtained by neglecting it. From this m is found, and l is then known.

If the latitude and longitude are given, and the Horizontal Force is required, we first find c approximately from the formula

$$c' = l - 49·83 + 0·157(\lambda - 5).$$

Subtracting from this

$$c'' = (\lambda - 5)\{0·0019 c' + 0·0155 \sin(50 c')\},$$

the difference is c, whence H is found.

Taking next the district north of the 1·7 line, there is no particular difficulty in finding similar equations to express the mean direction of the lines with great exactitude. If we write

$$1\cdot7 - \text{H} = 0\cdot036854\,c,$$
$$l - 53\cdot514 = -\,m\,(\lambda - 5) + c,$$
$$m = \cdot151 + \cdot07\sin(45\,c),$$

all that can be desired in this respect is attained.

These equations do not, however, give the forces at the central stations with the accuracy attained in the southern districts.   Far better results are obtained if we assume that the lines are parallel to the 1·7 line and are at equal distances from each other.   In this case m is constant, and, in addition to the first of the above equations, we have only

$$l - 53\cdot514 = -\,0\cdot151\,(\lambda - 5) + c.$$

The values obtained from the central stations by both formulæ are given in the following Table :—

District.	Horizontal Force at central stations in northern districts deduced from		
	Mean of all stations in district.	Formula with m variable.	Formula with m constant.
I.	1·5580	1·5583	1·5582
II.	1·6363	1·6390	1·6358
V.	1·6650	1·6608	1·6658
VI.	1·6979	1·6971	1·6972

We have, therefore, to choose between two formulæ.   The simpler represents the lines throughout Scotland as practically parallel, both with those which in the south extend right across the kingdom without considerable curvature and with those found by the district equations for the north of Scotland, and also reproduces the values of the Horizontal Force at the central stations with great accuracy.   The more complex expression represents the mean directions of the lines in the centre of Scotland and north of Ireland more satisfactorily, though these are obviously affected by some widespread disturbance, but it fails when tried by the test of the reproduction of the forces at the central stations.   We think there can be no question that, under these circumstances, the simpler formula, which assigns directions to the lines in harmony with those obtained in other parts of the kingdom, must be that selected ; and we have therefore employed it.

In the following Table the values of the Horizontal Forces at the central stations are compared with the calculated values given by the formulæ—

$$1\cdot 85 - H = c\,(1 - 0\cdot 001\ c)\,/24\cdot 47$$

and

$$l - 49\cdot 83 = c - m\,(\lambda - 5)$$

and

$$m = 0\cdot 157 - 0\cdot 0019\ c - 0\cdot 0155\ \sin\,(50\ c)$$

for values of $H > 1\cdot 7$, and by

$$1\cdot 7 - H = 0\cdot 036854\ c$$

and

$$l - 53\cdot 514 = c - 0\cdot 151\,(\lambda - 5)$$

for values of $H < 1\cdot 7$.

The difference between the observed and calculated results in District VIII. (South Ireland), is larger than in the other cases. It must, however, be remembered that it is absolutely very small. Thus, if the disturbed stations at Galway (which have been excluded from the calculations) were introduced, the difference would be reduced from ·0022 to ·0007. As the effect of one station is so great, it is remarkable that the discrepancies are not larger.

District.	Horizontal Force at central stations.		Difference in terms of 0·0001.
	Mean of values observed in district.	Calculated.	
I.	1·5580	1·5582	− 2
II.	1·6363	1·6358	+ 5
III.	1·7164	1·7180	−16
IV.	1·7970	1·7969	+ 1
V.	1·6650	1·6658	− 8
VI.	1·6979	1·6972	+ 7
VII.	1·8212	1·8218	− 6
VIII.	1·7053	1·7031	+22
IX.	1·7694	1·7698	− 4

We conclude this section with a Table, which gives the observed and calculated elements at every station.

In the calculations, graphic methods have been partly used, and, in some cases, this may have led to slight differences (which are, however, so small as to be quite unimportant) between the values given and those which would be deduced directly from the formulæ.

SUMMARY of Declinations, Inclinations, Horizontal and Vertical Forces, Observed and Calculated, with Differences.

Scotland.

No.	Name	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No. of Station
1	Aberdeen	20 16·3	20 21·0	− 4·7	71 12·3	71 7·7	+ 4·6	1·5734	1·5813	−·0079	4·6232	4·6260	−·0028	1
2	Arinagower (Coll)	23 40·4	22 58·2	+42·2	..	71 18·8	..	..	1·5772	..	..	..	..	2
3	L. Aylort	23 16·5	22 35·4	+41·1	71 24·0	71 21·5	+ 2·5	1·5663	1·5726	−·0063	4·6542	4·6617	−·0075	3
4	Ayr	21 17·9	21 26·8	− 8·9	70 21·4	70 27·8	− 6·4	1·6345	1·6300	+·0045	4·5792	4·5937	−·0145	4
5	Ballater	20 29·5	20 55·3	−25·8	71 15·4	71 10·0	+ 5·4	1·5714	1·5809	−·0095	4·6309	4·6350	−·0041	5
6	Banavie	22 6·7	22 9·5	− 2·8	71 11·4	71 17·0	− 5·6	1·5940	1·5764	+·0176	4·6797	4·6529	+·0268	6
7	Banff	21 4·5	20 46·1	+18·4	71 19·0	71 26·7	− 7·7	1·5684	1·5610	+·0074	4·6381	4·6505	−·0124	7
8	Berwick	19 36·4	19 57·9	−21·5	70 15·9	70 20·2	− 4·3	1·6483	1·6336	+·0147	4·5947	4·5717	+·0230	8
9	Boat of Garten	22 7·7	21 26·1	+41·6	71 16·3	71 21·2	− 4·9	1·5786	1·5693	+·0093	4·6562	4·6506	+·0056	9
10	L. Boisdale	22 53·3	23 39·2	−45·9	71 39·3	71 40·9	− 1·6	1·5310	1·5530	−·0220	4·6172	4·6908	−·0736	10
11	Bunnahabhain	23 10·3	22 29·0	+41·3	70 43·0	70 52·1	− 9·1	1·6243	1·6063	+·0180	4·6427	4·6305	+·0122	11
12	Callernish	23 40·6	23 40·9	− 0·3	72 7·1	72 9·4	− 2·3	1·5236	1·5191	+·0045	4·7223	4·7192	+·0031	12
13	Campbelton	22 8·1	22 1·2	+ 6·9	70 34·2	70 33·4	+ 0·8	1·6244	1·6259	−·0015	4·6050	4·6059	−·0009	13
14	Canna (mean)	23 13·0	23 6·5	+ 6·5	72 45·0	71 32·7	+72·3	1·5092	1·5607	−·0515	4·8605	4·6766	+·1839	14
15	Carstairs	20 52·2	20 56·8	− 4·6	70 15·7	71 29·0	−13·3	1·6448	1·6276	+·0172	4·5841	4·5920	−·0079	15
16	Crianlarich	21 50·6	21 43·2	+ 7·4	70 52·5	70 58·8	− 6·3	1·5980	1·5961	+·0019	4·6083	4·6302	−·0219	16
17	Crieff	21 33·6	21 14·1	+19·5	70 53·6	70 53·2	+ 0·4	1·6079	1·6010	+·0069	4·6416	4·6199	+·0217	17
18	Cumbrae	21 37·2	21 42·6	− 5·4	71 2·3	70 40·9	+21·4	1·5911	1·6159	−·0248	4·6310	4·6095	+·0215	18
19	Dalwhinnie	21 45·5	21 38·6	+ 6·9	71 0·1	71 14·0	−13·9	1·5909	1·5784	+·0125	4·6207	4·6454	−·0247	19
20	Dumfries	20 47·4	20 43·9	+ 3·5	70 2·6	70 6·2	− 3·6	1·6542	1·6519	+·0023	4·5555	4·5642	−·0087	20
21	Dundee	20 44·5	20 42·7	+ 1·8	70 52·2	70 50·5	+ 1·7	1·6002	1·6024	−·0022	4·6132	4·6122	+·0010	21
22	Edinburgh	20 47·2	20 44·4	+ 2·8	70 38·5	70 35·2	+ 3·3	1·6183	1·6195	−·0012	4·6061	4·5954	+·0107	22
23	Elgin	20 57·5	21 16·5	−19·0	71 32·0	71 31·2	+ 0·8	1·5577	1·5574	+·0003	4·6644	4·6599	+·0045	23
24	L. Eriboll	22 18·1	22 26·3	− 8·2	72 9·4	72 6·2	+ 3·2	1·5198	1·5190	+·0008	4·7214	4·7039	+·0175	24
25	Fairlie	..	21 40·4	..	70 42·8	71 39·3	+ 3·5	1·6172	1·6176	−·0004	4·6222	4·6087	+·0135	25
26	Fort Augustus	21 45·6	21 59·2	−13·6	71 27·7	71 23·8	+ 3·9	1·5641	1·5680	−·0039	4·6641	4·6584	+·0057	26
27	Gairloch	22 14·4	22 59·0	−44·6	71 44·3	71 49·9	− 5·6	1·5353	1·5417	−·0064	4·6528	4·6979	−·0451	27
28	Glasgow	21 11·5	21 22·4	−10·9	70 44·7	70 39·6	+ 5·1	1·6064	1·6169	−·0105	4·5988	4·6068	−·0080	28
29	Golspie	21 30·2	21 48·5	−18·3	71 46·7	71 45·8	+ 0·9	1·5382	1·5417	−·0035	4·6725	4·6790	−·0065	29
30	Hawick	20 16·0	20 22·0	+ 6·0	70 7·3	70 14·4	− 7·1	1·6487	1·6416	+·0071	4·5598	4·5697	−·0099	30
31	L. Inver	22 7·4	22 42·7	−35·3	72 0·2	71 59·6	+ 0·6	1·4990	1·5276	−·0286	4·6143	4·6996	−·0853	31
32	Inverness	21 43·3	21 48·3	− 5·0	71 31·1	71 31·4	+ 0·3	1·5642	1·5585	+·0057	4·6798	4·6642	+·0156	32

Scotland (continued)

No.	Name	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No. of Station
33	Iona	23 28·6	23 47·7	+ 40·9	70 55·8	71 8·7	− 12·9	1·6185	1·5883	+ ·0302	4·6711	4·6509	+ ·0202	33
34	Kirkwall	21 29·3	21 27·9	+ 1·4	72 12·8	72 11·0	+ 1·8	1·5108	1·5100	+ ·0008	4·7093	4·6984	+ ·0109	34
35	Kyle Akin	23 10·4	22 42·4	+ 28·0	71 38·5	71 34·8	+ 3·7	1·5465	1·5577	− ·0112	4·6603	4·6771	− ·0168	35
36	Lairg	21 50·3	22 6·3	− 16·0	71 50·3	71 50·1	+ 0·2	1·5356	1·5370	− ·0014	4·6811	4·6845	− ·0034	36
37	Lerwick	20 29·7	20 33·7	− 4·0	72 47·1	72 35·3	+ 11·8	1·4710	1·4770	− ·0060	4·7476	4·7098	+ ·0378	37
38	Lochgoilhead	21 54·2	21 49·5	+ 4·7	70 46·1	70 53·4	− 7·3	1·6021	1·6025	− ·0004	4·5924	4·6252	− ·0328	38
39	L. Maddy	23 18·0	23 34·3	− 16·3	71 52·1	71 54·1	− 2·0	1·5365	1·5372	− ·0007	4·6921	4·7035	− ·0114	39
40a	Oban	22 9·4	22 15·4	+ 6·0	70 53·2	71 5·4	− 12·2	1·6044	1·5904	+ ·0140	4·6297	4·6426	− ·0129	40a
40b	„ (Kerrera)	22 11·9	22 16·4	− 4·5	70 48·8	71 5·7	− 16·9	1·6103	1·5896	+ ·0207	4·6276	4·6416	− ·0140	40b
41	Pitlochrie	21 8·3	21 15·4	− 7·1	70 57·4	71 3·2	− 5·8	1·5899	1·5895	+ ·0004	4·6061	4·6303	− ·0242	41
42	Port Askaig	22 0·7	22 27·6	+ 33·1	70 36·2	70 50·6	− 14·4	1·6340	1·6073	+ ·0267	4·6409	4·6268	+ ·0141	42
43a	Portree	25 1·1	23 3·0	+ 118·1	72 18·4		+ 36·5	1·5177	1·5500	− ·0323				43a
43b	„	22 45·6	23 2·5	− 16·9	71 8·6	71 41·9	− 33·3	1·5876	1·5500	+ ·0376	4·7517	4·6863	+ ·0654	43b
43c	„	20 14·3	23 2·9	− 168·6	72 36·6		+ 54·7	1·5211	1·5500	− ·0289				43c
44	Row (Gairloch)	21 47·7	21 42·5	+ 5·2	70 51·0	70 47·5	+ 3·5	1·5978	1·6096	− ·0118	4·6012	4·6200	− ·0188	44
45	Scarnish (Tiree)	24 27·9	23 5·7	+ 82·2	71 19·4	71 16·8	+ 2·6	1·5909	1·5798	+ ·0111	4·7063	4·6619	+ ·0444	45
46	Soa	23 14·9	22 56·5	+ 18·4	71 59·6	71 34·0	+ 25·6	1·5072	1·5583	− ·0511	4·6368	4·6753	− ·0385	46
47	Stirling	21 28·6	21 13·9	+ 14·7	70 53·3	70 45·3	+ 8·0	1·5945	1·6097	− ·0152	4·6015	4·6107	− ·0092	47
48a	Stornoway (Ard Point)	24 16·3	23 29·3	+ 47·0	72 10·5	72 8·2	+ 2·3	1·5196	1·5193	+ ·0003	4·7259	4·7141	+ ·0118	48a
48b	Stornoway (Castle)	24 7·6	23 29·4	+ 38·2	72 9·1	72 8·4	+ 0·7	1·5146	1·5193	− ·0047	4·7038	4·7151	− ·0113	48b
49	Strachur	21 48·9	21 56·1	− 7·2	70 42·9	70 54·6	− 11·7	1·6095	1·6013	+ ·0082	4·5999	4·6268	− ·0269	49
50	Stranraer	21 35·0	21 31·9	+ 3·1	70 13·5	70 11·7	+ 1·8	1·6435	1·6481	− ·0046	4·5713	4·5765	− ·0052	50
51	Stronness	21 27·9	21 41·1	− 13·2	72 11·7	72 12·3	− 0·6	1·5149	1·5093	+ ·0056	4·7169	4·7023	+ ·0146	51
52	E. L. Tarbert	22 4·3	22 2·4	+ 1·9	70 46·8	70 46·7	+ 0·1	1·6053	1·6107	− ·0054	4·6047	4·6197	− ·0150	52
53	Thurso	21 38·4	21 42·9	− 4·5	72 1·1	72 2·4	− 1·3	1·5217	1·5212	+ ·0005	4·6883	4·6930	− ·0047	53
54	Wick	21 15·3	21 22·8	− 7·5	72 9·8	71 54·9	+ 14·9	1·5144	1·5290	− ·0146	4·7064	4·6822	+ ·0242	54

England.

No.	Name	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No. of Station
55	Aberystwith	19 56·5	20 11·2	−14·7	68 34·7	68 35·7	−1·0	1·7488	1·7506	−·0018	4·4575	4·4653	−·0078	55
56	Alderney	18 2·8	18 22·8	−20·0	66 38·9	66 37·2	+1·7	1·8691	1·8735	−·0044	4·3292	4·3335	−·0043	56
57	Alnwick	19 45·0	19 43·9	+1·1	70 3·6	70 6·7	−3·1	1·6511	1·6485	+·0026	4·5510	4·5560	−·0050	57
58	Alresford	18 9·7	18 18·0	−8·3	67 22·3	67 23·5	−1·2	1·8241	1·8221	+·0020	4·3761	4·3755	+·0006	58
59	Appleby	20 5·8	19 58·3	+7·5	69 44·9	69 42·3	+2·6	1·6690	1·6750	−·0060	4·5237	4·5293	−·0056	59
60	Barrow	20 9·3	20 16·1	−6·8	69 30·6	69 31·7	−1·1	1·6875	1·6885	−·0010	4·5157	4·5228	−·0071	60
61	Bedford	18 27·4	18 14·5	+12·9	68 7·3	67 58·7	+8·6	1·7705	1·7818	−·0113	4·4091	4·4052	+·0039	61
62	Birkenhead	19 58·3	19 58·9	−0·6	69 4·3	69 4·8	−0·5	1·7176	1·7160	+·0016	4·4912	4·4886	+·0026	62
63	Birmingham	18 44·0	19 4·7	−20·7	68 21·3	68 21·8	−0·5	1·7669	1·7605	+·0064	4·4525	4·4373	+·0152	63
64	Braintree	17 55·4	17 40·3	+15·1	67 45·4	67 41·2	+4·2	1·7942	1·7984	−·0042	4·3871	4·3821	+·0050	64
65	Brecon	19 38·6	19 41·1	−2·5	68 15·8	68 14·0	+1·8	1·7701	1·7727	−·0026	4·4399	4·4392	+·0007	65
66a	Bude Haven	19 56·5	19 49·8	+6·6	67 44·2	67 39·9	+4·3	1·8080	1·8120	−·0040	4·4164	4·4104	+·0060	66a
66b	„		19 50·0											66b
67	Cambridge	18 5·0	17 57·9	+7·1	68 2·4	67 56·5	+5·9	1·7784	1·7828	−·0044	4·4105	4·3995	+·0110	67
68	Cardiff	19 19·7	19 24·6	−4·9	67 52·3	67 55·0	−2·7	1·7944	1·7926	+·0018	4·4129	4·4182	−·0053	68
69	Cardigan	20 25·8	20 22·3	+3·5	68 31·3	68 28·6	+2·7	1·7535	1·7600	−·0065	4·4565	4·4617	−·0052	69
70	Carlisle	20 25·8	20 18·6	+7·2	69 54·0	69 56·8	−2·8	1·6625	1·6602	+·0023	4·5430	4·5482	−·0052	70
71	Chesterfield	19 11·9	19 2·4	+9·5	68 48·5	68 46·8	+1·7	1·7351	1·7321	+·0030	4·4752	4·4607	+·0145	71
72	Chichester	18 5·5	18 2·6	+2·9	67 11·6	67 10·8	+0·8	1·8395	1·8352	+·0043	4·3745	4·3615	+·0130	72
73	Clenchwarton	18 10·3	17 59·0	+11·3	68 17·9	68 16·0	+1·9	1·7662	1·7611	+·0051	4·4378	4·4171	+·0207	73
74	Clifton	19 10·7	19 7·5	+3·2	67 48·7	67 49·3	−0·6	1·7996	1·7974	+·0022	4·4124	4·4091	+·0033	74
75	Clovelly	19 53·8	19 49·5	+4·3	67 49·9	67 45·4	+4·5	1·8000	1·8059	−·0059	4·4177	4·4156	+·0021	75
76	Coalville	18 41·4	18 52·6	−11·2	68 24·1	68 27·7	−3·6	1·7534	1·7526	+·0008	4·4289	4·4398	−·0109	76
77	Colchester	17 55·2	17 29·9	+25·3	67 35·3	67 39·0	−3·7	1·8012	1·7998	+·0014	4·3675	4·3774	−·0099	77
78	Cromer	17 35·8	17 31·1	+4·7	68 20·0	68 15·1	+4·9	1·7603	1·7597	+·0006	4·4310	4·4102	+·0208	78
79	Dover	16 57·2	17 7·0	−9·8	67 8·0	67 5·4	+2·6	1·8336	1·8366	−·0030	4·3477	4·3456	+·0021	79
80	Falmouth	19 53·4	19 49·7	+3·7	67 15·0	67 17·5	−2·5	1·8323	1·8366	−·0043	4·3696	4·3886	−·0190	80
81	Gainsborough	18 36·4	18 43·3	−6·9	68 49·3	68 47·6	+1·7	1·7321	1·7296	+·0025	4·4705	4·4573	+·0132	81
82	Giggleswick	19 35·3	19 44·4	−9·1	69 22·3	69 23·4	−1·1	1·6962	1·6941	+·0021	4·5059	4·5047	+·0012	82
83	Gloucester	19 12·9	19 4·5	+8·4	68 4·3	68 2·3	+2·0	1·7811	1·7824	−·0013	4·4244	4·4198	+·0046	83
84	Grantham	18 29·0	18 32·5	−3·5	68 28·0	68 29·3	−1·3	1·7527	1·7490	+·0037	4·4419	4·4369	+·0050	84
85	Guernsey (L'Erée)	18 32·7	18 29·9	+2·8	66 34·1	66 29·9	+4·2	1·8746	1·8815	−·0069	4·3254	4·3267	−·0013	85
86	„ (Peter Port)	18 18·4	18 27·7	−9·3	66 32·4	66 29·4	+3·0	1·8796	1·8820	−·0024	4·3311	4·3262	+·0049	86
87	Harwich	17 18·9	17 18·8	0·0	67 38·4	67 38·2	+0·2	1·8031	1·8001	+·0030	4·3833	4·3754	+·0079	87

England (continued).

No.	Name.	Declination. Observed.	Calculated.	Difference.	Inclination. Observed.	Calculated.	Difference.	Horizontal Force. Observed.	Calculated.	Difference.	Vertical Force. Observed.	Calculated.	Difference.	No. of Station.
88	Harpenden	18 16·5	18 6·3	+10·2	67 52·4	67 45·4	+ 7·0	1·7916	1·7961	−·0045	4·4063	4·3917	+·0146	88
89	Haslemere	18 7·7	18 6·0	+ 1·7	67 20·6	67 21·1	− 0·5	1·8282	1·8238	+·0044	4·3797	4·3710	+·0087	89
90	Holyhead	20 51·1	20 47·2	+ 3·9	69 23·1	69 12·5	+10·6	1·6958	1·7109	−·0151	4·5080	4·5059	+·0021	90
91	Horsham	18 3·3	17 54·6	+ 8·7	67 15·2	67 16·9	− 1·7	1·8309	1·8275	+·0034	4·3669	4·3649	+·0020	91
92	Hull	18 57·8	18 35·8	+22·0	69 3·9	68 57·8	+ 6·1	1·7125	1·7182	−·0057	4·4764	4·4670	+·0094	92
93	Ilfracombe	19 46·5	19 46·3	+ 0·2	67 53·8	67 51·5	+ 2·3	1·7952	1·7987	−·0035	4·4203	4·4205	−·0002	93
94	Jersey (Grouville)	18 36·0	18 10·1	+25·9	66 8·7	66 14·3	− 5·6	1·9053	1·8965	+·0088	4·3086	4·3077	+·0009	94
95	" (S. Louis)	17 49·7	18 12·0	−22·3	66 13·0	66 14·8	− 1·8	1·8887	1·8961	−·0074	4·2856	4·3085	−·0229	95
96	" (S. Owen)	18 24·7	18 1·53	+ 9·4	66 15·5	66 16·8	− 1·3	1·8904	1·8942	−·0038	4·2979	4·3110	−·0131	96
97	Kenilworth	19 1·4	18 52·6	+ 8·8	68 28·8	68 15·2	+13·6	1·7576	1·7668	−·0092	4·4574	4·4287	+·0287	97
98	Kettering	18 36·0	18 27·6	+ 8·4	68 10·7	68 10·7	0·0	1·7656	1·7695	−·0039	4·4095	4·4187	−·0092	98
99	Kew	18 16·3	17 59·5	+16·8	67 37·4	67 31·8	+ 5·6	1·8093	1·8110	−·0017	4·3950	4·3788	+·0162	99
100a	King's Lynn	17 57·9	17 57·7	+ 0·2	68 17·8	68 15·9	+ 1·9	1·7656	1·7610	+·0046	4·4359	4·4164	+·0195	100a
100b	" (Gaywood)	18 1·7	17 56·6	+ 5·1	68 16·8	68 15·4	+ 1·4	1·7646	1·7615	+·0031	4·4298	4·4159	+·0139	100b
101	King's Sutton	18 51·8	18 37·5	+14·3	68 6·4	68 0·5	+ 5·9	1·7778	1·7819	−·0041	4·4240	4·4119	+·0121	101
102	Lampeter	18 55·3	20 4·8	− 9·5	68 25·5	68 25·0	+ 0·5	1·7559	1·7624	−·0065	4·4406	4·4543	−·0137	102
103	Leeds	19 8·9	19 17·4	− 8·5	69 10·8	69 10·3	+ 0·5	1·7082	1·7067	+·0015	4·4922	4·4863	+·0059	103
104	Leicester	18 23·6	18 43·1	−19·5	68 24·0	68 22·1	+ 1·9	1·7538	1·7581	−·0043	4·4296	4·4323	−·0027	104
105	Lincoln	18 18·9	18 33·3	−14·4	68 43·0	68 39·3	+ 3·7	1·7395	1·7385	+·0010	4·4654	4·4482	+·0172	105
106	Llandudno	20 51·5	20 22·2	+29·3	69 12·0	69 7·5	+ 4·5	1·7084	1·7149	−·0065	4·4974	4·4962	+·0012	106
107	Llangollen	20 8·4	19 54·3	+14·1	68 49·4	68 50·1	− 0·7	1·7331	1·7326	+·0005	4·4736	4·4745	−·0009	107
108	Llanidloes	19 53·8	19 55·1	− 1·3	68 33·8	68 33·5	+ 0·3	1·7501	1·7517	−·0016	4·4573	4·4595	−·0022	108
109	Loughborough	18 18·7	18 49·0	−30·3	68 27·7	68 28·5	− 0·8	1·7531	1·7513	+·0018	4·4419	4·4397	+·0022	109
110	Lowestoft	17 24·0	17 12·2	+11·8	68 0·4	67 54·6	+ 5·8	1·7797	1·7807	−·0010	4·4064	4·3872	+·0192	110
111	Mabelthorpe	18 16·5	18 9·9	+ 6·6	68 42·7	68 38·4	+ 4·3	1·7370	1·7373	−·0003	4·4578	4·4415	+·0163	111
	Malvern—													
112a	Colwall Green	19 3·6	19 11·7	− 8·1		68 10·5		1·7627	1·7737	−·0110				112a
112b	Great Malvern	19 33·0	19 10·5	+22·5	68 14·4	68 11·5	+ 2·9	1·7687	1·7722	−·0035	4·4310	4·4289	+·0021	112b
112c	Malvern Wells	19 22·4	19 10·1	+12·3		68 10·7		1·7682	1·7733	−·0051				112c
112d	Mathon	18 46·5	19 11·9	−25·4		68 11·7		1·7655	1·7722	−·0067				112d
	Manchester—													
113	Old Trafford	19 16·7	19 34·2	−17·5	69 3·9	69 1·4	+ 2·5	1·7125	1·7183	−·0058	4·4764	4·4814	−·0050	113
114	Manton	18 21·7	18 30·0	− 8·3	68 17·1	68 19·0	− 1·9	1·7661	1·7605	+·0056	4·4347	4·4268	+·0079	114
115	March	18 2·8	18 4·7	− 1·9	68 10·3	68 9·8	+ 0·5	1·7719	1·7684	+·0035	4·4237	4·4126	+·0111	115

England (continued).

No.	Name.	Declination Observed.	Declination Calculated.	Declination Difference.	Inclination Observed.	Inclination Calculated.	Inclination Difference.	Horizontal Force Observed.	Horizontal Force Calculated.	Horizontal Force Difference.	Vertical Force Observed.	Vertical Force Calculated.	Vertical Force Difference.	No. of Station.
116a	Melton Mowbray	19 10.9	18 37.7	+33.2	68 27.8	68 25.0	+2.8	1.7620	1.7546	+.0074	4.4648	4.4343	+.0305	116a
116b	,, ,,	19 4.2	18 38.2	+26.0	68 36.0	68 25.6	+10.4	1.7415	1.7539	−.0124	4.4438	4.4358	+.0080	116b
117	Milford Haven	20 8.8	20 21.9	−13.1	68 9.9	68 16.6	−6.7	1.7808	1.7738	+.0070	4.4445	4.4517	−.0072	117
118	Newark	18 46.2	18 40.7	+5.5	68 33.4	68 36.6	−3.2	1.7464	1.7416	+.0048	4.4464	4.4458	+.0006	118
119	Newcastle	19 30.3	19 35.6	−5.3	69 49.5	69 51.2	−1.7	1.6665	1.6641	+.0024	4.5356	4.5360	−.0004	119
120	Northampton	18 41.7	18 29.7	+12.0	68 9.4	68 5.2	+4.2	1.7667	1.7759	−.0092	4.4074	4.4143	−.0069	120
121	Nottingham	18 44.9	18 49.7	−4.8	68 37.6	68 34.5	+3.1	1.7470	1.7448	+.0022	4.4639	4.4460	+.0179	121
122	Oxford	18 33.7	18 34.7	−1.0	67 57.5	67 50.4	+7.1	1.7890	1.7927	−.0037	4.4187	4.4018	+.0169	122
123	Peterborough	18 21.9	18 15.8	+6.1	68 14.8	68 13.7	+1.1	1.7692	1.7651	+.0041	4.4338	4.4188	+.0150	123
124	Plymouth	19 31.6	19 28.8	+2.8	67 14.7	67 18.9	−4.2	1.8309	1.8334	−.0025	4.3651	4.3861	−.0210	124
125	Port Erin	20 55.4	21 7.4	−12.0	69 48.1	69 41.3	+6.8	1.6678	1.6800	−.0122	4.5333	4.5388	−.0055	125
126	Preston	19 52.3	19 52.8	+0.5	69 14.7	69 13.6	+1.1	1.7053	1.7051	+.0002	4.4999	4.4950	+.0049	126
127	Purfleet	17 54.5	17 43.1	+11.4	67 30.9	67 28.3	+2.6	1.8134	1.8136	−.0002	4.3812	4.3723	+.0089	127
128	Pwllheli	20 41.9	20 31.7	+10.2	68 50.9	68 55.8	−4.9	1.7407	1.7291	+.0116	4.4991	4.4877	+.0114	128
129	Ramsey	20 54.9	20 58.9	−4.0	69 55.0	69 46.9	+8.1	1.6617	1.6737	−.0120	4.5449	4.5445	+.0004	129
130	Ranmore	18 8.9	17 43.9	+25.0	67 20.0	67 23.6	−3.6	1.8261	1.8202	+.0059	4.3725	4.3714	+.0011	130
131a	Reading (Caversham)	18 15.2	18 19.3	−4.1	67 40.7	67 37.1	+3.6	1.8110	1.8067	+.0043	4.4109	4.3874	+.0235	131a
131b	Reading (Caversham)	18 11.6	18 19.3	−7.7	…	…	…	1.8141	1.8067	+.0074	…	…	…	131b
132a	Redcar	19 5.6	19 7.9		69 31.5	69 32.5	−1.0	1.6847	1.6828	+.0019	4.5119	4.5108	+.0011	132a
132b	,,		19 8.6	−2.6										132b
133	Ryde	18 1.6	18 12.1	−10.5	67 7.8	67 9.4	−1.6	1.8391	1.8375	+.0016	4.3602	4.3620	−.0018	133
134	St. Cyres (Exeter)	19 28.6	19 21.6	+7.0	67 26.2	67 30.5	−4.3	1.8260	1.8198	+.0062	4.3946	4.3951	−.0005	134
135	St. Leonards	17 24.8	17 25.5	+0.7	66 58.9	67 0.9	−2.0	1.8437	1.8433	+.0004	4.3396	4.3457	−.0061	135
136	Salisbury	18 23.9	18 36.5	−12.6	67 25.6	67 28.6	−3.0	1.8242	1.8179	+.0063	4.3881	4.3838	+.0043	136
137	Scarborough	18 48.3	18 43.0	+5.3	69 15.6	69 16.6	−1.0	1.7017	1.6978	+.0039	4.4939	4.4876	+.0063	137
138	Shrewsbury	19 41.2	19 36.1	+5.1	68 36.4	68 37.2	−0.8	1.7342	1.7457	−.0115	4.4266	4.4585	−.0319	138
139	Southend	17 44.4	17 29.9	+14.5	67 30.8	67 27.0	+3.8	1.8112	1.8138	−.0026	4.3755	4.3681	+.0074	139
140	Spalding	17 51.6	18 15.3	−23.7	68 23.1	68 20.8	+2.3	1.7512	1.7570	−.0058	4.4197	4.4245	−.0048	140
141	Stoke-on-Trent	19 22.7	19 23.3	+0.6	68 43.6	68 42.7	+0.9	1.7401	1.7384	+.0017	4.4693	4.4608	+.0085	141
142	Sutton Bridge	17 54.1	18 4.1	−10.0	68 21.1	68 17.4	+3.7	1.7620	1.7600	+.0020	4.4395	4.4195	+.0200	142
143	Swansea	19 45.6	19 51.1	−5.5	67 59.7	68 5.7	+6.0	1.7931	1.7830	+.0101	4.4370	4.4340	+.0030	143
144	Swindon		18 44.9		67 51.4	67 47.3	+4.1	1.7930	1.7975	−.0045	4.4061	4.4023	−.0038	144
145	Taunton	19 10.7	19 12.3	−1.6	67 32.7	67 36.0	−3.3	1.8154	1.8129	+.0025	4.3925	4.3984	−.0059	145

England (continued).

| Station | | Declination | | | Inclination | | | Horizontal Force. | | | Vertical Force. | | | No. of Station. |
No.	Name.	Observed.	Calculated.	Difference.	Observed.	Calculated.	Difference.	Observed.	Calculated.	Difference.	Observed.	Calculated.	Difference.	
146	Thetford	° ′ 17 41·2	° ′ 17 42·6	′ − 1·4	° ′ 68 1·4	° ′ 67 59·9	′ + 1·5	1·7791	1·7776	+·0015	4·4085	4·3991	+·0094	146
147	Thirsk	19 21·7	19 15·0	+ 6·7	69 28·3	69 22·7	+ 5·6	1·6912	1·6931	−·0019	4·5165	4·4992	+·0173	147
148	Tilney	17 58·1	18 0·3	− 2·2	68 20·7	68 14·5	+ 6·2	1·7655	1·7628	+·0027	4·4468	4·4157	+·0311	148
149	Tunbridge Wells	17 41·3	17 37·2	+ 4·1	67 10·8	67 14·1	− 3·3	1·8297	1·8292	+·0005	4·3485	4·3589	−·0104	149
150	Wallingford	18 21·6	18 25·7	− 4·1	67 48·4	67 43·4	+ 5·0	1·7986	1·8001	−·0015	4·4087	4·3942	+·0145	150
151	Weymouth	18 46·7	18 45·9	+ 0·8	67 11·7	67 14·9	− 3·2	1·8329	1·8342	−·0013	4·3593	4·3737	−·0144	151
152	Wheelock	··	··	··	68 49·0	68 49·7	− 0·7	··	··	··	··	··	··	152
153	Whitehaven	20 41·6	20 35·2	+ 6·4	69 47·6	69 48·9	− 1·3	1·6727	1·6699	+·0028	4·5447	4·5423	+·0024	153
154	Windsor	18 29·9	18 19·1	+10·8	67 38·8	67 35·4	+ 3·4	1·8084	1·8079	+·0005	4·3977	4·3841	+·0136	154
155	Wisbech	18 5·6	18 4·7	+ 0·9	68 19·0	68 14·6	+ 4·4	1·7653	1·7631	+·0022	4·4398	4·4171	−·0227	155
156	Worthing	17 59·0	17 51·1	+ 7·9	67 6·4	67 6·7	− 0·3	1·8402	1·8388	+·0014	4·3578	4·3554	+·0024	156
G	Greenwich	17 56·3	17 50·4	+ 5·9	67 28·6	67 29·9	− 1·3	1·8141	1·8124	+·0017	4·3746	4·3751	−·0005	G
S	Stonyhurst	19 42·6	19 46·7	− 4·1	69 11·4	69 16·5	− 5·1	1·7002	1·7024	−·0022	4·4734	4·4957	−·0223	S

NOTE.—In the case of a few places at which several sets of observations have been made, the numbers given above differ slightly from the means of those in the Table on pp. 270–273. The reason of this is that the disturbances given above were calculated from the mean values of the elements at the various sub-stations, and are thus not quite the same as the means of the disturbances deduced for each sub-station from the elements proper to it.

Ireland.

No.	Name	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No. of Station
157	Armagh	22 16·5	22 16·2	+ 0·3	69 57·6	70 3·1	− 5·5	1·6625	1·6600	+ ·0025	4·5578	4·5736	− ·0158	157
158	Athlone	22 26·7	22 38·3	− 11·6	69 40·0	69 40·8	− 0·8	1·6852	1·6865	− ·0013	4·5476	4·5544	− ·0068	158
159	Bagnalstown	21 55·0	21 47·8	+ 7·2	69 5·1	69 7·4	− 2·3	1·7208	1·7218	− ·0010	4·5028	4·5141	− ·0113	159
160	Ballina	23 26·9	23 34·9	− 8·0	70 25·8	70 12·9	+ 12·9	1·6323	1·6544	− ·0221	4·5917	4·5991	− ·0074	160
161	Ballywilliam	22 37·3	23 38·5	− 1·2	69 5·6	68 57·7	+ 7·9	1·7222	1·7326	− ·0104	4·5084	4·5039	+ ·0045	161
162	Bangor	21 44·4	21 48·9	− 4·5	70 1·3	70 7·8	− 6·5	1·6598	1·6534	+ ·0064	4·5657	4·5750	− ·0093	162
163	Bantry	22 40·3	22 36·6	+ 3·7	68 46·0	68 48·7	− 2·7	1·7529	1·7495	+ ·0034	4·5115	4·5127	− ·0012	163
164	Carrick-on-Shan-non	23 3·1	22 55·3	+ 7·8	69 53·3	69 59·6	− 6·3	1·6700	1·6668	+ ·0032	4·5607	4·5778	− ·0171	164
165	Castlereagh	23 11·1	23 3·6	+ 7·5	69 56·3	69 55·9	+ 0·4	1·6716	1·6713	+ ·0003	4·5773	4·5748	+ ·0025	165
166	Cavan	22 37·8	22 31·9	+ 5·9	69 57·3	69 55·9	+ 1·4	1·6629	1·6691	− ·0062	4·5577	4·5688	− ·0111	166
167	Charleville	22 30·8	22 31·8	− 1·0	69 5·3	69 7·3	− 2·0	1·7226	1·7255	− ·0029	4·5083	4·5235	− ·0152	167
168	Clifden	24 20·7	23 48·6	+ 32·1	70 4·8	69 57·6	+ 7·2	1·6631	1·6726	− ·0095	4·5891	4·5854	+ ·0037	168
169	Coleraine	22 36·9	22 33·1	+ 3·8	70 47·7	70 30·3	+ 17·4	1·6085	1·6310	− ·0225	4·6176	4·6070	+ ·0106	169
170	Cookstown Junc-tion	21 32·8	22 11·4	− 38·6	69 34·5	70 14·7	− 40·2	1·6830	1·6474	+ ·0356	4·5194	4·5871	− ·0677	170
171	Cork	22 18·1	22 13·3	+ 4·8	68 46·4	68 49·5	− 3·1	1·7506	1·7460	+ ·0046	4·5071	4·5068	+ ·0003	171
172	Donegal	23 20·1	23 12·7	+ 7·4	70 15·3	70 24·0	− 8·7	1·6449	1·6407	+ ·0042	4·5827	4·6077	− ·0250	172
173	Drogheda	21 54·7	21 52·9	+ 1·8	69 36·3	69 39·5	− 3·2	1·6895	1·6852	+ ·0043	4·5441	4·5455	− ·0014	173
174	Dublin	21 40·8	21 40·9	− 0·1	69 15·7	69 25·6	− 9·9	1·7087	1·6985	+ ·0102	4·5128	4·5254	− ·0126	174
175a	Enniskillen	23 5·3	22 50·5	+ 15·2	70 14·2	70 10·7	+ 3·5	1·6489	1·6543	− ·0054	4·5816	4·5894	− ·0078	175a
175b			22 50·1					1·6465		− ·0078				175b
176a	Galway	24 6·3	23 10·1	+ 56·2	69 53·0	69 43·1	+ 9·9	1·6583	1·6856	− ·0273	4·5274	4·5612	− ·0338	176a
176b	,,	23 29·8	23 10·2	+ 19·6	69 41·6	69 43·6	+ 2·0	1·6867		− ·0011	4·5581	4·5632	− ·0051	176b
177	Gort	22 50·5	22 56·8	− 6·3	69 31·7	69 34·2	− 2·5	1·7013	1·6954	+ ·0059	4·5571	4·5515	+ ·0056	177
178	Greenore	22 14·5	21 51·8	+ 22·7	69 42·2	69 48·8	− 6·6	1·6833	1·6756	+ ·0077	4·5514	4·5574	− ·0060	178
179	Kells	22 7·0	22 9·8	− 2·8	69 38·7	69 43·2	− 4·5	1·6925	1·6818	+ ·0107	4·5619	4·5513	+ ·0106	179
180	Kildare	21 0·4	21 57·8	+ 2·6	69 17·2	69 23·6	− 6·4	1·7073	1·7021	+ ·0052	4·5150	4·5268	− ·0118	180
181	Kilkenny	21 58·7	21 55·9	+ 2·8	69 5·0	69 7·9	− 2·9	1·7258	1·7218	+ ·0040	4·5154	4·5158	− ·0004	181
182	Killarney	21 55·8	21 50·1	+ 5·7	68 56·5	69 23·3	− 6·8	1·7405	1·7325	+ ·0080	4·5203	4·5257	− ·0054	182
183	Kilrush	23 11·4	23 5·5	+ 5·9	69 23·2	69 23·4	− 0·2	1·7090	1·7088	+ ·0002	4·5435	4·5439	− ·0004	183
184	Leenane	23 36·2	23 39·7	− 3·5	70 8·0	69 58·7	+ 9·3	1·6512	1·6707	− ·0195	4·5697	4·5848	− ·0151	184
185	Limerick	23 36·6	22 39·7	− 3·1	69 8·8	69 18·0	+ 9·2	1·7235	1·7131	− ·0104	4·5244	4·5333	− ·0089	185
186	Lisdoonvarna	22 58·5	23 10·5	− 12·0	69 31·8	69 36·0	− 4·2	1·7042	1·6941	+ ·0101	4·5653	4·5553	+ ·0100	186
187	Lismore	22 5·5	22 2·8	+ 2·7	68 48·6	68 54·3	− 5·7	1·7427	1·7395	+ ·0032	4·4952	4·5087	− ·0135	187

Ireland—continued.

No.	Name	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No. of Station.
188	Londonderry	22 50·5	22 52·5	− 2·0	70 26·9	70 31·1	− 4·2	1·6335	1·6313	+ ·0022	4·5997	4·6113	− ·0116	188
189	Oughterard	23 40·6	23 22·6	+18·0	69 56·7	69 50·5	+ 6·2	1·6762	1·6789	− ·0027	4·5916	4·5733	+ ·0183	189
190	Parsonstown	22 27·0	22 27·5	− 0·5	69 30·3	69 28·6	+ 1·7	1·6989	1·6997	− ·0008	4·5452	4·5404	+ ·0048	190
191	Sligo	23 4·6	23 16·2	−11·6	70 17·8	70 13·6	+ 4·2	1·6430	1·6522	− ·0092	4·5880	4·5959	− ·0079	191
192	Strabane	22 46·9	22 54·7	− 7·8	70 23·5	70 25·7	− 2·1	1·6352	1·6377	− ·0025	4·5901	4·6064	− ·0163	192
193	Tipperary	22 22·6	22 19·4	+ 3·2	69 4·9	69 8·2	− 3·3	1·7272	1·7236	+ ·0036	4·5188	4·5219	− ·0031	193
194	Tralee	...	23 2·5	...	69 9·4	69 12·2	− 2·8	...	1·7227	...	...	...	...	194
195	Valentia	23 16·0	23 8·5	+ 7·5	68 54·7	69 3·6	− 8·9	1·7448	1·7336	+ ·0112	4·5245	4·5298	− ·0053	195
196	Waterfoot	22 15·1	22 9·5	+ 5·6	70 29·0	70 23·6	+ 5·4	1·6293	1·6374	− ·0081	4·5968	4·5967	+ ·0001	196
197	Waterford	21 27·9	21 43·1	−15·2	68 53·7	68 53·9	− 0·2	1·7329	1·7374	− ·0045	4·4897	4·5015	− ·0118	197
198	Westport	23 15·1	23 38·4	−23·3	70 17·9	70 4·4	+13·5	1·6509	1·6642	− ·0133	4·6103	4·5907	+ ·0196	198
199	Wexford	21 18·1	21 23·7	− 5·6	68 56·2	68 51·6	+ 4·6	1·7324	1·7389	− ·0065	4·4982	4·4967	+ ·0015	199
200	Wicklow	21 21·4	21 26·4	− 5·0	69 9·9	69 11·3	− 1·4	1·7126	1·7151	− ·0025	4·5002	4·5119	− ·0117	200

Supplementary Stations.

No.	Name	Declination Observed	Declination Calculated	Declination Difference	Inclination Observed	Inclination Calculated	Inclination Difference	Horizontal Force Observed	Horizontal Force Calculated	Horizontal Force Difference	Vertical Force Observed	Vertical Force Calculated	Vertical Force Difference	No. of Station.
201	Chepstow	...	...	...	67 57·5	67 56·6	+ 0·9	...	...	...	...	...	...	201
202	Goodrich	...	...	...	68 9·0	68 5·1	+ 3·9	...	...	...	...	...	...	202
203	Hereford	...	...	...	68 15·6	68 12·6	+ 3·0	...	...	...	...	...	...	203
204	Ross	...	...	...	68 6·9	68 7·1	− 0·2	...	...	...	...	...	...	204
205	Tintern	...	...	...	68 1·3	67 59·1	+ 2·2	...	...	...	...	...	...	205

Local and Regional Disturbances.

We now come to the consideration of the local and regional disturbances which exist in many parts of the United Kingdom, and which we have investigated much more fully than has hitherto been done.

As the problem is one of difficulty, it is necessary that every step should be carefully considered. It may be attacked in three ways, which are not independent, but each of which is attended with special advantages and disadvantages, and which, when combined, afford in many cases the means of arriving at definite conclusions.

(1.) If the true isomagnetic curves are drawn as accurately as possible, without any attempt to smooth the irregularities, they present in disturbed districts distorted forms which enable us to judge of the nature and magnitude of the disturbance. The great advantage of this method is, that it is independent of calculation. It is not affected by errors possibly introduced by the method of determining the terrestrial curves. The conclusions arrived at are based directly upon the observations.

The objections that may be raised to it are, that it can only be used with effect in the case of considerable disturbances, and that it may tend to exaggerate the importance of those which, though of great local intensity, are of small range. A curve may be carried many miles from its true position, in order to pass through a station, the disturbance at which dies out within a very short distance.

When the stations are as numerous as ours, and when the curves are drawn with a due regard to the possibility of isolated maxima and minima, we do not think that the risk of error on this account is as great as it might at first sight appear, but in so far as it exists, it may be checked by the second method.

(2.) If the disturbances of the elements, *i.e.*, the differences between the observed values and those calculated from the general equations given above, are plotted down on a map, it is found that they are not scattered haphazard, but that certain districts exhibit definite peculiarities, such as that the observed value is always too large or too small. From a study of such maps deductions can be made as to the nature of the disturbing forces.

This method is open to the objection that the peculiarities in question may not correspond to physical realities, but may be due to the inadequacy of the formulæ. The calculated rate of increase of the declination with longitude, for instance, may be made a little too rapid in one part of the country, and a little too slow elsewhere. The observed declinations may, therefore, appear too large in the one district and too small in another, and thus a mere mathematical error may create a fictitious attractive or repulsive force.

To this objection, it may be answered that the observed disturbances are too large and too irregular to be thus explained. Taking, for instance, the English and Welsh stations which lie between long. 3 W. and long. 5 W., the mean disturbances of the Declinations for the groups indicated, are as follows :—

Four most northerly stations . . . . . . 14·4 W.
Intermediate group of eight stations . . 6·0 E.
Six most southerly stations . . . . . 3·0 W.

A glance at the isogonals (Plate II.) in this part of the kingdom, is sufficient to show that it is impossible to believe that the real terrestrial curve not only crosses and recrosses that which we have drawn between Plymouth and Holyhead (which is quite possible), but that the amplitude of the oscillation amounts to more than 20′ of Declination, or to about 45′ of longitude. If this were so, the terrestrial would be nearly as sinuous as the true curves (*cf.* Plate V.).

It is also to be observed, that even if there is a slight tendency of the kind supposed, it will be partly corrected by the first method. The objection to that is, that too much stress may be laid on the peculiarities of individual stations—to this, that too great weight may be given to the characteristics of districts.

If the two methods point to the same conclusion the two criticisms are mutually destructive.

A more serious objection is, that the indications are, at times, somewhat ambiguous. An increase in the Declination may be due to a small force acting at right angles to the magnetic meridian, or to a large one acting nearly parallel to it. The value of the disturbance of the Horizontal Force will often decide to which of these the effect is due, but nevertheless, it is not easy to deduce definite conclusions from a mere inspection of the maps. This consideration has, therefore, led us to supplement this method by another.

(3.) In this third method, we calculate the magnitude and direction of the disturbing force at each station. If δ be the observed, and δ_c the calculated value of the Declination, if a similar notation be used for the other elements, and if N, W, and Z be the northerly, westerly, and vertical components of the disturbing force, we have

$$N = H_c \cos \delta_c - H \cos \delta,$$
$$W = H_c \sin \delta_c - H \sin \delta,$$
$$Z = H_c \tan \theta_c - H \tan \theta,$$

whence the magnitude and direction of the disturbing force are known. Obviously the danger to be feared, in this case, is that the differences with which we deal are too small to give trustworthy results. The calculation must, however, lead to conclusions compatible with those to which the study of the disturbances points, and even if the results are only approximate, they are valuable as indicating, in a way which is more easily interpreted, the direction and magnitude of the forces under investigation.

Having thus described the general course which we propose to adopt, we shall now consider the methods more in detail. After that we shall apply them to certain districts, of which we have made a special study, and then to the whole area of the survey.

In attempting to discover the order which underlies the apparent irregularity of the disturbances, it is necessary to proceed on a working hypothesis, and we shall

postulate only the possibility of the existence of points or surfaces which exert magnetic forces. They will be called attractive or repulsive according as they attract or repel a north-seeking pole. If the existence or apparent existence of such centres is established the cause of the phenomenon will be a proper subject for enquiry.

The distortion produced by a symmetrical mass of magnetic matter on the isogonals depends upon the angle which these curves make with the magnetic meridian.

If we are dealing with a sufficiently small area, both the isogonals and the magnetic meridians may be represented by straight lines. If these lines are mutually perpendicular, and if the declination is westerly and increases with the latitude, the forms of the true isogonals in the neighbourhood of a symmetrical attracting mass, the centre of which is below the surface at A, will be of the type shown in Fig. 3, in which NS is the magnetic meridian through A.

Fig. 3.
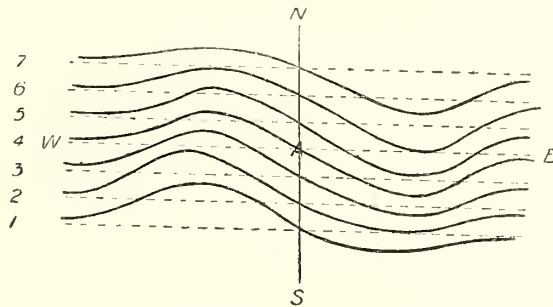
- - - - Terrestrial Isogonals running approximately east and west.
———— True Isogonals produced by centre of attraction at A.

In Japan the distribution of the isogonals and meridians approximates to this simple arrangement, and in the neighbourhood of the Fossa Magna according to Dr. NAUMANN ('Die Erscheinungen des Erdmagnetismus,' Stuttgart, 1887), a very remarkable disturbance of this kind is produced.

If the isogonals and meridians are very nearly coincident, and if the Declination increases with the longitude, the effect on the isogonals of a weak attractive centre will be of the type shown in Fig. 4. If, however, the disturbing force is sufficiently powerful to make the Declination, at some point to the east of the centre, greater than its undisturbed value at the centre, the form of the lines must approximate to that shown in Fig. 5. The Declination will increase rapidly with the longitude, attain a maximum value, fall to its normal value at points on the magnetic meridian which pass through the centre—on each side of which the attracting matter is supposed to be symmetrically situated—then fall to a minimum, and finally resume its normal rate of increase.

If, retaining all the other assumptions, we suppose that the isogonals and meridians are inclined at an angle of less than 90°, the distorted curves will assume forms similar to those in Fig. 6. It will be noticed that if the undisturbed isogonals be

rotated about A in the limiting cases when they coincide with AE or AN, the distorted curves will assume the forms shown in Figures 3, and 4, or 5.

Fig. 4.
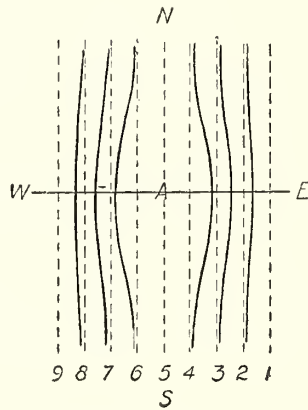
Fig. 5.
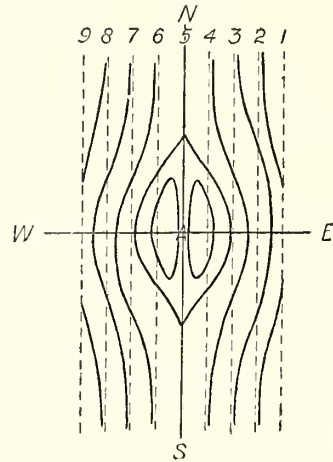
– – – – Terrestrial Isogonals running approximately north and south.

———— True Isogonals produced by (Fig. 4) a weak, and (Fig. 5) a strong centre of attraction at A.

Fig. 6.
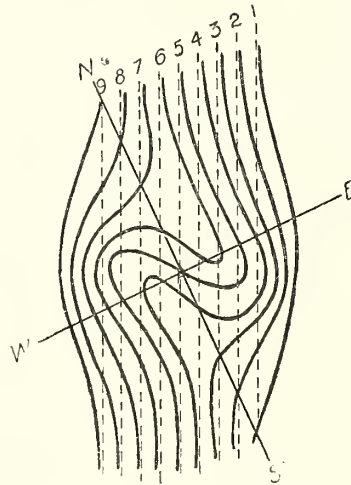
– – – – Terrestrial Isogonals approximately parallel.

———— True Isogonals produced by a centre of attraction at A.

It is, of course, quite possible that a single closed curve may sometimes be formed. An underground attracting surface, rising up steeply on the eastern side and falling away very gently to the west, might cause a maximum declination on the east without any corresponding minimum on the west; but in general the normal complement of a closed isogonal is another of the same type.

The curves shown in figs. 3 to 6 will not be realised in practice, but their main

characteristics may be reproduced, and a knowledge of their forms is very useful in interpreting maps on which the true isomagnetic curves are drawn.

These facts may also be illustrated by means of diagrams of another kind.   Thus if, in figs. 7 and 8, the slope of the line AD represents the normal rate of increase of the Declination with longitude, the effect of an attractive centre below A will be represented by the curves shown, which correspond to the two cases of a weak and strong attraction respectively.   If, however, the force is repulsive, the curves assume the forms shown in figs. 9 and 10.   In this case there may be two maxima and minima, following each other in order on opposite sides of the centre of repulsion.   The isogonals will be drawn together in its neighbourhood, and there may be two systems of loops, one on each side of the centre.

Figs. 7 to 10.

Fig. VII

Fig. VIII

Fig. IX

Fig. X
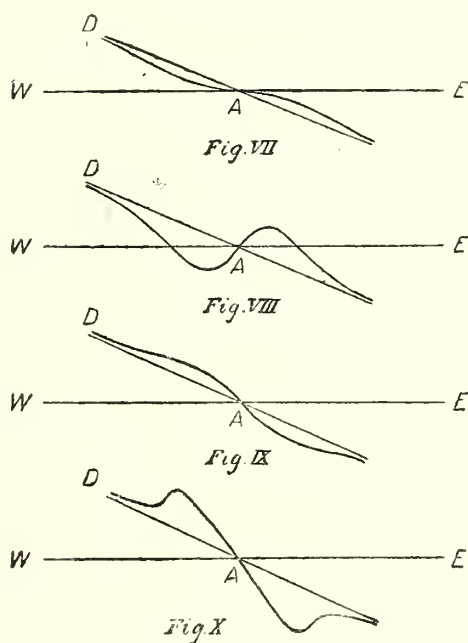
A centre of attraction will also be indicated by a convergence of the lines of equal Horizontal Force and Dip in its neighbourhood.

The observed will be greater than the calculated Force to the south, and smaller to north, of the centre.   The Dip, on the other hand, will be less and greater than its calculated values at stations to the south and north respectively of a point at which the directions of the normal and disturbing Forces coincide.

The phenomena are more complicated if the effect of the disturbance is to increase the value of the element on that side of the centre of attraction on which it would be normally the greater, instead of increasing it (as in the case of the Declinations) on that side (the east) on which it is normally the less.

Thus, if we represent a northward movement by progress along the line WE (fig. 10) from W to E, the effect of a strong centre of attraction would be to check

the ordinary decrease of the Horizontal Force with latitude. If the centre were sufficiently powerful, the decrease might be converted into an increase ; and since immediately over the centre the value must be normal, a minimum and maximum must follow in order. Another minimum and maximum may occur on the further side. The effect of a centre of attraction on the lines of equal Horizontal Force is thus the same as that of a centre of repulsion on the isogonals. In the case of the Dip, as in that of the Declination, there can only be two critical points on opposite sides of the point defined above.

The Vertical Force will increase with latitude at more than the normal rate to the south of a centre of attraction, and at less than the normal rate to the north of it. The lines of equal Vertical Force will thus be drawn southwards in the neighbourhood of such a centre.

If the Vertical Force attains a maximum value it will be at a point above the centre, and it must be followed by a minimum—at which, however, the Vertical Force will be greater than its calculated value.

It is evident from this discussion that there are a number of signs of a centre of attraction which may not all coexist, and which will be complicated in actual practice by irregularities in the distribution of the attracting masses, but which may nevertheless be of considerable practical help in a survey of local magnetic disturbances. (*Cf.* Lamont : 'Erdmagnetismus in Nord-Deutschland,' 1859, p. 21.)

We have dwelt on them, not because they present any difficulty, or even because they are altogether novel. Mr. Bennett Brough, A.R.S.M., has given a full account of the methods employed by the Swedish mining engineers in exploring for iron ore. (" Use of the Magnetic Needle," &c. 'Journal of the Iron and Steel Institute,' No. 1, 1887, pp. 289–303.) They are accustomed to map out the neighbourhood of a mass of ironstone with a magnet, in order to determine its exact position. We believe, however, that the systematic use of the forms of the isomagnetic curves has been largely overlooked in the case of surveys comparable with that described in this paper.

In Plates V. to VIII., the values of the Declinations, Horizontal Forces, Dips, and Vertical Forces determined at the various stations are entered and true magnetic curves are drawn. These curves have not been chosen for equal differences of the values of the element to which they refer, but those have been selected which exhibit the most marked peculiarities. In this way attention is best drawn to disturbed stations and districts which can then be studied in accordance with the plan above suggested.

Turning next to the investigation of the disturbances or differences between the observed and calculated values of the elements, we note that apart from the assurance they afford that the peculiarities studied are common to a district and do not depend only on a single station, they often supply additional information to that which can be gained from the true isomagnetics.

Thus, in the case of the Horizontal Force, a maximum followed by a minimum may be either to the north or the south of the centre of attraction. In the former case,

however, both the maximum and minimum values are less, while in the latter case they are greater than the normal values at the stations at which they occur, and if one maximum and minimum only are formed, the disturbances at once decide whether they are to the north or south of a centre of attraction.

The lines which separate regions of positive disturbance in which the observed is greater than the calculated value from those in which it is negative are also of importance. Thus, in the case of the Horizontal Force, if such a line is approximately perpendicular to the magnetic meridian, and if in passing over it from south to north we leave a region in which the disturbance is positive and enter one in which it is negative, we either pass over a centre or line of attraction or pass from the range of the influence of one centre of repulsion into that of another.

If the centre is not at a great depth below the surface, and if the magnetic matter is not widespread at its minimum depth, it will cause a sudden reversal in the sign of the disturbances which will, however, be large near the centre on both sides. On the other hand, passage from the region of influence of one centre to that of another will, if the distance between them is considerable, be marked by a transition from small positive to small negative disturbances, or *vice versâ*.

Similar remarks may be made with respect to the disturbances of the Declination, but they are most easy to interpret when the line which separates a positive from a negative region runs approximately north and south. If such a line meets another which separates a southern region of positive from a northern region of negative Horizontal Force disturbance, all the disturbing forces in the neighbourhood tend towards the point of intersection. It will be convenient to speak of such a point as a *peak*, and to call a line which divides regions of positive from those of negative disturbance of the Declination or Horizontal Force, so as to indicate attraction towards it, a *ridge line*. In like manner, a line which separates the regions of influence of two attractive centres may be called a *valley line*.

In the choice of these terms we are, no doubt, influenced by our views of the facts to be hereafter set forth, but they are convenient, quite apart from any theory of the cause of local magnetic forces. It will be proved beyond doubt that in some cases these forces emanate from matter below the surface of the earth. If this is so, an increment in their intensity must be due either to a closer approximation to the matter or to an increase in its magnetisation. We have no magnetic test to discrimate between the two, and therefore, without prejudice, adopt a nomenclature which is perhaps most consistent with the first hypothesis. If we wished to keep absolutely free from all expression of an opinion as to the causes of the phenomena, we might have called a peak (to which the lines of magnetic disturbing force converge), a magnetic sink, and so on, and for the present the terms we suggest may be taken as indicating merely points and lines from and to which such lines of force run.

The disturbances of the Declination and Horizontal Force, together with the ridge and valley lines are shown in Plates IX. and X.

The absolute value of the disturbance of the Vertical Force is especially uncertain. In the cases of the Declination and Horizontal Force a centre of attraction affects stations on opposite sides with disturbances of opposite signs, to that the true mean value for the district can be found, and the true disturbance at each station deduced. It is, however, possible that the disturbance of the Vertical Force may always be of the same sign. Thus, if the centres of force are rocks magnetised by induction, the north-seeking pole of a magnet would most frequently be attracted downwards in the northern hemisphere as the north-seeking poles of the rock magnets would be deeply buried in the earth. This hypothesis is that which experience has justified in Sweden. If this is so, the mean value of the Dip or Vertical Force in a district will be greater than its undisturbed value, and negative values do not necessarily indicate an upward force. It is therefore safer only to use the disturbances of the Vertical Force as a means of indicating relative maxima and minima.

If, however, centres of repulsion exist, they might be detected by the observation that a large negative (upward) disturbance of the Vertical Force was accompanied with a sudden reversal of the disturbing Horizontal Force in neighbouring stations. If the Horizontal Forces are small, the negative disturbing Vertical Force is more probably to be interpreted as indicating a downward attraction of less than average magnitude.

On turning to Plate XI., in which the disturbances of the Vertical Force are plotted down, it will be observed that there is, on the whole, an excess of positive values in the south-east, and that large negative values are more common in the north and west. It is obvious that in Scotland the disturbances are much greater than in England, and very large positive values of vertical disturbance occur in the western isles. Indeed, if we include the enormous value obtained at Canna, the average for the whole of Scotland is positive. Leaving out, however, this very abnormal station, it can hardly be doubted that negative values are more common in the north-west. If this result could be trusted as corresponding to physical fact, it might indicate that the country as a whole is magnetised in the direction of the magnetic meridian. We cannot, however, draw such a conclusion, more especially as it can be shown that the observed effect may probably be a result of our ignorance of the datum lines from which the disturbances of the Dips and Vertical Forces ought to be measured.

We have taken the Dip, as given by direct observation, as one of our fundamental elements, and in drawing the terrestrial curves which satisfy the conditions that the mean value of the Dip for each district when attributed to the central station is accurately reproduced by the formula, we have been compelled to ignore the possibility of an unbalanced downward Vertical Force acting at every station. If we suppose that in consequence of this the calculated Dips ought, all over the country, to be diminished by a positive quantity, we get for the corresponding decrement in Vertical Force

$$dV = H \sec^2 \theta \, d\theta.$$

If we take the values of H and θ at Wick and St. Leonards, and write for $d\theta$ the circular measure of 1' multiplied by x, the unknown number of minutes by which the calculated Dip has been taken too large, we get at

$$\text{Wick} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad dV = 0 \cdot 0048x,$$
$$\text{St. Leonards} \quad . \quad . \quad . \quad . \quad dV = 0 \cdot 0036x.$$

Hence in the north the error will be greater than in the south, that is, the excess of the calculated over the true undisturbed Vertical Force is greatest in the north. Consequently the difference between the observed and calculated Vertical Force will be more frequently negative in the north, the excess of the error amounting to $0 \cdot 0012$ metric unit for every minute by which the calculated Dip is too large.

If, then, apart from local attractions the country, as a whole, attracts the needle so that the Dip is everywhere 10' greater than it would be if the British Isles were replaced by sea, the error in the calculated Vertical Forces deduced from calculated Dips, obtained on the assumption that no such defect exists, would be $0 \cdot 0120$ greater in the north of Scotland than in the Channel, so that negative disturbances would largely predominate in the north.

As from the geological character of Scotland it is probable that the error in the calculated Dip would not be constant, but would be greater than in England, the validity of this explanation is even more probable than this calculation indicates.

It is not, therefore, safe to draw conclusions based on the relation between the Vertical disturbing Forces at distant stations.

In the calculation of the disturbing forces we are not aiming at, and could not attain to, results which would give more than a general idea of their direction and magnitude. It may be possible to conclude with certainty that an attractive mass exists near certain stations, even if the directions of the forces are wrong by 10° or 15° and their magnitudes are inaccurate by 50 per cent.

To show that a degree of accuracy of this kind is attained we collect here the results at stations where we have taken a full set of observations at two or more places, or on different occasions. There are of course localities where it would be hopeless to get a good result from a single observation. Such places are Canna and Portree. At these we should never have observed except for the sake of investigating the disturbances which we knew were very great, and we therefore do not now take them into account. It is sufficient if the agreement between different observations is satisfactory at stations which we regarded at the time of observation as normal.

At Stornoway we made four sets of measurements, the first in 1884 in the Castle Grounds, the other three in 1885 and 1888 on Ard Point, about a mile distant from the former station. The ground was not good, as when we observed simultaneously about 50 yards apart on Ard Point the declinations differed by 18'.

At Loch Aylort we observed in 1884 and 1888 as nearly as possible on the same station. At Oban we observed on the mainland in 1888, and on the island of Kerrera

which is about a mile distant, in 1884 and 1885. In the Sound of Islay we had two stations about four miles apart, viz., Port Askaig in 1884 and Bunnahabhain in 1888. All these places, though typical Scotch stations, were on ground which elsewhere would be considered but indifferently good for our purpose. At three other places more favourably situated we also observed twice, viz., at Stranraer in 1884 and 1888, at Reading in 1886 and 1888, and at Bude Haven in 1886. At the two former the stations were as nearly as possible the same on the two occasions, at Bude Haven they were less than half a mile apart.

The results are summed up in the following Table. The resultant Horizontal disturbing Force is indicated by F, and the angle which its direction makes with the geographical meridian by ϕ. The latter is taken as positive on the western side, due north being represented by $0°$.

Station.	Rock.	Date.	F.	ϕ.	Z.
				$\circ$	
Stornoway—	Gneiss	1884	·0175	129·6	— ·0113
(1) Castle Grounds		1885	·0179	109·2	Dip not observed
(2) Ard Point . .		1888 (T)	·0198	119·7	+ ·0046
		1888 (R)	·0278	113·5	+ ·0120
Loch Aylort . . .	Gneiss	1884	·0267	141·1	— ·0326
		1888	·0139	112·3	+ ·0178
Oban—	Trap	1888	·0143	10·9	— ·0129
(1) Mainland . .		1884	·0196	16·9	— ·0132
(2) Kerrera . . .		1885	·0219	16·2	— ·0124
Sound of Islay—	Primary	1884	·0309	53·1	+ ·0141
(1) Port Askaig .	Limestone	1888	·0265	69·9	+ ·0122
(2) Bunnahabhain.					
Stranraer	Clay, Slate	1884	·0038	178·5	— ·0072
		1888 (R)	·0065	181·8 ⎫	— ·0034
		„ (T)	·0062	175·4 ⎭	
Reading	Clay	1886	·0049	— 8·3	+ ·0235
		1888	·0084	—10·2	Dip not observed
Bude Haven . . .	Shale	1886	·0065	143·1	+ ·0091
			·0048	180·0	+ ·0030

The annexed small map (fig. 11) illustrates these numbers by showing the direction and magnitudes of the disturbing forces as determined on two occasions at stations in Scotland.

An inspection of this map and of the table justifies the statement that the magnitude and direction of the disturbing forces can be determined with an accuracy sufficient to enable us to draw conclusions from groups of stations even if it would not always be safe to argue from one.

Fig. 11.
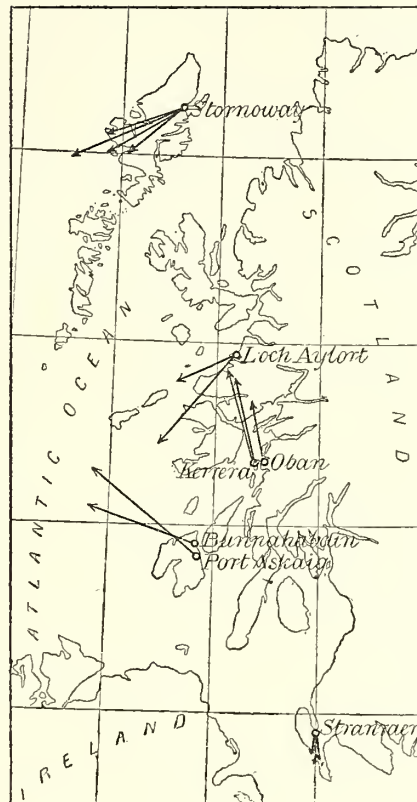
If the station is on good ground we may place more confidence in the results. Stranraer is interesting as being the only place where we both took complete sets of observations simultaneously on good ground. The results are in very close accord. But those obtained elsewhere leave no doubt as to the order of the magnitude of the disturbing force nor as to its direction to within (in unfavourable cases) 15° or 20°.

The difference in the signs of the Vertical Forces obtained at Loch Aylort and Stornoway in different years may in part be due to the uncertainty of the secular correction for the Dip, which appears to be very abnormal, especially at Loch Aylort (see p. 86). In this case we should be driven to the conclusion that a real change in the local Force had taken place. It is noticeable that the Vertical Force disturbance was apparently (algebraically) greater at Stornoway, Loch Aylort, Kerrera, and Stranraer on our second visits to these places.

The following Table contains the particulars as to the disturbing force at every station in accordance with the notation described above. On Plate XIII., the directions and magnitudes of the Horizontal disturbing Forces are shown, and regions of positive and negative Vertical Force disturbance are indicated, the former being shaded.

TABLE of Disturbing Forces.

No. of Station.	Name of Station.	F.	φ.	Z.
1	Aberdeen	82	−144	− 28
2	Arinagower	. .	. .	. .
3	Loch Aylort	{ 267	+141	− 326
		139	+112	+ 178
4	Ayr.	61	− 22	− 145
5	Ballater	151	−108	− 41
6	Banavie	176	+ 18	+ 268
7	Banff	112	+ 70	− 124
8	Berwick	179	− 15	+ 230
9 .	Boat of Garten	212	+ 86	+ 56
10	L. Boisdale	301	−114	− 736
11	Bunnahabhain	265	+ 70	+ 122
12	Callernish	45	+ 21	+ 31
13	Campbelton	36	+136	− 9
14	Canna	737	−112	+1839
15	Carstairs	173	+ 14	− 79
16	Crianlarich	39	+ 83	− 219
17	Crieff	114	+ 74	+ 217
18	Cumbrae	222	+174	+ 215
19	Dalwhinnie	127	+ 36	− 247
20	Dumfries	29	+ 56	− 87
21	Dundee	23	+180	+ 10
22	Edinburgh	17	+152	+ 107
23	Elgin	86	− 67	+ 45
24	L. Eriboll	37	− 56	+ 175
25	Fairlie	. .	. .	+ 135
26	Fort Augustus	74	−101	+ 57
27	Gairloch	210	− 85	− 451
28	Glasgow	117	−133	− 80
29	Golspie	89	− 92	− 65
30	Hawick	76	− 1	− 99
31	L. Inver	326	−129	− 853
32	Inverness	61	0	+ 156
33	Iona	326	+ 59	+ 202
34	Kirkwall	27	− 39	+ 109
35	Kyle Akin	140	+154	− 168
36	Lairg	73	− 79	− 34
37	Lerwick	62	−144	+ 378
38	Lochgoilhead	23	+122	− 328
39	Loch Maddy	73	− 72	− 114
40a	Oban	143	+ 11	− 129
40b	Oban (Kerrera)	{ 196	+ 17	− 132
		219	+ 16	− 124
41	Pitlochrie	33	− 63	− 242
42	Port Askaig	309	+ 53	+ 141
43	Portree	{ 619	+145	+ 712
		384	+ 11	− 378
		806	− 89	+1705
44	Row (Gairloch)	121	−170	− 188
45	Scarnish	394	+ 98	+ 444
46	Soa	518	−166	− 385
47	Stirling	166	+177	− 92
48a	Stornoway (Ard Point) . . .	{ 179	+109	. .
		198	+120	+ 46
		278	+114	+ 120
48b	Stornoway (Castle)	175	+130	− 113
49	Strachur	88	− 1	− 269

TABLE of Disturbing Forces—*continued*.

No. of Station.	Name of Station.	F.	ϕ.	Z.
50	Stranraer	$\begin{cases} 38 \\ 65 \\ 62 \end{cases}$	$\begin{matrix} +178 \\ -178 \\ +175 \end{matrix} \Big\}$	$\begin{matrix} -\ 72 \\ -\ 34 \end{matrix}$
51	Stromness	80	$-\ 25$	$+146$
52	E. Loch Tarbert	54	-167	-150
53	Thurso	20	$-\ 53$	$-\ 47$
54	Wick	150	-146	$+242$
55	Aberystwith	77	$-\ 83$	$-\ 78$
56	Alderney	117	$-\ 94$	$-\ 43$
57	Alnwick	29	$+\ 28$	$-\ 50$
58	Alresford	48	$-\ 47$	$+\ 6$
59	Appleby	69	$+168$	$-\ 56$
60	Barrow	35	$-\ 85$	$-\ 71$
61	Bedford	132	$+168$	$+\ 39$
62	Birkenhead	15	$+\ 8$	$+\ 26$
63	Birmingham	124	$-\ 40$	$+152$
64	Braintree	91	$+136$	$+\ 50$
65	Brecon	28	-135	$+\ 7$
66	Bude Haven	$\begin{cases} 65 \\ 48 \end{cases}$	$\begin{matrix} +143 \\ +180 \end{matrix}$	$\begin{matrix} +\ 91 \\ +\ 30 \end{matrix}$
67	Cambridge	57	$+158$	$+110$
68	Cardiff	32	$-\ 35$	$-\ 53$
69	Cardigan	67	-175	$-\ 52$
70	Carlisle	41	$+\ 77$	$-\ 52$
71	Chesterfield	56	$+\ 78$	$+145$
72	Chichester	46	$+\ 37$	$+130$
73	Clenchwarton	77	$+\ 67$	$+207$
74	Clifton	27	$+\ 57$	$+\ 33$
75	Clovelly	63	$+179$	$+\ 21$
76	Coalville	57	$-\ 63$	-109
77	Colchester	133	$+102$	$-\ 99$
78	Cromer	25	$+\ 92$	$+208$
79	Dover	60	-102	$+\ 21$
80	Falmouth	47	$+175$	-190
81	Gainsborough	42	$-\ 36$	$+132$
82	Giggleswick	49	$-\ 45$	$+\ 12$
83	Gloucester	46	$+126$	$+\ 46$
84	Grantham	41	$-\ 7$	$+\ 50$
85	Guernsey, L'Erée	101	-134	$-\ 13$
86	,, Peter Port	56	$-\ 96$	$+\ 49$
87	Harwich	30	$+\ 17$	$+\ 79$
88	Harpenden	70	$+148$	$+146$
89	Haslemere	44	$+\ 30$	$+\ 87$
90	Holyhead	153	-166	$+\ 21$
91	Horsham	57	$+\ 73$	$+\ 20$
92	Hull	124	$+136$	$+\ 94$
93	Ilfracombe	36	-162	$-\ 2$
94	Jersey, Grouville	168	$+\ 77$	$+\ 9$
95	,, S. Louis	144	-103	-229
96	,, S. Owen	65	$+144$	-131
97	Kenilworth	103	$+173$	$+287$
98	Kettering	59	$+150$	$-\ 92$
99	Kew	90	$+119$	$+162$
100a	King's Lynn	46	$+\ 19$	$+195$
100b	,, ,, (Gaywood) . .	40	$+\ 60$	$+139$
101	King's Sutton	85	$+138$	$+121$
102	Lampeter	81	-123	-137

TABLE of Disturbing Forces —*continued*.

No. of Station.	Name of Station.	F.	φ.	Z.
103	Leeds	45	− 51	+ 59
104	Leicester	108	− 95	− 27
105	Lincoln	89	− 48	+172
106	Llandudno	159	+134	+ 12
107	Llangollen	71	+107	− 9
108	Llanidloes	20	−140	− 22
109	Loughborough	155	− 65	+ 22
110	Lowestoft	61	+116	+192
111	Mablethorpe	34	+113	+163
112a	Malvern, Colwall	119	−140	. .
112b	„ Great Malvern . .	121	+127	+ 21
112c	„ Wells	81	+149	. .
112d	„ Mathon	147	− 98	. .
113	Manchester	105	−104	− 50
114	Manton	70	− 18	+ 79
115	March	36	+ 2	+111
116a	Melton Mowbray	185	+ 85	+305
116b	„ „	182	+152	+ 80
117	Milford Haven	98	− 23	− 72
118	Newark	56	+ 49	+ 6
119	Newcastle	35	− 27	− 4
120	Northampton	111	+165	− 69
121	Nottingham	33	− 27	+179
122	Oxford	38	−153	+169
123	Peterborough	52	+ 56	+150
124	Plymouth	30	+168	−210
125	Port Erin	135	−134	− 55
126	Preston	4	− 34	+ 49
127	Purfleet	60	+111	+ 89
128	Pwllheli	125	+ 45	+114
129	Ramsey	121	−150	+ 4
130	Ranmore	85	+ 65	+ 11
131a	Reading	49	− 8	+235
131b	„ (Caversham) . . .	84	− 10	. .
132	Redcar	23	− 15	+ 11
133	Ryde	59	− 56	− 18
134	St. Cyres (Exeter)	72	+ 50	− 5
135	St. Leonards	6	− 31	− 61
136	Salisbury	91	− 29	+ 43
137	Scarborough	47	+ 52	+ 63
138	Shrewsbury	118	−173	−319
139	Southend	81	+126	+ 74
140	Spalding	134	− 98	− 48
141	Stoke-on-Trent	17	+ 10	+ 85
142	Sutton Bridge	55	− 51	+200
143	Swansea	105	+ 4	+ 30
144	Swindon	. .	. .	+ 38
145	Taunton	26	0	− 59
146	Thetford	17	− 7	+ 94
147	Thirsk	38	+138	+173
148	Tilney	29	− 4	+311
149	Tunbridge Wells	22	+ 95	−104
150	Wallingford	26	−108	+145
151	Weymouth	13	180	−144
152	Wheelock	. .	. .	. .
153	Whitehaven	42	+ 69	+ 24
154	Windsor	58	+104	+136

TABLE of Disturbing Forces—*continued.*

No. of Station.	Name of Station.	F.	φ.	Z.
155	Wisbech	22	+ 30	+227
156	Worthing	44	+ 90	+ 24
G	Greenwich	36	+ 79	— 5
S	Stonyhurst	22	— 93	—223
157	Armagh	25	+ 27	—158
158	Athlone	59	— 81	— 68
159	Bagnalstown	38	+131	—113
160	Ballina	224	—146	— 74
161	Ballywilliam	104	—154	+ 45
162	Bangor	68	+ 3	— 93
163	Bantry	39	+ 52	— 12
164	Carrick-on-Shannon	49	+ 72	—171
165	Castlereagh	37	+107	+ 25
166	Cavan	68	+177	—111
167	Charleville	31	—147	—152
168	Clifden	183	+145	+ 37
169	Coleraine	226	—162	+106
170	Cookstown Junction	402	— 6	—677
171	Cork	52	+ 49	+ 3
172	Donegal	56	+ 63	—250
173	Drogheda	44	+ 33	— 14
174	Dublin	103	+ 21	—126
175	Enniskillen	107	+160	— 78
176a	Galway	386	+159	—338
176b	,, ...	97	+106	— 51
177	Gort	67	— 5	+ 56
178	Greenore	134	+ 78	— 60
179	Kells	108	+ 14	+106
180	Kildare	53	+ 36	—118
181	Kilkenny	43	+ 41	— 4
182	Killarney	85	+ 42	— 54
183	Kilrush	29	+110	— 4
184	Leenane	195	—151	—151
185	Limerick	105	+ 14	— 89
186	Lisdoonvarna	117	— 7	+100
187	Lismore	36	+ 46	—135
188	Londonderry	24	0	—116
189	Oughterard	92	+130	+183
190	Parsonstown	8	—140	+ 48
191	Sligo	107	—126	— 79
192	Strabane	45	—102	—163
193	Tipperary	38	+ 47	— 31
194	Tralee	. .	. .	. .
195	Valentia	118	+ 42	— 53
196	Waterfoot	85	—176	+ 1
197	Waterford	18	— 98	—118
198	Westport	174	—116	+196
199	Wexford	70	—134	+ 15
200	Wicklow	36	—114	—117

SURVEYS OF SELECTED DISTRICTS.

Having described and, as we hope, justified the methods by which we propose to investigate local and regional disturbances, we now proceed to discuss their application to districts of which we have made a special study. The results we shall arrive at will help us in the further elucidation of the magnetic state of the whole country.

The Malvern Hills.

The general nature and direction of the magnetization of igneous rocks is a problem on which comparatively few observations have been made. It is, indeed, known that when examined in detail they present great irregularities, and Commander CREAK, F.R.S., has shown ('Roy. Soc. Proc.,' vol. 40, 1886, p. 83) that when islands disturb the magnetic needle in the northern hemisphere they attract, and in the southern repel the north-seeking pole of a magnet. This is what would be expected if they were the upper extremities of magnetic masses magnetized by the Earth's induction.

In like manner in Sweden, where (as has already been stated) the method of searching for iron ore by means of the magnet has been carried to considerable perfection, the assumption made, and justified by experience, is that the upper parts of the beds of ironstone attract the north-seeking pole.

Observations somewhat similar to those of Commander CREAK can be carried out on land in cases where igneous rocks rise in the midst of sedimentary deposits, and such observations are specially interesting in cases, such as that of the Malvern Hills, in which the axis of the magnetic mass runs north and south.

If its depth is considerable with respect to its length, and if it is magnetized by induction, we should expect the upper visible parts to attract (in the northern hemisphere) the north-seeking pole. If, on the other hand, it is possible to conceive of a shallow mass of magnetic rock surrounded by non-magnetic matter, the northern end might repel the north-seeking pole. Finally, if the mass was itself magnetized independently of the present inductive action of the Earth, as is certainly the case with small masses of highly magnetized lodestone, its effect on a compass needle could only be determined by experiment.

With the view, then, of making a beginning towards the study of these questions in the United Kingdom, we determined the polarity of the northern end of the Malvern Hills.

This range consists of syenite and granite. The mass of igneous rocks runs due north and south for about eight miles, and is at the broadest part but little more than half a mile wide. On its eastern side is a great fault which extends many miles north and south of the range itself. Two stations, Great Malvern (112b) and Malvern Wells (112c), were taken on the eastern side of the hills. They were both on the Red Marl, and distant about a mile and a quarter from the centre of the range. Two

corresponding stations were also taken on the western side at about the same distance, and both on the Old Red Sandstone.   Of these Mathon (112d) was 0·3 mile to the north of the latitude of Great Malvern, and the line joining them cuts the range at about a mile from its northern end, passing close to its highest point.   Colwall (112a) is about three-quarters of a mile to the south of the latitude of Malvern Wells, and the line joining them cuts the range about $2\frac{1}{2}$ miles south of the northern termination of the igneous rocks.

The Declination and Force were determined at all these stations.   Time did not allow of the Dip being taken elsewhere than at Great Malvern.

The results are given in the following Table :—

		Declination.	Force.
Eastern Stations	Great Malvern	19 33·0	1·7687
	Malvern Wells	19 22·4	1·7682
Western Stations	Mathon	18 46·5	1·7655
	Colwall Green	19 3·6	1·7627

From these the disturbing forces were deduced, the notation being the same as that used on p. 268.

	F.	ϕ.
Great Malvern	·0121	126·6
Malvern Wells	·0081	148·7
Mathon	·0147	— 97·8
Colwall Green. . . .	·0119	−140·4

The accompanying Map shows the direction of these forces and their relative magnitudes.   The Worcestershire Beacon (1440 ft.) is the highest point on the range ; to the south of this the height diminishes, and then increases again to the Hereford-shire Beacon.   The directions of the disturbing forces tend towards these hills, and the results are, we think, only compatible with the view that the Malverns attract the north-seeking pole of the magnet.

Fig. 12.
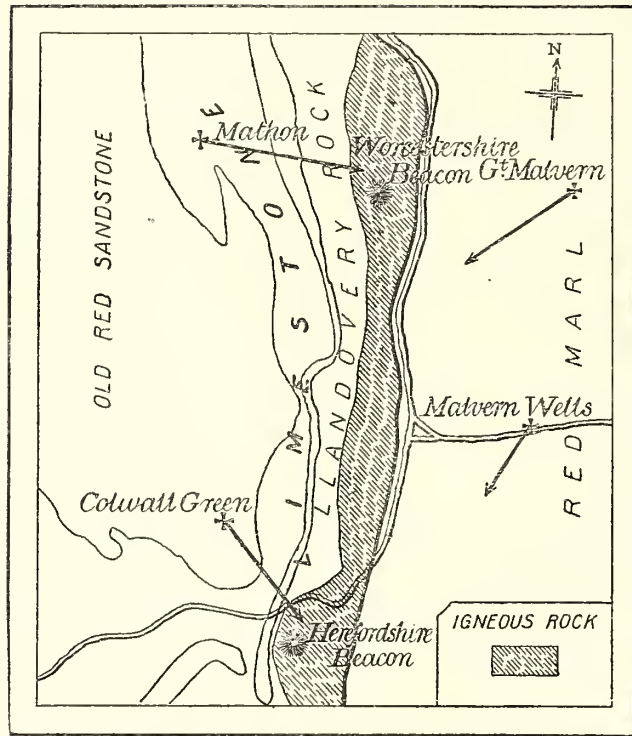
Disturbing Forces near the Malvern Hills.

The Island of Canna.

The attraction exerted by the Malverns having been demonstrated, it is convenient to discuss in the next place a locality where the disturbances are enormously greater.

Popular tradition has long attributed to the basaltic rocks of the Island of Canna the power of deviating the needle through very large angles. The compasses of passing ships are supposed to be affected by the eastern extremity of the island, on which stands Compass Hill.

Magnetic observations were made on the Island by Sir FREDERICK EVANS ('Phil. Trans.,' 1872, vol. 162, p. 325), but, we have, we believe, been able to add considerably to what was already known of its magnetic properties.

The island, which is about fifteen miles south-west of Skye, is about five miles long, its greatest length lying nearly due east and west. It is divided into two approximately equal portions by a neck of comparatively low elevation. The highest ground is in the eastern part where it rises to a height of 724 feet, and our observations have been confined to this portion. The cliffs on the north side are here some hundreds of feet in height and fall sheer into the sea. On the south side several small valleys lead to the shore, but the hills rise very steeply about a quarter of a mile inland, and the tops of several consist of irregular masses of basaltic columns. On Compass Hill,

which is the most easterly point on the island, the longer axis of the mass runs north and south. Its height on the inland side is from 20 to 40 feet, while it falls steeply towards the sea. On another hill which rises behind Kaill, the residence of R. THOM, Esq., the columnar mass is small and more clearly defined. The north or landward side is 15 or 20 feet high, the south side is a steep cliff. We roughly estimate the length at about 50 yards. In fig. 13 the position of this hill is indicated by B, that of the highest point on the island by A, and that of Compass Hill by C.

In 1884 we determined the magnetic elements by means of the Kew Magnetometer No. 60, and the Dip Circle No. 74, at the position marked P. The differences between the observed and calculated values are given in the following table :—

Date 1884.			
	Observed.	Calculated.	Difference.
Declination.	21 8·4	23 6·5	−1 58·1
Dip	72 45·0	71 32·7	1 12·3
Horizontal Force . .	1·5092	1·5607	− ·0515

Fig. 13.
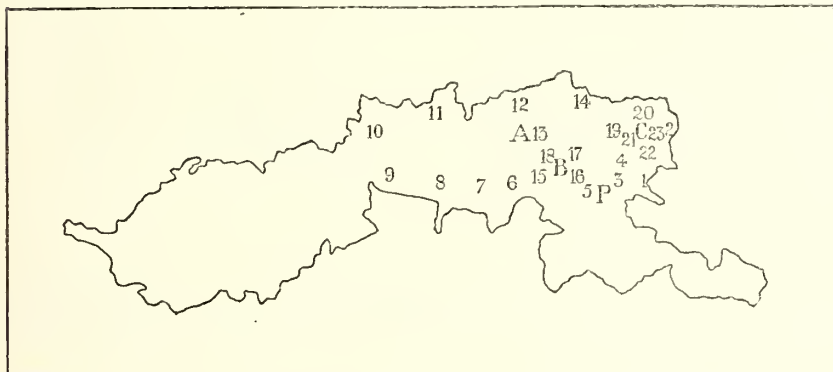
Stations on Canna.

Although these observations show that the station was highly disturbed, the effect on the Declination is not so great as to be detected, except by a careful observation. In 1888 we observed by means of a small azimuth compass at 23 stations which are indicated on the map, and determined at each the bearings of a number of distant points. The observations were made on August 16 and 17, bright sunny days with a northerly wind, on which the atmosphere was very transparent. We were therefore able to take the bearings, not only of prominent headlands on Skye and Rum, but also of some distant objects, such as the Ushinish Lighthouse on N. Uist in the Hebrides.

The azimuth compass was a small instrument which did not admit of great accuracy, but, as will be seen in the sequel, it was sufficient for the purpose we had in view.

We carried a chart with us, and marked the position of each station while on the spot. On the average, about four bearings were taken at each station, and at no station was the number less than two.

The bearings of the objects selected were afterwards taken from the chart, and, by comparing these with the observations, the Declination was determined. The agreement between the individual observations at each station was in general only moderately good, but, even in cases where only two observations were taken, we think the means are accurate to about half a degree.

Throwing out thirteen stations which were evidently highly disturbed, we took the mean of the Declinations at the other ten as giving the mean Declination for the easterly half of the island, which is 22°·8.

By subtracting this from the Declinations found at the stations, we obtained the disturbance of the Declination at each. The results are given in the following Table :—

Station.	Disturbance.	Station.	Disturbance.
I.	0°·4 W.	XIII.	0°·4 E.
II.	1·6 E.	XIV.	6·0 W.
III.	0·9 W.	XV. (W.)	1·9 W.
IV.	3·3 W.	XVI. (S.)	10·4 E.
V.	1·1 W.	XVII. (E.)	3·6 E.
VI.	1·6 E.	XVIII. (N.)	11·5 W.
VII.	1·5 W.	XIX.	5·8 E.
VIII.	1·3 W.	XX. (N.)	9·9 W.
IX.	8·1 W.	XXI. (W.)	25·8 E.
X.	0·6 E.	XXII. (S.)	10·3 E.
XI.	5·8 E.	XXIII. (E.)	23·8 E.
XII.	0·3 E.		

Stations XV. to XVIII. inclusive were taken round the basaltic mass on the summit of the hill behind Kaill, and on the sides indicated by the letters which follow the numbers.

Stations XX. to XXIII. were in like manner taken round the summit of Compass Hill. In both these cases the compass was generally within a foot or two of the basaltic columns. The observations, therefore, show that these are powerfully magnetic. The disturbing force at Station XXI. was nearly half that due to the horizontal intensity of the earth's magnetic field. On the other hand, it is evident that their influence diminishes very rapidly with the distance. Station VI. was only a few hundred yards from the hill behind Kaill, yet the compass was not affected by more than 1°·6.

Stations I., II., III., and IV. are grouped round the southern half of Compass Hill, at distances between 200 and 500 yards from the summit ; but the largest disturbance of the Declination is 3°·3 W.

Stations XXIII. and II. are situated one above the other on the eastern side of Compass Hill, the horizontal distance between the two being not more than 80 yards ;

yet the disturbance diminishes from 23°·8 E. at Station XXIII., near the top of the hill, to 1°·6 E. at Station II., near its base.

These conclusions are completely borne out by observations made on the "Coventina," in which we visited the island in 1884 and 1888.

On leaving Canna for Loch Boisdale in 1884 we sailed as close as possible to the northern face of the island, and took frequently the compass bearings of points on Skye. We were unable to detect the smallest deviation of the needle.

In 1888 observations were made under still more favourable circumstances. We approached the island from the north, and, when about three miles distant, the yacht was directed towards a mark on Rum, by which its course could be kept without reference to the compass. We were then sailing magnetic S. ¼ E., in the most favourable direction for the effect of Compass Hill (if any) to be detected. We passed it within 200 yards of the shore, but observed no deviation of the compass; and we are quite certain that, if there was any, it was less than one-eighth of a point, i.e., less than 1°·5.

The net result of our observations is that the basaltic cliffs of Canna are powerfully magnetic, and may deviate the needle of a compass placed near them by about two points, i.e., about 23°, but that the effect diminishes very rapidly with the distance, and is inappreciable on a ship's compass 200 yards from the base of the hill, to which tradition ascribes, and in which we have ourselves detected, the most powerful magnetic properties.

We have adopted 22°·8 as the mean value of the Declination at the less disturbed stations in Canna in August, 1888; this leads to 23° 13' for January 1, 1886, which is only 6' in excess of the calculated value. This is interesting, inasmuch as Plate IX. shows that neighbouring stations have Declination disturbances of opposite signs, and indicates that a line of no regional disturbance runs near to Canna.

Thus the four stations, Kyle Akin (No. 35), Soa (No. 46), Canna (No. 14), and Loch Boisdale (No. 10) lie very nearly in a straight line, and the disturbances of the Declinations vary continuously, being 28'·0, 18'·4, 6'·0, and − 45'·9, which proves that Canna lies near an attractive centre or ridge. This is in harmony with the fact that the disturbance of the Vertical Force is positive, and is enormously great, amounting to 0·1839, or about 0·04 of the whole Vertical Force.

A comparison of the results obtained at Malvern and Canna points very clearly to the otherwise probable conclusion that far-reaching effects are to be expected quite as much from the great mass and uniform magnetization of rocks as from their being highly magnetized. The basalt of Canna is far more susceptible and more magnetic than the Malvern syenite, but we doubt if a mile and a quarter from Compass Hill it would produce an effect on the Declination needle at all equal to that which we have shown is due at that distance to the Malverns.

The Eastern and South-Eastern Counties.

A problem of considerable interest has to be considered in connection with the Eastern and South-eastern counties. In the greater part of this district, the surface soil is such that it certainly can produce no marked effect upon the magnet, yet in this apparently "good ground," we have found magnetic disturbances of very wide range.

It is well known that the Declinations obtained at the Kew and Greenwich Observatories differ more widely than the difference of longitude will explain. The difference is not so great as appears at first sight, as the Kew results as published are not corrected for diurnal variation.

The published mean Declinations at these two stations in 1886 are, Greenwich 17° 54'·5, and Kew 18° 16'·9, of which the Kew result must be diminished by about 6'. This gives a difference of 16', which exceeds that corresponding to the difference of longitude by 10'. We are not aware that any attempt has hitherto been made to connect this discrepancy between the Declinations at these two important observatories with any regional disturbance in their neighbourhood.

The fact which first led us to believe that they lie within the area of such a disturbance was that there is not only a small decrease in the Declination between Kew and Reading (instead of an increase as the difference of longitude requires), but that there is also a very small difference between Worthing and Ryde.

The Declination at Reading should be about 20' greater than that at Kew, but the observation made there in 1886 proved that it is 1' less.

The Declination was again determined in 1888 at the same spot near Reading, with the result that the Declination, when reduced to epoch, came out 3'·6 lower than before, thus increasing the discrepancy between the actual and calculated differences between the two stations.

In like manner, the difference of longitude between Ryde and Worthing is equivalent to a change of 22' in the Declination, whereas the observed value at Ryde was only 2'·6 higher than that at the more easterly station.

In order to investigate these differences more fully, we determined to run a chain of stations along the valley of the Thames, to observe at another series half-way between the Thames and the Channel, and to interpolate a station between Worthing and Ryde.

The result is shown in the annexed diagram (fig. 14). Horizontal lengths indicate the longitudes of stations and vertical lines the Declination.

Three groups of stations, such that all places included in each are of nearly the same latitude are taken, thus forming three lines crossing the whole of the south of England. In the cases of Weymouth and Ryde, which lie considerably to the south of the line which nearly passes through the other stations corrections of 14' and 7' respectively have been added. Chichester is to the north of the line and its Declination has also been corrected by 7'.

It will be noticed that all three exhibit an anomaly on or immediately to the south

of a line joining Greenwich and Reading.   In the valley of the Thames the Declination attains a maximum value near Windsor and a well marked minimum near Reading.   To the south of this district there is a remarkable slackening in the rate of increase of the Declination.

An inspection of this figure is sufficient to prove that the anomaly in the Declination difference of Greenwich and Kew is not due to any accidental peculiarity of the position of either observatory, but is the result of a regional disturbance extending at least from that part of the valley of the Thames which lies between Greenwich and Reading to the south coast.

Fig. 14.
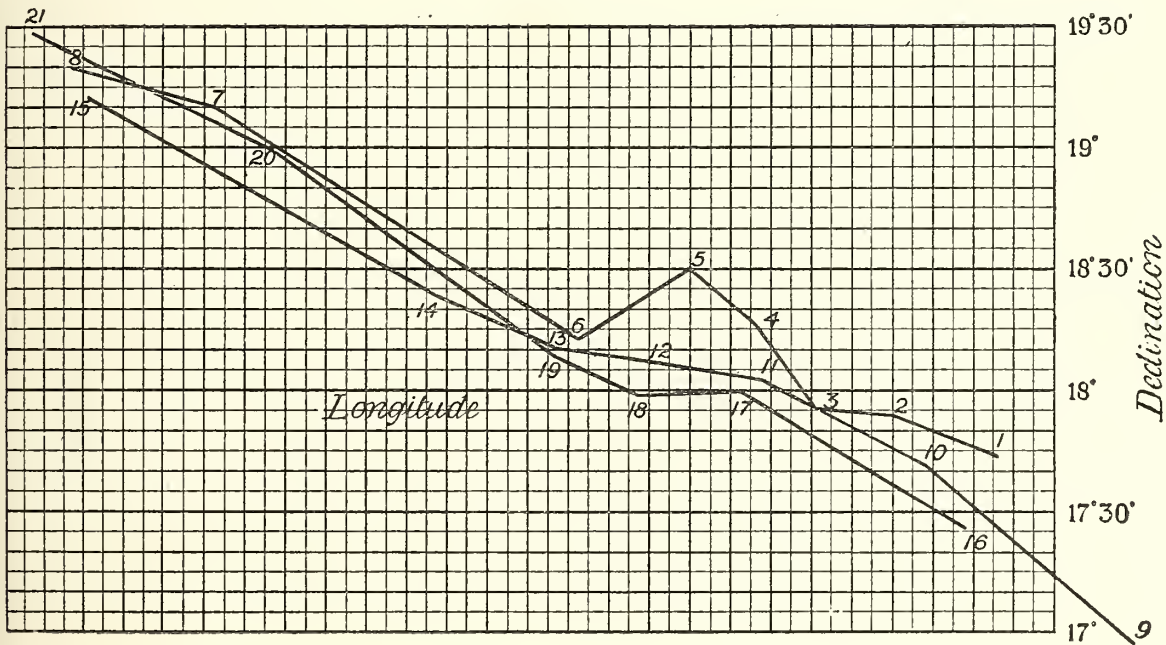
Declinations at—

1. Southend	7. Clifton	12. Haslemere	17. Worthing
2. Purfleet	8. Cardiff	13. Alresford	18. Chichester
3. Greenwich	9. Dover	14. Salisbury	19. Ryde
4. Kew	10. Tunbridge Wells	15. Taunton	20. Weymouth
5. Windsor	11. Horsham	16. St. Leonards	21. Exeter.
6. Reading			

The curves in the diagram when compared with figs. 7 and 8 (p. 263) are seen to be such as would be produced if a centre of force attracting the north-seeking pole of a magnet were situated near Windsor.   Immediately over such a centre the value of the Declination would be normal, while it would be too great and too small at stations to the east and west respectively.   If the centre were relatively weak the increase in the Declination with longitude instead of being represented by the slope of a straight line would be given by a curve of the same type as that in fig. 7, if it were strong we should have a curve like that in fig. 8.

Fig. 15 shows the true isogonals in this part of the country and proves that the form they assume corresponds to the last case.

The existence of a widespread disturbance is proved not only by the remarkable bend in the 18° 10′ isogonal between Kew and Reading, but also by several other stations. Thus the Declination at Harpenden is the same as that of Kew (18° 16′), though it is about thirty miles to the north of it and should therefore be 9′ greater. Again, Ranmore, Haslemere and Alresford, with Declinations of 18° 9′, 18° 8′ and 18° 10′ prove that the 18° 10′ line has really the great inclination to the meridian which the outer curves in fig. 6, p. 262, show. They are further supported by Horsham, Chichester, and Ryde, with Declinations of 18° 3′, 18° 5′ and 18° 2′ respectively.

The isoclinals and lines of equal Horizontal Force do not show any equally striking peculiarities in this part of the country. The latter, however, run a little too far north to the south of the Thames (see Plate VII.). This is as it should be, for to the south of an attracting centre the Horizontal Force will be abnormally great.

Fig. 15.
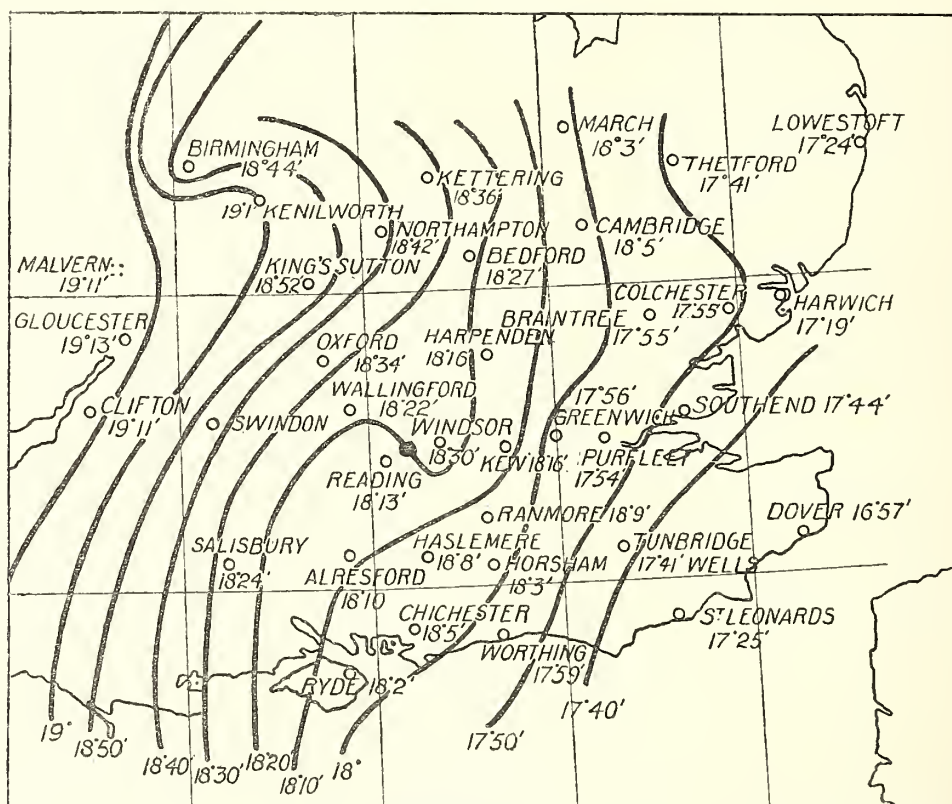
Isogonals in South-Eastern England.

The point at which the terrestrial and true isogonals would intersect if the district were not otherwise disturbed is marked by a dot, which will hereafter be called the *focus* of the Reading disturbance.

Let us now turn to the calculated disturbances of the elements. In the accompanying map (fig. 16) the figures represent the differences between the observed and calculated values of the Declination expressed in minutes of arc, and taken as positive when the needle is turned to the west. There is a sharply marked boundary between the regions of positive and negative disturbance which passes through the focus. To the west of it the needle is deflected to the east and *vice versâ*.

Fig. 16.

Declination disturbances in minutes of arc.
+ Indicates that the observed westerly Declination is greater than the calculated value.
———— Curve of no disturbance.

To the south of the focus the needle is oppositely deflected on each side of a line which runs nearly due north and south. To the north the effect on the Declination needle dies out.

If the cause of the disturbance were a mass of "magnetic matter" below the surface of the Earth symmetrical with respect to the isogonals, the line which divided easterly from westerly disturbances of the Declination would intersect that which divided positive from negative disturbances of the Horizontal Force over the focus or centre. The next map (fig. 17) shows that this condition is very nearly fulfilled, the two points being only 7 or 8 miles apart. Lastly, the maximum disturbance of the

Vertical Force should occur at the same point, and fig. 18 shows that we find it to be at Reading, which is our nearest station to the two points found as above described. All three elements then combine to indicate that the centre of the Thames Valley disturbance lies between Reading and Windsor, and is a few miles to the north and east of Reading.

Fig. 17.
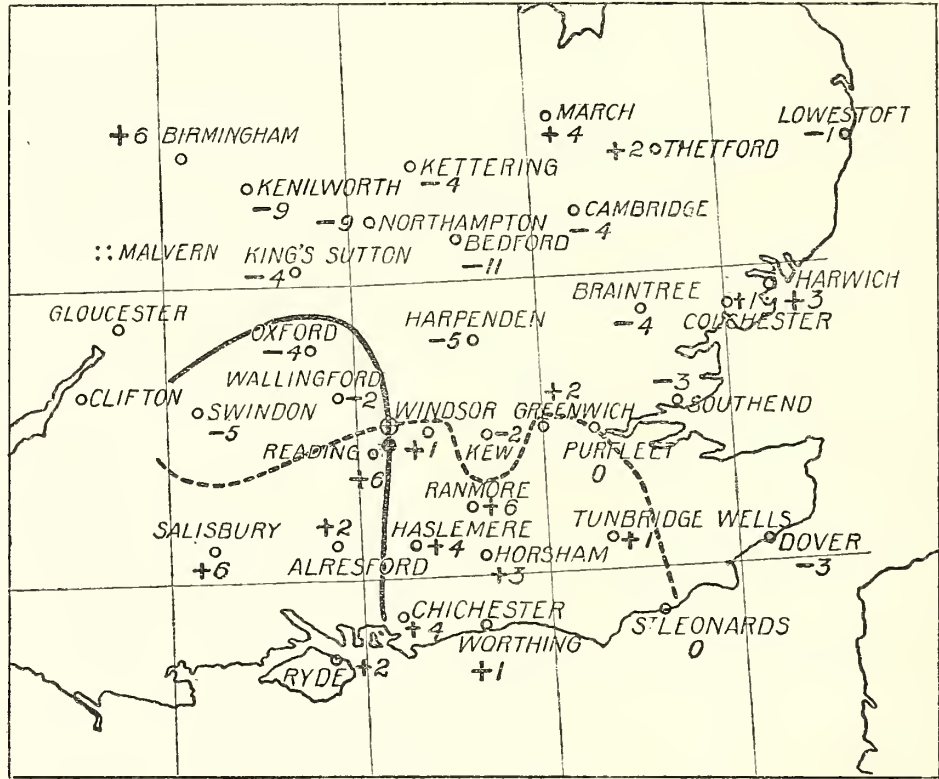
Horizontal Force disturbances in terms of 0·001 metric unit.
+ Indicates that the observed is greater than the calculated value of H.
——— Locus of no declination disturbance.
– – – – Locus of no horizontal force disturbance.

The Map of Vertical Force disturbances, however, teaches us a good deal more. If we draw contour lines to enclose all stations at which the Vertical disturbing Force is greater than 0·010 and 0·015 metric unit respectively, the first comprises two independent curves, the one embracing a large area around the focus, and the other surrounding Chichester.

If the cause of the phenomenon were an underground mass of igneous rock we might picture it as a sub-terrestrial mountain of which the peak is near the focus. The slope would be most rapid towards the south-east. Two ridges would run north-east and north-west towards Oxford and Cambridge, a third, less lofty, to Chichester, and a fourth, nearly due east, terminating very abruptly at Kew. If all this were so, the

disturbing forces at stations in the neighbourhood ought to be directed towards the lofty central mass.   Close to the peak it would itself form the centre of attraction. At stations near to, but not over, outlying ridges, the needle might be deflected towards them.   Immediately over a ridge the direction of the disturbance would change rapidly, and thus give an idea of instability.   The Horizontal Forces would be least over the peak, would increase up to a certain distance, and would finally die out.

Fig. 18.
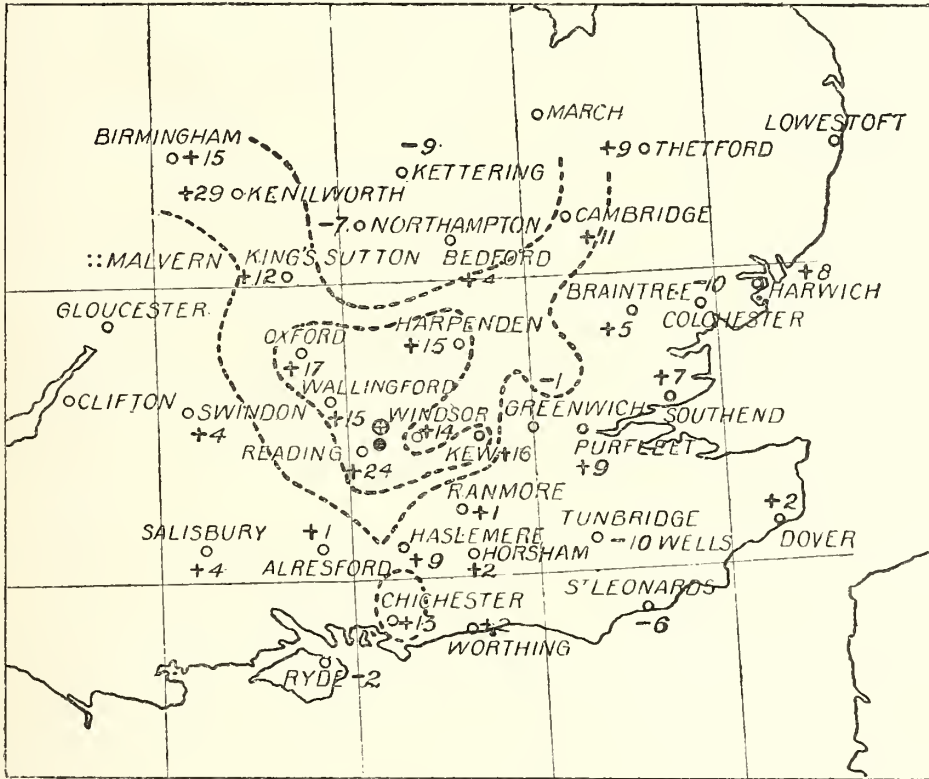
Vertical Force disturbances in terms of 0·001 metric unit.
+ Indicates that the observed is larger than the calculated value.
– – – – Contour lines of equal Vertical Force disturbance.

In fig. 19 we have depicted the disturbing Horizontal Forces.   They are drawn in the proper directions, and to a scale on which 0·9 mm. corresponds to 0·001 metric units.   They fulfil the above conditions exactly, and we think leave no doubt that in the south-east of England over an area of 10,000 square miles the lines of magnetic disturbing force tend to a centre which lies near to and probably between Twyford and Henley-on-Thames.   We have treated this district in great detail, because we rely upon these results to prove that the methods of calculation and deduction adopted are satisfactory, at all events in districts where the surface rock or soil is non-magnetic.   If the results attained elsewhere present greater difficulties

it must be due to the greater complications introduced by the interference of local with regional disturbances.

Fig. 19.

Disturbing Horizontal Magnetic Forces in South-Eastern England.

The maps which have been used to illustrate our discussion of the Reading disturbance give indications of other minor centres, and in particular the isogonals are considerably distorted in the north. It appears that if we draw a line through King's Lynn (No. 100), Spalding (No. 140), Melton Mowbray (No. 116), Loughborough (No. 109), Birmingham (No. 63), and Malvern (No. 112), the district through which it passes is the seat of local disturbances, which, though individually less widespread than that already discussed, are nevertheless of considerable intensity.

We will now investigate several points in this neighbourhood. It will not be necessary to do this in the same detail as before. The places to be considered lie so near to the borders of the district already studied, that methods which are so consistent in the one cannot be subject to any important error in the other.

The Wash.

We have discovered a remarkable disturbance in the neighbourhood of the Wash. No more unlikely region could *primâ facie* have been suggested, but the evidence for its existence is conclusive.

Our attention was first called to it by the fact that the Declination found in 1888 at Spalding is less than that observed at King's Lynn in 1886. Allowing for the difference of longitude, the Declination at Spalding ought to be about 16′ greater than at King's Lynn. The two values found and reduced to epoch were

$$\text{King's Lynn, 1886} \quad . \quad . \quad . \quad . \quad . \quad . \quad 17^{\circ} \; 57'\!\cdot\!9,$$
$$\text{Spalding, 1888} \quad . \quad . \quad . \quad . \quad . \quad . \quad 17 \; 51\!\cdot\!6.$$

Thus, if the King's Lynn value is normal, that at Spalding is 22′ too small.

Fig. 20.
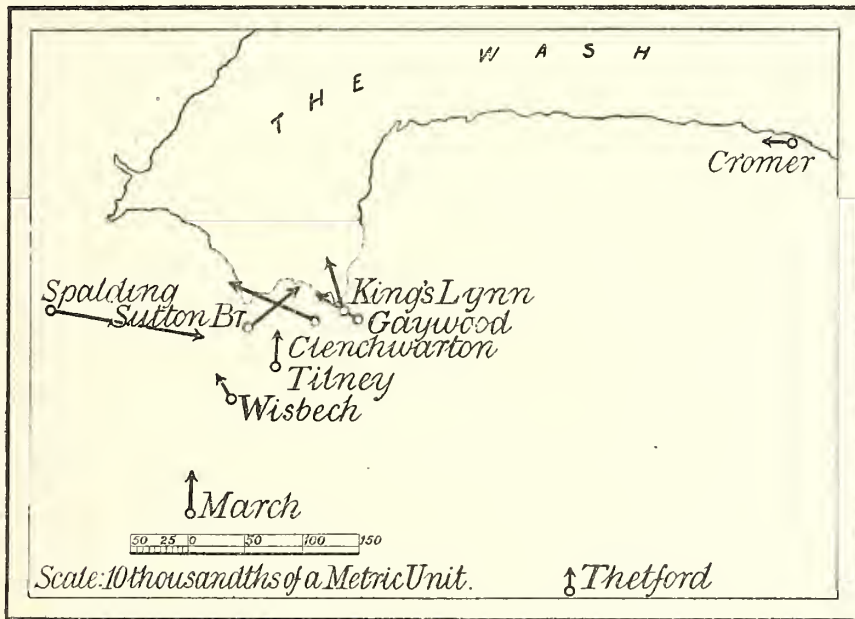
Disturbing Horizontal Magnetic Forces near the Wash.

A special survey was therefore made of the district with the following results.

A chain of stations was run along the edge of the Wash. The measurements at King's Lynn were repeated near to, but not on the same site, and observations were made at Clenchwarton and Sutton Bridge between King's Lynn and Spalding. The latitudes of these stations did not differ by more than 2′; the longitudes and Declinations were as follows :—

Stations.	Longitude.	Declination. Jan. 1, 1886.
King's Lynn (Gaywood 1888)	0° 26·0 E.	18° 1·7
King's Lynn (1886) . . .	0 24·3 E.	17 57·9
Clenchwarton (1888) . . .	0 21·3 E.	18 10·3
Sutton Bridge (1888) . . .	0 11·8 E.	17 54·1
Spalding (1888)	0 8·6 W.	17 51·6

These results make it certain that near the south of the Wash the Declination diminishes instead of increasing (as at normal stations) with the longitude.

The accompanying map shows the directions and magnitudes of the disturbing forces in this district. They indicate a centre of attraction to the north of the line which joins Spalding and King's Lynn.

The Leicestershire District.

Another series of local disturbances exists to the west of that which has just been described. The facts which first attracted our attention to it were that at two pairs of stations, viz., Birmingham and Northampton, Leicester and Peterborough, the observed Declination at the more westerly was only about 2′ greater than that at the more easterly station, though the calculated differences were as much as 35′ in the first case, and 27′ in the second case.

The observations indicated that in this district the isogonal lines run nearly east and west instead of nearly north and south, and we proceeded to investigate their forms more closely. We thought that the anomaly was probably connected with the fact that in Charnwood Forest, which is not very distant from Leicester, igneous rocks appear upon the surface, and observations were made round this district, though we were always careful that our station should be on what was apparently good observing ground.

We have thus confirmed the existence of a great easterly trend in the isogonal lines, and though the magnetic state of the district appears to be complicated, and to require further investigation, we have also established several facts which will probably prove to be of fundamental importance in the solution of the problems connected with it.

Three of the most interesting stations are Coalville (No. 76), Loughborough (No. 109), and Melton Mowbray (No. 116). Loughborough and Coalville are both on the Red Marl, with alluvium near to the streams. Between them lies Charnwood Forest, in which are masses of porphyry, greenstone, and syenite.

We should, perhaps, expect from the analogy of the Malverns that at these two stations the needle would be attracted towards the crystalline rocks. This does not appear to be the case, or, as is more probable, the stations are too far distant to be affected. At Loughborough, which is the more easterly station, the disturbance of the Declination is 30′·3 *towards the east*, while at Coalville it is in the same direction, but only to the extent of 11′·2.

About 12 or 13 miles further to the east is Melton Mowbray, situated on Lower Lias clay with argillaceous limestone at its base, yet a series of observations made here on April 22, 1888, gave a Declination disturbance of + 32′, *i.e.*, towards the west. Unfortunately, the observation for the geographical meridian could only be made near noon, as the sun was invisible during the rest of the day; but so remarkable did the result appear that the place was revisited on April 30. Another

station on the other side of the town, about a mile and a half distant from the first, was chosen, and the new observation was made under favourable conditions at about 5.45 P.M. The result indicated a Declination disturbance of $+ 26'$, which was in close accord with that previously obtained. At first sight, then, it appears that the peculiarities of the district might be explained by the hypothesis that a centre of force, powerful relatively to Charnwood Forest, exists somewhere between Melton Mowbray and Loughborough. The disturbances of the Declination at these two stations are in opposite directions and of nearly equal amounts. This view does not, however, correspond with the directions of the disturbing forces obtained at Melton Mowbray. If it were correct they should both have acted in nearly parallel directions towards the west. As a matter of fact, at the first station the disturbing force acts nearly due west, while at the second it is only 11° from south.

This indicates that the disturbance at Melton Mowbray is of a more local character, and cannot, in the manner suggested, be brought into relation with the oppositely directed disturbances at Coalville and Loughborough.

If, however, we turn from the Horizontal to the Vertical disturbing Forces we find that the peculiarities of these various stations may be connected, and that a magnetic map of the district can be drawn which may furnish the first rough outlines to which details may hereafter be added without rendering them substantially incorrect.

This explanation is based upon the fact that whereas in this part of England the disturbances of the Vertical Forces are for the most part positive, at four consecutive stations in this neighbourhood they are negative.

These stations are Coalville, Leicester, Kettering, and Northampton. Whether the low Vertical Forces are due to the presence of a repulsive centre, or to a deep cleft or valley in the attracting mass, we cannot tell. Indeed, the latter hypothesis might account for a repulsion if the mass of attracting matter which we have supposed to culminate in the Reading peak terminated abruptly on its northern edge. For, if it were magnetised by induction it is possible that some of the lines of force might escape upwards to the surface, diminish the Vertical Force, and urge the north pole northwards.

Contour lines drawn as in the accompanying map (fig. 21), indicate the possibility of a very sudden rise in the magnetic matter from a line drawn through Coalville, Leicester, and Kettering, to another which passes through Chesterfield, Nottingham, Melton Mowbray, and Peterborough. This view is supported by the fact that a ridge line, on passage across which the disturbance of the Declination changes sign, runs from near Chesterfield to near Melton Mowbray.

If this view were correct, Melton Mowbray would have to be regarded as near the summit of an extremely steep peak, as the Vertical Force changed from 0·0305 to 0·0080 in the small distance between the two stations. The direction of the Horizontal Forces would indicate a point a little to the west of both stations as the actual peak.

Fig. 21 is an attempt to realise the magnetic constitution of this part of England from this point of view.   In studying it we must remember that the direction of the Horizontal Force at a station where the Vertical Force is a maximum or minimum must be indeterminate in the sense that it cannot be deduced from the Vertical Forces

Fig. 21.
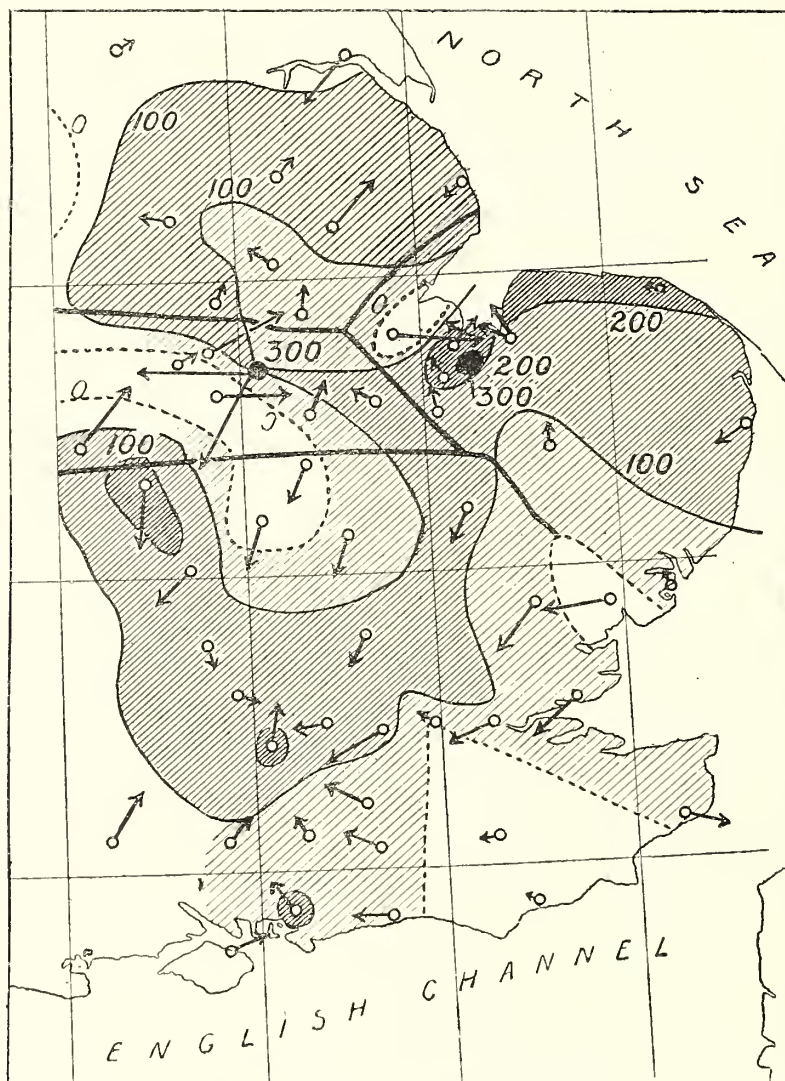
Contour lines of Vertical Disturbing Force and Horizontal Disturbing Forces in South-Eastern England.   Each darker tint corresponds to an increment of 0·0100 metric unit in the Vertical Disturbing Force.

at neighbouring stations.   All that we know when the Vertical Force is a maximum is, that a peak is probably in the neighbourhood, but on which side of the station we do not know, unless, as in the case of the Reading disturbance, other stations indi- cate it.

The whole district may apparently be divided into four parts, of which the boundaries are indicated on the map by heavy lines.

The southernmost is the region of the Reading disturbance. The tendency of the Horizontal Forces to act towards regions of high Vertical Force is unmistakeable. At King's Sutton the direction of the resultant appears to be affected by the region of high Vertical Force to the north, near Kenilworth. At Worthing and Ryde the Horizontal Forces point direct to the Chichester peak. At Purfleet and Southend the directions of the Forces are more southerly than the distribution of the Vertical Forces would have led us to suspect.

The most easterly district is that of the Wash disturbance. Near King's Lynn the Vertical Force is great. It is greatest at Tilney, which is the central station. The Forces at neighbouring stations converge to a point to the north of this, and the Horizontal Force at Tilney itself is directed northward. It is therefore likely that here, as in the case of Reading, the true peak, though near, is not at the spot at which we found the greatest Vertical Force. It probably lies to the north of it.

The central district is that of the Leicestershire disturbance. Here the phenomena are more complicated, and we wish it to be distinctly understood that we think it probable that Melton Mowbray does not occupy the position of unique importance which our observations allot to it. Nevertheless, we must point out that the hypothesis that a narrow ridge of attracting matter runs somewhat in the position we have assigned to it, is remarkably supported by the direction of the Horizontal Forces at Coalville, Loughborough, Leicester, and Manton, which would all be explained on this hypothesis. At Melton Mowbray the directions are, of course, indeterminate by means of the Vertical Force, and it is quite possible that the phenomena observed there may be due to some relatively small dyke, and not to an uprising of a part of a widespread mass of igneous rock. Until this district is more fully surveyed we are justified in adopting the view represented on the map, which is consistent with all the known facts.

There is, however, one station in the district, viz., Birmingham, which is not in harmony with the rest, as the direction of the Horizontal disturbing Force is toward the region of minimum Vertical Force. Perhaps this indicates that this region is not connected, as we have supposed, with the larger region of low Vertical Force to the west, but that the two are severed by a district of high Vertical Force running from Birmingham northward. Future investigation can alone decide this question.

The most northerly of the four districts is almost outside the region of our special surveys. We only introduce it to show that there is a large region of high Vertical Force to the north of Melton Mowbray, and that, therefore, there is nothing anomalous in the northerly directions of the Horizontal Forces at Nottingham and Grantham. If it be true that in the Leicestershire disturbance the attracting matter lies within narrow limits, it is quite possible that at these stations the predominant influence may be that of the larger northern mass.

On the whole, then, if we take the region bounded by the sea, by lat. 53° and long. 2° W., which includes about 50 stations, we think the Horizontal Forces unmistakeably tend to act toward the region of great Vertical Force. There is one striking exception at Birmingham, and one or two more doubtful ones on the lower reaches of the Thames, and the rule must be construed subject to the obvious condition that a true maximum of Vertical Force, though probably near to, is not necessarily at the station at which we happen to have found the largest among the Vertical Forces we have measured.

Subject to these exceptions and to this proviso the rule holds good.

General Results of the Investigation of the Local and Regional Disturbances.

Having described the results obtained in districts to which we have devoted special attention we now proceed to apply the same methods to the whole area of the survey. In adopting this course we are fully aware that the number of our stations is not sufficient to enable us to speak with any certainty as to the details of the magnetic peculiarities of the districts we are about to discuss, and it is quite possible that we may have arrived at some conclusions which must hereafter be modified. It appears to us that even under these conditions our work is much more likely to be useful if we give what only professes to be a first rough sketch map of the magnetic forces in play in the country than if we leave our successors to get what hints they can from observations which we ourselves have made no attempt to collate. Even, therefore, if our conclusions were much less certain than we believe them to be we think it would be better to state them.

Fortunately, however, we are able to take up a much stronger position than this. Our conclusions may be tested, (1) by the agreement of the results of the various methods of attacking the problem, (2) by the agreement of our results in Scotland with those which can be deduced from Mr. WELSH's survey, (3) by the establishment of relations between the magnetic phenomena at stations scattered over wide areas, and (4) by the establishment of a connexion between the magnetic and geological characteristics of various districts. In all these particulars we venture to assert that they will bear investigation, and we cannot but believe that we have detected the main directions of the lines of disturbing magnetic force.

The method we adopt is as follows :—

We draw the ridge and valley lines (see p. 265) which mark the centres and the boundaries of districts which are under the influence of a dominant locus of attraction. We take each district bounded by two valley lines, and study it by means of the true isomagnetics of the disturbances and disturbing forces, and lastly, we discuss the relations between its magnetic and geological characteristics.

The following are *a priori* probable consequences of the hypothesis that each district, bounded by two valley lines, is subject to an attraction tending towards the centre :—

(1.) Since the regional forces are weak near a valley line it is in such a position that we should expect the effects of such local forces as might exist to predominate, and discrepancies to occur more frequently than elsewhere.

(2.) The centre of a district, in the neighbourhood of the ridge line, should be a region of relatively high Vertical Force.   It will be remembered that the ridge line is drawn by means of the disturbances of the Declination and Horizontal Force only, and, therefore, an agreement between its position and that of a region of high Vertical Force is an independent confirmation of the accuracy of the theory that the forces in play in the district form a connected system, that is, they are regional and not merely local.

(3.) The directions of the Horizontal Forces at points of maximum or minimum Vertical Force may be indeterminate in the sense that they cannot be deduced from the distribution of the Vertical Forces.   This point has already been insisted on.

(4.) In crossing a valley line both sets of attractions in the region which it separates tend to produce a maximum and minimum of the Horizontal Force (see fig. 10, p. 263). If the southerly attraction were alone in play both of these would have values less than the normal.   If the northerly acted alone both values would be greater than the normal.   Hence, since the maximum will be to the north of the minimum it is probable that in producing it the effect of the northern attraction will predominate and the values will be greater than the normal, while in the case of the minimum they will probably be less.

Comparison with Mr. Welsh's *Survey of Scotland.*

No more severe test of the physical reality of the disturbing forces deduced by us from our observations can be applied than by the enquiry whether similar methods applied to Mr. Welsh's survey lead to similar results.   In part, the severity of the test lies in the fact that the methods cannot be precisely similar.   As England was so imperfectly surveyed in 1857, we are compelled to assume a linear function to express the isomagnetics in Scotland at that date.   The isogonals and isoclinals were calculated by Balfour Stewart, who used the geographical mile system of co-ordinates.   Unfortunately the third element selected by him for calculation was the Total Force, which throws but little light upon problems such as those we are now investigating.   We have, therefore, calculated the lines of equal Horizontal Force, using, as in our other calculations, the differences between the latitudes and longitudes and those of the central station as co-ordinates, and by combining these with Balfour Stewart's calculated values we have found the disturbing forces at the various stations.

Again, Mr. Welsh omitted the determination of one or more of the elements

at many stations. He apparently had chiefly the terrestrial isomagnetics in view, and did not consider that for the discovery of the local peculiarities of a station, it is essential that all three elements should be determined at it. In part this omission was due to misadventure. A number of Declination observations had to be rejected as the mirror was found to have been out of adjustment. If, therefore, the calculated magnitudes and directions of the disturbing forces depend to any great extent on the distribution of the stations or on the method of reduction, conclusions drawn from WELSH's survey and our own could not be in harmony.

In the case of the Declination comparison is possible, not only with the survey of 1857, but with the observations collected by Sir F. EVANS in 1872. We have measured the distances of the stations from the isogonals given by him ('Phil. Trans.,' 1872, vol. 162, p. 319), and have deduced the disturbances. In the accompanying map (fig. 22) we have combined the results of all three surveys.

The direction of a short line drawn through the stations, shows whether the north pole of the needle was deflected to the east or west. Observations made by Mr. WELSH are indicated by a dotted line, those due to the naval officers by two short lines with a dot between them, and our own by a continuous line.

A discrepancy between any two of the surveys is thus indicated by a cross.

Lines of no disturbance are drawn, separating districts in which the regional disturbance is of opposite signs, and passing, when possible, through stations at which different results have been obtained in different surveys.

Without insisting on the accuracy of the details of these curves, we think that a study of the map can only lead to the conclusion that the districts which they bound are affected by some common cause which produces a similar effect upon the needle. It can hardly be doubted for instance that the magnet is deviated in opposite directions in the neighbourhood of Elgin and Banff. All three surveys agree as to this fact. Again, the evidence is very strong that on the mainland to the north of the Caledonian Canal, except, perhaps, near Cape Wrath and the Pentland Firth, the needle is deflected by regional forces to the east, while in the Islands on the West Coast and the Mull of Cantyre it is deflected to the west. Of course it would be possible to find in these districts places, such as Canna or the Cuchullin Hills in Skye, where large deviations in both directions could be obtained within a few yards, but the great majority of survey stations are chosen too carefully for the local error to become so overwhelmingly predominant. We have omitted Portree because the ground there is known to be unfavourable to our purpose. The differences between the three results obtained are too great to make a mean value trustworthy. At Canna on the other hand, where it will be remembered we observed at twenty-three places, the mean value cleared of the larger disturbances agrees with the results obtained at neighbouring stations.

The following facts are also important.

Seven stations were common to the surveys of 1857, 1872, and 1886. One of

these, Lerwick, we disregard, as it lies so far from the rest of the district.   In five out of the remaining six the disturbance of the middle epoch as deduced from the isogonals given by Sir F. EVANS is intermediate to those which were obtained in 1857 and 1886.   At Oban there is an enormous discrepancy, which is not supported by

Fig. 22.
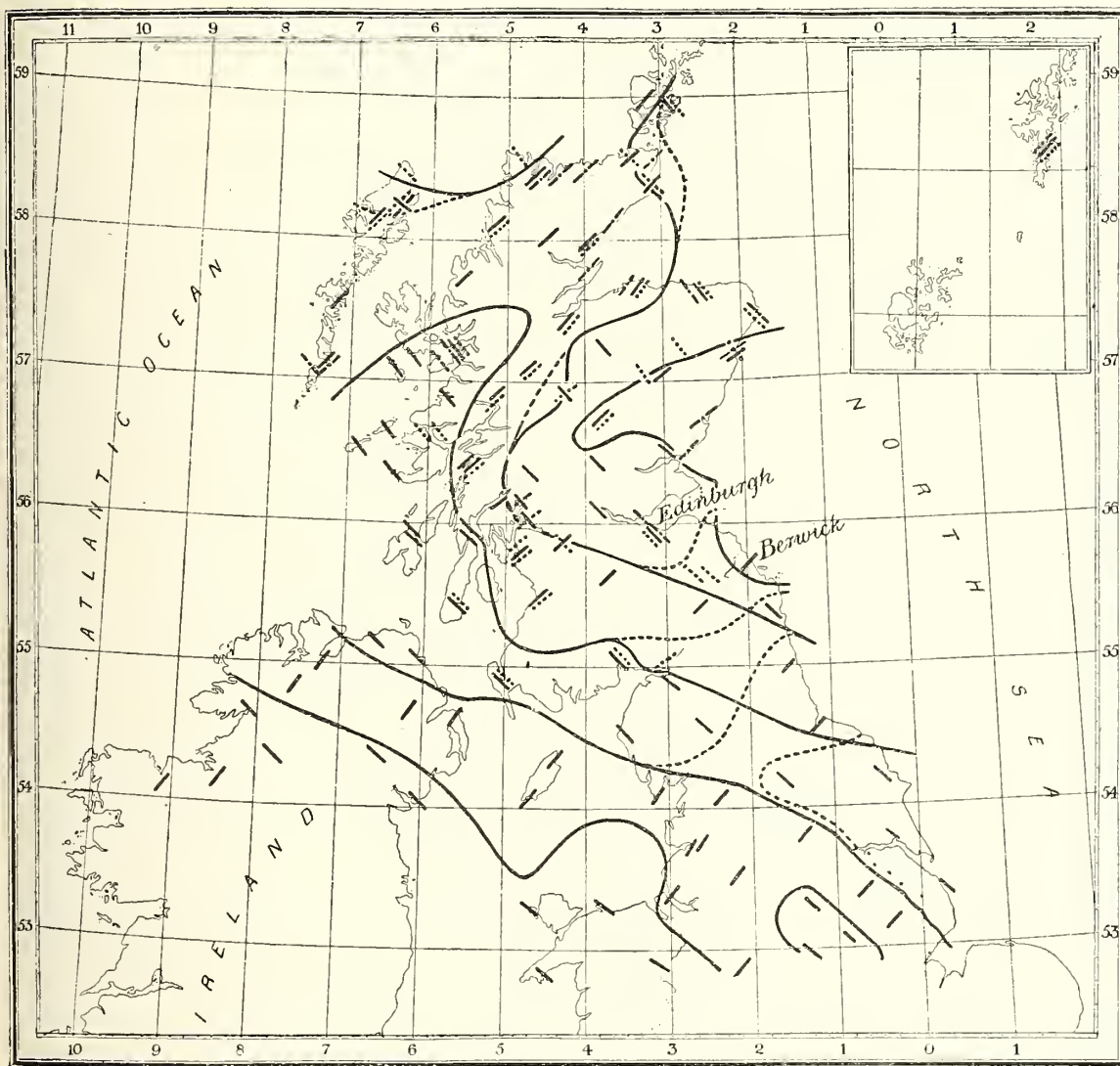
Directions of Declination Disturbances.

the observations made in 1872 and by ourselves on the neighbouring island of Kerrera.   There can be little doubt that the 1872 observation was subject to a powerful local disturbance.   The coincidence of the results in the other cases can hardly be accidental.   The figures are given in the following Table, Kerrera being added for the sake of comparison with Oban :—

DECLINATION Disturbances.

	1857.	1872.	1886.
Aberdeen	+ 5·1	− 4·0	− 6·7
Edinburgh	+ 19·8	+ 10·0	+ 4·2
Kyle Akin	+ 38·2	+ 27·0	+ 24·9
Oban	− 11·1	+ 55·0	− 8·3
Kerrera	..	− 5·0	− 6·8
Thurso	+ 13·8	+ 7·0	− 5·2
Wick	+ 11·6	+ 7·0	− 8·5

It is difficult to offer any satisfactory suggestion as to the causes of the apparently regular change. It may be due only to the fact that the three methods of deducing the disturbances by calculations are different, or it may be due to real changes in the local forces. Such alterations are not impossible since if the rocks were magnetised by induction on the earth's field the direction and intensity of the induced magetisation would be subject to secular change, but the differences are too great to be explained thus.

But whether local changes have taken place or not, there can be no doubt that the general distribution of the Declination disturbances in Scotland is the same now as it was in 1857.

In Map 22 the dotted lines are the curves of no disturbance which would have been drawn from our own observations alone. The differences introduced by the addition of WELSH's and EVANS's surveys are quite unimportant in the North. In the South they simplify the lines instead of introducing complications. As the disturbances at Edinburgh and Berwick are of opposite signs, we had to carry a line of no disturbance between them. Mr. WELSH, however, had two stations a little to the West of Berwick, viz., Makerstoun and Melrose, and at these the disturbance is of the same sign as at Edinburgh.

The introduction of these justifies us in treating Berwick as an isolated station, and the simple system represented by the continuous lines results. We have adopted these as the true lines of no Declination disturbance in this part of the Kingdom.

As Sir FREDERICK EVANS did not add any Dips or Horizontal Forces to the list of those already known, we proceed at once to the direct comparison of the disturbing forces deduced from our own and Mr. WELSH's surveys. The magnitudes and directions of these calculated by us as above described are as follows :—

DISTURBING Forces in Scotland from Mr. WELSH'S Survey in 1857.

Station.	Latitude.		Longitude.		F.	φ.	Z.
	°	′	°	′			
Aberdeen	57	9	2	05	163	−164°	− 227
Alford	57	14	2	45	109	− 43	− 72
Ardrishaig	56	1	5	27	43	+151	− 149
Ardrossan	55	39	4	47	..	..	+ 223
Ayr	55	28	4	38	59	−150	− 147
Balmacarra	57	17	5	39	127	+152	+ 51
Banff	57	39	2	31	77	+ 87	− 148
Braemar	57	1	3	25	184	+ 9	− 178
Bridgend	55	48	6	16	..	..	+ 309
Broadford	57	15	5	51	..	..	+ 417
Callernish	58	10	6	44	99	− 6	− 45
Corpach	56	51	5	8	100	− 3	+ 331
Cross	58	29	6	17	90	+ 67	+ 283
Dalwhinnie	56	56	4	17	107	+ 19	− 133
Dumfries	55	5	3	36	96	+123	− 172
Durness	58	34	4	44	112	+152	+ 190
Edinburgh	55	58	3	11	94	+131	− 93
Fort Augustus	57	9	4	40	80	− 87	+ 89
Glenmorven	56	38	5	58	410	+ 86	+1310
Golspie	57	58	3	58	7	− 90	+ 293
Gretna	55	1	3	3	172	−113	− 170
Helensburgh	56	2	4	43	149	−139	− 181
L. Inver	58	10	5	12	248	−148	− 608
Inverness	57	28	4	11	49	− 39	+ 163
Kintore	57	15	2	23	..	..	− 312
Kirkwall	58	59	2	58	153	+ 45	+ 250
Kyle Akin	57	16	5	44	174	+134	+ 28
Lamlash	55	31	5	5	..	..	− 1
Larbert	56	2	3	49	..	..	− 91
Lerwick	60	9	1	8	100	− 6	+ 238
Lochgoilhead	56	10	4	54	66	− 10	− 363
Makerstoun	55	35	2	31	65	+ 37	+ 162
Newton Stewart	54	56	4	28	155	−168	− 137
Oban	56	27	5	26	120	− 31	− 303
Peterhead	57	31	1	46	71	+174	− 141
Pitlochrie	56	42	3	43	58	−150	− 332
Port Askaig	55	52	6	8	..	..	+ 214
Stornoway	58	15	6	23	152	+ 7	+ 41
Stranraer	54	54	5	2	175	−166	− 333
Thurso	58	35	3	32	94	+ 65	+ 89
Wick	58	25	3	5	90	+172	+ 316

These values are shown in fig. 23. The arrows represent the Horizontal Forces in magnitude and direction. The numbers are the Vertical disturbing Forces in terms of 0·0001 metric unit. When negative they are underlined. Lines of no Vertical Force disturbance are also drawn. The shaded parts are regions of positive Vertical Force disturbance. A comparison of this with figs. 25 and 26 will suffice to show that there is a close agreement between the two. Thus, in the case of the Vertical Forces, we both find regions of high Vertical Force along the lines of the Caledonian Canal and the Western Isles, and on the East and West Coasts of South Scotland.

Fig. 23.
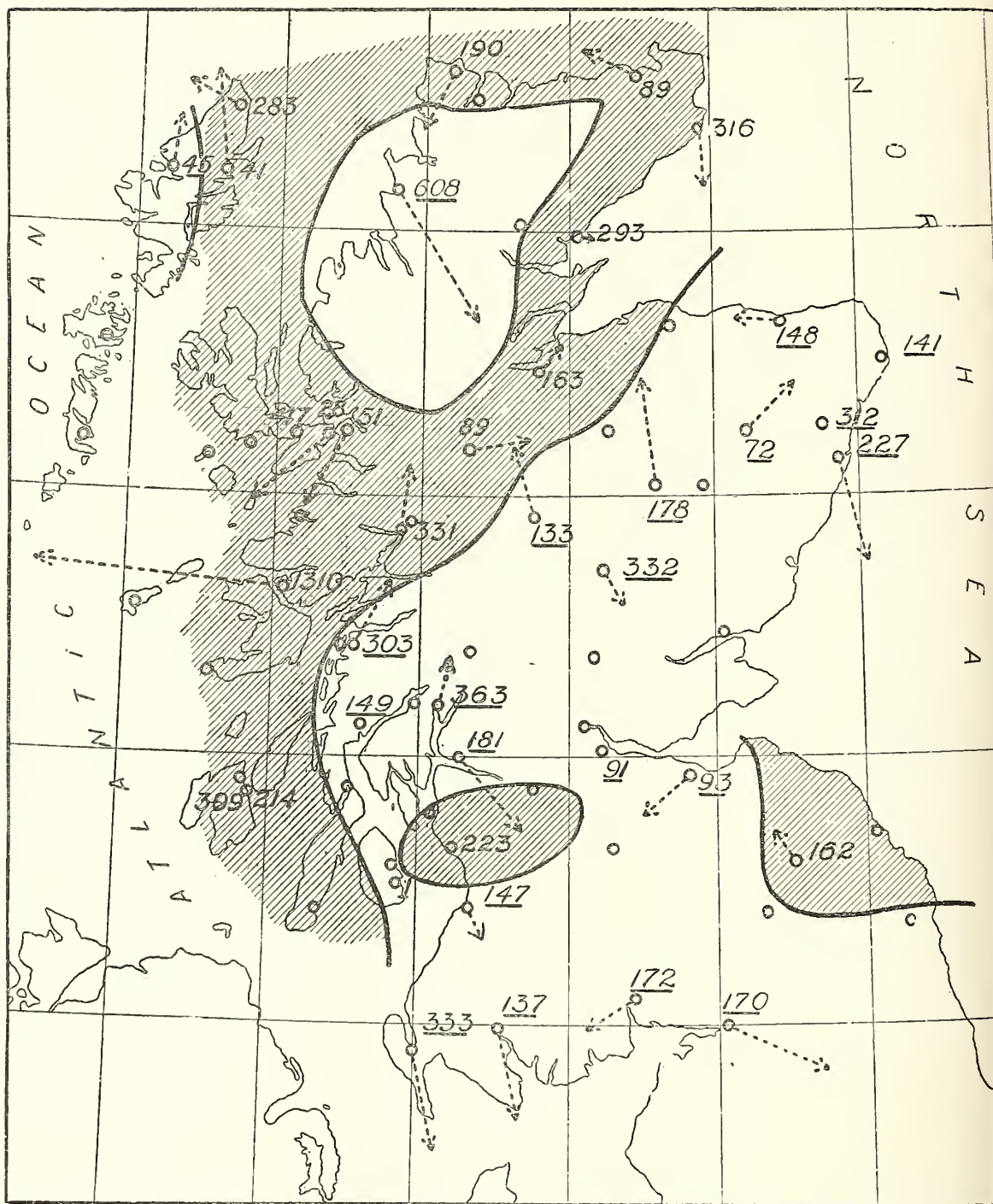
Disturbance map of Scotland, from Mr. WELSH's survey, 1857–58.

There are a few discrepancies in the direction of the Horizontal Forces, but the general agreement and occasional differences will be better discussed when we deal, as we are now about to do, with separate districts.   We propose to treat of the results of the two surveys simultaneously, and we think we shall succeed in showing that, although it would have been impossible to draw our conclusions from Mr. WELSH's less numerous stations, they are strongly confirmed by the deductions which we have made from his survey.

Fig. 24.
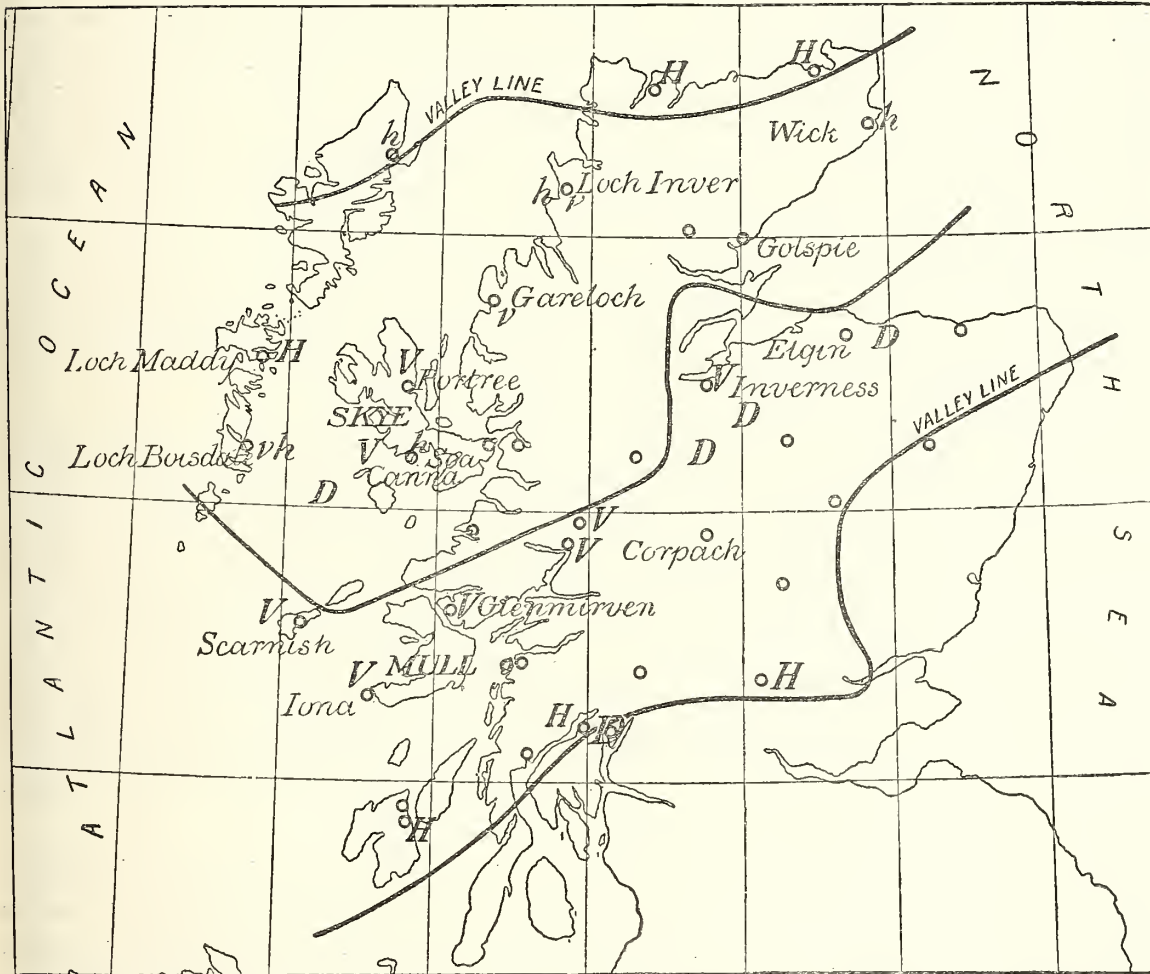
Highland District.

The Highland District.

In fig. 24 we have collected all the information which the isomagnetics in Plates V. to VIII. afford of this district. Those places only are named to which we actually refer. Where the more westerly of two stations has the smaller Declination, the isogonals are

distorted (Plate V.), and indicate a centre of attraction between the two.   All such points are marked *D*.

Loops in the Vertical Force isomagnetics indicate maxima and minima of Vertical Force, which are marked *V* and *v* respectively.   As no observations could be made to the west of the islands in the south-west of Scotland, it is impossible to prove formally that the Vertical Forces in this district are maxima.   The values at Scarnish and Iona are, however, the same as those at Wick and Golspie respectively, which are nearly two degrees farther north.   We have, therefore, felt justified in marking them as maxima, thereby indicating that they are very large.   Canna and Portree are so disturbed that but little reliance can be placed upon observations taken there, but they both give maximum values of the Vertical Force (Plate VIII).   At Corpach and Glenmorven we find that Mr. WELSH's observation gave maximum values of the Vertical Force.   At two of the most northern stations the Horizontal Force is a minimum ; at three of the most southern it is a maximum (Plate VIII.).   They are marked with *h* and *H* respectively.

From this map we can form an opinion as to the magnetic constitution of the district.   Points of high Vertical Force, and centres to which the needle is attracted, cluster thickly along a line which passes from Elgin to Inverness, thence along the line of the Caledonian Canal to Corpach, and so to Mull.

Another similar line runs north through Skye, and indications of a region of low Vertical Force, which separates the two, are given in the minimum values found at Gairloch and Loch Inver.   There is a subsidiary centre of attraction near Strachur.

The view that a general attraction is exerted toward the centre of the district is supported by the occurrence of maximum and minimum values of the Horizontal Force in the south and north respectively.   The minima at Loch Boisdale and Soa and the maximum at Loch Maddy (which necessarily follows) indicate a strong subsidiary centre of attraction to the south of these places.   BALFOUR STEWART, arguing from WELSH's results, placed such a centre to the south of Mull, but as we shall show, it is probably to the west of that island ('Brit. Assoc. Report,' 1859, p. 190).

It must be remembered that all these conclusions follow from the mere inspection of the results of the observation with no more calculation than is necessary to reduce them to the same epoch.   In the next map (fig. 25) we show the results of the calculations carried as far as possible, *i.e.*, to the point of deducing the disturbing forces.

The shaded parts are the regions of positive (*i.e.*, of great) Vertical Force.   The boundaries are fixed by the very rough method of assuming that between neighbouring stations the rate of change of the disturbing force is uniform.   The arrows represent the Horizontal disturbing Forces in magnitude and direction.   The dotted arrows represent the disturbing forces calculated from Mr. WELSH's survey.

There are only two or three discrepancies, and they all occur, as we anticipated, near valley lines.   The most remarkable is Stornoway.   There can be no doubt as to the

accuracy of our results obtained on three different occasions and at two stations, and we can only suppose either that there has been a real change or that Mr. WELSH's station was subject to some very great local disturbance. Lochgoilhead, which our observations place just over the valley line in the next district is, according to WELSH, just within this. At Pitlochrie there is a large angle between the two forces. It is doubtful whether the valley line is here correctly drawn. The region of high Vertical

Fig. 25.
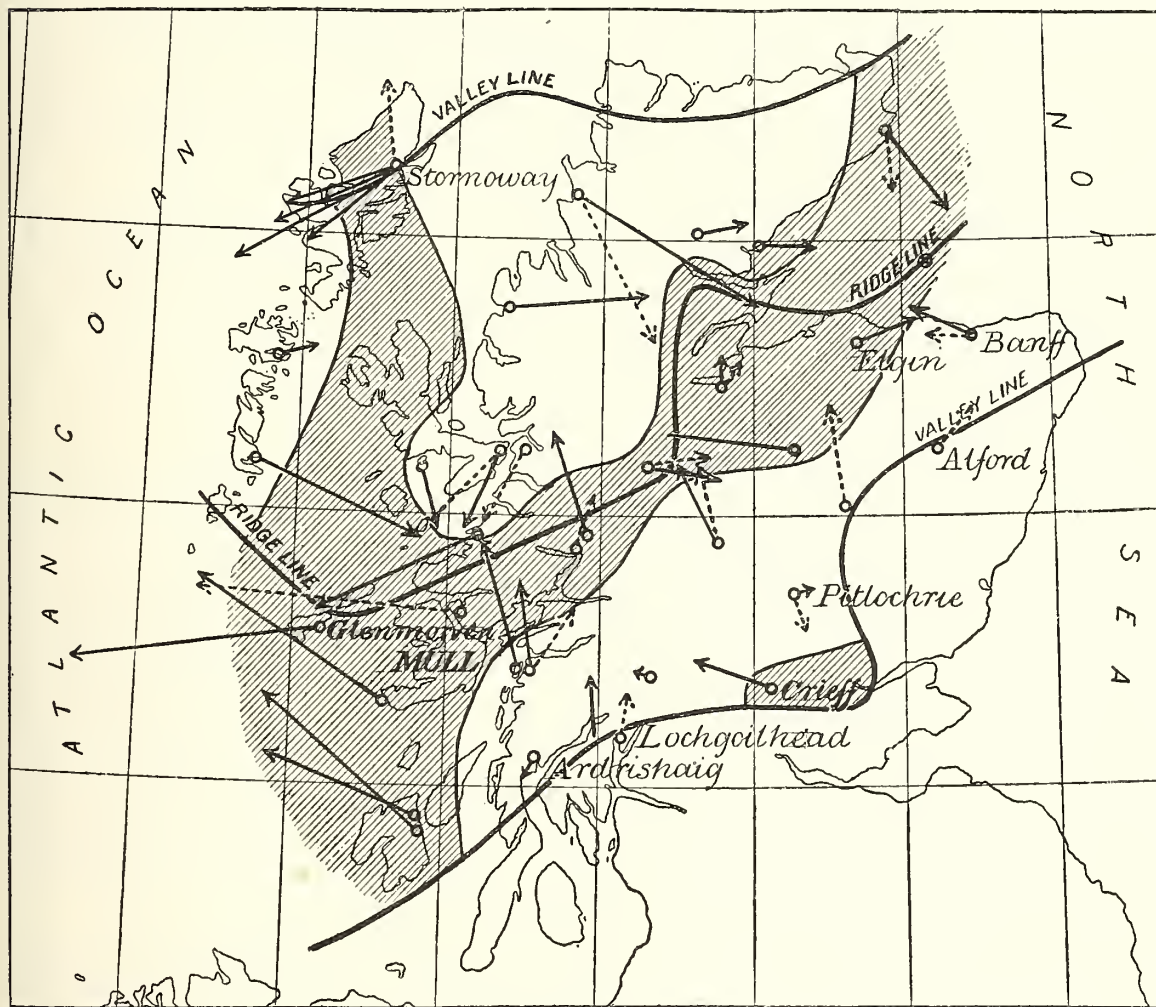
Highland District.

Force near Crieff seems to belong to the next district. At Crieff itself the disturbance of the Vertical Force is a maximum, and there is probably a peak in the neighbour-hood. The direction of the Horizontal Force cannot therefore be deduced from the Vertical Forces, and the fact that it happens to act northwards has perhaps led to its being wrongly included in this district. If the valley line runs direct from Alford to Lochgoilhead it would pass close to Pitlochrie. Our result would thus make its

relations uncertain, as the direction of the Horizontal Force would lie nearly along the valley line. The direction deduced from WELSH's observation would place it in the next district. The same remark applies, though more doubtfully, to Ardrishaig.

If, however, these three or four border stations be put aside, the map leads to a consistent view of the magnetic state of the district, which is in exact accord with that previously arrived at. The ridge line lies in the region of greatest Vertical Force disturbance. The Horizontal Forces tend towards that region, and at points within it are, on the whole, directed to the ridge line. The discrepancies between our results and those of Mr. WELSH are hardly, if at all, greater than those between our own, when repeated. It is remarkable that the south-western stations indicate a centre of attraction out at sea, but our results are completely confirmed by WELSH's at Glenmorven. We should certainly have expected *a priori* that Mull, which is highly basaltic, would be a centre of attraction.

Between Elgin and Banff there is a strong local centre. WELSH observed the Horizontal Force at Banff only, but the Declination obtained by him at Elgin is less than that at Banff, the difference being about 10', as against 7' given by our survey. The two sets of observations are thus in agreement.

The Scotch Coal-field District.

This district is bounded on the north by the valley line which forms the southern boundary of that which has just been discussed.

The southern boundary is inclined to the magnetic meridian at an angle which does not give any special advantage to either the Horizontal Force or the Declination as a means of determining its position. We have therefore taken the Declination line corrected (as described on p. 296) by WELSH's observations at Makerstoun and Melrose.

There is not a single abnormal station in this district, and the results of WELSH's observations fit in very well with ours. At Ayr, indeed, there is a very considerable discrepancy, and the forces at Alford and Gretna are larger than we should have expected if they are regarded as the resultants of the attractions of the two regions, close to the boundaries of which these stations are situated.

All the four stations at which the Horizontal Force is a minimum, viz. (taking them in order from the west), Cumbrae, Row, Stirling and Dundee, are situated close to the northern boundary of the Scotch Coal-field, in which there are large masses of basalt. The ridge line runs through the middle of this basaltic district and passes through or near two regions in which the disturbance of the Vertical Force is positive (*i.e.*, in which the force is great) to the east and west respectively. At Cumbrae the Vertical Force is a maximum, and the existence of a region of high Vertical Force in this neighbourhood is remarkably confirmed by the fact that WELSH's observations give a positive disturbance at Ardrossan a few miles further south, though at the neighbouring stations both to the south and north it is negative (fig. 23). On the whole then

there can be no doubt that in this district the ridge line passes through the region of greatest Vertical Force disturbance.   It obviously does so in the east.   In the west the region of positive disturbance is defined by a single station only, and its boundaries are uncertain.   If we included WELSH's observations at Ardrossan they would be pushed further south.   In the central region there are clear indications of a maximum disturbance, though the largest values are negative.   Thus at Row and Stirling the

Fig. 26.
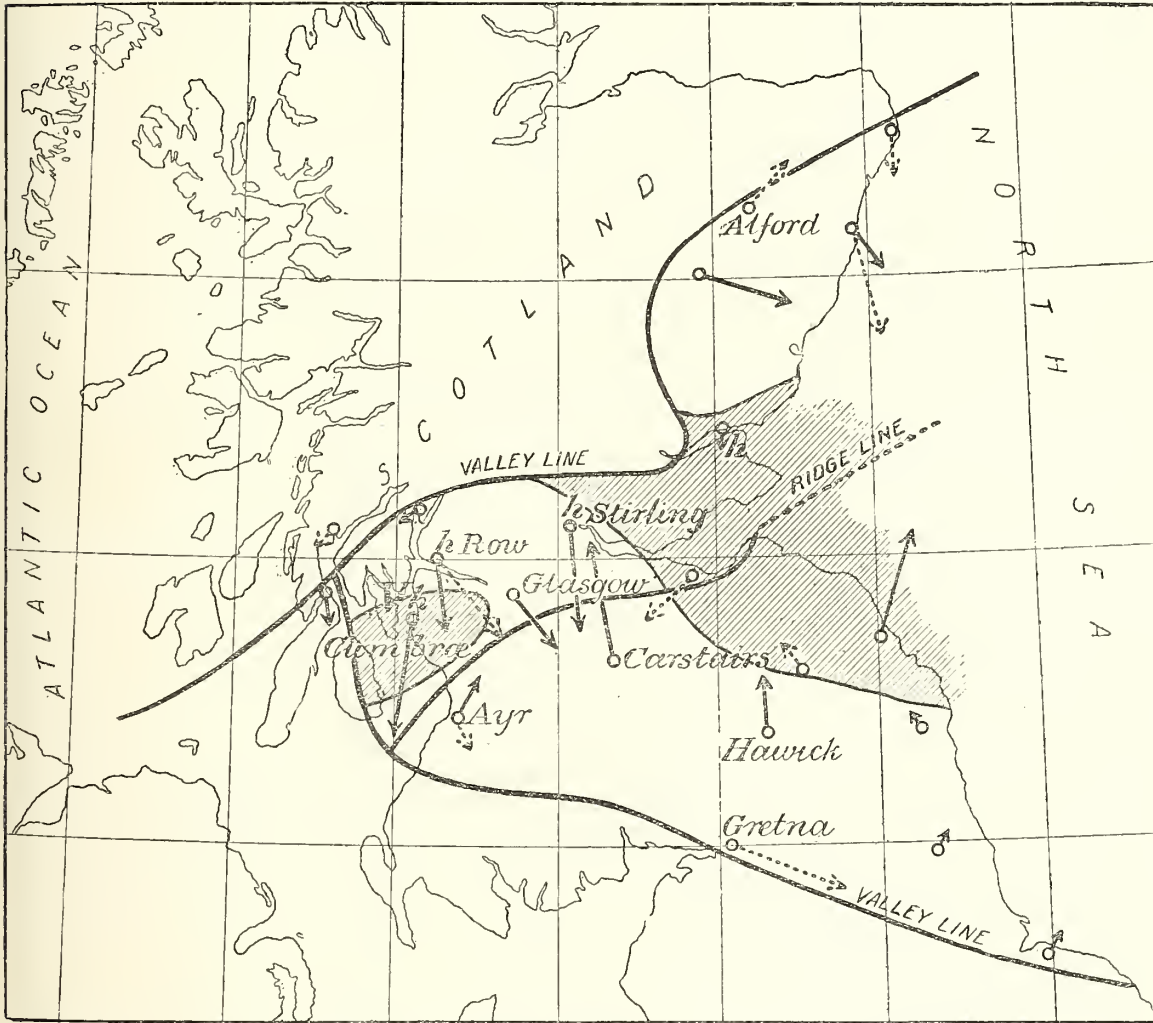
Scotch Coal-field District.

Vertical Force disturbances are — ·0188 and — ·0092 ; at Glasgow and Carstairs — ·0080 and — ·0079 ; and at Ayr and Hawick — ·0145 and — ·0099.   The two central stations which are nearest to the ridge line have algebraically the largest values.

The convergence of the Horizontal disturbing Forces towards the ridge line is unmistakeable.   The isogonals (Plate V.) give evidence of a subsidiary centre of attraction near Lochgoilhead.

North-Western England, North Wales, and Galloway.

The stations in this group are bounded on the west by the northern part of the Irish Sea. They are few in number, and so large a portion of the intervening spaces is covered with water that we cannot hope to unravel the intricacies of the district.

Fig. 27.
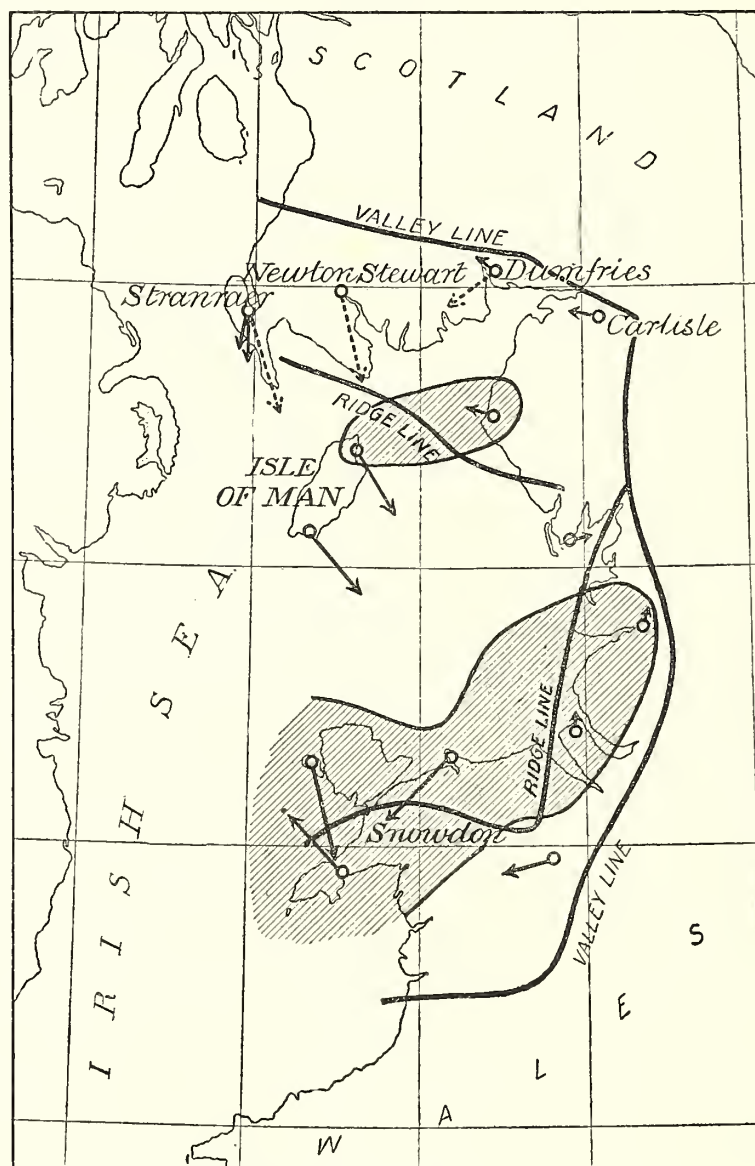
North-West England, North Wales, and Galloway.

The valley line which bounds it on the east has been drawn through the centre of the region of low Vertical Force which exists there (see Plate XI.). The southern part is obtained from the Horizontal Force disturbances (see Plate X.).

At Dumfries and Carlisle the disturbing forces are small, and coincide in direction with the valley line, *i.e.*, they fairly represent the resultants of two attractions exerted on each side of it.   At Stranraer, and in the Isle of Man, our disturbing Horizontal Forces act towards the south, and this result is confirmed by those

Fig. 28.
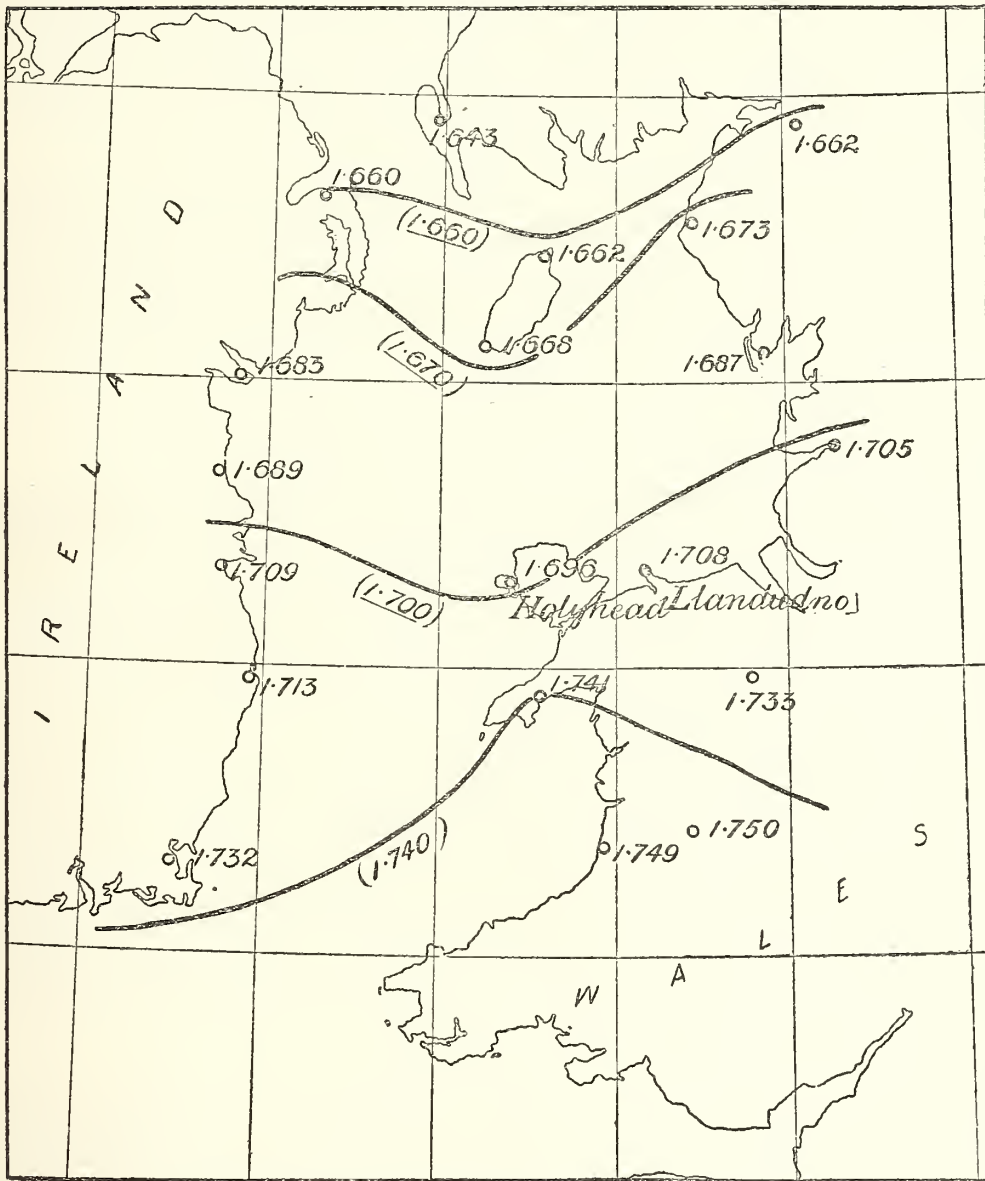
North-West England, North Wales, and Galloway.

deduced from WELSH's observations at Stranraer and Newton Stewart.   They are much larger than ours, and at Dumfries the southward tendency is more marked.   It must be remembered, however, that the calculations by which these results have been deduced from the 1857 survey have been conducted on the assumption that the

Scotch isomagnetics are linear functions of the coordinates which determine the geographical position, and that the errors involved in the assumption would chiefly affect stations such as these, which are on the border of the district.

If it is necessary to prove that these results are independent of the method of calculation, Fig. 28 is sufficient for the purpose.

The lines of equal Horizontal Force converge in a most remarkable way towards North Wales, which is one of the most certain indications of a centre of attraction. Again the Declinations at Holyhead and Llandudno differ only by 0′·4, while the difference of longitude corresponds to 25′. This again is strong evidence of the existence of a centre of attraction between them.

If, as appears probable, there are two regions of high Vertical Force in the district, it is possible that the northern one may be connected rather with the Cumberland Lakes than with North Wales. However this may be, and while fully admitting that it requires further study, we think that this district appears to obey the rules which hold good elsewhere.

The North-Eastern District.

This district is bounded on the north and west by valley lines which have been already described. On the south we have drawn a line which passes through the centres of two regions of negative Vertical Force disturbance.

The southern portion lies within the limits of our special surveys and has been so fully discussed that nothing need be added here. The northern part presents a feature of peculiar interest.

The results as directly observed do not furnish such clear indications as in a district where the presence of crystalline rocks is more obvious. The isogonals are, however, bent in the peculiar manner which indicates centres of attraction. Thus the 19° 6′ line (Fig. 29) points to such a centre to the south of Thirsk, the 18° 48′ isogonal points to another between Hull and Gainsborough. As both these lines depend in part upon the same stations, it is satisfactory to find that they are completely confirmed by the quite independent 18° 16′ line which runs nearly due east and west between Lincoln and Mablethorpe.

The Vertical Force isomagnetic which corresponds to 4·516 metric units bends southward to Thirsk in a way that is again an indication of a centre of attraction. There are thus signs of an attractive region lying between the following pairs of stations, viz., Mablethorpe and Lincoln, Hull and Gainsborough, Thirsk and Leeds.

On turning to the disturbances (Plates IX. and X.), we find that both the Declination and the Horizontal Force give nearly coincident ridge lines running along the line just indicated. The declination line passes northward to the Cumberland Lakes and southward to the Wash. The Horizonal Force line turns south amid the Yorkshire Hills and runs toward North Wales.

As we should expect from the close agreement of the Declination and Horizontal Force ridge lines the disturbing forces in this district are very easy to interpret. In no place in the kingdom is a locus of attraction more clearly indicated. At Appleby, Thirsk, Hull, and Mablethorpe the disturbing forces act in a south or south-easterly direction; at Giggleswick, Leeds, Gainsborough, and Lincoln they point north-east.

A well-marked ridge line thus runs from the Lincolnshire Wolds through Yorkshire and the limestone district of Westmoreland to the Cumberland Lakes.

Fig. 29.
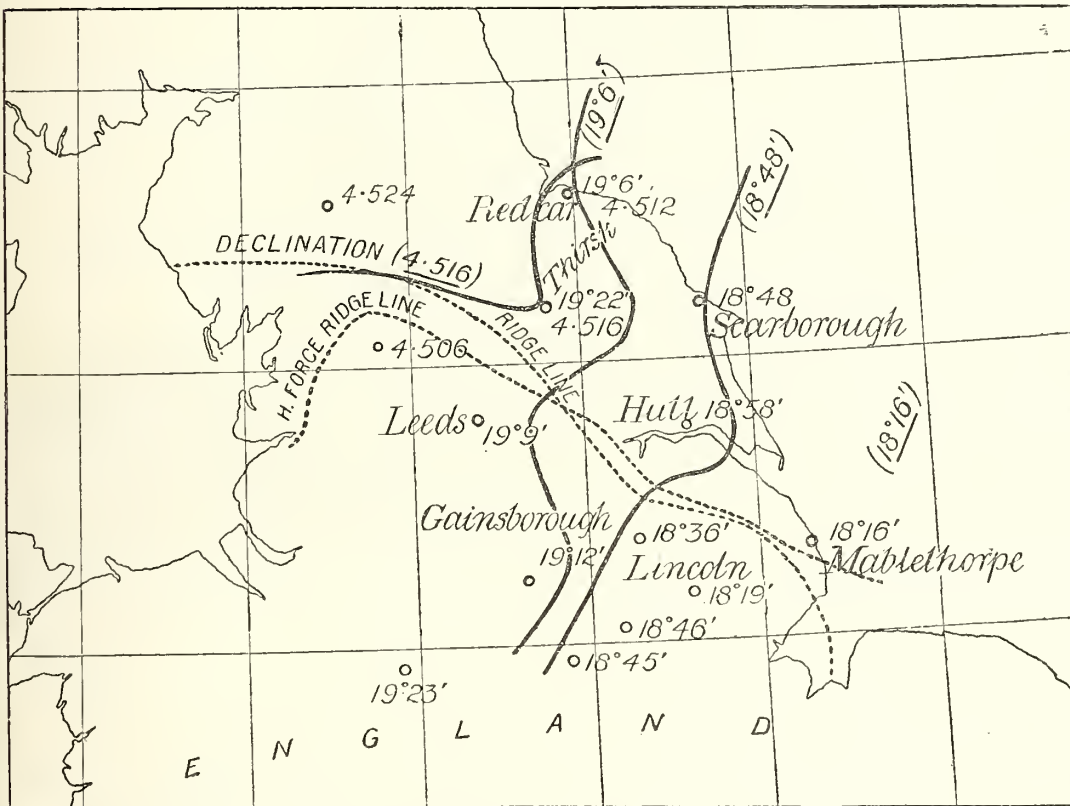
North-Eastern England.

In the southern part of its course it traverses a region of high Vertical Force, and passes near the station at which the positive disturbance is a maximum, terminating in the Wash peak.

The observations at Manchester and Chesterfield appear to indicate another centre of attraction in the limestone district of Derbyshire, but two stations are hardly sufficient to decide such a point. It is, however, very significant that the limestones of Derbyshire are intercalated with the basaltic rocks known locally as "toadstones," and although these do not cover a great area at the surface it is by no means

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improbable, as Professor JUDD informs us, that large masses of similar rocks occur at no great depth.

Fig. 30.
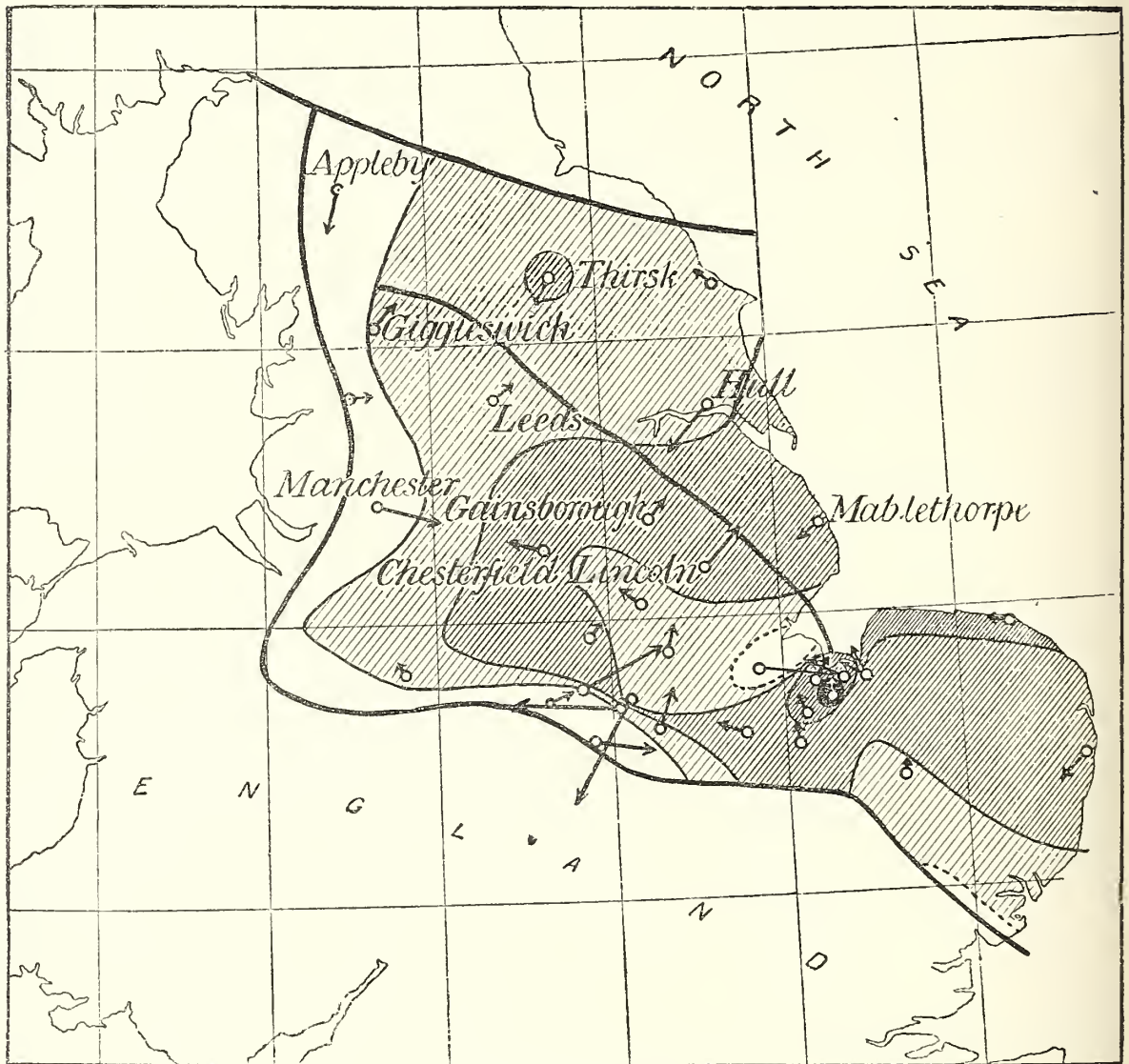
North-Eastern England.

The Southern District.

The Southern District contains those parts of England and Wales to the south of the regions which have already been discussed with the exception of Devonshire and Cornwall. In the latter outlying district the disturbing forces indicate on the whole an attraction to the south of the peninsula.

The Home Counties have been already discussed, and we only refer to the district again to point out that the ridge line which runs westward through the Reading peak

is continued into the Welsh coal-fields.   This fact is very important, and will be hereafter discussed when we consider the relations between the magnetic and geological peculiarities of this district.

Fig. 31.
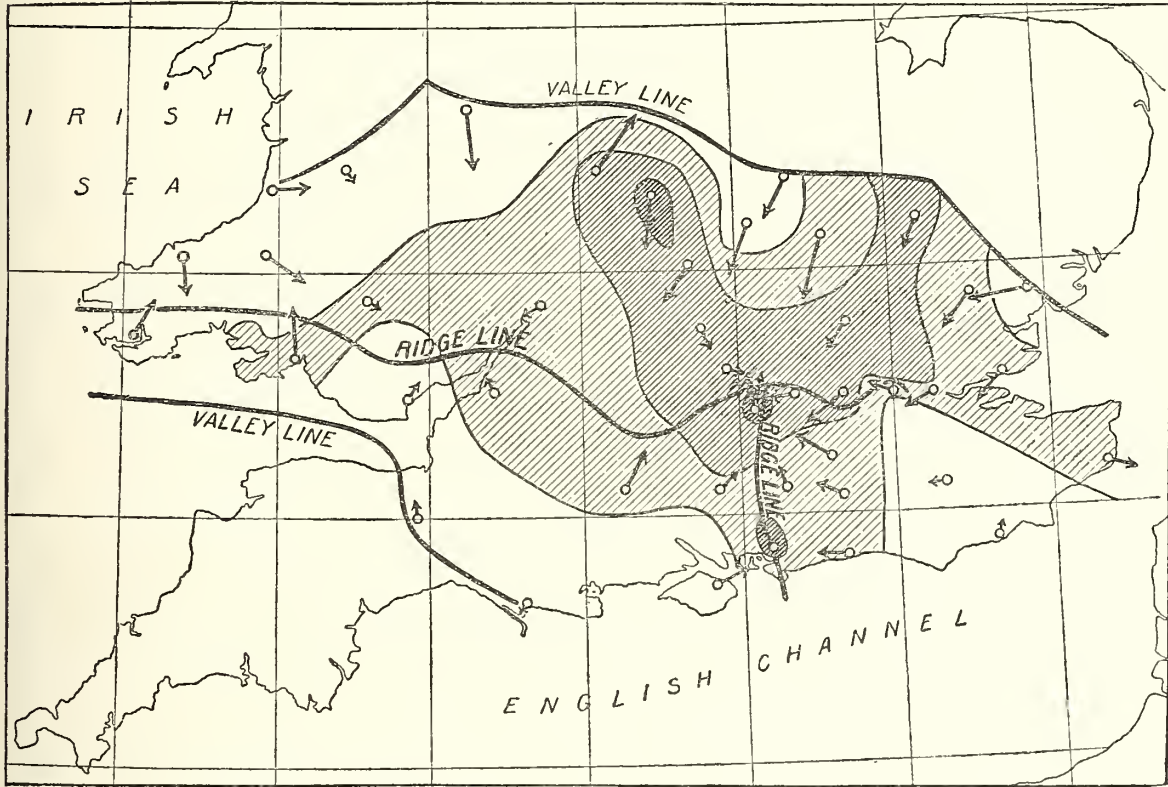
Southern England and Wales.

Ireland.

In Ireland, as elsewhere, the Horizontal Forces tend towards the regions of greatest Vertical Force.

At Coleraine and Waterfoot, stations to the north of the great mass of basaltic crystalline rocks in Antrim, the disturbing forces tend southward.   At Cookstown Junction and Bangor, on its southern borders, they are directed to the north.

Kells is a point of maximum Vertical Force, and the forces at neighbouring stations are all directed towards it.

A clearly marked ridge line runs through a region of positive Vertical Force disturbance in Connemara, and extends to the north beyond it.

Another series of stations of high Vertical Force extend eastwards from Clare, and they apparently dominate all the region to the south of them.   The direction of the disturbing force at Kilrush is anomalous.   At Charleville there is a minimum of

Vertical Force, and therefore the direction of the Horizontal Force cannot be determined from the rule.

The Wicklow and Arklow mountains, which are composed of granite, do not appear to exert any important effect on neighbouring stations, but there is a region of high Vertical Force in Wexford.

Magnetic Map of the United Kingdom.

In Plate XIII. we have attempted to represent the magnetic state of the whole country by bringing together the results of our studies of the magnetic districts into which it may be divided.

The valley and ridge lines are shown. The unshaded parts are regions of negative Vertical Force disturbance, and the three shades employed indicate that the disturbance is greater than 0 but less than ·01, greater than ·01 but less than ·02, and greater than ·02 metric unit respectively.

The lines representing the Horizontal Forces are drawn to a scale on which 1 mm. = 0·001 metric unit, except at Canna and Soa, where the disturbances are so large that the arrows would be inordinately long.

Considerations which have been adduced in the foregoing discussion have led us to depart, in a few minor points, from the strict rules by which the valley and ridge lines have been drawn. Thus, the line which separates the Highland and Scotch Coal-field Districts has been drawn so as to include Crieff in the latter.

The rather uncertain valley line in Mid-Wales, which was taken from the Horizontal Force disturbances (Plate X.), is replaced by the closely neighbouring ridge line, taken from the Declination disturbances (Plate X.). This would necessitate valley lines on each side of it, the position of which is uncertain.

In Ireland too, the ridge line which runs from Antrim to the neighbourhood of Kells, is not very definite, and it is best to consider the clearly marked portion of it in Antrim as only doubtfully connected with the centre of attraction near Kells. These points must be left for future investigation, but the unquestionable existence of widespread regional disturbance in the districts we have specially studied in England, together with the general agreement between the two maps of Scotland, deduced from the surveys of 1857 and 1886, leads us to hope that Plate XIII. gives the first approximation to a map of the disturbing magnetic forces in play over the whole kingdom.

On the Relation between the Magnetic and Geological Constitution of the Magnetic Districts.

Up to the present, we have discussed the various districts into which we have divided the country from the point of view of their magnetic peculiarities only.

It now remains to investigate the question whether any connection can be established between these and their geological characteristics.

It is well known that certain varieties of crystalline rocks are often more or less magnetic, and that when they are permanently magnetised the poles are sometimes very irregularly distributed.

We have, however, thought it worth while to investigate the magnetic state of some pieces of diorite and basalt, brought from Malvern and Canna respectively.

The fragments were cut into small rectangular blocks, delicately suspended and tested. We have to thank Dr. HOFFERT, Demonstrator in the Physical Laboratory of the Science Schools at South Kensington, for undertaking this part of the work. The observations were very tedious, and were, for the most part, made by Messrs. GRAY, ANDERSON, and WILKINSON, students in the laboratory.

Of several blocks from Malvern only one showed any polarity in its natural state, but when placed between the poles of an electromagnet it became magnetised by induction, so that the time of oscillation was reduced from 72^s to 56^s. The stone brought from Canna was part of a basaltic column and its upper, lower, east and west ends were marked. It showed polarity, but the upper end was a north-seeking pole, so that it was magnetised in a direction opposed to that which would be induced by the magnetic field of the earth. The moment due to the permanent magnetism was calculated by three different methods, viz.: (1) by the difference of time of oscillation when the direction of the field (about twelve times as strong as that of the earth) was reversed, (2) by the deflection when the stone was placed E. and W., and (3) by the difference of times when the position of the stone was reversed. The results obtained were

$$\cdot 0023 \quad \cdot 0027 \quad \text{and} \quad \cdot 0019 \text{ C.G.S. unit.}$$

The periods of oscillation were 93^s and 86^s in the earth's field and the artificial field above described. The general conclusion arrived at was that, as the volume was about 1 c.c., the permanent intensity of magnetisation was about 0·002 C.G.S. unit, and that in a field of strength F the induced intensity was about 0·0015 F.

We have examined the relations between the magnetic disturbing forces and the geology of the area of the survey by means of a geological map which has been specially prepared for us under the kind superintendence of Professor JUDD (Plate XIV.). Details are disregarded, but the principal masses of basaltic and non-basaltic crystalline rocks and the main groups of the sedimentary formations are clearly distinguished from each other. This may be compared with Plate XIII., in which the magnetic disturbing forces and the ridge and valley lines are shown.

As the ridge lines are drawn according to an arbitrary rule, they are only intended to draw attention to the districts in which loci of attraction probably exist. We have no real knowledge of their distances from the stations between which they run. In the geological map, therefore, the stations on each of the principal ridge lines have

been connected by lines, the spaces enclosed have been shaded, and thus the districts within which the main loci of attraction probably lie are clearly indicated.

It must be remembered that the outlines of these districts depend largely on the accidental positions of stations which were selected without reference to them, and that we must rather expect such rough indications of relations between magnetic and geological facts as may serve to guide future investigations than complete and unmistakeable harmony.

When regarded from this point of view, however, we think the results are very suggestive.

In the Highlands one region of attraction encloses the Caledonian Canal, another is evidently in close relation with the basaltic masses in Skye, Glenmorven, and Mull, though, as has been pointed out, the main centre of attraction appears to be to the west of these islands.

A third region in Scotland encloses the basaltic rocks of Arran and of the Scotch Coal-fields. The fact that the Fifeshire basalt lies outside it is probably due only to the accidental circumstance that we have no stations between Stirling (47) and Dundee (21).

A fourth region evidently consists of the Antrim basalt. There is a fifth in Connemara, where the rocks are granite.

With regard to North Wales, if we consider it as forming one district with Shropshire, we see that a line drawn through the centre of the basaltic rocks would first run from east to west, then nearly north, and finally turn to the west between Anglesea and Carnarvonshire. The district is thus very irregular and the ridge lines do not give much information, but we must point out that the Horizontal Forces at the stations which border it all tend inwards towards the axis above suggested. This is true of Holyhead (90), Llandudno (106), Llangollen (107), Shrewsbury (138), Aberystwith (55), and Pwllheli (128).

If this suggested relation is hereafter verified, every considerable mass of basaltic rock in the kingdom will be closely connected with a region of magnetic attraction. Of smaller masses it is to be noted that at Falmouth (80) the disturbing force acts southward toward the serpentine of the Lizard, and that the relatively small masses in Pembrokeshire and Wexford are within another region of attraction. Our measurements do not assign any particular importance to the outcrop of basaltic rock near Limerick (185) nor to the dykes in the north of England. It is, however, curious that a line drawn through Melton Mowbray and the Wash peak (indicated by circles) passes towards Wales through the only basalt in the Midlands.

Taking the evidence as a whole, we think we are justified in saying that large masses of basaltic rock indicate regions of magnetic attraction.

The other crystalline rocks appear to be much less important magnetically. Thus the Malverns, though a strong local centre, do not disturb a district of any considerable magnitude. If the effects of the two classes of rocks when they appear on the

surface are in such marked contrast, it may be open to question whether strong magnetic attraction in a district in which no crystalline rocks appear on the surface does not indicate not only crystalline but basic crystalline rocks beneath it.

However this may be, there are two regions of attraction which are not connected with basaltic rocks on the surface, at all events as the main cause of their peculiarities.

One of these runs westward from London to the South Wales Coal-field, and the directions of the disturbing forces in Wexford are such as would be caused if it crossed the Irish Channel. This extension is doubtful, but the line is most clearly marked right across England, and its general direction is such as to make it almost impossible to avoid the conclusion that it is connected with the palæozoic ridge, the existence of which, long ago predicted by Mr. GODWIN AUSTEN, has been proved by deep borings near London, and which is supposed to connect the Welsh and Belgian Coal-fields.

If the palæozoic rocks are nearer the surface here than elsewhere, the crystalline rocks may approach it also within a moderate distance, and if they are susceptible to magnetisation the observed results would follow. The palæozoic rocks are supposed to form basins, and if those beneath them have a similar outline, it would be possible to explain a centre of attraction such as the Reading peak.

The last region of attraction runs from the Wash through south-east Yorkshire toward the Cumberland Lakes, and, as Professor JUDD first pointed out to us, it includes the line to the north of the Humber, along which the oolitic and liassic strata thin out rapidly, and where, therefore, the crystalline rocks are probably suddenly brought much nearer to the surface. It is continued northward toward the igneous masses in Cumberland.

The centre of attraction which apparently exists at Kells, in Ireland, is not sufficiently accounted for by the surface geology of the district in which it is placed, and we have not felt ourselves justified in representing it as connected with the Antrim basalt, as the fact that this connection requires further confirmation has been already pointed out.

On the whole, then, we think we may assert that every region of magnetic attraction, with the possible exception of that near Kells, is marked either by the presence of basalt, or by some geological peculiarity which makes it possible or even probable that within it crystalline rocks, capable of affecting the magnet, are nearer the surface than elsewhere. This is possible as regards the line of fault marked by the Caledonian Canal, and probable as regards south-east Yorkshire and the south of England.

The former district is remarkable from the fact that it was far more strongly affected by the earthquake of Lisbon than the rest of the British Isles, and this may, perhaps, indicate that it has special relations to the primary rocks, which would account for its magnetic importance.

On comparing the geological map with Plate XIII., on which the disturbing forces are shown for the whole country, a difficulty arises from the fact that the Vertical disturbing Forces appear, on the whole, to be greater over the districts in the east and south of England, where the later sedimentary rocks occur, than over Wales and Ireland, where, from their absence, it would *primâ facie* appear probable that the downward attraction would be greater.

Of course many hypothetical explanations could be offered of the fact, such as that the primary rocks in England might possibly contain larger quantities of ferruginous matter, &c., but we must be content with observing that if a fairly uniform increase in the disturbing Vertical Force were to take place from east to west, it is very doubtful whether we should have detected it. Probably it would cause a disturbance of the first order, the terrestrial lines would be deflected, and the disturbances at distant points would not be comparable.

If the true Vertical Force isomagnetics could be prolonged beyond Wales into another district in which the tertiary strata re-appeared, a southward trend in Wales would indicate an increase in the force. On looking at Plate VIII. we think it will be admitted that the slope of the lines of equal Vertical Force is greater in Wales than in England, which is in harmony with the existence of a southward bend, but the fact that they are extremely irregular and terminate on the west in the sea makes any certain deduction impossible. We must, therefore, regard the distribution of the Vertical Force disturbances as presenting some difficulty, and must emphasise the necessity of using them only to compare neighbouring stations.

The Causes of Local and Regional Magnetic Forces.

It has long been known that distortions of the isomagnetics occur chiefly in the neighbourhood of crystalline rocks, and it has been generally assumed that this is due to so-called rock magnetism, the rocks being magnetised either permanently or by the inductive action of the earth's field.

Dr. NAUMANN, in the work already quoted ('Die Erscheinungen des Erdmagnetismus,' Stuttgart, 1887), has recently opposed this view.

The arguments which may be brought against it are : (1) That rocks brought from considerable depths do not exhibit magnetic qualities until they have been for some time upon the surface; (2) That the effects which rocks or mountains produce on the magnet, even if very great when the distance is small, diminish so rapidly as the distance increases that they are quite insufficient to account for the widespread effects which are attributed to them; (3) That extensive local magnetic disturbance is associated rather with geological faults than with the presence of igneous rocks; (4) That the cause of the phenomena is to be looked for in the effects produced on

earth-currents by dislocations of the strata rather than in rock magnetism. We will consider these arguments in the inverse order to that in which we have stated them.

Mr. Preece, F.R.S., Chief Electrician to the General Post Office, has been good enough to have measurements of the earth-currents made in several of the districts in which we have found large disturbances of the Declination. The currents flowing between various post-offices have been observed, and the directions and intensities of the currents noted.

The following Table gives the data obtained on telegraph lines between Melton Mowbray and stations in its neighbourhood. The letters P.D. signify Potential Difference.

Station.	Bearing from Melton.	Distance from Melton in miles.	Earth currents in milliampères.	Direction of current.	Resistance of circuit in ohms.	P.D. per mile in volts.
Long Clawson	N. 17° W.	6·0	0·062	From	912	0·009
Oakham	S. 50° E.	9·0	0·045	Melton	477	0·002
Asfordby	W.	2·5	0·070	in all	317	0·009
— „ (Private Line)	W. 15° N.	1·9	0·067	cases	317	0·011

These potential differences per mile are much smaller than those which occur during magnetic storms, and which on the earth-current theory must, we suppose, be regarded as the causes of the deflections of the Declination needle which then take place. During a very violent storm in 1881 Mr. Preece found a P.D. of 1·9 volt per mile. We will take for comparison a less extreme case. Yearly records of the movements of the Declination needle, and of the registers of the earth-currents apparatus, are published by the Greenwich Observatory, and the Astronomer-Royal informs us that half an inch on the earth-current registers corresponds approximately to a P.D. of one volt in circuit.

On September 10, 1886, an increase of 16′ took place in the Declination between 22^h and $22^h{\cdot}5$, and between $21^h{\cdot}8$ and $21^h{\cdot}2$ a change in the intensity of the current in one of the circuits occurred which corresponded to a change of P.D. of about 0·9 volt. The direction of the current was such as to produce the observed movement of the needle, and if we regard the current as its cause, since the distance between the earth plates is 3 miles, we find that a deflection of 16′ was produced by a potential difference of 0·3 volt per mile.

There are two earth-current circuits mutually at right angles, both of which are inclined at about 45° to the magnetic meridian. The second was but slightly affected, and, therefore, the effective difference of the potential was about 0·3 cos 45° = 0·2 volt per mile nearly.

At Melton Mowbray we find the disturbance of the Declination to be 33′ W. at one

of the two stations. It changes very rapidly, falling to 26′ W. within a mile and a quarter, and to nearly 30′ E. at Loughborough, which is thirteen miles distant. If currents produce it they must therefore be very local. On the other hand we know that the disturbances produced during magnetic storms are simultaneous over areas such as that of the United Kingdom, and if the currents produce the deflections, they also must be widespread, so that if we assume that the same earth-current is required at Melton to produce a permanent deflection as is observed at Greenwich when an equal temporary deflection takes place, we are not overstating the case.

Now, during the whole of the year 1886, there were only a few occasions on which the earth-currents were stronger than on that which we have selected as an example, and we may therefore say that on the earth-current theory there should be permanently at Melton a potential difference of 0·3 volt per mile (for the smaller deflection of 26′ is 1·6 times that observed at Greenwich on the selected occasion) an amount which is only registered at Greenwich during violent storms, and is nearly thirty times greater than that observed in a circuit only two miles in length in the neighbourhood of Melton itself.

So far we have dealt only with the magnitudes of the currents. The case becomes much stronger when we consider their directions. On all the circuits out of Melton Mowbray referred to in the table on p. 315, the earth-currents flowed from that station. Hence, on all the currents except that to Oakham, the direction of the current was such as to produce a deflection of the North Pole to the east, i.e., in the opposite direction to that which was actually observed, while the P.D. between Oakham and Melton was the smallest of those measured, being only 0·002 volt per mile.   We do not lay stress on this, as the distance was perhaps rather too great, the main fact being that the earth-currents between Asfordby and Melton, on the same side of the latter town as that on which our second station was placed, were not only much too small (if we may judge from what is observed at Greenwich) to produce the observed deflection, but that they were actually in the wrong direction.

To make certain that nothing in this argument depends upon the particular deflection obtained at Greenwich on the selected occasion, we have taken twenty other examples in which Declination disturbances of from 3′ to 22′ were accompanied by changes in the earth-currents. To avoid the necessity for correcting for diurnal variation they were chosen from the nocturnal hours when the magnet is normally steady, and occasions were selected on which one circuit only was appreciably affected. Except in these particulars they were chosen haphazard.

If we divide the E.M.F. per mile in volts (V) by the Declination disturbance (Δ) we get a number which expresses a relation between the two, and which, if they were cause and effect, ought to be constant. The following table shows that the selected example was not particularly favourable to our views.   As the first station at Melton was on the south side of the town we assume that the potential difference was the same as that between Melton and Oakham.   At the second station we take it to

be that between Melton and Asfordby. In both cases the whole is supposed to be effective, *i.e.*, it is not resolved along the magnetic meridian. A negative sign means that the current flowed in the direction opposite to that which would be required to produce the observed deflection.

Station.	
Greenwich . . . $\left\{\begin{array}{l}\\ \\ \\ \\ \end{array}\right.$	Largest value of V/Δ . 0·028
	Smallest „ „ . 0·008
	Mean „ „ . 0·016
	Value on selected occasion 0·013
Melton Mowbray . $\left\{\begin{array}{l}\\ \\ \end{array}\right.$	Station 1 . . . . . . 0·00006
	„ 2 . . . . . . 0·00040

We are quite aware that an argument of this sort is open to criticism. The exact relation between earth-currents and magnetic storms is uncertain; in short circuits the earth-currents may be masked by those due to the earth-plate, and we have been compelled to assume that the conditions at Greenwich and Melton are the same. We should not, therefore, have put the argument forward had the conclusion depended on any nice balance of figures. As it is, we think it supports the view that the permanent earth-currents at Melton Mowbray are very much less, and less extended, than the temporary currents observed at Greenwich during ordinary but considerable storms, though the permanent Declination disturbance which they are supposed to produce is of the same order as the temporary deflections which are observed simultaneously with the currents at Greenwich.

Similar results were obtained near to the Reading and to the Wash disturbances. If these are produced by earth-currents they must circulate round the peaks, and thus the potentials at points on opposite sides of them should be different. Near the Wash the direction of the telegraph wires is not very favourable for a test, but we owe to the kindness of Mr. PREECE, to whom we must again express our obligation, a series of measurements made at Reading and Windsor. Of these it is only necessary to say that, though Reading and Windsor are on opposite sides of the focus of disturbance, and though the needle is deflected 11′ to the west at Windsor and 6′ to the east at Reading, so that the assumed current circulating round the peak ought to run in opposite directions through them, the observed earth-currents were so small that no measure of their magnitude could be taken.

It might be urged, in answer to these arguments, that the earth-currents by which local disturbances are produced are not mere surface currents, but that they flow through the mass of the earth, possibly where the strata are (as recently suggested by Professors LAMB and SCHUSTER) of higher conductivity.

To this it may be replied that the greater the depth at which the currents are

supposed to be situated, the more difficult is it to account for the fact that their effects are so local.

A more conclusive answer is the fact that our observations show clearly that the needle is attracted to certain lines or points round which currents would have to circulate in the same direction (*i.e.*, with the hands of a watch) in order to account for the facts. If the magnetic field of the earth is due to currents flowing from east to west, it is easy to imagine that they might be deflected by layers of more or less than average conductivity ; but why should they form local eddies, in which the flow is always in the same direction ? They must circulate similarly round the Reading and Wash peaks, round Kells, probably also round the Antrim basalt and the Toad-stones of Derbyshire. In accordance with Commander CREAK's investigation, they must flow round magnetic islands in opposite directions in the two hemispheres. They must flow in different directions on the two sides of the palæozoic ridge in the southern counties, of the Malverns, of the Yorkshire ridge, of the Scotch coal-field, and of the Caledonian Canal. It may be possible to imagine physical causes which would account for such a state of things, but it does not appear easy to frame an hypothesis which shall be more probable than that involved in the theory of magnetic rocks.

As regards Dr. NAUMANN's view, that geological faults determine local magnetic action, it is disputed in its application to Japan by Dr. KNOTT, who has superintended the recent magnetic survey of that country. We find at Malvern a mass of crystalline rock bounded by a fault. Either the rock or the fault may be supposed to be the cause of the attraction toward the ridge which is undoubtedly exerted on the needle ; but whereas the rock is susceptible of magnetisation, and its effect on the magnet is precisely such as would be produced if it were magnetised by induction in the earth's field, there is no shred of direct evidence to prove that the fault is capable of causing or deflecting earth-currents, so as to make them flow in opposite directions on its opposite sides. It is more probable that the action of faults is due to the displacement of crystalline rocks in their neighbourhood ; and that a fault is, at all events, not a necessary cause of regional attractions is proved by our observations in Scotland. Two great fault lines traverse that country, one coinciding with the Caledonian Canal and the other running from Stonehaven to the mouth of the Clyde. The first of these is, and the second is not, associated with a locus of attraction.

Taking the next argument in order, we agree that mountains which exercise considerable influence on the magnet when close to them produce no effect at distances which are small relatively to the range of regional disturbances. It must, however, be remembered that the depth from the surface to magnetic rocks concealed by over-lying strata may not exceed the horizontal distances at which the Malverns and the Wash affect the Declination, and that their influence on the Vertical Forces might extend over vast areas. If, as would often be the case, they were not horizontal, but approached the surface by a gentle slope, the magnet would certainly tend to turn

towards or away from the nearer and more elevated portions, according to the nature of their magnetisation.

If the slope were less than the angle of Dip, the sides of the sub-terrestrial hill would be magnetised, so as to attract the north-seeking pole, and the empirical rule that the Horizontal Forces tend toward places of greatest Vertical Force would be amply and completely explained.

Dr. NAUMANN's argument loses a great part of its force if we regard a widespread disturbance of the isogonals around a mountain range as due not to the direct action of the mountains but to a far reaching mass of rocks of which they are the culminating point.

Lastly, as to the statement that rocks containing iron only become magnetic when brought to the surface—without impugning the actual observations, we can only say that so important a generalisation requires a much more extended basis of fact than any that is provided for it. If the Malvern granites produce in their immediate neighbourhood a magnetic effect, it is evident that rocks, the susceptibility of which can only be detected by refined methods, may in large masses deflect the lines of force in the earth's magnetic field to an appreciable extent.

On the whole then, while fully admitting that there is room for much further investigation on this head, and that any view is more or less hypothetical, we do not think that any theory has hitherto been proposed which offers fewer difficulties than that of rock magnetism.

If the cause of the magnetic disturbance is induced magnetism, it would vary with the direction and intensity of the earth's magnetic field, but the change could hardly be great relatively to the secular variations of the elements. It is therefore important to note that in the 'Phil. Trans.' for 1870 (vol. 160, p. 274), Sir E. SABINE gives the Declination at Greenwich and Kew as having been practically identical in 1842·5. His values are :—

$$\text{Greenwich} \quad . \quad . \quad . \quad . \quad . \quad . \quad 23° \, 13',$$
$$\text{Kew} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 23° \, 11'.$$

As according to his isogonals the Declination at Kew should, at that epoch have been 9' greater than that at Greenwich, whereas, according to his table, it was 2' less, the difference due to disturbance was $-11'$ (*i.e.*, the Kew value was too small), whereas in 1860 it was $+11'$, in 1865 $+10'$, and is now 10' (p. 270). If then we accept Sir E. SABINE's table as correct, we must assume that a disturbance difference amounting in all to 22' was established between 1842·5 and 1860, which has remained constant during the twenty-nine years which have elapsed since the latter epoch.

We, therefore, wrote to Mr. WHIPPLE asking him for information as to the observations given by SABINE, as having been made at Kew in 1842·5. It should be noted that the authority quoted by SABINE is " Ob^y," which indicates an official origin for the value given.

Mr. WHIPPLE writes, " I have now looked over all the MSS. papers I can find, referring to this Observatory in 1842, and I am quite unable to find any reference to the observation quoted in the ' Phil. Trans.' for 1870, p. 274. There was no Staff here at the date, nor any official publication, hence I do not see how " Observatory," or, as printed, " Ob^y," could be any authority. Again, in SABINE's contribution, ' Phil. Trans.,' 1849, p. 208, Greenwich and Bushey are both given with the names of the observers, and I think there can be no doubt that if SABINE had observed here in 1842, he would have certainly quoted the observations in that table. . . . I fear we must consider it somewhat too hypothetical to trust to."

On the whole then, it appears to us most likely that the number given by SABINE was not deduced from an observation made in 1842, but was reduced to that epoch from the Kew Observatory records, which did not begin till some years afterwards, and that in the deduction some numerical blunder was made.

SUMMARY.

We may now attempt to sum up the result of our studies of local and regional magnetic forces in the United Kingdom.

We have proved beyond possibility of doubt that two centres of attraction exist near Reading and the Wash respectively, and in the former case it is evident that the Horizontal disturbing Forces tend towards regions of maximum Vertical disturbing Force.

We have then gone through the United Kingdom district by district, and have shown that the same rule holds good everywhere. Putting aside stations of maximum or minimum Vertical Force and stations on valley lines (which do not furnish real exceptions) there are not half a dozen which are clearly anomalous in the whole 200 ; nor has this result been obtained by splitting the whole area of the survey up into a large number of small districts which have been carved out so as to secure an appearance of uniformity of behaviour. Omitting the border stations on the south of Scotland, in Devonshire and Cornwall, the Channel Isles and Dover, the magnetic constitution of the rest of Great Britain is accounted for by five principal ridge lines only.

The first is coincident for a great part of its length with the direction of the great fault of the Caledonian Canal. The second and third are connected with the masses of basalt in the Western Isles and the Scotch Coal-field respectively.

The fourth runs for 100 miles parallel to, if not coincident with, a line in Yorkshire and Cumberland along which geologists believe that crystalline rocks may occur near the surface, and sends out a branch toward the igneous rocks which occur in North Wales.

The fifth runs for nearly 200 miles from London to Milford Haven along another similar geological line, and sends out a branch to the south coast from near Reading.

It is probable that to these a sixth in North Wales and Shropshire should be added.

There are indications of other minor centres at Malvern, in Derbyshire, and in the neighbourhood of Charnwood Forest, but the number of stations involved is small. The vast majority are included in the few and simple systems above described.

These results are not the outcome of the calculations only, for all the principal conclusions can be drawn from the observations alone.   Mr. WELSH's survey, though including fewer stations at which all the elements were determined than ours, though omitting almost altogether stations in the Western Isles, though worked up (as regards the Dip and Declination) by means of a different system of geographical coordinates, and in the case of all three elements by means of an assumption as to the linearity of the isomagnetics which we have abandoned, confirms our conclusions. In four only of the 28 of his stations which fall within our districts, viz., Stornoway, Pitlochrie, Ayr, and Dumfries is there an important difference between us as to the direction of the disturbing forces.

In the most highly disturbed districts, at Loch Inver, Kyle Akin, Broadford, Glenmorven we agree as to the order of the magnitude and as to the direction of the forces. At Cumbrae and Fairlie we find closely neighbouring stations of positive surrounded by others of negative Vertical Force disturbance, and WELSH confirms us by a similar result at Ardrossan, not ten miles off.

These coincidences between the results of the two surveys, and between the magnetic peculiarities and the geological constitution of districts cannot be accidental, and we venture to assert that, throughout the kingdom, the lines of Horizontal magnetic disturbing Force tend towards regions of maximum Vertical disturbing Force, which are themselves defined by the presence of crystalline rocks, and especially of basalt, either visible on the surface or concealed by superimposed masses of sedimentary strata.

Beyond this general conclusion we do not wish at present to go. The detailed constitution of our principal magnetic districts (except in the case of the Reading and Wash disturbances) has yet to be investigated.   We have not discriminated between various possible causes of a decrease in the Vertical Force, such as the removal of the attracting mass to a greater depth, or the formation of repulsive poles, either by induction or permanent magnetism.   We do not think the last word has been said on the cause or causes of local and regional forces.   All these require and will, in the future, no doubt, receive attention.   We trust, however, that by following out to their logical conclusion the premises of a not improbable working hypothesis, we have succeeded in showing that local and regional forces obey certain simple laws, and that by means of these the kingdom can be divided into magnetic districts in which the relations between the direction of the disturbing forces and the main geological characteristics are so suggestive as to be worthy of careful statement and further investigation.

TABLES FOR THE CALCULATION OF THE MAGNETIC ELEMENTS IN THE FUTURE.

We conclude this paper with Tables, by means of which the values of the magnetic elements may be determined during the next few years for any place in the United Kingdom.   For this purpose we should know as accurately as possible—(1) the values of each element at the station at the epoch of the survey ; (2) the secular change ; (3) the local disturbance.   To determine the first of these, we give the next three Tables (VIII., IX., X.), in which the values of the elements deduced from the general formulæ are given for all intersections within the kingdom of lines of latitude and longitude which correspond to whole degrees, together with the variation per degree of latitude and longitude.   The method of using them requires no explanation.

The values in the Table of Horizontal Forces display some irregularities.   This is partly due to the fact, that a discontinuity is introduced along the line H = 1·7, on opposite sides of which we have used different formulæ.   The formula also, which is applied to the southern district, does not represent parallel lines, but lines of which the slope is a periodic function of the latitude.   We have slightly smoothed the irregularities, a process which will introduce some discrepancies between the numbers given by the Table and by the formulæ, but we have thought it better not to attempt to do away with all traces of the discontinuity.

TABLE VIII.—Declinations at Intersections of Degrees of Latitude and Longitude.

Latitude N.	10° W.		9° W.		8° W.		7° W.		6° W.		5° W.		4° W.		3° W.		2° W.		1° W.		0°.		1° E.		2° E.
60											23° 15'·2	42'·3	22° 32'·9	42'·4	21° 50'·5	42'·3	21° 8'·2	42'·4	20° 15'·8							
											23'·2		21'·8		20'·2		18'·8									
59											22° 52'·0	40'·9	22° 11'·1	40'·8	21° 30'·3	40'·9	20° 49'·4									
											22'·5		21'·0		19'·5		18'·0									
58					24° 27'·5	39'·3	23° 48'·2	39'·4	23° 8'·8	39'·3	22° 29'·5	39'·4	21° 50'·1	39'·3	21° 10'·8	39'·4	20° 31'·4									
					25'·1		23'·6		22'·1		20'·6		19'·1		17'·6		16'·4									
57					24° 2'·4	37'·8	23° 24'·6	37'·9	22° 46'·7	37'·8	22° 8'·9	37'·9	21° 31'·0	37'·8	20° 53'·2	37'·9	20° 15'·3									
					23'·2		21'·7		20'·2		18'·7		17'·2		15'·7		14'·2									
56			24° 15'·6	36'·4	23° 39'·2	36'·3	23° 2'·9	36'·4	22° 26'·5	36'·3	21° 50'·2	36'·4	21° 13'·8	36'·3	20° 37'·5	36'·4	20° 1'·1	36'·3	19° 24'·8							
			24'·0		22'·4		21'·0		19'·4		18'·0		16'·4		15'·0		13'·4		12'·0							
55	24° 26'·5	34'·9	23° 51'·6	34'·8	23° 16'·8	34'·9	22° 41'·9	34'·8	22° 7'·1	34'·9	21° 32'·2	34'·8	20° 57'·4	34'·9	20° 22'·5	34'·8	19° 47'·7	34'·9	19° 12'·8	34'·8	18° 38'·0					
	26'·2		24'·7		23'·2		21'·7		20'·2		18'·7		17'·2		15'·7		14'·2		12'·7		11'·2					
54	24° 0'·3	33'·4	23° 26'·9	33'·3	22° 53'·6	33'·4	22° 20'·2	33'·3	21° 46'·9	33'·4	21° 13'·5	33'·3	20° 40'·2	33'·4	20° 6'·8	33'·3	19° 33'·5	33'·4	19° 0'·1	33'·3	18° 26'·8					
	25'·1		26'·6		25'·1		23'·6		22'·1		20'·6		19'·1		17'·6		16'·1		14'·6		13'·1					
53	23° 32'·2	31'·9	23° 0'·3	31'·8	22° 28'·5	31'·9	21° 56'·6	31'·8	21° 24'·8	31'·9	20° 52'·9	31'·8	20° 21'·1	31'·9	19° 49'·2	31'·8	19° 17'·4	31'·9	18° 45'·5	31'·8	18° 13'·7	31'·9	17° 41'·8	31'·8	17° 10'·0
	30'·0		28'·5		27'·0		25'·5		24'·0		22'·5		21'·0		19'·5		18'·0		16'·5		15'·0		13'·5		12'·0
52	23° 2'·2	30'·4	22° 31'·8	30'·3	22° 1'·5	30'·4	21° 31'·1	30'·3	21° 0'·8	30'·4	20° 30'·4	30'·3	20° 0'·1	30'·4	19° 29'·7	30'·3	18° 59'·4	30'·4	18° 29'·0	30'·3	17° 58'·7	30'·4	17° 28'·3	30'·3	16° 58'·0
					26'·2		24'·8		23'·2		21'·8		20'·2		18'·8		17'·2		15'·8		14'·2		12'·8
51							21° 4'·9	28'·9	20° 36'·0	28'·8	20° 7'·2	28'·9	19° 38'·3	28'·8	19° 9'·5	28'·9	18° 40'·6	28'·8	18° 11'·8	28'·9	17° 42'·9	28'·8	17° 14'·1	28'·9	16° 45'·2
											22'·5		21'·0		19'·5		18'·0									
50											19° 44'·7	27'·4	19° 17'·3	27'·3	18° 50'·0	27'·4	18° 22'·6									

TABLE IX.—Horizontal Forces at Intersections of Degrees of Latitude and Longitude.

Latitude N.	10° W.	9° W.	8° W.	7° W.	6° W.	5° W.	4° W.	3° W.	2° W.	1° W.	0°.	1° E.	2° E.
60						1·4610	1·4665	1·4721	1·4776	1·4832			
						368	*369*	*368*	*369*				
59					1·4923	1·4978	1·5034	1·5089	1·5145				
						368	*369*	*368*	*369*	*368*			
58			1·5180	1·5236	1·5291	1·5347	1·5402	1·5458	1·5513				
			369	*368*	*369*	*368*	*369*	*368*	*369*				
57			1·5549	1·5604	1·5660	1·5715	1·5771	1·5826	1·5882				
			368	*369*	*368*	*369*	*368*	*369*	*368*				
56		1·5862	1·5917	1·5973	1·6028	1·6084	1·6139	1·6195	1·6250	1·6306			
		368	*369*	*368*	*369*	*368*	*369*	*368*	*369*	*368*			
55	1·6175	1·6230	1·6286	1·6341	1·6397	1·6452	1·6508	1·6563	1·6619	1·6674	1·6730		
	368	*369*	*368*	*369*	*368*	*369*	*368*	*369*	*368*	*369*	*370*		
54	1·6543	1·6599	1·6654	1·6710	1·6765	1·6821	1·6876	1·6932	1·6987	1·7043	1·7100		
	367	*368*	*373*	*378*	*384*	*388*	*391*	*392*	*393*	*393*	*391*		
53	1·6910	1·6967	1·7027	1·7088	1·7149	1·7209	1·7267	1·7324	1·7380	1·7436	1·7491		
	414	*417*	*417*	*414*	*410*	*406*	*404*	*403*	*404*	*405*	*409*		
52	1·7324	1·7384	1·7444	1·7502	1·7559	1·7615	1·7671	1·7727	1·7784	1·7841	1·7900	1·7960	1·8021
				407	*406*	*407*	*410*	*414*	*418*	*424*	*431*	*439*	*447*
51		1·8362		1·7909	1·7965	1·8022	1·8081	1·8141	1·8202	1·8265	1·8331	1·8399	1·8468
						409	*414*	*420*					
50						1·8431	1·8495	1·8561					

Table X.—Dips at the Intersections of Degrees of Latitude and Longitude.

Longitude.

Latitude N.	10° W.		9° W.		8° W.		7° W.		6° W.		5° W.		4° W.		3° W.		2° W.		1° W.		0°.		1° E.		2° E.
60															72°42'·0	*6'·0*	72°36'·0	*6'·0*	72°30'·0						
															30'·2		*30'·4*		*30'·6*						
59					72°42'·0	*6'·0*	72°36'·0	*6'·0*	72°30'·0	*6'·0*	72°24'·0	*6'·1*	72°17'·9	*6'·1*	72°11'·8	*6'·2*	72°5'·6	*6'·2*	71°59'·4						
					30'·2		*30'·4*		*30'·6*		*30'·8*		*31'·1*		*31'·3*		*31'·5*		*31'·7*						
58					72°11'·8	*6'·2*	72°5'·6	*6'·2*	71°59'·4	*6'·2*	71°53'·2	*6'·4*	71°46'·8	*6'·3*	71°40'·5	*6'·4*	71°34'·1	*6'·4*	71°27'·7						
					31'·3		*31'·5*		*31'·7*		*31'·9*		*32'·0*		*32'·3*		*32'·4*		*32'·6*						
57					71°40'·5	*6'·4*	71°34'·1	*6'·4*	71°27'·7	*6'·4*	71°21'·3	*6'·5*	71°14'·8	*6'·6*	71°8'·2	*6'·5*	71°1'·7	*6'·6*	70°55'·1						
					32'·3		*32'·4*		*32'·6*		*32'·9*		*33'·1*		*33'·2*		*33'·5*		*33'·7*						
56					71°6'·2	*6'·5*	71°1'·7	*6'·6*	70°55'·1	*6'·7*	70°48'·4	*6'·7*	70°41'·7	*6'·7*	70°35'·0	*6'·8*	70°28'·2	*6'·8*	70°21'·4						
					33'·2		*33'·5*		*33'·7*		*33'·8*		*34'·0*		*34'·2*		*34'·3*		*34'·5*						
55	70°48'·4	*6'·7*	70°41'·7	*6'·7*	70°35'·0	*6'·8*	70°28'·2	*6'·8*	70°21'·4	*6'·8*	70°14'·6	*6'·9*	70°7'·7	*6'·9*	70°0'·8	*6'·9*	69°53'·9	*7'·0*	69°46'·9	*7'·0*	69°39'·9				
	33'·8		*34'·0*		*34'·2*		*34'·3*		*34'·5*		*34'·7*		*34'·8*		*35'·0*		*35'·2*		*35'·4*		*35'·6*				
54	70°14'·6	*6'·9*	70°7'·7	*6'·9*	70°0'·8	*6'·9*	69°53'·9	*7'·0*	69°46'·9	*7'·0*	69°39'·9	*7'·0*	69°32'·9	*7'·1*	69°25'·8	*7'·1*	69°18'·7	*7'·2*	69°11'·5	*7'·2*	69°4'·3	*7'·2*	68°57'·1	
	34'·7		*34'·8*		*35'·0*		*35'·2*		*35'·4*		*35'·6*		*35'·8*		*36'·0*		*36'·2*		*36'·3*		*36'·5*		*36'·8*	
53	69°39'·9	*7'·0*	69°32'·9	*7'·1*	69°25'·8	*7'·1*	69°18'·7	*7'·2*	69°11'·5	*7'·2*	69°4'·3	*7'·2*	68°57'·1	*7'·3*	68°49'·8	*7'·3*	68°42'·5	*7'·3*	68°35'·2	*7'·4*	68°27'·8	*7'·5*	68°20'·3	*7'·5*	68°12'·8
	35'·6		*35'·8*		*36'·0*		*36'·2*		*36'·3*		*36'·5*		*36'·8*		*37'·0*		*37'·2*		*37'·5*		*37'·7*		*37'·9*		*38'·1*
52	69°4'·3	*7'·2*	68°57'·1	*7'·3*	68°49'·8	*7'·3*	68°42'·5	*7'·3*	68°35'·2	*7'·4*	68°27'·8	*7'·5*	68°20'·3	*7'·5*	68°12'·8	*7'·5*	68°5'·3	*7'·6*	67°57'·7	*7'·6*	67°50'·1	*7'·7*	67°42'·4	*7'·7*	67°34'·7
									37'·5		*37'·7*		*37'·9*		*38'·1*		*38'·4*		*38'·6*		*38'·9*		*39'·2*		*39'·5*
51									67°57'·7	*7'·6*	67°50'·1	*7'·7*	67°42'·4	*7'·7*	67°34'·7	*7'·8*	67°26'·9	*7'·8*	67°19'·1	*7'·9*	67°11'·2	*8'·0*	67°3'·2	*8'·0*	66°55'·2
									38'·6		*38'·9*		*39'·2*		*39'·5*		*39'·8*		*40'·2*		*40'·5*		*40'·8*		*41'·2*
50									67°19'·1	*7'·9*	67°11'·2	*8'·0*	67°3'·2	*8'·0*	66°55'·2	*8'·1*	66°47'·1	*8'·2*	66°38'·9	*8'·2*	66°30'·7	*8'·3*	66°22'·4	*8'·4*	66°14'·0

The determination of the rate of secular change is a more difficult matter, but at the completion of our work we have more data at our disposal than in the earlier stages, when coefficients had to be chosen for the reduction of observations. The plans adopted in the case of each element are as follows :—

1. *Declination.*

We have deduced from Sir F. EVANS's Map for 1872 (*loc. cit.*) the Declinations at 24 points distributed uniformly all over the kingdom, and have compared them with the values given by our Table VIII., on page 238. The secular corrections thus calculated show an increase from east to west amounting to about 0′·11 and 0′·14 per degree of longitude in latitudes 58° and 52° respectively.

There is also an increase with latitude above the latitude 52°.

The results are exhibited in the following Table :—

TABLE XI.—Mean Secular Change of Declination per annum between 1872 and 1886.

Latitude.	Longitude.					
	10° W.	8° W.	6° W.	4° W.	2° W.	0°.
60	..	..	..	9·1	9·0	..
58	..	9·6	9·2	8·8	8·9	..
56	..	9·1	8·6	8·3	8·2	..
54	9·3	8·5	8·1	7·8	7·8	7·9
52	9·4	8·9	8·1	7·8	7·8	7·7
50	..	..	..	8·3	8·0	..

This is in fair accord with the observations at individual stations given on p. 88, but as Sir FREDERICK's isomagnetics were not deduced by any definite system of calculation, but little importance must be attached to minor variations and discrepancies.

We do not, for instance, feel justified in assuming that there is a real increase of the secular coefficient to the south of lat. 52°, especially as M. MOUREAUX's values in France for corresponding intervals are less than ours.

It is also in accord with the results of the comparison of our observations with those of WELSH, which show that between 1857–1886 the secular change in Scotland increases with the latitude and with west longitude. We think, therefore, that the most that can be said at present is that the secular change at the time of our survey, as given by the mean of the Greenwich and Kew values (p. 91), was in the neighbourhood of London about 6′·5 per annum, and that a comparison with Sir F. EVANS's results shows that it is about 1′·5 greater in the south-west of Ireland, and about 2′ larger in the north-east of Scotland.

Hence by smoothing the irregularities in Table XI. above, and reducing all the numbers so as to give the present rate at London we get the figures in Table XII., p. 325.

These may be checked in future by means of the observations at Kew, Greenwich, Falmouth, Stonyhurst, and Valentia, but we do not think a comparison of two observatories only leads to trustworthy laws of rate of change of secular variation with geographical position. There can be no question that Declination change on the whole increases with the latitude, yet the secular coefficient is less at present at Stonyhurst than at Greenwich and Kew. It is therefore very much to be regretted that the observations so sedulously carried on by the late Provost LLOYD, at Trinity College, Dublin, have of late years been interrupted.

We may point out that the best position for a magnetic observatory would be in the centre of a widespread region of low Vertical Force. The disturbing rocks would probably produce less effect in such a situation, and all the phenomena might be expected to be more normal. Unfortunately Kew and Greenwich are within the range of the Reading disturbance, and Stonyhurst is in a region where the Vertical Force changes very rapidly both to the east and west of that place.

2. *Inclination.*

We have also in the case of the Dip found the secular change between 1842·5 and 1886 by comparing Table X. with Sir E. SABINE's map for the earlier epoch. The result confirms the conclusions previously arrived at, viz., that the secular change diminishes with latitude and increases with west longitude. As, however, the north-westerly stations are on the line of minimum change they probably do not form real exceptions to the general rule. Partly by this method and partly by the map of assumed values (p. 85) we have selected those given in Table XII.

3. *Horizontal Force.*

This element when treated in the same way gives since 1842·5 a greater rate of secular change in the south than in the north, and in the west than in the east. The latter conclusion is in accord with that of M. MOUREAUX.

We are now inclined to think that the high value of the secular change between the years 1883–85 at Greenwich (·0028) led us to assume rather too high a value (·0022) for the south of England.

The mean annual change for Greenwich between the years 1883–87 is ·0020 and for Kew ·0017. The values for Greenwich in 1857 and 1887 are 1·769 and 1·818, but in 1861 a new instrument was introduced, and to make the two values comparable we must subtract 0·016 from the first. This leads to a difference of 0·065 or a secular change of 0·0022.

Though this is a little greater than the present rate, we do not think the evidence is sufficient to justify us in altering the rates obtained for Scotland by a comparison for the interval 1857–87, and we assume that they are valid at the present time. For the neighbourhood of London we take 0·0019, and by the aid of SABINE's and WELSH's papers we have arrived at the values given in the following Table :—

TABLE XII.—Rates of Secular Change per Annum in the Declination, Inclination, and Horizontal Force, in terms of Minutes of Arc for the Declination and Dip, and of Metric Units for the Horizontal Force. (N.B.—The two former elements are diminishing, the Horizontal Force is increasing.)

Longitude.

Latitude.	10° W.	9° W.	8° W.	7° W.	6° W.	5° W.	4° W.	3° W.	2° W.	1° W.	0°	1° E.	2° E.
60°							8·6 / 1·0 / ·0015	8·4 / 0·9 / ·0015	8·3 / 0·9 / ·0015	8·1 / 0·9 / ·0014			
59°			9·0 / 1·2 / ·0018	8·9 / 1·1 / ·0017	8·7 / 1·1 / ·0017	8·6 / 1·1 / ·0017	8·4 / 1·0 / ·0016	8·3 / 1·0 / ·0016	8·1 / 1·0 / ·0016				
58°			8·8 / 1·3 / ·0019	8·6 / 1·2 / ·0018	8·5 / 1·2 / ·0018	8·3 / 1·2 / ·0018	8·2 / 1·1 / ·0017	8·0 / 1·1 / ·0017	7·9 / 1·1 / ·0017				
57°			8·6 / 1·4 / ·0020	8·5 / 1·3 / ·0019	8·3 / 1·3 / ·0019	8·2 / 1·3 / ·0019	8·0 / 1·2 / ·0018	7·9 / 1·2 / ·0018	7·7 / 1·2 / ·0018				
56°		8·4 / 1·6 / ·0021	8·4 / 1·4 / ·0020	8·2 / 1·4 / ·0019	8·1 / 1·4 / ·0019	7·9 / 1·3 / ·0019	7·8 / 1·3 / ·0018	7·6 / 1·3 / ·0018	7·5 / 1·2 / ·0018				
55°	8·3 / 1·7 / ·0021	8·1 / 1·6 / ·0021	8·2 / 1·5 / ·0021	8·1 / 1·5 / ·0020	7·9 / 1·5 / ·0020	7·7 / 1·4 / ·0020	7·6 / 1·4 / ·0019	7·5 / 1·4 / ·0019	7·3 / 1·3 / ·0019				
54°	8·1 / 1·8 / ·0022	8·0 / 1·7 / ·0022	8·0 / 1·6 / ·0021	7·8 / 1·6 / ·0020	7·7 / 1·5 / ·0020	7·5 / 1·5 / ·0020	7·4 / 1·5 / ·0019	7·2 / 1·4 / ·0019	7·1 / 1·4 / ·0019	6·9 / 1·4 / ·0018	6·8 / 1·3 / ·0018		
53°	7·9 / 1·9 / ·0022	7·7 / 1·8 / ·0022	7·8 / 1·7 / ·0022	7·7 / 1·7 / ·0021	7·5 / 1·6 / ·0021	7·4 / 1·6 / ·0021	7·2 / 1·6 / ·0020	7·1 / 1·5 / ·0020	6·9 / 1·5 / ·0020	6·8 / 1·5 / ·0020	6·6 / 1·4 / ·0019	6·5 / 1·4 / ·0019	6·4 / 1·4 / ·0018
52°			7·6 / 1·8 / ·0022	7·4 / 1·8 / ·0021	7·3 / 1·7 / ·0021	7·1 / 1·7 / ·0021	7·0 / 1·7 / ·0020	6·8 / 1·6 / ·0020	6·7 / 1·6 / ·0020	6·5 / 1·6 / ·0019	6·4 / 1·5 / ·0019	6·2 / 1·5 / ·0019	6·1 / 1·5 / ·0018
51°					7·1 / 1·8	7·0 / 1·8 / ·0021	6·8 / 1·8 / ·0020	6·7 / 1·7 / ·0020	6·5 / 1·7 / ·0020	6·4 / 1·7 / ·0019	6·2 / 1·6 / ·0019	6·1 / 1·6 / ·0019	5·9 / 1·6 / ·0018
50°						6·7 / 1·9 / ·0022	6·6 / 1·9 / ·0021	6·4 / 1·8 / ·0021	6·3 / 1·8 / ·0021				

The local disturbance at any place may be estimated from the disturbances at neighbouring stations as given in Plates IX., X., and XII. It is evident that this correction must be somewhat uncertain, but the maps will at all events give information as to whether the disturbance is likely to be large or small. As an example of the use of the tables and maps, we calculate the Declination for lat. 52° 30′ N. and long. 1° 30′ W. on July 1, 1889.

From Table VIII. it was

$$18° \ 29′·0 + 8′·2 + 15′·2 = 18° \ 53′·4 \text{ on January 1, 1886.}$$

From Table XII. the secular change is

$$- (6·5 + ·1 + ·15) \times 3·5 = - 23′·6$$

which reduces the value to 18° 29′·8.

From Plate IX. we see that the station is in a region of negative disturbance, and that the true Declination is probably less than the calculated amount by 15′ or 20′.

It is obvious that for this purpose much rougher methods of calculation would suffice, but the main reason for making the process as accurate as possible is that the values of disturbing forces can only be determined if the rates of secular change are carefully discussed and accurately known. As we hope that these will be further investigated, we give the fullest data at our disposal for the calculation of the undisturbed values of the elements.

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IV. *On the Effects of Pressure on the Magnetisation of Cobalt.*

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Communicated by Professor J. J. THOMSON, *F.R.S.*

[PLATES 15, 16.]

§ 1. IN August, 1888, I commenced an investigation into the effects of pressure on the magnetic properties of cobalt, at the suggestion of Professor J. J. THOMSON. In his "Applications of Dynamics to Physics and Chemistry," he has arrived at certain conclusions as to the relations of magnetisations and mechanical strains, and the primary object of the following investigations was to obtain experimental data, whereby these conclusions might be tested.

During the process of the experiments, numerous points presented themselves whose elucidation seemed of importance for a satisfactory comparison of theory and experiment. The experiments have thus included a wider range than was originally intended, and have occupied a considerable amount of time.

Phenomena resembling very closely in some points those to be presently discussed have been noticed by previous observers in iron and nickel. As frequent references to these will be found essential for an adequate discussion of the phenomena presented by cobalt, a brief outline of the more prominent characteristics is here interposed.

§ 2. If a bar of some magnetic metal, subjected to a constant longitudinal stress, be exposed to the action of a magnetising current in a surrounding spiral it becomes magnetised, and in general simultaneously alters its length. If the current cease to flow the bar retains in general part of its magnetisation, and it may not return to its original length. On the other hand, if the bar while under the action of a constant current in the magnetising spiral, be subjected to a longitudinal stress, it in general simultaneously alters its magnetisation. Also, if the stress cease to act the magnetisation may not return to its original value. Phenomena similar to the last are also shown by the magnetisation which is residual after the break of the current.

These phenomena all depend on the *intensity* of magnetisation in the bar. The mode of this dependence will be discussed in considerable detail presently, but erroneous ideas are not unlikely to arise unless the following practical considerations be kept clearly in view.

When a bar whose diameter of cross section is comparable with its length is magnetised in a uniform field, there seems good reason to believe that the magnetisation it possesses is very decidedly less in the interior near the axis than it is near the surface, and when the bar is possessed merely of residual magnetism, the difference between the interior and surface distributions seems completely proved. When, as usual, the magnetising field is not uniform, there is in general still less reason to expect a uniform intensity of magnetisation. Thus, in general, a bar may be regarded as composed of a large number of elements, throughout each of which the magnetisation may be treated as sensibly uniform, but which may differ widely amongst themselves. The term *intensity* applied to the magnetisation of such a bar merely means some sort of average of the intensities throughout the several elements.

Now the application of a given stress to such a bar may affect very differently the magnetism of different elements. It may, for instance, increase the magnetisation of some of them and diminish that of others. Experiments as a rule give merely a sort of integral of the effects on the different elements, and this ought to be clearly recognised.

In particular I would point out that the magnetic moment of a bar, as determined by some definite method, may have the same value when the bar is exposed to the action of a certain current in a spiral, as it has after the break of a larger current in the spiral, and that notwithstanding there may be an important difference between the distributions of magnetism in the two cases. There is thus no *à priori* reason to expect the same phenomena to follow the application of a given longitudinal stress in the two cases.

Effects of Magnetism on the Length of Bars.

§ 3. Such effects have been observed by JOULE,[*] MAYER,[†] Professor BARRETT,[‡] and Mr. SHELFORD BIDWELL.[§]

JOULE's experiments dealt with iron and steel. His method of experiment consisted in running a current through a magnetising spiral and observing the instantaneous changes in the length of the bar accompanying the make and break of the current. Observations were taken with several currents gradually rising in strength. Apparently JOULE did not demagnetise his rods in the course of his experiments, nor does he seem to have examined the effect of making and breaking the same current more than once. He concluded that in iron or soft steel bars free from stress the institution of a current in a surrounding spiral is followed by an increase in the length of the bar proportional to the square of the intensity of the magnetisation

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[*] 'Phil. Mag.,' vol. 30 (3rd series), 1847, pp. 76–87 and 225–241.

[†] *Ibid.*, vol. 46 (4th series), 1873, pp. 177–201.

[‡] *Ibid.*, vol. 47 (4th series), 1874, p. 51; also 'Nature,' vol. 26, 1882, p. 585.

[§] 'Phil. Trans.,' A, vol. 179, 1888, pp. 205–230. Also 'Roy. Soc. Proc.,' vol. 40, 1886, pp. 109–133 and 257–266.

induced in the bar, and that on the break of the current there remains an increase in length proportional to the square of the intensity of the residual magnetisation. JOULE also found that iron and soft steel under tension lengthened or shortened when magnetised according as the tension was small or great. He believed the shortening proportional to the product of the bar's magnetisation into the strength of the magnetising current. This last conclusion, however, is certainly not warranted by the results of his experiments.

On JOULE's p. 237, are given the results of an experiment, numbered 23, in which a soft steel wire exposed to a certain tension showed an increased length while exposed to a certain current, lengthened further when the current was broken, and then shortened on the make of a stronger current. This suggests that for the wire in question under a given load there existed a critical intensity of magnetisation, and that the bar lengthened or shortened according as the magnetisation was less or greater than the critical value. The true interpretation of this experiment seems to have been entirely missed by JOULE, and it would not appear to have attracted the notice of subsequent observers. It is thus only within the last few years that the existence of a critical field, or state of magnetisation, has been brought to light by the decisive experiments of Mr. SHELFORD BIDWELL.

§ 4. MAYER experimented with rods free from longitudinal stress, and he gives the results of the make and break of only one strength of current. The current was, however, twice made and broken in some at least of the experiments, and the results of both the magnetic cycles are recorded. Some of the results, given in a table on his p. 200, are so suggestive in the light of our present knowledge that I reproduce them here in a form slightly altered to suit the present inquiry. The numbers in the first column distinguish the rods used, all the other numbers give the lengthening, or when with a minus sign the shortening, of the rods in terms of an arbitrary unit.

Rod No.		Increase in length of rods.		
		First make of circuit.	After first break.	Second make of circuit.
Iron.	1	1·15*	·4	1·1
	2	1·6	·4	1·6
	3	2·0	1·1	2·5
	4	2·5	1·35	2·5
	5	1·65	1·05	2·05
	6	1·4	·55	1·45
Steel.	7	·8	1·4	1·15
	8	— ·25	·25	— ·25
	9	— ·4	— ·15	— ·35

* In original 1·25, but this seems a misprint.

On breaking the second currents the rods all returned to the length they possessed on the break of the first currents, so that the changes in the lengths of the rods became cyclic after a single make and break of the current. The last three experiments are, it will be observed, very far from agreeing with JOULE'S laws.

The experiments suggest that it is mainly on the intensity of the existing magnetisation that the phenomena depend, but the divergence between the figures in the second and fourth columns indicates that the previous history of the material may, in some cases at least, be of considerable importance.

In all the iron rods the induced magnetisation is clearly much below the critical. In 7, the softest of the steel rods, the induced magnetisation though below is clearly approaching the critical. In rod 8, the critical magnetisation lies between the induced and the residual; and if a critical magnetisation existed in the case of rod 9 it must have been less than the residual.

§ 5. Of Mr. SHELFORD BIDWELL'S experiments those described in the 'Philosophical Transactions,' appear the most complete. In these, he first demagnetised the specimen, and then sent through a surrounding spiral a series of currents gradually rising in strength. For each strength of current the alterations in the length of the rod accompanying two successive makes were observed and recorded. The Tables III. and IV., pp. 220 and 221, for an iron rod show a small but unmistakeable difference between the effects of the first and second makes of a current of given strength. In Table V. p. 222, for a cobalt rod the differences might well be attributed to experimental errors.

Mr. SHELFORD BIDWELL found critical fields both for iron and cobalt. Iron he found to lengthen or shorten when magnetised according as the magnetising field was below or above the critical, while cobalt presented exactly the opposite phenomena. A nickel rod shortened in all fields up to 1400 C.G.S. units, and the experiments left the existence of a critical field an open question. The critical fields for several specimens of iron varied from about 270 to 380 C.G.S. units, and the maximum elongations appeared in fields of about 80 C.G.S. units. In one cobalt rod the maximum shortening occurred in a field of about 400 C.G.S. units, and the critical field considerably exceeded 800 C.G.S. units. A second softer specimen showed a maximum shortening in a field of about 300 C.G.S. units, and had a critical field of 750 C.G.S. units. In these experiments, the rods were free from stress. Mr. SHELFORD BIDWELL'S experiments, however, described in the 'Proceedings of the Royal Society,' show that the critical fields for iron are lowered by tension, a result in exact accordance with JOULE'S experiments.

§ 6. Professor BARRETT found a cobalt rod to lengthen when exposed to a comparatively weak field. The data he supplies are too limited to enable one to judge whether his later experiments on the subject were wholly free from the objections which he himself believed to have invalidated the conclusions based on his earlier experiments. I scarcely think his results afford any reasonable ground for doubting

the accuracy of Mr. SHELFORD BIDWELL's conclusions, at least for the actual cobalt rods of his experiments.

Effects of Longitudinal Stress on the Magnetisation of Rods and Wires.

§ 7. A preliminary general idea of these effects may be obtained from a consideration of the phenomena presented by a soft iron wire stretched beyond its original limit of perfect elasticity, and then subjected, in a weak magnetic field, to a stress cycle consisting of the application and removal of a definite tension for which it is perfectly elastic.

The first application of the tension causes a large increase in the magnetisation of the wire, and on the removal of the tension this increase nearly all remains, or may even be added to. The next one or two tension cycles cause a distinct progressive increase in the magnetisation. Presently, answering to the tension cycles, there appears a nearly cyclic change in the magnetisation, in which the maximum magnetisation appears when the tension is "on." Also the magnitude of the cyclic effect remains, at least approximately, constant for, at all events, a large number of tension cycles.

Subtracting the effect of the final cyclic change of magnetisation from the change accompanying the first application of tension, we get what may be fairly regarded as that portion of the effect of the first tension which is of a permanent and non-cyclic character. This quantity I have here termed the "shock-effect." If the magnetising current be broken, the first application of tension or of pressure has a large shock-effect, which invariably consists in a diminution of the residual magnetisation. The effects of beating,[*] or other mechanical agitation of a rod, are in many respects very similar.

The experiments on the effects of stress on magnetism which are of most importance for our present purpose are those of VILLARI,[†] Sir W. THOMSON,[‡] Mr. SHIDA,[§] and Professor EWING.[||] All these observers have, at least in practice, recognised the distinction between the effect on a bar's magnetisation of the first few applications of a given longitudinal stress and its subsequent applications.

§ 8. VILLARI unfortunately gives no clue to the strength of his magnetising fields, save the number and relative size of the DANIELL or BUNSEN elements he employed. He discovered that the non-cyclic effects of the first tension and the cyclic effects of the later tension cycles in bars—or rather stout wires—of iron in weak magnetic fields are of the character indicated in the last paragraph. In stronger fields he found

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[*] See WIEDEMANN's 'Elektricität,' vol. 3, p. 666, &c.
[†] POGGENDORFF's 'Annalen,' vol. 126, 1865, pp. 87–122.
[‡] 'Phil. Trans.,' 1879, pp. 55–85; or 'Mathematical and Physical Papers,' vol. 2.
[§] 'Roy. Soc. Proc.,' vol. 35, 1883, pp. 404–454.
[||] 'Phil. Trans.,' 1885, pp. 523–640; 1888, A., pp. 325–337.

that, while the first application of the tension still increased the bar's magnetisation, the cyclic effect had changed sign. On his p. 91, in experiment 3, is given an instance where a soft iron bar, exposed to one of his strongest fields, lost magnetisation on the very first application of the tension. From the data supplied, however, it will be found that the shock-effect of this first tension was still an increase of magnetisation. Similar cases are also referred to on his p. 97, but no data are given from which the sign of the shock-effect can be deduced.

From VILLARI's experiments it obviously follows that a critical field, or intensity of magnetisation, must exist where the cyclic effects of tension vanish. The field in which the effect vanishes is called by Sir W. THOMSON the "VILLARI critical point." He and Professor EWING have made it clear that the field in which this phenomenon occurs depends on the amount of the tension which is applied and removed, and also on the existence of any permanent load. Professor EWING also prefers to define the VILLARI point by reference to the magnetisation existing in the rod, and not to the field producing it. Some of VILLARI's steel bars behaved like iron, but in others a critical field, if existent, was lower than the lowest field he employed.

VILLARI also observed the fall in the residual magnetisation caused by the first tension after the break of the current. He further observed that after several tension cycles there appeared a cyclic change in the residual magnetisation. Without making any restriction as to the strength of the pre-existing fields, he lays down the law, that in soft iron the residual magnetisation in the cyclic state is greatest when the tension is "on," while in hard iron and steel, it is greatest when the tension is "off." I can find no clue as to the actual strength of the pre-existing field or fields he employed.

Fitting an iron core inside a coaxial iron tube of the same length, VILLARI found that in fields between certain limits, the core and the tube showed, in response to cyclic changes of tension, cyclic changes of magnetisation of opposite signs. The phenomena, as VILLARI himself pointed out, are exactly in accordance with the view, stated in § 5, that the intensity of magnetisation is greater on the surface of a bar than in its interior.

§ 9. Sir W. THOMSON experimenting on a soft iron wire permanently stretched, and then exposed to cycles "on" and "off" of some weight causing no further permanent extension, found a critical field which was in general higher the smaller the weight employed in the tension cycles. The critical fields obtained by the magneto-metric method—calculated presumably for the centre of the magnetising coil—averaged about 20 C.G.S. units, and the fields at which the cyclic effect was a maximum, were roughly about a quarter of the critical.

By the ballistic method with an induction coil at the centre of a much longer magnetising coil, critical fields such as 6·5 C.G.S. units were obtained. This Sir W. THOMSON apparently assigns to the experimental wire having its magnetisation greatest at the centre of the magnetising spiral, so that this portion of the wire would

be brought into the critical state by much weaker currents than the portions under the ends of the spiral.

In his § 240, Sir W. Thomson states that a Villari critical point was found in the central portions of a nickel bar, but from a note to § 239, it would appear that with the magnetometer opposite an end of the bar, no trace of a neutral field was obtained. With a magnetometer as usual opposite one end, he found in nickel and cobalt a cyclic effect the reverse of that occurring in iron in fields below the Villari point, *i.e.*, the magnetisation was least when tension was "on."

§ 10. Mr. Shida obtained Villari critical fields of 15 and 10 C.G.S. units respectively for two different specimens of soft iron wire, which had been permanently stretched, and on which the tension cycles were performed with weights not much less than those causing the permanent extension. With a pianoforte steel wire, he found that in the cyclic state tension "on" co-existed with a minimum of magnetisation in all his fields.

§ 11. The first thing in Professor Ewing's researches that concerns us is his method of demagnetising wires. This consists in subjecting them to the action of a series of rapidly reversed currents diminishing in intensity. In § 19 of the first of his papers referred to above, he states that his wires were thereby reduced to a standard condition—different possibly, he admits, from their condition previous to their first magnetisation.

In his § 107 Professor Ewing defines the Villari critical point as " that value of the magnetisation $\Im$ at which reversal occurs in the sign of the magnetic difference produced by two (assigned) states of longitudinal stress." He uses the term as applicable (1) to that value of $\Im$ at which no magnetic change accompanies the *repeated* alternation from one to the other of two assigned states of stress, (2) to the value of $\Im$ answering to the intersection of the two curves giving the relations of $\Im$ to $\mathfrak{H}$—the field,—when $\mathfrak{H}$ is gradually raised from zero, and separate experiments are completed with the wire in each of the assigned states of stress. He also proposes that one of the assigned states of stress should be that answering to no load. It will be noticed that Professor Ewing's first usage of the " Villari critical field " is that which accords with Sir W. Thomson's.

In originally soft annealed iron wires stretched beyond the limit of perfect elasticity, Professor Ewing found in every case distinct Villari points of the first kind. These existed in lower fields the greater the weight producing the second assigned state of stress. If in a certain field the cyclic state has been reached for the loads 0 and Q, for which the magnetisation has the critical value, and the load be gradually raised from 0, then there ensues a rise in the magnetisation until the load reaches some value P less than Q, and then an equal fall as the load is increased to Q. Even in the weakest fields Professor Ewing seldom found the magnetisation increase continually when the load was raised nearly to the limit of perfect elasticity.

From the figures on Professor Ewing's Plate 63, it will be seen that the first

application of load has effects very similar to those just described. It will appear, however, from fig. 42, § 88, that the critical fields at which the first application of a given load has no effect on the magnetisation are distinctly higher than the VILLARI points for the cyclic applications of these loads. This figure also shows that the first application of a moderate load produces a very large increase of the magnetisation in weak fields, whereas in the strongest field of 34 C.G.S. units the effect is trifling.

From fig. 40 it will be seen that in the cyclic state the effects of tension cycles on the magnetisation residual after a field of 3·33 C.G.S. units were of the same general character as appeared with an equal induced magnetisation.

Soft annealed wires, when none but very small loads were used, were found by Professor EWING to possess in weak fields a maximum of magnetisation with tension " on," agreeing so far with stretched wires. With greater loads the effects become reversed after the first application of the load. Professor EWING attributes this to *hysteresis*, and shows that when tapped these wires behave as stretched wires with low VILLARI points.

From fig. 43, Plate 64, it will be seen that soft wires respond to the first application of tension in a precisely similar way to stretched wires, but have much lower critical fields. The phenomena accompanying the application of load cycles to a soft annealed wire possessed of the magnetism residual after the break of a field of ·34 C.G.S. unit, are discussed in §§ 83–85 and shown in fig. 39. They are unquestionably of the same general character as those occurring with an equal induced magnetisation.

Iron wires, both when stretched and annealed, and a pianoforte steel wire stretched after annealing, were exposed by Professor EWING to gradually increasing magnetising currents, and the curves connecting $\mathfrak{I}$ and $\mathfrak{H}$ are given in his Plates 64 and 67.

The specimens all showed VILLARI points of the second kind, which agreed with those of the first kind in being as a rule lower, the heavier the load. These critical fields, or magnetisations, of the second kind were distinctly higher than the corresponding ones of the first kind.

Simultaneously observations were taken of the residual magnetisation whose results are shown in figures in Plates 67 and 68. In all the specimens the magnetisation residual after weak fields, and also its ratio to the induced magnetisation, were found to be largely increased by the presence of a load, the increase being greatest with the heaviest loads used. As the fields were raised the influence of the loads diminished, and in the annealed iron and the steel wires the residual magnetisation and its ratio to the induced became eventually less under the heavier loads than in the absence of a load.

§ 12. In his papers in ' Phil. Trans.,' A., 1888, Professor EWING found that in fields from 0 to 116 C.G.S. units, the presence of a tension on a thin nickel wire diminished its induced and residual magnetisations, and also the ratio of the residual to the induced. Also cyclic changes of tension " on " and " off " were accompanied by

cyclic changes of magnetisation, in which the magnetisation, whether induced or residual, was least when the tension was " on."

The effects of pressure on a nickel bar were exactly the opposite of those of tension, and the increase in the susceptibility was extremely large, especially near the " Wendepunkt "—or point where $\mathfrak{I}/\mathfrak{H}$ is a maximum. Professor EWING looked for a VILLARI point in low fields but found no trace of its existence.

Theoretical Considerations.

§ 13. The experimental results already stated will enable Professor J. J. THOMSON's theoretical conclusions to be understood.

Let e denote the strain in an isotropic elastic cylinder parallel to its axis, f and g the strains in the cross section perpendicular to and along the radius vector. Also let $\mathfrak{H}$, $\mathfrak{I}$ and κ denote respectively the strength of the magnetic field, the intensity of magnetisation, and the coefficient of magnetic induction, assumed everywhere the same throughout the cylinder.

In the case of a uniform longitudinal stress $g = f$. Thus, assuming

$$\frac{d\mathfrak{I}}{dg} = \frac{d\mathfrak{I}}{df} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

Professor THOMSON obtains for this case, on his p. 51, two equations (48), of which the first may be put in the slightly altered form

$$\kappa \frac{de}{d\mathfrak{I}} = \mathrm{C} \left(1 - \mathfrak{H}\frac{d\kappa}{d\mathfrak{I}}\right)\left(\frac{d\mathfrak{I}}{de} - 2\sigma\frac{d\mathfrak{I}}{df}\right) \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Here C denotes a positive constant depending only on the elastic properties of the cylinder, and σ is POISSON's ratio.

All the differential coefficients must of course apply to some one definite state of the cylinder, and in determining $d\mathfrak{I}/de$ and $d\mathfrak{I}/df$ the strength of the field must be kept constant.

It is pointed out by Professor THOMSON that according to EWING's experiments $1 - \mathfrak{H}\, d\kappa/d\mathfrak{I}$ is positive for iron. It is obviously positive for any magnetic metal in any field exceeding that which answers to its " Wendepunkt." It also appears to be positive for nickel and cobalt throughout the experiments of Professor ROWLAND.[*] Thus it is probably safe to regard it as essentially positive.

Under a uniform longitudinal stress

$$g = f = -\sigma e.$$

* ' Phil. Mag.,' vol. 48, 1874, p. 321.

If then, through a small increase in the stress, there be increases δe, δf, δg in the strains, we must have

$$\delta g = \delta f = -\sigma \delta e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

If $\delta \Im$ be the consequent increase in the magnetisation, the field remaining constant,

$$\delta \Im = \frac{d\Im}{de}\,\delta e + \frac{d\Im}{df}\,\delta f + \frac{d\Im}{dg}\,\delta g,$$

and so by (1) and (3)

$$\delta \Im = \delta e\left(\frac{d\Im}{de} - 2\sigma\,\frac{d\Im}{df}\right).$$

Thus from (2)

$$\kappa\frac{de}{d\Im} = C\left(1 - \mathfrak{H}\,\frac{d\kappa}{d\Im}\right)\frac{\delta\Im}{\delta e} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

where $\delta \Im$ and δe denote the small increments in the magnetisation and longitudinal strain which follow a small change in the longitudinal stress.

If (4) were a complete representation of the state of matters, then $de/d\Im$ and $\delta\Im/\delta e$ should always have the same sign, and should vanish for the same value of the magnetisation.

It should be carefully noticed that in employing (4) we tacitly assume the changes in the strains to be very small, and thus the value of $\delta\Im/\delta e$ should be obtained by experiments in which the change in the longitudinal stress is very small.

Thus, in particular, the magnetisation at which $\delta\Im/\delta e$ vanishes is the VILLARI point, as determined by experiments in which the difference between the two assigned states of stress is very small. And thus, in accordance with the experiments of Sir W. THOMSON and Professor EWING, this theoretical VILLARI point may well be very much higher than those hitherto found by experiment.

It may also be as well to point out that the magnetisation answering to the vanishing of $de/d\Im$ is not the critical magnetisation observed by Mr. SHELFORD BIDWELL where the rod resumes its original length, but the much lower magnetisation where the rod, if iron, has its greatest extension, if cobalt, its greatest shortening.

Comparison of Theory and Experiment.

§ 14. The experiments already recorded sufficiently show that in iron, there is in weak fields, and again, in very strong fields, an agreement in sign between $de/d\Im$ and $\delta\Im/\delta e$, both expressions changing from positive to negative as the strength of the field is raised. It will be noticed, however, that the magnetisations obtained experimentally for the VILLARI point are much lower than those obtained for the point of maximum lengthening of the rod. This difference is certainly in some measure accounted for

by the lowering of the VILLARI point by the comparatively large stresses practically employed. Professor THOMSON also has pointed out that MAXWELL's distribution of stress should produce an independent system of strains in the bar, the strain in the direction of the lines of force being an extension. Though apparently a very small effect this would tend in iron to raise the magnetisation where $de/d\mathfrak{J}$ vanishes somewhat over the VILLARI point.

In nickel we see from the experiments of Sir W. THOMSON, Mr. SHELFORD BIDWELL, and Professor EWING, that $de/d\mathfrak{J}$ and $\delta\mathfrak{J}/\delta e$ are both negative in all ordinary fields. If the results obtained by Sir W. THOMSON in the central portions of a nickel bar, in a strong field, be accepted, then there does actually exist a very high critical magnetisation at which $\delta\mathfrak{J}/\delta e$ vanishes. There should accordingly be, not in weak fields where Professor EWING looked for it, but in very high fields, a critical magnetisation where a nickel bar would cease to contract. Such a phenomenon cannot be said to have its existence demonstrated by Mr. SHELFORD BIDWELL, but at the same time it is certainly not disproved by his experiments.

In cobalt, according to Sir W. THOMSON, $\delta\mathfrak{J}/\delta e$ is negative in weak fields, and so also is $de/d\mathfrak{J}$ if we accept the results of Mr. SHELFORD BIDWELL. According to the latter observer, however, $de/d\mathfrak{J}$ changes sign in fields much higher, it is true, than the corresponding fields for iron, but still easily obtainable. Further, the cobalt not only recovers from the very considerable shortening it has experienced, but lengthens considerably as the field is raised. It would thus appear impossible to assign the phenomenon in any essential degree to the MAXWELL effect. If then Professor THOMSON's theoretical conclusions are sound, $\delta\mathfrak{J}/\delta e$ should change sign from negative to positive in a cobalt rod for a magnetisation quite within the reach of experiment.

Experimental Verification of Theory in Cobalt.

§ 15. The question whether or not a VILLARI point of the above kind exists in cobalt was the primary object of the following experiments, and a decided answer in the affirmative was obtained. Under the moderate stress employed, $\delta\mathfrak{J}/\delta e$ was negative in weak fields and positive in strong, and an unmistakeable VILLARI point appeared in a field of about 120 C.G.S. units.

This, it will be noticed, is a much lower *field* than those which Mr. SHELFORD BIDWELL obtained for the vanishing point of $de/d\mathfrak{J}$ in either of the specimens he employed, and the MAXWELL effect would here tend to lower these latter fields. In my specimen, however, as the field was raised from 120 to 300 C.G.S. units the *magnetisation* increased by less than 20 per cent. The accordance of theory and experiment would thus, in reality, appear to be closer in cobalt than in any recorded experiments on iron.

Preliminary Sketch of Phenomena Observed.

§ 16. As the phenomena observed are somewhat complex, their mutual relationships might be hidden under the multiplicity of facts, which the full discussion of each separate phenomenon introduces. A brief outline of some of the more important and certain results will, it is hoped, serve as a key to the subsequent more complete discussion.

It is important to notice that all the phenomena were observed in a given specimen of cobalt, magnetised by a current in a given coil, and that it was found by repeating several of the sets of observations at various periods of the research, that the character of the specimen had not to all appearance been altered during the process of the experiments. The importance of this, when an attempt shall be made to connect the phenomena together, and explain them as consequences of one or more fundamental principles, will be at once obvious to any one who has noticed the variety in the phenomena observed in different specimens of iron.

At the same time, when various specimens of the same magnetic metal—or even of different magnetic metals—are exposed to gradually increasing magnetic fields, and curves drawn in which the abscissæ give the strength of the field and the ordinates the intensity of magnetisation, experiment shows that the fields in which corresponding points of the several curves occur may widely differ, but that there exists a general resemblance in the form of the curves, which may be made much closer by properly altering the scale of abscissæ.

Thus, while the particular field at which a certain phenomenon presents itself in a particular specimen may so largely depend on the individual peculiarities of the specimen that it may be of little value as an isolated fact, yet the determination of the point on the curve where the phenomenon occurs, and the position it occupies relative to the points where other phenomena occur, may lead to the recognition of general laws.

Phenomena Observed in the Induced Magnetisation.

§ 17. The "Wendepunkt," or point where the coefficient of induced magnetisation is a maximum, is a point in the curves that can easily be recognised. Very probably the most satisfactory comparison of the phenomena observed in different specimens would be obtained by expressing the magnetisation of each specimen in terms of the magnetisation it possesses at its Wendepunkt as a unit. The Wendepunkt of the specimen employed occurred in a field of about 35 C.G.S. units.

For the reasons just stated, the relations of both the induced and residual magnetisations to the strength of the field, though not a primary object of investigation, were observed under various conditions as to pressure. These relations, in the case of the induced magnetisation, are shown in figs. 1–4 (Plate 15).

The difference between the effects on the induced magnetisation of the first few and

subsequent applications of longitudinal stress, referred to in § 7 as occurring in iron, is equally characteristic of cobalt, so long as the magnetisation is not much in excess of that of the critical or VILLARI point. In the present experiments the stress applied was a pressure, and so δe being always negative, and the magnetisation in the weaker fields greatest when the rod was under pressure, $\delta\Im/\delta e$ was, as already stated, negative in fields below the critical.

The effect of the first application of pressure may, as explained in § 7, be regarded as the sum of the cyclic effect and of a non-cyclic or shock-effect. In all fields below 200 C.G.S. units the shock-effect gave an unmistakeable increase of magnetisation. In all stronger fields, up to at least 725 C.G.S. units, the magnitude of the shock-effect was so small that its sign even was a doubtful matter. Thus, in cobalt, the shock-effect, so long at least as it is an appreciable quantity, gives an increase of magnetisation. This consequence of the shock-effect, even in fields much above the VILLARI point, is known from VILLARI's experiments, recorded in § 8, to be also characteristic of iron.

The absolute measures, both of the shock-effect of the first pressure and of the cyclic effect, possessed maxima in the neighbourhood of the Wendepunkt. Relative, however, to the magnetisation existing prior to pressure, both these effects increased continually in importance as the strength of the field was reduced within the actual limits of the experiments.

The total effect of the first application of pressure had a critical field of about 160 C.G.S. units; which, it will be noticed, is decidedly higher than the critical field for the cyclic effect. As stated in § 11, in discussing Professor EWING's experiments, the existence of a higher critical field for the first application of stress than for the cyclic application, is equally characteristic of iron.

As it is important to know how far, if at all, the nature and magnitude of the cyclic effect of pressure depend on the circumstances attending the process of magnetisation, the rod was sometimes subjected to pressure before and during its introduction into the magnetising spiral. It was found that there was no material alteration in the cyclic effect. The mode of variation of the effects of the first and of the cyclic application of pressure with the strength of the field is exhibited in figs. 5–8 (Plate 15).

Phenomena Observed in the Residual Magnetisation.

§ 18. Comparing figs. 1 and 9, it will be seen that the residual magnetisation approaches "saturation" in much lower fields than does the induced.

The intensity of the residual, as of the induced, magnetisation was found to depend on the treatment of the rod during the flow and the break of the current. When the rod was free from pressure at the instant the current was broken, the residual magnetisation in all fields below 120 or 130 C.G.S. units was increased by the appli-

cation of pressure cycles during the flow of the current.   In stronger fields no certain effect could be attributed to the pressure cycles.   The effect, as may be seen from fig. 10, was greatest in very weak fields.

The existence of pressure during the break of the current proved to have a considerable influence on the intensity of the residual magnetisation.   As might have been anticipated it increased the residual magnetisation in weak fields, but the effect changed its sign in fields far below the VILLARI point, and showed no signs of vanishing even in a field of 400 C.G.S. units.

The ratios of the residual magnetisation to the strength of the pre-existing field and to the intensity of the induced magnetisation under various conditions as to pressure are shown in figs. 11 and 12 (Plate 16).   From these it appears that in the absence of all pressure the intensity of the residual magnetisation is extremely small in weak fields.   In this case the phenomena closely resemble those noticed by many observers in iron.   The only essential difference apparently is that the fields at which certain phenomena appear in cobalt are much higher than the fields at which the corresponding phenomena appear in ordinary iron.   This is in exact agreement with the views already expressed as to the relative positions of the Wendepunkt.

The application of pressure cycles during the flow of the current increased immensely the intensity of the residual magnetisation in weak fields.   The increase, in fact, due to this cause in the residual magnetisation in the weakest experimental fields, was even greater proportionally than in the induced, so that the ordinates of curves c and d of fig. 12 continually increase as the strength of the field is reduced.

The effects of the application or removal of pressure in shaking out the residual magnetisation were also observed, and are shown in Curve II., fig. 13, and in figs. 14 and 15.   From the first of these curves it appears that the percentage of residual magnetisation shaken out by a series of pressure cycles continually diminished as the strength of the field was raised.

When the rod was free from pressure during the break of the current, the percentage of the residual magnetisation, shaken out by the first application of pressure, was in weak fields decidedly diminished by the previous application of pressure cycles during the flow of the current.   In fields over 70 or 80 C.G.S. units, however, the effect of previous pressure cycles was extremely small.   In weak fields the removal of a pressure that had existed during the break of the current was fully as effective in shaking out residual magnetisation as was the application of a pressure when the rod, during the break of the current, remained free from pressure.   In fields over 30 or 40 C.G.S. units, however, the removal of pressure became decidedly less effective than the application.

As has been stated in §§ 8 and 11, VILLARI and Professor EWING found that cyclic applications of stress were followed by cyclic changes in the residual magnetisation of the same character as those produced in the induced magnetisation.   As it seemed important to test the generality of these conclusions, the cyclic effects of pressure on

the residual magnetisation were here observed when the treatment of the rod during the flow and break of the current was varied.

When pressure cycles were applied during the flow of the current, and pressure was "on" during its break, the cyclic effect had invariably the same sign while the strength of the field was raised from 11 to 400 C.G.S. units. The effect in this case was in the same direction as in the induced magnetisation below the VILLARI point, *i.e.*, the magnetisation was greatest when pressure was " on."

When, however, the rod was free from pressure during the break of the current, a critical field was found to exist. In the magnetisation residual after fields above the critical, the magnetisation was least when pressure was "on." This critical field was raised by the application of pressure cycles during the flow of the current, but it was in any case much lower than the critical field for the induced magnetisation. It follows, of course, that the intensity of the critical residual magnetisation was very much less than that of the critical induced.

The amounts of the cyclic effects under the various conditions are shown in the fifth columns of Tables II. and VIII., and in the sixth columns of Tables VI. and VII. These results show conclusively that for cobalt, and so in all probability for iron, the relations between stress and residual magnetisation must be largely dependent on circumstances other than the so-called intensity of magnetisation. This, as pointed out in § 2, was *à priori* far from improbable, and should act as a warning against extending to residual magnetisation laws proved only for the induced.

The fact that in fields over 60 or 70 C.G.S. units both the intensity of the residual magnetisation and the character of the cyclic effect of pressure are very little affected by the application of pressure cycles during the flow of the current, while markedly influenced by the existence of pressure during the break of the current, seems worthy of special notice.

All the phenomena observed in the residual magnetisation apparently altered but little in magnitude, as the strength of the field was raised from 100 to 400 C.G.S. units. This further emphasises the difference between them and the phenomena observed in the induced magnetisation.

The Apparatus.

§ 19. The rod experimented on was supplied by Messrs. JOHNSON and MATTHEY. The mean of a series of measurements gave 16·98 cms. for its length and ·546 sq. cm. for the area of its cross section. During the experiments the rod was contained in a brass tube, the closed end of which was sufficiently thick to afford an unyielding resistance to the rod when under pressure. The rod could be slid in or out of the tube, but the fit was tight enough to prevent any lateral movement. On the outside of the brass tube, at its open end, a screw of small pitch was cut, answering to the screw cut on the inside of a brass cap. The cap carried a projecting arm, forming a

diameter of a cross section, by means of which a considerable couple could be exerted. The pressure was applied to the cobalt by means of a sort of ram-rod rigidly attached to the inside of the brass cap. When the cap was screwed on this ram-rod advanced into the brass tube, and came to bear on the end of the cobalt rod.
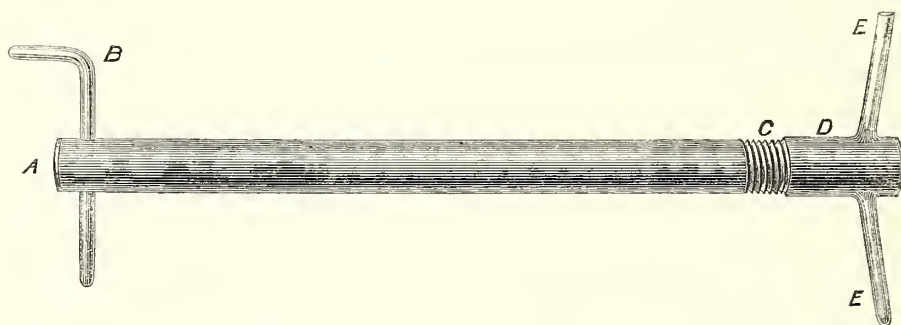
Drawing of the brass tube containing the cobalt rod, removed from the coil.

A Closed end of brass tube.
B Pin which can be slipped through the holes in the cheeks attached to one face of the coil and, as in the figure, through the diametral hole bored in the solid end of the brass tube.
C Screw cut on the outside of the brass tube at its open end.
D Brass cap with screw cut on its inside answering to C, and carrying ram-rod arrangement for applying pressure to the cobalt rod.
EE Projecting arm, by means of which the cap is screwed on.

In the original experiments, conducted in August, 1888, the brass tube was firmly imbedded in a block of wood, fixed inside a magnetising coil. This coil had a length of 19·8 cms., and consisted of five layers of very thick copper wire. A current of one ampère produced a field varying from 9·26 C.G.S. units at the centre of the rod to 6·85 units at its ends. This, it must be admitted, is not a very uniform field, but the fall occurred mainly within a centimetre or two of the ends of the rod.

In this preliminary investigation the observations were taken somewhat roughly. The reason they are referred to here is that they clearly showed the existence of a cyclic and a non-cyclic effect of pressure, and also the existence of a VILLARI point for the former effect. Further, the result of experiments on several different occasions agreed in giving 120 C.G.S. units as a close approximation to the strength of the field due to the current at the centre of the rod when the VILLARI point appeared. The mode of variation of the several effects with the strength of the field showed also a general agreement with the subsequent more accurate observations taken with a different coil. This seems to warrant the belief that the phenomena observed do not owe any of their essential features to peculiarities of the apparatus.

When it was attempted to obtain accurate results with the original apparatus various difficulties were encountered. In overcoming these I am much indebted to the ingenuity of Mr. BARTLETT, the assistant at the Cavendish Laboratory, who constructed the new apparatus.

The new coil had a length of 18·15 cm., an internal diameter of 1·4 cm., and an external diameter of 7 cm. It consisted of thirteen layers of copper wire of ·24 cm. diameter when covered. In the solid end of the brass tube holding the cobalt a diametral hole was bored at right angles to the length of the tube, and a corresponding hole was made in each of two cheeks projecting from one of the faces of the coil. By slipping a pin through these holes the tube was firmly secured in a fixed position relative to the coil, and when the pin was withdrawn the tube and its contained cobalt could be removed and replaced without the least risk of disturbing the remainder of the apparatus.

A current of 1 ampère in the new coil produces a field varying from 65·7 C.G.S. units at the centre to 43·9 units at the ends of the cobalt rod. Throughout the greater portion of the rod, however, the field is very nearly the same as at the centre. These figures refer entirely to the action of the current, no attempt having been made to allow for the action of the rod itself.

In all the following calculations the values assigned to the field $\mathfrak{H}$ are calculated from the action of the current alone at the centre of the rod, and so are somewhat higher than the mean values of the actual fields. The difficulty of determining the actual fields at the different elements of the rod would be very great even treating the permeability as constant, a most erroneous supposition; and, considering the differences to be expected between different specimens of the same metal, an attempt at great accuracy in this department seems entirely supererogatory.

§ 20. Throughout the whole investigation the ordinary magnetometric method was employed. The axis of the coil was perpendicular to the magnetic meridian, and the magnetometer was situated in the direction of this axis produced. The magnetometer needle carried a mirror which formed on a millimetre scale, set parallel to the magnetic meridian, an image of a vertical wire placed across a slit in the centre of the scale behind which a lamp stood. The direct action of the coil current was neutralised by a compensating coil traversed by the same current. The strength of the current was recorded by a Thomson graded ammeter in circuit with the coil. The current was derived from the storage cells of the laboratory, and its strength was varied by changing the number of cells and the resistance in a wire bridge in circuit with the coil.

§ 21. In order to render apparent the relative magnitudes of the several phenomena some common system of magnetic measurements was essential. Thus all the scale readings have been reduced, and in the following tables and diagrams only C.G.S. units appear. From the previous remarks as to the method of calculating $\mathfrak{H}$ it will be clearly understood that the values given for the strengths of the fields, and consequently for the coefficients of magnetic induction κ, cannot rigidly be held to apply to the actual condition of the rod, in which $\mathfrak{H}$ and κ varied from point to point, but must be regarded as giving an approximation to the mean state of the rod, and more

particularly as indicating by their variation the amounts of the changes actually occurring.

It should also be understood that in calculating the intensity of magnetisation $\mathfrak{J}$ the rod was treated as magnetised solenoidally. In some of the experiments, e.g., those on residual magnetism, a direct calculation on this basis might have introduced a considerable error, owing to the small distance of the rod from the magnetometer. The actually observed readings, however, were first reduced by direct comparison of the readings obtained in an independent set of experiments, in which the rod in a given magnetised condition was placed first of all in the positions it was about to occupy in the course of the main observations, and then in a certain standard position, where its distance from the magnetometer was so great that the precise position of the " poles " was of comparatively little moment. As stated in § 2, $\mathfrak{J}$ in reality doubtless varies from point to point of the rod, and the values here recorded indicate merely a sort of average.

Disturbing Agencies.

§ 22. Before discussing the experiments in detail it will be as well to refer to two disturbing agencies, the one of importance mainly in weak fields, the other in strong.

The former, the residual effects of previous magnetisations, affected very considerably some of the earlier experiments here recorded. In weak fields so important was this that the rod was sometimes found, after showing a considerable induced magnetisation, to possess, on breaking the circuit, residual magnetism of opposite sign. The application of pressure in general increased this residual magnetisation of opposite sign, sometimes to a very considerable extent. Or, supposing the residual magnetisation to be at first of the same sign as the preceding induced, it might change sign on the application of pressure. These effects were produced at pleasure with the greatest ease when the rod was treated in accordance with the following hypothesis :—

Calling the ends of the rod A and B, a weak current makes A, say, a north pole, and on breaking the current A is left with a quantity, N_1', of northern polarity. A smaller reverse current shakes out only so much of this residual magnetism, leaving a quantity, N_2, in A, which exists alongside of a larger quantity, S_1, of southern polarity. The end A thus appears a south pole of strength $S_1 - N_2$. On breaking this second current, A is left with a quantity S_1', less than S_1 of residual southern magnetism, and a quantity N_2', probably less than N_2, of residual northern magnetism. Whether A appears a south or a north pole depends on whether S_1' or N_2' is the greater. On applying pressure, a considerably greater proportion of S_1' is shaken out than of N_2', and the polarity of the end A may thus change sign.

Similar phenomena, proceeding doubtless from the same cause, are described by WIEDEMANN as occurring in iron. They fit in well enough either with WEBER's theory or with the view that the deep seated magnetic molecules are less affected by weak fields than are the surface molecules.

The total removal of this disturbing agency would require the complete demagnetisation of the rod. The method of demagnetisation employed at first consisted of applying a few reverse currents gradually diminishing in intensity. The magnetisation left in the rod seemed infinitesimal, but the unsatisfactory character of the result will be easily seen on reference to Tables I. and II. The method finally adopted was to expose the rod to a succession of diminishing reverse currents—the first current in the case of a weak field considerably exceeding that in existence during the pressure cycles—and then to tap it vigorously. The first attempt was by no means always successful, and the loss of time was a distinct objection to the method.

It is, of course, impossible to be sure that the rod was ever demagnetised, in the sense of being restored to its condition prior to its first magnetisation, but it was, at all events, reduced to such a condition that successive experiments made with the same field showed an excellent agreement. In strong fields demagnetisation appeared to be, at least for many purposes, of very little importance.

§ 23. The second disturbing agency, the heating of the coil and thence of the rod, by means of the current, was troublesome only in fields over 400 C.G.S. units. The heating of the coil wires increases their resistance, and so tends to diminish the strength of the field; while the heating of the rod increases its permeability, as has been shown by Professor Rowland.[*] For a rise of temperature of 225° C. he found the maximum value of the coefficient of induced magnetisation to increase by about 70 per cent., and to appear in a distinctly lower field.

This explains why, in some of my stronger fields, the scale reading kept altering in a direction indicating a progressive increase in the rod's magnetisation. This did not, of course, annul the cyclic changes accompanying cyclic changes of pressure, but it rendered impossible any very great accuracy in the determination of their magnitudes. The determination of the effect of the first application of pressure was found particularly difficult.

Lest it should be supposed that the cyclic changes in the magnetisation may in reality be due to the heating and cooling of the rod produced by the application and removal of pressure, I would here point out that the cyclic changes of magnetisation alter in sign in a comparatively low field, while the increase in the permeability accompanying heating was observed by Rowland in fields of all strengths, from 50 to 1470 C.G.S. units.

First Series of Experiments.

§ 24. With the new apparatus four principal series of experiments were performed. The first series took place in December, 1888, and the results are given in Tables I. and II. The general order of conducting these experiments was as follows :—

The rod, after being demagnetised by reverse currents alone without tapping, was removed to a distance. The number of storage cells and the bridge resistance were

* 'Phil. Mag.,' 4th series, vol. 48, 1874, p. 321.

then adjusted to give the field desired. When the ammeter and the scale reading became steady, the rod was brought up and pushed rapidly into the coil, and the pin secured. The difference between the scale readings when stationary, before and after the introduction of the rod, supplied the data for calculating the initial value of the induced magnetisation.

A considerable number of cycles of pressure " on " and " off " were then performed, the scale reading, when become stationary, being usually taken after each " on " and " off." Sometimes when readings had been taken of the effects of the first four or five pressure cycles, some five or six cycles were applied in rapid succession and then the taking of readings once more resumed. When a sufficient number of readings had been taken for the calculation of the average effect of a pressure cycle, the circuit was suddenly broken, the rod being always free from pressure during the break.

When the scale reading had become stationary after the break and had been observed, some six pressure cycles were applied, readings being taken after each " on " and " off." A series of six pressure cycles were then applied whose effects were not observed. These were followed by some six or seven more pressure cycles whose effects were observed. Finally the rod was removed to a distance and the magnetometer zero taken. The difference between the scale reading taken immediately after the break of the current and that taken after the removal of the rod supplied the data for calculating the initial value of the residual magnetisation.

TABLE I.

$\mathfrak{H}$.	$\mathfrak{I}_1$.	κ_1.	$\mathfrak{I}_2 - \mathfrak{I}_1$	κ_2.	First "on."	Cyclic "on"—"off."	Shock-effect $\dfrac{}{\mathfrak{I}_1}$	Cyclic "on"—"off" $\dfrac{}{\mathfrak{I}_2}$
·80 S (13)	1·54	1·9	1·08	3·3	·77	..	..	..
1·15 N (12)	4·01	3·5	3·09	6·2	3·09	·80	·57	·11
1·44 N (15)	4·63	3·2	2·47	4·9	2·93	·65	·49	·09
1·61 S (14)	3·55	2·2	2·32	3·6	1·39	·11	·36	·02
1·9 N (11)	8·9	4·7	2·8	6·2	3·7	1·2	·27	·10
4·3 N (10)	19·6	4·5	6·9	6·1	7·4	1·6	·30	·059
7·8 N (9)	38·0	4·8	9·7	6·1	10·0	2·8	·19	·059
9·9 N (8)	50·0	5·0	13·1	6·4	13·9	3·0	·21	·048
18·4 N (2)	107·5	5·8	15·7	6·7	19·1	5·0	·13	·041
19·2 S (1)	113·7	5·9	13·9	6·6	16·7	4·8	·11	·037
25·6 S (3)	148·5	5·8	16·8	6·5	24·4	5·8	·12	·035
36·8 S (4)	218·1	5·9	15·2	6·3	20·6	7·2	·06	.031
66·9 S (5)	368·8	5·5	10·6	5·7	15·0	4·4	·029	·012
100·7 N (6)	..	..	..	..	8·4	1·4	·015*	·003*
126·6 N (7)	..	..	..	..	4·0	− 0·7	·009*	− ·001*

* The values employed here for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are derived from subsequent tables.

TABLE II.

$\mathfrak{H}$.	$\mathfrak{I}_1'$.	$\mathfrak{I}_2'$.	First "on."	Cyclic "on"−"off."	$\mathfrak{I}_1'/\mathfrak{I}_2$	Shock-effect / $\mathfrak{I}_1'$	Cyclic "on"−"off" / $\mathfrak{I}_2'$
1·15 N (12)	3·40	2·17	− ·62	+ ·46	·48	− ·32	·21
1·44 N (15)	2·62	1·23	·62	+ ·51	·37	·43	·41
1·61 S (14)	4·32	N	4·32	..	·74	1·0	..
1·9 N (11)	5·9	3·4	1·1	+ ·42	·50	·26	·124
4·3 N (10)	8·6	4·9	2·0	+ ·49	·33	·29	·100
7·8 N (9)	13·1	6·9	3·7	+ ·54	·28	·32	·078
9·9 N (8)	16·4	9·0	4·6	+ ·62	·26	·32	·069
18·4 N (2)	27·9	15·7	8·6	+ ·26	·23	·32	·017
19·2 S (1)	29·9	19·0	5·4	+ ·15	·23	·19	·008
25·6 S (3)	39·4	25·3	9·5	+ ·15	·24	·25	·006
36·8 S (4)	51·0	34·6	12·9	− ·22	·22	·25	− ·006
66·9 S (5)	66·1	40·4	17·2	− 1·07	·18	·24	− ·027
100·7 N (6)	68·5	45·6	16·9	− 1·20	·14	·23	− ·026
126·6 N (7)	70·9	48·6	17·8	− 1·38	·13	·23	− ·028

§ 25. In Table I. $\mathfrak{H}$ is the strength of the field calculated as is explained in § 19, $\mathfrak{I}_1$ is the intensity and κ_1 the coefficient of induced magnetisation in the rod previous to the application of any pressures; while $\mathfrak{I}_2$ is the intensity and κ_2 the coefficient of induced magnetisation in the rod free from pressure after all the pressures cycles have been applied. Thus $\mathfrak{I}_2 - \mathfrak{I}_1$ is the increase in the magnetisation taking place during the application of the pressure cycles. Their application occupied a considerable time, during which absolute steadiness in the current could hardly be expected; and so $\mathfrak{I}_2 - \mathfrak{I}_1$ is probably in no case an absolutely exact measure of the increase in the magnetisation due to the pressure cycles. By *first "on"* is meant the increase in the magnetisation caused by the first application of pressure. *Cyclic "on" — "off"* gives the average algebraic excess of the magnetisation existing when pressure is "on" over that existing when pressure is "off." As already explained the *shock-effect* is the algebraic excess of the *first "on"* over the *cyclic "on" — "off."*

The second last column gives the ratio of the shock-effect to the intensity of the magnetisation prior to pressure, while the last column gives the ratio of the cyclic "on" — "off" to the intensity of magnetisation in the cyclic state. Thus the figures in these two columns may fairly be regarded as measuring the importance in the different fields of the shock-effect and the cyclic effect respectively.

In the weakest field the last three columns have no entries, because the experiments determining the cyclic effect of pressure failed to indicate a clear result.

The entries in the two last fields indicate, as they do elsewhere, that no observations of the corresponding quantities were obtained. In these fields when the compensating coil exactly balanced the coil current so that in the absence of the rod the spot was at the centre of the scale, the introduction of the rod drove the spot off the scale. The compensating coil was thus moved so as to neutralise part of the effect

of the rod, and so bring the spot near the centre of the scale. In the absence of the rod the spot was now off or nearly off the scale, so that accurate measures of the induced magnetisation were impossible.

The effect of the changed position of the compensating coil was allowed for in reducing the observations. A similar use of the compensating coil was subsequently made in some of the higher fields. It had the disadvantage of making the scale reading more disturbed by slight irregularities in the current. In these strong fields, however, the cyclic changes were so small compared to the total induced magnetisation that the alternative of increasing the distances of the coils from the magnetometer possessed greater disadvantages.

§ 26. In Table II., $\mathfrak{H}$ is the field existing prior to the break of the current. $\mathfrak{I}_1'$ is the residual magnetisation prior to the application of any pressure, $\mathfrak{I}_2'$ that finally existing after what was approximately a constant number of pressure cycles. A minus sign represents, as elsewhere, a diminution in the magnetisation, and when placed at the head of a column it applies to all the entries in that column. The *shock-effect* is obtained by subtracting from the algebraic value of the total effect of the first pressure the algebraic value of the *cyclic "on" — "off."*

The second last column gives the ratio of the *shock-effect* to the intensity of the residual magnetisation prior to pressure, while the last column gives the ratio of the *cyclic "on" — "off"* to the intensity of the residual magnetisation in the cyclic state.

§ 27. In both the Tables I. and II. the letter N signifies that a certain end of the rod, which we shall call A, was a north pole ; S, that it was a south pole. The numbers in brackets in the first columns give the order in which the experiments took place.

It was intended to apply as nearly as possible the same degree of pressure in every single case. Still, as the pressure was applied by hand, a certain amount of irregularity was bound to exist. The smallness in the variations shown by the individual observations of the cyclic effect in each separate field, and the smoothness of the several curves obtained prove that throughout any single series of experiments a pretty uniform standard must have been maintained. As a considerable interval of time elapsed between some of the successive series of experiments, notably those made in December and January, the standard probably varied somewhat from one series to another ; fig. 7, at all events suggests that in February the standard pressure was somewhat less than in December.

§ 28. The results of Table I. and II. will be presently discussed along with the corresponding results from subsequent tables, but certain peculiarities illustrative of the first difficulty stated in § 22, claim a special attention.

A glance at the values of κ_1 and κ_2 in Table I., shows that the supposed demagnetisation following the strong field (7) had not sufficed to remove all its effects. A comparison of the fields (14) and (15), or of (12) and (13), shows a much smaller susceptibility for currents making A a south pole, than for those making it as in (7) a north pole. From the results subsequently obtained with a more perfect system of

demagnetisation, and contained in Tables IX. and X., it appears that in those of the eight weakest fields of Table I., in which A was a north pole, the values obtained for κ_1, κ_2, and the cyclic effect are all undoubtedly somewhat too great.

The effects of the residual polarity appear still more decidedly in Table II. in the case of field (14). Here, what must be regarded as an abnormally large amount of residual magnetism, was originally present. Every trace of this was, however, removed by a single pressure, and with subsequent pressures magnetism of opposite sign became apparent. An even more striking case, that of field (13), does not appear in Table II. because no absolute measures were taken. In this case, the residual magnetism on the break even of the current was of opposite sign to the induced, and it increased on the application of pressure.

Second Series of Experiments.

§ 29. In the next principal series of experiments—taken in January, 1889—the method of taking the observations was much the same as in the first series, but the distances between the pieces of the apparatus were varied. Thus the distance of the centre of the rod from the magnetometer needle was 73·6 cm., 50 cm., or only 39·2 cm., according as the quantity under examination was the intensity of the induced magnetisation, the effect of pressure on the induced magnetisation, or the residual magnetisation. The compensating coil and the scale remained screwed to the beam carrying the apparatus, while the three positions of the magnetising coil and the corresponding positions of the magnetometer were indicated by pencil marks on the beam. There was no shaking of the rod, or variation in the current caused by the movements of the coil and magnetometer.

The fields in this series of experiments varied from 84·8 to 725 C.G.S. units. Preliminary experiments in the weakest of these fields, showed that demagnetising the rod had no appreciable influence on the magnitude of the cyclic effect of pressure. There was thus in none of the recorded experiments, as given in Tables III. and IV., any attempt at demagnetisation; but the weaker fields were taken first so as to avoid the disturbing influence of any residual effects that might originate with very strong currents.

TABLE III.

$\mathfrak{H}$.	$\mathfrak{I}_1$.	κ_1.	$\mathfrak{I}_2 - \mathfrak{I}_1$.	First "on."	Cyclic "on"–"off."	$\dfrac{\text{Shock-effect}}{\mathfrak{I}_1}$.	$\dfrac{\text{Cyclic "on"–"off"}}{\mathfrak{I}_2}$.
84·8	409	4·8	7·4	+ 9·2	+ 2·2	+ ·017	+ ·005
106·4	472	4·4	4·6	+ 5·0	+ 0·7	+ ·009	+ ·001
126·6	526	4·1	..	..	..	..	..
155 m	554	3·6	2·3	− 1·2	− 2·1	+ ·002	− ·004
181	582	3·2	..	− 1·8	− 2·4	+ ·001	− ·004
210	611	2·9	3·1	..	− 3·2	..	− ·005
270 m	629	2·3	1·3	− 2·4	− 3·6	+ ·002	− ·006
282	624	2·2	..	..	..	..	..
293	613	2·1	1·1	− 2·4	− 2·9	+ ·001	− ·005
339	654	1·9	..	..	..	..	..
357	669	1·9	..	− 4·5	− 3·2	− ·002	− ·005
406 m	690	1·7	7·9	− 2·8	− 3·8	+ ·001	− ·005
541	707	1·3	17·0	− 1·5	− 2·3	+ ·001	− ·003
633	777	1·2	34	− 5·2	− 3·4	− ·002	− ·004
725	791	1·1	$\left\{\begin{array}{c}22\cdot9\\1\cdot3\end{array}\right\}$	..	− 2·2	..	− ·003

TABLE IV.

Number of fields.	Limiting fields.	Average first "on."	Average cyclic "on"–"off."	Average shock-effect $\div \mathfrak{I}_1'$.
7	85–210	− 16·4	− 1·14	− ·21
8	250–355	17·4	1·09	·21
4	400–725	14·6	·80	·19

§ 30. In Table III., which gives the observations on the induced magnetism, the headings have the same meanings as in Table I. When m is attached to the strength of the field, two or three separate experiments were conducted with fields of approximately this strength, and the results given in the table are the averages of those obtained. The blanks are mainly due to unsteadiness in the scale-readings brought about by fluctuations in the strength of the magnetising current.

In no case probably, as already explained, does the column headed $\mathfrak{I}_2 - \mathfrak{I}_1$ give with perfect accuracy the change in the magnetisation produced by the pressure cycles alone, and in fields over 400 units the changes under this head must mainly be due to a totally different cause, viz., the heating of the rod. In these high fields the scale-reading did not remain stationary in the absence of pressure, but showed a progressive increase of magnetisation. To obtain the cyclic effect, the pressures were applied and removed at as nearly uniform intervals as possible, and the changes in the magnetisation occurring in successive intervals were compared.

In the strongest field, after the application of a large number of pressure cycles,

during which an increase of 22·9 units of magnetisation appeared, the current was broken, and after a short interval re-made, before the rod had cooled much. During a second almost equally numerous series of pressure cycles an increase of only 1·3 unit of magnetisation appeared. At the commencement of the second part of the experiment most of the coil was doubtless colder than the contained rod, so that the temperature of the rod would not alter much for some time after the re-make of the current. This last experiment indicates, I think conclusively, that in fields of this strength the permanent change in the magnetisation brought about by pressure cycles must be small, whatever its sign may be.

The values given for the *First "on"* in the strongest fields cannot claim great accuracy. The rod began to get heated before the oscillations of the magnetometer needle following the introduction of the rod into the coil had sufficiently subsided to permit of a reading being taken. The progressive increase of magnetisation was, in fact, going on most rapidly when the first pressure was applied. There had thus occurred a considerable progressive change before the next scale-reading could be taken, so that it was very difficult to deduce the true effect of the first pressure.

The values obtained in these high fields for the intensity of the induced magnetisation must also have been to some extent unduly raised by the heating, so that the curve of fig. 1 is, in its higher portions, not so flat as it ought to be.

§ 31. Table IV. gives merely certain average results, calculated from the large number of observations actually made on the residual magnetisation. On comparing the results after the conclusion of the experiments, it became apparent that the magnitudes of the several phenomena varied so little with the strength of the pre-existing field within the limits of the observations, that trustworthy deductions as to the exact modes of their variation were rendered impossible by the small variations in the distances of the pieces of the apparatus which the method of experiment was sure to introduce. The measurements of the effects of pressure are, in addition, exposed to possible irregularities in the magnitude of the pressure, but this ought not to affect sensibly the average results recorded in the Table.

Third Series of Experiments.

§ 32. In the next series of experiments, extending from January to February, the residual magnetisation alone was under investigation. Every single piece of apparatus was bolted or screwed to the supporting beam, so as to prevent any relative movement. The magnetometer needle was distant 82 cm. from the scale and 38 cm. from the middle of the rod.

First of all a set of observations were taken of the initial amount of the residual magnetisation as the strength of the pre-existing field was raised by steps from 1·7 to 79·4 C.G.S. units. The rod was apparently very nearly demagnetised by reversed currents previous to the first experiment, but no subsequent demagnetisation was

performed.   No pressure or tapping was applied to the rod throughout these observations.   The current was always in the direction making the end A the north pole.

The results, as recorded in Table V., are deduced from the scale readings taken (1) with the rod in its place inside the coil immediately after the break of the current, (2) with the rod removed to a distance.   In general, two observations were taken for each strength of field.   The mean of the two observations is the $\mathfrak{J}_1'$ of the Table.   By $\mathfrak{H}$ is meant the strength of the field due to the current just broken, and by κ_1' the ratio of $\mathfrak{J}_1'$ to $\mathfrak{H}$.   The mode of variation of $\mathfrak{J}_1'$ with $\mathfrak{H}$ is shown in curve a, fig. 10 ; while the mode of variation of κ_1', which is subsequently termed the *residual susceptibility*, is shown in curve a, fig. 11.

Table V.

$\mathfrak{H}$.	$\mathfrak{J}_1'$.	κ_1'.	$\mathfrak{H}$.	$\mathfrak{J}_1'$.	κ_1'.
1·7	·22	·13	14·1	12·9	·91
2·6	·29	·11	16·1	15·8	·98
3·5	·56	·16	18·0	21·4	1·20
4·4	·95	·22	22·3	27·5	1·23
5·0	1·3	·25	25·0	28·5	1·14
5·6	2·0	·36	27·3	31·5	1·15
6·0	2·4	·40	38·0	43·0	1·13
7·0	3·8	·55	45·4	50·1	1·10
9·4	6·6	·70	54·6	56·4	1·03
11·5	9·1	·79	70·4	61·9	·88
12·2	10·2	·84	79·4	63·8	·80

§ 33.   In continuation of these experiments a much more elaborate set of observations were taken, with the object of determining whether the amount of the residual magnetisation or its properties depended on the treatment of the rod during the flow or break of the current.   In one set of experiments, spoken of in future as the L type, no pressure at all was applied during the flow of the current.   In a second set, the M type, six cycles of pressure " on " and " off" were applied, the current being broken when pressure was " off."   In a third set, the N type, a pressure was applied after six pressure cycles, the current being broken when pressure was " on."   In these, as in all the other experiments, the rod was introduced into the coil after the scale reading showed that the current had become steady.   The experiments of the several types were not conducted separately, but on the following plan.   Starting with a certain number of storage cells and a certain resistance in circuit, complete observations were made for the corresponding field with each of the three types in the order L, M, N, say.   Then with a greater number of cells, or a reduced resistance, observations were taken for the next higher field, in the order M, N, L, and so on, in cyclic order.   Care in fact was taken that the experiments of any given type should not unduly often be either the first experiment of the day or the first

experiment with a given strength of field. This variation in the order was adopted lest during the application for several hours of a succession of pressure cycles, interrupted only by the demagnetisations, the rod might become less responsive to pressure, developing a sensible amount of what may be termed *fatigue*. Under the actual conditions, if this did happen, it would not affect the experiments of one type more than those of another.

The strength of the pre-existing fields was raised step by step from 1 to 400 C.G.S. units, and until fields of 350 units, or thereby, were reached, the rod was demagnetised, before each single experiment, by reversed currents and vigorous tapping. In the stronger fields the rod was merely exposed to a preliminary reverse current, of the same strength as that about to be used in the experiment. During the actual observations the end A was invariably the north pole.

In each experiment of each of the three types, with the exception of some of the weakest fields, the following operations were conducted. When the spot on the scale had become stationary, after the break of the current, a reading was taken. The pin was then withdrawn and the rod carefully removed, being kept at right angles to the magnetic meridian. The scale reading when stationary having been observed, the rod was gently restored to its place and the pin secured. With a little practice this operation was accomplished without shaking out any sensible amount of magnetism, as the coincidence of the scale readings before the rod's removal and after its restoration sufficiently testified. Pressure was then applied once and removed in the experiments of types L and M, being simply removed in those of type N, and the corresponding scale readings observed. Twelve pressures cycles were then applied, the effects of the last six only being observed. The rod was then removed and the constancy of the magnetometer zero tested.

The various observations obviously supplied sufficient data for calculating the *initial residual* magnetisation, the *first "on"* or *first "off"* of pressure, the *cyclic effect* of pressure, and the *final residual* magnetisation, *i.e.*, the residual surviving a definite number—13 for types L and M—of pressure cycles.

§ 34. The results of the experiments of type L are given in Table VI. All the headings have been already explained, except *next "off."* This means the change in the magnetisation accompanying the removal of the first pressure applied after the break of the current. In the two weakest fields the amount of the initial residual magnetisation alone was observed, and in the three next weakest fields the effects of only one pressure cycle were taken. Thus no data existed for calculating the cyclic effect in these fields; and the first three results given in the last column are unduly large, because the values there assigned to $\mathfrak{J}_2'$ are the magnetisations existing after the application of only a single pressure cycle.

In the fields over 90 C.G.S. units the changes in the properties of the residual magnetism accompanying the rise in the fields are so small that the small irregularities in the magnitudes of the pressures applied become important. Thus the

stronger fields are grouped into two sets, and the average effects for each set alone are given.

In like manner the results of the experiments of type M are given in Table VII., and those of type N in Table VIII. In these, too, the cyclic effects of pressure were unfortunately not observed in the weakest fields. In the six weakest fields of Table VII. the value assigned to $\mathfrak{I}_2'$ is the residual magnetisation when only one pressure cycle has been applied, while in the seven weakest fields of Table VIII. the value assigned it is the residual left on the removal of the pressure existing during the break of the current. The corresponding results in the last columns of these tables are consequently all unduly large. In the strongest fields average results alone are given as in Table VI.

The results of Tables VI., VII., and VIII. supply the data from which the curves b, c, d, respectively, of figs. 9 to 15 (Plate 16) are drawn.

Table VI.

$\mathfrak{H}$.	$\mathfrak{I}_1'$.	κ_1'.	First "on."	Next "off."	Cyclic "on"−"off."	Ratio of Shock-effect of first "on" to $\mathfrak{I}_1'$.	Ratio of Shock-effect of next "off" to Shock-effect of first "on."	$\mathfrak{I}_2' \div \mathfrak{I}_1'$.
2·6	1·26	·48	..	..	..	..	..	..
4·2	2·43	·58	..	..	..	..	..	..
5·6	5·0	·89	−3·0	−·15	..	−·60	·05	·37
8·9	8·4	·95	4·7	·29	..	·56	·06	·40
11·5	10·7	·93	5·8	·29	..	·54*	·05*	·43*
18·0	24·4	1·36	8·5	·95	0	·35*	·11*	·57*
24·0	32·7	1·36	10·7	·87	−·32	·32	·11	·61
35·0	44·2	1·26	13·7	·94	·62	·28	·12	·61
46·7	49·7	1·06	13·5	1·00	·49	·26	·11	·65
59·0	57·3	·97	14·3	·86	·57	·24	·10	·67
74·8	61·7	·82	14·4	1·22	·34	·23	·11	·71
93·2	64·9	·70						
109	65·7	·60						
122	66·3	·54						
133	68·2	·51	15·2	·85	·53	·22	·09	·72
147	68·0	·47						
173	68·5	·40						
207	68·7	·33						
242	70·2	·29						
292	70·5	·24						
353	70·6	·20	14·1	·69	·49	·19	·08	·75
385	71·4	·18						
414	71·1	·17						

* In the entries above this point, in absence of experimental data, the *cyclic* "on"−"off" is neglected. Thus the results above this point are not strictly comparable with those below.

Table VII.

$\mathfrak{H}$.	$\mathfrak{I}_1'$.	κ_1'.	First "on."	Next "off."	Cyclic "on"—"off."	Ratio of Shock-effect of first "on" to $\mathfrak{I}_1'$.	Ratio of Shock-effect of next "off" to Shock-effect of first "on."	$\mathfrak{I}_2' \div \mathfrak{I}_1'$.
0·95	2·32	2·44	− 1·2	− ·15	..	− ·51	·13	·43
1·27	2·65	2·09	1·0	·44	..	·39	·43	·44
1·84	3·53	1·92	1·5	·46	..	·42	·33	·46
2·7	3·9	1·44	1·9	·29	..	·49	·15	·43
6·0	9·0	1·50	4·2	·77	..	·46	·18	·45
8·9	13·8	1·51	5·2	1·06	..	·39	·20	·54
11·6	19·4	1·68	6·4	1·86	+ ·53	·36*	·19*	·48*
18·0	30·2	1·68	7·9	2·04	+ ·47	·28	·20	·61
22·8	36·1	1·58	8·8	1·63	+ ·21	·25	·16	·62
34·9	50·4	1·45	12·8	1·72	− ·27	·25	·16	·61
46·7	53·8	1·15	12·2	1·65	− ·14	·23	·15	·67
57·5	59·7	1·04	14·2	1·27	− ·32	·23	·11	·69
76·2	65·4	·86	14·3	1·12	− ·54	·21	·12	·68
94·2	66·1	·70						
109	66·4	·61						
121	66·9	·55						
134	68·2	·51	>14·0	·97	− ·56	·20	·11	·73
147	68·7	·47						
173	68·7	·40						
207	68·7	·33						
242	70·2	·29						
301	70·2	·23						
350	70·6	·20	>12·2	·79	− ·48	·17	·11	·77
397	70·6	·18						

* In the entries above this point, in absence of experimental data, the *cyclic* "*on*" − "*off*" is neglected. Thus the results above this point are not strictly comparable with those below.

TABLE VIII.

$\mathfrak{H}$.	$\mathfrak{J}_1'$.	κ_1'.	First "off."	Cyclic "on"—"off."	Shock-effect of first "off" / $\mathfrak{J}_1'$	$\mathfrak{J}_2' \div \mathfrak{J}_1'$.
·98	2·05	2·09	−1·17	..	−·57	·43
1·61	3·52	2·19	2·27	..	·65	·35
1·96	4·26	2·17	2·64	..	·62	·41
2·65	4·84	1·83	3·01	..	·62	·39
3·8	6·2	1·64	3·5	..	·57	·43
6·1	10·1	1·66	5·6	..	·55	·45
8·7	13·9	1·59	6·3	..	·45*	·53*
11·5	20·2	1·76	8·8	+·58	·41	·47
17·3	31·4	1·82	12·1	·69	·36	·52
22·9	37·6	1·64	11·3	·54	·29	·62
36·0	49·4	1·38	11·4	·35	·23	·68
46·7	53·3	1·14	12·5	·36	·23	·69
57·5	58·1	1·01	10·6	·09	·18	·69
79·1	61·3	·76	12·4	·20	·20	·73
93·6	63·6	·68				
111	64·1	·58				
124	65·3	·53				
134	65·3	·49	11·6	·17	·18	·75
147	65·6	·45				
180	66·2	·37				
213	66·9	·32				
224	66·8	·30				
299	67·4	·23	11·9	·20	·18	·77
348	67·9	·19				
400	67·8	·17				

Fourth Series of Experiments.

§ 35. In the last series of experiments, which extended into March, the effects of pressure on the induced magnetism were more particularly under investigation. Except on one or two occasions, the magnetometer was distant 50 cms. from the centre of the magnetising coil, and 81 cms. from the scale. Previous to each experiment the rod was demagnetised by reversed currents followed by vigorous tapping.

After the introduction of the rod into the coil, fourteen pressure cycles were in general applied, readings being taken of the effects of the first four and the last four cycles. From these the effect of the first pressure and the cyclic effect are calculated. Then, leaving the current untouched, the rod was usually removed to a distance and replaced, scale-readings being taken. The comparison of the reading taken when the rod was out with that taken prior to its original introduction into the coil, tested the constancy of the magnetometer zero. From the readings taken before and after the rod's removal, we find the magnetisation $\mathfrak{J}_2$ of the rod at the end of the pressure cycles; while from the readings taken before and after the rod's re-introduction we find the magnetisation $\mathfrak{J}_3$ on the second introduction of the rod into the coil. Thus, $\mathfrak{J}_2 - \mathfrak{J}_1$ is, as in former experiments, the increase in the magnetisation during the application of the pressure cycles, while $\mathfrak{J}_2 - \mathfrak{J}_3$ is the magnetisation lost during the removal of the rod from the coil and its re-introduction.

* In the entries above this point, in absence of experimental data, the *cyclic* "on" — "off" is neglected. Thus the results above this point are not strictly comparable with those below.

The compensating coil was intentionally placed so as to somewhat more than balance the magnetising coil, and so, in some of the strongest fields, the spot was off the scale when the rod was removed. Thus, in these fields, there were no data for calculating $\mathfrak{I}_1$, and in calculating $\mathfrak{I}_2 - \mathfrak{I}_1$ and $\mathfrak{I}_2 - \mathfrak{I}_3$ it had to be assumed that the magnetometer zero had not altered in the intervals that elapsed during the application of the pressure cycles and during the absence of the rod from the coil. The latter interval was always short, so that any sensible alteration during it was improbable. During these observations the end A of the rod was always the north pole.

There then followed in general, while the current remained unchanged, some observations intended to test the residual effects of the pressure cycles. In these the rod was repeatedly removed from the coil, it might be when free from or when subjected to pressure, and while out its state, as regards pressure, might or might not be changed. Not infrequently, too, the ends were changed while it was out, so that on its re-introduction the end A might be either a north or a south pole. Sometimes on the re-introduction of the rod *first* "*ons*" or *first* "*offs*" of pressure were taken, with the object of determining whether their magnitude was affected by repetition or by a change in the end of the rod. The results of this last series of experiments are contained in Tables IX. and X. They are the results on which most of the curves in figs. 1–8 (Plate 15) are based.

TABLE IX.

$\mathfrak{H}$.	$\mathfrak{I}_1$.	κ_1.	$\mathfrak{I}_2 - \mathfrak{I}_1$.	κ_2.	$\mathfrak{I}_2 - \mathfrak{I}_3$.	$\mathfrak{I}_A$ "on"—"off."	$\mathfrak{I}_B$ "on"—"off."	"Off" $\mathfrak{I}_A - \mathfrak{I}_B$.	"On" $\mathfrak{I}_A - \mathfrak{I}_B$.
1·8	4·1	2·3	2·4	3·6	..	..	1·4	..	..
2·2 p	8·2	3·8	1·9	4·7	1·4	..	..	·2	..
*3·4	11·0	3·2	3·4	5·0	2·7	..	..	3·8	..
4·3 p	20·2	4·7	·3	4·8	2·7	1·7	..	..	..
8·1 p	41·5	5·1	5·0	5·8	5·2	2·4	..	9·9	..
8·6	37·3	4·3	10·2	5·5	5·1	..	..	9·7	..
*12·5	61·7	4·6	5·1	6·2	..	..	..	..	10·3
23·4 p	147·1	6·3	3·4	6·4	13·5	..	..	..	..
23·7	141·1	6·0	13·2	6·5	11·6	..	..	5·3	..
35·2 p	253·8	7·2	6·2	7·4	16·5	..	..	..	..
35·7	233·4	6·5	13·3	6·9	10·4	..	..	2·9	..
46·0	290·0	6·3	12·3	6·6	12·0	9·0	..	..	..
47·5 p	312·3	6·6	4·7	6·7	9·2	..	..	·5	..
57·5	339·5	5·9	10·6	6·1	9·5	..	..	..	..
57·5 p	375·7	6·5	−4·1	6·5	11·6	8·1	..	·5	..
72·6	397·3	5·5	6·9	5 6	10·5	7·4	..	·6	..
79·8	405·8	5·1	8·6	5·2	10·2	6·3	..	1·9	..
92·0	427·0	4·6	8·1	4·7	8·4	7·3	..	·6	..
103·5	491·5	4·7	6·1	4·8	4·5	4·2	..	3·0	..
115	510·4	4·4	3·8	4·5	4·7	1·8	2·1	2·2	..
128	530	4·1	2·4	4·2	4·4	2·1	..	1·8	..
144	550	3·8	3·0	3·8	..	1·4	..	..	..
158	554	3·5	2·8	3·5	2·8	·5	..	..	..
174	..	..	1·0	..	1·1	−0·6	..	0	..
198	..	..	1·6	..	1·6	−1·6	..	−0·4	−1·1
224	..	..	..	..	1·4	−1·6	..	−0·7	..
270	..	..	1·2	..	0·2	−4·1	−2·2	−4·4	−5·0
351	..	..	..	..	..	−4·1	−3·1	−5·1	−5·6

* Current fell markedly during pressure cycles. κ_1 is calculated on its initial, κ_2 on its final value.

TABLE X.

$\mathfrak{H}$	First "on" A_1	First "on" A_2	First "on" B_2	First "off" A_1	First "off" A_2	First "off" B_2	Cyclic "on" − "off"	First "on" Shock-effect ÷ $\mathfrak{H}_1$ A_1	First "on" Shock-effect ÷ $\mathfrak{H}_1$ A_2	First "off" Shock-effect ÷ $\mathfrak{H}_1$ A_1	First "off" Shock-effect ÷ $\mathfrak{H}_1$ A_2	Cyclic "on" − "off" ÷ $\mathfrak{H}_2$
1·8	2·4	‥	‥	‥	‥	‥	·29	·51	‥	‥	‥	·044
2·2 p	‥	‥	‥	·1·3	‥	‥	·34	‥	‥	·2	‥	·034
3·4	4·8	‥	‥	‥	‥	‥	‥	·34	‥	·04	‥	‥
4·3 p	‥	‥	‥	−·1	‥	‥	·72	‥	‥	·098	‥	·035
8·1 p	‥	5·8	7·0	2·3	‥	‥	1·7	‥	‥	‥	‥	·036
8·6	10·1	‥	‥	‥	·3	‥	1·2	·24	·10	‥	·24	·026
12·5	9·9	‥	‥	‥	‥	‥	1·8	·13	‥	‥	‥	·027
23·4 p	‥	‥	16·1	·3	‥	1·5	2·6	‥	‥	·02	‥	·017
23·7	16·3	‥	17·3	‥	‥	‥	2·5	·09	‥	‥	‥	·016
35·2 p	‥	‥	‥	−2·6	‥	‥	4·4	‥	‥	‥	‥	·017
35·7	16·6	13·8	12·7	‥	1·0	‥	5·3	·048	·032	·007	·012	·021
46·0	17·0	12·4	14·8	‥	‥	‥	4·6	·043	·028	‥	‥	·015
47·5 p	‥	12·9	12·5	−1·4	−1·6	‥	3·6	‥	·026	·007	·007	·011
57·5	16·5	10·0	10·5	‥	−1·2	‥	4·1	·036	·018	‥	·005	·012
57·5 p	‥	10·5	‥	−2·3	−1·2	‥	3·3	‥	·020	·003	·003	·009
72·6	10·9	10·5	‥	‥	−1·1	‥	2·4	·022	·020	‥	·003	·006
79·9	11·7	10·0	‥	‥	−·8	‥	2·4	·023	·018	‥	·003	·006
92·0	11·0	5·5	4·7	‥	−·7	‥	2·1	·021	·009	‥	·003	·005
103·5	5·2	‥	‥	‥	+1·1	‥	·7	·009	‥	‥	·003	·001
115·0	4·7	2·1	1·8	‥	+1·1	‥	−·45	·008	·005	‥	·001	−·001
128	2·6	0·9	0·9	‥	+1·6	‥	−·51	·006	·003	‥	·001	−·001
144	1·2	0·6	‥	‥	+2·0	‥	−·9	·004	·003	‥	·001	−·002
158	0·9	−0·4	−0·3	‥	+2·1	‥	−1·1	·004	·002	‥	·002	−·002
174	−1·4	−0·5	1·1	‥	+2·8	‥	−1·4	0	·002	‥	·002	−·0025
198	−2·4	‥	−2·2	‥	+2·6	‥	−1·7	−·001	‥	‥	·001	−·003
224	−1·3	−1·8	−2·4	‥	+3·0	+3·0	−1·6	·001	−·001	‥	·001	−·003
270	−2·7	−2·7	‥	‥	+3·4	+3·4	−2·2	−·001	0	‥	·001	−·004
351	‥	‥	‥	‥	‥	‥	−2·7	‥	‥	‥	‥	−·004

§ 36. In Table IX. the first six headings have been already explained. The letter p after the numerical measure of the field in the first column signifies that the rod was under pressure previous to and during its introduction into the coil. In such a case the entries under $\mathfrak{I}_1$ and κ_1 refer to the magnetisation existing before this pressure was removed, and so are on a different footing from the other entries under these headings. The entries under the other headings are also in such a case doubtless indirectly affected, but to a much smaller degree.

In the last four headings the suffix indicates which end of the rod was the north pole when the magnetisation in question was measured. The results under these headings are deduced from the readings obtained by removing and replacing the rod subsequent to the application of the pressure cycles, during which application it must be remembered A was always the north pole.

The seventh column gives the excess of the magnetisation existing when the rod was under pressure when reintroduced into the coil over that existing when it was free from pressure when reintroduced, the end A being in each case the north pole of the reintroduced rod. The eighth column differs from the seventh only in that the end B was in each case the north pole of the reintroduced rod. The ninth column gives the excess of the magnetisation found when A was the north pole of the reintroduced rod over that found when B was the north pole, the rod in each of its reintroductions being free from pressure. The tenth column differs from the ninth only in that the rod during each of its reintroductions was under pressure.

§ 37. In Table X. *First "on"* is, as previously, the total change in the magnetisation—cyclic and non-cyclic—due to the first application of pressure. A_1 gives the effect when the rod, immediately after being demagnetised, has been exposed for the first time to the field in question. A_2 gives the effect of the first pressure applied after the *reintroduction* of the rod in a state free from pressure, and with the end A a north pole. In the case of the results under A_2, there have intervened since demagnetisation a complete series of pressure cycles applied with A a north pole, and at least one removal and reintroduction of the rod. In some cases there were several removals and reintroductions, and the mean of the observations is recorded. Under B_2 we get the effect when, on its reintroduction, the rod has B for its north pole. In this case it must be borne in mind that the pressure cycles applied since the preceding demagnetisation occurred when the opposite end A was the north pole.

First "off" is the total effect of the removal of a pressure existing previous to and during the introduction of the rod into the coil. A_1 refers to the case when demagnetisation has immediately preceded the observation, A_2 and B_2 to the cases when a complete series of pressure cycles, with A the north pole, and at least one removal and reintroduction have intervened since demagnetisation. The distinction between A and B is the same as in the case of the *First "on."*

Cyclic "on"—"off" has its usual meaning. Under the next two headings are given the non-cyclic portions of the effects of the first application of pressure and of the first

removal of a pre-existing pressure. A_1 and A_2 refer to the results recorded in the first two columns under the headings *First " on "* and *First " off."* The last heading has been already explained. As usual .. indicates that no observations were taken. When no algebraic sign is attached to a figure $+$ is understood.

§ 38. To a casual observer the fluctuations that appear in the results of the foregoing tables may seem excessive. It should, however, be borne in mind that most of the pressure effects are very small compared to the total amount of the induced magnetisation—sometimes as little as one-thousandth part. Thus the small irregularities in the magnetising current which are produced by the slightest want of constancy in the cells, or by the vibrations communicated to the resistance wires from shakings of the floor, may produce commensurable effects. Such irregularities were frequently recognised through a continual quivering motion of the spot on the scale, while the ammeter reading appeared steady enough. Of course, when the unsteadiness was very marked, observations were suspended, but comparatively little of the work done would have been accomplished in the time if observations had been taken only when absolute steadiness prevailed.

§ 39. A separate discussion of the results of each table would involve a good deal of repetition. It has thus appeared best, as a rule, to embody the most trustworthy results in curves, and to discuss particular points in connection with the features of the individual curves. In every curve the horizontal coordinate gives the strength of the field, $\mathfrak{H}$, calculated after the manner explained in § 19. In the case of the residual magnetism, $\mathfrak{H}$ is of course the strength of the pre-existing field.

In fig. 1 (Plate 15), the ordinates give the initial magnetisation when the rod is introduced into the magnetising coil free from pressure. In fig. 3 the ordinates give the coefficient of induced magnetisation under the same conditions. The first portions of both curves, in which the individual observations are indicated by dots, are based on Table IX., while the second portions, in which the individual observations are indicated by crosses are based on Table III.

These curves are of the same general character as those obtained by many observers for iron. In fields below 8 or 9 C.G S. units, the magnetisation increases comparatively slowly as the strength of the field is raised. There then ensues, as is most clearly shown by the steepness of the commencing portion of the curve of fig. 3, a much more rapid increase of magnetisation. The rate of increase attains a maximum, as shown in fig. 3, in a field of about 35 C.G.S. units ; which, accordingly, is the *Wendepunkt* for the specimen. The rate of increase of the magnetisation then falls off, but at first in a comparatively gradual manner. There is thus no very clear indication of an approach to " saturation " until the strength of the field approaches 200 C.G.S. units. Even in the strongest experimental fields the rate of increase is by no means infinitesimal, though, as already stated, this is probably in part accounted for by the heating of the rod.

§ 40. The smallness of the scale of fig. 1, does not allow the effect of pressure on

the induced magnetisation to be shown, but the enlarged scale of fig. 2 shows the effect in fields below 30 C.G.S. units, where it is of most importance.

The thick line a, with the individual observations indicated by dots, is merely the initial portion of fig. 1 on an enlarged scale. The dotted line b, with the individual observations indicated by circles, gives the induced magnetisation after the application of the pressure cycles. The curves, a and b, are based on the same experiments, and the difference between their ordinates answers to the values in the column $\mathfrak{I}_2 - \mathfrak{I}_1$ of Table IX. in those experiments in which the rod was originally free from pressure.

The third curve c, with the individual observations indicated by crosses, gives the induced magnetisation prior to the application of pressure cycles, in these experiments denoted by a p in Table IX., in which the rod was under pressure when introduced into the coil.

In fig. 4 the ordinates give the coefficients of magnetic induction in the three cases of fig. 2. The data are taken from the same table, and the letters, &c., have the same significations.

The difference between the curves a and b shows the very large effect of the pressure cycles in increasing the magnetisation in the weaker fields. It should be remembered that between the corresponding pairs of observations on which these curves are based there intervened no break of the current, nor any change in the resistance, or in the position of the ammeter. Thus any error in the zero or orientation of the ammeter would affect each curve alike, and leave the difference of the ordinates practically unaffected. Variations, it is true, in the strength of the current during the application of pressure cycles, would affect curve b without affecting curve a. Such variations, however, could escape notice in the weaker fields only when very small, because the ammeter was then in its most sensitive position.

The comparison of the curves c with the others is not so satisfactory. During the time of most of the experiments the ammeter was being used for other purposes, and so had to be set up afresh every other day. Thus, though its position was carefully adjusted, small differences were certain to occur.

For these and other reasons it would be unsafe to draw any conclusions from the crossing of the curves b and c in fig. 4. It may, however, be regarded as perfectly certain that in the weaker fields the curve c lies between the curves a and b, and that within the limits of fig. 4 the curve c lies very distinctly above the curve a.

The form of all these curves of fig. 4, suggests that under all conditions as to pressure the coefficient of magnetic induction becomes extremely small in very weak fields. There may, however, be a turning point in one or all of the curves in fields lower than those experimented on.

§ 41. In curve a of fig. 5 the ordinates give the total change in the magnetisation produced by the first pressure in those experiments of Table IX. in which the rod was free from pressure when introduced into the coil. The change in magnetisation is measured with the pressure " on," and so is really the algebraic sum of the cyclic and

non-cyclic effects. The individual observations are indicated by dots, whose distances from the curve are in no case serious.

The curve shows that the total effect of the first pressure attains a distinct maximum in a field of about 45 C.G.S. units—which somewhat exceeds the Wendepunkt—and then diminishes somewhat rapidly as the strength of the field is raised. An unmistakeable critical field, where the effect vanishes, appears at about 160 C.G.S. units, and in stronger fields the total effect of the first pressure is a diminution of magnetisation. From the form of the curve it would appear that the diminution does not increase indefinitely as the strength of the field is raised. But the experiments leave it uncertain whether this diminution of magnetisation attains a maximum in a field of about 350 C.G.S. units, or whether it continually approaches an asymptotic value.

§ 42. In curve b of fig. 5, the ordinates give the loss in the induced magnetisation accompanying the removal of the rod from the coil, after the application of the pressure cycles, and its reintroduction. The curve is based on the column $\mathfrak{I}_2 - \mathfrak{I}_3$ of Table IX.

The ordinate corresponds more or less closely to the non-cyclic effect of the sum of the pressure cycles applied during the flow of the current prior to the rod's removal. There are two reasons however, why it cannot be regarded as an exact measure of this effect, more especially in the weaker fields. In the first place, even when no pressures are applied it is well known that the magnetisation of iron is not exactly the same on the second exposure to a certain field as it is on the first. Some occasional observations on the cobalt itself gave a somewhat greater magnetisation on a second exposure than on the first. The difference was, however, unimportant, and in fields over the Wendepunkt it was, if existent, extremely small. In the second place part of the effect of the pressure cycles unquestionably survives a removal of the rod from the magnetic field. As will be seen from § 65, this residual effect is of considerable importance in fields below 30 units. Owing to both these causes the effect of the pressure cycles in fields below 30 or 40 units must be decidedly greater than the curve b would indicate. No observations exist to show whether or not the curve b eventually crosses the axis of abscissæ. Not improbably this would be a very difficult point to settle owing to the heating of the rod.

The reason for drawing a curve for $\mathfrak{I}_2 - \mathfrak{I}_3$ in preference to one for $\mathfrak{I}_2 - \mathfrak{I}_1$, is that only a few minutes elapsed between the taking of the readings giving $\mathfrak{I}_2$ and $\mathfrak{I}_3$, while the reading which gives $\mathfrak{I}_1$ might have been taken over an hour previously. Thus under ordinary circumstances the variation of the current could not but be utterly insignificant in the one instance, whereas in the other it occasionally reached a measurable quantity.

In Table IX. omitting of course the fields marked p, the values of $\mathfrak{I}_2 - \mathfrak{I}_1$ and $\mathfrak{I}_2 - \mathfrak{I}_3$ show a very fair agreement. In fields below the Wendepunkt the former quantity is, as the above reasoning suggests, distinctly the larger ; but in stronger

fields the one is sometimes the larger and sometimes the other. In no single case was a negative value obtained for either quantity, though in fields over 200 C.G.S. units, both become unquestionably very small.

The only safe conclusion appears to be that if a critical field existed for the non-cyclic or shock-effect of the pressure cycles it exceeded 275 C.G.S. units. Considering, however, that the critical field for the cyclic effect of the pressure cycles is only about 120 C.G.S. units, this seems an important result.

§ 43. In the curve of fig. 6, which is based on Table X., the ordinates give the ratio of the increase in the induced magnetisation produced by the non-cyclic part of the first pressure relative to the amount of the induced magnetisation existing previous to the pressure. The ordinates may thus be held to measure the relative importance in fields of various strengths of the non-cyclic portion of the effect of the first pressure.

The curve shows in a striking manner how the relative importance of the non-cyclic portion of the effect of the first pressure continually diminishes as the strength of the field increases. In the weakest field employed, viz., 1·8 C.G.S. units, the first pressure permanently increases the magnetisation by one half its original value, whereas in a field of 160 C.G.S. units the increase is certainly less than one part in two hundred.

In stronger fields, as appears from the ninth and tenth columns of Table X., this shock-effect becomes extremely small, and continues to be so within the range of the experiments.

The experiments cannot be said to settle conclusively the sign even of the effect, but the evidence is decidedly in favour of its remaining positive in fields of at least 270 C.G.S. units. It will, in fact, be observed that in the tenth column of Table X., a decrease of magnetisation was in no single case obtained. Now in the higher fields the results in this column are calculated from the mean of two or three observations, whereas those in the preceding column answer, of course, to only one observation.

Effects of the Removal of Pressure.

§ 44. It might appear at first sight that the increase in the magnetisation invariably found to accompany the application of the sum of the pressure cycles in fields up to 270 C.G.S. units is in itself sufficient proof that the non-cyclic effect of the first pressure must be within the same limits an increase of magnetisation. The increase, however, in the former case might be due to the removals not the applications of pressure.

The removal of pressure from a rod occasions relative displacements of its parts to pretty much the same extent as does the application of pressure, and so it too may be expected to have some permanent influence on the magnetisation. This effect was actually found to exist, and its magnitude relative to the original magnetisation is

given in the eleventh and twelfth columns of Table X. Comparing these with the two previous columns, it will be seen that in weak fields the non-cyclic effect of the removal of the original pressure, though much less important than the corresponding effect of the application of the first pressure, is by no means a negligible quantity. It may, however, be easily overlooked, because in the weaker fields, where its importance is greatest, it is in the opposite direction to the cyclic effect of the removal of pressure. Also in fields between 25 and 95 C.G.S. units, as appears from the thirteenth column of Table X., the cyclic part of the effect is numerically the larger, and so between these limits the removal of an original pressure appears to be accompanied by a diminution in the magnetisation. In fields over 120 C.G.S. units the non-cyclic effect has the same sign as the cyclic, and so the increase in magnetisation accompanying the removal of an original pressure is fairly conspicuous.

Cyclic Effect of Pressure on the Induced Magnetisation.

§ 45. In fig. 7 the ordinates give the cyclic change in the induced magnetisation accompanying the pressure cycles. The thick line a, in which the individual observations are indicated by dots, is based on Table X., the broken line b with individual observations indicated by crosses, on Table I., and the dotted line c with individual observations indicated by circles, on Table III.

These curves agree in showing a critical or VILLARI field of about 120 C.G.S. units. The magnetisation is greatest or least when the rod is under pressure according as the strength of the field is less or greater than that of the critical field.

It will be seen from fig. 1 that the magnetisation of the rod in the critical field is about 520 units. According to both the curves, a and b, the absolute magnitude of the cyclic effect attains a maximum in a field of about 35 C.G.S. units. This, it will be remembered, is the field found for the Wendepunkt in § 39. In general form the two curves a and b could hardly agree better than they do, and the difference in the absolute lengths of their ordinates might well be due to a difference between the standard pressures adopted at the times. Both curves, it will be noticed, pass through almost all their experimental points. Near the critical field all three curves would lie so close together that only the experimental points of c are there shown.

The highest field in Table IX. is 351 C.G.S. units, and the form of the curve a leaves it uncertain whether the cyclic effect attains an algebraic minimum in a field somewhat higher than this and then diminishes numerically, or whether it continually approaches an asymptotic value. The former alternative is unquestionably supported by the form of the curve c, which extends to a field of 725 C.G.S. units. The deviations of the experimental points from this curve are, however, so large that it would be rash to attach much weight to its precise form. Probably all we are entitled to infer is that, if an asymptotic value exists for the cyclic effect, its numerical value cannot much exceed that found in a field of 400 C.G.S. units; whereas, if a second

critical field exists, where the effect changes sign a second time, its value must very considerably exceed 700 C.G.S. units.

It should be noticed that the ordinates in fig. 7 are drawn on double the scale of those in fig. 5. Thus, comparing in those figures the two curves a, which are based on the same series of experiments, we see that, in fields below the VILLARI point of fig. 7, the total effect of the first pressure is invariably very considerably larger than the cyclic effect of pressure. The maximum value attained by the former effect is about thrice that attained by the latter.

§ 46. In fig. 8 the ordinates give the ratio of the cyclic increase in magnetisation accompanying pressure " on " to the amount of the induced magnetisation existing when the cyclic state is reached. They thus show what proportion of the magnetisation takes part in the cyclic change.

The curves a and c are derived from the same experiments as the corresponding curves a and c of fig. 7. The form of a, near the vertical axis, shows how much the relative importance of the cyclic effect increases as the strength of the field is reduced.

In comparing the relative magnitude of the cyclic effect and of the shock effect of the first pressure, it should be noticed that the ordinates of fig. 8 are drawn on a scale ten times that on which the ordinates of fig. 6 are drawn. Thus, in the weakest fields, the shock effect is fully ten times the cyclic effect.

Residual Magnetisation.

§ 47. In figs. 9 and 10 (Plate 16) the ordinates give the amount of the residual magnetisation existing immediately after the break of the current. The curves b and d are shown in both figures, but on a different scale. The curve a is based on Table V. The rod was here demagnetised before exposure to the weakest field, but not subsequently, and remained entirely free from pressure. The curve b is based on Table VI. The rod was here demagnetised by reverse currents and vigorous tapping before each introduction into the coil, but no pressures were applied while it was under the influence of the magnetising currents. The curves c and d are based on Tables VII. and VIII. respectively. In both cases the rod was demagnetised, as in the case of curve b, but was subjected to six pressure cycles while under the influence of the magnetising current. In the case of c the pressure was " off " when the current was broken, whereas in the case of d the pressure was " on." The individual observations are indicated by dots for the curves b, by circles for the curve c, and by crosses for the curves a and d. With the scale of fig. 9 the curve c would be indistinguishable from the curve d in weak fields, and from the curve b in strong fields.

Effects of Pressure on the Amount of the Residual Magnetisation.

§ 48. The curves a and b of fig 10 agree in showing only a small amount of residual magnetisation on the break of weak fields ; but in fields below 5 C.G.S. units

the ordinates of b are at least double those of a. The difference is surprisingly great, considering the similarity of the conditions under which the corresponding experiments were conducted. It must, presumably, be mainly attributed to the residual effect of the shocks employed in completing the demagnetisation in the experiments on which the curve b is based.

In weak fields, as the difference between the curves c and d on the one hand, and the curves a and b on the other abundantly proves, the effect of pressures applied during the flow of the current in increasing the residual magnetisation is simply enormous. It would thus appear a $priori$ probable that the application of shocks immediately before the starting of the current should have some appreciable tendency in the same direction. Fortunately I happened to observe the induced as well as the residual magnetisation for the field 2·6 C.G.S. units in the experiments on which curve a is based. The numerical value of the induced magnetisation was 8·3, which is almost exactly the value obtained by interpolation for the same field from Table IX. Consequently, in this field, and so presumably in other weak fields, the difference between the curves a and b cannot be attributed, in any important degree, to an increase in the induced magnetisation brought about by the process of demagnetisation employed in the case of curve b.

As the strongest field in Table V. is only 79 C.G.S. units, no data exist for drawing the curve a in higher fields. It is only, however, in fields below 45 or 50 C.G.S. units that there is any clear difference between the results of Tables V. and VI. We are thus probably justified in concluding that the process of demagnetisation has an appreciable effect on the amount of the residual magnetisation only in fields below 50 C.G.S. units.

As already stated, the results of Table V. are each the mean of two observations taken in close succession with the same current. Almost invariably the second observation showed a slightly larger amount of residual magnetisation than the first. This property has been noticed by several observers in iron.

The difference between the curves b and c of fig. 10, which are based on experiments in which the same process of demagnetisation was applied, shows how effective pressure cycles are in increasing the amount of the residual magnetisation in weak fields. In stronger fields than those of fig. 10 the curves gradually approach one another, and in fields exceeding 120 or 130 C.G.S. units they cannot with certainty be said to differ. Thus, in fields below 120 or 130 C.G.S. units, the application of pressure cycles during the flow of the current increases the amount of the residual magnetisation, but in stronger fields no effect can with certainty be said to exist.

In fields below 30, or at all events 20 C.G.S. units, the difference between the curves c and d is, though small, perfectly clear and incontestable. The existence of pressure during the break of the current is the only point in which the experiments on which curve d is based differed from those on which curve a is based. As the effect of pressure " on " during the flow of the current is an increase of induced magnetisation

in fields below 120 C.G.S. units, an excess in the ordinates of curve d over those of curve c in weak fields might reasonably have been expected. The difference, however, as will more fully be seen presently, is somewhat greater than might have been anticipated.

The crossing of the curves c and d in a field of from 35 to 40 C.G.S. units, and thus far below the VILLARI field, appears a somewhat striking fact. In fields over 120 C.G.S. units the curve c, if shown in fig. 9, could not be distinguished from curve b, and the difference between its ordinates and those of curve d seems then truly remarkable. In fields over 150 C.G.S. units this difference remains nearly constant, and amounts to about 4 per cent. of the ordinates of c. If it be remembered that according to the last column of Table X. the percentage of the induced magnetisation, which is cyclic for the same pressure " on " and " off," is in fields between 150 and 400 C.G.S. units never in excess of ·4, the significance of this result will be more fully understood.

§ 49. The ratio of the residual magnetisation existing in the rod immediately after the break of the current to the numerical value of the strength of the pre-existing field will, for shortness, be here termed the *residual susceptibility*. It is denoted by κ_1' in Tables V. to VIII., and its values are represented graphically by the ordinates of the curves of fig. 11. In this and in the subsequent figures the letters a, b, c, d denote curves obtained from the same series of experiments as are the respective curves a, b, c, d of fig. 10. In fig. 11 and the subsequent figures, however, the individual observations are indicated by crosses for curves a and c, by dots for curve b, and by circles for curve d.

In the absence of pressure, as shown by curves a and b, the residual susceptibility is very small in weak fields, but increases rapidly, attaining a maximum in a field a little over 20 C.G.S. units. This, it will be noticed, is not much over half the field which gives the Wendepunkt for the induced magnetisation. The greatest value actually observed for κ_1' in the case of curve b, was only 1·36. After the maximum is passed there is a continuous, but gradual, diminution in the residual susceptibility.

The difference between the commencing portions of curves a and b shows, even more clearly than in fig. 10, the large effect in weak fields of the process of demagnetisation. In fields over 50 C.G.S. units the two curves practically coincide, and are drawn as one.

The crossing points for the various curves in fig. 11 are, of course, the same as in figs. 9 and 10. Within the range of the figure the curve c lies distinctly above the curve b. In fields over 55 C.G.S. units, the curve d is not drawn, as it could hardly within the remaining limits of the figure be distinguished from curve b.

Commencing with the strongest experimental fields, the ordinates of the curves c and d increase gradually as the strength of the field is reduced, and show maxima values in somewhat lower fields than do the curves a and b. As the strength of the fields is further reduced, the ordinates of c and d pass through distinct minima values, and then increase rapidly as the strength of the fields approaches the lowest experi-

mental value. The value of κ_1' cannot, of course, exceed that of κ_1; so either the ordinates of c and d must be rapidly approaching second maxima, or else the ordinates of the curves b and c of fig. 4 must possess minima values in fields lower than were experimented on.

§ 50. In comparing the amounts of the residual and induced magnetisations, the employment of the term *retentiveness* in an exact way will be found serviceable. It is here used in accordance with the following definition :—

The retentiveness is the ratio of the residual to the induced magnetisation, both quantities being measured with the rod exposed to one and the same state of mechanical stress, and the stress remaining constant during the interval that elapses between the measurements.

In fig. 12 the ordinates give the retentiveness of the rod in the cases illustrated by the curves b, c, d. As the residual magnetisation alone was observed in the experiments recorded in Tables VI., VII., and VIII., the necessary values of the induced magnetisation were derived by interpolation from Tables IX. and X. In curve b the induced magnetisation is that existing prior to the application of pressure, or is the $\mathfrak{J}_1$ of Table IX. In curve c the induced magnetisation is $\mathfrak{J}_2$; for after six pressure cycles the increase in the magnetisation accompanying further pressure cycles is, for our present purpose, quite negligible. In the experiments answering to curve d, the rod, after experiencing six pressure cycles, was under pressure when the current was broken. Thus, for curve d the induced magnetisation is got by adding to $\mathfrak{J}_2$ the algebraic value of the *cyclic " on"—" off"* of Table X.

The curve b shows that, in the absence of pressure, the retentiveness attains a maximum in a field of about 15 C.G.S. units, a field somewhat below that at which the maximum residual susceptibility occurs in the corresponding curve of fig. 11. A great similarity exists between curve b and the curve for zero load in Professor Ewing's fig. 57. The principal difference is that the field at which the maximum retentiveness appears is considerably higher in cobalt than in iron.

The curves c and d of fig. 12 bear a considerable resemblance to the corresponding curves of fig. 11 ; but there are no longer distinct maxima or minima. Comparing these curves with curve b we see that in the weakest experimental fields pressure cycles increase the retentiveness in the ratio of four or five to one. The tendency to become tangential to the vertical axis shown by the curves c and d—which cannot, however, go on indefinitely as the fields are further reduced—is not exhibited by any of the curves for the retentiveness of loaded iron-wire occurring in Professor Ewing's fig. 57. Possibly if he had employed lower fields he might have found similar phenomena, so it would not be safe to assume that iron and cobalt actually differ in this particular.

The ordinates of curve d being greater in weak fields than those of curve c, it follows that in fields below 30 C.G.S. units, where the curves cross, the retentiveness of the rod is greatest when it is under pressure while the current is broken. In fields

over 30 C.G.S. units the retentiveness is least when the current is broken with pressure " on," and in fields over 50 units the difference is not inconsiderable. In fields over the VILLARI point, of 120 C.G.S. units, the induced magnetisation is greater for curve *c* than for curve *d*, so that in such fields the excess of the ordinates of curve *c* over those of curve *d* would, relatively to their absolute magnitude, be somewhat less in the case of fig. 12 than in the case of fig. 11.

Similar considerations lead to the conclusion that in fields over 130 C.G.S. units the curve *b* would be, if anything, slightly above the curve *c*. In other words, in fields over 130 C.G.S. units the effect of pressure cycles is, if anything, slightly to diminish the retentiveness. This effect is, however, insignificant compared to that proceeding from the existence of pressure " on " during the break of the current.

The curves *b* and *d* cross in a field of about 40 C.G.S. units. The curve *d* would then become the lower, but it would be for a time so close to *b* that it is not attempted to show it separately in the figure.

The conclusions which the preceding considerations lead to are the following :—

In fields below 30 C.G.S. units the application of cycles of pressure " on " and " off " during the flow of the current, and the existence of pressure " on " during the break of the current, both tend to increase the retentiveness, the former agency being in fields below 10 C.G.S. units far the more important. In fields between 30 and 120 C.G.S. units, or thereby, the application of cycles of pressure still increases the retentiveness, but the effect continually diminishes as the strength of the field is raised. In fields over 130 C.G.S. units up to at least 300 or 400 C.G.S. units the effect of pressure cycles is very small, but is, if anything, a diminution of the retentiveness. In fields from 30 C.G.S. units up to at least 300 or 400 C.G.S. units the existence of pressure " on " during the break of the current diminishes the retentiveness, and the effect is of the same order of magnitude as that due to the same cause in fields below 30 C.G.S. units. In fields between 30 and 40 C.G.S. units, or thereby, the increase in the retentiveness occasioned by the pressure cycles exceeds the diminution occasioned by the existence of pressure " on," but in stronger fields the latter effect is the larger, and its superiority appears continually to increase with the strength of the field.

§ 51. Curve *b* of fig. 12 is repeated in Curve I. of fig. 13, in order that its form in fields over 75 C.G.S. units may be seen. There appears a continual but very slow diminution in the retentiveness as the field is raised to 400 C.G.S. units. Curve II. of fig. 13 will be discussed presently.

§ 52. In fig. 14 the ordinates give the absolute amount of the residual magnetisation which seems to disappear with the first alteration of stress after the break of the current.

Comparing the curves *b* and *c*, in which the loss of magnetisation accompanies the application of pressure, with the curve *d*, in which it accompanies the removal of the pressure existing during the break of the current, we see that the effect must be to a

large extent independent of the precise character of the change of stress, and would doubtless follow more or less any mechanical agitation of the rod.

The curve b is drawn through nearly all the experimental points, but in the stronger fields of c and d, especially the latter, the experimental points are somewhat widely scattered. These latter curves accordingly cannot be trusted in their minute details. The curves are based on the data in the fourth columns of Tables VI., VII., and VIII. These tables give only certain average results for fields over 90 C.G.S. units, but they indicate that the ordinates of b and c would attain maxima values in fields somewhere between 100 and 200 C.G.S. units. Thereafter the ordinates of c would apparently diminish the more rapidly of the two, so that in fields of 400 C.G.S. units the curve c would have approached pretty close to the curve d, which remains apparently nearly horizontal.

§ 53. The application of cycles of pressure causes cyclic changes in the magnetisation, whose character will presently be discussed. In consequence of this the effects of the first change of stress on the residual magnetisation are only, in part, of a non-cyclic character. To get the exact quantity of magnetisation whose *permanent* disappearance is secured by the first change of stress, the cyclic effect must be allowed for.

Suppose, for instance, a field of 50 C.G.S. units is broken when pressure is " off." The cyclic effect is then a diminution of magnetisation when pressure is " on." Consequently the loss of magnetisation accompanying the first application of pressure exceeds the quantity whose permanent disappearance is secured by the amount which takes part in the cyclic change. Subtracting from the total loss the numerical value of the cyclic effect, we get what may be regarded as the true loss of residual magnetisation due to the first application of pressure. This quantity is that here termed the shock-effect of the first pressure.

In the case of residual magnetism the cyclic effect is in general small compared to the shock-effect, so that a considerable error in the cyclic effect will seldom seriously affect the calculation of the shock-effect.

§ 54. In fig. 15 the ordinates represent the ratio of the shock-effect of the first alteration of pressure to the amount of the pre-existing residual magnetisation. They may thus be regarded as measuring the efficacy of the first change of pressure in shaking out the residual magnetisation. Unfortunately no readings were taken of the cyclic effects of pressure in the weaker fields in the series of experiments on which the curves are based. Thus up to the points where the transverse lines are drawn, the ordinates represent the ratio of the *total* effect of the first change of pressure to the initial residual magnetisation. Consequently, as will be seen presently, in the commencing portions of the curves the ordinates of b and c are probably somewhat underestimated, and those of d on the other hand overestimated. The corrections could, however, hardly modify the relative positions of the curves.

The curves all show that the efficacy of the first change of pressure in shaking out

the residual magnetisation, as measured by the percentage got rid of, continually diminishes as the strength of the field is increased. According to the seventh columns of Tables VI. and VII., and the sixth column of Table VIII., on which the curves are based, this diminution most probably goes on up to fields of 400 C.G.S. units at least. The difference, however, between the percentages shaken out in the case of fields of 100 and 400 C.G.S. units is under all conditions very small.

Comparing curves b and c, in figs. 14 and 15, and the seventh columns of Tables VI. and VII., we see that while in fields below 15 C.G.S. units an absolutely larger amount of residual magnetisation is shaken out by the first pressure when pressure cycles have been applied during the flow of the current, still in these and in all higher fields within the experimental limits a greater percentage is removed by the first pressure when no pressures have been applied during the current.

We thus see that in fields below 100 C.G.S. units, pressure cycles during the flow of the current diminish simultaneously the percentage of the induced magnetisation got rid of by breaking the current, and the percentage of residual magnetisation got rid of by the subsequent application of a pressure. In fields from 130 to at least 400 C.G.S. units pressure cycles during the flow of the current appear slightly to increase the percentage of induced magnetisation got rid of by breaking the current, while simultaneously diminishing the percentage of residual magnetisation got rid of by the subsequent application of a pressure. In these strong fields, however, both these effects are comparatively insignificant.

§ 55. In the experiments on which the curves b and c are based the effects of removing the first pressure applied after the break of the current were also observed. The invariable result was a loss of magnetisation, and its magnitude is recorded in the fifth columns of Tables VI. and VII. In considering how much of this loss is permanent, the cyclic effect of the change of pressure must be allowed for. This has been done in calculating the eighth columns of Tables VI. and VII., which give the relative importance of the first pressure and its removal in permanently reducing the magnetisation.

When no pressures have been applied during the flow of the current we see from Table VI. that, in fields between 18 and 75 C.G.S. units, the removal of the first pressure produces uniformly about one-ninth of the effect of its application. In the higher fields the relative importance of the removal of pressure shows a distinct diminution.

Comparing Tables VI. and VII. we see that in fields below 50 C.G.S. units the importance of the removal of the first pressure after the break of the current relative to its application is largely increased by the application of pressure cycles during the flow of the current. In fields over 50 C.G.S. units the effect of the removal bears to that of the application of the first pressure in the case of Table VII. the apparently nearly constant ratio of $1:9$, which still somewhat exceeds the ratios found in the corresponding fields of Table VI.

The fact that the importance of the removal of the first pressure relative to its application is greatest in weak fields must, I think, be connected with the fact that in weak fields, as shown by curves c and d, fig. 15, a larger proportion of the residual magnetisation is shaken out by the removal of a pressure existing during the break of the current, than by the application of an equal pressure when the rod is free from pressure during the break of the current. In the weak fields pressure cycles, which exist in the case of curves c and d, cause a large permanent increase in the induced magnetisation, and, of this, a much larger proportion is due to the application than to the removal of pressure. Thus, the phenomena suggest that the residual magnetisation arising from the portion of the induced magnetisation which the application of pressure enables the rod to acquire is more easily shaken out by the removal of pressure than is the remainder of the residual magnetisation.

It may also be worth mentioning that by increasing the magnitude of the first pressure after the break of the current the effect of its application was increased, but the effect of its removal diminished absolutely as well as relatively.

Cyclic effect of Pressure on the Residual Magnetisation.

§ 56. The cyclic effects of pressure cycles on the residual magnetisation during the experiments on which the curves b, c, d are based, are shown in the sixth columns of Tables VI. and VII. and the fifth column of Table VIII. The cyclic effects were also observed in the December experiments, and are recorded in the fifth column of Table II. In the experiments of Table II. the demagnetisation was incomplete, but otherwise the conditions were identical with those of the experiments of Table VII.

The cyclic effect was found very difficult to measure accurately, partly on account of its smallness, and partly, doubtless, on account of its sensitiveness to small variations in the treatment of the rod during the flow, and more especially the break of the current. In the weaker fields the cyclic effects were observed only in the experiments of Table II., which were inferior in accuracy to the later observations.

For these reasons the numerical values attributed to the cyclic effects in the tables cannot claim to be more than somewhat rough approximations. Thus, while believing the results to represent correctly the general features of the phenomena, I have not embodied them in curves, as the precise forms of these might have owed too much to the imagination.

§ 57. Taking first the case when pressure cycles were applied during the flow of the current but no pressure existed during the break, we see from Tables II. and VII. that the cyclic effect in the residual magnetisation is an increase or a decrease of magnetisation when pressure is "on," according as the pre-existing field is below or above a certain critical field. So far the parallelism between the residual and induced magnetisations seems complete. The critical field, however, for the residual magnetisation is according to both tables only about 30 C.G.S. units, and so only about a quarter of

that for the induced magnetisation, and the intensity of the critical magnetisation in the former case is less than an eleventh of that in the latter. Table VII. shows no sensible variation in the magnitude of the effect as the field is raised from 70 to 400 C.G.S. units.

Taking next the case when no pressures were applied during the flow of the current, we see from Table VI. that in a field of 18 C.G.S. units—the lowest in which observations of the effect were taken—no sure cyclic effect was detected. Thus a critical field must exist at or very close to this field. In higher fields there is a clear diminution of magnetisation accompanying pressure " on," and in fields over 100 C.G.S. units the results agree as closely with those of Table VII. as they well could. Thus, the only conspicuous difference in the cyclic effect produced by pressure cycles during the flow of the current is a rising of the critical field.

Table VIII., however, shows that a truly remarkable difference occurs in the phenomena when pressure exists during the break of the current. In this case, in fields from 11·5 to 400 C.G.S. units, the residual magnetisation in the cyclic state is invariably greater when pressure is " on " than when it is " off." The magnitude of the cyclic effect apparently diminishes at first as the field is raised, but in fields of from 70 to 400 C.G.S. units, it is at least approximately constant.

It thus appears that the sign even of the cyclic change of the residual magnetisation accompanying cycles of pressure, is altered by such a seemingly trifling circumstance as the existence of pressure during the break of the pre-existing current.

It seems almost unnecessary to point out that the facts stated in this paragraph convincingly show that the effect on the magnetisation of cyclic applications of pressure is not determined solely, or in some circumstances even principally, by the measure of the rod's magnetic moment, or its so-called intensity of magnetisation.

§ 58. As the conclusions of the last paragraph as to the cyclic effect seem of considerable importance, it may not be out of place to supply data by which the value of the evidence on which they are based may be fairly judged.

The precautions taken in varying the order of the experiments have been already mentioned in explaining the Tables VI., VII., and VIII.

Now in fields stronger than the critical field of Table VII., there were, in all, 46 separate experiments of the types L, M, and N. Answering to pressure " on," there appeared a minimum of magnetisation in each of the 31 experiments of the types L and M, a maximum in each of the 15 experiments of type N. As the difference was wholly unexpected, and the observer had no preconceived ideas on the subject, it is difficult to conceive how the evidence could be stronger.

§ 59. The ordinates of Curve II. of fig. 13, give the fraction of the residual magnetisation, which is removed by a definite number of pressure cycles in the experiments of Table VI., or type L.

The percentage removed continually diminishes as the strength of the pre existing field rises. The rate of diminution in the percentage removed is very rapid as the

strength of the field is raised to 30 or 40 C.G.S units, but very slow when the field is raised over 150 C.G.S. units.

A comparison of this curve with the curves of figs. 4 and 6 brings out clearly the fact that the residual magnetisation, which is most easily removed by pressure cycles, is that left on the break of those fields in which pressure cycles have the greatest effect on the induced magnetisation.

The data in the last columns of Tables VII. and VIII. would lead to curves very closely resembling Curve II. Thus the preceding remarks apply equally to the experiments of types M and N.

§ 60. Comparing Curves I. and II. of fig. 13, we see that in fields over 30 C.G.S. units, the increase in the percentage of the induced magnetisation removed by breaking the current, and the diminution in the percentage of the residual magnetisation removed by a definite number of pressure cycles, go hand in hand.

It has been already mentioned that the application of a particularly severe first pressure after the break of a current lessened the effect of the removal of the pressure, and it also lessened the joint effect of succeeding pressure cycles, in which the pressure was of the normal intensity.

Thus the close resemblance of Curves I. and II. in the stronger fields suggests that the sudden break of a current acts in some respects as a mechanical shock, and that the magnitude of this shock-effect continually increases as the strength of the field is raised.

Separation of the Effects of the Application and Existence of Pressure.

§ 61. There are still some points connected with the fundamental character of the effect on the rod's magnetisation produced by the application of pressure, on which Tables IX. and X. throw some light.

We have seen that in general the first application of pressure after the rod's introduction into a given field, causes a change in the magnetisation. Further, as a rule, the greater portion of this change survives the removal of the pressure, so that for its continued existence the existence of pressure is not essential. It thus seems of importance to determine whether the change depends solely on the existence of pressure for a finite time during the flow of the current, or is due in whole or in part to the actual application of pressure.

Taking WEBER'S hypothesis for illustration, the change must consist either in a sub-permanent increase in the general mobility of the ultimate magnetic molecules, or in a swinging round of some of them into extreme positions, where they are kept by molecular friction even after the removal of the pressure. The application of pressure produces doubtless molecular movements, during which the rotations of the ultimate magnets might be expected to go on more freely, so that on this theory we should a priori be disinclined to attribute the entire effect to the mere existence of pressure.

In practice it is difficult if not impossible wholly to separate the effects of the application of pressure from those of its mere existence. When the rod is put under pressure before its introduction into the magnetising coil, and not introduced until sufficient time has elapsed for any molecular motion to die out, the magnetisation might at first sight be supposed to differ from that possessed by the rod when the pressure is applied after its introduction by an amount which exactly measures the effect of the actual application of the pressure. It must be remembered, however, that sudden magnetisation, altering as it does the rod's length, must act in part as a mechanical shock and set up molecular movements, whose character and amplitude would naturally depend to some extent on the state of the rod as to pressure.

In the following remarks the difference between the magnetisations of the rod under pressure in the two circumstances of its introduction is entirely attributed to the effect of the application of pressure, but the previous statements should be taken as a warning against accepting this as more than an approximation to the truth.

§ 62. Certain experiments bearing on the question have been already referred to in § 40 in discussing the curves of fig. 4. These experiments were limited to fields under 60 C.G.S. units; but within that range they showed conclusively that in a freshly demagnetised bar the existence of pressure applied before the rod's introduction into the magnetising coil increases the susceptibility. In the weaker fields of fig. 4 the curves show a greater susceptibility when the application of the pressure has succeeded than when it has preceded the introduction of the rod into the coil, and the evidence for this was considered satisfactory. In fields from 30 to 60 C.G.S. units the reverse is the case according to the curves, but reasons were given for regarding the evidence on this point as insufficient.

In these experiments when pressure existed during the introduction of the rod into the coil it had always been applied some minutes previously, so that any molecular movements set a-going by the process had presumably pretty well subsided. Further, any such movements must have been inconsiderable compared to those occasioned by the vigorous tapping which accompanied the demagnetisation of the rod shortly before its introduction in both sets of experiments.

We are thus in all probability entitled to conclude from these experiments that in a freshly demagnetised rod the mere existence of pressure, apart from the consequences of its application, has, in fields up to at least 60 C.G.S. units, the effect of permanently increasing the induced magnetisation, while in the weakest fields there can be no doubt that the actual application of pressure has an independent effect in the same direction.

§ 63. The results of a more complete series of observations bearing on the same point are given in the seventh column of Table IX., under the heading $\mathfrak{J}_A$ "on"—"off." This signifies, as already stated, the excess of magnetisation possessed by the rod when introduced into the coil under pressure, over that it possesses when introduced free from pressure. The circumstances of the rod are, except as regards the pressure,

identical in the two cases.   The rod was not, as in the experiments discussed in the last paragraph, in a freshly demagnetised condition, but had just been exposed to numerous pressure cycles in a field of the same strength and sign.

This column agrees with fig. 4 in showing in weak fields an increase of magnetisation due to the existence of pressure.   An unmistakeable critical field, however, ensues, the effect eventually changing sign.   Thus the character of the effect of the mere existence of pressure varies in the same way as does that of the joint effect of the application and existence of pressure, which we have already discussed in § 41.

In this case an exact basis of comparison exists, because in the third column of Table X., under the heading *First " on "* A$_2$, we have the combined effect of the application and existence of pressure when the previous treatment of the rod was the same as in the experiments of the seventh column of Table IX.

A comparison of these two columns shows that the fields in which the effects attain their maxima, and the critical fields must be nearly the same, if not absolutely identical in the two cases.   The magnitudes of the effects are, however, decidedly different.   There can be no question that in the weaker fields the effect is very considerably greatest when the application of the pressure succeeds the introduction of the rod into the coil, and this effect of the application of the pressure clearly continues until the field of 128 units is reached.   From this to somewhat over the critical field neither the application nor the existence of pressure has effects large enough for satisfactory determination.   In the fields of 198 units and upwards the diminution of magnetisation due to pressure was in every case found greatest when its application preceded the introduction of the rod.

Thus in all fields up to 128 C.G.S. units the actual application of pressure during the flow of the current produces in the condition of the rod during these experiments an unmistakeable increase in the magnetisation, and not improbably this tendency to increase the magnetisation exists in all the fields within the range of the experiments.

In the eighth column of Table IX are a few results from experiments on the same point.   The circumstances of these experiments differed from those of the preceding column, only in that the previous fields of equal strength in which pressure cycles had been applied, were of opposite sign to those in which the effects were observed.   The phenomena are clearly of the same general character as those of the previous column, and the existence of a critical field is unmistakeable.   The observations are insufficient to determine the exact position of the critical field, but it is obviously not far from the critical field of the preceding column.

The Polar or Non-polar Character of the Effects of Pressure.

§ 64.   Another very important question is whether the effects of the application and removal of pressure on the magnetisation are of a *polar* or a *non-polar* character. The meaning of these terms will be best understood from an illustration.

Let us suppose the induced magnetisation increased by a series of pressure cycles applied during the flow of a current, which makes A the north pole. The effect is non-polar if the rod be brought into the condition of an unstrained rod of a different material possessed of a higher susceptibility. The effect is polar if the susceptibility, while increased in fields making A a north pole, is equally diminished in fields of opposite sign—supposing for the moment that an opposite field could be introduced without altering the rod's temporary properties. If the susceptibility be simply altered by different amounts in fields of opposite sign the effect may be regarded as a combination of a polar and a non-polar effect.

It is probably impossible to arrive at a complete knowledge of the character of the effect, because the introduction of an opposite field implies a break and make of the magnetised current affecting the temporary properties of the rod. It is only in so far as the effects of pressure cycles are residual that their polar or non-polar character admits of direct investigation.

If, after the application of pressure cycles in a certain field, an increase of susceptibility, not otherwise accounted for, be found in fields both of the same and of opposite polarity, the existence of a non-polar effect would seem to be established. The same conclusion follows if there be a decrease of susceptibility irrespective of the sign of the field. If the susceptibility be found less in fields of the same name as that existing during the pressure cycles, than in fields of opposite name, it is difficult to think of any probable explanation other than a *true* polar effect.

If, however, the susceptibility be greater in fields of the same sign as that existing during the pressure cycles than in fields of opposite sign, the phenomenon may be explained without assuming a *true* polar effect.

The reasons for these statements will probably be most clearly presented by embodying them in language literally applicable only to WEBER's theory of magnetism. According to this theory, the only way in which we can imagine the application or removal of pressure to produce a permanent change in the magnetisation is either by endowing the ultimate magnetic molecules with an increased or diminished mobility, or by enabling them to swing round into extreme positions, where they stick. The change in the molecular mobility is non-polar if the molecules turn as freely in fields of one sign as in fields of the opposite sign, otherwise the change is polar.

Now when the susceptibility is found increased in fields both of the same and of opposite sign to that existing during the pressure cycles, or when it is diminished irrespective of the sign of the field, or, lastly, when it is less in a field of the same sign as that during which the pressures were applied than in a field of opposite sign, it is clear that the phenomena cannot be attributed to the magnetic molecules having stuck in positions into which they swung while under the directive influence of the original field. Thus, in these cases, the phenomena must be ascribed to some change in the mobility of the magnetic molecules, and so there must be a true polar

or non-polar effect. It is obvious, however, that if the magnetic molecules swung round and stuck in the way imagined, then the magnetisation would naturally be greater in a field of the same sign as the original than in a field of opposite sign. This is not regarded here as a *true* polar effect, though producing in the induced magnetisation phenomena identical with those which would arise from a true polar change in the molecular mobility. The turning round of the molecules has been already discussed in §§ 22 and 28, where its occurrence was demonstrated. The effect on the induced magnetisation will here be referred to as a *quasi* polar effect.

§ 65. The ninth column of Table IX. may be regarded as measuring the combined *true* and *quasi* polar effects of a numerous series of pressure cycles, in so far as they survive the break of the field during which the pressures were applied, and the make of an equal field of the same or opposite sign. The exact quantity measured is the algebraic excess of the magnetisation in a field of the same sign as that existing during the pressure cycles over that in a field of opposite sign, the rod being in each case introduced free of pressure into the second fields.

The results obtained with the weakest field of the table, 1·8 C.G.S. units, differed so much from the others that it was inconvenient to include them in the table. They were as follows :—

Circumstances of rod.	End a north pole.	Pressure.	$\mathfrak{I}$.
First introduction	A	off	4·1
After pressure cycles . . .	,,	,,	6·5
Out, re-introduced. . . .	B	,,	4·8
Kept in	,,	on	6·5
Out, re-introduced. . . .	,,	,,	6·2
,, ,, ...	A	off	4·1
,, ,, ...	B	,,	5·5
,, ,, ...	A	,,	4·1
Kept in	,,	on	6·5
Out, re-introduced. . . .	,,	off	5 3
,, . ,, ...	B	,,	5·1

There are here two or three instances of an increase of susceptibility of a non-polar character, and also clear indications of a polar effect. Considering how small the total induced magnetisation is, both the polar and non-polar effects are of considerable importance.

In the next field, 2·2 C.G.S. units, the rod was originally under pressure, and yet $\mathfrak{I}_2 - \mathfrak{I}_1$ exceeds $\mathfrak{I}_2 - \mathfrak{I}_3$. Thus, as $\mathfrak{I}_A - \mathfrak{I}_B$ is only ·2, there occurred a very decided residual increase of susceptibility, mainly of a non-polar character.

In all the fields, from 3·4 to 35·7 C.G.S. units, the residual effect is largely of a polar character. Take, for instance, the field 8·6. Here the pressure cycles raise the magnetisation by 10·2. Of this, 5·1 survives the break and re-make of the current, and so only about the half of the increase is lost.

The difference, however, in this case between $\mathfrak{I}_A$ and $\mathfrak{I}_B$ amounts to no less than 9·7, and thus the susceptibility, when B is the north pole, is lowered by nearly as much as the susceptibility, when A is a north pole, is increased. The residual effect is, in fact, almost entirely of a polar character. This appears to be about the strength of field at which the polar effect is largest.

In the fields from 35·7 to 128 C.G.S. units the results show too large fluctuations to lay any claims to great accuracy. They may, however, be regarded as satisfactorily proving that within these limits the polar residual effect is small, and that it has the same sign as in the weaker fields.

In the field of 174 C.G.S. units no residual polar effect could be detected, and in stronger fields the effect changes sign. There is thus a neutral field, which if not identical with, is certainly very near that at which the total effect of the first pressure on the induced magnetisation vanishes. In weaker fields the susceptibility of the rod is greatest when the field is of the same sign as that existing during the pressure cycles, in stronger fields the susceptibility is in these circumstances least.

In fields above this neutral field there is thus, according to the previous reasoning, a *true* polar effect. In fields again, such as 12·5 C.G.S. units, a *quasi* polar effect unquestionably existed, because there could then be obtained the phenomenon of the reversal of the rod's polarity on the break of an equal reverse current described in § 22. This phenomenon could not be obtained in a field of 23·7 C.G.S. units.

There is thus positive proof that a *true* polar effect exists in strong fields and a *quasi* polar effect in weak fields, and most probably in all but certain critical fields they exist simultaneously.

§ 66. We have seen that the residual polar effect has practically the same critical field as the total effect of the first pressure, or *First* "*on*" of Table X. Also both these effects are relatively very large in weak fields, and it might thus be supposed they stood in some intimate relationship to one another. In fields of 30 to 50 C.G.S. units, however, the residual polar effect is very small, while the *First* "*on*" is about its maximum.

An explanation of the discrepancy is supplied by a consideration of the curves *b* and *c* of fig. 12. The difference between the ordinates of these curves in a field of 8 C.G.S. units is great compared to the difference in a field of 30 C.G.S. units. Thus in the former field pressure cycles are much more effective in increasing the retentiveness than in the latter. A consideration of the numerical data shows that of the additional magnetisation which the pressure cycles enable the rod to acquire the loss on the break of the current is proportionally much less in a field of 8 than in a field of 30 C.G.S. units. Thus the *quasi* polar residual effect must be much smaller in the latter field than in the former, which doubtless accounts for the rapid falling off in the value of the total polar effect as the field is raised.

In the last column of Table IX. there are a few results bearing on the same point. These are derived from experiments in which the rod was under pressure when

reintroduced into the coil after being subjected to a series of pressure cycles in a previous field of the same strength. So far as they go the results are very similar to those of the previous column, from which it follows that the residual polar effect is, at least in its general character, independent of the state of the reintroduced rod as to pressure.

§ 67. Further results bearing on the polar character of the residual effects of pressure are given in the second, third, and fourth columns of Table X. These give the total algebraic increase of the induced magnetisation due to the first application of pressure. The headings A_1, A_2, B_2 show the condition of the rod as explained in § 37.

Comparing the columns headed A_1 and A_2, we see that in fields below 50 or 60 C.G.S. units the increase in the magnetisation caused by the application of the first pressure is very considerably greater when the rod is in a freshly demagnetised condition than when it has recently been subjected to a numerous series of pressure cycles in a field of the same strength and sign. In fields between 60 and 90 C.G.S. units the effect of pressure is still distinctly larger in the freshly demagnetised rod, but whether considered absolutely or relatively the difference between the results in the two columns is small. In stronger fields the numerical value of A_1, whether it signify an increase or a decrease of magnetisation, appears almost invariably greater than that of A_2. Both quantities, however, are then small, and in the strongest fields the effects of the first pressure were difficult to measure exactly, as the irregularities in the column headed A_1 would suggest. It would thus be unsafe to build too much upon the differences between the two columns in fields over 90 C.G.S. units. The evidence, however, clearly supports the conclusion that even in the strongest fields the numerical value of the change in the magnetisation due to the first pressure is, *ceteris paribus*, greatest in a freshly demagnetised bar.

The observed differences between the columns headed A_1 and A_2 are exactly of the character we should have expected from the already discussed experiments on the residual effects of pressure. In the weaker fields, we saw that the residual effect is very considerable. Thus the rod on its second exposure to a given field, under the circumstances of the experiment, is more or less in the state of a rod already subjected to pressure. Thus its condition, when the pressure A_2 is applied, is somewhat intermediate between its conditions before and after the application of the pressure A_1. Now the effect of the second pressure, after the introduction of the rod into the coil, was invariably, in the weaker fields, small compared to the effect of the first pressure. Consequently, in such fields, the pressure A_2 would naturally have a smaller effect than the pressure A_1, which is precisely the phenomenon observed.

Comparing the fourth column of Table X. with the previous two, we see that in fields up to 60 or 70 C.G.S. units, the increase in the magnetisation of the reintroduced rod, due to the first pressure, is distinctly greater when the field is of the

opposite sign to that existing during the pressure cycles, than when it is of the same sign. In stronger fields, the sign of the field existing during the pressure cycles appears practically immaterial.

A difference between the effects of the pressures A_1 and A_2, or A_1 and B_2, might be due either to a polar or a non-polar residual effect of pressure, but a difference between the effects of the pressures A_2 and B_2 must be due entirely to a polar residual effect. Thus the experiments on the first pressures afford independent proof of the existence of a residual polar effect in fields up to 60 or 70 C.G.S. units. This is in accordance with the results of § 65.

There would also appear to be evidence of the existence of a non-polar effect in the weaker fields. In such a field, we have already shown the existence of a very considerable *quasi* polar residual effect. Now the application of a pressure in a field of opposite sign, would unquestionably shake out part of this *quasi* polar effect, and might be expected to produce, in addition, a polar effect of its own, equal or nearly equal to that of the first pressure in the original field. Thus, if there were not a non-polar residual effect, we should expect the pressure B_2 to have, in weak fields, a larger effect than the pressure A_1. This does not, however, appear to be the case.

The positions of the critical fields for the vanishing of the total effect of the first application of pressure, indicated by the second, third, and fourth columns of Table X. are in good agreement. Consequently, it would appear that the position of this critical field depends but little on the treatment of the rod since its last demagnetisation.

§ 68. The results given under the heading *First "off"* in Table X., show that in the strongest fields of that table the residual polar effects, due to the application of pressure cycles in pre-existing fields of the same strength, have no appreciable influence on the magnitude of the increase of magnetisation which accompanies the removal of the pressure existing during the re-introduction of the rod into the magnetising coil.

Combining this phenomenon with the close agreement in the stronger fields of the magnitudes of the effects of the first pressures A_2 and B_2, we seem entitled to conclude that the residual polar effect of pressure cycles in a strong field, of whose existence the two last columns in Table IX. give clear proof, is not sensibly diminished by the application or removal of a single pressure in a subsequent equal field, whether of the same, or of opposite sign.

Resemblances between the Phenomena shown by Iron and Cobalt.

§ 69. Most of the following resemblances between the phenomena shown by the cobalt rod and those shown by the stretched iron wires of Professor EWING's experiments, have been already incidentally noticed, but a formal statement and comparison may be useful. In the following statements, *stress* signifies *tension* in the case of the iron wires, and *pressure* in the case of the cobalt.

1. The strengths of the fields in which the changes in the magnetisation produced by the first application of a given stress, or by its cyclic repetitions, bear the greatest ratios to the pre-existing magnetisations, are very low, if indeed they possess a finite value.

2. The fields in which the changes in the magnetisation produced by the first application of a given stress, or by its cyclic repetitions, are absolutely greatest, are not very far from the region of the Wendepunkt.

3. The maximum cyclic effect appears, however, in a distinctly lower field than does the maximum effect of the first application of stress.

4. Both the cyclic effect and the effect of the first application of stress possess critical or VILLARI points, and the critical magnetisations are both considerably greater than that answering to the Wendepunkt.

5. The critical field, however, for the cyclic effect is decidedly lower than that for the first application of stress.

6. The residual magnetisation approaches "saturation" in much lower fields than does the induced.

7. The *residual susceptibility*—or the ratio of the residual magnetisation to the strength of the pre-existing field—when the specimen is free from stress, attains a maximum when the pre-existing field is very considerably lower than that answering to the Wendepunkt.

8. The *retentiveness*—or the ratio of the residual to the induced magnetisation—in the absence of stress, attains a maximum in a field decidedly lower than that in which the *residual susceptibility* is a maximum.

9. Stress increases largely the residual susceptibility and the retentiveness in weak fields, and decidedly lowers the fields in which these magnitudes present their maxima.

§ 70. If the magnetisation answering to its Wendepunkt be taken as the unit for each individual specimen, the following Table gives approximately the induced magnetisations existing during certain critical events in the magnetisation of the cobalt rod, and three samples of stretched iron wire examined by Professor EWING.

TABLE XI.

	Maximum Retentiveness.	Maximum residual susceptibility.	Critical point for cyclic effect.	Critical point for first stress.
Cobalt	·40	·58	2·31	2·53
1st iron wire . .	..	..	1·60	1·87
2nd ,, . .	·53	·70		
3rd ,, . .	·55	·70		

This will serve to show the kind of evidence on which the above statements as to the relations between cobalt and iron are based.   Considering also that the intensities

of the magnetisations answering to the Wendepunkte of the stretched iron wires were on an average nearly thrice that answering to the Wendepunkt in the cobalt rod, that the wires were practically of infinite length, and that the critical fields observed in them answer, as a rule, to stresses considerably more severe than the normal stress applied to the cobalt, it seems not unlikely that by taking the position of the Wendepunkt in the absence of stress as a standard for comparison, it may be possible to trace general laws underlying the varied phenomena occurring in the different magnetic metals, and thus throw light on the material changes in which the process of magnetisation consists.

NOTE.

[The referees to whom the paper was submitted having suggested the desirability of an examination into the chemical character of the rod, I add the following analysis, for which I am indebted to Mr. R. H. ADIE, of Trinity College, Cambridge, who kindly undertook the work :—

	Percentages.
Cobalt	91·99
Nickel	4·36
Manganese	1·86
Iron	1·37
Carbon, &c.	·42
	100·00

The rod is thus not a pure specimen of cobalt. The principal impurity, however, is nickel, which closely resembles cobalt in its magnetic properties, though in general regarded as less strongly magnetic. The effects of pressure on the magnetisation of nickel are of the same general character as in the case of cobalt in fields below the critical, though probably more intense. One would thus expect the critical field to be slightly raised in consequence of the presence of the nickel.—April 16, 1890.]

Index.

(Numbers refer to paragraphs.)

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V. *On the Theory of Free Stream Lines.*

By J. H. MICHELL, *Trinity College, Cambridge.*

Communicated by Professor J. J. THOMSON, *F.R.S.*

Introduction.

THE attention of mathematicians was first called to the subject of the present paper by a memoir of HELMHOLTZ's in 1868, on " Discontinuous Fluid Motion."[*]

In discussing the steady motion of liquids past salient edges of fixed obstacles, it is found that the assumption of continuity of the motion leads to negative pressures in the liquid. HELMHOLTZ showed, in the paper above-mentioned, that some cases of this kind could be solved by assuming a surface of discontinuity, on one side of which the liquid is at rest, and he gave a mathematical solution of one case where the motion is in two dimensions.

The next advance in the subject was made by KIRCHHOFF who, in 1869, in a paper entitled " Zur Theorie freier Flüssigkeitsstrahlen " in ' CRELLE's Journal,' gave a generalization of the method which HELMHOLTZ had used, and obtained thereby the solution of three new interesting cases. Subsequently in his ' Vorlesungen über mathematische Physik,' he published another method and worked out the same problems by means of it, but gave no new ones.

RAYLEIGH in the ' Philosophical Magazine,' December, 1876, discussed the solutions of KIRCHHOFF, and gave a drawing of the bounding free stream lines in one case.

As far as I know, these are the only investigations published on the mathematical side respecting a branch of hydrodynamics of great theoretical and practical interest.

In considering the method of transformation of polygons given independently by SCHWARZ and CHRISTOFFEL I have been led to a new transformation, which together with theirs, gives a general solution of the problem of free non-reentrant stream lines with plane rigid boundaries.

A considerable number of the cases of high interest prove to be of a tolerably simple nature, and I have worked out several in detail.

These problems occupy the first part of the paper. In the second part I have given some extensions of the transformation formulæ, which are applicable to problems of condensers and the form of hollow vortices in certain cases.

[*] ' Berlin Monatsberichte,' 1868 ; and ' Gesamm. Abhandl.,' vol. 1.

The general Theory of Transformation.

Let x, y be two conjugate functions with respect to the two variables ϕ, ψ, so that

$$x + iy = f(\phi + i\psi),$$

and write

$$x + iy = z \qquad x - iy = z',$$
$$\phi + i\psi = w \qquad \phi - i\psi = w'.$$

x, y may be regarded as the rectangular coordinate of a point in a plane which we shall call the z plane, and similarly ϕ, ψ are the coordinates of a point in the w plane.

Consider the functions

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'},$$

$$W = -i \log \frac{dz}{dw} \Big/ \frac{dz'}{dw'}.$$

Since they can be written in the form

$$V = \log \frac{dz}{dw} + \log \frac{dz'}{dw'},$$

$$W = -i \left[\log \frac{dz}{dw} - \log \frac{dz'}{dw'} \right],$$

they both satisfy LAPLACE's equation, and we have

$$V + iW = 2 \log \frac{dz}{dw},$$

so that V, W are conjugate functions with respect to x, y or ϕ, ψ.

The transformations of the present paper will be deduced from the properties of the function V, so that its nature must be considered in detail. We have as alternative forms of V

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'} = \log \left\{ \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 \right\} = -\log \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 \right\}.$$

If the element of arc of ψ constant in the z plane be given by

$$ds_1 = \frac{d\phi}{h},$$

then

$$h^2 = \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2,$$

so that

$$V = -\log h^2.$$

And

$$i\mathrm{W} = \log \frac{\dfrac{dx}{d\phi} + i\dfrac{dy}{d\phi}}{\dfrac{dx}{d\phi} - i\dfrac{dy}{d\phi}} = 2i\theta,$$

where θ is the angle the tangent to the curve ψ makes with the axis of x.

[Mr. BRILL has used the function W as a means of transformation ('Cambridge Phil. Soc. Proc.,' vol. 6, and 'Messenger of Math.,' August, 1889), and has thus anticipated me in one of the general theorems given in the latter part of this paper as I shall notice in the proper place. I was not acquainted with his work when I developed the method here given.]

Let k be the curvature of the curve ψ at ϕ. We have the well-known formula

$$k = \frac{dh}{d\psi}$$

which is

$$k = \frac{d}{d\psi} e^{-\frac{1}{2}\mathrm{V}} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now let the arc of the curve ψ be connected with k by the equation

$$s = f(k)$$

so that

$$\frac{ds}{d\phi} = \frac{d}{d\phi} f(k),$$

or from (1)

$$e^{\frac{1}{2}\mathrm{V}} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{1}{2}\mathrm{V}}\right). \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If ψ consist of parts of straight lines we have simply

$$\frac{d\mathrm{V}}{d\psi} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The formula (2) suggests a general method for finding a transformation

$$z = f(\phi + i\psi),$$

such that ψ_0 is an arbitrary curve in the z plane.

If the region within ψ_0 corresponds, point for point, to the part of the ω plane lying above $\psi = \psi_0$, the problem is reduced to finding a potential function V, which is continuous throughout the space bounded by a straight line, and such that

$$e^{\frac{1}{2}\mathrm{V}} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{1}{2}\mathrm{V}}\right)$$

over that straight line

We have still to discuss the question of the singular points of V.

The function V will be finite and continuous for all points except where two branches of a ψ curve (or a ϕ curve) cut.

At such a point dz/dw is either zero or infinite, and in either case V is infinite.

It will be sufficient for our purpose to consider only the simplest singularity, that is, in which we have in the neighbourhood of a point $(\phi_0 \ \psi_0)$ of this nature

$$V = n \log \left[(\phi - \phi_0)^2 + (\psi - \psi_0)^2 \right] + C,$$

and therefore

$$\frac{dz}{dw} = A \, (w - w_0)^n,$$

where the value of n will depend on the nature of the singularity in question, and will be seen from the particular problems to which we proceed.

PROBLEM I.

To find the transformation $z = f(\phi + i\psi)$, which makes the area for which ψ is positive in the w plane correspond point for point to the area inside a given rectilinear polygon in the z plane.   (The problem of SCHWARZ[*] and CHRISTOFFEL[†].)

Consider the conditions which the function V must satisfy in the ω plane.

(a) There are to be no singular points for ψ positive.

(b) Along $\psi = 0$ we have $dV/d\psi = 0$.

(c) At certain points of $\psi = 0$, which correspond to the angular points of the polygon in the z plane, we have V infinite.

It is plain from this specification that the function V is (to a constant) merely the potential of masses at the singular points ϕ_1, ϕ_2 . . . along $\psi = 0$; and, therefore,

$$V = \log \Pi \left\{ (\phi - \phi_r)^2 + \psi^2 \right\}^{n_r} + C,$$

so that

$$\frac{dz}{dw} = \Pi A \, (w - \phi_r)^{n_r}$$

where Π is the product symbol.

It remains to find the quantities n_r.   Draw a small semicircle of radius R around the point ϕ_r,
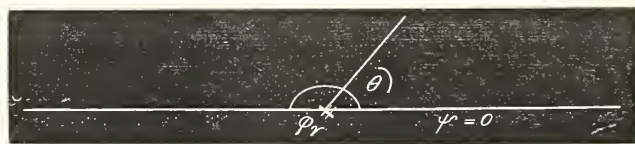
* " Ueber einige Abbildungsaufgaben " ('CRELLE,' vol. 70, 1869).

† " Sul problema delle temperature stazionarie," &c. ('Annali di Matematica,' vol. 1, 1867).

and on this semicircle let

$$w - \phi_r = \mathrm{R}\left(\cos \theta + i \sin \theta\right),$$

so that

$$\frac{dz}{dw} = \mathrm{A R}^{n_r}\left(\cos n_r\, \theta + i \sin n_r\theta\right).$$

Consequently, as we pass from $\theta = \pi$ to $\theta = 0$, dz/dw goes from

$$\mathrm{A R}^{n_r}\left(\cos n_r\pi + i \sin n_r\pi\right)$$

to

$$\mathrm{A R}^{n_r}.$$

The amplitude of dz/dw has, therefore, decreased by $n_r\pi$.

But the increase of amplitude is $\pi - \alpha_r$ where α_r is the internal angle of the polygon which corresponds to ϕ_r.

Therefore,

$$\pi - \alpha_r = -n_r\pi,$$

or

$$n_r = \frac{\alpha_r}{\pi} - 1,$$

so that the transformation becomes

$$\frac{dz}{dw} = \mathrm{A}\Pi\left(w - \phi_r\right)^{\alpha_r/\pi - 1},$$

which is the formula given by SCHWARZ and CHRISTOFFEL.

PROBLEM I.—SPECIAL CASE.

For the study of non-reentrant free stream lines we require a special case of this formula.

Suppose the polygon to consist of a series of straight line sinfinite in one direction, all parallel to $y = 0$, so that the angles of the polygon are either 0 or 2π, and, therefore, $n_r = +1$ or -1.

Let ϕ_{1r} correspond to an angle 2π, and, therefore, to an end of a line within a finite distance of the origin, and let ϕ_{2r} correspond to an angle 0, and, therefore, to the adjacent ends of two lines at an infinite distance from the origin.

SCHWARZ's formula then becomes

$$\frac{dz}{dw} = \Pi \mathrm{A}\, \frac{w - \phi_{1r}}{w - \phi_{2r}}$$

It is plain that there cannot be more than one factor more in the numerator than in the denominator, so that we can write

$$\frac{dz}{dw} = Aw + B + \Sigma \frac{C_r}{w - \phi_{2r}},$$

and, therefore, on integrating

$$z = \tfrac{1}{2} Aw^2 + Bw + D + \Sigma C_r \log (w - \phi_{2r}),$$

we may at once determine the distances between consecutive lines in terms of the ϕ_r, or in terms of the D_r.

For consider the passage of w around the small semicircle R above described.

We have

$$w - \phi_r = Re^{i\theta},$$

therefore

$$z_1 - z_0 = \int_\pi^0 \frac{C_r}{Re^{i\theta}} d (Re^{i\theta}),$$

$$= iC_r \int_\pi^0 d\theta,$$

$$= - i\pi C_r.$$

So that y increases by $- \pi C_r$ in passing the point ϕ_{2r}, and, therefore, the distance between the parallel lines r and $r + 1$ is $- \pi C_r$, or, in terms of the ϕ_r, is

$$- \pi A \Pi \frac{\phi_r - \phi_{1s}}{\phi_r - \phi_{2s}}.$$

After we have fixed on the angle which is to correspond to $\phi = \pm \infty$, we can in general choose the position of two other points ϕ_r, ϕ_s, and then the transformation-formula is determined to an additive constant.

For example, take the case of two doubly infinite lines AB, CD, with a semi-infinite line EF between them.

C   D

    E   F

A   B

We may take the zero angle (A, C) to be $\phi = \pm \infty$, and the angles (B, F) (F, D) to be $\phi = - 1$, $\phi = + 1$. Then if the angle E is $\phi = c$, we have

$$\frac{dz}{dw} = \frac{A (w - c)}{(w - 1)(w + 1)}.$$

Let the distance between AB and EF be d_1 and that between EF and CD d_2.
Then integrating past the points $\omega = -1$, $\omega = 1$, we find

$$d_1 = -\pi \frac{A(1 - c)}{2},$$

$$d_2 = -\pi \frac{A(1 + c)}{2},$$

which determine A and c, and, therefore, make the formula definite.

Other examples will occur in the physical application.

PROBLEM II.

The second transformation which we need may be stated most simply as an electrical problem.

Let there be any number of infinitely long plane conductors, all in the same plane, and with parallel edges.

It is required to find the potential at any point when these conductors are raised to given potentials.

A ———— B   C ———— D   E ———— F

Let AB, CD, EF, . . . be the sections of the conductors by the plane (xy).

Everything is symmetrical with regard to the line ABF which we take to be $y = 0$.

Consider the specifications of the transformation-function V where ψ is the potential and ϕ are the lines of force.

We must have $dV/d\psi = 0$ over the conductors, since they are straight, and, therefore, also $dV/dy = 0$.

There will be infinite points at the edges A, B . . . of the conductors, and also at points in the field corresponding to branch points of ψ (or ϕ). These last will be distributed symmetrically with respect to $y = 0$.

From these conditions it is plain that the solution is that V is the potential of masses at the singular points in question, so that we may write

$$V = \log \Pi \left\{ (x - x_r)^2 + (y - y_r)^2 \right\}^{n_r} + C,$$

and, therefore,

$$\frac{dz}{dw} = A\Pi (z - z_r)^{n_r}.$$

It remains to find the quantities n_r.

For a singular point in the field where m branches of ψ meet we have simply

$$n_r = -m.$$

3 E 2

For an edge of the conductor we may proceed thus :—

Let

$$z - z_0 = \mathrm{R}e^{i\theta},$$

then

$$(z - z_0)^n = \mathrm{R}^n \left(\cos n\theta + i \sin n\theta \right),$$

so that as θ goes from $-\pi$ to $+\pi$, $(z - z_0)^n$ goes from

$$\mathrm{R}^n \left[\cos \left(-n\pi \right) + i \sin \left(-n\pi \right) \right]$$

to

$$\mathrm{R}^n \left[\cos \left(n\pi \right) + i \sin \left(n\pi \right) \right],$$

and, therefore, the amplitude of dz/dw increases by $2n\pi$. Now the amplitude goes from 0 to π, therefore, $n = \frac{1}{2}$.

Hence

$$\frac{dw}{dz} = \Pi \frac{(z - z_r)^{m_r}}{(z - x_s)^{\frac{1}{2}}},$$

where r refers to a point in the field and s to the edge of a conductor.

If one of the conductors reduce to a line, we have two of the x_s equal, say $x_s = x_{s+1}$, and there is a factor $z - x_s$ in the denominator, and so for any number of line conductors.

There are many other cases for which a formula like the above applies; it is not always necessary that the conductors should be in the same plane.

It is very easy to perceive such cases by considering whether the equation $dV/dn = 0$ is satisfied over the conductors where dn is an element of a normal to a conductor.

By combining this transformation with that of SCHWARZ and CHRISTOFFEL we get a general solution for the free-stream-line problem, as I shall presently show.

It is necessary first to deduce some special formulæ, which will be continually used hereafter.

(a) Take first the case of one conductor and one line—
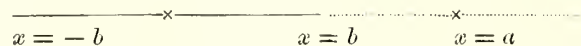
Let the conductor extend from $x = -b$ to $x = +b$ and the line distribution be at $x = a$.

Then the general formula reduces to

$$\frac{dw}{dz} = \frac{Ai}{(z - a)(z^2 - b^2)^{\frac{1}{2}}}$$

where A is real, supposing that there is no singular point in the field.

The nature of the multiplier Ai is obtained by considering that between

$$x = -b \text{ and } x = b \quad \frac{d\psi}{dx} = 0.$$

The integral of this is

$$w = Ai \log \frac{az - b^2 + \sqrt{(a^2 - b^2)}\sqrt{(z^2 - b^2)}}{b(z - a)}$$

supposing the potential of the conductor is zero.

(b) Now let the conductor be two semi-infinite planes, with a gap from $x = -b$ to $x = b$ between them, and let there be a line distribution at $x = a$ where $b > a > -b$.

$$x = a$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$x = -b \qquad\qquad x = b$$
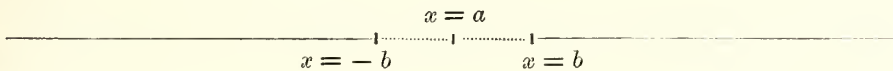
Then

$$\frac{dw}{dz} = \frac{A}{(z - a)(z^2 - b^2)^{\frac{1}{2}}}$$

where A is real.

And

$$w = Ai \log \frac{b^2 - az + \sqrt{(b^2 - a^2)}\sqrt{(b^2 - z^2)}}{b(z - a)},$$

the potential of the conductor being zero.

(c) If we put $z = b + z'$ in the result of (a), and then make $b = \infty$, we get for the case of a semi-infinite conductor $x = 0$ to $x = -\infty$ with a line distribution at $x = a$,

$$w = Ai \log \frac{\sqrt{z} + \sqrt{a}}{\sqrt{z} - \sqrt{a}},$$

the potential of the conductor being 0.

These results (a), (b), (c), could of course be deduced from the known formulæ for elliptical conductors.

On the Theory of Non-reentrant Free Stream Lines.

The presence of sharp salient edges in a moving liquid always implies surfaces of discontinuity, but these may be closed or unclosed according to circumstances. It would be difficult to give a rule as to the kind of motion in a given case, for an alteration in the relative sizes of the solids concerned will totally alter the character of the motion.

As an illustration of this take the case of two parallel planes of finite breadth placed symmetrically one behind the other in a broad stream. If the second plane be of less than a certain width, free stream lines will proceed from the edges of the first, and the second will be in still water.

Now suppose the second plane broadened until it cuts the stream lines from the first plane.

The character of the motion is changed. Two vortices will appear behind the first plane, and in addition there will be free stream lines from the edge of the second plane.

No method has yet been discovered which will give solutions of cases where there is motion on both sides of a surface of discontinuity. In the problems treated in the present paper there is always still water on one side of a free stream line.

In the present section the motion considered is in two dimensions, the boundaries are plane, and the free stream lines are non-reentrant.

Let x, y be the coordinates of a point in the liquid, ϕ, ψ the potential and stream functions respectively. The region in the w plane corresponding to moving liquid in the z plane will be bounded by straight lines ψ, infinite in one direction at least and parallel to $\psi = 0$.

The area in the w plane is therefore of the nature treated in Problem I. (a), that is, it is bounded by a polygon whose angles are alternately four right angles and zero.

This area, then, by means of Problem I. (a), may be transformed into the part of a new u plane in which q is positive, where $u \equiv p + iq$.

In this u plane the boundaries of the liquid, both the plane boundaries and the free stream lines, are represented by the line $q = 0$.

Let, as before,

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'} = -\log \left[\left(\frac{d\phi}{dx} \right)^2 + \left(\frac{d\phi}{dy} \right)^2 \right].$$

We have seen that V is a potential function, considered as a function of ϕ, ψ, and, therefore, it is also a potential function considered as a function of p, q, for ϕ, ψ are conjugate with respect to p, q.

Further, we have seen that along a straight boundary $\psi = $ constant we have $dV/d\psi = 0$, and, since all the straight boundaries correspond to portions of $q = 0$, we

must have $dV/dq = 0$ along all these portions.   Along a free stream line the pressure is constant, since it must be equal to the pressure on the liquid which is at rest.

Now, in steady motion, we have BERNOULLI's pressure equation

$$\frac{p}{\rho} + \tfrac{1}{2}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right] = \text{constant.}$$

Therefore, along a free stream line $\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2$ is constant, and, therefore, V is constant.

All the portions of $q = 0$ which do not correspond to plane boundaries correspond to free stream lines, for which V is constant.

Lastly $\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2$ is zero at reentrant angles of the boundary, and also at points where stream lines branch.

For these points V is infinite and positive.

In all the cases we shall consider, these infinite points will be along $q = 0$, so that a stream line only branches at the boundary.

We have now reduced the problem to finding a function V which satisfies LAPLACE's equation ; is finite and continuous in that half of the plane u for which q is positive ; is constant along parts of $q = 0$; along the other parts satisfies $dV/dq = 0$; and at points along it is $+\infty$.

This problem has an obvious solution.

It is plain that V is merely the potential due to conductors coinciding with those parts of $q = 0$ for which V is constant, and having that constant as potential, together with masses at the points for which V is infinite.

The general solution of this has been given in Problem II.

Let U be the conjugate of V, so that $U + iV = f(p + iq)$.   Then, translating Problem II. into the present notation we have

$$\frac{d(U + iV)}{d(p + iq)} = \Pi A \frac{(u - u_r)^{m_r}}{(u - p_s)^{k}}$$

and for a point mass two factors of the denominator coincide.

Write this

$$\frac{d(U + iV)}{du} = f'(u),$$

so that

$$U + iV = \int f'(u)\, du.$$

Now

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'},$$

therefore

$$U = i \log \left(\frac{dz}{dw} \Big/ \frac{dz'}{dw'}\right)$$

and

$$U + iV = 2i \log \frac{dz}{dw},$$

Therefore

$$\frac{dz}{dw} = e^{-i/2 \int f(u)\,du},$$

Now we have obtained the transformation from the w to the u plane in the form

$$\frac{dw}{du} = \phi(u) \qquad [\text{Problem I. } (a)].$$

Therefore

$$\frac{dz}{du} = \frac{dz}{dw} \cdot \frac{dw}{du} = \phi(u)\, e^{-i/2 \int f(u)\,du},$$

which gives z as a function of u.

This is the general solution of the hydrodynamical problem before us.

Near an angle of the boundary or a branching of a stream line we shall have

$$dz/du = A\,(u - u_0)^n.$$

The determination of the index n rests on principles already used.

If the internal angle of the boundary, or the angle between the two branches of the stream line be a, then

$$n = \frac{a}{\pi} - 1.$$

For example, if a stream line divide on a plane wall, $n = 0$, and the point of division is not a singular point for dz/du, although it is for dz/dw. We may then lay down the following rule :—

Near a singular point

$$\frac{dz}{dw} = A\,(u - u_0)^{\frac{a}{\pi} - 1}$$

where a is the internal angle of the boundary except when this point is a point of branching of a stream line, in which case

$$\frac{dz}{dw} = A\,(u - u_0)^{\frac{a}{\pi} - 2},$$

for at that point

$$\frac{dz}{du} = A\,(u - u_0)^{\frac{a}{\pi} - 1},$$

and

$$dw/du = B\,(u - u_0). \qquad [\text{Problem I. } (a)].$$

We shall now go on to consider such cases as are susceptible of tolerably simple treatment.

We shall suppose there are only two free stream lines, and throughout take the velocity along them to be 1, so that $V = 0$.

CASE I. *A Single Jet from a Vessel.*

There will be but two bounding stream lines, which we may take to be $\psi = 0$, $\psi = \pi$, both extending from $+ \infty$ to $- \infty$.

The diagram in the w plane consists merely of two parallel infinite straight lines, AB, CD, at a distance π apart.
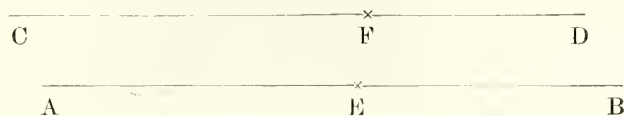
A portion of each, say EB, FD, will correspond to the boundaries of the jet.

If now we transform to the u plane so that the ends D, B go to $p = \pm \infty$, we may choose the points $u = -1$, $u = 1$, to correspond to the edges of the aperture from which the jet issues.

The point (C, A) will then be in a definite position, $u = c$, and the formula of transformation from w to u is

$$\frac{dw}{du} = \frac{A}{u - c}.$$

Remembering that π is the distance between the two stream lines, we see that $A = 1$, and therefore

$$\frac{dw}{du} = \frac{1}{u - c}.$$

Let $u = a_n$ correspond to an angle a_n of the vessel, then the points along $q = 0$ are arranged in the manner of the figure

free stream line.		rigid.		rigid.		free.
$\psi = \pi$	$u = -1$	$\psi = \pi$ $u = a_n$	$u = c$	$\psi = 0$	$u = 1$	$\psi = 0$
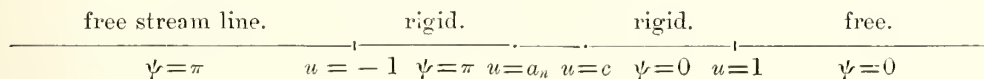
The appropriate formula is then that of Problem II. (*b*), viz.,

$$\frac{dz}{dw} = \Pi \left[\frac{1 - a_n u + \sqrt{(1 - a_n^2)} \sqrt{(1 - u^2)}}{u - a_n} \right]^{1 - \frac{a_n}{\pi}}$$

and therefore

$$\frac{dz}{du} = \frac{1}{u - c} \Pi \left[\frac{1 - a_n u + \sqrt{(1 - a_n^2)} \sqrt{(1 - u^2)}}{u - a_n} \right]^{1 - \frac{a_n}{\pi}}$$

Example I.—A rectangular vessel of given width has an aperture in the bottom.
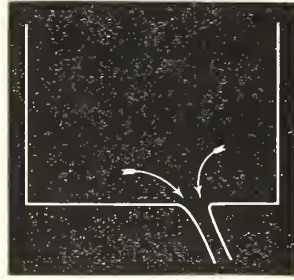
Here there are only two angles, each of which $= \frac{1}{2}\pi$, and therefore $1 - \alpha_n/\pi = \frac{1}{2}$. Let the two angles be $u = a$, $u = b$, where $1 > b > c > a > -1$.

Then, observing that

$$\{1 - au + \sqrt{1 - a^2}\sqrt{1 - u^2}\} = \{\sqrt{\tfrac{1}{2}(1 - a)}\sqrt{1 + u} + \sqrt{\tfrac{1}{2}(1 + a)}\sqrt{1 - u}\}^2$$

we get

$$\frac{dz}{du} = \frac{1}{u - c}\frac{\{\sqrt{\tfrac{1}{2}(1 - a)}\sqrt{1 + u} + \sqrt{\tfrac{1}{2}(1 + a)}\sqrt{1 - u}\}\{\sqrt{\tfrac{1}{2}(1 - b)}\sqrt{1 + u} + \sqrt{\tfrac{1}{2}(1 + b)}\sqrt{1 - u}\}}{\sqrt{(u - a \cdot u - b)}}$$

or

$$\frac{dz}{du} = \frac{Au + B + C\sqrt{(1 - u^2)}}{(u - c)\sqrt{(u - a) \cdot (u - b)}}.$$

Where

$$A = \sqrt{\tfrac{1}{4}(1 - a)(1 - b)} - \sqrt{\tfrac{1}{4}(1 + a)(1 + b)},$$

$$B = \sqrt{\tfrac{1}{4}(1 - a)(1 - b)} + \sqrt{\tfrac{1}{4}(1 + a)(1 + b)},$$

$$C = \sqrt{\tfrac{1}{4}(1 + a)(1 - b)} + \sqrt{\tfrac{1}{4}(1 - a)(1 + b)}.$$

Between $u = -1$ and $u = a$, that is along the bottom of the vessel from the edge of the aperture to the angle on the right, we have

$$\frac{dy}{dp} = 0,$$

$$\frac{dx}{dp} = \frac{Ap + B + C\sqrt{(1 - p^2)}}{(p - c)\sqrt{(p - a)(p - b)}};$$

therefore the distance between the two points is

$$l_1 = \int_{-1}^{a} \frac{Ap + B + C\sqrt{(1 - p^2)}}{(c - p)\sqrt{(p - a)(p - b)}}\,dp.$$

Similarly the other piece of the bottom of the vessel is of length

$$l_2 = \int_b^1 \frac{Ap + B + C\sqrt{(1 - p^2)}}{(p - c)\sqrt{(p - a)(p - b)}}\, dp.$$

When $\phi = -\infty$, that is, when $u = c$

$$\frac{dz}{dw} = -i\frac{Ac + B + C\sqrt{(1 - c^2)}}{\sqrt{(c - a)(b - c)}}.$$

Therefore the velocity in the vessel at a distance from the aperture is

$$\frac{\sqrt{(c - a)(b - c)}}{Ac + B + C\sqrt{(1 - c^2)}},$$

and therefore the breadth of the vessel is

$$\pi\frac{Ac + B + C\sqrt{(1 - c^2)}}{\sqrt{(c - a)(b - c)}} = d \text{ (say)}.$$

The breadth of the aperture is

$$d - l_1 - l_2,$$

and the breadth of the jet is ultimately π.

The question is now reduced to a matter of the integration of l_1, l_2.

For the general case elliptic integrals occur, and the expressions for our purpose may as well be left in their present form. If, however, the aperture be in the centre of the vessel, the integrals will work out, and we get a simple expression for the contraction of the jet. To this we now proceed.

Sub-Example I.—Jet from an aperture in the centre of the bottom of a rectangular vessel.

In this case we may take the angles of the vessel at $u = -a$, $u = a$, $a < 1$.
The expression for dz/du then reduces to

$$\frac{dz}{du} = \frac{1}{u}\frac{\sqrt{(1 - a^2)} + \sqrt{(1 - u^2)}}{\sqrt{u^2 - a^2}}.$$

We now have

$$l_2 = l_1 = \int_a^1 \frac{1}{p}\frac{\sqrt{(1 - a^2)} + \sqrt{(1 - p^2)}}{\sqrt{p^2 - a^2}}\, dp.$$

Now

$$\int_a^1 \frac{1}{p}\frac{1}{\sqrt{(p^2 - a^2)}}\, dp = -\int_1^{\frac{1}{a}} \frac{d\lambda}{\sqrt{(1 - a^2\lambda^2)}}\quad \left(\lambda = \frac{1}{p}\right),$$

$$= -\frac{1}{a}\left[\sin^{-1} a\lambda\right]_1^{\frac{1}{a}},$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \sin^{-1} a\right],$$

and

$$\int_a^1 \frac{\sqrt{(1-p^2)}}{p\,\sqrt{(p^2-a^2)}}\,dp = -\int \frac{r^2 dr}{(1-r^2)\,\sqrt{(1-a^2-r^2)}} \quad (p^2 = 1 - r)$$

$$= \int \frac{dr}{\sqrt{(1-a^2-r^2)}} - \int \frac{dr}{(1-r^2)\,\sqrt{(1-a^2-r^2)}}$$

$$= \left[\sin^{-1}\frac{r}{\sqrt{(1-a^2)}} - \frac{1}{a}\,\tan^{-1}\frac{ar}{\sqrt{(1-a^2-r^2)}} \right]_{\sqrt{(1-a^2)}}^0$$

$$= \frac{\pi}{2}\frac{1-a}{a},$$

therefore

$$l_2 = l_1 = \frac{\pi}{2}\frac{1-a}{a} + \frac{\sqrt{(1-a^2)}}{a}\left[\frac{\pi}{2} - \sin^{-1}a\right].$$

When $u = 0$ that is, at $\phi = -\infty$, we have

$$\frac{dz}{dw} = -i\frac{\sqrt{(1-a^2)}+1}{a}.$$

Therefore the velocity in the vessel at a distance from the aperture is

$$\frac{a}{1 + \sqrt{(1-a^2)}}$$

Therefore the breadth of the vessel is

$$\pi\,\frac{1 + \sqrt{(1-a^2)}}{a} = d\ \text{(say)}.$$

So that the breadth of the aperture is

$$d - 2l_1 = \pi + 2\,\frac{\sqrt{(1-a^2)}}{a}\,\sin^{-1}a\ .$$

Hence, since the final breadth of the jet is π, we have as the ratio of the breadth of the jet to that of the aperture

$$\frac{\pi}{\pi + 2\,\dfrac{\sqrt{(1-a^2)}}{a}\,\sin^{-1}a}$$

Now

$$\frac{1 + \sqrt{(1-a^2)}}{a} = \frac{d}{\pi},$$

therefore

$$\frac{1 - \sqrt{(1-a^2)}}{a} = \frac{\pi}{d},$$

and

$$2\,\frac{\sqrt{(1-a^2)}}{a} = \frac{d}{\pi} - \frac{\pi}{d}.$$

Also

$$\sin^{-1} a = \tan^{-1} \frac{a}{\sqrt{(1 - a^2)}} = \tan^{-1} \frac{2\pi d}{d^2 - \pi^2}.$$

Therefore the contraction ratio is

$$\frac{\pi}{\pi + \left(\dfrac{d}{\pi} - \dfrac{\pi}{d}\right) \tan^{-1} \dfrac{2\pi d}{d^2 - \pi^2}}.$$

Now, getting rid of the special units, let d be the breadth of the vessel as before, k the breadth of the aperture, and c that of the jet, then

$$k = c \left[1 + \frac{1}{\pi} \left(\frac{d}{c} - \frac{c}{d} \right) \tan^{-1} \frac{2dc}{d^2 - c^2} \right].$$

If d be very large compared with k, we get

$$k = c \left[1 + \frac{1}{\pi} \ \underset{c}{\text{Lim.}} \frac{d}{c} \tan^{-1} 2 \frac{c}{d} \right] = c \ \frac{\pi + 2}{\pi}.$$

This is the result obtained by RAYLEIGH[*] from KIRCHHOFF's solution for the case of an aperture in an infinite plane bounding wall.

As d/k decreases from infinity the contraction also continually decreases, until when $d = k$ the contraction is zero.

In order to get some idea of how soon the finite breadth of the vessel affects the contraction ratio perceptibly, consider the case when

$$d^2 - c^2 = 2cd$$

or

$$d = (1 + \sqrt{2})\, c.$$

The contraction ratio is then

$$\frac{1}{1 + \dfrac{2}{\pi} \cdot \dfrac{\pi}{4}} = \tfrac{2}{3}$$

and

$$d = (1 + \sqrt{2})\, \tfrac{2}{3} k,$$

so that the finiteness of the vessel has very little effect on the jet if the breadth is more than twice that of the aperture.

The equations to the free stream line $\psi = \pi$ are

$$\left.\begin{aligned}
\frac{dx}{dp} &= -\sqrt{(1 - a^2)} \frac{1}{p} \frac{1}{\sqrt{(p^2 - a^2)}} \\
\frac{dy}{dp} &= -\frac{1}{p} \frac{\sqrt{(p^2 - 1)}}{\sqrt{(p^2 - a^2)}}
\end{aligned}\right\}$$

[*] ' Phil. Mag.,' Dec., 1876.

The former gives

$$x = \frac{\pi}{2} - \frac{\sqrt{(1-a^2)}}{a} \sin^{-1} \frac{u}{p}.$$

Now

$$p = -e^{\phi} = -e^{s},$$

where s is the arc.   Therefore

$$x = \frac{\pi}{2} + \frac{\sqrt{(1-a^2)}}{a} \sin^{-1} ae^{-s}.$$

When $a = 0$ we have

$$x = \frac{\pi}{2} + e^{-s},$$

which is the form given by RAYLEIGH ('Phil. Mag.,' 1876) for KIRCHHOFF's case.

Example II.—Tube projecting far into the bottom of a vessel of given breadth.

The tube is supposed to project so far into the vessel that the motion at its bottom may be neglected.

There are here two singular points only, $u = -a$, $u = a$, corresponding to the coalescing angles of the bottom of the vessel.

The appropriate transformation is, therefore,

$$\frac{dz}{dw} = A \frac{1 - au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a} \cdot \frac{1 + au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u+a}$$

$$= A \frac{2 - a^2 - u^2 + 2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u^2 - a^2}.$$

Since, when $u = \infty$, $dz/dw = -i$, we have $A = i$.
So that

$$\frac{dz}{dw} = i \frac{2 - a^2 - u^2 + 2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u^2 - a^2}$$

and

$$\frac{dw}{du} = \frac{1}{u} \text{ as before ;}$$

therefore

$$\frac{dz}{du} = \frac{i}{u} \frac{2 - a^2 - u^2 + 2\sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{u^2 - a^2} .$$

When $u = 0$, that is, at a distance upwards from the tube,

$$\frac{dz}{dw} = -\frac{i}{a^2} \left(2 - a^2 - 2\sqrt{1 - a^2}\right).$$

therefore the breadth of the vessel is

$$\frac{\pi a^2}{2 - a^2 - 2\sqrt{(1 - a^2)}} = d \; (\text{say}).$$

In passing over $u = -a$, x increases by

$$2\frac{\pi}{a^2}\left(1 - a^2\right),$$

therefore the breadth of the tube is

$$\frac{\pi a^2}{2 - a^2 - 2\sqrt{(1 - a^2)}} - \frac{4\pi}{a^2}\left(1 - a^2\right) = \frac{\pi}{a^2}\left\{3a^2 - 2 + 2\sqrt{(1 - a^2)}\right\} = k \,(\text{say}).$$

Hence

$$d - k = \frac{4\pi}{a^2}\left(1 - a^2\right),$$

therefore

$$\frac{d - k + 4\pi}{4\pi} = \frac{1}{a^2} .$$

Substituting in the value of d we have

$$d + \pi = \pi \frac{d - k + 4\pi}{2\pi} + \sqrt{\left(\frac{d - k}{d - k + 4\pi}\right)} \cdot \frac{d - k + 4\pi}{2}$$

or

$$d + k - 2\pi = \sqrt{\left\{(d - k)(d - k + 4\pi)\right\}}.$$

On squaring this gives

$$dk - 2\pi d + \pi^2 = 0,$$

or, getting rid of the special unit of length, if c be the breadth of the jet, we have

$$dk - 2cd + c^2 = 0,$$

or

$$(d - c)^2 = d(d - k).$$

When d is very great we get

$$k = 2c,$$

and the ratio of contraction is $\frac{1}{2}$.

This particular case was the first solution of free stream lines given, and by HELMHOLTZ.

Lord RAYLEIGH[*] has given the equation

$$\frac{2}{k} = \frac{1}{c} + \frac{1}{d}$$

for the coefficient of contraction when there is a tube projecting inwards in a vessel of finite breadth.

The assumption made is, however, not that of this example, for he has taken the velocity along the bottom of the vessel to be the same as that at a distance upwards from the tube.

This is one of the very few cases in which the contraction can be determined accurately from elementary principles.

It is scarcely worth while to put the proof of this down, but it is worth remarking that the corresponding case in three dimensions, can also be worked out, viz., the case of one circular cylinder projecting far into another.

Let r_1, r_2 be the radii of the cylinders $r_1 > r_2$, and r_3 the radius of the jet, then

$$r_1^2 r_2^2 - 2r_1^2 r_3^2 + r_3^4 = 0.$$

This includes the case first noticed, I believe, by BORDA,[†] of a cylindrical tube in an infinite plane wall, the proof being just the same.

Example III.—Tube projecting into a vessel of great breadth.

If we take the case of a tube projecting into a vessel of great breadth, we do not get such simple expressions.
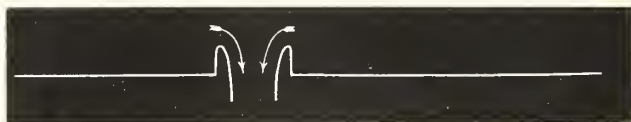
The transformation to the u plane is the same as before, but there is now a singular point at $u = 0$, corresponding to $\phi = -\infty$. We arrive at the correct result by first supposing the vessel of finite breadth, so that there are points at $u = -a$, $u = -b$, $u = b$, $u = a$, for each of which $\alpha/\pi - 1 = -\frac{1}{2}$, and then making b vanish.

We have, therefore,

$$\frac{dz}{du} = \frac{i}{\sqrt{(u^2 - a^2)}} \{ \sqrt{(1 - a^2)} + \sqrt{(1 - u^2)} \} \frac{1 + \sqrt{(1 - u^2)}}{u},$$

and

$$\frac{dz}{du} = \frac{i}{u^2 \sqrt{(u^2 - a^2)}} \{ \sqrt{(1 - a^2)} + 1 - u^2 + (1 + \sqrt{1 - a^2}) \sqrt{(1 - u^2)} \}.$$

* "The Contracted Vein," 'Phil. Mag.,' Dec. 1876.
† 'Mém. de l'Acad.,' 1766, Paris. I owe this reference to the kindness of Lord RAYLEIGH.

Between $u = a$ and $u = 1$ we have

$$\frac{dx}{dp} = 0$$

$$\frac{dy}{dp} = \frac{1}{p^2 \sqrt{(p^2 - a^2)}} \{\sqrt{(1 - a^2)} + 1 - p^2 + \sqrt{(1 - p^2)}(1 + \sqrt{1 - a^2})\},$$

therefore the length of the pipe is

$$(1 + \sqrt{1 - a^2}) \int_a^1 \frac{dp}{p^2 \sqrt{(p^2 - a^2)}} - \int_a^1 \frac{dp}{\sqrt{(p^2 - a^2)}} + (1 + \sqrt{1 - a^2}) \int_a^1 \frac{\sqrt{(1 - p^2)}}{p^2 \sqrt{(p^2 - a^2)}} \, dp.$$

Now

$$\int_a^1 \frac{dp}{p^2 \sqrt{(p^2 - a^2)}} = \frac{1}{a^2} \left[\sin \theta \right]_0^{\cos^{-1} a} \qquad (p = a \sec \theta) = \frac{1}{a^2} \sqrt{(1 - a^2)}$$

$$\int_a^1 \frac{dp}{\sqrt{(p^2 - a^2)}} = \log \frac{1 + \sqrt{(1 - a^2)}}{a} = \cosh^{-1} \frac{1}{a}$$

$$\int_a^1 \frac{\sqrt{(1 - p^2)} \, dp}{p^2 \sqrt{(p^2 - a^2)}} = L(a) \text{ (say)},$$

an elliptic integral which will be brought to the standard form below.

Therefore the length of the pipe is

$$\frac{(1 + b) b}{a^2} - \log \frac{1 + b}{a} + (1 + b) L(a),$$

where

$$b \equiv \sqrt{(1 - a^2)}.$$

From

$$u = 1 \text{ to } u = \infty$$

we have

$$\int_1^\infty \frac{dx}{dp} \, dp = (1 + b) \int_1^\infty \frac{\sqrt{(p^2 - 1)}}{p^2 \sqrt{(p^2 - a^2)}} \, dp = (1 + b) M(a) \text{ (say)},$$

where M is another elliptic integral to be reduced below.

Summing up, the breadth of jet is π; that of the aperture is

$$\pi + 2(1 + \sqrt{1 - a^2}) M(a),$$

so that the contraction ratio is

$$\frac{\pi}{\pi + 2(1 + \sqrt{1 - a^2}) M(a)},$$

where the length of the pipe is

$$\frac{1 + \sqrt{(1 - a^2)}}{a} \sqrt{(1 - a^2)} - \log \frac{1 + \sqrt{(1 - a^2)}}{a} + (1 + \sqrt{1 - a^2}) L(a).$$

To reduce the elliptic integrals, we have

$$\int_a^1 \frac{1}{p}\,\frac{\sqrt{(1-p^2)}}{\sqrt{(p^2-a^2)}}\,dp = \int_1^{1/a}\frac{\sqrt{(r^2-1)}}{\sqrt{(1-a^2r^2)}}\,dr \qquad rp=1$$

$$= \int_1^{1/a}\frac{r^2-1}{\sqrt{(r^2-1\cdot 1-a^2r^2)}}\,.$$

Now (CAYLEY, ' Elliptic Functions,' p. 315)

$$\frac{dr}{\sqrt{(r^2-1\,\cdot\,1-a^2r^2)}} = \frac{dx}{\sqrt{(1-x^2\,\cdot\,1-k^2x^2)}} \quad \text{where } k^2 = \frac{1-a^2}{a^2} \text{ and } r^2 = \frac{1}{1-(1-a^2)x^2}.$$

Thus

$$\mathrm{L}\,(a) = \int_0^1\left[\frac{1}{1-(1-a^2)x^2}-1\right]\frac{dx}{\sqrt{(1-x^2\,\cdot\,1-k^2x^2)}}$$

$$= \Pi_1\,(k,\ a^2-1) - \mathrm{F}_1\,(k),$$

where Π and F are the third and first elliptic integrals.　And

$$\mathrm{M}\,(a) = \int_1^\infty \frac{\sqrt{(p^2-1)}}{p^2\sqrt{(p^2-a^2)}}\,dp = \int_0^1\frac{\sqrt{(1-r^2)}}{\sqrt{(1-a^2r^2)}}\,dr$$

$$= \left(1-\frac{1}{a^2}\right)\int_0^1\frac{dr}{\sqrt{(1-r^2\,\cdot\,1-a^2r^2)}} + \frac{1}{a^2}\int_0^1\frac{dr\,\sqrt{(1-a^2r^2)}}{\sqrt{(1-r^2)}}$$

$$= \left(1-\frac{1}{a^2}\right)\mathrm{F}_1\,(a) + \frac{1}{a^2}\,\mathrm{E}_1\,(a),$$

where E is the second elliptic integral.

Suppose now the length of the pipe is small, so that a is nearly unity.
We have

$$\mathrm{L}\,(a) = \int_a^1\frac{\sqrt{(1-p^2)}}{p^2\sqrt{(p^2-a^2)}}\,dp.$$

If we put $p^2=x$, $2p\,dp=dx$, we may put the factor p in the integral equal to unity, and so get

$$\mathrm{L}\,(a) = \tfrac{1}{2}\int_{a^2}^1\frac{\sqrt{(1-x)}}{\sqrt{(x-a^2)}}\,dx$$

$$= \tfrac{1}{4}\pi\,(1-a^2).$$

So that the length of pipe becomes

$$l = \left(\frac{\pi}{4}+1\right)(1-a^2) \quad \text{to the first order;}$$

while the aperture becomes

$$k = \pi + 2\left(1 + \sqrt{1 - a^2}\right)\left[\left(1 - \frac{1}{a^2}\right)\log\frac{4}{\sqrt{(1 - a)}} + 1\right]$$

$$= \pi + 2 + 2\sqrt{(1 - a^2)}$$

to the first order, and the breadth of jet is $c = \pi$.

So that

$$\frac{l}{k} = \frac{\pi + 4}{4(\pi + 2)}(1 - a^2),$$

and

$$\frac{c}{k} = \frac{\pi}{\pi + 2}\left(1 - \frac{2}{\pi + 2}\sqrt{1 - a^2}\right),$$

therefore

$$\frac{c}{k} = \frac{\pi}{\pi + 2}\left(1 - \frac{4}{\pi + 2}\sqrt{\frac{\pi + 2}{\pi + 4}\frac{l}{k}}\right),$$

which is the approximate expression required.

Example IV.— Flow from an aperture in a pipe in which the water is at rest.

By considering the symmetry of the motion this can be reduced to the case of flow from a rectangular vessel in which the aperture extends from the bottom up, and the bottom is continued into a horizontal plane, as in the figure.
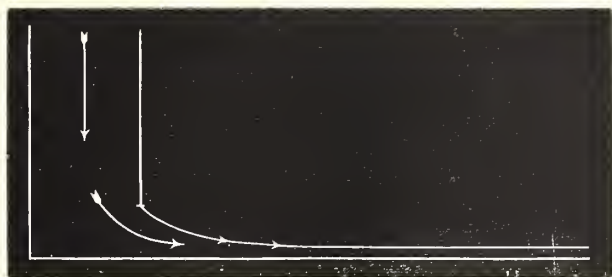
The w diagram consists of two doubly infinite straight lines, $\psi = 0$, $\psi = \pi$, as before.

But if in the u plane we make $u = -\infty$ correspond to $\phi = \infty$, the arrangement of points is not the same as before and the notation must be altered.

We take, then, $u = 0$ for the edge of the aperture, $u = 1$ the right angle of the vessel, and then put $u = a$ for $\phi = -\infty$.

We then have

$$\frac{dw}{du} = \frac{1}{u - a}$$

and the arrangement of points on $q = 0$ is as in the figure.

free	rigid	rigid		rigid
$\psi = \pi$	$u = 0$ $\psi = \pi$	$u = a$ $\psi = 0$	$u = 1$	
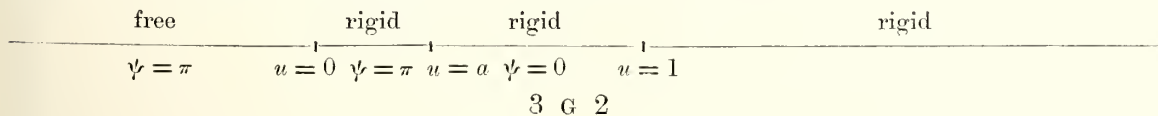
The only singular point for dz/dw is at $u = 0$, and, therefore, from Problem II. (c),

$$\frac{dz}{dw} = \left(\frac{\sqrt{u} + 1}{\sqrt{u} - 1}\right)^{\frac{1}{2}}$$

$$\frac{dz}{du} = \frac{1}{u - a} \left(\frac{\sqrt{u} + 1}{\sqrt{u} - 1}\right)^{\frac{1}{4}}$$

Along the free stream line

$$\frac{dz}{dp} = \frac{1}{p - a}\left(\frac{p + 1 + 2i\sqrt{(-p)}}{p - 1}\right)^{\frac{1}{2}}$$

where p lies between 0 and $-\infty$.

If

$$\frac{dy}{dx} = \tan\psi$$

we have

$$\frac{p + 1}{p - 1} = \cos 2\psi$$

$$\frac{2\sqrt{(-p)}}{p - 1} = \sin 2\psi$$

$$p = -\cot^2\psi.$$

Therefore

$$\frac{dx}{d\psi} = 2\frac{\cos^2\psi \, \operatorname{cosec}^3\psi}{\cot^2\psi + a}$$

$$\frac{dy}{d\psi} = 2\frac{\cos\psi \, \operatorname{cosec}^2\psi}{\cot^2\psi + a}.$$

If we put $\cos\psi = \lambda$ in the first we get

$$x = -2\int \frac{\lambda^2}{1 - \lambda^2}\frac{1}{a + (1 - a)\lambda^2}\,d\lambda,$$

$$= -2\int \frac{d\lambda}{1 - \lambda^2} - 2\frac{a}{1 - 2a}\int \frac{d\lambda}{a + (1 - a)\lambda^2},$$

$$= 2\log\frac{1 - \lambda}{1 + \lambda} - 2\frac{\sqrt{a}}{1 - 2a}\cdot\frac{1}{\sqrt{(1 - a)}}\tan^{-1}\frac{\lambda}{\sqrt{a}}\sqrt{(1 - a)} + C,$$

and, in the second, putting

$$\sin\psi = \mu$$

we have

$$y = 2\int \frac{d\mu}{1 - (1 - a)\mu^2},$$

$$= -\frac{1}{\sqrt{(1 - a)}}\log\frac{1 - \sqrt{(1 - a)}\mu}{1 + \sqrt{(1 - a)}\mu} + \pi.$$

Hence when $\psi = \frac{1}{2}\pi$, $i.e.$, at the edge of the aperture

$$y = \frac{1}{\sqrt{(1-a)}} \log \frac{1 + \sqrt{(1-a)}}{1 - \sqrt{(1-a)}} + \pi,$$

therefore the ratio of the jet to the aperture is

$$\frac{c}{k} = \frac{\pi}{\pi + \dfrac{1}{\sqrt{(1-a)}} \log \dfrac{1 + \sqrt{(1-a)}}{1 - \sqrt{(1-a)}}}.$$

The velocity at $\phi = -\infty$ is

$$\left(\frac{1 - \sqrt{a}}{1 + \sqrt{a}}\right)^{\frac{1}{2}},$$

therefore the width of the vessel is

$$d = \pi \left(\frac{1 + \sqrt{a}}{1 - \sqrt{a}}\right)^{\frac{1}{2}},$$

therefore

$$\frac{d}{c} = \left(\frac{1 + \sqrt{a}}{1 - \sqrt{a}}\right)^{\frac{1}{2}}.$$

From which we obtain

$$\frac{1 + \sqrt{a}}{2} = \frac{1}{\dfrac{c^2}{d^2} + 1},$$

$$\frac{1 - \sqrt{a}}{2} = \frac{1}{\dfrac{d^2}{c^2} + 1},$$

therefore

$$k = c \left[1 + \frac{1}{2\pi} \left(\frac{c}{d} + \frac{d}{c}\right) \log \frac{\frac{1}{2}\left(\frac{c}{d} + \frac{d}{c}\right) + 1}{\frac{1}{2}\left(\frac{c}{d} + \frac{d}{c}\right) - 1} \right],$$

or

$$k = c \left[1 + \frac{1}{\pi} \left(\frac{d}{c} + \frac{c}{d}\right) \log \frac{d + c}{d - c} \right].$$

This is for the vessel considered.   For the pipe

$$k = c \left[1 + \frac{1}{\pi} \left(\frac{2d}{c} + \frac{c}{2d}\right) \log \frac{2d + c}{2d - c} \right].$$

CASE II.—*Impact of a stream against a plane.*

A stream of given breadth impinges at a given angle against a plane.
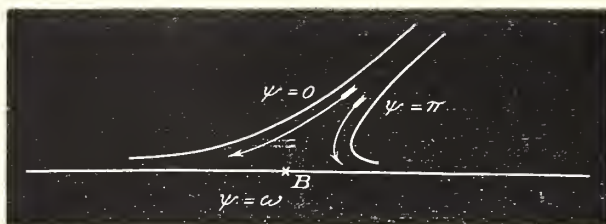
The impinging stream is bounded by the stream lines $\psi = 0$, $\psi = \pi$.

The stream line which branches at the point B on the plane is $\psi = \varpi$.

The diagram in the w plane consists of two infinite straight lines with a semi-infinite one between them, as in the figure.

$$\psi = \pi$$

$$\psi = \varpi$$

$$\psi = 0.$$
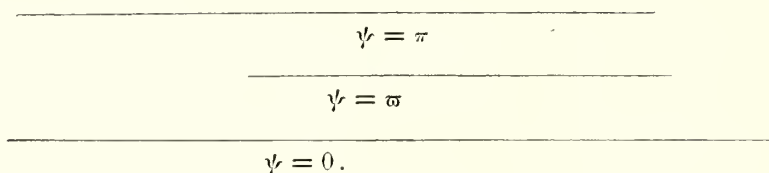
In transforming to the u plane we suppose that $\phi = -\infty$ corresponds to $u = \pm \infty$, and that $u = -1$, $u = 1$ are the extremities of the plane.

We must then take $u = a$, an unknown constant for the point B, where the stream line $\psi = \varpi$ divides, and observe that $a < 1 > -1$.

We then have

$$\frac{dw}{du} = A \frac{u - a}{(u - 1)(u + 1)}$$

where

$$\left. \begin{array}{l} -\pi A \dfrac{1 + a}{2} = \varpi \\[2mm] -\pi A \dfrac{1 - a}{2} = \pi - \varpi \end{array} \right\} ,$$

therefore

$$A = -1 \quad \text{and} \quad a = \frac{2\varpi}{\pi} - 1.$$

Along $q = 0$ we have, therefore, the following arrangement of points :—

free			rigid		rigid		free	
$\phi = -\infty$ $\psi = 0$		$u = -1$	$\psi = \varpi$	$u = a$		$u = 1$	$\psi = \pi$	$\phi = -\infty$
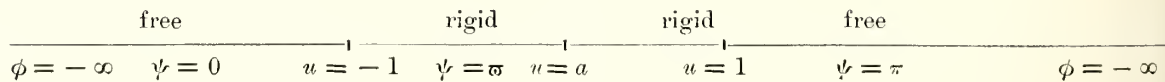
The only singular point of dz/dw is at $u = a$.
Hence

$$\frac{dz}{dw} = \frac{1 - au + \sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{u - a},$$

and, therefore,

$$\frac{dz}{du} = -\frac{1 - au + \sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{(u - 1)(u + 1)}$$

$$= \frac{1 - au}{1 - u^2} + \sqrt{(1 - a^2)}\frac{1}{\sqrt{(1 - u^2)}}.$$

When $u = \pm \infty$, we have

$$\frac{dx}{d\phi} + i\frac{dy}{d\phi} = -a - i\sqrt{(1 - a^2)}.$$

Therefore, if θ be the inclination of the stream to the plane, we have

$$\tan \theta = \frac{\sqrt{(1 - a^2)}}{a} \quad \text{or} \quad \cos \theta = a.$$

Since we have before found

$$a = \frac{2\varpi}{\pi} - 1,$$

this gives

$$\pi \cos \theta = 2\varpi - \pi.$$

This equation merely expresses that the momentum parallel to the plane is unaltered by impact.

Along the stream line $\psi = \pi$ we have

$$\left.\begin{aligned} \frac{dx}{dp} &= -\frac{1 - ap}{p^2 - 1} \\ \frac{dy}{dp} &= \sqrt{(1 - a^2)}\frac{1}{\sqrt{(p^2 - 1)}}, \end{aligned}\right\}$$

therefore

$$\left.\begin{aligned} x &= \frac{a}{2}\log(p^2 - 1) + \tfrac{1}{2}\log\frac{p + 1}{p - 1} + A \\ &= \frac{1 + a}{2}\log(p + 1) - \frac{1 - a}{2}\log(p - 1) + A \\ y &= \sqrt{(1 - a^2)}\log(p + \sqrt{p^2 - 1}) + \pi - \varpi. \end{aligned}\right\}$$

where p lies between $+1$ and ∞; and for $\psi = 0$ we have

$$\frac{dx}{dp} = -\frac{1 - ap}{p^2 - 1} \qquad \frac{dy}{dp} = -\sqrt{(1 - a^2)}\frac{1}{\sqrt{(p^2 - 1)}},$$

therefore

$$x = \frac{1+a}{2} \log -(1+p) - \frac{1-a}{2} \log (1-p) + A' \Bigg\}$$
$$y = \sqrt{(1-a^2)} \log (\sqrt{p^2-1} - p) + \varpi$$

where p lies between -1 and $-\infty$.

If we put $p = \dfrac{1}{\cos \theta}$ in the first and $p = -\dfrac{1}{\cos \theta}$ in the second, we get for $\psi = \pi$

$$x_1 = (1+a) \log \cos \frac{\theta}{2} - (1-a) \log \sin \frac{\theta}{2} - a \log \cos \theta + A \Bigg\}$$
$$y_1 = \sqrt{(1-a^2)} \log \cot \tfrac{1}{2}\left(\frac{\pi}{2} - \theta\right) + \pi - \varpi,$$

and for $\psi = 0$

$$x_2 = (1+a) \log \sin \frac{\theta}{2} - (1-a) \log \cos \frac{\theta}{2} - a \log \cos \theta + A' \Bigg\}$$
$$y_2 = \sqrt{(1-a^2)} \log \cot \tfrac{1}{2}\left(\frac{\pi}{2} - \theta\right) + \varpi$$

where in both cases θ lies between 0 and $\tfrac{1}{2}\pi$.   When $\theta = \tfrac{1}{2}\pi$ in both we get

$$x_1 - x_2 = A - A' \Bigg\}$$
$$y_1 - y_2 = \pi - 2\varpi$$

therefore

$$A - A' = \frac{\pi + (\pi - 2\varpi)a}{\sqrt{(1-a^2)}} = \pi \sqrt{(1-a^2)},$$

and the equations of the bounding stream lines are now completely determined.

CASE III.—*Flow of a broad stream past a plane wall in which there is an aperture.*

Let BC be the aperture in a plane wall ABCD, and let the stream flow from left to right.
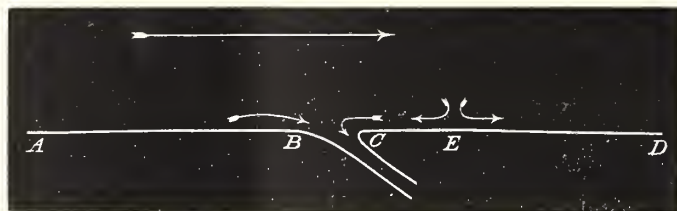
The left boundary of the issuing jet will be the continuation of the stream line AB. The right boundary will be one branch of a stream line which divides on the plane CD at some point E.

The diagram in the w plane will then consist of one infinite straight line and one semi-infinite, as in the figure.

$$\psi = \pi$$

$$\psi = 0$$
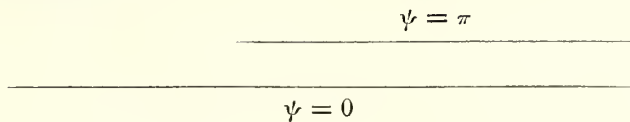
We suppose $\psi = 0$, $\psi = \pi$ to be the bounding stream lines. In transforming to the u plane we take the point $\psi = 0$, $\phi = -\infty$ to be $u = -\infty$; $\psi = 0$, $\phi = \infty$ to be $u = a$, and then use the two arbitrary constants at our disposal by making the edges of the aperture to be $u = -b$, $u = b$, and the branch point of $\psi = \pi$ to be $u = 1 + a$, where $1 + a > b > a$.

We then have

$$\frac{dw}{du} = A \frac{u - a - 1}{u - a},$$

and, remembering that the bounding stream lines are $\psi = 0$, $\psi = \pi$, we obtain $A = 1$, so that

$$\frac{dw}{du} = \frac{u - a - 1}{u - a}.$$

There is only one singular point for dz/dw, viz., $u = a + 1$, and at this point $\alpha/\pi - 1 = 0$, so that no factor $u - a - 1$ appears in dz/du.

The arrangement of points on $q = 0$ is as in the figure

rigid		free				rigid		
$\psi = 0$	$u = -b$	$u = a$	$\psi = \pi$	$u = b$	$u = a + 1$	$\psi = \pi$		
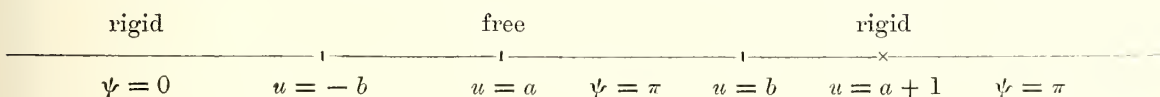
and hence

$$\frac{dz}{dw} = \frac{(a + 1)\,u - b^2 + \sqrt{(a + 1)^2 - b^2}\,\sqrt{(u^2 - b^2)}}{b\,(u - a - 1)},$$

so that

$$\frac{dz}{du} = \frac{(a + 1)\,u - b^2 + \sqrt{(a + 1)^2 - b^2}\,\sqrt{(u^2 - b^2)}}{b\,(u - a)}.$$

When $u = \pm \infty$, we have

$$\frac{dz}{dw} = \frac{a + 1 + \sqrt{(a + 1)^2 - b^2}}{b},$$

and, therefore, the velocity of the stream is

$$\frac{b}{(a + 1) + \sqrt{(a + 1)^2 - b^2}}.$$

When $u = a$, that is at a great distance along the jet,

$$\frac{dz}{dw} = -\frac{a(a+1) - b^2 + i\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}}{b},$$

so that the inclination of the jet to the wall is ultimately

$$\cos^{-1}\frac{b^2 - a(a+1)}{b},$$

the final breadth of the jet being π.

There appear to be two constants here, whereas there ought only to be one.

The explanation is that we have not yet expressed that the two parts of the boundary are in the same plane. This will give a relation between a and b.

Between $u = -b$ and $u = a$ we have

$$\frac{dx}{dp} = \frac{1}{b}\frac{(a+1)p - b^2}{p - a},$$

$$\frac{dy}{dp} = \frac{\sqrt{(a+1)^2 - b^2}}{b}\frac{\sqrt{(b^2 - p^2)}}{p - a}.$$

Now

$$\int\frac{\sqrt{(b^2 - p^2)}}{p - a} = \int\frac{1}{\sqrt{(b^2 - p^2)}}\left[\frac{b^2 - a^2}{p - a} - a - p\right],$$

and

$$\int\frac{dp}{(p - a)\sqrt{(b^2 - p^2)}}$$

$$= -\int\frac{dr}{\sqrt{(b^2 - a^2)r^2 + 2ar - 1}}, \qquad \text{where } p - a = -\frac{1}{r},$$

$$= -\int\frac{dr}{\sqrt{\left(\sqrt{b^2 - a^2}\,r + \frac{a}{\sqrt{(b^2 - a^2)}}\right)^2 - \frac{b^2}{b^2 - a^2}}},$$

$$= -\frac{1}{\sqrt{(b^2 - a^2)}}\log\left[\sqrt{(b^2 - a^2)}\,r + \frac{a}{\sqrt{(b^2 - a^2)}} + \sqrt{(b^2 - a^2)r^2 + 2ar - 1}\right],$$

$$= -\frac{1}{\sqrt{(b^2 - a^2)}}\log\frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{a - p}.$$

Thus along $\psi = 0$, measuring from the edge A, we have

$$x - x_A = \frac{a+1}{b}(p + b) + \frac{a(a+1) - b^2}{b}\log\frac{a - p}{a + b},$$

$$y - y_A = \frac{\sqrt{(a+1^2) - b^2}}{b}\left[-\sqrt{(b^2 - a^2)}\log\frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{(a - p)b}\right.$$

$$\left. + a\sin^{-1}.\frac{-p}{b} - a\frac{\pi}{2} + \sqrt{(b^2 - p^2)}\right].$$

In like manner between $u = a$ and $u = b$ on $\psi = \pi$, we have

$$x - x_{\mathrm{B}} = \frac{a+1}{b}(p-b) + \frac{a(a+1) - b^2}{b} \log \frac{p-a}{b-a},$$

$$y - y_{\mathrm{B}} = \frac{\sqrt{(a+1)^2 - b^2}}{b}\left[-\sqrt{(b^2 - a^2)} \log \frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{(p-a)b} \right.$$

$$\left. - a \sin^{-1}\frac{p}{b} + a\frac{\pi}{2} + \sqrt{(b^2 - p^2)} \right].$$

To find $x_{\mathrm{A}} - x_{\mathrm{B}}$ and $y_{\mathrm{A}} - y_{\mathrm{B}}$ we must put $p = a - \epsilon$ and $p = a + \epsilon$ respectively in the above, and then decrease ϵ indefinitely.

If we then remember that, in passing the point $p = a$, $x + iy$ increases by

$$-\frac{i\pi}{b}\left\{ (a+1)a - b^2 + i\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)} \right\}$$

we get

$$x_{\mathrm{B}} - x_{\mathrm{A}} = 2(a+1) + \frac{a(a+1) - b^2}{b}\log\frac{b-a}{b+a} + \frac{\pi}{b}\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}$$

and

$$y_{\mathrm{B}} - y_{\mathrm{A}} = -\pi a\frac{\sqrt{(a+1)^2 - b^2}}{b} + \frac{\pi}{b}\{b^2 - a(a+1)\}.$$

In the case we are considering $y_{\mathrm{B}} = y_{\mathrm{A}}$, and, therefore,

$$b^2 - a(a+1) = a\sqrt{(a+1)^2 - b^2},$$

or

$$b^2 = (a+1)^2 - 1,$$

so that

$$\left.\begin{array}{l} (a+1)^2 - b^2 = 1 \\ a(a+1) - b^2 = -a. \end{array}\right\}$$

Using these equalities, we now get for the breadth of the aperture

$$x_{\mathrm{B}} - x_{\mathrm{A}} = 2(a+1) + \frac{a}{b}\log\frac{b+a}{b-a} + \pi\frac{\sqrt{2a}}{b}$$

where $b/(a+2)$ is the velocity of the stream, π the breadth of the jet, and $\cos^{-1}(a/b)$ the inclination of the jet to the plane.

To get rid of the special units let v_1 be the velocity of the stream, v_2 of the jet, k the breadth of the aperture, c that of the jet.
Then

$$\frac{v_1}{v_2} = \frac{b}{a+2},$$

therefore

$$\left(\frac{v_1}{v_2}\right)^2 = \frac{b^2}{(a+2)^2} = \frac{a}{a+2}.$$

So that

$$a = \frac{2v_1^2}{v_2^2 - v_1^2},$$

and

$$b = \frac{2v_2 v_1}{v_2^2 - v_1^2},$$

therefore

$$a + b = \frac{2v_1(v_2 + v_1)}{v_2^2 - v_1^2}$$

$$-a + b = \frac{2v_1(v_2 - v_1)}{v_2^2 - v_1^2}.$$

Hence

$$\frac{k}{c}\pi = 2\frac{v_2^2 + v_1^2}{v_2^2 - v_1^2} + \frac{v_1}{v_2}\log\frac{v_2 + v_1}{v_2 - v_1} + \pi\sqrt{\frac{v_2^2 - v_1^2}{v_2^2}},$$

and the inclination of the jet to the plane is

$$\cos^{-1}\frac{v_1}{v_2}.$$

An interesting element not yet calculated is the distance of the branching of the stream line $\psi = \pi$ from the edge of the aperture.

This distance is

$$\int_b^{a+1}\left[\frac{1}{b}\frac{(a+1)p - b^2 + \sqrt{(p^2 - b^2)}}{p - a}\right]dp$$

$$= \int_b^{a+1}\left[\frac{a+1}{b} - \frac{a}{b}\frac{1}{p-a} + \frac{1}{b}\left\{p + a - \frac{2a}{p-a}\right\}\frac{1}{\sqrt{(p^2 - b^2)}}\right]dp,$$

where we have used the relation between a and b to simplify the expression,

$$= \left[\frac{a+1}{b}p - \frac{a}{b}\log(p-a) + \frac{1}{b}\sqrt{(p^2 - b^2)} + \frac{a}{b}\log(p + \sqrt{p^2 - b^2})\right.$$

$$\left. + \frac{\sqrt{2a}}{b}\sin^{-1}\left\{\frac{b^2 - ap}{b(p-a)}\right\}\right]_b^{a+1}$$

$$= \frac{a+1}{b}(a + 1 - b) + \frac{a}{b}\log(b - a) + \frac{1}{b} + \frac{a}{b}\log\frac{a+2}{b}$$

$$+ \frac{\sqrt{2a}}{b}\left(\sin^{-1}\frac{a}{b} - \frac{\pi}{2}\right).$$

Therefore, in terms of v_1/v_2 and c, this distance is l where

$$\frac{l}{c}\pi = \left[\frac{v_2^2 + v_1 v_2 + v_1^2}{v_1 v_2} \times \frac{v_2 - v_1}{v_2 + v_1} + \frac{v_1}{v_2} \log \frac{2v_2}{v_2 + v_1} + \sqrt{\frac{v_2^2 - v_1^2}{v_2^2}} \left(\sin^{-1} \frac{v_1}{v_2} - \frac{\pi}{2} \right) \right].$$

For example, let the stream have half the velocity of the issuing jet, so that $v_1 = \frac{1}{2} v_2$. Then the jet makes an angle of $60°$ with the plane, and its breadth is $\frac{1}{2} k$ approximately, while l is about $\frac{1}{4} k$.

CASE IV.—*Jet from a pipe along which liquid is flowing.*

The liquid is flowing along a pipe bounded by the plane walls AC, DE, and there is an aperture AB in the former.
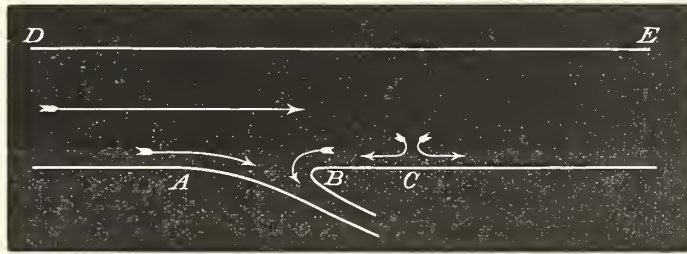
The left boundary of the jet is $\psi = 0$, the right boundary is a stream line $\psi = \varpi$, which branches at a point C on BC, and DE is $\psi = \pi$.

The w diagram is as in the figure, consisting of two infinite lines, $\psi = 0$, $\psi = \pi$, with a semi-infinite line $\psi = \varpi$ between them.
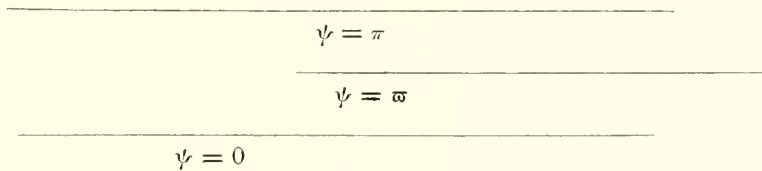
In transforming to the u plane, we suppose that $\phi = -\infty$ corresponds to $u = -\infty$, and that $u = -1$, $u = +1$ are the edges of the aperture.

The constants of the transformation are then determined, and we take $u = a$ for the branch point, $u = b$ for the jet at an infinite distance, $u = c$ for $\phi = \infty$ in the pipe. Then

$$\frac{dw}{du} = A \frac{u - a}{(u - b)(u - c)} \quad b < a < c,$$

where

$$- A \frac{b - a}{b - c} \pi = \varpi$$

$$- A \frac{c - a}{b - c} \pi = \varpi - \pi,$$

and therefore

$$A = -1$$

$$\frac{\varpi}{\pi} = \frac{b-a}{b-c}$$

$$\frac{\pi - \varpi}{\pi} = -\frac{c-a}{b-c}.$$

The arrangement of points on $q = 0$ is as in the figure

rigid	free	free	rigid	rigid
$\psi = 0$	$u=-1$ $\psi=0$ $u=b$	$\psi=\varpi$ $u=1$	$u=a$	$u=c$ $\psi=\pi$
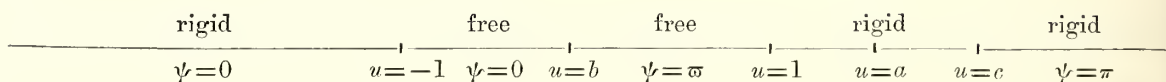
and from Problem II. (a) we have

$$\frac{dz}{dw} = \frac{au - 1 + \sqrt{(a^2-1)}\,\sqrt{(u^2-1)}}{u-a},$$

so that

$$\frac{dz}{du} = -\frac{au - 1 + \sqrt{(a^2-1)}\,\sqrt{(u^2-1)}}{(u-b)\,(u-c)}.$$

Hence, in the pipe, the velocity at $\phi = -\infty$ is

$$\frac{1}{a + \sqrt{(a-1)}} = a - \sqrt{(a^2-1)},$$

and at $\phi = \infty$ is

$$\frac{c-a}{ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)}}.$$

The breadth of the pipe is, therefore,

$$\pi\left(a + \sqrt{a^2-1}\right) = (\pi - \varpi)\,\frac{ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)}}{c-a}$$

Now

$$\pi - \varpi = \pi\,\frac{c-a}{c-b},$$

therefore

$$(c-b)\left(a + \sqrt{a^2-1}\right) = ac - 1 + \sqrt{(a^2-1)}\,\sqrt{(c^2-1)}.$$

This is the relation between a, b, and c. To find the size of the aperture we must integrate between $u = -1$ and $u = 1$. Within these limits we have

$$\frac{dx}{dp} = \frac{1 - ap}{(b-p)\,(c-p)} = \frac{1 - ab}{c-b}\,\frac{1}{b-p} + \frac{ac - 1}{c-b}\,\frac{1}{c-p}$$

$$\frac{dy}{dp} = -\sqrt{(a^2-1)}\,\frac{\sqrt{(1-p^2)}}{(b-p)\,(c-p)} = -\frac{\sqrt{(a^2-1)}}{c-b}\,\sqrt{1-p^2}\left[\frac{1-ab}{b-p} + \frac{ac-1}{c-p}\right].$$

The former gives

$$x - x_A = -\frac{1-ab}{c-b}\log\frac{b-p}{b+1} - \frac{ac-1}{c-b}\log\frac{c-p}{c+1} \quad \text{between } p = -1 \text{ and } p = b,$$

and

$$x - x_B = \frac{ab-1}{c-b}\log\frac{p-b}{1-b} - \frac{ac-1}{c-b}\log\frac{c-p}{c-1} \quad \text{between } p = b \text{ and } p = 1.$$

Also in passing the point $u = b$, x changes by

$$\pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b};$$

therefore, putting $p = b - \epsilon$ in $x - x_A$, $p = b + \epsilon$ in $x - x_B$, and proceeding to the limit, we get as the breadth of the aperture,

$$\frac{1-ab}{c-b}\log\frac{1+b}{1-b} + \frac{ac-1}{c-b}\log\frac{c+1}{c-1} + \pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}.$$

The direction of the jet makes an angle

$$\cos^{-1}\left[\frac{1-ab}{a-b}\right]$$

with the bounding planes.   The breadth of the jet is

$$\pi\frac{a-b}{c-b}.$$

To sum up : Let d be the breadth of the pipe, k of the aperture, l of the jet.   Let, further, v_1 be the velocity at $\phi = -\infty$, v_2 that at $\phi = \infty$, v_3 that of the jet.

$$\frac{v_3}{v_1} = a + \sqrt{(a^2-1)} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{v_1}{v_2} = \frac{c-b}{c-a} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and therefore

$$\frac{v_1}{v_1-v_2} = \frac{c-b}{a-b} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{d}{l} = \frac{v_3}{v_1-v_2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{k}{l} = \frac{c-b}{\pi(a-b)}\left\{\frac{1-ab}{c-b}\log\frac{1+b}{1-b} + \frac{ac-1}{c-b}\log\frac{1+c}{-1+c} + \pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}\right\} . \quad (5)$$

with

$$(c-b)(a+\sqrt{a^2-1}) = ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)} \quad . \quad . \quad . \quad (6)$$

The equation (6) may be written

$$(c - a)\frac{v_3}{v_2} = (ac - 1) + \sqrt{(a^2 - 1)}\,\sqrt{(c^2 - 1)}.$$

Now

$$(c - a)^2 \equiv (ac - 1)^2 - (a^2 - 1)(c^2 - 1),$$

therefore

$$(c - a)\frac{v_2}{v_3} = (ac - 1) - \sqrt{(a^2 - 1)}\,\sqrt{(c^2 - 1)};$$

and, by addition and subtraction,

$$(c - a)\left(\frac{v_3}{v_2} + \frac{v_2}{v_3}\right) = 2(ac - 1),$$

$$(c - a)\left(\frac{v_3}{v_2} - \frac{v_2}{v_3}\right) = 2\sqrt{(a^2 - 1)}\,\sqrt{(c^2 - 1)}.$$

From (1)

$$a = \tfrac{1}{2}\left(\frac{v_3}{v_1} + \frac{v_1}{v_3}\right)$$

$$\sqrt{(a^2 - 1)} = \tfrac{1}{2}\left(\frac{v_3}{v_1} - \frac{v_1}{v_3}\right).$$

Therefore

$$c\left[\frac{v_3}{v_2} + \frac{v_2}{v_3} - \frac{v_3}{v_1} - \frac{v_1}{v_3}\right] = \tfrac{1}{2}\left(\frac{v_3}{v_2} + \frac{v_2}{v_3}\right)\left(\frac{v_3}{v_1} + \frac{v_1}{v_3}\right) - 2,$$

or,

$$c(v_1 - v_2)(v_3^2 - v_1 v_2) = \tfrac{1}{2}(v_1 - v_2)^2 v_3 + \tfrac{1}{2}(v_3^2 - v_1 v_2)^2 \frac{1}{v_3};$$

so that

$$\frac{1 + c}{-1 + c} = \frac{[(v_1 - v_2)v_3 + (v_3^2 - v_1 v_2)]^2}{[-(v_1 - v_2)v_3 + (v_3^2 - v_1 v_2)]^2}$$

$$= \frac{(v_3 - v_2)^2 (v_3 + v_1)^2}{(v_3 + v_2)^2 (v_3 - v_1)^2}.$$

And

$$\frac{ac - 1}{a - b} = \tfrac{1}{2}\left(\frac{v_3}{v_2} + \frac{v_2}{v_3}\right)\frac{v_2}{v_1 - v_2}.$$

Also

$$(c - a)\frac{v_1}{v_2} = c - b \quad \text{from (2)},$$

therefore

$$b = \tfrac{1}{2}\frac{v_3^2 + v_1^2}{v_2 v_3} - \tfrac{1}{2}\frac{(v_1 - v_2)^2 v_3^2 + (v_3^2 - v_1 v_2)^2}{v_2 v_3 (v_3^2 - v_1 v_2)}$$

$$= \tfrac{1}{2}\frac{(v_1^2 + v_1 v_2)(v_3^2 - v_1 v_2) - (v_1 - v_2)^2 v_3^2}{v_2 v_3 (v_3^2 - v_1 v_2)},$$

so that

$$1 + b = \tfrac{1}{2} \frac{(v_1^2 + v_1 v_2 + 2v_2 v_3)(v_3^2 - v_1 v_2) - (v_1 - v_2)^2 v_3^2}{v_2 v_3 (v_3^2 - v_1 v_2)},$$

and

$$1 - b = \tfrac{1}{2} \frac{(2v_2 v_3 - v_1^2 - v_1 v_2)(v_3^2 - v_1 v_2) + (v_1 - v_2)^2 v_2^2}{v_2 v_3 (v_3^2 - v_1 v_2)}.$$

These may be reduced to the form

$$1 + b = \tfrac{1}{2} \frac{(v_3 + v_1)^2}{v_3^2 - v_1 v_2} \cdot \frac{2v_3 - v_1 - v_2}{v_3},$$

$$1 - b = \tfrac{1}{2} \frac{(v_3 - v_1)^2}{v_3^2 - v_1 v_2} \cdot \frac{2v_3 + v_1 + v_2}{v_3}.$$

Further,

$$a - b = \tfrac{1}{2} \frac{(v_3^2 v_2 - v_1^3)(v_3^2 - v_1 v_2) + (v_1 - v_2)^2 v_3^2 v_1}{v_1 v_2 v_3 (v_3^2 - v_1 v_2)},$$

$$= \tfrac{1}{2} \frac{(v_3^2 - v_1^2)^2}{v_1 v_3 (v_3^2 - v_1 v_2)},$$

$$1 - ab = \tfrac{1}{4} \frac{\{4 v_3^2 v_1 v_2 - (v_3^2 + v_1^2)(v_1^2 + v_1 v_2)\}\{v_3^2 - v_1 v_2\} + (v_3^2 + v_1^2)(v_1 - v_2)^2 v_3^2}{v_1 v_2 v_3^2 (v_3^2 - v_1 v_2)},$$

$$= \tfrac{1}{4} \frac{(v_1 + v_2)(v_3^2 - v_1^2)^2}{v_1 v_3^2 (v_3^2 - v_1 v_2)}.$$

Whence, finally,

$$\frac{1 - ab}{a - b} = \tfrac{1}{2} \frac{v_1 + v_2}{v_3},$$

$$\frac{1 + b}{1 - b} = \frac{(v_3 + v_1)^2}{(v_3 - v_1)^2} \frac{2v_3 - v_1 - v_2}{2v_3 + v_1 + v_2},$$

$$\frac{\sqrt{(a^2 - 1)}\sqrt{(1 - b^2)}}{c - b} = \tfrac{1}{2} \frac{\sqrt{4v_3^2 - (v_1 + v_2)^2}}{v_3}.$$

So that we get for k/l

$$\frac{k}{l} \pi = \tfrac{1}{2} \frac{v_1 + v_2}{v_3} \log \frac{(v_3 + v_1)^2}{(v_3 - v_1)^2} \frac{2v_3 - v_1 - v_2}{2v_3 + v_1 + v_2}$$

$$+ \frac{v_3^2 + v_2^2}{v_3 (v_1 - v_2)} \log \frac{(v_3 - v_2)(v_3 + v_1)}{(v_3 + v_2)(v_3 - v_1)}$$

$$+ \tfrac{1}{2} \pi \left\{ \frac{4v_3^2 - (v_1 + v_2)^2}{v_3^2} \right\}^{\frac{1}{2}}.$$

An interesting special case is the flow from an aperture in the side of a rectangular vessel, of which the bottom is at a considerable distance from the aperture.

For this case $v_2 = 0$, and, therefore,

$$\frac{d}{l} = \frac{v_3}{v_1},$$

expressing the equality of inflow and outflow.

So that

$$\frac{k}{l}\pi = \frac{1}{2}\frac{v_1}{v_3}\log\left(\frac{v_3 + v_1}{v_3 - v_1}\right)^2 \cdot \frac{2v_3 - v_1}{2v_3 + v_1} + \frac{v_3}{v_1}\log\frac{v_3 + v_1}{v_3 - v_1} + \frac{\pi}{2}\left\{4 - \frac{v_1^2}{v_3^2}\right\}^{\frac{1}{2}}$$

$$= \left(\frac{v_1}{v_3} + \frac{v_3}{v_1}\right)\log\frac{v_3 + v_1}{v_3 - v_1} + \frac{1}{2}\frac{v_1}{v_3}\log\frac{2v_3 - v_1}{2v_3 + v_1} + \frac{\pi}{2}\left\{4 - \frac{v_1^2}{v_3^2}\right\}^{\frac{1}{2}},$$

or

$$\frac{k}{l}\pi = \left(\frac{d}{l} + \frac{l}{d}\right)\log\frac{d + l}{d - l} + \frac{1}{2}\frac{l}{d}\log\frac{2d - l}{2d + l} + \frac{\pi}{2}\left(4 - \frac{l^2}{d^2}\right)^{\frac{1}{2}},$$

which is the simplest expression for this case.

The angle the jet makes with the bounding wall is

$$\cos^{-1}\frac{1 - ab}{a - b},$$

or

$$\cos^{-1}\frac{1}{2}\frac{v_1 + v_3}{v_3}$$

in the general case, while in the particular case it is simply

$$\cos^{-1}\frac{1}{2}\frac{v_1}{v_3}.$$

PART II.

We will now go on to consider problems in which the region of (x, y) is not simply connected, and consequently SCHWARZ's transformation does not apply.

First consider the area outside a closed polygon.

We may state the problem electrically thus :—

PROBLEM III.

To find the potential due to a polygonal prismatic conductor at a given potential which we may take to be zero, the field at infinity being at an infinite potential.

Let ψ be the potential, ϕ the lines of force, and let ϕ increase by $2l$ in going round the polygon.

Then the area in the ω plane is a rectangle bounded by $\psi = 0$, $\psi = \infty$, $\phi = -l$, $\phi = l$.
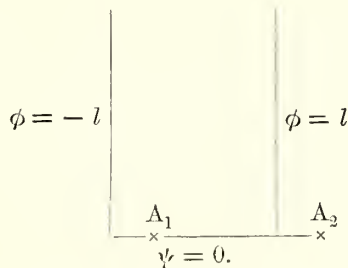
The conditions which the transformation function V satisfy are

(a)
$$\frac{d\,V}{d\,\psi} = 0 \text{ over } \psi = 0.$$

(b) V finite and continuous at all points within a finite distance in the rectangle.

(c) V periodic in ϕ so that
$$V(\phi + 2l) = V(\phi).$$

(d) V infinite at points $A_1 \ldots$ along $\psi = 0$.

We can determine V from these specifications by means of W. THOMSON's method of images. For if we repeat the points $A_1 \ldots$ at equal distances $2l$ along $\psi = 0$, and make V the potential of these points, the conditions will clearly be satisfied.

Hence
$$V = A\Sigma_r \log \prod_{-\infty}^{+\infty} \{(\phi - \phi_r - 2nl)^2 + \psi^2\}^m,$$

and

$$\frac{d\,z}{d\,w} = A\Pi_r \prod_{-\infty}^{+\infty} \{w - \phi_r - 2nl\}^m$$

$$= B\Pi_r \left\{ \sin(w - \phi_r)\frac{\pi}{2l} \right\}^m,$$

and exactly as in the treatment of SCHWARZ's formula we have $m = \alpha_r/\pi - 1$, where α_r is the exterior angle of the polygon corresponding to ϕ_r.

So that, finally,[*]

$$\frac{dz}{dw} = \text{B}\Pi\left\{\sin(w - \phi_r)\frac{\pi}{2l}\right\}^{\alpha_r/\pi - 1}$$

As an example, take the case of a rectangle.

The four singular points may be taken to be

$$\phi = -a, \qquad \phi = a, \qquad \phi = l - a, \qquad \phi = l + a,$$

and we have, taking for simplicity $l = \frac{1}{2}\pi$,

$$\frac{dz}{dw} = \text{A}\sqrt{-\sin(w - a)\sin(w + a)\cos(w - a)\cos(w + a)},$$

$$= \tfrac{1}{2}\text{A}\sqrt{(\sin^2 2a - \sin^2 2w)}.$$

Hence

$$z = \tfrac{1}{2}\text{A}\int\sqrt{(\sin^2 2a - \sin^2 2w)}\,dw + \text{B},$$

and z is an elliptic integral of w.

PROBLEM IV.

Suppose now there are two polygonal prismatic conductors, one at potential $\psi = -k$, the other at $\psi = k$, and at first (A) suppose that one of the conductors is within the other.

Let ϕ increase by $2l$ in going round either polygon.

The area in the w plane is now a finite rectangle bounded by $\phi = -l$, $\phi = l$, $\psi = -k$, $\psi = k$.
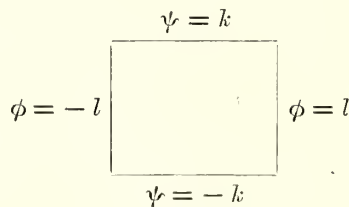
In order to satisfy the condition $d\text{V}/d\psi = 0$ over $\psi = -k$ and $\psi = k$ we must have a double system of images of the singular points, viz., at

$$\phi_0 + 2ml, \qquad \psi_0 + 4nk,$$

$$\phi_0 + 2ml, \qquad \psi_0 + (2n + 1)2k.$$

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* Mr. BRILL has already given this formula ('Messenger of Math.,' August, 1889), and I only insert it here for the sake of completeness.

NOTE.—April 29. Since the above paper was read, Mr. BRILL has given the next transformation in the same journal.

Thus each singular point gives a factor

$$\prod_{-\infty}^{+\infty} \prod_{-\infty}^{+\infty} \{w - w_0 - 2ml - 4nk\}^M \{w - w_0 - 2ml - (2n + 1)\,2k\}^M,$$

or, what is the same thing,

$$\Theta \left[a\,(w - w_0)\right]^M \mathrm{H} \left[a\,(w - w_0)\right]^M,$$

where Θ, H are Jacobi's functions so indicated, and, therefore,

$$\frac{dz}{dw} = \prod \left\{\Theta \left[a\,(w - w_0)\right] \mathrm{H} \left[a\,(w - w_0)\right]\right\}^M,$$

where

$$\left. \begin{array}{l} al = \mathrm{K} \\ 2ak = \mathrm{K}' \end{array} \right\}$$

K, K' being the complete elliptic integrals usually so denoted.

If α_0 be the internal angle of the figure corresponding to w_0, we have, as before,

$$M = \frac{\alpha_0}{\pi} - 1,$$

so that, finally,

$$\frac{dz}{dw} = \prod_r \left\{\Theta \left[a\,(w - w_r)\right] \mathrm{H} \left[a\,(w - w_r)\right]\right\}^{\alpha_r/\pi - 1}.$$

(B) Suppose now that one conductor is outside the other, and that the potential at infinity is zero, that of the conductors being $-k$ and $+k$. We suppose equal and opposite quantities of electricity on the conductors, $2l$ being the cyclic constant, as before.

The terms corresponding to angles of the polygon will be the same as in (A).

But there is now in addition a singular point in the field which we proceed to determine.

At a great distance from the prisms the potential will be the same as for two line distributions at the centres of mass, say at $z = a$, $z = -a$.

So that

$$w = \mathrm{M} \log \frac{z - a}{z + a} = -2 \frac{\mathrm{M}a}{z}$$

ultimately, and

$$\frac{dw}{dz} = 2\mathrm{M}az^{-2} = \frac{w^2}{2\mathrm{M}a}$$

or

$$\frac{dz}{dw} = \frac{2\mathrm{M}a}{w^2},$$

and, therefore, there is a point of order -2 at the point in the w rectangle corresponding to the potential at infinity.

Let this point be ϕ_0, ψ_0.   Then we have

$$\frac{dz}{dw} = \frac{\Pi_r \{\Theta\left[a\left(w - w_r\right)\right] \mathrm{H}\left[a\left(w - w_r\right)\right]\}^{a_r/\pi - 1}}{\{\Theta\left[a\left(w - w_0\right)\right] \mathrm{H}\left[a\left(w - w_0\right)\right]\}^2}$$

as the general expression for the potential of two polygonal prismatic conductors exterior to each other.

Hollow Vortices.

In conclusion I shall show how the methods of this paper may be applied to find the form of hollow vortices.

Inside a vessel bounded by plane walls, let there be a hollow vortex in steady motion.

Let $\psi = 0$ be the free steam line of the vortex, $\psi = 2k$ the rigid boundary.

In the w plane the area is bounded by $\psi = 0$, $\psi = 2k$, $\phi = -l$, $\phi = l$, $2l$ being the circulation round the vortex.
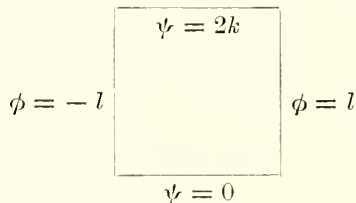
The function V satisfies the following conditions :—

(a.) V $= 0$ over $\psi = 0$, if the velocity along the free steam line be unity.

(b.) $dV/d\psi = 0$ over $\psi = 2k$.

(c.) V is periodic with respect to ϕ, so that $V(\phi + 2l) = V(\phi)$.

These conditions are to be satisfied by taking equal singular points at distances $2l$ along $\psi = 2k$, and then continually reflecting these points in the two planes $\psi = 0$, $\psi = 2k$, but in reflecting in $\psi = 0$ the image is of opposite sign to the object.

Corresponding, then, to a point M at $(\phi_0, 2k)$ we have positive images at

$$\phi_0 + 2ml, \quad 2k + 2m \cdot 4k,$$

and negative images at

$$\phi_0 + 2ml, \quad -2k + 2m \cdot 4k.$$

Therefore, corresponding to this point M, we have a factor

$$\frac{\mathrm{H}^{\mathrm{M}} a\left(w - \phi_0 - 2ik\right)}{\Theta^{\mathrm{M}} a\left(w - \phi_0 - 2ik\right)} \quad \text{in} \quad \frac{dz}{dw},$$

where

$$\left.\begin{array}{c} al = \mathrm{K} \\ 4ak = \mathrm{K}' \end{array}\right\}$$

and

$$M = \frac{\alpha_0}{\pi} - 1.$$

It is quite clear that all but the simplest cases will be of quite unmanageable complexity.

One of the very simplest cases may be taken as an example.
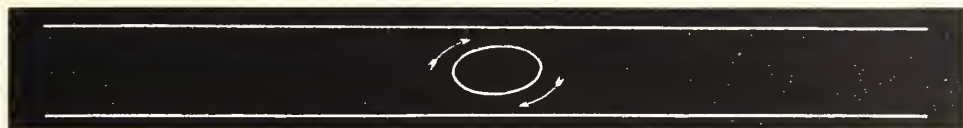
Example.—Hollow vortex between two parallel planes.

Here for the two singular points $\alpha = 0$.

If for simplicity we put $a = 1$, and, therefore, $4k = K'$, the singular points are at $\phi = 0$, $\phi = K$.

Hence

$$\frac{dz}{dw} = C \frac{\Theta\left(w - \frac{iK'}{2}\right) \Theta\left(w - \frac{iK'}{2} - K\right)}{H\left(w - \frac{iK'}{2}\right) H\left(w - \frac{iK'}{2} - K\right)}$$

$$= C' \frac{1}{sn\left(w - \frac{iK'}{2}\right) sn\left(w - \frac{iK'}{2} - K\right)}$$

$$= C'' \frac{dn\left(w - \frac{iK'}{2}\right)}{sn\left(w - \frac{iK'}{2}\right) cn\left(w - \frac{iK'}{2}\right)}.$$

This integrates at once, giving

$$z = C'' \log \frac{sn\left(w - \frac{1}{2}iK'\right)}{cn\left(w - \frac{1}{2}iK'\right)} + C'''$$

$$\equiv C'' \log tn\left(w - \frac{1}{2}iK'\right) + C'''$$

The equation to the vortex is, therefore,

$$z = C'' \log tn\left(\phi - \frac{1}{2}iK'\right) + C'''.$$

VI. *On the Extension and Flexure of Cylindrical and Spherical Thin Elastic Shells.*
By A. B. BASSET, *M.A., F.R.S.*

1. THE various theories of thin elastic shells which have hitherto been proposed have been discussed by Mr. LOVE[*] in a recent memoir, and it appears that most, if not all of them, depend upon the assumption that the three stresses which are usually denoted by R, S, T are zero ; but, as I have recently pointed out,[†] a very cursory examination of the subject is sufficient to show that this assumption cannot be rigorously true. It can, however, be proved that, when the external surfaces of a plane plate are not subjected to pressure or tangential stress, these stresses depend upon quantities proportional to the square of the thickness, and whenever this is the case they may be treated as zero in calculating the expression for the potential energy due to strain, because they give rise to terms proportional to the fifth power of the thickness, which may be neglected, since it is usually unnecessary to retain powers of the thickness higher than the cube. It will also, in the present paper, be shown by an indirect method that a similar proposition is true in the case of cylindrical and spherical shells, and, therefore, the fundamental hypothesis upon which Mr. LOVE has based his theory, although unsatisfactory as an assumption, leads to correct results. A general expression for the potential energy due to strain in curvilinear coordinates has also been obtained by Mr. LOVE, and the equations of motion and the boundary conditions have been deduced therefrom by means of the Principle of Virtual Work, and if this expression and the equations to which it leads were correct, it would be unnecessary to propose a fresh theory of thin shells ; but although those portions of Mr. LOVE's results which depend upon the thickness of the shell are undoubtedly correct, yet, for reasons which will be more fully stated hereafter, I am of opinion that the terms which depend upon the cube of thickness are not strictly accurate, inasmuch as he has omitted to take into account several terms of this order, both in the expression for the potential energy and elsewhere. His preliminary analysis is also of an exceedingly complicated character.

2. Throughout the present paper the notation of THOMSON and TAIT's "Natural Philosophy" will be employed for stresses and elastic constants, but, for the purpose

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[*] 'Phil. Trans.,' A, 1888, p. 491.
[†] 'London Math. Soc. Proc.,' vol. 21, p. 33.

of facilitating comparison, Mr. LOVE's notation will be employed for strains and directions. It will also be convenient to denote the values of the various quantities involved, at a point P on the middle surface of the shell by *unaccented* letters; and the values of the same quantities at a point P' on the normal at P, whose distance from P is h', by *accented* letters. The radius of the shell will also be denoted by a, and its thickness by $2h$.

The theory which it is proposed to develop for cylindrical and spherical shells is identical, except in matters of detail, with the theory of plane plates which I recently communicated to the London Mathematical Society,* but for the sake of completeness a short outline will be given.
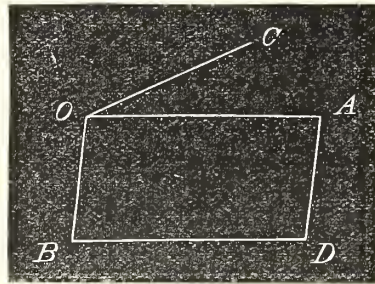
In the figure let OADB be a small curvilinear rectangle described on the middle surface of the shell, of which the sides are lines of curvature; and let us consider a small element of the shell bounded by the external surfaces, and the four planes passing through the sides of this rectangle, which are perpendicular to the middle surface.

The resultant stresses per unit of length which act upon the element, and which are due to the action of contiguous portions of the shell, are completely specified by the following quantities; viz., across the section AD,

T_1 = a tension across AD parallel to OA,
M_2 = a tangential shearing stress along AD,
N_2 = a normal shearing stress parallel to OC,
G_2 = a flexural couple from C to A, whose axis is parallel to AD,
H_1 = a torsional couple from B to C, whose axis is parallel to OA.

Similarly the resultant stresses per unit of length which act across the section BD are,

T_2 = a tension across BD parallel to OB,
M_1 = a tangential shearing stress along BD,
N_1 = a normal shearing stress parallel to OC,
G_1 = a flexural couple from B to C, whose axis is parallel to BD,
H_2 = a torsional couple from C to A, whose axis is parallel to OB.

* 'London Math. Soc. Proc.,' vol. 21, p. 33.

If the edges AD, BD were of finite length, there would also be a couple whose axis is parallel to the normal, but since this couple is proportional to the cube of the edge, it vanishes in comparison with the other stresses when the rectangle OADB is indefinitely diminished.

We shall denote the components of the bodily forces per unit of mass in the directions OA, OB, OC by X, Y, Z ; but for reasons which will be more fully explained hereafter, we shall suppose that these forces arise solely from external causes, such as gravity and the like.   All forces arising from pressures or tangential stresses applied to the surface of the shell will be expressly excluded.

The first step is to write down the equations of motion of an element of the shell in terms of the sectional stresses,* which can be done by the usual methods ; we shall thus obtain six equations, three of which are formed by resolving the forces parallel to OA, OB, OC, and three more by taking moments about these lines.

These equations will not, however, enable us to solve any statical or dynamical problems ; in order to do this we require the equations of motion in terms of the displacement of a point on the middle surface and their space variations with respect to the coordinates of that point.

3. The values at P′ of all the quantities with which we are concerned are functions of the position of P′, and are, therefore, functions of (r, z, ϕ) or (r, θ, ϕ), according as the shell is cylindrical or spherical.   If, therefore, $\mathcal{Q}'$ be the value of any such quantity at P′, and $\mathcal{Q}$ the value of the same quantity at the point P, which is the projection of P′ on the middle surface, it follows that

$$\mathcal{Q}' = \mathrm{F}(r) = \mathrm{F}(a + h')$$
$$= \mathcal{Q} + h'\left(\frac{d\mathcal{Q}}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\mathcal{Q}}{dr^2}\right) + \ldots \quad \ldots \ldots \ldots \quad (1)$$

by Taylor's theorem, where the brackets are employed, as will be done throughout this paper, to denote the values of the differential coefficients at the middle surface where $r = a$.

Objections have been raised by Saint-Venant and endorsed by Mr. Love, to the supposition that the first few terms of the expansion by Taylor's theorem of the quantities involved may be taken as a sufficient approximation.   If, however, this objection were valid, it would appear to me to upset the greater part of most physical investigations ; inasmuch as it is always assumed as a general principle, that when a quantity is known to be a function of the position of a point P, its value at a neighbouring point P′ may be obtained by Taylor's theorem, unless some physical discontinuity exists in passing from P to P′.   If, therefore, we put $\mathcal{Q} = R$, we may write

$$\mathrm{R}' = \mathrm{A} + \mathrm{A}_1 h' + \tfrac{1}{2}\mathrm{A}_2 h'^2 + \ldots \quad \ldots \ldots \ldots \quad (2)$$

where the A's are functions of the position of P and also of the thickness of the shell.

* See Besant "On the Equilibrium of a Bent Lamina," 'Quart. Journ. Math.,' vol. 4, p. 12.

A question which is of fundamental importance in the theory now arises, as to the way in which the A's depend upon h.

If R′ were of the order of the square of the thickness, it is evident that A and A_1 could not contain any powers of h lower than the second and first respectively, whilst A_2 could not contain any negative power of h. The A's are entirely unknown quantities, and as there appears to be no possibility of determining them by an *a priori* method, it seems hopeless to attempt to construct any theory of thin shells without the aid of some assumption which will enable us to get rid of them. *If, however, we assume, as has been practically done by previous writers, that, when the surfaces of the shell are not subjected to any surface forces such as pressures or tangential stresses,* R′ *and also* S′ *and* T′, *so far as they depend on* h *and* $h′$, *are capable of being expressed in the form*

$$u_2 + u_3 + \ldots u_n + \ldots$$

where u_n *is a homogeneous n-tic function of* h *and* $h′$, *the problem can be completely solved without attempting to determine by any* a priori *method the values of any unknown quantities, and upon this fundamental hypothesis the theory of the present paper will be based.*

There is some direct evidence of the truth of this hypothesis. In the case of a plane plate of infinite extent, it can be proved to be true by means of the general equations of motion of an elastic solid;[*] and for the purpose of testing the hypothesis in the case of a curved shell, I have recently investigated to a second approximation, so as to obtain the term in h^2, the period of the radial vibrations of an indefinitely long cylindrical shell, by means of the general equations, and also by means of the theory of thin shells, and both results agree.[†] But far the most conclusive evidence in favour of the truth of the hypothesis is furnished by the results to which it leads; and I have, therefore, conducted the following investigation in such a manner as to furnish a test of the correctness of the final results, and consequently of the fundamental hypothesis by means of which they are deduced.

Having obtained the equations of motion of a cylindrical and a spherical shell in terms of the sectional stresses, all these stresses are then calculated by a direct method, with the exception of the tensions T_1, T_2, which cannot be calculated directly, since they involve the unknown quantities A and A_2. After that the potential energy and the other constitutents of the variational equation are calculated, and the variation worked out by the usual methods. The final result, as is always the case in such investigations, consists of a line integral and a surface integral, the former of which determines the values of the sectional stresses in terms of the displacements, and the latter of which determines in the same form the three equations of motion. Now, if the work and the fundamental hypothesis upon which the theory is based are correct,

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[*] Lord RAYLEIGH, 'London Math. Soc. Proc.,' vol. 20, p. 225; see also vol. 21, p. 33.

[†] 'London Math. Soc. Proc.,' vol. 21, p. 53.

the variational equation will give the correct values of the tensions T_1, T_2, which are unknown, and will also reproduce the values of the other stresses which have been obtained directly. This is the first test. The second test is furnished by the consideration that, if we substitute the values of the sectional stresses which we have obtained from the variational equation, in the first three of our original equations of motion in terms of these stresses, we ought to reproduce the equations of motion in terms of the displacements, which have been obtained from the variational equation. This is found to be the case both when the shell is cylindrical and when it is spherical; and I therefore think that the fundamental hypothesis is sufficiently established. Having obtained the values of the sectional stresses, the boundary conditions can be deduced by means of STOKES' theorem, which enables us to prove that it is possible to apply a certain distribution of stress to the edge of a thin shell, without producing any alteration in the potential energy due to strain.

The fundamental hypothesis that R', S', T' may be treated as zero is not true when the surfaces of the shell are subjected to external pressures or tangential stresses ; for if the convex and concave surfaces of the shell were subjected to pressures Π_1, Π_2, the value of R' as we pass through the substance of the shell from its exterior to its interior surface, must vary from $-\Pi_1$ to $-\Pi_2$, and consequently (excepting in very special cases) R will contain a term independent of the thickness. Hence the theory developed in the present paper is not applicable to problems relating to the collapse of boiler flues, or to the communication of the vibrations of a vibrating body to the atmosphere. In order to obtain a theory which would enable such questions to be mathematically investigated, it would be necessary to find the values of the additional terms in the variational equation of motion, which depend upon the external pressures ; and this is a problem which awaits solution.

It will be convenient briefly to state the notation employed.

In the case of a cylindrical shell, OA is measured along a generating line, and OB along a circular section. In the case of a spherical shell, OA is measured along a meridian, and OB along a parallel of latitude.

The three extensional strains along OA, OB, OC are denoted by σ_1, σ_2, σ_3 ; and the three shearing strains about those lines by ϖ_1, ϖ_2, ϖ_3. We shall also use the letters λ, μ, p ; λ', μ', p' to denote the first and second differential coefficients of σ_1, σ_2, ϖ_3 with respect to r, when $r = a$. We shall also write

$$\mathrm{E} = (m - n)/(m + n), \qquad \mathrm{K} = \sigma_1 + \sigma_2,$$
$$\mathfrak{A} = \sigma_1 + \mathrm{E}\,(\sigma_1 + \sigma_2), \qquad \mathfrak{B} = \sigma_2 + \mathrm{E}\,(\sigma_1 + \sigma_2),$$
$$\mathfrak{E} = \lambda + \mathrm{E}\,(\lambda + \mu), \qquad \mathfrak{F} = \mu + \mathrm{E}\,(\lambda + \mu),$$
$$\mathfrak{E}' = \lambda' + \mathrm{E}\,(\lambda' + \mu'), \qquad \mathfrak{F}' = \mu' + \mathrm{E}\,(\lambda' + \mu').$$

Cylindrical Shells.

4. Before we can obtain the equations of motion or the potential energy, it will be necessary to ascertain the values of the first and second differential coefficients of the displacements with respect to r when $r = a$. We shall, therefore, proceed to calculate these quantities.

Putting λ, μ; λ', μ' for the values of $(d\sigma_1/dr)$, $(d\sigma_2/dr)$; $(d^2\sigma_1/dr^2)$, $(d^2\sigma_2/dr^2)$ when $r = a$, we have

$$
\begin{aligned}
\mathrm{R}' &= (m + n)\,\sigma'_3 + (m - n)\,(\sigma'_1 + \sigma'_2) \\
&= (m + n)\,\sigma_3 + (m - n)\,(\sigma_1 + \sigma_2) \\
&\quad + \left\{(m + n)\left(\frac{d\sigma_3}{dr}\right) + (m - n)(\lambda + \mu)\right\}h' \\
&\quad + \tfrac{1}{2}\left\{(m + n)\left(\frac{d^2\sigma_3}{dr^2}\right) + (m - n)(\lambda' + \mu')\right\}h'^2 + \cdots \quad \cdots \quad (3).
\end{aligned}
$$

But from (2),

$$
\mathrm{R}' = \mathrm{A} + \mathrm{A}_1 h' + \tfrac{1}{2}\mathrm{A}_2 h'^2 + \cdots \quad \cdots \quad (4),
$$

whence

$$
\left.
\begin{aligned}
\mathrm{A} &= (m + n)\,\sigma_3 + (m - n)\,(\sigma_1 + \sigma_2) \\
\mathrm{A}_1 &= (m + n)\left(\frac{d\sigma_3}{dr}\right) + (m - n)\,(\lambda + \mu) \\
\mathrm{A}_2 &= (m + n)\left(\frac{d^2\sigma}{dr^2}\right) + (m - n)\,(\lambda' + \mu')
\end{aligned}
\right\} \quad \cdots \quad (5),
$$

where A, A_1 do not contain any lower powers of h, than h^2 and h respectively, and A_2 does not contain any negative powers of h.

If u', v', w' be the component displacements of any point of the substance of the shell in the direction z, ϕ, r, the equations connecting the displacements and strains are

$$
\left.
\begin{aligned}
\sigma'_1 &= \frac{du'}{dz} \\
\sigma'_2 &= \frac{1}{r}\left(\frac{dv'}{d\phi} + w'\right) \\
\sigma'_3 &= \frac{dw'}{dr} \\
\varpi'_1 &= \frac{dv'}{dr} - \frac{v'}{r} + \frac{1}{r}\frac{dw'}{d\phi} \\
\varpi'_2 &= \frac{dw'}{dz} + \frac{du'}{dr} \\
\varpi'_3 &= \frac{1}{r}\frac{du'}{d\phi} + \frac{dv'}{dz}
\end{aligned}
\right\} \quad \cdots \cdots \quad (6),
$$

whence if

$$
\mathrm{E} = \frac{m - n}{m + n}, \qquad \mathrm{K} = \sigma_1 + \sigma_2 \quad \cdots \cdots \quad (7).
$$

we obtain

$$\left(\frac{du}{dr}\right) = \varpi_2 - \frac{dw}{dz}, \quad \left(\frac{dv}{dr}\right) = \varpi_1 + \frac{v}{a} - \frac{1}{a}\frac{dw}{d\phi}, \quad \left(\frac{dw}{dr}\right) = \frac{A}{m+n} - \mathrm{E}\mathrm{K} \qquad (8),$$

and

$$\left.\begin{aligned}
\left(\frac{d^2u}{dr^2}\right) &= \left(\frac{d\varpi_2}{dr}\right) - \left(\frac{d^2w}{dz\,dr}\right) = \left(\frac{d\varpi_2}{dr}\right) - \frac{1}{m+n}\frac{dA}{dz} + \mathrm{E}\frac{d\mathrm{K}}{dz} \\
\left(\frac{d^2v}{dr^2}\right) &= \left(\frac{d\varpi_1}{dr}\right) + \frac{\varpi_1}{a} - \frac{1}{a(m+n)}\frac{dA}{d\phi} + \frac{\mathrm{E}}{a}\frac{d\mathrm{K}}{d\phi} \\
\left(\frac{d^2w}{dr^2}\right) &= \frac{A_1}{m+n} - \mathrm{E}\,(\lambda + \mu)
\end{aligned}\right\} \qquad \cdots \quad (9).$$

5. We can now obtain the equations of motion in terms of the sectional stresses.

Let dS be an element of the middle surface whose coordinates are (a, z, ϕ), and dS' an element of a layer of the shell whose coordinates are $(a + h', z, \phi)$; then $dS' = (1 + h'/a)\,dS$. If we consider a small element of volume bounded by the two external surfaces of the shell, and the four planes passing through the sides of dS, which are perpendicular to the middle surface, we obtain by resolving parallel to OA,

$$\frac{d}{dz}(\mathrm{T}_1 a\,\delta\phi)\,\delta z + \frac{d}{d\phi}(\mathrm{M}_1\,\delta z)\,\delta\phi = \rho\,dS\int_{-h}^{h}(\ddot{u}' - \mathrm{X})(1 + h'/a)\,dh' \qquad (10).$$

But

$$u' = u + h'\left(\frac{du}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2u}{dr^2}\right);$$

accordingly if we substitute the values of (du/dr) and (d^2u/dr^2) from (8) and (9) and recollect that all terms which vanish with h may be omitted when multiplied by h^3, the right hand side of (10) becomes

$$\rho\,dS\left\{2h\,(\ddot{u} - \mathrm{X}) + \tfrac{1}{3}h^3\,\mathrm{E}\frac{d\ddot{\mathrm{K}}}{dz} - \frac{2h^3}{3a}\frac{d\ddot{w}}{dz}\right\}.$$

Resolving parallel to OB, OC, and then taking moments about OA, OB, OC, we shall obtain in a similar way five other equations, which, together with (10), may be written

$$\left.\begin{aligned}
\frac{d\mathrm{T}_1}{dz} + \frac{1}{a}\frac{d\mathrm{M}_1}{d\phi} &= \rho\left\{2h\,(\ddot{u} - \mathrm{X}) + \tfrac{1}{3}h^3\mathrm{E}\frac{d\ddot{\mathrm{K}}}{dz} - \frac{2h^3}{3a}\frac{d\ddot{w}}{dz}\right\} \\
\frac{1}{a}\frac{d\mathrm{T}_2}{d\phi} + \frac{\mathrm{N}_1}{a} + \frac{d\mathrm{M}_2}{dz} &= \rho\left\{2h\,(\ddot{v} - \mathrm{Y}) + \frac{h^3}{3a}\mathrm{E}\frac{d\ddot{\mathrm{K}}}{d\phi} + \frac{2h^3}{3a^2}\left(\ddot{v} - \frac{d\ddot{w}}{d\phi}\right)\right\} \\
\frac{d\mathrm{N}_2}{dz} + \frac{1}{a}\frac{d\mathrm{N}_1}{d\phi} - \frac{\mathrm{T}_2}{a} &= \rho\left\{2h\,(\ddot{w} - \mathrm{Z}) - \tfrac{1}{3}h^3\mathrm{E}\,(\ddot{\lambda} + \ddot{\mu}) - \frac{2h^3}{3a}\mathrm{E}\ddot{\mathrm{K}}\right\} \\
\frac{1}{a}\frac{d\mathrm{G}_1}{d\phi} + \frac{d\mathrm{H}_1}{dz} + \mathrm{N}_1 &= \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right) \\
\frac{d\mathrm{G}_2}{dz} + \frac{1}{a}\frac{d\mathrm{H}_2}{d\phi} - \mathrm{N}_2 &= -\tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a}\right)
\end{aligned}\right\} \qquad (11).$$

$$(\mathrm{M}_2 - \mathrm{M}_1)\,a - \mathrm{H}_2 = 0.$$

These equations will not enable us to solve the problem in hand; in order to do this we require the equations of motion in terms of the displacements, and also the values of the sectional stresses in terms of the same quantities.

6. The values of the couples, and also the values of M_1, M_2, can be obtained by direct calculation; but the values of T_1, T_2 cannot be so obtained, since they involve the quantities Ah and A_2h^3, which are unknown, and which cannot be neglected. We shall, therefore, be compelled to find the expression for the potential energy, and employ the Calculus of Variations.

The following results will, however, be necessary hereafter. If P', Q', R', S', T', U', be the stresses at the point $a + h'$, z, ϕ, we have

$$T_1 a\, \delta\phi = \int_{-h}^{h} P'\,(a + h')\, \delta\phi\, dh'$$

$$= \int_{-h}^{h} \left\{ P + h'\left(\frac{dP}{dr}\right) + \tfrac{1}{2} h'^2 \left(\frac{d^2P}{dr^2}\right) \right\} \left(1 + \frac{h'}{a}\right) a\, \delta\phi\, dh',$$

whence

$$
\left.
\begin{aligned}
T_1 &= 2hP + \tfrac{1}{3} h^3 \left(\frac{d^2P}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{dP}{dr}\right) \\[6pt]
T_2 &= 2hQ + \tfrac{1}{3} h^3 \left(\frac{d^2Q}{dr^2}\right) \\[6pt]
M_2 &= 2nh\varpi_3 + \tfrac{1}{3} nh^3 \left(\frac{d^2\varpi_3}{dr^2}\right) + \frac{2nh^3}{3a}\left(\frac{d\varpi_3}{dr}\right) \\[6pt]
M_1 &= 2nh\varpi_3 + \tfrac{1}{3} nh^3 \left(\frac{d^2\varpi_3}{dr^2}\right) \\[6pt]
G_1 &= -\tfrac{2}{3} h^3 \left(\frac{dQ}{dr}\right) \\[6pt]
G_2 &= \tfrac{2}{3} h^3 \left\{ \left(\frac{dP}{dr}\right) + \frac{P}{a} \right\} \\[6pt]
H_1 &= -\tfrac{2}{3} nh^3 \left\{ \left(\frac{d\varpi_3}{dr}\right) + \frac{\varpi_3}{a} \right\} \\[6pt]
H_2 &= \tfrac{2}{3} nh^3 \left(\frac{d\varpi_3}{dr}\right)
\end{aligned}
\right\} \quad \cdots \cdots \quad (12).
$$

From the third, fourth, and last of these we see that $(M_2 - M_1)\, a = H_2$, as ought to be the case.

Let

$$
\left.
\begin{aligned}
\mathfrak{A} &= \sigma_1 + E(\sigma_1 + \sigma_2), & \mathfrak{B} &= \sigma_2 + E(\sigma_1 + \sigma_2) \\
\mathfrak{E} &= \lambda + E(\lambda + \mu), & \mathfrak{F} &= \mu + E(\lambda + \mu)
\end{aligned}
\right\} \quad \cdots \quad (13).
$$

Then, in the terms multiplied by h^3, we may put

$$P = 2n\mathfrak{A}, \qquad Q = 2n\mathfrak{B},$$

$$\left(\frac{dP}{dr}\right) = 2n\mathfrak{E}, \qquad \left(\frac{dQ}{dr}\right) = 2n\mathfrak{F},$$

whence, if $p = (d\varpi_3/dr)$, the last four of (12) become

$$
\left.
\begin{array}{ll}
G_1 = -\tfrac{4}{3} nh^3 \mathfrak{F}, & G_2 = \tfrac{4}{3} nh^3 \left(\mathfrak{E} + \dfrac{\mathfrak{A}}{a} \right) \\[2ex]
H_1 = -\tfrac{2}{3} nh^3 \left(p + \dfrac{\varpi_3}{a} \right), & H_2 = \tfrac{2}{3} nh^3 p
\end{array}
\right\}
\quad \dots \quad (14).
$$

Since the couples are proportional to the cube of the thickness, it follows from the fourth and fifth of (11), that the normal shearing stresses N_1, N_2 are also proportional to the cube of the thickness, and therefore the terms of lowest order in the expressions for the shearing strains ϖ_1', ϖ_2' are quadratic functions of h and h', since such functions when integrated through a section of the shell, give rise to quantities proportional to the cube of the thickness. This is consistent with the fundamental hypothesis.

The next thing is to calculate the values of the quantities λ, μ, p.

From the first and fifth of (6) we obtain

$$
\lambda = \left(\frac{d\sigma_1}{dr} \right) = \frac{d\varpi_2}{dz} - \frac{d^2 w}{dz^2},
$$

and, since the terms in λ are all multiplied by h^3, we may put

$$
\lambda = -\frac{d^2 w}{dz^2} \quad \dots \dots \dots \dots \quad (15).
$$

Similarly from the second and fourth of (6) we obtain

$$
\mu = -\frac{1}{a^2} \left(\frac{d^2 w}{d\phi^2} + w \right) - \frac{E}{a} (\sigma_1 + \sigma_2) \quad \dots \dots \dots \quad (16).
$$

Lastly,

$$
p = \left(\frac{d\varpi_3}{dr} \right) = \frac{1}{a} \left(\frac{d^2 u}{dr\, d\phi} \right) + \left(\frac{dv}{dr\, dz} \right) - \frac{1}{a^2} \frac{du}{d\phi},
$$

or

$$
p = -\frac{2}{a} \frac{d^2 w}{dz\, d\phi} + \frac{1}{a} \frac{dv}{dz} - \frac{1}{a^2} \frac{du}{d\phi} \quad \dots \dots \dots \quad (17).
$$

We have, therefore, completely determined the values of the couples in terms of known quantities.

We shall also require the values of $(d^2\sigma_1/dr^2)$, $(d^2\sigma_2/dr')$, $(d^2\varpi_3/dr')$, the first two of which we have denoted by λ', μ'; and the last of which we shall denote by p'. The values of these quantities can, by a similar process, be shown to be

$$
\left.
\begin{array}{l}
\lambda' = E \dfrac{d^2 K}{dz^2} \\[2ex]
\mu' = -\dfrac{2\mu}{a} + \dfrac{E}{a^2} \dfrac{d^2 K}{d\phi^2} - \dfrac{E}{a} (\lambda + \mu) \\[2ex]
p' = -\dfrac{p}{a} + \dfrac{\varpi_3}{a^2} + \dfrac{2E}{a} \dfrac{d^2 K}{dz\, d\phi}
\end{array}
\right\}
\quad \dots \dots \dots \quad (18).
$$

Equation (17) and the last of (18), combined with the third and fourth of (12), determine the values of M_1, M_2.

It will be desirable to point out at this stage of the investigation, that we have obtained materials for the complete solution of any problem in which T_1 and u are zero, and none of the quantities are functions of z. The boundary conditions at a free edge will be discussed in § 11, and the reader who does not wish to be troubled with the long analytical process of finding the potential energy and working out the variational equation of motion, may pass at once to § 10, and the following sections where certain problems of a fairly simple kind are discussed.

7. We must now find the potential energy due to strain.

By the ordinary formula, the potential energy of a portion of the shell is

$$
\begin{aligned}
W = \tfrac{1}{2} \iiint_{-h}^{h} & [(m+n)\,\Delta'^2 \\
& + n\{\varpi_1'^2 + \varpi_2'^2 + \varpi_3'^2 - 4(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1')\}](1 + h'/a)\,dh'\,dS \quad (19),
\end{aligned}
$$

where the integration with respect to z and ϕ extends over the middle surface of the portion considered. In evaluating this expression we may at once omit ϖ_1', ϖ_2', for since they are quadratic functions of h and h', they will give rise to terms which are proportional to h^5, which are to be neglected.

Since

$$
\Delta' = \Delta + h'\left(\frac{d\Delta}{dr}\right) + \tfrac{1}{2} h'^2 \left(\frac{d^2\Delta}{dr^2}\right) + \cdots
$$

it follows that

$$
\begin{aligned}
\tfrac{1}{2}(m+n)\int_{-h}^{h} & \Delta'^2 \left(1 + \frac{h'}{a}\right) dh' \\
& = (m+n)\left\{ h\Delta^2 + \tfrac{1}{3}h^3\left(\frac{d\Delta}{dr}\right)^2 + \tfrac{1}{3}h^3\Delta\left(\frac{d^2\Delta}{dr^2}\right) + \frac{2h^3}{3a}\Delta\left(\frac{d\Delta}{dr}\right)\right\},
\end{aligned}
$$

from which it is seen that W is expressible in a series of odd powers of h.

From (5) we obtain

$$
\Delta = (1 - E)(\sigma_1 + \sigma_2) + \frac{A}{m+n},
$$

$$
\left(\frac{d\Delta}{dr}\right) = (1 - E)(\lambda + \mu) + \frac{A_1}{m+n},
$$

$$
\left(\frac{d^2\Delta}{dr^2}\right) = (1 - E)(\lambda' + \mu') + \frac{A_2}{m+n},
$$

and, therefore, the portion of W per unit of area of the middle surface, which depends upon Δ', is

$$\frac{4n^2}{m+n}\Big\{h\left(\sigma_1+\sigma_2+\mathrm{A}/2n\right)^2+\tfrac{1}{3}h^3\left(\lambda+\mu+\mathrm{A}_1/2n\right)^2$$

$$+\tfrac{1}{3}h^3\left(\sigma_1+\sigma_2+\mathrm{A}/2n\right)\left(\lambda'+\mu'+\mathrm{A}_2/2n\right)+\frac{2h^3}{3a}\left(\sigma_1+\sigma_2+\mathrm{A}/2n\right)\left(\lambda+\mu+\mathrm{A}_1/2n\right)\Big\}\quad(20)$$

in which in the last three terms we may omit the A's since they are multiplied by h^3.

Again

$$\sigma_1'\sigma_2'\left(1+h'/a\right)=\sigma_1\sigma_2+\lambda\mu h'^2+\tfrac{1}{2}\left(\lambda'\sigma_2+\mu'\sigma_1\right)h'^2+\left(\lambda\sigma_2+\mu\sigma_1\right)h'^2/a+\;\cdot\;\;\cdot$$

whence

$$2n\int_{-h}^{h}\sigma_1'\sigma_2'\left(1+h'/a\right)dh'=4nh\sigma_1\sigma_2+\tfrac{4}{3}nh^3\lambda\mu+\tfrac{2}{3}nh^3\left(\lambda'\sigma_2+\mu'\sigma_1\right)$$

$$+\frac{4nh^3}{3a}\left(\lambda\sigma_2+\mu\sigma_1\right)\qquad(21).$$

Also

$$\left(\sigma_1'+\sigma_2'\right)\sigma_3'\left(1+h'/a\right)=\left(\sigma_1+\sigma_2\right)\sigma_3+h'^2\left(\lambda+\mu\right)\left(\frac{d\sigma_3}{dr}\right)+\tfrac{1}{2}h'^2\left(\lambda'+\mu'\right)\sigma_3$$

$$+\tfrac{1}{2}h'^2\left(\sigma_1+\sigma_2\right)\left(\frac{d^2\sigma_3}{dr^2}\right)+\frac{h'^2}{a}\Big\{\left(\lambda+\mu\right)\sigma_3+\left(\sigma_1+\sigma_2\right)\left(\frac{d\sigma_3}{dr}\right)\Big\},$$

whence

$$2n\int_{-h}^{h}\left(\sigma_1'+\sigma_2'\right)\sigma_3'\left(1+h'/a\right)dh'=4nh\left(\sigma_1+\sigma_2\right)\Big\{\frac{\mathrm{A}}{m+n}-\mathrm{E}\left(\sigma_1+\sigma_2\right)\Big\}$$

$$-\tfrac{4}{3}nh^3\mathrm{E}\left(\lambda+\mu\right)^2-\tfrac{4}{3}nh^3\mathrm{E}\left(\sigma_1+\sigma_2\right)\left(\lambda'+\mu'\right)-\frac{8nh^3}{3a}\mathrm{E}\left(\lambda+\mu\right)\left(\sigma_1+\sigma_2\right)\qquad(22).$$

Lastly

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi_3'^2\left(1+h'/a\right)dh'=nh\varpi_3^2+\tfrac{1}{3}nh^3p^2+\tfrac{1}{3}nh^3\varpi_3p'+\frac{2nh^3}{3a}\varpi_3p\qquad(23).$$

Substituting from (20), (21), (22), (23) in (19), it will be found that the term $\mathrm{A}h$, which is (or at any rate may be) proportional to h^3, disappears; and thus the value of the potential energy per unit of the area of the middle surface is

$$\mathrm{W}=2nh\left\{\sigma_1^2+\sigma_2^2+\mathrm{E}\left(\sigma_1+\sigma_2\right)^2+\tfrac{1}{2}\varpi^2\right\}$$

$$+\tfrac{2}{3}nh^3\left\{\lambda^2+\mu^2+\mathrm{E}\left(\lambda+\mu\right)^2+\tfrac{1}{2}p^2\right\}$$

$$+\tfrac{2}{3}nh^3\left(\mathfrak{A}\lambda'+\mathfrak{B}\mu'+\tfrac{1}{2}\varpi p'\right)$$

$$+\tfrac{4}{3}\frac{nh^3}{a}\left(\mathfrak{A}\lambda+\mathfrak{B}\mu+\tfrac{1}{2}\varpi p\right)\;.\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\quad(24),$$

in which ϖ is written for ϖ_3, the suffix being no longer required.

This is the expression for the potential energy as far as the term involving h^3. The first term depends solely upon the extension of the middle surface; the second term depends principally upon the quantities by which the bending is specified, and the

third and fourth consist of the products of the extensions and the quantities which principally depend upon the bending.

8. Having obtained the value of the potential energy, we must in the next place form the variational equation of motion. This equation may symbolically be written

$$\delta W + \delta \mathfrak{T} = \delta U + \delta \mathfrak{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

where $\delta \mathfrak{T}$ is the term which depends upon the time variations of the displacements, δU is the work done by the bodily forces, and $\delta \mathfrak{L}$ represents the work done upon the edges of the portion of the shell considered, in producing the displacements, $\delta u, \delta v, \delta w$, by the forces arising from the action of contiguous portions of the shell. It, therefore, follows that $\delta \mathfrak{L}$ is a line integral taken round the edge of the portion of the shell which is being considered; and as one of our objects is to calculate the values of the sectional stresses in terms of the displacements by means of (25), it will be convenient to apply the variational equation to a curvilinear rectangle bounded by four lines of curvature.

We must now calculate $\delta \mathfrak{T}$. We have

$$\delta \mathfrak{T} = \rho \iiint_{-h}^{h} (\ddot{u}' \, \delta u' + \ddot{v}' \, \delta v' + \ddot{w}' \, \delta w') \, (1 + h'/a) \, dh' \, dS.$$

Now

$$\int_{-h}^{h} \ddot{u}' \, \delta u' \, (1 + h'/a) \, dh' = 2h \ddot{u} \, \delta u + \tfrac{2}{3} h^3 \frac{d\ddot{u}}{dr} \frac{d\delta u}{dr} + \tfrac{1}{3} h^3 \left(\ddot{u} \frac{d^2 \delta u}{dr^2} + \frac{d^2 \ddot{u}}{dr^2} \delta u \right)$$
$$+ \frac{2h^3}{3a} \left(\ddot{u} \frac{d\delta u}{dr} + \frac{d\ddot{u}}{dr} \delta u \right)$$

$$= 2h \ddot{u} \, \delta u + \tfrac{2}{3} h^3 \frac{d\ddot{w}}{dz} \frac{d\delta w}{dz} + \tfrac{1}{3} h^3 E \left(\ddot{u} \frac{d\delta K}{dz} + \frac{d\ddot{K}}{dz} \delta u \right)$$
$$- \frac{2h^3}{3a} \left(\ddot{u} \frac{d\delta w}{dz} + \frac{d\ddot{w}}{dz} \delta u \right)$$

by (8) and (9). Treating the other terms in a similar way, we shall find that the value of $\delta \mathfrak{T}$ is

$$\delta \mathfrak{T} = 2\rho h \iint (\ddot{u} \, \delta u + \ddot{v} \, \delta v + \ddot{w} \, \delta w) \, dS$$

$$+ \tfrac{2}{3} \rho h^3 \iint \left\{ \frac{d\ddot{w}}{dz} \frac{d\delta w}{dz} + \frac{1}{a^2} \left(\frac{d\ddot{w}}{d\phi} - \ddot{v} \right) \left(\frac{d\delta w}{d\phi} - \delta v \right) + E^2 \ddot{K} \, \delta K \right\} dS$$

$$+ \tfrac{1}{3} \rho h^3 E \iint \left\{ \ddot{u} \frac{d\delta K}{dz} + \frac{\ddot{v}}{a} \frac{d\delta K}{d\phi} - \ddot{w} \, (\delta \lambda + \delta \mu) + \frac{d\ddot{K}}{dz} \delta u + \frac{1}{a} \frac{d\ddot{K}}{d\phi} \delta v - (\ddot{\lambda} + \ddot{\mu}) \, \delta w \right\} dS$$

$$- \frac{2\rho h^3}{3a} \iint \left\{ \ddot{u} \frac{d\delta w}{dz} + \frac{d\ddot{w}}{dz} \delta u + \frac{\ddot{v}}{a} \left(\frac{d\delta w}{d\phi} - \delta v \right) + \frac{1}{a} \left(\frac{d\ddot{w}}{d\phi} - \ddot{v} \right) \delta v \right.$$
$$\left. + E \left(\ddot{w} \, \delta K + \ddot{K} \, \delta w \right) \right\} dS \quad . \quad . \quad (26).$$

We must, in the next place, calculate $\delta\mathfrak{L}$. We have

$$\delta\mathfrak{L} = \iint_{-h}^{h} (P'\,\delta u' + U'\,\delta v')\,(d+h')\,dh'\,d\phi + \iint_{-h}^{h} (Q'\,\delta v' + U'\,\delta u')\,dh'\,dz$$

$$+ \int N_2 \delta w/a\,d\phi + \int N_1\,\delta w\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

From the way in which $\delta\mathfrak{C}$ has been calculated, we see from (8), (9), and (12), that

$$\int_{-h}^{h} P'\,\delta u'\,(1+h'/a)\,dh' = T_1\,\delta u + G_2\,\frac{d\delta u}{dr} + \tfrac{1}{3}h^3 P\,\frac{d^2\delta u}{dr^2}$$

$$= T_1\,\delta u - G_2\,\frac{d\delta w}{dz} + \tfrac{2}{3}nh^3 E\mathfrak{A}\,\frac{d\delta K}{dz}.$$

Treating the other terms in a similar way, we find

$$\delta\mathfrak{L} = \int\left\{ T_1\,\delta u + M_2\,\delta v + N_2\,\delta w - G_2\,\frac{d\delta w}{dz} + \frac{H_1}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right.$$

$$\left. + \tfrac{2}{3}nh^3 E\mathfrak{A}\,\frac{d\delta K}{dz} + \frac{nh^3 E\varpi}{3a}\,\frac{d\delta K}{d\phi}\right\}a\,d\phi$$

$$+ \int\left\{ M_1\,\delta u + T_2\,\delta v + N_1\,\delta w + \frac{G_1}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) - H_2\,\frac{d\delta w}{dz}\right.$$

$$\left. + \frac{2nh^3}{3a}\,E\mathfrak{B}\,\frac{d\delta K}{\delta\phi} + \tfrac{1}{3}nh^3 E\varpi\,\frac{d\delta K}{dz}\right\}dz \quad . \quad . \quad (28).$$

Lastly, since the shell is supposed to be so thin that X, Y, Z, may be treated as constants during the integration with respect to h',

$$\delta U = \rho\iiint_{-h}^{h} (X\,\delta u' + Y\,\delta v' + Z\,\delta w')\,(1+h'/a)\,dh'\,dS$$

$$= 2\rho\,h\iint(X\,\delta u + Y\,\delta v + Z\,\delta w)\,dS$$

$$+ \tfrac{1}{3}\rho\,h^3 E\iint\left\{X\,\frac{d\delta K}{dz} + \frac{Y}{a}\,\frac{d\delta K}{d\phi} - Z\,(\delta\lambda + \delta\mu)\right\}dS$$

$$- \frac{2\rho h^3}{3a}\iint\left\{X\,\frac{d\delta w}{dz} + \frac{Y}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) + ZE\,\delta K\right\}dS \quad . \quad . \quad . \quad (29).$$

9. We have now obtained all the materials for the complete solution of the problem, and we shall proceed to work out the variation in the ordinary way.

Let us denote the four terms of the expression for W given in (24), when integrated over a curvilinear rectangle bounded by four lines of curvature, by W_1, W_2, W_3, W_4. Then

$$\delta W_1 = 4nh\iint(\mathfrak{A}\delta\sigma_1 + \mathfrak{B}\delta\sigma_2 + \tfrac{1}{2}\varpi\delta\varpi)\,a\,dz\,d\phi.$$

Substituting the values of σ_1, σ_2, ϖ from the first, second, and sixth of (6), and integrating by parts we shall obtain

$$\delta W_1 = 4nh \int (\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\delta v)\, a\, d\phi + 4nh \int (\tfrac{1}{2}\varpi\delta u + \mathfrak{B}\delta v)\, dz$$

$$- 4nh \iint \left\{ \left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right)\delta u + \left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\varpi}{dz}\right)\delta v - \frac{\mathfrak{B}}{a}\delta w \right\} a\, dz\, d\phi \quad . \quad . \quad (30).$$

Now δW_2, δW_3, δW_4 depend upon h^3; if therefore we substitute in (25) the value of δW_1 from (30), and the portions of $\delta\mathfrak{C}$, δU, and $\delta\mathfrak{L}$, which depend upon h, we shall obtain the approximate equations

$$\left. \begin{array}{ll} T_1 = 4nh\,\mathfrak{A}, & T_2 = 4nh\,\mathfrak{B} \\ M_1 = M_2 = 2nh\varpi & \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

and

$$\left. \begin{array}{l} \rho\ddot{u} = 2n\left(\dfrac{d\mathfrak{A}}{dz} + \dfrac{1}{2a}\dfrac{d\varpi}{d\phi}\right) + \rho X \\[2mm] \rho\ddot{v} = 2n\left(\dfrac{1}{a}\dfrac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\dfrac{d\varpi}{dz}\right) + \rho Y \\[2mm] \rho\ddot{w} = -2n\,\mathfrak{B}/a + \rho Z \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (32).$$

These equations are the same as those obtained by Mr. Love,[*] and which are employed by him in discussing the vibrations of a cylindrical shell. The complete equations giving $\rho\ddot{u}$, $\rho\ddot{v}$, $\rho\ddot{w}$ in terms of the displacements and their space variations contain certain additional terms involving h^2 (since the common factor h disappears) which it is our object to determine; but, since we do not retain terms higher than h^3, we may, if convenient, substitute the above approximate values in all terms of (25) which are multiplied by h^3.

Again

$$\delta W_2 = \tfrac{4}{3}nh^3 \iint (\mathfrak{E}\,\delta\lambda + \mathfrak{F}\,\delta\mu + \tfrac{1}{2}p\,\delta p)\, a\, dz\, d\phi.$$

Substituting the values of λ, μ, p from (15), (16), and (17), we obtain

$$\iint \mathfrak{E}\,\delta\lambda\, dz\, d\phi = -\iint \mathfrak{E}\,\frac{d^2\,\delta w}{dz^2}\, dz\, d\phi$$

$$= \int \left(\frac{d\mathfrak{E}}{dz}\,\delta w - \mathfrak{E}\,\frac{d\,\delta w}{dz}\right) d\phi - \iint \frac{d^2\,\mathfrak{E}}{dz^2}\,\delta w\, dz\, d\phi \quad . \quad . \quad . \quad . \quad (33),$$

also

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[*] 'Phil. Trans.,' A., 1888, pp. 538 and 540. Equations (32) correspond to Love's equations (86), (87), and (88); and (31) to (101).

$$\iint \mathfrak{F}\,\delta\mu\,dz\,d\phi = -\frac{1}{a^2}\iint \mathfrak{F}\left\{\frac{d^2\delta w}{d\phi^2} + \delta w + \mathrm{E}\left(a\frac{d\,\delta u}{dz} + \frac{d\,\delta v}{d\phi} + \delta w\right)\right\}dz\,d\phi$$

$$= -\frac{\mathrm{E}}{a}\int \mathfrak{F}\,\delta u\,d\phi - \frac{1}{a^2}\int\left(\mathrm{E}\mathfrak{F}\,\delta v - \frac{d\mathfrak{F}}{d\phi}\delta w + \mathfrak{F}\frac{d\delta w}{d\phi}\right)dz$$

$$+ \frac{1}{a^2}\iint\left\{\mathrm{E}a\frac{d\mathfrak{F}}{dz}\delta u + \mathrm{E}\frac{d\mathfrak{F}}{d\phi}\delta v - \left(\frac{d^2\mathfrak{F}}{d\phi^2} + \mathfrak{F} + \mathrm{E}\mathfrak{F}\right)\delta w\right\}dz\,d\phi \quad (34),$$

and

$$\tfrac{1}{2}\iint p\,\delta p\,dz\,d\phi = \frac{1}{2a^2}\iint p\left(a\frac{d\delta v}{dz} - \frac{d\delta u}{d\phi} - 2a\frac{d^2\delta w}{dz\,d\phi}\right)dz\,d\phi$$

$$= \frac{1}{2a}\int p\,\delta v\,d\phi - \frac{1}{2a^2}\int p\,\delta u\,dz + \frac{1}{2a^2}\iint\left(\frac{dp}{d\phi}\delta u - a\frac{dp}{dz}\delta v\right)dz\,d\phi$$

$$- \frac{1}{a}\iint p\frac{d^2\delta w}{dz\,d\phi}\,dz\,d\phi.$$

The last integral can be evaluated in two different ways, according as we integrate, first, with respect to ϕ, and secondly, with respect to z; or first with respect to z, and secondly with respect to ϕ. The proper way to deal with such a term is, to evaluate the integral in both ways, and then multiply the two values by β and $1 - \beta$, and add, where β is a quantity which must be determined from the conditions of the problem in hand. We shall thus find that the value of β is $\tfrac{1}{2}$; we therefore obtain

$$\tfrac{1}{2}\iint p\,\delta p\,dz\,d\phi = \frac{1}{2a}\int\left(p\,\delta v + \frac{dp}{d\phi}\delta w - p\frac{d\delta w}{d\phi}\right)d\phi$$

$$- \frac{1}{2a^2}\int\left(p\,\delta u - a\frac{dp}{dz}\delta w + ap\frac{d\delta w}{dz}\right)dz$$

$$+ \frac{1}{2a^2}\iint\left(\frac{dp}{d\phi}\delta u - a\frac{dp}{dz}\delta v - 2a\frac{d^2p}{dz\,d\phi}\delta w\right)dz\,d\phi \quad \ldots\ldots (35).$$

Collecting all the terms together from (33), (34), and (35), we finally obtain

$$\delta\mathrm{W}_2 = \tfrac{4}{3}nh^3\int\left\{-\frac{\mathrm{E}\mathfrak{F}}{a}\delta u + \left(\frac{d\mathfrak{E}}{dz} + \frac{1}{2a}\frac{dp}{d\phi}\right)\delta w - \mathfrak{E}\frac{d\delta w}{dz} - \frac{p}{2a}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}a\,d\phi$$

$$+ \tfrac{4}{3}nh^3\int\left\{-\frac{p}{2a}\delta u - a\frac{(1+\mathrm{E})\mathfrak{F}}{a}\delta v + \left(\frac{1}{a}\frac{d\mathfrak{F}}{d\phi} + \frac{1}{2}\frac{dp}{dz}\right)\delta w\right.$$
$$\left. - \frac{\mathfrak{F}}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) - \tfrac{1}{2}p\frac{d\delta w}{dz}\right\}dz$$

$$+ \frac{4nh^3}{3a}\iint\left[\left(\mathrm{E}\frac{d\mathfrak{F}}{dz} + \frac{1}{2a}\frac{dp}{d\phi}\right)\delta u + \left(\frac{\mathrm{E}}{a}\frac{d\mathfrak{F}}{d\phi} - \tfrac{1}{2}\frac{dp}{dz}\right)\delta v\right.$$
$$\left. - \left\{a\frac{d\mathfrak{E}}{dz^2} + \frac{1}{a}\left(\frac{d^2\mathfrak{F}}{d\phi^2} + \mathfrak{F} + \mathrm{E}\mathfrak{F}\right) + \frac{d^2p}{dz\,d\phi}\right\}\delta w\right]a\,dz\,d\phi \quad (36).$$

Let

$$\mathfrak{E}' = \lambda' + \mathrm{E}(\lambda' + \mu'), \qquad \mathfrak{F}' = \mu' + \mathrm{E}(\lambda' + \mu') \quad . \quad . \quad . \quad . \quad (37),$$

then

$$\delta \mathrm{W}_3 = \tfrac{2}{3} nh^3 \iint \left(\mathfrak{E}'\delta\sigma_1 + \mathfrak{F}'\delta\sigma_2 + \tfrac{1}{2} p'\delta\varpi + \mathfrak{A}\delta\lambda' + \mathfrak{B}\delta\mu' + \tfrac{1}{2}\varpi\delta p' \right) dS;$$

from this result, together with (24), it is seen that $\delta \mathrm{W}_3$ and $\delta \mathrm{W}_4$ each consist of two parts, which may be denoted by $\delta \mathrm{W}_3'$, $\delta \mathrm{W}_3''$ and $\delta \mathrm{W}_4'$, $\delta \mathrm{W}_4''$ respectively. The values of $\delta \mathrm{W}_3'$ and $\delta \mathrm{W}_4'$ may at once be written down from (30), by changing $\mathfrak{A}$, $\mathfrak{B}$, ϖ into $\mathfrak{E}'$, $\mathfrak{F}'$, p' and $\mathfrak{E}$, $\mathfrak{F}$, p respectively, and by altering the coefficients from $4nh$ into $\tfrac{2}{3}nh^3$ and $4nh^3/3a$ respectively. With regard to $\delta \mathrm{W}_3''$ we have

$$\delta \mathrm{W}_3'' = \tfrac{2}{3} nh^3 \iint \left(\mathfrak{A}\,\delta\lambda' + \mathfrak{B}\,\delta\mu' + \tfrac{1}{2}\varpi\delta p' \right) dS.$$

Substituting the value of λ' from (18) and integrating once by parts, we obtain

$$\iint \mathfrak{A}\,\delta\lambda'dS = \iint \mathrm{E}\,\mathfrak{A}\,\frac{d^2\delta\mathrm{K}}{dz^2}\,dS$$

$$= \mathrm{E}\int \mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz}\,ad\phi - \mathrm{E}\iint\frac{d\mathfrak{A}}{dz}\,\frac{d\delta\mathrm{K}}{dz}\,dS.$$

Treating the other terms in a similar way, we shall finally obtain

$$\delta \mathrm{W}_3'' = \int\left(\tfrac{2}{3}nh^3\mathrm{E}\mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz} + \frac{nh^3}{3a}\mathrm{E}\varpi\,\frac{d\delta\mathrm{K}}{d\phi}\right)ad\phi + \int\left(\frac{2nh^3}{3a}\mathrm{E}\mathfrak{B}\,\frac{d\delta\mathrm{K}}{d\phi} + \tfrac{1}{3}nh^3\mathrm{E}\varpi\,\frac{d\delta\mathrm{K}}{dz}\right)dz$$

$$- \tfrac{2}{3}nh^3\iint\left\{\mathrm{E}\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right)\frac{d\delta\mathrm{K}}{dz} + \frac{\mathrm{E}}{a}\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\varpi}{dz}\right)\frac{d\delta\mathrm{K}}{d\phi} - \frac{\mathrm{E}\mathfrak{B}}{a}(\delta\lambda + \delta\mu)\right\}dS$$

$$+ \tfrac{2}{3}\frac{nh^3}{a}\iint\left\{-2\mathfrak{B}\delta\mu + \frac{\varpi}{2a}(\delta\varpi - a\delta p)\right\}dS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

If in the first surface integral in this equation, we substitute the approximate values of the coefficients of $d\delta\mathrm{K}/dz$, &c., from (32), which we may do, since this integral is multiplied by h^3, and then substitute the values of $\delta \mathrm{W}_3''$, $\delta\mathfrak{T}$, $\delta\mathrm{U}$, and $\delta\mathfrak{U}$ in (25), it will be found that all the terms involving $d\delta\mathrm{K}/dz$, $d\delta\mathrm{K}/d\phi$, and $\delta\lambda + \delta\mu$ cut out; we are, therefore, no longer concerned with them, and the value of $\delta \mathrm{W}_3''$ reduces to the last line. On this understanding we may, therefore, write

$$\delta \mathrm{W}_3'' + \delta \mathrm{W}_4'' = \frac{4nh^3}{3a}\iint\left(\mathfrak{A}\,\delta\lambda + \frac{\varpi}{4a}\delta\varpi + \tfrac{1}{4}\varpi\delta p\right)dS$$

$$= -\frac{4nh^3}{3a}\int\left\{\mathfrak{A}\frac{d\delta w}{dz} + \frac{\varpi}{2a}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}ad\phi$$

$$+ \frac{4nh^3}{3a}\iint\left\{\frac{d\mathfrak{A}}{dz}\frac{d\delta w}{dz} + \frac{1}{2a}\frac{d\varpi}{dz}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}dS \quad . \quad . \quad . \quad (39).$$

We are now in a position to test the correctness of some of our work, for picking out the terms involving $d\delta w/d\phi - \delta v$, $d\delta w/dz$ in the line integrals in (36) and (39), and equating them to the corresponding terms in the value of $\delta \mathfrak{L}$ which is given by (28), we see that we have reproduced the values of the couples, which we have already obtained in equation (14). We may therefore leave the couple terms out henceforth.

Collecting all our results from (26), (28), (29), and (39) the variational equation becomes

$$\delta W_1 + \delta W_2 + \delta W_3{}' + \delta W_4{}' + \tfrac{4}{3}\frac{nh^3}{a} \iint \left\{ \frac{d\mathfrak{A}}{dz}\frac{d\delta w}{dz} + \frac{1}{2a}\frac{d\varpi}{dz}\left(\frac{d\delta w}{d\phi} - \delta v\right) \right\} dS$$

$$+ 2\rho h \iint (\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w)\, dS$$

$$+ \tfrac{2}{3}\rho h^3 \iint \left\{ \left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a}\right)\frac{d\delta w}{dz} + \frac{1}{a^2}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right)\left(\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{E}\left(\mathrm{E}\ddot{K} - \ddot{w}/a\right)\delta K \right\} dS$$

$$+ \tfrac{1}{3}\rho h^3 \mathrm{E} \iint \left\{ \frac{d\ddot{K}}{dz}\delta u + \frac{1}{a}\frac{d\ddot{K}}{d\phi}\delta v - (\ddot{\lambda} + \ddot{\mu})\,\delta w \right\} dS$$

$$- \frac{2\rho h^3}{3a} \iint \left\{ \frac{d\ddot{w}}{dz}\delta u + \frac{1}{a}\left(\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + \mathrm{E}\ddot{K}\,\delta w \right\} dS$$

$$= 2\rho h \iint (\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w)\, dS - \frac{2\rho h^3}{3a} \iint \left\{ \mathrm{X}\frac{d\delta w}{dz} + \frac{\mathrm{Y}}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{ZE}\,\delta K \right\} dS$$

$$+ \int (\mathrm{T}_1\,\delta u + \mathrm{M}_2\,\delta v + \mathrm{N}_2\,\delta w)\, a\, d\phi + \int (\mathrm{M}_1\,\delta u + \mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w)\, dz \quad . \quad . \quad (40)$$

where the values of δW_1, δW_2 are given by (30) and (36), and the values of $\delta W_3{}'$, $\delta W_4{}'$ are obtained from (30) by changing certain letters as we have explained above.

We have now got rid of all the terms involving the second differential coefficients of δu, δv, δw, and all that now remains to be done is to integrate by parts the terms which involve the first differential coefficients. Putting

$$\alpha = \frac{2n}{\rho a}\frac{d\mathfrak{A}}{dz} + \frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a},$$

$$\beta = \frac{n}{\rho a}\frac{d\varpi}{dz} + \frac{1}{a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right) + \frac{\mathrm{Y}}{a},$$

$$\gamma = \mathrm{E}\left(\mathrm{E}\ddot{K} - \ddot{w}/a + \mathrm{Z}/a\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

we have

$$\tfrac{2}{3}\,\rho h^3 \iint \left(\alpha\,\frac{d\,\delta w}{dz} + \frac{\beta}{a}\frac{d\,\delta w}{d\phi} + \gamma\,\delta \mathrm{K} \right) d\mathrm{S}$$

$$= \tfrac{2}{3}\,\rho h^3 \int (\gamma\,\delta u + \alpha\,\delta w)\, a\,d\phi + \tfrac{2}{3}\,\rho h^3 \int (\gamma\,\delta v + \beta\,\delta w)\,dz$$

$$- \tfrac{2}{3}\,\rho h^3 \iint \left\{ \frac{d\gamma}{dz}\,\delta u + \frac{1}{a}\frac{d\gamma}{d\phi}\,\delta v + \left(\frac{d\alpha}{dz} + \frac{1}{a}\frac{d\beta}{d\phi} - \frac{\gamma}{a} \right)\delta w \right\} d\mathrm{S} \quad . \quad . \quad . \quad (42).$$

Substituting the values of $\delta\mathrm{W}_1$, $\delta\mathrm{W}_2$, $\delta\mathrm{W}_3{}'$, $\delta\mathrm{W}_4{}'$, and the right hand side of (42) in (40), and picking out the line integral terms, we obtain the following equations for the sectional stresses, viz.,

$$\mathrm{T}_1 = 4nh\mathfrak{A} - \frac{4nh^3}{3a}\mathrm{E}\mathfrak{F} + \tfrac{2}{3}nh^3\mathfrak{E}' + \frac{4nh^3}{3a}\mathfrak{E} + \frac{2\rho h^3}{3a}\mathrm{E}\left(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z}\right)$$

$$\mathrm{M}_2 = 2nh\varpi + \tfrac{1}{3}nh^3 p' + \frac{2nh^3}{3a}\,p$$

$$\mathrm{N}_2 = \tfrac{4}{3}nh^3\left(\frac{d\mathfrak{E}}{dz} + \frac{1}{2a}\frac{dp}{d\phi}\right) + \frac{4nh^3}{3a}\frac{d\mathfrak{A}}{dz} + \tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a}\right)$$

$$\mathrm{G}_2 = \tfrac{4}{3}nh^3\left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right)$$

$$\mathrm{H}_1 = -\tfrac{2}{3}nh^3\left(p + \frac{\varpi}{a}\right)$$

$$\left. \right\rbrace \quad (43)$$

which give the values of the sectional stresses across a circular section; and

$$\mathrm{M}_1 = 2nh\varpi + \tfrac{1}{3}nh^3 p'$$

$$\mathrm{T}_2 = 4nh\mathfrak{B} - \frac{4nh^3}{3a}\mathrm{E}\mathfrak{F} + \tfrac{2}{3}nh^3\mathfrak{F}' + \frac{2\rho h^3}{3a}\mathrm{E}\left(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z}\right)$$

$$\mathrm{N}_1 = \tfrac{4}{3}nh^3\left(\frac{1}{a}\frac{d\mathfrak{F}}{d\phi} + \tfrac{1}{2}\frac{dp}{dz}\right) + \frac{2nh^3}{3a}\frac{d\varpi}{dz} + \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right)$$

$$\mathrm{G}_1 = -\tfrac{4}{3}nh^3\mathfrak{F}$$

$$\mathrm{H}_2 = \tfrac{2}{3}nh^3 p$$

$$\left. \right\rbrace \quad . \quad . \quad . \quad (44)$$

which give the values of the sectional stresses across a meridian.

If we compare these equations with the third and fourth of (12), with (14), and with the fourth and fifth of (11), we see that we have reproduced (i.) the values of M_1, M_2 given by (12); (ii.) the values of the couples given by (14); (iii.) the values of the normal shearing stresses which are obtained from the fourth and fifth of (11), by substituting the values of the couples from (14). We have thus subjected our fundamental hypothesis to a fairly searching test. It is, however, in our power to subject it to a still further test; for if we equate the coefficients of δu, δv, δw in the surface integrals in (40) and (42), we shall obtain the equations of motion in terms of the displacements,

and on substituting the values of the sectional stresses from (43) and (44), in the first three of (11), we ought to reproduce the equations of motion in terms of the displacements which we have obtained from the variational equation.

From (30), (36), (40), and (42) it follows that these equations are

$$\rho \left\{ 2\left(\ddot{u} - \mathrm{X}\right) + \tfrac{1}{3}h^2\mathrm{E}\frac{d\ddot{\mathrm{K}}}{dz} - \frac{2h^2}{3a}\frac{d\ddot{w}}{dz} \right\} = 4n\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right) - \frac{4nh^2}{3a}\mathrm{E}\frac{d\mathfrak{F}}{dz}$$
$$+ \tfrac{2}{3}nh^2\left(\frac{d\mathfrak{E}'}{dz} + \frac{1}{2a}\frac{dp'}{d\phi}\right) + \frac{4nh^2}{3a}\frac{d\mathfrak{E}}{dz} + \tfrac{2}{3}\rho h^2\frac{d\gamma}{dz} \qquad (45).$$

$$\rho\left\{ 2\left(\ddot{v} - \mathrm{Y}\right) + \frac{h^2}{3a}\mathrm{E}\frac{d\ddot{\mathrm{K}}}{d\phi} + \frac{2h^2}{3a}\left(\ddot{v} - \frac{d\ddot{w}}{d\phi}\right) \right\}$$
$$= 4n\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\varpi}{dz}\right) + \frac{4nh^2}{3a^2}\left\{ (1-\mathrm{E})\frac{d\mathfrak{F}}{d\phi} + a\frac{dp}{dz} + \tfrac{1}{2}\frac{d\varpi}{dz} \right\}$$
$$+ \tfrac{2}{3}nh^2\left(\frac{1}{a}\frac{d\mathfrak{F}'}{d\phi} + \tfrac{1}{2}\frac{dp'}{dz}\right) + \frac{2\rho h^2}{3a}\frac{d\gamma}{d\phi} + \frac{2\rho h^2}{3a^2}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right) \quad . \quad . \quad (46).$$

$$\rho\left\{ 2\left(\ddot{w} - \mathrm{Z}\right) - \tfrac{1}{3}h^2\mathrm{E}\left(\ddot{\lambda} + \ddot{\mu}\right) - \frac{2h^2}{3a}\mathrm{E}\ddot{\mathrm{K}} \right\}$$
$$= -\frac{4n\mathfrak{B}}{a} + \frac{4nh^2}{3a^2}\left\{ a^2\frac{d^2\mathfrak{E}}{dz^2} + \frac{d^2\mathfrak{F}}{d\phi^2} + \mathrm{E}\mathfrak{F} + a\frac{d^2p}{dz\,d\phi} \right\}$$
$$- \frac{2nh^2}{3a}\mathfrak{F}' + \frac{4nh^2}{3a}\frac{d}{dz}\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right) + \tfrac{2}{3}\rho h^2\frac{d}{dz}\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a}\right)$$
$$+ \frac{2\rho h^2}{3a^2}\frac{d}{d\phi}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right) - \frac{2\rho h^2}{3a^2}\mathrm{E}\left(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z}\right) \quad . \quad . \quad . \quad . \quad (47).$$

If we compare these equations with the equations obtained by substituting the values of the sectional stresses in the first line of (11), it will be found that they agree in every respect.

10. It will hereafter be necessary to consider certain problems in which the middle surface is supposed to experience no extension or contraction throughout the motion; and it will, therefore, be necessary to obtain the requisite equations when this is supposed to be the case. .

The conditions of inextensibility are

$$\sigma_1 = 0, \quad \sigma_2 = 0, \quad \varpi = 0\,;$$

or

$$\frac{du}{dz} = 0, \qquad \frac{dv}{d\phi} + w = 0, \qquad \frac{du}{d\phi} + a\frac{dv}{dz} = 0 \quad . \quad . \quad . \quad . \quad (48),$$

which require that

$$u = \mathrm{U}a, \qquad v = -z\frac{d\mathrm{U}}{d\phi} + \mathrm{V}, \qquad w = z\frac{d^2\mathrm{U}}{d\phi^2} - \frac{d\mathrm{V}}{d\phi} \quad . \quad . \quad . \quad (49),$$

where U and V are functions of ϕ alone.

In this case the potential energy reduces to the second line alone, and from (15), (16), and (17) we obtain

$$\left.\begin{aligned}
\lambda &= 0 \\[4pt]
\mu &= -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right) \\[4pt]
p &= -\frac{2}{a}\left(\frac{d^2w}{dz\,d\phi} - \frac{dv}{dz}\right)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and from (24)

$$\mathrm{W} = \frac{4nh^3}{3a^2}\left\{\frac{m}{a^2\,(m+n)}\left(\frac{d^2w}{d\phi^2} + w\right)^2 + \left(\frac{d^2w}{dz\,d\phi} - \frac{dv}{dz}\right)^2\right\} \quad . \quad . \quad . \quad (51),$$

which agrees with the expression obtained by Lord RAYLEIGH.[*]

Also from (14)

$$\left.\begin{aligned}
\mathrm{G}_1 &= \frac{4nh^3}{3a^2}\,(1+\mathrm{E})\left(\frac{d^2w}{d\phi^2} + w\right) \\[4pt]
\mathrm{G}_2 &= -\frac{4nh^3}{3a^2}\,\mathrm{E}\left(\frac{d^2w}{d\phi^2} + w\right) \\[4pt]
\mathrm{H}_1 &= -\mathrm{H}_2 = -\tfrac{2}{3}\,nh^3 p
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

The values of the stresses M_1, M_2 may be obtained either from (43) and (44), or from (12) combined with (15), (16), (17), and (18) by introducing the conditions of inextensibility; and the values of T_1, T_2 might be calculated by taking the variation subject to the conditions of inextensibility, and using indeterminate multipliers. This process would not, however, be of much assistance, inasmuch as it would introduce two undetermined quantities into the values of T_1, T_2, which depend upon the boundary conditions; whereas in this case the values of T_1, T_2 can be obtained directly from the first and third of (11) combined with (49). The values of N_1, N_2 are given by the fourth and fifth of (11) combined with (50) and (52).

11. We must lastly consider the boundary conditions.

Equations (43) and (44) determine the stresses on the line elements $a\,d\phi$ and dz respectively, which are produced by the action of contiguous portions of the shell; and it might at first sight appear, as was supposed by POISSON,[†] that when a shell

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[*] 'Roy. Soc. Proc.,' vol. 45, p. 116.

[†] 'Paris, Acad. des Sciences, Mémoires,' 1829, vol. 8, p. 357.

of finite dimensions is under the influence of forces and couples applied to its edges, these equations would give the values of such forces or couples, and that the conditions to be satisfied at a free edge would require that each of the above five stresses should vanish at a free edge. KIRCHHOFF[*] has, however, shown that this is not the case, but that the boundary conditions are only *four* in number; and the reason of this is, that it is possible to apply a certain distribution of stress to the edge of a shell, without producing any alteration in the potential energy.

By STOKES' theorem,

$$\int \left(\frac{d\mathrm{H}'}{d\phi} \delta w + \mathrm{H}' \frac{d\delta w}{d\phi} \right) d\phi + \int \left(\frac{d\mathrm{H}'}{dz} \delta w + \mathrm{H}' \frac{d\delta w}{dz} \right) dz = 0 ;$$

the integration extending round any curvilinear rectangle bounded by four lines of curvature OA, AD, DB, BA. If, therefore, we apply to the side AD the stresses

$$\mathrm{M}_2' = \mathrm{H}'/a, \qquad \mathrm{N}_2' = \frac{1}{a} \frac{d\mathrm{H}'}{d\phi}, \qquad \mathrm{H}_1' = \mathrm{H}' ;$$

to the side DB the stresses

$$\mathrm{N}_1' = \frac{d\mathrm{H}'}{dz}, \qquad \mathrm{H}_2' = - \mathrm{H}',$$

and to the sides BO, OA, corresponding and opposite stresses respectively, the preceding integral becomes

$$\int \left\{ \mathrm{M}_2' \delta v + \mathrm{N}_2' \delta w + \frac{\mathrm{H}_1'}{a} \left(\frac{d\delta w}{d\phi} - \delta v \right) \right\} a\,d\phi + \int \left(\mathrm{N}_1' \delta w - \mathrm{H}_2' \frac{d\delta w}{dz} \right) dz = 0,$$

which shows that the work done by these stresses is zero. Such a system of stresses may, therefore, be applied or removed without interfering with the equilibrium or motion of the shell.

Let us now suppose that the rectangle OADB, instead of being under the action of the remainder of the shell, is isolated, and that its state of strain is maintained by means of constraining stresses applied to its edges; then it follows that if, instead of the torsional couples H_1, H_2, due to the action of contiguous portions of the shell, we apply torsional couples $\mathfrak{M}_1$, $\mathfrak{M}_2$, where

$$\mathfrak{M}_1 = \mathrm{H}_1 + \mathrm{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53),$$

$$\mathfrak{M}_2 = \mathrm{H}_2 - \mathrm{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

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[*] 'CRELLE,' vol. 40, p. 51, 1850, and Collected Works, p. 237.

the state of strain will remain unchanged, provided we apply in addition the stresses

$$
\left.\begin{aligned}
\mathfrak{M}_2 &= \mathrm{M}_2 + \mathrm{H}'/a \\
\mathfrak{P}_2 &= \mathrm{N}_2 + \frac{1}{a}\frac{d\mathrm{H}'}{d\phi}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)
$$

and

$$
\mathfrak{P}_1 = \mathrm{N}_1 + \frac{d\mathrm{H}'}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56),
$$

whence, eliminating H′ between (53), (55), and (54), (56) respectively, we obtain

$$
\left.\begin{aligned}
\mathfrak{M}_2 a - \mathfrak{P}_1 &= \mathrm{M}_2 a - \mathrm{H}_1 \\
\mathfrak{P}_2 - \frac{1}{a}\frac{d\mathfrak{P}_1}{d\phi} &= \mathrm{N}_2 - \frac{1}{a}\frac{d\mathrm{H}_1}{d\phi}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),
$$

and

$$
\mathfrak{P}_1 + \frac{d\mathfrak{P}_2}{dz} = \mathrm{N}_1 + \frac{d\mathrm{H}_2}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).
$$

In these equations the Roman letters denote the stresses due to the action of contiguous portions of the shell, whose values are given by (43) and (44), whilst the Old English letters denote the values of the actual stresses applied to the boundary. If, therefore, the shell consists of a portion of a cylinder which is bounded by four lines of curvature and whose edges are free, the boundary conditions along the circular edges are obtained by equating the right hand sides of the first and fourth of (43), and the right hand sides of (57) to zero, the first two of which express the condition that the tension perpendicular to, and the flexural couple about, a line element of the circular edge must vanish when the edge is free ; and the boundary conditions along the straight edge are similarly obtained by equating the right hand sides of the first, second, and fourth of (44), and the right hand side of (58) to zero, the first three of which express the conditions that the tangential shearing stress, the tension and the flexural couple must vanish when the free edge is a generating line. We may also, if we do not wish to introduce the time and the bodily forces into these equations, substitute for $\ddot{u} - \mathrm{X}$, $\ddot{v} - \mathrm{Y}$, $\ddot{w} - \mathrm{Z}$ their approximate values from (32).

12. We have now obtained all the materials we require, for a perfectly accurate approximate solution of any problem relating to the vibrations of a thin cylindrical shell as far as the terms involving the cube of the thickness, but before proceeding to discuss any problems, it will be necessary to make some remarks respecting Mr. Love's paper. The first line of my expression for the potential energy which is given in (24), and which involves h and not h^3, agrees with the expression obtained by Mr. Love and other writers; also the approximate equations of motion (32) agree, as has been already pointed out, with the corresponding equations obtained by him, and by means of which he has discussed the extensional vibrations of a cylinder. It will

also hereafter appear, that observations of a precisely similar character apply to the corresponding equations which determine to a first approximation the extensional vibrations of a spherical shell. This portion of his paper therefore appears to be perfectly satisfactory; but that portion which involves the terms depending upon the cube of the thickness is open to criticism.

In the first place, he appears to have employed a system of *rectangular* axes, consisting of the normal at a point on the middle surface, and the tangents to the two lines of curvature through that point. Now, although it is immaterial, so long as we confine our attention to infinitesimals of the first order, whether a quantity is measured along the tangents to three orthogonal curves or along the curves themselves, yet when it is necessary to take into consideration infinitesimals of higher orders, which is always the case whenever an investigation involves changes of curvature, a method in which everything is referred to rectangular axes requires care; and on comparing the terms in h^3 in (24) with the corresponding terms in Mr. Love's expression for the potential energy, it will be seen that he has omitted several terms which involve the extensions of the middle surface, which partly, although not entirely, arises from his having omitted the factor $1 + h'/a$. It is not improbable that these terms may be small, but at the same time we are not at liberty to neglect them altogether; for it is quite evident that a term such as $\delta(\mathfrak{B}\mu)$ in the variational equation, will give rise to terms in the equations of motion and the equations giving the values of the sectional stresses, which do not involve the extension of the middle surface.

In the second place, on comparing Mr. Love's variational equation of motion[*] with my equations (25), (26), (28), and (29), it will be seen that he has omitted several terms in the expressions for $\delta\mathfrak{T}$, δU, and $\delta\mathfrak{L}$.

In the third place he states (p. 521) that the extensional quantities "σ_1, σ_2, ϖ may not, in general, be regarded as of a higher order of small quantities than κ_2, λ_1, κ_1," which are the quantities upon which the bending depends. The argument of Lord Rayleigh[†] appears to me to show, that at points whose distance from the edge is large in comparison with the thickness, the extensional terms are usually small in comparison with the terms upon which the bending depends. It must be obvious to every one, that a thin plate of metal or a steel spring can be bent with the greatest ease by means of the fingers, whereas the production of any extension of the middle surface which would be capable of measurement, would involve considerable muscular effort. These considerations indicate that when a thin shell is vibrating, the change of curvature is so greatly in excess of the extension of the middle surface, that notwithstanding the smallness of h^3 compared with h, the product $h^3 (\delta\rho^{-1})^2$ is large[‡] compared with

[*] 'Phil. Trans.,' A., 1888, p. 514, equation (19).

[†] 'Roy. Soc. Proc.,' vol. 45, p. 105.

[‡] The problem discussed in § 14 shows that the product $h\sigma^2$ *may be* of the order $h^5 (\delta\rho^{-1})^2$, except in the neighbourhood of a free edge; but in the equations of motion we have to deal with the quantities $h\sigma$ and $h^3 \delta\rho^{-1}$.

the product $h\sigma^2$. At the same time, inasmuch as the production of change of curvature involves some extension or contraction of all but the central layers, and consequently of those portions of the shell which are near its external surface, it does not seem unreasonable to suppose that in the neighbourhood of a free edge, an extension or contraction of the middle surface may take place, which is comparable with the change of curvature.

In the fourth place, Mr. Love appears to have argued as if the equations of motion of a shell, whose middle surface undergoes no extension or contraction throughout the motion, might be obtained from his general equations (30), (31), (32), by putting $\sigma_1 = \sigma_2 = \varpi = 0$; but it has already been pointed out, that the correct equations for this kind of motion must be obtained by taking the variation subject to the conditions of inextensibility, and introducing indeterminate multipliers. It will be shown in the next section, that in the case of the flexural vibrations of an indefinitely long complete cylindrical shell considered by Hoppe and Lord Rayleigh,[*] the differential equation for the tangential displacement v is of the sixth degree, and that when the cross section of the shell consists of a circular arc, this equation contains sufficient constants to enable all the conditions of the problem to be satisfied.

13. The first problem which we shall consider will be that of the flexural vibrations of an indefinitely long cylinder, in which the displacement of every element lies in a plane perpendicular to the axis of the cylinder, and which has been discussed by Hoppe and Lord Rayleigh.

In this problem the middle surface is supposed to undergo no extension or contraction throughout the motion, and the solution is most easily obtained by means of the general equations (11). In these equations we must omit all the terms on the right hand sides which involve h^3, for they would, if retained, give rise to a term involving h^4 in the period equation, which must be rejected, since we do not carry the approximation further than h^2 in determining the period.

We evidently have[†] $M_1 = N_2 = H_2 = 0$; also none of the quantities are functions of z. The equations of motion are thus

$$\frac{dT_2}{d\phi} + N_1 = 2\rho h a \ddot{v},$$

$$\frac{dN_1}{d\phi} - T_2 = 2\rho h a \ddot{w},$$

$$\frac{dG_1}{d\phi} + N_1 a = 0,$$

* 'Theory of Sound,' vol. 1, p. 324; 'Roy. Soc. Proc.,' vol. 45, p. 129. Equation (51).
† We shall presently see that these conditions imply a constraint at infinity.

also the condition of inextensibility gives

$$\frac{dv}{d\phi} + w = 0.$$

Eliminating N_1, T_2, and w, and substituting the value of G_1 from the first of (52), we obtain

$$\frac{4mnh^2}{3\rho a^4 (m+n)} \left(\frac{d^3}{d\phi^3} + \frac{d}{d\phi} \right)^2 v + \frac{d^2\ddot{v}}{d\phi^2} - \ddot{v} = 0 \quad \ldots \quad \ldots \quad (59),$$

whence, putting

$$v = A\epsilon^{\iota pt + \iota s\phi},$$

we obtain

$$p^2 = \frac{4mnh^2 s^2 (s^2-1)^2}{3\rho a^4 (m+n)(s^2+1)} \quad \ldots \quad \ldots \quad \ldots \quad (60),$$

which is the required result.

If the cylinder is complete, s is any integer, unity excluded, but if the cross-section of the cylinder consists of a circular arc of length $2a\alpha$, s will not be an integer. Its values in terms of p are the six roots of (60), but in order to obtain the frequency equation, the value of s in terms of the dimensions and elastic constants is required. The additional equations are obtained from the boundary conditions, which have to be satisfied along the straight edges of the shell, and these require that the tension T_2, the normal shearing stress N_1, and the flexural couple G_1 should vanish at the edges where $\phi = \pm \alpha$.

Since

$$G_1 = - \frac{8mnh^3}{3(m+n)} \mu,$$

where

$$\mu = \frac{1}{a^2} \left(\frac{d^3v}{d\phi^3} + \frac{dv}{d\phi} \right),$$

the boundary conditions are obviously

$$\mu = 0,$$

$$\frac{d\mu}{d\phi} = 0,$$

$$\frac{4mnh^2}{3\rho a^2 (m+n)} \frac{d^2\mu}{d\phi^2} + \frac{d^3v}{d\phi\, dt^2} = 0.$$

These conditions have to be satisfied at each of the edges of the shell where $\phi = \pm \alpha$, and there are, therefore, six equations of condition; hence the six constants

which appear in the solution of (59) can be eliminated, and the resulting determinantal equation, combined with (60), will give the frequency.[*]

If a complete cylinder of *finite length* were vibrating in this manner, it would be necessary to satisfy the conditions at the circular ends, and this would require that $T_1 = 0$, $G_2 = 0$ at the ends for all values of ϕ; and from the first and fourth of (43) we see that this requires that $\mu = 0$, or

$$\frac{d^2w}{d\phi^2} + w = 0,$$

whence

$$w = A \cos \phi + B \sin \phi$$

for all values of ϕ. Since it is impossible to satisfy this condition for the kind of motion considered, it follows that when the cylinder is of finite length it would be necessary to apply at every point of the circular boundary a tension T_1 and a couple G_2 of the requisite amount.

This is the question upon which Lord RAYLEIGH and Mr. LOVE are at issue; and the preceding investigation shows that Mr. LOVE is right in supposing that it is impossible to satisfy the boundary conditions along the curved edges of a cylindrical shell when these edges are free, although he does not appear to have noticed that it is possible to satisfy these conditions when the free edges are generating lines. In order to obtain a complete mathematical solution of this question, it would be necessary to work out the problem of the free vibrations of a complete cylindrical shell of given length $2l$, which is deformed in such a manner that $dv/d\phi + w = 0$, where v and w are functions of ϕ alone, and is then let go, without assuming that the middle surface remains unextended during the subsequent motion.

Owing unfortunately to the extremely complicated nature of the general equations, a rigorous solution of this problem would be exceedingly difficult. We shall, however, be able to throw some light upon this question, by solving and discussing the following much simpler statical problem.

14. Let us consider a heavy cylindrical shell, whose cross section is a semicircle, and which is suspended by means of vertical bands attached to its straight edges, so that its axis is horizontal; and let us investigate the state of strain produced by its own weight.

In order to simplify the problem as much as possible, we shall suppose that the displacement of every point of the middle surface lies in a plane perpendicular to the axis, and we shall afterwards investigate the stresses which must be applied to the circular edges, in order to maintain this state of things.

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[*] [This problem is of a similar character to that of a bar, whose natural form is circular, and which has been discussed by LAMB. 'London Math. Soc. Proc.,' vol. 19, p. 365.—June, 1890.]
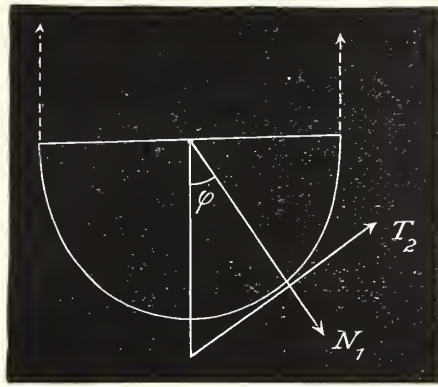
We have

$$Y = - g \sin \phi, \quad Z = g \cos \phi ;$$

whence, if $W = 2g\rho ah$, the equations of equilibrium are

$$\frac{dT_2}{d\phi} + N_1 = W \sin \phi ,$$

$$\frac{dN_1}{d\phi} - T_2 = - W \cos \phi ,$$

$$\frac{dG_1}{d\phi} + N_1 a = - \frac{h^2}{3a} W \sin \phi ;$$

from which we obtain

$$\frac{d^2 T_2}{d\phi^2} + T_2 = 2W \cos \phi ,$$

the integral of which is

$$T_2 = A \cos \phi + B \sin \phi + W\phi \sin \phi ,$$

and, therefore,

$$N_1 = A \sin \phi - B \cos \phi - W\phi \cos \phi.$$

Since $N_1 = 0$ when $\phi = \frac{1}{2}\pi$, $A = 0$; also since $T_2 = \frac{1}{2}\pi W$ when $\phi = \frac{1}{2}\pi$, $B = 0$; whence

$$T_2 = W\phi \sin \phi, \qquad N_1 = - W\phi \cos \phi \quad . \quad . \quad . \quad . \quad . \quad (61),$$

and, therefore,

$$\frac{dG_1}{d\phi} = Wa \left(\phi \cos \phi - \frac{h^2}{3a^2} \sin \phi \right),$$

whence

$$G_1 = Wa \left(\phi \sin \phi + \cos \phi + \frac{h^2}{3a^2} \cos \phi \right) + C.$$

Since $G_1 = 0$ when $\phi = \frac{1}{2}\pi$, $C = - \frac{1}{2}W\pi a$; accordingly

$$G_1 = Wa \left\{ \phi \sin \phi + \left(1 + \frac{h^2}{3a^2} \right) \cos \phi - \frac{1}{2}\pi \right\} \quad . \quad . \quad . \quad . \quad (62).$$

But

$$G_1 = -\tfrac{4}{3} nh^3 \mathfrak{F} = -\tfrac{4}{3} nh^3 (1 + E) \mu \quad . \quad . \quad . \quad . \quad . \quad (63),$$

whence

$$\tfrac{4}{3} nh^3 \mu = -\frac{Wa}{1 + E}\left\{\phi \sin\phi + \left(1 + \frac{h^2}{3a^2}\right)\cos\phi - \tfrac{1}{2}\pi\right\}. \quad . \quad . \quad (64).$$

Again, if R denote the change of curvature along a circular section, so that

$$R = -\frac{1}{a^2}\left(\frac{d^2 w}{d\phi^2} + w\right)$$

we have

$$\mu = R - E\sigma_2/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Also by (18)

$$\mu' = -(2 + E)\mu/a + \frac{E}{a^2}\frac{d^2\sigma_2}{d\phi^2}$$

and, therefore,

$$\mathfrak{F}' = (1 + E)\mu' = -(1 + E)(2 + E)\mu/a + \frac{(1 + E)E}{a^2}\frac{d^2\sigma_2}{d\phi^2},$$

whence, by the second of (44),

$$T_2 = 4nh(1 + E)\sigma_2 - \frac{2nh^3}{3a}(1 + E)(3 + 2E)\mu + \frac{2nh^3}{3a^2}(1 + E)E\frac{d^2\sigma_2}{d\phi^2} + \frac{h^2}{3a^2}EW\cos\phi.$$

Substituting the values of T_2 and μ from (61) and (64), we obtain

$$2nh(1 + E)\left\{2\sigma_2 + \frac{h^2 E}{3a^2}\frac{d^2\sigma_2}{d\phi^2}\right\} + \tfrac{1}{2}(2 + 3E)\left\{\phi \sin\phi + \left(1 + \frac{h^2}{3a^2}\right)\cos\phi - \tfrac{1}{2}\pi\right\}$$
$$+ \frac{h^2 E}{3a^2}W\cos\phi - W\phi\sin\phi = 0.$$

This equation might, if necessary, be solved by successive approximation, but a first approximation will be sufficient. Omitting the terms in h^2, and recollecting that W involves h as a factor, we obtain

$$4nh(1 + E)\sigma_2 + W(\cos\phi - \tfrac{1}{2}\pi) + \tfrac{3}{2}EW(\phi\sin\phi + \cos\phi - \tfrac{1}{2}\pi) = 0 \quad . \quad (66),$$

whence from (64), (65), and (66), we obtain

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a(\tfrac{1}{2}\pi - \phi\sin\phi - \cos\phi)}{h^2\{\tfrac{1}{2}\pi - \cos\phi + \tfrac{3}{2}E(\tfrac{1}{2}\pi - \phi\sin\phi - \cos\phi)\}} \quad . \quad . \quad . \quad (67).$$

Since the numerator of this fraction is an even function of ϕ, it does not change sign with ϕ; also the numerator is always positive between the limits $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, and its maximum value occurs when $\phi = 0$ and is equal to $\frac{1}{2}\pi - 1$, and its minimum value occurs when $\phi = \frac{1}{2}\pi$ and is equal to zero.   We, therefore, see that when $\phi = 0$,

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a}{h^2};$$

and when $\phi = \frac{1}{2}\pi$,

$$\frac{R}{\sigma_2} = \frac{E}{a}.$$

Since the thickness of the shell is supposed to be small compared with its radius, it follows that the change of curvature is large compared with the extension of the middle surface, except when $a\left(\frac{1}{2}\pi - \phi\right)$ is comparable with h, i.e., in the neighbourhood of the straight edges of the shell; and therefore at all points of the shell whose distances from the edges are large in comparison with its thickness, the terms depending upon the product of the change of curvature and the cube of the thickness, i.e., the terms upon which the bending depends, are of the same order as the terms depending upon the product of the extension of the middle surface and the thickness; but at points whose distances from the edge are comparable with the thickness of the shell, the extension of the middle surface is of the same order as the change of curvature, and therefore the terms depending upon the product of the change of curvature and the cube of the thickness are small in comparison with the terms depending upon the product of the extension and the thickness.

We shall now calculate the stresses which must be applied to the circular edges in order to maintain this particular kind of strain.   From (43) we have

$$G_2 = \tfrac{4}{3}\, nh^3\, E\, (\mu + \sigma_2/a).$$

Substituting the values of μ and σ_2 from (62), (63), and (64), we see that the terms in σ_2 may be omitted, and we obtain

$$G_2 = \frac{EWa}{1+E}\left(\tfrac{1}{2}\pi - \phi\sin\phi - \cos\phi\right) \quad . \quad . \quad . \quad . \quad . \quad (68),$$

which shows that G_2 is positive.

Also

$$T_1 = 4nh\, E\, \sigma_2 - \frac{2nh^3}{3a}\, E\, (2 + 3E)\, \mu$$

$$= \frac{EW}{1+E}\, \phi\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69),$$

which shows that T_1 is positive.

Comparing (68) and (69) with (61) and (62) we see that ratios of the tension T_1 and the couple G_2, to T_2 and G_1 are numerically equal to $E/(1 + E)$; we further

see that G_1 is negative, and, therefore, the strain tends to increase the curvature of the circular sections. Now when a cylindrical shell is bent about a generating line in such a manner that its curvature is increased, all lines parallel to the axis which lie on the convex side of the middle surface will be contracted, whilst all such lines which lie on the concave side will be extended, and this contraction and extension will give rise to a couple about the circular sections which tends to produce anticlastic curvature of the generating lines. In order to prevent this taking place it is necessary to apply at every point of the circular edges a couple G_2 tending to produce synclastic curvature, and a tension T_1 parallel to the axis, whose values are given by (68) and (69). If this couple and tension were removed, the middle surface would bend about its circular sections, and anticlastic curvature of the generating lines would be produced, and this would necessarily involve extension or contraction parallel to the axis, so that the problem could no longer be treated as one of two dimensions.

It must, however, be within the experience of everyone that when a thin cylindrical shell of finite length, whose cross section is the arc of a circle, is bent about its generating lines, the shell does not assume a saddle-back form, and consequently the anticlastic curvature of the generating lines must be so small as to be inappreciable. This circumstance furnishes an additional argument in favour of the supposition that the extension of the middle surface is only sensible in the neighbourhood of the free edges.

We therefore conclude that if the circular edges were free, some extension or contraction of the middle surface must necessarily take place, but that this extension or contraction is small compared with the change of curvature along a circular section, except just in the neighbourhood of the edges. From these considerations the inference is, that if by means of proper constraints applied to the circular edges, a cylindrical shell were enabled to execute the non-extensional vibrations discussed in § 13, the vibrations would cease to be non-extensional if the constraints were removed ; but that the amplitudes of those portions of the displacements upon which the extension depends, would be very small compared with the amplitudes of those portions upon which the change of curvature along a circular section depends, except just in the neighbourhood of the edges. Moreover, the theory of plane plates shows, that the frequency of the extensional vibrations is expressible* by means of a series of even powers of h, *commencing with a term independent of* h ; whilst the frequency of the flexural vibrations is expressible by means of a similar series *commencing with* h^2. It therefore follows, that the pitch of the notes arising from the former class of vibrations, is high compared with the pitch of those arising from the latter class. And although, except under special circumstances, it is not possible in the case of curved shells whose edges are free, for these two classes of vibrations to coexist independently, as in the case of a plane plate ; yet recent investigations show, that the pitch of

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* Lord RAYLEIGH, ' London Math. Soc. Proc.,' vol. 20, p. 225. See especially equations (38), (45), and (53).

notes which mainly depend upon the extension is usually high in comparison with the pitch of notes which mainly depend upon bending, and consequently the notes arising from the former cause, both on account of the smallness of their amplitudes and the highness of their pitch, would probably be so feeble in comparison with those which arise from the latter cause, as to be scarcely capable of producing any appreciable effect upon the ear. Judging from the usual course of such investigations, the probable form of the exact solution of the problems suggested at the end of § 13 would be that of an infinite series, the periods of the different components of which would satisfy a transcendental equation having an infinite number of roots; but the preceding considerations point to the conclusion that the frequency of the gravest* note given by (60), viz., $p^2 = 48mnh^2/5\rho a^4\,(m+n)$, although perhaps not rigorously accurate, is a close approximation to the truth.

Spherical Shells.

15. The fundamental equations for a spherical shell can be investigated in precisely the same manner as in the case of a cylindrical shell.

If u', v', w' be the component displacements at any point of the substance of the shell in the directions, θ, ϕ, r, the equations connecting the displacements and strains are
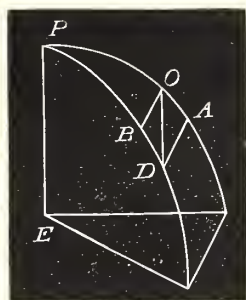
$$
\left.\begin{aligned}
\sigma'_1 &= \frac{1}{r}\left(\frac{du'}{d\theta} + w'\right) \\[4pt]
\sigma'_2 &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{dv'}{d\phi} + u'\cot\theta + w'\right) \\[4pt]
\sigma'_3 &= \frac{dw'}{dr} \\[4pt]
\varpi'_1 &= \frac{1}{r\sin\theta}\frac{dw'}{d\phi} + \frac{dv'}{dr} - \frac{v'}{r} \\[4pt]
\varpi'_2 &= \frac{du'}{dr} - \frac{u'}{r} + \frac{1}{r}\frac{dw'}{d\theta} \\[4pt]
\varpi'_3 &= \frac{1}{r}\left(\frac{dv'}{d\theta} - v'\cot\theta + \frac{1}{\sin\theta}\frac{du'}{d\phi}\right)
\end{aligned}\right\} \quad \cdots \quad (1),
$$

* [The experiments of Lord Rayleigh, 'Phil. Mag.,' Jan., 1890, show that the effective pitch of a bell is usually not the same as that of its gravest tone; and, in the bells which he examined, the fifth tone in order was the one which agreed with the nominal pitch of the bell.—June, 1890.]

whence

$$\left.\begin{aligned}
\left(\frac{du}{dr}\right) &= \varpi_2 + \frac{u}{a} - \frac{1}{a}\frac{dw}{d\theta} \\[6pt]
\left(\frac{dv}{dr}\right) &= \varpi_1 + \frac{v}{a} - \frac{1}{a\sin\theta}\frac{dw}{d\phi} \\[6pt]
\left(\frac{dw}{dr}\right) &= \frac{A}{m+n} - EK
\end{aligned}\right\} \quad \cdots \cdots \cdots \quad (2),$$

also

$$\left.\begin{aligned}
\left(\frac{d^2u}{dr^2}\right) &= \left(\frac{d\varpi_2}{dr}\right) + \frac{\varpi_2}{a} - \frac{1}{a\,(m+n)}\frac{dA}{d\theta} + \frac{E}{a}\frac{dK}{d\theta} \\[6pt]
\left(\frac{d^2v}{dr^2}\right) &= \left(\frac{d\varpi_1}{dr}\right) + \frac{\varpi_1}{a} - \frac{1}{a\,(m+n)\sin\phi}\frac{dA}{d\phi} + \frac{E}{a\sin\theta}\frac{dK}{d\phi} \\[6pt]
\left(\frac{d^2w}{dr^2}\right) &= \frac{A_1}{m+n} - E\,(\lambda+\mu)
\end{aligned}\right\} \quad \cdots \cdot \quad (3).$$

16. We can now obtain the equations of motion in terms of the sectional stresses.

If dS be an element of the middle surface whose coordinates are $(a,\ \theta,\ \phi)$, and dS' an element of a layer of the shell whose coordinates are $(a + h',\ \theta,\ \phi)$, then $dS' = (1 + h'/a)^2\, dS$; whence, if in the figure OA, OB respectively coincide with the meridians and circular sections, we obtain by resolving parallel to OA,

$$\frac{d}{d\theta}(T_1 a \sin\theta\,\delta\phi)\,\delta\theta - T_2 a \cos\theta\,\delta\theta\,\delta\phi + \frac{d}{d\phi}(M_1 a\,\delta\theta)\,\delta\phi + N_2 a \sin\theta\,\delta\theta\,\delta\phi$$

$$= \rho\,dS\int_{-h}^{h}(\ddot{u}' - X)\,(1 + h'/a)^2\,dh' \quad \cdot \quad \cdot \quad (4).$$

But

$$u' = u + h'\left(\frac{du}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2u}{dr^2}\right);$$

accordingly if we substitute the values of (du/dr) and (d^2u/dr^2) from (2) and (3), and recollect that all quantities which vanish with h may be omitted when multiplied by h^3, the right hand side of (4) becomes

$$\rho\,dS\left\{2h\left(1 + \frac{h^2}{a^2}\right)\ddot{u} + \frac{h^3 E}{3a}\frac{d\dot{K}}{d\theta} - \frac{4h^3}{3a^2}\frac{d\ddot{w}}{d\theta} - 2h\left(1 + \frac{h^2}{3a^2}\right)X\right\}.$$

Resolving parallel to OB, OC, and then taking moments about OA, OB, OC we shall obtain in a similar way five other equations, which, together with (4), may be written,

$$\frac{d}{d\theta}(T_1 \sin\theta) - T_2 \cos\theta + \frac{dM_1}{d\phi} + N_2 \sin\theta$$

$$= \left\{2h\left(1 + \frac{h^2}{a^2}\right)\ddot{u} + \frac{h^3 E}{3a}\frac{d\ddot{K}}{d\theta} - \frac{4h^3}{3a^2}\frac{d\ddot{w}}{d\theta} - 2h\left(1 + \frac{h^2}{3a^2}\right)X\right\}\rho a \sin\theta,$$

$$\frac{dT_2}{d\phi} + \frac{d}{d\theta}(M_2 \sin\theta) + M_1 \cos\theta + N_1 \sin\theta$$

$$= \left\{2h\left(1 + \frac{h^2}{a^2}\right)\ddot{v} + \frac{h^3 E}{3a\sin\theta}\frac{d\ddot{K}}{d\phi} - \frac{4h^3}{3a^2\sin\theta}\frac{d\ddot{w}}{d\phi} - 2h\left(1 + \frac{h^2}{3a^2}\right)Y\right\}\rho a \sin\theta,$$

$$\frac{d}{d\theta}(N_2 \sin\theta) + \frac{dN_1}{d\phi} - (T_1 + T_2)\sin\theta$$

$$= \left\{2h\left(1 + \frac{h^2}{3a^2}\right)\ddot{w} - \tfrac{1}{3}h^3 E(\ddot{\lambda} + \ddot{\mu}) - \frac{4h^3}{3a}E\ddot{K} - 2h\left(1 + \frac{h^2}{3a^2}\right)Z\right\}\rho a \sin\theta, \quad (5)$$

$$\frac{dG_1}{d\phi} + N_1 a \sin\theta + \frac{d}{d\theta}(H_1 \sin\theta) - H_2 \cos\theta$$

$$= \tfrac{2}{3}\rho h^3\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\theta} - 3\ddot{v} + 2Y\right)\sin\theta,$$

$$\frac{d}{d\theta}(G_2 \sin\theta) + G_1 \cos\theta - N_2 a \sin\theta + \frac{dH_2}{d\theta}$$

$$= -\tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{d\theta} - 3\ddot{u} + 2X\right)\sin\theta,$$

$$(M_1 - M_2)a - H_1 - H_2 = 0.$$

17. We shall now (as in the case of a cylindrical shell) proceed to obtain the values of the couples and the stresses M_1, M_2 by direct calculation.

We have

$$T_1 a \sin\theta \, \delta\phi = \int_{-h}^{h} P'(a + h')\sin\theta \, \delta\phi \, dh' ;$$

whence

$$T_1 = 2hP + \tfrac{1}{3}h^3\left(\frac{d^2 P}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{dP}{dr}\right)$$

$$T_2 = 2hQ + \tfrac{1}{3}h^3\left(\frac{d^2 Q}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{dQ}{dr}\right)$$

$$M_1 = M_2 = 2nh\varpi + \tfrac{1}{3}nh^3\left(\frac{d^2\varpi_3}{dr^2}\right) + \frac{2nh_3}{3a}\left(\frac{d\varpi_3}{dr}\right)$$

$$G_1 = -\tfrac{2}{3}h^3\left\{\left(\frac{dQ}{dr}\right) + \frac{Q}{a}\right\} \qquad \qquad (6).$$

$$G_2 = \tfrac{2}{3}h^3\left\{\left(\frac{dP}{dr}\right) + \frac{P}{a}\right\}$$

$$H_1 = -H_2 = -\tfrac{2}{3}nh^3\left\{\left(\frac{d\varpi_3}{dr}\right) + \frac{\varpi_3}{a}\right\}$$

The third and sixth of these equations satisfy the last of (5) as ought to be the case. Also, employing our previous notation, we see that

$$
\left.
\begin{aligned}
\mathrm{G}_1 &= -\tfrac{4}{3} n h^3 \left(\mathfrak{F} + \frac{\mathfrak{B}}{a} \right), \quad \mathrm{G}_2 = \tfrac{4}{3} n h^3 \left(\mathfrak{E} + \frac{\mathfrak{A}}{a} \right) \\
\mathrm{H}_1 &= -\mathrm{H}_2 = -\tfrac{2}{3} n h^3 \left(p + \frac{\varpi_3}{a} \right)
\end{aligned}
\right\} \quad \dots \dots \quad (7).
$$

Since the couples are proportional to the cube of the thickness, it follows from the fourth and fifth of (5) that the normal shearing stresses N_1, N_2 are also proportional to the cube of the thickness, and, therefore, that the shearing strains ϖ'_1, ϖ'_2 are quadratic functions of h and h'.

Employing our previous notation, the next thing is to calculate the quantities λ, μ, p, λ', μ', p'. We have

$$
\left.
\begin{aligned}
\lambda &= \left(\frac{d\sigma_1}{dr} \right) = -\frac{1}{a^2} \left(\frac{d^2w}{d\theta^2} + w \right) - \frac{\mathrm{E}}{a} (\sigma_1 + \sigma_2) \\
\mu &= \left(\frac{d\sigma_2}{dr} \right) = -\frac{1}{a^2} \left(\frac{1}{\sin^2\theta} \frac{d^2w}{d\phi^2} + \cot\theta \frac{dw}{d\theta} + w \right) - \frac{\mathrm{E}}{a} (\sigma_1 + \sigma_2) \\
p &= \left(\frac{d\varpi_3}{dr} \right) = \frac{2}{a^2 \sin\theta} \left(\cot\theta \frac{dw}{d\phi} - \frac{d^2w}{d\theta\, d\phi} \right)
\end{aligned}
\right\} \quad \dots \quad (8),
$$

in which equations we have omitted all quantities which vanish with h, because λ, μ, p occur in expressions which are multiplied by h^3. Similarly

$$
\left.
\begin{aligned}
\lambda' &= -\frac{2\lambda}{a} - \frac{\mathrm{E}}{a} (\lambda + \mu) + \frac{\mathrm{E}}{a^2} \frac{d^2\mathrm{K}}{d\theta^2} \\
\mu' &= -\frac{2\mu}{a} - \frac{\mathrm{E}}{a} (\lambda + \mu) + \frac{\mathrm{E}}{a^2} \left(\frac{1}{\sin^2\theta} \frac{d^2\mathrm{K}}{d\phi^2} + \cot\theta \frac{d\mathrm{K}}{d\theta} \right) \\
p' &= -\frac{2p}{a} - \frac{2\mathrm{E}}{a^2 \sin\theta} \left(\cot\theta \frac{d\mathrm{K}}{d\phi} - \frac{d^2\mathrm{K}}{d\theta\, d\phi} \right)
\end{aligned}
\right\} \quad \dots \quad (9).
$$

18. The variational equation may be written

$$
\delta \mathrm{W} + \delta \mathfrak{T} = \delta \mathrm{U} + \delta \mathfrak{L} \quad \dots \dots \dots \quad (10),
$$

and we must now calculate the values of the four terms in it, and we shall begin with W.

Since we may omit ϖ'_1, ϖ'_2, and may, therefore, write ϖ for ϖ_3, the potential energy of any portion of the shell is

$$
\mathrm{W} = \tfrac{1}{2} \iiint_{-h}^{h} [(m+n)\Delta'^2 + n\{\varpi'^2 - 4(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1)\}](1 + h'/a)^2\, dh'\, d\mathrm{S} \quad (11)
$$

where the integration with respect to S extends over the middle surface of the portion considered. Since

$$\Delta' = \Delta + h'\left(\frac{d\Delta}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\Delta}{dr^2}\right) + \ \cdots$$

we obtain

$$\tfrac{1}{2}(m+n)\int_{-h}^{h}\Delta'^2\left(1 + h'/a\right)^2 dh'$$

$$= (m+n)\left\{h\left(1 + \frac{h^2}{3a^2}\right)\Delta^2 + \tfrac{1}{3}h^3\left(\frac{d\Delta}{dr}\right)^2 + \tfrac{1}{3}h^3\Delta\left(\frac{d^2\Delta}{dr^2}\right) + \frac{4h^3}{3a}\Delta\left(\frac{d\Delta}{dr}\right)\right\}$$

$$= \frac{4n^2}{m+n}\left\{h\left(1 + \frac{h^2}{3a^2}\right)(\sigma_1 + \sigma_2 + A/2n)^2 + \tfrac{1}{3}h^3(\lambda + \mu)^2\right.$$

$$\left. + \tfrac{1}{3}h^3(\sigma_1 + \sigma_2)(\lambda' + \mu') + \frac{4h^3}{3a}(\sigma_1 + \sigma_2)(\lambda + \mu)\right\} \quad\ldots\ldots \quad (12),$$

also

$$2n\int_{-h}^{h}\sigma'_1\sigma'_2\left(1 + h'/a\right)^2 dh' = 4nh\left(1 + \tfrac{1}{3}h^2/a^2\right)\sigma_1\sigma_2 + \tfrac{4}{3}nh^3\lambda\mu$$

$$+ \tfrac{2}{3}nh^3(\lambda'\sigma_2 + \mu'\sigma_1) + \frac{8nh^3}{3a}(\lambda\sigma_2 + \mu\sigma_1) \quad . \quad . \quad (13),$$

and

$$2n\int_{-h}^{h}(\sigma'_1 + \sigma'_2)\sigma'_3\left(1 + h'/a\right)^2 dh'$$

$$= 4nh\left(1 + \tfrac{1}{3}h^2/a^2\right)\left\{\frac{A}{m+n} - E(\sigma_1 + \sigma_2)\right\}(\sigma_1 + \sigma_2) - \tfrac{4}{3}nh^3 E(\lambda + \mu)^2$$

$$- \tfrac{4}{3}nh^3 E(\sigma_1 + \sigma_2)(\lambda' + \mu') - \frac{16nh^3}{3a}E(\lambda + \mu)(\sigma_1 + \sigma_2) \quad . \quad . \quad . \quad (14);$$

lastly

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi'^2\left(1 + h'/a\right)^2 dh' = nh\left(1 + \tfrac{1}{3}h^2/a^2\right)\varpi^2 + \tfrac{1}{3}nh^3 p^2 + \tfrac{1}{3}nh^3\varpi p' + \frac{4nh^3}{3a}\varpi p \quad (15).$$

Substituting from (12), (13), (14) and (15) in (11), the value of W per unit of area of the middle surface is,

$$W = 2nh\left(1 + \frac{h^2}{3a^2}\right)\{\sigma_1^2 + \sigma_2^2 + E(\sigma_1 + \sigma_2)^2 + \tfrac{1}{2}\varpi^2\}$$

$$+ \tfrac{2}{3}nh^3\{\lambda^2 + \mu^2 + E(\lambda + \mu)^2 + \tfrac{1}{2}p^2\}$$

$$+ \tfrac{2}{3}nh^3\left(\mathfrak{A}\lambda' + \mathfrak{B}\mu' + \tfrac{1}{2}\varpi p'\right)$$

$$+ \frac{8nh^3}{3a}\left(\mathfrak{A}\lambda + \mathfrak{B}\mu + \tfrac{1}{2}\varpi p\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

We must now obtain $\delta\mathbb{T}$. We have

$$\delta\mathbb{T} = \rho \iiint_{-h}^{h} (\ddot{u}'\,\delta u' + \ddot{v}'\,\delta v' + \ddot{w}'\,\delta w')(1 + h'/a)^2\, dh'\, dS\,;$$

also

$$\int_{-h}^{h} \ddot{u}'\,\delta u'\,(1 + h'/a)^2\, dh' = 2h\left(1 + \frac{h^2}{3a^2}\right)\ddot{u}\,\delta u + \tfrac{2}{3}\,h^3\,\frac{d\ddot{u}}{dr}\frac{d\delta u}{dr} + \tfrac{1}{3}\,h^3\left(\ddot{u}\,\frac{d^2\delta u}{dr^2} + \frac{d^2\ddot{u}}{dr^2}\,\delta u\right)$$

$$+ \frac{4h^3}{3a}\left(\ddot{u}\,\frac{d\delta u}{dr} + \frac{d\ddot{u}}{dr}\,\delta u\right)$$

$$= 2h\left(1 + \frac{h^2}{3a^2}\right)\ddot{u}\,\delta u + \frac{2h^3}{3a^2}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\left(\frac{d\delta w}{d\theta} - \delta u\right)$$

$$+ \frac{h^3 E}{3a}\left(\ddot{u}\,\frac{d\delta K}{d\theta} + \frac{d\ddot{K}}{d\theta}\,\delta u\right) - \frac{4h^3}{3a^2}\left\{\ddot{u}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u\right\}$$

by (2) and (3). Treating the other terms in a similar way, we obtain

$$\delta\mathbb{T} = 2\rho h\left(1 + \tfrac{1}{3}\,h^2/a^2\right)\iint(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\,dS$$

$$+ \tfrac{2}{3}\,\rho h^3 \iint \left\{\frac{1}{a^2}\left(\frac{d\ddot{w}}{d\theta} - 3\ddot{u}\right)\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{1}{a^2}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 3\ddot{v}\right)\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right)\right.$$

$$\left. + E\left(E\ddot{K} - \frac{2\ddot{w}}{a}\right)\delta K\right\}\,dS$$

$$+ \tfrac{1}{3}\,\rho h^3 E \iint \left\{\frac{\ddot{u}}{a}\frac{d\delta K}{d\theta} + \frac{\ddot{v}}{a\sin\theta}\frac{d\delta K}{d\phi} - \ddot{w}(\delta\lambda + \delta\mu) + \frac{1}{a}\frac{d\ddot{K}}{d\theta}\,\delta u + \frac{1}{a\sin\theta}\frac{d\ddot{K}}{d\phi}\,\delta v\right.$$

$$\left. - (\ddot{\lambda} + \ddot{\mu})\,\delta w\right\}\,dS$$

$$- \frac{4\rho h^3}{3a}\iint \left\{\frac{1}{a}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u + \frac{1}{a}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + E\ddot{K}\,\delta w\right\}\,dS \quad . \quad . \quad . \quad . \quad (17).$$

We must next find $\delta\mathbb{L}$.
We have

$$\delta\mathbb{L} = \iint_{-h}^{h}(\mathrm{P}'\,\delta u' + \mathrm{U}'\,\delta v')(a + h')\sin\theta\, dh'\, d\phi + \iint_{-h}^{h}(\mathrm{Q}'\,\delta v' + \mathrm{U}'\,\delta u')(a + h')\, dh'\, d\theta$$

$$+ \int \mathrm{N}_2 a\sin\theta\,\delta w\, d\phi + \int \mathrm{N}_1 a\,\delta w\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

whence

$$\delta\mathfrak{L} = \int\left\{ T_1\,\delta u + M_2\,\delta v + N_2\,\delta w - \frac{G_2}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{H_1}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right)\right.$$
$$\left. + \frac{2nh^3 E\mathfrak{A}}{3a}\frac{d\delta K}{d\theta} + \frac{nh^3 E\varpi}{3a\sin\theta}\frac{d\delta K}{d\phi}\right\} a\sin\theta\,d\phi$$
$$+ \int\left\{ M_1\,\delta u + T_2\,\delta v + N_1\,\delta w + \frac{G_1}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right) - \frac{H_2}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right)\right.$$
$$\left. + \frac{2nh^3 E\mathfrak{B}}{3a\sin\theta}\frac{d\delta K}{d\phi} + \frac{nh^3 E\varpi}{3a}\frac{d\delta K}{d\theta}\right\} a\,d\theta \quad . \quad . \quad (19).$$

Lastly,

$$\delta U = \rho\iiint_{-h}^{h} (X\,\delta u' + Y\,\delta v' + Z\,\delta w')(1 + h'/a)^2\,dh'dS$$
$$= 2\rho h\left(1 + \frac{h^2}{3a^2}\right)\iint (X\,\delta u + Y\,\delta v + Z\,\delta w)\,dS$$
$$+ \tfrac{1}{3}\rho h^3 E\iint\left\{\frac{X}{a}\frac{d\delta K}{d\theta} + \frac{Y}{a\sin\theta}\frac{d\delta K}{d\phi} - Z(\delta\lambda + \delta\mu)\right\}dS$$
$$- \frac{4\rho h^3}{3a}\iint\left\{\frac{X}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{Y}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right) + ZE\,\delta K\right\}dS \quad . \quad (20).$$

19. We shall, as in the case of a cylindrical shell, denote the four lines of W by W_1, W_2, W_3, W_4. Whence

$$\delta W_1 = 4nh\left(1 + \tfrac{1}{3}h^2/a^2\right)\iint (\mathfrak{A}\delta\sigma_1 + \mathfrak{B}\delta\sigma_2 + \tfrac{1}{2}\varpi\delta\varpi)\,dS$$
$$= 4nh\left(1 + \frac{h^2}{3a^2}\right)\left\{\int(\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\,\delta v)a\sin\theta\,d\phi + \int(\mathfrak{B}\,\delta v + \tfrac{1}{2}\varpi\,\delta u)a\,d\theta\right\}$$
$$- 4nh\left(1 + \frac{h^2}{3a^2}\right)\iint\left[\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}\delta u\right.$$
$$+ \left\{\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d}{d\theta}(\varpi\sin\theta) + \tfrac{1}{2}\varpi\cos\theta\right\}\delta v - (\mathfrak{A} + \mathfrak{B})\sin\theta\,\delta w\Big]a\,d\theta\,d\phi \quad (21),$$

from which we obtain the approximate equations

$$\left.\begin{array}{cc} T_1 = 4nh\mathfrak{A}, & T_2 = 4nh\mathfrak{B} \\ M_1 = M_2 = 2nh\varpi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22).$$

$$\left.\begin{array}{l} \rho\ddot{u} = \dfrac{2n}{a\sin\theta}\left\{\dfrac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\dfrac{d\varpi}{d\phi}\right\} + \rho X \\[2mm] \rho\ddot{v} = \dfrac{2n}{a\sin\theta}\left\{\dfrac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\dfrac{d}{d\theta}(\varpi\sin\theta) + \tfrac{1}{2}\varpi\cos\theta\right\} + \rho Y \\[2mm] \rho\ddot{w} = -\dfrac{2n}{a}(\mathfrak{A} + \mathfrak{B}) + \rho Z \end{array}\right\} \quad . \quad . \quad (23).$$

These are the equations which have been obtained by Mr. Love,[*] and which have been employed by him in discussing the extensional vibrations of a spherical shell.

Again

$$\delta W_2 = \tfrac{4}{3} nh^3 \iint (\mathfrak{E}\delta\lambda + \mathfrak{F}\delta\mu + \tfrac{1}{2} p\delta p)\, dS.$$

Substituting the values of λ, μ, p from (8) we obtain

$$\iint \mathfrak{E}\delta\lambda\, dS = -\iint \mathfrak{E}\left\{ \frac{d^2\delta w}{d\theta^2} + \delta w + \mathrm{E}\left(\frac{d\delta u}{d\theta} + \frac{1}{\sin\theta}\frac{d\delta v}{d\phi} + \delta u \cot\theta + 2\delta w \right) \right\} \sin\theta\, d\theta\, d\phi$$

$$= -\int \left\{ \mathrm{E}\mathfrak{E} \sin\theta \delta u - \frac{d}{d\theta}(\mathfrak{E}\sin\theta)\delta w + \mathfrak{E}\sin\theta\frac{d\delta w}{d\theta} \right\} d\phi - \int \mathrm{E}\mathfrak{E}\delta v\, d\theta$$

$$+ \iint \left[\mathrm{E}\sin\theta\frac{d\mathfrak{E}}{d\theta}\delta u + \mathrm{E}\frac{d\mathfrak{E}}{d\phi}\delta v \right.$$

$$\left. - \left\{ \frac{d^2}{d\theta^2}(\mathfrak{E}\sin\theta) + (1+2\mathrm{E})\mathfrak{E}\sin\theta \right\} \delta w \right] d\theta\, d\phi \quad (24),$$

also

$$\iint \mathfrak{F}\delta\mu\, dS = -\iint \mathfrak{F}\left\{ \frac{1}{\sin\theta}\frac{d^2\delta w}{d\phi^2} + \cos\theta\frac{d\delta w}{d\theta} + \delta w \sin\theta \right.$$

$$\left. + \mathrm{E}\left(\frac{d\delta u}{d\theta}\sin\theta + \frac{d\delta v}{d\phi} + \delta u\cos\theta + 2\delta w\sin\theta \right) \right\} d\theta\, d\phi$$

$$= -\int (\mathrm{E}\mathfrak{F}\sin\theta\delta u + \mathfrak{F}\cos\theta\delta w)\, d\phi$$

$$- \int \left(\mathrm{E}\mathfrak{F}\delta v - \frac{1}{\sin\theta}\frac{d\mathfrak{F}}{d\phi}\delta w + \frac{\mathfrak{F}}{\sin\theta}\frac{d\delta w}{d\phi} \right) d\theta$$

$$+ \iint \left[\mathrm{E}\sin\theta\frac{d\mathfrak{F}}{d\theta}\delta u + \mathrm{E}\frac{d\mathfrak{F}}{d\phi}\delta v \right.$$

$$\left. - \left\{ \frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2} - \frac{d}{d\theta}(\mathfrak{F}\cos\theta) + (1+2\mathrm{E})\mathfrak{F}\sin\theta \right\} \delta w \right] d\theta\, d\phi \quad (25).$$

In the last term $p\,\delta p$, we must treat the integral which involves $d^2\delta w/d\theta\, d\phi$ exactly in the same way as in the corresponding case of a cylindrical shell, and we shall thus obtain

$$\tfrac{1}{2}\iint p\,\delta p\, dS = \iint p \left(\cot\theta\frac{d\delta w}{d\phi} - \frac{d^2\delta w}{d\theta d\phi} \right) d\theta\, d\phi$$

$$= \tfrac{1}{2}\int \left(\frac{dp}{d\phi}\delta w - p\frac{d\delta w}{d\phi} \right) d\phi + \int \left\{ \left(p\cot\theta + \tfrac{1}{2}\frac{dp}{d\theta} \right)\delta w - \tfrac{1}{2}p\frac{d\delta w}{d\theta} \right\} d\theta$$

$$- \iint \left(\cot\theta\frac{dp}{d\phi} + \frac{d^2p}{d\theta d\phi} \right) \delta w\, d\theta\, d\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

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[*] 'Phil. Trans.,' A, 1888, p. 527. Equation (23) corresponds to Love's equations (46), (47), and (48) and (22) to (72).

Adding (24), (25), and (26), we finally obtain

$$\delta W_2 = \frac{4nh^3}{3a}\int\Big[-\{\mathfrak{E}+E(\mathfrak{E}+\mathfrak{F})\}\,\delta u - \tfrac{1}{2}p\,\delta v + \frac{1}{\sin\theta}\Big\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta)-\mathfrak{F}\cos\theta$$

$$+\tfrac{1}{2}\frac{dp}{d\phi}\Big\}\,\delta w - \mathfrak{E}\Big(\frac{d\delta w}{d\theta}-\delta u\Big)-\tfrac{1}{2}p\Big(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi}-\delta v\Big)\Big]\,a\sin\theta\,d\phi$$

$$+\frac{4nh^3}{3a}\int\Big[-\tfrac{1}{2}p\,\delta u-\{\mathfrak{F}+E(\mathfrak{E}+\mathfrak{F})\}\,\delta v+\frac{1}{\sin\theta}\Big(\frac{d\mathfrak{F}}{d\phi}+p\cos\theta+\tfrac{1}{2}\sin\theta\frac{dp}{d\theta}\Big)\,\delta w$$

$$-\mathfrak{F}\Big(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi}-\delta v\Big)-\tfrac{1}{2}p\Big(\frac{d\delta w}{d\theta}-\delta u\Big)\Big]\,a\,d\theta$$

$$+\tfrac{4}{3}nh^3\iint\Big[E\sin\theta\frac{d}{d\theta}(\mathfrak{E}+\mathfrak{F})\,\delta u + E\frac{d}{d\phi}(\mathfrak{E}+\mathfrak{F})\,\delta v$$

$$-\Big\{\frac{d^2}{\delta\theta^2}(\mathfrak{E}\sin\theta)+(1+2E)(\mathfrak{E}+\mathfrak{F})\sin\theta+\frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2}-\frac{d}{d\theta}(\mathfrak{F}\cos\theta)$$

$$+2(1+E)\mathfrak{F}\sin\theta+\cot\theta\frac{dp}{d\phi}+\frac{d^2p}{d\theta\,d\phi}\Big\}\,\delta w\Big]\,d\theta\,d\phi \qquad (27).$$

The expressions for W_3, W_4 may, as in the case of a cylindrical shell, be divided into two parts W_3', W_3'', W_4', W_4''. The values of $\delta W_3'$, $\delta W_4'$, may at once be written down from (21) by changing $\mathfrak{A}$, $\mathfrak{B}$, ϖ into $\mathfrak{E}'$, $\mathfrak{F}'$, p' and $\mathfrak{E}$, $\mathfrak{F}$, p respectively, and by altering the coefficient into $\tfrac{2}{3}nh^3$ and $8nh^3/3a$ respectively. With regard to W_3'' we have

$$\delta W_3'' = \tfrac{2}{3}nh^3\iint(\mathfrak{A}\,\delta\lambda'+\mathfrak{B}\,\delta\mu'+\tfrac{1}{2}\varpi\,\delta p')\,dS.$$

Substituting the value of λ' from (9) and integrating once by parts, we shall obtain

$$\iint\mathfrak{A}\,\delta\lambda'\,dS = E\int\mathfrak{A}\,\frac{d\,\delta K}{d\theta}\sin\theta\,d\phi$$

$$-\iint\Big[E\frac{d}{d\theta}(\mathfrak{A}\sin\theta)\frac{d\,\delta K}{d\theta}+\mathfrak{A}a\sin\theta\{2\,\delta\lambda+E(\delta\lambda+\delta\mu)\}\Big]\,d\theta\,d\phi.$$

Treating the other terms in a similar manner, we shall finally obtain

$$\delta W_3'' = \int\Big\{\frac{2nh^3 E\mathfrak{A}}{3a}\frac{d\,\delta K}{d\theta}+\frac{nh^3 E\varpi}{3a\sin\theta}\frac{d\,\delta K}{d\phi}\Big\}\,a\sin\theta\,d\phi$$

$$+\int\Big\{\frac{2nh^3 E\mathfrak{B}}{3a\sin\theta}\frac{d\,\delta K}{d\phi}+\frac{nh^3 E\varpi}{3a}\frac{d\,\delta K}{d\theta}\Big\}\,a\,d\theta$$

$$-\frac{2nh^3}{3a^2}\iint\Big[\frac{E}{\sin\theta}\Big\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta)-\mathfrak{B}\cos\theta+\tfrac{1}{2}\frac{d\varpi}{d\phi}\Big\}\frac{d\,\delta K}{d\theta}$$

$$+\frac{E}{\sin^2\theta}\Big(\frac{d\mathfrak{B}}{d\phi}+\tfrac{1}{2}\frac{d\varpi}{d\theta}\sin\theta+\varpi\cos\theta\Big)\frac{d\,\delta K}{d\phi}+Ea(\mathfrak{A}+\mathfrak{B})(\delta\lambda+\delta\mu)\Big]\,dS$$

$$-\frac{4nh^3}{3a}\iint(\mathfrak{A}\,\delta\lambda+\mathfrak{B}\,\delta\mu+\tfrac{1}{2}\varpi\,\delta p)\,dS.$$

If in the first surface integral in this equation we substitute the approximate values of the coefficients of $d\,\delta K/d\theta$ &c. from (23), which we may do since this integral is multiplied by h^3, and then substitute the values of $\delta W_3''$, $\delta\mathfrak{T}$, $\delta\mathfrak{L}$, and δU in (10), it will be found that all the terms involving $d\,\delta K/d\theta$, $d\,\delta K/d\phi$, and $\delta\lambda + \delta\mu$ cut out; we are therefore no longer concerned with them, and the value of $\delta W_3''$ reduces to the last line; on this understanding we may write

$$\delta W_3'' + \delta W_4'' = \frac{4nh^3}{3a} \iint (\mathfrak{A}\,\delta\lambda + \mathfrak{B}\,\delta\mu + \tfrac{1}{2}\,\varpi\,\delta p)\,dS \quad . \quad . \quad . \quad (28).$$

The variation of the right-hand side of (28) might at once be written down from (27) by substituting $\mathfrak{A}$, $\mathfrak{B}$, and ϖ for $\mathfrak{E}$, $\mathfrak{F}$, and p; but it will be more convenient to present the results in another form. Taking the first term, and integrating the second differential coefficients *once* by parts, we obtain

$$\iint \mathfrak{A}\,\delta\lambda\,dS = -\int \mathfrak{A}\,\frac{d\,\delta w}{d\theta}\sin\theta\,d\phi + \iint \left\{ \frac{d}{d\theta}(\mathfrak{A}\sin\theta)\frac{d\,\delta w}{d\theta} - \mathfrak{A}\sin\theta\,\delta w \right.$$
$$\left. - \mathrm{E}\mathfrak{A}\,a\sin\theta\,\delta K \right\}\,d\theta\,d\phi.$$

Treating the other terms in a similar way, and adding to the result from (21) that portion of δW_1 which depends upon h^3, and finally replacing the coefficients of $d\,\delta w/d\theta - \delta u$, &c., by their approximate values from (23), the final result will be

$$-\frac{4nh^3}{3a^2}\int \left\{ \mathfrak{A}\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + \tfrac{1}{2}\varpi\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right) \right\} a\sin\theta\,d\phi$$
$$-\frac{4nh^3}{3a^2}\int \left\{ \mathfrak{B}\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right) + \tfrac{1}{2}\varpi\left(\frac{d\,\delta w}{d\theta} - \delta u\right) \right\} a\,d\theta$$
$$+\frac{2\rho h^3}{3a^2}\iint \left\{ (\ddot{u} - \mathrm{X})\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + (\ddot{v} - \mathrm{Y})\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right) \right.$$
$$\left. + \mathrm{E}a\,(\ddot{w} - \mathrm{Z})\,\delta K \right\}\,dS \quad (29).$$

This result enables us to test the accuracy of a portion of our work, and the fundamental hypothesis on which the theory is based; for if we substitute in (10) the expression (29), and also the value of $\delta\mathfrak{L}$ from (19), it will be seen that we have reproduced the values of the couples which are given by (7); also comparing with $\delta\mathfrak{L}$, the line integral parts of δW_2, given by (27), the line integral parts of $\delta W_3'$ and $\delta W_4'$, which, as we have explained above, are obtained from (21) by changing certain letters, we see that we have also reproduced the values of M_1, M_2, given by the third of (6). We may, therefore, omit the couple terms, and also the terms in M; also, since we

have disposed of the terms in δW_1, which involve h^3, we shall write $\delta W_1'$ for the remaining portion which depends upon h, and the variational equation finally becomes

$$\delta W_1' + \delta W_2' + \delta W_3' + \delta W_4' + 2\rho h \left(1 + \tfrac{1}{3} h^2/a^2\right) \iint \left(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w\right) dS$$

$$+ \tfrac{2}{3}\rho h^3 \iint \left\{ \frac{1}{a^2}\left(\frac{d\ddot{w}}{d\theta} - 2\ddot{u}\right)\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + \frac{1}{a^2}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right)\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right) \right.$$

$$\left. + E\left(E\ddot{K} - \frac{\ddot{w}}{a}\right)\delta K \right\} dS$$

$$+ \tfrac{1}{3}\rho h^3 E \iint \left\{ \frac{1}{a}\frac{d\ddot{K}}{d\theta}\delta u + \frac{1}{a\sin\theta}\frac{d\ddot{K}}{d\phi}\delta v - (\ddot{\lambda} + \ddot{\mu})\,\delta w \right\} dS$$

$$- \frac{4\rho h^3}{3a} \iint \left\{ \frac{1}{a}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u + \frac{1}{a}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + E\ddot{K}\,\delta w \right\} dS$$

$$= 2\rho h \left(1 + \frac{h^2}{3a^2}\right) \iint \left(X\,\delta u + Y\,\delta v + Z\,\delta w\right) dS$$

$$- \frac{2\rho h^3}{3a^2} \iint \left\{ X\left(\frac{d\delta w}{d\theta} - \delta u\right) + Y\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right) + ZEa\,\delta K \right\} dS$$

$$+ \int (T_1\,\delta u + N_2\,\delta w)\, a\sin\theta\, d\phi + \int (T_2\,\delta v + N_1\,\delta w)\, a\, d\theta \quad \ldots \ldots \quad (30).$$

We have now got rid of all the terms involving the second differential coefficients of δu, δv, δw; and all that remains to be done is to integrate by parts the terms which involve the first differential coefficients. Putting

$$\alpha = \frac{d\ddot{w}}{d\theta} - 2\ddot{u} + X, \qquad \beta = \frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + Y, \qquad \gamma = E\left(aE\ddot{K} - \ddot{w} + Z\right) \quad (31),$$

we have

$$\frac{2\rho h^3}{3a^2} \iint \left\{ \alpha \frac{d\delta w}{d\theta} + \frac{\beta}{\sin\theta}\frac{d\delta w}{d\phi} + a\gamma\delta K \right\} dS$$

$$= \frac{2\rho h^3}{3a} \int (\gamma\delta u + \alpha\delta w)\, a\sin\theta d\phi + \frac{2\rho h^3}{3a} \int (\gamma\delta v + \beta\delta w)\, a d\theta$$

$$- \frac{2\rho h^3}{3a^2} \iint \left[\left\{ \frac{d}{d\theta}(\gamma\sin\theta) - \gamma\cos\theta \right\}\delta u + \frac{d\gamma}{d\phi}\delta v + \left\{ \frac{d}{d\theta}(\alpha\sin\theta) + \frac{d\beta}{d\phi} - 2\gamma\sin\theta \right\}\delta w \right] \frac{dS}{\sin\theta} \quad (32).$$

Substituting the values of $\delta W_1'$, $\delta W_2'$, $\delta W_3'$, $\delta W_4'$, and the right hand side of (32) in (30), and picking out the line integral terms, we obtain the following equations for the sectional stresses, viz.,

$$T_1 = 4nh\mathfrak{A} + \frac{4nh^3}{3a}\{\mathfrak{E} - E(\mathfrak{E} + \mathfrak{F})\} + \tfrac{2}{3}nh^3\mathfrak{E}' + \frac{2\rho h^3}{3a}E(a E\ddot{K} - \ddot{w} + z)$$

$$M_2 = 2nh\varpi + \frac{2nh^3}{3a}p + \tfrac{1}{3}nh^3 p'$$

$$N_2 = \frac{4nh^3}{3a\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta + \tfrac{1}{2}\frac{dp}{d\phi}\right\} + \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\theta} - 2\ddot{u} + X\right)$$

$$G_2 = \tfrac{4}{3}nh^3(\mathfrak{E} + \mathfrak{A}/a)$$

$$H_1 = -\tfrac{2}{3}nh^3(p + \varpi/a)$$

$$\left.\phantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}\quad (33)$$

which give the values of the sectional stresses across a parallel of latitude; and

$$M_1 = 2nh\varpi + \frac{2nh^3}{3a}p + \tfrac{1}{3}nh^3 p'$$

$$T_2 = 4nh\mathfrak{B} + \frac{4nh^3}{3a}\{\mathfrak{F} - E(\mathfrak{E} + \mathfrak{F})\} + \tfrac{2}{3}nh^3\mathfrak{F}' + \frac{2\rho h^3}{3a}E(a E\ddot{K} - \ddot{w} + Z)$$

$$N_1 = \frac{4nh^3}{3a\sin\theta}\left(\frac{d\mathfrak{F}}{d\phi} + p\cos\theta + \tfrac{1}{2}\sin\theta\frac{dp}{d\theta}\right) + \frac{2\rho h^3}{3a}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + Y\right)$$

$$G_1 = -\tfrac{4}{3}nh^3(\mathfrak{F} + \mathfrak{B}/a)$$

$$H_2 = \tfrac{2}{3}nh^3(p + \varpi/a)$$

$$\left.\phantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}\quad (34)$$

which give the values of the sectional stresses across a meridian.

In these equations we may, if we please, substitute the approximate values of $\ddot{u}$, $\ddot{v}$, $\ddot{w}$ from (23), and by means of these values it can be shown that the values of N_1, N_2 agree with the values which are obtained by substituting the values of the couples in the fourth and fifth of (5).

Picking out the coefficients of δu, δv, δw, in the surface integrals, we obtain the equations of motion, which are

$$\left\{2\left(1 + \frac{h^2}{a^2}\right)\ddot{u} + \frac{h^2 E}{3a}\frac{d\ddot{K}}{d\theta} - \frac{4h^2}{3a^2}\frac{d\ddot{w}}{d\theta} - 2\left(1 + \frac{h^2}{3a^2}\right)X\right\}\rho a\sin\theta$$

$$= 4n\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}$$

$$- \frac{4nh^2}{3a^2}E\sin\theta\frac{d}{d\theta}(\mathfrak{E} + \mathfrak{F}) + \tfrac{2}{3}nh^2\left\{\frac{d}{d\theta}(\mathfrak{E}'\sin\theta) - \mathfrak{F}'\cos\theta + \tfrac{1}{2}\frac{dp'}{d\phi}\right\}$$

$$+ \frac{8nh^2}{3a}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta + \tfrac{1}{2}\frac{dp}{d\phi}\right\} + \frac{2\rho h^2}{3a}\left\{\alpha\sin\theta + \frac{d\gamma}{d\theta}\sin\theta - \gamma\cos\theta\right\}\ (35),$$

$$\left\{2\left(1+\frac{h^2}{a^2}\right)\ddot{v}+\frac{h^2\mathrm{E}}{3a\sin\theta}\frac{d\ddot{\mathrm{K}}}{d\phi}-\frac{4h^2}{3a^2\sin\theta}\frac{d\ddot{w}}{d\phi}-2\left(1+\frac{h^2}{3a^2}\right)\mathrm{Y}\right\}\rho a\sin\theta$$

$$=4n\left\{\frac{d\mathfrak{B}}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}\left(\varpi\sin\theta\right)+\tfrac{1}{2}\varpi\cos\theta\right\}$$

$$-\frac{4nh^2}{3a}\,\mathrm{E}\,\frac{d}{d\phi}\left(\mathfrak{E}+\mathfrak{F}\right)+\tfrac{2}{3}nh^2\left\{\frac{d\mathfrak{F}'}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}\left(p'\sin\theta\right)+\tfrac{1}{2}p'\cos\theta\right\}$$

$$+\frac{8nh^2}{3a}\left\{\frac{d\mathfrak{F}}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}\left(p\sin\theta\right)+\tfrac{1}{2}p\cos\theta\right\}+\frac{2\rho h^2}{3a}\left(\beta\sin\theta+\frac{d\gamma}{d\phi}\right)\quad . \quad (36).$$

$$\left\{2\left(1+\frac{h^2}{3a^2}\right)\left(\ddot{w}-\mathrm{Z}\right)-\tfrac{1}{3}h^2\mathrm{E}\left(\ddot{\lambda}+\ddot{\mu}\right)-\frac{4h^2}{3a}\,\mathrm{E}\ddot{\mathrm{K}}\right\}\rho a\sin\theta$$

$$=-4n\left(\mathfrak{A}+\mathfrak{B}\right)\sin\theta+\frac{4nh^2}{3a}\left\{\frac{d^2}{d\theta^2}\left(\mathfrak{E}\sin\theta\right)+\left(1+2\mathrm{E}\right)\left(\mathfrak{E}+\mathfrak{F}\right)\sin\theta\right.$$

$$+\frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2}-\frac{d}{d\theta}\left(\mathfrak{F}\cos\theta\right)+\cot\theta\frac{dp}{d\phi}+\frac{d^2p}{d\theta\,d\phi}\right\}$$

$$-\tfrac{2}{3}nh^2\left(\mathfrak{E}'+\mathfrak{F}'\right)\sin\theta-\frac{8nh^2}{3a}\left(\mathfrak{E}+\mathfrak{F}\right)\sin\theta$$

$$+\frac{2\rho h^2}{3a}\left\{\frac{d}{d\theta}\left(\alpha\sin\theta\right)+\frac{d\beta}{d\phi}-2\gamma\sin\theta\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

The correctness of these equations may be tested by substituting the values of the sectional stresses from (33) and (34) in the first three of (5), when it will be found that we shall reproduce (35), (36), and (37).

20. The boundary conditions for a spherical shell may be investigated in exactly the same manner as in the case of a cylindrical shell, by means of STOKES' theorem ; for in the present case the theorem may be written

$$\int\left(\frac{d\mathrm{H}'}{d\phi}\delta w+\mathrm{H}'\frac{d\delta w}{d\phi}\right)d\phi+\int\left(\frac{d\mathrm{H}'}{d\theta}\delta w+\mathrm{H}'\frac{d\delta w}{d\theta}\right)d\theta=0,$$

the integration extending round any curvilinear rectangle bounded by two meridians and two parallels of latitude. If, therefore, in the figure we apply to the side AD the stresses,

$$\mathrm{M}_2'=\mathrm{H}'/a, \qquad \mathrm{N}_2'=\frac{1}{a\sin\theta}\frac{d\mathrm{H}'}{d\phi}, \qquad \mathrm{H}_1'=\mathrm{H}';$$

to the side BD the stresses

$$\mathrm{M}_1'=\mathrm{H}'/a, \qquad \mathrm{N}_1'=\frac{1}{a}\frac{d\mathrm{H}'}{d\theta}, \qquad \mathrm{H}_2'=-\mathrm{H}';$$

and to the sides OB, OA, corresponding and opposite stresses respectively, the preceding integral becomes

$$\int \left\{ M_2' \, \delta v + N_2' \, \delta w + \frac{H_1'}{a} \left(\frac{1}{\sin \theta} \frac{d \delta w}{d \phi} - \delta v \right) \right\} a \sin \theta \, d\phi$$

$$+ \int \left\{ M_1' \, \delta u + N_1' \, \delta w - \frac{H_2'}{a} \left(\frac{d \delta w}{d \theta} - \delta u \right) \right\} a \, d\theta = 0 ,$$

which shows that the work done by this system of stresses is zero.

If, therefore, we suppose that the rectangle OADB, instead of being under the action of the remainder of the shell, is isolated, and that its state of strain is maintained by stresses applied to its edges, then it follows that if instead of the torsional couples H_1, H_2, due to the action of contiguous portions of the shell, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\mathfrak{H}_1 = H_1 + H' \quad \ldots \quad \ldots \quad \ldots \quad (38),$$

$$\mathfrak{H}_2 = H_2 - H' \quad \ldots \quad \ldots \quad \ldots \quad (39),$$

the state of strain will remain unchanged, provided we apply in addition the stresses

$$\left. \begin{array}{l} \mathfrak{M}_2 = M_2 + H'/a \\ \mathfrak{N}_2 = N_2 + \dfrac{1}{a \sin \theta} \dfrac{dH'}{d\phi} \end{array} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

and

$$\left. \begin{array}{l} \mathfrak{M}_1 = M_1 + H'/a \\ \mathfrak{N}_1 = N_1 + \dfrac{1}{a} \dfrac{dH'}{d\theta} \end{array} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (41),$$

whence eliminating H' between (38) and (40), and between (39) and (41) respectively, we obtain

$$\left. \begin{array}{l} \mathfrak{M}_2 a - \mathfrak{H}_1 = M_2 a - H_1 \\ \mathfrak{N}_2 a \sin \theta - \dfrac{d \mathfrak{H}_1}{d\phi} = N_2 a \sin \theta - \dfrac{dH_1}{d\phi} \end{array} \right\} \quad \ldots \quad \ldots \quad (42)$$

and

$$\left. \begin{array}{l} \mathfrak{M}_1 a + \mathfrak{H}_2 = M_1 a + H_2 \\ \mathfrak{N}_1 a + \dfrac{d \mathfrak{H}_2}{d\theta} = N_2 a + \dfrac{dH_2}{d\theta} \end{array} \right\} \quad \ldots \quad \ldots \quad (43)$$

in which we are to remember that $\mathfrak{H}_1 = - \mathfrak{H}_2$, and $H_1 = - H_2$.

In these equations the Roman letters denote the stresses due to the action of contiguous portions of the shell, whilst the Old English letters denote the values of the actual stresses applied to the boundary. If, therefore, the shell consists of a

portion of a sphere bounded by two meridians and two parallels of latitude, and whose edges are free, the boundary conditions along a parallel of latitude are obtained by equating the right hand sides of the first and fourth of (33) and of (42) to zero ; whilst the boundary conditions along a meridian are similarly obtained by equating the right hand sides of the first and fourth of (34) and of (43) to zero.

21. If the shell is supposed to vibrate in such a manner, that its middle surface does not experience any extension or contraction throughout the motion, the equations of motion can be obtained by taking the variation subject to the conditions of inextensibility, and introducing indeterminate multipliers.

22. It will now be convenient to make a short digression for the purpose of considering some of the quantities involved.

Let P be any point on the deformed middle surface whose undisplaced coordinates are (a, θ, ϕ). The coordinates of P after deformation are

$$R = a + w, \qquad \Theta = \theta + u/a, \qquad \Phi = \phi + v/a \sin \theta \quad . \quad . \quad . \quad (44),$$

and since u, v, w are functions of θ and ϕ, the elimination of the latter quantities from (44) will give a relation between R, Θ, Φ, which is the equation of the deformed middle surface.

If ρ_1 be the radius of curvature at any point of a meridian section after deformation, and P the perpendicular from the centre on to the tangent at that point to the deformed section,

$$\frac{1}{\rho_1} = \frac{1}{R} \frac{dP}{dR}.$$

Now

$$\frac{1}{P^2} = \frac{1}{R^2}\left\{1 + \left(\frac{dR}{Rd\Theta}\right)^2\right\}$$

$$= \frac{1}{R^2}\left\{1 + \frac{(dw/d\theta)^2}{R^2(1 + du/ad\theta)^2}\right\},$$

and therefore, neglecting cubes of displacements,

$$P = a + w - \frac{1}{2a}\left(\frac{dw}{d\theta}\right)^2.$$

Also

$$dR = \frac{dw}{d\theta}\, d\theta,$$

whence

$$\frac{1}{\rho_1} - \frac{1}{a} = -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (45),$$

which gives the change of curvature along a meridian.

We shall now find an expression for the change of curvature along any great circle which makes an angle γ with a meridian.

In the figure on p. 463 join OD, and let the angle OED $= \chi$, and the angle DOA $= \gamma$; then by (45) the change of curvature along OD is

$$- \frac{1}{a^2} \left(\frac{d^2w}{d\chi^2} + w \right).$$

If $w + \delta w$ be the normal displacement at D, it follows by equating the two values of δw that

$$\frac{dw}{d\chi} \delta\chi + \frac{1}{2} \frac{d^2w}{d\chi^2} \delta\chi^2 = \frac{dw}{d\theta} \delta\theta + \frac{dw}{d\phi} \delta\phi + \frac{1}{2} \left(\frac{d^2w}{d\theta^2} \delta\theta^2 + 2 \frac{d^2w}{d\theta d\phi} \delta\theta \, \delta\phi + \frac{d^2w}{d\phi^2} \delta\phi^2 \right) \quad (46).$$

From the spherical triangle ODP we have

$$- \cos\gamma = \frac{\cos \text{PD} - \cos\theta \cos\delta\chi}{\sin\theta \sin\delta\chi},$$

whence

$$\sin\theta \, \delta\theta = \cos\gamma \sin\theta \, \delta\chi + \tfrac{1}{2} \cos\theta \, \delta\chi^2 - \tfrac{1}{2} \cos\theta \, \delta\theta^2$$
$$= \cos\gamma \sin\theta \, \delta\chi + \tfrac{1}{2} \cos\theta \sin^2\gamma \, \delta\chi^2.$$

Again

$$\frac{\sin\delta\phi}{\sin\delta\chi} = \frac{\sin\gamma}{\sin(\theta + \delta\theta)},$$

whence

$$\delta\phi = \frac{\sin\gamma}{\sin\theta} (\delta\chi - \cot\theta \cos\gamma \, \delta\chi^2).$$

Substituting these values of $\delta\theta$, $\delta\phi$ in (46) and equating coefficients of $\delta\chi^2$, we obtain

$$\frac{d^2w}{d\chi^2} = \frac{\sin^2\gamma}{\sin^2\theta} \frac{d^2w}{d\phi^2} + \frac{\sin 2\gamma}{\sin\theta} \frac{d^2w}{d\theta d\phi} + \cos^2\gamma \frac{d^2w}{d\theta^2} - \frac{\sin 2\gamma \cos\theta}{\sin^2\theta} \frac{dw}{d\phi} + \sin^2\gamma \cot\theta \frac{dw}{d\theta} \quad (47).$$

Whence it follows, that if ρ_1, ρ_2 are the principal radii of curvature along and perpendicular to a meridian

$$\left. \begin{aligned} \frac{1}{\rho_1} - \frac{1}{a} &= -\frac{1}{a^2} \left(\frac{d^2w}{d\theta^2} + w \right) \\ \frac{1}{\rho_2} - \frac{1}{a} &= -\frac{1}{a^2} \left(\frac{1}{\sin^2\theta} \frac{d^2w}{d\phi^2} + \cot\theta \frac{dw}{d\theta} + w \right) \end{aligned} \right\} \quad \cdots \quad (48).$$

23. When the middle surface is inextensible, it has been shown by Lord RAYLEIGH[*] that the displacements are given by the equations

$$
\left.
\begin{aligned}
u &= - \Sigma A_s \, \epsilon^{\iota s\phi} \sin \theta \tan^s \tfrac{1}{2} \theta \\
v &= \quad \Sigma \iota A_s \, \epsilon^{\iota s\phi} \sin \theta \tan^s \tfrac{1}{2} \theta \\
w &= \quad \Sigma A_s \, \epsilon^{\iota s\phi} (s + \cos \theta) \tan^s \tfrac{1}{2} \theta
\end{aligned}
\right\} \quad \dots \dots \quad (49).
$$

where $s = 2, 3, 4 \dots$ and A_s is a complex function of the time. From these equations it can easily be shown by means of (48) that

$$
\frac{1}{\rho_1} - \frac{1}{a} = - \Sigma \frac{A_s (s^3 - s) \, \epsilon^{\iota s\phi} \tan^s \tfrac{1}{2} \theta}{a \sin^2 \theta} = - \left(\frac{1}{\rho_2} - \frac{1}{a} \right) \quad \dots \dots \quad (50).
$$

The value of the potential energy is given by the second line of (16); also by the first two of (8) and by (48)

$$
\lambda = - \mu = \frac{1}{\rho_1} - \frac{1}{a} \quad \dots \dots \dots \quad (51),
$$

and by the last of (8)

$$
p = - 2\iota \Sigma \frac{A_s (s^3 - s) \, \epsilon^{\iota s\phi} \tan^s \tfrac{1}{2} \theta}{a \sin^2 \theta},
$$

whence

$$
W = \frac{4nh^3}{3a^2 \sin^4 \theta} \left[\{ \Sigma \, A_s (s^3 - s) \cos s\phi \tan^s \tfrac{1}{2} \theta \}^2 + \{ \Sigma \, A_s (s^3 - s) \sin s\phi \tan^s \tfrac{1}{2} \theta \}^2 \right] \quad (52),
$$

which agrees with Lord RAYLEIGH's result.

Let us now suppose that a bell which consists of a spherical shell, bounded by a small circle whose latitude is $\tfrac{1}{2}\pi - \alpha$, is vibrating in such a manner that its middle surface does not undergo any extension or contraction throughout the motion. One of the boundary conditions requires that the flexural couple G_2 should vanish along the circle of latitude which constitutes the free edge of the bell. By (7) and (51)

$$
\begin{aligned}
G_2 &= \tfrac{4}{3} nh^3 \bar{\Sigma} = \tfrac{4}{3} nh^3 \{ \lambda + E (\lambda + \mu) \} \\
&= \tfrac{4}{3} nh^3 \left(\frac{1}{\rho_1} - \frac{1}{a} \right).
\end{aligned}
$$

From (50) we see that G_2 cannot vanish for any value of θ except $\theta = 0$, that is, at the pole, provided $s > 2$. It, therefore, follows that a spherical bell whose edge is free cannot vibrate in this manner if the middle surface is supposed to remain absolutely inextensible throughout the motion. If, however, extension or contraction

* ' London Math. Soc. Proc ,' vol 13, p. 4 (1881).

were to take place in the neighbourhood of the edge, it would be possible for G_2 to vanish there, and also to satisfy the other boundary conditions.

There seems no reason to doubt that the argument which has been employed in the case of a cylindrical shell, would apply equally to the case of a spherical shell, and probably also to a shell of any shape; in which case, the portions of the displacements upon which extension principally depends, would be small compared with the portions upon which bending principally depends, except at points whose distances from a free edge are comparable with the thickness. At the same time it would be very desirable to obtain the solution of some problem relating to the vibrations of a shell whose edges are free, in which no supposition is made as to the relative magnitudes of the extensional and flexural terms.

VII. *Memoir on Symmetric Functions of the Roots of Systems of Equations.*

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§ 1. *Preliminary Ideas.*

1. The theory of the symmetrical functions of a single system of quantities has been investigated in a large number of memoirs, but so far, only a few attempts have been made to develop an analogous theory with regard to several systems of quantities. The chief authors are Schläfli[*] and Cayley,[†] both of whom have, however, restricted themselves to the outlines of the commencement of such a theory. In the theory of the single system it is found convenient to regard the quantities as the roots of an equation, since the coefficients of such an equation are themselves those particular symmetric functions of the quantities which have been variously termed fundamental, elementary, and unitary; they are fundamental because all other rational integral functions are expressible by their products of the same or lower degree ; elementary because they are those which, first of all, naturally arise ; unitary because their partitions are composed wholly of units. The left hand side of the equation referred to is a product of binomial linear functions of a single variable x, so that, $\alpha_1, \alpha_2, \ldots \alpha_n$ being the quantities which compose the system, the fundamental relation may be written

$$(1 + \alpha_1 x)(1 + \alpha_2 x) \ldots (1 + \alpha_n x) = 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$
$$= 1 + (1) x + (1^2) x^2 + \ldots + (1^n) x^n,$$

in the ordinary partition notation.

In a general discussion it is convenient and advantageous to suppose the number of quantities infinite, so that the relation becomes

$$(1 + \alpha_1 x)(1 + \alpha_2 x) \ldots = (1 + a_1 x + a_2 x^2 + \ldots) = 1 + (1) x + (1^2) x^2 + \ldots$$

[*] " Ueber die Resultante eines Systemes mehrerer algebraischen Gleichungen." ' Vienna Academy Denkschriften,' vol. 4, 1852.

[†] " On the Symmetric Functions of the Roots of certain Systems of Two Equations." ' Phil. Trans.,' vol. 147 (1857).

2. Instead of taking a product of binomial linear functions of one variable, as above, we can, for m systems of quantities, take a product of non-homogeneous linear functions of m variables, and each such linear function may be taken of the form

$$1 + \alpha_{s1}x_1 + \alpha_{s2}x_2 + \ldots + \alpha_{sm}x_m.$$

As indicative of this general case it is sufficient to consider merely the case of two systems of quantities. Complexity of formulas is thereby avoided, but it must be distinctly borne in mind that all the succeeding theorems can be at once extended to the general case of m systems by an easy enlargement of the nomenclature and notation.

I consider, then, two systems of quantities

$$\alpha_1, \ \alpha_2, \ \ldots \ \alpha_n \,;$$
$$\beta_1, \ \beta_2, \ \ldots \ \beta_n \,;$$

as connected with two non-homogeneous equations, in two variables, in such wise that the values α_s, β_s, of the variables respectively constitute one solution of the two simultaneous equations. In order to avoid identical relations between fundamental forms, as well as for other reasons, which will appear, I take the number of quantities n in each system to be infinite.

By analogy the fundamental relation is written

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_s x + \beta_s y) \ldots$$
$$= 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots + a_{pq}x^p y^q + \ldots$$

As shown by SCHLÄFLI this equation may be directly formed and exhibited as the resultant of the two given equations, and an arbitrary, linear, non-homogeneous equation in two variables. Beyond the preliminary idea this investigation has little to do with the original equations or with the theory of resultants. It starts with the fundamental equation just written, the right-hand side of which may be put into the form

$$1 + \Sigma\alpha_1 x + \Sigma\beta_1 y + \Sigma\alpha_1\alpha_2.x^2 + \Sigma\alpha_1\beta_2.xy + \Sigma\beta_1\beta_2.y^2 + \ldots$$

The most general symmetric function to be considered is

$$\Sigma\alpha_1^{p_1}\beta_1^{q_1}\alpha_2^{p_2}\beta_2^{q_2}\alpha_3^{p_3}\beta_3^{q_3}\ldots$$

which I represent symbolically by

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3}\cdots).$$

Observe that the summation is in regard to the expressions obtained by permuting the n suffices

$$1, 2, 3, \ldots n.$$

The weight of the function must be considered as bipartite; it consists of the two numbers

$$p_1 + p_2 + p_3 + \ldots = \Sigma p,$$
$$q_1 + q_2 + q_3 + \ldots = \Sigma q,$$

and I speak of the biweight $\Sigma p, \Sigma q$.

The sum $\Sigma p + \Sigma q$ may be called the whole weight, or simply the weight. Associated with any number w there will be a weight w and a biweight corresponding to every composition of w by means of two numbers, including zero as a number. By composition is meant partition, in which regard is paid to the order of the parts; for instance, 21 and 12 are different binary compositions of 3, and 30, 21, 12, 03 constitute the system.

3. It is necessary to introduce the notion of the partition of the bipartite number which denotes the biweight.

Thus of the biweight $\Sigma p, \Sigma q$ the expression

$$(\overline{p_1 q_1 \, p_2 q_2 \, p_3 q_3 \cdots})$$

may be termed a partition.

The dual symbols $\overline{p_1 q_1}, \overline{p_2 q_2}, \overline{p_3 q_3}, \ldots$ are the parts of this partition; the parts are themselves bipartite and may be termed biparts.

We have thus a biweight denoted by a bipartite number partitioned into a number of bipartite numbers termed biparts.

It is convenient to arrange the biparts so that the sums of the symbols which compose them are in descending order of magnitude from left to right.

According to usual practice repetitions of biparts are denoted by power symbols; thus

$$(\overline{p_1 q_1^2}) \equiv (\overline{p_1 q_1 \, p_1 q_1}).$$

4. In the notation just explained the fundamental relation is written

$$(1 + a_1 x + \beta_1 y)(1 + a_2 x + \beta_2 y) \ldots$$
$$= 1 + (\overline{10}) x + (\overline{01}) y + (\overline{10^2}) x^2 + (\overline{10\,01}) xy + (\overline{01^2}) y^2$$
$$+ (\overline{10^3}) x^3 + (\overline{10^2\,01}) x^2 y + (\overline{10\,01^2}) xy^2 + (\overline{01^3}) y^3 + \ldots$$

where $(\overline{10\,01^2})$ denotes $\Sigma\, a_1 \beta_2 \beta_3$ and in general $a_{pq} = (\overline{10^p\,01^q})$.

Observe that here the number of quantities in each system is considered to be infinite, and that the right hand side of the equation is taken with unit and not

multinomial coefficients (*cf.* CAYLEY, *loc. cit.*). This is done because it is the universal practice in the theory of the single system, and because otherwise it appears to possess undoubted advantages.

The symmetric functions which appear in the relation are fundamental since, as will appear, they serve to express all other rational integral symmetric functions, and they may be further termed single-unitary, in that, not only is each composed entirely of units, but also each bipart comprises but a single unit.

It is obvious that the number of biweights connected with the weight w is $w + 1$.

5. It may be asked in how many ways it is possible to partition a biweight into biparts.

In the ordinary theory of partitions the number of partitions of a number w is the coefficient of x^w in the ascending expansion of

$$\frac{1}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3 \,.\, 1 - x^4 \ldots}.$$

In the present case, the number of partitions of the biweight pq into biparts is the coefficient of $x^p y^q$ in the ascending expansion of

$$\frac{1}{1 - x \,.\, 1 - y \,.\, 1 - x^2 \,.\, 1 - xy \,.\, 1 - y^2 \,.\, 1 - x^3 \,.\, 1 - x^2 y \,.\, 1 - xy^2 \,.\, 1 - y^3 \ldots}.$$

or, putting y equal to x, we see that the whole number of partitions of the weight $p + q$ into biparts is the coefficient of x^{p+q} in the ascending expansion of

$$\frac{1}{(1 - x)^2 \, (1 - x^2)^3 \, (1 - x^3)^4 \ldots}.$$

Further, it is clear that the number of partitions of the biweight pq into exactly μ biparts is the coefficient of $a^\mu x^p y^q$ in the expansion of

$$\frac{1}{1 - ax \,.\, 1 - ay \,.\, 1 - ax^2 \,.\, 1 - axy \,.\, 1 - ay^2 \,.\, 1 - ax^3 \,.\, 1 - ax^2 y \,.\, 1 - axy^2 \,.\, 1 - ay^3 \ldots}.$$

6. It is convenient now to have before us a list of the symmetric functions up to weight 4 inclusive.

The expanded generating function is

$$1 + x + y + 2x^2 + 2xy + 2y^2 + 3x^3 + 4x^2 y + 4xy^2 + 3y^3$$
$$+ 5x^4 + 7x^3 y + 9x^2 y^2 + 7xy^3 + 5y^4 + \ldots,$$

and we have

WEIGHT 1,

Biweight 10,   Biweight 01,

$(\overline{10})$; $(\overline{01})$;

WEIGHT 2,

Biweight 20,   Biweight 11,   Biweight 02,

$(\overline{20})$ $(\overline{11})$ $(\overline{02})$

$(\overline{10^2})$; $(\overline{10}\,\overline{01})$; $(\overline{01^2})$;

WEIGHT 3,

Biweight 30,   Biweight 21,   Biweight 12,   Biweight 03,

$(\overline{30})$ $(\overline{21})$ $(\overline{12})$ $(\overline{03})$

$(\overline{20}\,\overline{10})$ $(\overline{20}\,\overline{01})$ $(\overline{10}\,\overline{02})$ $(\overline{01}\,\overline{02})$

$(\overline{10^3})$; $(\overline{11}\,\overline{10})$ $(\overline{01}\,\overline{11})$ $(\overline{01^3})$;

$(\overline{10^2}\,\overline{01})$; $(\overline{10}\,\overline{01^2})$;

WEIGHT 4,

Biweight 40,   Biweight 31,   Biweight 22,   Biweight 13,   Biweight 04,

$(\overline{40})$ $(\overline{31})$ $(\overline{22})$ $(\overline{13})$ $(\overline{04})$

$(\overline{30}\,\overline{10})$ $(\overline{21}\,\overline{10})$ $(\overline{21}\,\overline{01})$ $(\overline{12}\,\overline{01})$ $(\overline{03}\,\overline{01})$

$(\overline{20^2})$ $(\overline{30}\,\overline{01})$ $(\overline{12}\,\overline{10})$ $(\overline{03}\,\overline{10})$ $(\overline{02^2})$

$(\overline{20}\,\overline{10^2})$ $(\overline{20}\,\overline{11})$ $(\overline{20}\,\overline{02})$ $(\overline{02}\,\overline{11})$ $(\overline{02}\,\overline{01^2})$

$(\overline{10^4})$; $(\overline{20}\,\overline{10}\,\overline{01})$ $(\overline{11^2})$ $(\overline{02}\,\overline{10}\,\overline{01})$ $(\overline{01^4})$.

$(\overline{11}\,\overline{10^2})$ $(\overline{20}\,\overline{01^2})$ $(\overline{11}\,\overline{01^2})$

$(\overline{10^3}\,\overline{01})$; $(\overline{02}\,\overline{10^2})$ $(\overline{10}\,\overline{01^3})$;

$(\overline{11}\,\overline{10}\,\overline{01})$

$(\overline{10^2}\,\overline{01^2})$;

§ 2. *Preliminary Algebraic Theory.*

7. The partitions with one bipart correspond to the sums of the powers in the single system, or unipartite theory. They are easily expressed in terms of the fundamental symmetric functions.

The right hand side of the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots,$$

may be written

$$\overline{\exp}\,(a_{10}x + a_{01}y),$$

or, since it is convenient to write the symmetric function $(\overline{pq})$ in the form s_{pq}, this is

$$\overline{\exp}\,(s_{10}x + s_{01}y),$$

where the bar over exp indicates a symbolism by which $\frac{s_{10}{}^p s_{01}{}^q}{p!\,q!}$ denotes $(\overline{10^p}\,\overline{01^q}) \equiv a_{pq}$.

Hence the relation

$$1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots = \overline{\exp}\,(s_{10}x + s_{01}y),$$

which is important in connection with the collateral theory of operations to be presently brought into view.

8. Taking logarithms of both sides of the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots,$$

there results

$$s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2) + \tfrac{1}{3}(s_{30}x^3 + 3s_{21}x^2 y + 3s_{12}xy^2 + s_{03}y^3) - \ldots *$$
$$= \log(1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots).$$

Hence

$$1 + a_{10}x + a_{01}y + \ldots a_{pq}x^p y^q + \ldots = \overline{\exp}\,(s_{10}x + s_{01}y)$$
$$= \exp\{s_{10}x + s_{01}y - \tfrac{1}{2}(s_{20}x^2 + 2s_{11}xy + s_{02}y^2)$$
$$+ \tfrac{1}{3}(s_{30}x^3 + 3s_{21}x^2 y + 3s_{12}xy^2 + s_{03}y^3) - \ldots\}.$$

Also we have the series of relations :—

$$\begin{cases} s_{10} = a_{10}, \\ s_{01} = a_{01}, \end{cases}$$

$$\begin{cases} s_{20} = a_{10}{}^2 - 2a_{20}, \\ s_{11} = a_{10}a_{01} - a_{11}, \\ s_{02} = a_{01}{}^2 - 2a_{02}, \end{cases}$$

$$\begin{cases} s_{30} = a_{10}{}^3 - 3a_{20}a_{10} + 3a_{30}, \\ s_{21} = a_{10}{}^2 a_{01} - a_{20}a_{01} - a_{11}a_{10} + a_{21}, \\ s_{12} = a_{01}{}^2 a_{10} - a_{02}a_{10} - a_{11}a_{01} + a_{12}, \\ s_{03} = a_{01}{}^3 - 3a_{02}a_{01} + 3a_{03}, \end{cases}$$

$$\vdots$$

---

* Viz :– $s_{10}x + s_{01}y = \Sigma(\alpha_1 x + \beta_1 y)$; $s_{20}x^2 + 2s_{11}xy + s_{02}y^2 = \Sigma(\alpha_1 x + \beta_1 y)^2$; &c.

and in general

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi-1} \frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots} a_{p_1q_1}^{\pi_1} a_{p_2q_2}^{\pi_2}\dots$$

9. Moreover, the fundamental symmetric functions are expressed in the terms of the forms s_{pq} by the formula

$$(-)^{p+q-1} a_{pq} = \Sigma \left\{ \frac{(p_1+q_1-1)!}{p_1!\,q_1!} \right\}^{\pi_1} \left\{ \frac{(p_2+q_2-1)!}{p_2!\,q_2!\dots} \right\}^{\pi_2} \dots \frac{(-)^{\Sigma\pi-1}}{\pi_1!\,\pi_2!\dots} s_{p_1q_1}^{\pi_1} s_{p_2q_2}^{\pi_2}\dots,$$

as will be evident by simply applying the multinomial theorem to one of the above written general identities of Art. 8.

10. The single-bipart functions having been actually expressed in terms of the fundamental symmetric functions, it remains to show that all other rational algebraic symmetric functions are also so expressible. SCHLÄFLI (*loc. cit.*) has established this by induction, and it is not necessary to further discuss the theorem here. In Art. 43 of the present memoir, will be found the actual expression of a given symmetric function by means of single-bipart forms, a formula which, combined with one given above, Art. 8, serves to establish the theorem conclusively.

The Symmetric Function h_{pq}.

11. Write

$$(1 + a_1 x + \beta_1 y)(1 + a_2 x + \beta_2 y)\dots = 1 + a_{10}x + a_{01}y + \dots + a_{pq}x^p y^q + \dots$$

$$= \frac{1}{1 - h_{10}x - h_{01}y + \dots + (-)^{p+q} h_{pq}x^p y^q + \dots},$$

as the definition of the function h_{pq}.

Writing $-x, -y$ for x, y, we have

$$1 + h_{10}x + h_{01}y + \dots + h_{pq}x^p y^q = \frac{1}{(1 - a_1 x - \beta_1 y)(1 - a_2 x - \beta_2 y)\dots},$$

and expanding the right hand in ascending powers of x and y

$$h_{pq} = \Sigma \frac{(p_1+q_1)!}{p_1!\,q_1!} \frac{(p_2+q_2)!}{p_2!\,q_2!} \dots (\overline{p_1 q_1} \; \overline{p_2 q_2} \dots),$$

the summation being for all partitions of the biweight. Changing the signs of x and y in the relation first written down, we obtain

$$1 + h_{10}x + h_{01}y + \dots + h_{pq}x^p y^q + \dots = \frac{1}{1 - a_{10}x - a_{01}y + \dots + (-)^{p+q} a_{pq}x^p y^q + \dots},$$

an identity which arises from the former by interchanging the letter h with the letter a.

Hence, if f and ϕ be any two functions, such that

$$f\left(a_{10}, a_{01}, \ldots a_{pq}, \ldots\right) = \phi\left(h_{10}, h_{01}, \ldots h_{pq}, \ldots\right),$$

then also

$$\phi\left(a_{10}, a_{01}, \ldots a_{pq}, \ldots\right) = f\left(h_{10}, h_{01}, \ldots h_{pq}, \ldots\right);$$

and, in general, in any relation connecting the functions a with the functions h, an identity will still remain if the letters a and h be transposed.

By the multinomial theorem

$$(-)^{p+q-1} h_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi)!}{\pi_1! \, \pi_2! \ldots} a_{p_1 q_1}^{\pi_1} a_{p_2 q_2}^{\pi_2} \ldots$$

From a previous result in this article, by taking logarithms and expanding

$$\frac{(p+q-1)!}{p! \, q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1! \, \pi_2! \ldots} h_{p_1 q_1}^{\pi_1} h_{p_2 q_2}^{\pi_2} \ldots,$$

which is to be compared with the formula

$$(-)^{p+q-1} \frac{(p+q-1)!}{p! \, q!} s_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1! \, \pi_2! \ldots} a_{p_1 q_1}^{\pi_1} a_{p_2 q_2}^{\pi_2} \ldots;$$

and it will be noticed that s_{pq} remains unchanged when h is written for a, except for a change of sign, when the weight $p + q$ is even.

§ 3. The Differential Operations.

12. The beautiful properties of these symmetric functions are most easily established by means of the differential operations whose theory I proceed to establish.

Consider the identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_n x + \beta_n y)$$
$$= 1 + a_{10} x + a_{01} y + a_{20} x^2 + a_{11} xy + a_{02} y^2 + \ldots,$$

where n may be as large as we please.

Multiply each side by $(1 + \mu x + \nu y)$.

The right hand side becomes

$$1 + (a_{10} + \mu)\, x + (a_{01} + \nu)\, y + (a_{20} + \mu a_{10})\, x^2 + (a_{11} + \mu a_{01} + \nu a_{10})\, xy$$
$$+ (a_{02} + \nu a_{01})\, y^2 + \ldots,$$

and, in general, a_{pq} becomes converted into

$$a_{pq} + \mu a_{p-1, q} + \nu a_{p, q-1}.$$

Hence any rational integral function of the coefficients

$$a_{10}, \ a_{01}, \ a_{20}, \ a_{11}, \ a_{02}, \ \ldots \ ,$$

viz.,

$$f(a_{10}, \ a_{01}, \ a_{20}, \ a_{11}, \ a_{02}, \ \ldots) \equiv f,$$

is converted into

$$f + (\mu g_{10} + \nu g_{01})f + \frac{1}{2!}(\mu g_{10} + \nu g_{01})^2 f + \frac{1}{3!}(\mu g_{10} + \nu g_{01})^3 f + \ldots ,$$

where

$$g_{10} = \Sigma a_{p-1, q}\, \partial_{a_{pq}}; \quad g_{01} = \Sigma a_{p, q-1}\, \partial_{a_{pq}};$$

and the multiplication of operators is symbolic.*

The new value of f is

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01})\,f,$$

where the bar is placed over exp to denote that the multiplication of operators is symbolic (*vide* Art. 7).

13. Write

$$G_{pq} = \frac{1}{p!\, q!}\,\overline{g_{10}{}^p\, g_{01}{}^q},$$

the bar denoting symbolic multiplication, then

$$\overline{\exp}\,(\mu g_{10} + \nu g_{01})f$$
$$= (1 + \mu G_{10} + \nu G_{01} + \mu^2 G_{20} + \mu\nu G_{11} + \nu^2 G_{02} + \ldots + \mu^p \nu^q G_{pq} + \ldots)f.$$

(Compare Art. 7.)

Now suppose the symmetric function f expressed in terms of

$$\alpha_1, \ \beta_1, \ \alpha_2, \ \beta_2, \ \alpha_3, \ \beta_3, \ \ldots \ \alpha_n, \ \beta_n$$

to be

$$(\,\overline{p_1 q_1}\ \overline{p_2 q_2}\ \overline{p_3 q_3}\, \ldots).$$

The introduction of the new quantities μ, ν results in the addition to

$$(\,\overline{p_1 q_1}\ \overline{p_2 q_2}\ \overline{p_3 q_3}\, \ldots)$$

of the terms

$$\mu^{p_1}\nu^{q_1}(\,\overline{p_2 q_2}\ \overline{p_3 q_3}\, \ldots) + \mu^{p_2}\nu^{q_2}(\,\overline{p_1 q_1}\ \overline{p_3 q_3}\, \ldots) + \mu^{p_3}\nu^{q_3}(\,\overline{p_1 q_1}\ \overline{p_2 q_2}\, \ldots) + \ldots ;$$

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* By "symbolic" is to be understood "non-operational," as in what is commonly known as the "symbolic" form of TAYLOR'S Theorem.

and hence

$$f + \mu^{p_1}\nu^{q_1}\left(\overline{p_2 q_2}\ \overline{p_3 q_3}\ldots\right) + \mu^{p_2}\nu^{q_2}\left(\overline{p_1 q_1}\ \overline{p_3 q_3}\ldots\right) + \mu^{p_3}\nu^{q_3}\left(\overline{p_1 q_1}\ \overline{p_2 q_2}\ldots\right) + \ldots$$
$$= \left(1 + \mu \mathrm{G}_{10} + \nu \mathrm{G}_{01} + \mu^2 \mathrm{G}_{20} + \mu\nu \mathrm{G}_{11} + \nu^2 \mathrm{G}_{02} + \ldots\right)f\,;$$

and equating coefficients of like products $\mu^p \nu^q$, we find

$$\mathrm{G}_{p_1 q_1}\left(\overline{p_1 q_1}\ \overline{p_2 q_2}\ \overline{p_3 q_3}\ldots\right) = \left(\overline{p_2 q_2}\ \overline{p_3 q_3}\ldots\right),$$

$$\mathrm{G}_{p_2 q_2}\left(\overline{p_1 q_1}\ \overline{p_2 q_2}\ \overline{p_3 q_3}\ldots\right) = \left(\overline{p_1 q_1}\ \overline{p_3 q_3}\ldots\right),$$

$$\mathrm{G}_{p_3 q_3}\left(\overline{p_1 q_1}\ \overline{p_2 q_2}\ \overline{p_3 q_3}\ldots\right) = \left(\overline{p_1 q_1}\ \overline{p_2 q_2}\ldots\right),$$

$$\mathrm{G}_{p_1 q_1}\left(\overline{p_1 q_1}\right) = 1,$$

$$\mathrm{G}_{p_1 q_1}\mathrm{G}_{p_2 q_2}\ldots \mathrm{G}_{p_n q_n}\left(\overline{p_1 q_1}\ \overline{p_2 q_2}\ldots \overline{p_n q_n}\right) = 1,$$

and $\mathrm{G}_{rs}f = 0$, unless the bipart $\overline{rs}$ is involved in the expression of f.

From the above we gather the very important fact that the effect of the operation G_{pq} upon a partition is to obliterate one bipart $\overline{pq}$ when such bipart is present, and to annihilate the partition if it contains no bipart $\overline{pq}$.

14. I return to the result

$$1 + \mu\mathrm{G}_{10} + \nu\mathrm{G}_{01} + \ldots + \mu^p\nu^q\mathrm{G}_{pq} + \ldots = \overline{\exp}\left(\mu g_{10} + \nu g_{01}\right),$$

wherein be it remembered the multiplication of operators in the right hand expression is symbolic. I seek to replace $\overline{\exp}\left(\mu g_{10} + \nu g_{01}\right)$ by an expression containing products of linear partial differential operations in which the multiplication is not symbolic.

We have by definition

$$g_{10} = \partial_{a_{10}} + a_{10}\,\partial_{a_{20}} + a_{01}\,\partial_{a_{11}} + \ldots,$$
$$g_{01} = \partial_{a_{01}} + a_{01}\,\partial_{a_{02}} + a_{10}\,\partial_{a_{11}} + \ldots;$$

let further

$$g_{pq} = \partial_{a_{pq}} + a_{10}\,\partial_{a_{p+1,q}} + a_{01}\,\partial_{a_{p,q+1}} + \ldots + a_{rs}\,\partial_{a_{p+r,q+s}} + \ldots,$$

a definition which includes the former.

15. I will establish the relation

$$\overline{\exp}\left(m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots\right)$$
$$= \exp\left(\mathrm{M}_{10}g_{10} + \mathrm{M}_{01}g_{01} + \ldots + \mathrm{M}_{pq}g_{pq} + \ldots\right),$$

where on the left and right hand sides the multiplications of operators are respectively symbolic and not symbolic, and

$$\exp\left(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots\right)$$
$$= 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots,$$

where ξ, η are the undetermined algebraic quantities.

For the multiplication of two operators $g_{p_1q_1}$, $g_{p_2q_2}$, we have the formula

$$g_{p_1q_1}\,g_{p_2q_2} = \overline{g_{p_1q_1}\,g_{p_2q_2}} + g_{p_1q_1}\,\underset{|}{+}\,g_{p_2q_2},$$

wherein the symbol $\underset{|}{+}$ denotes explicit operation upon the operand, regarding the latter as a function of symbols of quantity only, and not of the differential inverses.

Also

$$g_{p_1q_1}\,\underset{|}{+}\,g_{p_2q_2} = g_{p_1+p_2,\,q_1+q_2}.$$

Put

$$u_1 = m_{10}g_{10} + m_{01}g_{01} + \ldots + m_{pq}g_{pq} + \ldots,$$

which may be written

$$u_1 = \left(m_{10} + m_{01} + \ldots + m_{pq} + \ldots\right)g,$$

in which $m_{pq}g_{pq}$ is symbolically written $m_{pq}g$.

But

$$u_2 = u_1\,\underset{|}{+}\,u_1 = \left(m_{10} + m_{01} + \ldots + m_{pq} + \ldots\right)^2 g,$$

where, after expansion of the right hand side, $m_{p_1q_1}m_{p_2q_2}g$ is to be written

$$m_{p_1q_1}m_{p_2q_2}g_{p_1+p_2,\,q_1+q_2}.$$

Then, with a similar convention,

$$u_s = u_1\,\underset{|}{+}\,u_{s-1} = \left(m_{10} + m_{01} + \ldots + m_{pq} + \ldots\right)^s g.$$

Further, it is easy to prove the relation

$$u_s\,\underset{|}{+}\,u_t = u_{s+t}.$$

But for a series of linear partial differential operators enjoying this property, it is a well known and easily established theorem of SYLVESTER's that

$$\overline{\exp}\,u_1 = \exp\left(u_1 - \tfrac{1}{2}u_2 + \tfrac{1}{3}u_3 - \ldots\right).$$

Hence, substituting, we easily reach the relation

$$\exp\left(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots\right) = 1 + m_{10}\xi + m_{01}\eta + \ldots + m_{pq}\xi^p\eta^q + \ldots,$$

wherein ξ and η are undetermined algebraic quantities.

This establishes the theorem.

16. To apply it to the case in hand, put

$$m_{10} = \mu, \qquad m_{01} = \nu, \qquad m_{pq} = 0 \text{ in other cases.}$$

Then

$$M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots = \log(1 + \mu\xi + \nu\eta)$$

$$\therefore \quad M_{pq} = (-)^{p+q+1} \frac{(p+q-1)!}{p!\,q!} \mu^p\nu^q,$$

and the result is

$$\overline{\exp}(\mu g_{10} + \nu g_{01}) = \exp\{\mu g_{10} + \nu g_{01} - \tfrac{1}{2}(\mu^2 g_{20} + 2\mu\nu g_{11} + \nu^2 g_{02})$$
$$+ \tfrac{1}{3}(\mu^3 g_{30} + 3\mu^2\nu g_{21} + 3\mu\nu^2 g_{12} + \nu^3 g_{03}) - \ldots\}.$$

Combining with a former result

$$\mu g_{10} + \nu g_{01} - \tfrac{1}{2}(\mu^2 g_{20} + 2\mu\nu g_{11} + \nu^2 g_{02}) + \tfrac{1}{3}(\mu^3 g_{30} + 3\mu^2\nu g_{21} + 3\mu\nu^2 g_{12}\, \nu^3 g_{03}) - \ldots$$
$$= \log(1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p\nu^q G_{pq} + \ldots).$$

(Compare Art. 8.)

17. By expanding the right hand side we can express each linear operator of the form g_{pq} in terms of products of the obliterating operators which have the form G_{pq}.

The law is identical with that which expresses the functions containing one bipart in terms of the fundamental symmetric functions.   We find

$$\begin{cases} g_{10} = G_{10} \\ g_{01} = G_{01} \end{cases}$$

$$\begin{cases} g_{20} = G_{10}{}^2 - 2G_{20} \\ g_{11} = G_{10}G_{01} - G_{11} \\ g_{02} = G_{01}{}^2 - 2G_{02} \end{cases}$$

$$\begin{cases} g_{30} = G_{10}{}^3 - 3G_{20}G_{10} + 3G_{30} \\ g_{21} = G_{10}{}^2 G_{01} - G_{20}G_{01} - G_{11}G_{10} + G_{21} \\ g_{12} = G_{01}{}^2 G_{10} - G_{02}G_{10} - G_{11}G_{01} + G_{12} \\ g_{03} = G_{01}{}^3 - 3G_{02}G_{01} + 3G_{03} \end{cases}$$
$$\vdots$$

and in general

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} G_{p_1q_1}^{\pi_1}\, G_{p_2q_2}^{\pi_2}\, \ldots$$

while

$$(-)^{p+q-1} G_{pq} = \Sigma_\pi \left\{\frac{(p_1 + q_1 - 1)!}{p_1!\,q_1!}\right\}^{\pi_1} \left\{\frac{(p_2 + q_2 - 1)!}{p_2!\,q_2!}\right\}^{\pi_2} \ldots \frac{(-)^{\Sigma\pi - 1}}{\pi_1!\,\pi_2!\ldots} g_{p_1q_1}^{\pi_1}\, g_{p_2q_2}^{\pi_2}\, \ldots$$

(Compare Art. 8.)

18. By comparison of these relations with the corresponding algebraic ones to which reference has been made, it is manifest that g_{pq} and G_{pq} are respectively in correlation with s_{pq} and a_{pq}. In other words these operations respectively correspond to the partitions $(\overline{pq})$ and $(\overline{10^p \ 01^q})$. It is necessary to find the operations which correspond to the remaining partitions which symbolize symmetric functions.

We have the easily derivable results in operations

$$g_{p_1q_1} g_{p_2q_2} = \overline{g_{p_1q_1} g_{p_2q_2}} + g_{p_1+p_2,\, q_1+q_2},$$

$$g_{p_1q_2}{}^2 = \overline{g_{p_1q_2}{}^2} + g_{2p_1,\, 2q_1},$$

$$g_{p_1q_1} g_{p_2q_2} g_{p_3q_3} = \overline{g_{p_1q_1} g_{p_2q_2} g_{p_3q_3}} + \overline{g_{p_1q_1} g_{p_2+p_3,\, q_2+q_3}} + \overline{g_{p_2q_2} g_{p_3+p_1,\, q_3+q_1}}$$
$$+ \overline{g_{p_3q_3} g_{p_1+p_2,\, q_1+q_2}} + g_{p_1+p_2+p_3,\, q_1+q_2+q_3},$$

$$g_{p_1q_1}{}^2 g_{p_2q_2} = \overline{g_{p_1q_1}{}^2 g_{p_2q_2}} + 2\overline{g_{p_1q_1} g_{p_1+p_2,\, q_1+q_2}} + \overline{g_{2p_1,\, 2q_1} g_{p_2q_2}} + g_{2p_1+p_2,\, 2q_1+q_2},$$

$$g_{p_1q_1}{}^3 = \overline{g_{p_1q_1}{}^3} + 3\overline{g_{2p_1,\, 2q_1} g_{p_1q_1}} + g_{3p_1,\, 3q_1};$$

where as usual the bar denotes symbolic multiplication; and comparing these with the algebraic formulæ

$$(\overline{p_1q_1})\,(\overline{p_2q_2}) = (\overline{p_1q_1\ p_2q_2}) + (\overline{p_1+p_2,\, q_1+q_2})$$

$$(\overline{p_1q_1})^2 = 2\,(\overline{p_1q_1{}^2}) + (\overline{2p_1,\, 2q_1})$$

$$(\overline{p_1q_1})\,(\overline{p_2q_2})\,(\overline{p_3q_3}) = (\overline{p_1q_1\ p_2q_2\ p_3q_3}) + (\overline{p_1q_1\ p_2+p_3,\, q_2+q_3}) + (\overline{p_2q_2\ p_3+p_1,\, q_3+q_1})$$
$$+ (\overline{p_3q_3\ p_1+p_2,\, q_1+q_2}) + (\overline{p_1+p_2+p_3,\, q_1+q_2+q_3})$$

$$(\overline{p_1q_1})^2\,(\overline{p_2q_2}) = 2\,(\overline{p_1q_1{}^2\ p_2q_2}) + 2\,(\overline{p_1q_1\ p_1+p_2,\, q_1+q_2}) + (\overline{2p_1,\, 2q_1\ p_2q_2})$$
$$+ (\overline{2p_1+p_2,\, 2q_1+q_2})$$

$$(\overline{p_1q_1})^3 = 6\,(\overline{p_1q_1{}^3}) + 3\,(\overline{2p_1,\, 2q_1\ p_1q_1}) + (\overline{3p_1,\, 3q_1})$$

it is evident that the operations

$$\frac{\overline{g_{p_1q_1}{}^2}}{2!}, \qquad \frac{\overline{g_{p_1q_1}{}^2 g_{p_2q_2}}}{2!}, \qquad \frac{\overline{g_{p_1q_1}{}^3}}{3!},$$

are produced according to the same *law* as the symmetric functions

$$(\overline{p_1q_1{}^2}), \qquad (\overline{p_1q_1{}^2 p_2q_2}), \qquad (\overline{p_1q_1{}^3});$$

further the *law* is perfectly general and indicates that the operation

$$\frac{1}{\pi_1!}\,\frac{1}{\pi_2!}\cdots \overline{g_{p_1q_1}{}^{\pi_1} g_{p_2q_2}{}^{\pi_2}}\cdots,$$

is in co-relation with the symmetric function

$$\left(\overline{p_1 q_1}^{\pi_1} \ \overline{p_2 q_2}^{\pi_2} \dots\right).$$

19. There is thus complete correspondence between quantity and operation, and any formula of quantity may be at once translated into a formula of operation.

Observe that a product of symmetric functions

$$\left(\overline{p_1 q_1}^{\pi_1} \ \overline{p_2 q_2}^{\pi_2} \dots\right) \left(\overline{r_1 s_1}^{\rho_1} \ \overline{r_2 s_2}^{\rho_2} \dots\right)$$

is in correspondence with the operation

$$\frac{1}{\pi_1!} \ \frac{1}{\pi_2!} \dots \overline{g_{p_1 q_1}^{\pi_1} g_{p_2 q_2}^{\pi_2} \dots} \ \frac{1}{\rho_1!} \ \frac{1}{\rho_2!} \dots \overline{g_{r_1 s_1}^{\rho_1} g_{r_2 s_2}^{\rho_2} \dots}$$

the notation indicating that the two operations

$$\frac{1}{\rho_1!} \ \frac{1}{\rho_2!} \dots \overline{g_{r_1 s_1}^{\rho_1} g_{r_2 s_2}^{\rho_2} \dots} \quad \text{and} \quad \frac{1}{\pi_1!} \ \frac{1}{\pi_2!} \dots \overline{g_{p_1 q_1}^{\pi_1} g_{p_2 q_2}^{\pi_2} \dots}$$

are to be successively performed.

For an example take the algebraic formula

$$(\overline{31} \ \overline{01}) = -\tfrac{1}{2}(\overline{21} \ \overline{10})(\overline{01}) + \tfrac{1}{2}(\overline{21} \ \overline{01})(\overline{10}) + \tfrac{1}{2}(\overline{10} \ \overline{01})(\overline{21}) - \tfrac{1}{2}(\overline{21} \ \overline{10} \ \overline{01}),$$

which is translated into the operator formula

$$\overline{g_{31} g_{01}} = -\tfrac{1}{2} \overline{g_{21} g_{10}} \cdot g_{01} + \tfrac{1}{2} \overline{g_{21} g_{01}} \cdot g_{10} + \tfrac{1}{2} \overline{g_{10} g_{01}} \cdot g_{21} - \tfrac{1}{2} \overline{g_{21} g_{10} g_{01}}.$$

20. It is now necessary to enquire into the laws which appertain to the performance of these operations upon symmetric functions. We have seen, *ante* Art. 13, the law by which the obliterator G_{pq} is performed upon a monomial symmetric function.

Since

$$(-)^{p+q-1} \frac{(p+q-1)!}{p! \, q!} g_{pq} = \Sigma_\pi (-)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1! \, \pi_2!} G_{p_1 q_1}^{\pi_1} G_{p_2 q_2}^{\pi_2} \dots,$$

we can operate with g_{pq} upon a monomial form by operating independently with the successive G products on the right, and adding the results together. As a particular result, observe that a term on the right is G_{pq}, and hence

$$(-)^{p+q-1} \frac{(p+q-1)!}{p! \, q!} g_{pq} s_{pq} = G_{pq} s_{pq} = G_{pq} (\overline{pq}) = 1,$$

or

$$g_{pq} s_{pq} = (-)^{p+q-1} \frac{p! \, q!}{(p+q-1)!};$$

and g_{pq} causes every other single bipart function to vanish; it must, indeed, cause any monomial function to vanish which does not comprise one of the partitions of the biweight pq amongst its biparts.

21. The relation just obtained yields the equivalence

$$g_{pq} = (-)^{p+q-1} \frac{p! \, q!}{(p+q-1)!} \, \partial_{s_{pq}},$$

and further results of the nature

$$\overline{g_{p_1 q_1} g_{p_2 q_2}} = (-)^{p_1 + p_2 + q_1 + q_2} \left\{ \frac{p_1! \, q_1! \, p_2! \, q_2!}{(p_1 + q_1 - 1)! \, (p_2 + q_2 - 1)!} \, \partial_{s_{p_1 q_1}} \partial_{s_{p_2 q_2}} \right.$$
$$\left. + \frac{(p_1 + p_2)! \, (q_1 + q_2)!}{(p_1 + p_2 + q_1 + q_2 - 1)!} \, \partial_{s_{p_1 + p_2, \, q_1 + q_2}} \right\},$$

which are of use in connexion with the theory of function with single biparts.

Since every symmetric function is expressible in terms of the fundamental symmetric functions, every operation $\overline{g_{p_1 q_1}{}^{\pi_1} \, g_{p_2 q_2}{}^{\pi_2}}$ is necessarily expressible as a sum of G products and can be performed upon a monomial symmetric function.

22. The solutions of the partial differential equation

$$g_{pq} = 0,$$

are the single bipart forms omitting s_{pq} (Art. 21), while the solution of the partial differential equation

$$G_{pq} = 0,$$

are those monomial symmetric functions in which the bipart $\overline{pq}$ is absent (Art. 13).

23. The operation $\partial_{a_{pq}}$ is expressible by means of the operations g_{pq}. Reversing the formula

$$g_{pq} = \partial_{a_{pq}} + a_{10} \partial_{a_{p+1, q}} + a_{01} \partial_{a_{p, q+1}} + \ldots + a_{rs} \partial_{a_{p+r, q+s}} + \ldots,$$

we obtain

$$\partial_{a_{pq}} = g_{pq} - h_{10} g_{p+1, q} - h_{01} g_{p, q+1} + \ldots + (-)^{r+s} h_{rs} g_{p+r, q+s} + \ldots,$$

where as before (Art. 11),

$$(-)^{r+s-1} h_{rs} = \Sigma_\rho (-)^{\Sigma \rho - 1} \frac{(\Sigma \rho)!}{\rho_1! \, \rho_2!} a_{r_1 s_1}{}^{\rho_1} \, a_{r_2 s_2}{}^{\rho_2} \, \ldots \ldots$$

§ 4. *The Theory of Three Identities.*

24. The course of the investigation at this point necessitates the introduction of two identities similar to, and in addition to, the fundamental identity.

Let

$$1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots = (1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots \quad . \quad . \quad \text{(I.)}$$

$$1 + b_{10}x + b_{01}y + \ldots + b_{pq}x^p y^q + \ldots = (1 + \alpha_1^{(1)}x + \beta_1^{(1)}y)(1 + \alpha_2^{(1)}x + \beta_2^{(1)}y) \ldots \text{ (II.)}$$

$$1 + c_{10}x + c_{01}y + \ldots + c_{pq}x^p y^q + \ldots = (1 + \alpha_1^{(2)}x + \beta_1^{(2)}y)(1 + \alpha_2^{(2)}x + \beta_2^{(2)}y) \ldots \text{ (III.)}$$

wherein x and y may be regarded as undetermined quantities and the identities as merely expressing the relations between the coefficients on the left and quantities α, β on the right.

Assume the coefficients and quantities in the first two identities to be given and the coefficients in the third identity to be then determined by the relations :—

$$1 + c_{01}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p \eta^q + \ldots$$
$$= \Pi_s (1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s^p \beta_s^q b_{pq}\xi^p \eta^q + \ldots),$$

ξ and η being undetermined quantities.

Multiplying out the right-hand side of this relation, it is found to be equivalent to the series of relations :—

$$c_{10} = (\overline{10})\, b_{10},$$

$$c_{01} = (\overline{01})\, b_{01},$$

$$c_{20} = (\overline{20})\, b_{20} + (\overline{10^2})\, b_{10}^2,$$

$$c_{11} = (\overline{11})\, b_{11} + (\overline{10}\ \overline{01})\, b_{10}b_{01},$$

$$c_{02} = (\overline{02})\, b_{02} + (\overline{01^2})\, b_{01}^2,$$

$$c_{30} = (\overline{30})\, b_{30} + (\overline{20}\ \overline{10})\, b_{20}b_{10} + (\overline{10^3})\, b_{10}^3,$$

$$c_{21} = (\overline{21})\, b_{21} + (\overline{20}\ \overline{01})\, b_{20}b_{01} + (\overline{11}\ \overline{10})\, b_{11}b_{10} + (\overline{10^2}\ \overline{01})\, b_{10}^2 b_{01},$$

$$c_{12} = (\overline{12})\, b_{12} + (\overline{02}\ \overline{10})\, b_{02}b_{10} + (\overline{11}\ \overline{01})\, b_{11}b_{01} + (\overline{10}\ \overline{01^2})\, b_{10}b_{01}^2,$$

$$c_{03} = (\overline{03})\, b_{03} + (\overline{02}\ \overline{01})\, b_{02}b_{01} + (\overline{01^3})\, b_{01}^3;$$

and generally in the expression of c_{pq} every symmetric function of biweight pq of the quantities in the first identity occurs, each attached to the corresponding product of coefficients from the second identity.

25. Represent the symmetric functions of the quantities occurring in the second and third identities by partitions in brackets $(\quad)_1$, $(\quad)_2$, respectively.

Now

$$\Pi_s\left(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s{}^p\beta_s{}^q b_{pq}\xi^p\eta^q + \ldots\right)$$

is from the identity II. equal to

$$\Pi_s\left[\left(1 + \alpha_s\alpha_1{}^{(1)}\xi + \beta_s\beta_1{}^{(1)}\eta\right)\left(1 + \alpha_s\alpha_2{}^{(1)}\xi + \beta_s\beta_2{}^{(1)}\eta\right)(\ldots)\ldots\right],$$

which is

$$\Pi_s\Pi_t\left(1 + \alpha_s\alpha_t{}^{(1)}\xi + \beta_s\beta_t{}^{(1)}\eta\right).$$

26. Hence the assumed relation becomes on taking logarithms

$$\Sigma_s \log\left(1 + \alpha_s{}^{(2)}\xi + \beta_s{}^{(2)}\eta\right) = \Sigma_s\Sigma_t \log\left(1 + \alpha_s\alpha_t{}^{(1)}\xi + \beta_s\beta_t{}^{(1)}\eta\right),$$

and expanding and equating coefficients of $\xi^p\eta^q$

$$(\overline{pq})_2 = (\overline{pq})(\overline{pq})_1;$$

an important relation which shows that the assumed relation is unaltered when the set of quantities α is interchanged with the set $\alpha^{(1)}$, in such wise that α_s and $\alpha_s{}^{(1)}$ are transposed. It is indeed of fundamental importance, and will be brought prominently forward in the sequel. Its consideration must be postponed until a further step has been taken in the theory of the operators.

27. Let the operators

$$g_{pq}, \quad G_{pq},$$
$$g_{pq}', \quad G_{pq}',$$
$$g_{pq}'', \quad G_{pq}'',$$

refer to identities I., II., III., respectively.

Writing the relation

$$1 + c_{10}\xi + c_{01}\eta + \ldots + c_{pq}\xi^p\eta^q + \ldots$$
$$= \Pi_s\left(1 + \alpha_s b_{10}\xi + \beta_s b_{01}\eta + \ldots + \alpha_s{}^p\beta_s{}^q b_{pq}\xi^p\eta^q + \ldots\right),$$

in the abbreviated form

$$U = u_{\alpha_1\beta_1}u_{\alpha_2\beta_2}u_{\alpha_3\beta_3}\ldots,$$

and performing the operation

$$g_{pq}{}' = \partial_{b_{pq}} + b_{10}\,\partial_{b_{p+1,q}} + b_{01}\,\partial_{b_{p,q+1}} + \;..+ b_{rs}\,\partial_{b_{p+r,q+s}} + \cdots$$

we have

$$g_{pq}{}'\mathrm{U} = (g_{pq}{}'u_{a_1\beta_1})\,u_{a_2\beta_2}u_{a_3\beta_3}\cdots + u_{a_1\beta_1}(g_{pq}{}'u_{a_2\beta_2})\,u_{a_3\beta_3} + \cdots$$

Moreover,

$$g_{pq}{}'u_{a_s\beta_s} = \alpha_s{}^p\beta_s{}^q\xi^p\eta^q u_{a_s\beta_s}\,;$$

hence

$$g_{pq}{}'\mathrm{U} = (\overline{pq})\xi^p\eta^q\mathrm{U},$$

and replacing U by its value, we have

$$g_{pq}{}'c_{pq} = (\overline{pq})\,;$$

while in general

$$g_{pq}{}'c_{rs} = (\overline{pq})\,c_{r-p,s-q}.$$

Now regarding the coefficients b_{pq} as functions of the coefficients c_{pq} only, we have

$$g_{pq}{}' = (g_{pq}{}'c_{pq})\,\partial_{c_{pq}} + \cdots + (g_{pq}{}'c_{rs})\,\partial_{c_{rs}} + \cdots$$
$$= (\overline{pq})\,(\partial_{c_{pq}} + c_{10}\,\partial_{c_{p+1,q}} + c_{01}\,\partial_{c_{p,q+1}} + \cdots + c_{r-p,s-q}\,\partial_{c_{rs}} + \cdots)$$

Thus

$$g_{pq}{}' = (\overline{pq})\,g_{pq}{}''.$$

But the assumed relation is symmetrical as regards the quantities in the first two identities; hence also

$$g_{pq} = (\overline{pq})_1\,g_{pq}{}'',$$

and thence, since $(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1$, we have

$$(\overline{pq})_2\,g_{pq}{}'' = (\overline{pq})_1\,g_{pq}{}' = (\overline{pq})\,g_{pq}.$$

If we then regard the assumed relation as defining a transformation of the quantities occurring in the identity III. into either of the sets of quantities associated with I. or II., the operation

$$(\overline{pq})_2\,g_{pq}{}''$$

is an invariant.

28. Since $g_{pq}{}' = (\overline{pq})\,g_{pq}{}''$

$$\xi g_{10}{}' + \eta g_{01}{}' - \tfrac{1}{2}(\xi^2 g_{20}{}' + 2\xi\eta g_{11}{}' + \eta^2 g_{02}{}') + \tfrac{1}{3}(\xi^3 g_{30}{}' + 3\xi^2\eta g_{21}{}' + 3\xi\eta^2 g_{12}{}' + \eta^3 g_{03}{}') - \cdots$$
$$= \xi\,(\overline{10})\,g_{10}{}'' + \eta\,(\overline{01})\,g_{01}{}'' - \tfrac{1}{2}\{\xi^2\,(\overline{20})\,g_{20}{}'' + 2\xi\eta\,(\overline{11})\,g_{11}{}'' + \eta^2\,(\overline{02})\,g_{02}{}''\}$$
$$+ \tfrac{1}{3}\{\xi^3\,(\overline{30})\,g_{30}{}'' + 3\xi^2\eta\,(\overline{21})\,g_{21}{}'' + 3\xi\eta^2\,(\overline{12})\,g_{12}{}'' + \eta^3\,(\overline{03})\,g_{03}{}''\} - \cdots$$

The left hand side of this equation is

$$\log \left(1 + \xi G_{10}' + \eta G_{01}' + \ldots + \xi^p \eta^q G_{pq}' + \ldots\right) \; (\textit{vide Art. 16}),$$

while if the operators g'' be replaced by their expressions in terms of the operators G'', it is easily seen that the right side is

$$\Sigma_s \log \left(1 + \alpha_s G_{10}'' \xi + \beta_s G_{01}'' \eta + \ldots + \alpha_s^p \beta_s^q G_{pq}'' \xi^p \eta^q + \ldots\right).$$

29. Hence the operator relation

$$1 + G_{10}' \xi + G_{01}' \eta + \ldots + G_{pq}' \xi^p \eta^q + \ldots$$
$$= \Pi_s \left(1 + \alpha_s G_{10}'' \xi + \beta_s G_{01}'' \eta + \ldots + \alpha_s^p \beta_s^q G_{pq}'' \xi^p \eta^q + \ldots\right).$$

This result must be compared with the relation (Art. 24)

$$1 + c_{10} \xi + c_{01} \eta + \ldots + c_{pq} \xi^p \eta^q + \ldots$$
$$= \Pi_s \left(1 + \alpha_s b_{10} \xi + \beta_s b_{01} \eta + \ldots + \alpha_s^p \beta_s^q b_{pq} \xi^p \eta^q + \ldots\right).$$

30. I say that such a comparison yields the following theorem :—
" In any relation connecting the quantities c_{pq} with the quantities b_{pq}, we are at liberty to substitute

$$G_{pq}' \text{ for } c_{pq}, \quad \text{and} \quad G_{pq}'' \text{ for } b_{pq};$$

and we in this manner obtain a relation between operators in correspondence."

To explain this further, observe that ξ, η being undetermined quantities in the assumed relation which connects the quantities of the three identities I., II., III., we are able to express any product whatever of the coefficients $c_{10}, c_{01}, \ldots c_{pq}, \ldots$ in terms of products of coefficients $b_{10}, b_{01}, \ldots b_{pq}, \ldots$ and of symmetrical functions of the quantities $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ The substitution in question can be made in any equation thus formed.

31. With regard to the relation of Art. 24, viz. :—

$$1 + c_{10} \xi + c_{01} \eta + \ldots + c_{pq} \xi^p \eta^q + \ldots$$
$$= \Pi_s \left(1 + \alpha_s b_{10} \xi + \beta_s b_{01} \eta + \ldots + \alpha_s^p \beta_s^q b_{pq} \xi^p \eta^q + \ldots\right),$$

two important facts have been established—
(i.) That the relation is unaltered when the quantities occurring in the first identity

$$1 + a_{10} x + a_{01} y + \ldots + a_{pq} x^p y^q + \ldots = (1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots$$

are exchanged with those occurring in the second identity

$$1 + b_{10}x + b_{01}y + \ldots + b_{pq}x^p y^q + \ldots = (1 + \alpha_1^{(1)}x + \beta_1^{(1)}y)(1 + \alpha_2^{(1)}x + \beta_2^{(1)}y)\ldots$$

each with each.   (Art. 26.)

(ii.) That we can always proceed to a relation between the operations by writing

$$\text{G}_{pq}{}' \text{ for } c_{pq} \quad \text{and} \quad \text{G}_{pq}{}'' \text{ for } b_{pq}. \quad \text{(Art. 30.)}$$

I will refer to these facts as the first and second properties of the relation respectively.

§ 5.　*The First Law of Symmetry.*

32. By means of the equality

$$(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1$$

which has been established *ante* (Art. 26), it is clear that any symmetric function expressed in a bracket ( )$_2$ can be expressed as a linear function of products of symmetric functions of the form ( ) ( )$_1$; it is also clear from the first property above defined, that such expression will remain unaltered when the brackets ( ) and ( )$_1$ are interchanged; it must, therefore, be a symmetric function in regard to these brackets.

We may, therefore, suppose an equation

$$(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2}\ldots)_2 = \ldots + \text{J}\,(\overline{a_1 b_1}^{\alpha_1}\,\overline{a_2 b_2}^{\alpha_2}\ldots)(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2}\ldots)_1$$

$$+ \text{J}\,(\overline{a_1 b_1}^{\alpha_1}\,\overline{a_2 b_2}^{\alpha_2}\ldots)_1(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2}\ldots) + \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

Moreover, we can express any product of the coefficients $c_{10}, c_{01}, \ldots c_{pq}, \ldots$ as a linear function of expressions each of which contains a monomial symmetric function of the quantities α_1, β_1; α_2, β_2; $\ldots$ and a product of coefficients $b_{10}, b_{01}, \ldots b_{pq}, \ldots$

Assume then

$$c_{p_1 q_1}^{\pi_1} c_{p_2 q_2}^{\pi_2}\ldots = \ldots + \text{L}\,(\overline{a_1 b_1}^{\alpha_1}\,\overline{a_2 b_2}^{\alpha_2}\ldots)\,b_{r_1 s_1}^{\rho_1} b_{r_2 s_2}^{\rho_2}\ldots + \ldots, \quad \ldots \quad \text{(B)}$$

$$c_{a_1 b_1}^{\alpha_1} c_{a_2 b_2}^{\alpha_2}\ldots = \ldots + \text{M}\,(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2}\ldots)\,b_{r_1 s_1}^{\rho_1} b_{r_2 s_2}^{\rho_2}\ldots + \ldots, \quad \ldots \quad \text{(C)}$$

From equation (B) is derived by the second property the operator relation

$$\text{G}'_{p_1 q_1}{}^{\pi_1} \text{G}'_{p_2 q_2}{}^{\pi_2}\ldots = \ldots + \text{L}\,(\overline{a_1 b_1}^{\alpha_1}\,\overline{a_2 b_2}^{\alpha_2}\ldots)\,\text{G}''_{r_1 s_1}{}^{\rho_1} \text{G}''_{r_2 s_2}{}^{\rho_2}\ldots + \ldots,$$

and performing each side of this equation upon the opposite side of the equation (A) we obtain, after cancelling of $(\overline{a_1 b_1}^{a_1} \, \overline{a_2 b_2}^{a_2} \ldots)$,

$$\mathrm{L}\, G''^{\rho_1}_{r_1 s_1} \, G''^{\rho_2}_{r_2 s_2} \ldots (\overline{r_1 s_1}^{\rho_1} \, \overline{r_2 s_2}^{\rho_2} \ldots)_2 = \mathrm{J}\, G''^{\pi_1}_{p_1 q_1} \, G''^{\pi_2}_{p_2 q_2} \ldots (\overline{p_1 q_1}^{\pi_1} \, \overline{p_2 q_2}^{\pi_2} \ldots)_1,$$

no other terms surviving the operations, or

$$\mathrm{L} = \mathrm{J} \, ;$$

since the symmetric function on either side is reduced to unity by the operation.

Similarly the equation (C) yields the equation of operators

$$G'^{a_1}_{a_1 b_1} \, G'^{a_2}_{a_2 b_2} \ldots = \ldots + \mathrm{M}\, (\overline{p_1 q_1}^{\pi_1} \, \overline{p_2 q_2}^{\pi_2} \ldots)\, G''^{\rho_1}_{r_1 s_1} \, G''^{\rho_2}_{r_2 s_2} \ldots + \ldots ,$$

and this when performed on opposite sides of equation (A) gives

$$\mathrm{M} = \mathrm{J}.$$

Hence

$$\mathrm{L} = \mathrm{M},$$

and we have the law of symmetry expressed by the two relations

$$c^{\pi_1}_{p_1 q_1} \, c^{\pi_2}_{p_2 q_2} \ldots = \ldots + \mathrm{L}\, (\overline{a_1 b_1}^{a_1} \, \overline{a_2 b_2}^{a_2} \ldots)\, b^{\rho_1}_{r_1 s_1} \, b^{\rho_2}_{r_2 s_2} \ldots + \ldots ,$$

$$c^{a_1}_{a_1 b_1} \, c^{a_2}_{a_2 b_2} \ldots = \ldots + \mathrm{L}\, (\overline{p_1 q_1}^{\pi_1} \, \overline{p_2 q_2}^{\pi_2} \ldots)\, b^{\rho_1}_{r_1 s_1} \, b^{\rho_2}_{r_2 s_2} \ldots + \ldots ,$$

viz., if in the first of these relations the partitions $(\overline{p_1 q_1}^{\pi_1} \, \overline{p_2 q_2}^{\pi_2} \ldots)$, $(\overline{a_1 b_1}^{a_1} \, \overline{a_2 b_2}^{a_2} \ldots)$ be interchanged the numerical coefficient L remains unaltered.

The theorem is a consequence of the two properties that have been established in regard to the three identities and the relation assumed to exist between the quantities involved in them.

It appears to be the most important theorem in symmetrical algebra.

33. I now pass to certain consequences which flow straight from the theorem.

It is necessary to make a few definitions.

Definition.

"A partition is separated into separates by writing down a set of partitions, each separate partition in its own brackets, so that when all the parts of these partitions are assembled in a single bracket, the partition which is separated is reproduced."

For the purpose of this portion of the memoir alone it would have been expedient to use the word bipart in lieu of the word part in the foregoing definition, but I have retained the word part for the reason that the definition remains valid whatever be the order of multiplicity of the parts.

Of a partition $(\overline{p_1 q_1} \; \overline{p_2 q_2} \; \overline{p_3 q_3})$ the product $(\overline{p_1 q_1} \; \overline{p_3 q_3})(\overline{p_2 q_2})$ is a separation composed of the separates $(\overline{p_1 q_1} \; \overline{p_3 q_3})$ and $(\overline{p_2 q_2})$.

Definition.

" A partition *quâ* its separations is termed a separable partition."

Definition.

" If the successive biweights of the separates of a separation be

$$w_1^{(1)} w_2^{(1)}, \quad w_1^{(2)} w_2^{(2)}, \quad w_1^{(3)} w_2^{(3)} \ldots,$$

the separation is said to have the *specification*

$$(\overline{w_1^{(1)} w_2^{(1)}} \; \overline{w_1^{(2)} w_2^{(2)}} \; \overline{w_1^{(3)} w_2^{(3)}} \ldots).$$"

Observe that the biweights of the separation and of its specification are necessarily the same, and identical with the biweight of the separable partition.

Observation.

The separable partition is counted as one amongst its own separations.

34. To take a concrete example of these definitions, consider a separable partition $(\overline{20} \; \overline{10} \; \overline{01})$.

We have

Separations.	Specifications.
$(\overline{20} \; \overline{10} \; \overline{01})$	$(\overline{31})$,
$(\overline{20} \; \overline{10})(\overline{01})$	$(\overline{30} \; \overline{01})$,
$(\overline{20} \; \overline{01})(\overline{10})$	$(\overline{21} \; \overline{10})$,
$(\overline{10} \; \overline{01})(\overline{20})$	$(\overline{11} \; \overline{20})$,
$(\overline{20})(\overline{10})(\overline{01})$	$(\overline{20} \; \overline{10} \; \overline{01})$.

35. I will discuss the law of symmetry that has been established in the light of these definitions.

I recall the relations

$$c_{10} = (\overline{10})\, b_{10},$$

$$c_{01} = (\overline{01})\, b_{01},$$

$$c_{20} = (\overline{20})\, b_{20} + (\overline{10^2})\, b_{10}^{\,2},$$

$$c_{11} = (\overline{11})\, b_{11} + (\overline{10\ 01})\, b_{10} b_{01},$$

$$c_{02} = (\overline{02})\, b_{02} + (\overline{01^2})\, b_{01}^{\,2},$$

$$\vdots \qquad \ldots \ldots \ldots$$

In the expression of c_{pq} each partition in brackets ( ) has the biweight pq, and each partition is attached to a product of quantities b_{pq}, such that each factor corresponds to a single bipart in the partition.

Hence on proceeding to form a relation

$$c_{p_1 q_1}^{\pi_1}\, c_{p_2 q_2}^{\pi_2} \ldots = \ldots + P\, b_{r_1 s_1}^{\rho_1}\, b_{r_2 s_2}^{\rho_2} \ldots + \ldots$$

wherein P represents the complete symmetric function cofactor of $b_{r_1 s_1}^{\rho_1}\, b_{r_2 s_2}^{\rho_2} \ldots$ it is clear that P is a linear function of symmetric function products, each of which has a specification

$$(\overline{p_1 q_1}^{\pi_1}\, \overline{p_2 q_2}^{\pi_2} \ldots),$$

and is also a separation of the separable partition

$$(\overline{r_1 s_1}^{\rho_1}\, \overline{r_2 s_2}^{\rho_2} \ldots).$$

The partitions $(\overline{p_1 q_1}^{\pi_1}\, \overline{p_2 q_2}^{\pi_2} \ldots)$, $(\overline{r_1 s_1}^{\rho_1}\, \overline{r_2 s_2}^{\rho_2} \ldots)$, as well as each of the separations which present themselves in the linear function P are of the same biweight.

When the separations in the function P are all expanded into a sum of monomial symmetric functions, each of the latter has the same biweight.

Taking the separable partition $(\overline{r_1 s_1}^{\rho_1}\, \overline{r_2 s_2}^{\rho_2} \ldots)$ as fixed, a definite number of specifications appertain to the separations. Forming then c products in correspondence with each of these specifications, the law of symmetry indicates that the *same number* of different monomial symmetric functions will appear in the developments of the several linear functions P. Further, the partitions of these monomial symmetric functions will be, in some order, identical with the several specifications of the separations of the fixed separable partition.

Assume the specifications to be, in any order

$$\theta_1,\, \theta_2,\, \ldots \theta_k,$$

and write the identity

$$c_{p_1 q_1}^{\pi_1}\, c_{p_2 q_2}^{\pi_2} \ldots = \ldots + P\, b_{r_1 s_1}^{\rho_1}\, b_{r_2 s_2}^{\rho_2} \ldots + \ldots$$

in the abbreviated notation

$$c_\theta = \ldots + P_\theta b_r + \ldots$$

We have then the relations :—

$$c_{\theta_k} = \ldots + P_{\theta_k} b_r + \ldots \qquad (k = 1, 2, 3, \ldots k),$$

and

$$P_{\theta_1} = m_{11}\theta_1 + m_{12}\theta_2 + \ldots + m_{1k}\theta_k,$$
$$P_{\theta_2} = m_{21}\theta_1 + m_{22}\theta_2 + \ldots + m_{2k}\theta_k,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$P_{\theta_k} = m_{k1}\theta_1 + m_{k2}\theta_2 + \ldots + m_{kk}\theta_k,$$

the quantities m being numerical.

The determinant of the numbers m is symmetrical, for by the law of symmetry

$$m_{rs} = m_{sr}.$$

Hence, the coefficient of symmetric function θ_s in the development of the assemblage of separations P_{θ_r} is identical with the coefficient of symmetric function θ_r in the development of the assemblage of separations P_{θ_s}.

We may now proceed to express the symmetric functions θ as linear functions of the assemblages of separations P_θ and by elementary theory of determinants, the determinant of the system of results is symmetrical. Hence

$$\theta_1 = \mu_{11}P_{\theta_1} + \mu_{12}P_{\theta_2} + \ldots + \mu_{1k}P_{\theta_k},$$
$$\theta_2 = \mu_{21}P_{\theta_1} + \mu_{22}P_{\theta_2} + \ldots + \mu_{2k}P_{\theta_k},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\theta_k = \mu_{k1}P_{\theta_1} + \mu_{k2}P_{\theta_2} + \ldots + \mu_{kk}P_{\theta_k},$$

wherein

$$\mu_{rs} = \mu_{sr}.$$

36. From this are deduced two important theorems, the one a theorem of expressibility, and the other a theorem of symmetry. Any one of the monomial symmetric functions θ is expressed by a partition which is a specification of a separation of the partition $(\overline{r_1 s_1}^{\rho_1} \ \overline{r_2 s_2}^{\rho_2} \ldots)$. This implies that the biparts occurring in the partition of θ can be so partitioned into biparts that when assembled together they will be identical with the biparts of the partition $(\overline{r_1 s_1}^{\rho_1} \ \overline{r_2 s_2}^{\rho_2} \ldots)$. Hence the theorem of expressibility :—

37. *Theorem.*

" The biparts of the partition of a monomial symmetric function θ are partitioned in

any manner into biparts, which, when all assembled together in a single bracket, are represented by

$$(\overline{r_1 s_1}^{\rho_1}\, \overline{r_2 s_2}^{\rho_2} \ldots).$$

The symmetric function θ is expressible as a linear function of assemblages of separations of the symmetric function $(\overline{r_1 s_1}^{\rho_1}\, \overline{r_2 s_2}^{\rho_2} \ldots).$"

38. The theorem of symmetry is as follows :—

Theorem.

" When the monomial symmetric function θ_r is expressed as a linear function of the assemblages of separations P_{θ_1}, P_{θ_2}, $\ldots P_{\theta_k}$, the coefficient of the assemblage P_{θ_s} is the same as the coefficient of the assemblage P_{θ_r} when θ_s is so expressed."

39. This theorem enables us to form a pair of symmetrical tables in regard to every partition of every biweight. The number of tables is therefore twice the number of partitions, the generating function for which has been already given.

§ 6. *The Functions composed of One Part.*

40. I will now establish a law by means of which any symmetric function expressed by a partition with a single bipart may be at once expressed in terms of separations of any partition of its biweight. It is merely necessary to interpret a result already obtained.

I recall the formula of Art. 26,

$$(\overline{pq})_2 = (\overline{pq})\,(\overline{pq})_1,$$

which may also be written by Art. 8,

$$\Sigma_\pi \frac{(-)^{\Sigma\pi - 1}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} c_{p_1 q_1}^{\pi_1} c_{p_2 q_2}^{\pi_2} \ldots = (\overline{pq})\,\Sigma_\pi \frac{(-)^{\Sigma\pi - 1}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} b_{p_1 q_1}^{\pi_1} b_{p_2 q_2}^{\pi_2} \ldots$$

Let us compare the cofactor of $b_{p_1 q_1}^{\pi_1} b_{p_2 q_2}^{\pi_2} \ldots$ in the development of the left hand side with its cofactor on the right hand side.

When the left hand side is multiplied out each symmetric function product which multiplies the term $b_{p_1 q_1}^{\pi_1} b_{p_2 q_2}^{\pi_2} \ldots$ is necessarily a separation of the symmetric function $(\overline{p_1 q_1}^{\pi_1}\, \overline{p_2 q_2}^{\pi_2} \ldots)$. The result of the comparison will therefore be the expression of the function $(\overline{pq})$ in terms of such separations.

41. Let S_{pq} be the value assumed by c_{pq} when b_{pq} and other quantities b are put equal to unity.

Further, let $s_{(\overline{p_1 q_1}^{\pi_1}\, \overline{p_2 q_2}^{\pi_2} \ldots)}$ denote the expression of $s_{pq} = (\overline{pq})$ by means of separations of the symmetric function $(\overline{p_1 q_1}^{\pi_1}\, \overline{p_2 q_2}^{\pi_2} \ldots)$.

Then we may write

$$\Sigma_\pi \left(-\right)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2!\, \ldots}\, s_{(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2}\, \ldots)} = \Sigma_\pi \left(-\right)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2!\, \ldots}\, S_{p_1q_1}^{\pi_1}\, S_{p_2q_2}^{\pi_2} \cdots$$

where S_{pq} denotes the sum of all the symmetric functions of biweight pq.

Represent the different separates of the partition $(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2}\, \ldots)$ by $(J_1), (J_2), \ldots$ and any separation by $(J_1)^{j_1} (J_2)^{j_2} \ldots$; substitute for the quantities S_{pq} their values in terms of symmetric functions; apply the multinominal theorem and equate corresponding portions of the two sides and there results the formula

$$\left(-\right)^{\Sigma\pi - 1} \frac{(\Sigma\pi - 1)!}{\pi_1!\, \pi_2!\, \ldots}\, s_{(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2}\, \ldots)} = \Sigma_j \left(-\right)^{\Sigma j - 1} \frac{(\Sigma j - 1)!}{j_1!\, j_2!\, \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots$$

where the summation is taken for every separation of the given partition.

42. This important result is a generalisation of the VANDERMONDE-WARING law for the expression of the sums of the powers of the roots of an equation in terms of the coefficients.

43. The formula may be reversed so as to exhibit any symmetric function whatever in terms of single bipart functions. The result easily reached is

$$\left(-\right)^{\Sigma\pi - 1} \left(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2} \cdots\right)$$

$$= \Sigma_j \left(-\right)^{\Sigma j - 1} \frac{(\Sigma\pi_1 - 1)!\, (\Sigma\pi_2 - 1)! \ldots}{j_1!\, j_2!\, \ldots\, \pi_{11}!\, \pi_{12}!\, \ldots\, \pi_{21}!\, \pi_{22}!\, \ldots}\, s_{(\overline{p_{11}q_{11}}^{\pi_{11}}\, \overline{p_{12}q_{12}}^{\pi_{12}} \ldots)}^{j_1}\, s_{(\overline{p_{21}q_{21}}^{\pi_{21}}\, \overline{p_{22}q_{22}}^{\pi_{22}} \ldots)}^{j_2} \cdots,$$

the summation being for every separation

$$\left(\overline{p_{11}q_{11}}^{\pi_{11}}\, \overline{p_{12}q_{12}}^{\pi_{12}} \cdots\right)^{j_1} \left(\overline{p_{21}q_{21}}^{\pi_{21}}\, \overline{p_{22}q_{22}}^{\pi_{22}} \cdots\right)^{j_2} \cdots$$

of the symmetric function

$$\left(\overline{p_1q_1}^{\pi_1}\, \overline{p_2q_2}^{\pi_2} \cdots\right).$$

§7. Second Law of Symmetry.

44. The operation

$$g_{pq} = \partial_{a_{pq}} + a_{10}\partial_{a_{p+1,q}} + a_{01}\partial_{a_{p,q+1}} + \ldots + a_{rs}\partial_{a_{p+r,q+s}} + \cdots$$

may be said to be of biweight pq, since it lowers the weight of a symmetric function by the biweight pq. Further, its degree is zero, since it does not in general lower the degree of a symmetric function. If, however, g_{pq} operates upon a symmetric function of its own biweight, it is equivalent to the simple differential operation $\partial_{a_{pq}}$, and is of degree unity.

Similarly, the operation

$$\frac{\overline{g_{p_1 q_1}{}^{\pi_1} g_{p_2 q_2}{}^{\pi_2} \cdots}}{\pi_1! \; \pi_2! \cdots}$$

will be regarded as being of weight (biweight) pq, where $(\overline{p_1 q_1}^{\pi_1} \, \overline{p_2 q_2}^{\pi_2} \cdots)$ is a partition of the biweight pq ; and if, as a particular case, the operand be of the same biweight pq, it will be equivalent to the operation

$$\frac{\partial_{a_{p_1 q_1}}{}^{\pi_1} \partial_{a_{p_2 q_2}}{}^{\pi_2} \cdots}{\pi_1! \; \pi_2! \cdots} \, ,$$

and will be of degree equal to

$$\pi_1 + \pi_2 + \ldots = \Sigma \pi.$$

Since therefore

$$\frac{\partial_{a_{p_1 q_1}}{}^{\pi_1} \partial_{a_{p_2 q_2}}{}^{\pi_2} \cdots}{\pi_1! \; \pi_2!} \, a_{p_1 q_1}{}^{\pi_1} a_{p_2 q_2}{}^{\pi_2} \cdots = 1,$$

we have the result

$$\frac{\overline{g_{p_1 q_1}{}^{\pi_1} g_{p_2 q_2}{}^{\pi_2} \cdots}}{\pi_1! \; \pi_2!} \, a_{p_1 q_1}{}^{\pi_1} a_{p_2 q_2}{}^{\pi_2} \cdots = 1 \, ;$$

assuming then a result

$$(\overline{p_1 q_1}^{\pi_1} \, \overline{p_2 q_2}^{\pi_2} \cdots)_2 = \ldots + P b_{r_1 s_1}{}^{\rho_1} b_{r_2 s_2}{}^{\rho_2} \cdots + \cdots$$

derived from the three initial identities of Art. 24 and the relation assumed to exist between the quantities involved, we are at once led to the operator relation

$$\frac{\overline{g'_{p_1 q_1}{}^{\pi_1} g'_{p_2 q_2}{}^{\pi_2} \cdots}}{\pi_1! \; \pi_2! \cdots} = \ldots + P G''_{r_1 s_1}{}^{\rho_1} G''_{r_2 s_2}{}^{\rho_2} \cdots + \cdots$$

where P consists entirely of symmetric functions of quantities which occur in the first identity.   Further suppose a second result

$$(\overline{r_1 s_1}^{\rho_1} \, \overline{r_2 s_2}^{\rho_2} \cdots)_2 = \ldots + Q b_{p_1 q_1}{}^{\pi_1} b_{p_2 q_2}{}^{\pi_2} \cdots + \cdots$$

Hence, operating on the left and right of this result with right and left sides of the foregoing operator relation, we obtain

$$(\ldots + P G''_{r_1 s_1}{}^{\rho_1} G''_{r_2 s_2}{}^{\rho_2} \cdots + \cdots)(\overline{r_1 s_1}^{\rho_1} \, \overline{r_2 s_2}^{\rho_2} \cdots)$$

$$= \frac{\overline{g'_{p_1 q_1}{}^{\pi_1} g'_{p_2 q_2}{}^{\pi_2} \cdots}}{\pi_1! \; \pi_2! \cdots} (\ldots + Q b_{p_1 q_1}{}^{\pi_1} b_{p_2 q_2}{}^{\pi_2} \cdots + \cdots),$$

or from theorems established above (Arts. 13, 44)

$$P = Q,$$

no other terms surviving the operation.

45. Hence a theorem of symmetry :—

Theorem.

" If

$$(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2}\ldots)_2 = \ldots + \mathrm{P}\,b_{r_1 s_1}^{\rho_1}\,b_{r_2 s_2}^{\rho_2}\ldots + \ldots,$$

the cofactor symmetric function P is unaltered when the partitions $(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2}\ldots)$, $(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2}\ldots)$, are interchanged."

The function P presents itself in the first place as a linear function of separations of the partition of the b product to which it is attached. The theorem supplies linear functions of separations of any two partitions $(\overline{p_1 q_1}^{\pi_1}\,\overline{p_2 q_2}^{\pi_2}\ldots)$, $(\overline{r_1 s_1}^{\rho_1}\,\overline{r_2 s_2}^{\rho_2}\ldots)$ respectively, of the same biweight, which are equal to one another.

46. To make the matter clear, form a table of biweight 21 as follows :—

	b_{21}	$b_{20}b_{01}$	$b_{11}b_{10}$	$b_{10}^2 b_{01}$
$(\overline{21})_2$	$(\overline{21})$	$(\overline{20}\,\overline{01})$ $-\,(\overline{20})\,(\overline{01})$	$(\overline{11}\,\overline{10})$ $-\,(\overline{11})\,(\overline{10})$	$(\overline{10^2}\,\overline{01})$ $-\,(\overline{10^2})\,(\overline{01})$ $-\,(\overline{10}\,\overline{01})\,(\overline{10})$ $+\,(\overline{10})^2\,(\overline{01})$
$(\overline{20}\,\overline{01})_2$	$-\,(\overline{21})$	$-\,(\overline{20}\,\overline{01})$ $-\,(\overline{20})\,(\overline{01})$	$-\,(\overline{11}\,\overline{10})$ $+\,(\overline{11})\,(\overline{10})$	$-\,(\overline{10^2}\,\overline{01})$ $-\,(\overline{10^2})\,(\overline{01})$ $+\,(\overline{10}\,\overline{01})\,(\overline{10})$
$(\overline{11}\,\overline{10})_2$	$-\,(\overline{21})$	$-\,(\overline{20}\,\overline{01})$ $+\,(\overline{20})\,(\overline{01})$	$-\,(\overline{11}\,\overline{10})$	$-\,(\overline{10^2}\,\overline{01})$ $+\,(\overline{10^2})\,(\overline{01})$
$(\overline{10^2}\,\overline{01})_2$	$(\overline{21})$	$(\overline{20}\,\overline{01})$	$(\overline{11}\,\overline{10})$	$(\overline{10^2}\,\overline{01})$

which is to be read by rows.*

Each term in a column is a separation of the partition of the b product at the head of the column.

The separations in each line of terms as written possess the same specifications, and also the same numerical coefficients. In the right hand column the partition separated is a fundamental symmetric function, and hence each separate therein appearing is so also. Each block of separations in the right hand column is the expression by means of fundamental symmetric functions of the monomial symmetric function of the same elements whose partition appears to the left of the same line. The terms of the first three columns may be regarded as being formed according to the same *law* as the right hand column, and therefore according to a law defined by

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* Each term in the left hand column is equal to the aggregate of terms in *any* block in the same row.

the monomial function at the left of the same line. For example, the terms in the second column and third line are separations of $(\overline{20}\ \overline{01})$ formed according to the law of the function $(\overline{11}\ \overline{10})$. Also the terms in the third column and second line are separations of $(\overline{11}\ \overline{10})$ formed according to the law of the function $(\overline{20}\ \overline{01})$. Now observe that the law of symmetry establishes that the table enjoys row and column symmetry. Hence the assemblage of separations of $(\overline{20}\ \overline{01})$ formed according to the law of $(\overline{11}\ \overline{10})$ is equal to the assemblage of separations of $(\overline{11}\ \overline{10})$ formed according to the law of $(\overline{20}\ \overline{01})$.

47. Hence in general the theorem :—

"The assemblage of separations of $(\overline{r_1 s_1}^{\rho_1}\ \overline{r_2 s_2}^{\rho_2} \ldots)$, formed according to the law of $(\overline{p_1 q_1}^{\pi_1}\ \overline{p_2 q_2}^{\pi_2} \ldots)$, is equal to the assemblage of separations of $(\overline{p_1 q_1}^{\pi_1}\ \overline{p_2 q_2}^{\pi_2} \ldots)$, formed according to the law of $(\overline{r_1 s_1}^{\rho_1}\ \overline{r_2 s_2}^{\rho_2} \ldots)$."

In the particular case considered the equality is

$$- (\overline{20}\ \overline{01}) + (\overline{20})(\overline{01}) = - (\overline{11}\ \overline{10}) + (\overline{11})(\overline{10}).$$

To actually form separations of $(\overline{r_1 s_1}^{\rho_1}\ \overline{r_2 s_2}^{\rho_2} \ldots)$, according to the law of $(\overline{p_1 q_1}^{\pi_1}\ \overline{p_2 q_2}^{\pi_2} \ldots)$, the separations of the former must be written down, and also the expression of the latter, by means of fundamental symmetric functions. The separations are then given the same coefficients as the products of fundamental symmetric functions which possess the same specifications.

§ 8. *Third Law of Symmetry.*

48. From the relation

$$c_{p_1 q_1}^{\pi_1} c_{p_2 q_2}^{\pi_2} \ldots = \ldots + L b_{r_1 s_1}^{\rho_1} b_{r_2 s_2}^{\rho_2} \ldots + \ldots$$

is derived the operator relation

$$G'^{\pi_1}_{p_1 q_1} G'^{\pi_2}_{p_2 q_2} \ldots = \ldots + L G''^{\rho_1}_{r_1 s_1} G''^{\rho_2}_{r_2 s_2} \ldots + \ldots$$

and, thence, by the method already employed,

$$(\overline{r_1 s_1}^{\rho_1}\ \overline{r_2 s_2}^{\rho_2} \ldots)_2 = \ldots + L (\overline{p_1 q_1}^{\pi_1}\ \overline{p_2 q_2}^{\pi_2} \ldots)_1 + \ldots$$

This law of symmetry is of considerable importance and interest, but I do not stop to further discuss it.*

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* *Vide* 'American Journal of Mathematics.' "Third Memoir on a New Theory of Symmetric Functions," now in progress in vols. 11, 12, and succeeding volumes.

§ 9. *The linear Partial Differential Operations of the Theory of Separations.*

49. For purposes of calculation it is necessary to adapt the operations

$$g_{10}, \; g_{01}, \; \ldots \; g_{pq} \; \ldots,$$

so that they may be performed on a symmetric function when the latter is expressed in terms of the separations of any given partition.

Of any partition $(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)$ separates (*vide* Definitions) are formed by taking all possible combinations of the parts. These are precisely

$$(\pi_1 + 1)\,(\pi_2 + 1) \ldots \; - 1,$$

distinct separates which must be regarded as independent variables.

Put

$$(\overline{10}^{\pi_{10}+\rho_{10}} \overline{01}^{\pi_{01}+\rho_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}+\rho_{p_1 q_1}} \ldots)$$

for any separate of a given separable partition P.

Then by a known theorem

$$g_{pq} = \Sigma g_{pq} \left(\overline{10}^{\pi_{10}+\rho_{10}} \overline{01}^{\pi_{01}+\rho_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1} + \rho_{p_1 q_1}} \ldots \right) \partial_{\left(\overline{10}^{\pi_{10}+\rho_{10}} \overline{01}^{\pi_{01}+\rho_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}+\rho_{p_1 q_1}} \ldots \right)},$$

the summation being in regard to all the separates.

Moreover (Art. 17)

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!}\, g_{pq} = \Sigma \frac{(-)^{\Sigma\pi-1}\,(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}! \ldots \pi_{p_1 q_1}!}\, G_{10}^{\pi_{10}} G_{01}^{\pi_{01}} \ldots G_{p_1 q_1}^{\pi_{p_1 q_1}} \ldots,$$

the summation being in regard to all the partitions $(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)$ of the biweight pq; and also (Art. 13)

$$G_{10}^{\pi_{10}} G_{01}^{\pi_{01}} \ldots G_{p_1 q_1}^{\pi_{p_1 q_1}} \ldots (\overline{10}^{\pi_{10}+\rho_{10}} \overline{01}^{\pi_{01}+\rho_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}+\rho_{p_1 q_1}} \ldots) = (\overline{10}^{\rho_{10}} \overline{01}^{\rho_{01}} \ldots \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots).$$

50. Hence

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q! \ldots}\, g_{pq}$$

$$= \Sigma_\pi \Sigma_\rho \frac{(-)^{\Sigma\pi-1}\,(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}! \ldots \pi_{p_1 q_1}! \ldots} (\overline{10}^{\rho_{10}} \overline{01}^{\rho_{01}} \ldots \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots) \partial_{\left(\overline{10}^{\pi_{10}+\rho_{01}} \overline{01}^{\pi_{01}+\rho_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots \right)}$$

the summation being in regard to

(1) Every separate of the given separable partition;

(2) Every partition of the biweight pq.

The right hand side of this relation may be broken up into fragments in each of which all the numbers $\pi_{10}, \pi_{01}, \ldots \pi_{p_1 q_1} \ldots$ are constant.

In fact we may write

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq}$$

$$= \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1 q_1}!\ldots} \Sigma_\rho \left(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1 q_1}^{\rho_{p_1 q_1}}\ldots\right)\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}+\rho_{p_1 q_1}}\ldots)},$$

wherein, following the summation sign Σ_ρ, the numbers $\pi_{10}, \pi_{01}, \ldots \pi_{p_1 q_1} \ldots$ are constant, and the operator

$$\Sigma_\rho \left(\overline{10}^{\rho_{10}}\,\overline{01}^{\rho_{01}}\ldots\overline{p_1 q_1}^{\rho_{p_1 q_1}}\ldots\right)\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}+\rho_{p_1 q_1}}\ldots)},$$

is one of the fragments above mentioned.

This operation has a biweight pq, and may be defined also in regard to the partition $(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}}\ldots)$ of the biweight pq.

51. Write then for brevity and convenience

$$\Sigma_\rho \left(\overline{10}^{\rho_{10}}\overline{01}^{\rho_{01}}\ldots\overline{p_1 q_1}^{\rho_{p_1 q_1}}\ldots\right)\partial_{(\overline{10}^{\pi_{10}+\rho_{10}}\,\overline{01}^{\pi_{01}+\rho_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}+\rho_{p_1 q_1}}\ldots)} = g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}}\ldots)};$$

so that we may write

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}g_{pq} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_{10}!\,\pi_{01}!\ldots\pi_{p_1 q_1}!\ldots}g_{(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}}\ldots)},$$

where the summation is in regard to every partition $(\overline{10}^{\pi_{10}}\,\overline{01}^{\pi_{01}}\ldots\overline{p_1 q_1}^{\pi_{p_1 q_1}}\ldots)$ of the biweight pq.

52. In general not every partition of the biweight pq will occur in the given separable partition, but it is convenient to consider the general result just written down as including every such partition. It will be seen later that this result is of great importance in the theory.

I remark that on the left-hand side we have a linear partial differential operation g_{pq} whose expression by means of the fundamental symmetric functions and their differential inverses is well known by what has preceded. Such expression is all that is needed so long as we are concerned only with the fundamental forms which, as they appear in the expression of a monomial symmetric function of biweight pq, present themselves in products which are separations of the symmetric function $(\overline{10}^p\,\overline{01}^q)$. In the present broader theory in which the leading idea is the consideration of any partition at pleasure of the biweight as the separable partition, we bring into view the exhibition of the operation g_{pq} as a linear function of operations, each of which is in correspondence with a partition of the biweight.

We have, in fact, a biweight operator g_{pq} decomposed into à full number of bipart partition operators

$$g_{(\overline{10}^{\pi_{10}}\ \overline{01}^{\pi_{01}}\ \ldots\ \overline{p_1 q_1}^{\pi}\, p_1 q_1 \ldots)}.$$

Observe that the whole theory of separations is a generalisation from a weight to a partition of a weight. Here we have generalised from a weight operation to a partition operation, and I henceforward regard the partition operator as the essential linear partial differential operator of the theory. The biweight operator g_{pq} has been expressed *ante* (Art. 17) in terms of the obliterating operators of the form G_{pq}. These operations G_{pq} are equally available in the theory of the separable partition in general. The mode of their operation upon a symmetric function product will be subsequently explained (in § 11). So far I have merely considered their operation upon monomial forms.*

53. I observe that the biweight operator g_{pq} is expressed as a linear function of the partition operators of the same biweight, according to the same laws as—

(1) The operator g_{pq} is expressed in terms of the operations G_{pq} (Art. 17).

(2) The symmetric functions, containing one part only, are expressed in terms of the fundamental or single-unitary forms (Art. 8).

E.g., compare the three results (the first slightly modified) :—

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq} = \Sigma_\pi\ \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,g_{(\overline{p_1 q_1}^{\pi_1}\ \overline{p_2 q_2}^{\pi_2}\ \ldots)},$$

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,g_{pq} = \Sigma_\pi\ \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,\mathrm{G}_{p_1 q_1}^{\pi_1}\ \mathrm{G}_{p_2 q_2}^{\pi_2}\cdots,$$

$$(-)^{p+q-1}\frac{(p+q-1)!}{p!\,q!}\,(\overline{pq}) = \Sigma_\pi\ \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,a_{p_1 q_1}^{\pi_1}\ a_{p_2 q_2}^{\pi_2}\cdots.$$

54. For convenience of reference, I write down the particular simplest cases of the decomposition.

$$g_{10} = g_{(\overline{10})},$$

$$g_{01} = g_{(\overline{01})},$$

$$g_{20} = g_{(\overline{10}^2)} - 2g_{(\overline{20})},$$

$$g_{11} = g_{(\overline{10}\ \overline{01})} - g_{(\overline{11})},$$

$$g_{02} = g_{(\overline{01}^2)} - 2g_{(\overline{02})},$$

$$g_{30} = g_{(\overline{10}^3)} - 3g_{(\overline{20}\ \overline{10})} + 3g_{(\overline{30})},$$

$$g_{21} = g_{(\overline{10}^2\ \overline{01})} - g_{(\overline{20}\ \overline{10})} - g_{(\overline{11}\ \overline{10})} + g_{(\overline{21})},$$

$$g_{12} = g_{(\overline{01}^2\ \overline{10})} - g_{(\overline{02}\ \overline{10})} - g_{(\overline{11}\ \overline{01})} + g_{(\overline{12})},$$

$$g_{03} = g_{(\overline{01}^3)} - 3g_{(\overline{02}\ \overline{01})} + 3g_{(\overline{03})}.$$

* The decomposition of the obliterating operation G_{pq} into partition obliterating operations is given *post* § 10.

55. I also give the developed expressions of a few of the partition operators. Thus

$$g_{(\overline{10})} = \partial_{(\overline{10})} + (\overline{10})\, \partial_{(\overline{10}^2)} + (\overline{01})\, \partial_{(\overline{10}\ \overline{01})} + (\overline{20})\, \partial_{(\overline{20}\ \overline{10})} + (\overline{10}^2)\, \partial_{(\overline{10}^3)}$$

$$+ (\overline{11})\, \partial_{(\overline{11}\ \overline{10})} + (\overline{10}\ \overline{01})\, \partial_{(\overline{10}^2\ \overline{01})} + (\overline{02})\, \partial_{(\overline{10}\ \overline{02})} + (\overline{01}^2)\, \partial_{(\overline{10}\ \overline{01}^2)}$$

$$+ \ldots ,$$

$$g_{(\overline{20}\ \overline{01})} = \partial_{(\overline{20}\ \overline{01})} + (\overline{10})\, \partial_{(\overline{20}\ \overline{10}\ \overline{01})} + (\overline{01})\, \partial_{(\overline{20}\ \overline{01}^2)} + (\overline{20})\, \partial_{(\overline{20}^2\ \overline{01})}$$

$$+ (\overline{10}^2)\, \partial_{(\overline{20}\ \overline{10}^2\ \overline{01})} + (\overline{11})\, \partial_{(\overline{20}\ \overline{11}\ \overline{01})} + (\overline{10}\ \overline{01})\, \partial_{(\overline{20}\ \overline{10}\ \overline{01}^2)}$$

$$+ (\overline{02})\, \partial_{(\overline{20}\ \overline{01}\ \overline{02})} + (\overline{01}^2)\, \partial_{(\overline{20}\ \overline{01}^3)} + \ldots$$

The mode of operation of the biweight operators in the separation theory is now manifest.

56. Let

$$g_{(\overline{10}^{\pi_{10}}\ \overline{01}^{\pi_{01}} \ldots\ \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)}, \qquad g_{(\overline{10}^{\rho_{10}}\ \overline{01}^{\rho_{01}} \ldots\ \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots)},$$

be any two partition operators of the same or different biweights. Representing them for brevity by

$$g_{(\pi)}, \qquad g_{(\rho)},$$

we have

$$g_{(\pi)} g_{(\rho)} = \overline{g_{(\pi)} g_{(\rho)}} + g_{(\pi)} \underset{\mid}{\dagger} g_{(\rho)},$$

wherein the multiplication on the left denotes successive operation, the bar on the right denotes symbolical multiplication, and the symbol $\underset{\mid}{\dagger}$ denotes explicit differentiation on the operand regarded as a function of symbols of quantity only.

It is easy to establish the result

$$g_{(\pi)} \underset{\mid}{\dagger} g_{(\rho)} = g_{(\overline{10}^{\pi_{10} + \rho_{10}}\ \overline{01}^{\pi_{01} + \rho_{01}} \ldots\ \overline{p_1 q_1}^{\pi_{p_1 q_1} + \rho_{p_1 q_1}} \ldots)}$$

$$= g_{(\pi + \rho)} \text{ for brevity.}$$

Hence

$$g_{(\pi)} \underset{\mid}{\dagger} g_{(\rho)} = g_{(\rho)} \underset{\mid}{\dagger} g_{(\pi)} = g_{(\pi + \rho)},$$

and

$$g_{(\pi)} g_{(\rho)} = \overline{g_{(\pi)} g_{(\rho)}} + g_{(\pi + \rho)},$$

or at full length

$$g_{(\overline{10}^{\pi_{10}}\ \overline{01}^{\pi_{01}} \ldots\ \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)}\, g_{(\overline{10}^{\rho_{10}}\ \overline{01}^{\rho_{01}} \ldots\ \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots)} = \overline{g_{(\overline{10}^{\pi_{10}}\ \overline{01}^{\pi_{01}} \ldots\ \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)}\, g_{(\overline{10}^{\rho_{10}}\ \overline{01}^{\rho_{01}} \ldots\ \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots)}},$$

$$+ g_{(\overline{10}^{\pi_{10} + \rho_{10}}\ \overline{01}^{\pi_{01} + \rho_{01}} \ldots\ \overline{p_1 q_1}^{\pi_{p_1 q_1} + \rho_{p_1 q_1}} \ldots)},$$

the fundamental law of multiplication (compare Art. 15).

Also

$$g_{(\pi)} g_{(\rho)} - g_{(\rho)} g_{(\pi)} = 0,$$

or

$$g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)} g_{(\overline{10}^{\rho_{10}} \overline{01}^{\rho_{01}} \ldots \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots)} - g_{(\overline{10}^{\rho_{10}} \overline{01}^{\rho_{01}} \ldots \overline{p_1 q_1}^{\rho_{p_1 q_1}} \ldots)} g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)} = 0,$$

shewing that any two partition operators are commutative.

57. The left hand side of the result just reached is called by Sophus Lie[*] the " Zusammensetzung " or " Combination " of the operators which appear. Sylvester[†] has also called it the 'Alternant' of the two operators.

The whole system of partition operators forms an infinite group in co-relation with an infinite group of transformations—

We can state the theorem :—

Theorem.

" The Combination or Alternant of any two partition operators vanishes."
Considering the partial differential equation

$$g_{(\pi)} = 0,$$

and ϕ any function which is a solution, then must $g_{(\rho)} \phi$ be also a solution, since

$$g_{(\pi)} g_{(\rho)} \phi - g_{(\rho)} g_{(\pi)} \phi = 0.$$

Theorem.

" If $g_{(\pi)}$ and $g_{(\rho)}$ be any two partition operators, and ϕ a solution of the equation $g_{(\pi)} = 0$; then will $g_{(\rho)} \phi$ be also a solution of the same equation."

58. Consider now the partial differential equation of Art. 51,

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} g_{pq} = \Sigma_\pi \frac{(-)^{\Sigma\pi - 1} (\Sigma\pi - 1)!}{\pi_{10}!\,\pi_{01}! \ldots \pi_{p_1 q_1}! \ldots} g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)} = 0.$$

Assume the separable partition to be

$$(\overline{10}^{\sigma_{10}} \overline{01}^{\sigma_{01}} \ldots \overline{p_1 q_1}^{\sigma_{p_1 q_1}} \ldots),$$

so that the operand is a linear function of separations of this partition.
The effect of the partition operator

$$g_{(\overline{10}^{\pi_{10}} \overline{01}^{\pi_{01}} \ldots \overline{p_1 q_1}^{\pi_{p_1 q_1}} \ldots)},$$

is the production of terms each of which is a separation of the partition

$$(\overline{10}^{\sigma_{10} - \pi_{10}} \overline{01}^{\sigma_{01} - \pi_{01}} \ldots \overline{p_1 q_1}^{\sigma_{p_1 q_1} - \pi_{p_1 q_1}} \ldots).$$

Observe that separations of this partition cannot be produced by any *other* of the partition operators which present themselves on the left hand side of the differential equation. Hence if the operand satisfies the differential equation

$$g_{pq} = 0,$$

it must also satisfy the differential equation

$$g_{(\overline{10}^{\pi_{10}} \, \overline{01}^{\pi_{01}} \, \ldots \, \overline{p_1 q_1}^{\pi} p_1 q_1 \ldots)} = 0.$$

59. This important theorem may be enunciated as follows :—

Theorem.

" If a function, expressed in terms of separations of a given monomial symmetric function, be annihilated by a biweight operator it must also be annihilated by every partition operator of that biweight."

As regards the calculation of Tables of Separations of Symmetric Functions, this is the cardinal theorem.

As an example of its application I propose to utilise it for the purpose of exhibiting the function $(\overline{31} \, \overline{01})$ as a linear function of separations of $(\overline{21} \, \overline{10} \, \overline{01})$. The law of expressibility shows this to be possible, for $(\overline{21} \, \overline{10})$ is a partition of the biweight 31. Remarking that the separation $(\overline{21}) \, (\overline{10}) \, (\overline{01})$ cannot occur in the expression, since it is the only separation which produces the monomial $(\overline{32})$ when multiplied out, I put

$$(\overline{31} \, \overline{01}) = A \, (\overline{21} \, \overline{10}) \, (\overline{01}) + B \, (\overline{21} \, \overline{01}) \, (\overline{10}) + C \, (\overline{10} \, \overline{01}) \, (\overline{21}) + D \, (\overline{21} \, \overline{10} \, \overline{01}).$$

A monomial symmetric function is caused to vanish by means of the operation of the biweight operator g_{pq} if no partition of the biweight pq is comprised amongst its parts. In consequence of this, the only biweight operators which do not cause it to vanish are g_{31}, g_{01}, and g_{32}. Hence all the partition operators of every other biweight operator annihilate the function $(\overline{31} \, \overline{01})$. It suffices to employ as annihilators the two partition operators $g_{(\overline{01})}$ and $g_{(\overline{21})}$.

Hence, retaining only significant terms,

$$\{\partial_{(\overline{10})} + (\overline{01}) \, \partial_{(\overline{10} \, \overline{01})} + (\overline{21}) \, \partial_{(\overline{21} \, \overline{10})} + (\overline{21} \, \overline{01}) \, \partial_{(\overline{21} \, \overline{10} \, \overline{01})}\} \, (\overline{31} \, \overline{01}) = 0,$$

$$\{\partial_{(\overline{21})} + (\overline{10}) \, \partial_{(\overline{21} \, \overline{10})} + (\overline{01}) \, \partial_{(\overline{21} \, \overline{01})} + (\overline{10} \, \overline{01}) \, \partial_{(\overline{21} \, \overline{10} \, \overline{01})}\} \, (\overline{31} \, \overline{01}) = 0,$$

leading to

$$A + C = 0, \quad B + D = 0,$$
$$C + D = 0, \quad A + B = 0,$$

or

$$D = -C = -B = A.$$

3 U 2

Hence

$$(\overline{31}\ \overline{01}) = A\ \{(\overline{21}\ \overline{10})\ (\overline{01}) - (\overline{21}\ \overline{01})\ (\overline{10}) - (\overline{10}\ \overline{01})\ (\overline{21}) + (\overline{21}\ \overline{10}\ \overline{01})\}\ ;$$

and it is easy to see that $A = -\frac{1}{2}$, for each of the products $(\overline{21}\ \overline{01})\ (\overline{10})$, $(\overline{10}\ \overline{01})\ (\overline{21})$ on multiplication produces a term $+ (\overline{31}\ \overline{01})$, and this monomial is not produced by the development of either of the other two products. The value of A may, however, be instructively obtained by means of the operator g_{01}.

For

$$g_{01}\ (\overline{31}\ \overline{01}) = (\overline{31}),$$

and

$$g_{01} = g_{(\overline{01})} = \partial_{(\overline{01})} + (\overline{10})\ \partial_{(\overline{10}\ \overline{01})} + (\overline{21})\ \partial_{(\overline{21}\ \overline{01})} + (\overline{21}\ \overline{10})\ \partial_{(\overline{21}\ \overline{10}\ \overline{01})}. \quad \text{(Art. 53.)}$$

Hence

$$(\overline{31}) = 2A\ \{- (\overline{21})\ (\overline{10}) + (\overline{21}\ \overline{10})\},$$

and now A is obviously equal to $-\frac{1}{2}$.

But we may further employ the operator g_{31}.

For

$$g_{31} = - G_{31} + \ldots = + g_{(\overline{21}\ \overline{10})} + \ldots \quad \text{(Arts. 17, 53)},$$

significant terms only being retained; hence $- G_{31}$ and $g_{(\overline{21}\ \overline{10})} \equiv \partial_{(\overline{21}\ \overline{10})}$ are equivalent operations in the present case, and performing them on their own sides $- 1 = 2A$ or $A = -\frac{1}{2}$.

Thus

$$(\overline{31}\ \overline{01}) = - \tfrac{1}{2} (\overline{21}\ \overline{10})\ (\overline{01}) + \tfrac{1}{2} (\overline{21}\ \overline{01})\ (\overline{10}) + \tfrac{1}{2} (\overline{10}\ \overline{01})\ (\overline{21}) - \tfrac{1}{2} (\overline{21}\ \overline{10}\ \overline{01}).$$

§ 10. *The Partition obliterating Operators.*

60. In the foregoing a generalisation has been made from a number to the partition of a number in the case of the operations g_{10}, g_{01}, $\ldots g_{pq}$, $\ldots$ The possibility of the like generalisation in respect of the obliterating operators G_{10}, G_{01}, $\ldots G_{pq}$, $\ldots$ is naturally presented as a subject for enquiry.

Consider a symmetric function

$$f\ (a_{10},\ a_{01},\ \ldots a_{pq},\ \ldots\ ) = f$$

to be the product of m monomial functions, and write

$$f = f_1 f_2 \ldots f_m.$$

Supposing a_{pq} changed into $a_{pq} + \mu a_{p-1,\,q} + \nu a_{p,\,q-1}$, we have from previous work

$$(1 + \mu G_{10} + \nu G_{01} + \ldots\ldots\ldots + \mu^p \nu^q G_{pq} + \ldots) f$$
$$= (1 + \mu G_{10} + \nu G_{01} + \ldots\ldots + \mu^p \nu^q G_{pq} + \ldots) f_1$$
$$\times\, (1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f_2$$
$$\times \ldots \qquad\qquad\qquad\qquad \vdots$$
$$\times\, (1 + \mu G_{10} + \nu G_{01} + \ldots + \mu^p \nu^q G_{pq} + \ldots) f_m.$$

Expanding the right hand side and equating coefficients of like products of powers μ and ν, we get

$$G_{10} f = \Sigma\, (G_{10} f_1)\, f_2 f_3 \ldots f_m,$$
$$G_{20} f = \Sigma\, (G_{10} f_1)\, (G_{10} f_2)\, f_3 \ldots f_m + \Sigma\, (G_{20} f_1)\, f_2 f_3 \ldots f_m,$$
$$G_{11} f = \Sigma\, (G_{10} f_1)\, (G_{01} f_2)\, f_3 \ldots f_m + \Sigma\, (G_{11} f_1)\, f_2 f_3 \ldots f_m,$$
$$G_{30} f = \Sigma\, (G_{10} f_1)\, (G_{10} f_2)\, (G_{10} f_3)\, f_4 \ldots f_m + \Sigma\, (G_{20} f_1)\, (G_{10} f_2)\, f_3 \ldots f_m + \Sigma\, (G_{30} f_1)\, f_2 \ldots f_m,$$
$$G_{21} f = \Sigma\, (G_{10} f_1)\, (G_{10} f_2)\, (G_{01} f_3)\, f_4 \ldots f_m + \Sigma\, (G_{11} f_1)\, (G_{10} f_2)\, f_3 \ldots f_m,$$
$$= \Sigma\, (G_{20} f_1)\, (G_{01} f_2)\, f_3 \ldots f_m + \Sigma\, (G_{21} f_1)\, f_2 f_3 \ldots f_m,$$

and so forth, where the summations are, in regard to the different terms, obtained by permutation of the m suffixes of the functions $f_1, f_2, \ldots f_m$.

In general in the expression for $G_{pq} f$ there will occur a summation corresponding to each partition of the biweight pq. If a partition be $(\overline{p_1 q_1}\ \overline{p_2 q_2} \ldots \overline{p_s q_s})$ the summation is

$$\Sigma\, (G_{p_1 q_1} f_1)\, (G_{p_2 q_2} f_2) \ldots (G_{p_s q_s} f_s)\, f_{s+1} \ldots f_m.$$

61. Thus, when performed upon a product of functions, the operator G_{pq} breaks up into as many distinct operations as the biweight pq possesses partitions. It is convenient to denote the operation indicated by the summation

$$\Sigma\, (G_{p_1 q_1} f_1)\, (G_{p_2 q_2} f_2) \ldots (G_{p_s q_s})\, f_s f_{s+1} \ldots f_m,$$

by

$$G_{(\overline{p_1 q_1}\ \overline{p_2 q_2} \ldots \overline{p_s q_s})},$$

and to speak of it as a partition operator.

62. We may now write down an equivalence

$$G_{pq} = \Sigma G_{(\overline{p_1 q_1}\ \overline{p_2 q_2} \ldots \overline{p_s q_s})},$$

where the summation is in regard to every partition of the biweight pq. This is, in fact, a theorem for operating with G_{pq} upon a product of symmetric functions, and it is consistent with the more simple law previously established.

In particular

$$G_{10} = G_{(\overline{10})},$$
$$G_{01} = G_{(\overline{01})},$$
$$G_{20} = G_{(\overline{10}^2)} + G_{(\overline{20})},$$
$$G_{11} = G_{(\overline{10}\,\overline{01})} + G_{(\overline{11})},$$
$$G_{02} = G_{(\overline{01}^2)} + G_{(\overline{02})}.$$
$$\vdots$$

63. The relations between the partition g operators and the partition G operators are of great interest.

Recalling the equivalence (Arts. 53 and 41)

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} g_{(\overline{p_1q_1}^{\pi_1}\ \overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} G_{p_1q_1}^{\pi_1} G_{p_2q_2}^{\pi_2}\ldots,$$

which should be compared with the algebraical result

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} s_{(\overline{p_1q_1}^{\pi_1}\ \overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} S_{p_1q_1}^{\pi_1} S_{p_2q_2}^{\pi_2}\ldots,$$

there arise the relations :—

$$g_{(\overline{10})} = G_{10} = G_{(\overline{10})},$$
$$g_{(\overline{01})} = G_{01} = G_{(\overline{01})},$$
$$g_{(\overline{10}^2)} - 2g_{(\overline{20})} = G_{10}^2 - 2G_{20} = G_{(\overline{10})}^2 - 2G_{(\overline{10}^2)} - 2G_{(\overline{20})},$$
$$g_{(\overline{10}\,\overline{01})} - g_{(\overline{11})} = G_{10}G_{01} - G_{11} = G_{(\overline{10})}G_{(\overline{01})} - G_{(\overline{10}\,\overline{01})} - G_{(\overline{11})},$$
$$g_{(\overline{10}^3)} - 3g_{(\overline{20}\,\overline{10})} + 3g_{(\overline{30})} = G_{10}^3 - 3G_{20}G_{10} + 3G_{30},$$
$$= G_{(\overline{10})}^3 - 3G_{(\overline{10}^2)}G_{(\overline{10})} + 3G_{(\overline{10}^3)} - 3\{G_{(\overline{20})}G_{(\overline{10})} - G_{(\overline{20}\,\overline{10})}\} + 3G_{(\overline{30})},$$

and so forth.

64. Now consider the relation last written.

I say that it may be broken up into three relations, viz.:—

$$g_{(\overline{10}^3)} = G_{(\overline{10})}^3 - 3G_{(\overline{10}^2)}G_{(\overline{10})} + 3G_{(\overline{10}^3)},$$
$$g_{(\overline{20}\,\overline{10})} = G_{(\overline{20})}G_{(\overline{10})} - G_{(\overline{20}\,\overline{10})},$$
$$g_{(\overline{30})} = G_{(\overline{30})}\,;$$

for suppose an operand to be composed of separations of a separable partition $(\overline{10}^{\pi_{10}}\ \overline{20}^{\pi_{20}}\ \overline{30}^{\pi_{30}}\ldots)$, the performance of the operations on the two sides of the relation

produces the same result identically. This result is composed of three portions containing separations of the partitions

$$(\overline{10}^{\pi_{10}-3}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}}\ldots), \quad (\overline{10}^{\pi_{10}-1}\,\overline{20}^{\pi_{20}-1}\,\overline{30}^{\pi_{30}}\ldots), \quad (\overline{10}^{\pi_{10}}\,\overline{20}^{\pi_{20}}\,\overline{30}^{\pi_{30}-1}\ldots)$$

respectively. Hence the operations which produce the three identically equal portions of the result must be equivalent, and the three relations between the operators therefore follow.

In general, we may say that of the biweight pq there are as many relations between partition operations as there are partitions of the biweight pq.

65. The general law of the coefficients will be now investigated.

In the result of Art. 53, viz.,

$$\Sigma_\pi \frac{(-)^{\Sigma\pi-1}\,(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_\pi \frac{(-)^{\Sigma\pi-1}\,(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,G_{p_1q_1}^{\pi_1}\,G_{p_1q_1}^{\pi_2}\ldots\,,$$

we have to substitute for $G_{p_1q_1}$, $G_{p_2q_2}\ldots$, the sums of the partition G operators of weights p_1q_1, $p_2q_2\ldots$, respectively; we have then to collect on the right all the G products which are associated with separations of one and the same partition, and to equate them to the corresponding g operator on the left. It is evident that this process does not alter the *law* of the coefficients, and that representing the different separates of the given partition by (J_1), $(J_2)\ldots$, and any separation by $(J_1)^{j_1}\,(J_2)^{j_2}\ldots$, we may write

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,g_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}\,G_{(J_1)}^{j_1}\,G_{(J_2)}^{j_2}\ldots$$

Observe that this is precisely the law which gives the expression of a single bipart function in terms of separations of the partition $(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)$. I recall the result of Art. 41, viz.,

$$(-)^{\Sigma\pi-1}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots}\,s_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)} = \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma j-1)!}{j_1!\,j_2!\ldots}\,(J_1)^{j_1}\,(J_2)^{j_2}\ldots\,,$$

which renders the correspondence between the algebraic and differential theories very striking.

66. Reversing the formula as in the algebraic theory we get the important formula :—

$$(-)^{\Sigma\pi-1}G_{(\overline{p_1q_1}^{\pi_1}\,\overline{p_2q_2}^{\pi_2}\ldots)}$$

$$= \Sigma_j(-)^{\Sigma j-1}\frac{(\Sigma\pi_1-1)!\,(\Sigma\pi_2-1)!\ldots}{j_1!\,j_2!\ldots\pi_{11}!\,\pi_{12}!\ldots\pi_{21}!\,\pi_{22}!\ldots}\,g_{(\overline{p_{11}q_{11}}^{\pi_{11}}\,\overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}^{j_1}\,g_{(\overline{p_{21}q_{21}}^{\pi_{21}}\,\overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}^{j_2}\ldots\,,$$

the summation being for every separation

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots$$

of the partition $(\overline{p_1q_1}^{\pi_1}\ \overline{p_2q_2}^{\pi_2}\ldots)$. (Compare Art. 43.)

Observe that in these formulæ the multiplications of operations are non-symbolic and denote successive operations.

67. Remark the results of operations :—

$$\mathrm{G}_{(\overline{p_1q_1}^{\pi_1}\ \overline{p_2q_2}^{\pi_2}\ldots)}\,s_{\overline{p_1q_1}}{}^{\pi_1}\,s_{\overline{p_2q_2}}{}^{\pi_2}\ldots=1,$$

$$\frac{1}{j_1!}\frac{1}{j_2!}\cdots\,g_{(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)}^{j_1}\,g_{(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)}^{j_2}\cdots$$

$$\cdots(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)^{j_1}(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)^{j_2}\ldots=1.$$

§ 11. *The Multiplication of Symmetric Functions.*

68. The partition G operators are of great service in multiplication. An example will make this clear. It is required to find the coefficient of $(\overline{11^2})$ in the product $(\overline{10^2})(\overline{01})^2$.

Put

$$(\overline{10^2})(\overline{01})^2=\ldots+\mathrm{A}(\overline{11^2})+\ldots$$

On operating with $\overset{2}{\mathrm{G}_{11}}$ on the right the result is A, since every other term is annihilated ; and since

$$\mathrm{G}_{11}=\mathrm{G}_{(\overline{11})}+\mathrm{G}_{(\overline{10}\ \overline{01})},$$

we have

$$\overset{2}{\mathrm{G}_{11}}(\overline{10^2})(\overline{01})^2=\{\mathrm{G}_{(\overline{11})}+\mathrm{G}_{(\overline{10}\ \overline{01})}\}^2(\overline{10^2})(\overline{01})^2=\mathrm{A},$$

therefore

$$\{\mathrm{G}_{(\overline{11})}+\mathrm{G}_{(\overline{10}\ \overline{01})}\}\,.\,2\,(\overline{10})(\overline{01})=\mathrm{A},$$

hence

$$\mathrm{A}=2.$$

Similarly putting

$$(\overline{p_{11}q_{11}}^{\pi_{11}}\ \overline{p_{12}q_{12}}^{\pi_{12}}\ldots)(\overline{p_{21}q_{21}}^{\pi_{21}}\ \overline{p_{22}q_{22}}^{\pi_{22}}\ldots)(\ldots)=\ldots+\mathrm{A}(\overline{r_1s_1}^{\rho_1}\ \overline{r_2s_2}^{\rho_2}\ldots)+\ldots$$

we have merely to operate on the left-hand side with the partition operators equivalent to $\mathrm{G}_{r_1s_1}^{\rho_1}\,\mathrm{G}_{r_2s_2}^{\rho_2}\ldots$ in order to find A.

§ 12. *Symmetric Functions of Differences.*

69. In the unipartite theory there is a transformation which connects the Symmetric Functions of the Differences of the roots of the equation

$$x^n - na_1 x^{n-1} + n(n-1)a_2 x^{n-2} - \ldots + (-)^n n! a_n = 0$$

with the non-unitary symmetric functions of the roots of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots + (-)^n a_n = 0. \qquad (n = \infty)$$

In fact, the annihilating operator is in each case found to be

$$g_1 = \partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \ldots$$

The theory of the Invariants and Covariants of a Binary quantic may be thus brought to depend upon non-unitary symmetric functions. (*Vide* 'American Journal of Mathematics,' vol. 6, p. 131.)

In the present case, there is also a transformation. For the purpose in hand, write the fundamental identity in the form

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_n x + \beta_n y)$$
$$= 1 + na_{10}x + na_{01}y + \ldots + \frac{n!}{(n-p-q)!} a_{pq} x^p y^q + \ldots$$

Any function of the differences of the quantities on the left remains unaltered, when we write for the quantities α_s, β_s, respectively $\alpha_s + h$ and $\beta_s + h$. The coefficient of $x^p y^q$ on the right then becomes

$$\Sigma (\alpha_1 + h)(\alpha_2 + h) \ldots (\alpha_p + h)(\beta_{p+1} + h)(\beta_{p+2} + h) \ldots (\beta_{p+q} + h),$$

which is

$$(\overline{10}^p \, \overline{01}^q) + (n-p-q+1)\{(\overline{10}^{p-1}\,\overline{01}^q) + (\overline{10}^p\,\overline{01}^{q-1})\}h$$
$$+ \frac{(n-p-q+1)(n-p-q+2)}{2!}\{(\overline{10}^{p-2}\,\overline{01}^q) + 2(\overline{10}^{p-1}\,\overline{01}^{q-1}) + (\overline{10}^p\,\overline{01}^{q-2})\}h^2$$
$$+ \ldots$$

But

$$(\overline{10}^p\,\overline{01}^q) = \frac{n!}{(n-p-q)!}a_{pq}.$$

Hence a_{pq} is transformed into

$$a_{pq} + (a_{p-1,q} + a_{p,q-1})h + (a_{p-2,q} + 2a_{p-1,q-1} + a_{p,q-2})\frac{h^2}{2!} + \ldots,$$

the general term being

$$\frac{a_{p-s,\,q-t}\,h^{s+t}}{s!\,t!}.$$

Hence any symmetrical function

$$f(a_{10}, a_{01}, \ldots a_{pq} \ldots) \equiv f,$$

is transformed into

$$f\{a_{10} + h, \ a_{01} + h, \ldots a_{pq} + (a_{p-1, q} + a_{p, q-1})h + \ldots \ldots\},$$

or writing

$$g_{pq} = \partial_{a_{pq}} + a_{10}\partial_{a_{p+1, q}} + a_{01}\partial_{a_{p, q+1}} + \ldots,$$

this is

$$\overline{\exp} \left\{(g_{10} + g_{01})h + (g_{20} + 2g_{11} + g_{02})\frac{h^2}{2!} + \ldots\right\}f,$$

the bar over exp denoting that the multiplications of operators, which arise, are symbolic.

Now, by the theorem of Art. 15, this is

$$\exp\{M_{10}g_{10} + M_{01}g_{01} + \ldots + M_{pq}g_{pq} + \ldots\}f,$$

the multiplication denoting successive operations, and identically

$$\exp(M_{10}\xi + M_{01}\eta + \ldots + M_{pq}\xi^p\eta^q + \ldots)$$

$$= 1 + \xi + \eta + \frac{1}{2!}(\xi + \eta)^2 + \frac{1}{3!}(\xi + \eta)^3 + \ldots$$

$$= \exp(\xi + \eta).$$

Hence $M_{10} = M_{01} = 1$ and the other coefficients M are zero.

Hence the symmetric function f is converted into

$$\exp(g_{10} + g_{01})h \cdot f,$$

and, if f be a function of the differences

$$\exp(g_{10} + g_{01})h \cdot f = f.$$

Hence the necessary and sufficient condition, that f may be a function of the differences, is the satisfaction of the linear partial differential equation

$$g_{10} + g_{01} = 0.$$

70.   These operators g_{10} and g_{01} have been previously met with in the discussion of the symmetric functions connected with the fundamental identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots = 1 + a_{10}x + a_{01}y + \ldots + a_{pq}x^p y^q + \ldots,$$

but then they played a different rôle.*

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* Two simple cases of this important transformation should be verified by the reader.   For $n = 2$, $(\overline{11})$ is transformed into $\frac{1}{2}(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$ connected with the ternary quadric.   For $n = 3$, $(\overline{21})$ is transformed into $\frac{1}{18}\Sigma(\alpha_1 - \alpha_2)\{(\alpha_1 - \alpha_3) + (\alpha_2 - \alpha_3)\}(\beta_1 - \beta_2)$ connected with the ternary cubic.

In that case g_{10} and g_{01} were shewn to annihilate all functions in which the biparts $\overline{10}, \overline{01}$, respectively, were absent. Hence, expressing all such functions in terms of $a_{10}, a_{01}, \ldots a_{pq}, \ldots$ we have at once a number of symmetric functions of difference of the quantities in the identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) \ldots (1 + \alpha_n x + \beta_n y)$$

$$= 1 + n a_{10} x + n a_{01} y + \ldots + \frac{n!}{(n-p-q)!} a_{pq} x^p y^q .$$

71. The differential equation

$$g_{10} + g_{01} = 0,$$

is, as a particular case, satisfied by the solutions of the simultaneous equations

$$g_{10} = 0, \quad g_{01} = 0.$$

In correspondence therewith we have functions composed of differences $\alpha_s - \alpha_t$, $\beta_s - \beta_t$, but not of differences $\alpha_s - \beta_s, \alpha_s - \beta_t$. The functions of differences $\alpha_s - \alpha_t$, $\beta_s - \beta_t$ are represented by the infinite series of monomial symmetric functions whose partitions contain neither of the biparts $\overline{10}, \overline{01}$.

The generating function for the number of biweight pq is

$$\frac{1}{(1 - x^2)(1 - xy)(1 - y^2)(1 - x^3)(1 - x^2 y)(1 - xy^2)(1 - y^3) \ldots} .$$

The remaining functions of differences correspond to those solutions of $g_{10} + g_{01} = 0$ which are not simultaneous solutions of $g_{10} = 0$ and $g_{01} = 0$.

Denoting by N any aggregate of biparts from which both $\overline{10}$ and $\overline{01}$ are excluded, we have the system of solutions

$$(\overline{10}\,\mathrm{N}) - (\overline{01}\,\mathrm{N})$$

$$(\overline{10}^2\,\mathrm{N}) - (\overline{10}\,\overline{01}\,\mathrm{N}) + (\overline{01}^2\,\mathrm{N}),$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$(\overline{10}^{p+q}\,\mathrm{N}) - (\overline{10}^{p+q-1}\,\overline{01}\,\mathrm{N}) + \ldots + (-)^q (\overline{10}^p\,\overline{01}^q\,\mathrm{N}) + \ldots,$$

for on operating with $g_{10} + g_{01} = \mathrm{G}_{10} + \mathrm{G}_{01}$ the terms destroy each other in pairs.

Observe that these solutions are of the same weight but not of the same biweight in every term.

The number of solutions of a given weight is given by the generating function

$$\frac{x}{(1 - x)(1 - x^2)^3(1 - x^3)^4 \ldots (1 - x^\mu)^{\mu+1} \ldots} .$$

3 x 2

Hence the *whole* number of asyzygetic functions of differences of a given weight is given by

$$\frac{1}{(1-x)(1-x^2)^3(1-x^3)^4\ldots(1-x^\mu)^{\mu+1}\ldots}.$$

§ 13. *Special Fundamental Identity of Finite Order.*

72. By taking the fundamental identity of infinite order syzygetic relations between monomial symmetric functions were avoided. Whenever the fundamental identity is taken of a finite order > 1 certain such relations of necessity arise.

Professor CAYLEY ('Collected Papers,' vol. 2, p. 454, and 'Phil. Trans.,' 1857) takes a fundamental identity equivalent to

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + 2hx + 2gy^2 + bx^2 + 2fxy + cy^2,$$

and finds identically

$$bc - f^2 - bg^2 - ch^2 + 2fgh = 0,$$

the condition that the expression to the right shall break up into two linear factors.

I take as the fundamental identity

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$

and observe that the syzygetic relation must connect monomial symmetric functions, each of which is symbolised by a partition containing more than two biparts. The symmetric functions must be of the same biweight of the form pp since the quantities α must occur symmetrically with the quantities β. Of the biweight 1, 1, there exists no partition containing more than two biparts. Of the biweight 2,2, we have the four partitions

$$(\overline{20}\ \overline{01}^2),\ (\overline{02}\ \overline{10}^2),\ (\overline{11}\ \overline{10}\ \overline{01}),\ (\overline{10}^2\ \overline{01}^2),$$

and if the corresponding symmetric functions can be linearly connected, so that no fundamental symmetric function of weight greater than 2 occurs, the linear function must vanish.

From the tables, *post* § 14, biweight 22, partition $(\overline{10}^2\ \overline{01}^2)$, we find

$$(\overline{20}\ \overline{01}^2) - (\overline{11}\ \overline{10}\ \overline{01}) + (\overline{02}\ \overline{10}^2) = -4a_{20}a_{02} + a_{11}^2 + a_{20}a_{01}^2 + a_{02}a_{10}^2 - a_{11}a_{10}a_{01};$$

the terms involving a_{22}, a_{21}, and a_{12} disappearing.

73. This is right, and shows (changing sign) that the well-known expression (discriminant)

$$bc - f^2 - bg^2 - ch^2 + 2fgh,$$

is equal to

$$- (\overline{20}\ \overline{01}^2) + (\overline{11}\ \overline{10}\ \overline{01}) - (\overline{02}\ \overline{10}^2),$$

a form which, for the ternary quadric, vanishes at sight.

Another form is

$$- s_{20}a_{02} + s_{11}a_{11} - s_{02}a_{20}.$$

74. The expression

$$4a_{20}a_{02} - a_{11}^2 - a_{20}a_{01}^2 - a_{02}a_{10}^2 + a_{11}a_{10}a_{01}$$

satisfies the partial differential equation which appertains to the differences of the quantities in the relation

$$(1 + \alpha_1 x + \beta_1 y)(1 + \alpha_2 x + \beta_2 y) = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2.$$

This equation is

$$2\,\partial_{a_{10}} + a_{10}\,\partial_{a_{20}} + a_{01}\,\partial_{a_{11}} + 2\,\partial_{a_{01}} + a_{10}\,\partial_{a_{11}} + a_{01}\,\partial_{a_{02}} = 0.$$

It is not, as a fact, expressible as a function of differences of α_1, β_1, α_2, β_2, because it vanishes altogether for a fundamental identity of the order 2.

In relation to a fundamental identity of order greater than 2, the expression does not satisfy the equation of differences. Although it may be regarded from the above as a vanishing function of the differences, it is convertible into a non-vanishing function by the transformation before given. The transformed expression is

$$(\overline{20}\ \overline{02}) - 2\,(\overline{11}^2) \quad \text{or} \quad (s_{20}s_{02} - s_{11}^2),$$

which visibly satisfies the differential equation

$$g_{10} + g_{01} \equiv \partial_{s_{10}} + \partial_{s_{01}} = 0.$$

§ 14. *The Construction of Symmetrical Tables.*

75. From the first law of symmetry it has been established that it is possible to form two symmetrical tables in connexion with every partition of every biweight. As illustrations, I give certain of the results as far as weight 4, inclusive. We have presented for the weight 4 the biweights 40, 31, 22, 13, 04. The theory of the biweights 40 and 04 is precisely the same as that of the weight 4 in the unipartite theory.* The one is, in fact, concerned only with the single system of quantities

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* *Vide* 'American Journal of Mathematics,' vol. 11, and succeeding volumes.

α_1, α_2, α_3, ... and the other only with the single system β_1, β_2, β_3, ... We may, therefore, suppress altogether the zero elements in the biparts and then proceed to form the tables for the several partitions (unipart) of the weight which I have already set forth in the 'American Journal of Mathematics' (vol. 11).

Of the remaining biweights 31, 22, 13, it is merely necessary to calculate the two former, since the tables for the biweight 13 are obviously immediately obtainable from those of the biweight 31 by interchanging the elements of each bipart :—e.g., by writing $\overline{qp}$ for $\overline{pq}$.

There are seven partitions of biweight 31, viz. :—

$$(\overline{10^3\ 01}),\ (\overline{11}\ \overline{10^2}),\ (\overline{20}\ \overline{10}\ \overline{01}),\ (\overline{20}\ \overline{11}),\ (\overline{21}\ \overline{10}),\ (\overline{30}\ \overline{01}),\ (\overline{31}),$$

and nine of the biweight 22, viz. :—

$$(\overline{10^2\ 01^2}),\ (\overline{11}\ \overline{10}\ \overline{01}),\ (\overline{11^2}),\ (\overline{02}\ \overline{10^2}),\ (\overline{20}\ \overline{02}),\ (\overline{12}\ \overline{10}),\ (\overline{22}).$$
$$(\overline{20}\ \overline{01^2}),\qquad\qquad (\overline{21}\ \overline{01}),$$

Of these the table of $(\overline{20}\ \overline{01^2})$ gives also the table of $(\overline{02}\ \overline{10^2})$ by transposing the elements of the biparts, and similarly the table of $(\overline{21}\ \overline{01})$ gives that of $(\overline{12}\ \overline{10})$. We have thus 28 tables; but of these, the four corresponding to the partitions $(\overline{31})$ and $(\overline{22})$ are mere identities, so that the number is reduced to 24. The earlier tables which are necessary are those of the partitions $(\overline{10}\ \overline{01})$, $(\overline{20}\ \overline{01})$, $(\overline{11}\ \overline{10})$, $(\overline{10^2}\ \overline{01})$. These are now given. Each table is read according to the lines.

BIWEIGHT 11.

Partition $(\overline{10}\ \overline{01})$.

	$(\overline{10}\ \overline{01})$	$(\overline{10})\ (\overline{01})$	
$(\overline{11})$	— 1	. 1	± 1
$(\overline{10}\ \overline{01})$	1		

	$(\overline{11})$	$(\overline{10}\ \overline{01})$
$(\overline{10}\ \overline{01})$		1
$(\overline{10})\ (\overline{01})$	1	1

BIWEIGHT 21.

Partition $(\overline{20}\ \overline{01})$.

	$(\overline{20}\ \overline{01})$	$(\overline{20})\ (\overline{01})$	
$(\overline{21})$	— 1	1	± 1
$(\overline{20}\ \overline{01})$	1		

	$(\overline{21})$	$(\overline{20}\ \overline{01})$
$(\overline{20}\ \overline{01})$		1
$(\overline{20})\ (\overline{01})$	1	1

Biweight 21.

Partition $(\overline{11}\,\overline{01})$.

	$(\overline{11}\,\overline{10})$	$(\overline{11})(\overline{10})$	
$(\overline{21})$	-1	1	± 1
$(\overline{11}\,\overline{10})$	1		

	$(\overline{21})$	$(\overline{11}\,\overline{10})$
$(\overline{11}\,\overline{10})$		1
$(\overline{11})(\overline{10})$	1	1

Biweight 21.

Partition $(\overline{10}^2\,\overline{01})$.

	a_{21}	$a_{20}a_{01}$	$a_{11}a_{10}$	$a_{10}{}^2 a_{01}$	
$(\overline{21})$	1	-1	-1	1	± 2
$(\overline{20}\,\overline{01})$	-1	-1	1		
$(\overline{11}\,\overline{10})$	-1	1			± 1
$(\overline{10}^2\,\overline{01})$	1				

	$(\overline{21})$	$(\overline{20}\,\overline{01})$	$(\overline{11}\,\overline{10})$	$(\overline{10}^2\overline{01})$
a_{21}				1
$a_{20}a_{01}$			1	1
$a_{11}a_{10}$		1	1	2
$a_{10}{}^2 a_{01}$	1	1	2	2

Biweight 31.

Partition $(\overline{30}\,\overline{01})$.

	$(\overline{30}\,\overline{01})$	$(\overline{30})(\overline{01})$	
$(\overline{31})$	-1	1	± 1
$(\overline{30}\,\overline{01})$	1		

	$(\overline{31})$	$(\overline{30}\,\overline{01})$
$(\overline{30}\,\overline{01})$		1
$(\overline{30})(\overline{01})$	1	1

Partition $(\overline{20}\,\overline{10})$.

	$(\overline{21}\,\overline{10})$	$(\overline{21})(\overline{10})$	
$(\overline{21})$	-1	1	± 1
$(\overline{21}\,\overline{10})$	1		

	$(\overline{31})$	$(\overline{21}\,\overline{10})$
$(\overline{21}\,\overline{10})$		1
$(\overline{21})(\overline{10})$	1	1

Partition $(\overline{20}\,\overline{11})$.

	$(\overline{20}\,\overline{11})$	$(\overline{20})(\overline{11})$	
$(\overline{31})$	-1	1	± 1
$(\overline{20}\,\overline{11})$	1		

	$(\overline{31})$	$(\overline{20}\,\overline{11})$
$(\overline{20}\,\overline{11})$		1
$(\overline{20})(\overline{11})$	1	1

Partition $(\overline{20}\,\overline{10}\,\overline{01})$.

	$(\overline{20}\,\overline{10}\,\overline{01})$	$(\overline{20}\,\overline{10})(\overline{01})$	$(\overline{20}\,\overline{01})(\overline{10})$	$(\overline{10}\,\overline{01})(\overline{20})$	$(\overline{20})(\overline{10})(\overline{01})$	
$(3\overline{1})$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	1	$\pm1\tfrac{1}{2}$
$(30\,\overline{01})$	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	$\tfrac{1}{2}$		±1
$(2\overline{1}\,\overline{10})$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$		±1
$(20\,\overline{11})$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$		±1
$(\overline{20}\,\overline{10}\,\overline{01})$	1					

	$(3\overline{1})$	$(30\,\overline{01})$	$(2\overline{1}\,\overline{10})$	$(20\,\overline{11})$	$(\overline{20}\,\overline{10}\,\overline{01})$
$(\overline{20}\,\overline{10}\,\overline{01})$					1
$(\overline{20}\,\overline{10})(\overline{01})$			1	1	1
$(\overline{20}\,\overline{01})(\overline{10})$		1		1	1
$(\overline{10}\,\overline{01})(\overline{20})$		1	1		1
$(\overline{20})(\overline{10})(\overline{01})$	1	1	1	1	1

Partition $(\overline{11}\,\overline{10}{}^2)$.

	$(\overline{11}\,\overline{10}{}^2)$	$(\overline{10}{}^2)(\overline{11})$	$(\overline{11}\,\overline{10})(\overline{10})$	$(\overline{11})(\overline{10})^2$	
$(3\overline{1})$	1	-1	-1	1	±2
$(20\,\overline{11})$	-1	-1	1		
$(2\overline{1}\,\overline{10})$	-1	1			±1
$(\overline{11}\,\overline{10}{}^2)$	1				

	$(3\overline{1})$	$(20\,\overline{11})$	$(2\overline{1}\,\overline{10})$	$(\overline{11}\,\overline{10}{}^2)$
$(\overline{11}\,\overline{10}{}^2)$				1
$(\overline{10}{}^2)(\overline{11})$			1	1
$(\overline{11}\,\overline{10})(\overline{10})$		1	1	2
$(\overline{11})(\overline{10})^2$	1	1	2	2

Partition ($\overline{10}^3\,\overline{01}$).

	a_{31}	$a_{30}a_{01}$	$a_{21}a_{10}$	$a_{20}a_{11}$	$a_{20}a_{10}a_{01}$	$a_{11}a_{10}^2$	$a_{10}^3a_{01}$	
($\overline{31}$)	-1	1	1	1	-2	-1	1	$\pm\,4$
($\overline{30}\,\overline{01}$)	1	2	-1	-1	-1	1		
($\overline{21}\,\overline{10}$)	1	-1	0	-1	1			$\pm\,2$
($\overline{20}\,\overline{11}$)	1	-1	-1	1				$\pm\,2$
($\overline{20}\,\overline{10}\,\overline{01}$)	-2	-1	1					
($\overline{11}\,\overline{10}^2$)	-1	1						$\pm\,1$
($\overline{10}^3\,\overline{01}$)	1							

	($\overline{31}$)	($\overline{30}\,\overline{01}$)	($\overline{21}\,\overline{10}$)	($\overline{20}\,\overline{11}$)	($\overline{20}\,\overline{10}\,\overline{01}$)	($\overline{11}\,\overline{10}^2$)	($\overline{10}^3\,\overline{01}$)
a_{31}							1
$a_{30}a_{01}$						1	1
$a_{21}a_{10}$					1	1	3
$a_{20}a_{11}$				1	1	2	3
$a_{20}a_{10}a_{01}$			1	1	1	3	3
$a_{11}a_{10}^2$		1	1	2	3	4	6
$a_{10}^3a_{01}$	1	1	3	3	3	6	6

BIWEIGHT 22.

Partition ($\overline{21}\,\overline{01}$).

	($\overline{21}\,\overline{01}$)	($\overline{21}$)($\overline{01}$)	
($\overline{22}$)	-1	1	$\pm\,1$
($\overline{21}\,\overline{01}$)	1		

	($\overline{22}$)	($\overline{21}\,\overline{01}$)
($\overline{21}\,\overline{01}$)		1
($\overline{21}$)($\overline{01}$)	1	1

Partition $(\overline{20}\,\overline{02})$.

	$(\overline{20}\,\overline{02})$	$(\overline{20})(\overline{02})$	
$(\overline{22})$	-1	1	± 1
$(\overline{20}\,\overline{02})$	1		

	$(\overline{22})$	$(\overline{20}\,\overline{02})$
$(\overline{20}\,\overline{02})$		1
$(\overline{20})(\overline{02})$	1	1

Partition $(\overline{11}^2)$.

	$(\overline{11}^2)$	$(\overline{11})^2$
$(\overline{22})$	-2	1
$(\overline{11}^2)$	1	

	$(\overline{22})$	$(\overline{11}^2)$
$(\overline{11}^2)$		1
$(\overline{11})^2$	1	2

Partition $(\overline{20}\,\overline{01}^2)$.

	$(\overline{20}\,\overline{01}^2)$	$(\overline{20})(\overline{01}^2)$	$(\overline{20}\,\overline{01})(\overline{01})$	$(\overline{20})(\overline{01})^2$	
$(\overline{22})$	1	-1	-1	1	± 2
$(\overline{20}\,\overline{02})$	-1	-1	1		
$(\overline{21}\,\overline{01})$	-1	1			± 1
$(\overline{20}\,\overline{01}^2)$	1				

	$(\overline{22})$	$(\overline{20}\,\overline{02})$	$(\overline{21}\,\overline{01})$	$(\overline{20}\,\overline{01}^2)$
$(\overline{20}\,\overline{01}^2)$				1
$(\overline{20})(\overline{01}^2)$			1	1
$(\overline{20}\,\overline{01})(\overline{01})$		1	1	2
$(\overline{20})(\overline{01})^2$	1	1	2	2

Partition $(\overline{11}\ \overline{10}\ \overline{01})$.

	$(\overline{11}\ \overline{10}\ \overline{01})$	$(\overline{11}\ \overline{10})(\overline{01})$	$(\overline{11}\ \overline{01})(\overline{10})$	$2(\overline{10}\ \overline{01})(\overline{11})$	$(\overline{11})$	$(\overline{10})(\overline{01})$
$(\overline{2}\overline{2})$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{1}{4}$	1	$\pm\tfrac{3}{2}$
$(\overline{21}\ \overline{01})$	$-\tfrac{1}{2}$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	$\tfrac{1}{4}$		± 1
$(\overline{12}\ \overline{10})$	$-\tfrac{1}{2}$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	$\tfrac{1}{4}$		± 1
$(\overline{11}^2)$	$-\tfrac{1}{4}$	$\tfrac{1}{4}$	$\tfrac{1}{4}$	$-\tfrac{1}{8}$		$\pm\tfrac{1}{2}$
$(\overline{11}\ \overline{10}\ \overline{01})$	1					

	$(\overline{2}\overline{2})$	$(\overline{21}\ \overline{01})$	$(\overline{12}\ \overline{10})$	$(\overline{11}^2)$	$(\overline{11}\ \overline{10}\ \overline{01})$
$(\overline{11}\ \overline{10}\ \overline{01})$					1
$(\overline{11}\ \overline{10})(\overline{01})$			1	2	1
$(\overline{11}\ \overline{01})(\overline{10})$		1	0	2	1
$2(\overline{10}\ \overline{01})(\overline{11})$		2	2	0	2
$(\overline{11})(\overline{10})(\overline{01})$	1	1	1	2	1

Partition $(\overline{10}^2\,\overline{01}^2)$.

	a_{22}	$a_{21}a_{01}$	$a_{12}a_{10}$	$a_{20}a_{02}$	a_{11}^2	$a_{20}a_{01}^2$	$a_{02}a_{10}^2$	$a_{11}a_{10}a_{01}$	$a_{10}^2a_{01}^2$	
$(\overline{22})$	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	1	$\pm\frac{10}{3}$
$(\overline{21}\,\overline{01})$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$		± 2
$(\overline{12}\,\overline{01})$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$		± 2
$(\overline{20}\,\overline{02})$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$		
$(\overline{11}^2)$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		$\pm\frac{4}{3}$
$(\overline{20}\,\overline{01}^2)$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		
$(\overline{02}\,\overline{10}^2)$	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$		
$(\overline{11}\,\overline{10}\,\overline{01})$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$		$\pm\frac{7}{3}$
$(\overline{10}^2\,\overline{01}^2)$	1									

	$(\overline{22})$	$(\overline{21}\,\overline{01})$	$(\overline{12}\,\overline{10})$	$(\overline{20}\,\overline{02})$	$(\overline{11}^2)$	$(\overline{20}\,\overline{01}^2)$	$(\overline{02}\,\overline{10}^2)$	$(\overline{11}\,\overline{10}\,\overline{01})$	$(\overline{10}^2\,\overline{01}^2)$
a_{22}									1
$a_{21}a_{01}$							1	1	2
$a_{12}a_{10}$						1	0	1	2
$a_{20}a_{02}$					1	0	0	1	1
a_{11}^2				1	2	2	2	2	4
$a_{20}a_{01}^2$			1	0	2	0	1	2	2
$a_{02}a_{10}^2$		1	0	0	2	1	0	2	2
$a_{11}a_{10}a_{01}$		1	1	1	2	2	2	3	4
$a_{10}^2a_{01}^2$	1	2	2	1	4	2	2	4	4

§ 15. *Property of the Coefficients in the Tables.*

76. There is in regard to the coefficients a very simple and important property which does not come into view with the unipartite theory so long as the tables are restricted to the particular cases in which the separable partitions are composed entirely of units. The property appears the instant we consider a separable partition composed of parts which are *not* all similar. The law is the same whether the symmetric functions are unipartite, bipartite, or in general m-partite. It depends upon the possibility of grouping the various separations in a particular manner. To make this clear suppose we are presented with a separable partition $(\overline{10}^2\ \overline{01}^2)$. The nine separations may be written down in four groups, as follows :—

Group 1.	Group 2.	Group 3.	Group 4.
$(\overline{10})^2\,(\overline{01})^2$	$(\overline{10}\ \overline{01}^2)\,(\overline{10})$	$(\overline{10}^2\ \overline{01})\,(\overline{01})$	$(\overline{10}^2\ \overline{01}^2)$
$(\overline{11})\,(\overline{10})\,(\overline{01})$	$(\overline{01}^2)\,(\overline{10})^2$	$(\overline{10}^2)\,(\overline{01})^2$	$(\overline{10}^2)\,(\overline{01}^2)$
$(\overline{11})^2$			

In Group 2 it will be seen that the parts $(\overline{10}^2)$ of the separable partition occur in the separation $(\overline{10})^2$, while the parts $(\overline{01}^2)$ occur in the separation $(\overline{01}^2)$, so that the expression $\{(\overline{10})^2, (\overline{01}^2)\}$ may be taken as defining a certain separation property of the separations of the group. The group in question may be denoted by $Gr\ \{(\overline{10})^2, (\overline{01}^2)\}$ and on the same principle the Groups 1, 3, and 4 may be denoted by

$$Gr\ \{(\overline{10})^2, (\overline{01})^2\}, \quad Gr\ \{(\overline{10}^2), (\overline{01})^2\}, \quad Gr\ \{(\overline{10}^2), (\overline{01}^2)\}$$

respectively. In the separable partition $(\overline{10}^2\ \overline{01}^2)$ the parts $(\overline{10})$ and $(\overline{01})$ occur each twice, and a group results from every combination of a partition of 2 with a partition of 2. If P_2 denote the number of partitions of 2, the number of groups will be $P_2^2 = 4$. In general if the different parts of the separable partition occur $a, b\ c, \dots$ times the number of groups of separations is $P_a P_b P_c \dots$.

77. The leading property that has been adverted to is that in the expression of a single-bipart function by means of separations of a partition composed of dissimilar parts, the algebraic sum of the coefficients in each group of separations is zero. A corollary at once follows which will be given in its proper place.

78. From the identity of Art. 24, viz. :—

$$1 + c_{10}\xi + c_{01}\eta + \dots + c_{pq}\xi^p\eta^q + \dots = \Pi_s\left(1 + \alpha_s b_{10}\xi + \dots + \alpha_s{}^p\beta_s{}^q b_{pq}\xi^p\eta^q + \dots\right)$$

is derived a series of relations which express the quantities c in terms of the

quantities b and symmetric functions of the quantities a, β. These are given in Art. 24.

To put the *group* in evidence it is necessary to modify these relations by writing $b_{\overline{r_1 s_1}^{\rho_1}}$ for $b_{r_1 s_1}^{\rho_1}$, so that for examples the expressions for c_{20} and c_{11} become

$$(\overline{20})\, b_{20} + (\overline{10^2})\, b_{\overline{10}^2}$$

$$(\overline{11})\, b_{11} + (\overline{10}\ \overline{01})\, b_{10} b_{01}$$

respectively. In any product $c_{p_1 q_1}{}^{\pi_1} c_{p_2 q_2}{}^{\pi_2} \ldots$ the cofactor of the product

$$b_{\overline{r_1 s_1}^{\rho_1}}^{\sigma_1}\, b_{\overline{r_1 s_1}^{\rho_2}}^{\sigma_2} \ldots b_{\overline{t_1 u_1}^{\tau_1}}^{v_1}\, b_{\overline{t_1 u_1}^{\tau_2}}^{v_2}\,,$$

is composed of symmetric function products each of which appertains to the group

$$Gr\ \{(\overline{r_1 s_1}^{\rho_1})^{\sigma_1}\, (\overline{r_1 s_1}^{\rho_2})^{\sigma_2} \ldots,\ (\overline{t_1 u_1}^{\tau_1})^{v_1}\, (\overline{t_1 u_1}^{\tau_2})^{v_2} \ldots\}.$$

The sum of the coefficients attached to the members of this group is obtained by putting each monomial symmetric function equal to unity. The sum in question then appears as the numerical coefficient of the b product above written.

Write then

$$c_{10}{}^1 = b_{10}$$
$$c_{01}{}^1 = b_{01}$$
$$c_{20}{}^1 = b_{20} + b_{\overline{10}^2}$$
$$c_{11}{}^1 = b_{11} + b_{10} b_{01}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

so that, ξ and η being arbitrary,

$$1 + c_{10}{}^1 \xi + c_{01}{}^1 \eta + \ldots + c_{pq}{}^1 \xi^p \eta^q + \ldots$$
$$= (1 + b_{10}\xi + b_{\overline{10}^2}\xi^2 + b_{\overline{10}^3}\xi^3 + \ldots) \ldots (1 + b_{pq}\xi^p \eta^q + b_{\overline{pq}^2}\xi^{2p}\eta^{2q} + \ldots) \ldots$$

a factor appearing on the right for each biweight.

To find the sum of the coefficients in each group in the case of the expression of the single-bipart functions we have now merely to take logarithms when (Art. 26) the functions being s_{pq} or $(\overline{pq})$, the sum in each case presents itself multiplied by $(-)^{p+q-1}(1/p!\,q!)(p+q-1)!$. Expanding the right hand side after taking logarithms we see that only terms of the form

$$b_{\overline{rs}^{\rho_1}}^{\sigma_1}\, b_{\overline{rs}^{\rho_2}}^{\sigma_2}\, ..$$

can appear. Hence the theorem :—

"In the expression of symmetric function $(\overline{pq})$ by means of separations of any partition of the same biweight, the partition consisting of dissimilar parts, the algebraic sum of the coefficients in each group of separations is zero."

As regards the remaining cases where the separable partition does not contain dissimilar parts, the *group* obviously contains but a single separation and *quâ group* has no existence.

We have in fact the expression of $(\overline{pq})$ by means of separations of $(\overline{rs^k})$ where $kr = p$, $ks = q$.

The result is clearly

$$(-)^{p+q-1} \frac{(p+q-1)!}{p!\,q!} (\overline{pq}) = \Sigma (-)^{\Sigma\sigma - 1} \frac{(\Sigma\sigma - 1)!}{\sigma_1!\,\sigma_2!\ldots} (\overline{rs^{\rho_1}})^{\sigma_1} (\overline{rs^{\rho_2}})^{\sigma_2} \ldots$$

79. The law of group of separations may be verified from the tables. It is a very satisfactory aid to calculation, particularly in the detection of missing separations.

Moreover the law embraces symmetric functions other than those symbolised by a single bipart. Suppose the function expressed in terms of single-bipart functions. The latter may be separately expressed in terms of separations of partitions in such wise that the function in question will be represented by means of separations of any given partition of its biweight. The law of the group will hold for the single-bipart functions whenever the separable partition contains dissimilar parts, and moreover, in a product of single-bipart functions the law will hold if one or more of the factors is expressed in terms of separations of a partition containing dissimilar parts. Hence the only exception occurs when we find presented a product of the form

$$S_{(\overline{r_1 s_1}^{\rho_1})} \; S_{(\overline{r_2 s_2}^{\rho_2})} \; S_{(\overline{r_3 s_3}^{\rho_3})} \ldots ;$$

now if the symmetric function, say $(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots)$, whose expression we are considering in connection with a given separable partition, say $(\overline{a_1 b_1}^{a_1} \overline{a_2 b_2}^{a_2} \ldots)$, itself possesses a separation of specification

$$(\overline{a_1 a_1}, \; \overline{a_1 b_1} \; \overline{a_2 a_2}, \; \overline{a_2 b_2}, \; \ldots)$$

a product of this form will certainly occur, but not otherwise.

Hence the theorem :—

80. "In the expression of symmetric function

$$(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \ldots),$$

by means of separations of

$$(\overline{a_1 b_1}^{a_1} \overline{a_2 b_2}^{a_2} \ldots),$$

the algebraic sum of the coefficients in each group of separations is zero if the partition

$(\overline{p_1 q_1}^{\pi_1} \overline{p_2 q_2}^{\pi_2} \dots)$ possesses no separations of specification $(\overline{a_1 a_1, a_1 b_1}, \overline{a_2 a_2, a_2 b_2} \dots)$ but not otherwise."

The law may be verified in the case of the table of separations of $(\overline{10^2}\ \overline{01^2})$, for the symmetric functions $(\overline{22})$, $(\overline{21}\ \overline{01})$, $(\overline{12}\ \overline{10})$, $(\overline{11^2})$, $(\overline{11}\ \overline{10}\ \overline{01})$ for none of these five functions can be separated so that the specification is $(\overline{20}\ \overline{02})$. On the other hand the group law does not hold for $(\overline{20}\ \overline{01^2})$ because the separation $(\overline{20})\ (\overline{01^2})$ has a specification $(\overline{20}\ \overline{02})$.

§ 16. *Conclusion.*

81. All the preceding results can be easily extended to the m-partite theory connected with m systems. The weights are m-partite as also the parts of the partitions. As a general rule m suffices appear in the symbols. The laws of symmetry and their consequences, the symmetrical tables, the correspondences between the algebras of quantity and differential operation, the partition linear and obliterating operators, the law of groups of coefficients (and in fact the whole investigation here presented) proceed *pari-passu* with the bipartite theory above set forth. The unipartite or ordinary theory of the single system is also absolutely included in every respect.

In its applications, the results will be chiefly of use in the theory of elimination in the most general case. In this regard SCHLÄFLI's memoir (*loc. cit.*) may be consulted.

VIII. *Account of recent Pendulum Operations for determining the relative Force of Gravity at the Kew and Greenwich Observatories.*

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(*Communicated at the request of the Kew Committee.*)

The recent pendulum observations for the purpose of determining the gravity connexion between the Royal Observatory at Greenwich and the Royal Society's Observatory at Kew, were undertaken in order to improve and strengthen the connexion between the Indian series of pendulum operations and other series taken in other parts of the world.

The Indian series had been carried out in the years 1865 to 1873, when two invariable pendulums, the property of the Royal Society, which had been designed by Captain Kater for the purpose of investigating the relative force of gravity in different latitudes, were swung at the Kew Observatory, and at various places in and on the way to India, in the course of the operations of the Great Trigonometrical Survey of India. The work was originated at the suggestion of the President of the Royal Society, General Sir E. Sabine; the greater portion was performed by Captain J. P. Basevi, R.E., who lost his life from exposure while operating on the high table-lands of the Himalayan Mountains; the remainder was completed by Captain W. J. Heaviside, R.E.; both officers acted under the personal superintendence of General J. T. Walker, the Superintendent of the Great Trigonometrical Survey.

The points at which the pendulums were swung and the number of vibrations they · made in 24 hours were determined, were mostly stations of the Central Meridional Arc of the Survey which extends from Cape Comorin to the Himalayan Mountains; a few stations were added on the East and West Coasts of India, and on neighbouring islands, and also at Aden and Ismailia. The base station of the entire series of operations—that is to say, the one at which they were commenced and concluded—was the Royal Society's Observatory at Kew, near Richmond, Surrey.

With a view to effecting a connexion between the operations in India and similar operations recently completed in Russia, and also for other reasons, two reversible pendulums, the property of the Russian Imperial Academy of Sciences, which had been employed in Russia, were sent out to India and swung at some of the Indian stations, *pari passu*, with the pendulums of the Royal Society.

For the purpose of connecting the Indian operations with those taken by KATER, SABINE, FOSTER, and other observers in various parts of the globe, it was intended to swing the Royal Society's pendulums, on their return to England, at the Royal Observatory, Greenwich, which was a station of various important series of operations. But when the time arrived, in 1873, it was found that extensive preparations were being made in the Observatory for several expeditions which were being outfitted for the observation of the approaching transit of Venus, so that no space was left available for the pendulum operations. It was therefore decided to make the desired connexion by swinging KATER's convertible pendulum, for determining the absolute length of the seconds' pendulum in any latitude, at Kew, as already it had been swung by SABINE at Greenwich. This was done by Captain HEAVISIDE, and was the last stage of the Indian pendulum operations; the results were published in vol. 5 of the ' Account of the Operations of the Great Trigonometrical Survey,' Dehra Dun, 1879, which gives full details of all the operations, including the swings with the Russian pendulums.

The absolute length of the seconds' pendulum, *in vacuo*, at Kew, was found to be 39·14008 inches of the British standard yard, in 1873, whereas at Greenwich, SABINE had found it to be 39·13734 inches of Sir GEORGE SHUCKBURGH's standard scale, in 1831.* Thus it would seem that a seconds' pendulum will make about three more vibrations in 24 hours at Kew than at Greenwich. But the two Observatories are nearly in the same latitude, and differ very little in height, and are only ten miles apart; thus this difference is much too large to be accepted as trustworthy.

In his work on Geodesy, Colonel CLARKE, C.B., R.E., of the Ordnance Survey of Great Britain, employs a large number of pendulum determinations in various parts of the globe to investigate the figure of the earth. He remarks that the selection of Kew, instead of London or Greenwich, as the base station for the Indian series of swings, was unfortunate; and, disregarding the connexion of Kew with Greenwich by the two determinations of the length of the seconds' pendulum, he employs the ratio of Madras to London from the observations of GOLDINGHAM and SABINE, and that of Kew to St. Petersburgh, by HEAVISIDE and SAWITSCH. Then he makes four different combinations of his data, from which he obtains as many values of the earth's ellipticity; and for each value he finds the corresponding system of quantities, *x*, indicating the apparent excess or defect of the observed over the theoretical force of gravity at each station of observation; one combination gives Kew an excess of 6·06 vibrations over London; another gives it an excess of 5·12 vibrations over Greenwich, but reduces the excess over London to 3·10 vibrations. These figures indicate variations in the vibration numbers such as are usually met with on changes of latitude of 1 to 2 degrees, and they show that the actual relation of Greenwich to

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* It has recently been ascertained that very little, if any, of the difference can be due to error of the unit of the SHUCKBURGH scale as compared with the standard yard. See No. 288 of the ' Proceedings of the Royal Society ' (vol. 47, 1890, p. 186.)

Kew had not yet been precisely determined, and that special observations were still required for the purpose.

In 1881, Colonel HERSCHEL, R.E., was deputed by the Secretary of State for India to take pendulum observations at the Greenwich and the Kew Observatories; also at some places in America, with a view to making a connexion with the pendulum operations of the Coast and Geodetic Survey of the United States. He employed the two pendulums of the Royal Society which had been used in the Indian operations, and also a third pendulum of precisely similar construction which had been deposited in the Kew Observatory by the Admiralty, the experience already gained in India having shown that the employment of a third pendulum was desirable. After completing his swings in England and America, he made over the three pendulums to officers of the United States' Survey, who took them round the world, and swung them at Auckland, Sydney, Singapore, Tokio, San Francisco, and finally at Colonel HERSCHEL's station in Washington.*

When the observations came to be finally reduced, it was found that the results between Kew and Greenwich by the three pendulums were largely discordant, one giving Kew an excess of 1·97 vibrations, another an excess of 1·39, while the third gave a defect of 4·98 vibrations. It was obviously necessary that the pendulums should be again swung at the two places, in order to obtain a more satisfactory determination of the relative vibration numbers. Fresh swings were therefore made at Kew in 1888, and at Greenwich in the following year. The operations were performed by members of the Observatory staff at each place, Mr. HOLLIS taking the lead and responsibility at Greenwich, under the direction of the Astronomer Royal, and Mr. CONSTABLE at Kew, under the Superintendent of the Kew Observatory. The final results give a vibration number for Kew which differs by less than one vibration from that at Greenwich, and may be accepted as very fairly probable.

It is the object of the present paper to give an abridged account of the above operations, both the primary by Colonel HERSCHEL, and the revisionary by Messrs. HOLLIS and CONSTABLE.† For this purpose it is necessary, in the first instance, to give brief descriptions of the pendulums, and of the *modus operandi* adopted by the different observers.

Description of the Pendulums.

All three pendulums are of KATER's Invariable Pattern; they are made of brass, with a steel knife-edge at the head of each pendulum, and they are of very nearly the same dimensions. One is numbered 4 and another 11; the third has

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* Full details of the operations and their results are given in Appendix No. 14 of the ' Report of the United States Coast and Geodetic Survey ' for 1884.

† Full details of Colonel HERSCHEL's operations, in manuscript, were made over to the Royal Society for record, by the Secretary of State for India; the details of the other operations are recorded in the observatories in which they respectively took place.

no such distinguishing number, but is marked 1821, presumably the year in which it was constructed ; Colonel HERSCHEL believes that it is probably No. 6 of the series, and has so called it. No. 4 was employed by SABINE in his operations between the parallels of 13° South and 80° North Latitude, in 1822–23 ; and No. 6 (1821) was used by the late Astronomer Royal, Sir GEORGE AIRY, in experiments in the Harton Colliery Pit, in 1854, to determine the earth's mean density ; these two are the pendulums of the Royal Society which were employed throughout the operations in India. No. 11 was used by BAILEY, in London, in 1832, and by MACLEAR, at the Cape of Good Hope, in 1839 ; it was afterwards lent for a while to the Admiralty, and eventually deposited in the Kew Observatory.

Each pendulum is furnished with a pair of agate planes, on which it is intended to be swung. The planes are set on either side of a half-inch opening in a solid brass frame, which is mounted on a plate at the head of the receiver, and is provided with three levelling screws; outside the frame there is a pair of moveable arms carrying Y's, in which the pendulum rests while not vibrating, and on lowering which the knife edge comes in contact with the agate planes for vibration. The pendulum is placed midway between the supporting planes by hand and eye estimate, but it is always brought by the Y's down on to the same line across the planes, in all positions of the pendulum, whether the marked face is pointing towards the observer or towards the clock.

The length of the pendulum is invariable, excepting from change of temperature for which the correction to the vibration-number is known. The shape is that of a flexible bar of plate brass, 62 inches long, 1·7 inch broad, and 0·13 inch thick from the knife edge downwards for a distance of about 40 inches, where a flat circular bob, 6 inches in diameter and 1·3 inch thick, with a bevelled edge, is soldered on to the bar ; the tail piece, below the bob, is reduced to a breadth of 0·7 inch, and is about 16 inches long. The bar is necessarily very flexible, its thickness being less than a tenth of its breadth, and this flexibility is greatly in contrast with the rigidity of the German and French pendulums. KATER is believed to have adopted a flexible form of bar in preference to a rigid bar designedly, under the impression that it was less likely to become permanently bent by accident, and more likely to acquire exact verticality when its knife-edge is resting on the agate planes during the course of the vibrations.

The Processes of Manipulation.

When the pendulums were sent out to India, it was intended that they should always be swung as nearly as possible in a vacuum. For this purpose a receiver of sheet copper, mounted on a substantial and well braced wooden stand, was furnished for the pendulums to be swung in ; the receiver was closed above by a hemispherical glass cap, which could be removed at pleasure for the insertion or withdrawal of a pendulum. Two thermometers were fixed in a dummy pendulum, of the same size as

the vibrating pendulum less the tail-piece, which was fixed inside the receiver. In the Indian operations, the air was always exhausted to the lowest pressure attainable, and the vibration-number obtained at that pressure was reduced to the vacuum by special corrections, which will be afterwards described. At first the receiver was found to leak slightly, the pressure rising about an inch in the course of a day's swings; but eventually this leakage was traced to the two stuffing boxes, through which one rod passes for lowering the knife-edge on, or raising it off, the agate planes, and another rod for setting the pendulum in motion; the leaks were stopped by fitting cups round the necks of the rods, and keeping them filled with oil. When the apparatus came into Colonel HERSCHEL's hands much leakage was met with at the lower pressures. He continued to use the receiver, but he did not attempt to obtain low pressures, and he abandoned the reduction to the vacuum; his pressures ranged between 26 and 28 inches, and his observations were reduced to an adopted standard pressure of 26 inches, with the temperature of the air at 32° F. The same procedure was adopted when the pendulums were swung by the officers of the United States' Coast and Geodetic Survey. But when the apparatus was returned to England, the receiver was repaired and made thoroughly air-tight before the revisionary swings at Kew and Greenwich were commenced; then half the swings at each place were taken under a pressure of about 2 inches, and the other half at about 27 inches; the results were reduced to a vacuum in both cases, by the same formula, which will be given hereafter.

For each invariable pendulum the vibration-number is determined by swinging it in front of the pendulum of a clock and observing the times of the first and last coincidences of the two pendulums, at the beginning·and end of a set of swings; also the time of a coincidence immediately following the first, in order to get an approximate value of the interval between successive coincidences; thus the number of swings made by the invariable pendulum during a given amount of clock time, is ascertained by observation, and from it the number of swings in 24 hours is calculated, after making due allowance for clock rate, arc of vibration, temperature, and pressure. The clock employed in the Indian and subsequent operations was one by SHELTON, which had been used for the same purpose by SABINE. The invariable pendulums make 200 to 300 vibrations daily less than the pendulum of a clock regulated to solar time, and about twice that number to siderial time.

For the observations of coincidence, in the Indian operations, a circular disc of white metal was mounted on the bob of the clock pendulum, and an image of it, made very slightly less in diameter than the breadth of the tail-piece of the invariable pendulum, was produced, by an intermediate lens, in the plane of the tail-piece. The image would disappear for a moment and then reappear behind the tail-piece at every apparent conjunction of the two pendulums; and these conjunctions occur when both pendulums are swinging in the same direction, the intermediate conjunctions, with the swings in opposite directions, being unobservable. The exact moment of coincidence

was at first deduced from observations of the disappearance and reappearance of both edges of the disc; but after a short time, this was considered unnecessary, and one edge only was observed, the same throughout each set of swings.

Colonel HERSCHEL substituted for the single large disc a system of multiple discs consisting of several pairs of small circles, arranged symmetrically on opposite sides of a central vertical line, and painted white on a piece of black cardboard which was attached to the bob of the clock pendulum. He designed a large variety of systems, one of which is here shown. He observed the times of disappearance and re-appearance of several pairs of discs, eventually retaining five pairs only, of which the general
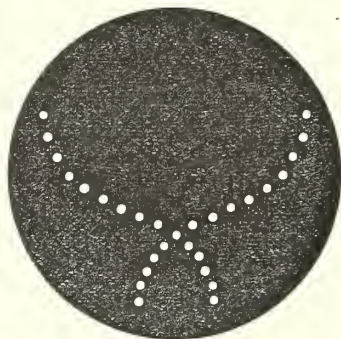
mean was taken as the moment of coincidence. The United States' officers adopted one of Colonel HERSCHEL's discs, but observed on only one side of the tail-piece and not on both sides as he had done. In the revisionary operations at Kew and Greenwich, a single large disc, of which the image was made of nearly the same diameter as the tail-piece, was employed, and observations of disappearance and re-appearance were made on one side only, as in India.*

In the Indian and the revisionary operations the times of the three first and the three last coincidences in each set of swings were observed, and the means were employed to indicate the moments of commencement and conclusion of the set; the observed intervals between successive coincidences gave the divisor to the duration of the set to find the total number of intervals which is wanted in calculating the vibration-number. In Colonel HERSCHEL's operations one or two discs were observed

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* Very great precision in the determination of the moment of coincidence is unnecessary. If V be the vibration-number of a pendulum, R the clock vibrations in a mean solar day, and N the clock vibrations during a set of swings in which there are n intervals between visible coincidences, then

$$V = R\left(1 - \frac{2n}{N}\right) \quad \text{and} \quad dV = 2R\frac{n}{N^2}dN.$$

Let R = 86,630, let the duration of the set of swings be 6 hours and the interval between coincidences 6 minutes, giving $n = 60$, then

$$dV = \cdot 022\, dN.$$

Thus an error of 4 seconds in N. which is improbably large, would not affect the vibration-number by as much as 0·1, which is but a fraction of the probable error from other causes.

immediately after the initial observations of coincidence, to obtain the interval between successive coincidences.

In the Indian operations the agate planes were always carefully levelled before the pendulum was set up on them, and the level readings, as taken before and after the swings by each pendulum, were recorded and published. As the pendulum is necessarily made to vibrate with its tail-piece a short distance in front of the scale for measuring the arc of vibration, this distance was read off on a scale fixed at right angles to the arc-scale, to enable the observed arc reading to be duly corrected ; and it was measured in the two positions of the instruments, both with the marked face towards the observer and towards the clock. Thus, as the knife-edge was always lowered by the Y's down to the same line on the agate planes, and as the planes were always horizontal, half the difference between the distances of the tail-piece from the arc-scale, in the two positions, indicates the magnitude of any deviation of the bar of the pendulum, at the tail-end, from perpendicularity to the knife-edge. These distances were always recorded, and they show that both the pendulums were bent, but that the bending was practically constant throughout the whole of the operations ; the marked face of Pendulum No. 4 was deflected ·5 inch outwards, and that of No. 6 (1821) ·3 inch backwards, at the tail-end. As, however, a general constancy was preserved throughout, the whole of the results were truly differential.

Colonel HERSCHEL commenced his operations at Kew by swinging No. 4 pendulum in the condition in which he found it. He soon noticed bends in both the two pendulums, and also found that the knife-edges were somewhat rusted. There was reason to suspect that the pendulums might have received some injuries when set up at one of the great Annual Exhibitions in South Kensington which was held a few years after their return from India. Consequently both the pendulums were straightened and their knife-edges were re-ground. This, of course, causes a break of continuity with the antecedent operations with these pendulums, and destroys the differentiality of the vibration numbers obtained before and afterwards.

In the revisionary operations at Kew and Greenwich the pendulums were swung in the same condition as when employed by Colonel HERSCHEL and the officers of the United States.

In the Indian operations each set of swings was usually of about 9 hours' duration, from 8 A.M. to 5 P.M., intermediate readings of the thermometers and observations of coincidence being taken at intervals of about $1\frac{1}{2}$ hour apart. When all the observations were finally reduced it was seen that, whenever the daily variation of temperature was considerable, the clock rate at different hours of the day varied sensibly from the mean daily rate ; thus it was evident that the vibration-number, which depends on the actual clock rate during the set of swings, but is deduced from the mean daily rate which is derived from successive transits of the same stars, might be much influenced by variations of rate occurring during the part of the day when the pendulum is under vibration.

Colonel HERSCHEL got over this difficulty, and eliminated the influence of hourly variations of clock rate, by linking successive sets of swings together so as to fill up the whole 24 hours. He made the duration of each set of his swings somewhat less than 6 hours, taking all necessary observations of temperature, coincidence, &c. at the commencement and conclusion of the set, but without any intermediate observations; immediately after the conclusion of one set he commenced the next set. In this way observations were sometimes carried on continuously for two or three days by himself and his assistant. In the revisionary operations at Kew a similar procedure was adopted; the temperatures and coincidences in the intervals between the beginning and end of each set were generally observed also, and the temperatures when not observed were recorded on a thermograph. At Greenwich observations were made at 10 A.M., 4 P.M., and 10 P.M.; the swings at low pressure were divided into two sets of 6 hours each and one set of 12 hours, so as to fill up the 24 hours; those at high pressure into two sets of 6 hours each, to fill up 12 hours. The temperatures between the thermometer readings were registered by a thermograph. The daily range of temperature was very small at Kew, and rarely as much as 1° F. at Greenwich. At both places the time was derived electrically from the siderial standard clock at Greenwich, which is fixed in the basement of the Observatory, where there is no sensible diurnal variation of temperature. Under the actual circumstances there was no real necessity for continuous observations throughout the 24 hours to control the clock rate.

The differences in procedure and manipulation which have been pointed out thus far are not such as to have affected the results sensibly; but in one other matter there was a difference of procedure which might have materially influenced the results. Colonel HERSCHEL did not maintain the agate planes in exact horizontality; he believed that when the planes were truly level, and the pendulum was set up on them, the knife-edge, if pressed down against them by hand, was found to be not truly in contact with the planes throughout the line of bearing; consequently he dislevelled the planes until the contact, as judged by touch, was thorough, and then he commenced swinging the pendulum. Such imperfect contact was never noticed in the Indian operations; it is possible with pendulums having a rigid bar, when the knife-edge is not truly perpendicular to the bar; but with pendulums such as these, which have a very flexible bar, it seems scarcely possible, at least without a grosser displacement of the knife-edge from the perpendicular than is at all probable. The officers of the United States who swung these pendulums at several stations have been questioned on this point, and Mr. EDWIN SMITH reports that after levelling he " tested the contact of the knife-edges with the agate planes by touch and found it impracticable to make any change, so the pendulums were always swung with the planes levelled with the spirit level." In the revisionary swings this was done also.

Colonel HERSCHEL does not appear to have measured the actual dislevelment of the agate planes which was caused by his method of treatment; had he done so and his surmise been correct, the magnitude of the dislevelment would have been the same

after as it was before each transposition of the pendulum, but its sign would have changed, because its direction would have been reversed.   The record of the observations gives no level readings, nor does it give the distances between the tail-piece and the arc-scale in the two positions of vibration.   All that is known is that the agate planes were not maintained in a position of constant horizontality, as in all other operations.   Thus the results are not strictly differential, and it is to be inferred that the large discrepancies occasionally met with between different groups of results, even when the individual results in each group are highly accordant, are due to this cause.

The results of the several operations will now be given.

COLONEL HERSCHEL'S RESULTS, REDUCED TO TEMPERATURE OF 62° F., AND TO THE DENSITY OF THE AIR UNDER THE PRESSURE OF 26 INCHES AT THE TEMPERATURE OF 32° F.

Determinations at Kew Observatory.

PRELIMINARY Vibration-numbers of Pendulum No. 4, before straightening the bar and re-grinding the knife-edge.

Marked face, M, to front.		Plain face, P, to front.		Means.	
Set.	V.	Set.	V.		
2	86158·74	15	86161·44		
3	86158·84	16	86161·02		
4	86158·05	17	86160·87		
5	86159·21	18	86161·25	Face M .	86158·38
6	86157·31	19	86160·60	Face P .	86160·98
8	86158·41	20	86160·24		
9	86158·25	21	86160·86		86159·68
10	86158·32	22	86161·26		
11	86158·31	23	86160·12		
12	86158·91	24	86160·87		
13	86158·83	25	86161·52		
14	86157·41	26	86161·69		

When this result is reduced to a vacuum, it may be compared with the results of the swings by the same pendulum at the same place which were obtained for the Indian operations.   The reduction to the vacuum may be approximately taken as + 8·32, which gives the vibration-number 86168·00, to compare with 86169·45 obtained in 1864, before the pendulum was sent out to India, and 86169·57 obtained in 1873, after its return from India.

Colonel HERSCHEL's *Determinations at Kew Observatory, continued.*

VIBRATION-NUMBERS by all three Pendulums, obtained after the bars were straightened and the knife-edges re-ground.

Pendulum No. 4.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
74	86157·07	94	86157·16		
75	86157·62	95	86157·08		
76	86158·21	96	86157·75		
77	86157·05	97	86157·26		
78	86156 75	98	86157·54		
79	86157·74	99	86157·78		
80	86158·13	100	86158·45		
81	86157·88	101	86157·84	Face M .	86157·42
82	86157·44	102	86157·27	Face P .	86157·87
83	86157·67	103	86157·52		
84	86157·54	104	86157·65		86157·64
85	86157·06	105	86157·42		
86	86157·01	106	86157·62		
87	86156·39	107	86158·47		
88	86159·39	108	86158·87		
89	86158·01	109	86159·01		
90	86155·74	110	86158·63		
91	86157·15	111	86158 07		
92	86157·35	112	86158·21		
93	86156·94				

Colonel HERSCHEL'S *Determinations at Kew Observatory, continued.*

Pendulum No. 6 (1821).

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
28	86057·21	51	86056·54		
29	86056·64	52	86055·95		
30	86056·28	53	86055·88		
31	86056·00	54	86057·15		
32	86055·05	55	86057·11		
33	86055·25	56	86056·72		
34	86055·87	57	86056·71		
35	86055·64	58	86056·78		
36	86055·82	59	86056·75	Face M .	86056·32
38	86058·82	60	86056·42	Face P .	86056·74
39	86058·11	61	86056·54		
40	86056·84	62	86057·08		86056·53
41	86056·65	63	86057·18		
42	86053·87	64	86057·18		
43	86054·77	65	86057·19		
44	86056·93	66	86057·33		
45	86056·62	67	86057·01		
46	86056·48	68	86056·93		
47	86056·52	69	86056·68		
48	86056·66	70	86057·11		
49	86056·80	71	86055·21		

Pendulum No. 11.

Face M.		Mean.	
Set.	V.		
115	86099·64		
116	86099·18		
117	86099·11		
118	86099·67		
119	86100·60		
120	86099·83		
121	86101·54		
122	86100·76		
123	86099·87		
124	86100·44	Face M .	86100·57
125	86100·17		
126	86100·78		
127	86100·42		
128	86100·83		
129	86100·86		
130	86101·10		
131	86100·81		
132	86101·61		
134	86103·58		

Colonel HERSCHEL's *Determinations at Greenwich Observatory.*

Pendulum No. 4.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
135	86153·96	151	86156·34		
136	86153·46	152	86156·04		
137	86153·54	153	86156·24		
138	86153·10	154	86156·38	Face M, 1st Series . . .	86153·39
139	86153·99	155	86156·52	Face M, 2nd and 3rd Series	86156·65
140	86153·13	156	86156·24		
141	86153·20	157	86156·78		86155·02
142	86153·53	158	86156·49	Face P	86156·31
143	86153·35	159	86156·26		
144	86153·40	160	86155·84		86155·67
145	86153·04	161	86156·21		
146	86153·38	162	86156·58		
147	86153·29	163	86155·98		
148	86153·34	164	86156·43		
149	86153·33				
150	86153·12				
165	86156·78				
166	86156·62				
167	86157·18				
168	86156·68				
169	86157·29				
170	86157·06				
171	86157·20				
212	86156·57				
213	86156·42				
214	86156·31				
215	86156·51				
216	86156·24				
217	86156·28				
218	86156·34				
219	86156·30				

Colonel HERSCHEL's *Determinations at Greenwich Observatory, continued.*

Pendulum No 6 (1821).

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
172	86054·82	184	86054·89		
173	86055·04	185	86055·91		
174	86054·91	186	86055·70		
175	86054·81	187	86054·97		
776	86054·54	188	86055·82	Face M	86054·87
177	86055·05	189	86055·41	Face P	86055·41
178	86055·17	190	86055·34		
179	86054·49	191	86055·28		86055·14
180	86055·23	192	86055·74		
181	86054·79	193	86055·38		
182	86054·48	194	86055·24		
183	86055·06	195	86055·24		

Pendulum No. 11.

Face M.		Mean.
Set.	V.	
196	86104·54	
197	86105·40	
198	86105·82	
199	86104·80	
200	86105·29	
201	86105·23	
202	86105·25	
203	86105·15	
204	86105·17	Face M . 86105·55
205	86105·80	
206	86106·16	
207	86105·24	
208	86106·04	
209	86105·69	
210	86106·40	
211	86106·87	

Colonel HERSCHEL's observations give the following values of the differences between the vibration-numbers at Kew and Greenwich.

$$\text{Kew—Greenwich} = +\ 1\text{·}97 \text{ by Pendulum No. 4.}$$
$$= +\ 1\text{·}39 \qquad ,, \qquad \text{No. 6 (1821)}.$$
$$= -\ 4\text{·}98 \qquad ,, \qquad \text{No. 11}$$

REVISIONARY RESULTS, REDUCED TO THE TEMPERATURE OF 62° F., AND TO A VACUUM.

Mr. CONSTABLE'S Determinations at Kew under Low Pressures, about 2 inches.

Pendulum No. 4.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
4	86167·56	1	86167·12		
5	86167·07	2	86167·47	Face M . 86167·31	
6	86167·31	3	86167·36	Face P . 86166·69	
		7	86165·91		
		8	86166·53	86167·00	
		9	86165·73		

Pendulum No. 6 (1821).

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
10	86066·74	13	86067·15	Face M . 86066·25	
11	86065·97	14	86067·00	Face P . 86067·14	
12	86066·04	15	86067·58		
		16	86066·83	86066·70	

Pendulum No. 11.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
17	86117·41	20	86116·82	Face M . 86117·43	
18	86117·11	21	86117·24	Face P . 86117·17	
19	86117·78	22	86117·44		
				86117·30	

Mr. Constable's Determinations at Kew under High Pressures, about 27 inches.

Pendulum No. 4.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
39	86165·20	43	86165·75	Face M .	86164·80
40	86164·81	44	86165·70	Face P .	86165·74
41	86164·44	45	86165·73		
42	86164·75	46	86165·80		86165·27

Pendulum No. 6 (1821).

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
31	86065·98	35	86065·80	Face M .	86066·07
32	86066·22	36	86065·57	Face P .	86065·54
33	86066·24	37	86065·43		
34	86065·84	38	86065·35		86065·80

Pendulum No. 11.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
23	86115·88	27	86115·82	Face M .	86116·14
24	86116·25	28	86115·92	Face P .	86115·94
25	86116·10	29	86116·04		
26	86116·33	30	86115·97		86116·04

Mr. Hollis's Determinations at Greenwich under Low Pressures, about 2 inches.

Pendulum No. 4.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
1	86165·08	5	86165·24	Face M .	86165·06
2	86165·03	6	86165·25	Face P .	86165·16
3	86165·09	7	86165·10		
4	86165·02	8	86164·98		86165·11
		9	86165·25		

Pendulum No. 6 (1821).

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
10	86065·61	16	86065·60		
11	86065·47	17	86065·17	Face M .	86065·64
12	86065·55	18	86065·58	Face P .	86065·47
13	86065·56	19	86065·54		
14	86065·71	20	86065·51		86065·56
15	86065·91	21	86065·43		

Pendulum No. 11.

Face M.		Face P.		Means.	
Set.	V.	Set.	V.		
22	86116·73	27	86116·22		
23	86116·65	28	86116·41	Face M .	86116·68
24	86116·72	29	86116·39	Face P .	86116·30
25	86116·79	30	86116·18		
26	86116·50				86116·49

Mr. Hollis's determinations at Greenwich under High Pressures, about 27 inches.

Pendulum No. 4.

Face M.		Face P.		Means.		
Set.	V.	Set.	V.			
39	86163·84	43	86163·99	Face M	.	86164·11
40	86163·98	44	86163·78	Face P	.	86163·97
41	86164·30	45	86164·07			
42	86164·33	46	86164·00			86164·04
		47	86164·03			

Pendulum No. 6 (1821).

Face M.		Face P.		Means.		
Set.	V.	Set.	V.			
48	86064·47	52	86063·77	Face M	.	86064·23
49	86064·45	53	86063·74	Face P	.	86063·65
50	86063·93	54	86063·55			
51	86064·09	55	86063·54			86063·94

Pendulum No. 11.

Face M.		Face P.		Means.		
Set.	V.	Set.	V.			
35	86116·07	31	86115·20	Face M	.	86116·14
36	86116·26	32	86115·21	Face P	.	86115·22
37	86116·17	33	86115·21			
38	86116·07	34	86115·26			86115·68

Thus the revisionary operations give the following values of the differences between the vibration-numbers at Kew and Greenwich :—

$$\begin{array}{lll} & \text{Low pressures.} & \text{High pressures.} \\ \text{Kew—Greenwich} = +1{\cdot}89 & +1{\cdot}23 & \text{by pendulum No. 4.} \\ = +1{\cdot}14 & +1{\cdot}86 & ,, \quad \text{No. 6 (1821).} \\ = +0{\cdot}81 & +0{\cdot}36 & ,, \quad \text{No. 11.} \\ \text{General mean} = +1{\cdot}22 \pm {\cdot}16. \end{array}$$

It will be seen that the mean value is fairly in accordance with the values derived from Colonel HERSCHEL's observations with pendulums No. 4 and No. 6 (1821). His swings with those pendulums, at Kew, were made in the basement of the Kew Observatory, within a few feet of the spot at which the revisionary swings, with all three pendulums, were made ; but he swung pendulum No. 11 in a shed outside the Observatory, under circumstances of great disadvantage as regards the stability and firmness of the support of the stand of the invariable pendulum and also of the support of the clock. Thus his observations at Kew, with pendulum No. 11, though generally very accordant *inter se*, are very probably burdened with a large constant error, and must therefore be rejected.

On the Reduction to a Vacuum.

In all pendulum experiments—even those of a purely differential character, as with invariable pendulums—it has been generally customary to apply a correction for the retardation which is caused by the air, in order to obtain results such as would have been obtained if the pendulum had been swung in a vacuum. This correction was originally determined by calculating the influence of the buoyancy of the atmosphere in diminishing the weight—and consequently the accelerating force—of the pendulum. Afterwards BESSEL showed that the correction thus obtained was too small, for the pendulum is accompanied in its oscillation by a certain amount of air, varying with its form, which increases the mass in vibration and the moment of inertia. Thus the buoyancy correction has to be multiplied by a factor, $1 + k$, which can be computed mathematically for pendulums of certain simple forms, but must be determined experimentally, by swings at high and low pressures, when the form is not susceptible of being brought under mathematical treatment. The buoyancy correction, thus augmented, is usually called the pressure correction.

The buoyancy correction and the pressure correction have been investigated for pendulums No. 4 and No. 6 (1821) by special and laborious series of operations which are fully set forth in vol. 5 of the 'Account of the Operations of the Great Trigonometrical Survey of India.' Nothing of the kind is known to have been done for No. 11 ; but the results obtained for the two first pendulums are so closely accordant that they may be applied without objection to the third, which is almost identical with them in form and construction.

The buoyancy correction $= \cdot 23 \dfrac{\beta}{1 + \cdot 0023\,(t - 32°)}$ in which β is the pressure in inches, and t the temperature in degrees of FAHRENHEIT.

The pressure correction was found to be $\cdot 32 \dfrac{\beta}{1 + \cdot 0028\,(t - 32°)}$ by experimental swings which were made specifically for the purpose at Kew, under extreme high and low pressures, immediately before the pendulums were sent out to India. Corrections determined by this formula were applied, provisionally, to the whole of the

swings in India ; and this has also been done to the revisionary swings at Kew and Greenwich, to produce the vibration-numbers which have already been set forth.

But, in the course of the operations in India, Captain BASEVI reinvestigated both the temperature and the pressure corrections of his two pendulums, those of No. 4 with great elaboration.   A series of several sets of swings was made with it at each of the successive pressures of 0·6, 1·9, 4·2, 10·0, 17·5, and 27·7 inches, at the temperature of about 101° F ; another series at the same pressures, at the temperature of about 53° ; and a third at the pressures of 1·9 and 4·2 inches and temperature of about 80°.   He came to the conclusion that the pressure correction is best represented by an empirical formula of three terms,

$$A\beta^{\frac{1}{2}} + B\beta + C\beta^{\frac{3}{2}},$$

in which the second term is the correction for buoyancy.   Then he assumed A to be $= x \sqrt{461° + t}$ and $C = y \div \sqrt{461° + t}$ — 461° being the absolute zero of the air thermometer—and formed a corresponding series of equations for the determination of x and y from his fourteen sets of observations.   The solution of these equations gave $x = $ ·022 $\pm$ ·002, and $y = $ ·123 $\pm$ ·025, which values satisfied the equations of condition very satisfactorily.

But the subject is one of great complexity and difficulty, as will be seen in consulting Chapter VI. of the Indian pendulum volume.   Something appears to be wanting to explain the inconsistencies between vibration-numbers derived from different series of very accordant observations which are occasionally met with. Possibly it may be necessary to take cognisance of the atmospheric humidity during the observations, which has never been done hitherto.   Or it may be that the inconsistencies arise from changes in the relative conditions of the bearing surfaces of the knife-edge and the agate planes, which are met with on successive transpositions of the pendulum, and which the observer cannot control.

Transposition is almost invariably attended with a change in the vibration-number ; but in the Indian operations it was found that the changes were not constant in either sign or magnitude ; it is shown, at page 114 of the volume already cited, that, for the whole of the 34 stations of observation, the mean value of M — P ranges from + ·54 to — ·52, and has an average value of + ·07 $\pm$ ·03 for Pendulum No. 4, and ranges from + ·67 to — ·59, with the average value — ·04 $\pm$ ·03 for Pendulum No. 6 (1821).

In reducing the Indian swings for investigating the pressure correction, Captain BASEVI's observations of the vibration-numbers at different pressures were employed directly, without having recourse to his empirical formula.   The observed vibration-numbers at each pressure were reduced to a vacuum by the Kew formula with the numerical constant ·32, and then the results for the higher pressures were compared with the result for the lowest pressure, which, being 0·6 of an inch, was very close to the vacuum ; and it was found that the higher pressures required residual positive corrections, increasing with the temperature as well as the pressure ; at the highest

pressure, 27·7 inches, the correction amounted to ·53 under the temperature of 53° and to 1·34 under the temperature of 100°. Corresponding corrections were therefore applied to the whole of the Indian swings, as they had already been provisionally reduced by the Kew formula.

This must now be done for the revisionary swings at Kew and Greenwich, the results of which, as hitherto presented, have been reduced to the vacuum by the Kew formula only. It will be seen that the vibration-numbers at the pressure of 27 inches are less than those at the pressure of 2 inches by 1·30 at Kew and 1·16 at Greenwich; but it appears from Captain Basevi's investigations that the high pressure results at both places should be increased by about 0·7, which will reduce the discordances with the low pressure results to 0·60 and 0·46.

When pendulum operations are differential, and the variations of atmospheric pressure, at different stations, are small, the value of the correction for pressure does not require to be known with much accuracy. Reducing to a vacuum is also then unnecessary, and any convenient standard of atmospheric density may be employed instead. Thus Colonel Herschel has sufficiently provided for the elimination of the effects of variations of pressure at his stations, which were all of low elevation, by reducing his swings to the standard pressure of 26 inches under the temperature of 32° F., instead of to a vacuum.

The Indian swings were invariably reduced to a vacuum, in accordance with previous procedure. They were made in an exhausted receiver, under the lowest pressure attainable, partly because this enabled them to be extended over a longer time, and thus be less influenced by hourly variations of clock-rate, than if taken under full pressure; partly because the receiver would protect the pendulum from the action of currents of air; and partly to obtain as nearly a uniform pressure at all the stations as possible, and thus secure strictly differential results; for it was intended that the pendulums should be swung both at low levels in the neighbourhood of the ocean and at the highest attainable elevations, as on the table-lands of the Himalayan mountains, where the pressure of the atmosphere is halfway down to the vacuum, so that a considerable range of pressure had to be met with and provided for in the best way possible. By exhausting the receiver, the pressures under which the swings were actually taken were generally maintained between 1 and 2 inches, excepting at first, when slight leakage was met with, the locus of which was not immediately detected, and then the pressures ranged from 1 to 4 inches. But these differences of pressure are so small that the uncertainty as to the precise amount of the pressure correction cannot exert an appreciable influence on the differential results which have already been deduced, and which are the ultimate object of the observations. And this is the case also both in Colonel Herschel's and in the revisionary operations, the range of pressure being always under 2 inches, whether the swings were taken under high pressures only or under both high and low pressures.

Thus, a more exact knowledge of the correction for pressure might sensibly affect

the vibration-numbers for Kew and Greenwich which have already been presented; but it would not affect the differences between those numbers, which are what is really wanted, to a degree that is at all comparable with the errors to which pendulum observations are liable from other causes.

REDUCTION TO THE SEA-LEVEL.

The pendulums were swung at an elevation above the mean sea-level of about 15 feet at Kew and 157 feet at Greenwich. The vibration-numbers must be correspondingly increased.

The well-known formula for the correction for height is

$$\text{Correction} = V\frac{h}{r},$$

V being the vibration-number, h the height, and r the radius of the earth. Dr. YOUNG has suggested that account should also be taken of the continental mass which is situated between the level of the sea and that of the station of observation, in increasing the force of attraction and consequently the vibration-number. Thus, in accordance with his views, the usually-accepted correction takes cognisance of both height and mass, and is

$$= \frac{5}{8}V\frac{h}{r}.$$

Thus for these pendulums, when h is expressed in feet, we have

$$\text{Correction for height only} = \frac{h}{243},$$

$$\text{Correction for height and mass} = \frac{h}{391}.$$

It is now, however, very questionable whether any reduction for mass is allowable. The pendulum operations in India have thrown much light on the constitution of the earth's crust, and shown that there is a marked deficiency of density under the Himalayan mountains, and an increase of density under the bed of the Indian Ocean. Thus continental matter above the sea level may be conceived as appertaining to the strata underneath, immediately below the sea level, which are correspondingly attenuated. In this case, the excess of matter above would probably compensate for the deficiency of matter below, and would not form an attracting force to be independently allowed for while no cognisance is taken of the deficiency below.

If the vibration-numbers at Kew and Greenwich are corrected for height only, the correction to be applied to their difference will be

$$= \frac{15 - 157}{243} = -0{\cdot}58,$$

whence we obtain

$$\text{Kew—Greenwich} = 1{\cdot}22 - 0{\cdot}58 = 0{\cdot}64$$

as the result of the revisionary operations.

ADDENDUM.

RESULTS of the swings at other stations which were taken with the same pendulums, by Colonel HERSCHEL, and by Mr. EDWIN SMITH of the United States Coast and Geodetic Survey. They are reduced to the temperature of 62° F., and to the density of air under the pressure of 26 inches at the temperature of 32° F.; but they are not reduced to the sea-level.

Swings by Colonel HERSCHEL.

Stations.	Height in feet above sea.	No. 4.	No. 6 (1821).	No. 11.
London,* Langham Place, Cellar .	85	86157·31	86055·97	86109·59
Washington, Smithsonian Institute	34	86109·45	86008·35	86060·93
Hoboken, Stevens Institute . . .	30	86115·23	86014·06	86066·93
Whence				
Greenwich—London . . . . =	..	+ 1·64	+ 0·83	+ 4·05
London—Washington . . . . =	..	+ 47·86	+ 47·62	+ 48·66
Washington—Hoboken . . . =	..	− 5·78	− 5·71	− 6·00

Swings by Mr. EDWIN SMITH.

Stations.	Height in feet.	No. 4.	No. 6 (1821).	No. 11.
Auckland	261	86102·75	86002·11	86054·13
Sydney	140	86090·93	85990·32	86042·08
Singapore	45	86021·13	85919·97	85971·34
Tokio†	20	86099·83	85995·17	86046·91
San Francisco	375	86103·77	86003·13	86055·66
Washington	34	86109·31	86009·29	86061·41
Whence				
Auckland—Sydney . . . . =	..	+ 11·82	+ 11·79	+ 12·05
Sydney—Singapore . . . . =	..	+ 69·80	+ 70·35	+ 70·74
Singapore—Tokio . . . . . =	..	− 78·70	− 75·20	− 75·57
Tokio—San Francisco . . . =	..	− 3·94	− 7·96	− 8·75
San Francisco—Washington . =	..	− 5·54	− 6·16	− 5·75

* This station is at No. 1, All Souls Place, Langham Place; it is about 380 feet S.E. of, and 14 feet lower in level than, Mr. BROWNE's house in Portland Place, which was KATER's station. The value of Greenwich—London (Portland Place) = + 0·48, with Invariable Pendulum No. 12; it was determined by SABINE in 1828, see ' Phil. Trans.' for 1829, p. 87.

† Mr. EDWIN SMITH remarks of the observations at Tokio, that "the vibration-number of Pendulum No. 4 is between three and four vibrations too great. I can only explain this discrepancy by the supposition that some foreign material was adhering to the pendulum during these observations. Great care was always taken in wiping the pendulums before suspending them."

IX. *On the Alleged Slipping at the Boundary of a Liquid in Motion.*

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In treatises on hydrodynamics, the flow of a liquid through a straight tube is investigated on the supposition that there may be finite slipping between the walls of the tube and the outermost layer of liquid. This leads to the introduction of a " slipping coefficient " which vanishes when there is no relative motion between them.

Let r denote the radius of the tube,

p_1 the pressure at one end,

p_2 ,, ,, the other,

μ the coefficient of viscosity,

l the length of the tube,

ρ the density of the liquid.

Then it may be shown (Lamb's ' Hydrodynamics, p. 222) that when the motion is linear the flux is given by

$$\frac{1}{8} \frac{\pi r^4}{\mu \rho} \frac{p_1 - p_2}{l} + \frac{1}{2} \frac{\pi r^3}{\beta} \frac{p_1 - p_2}{l}$$

or

$$\frac{1}{8} \frac{\pi (p_1 - p_2)}{\mu \rho l} \left\{ r^4 + 4 \mu \rho \frac{1}{\beta} r^3 \right\},$$

where $1/\beta$ may be defined as the slipping coefficient.

The experiments of Poiseuille[*] showed that the coefficient was certainly zero for glass tubes, but there was doubt whether this held for all materials.

Helmholtz and Piotrowski[†] attacked the problem in another way. They suspended bifilarly an accurately worked sphere, whose inner surface was gilded and polished, and by observing the time of swing and the logarithmic decrement when the sphere was filled with water and various other liquids, deduced a value for the

[*] ' Mémoires des Savants Étrangers,' 1846.

[†] ' Sitzungsber. der k. Akad. in Wien,' vol. 40, 1860.

coefficient of viscosity and for the slipping coefficient, from the theory of spheres oscillating in a viscous medium, as worked out by STOKES and HELMHOLTZ.

For distilled water their value of the factor $\mu\rho\ 1/\beta$ in the above expression is $\lambda = 2\cdot3534$ mm.

If we apply this to the case of a tube we get a somewhat startling result. From equation (i.) it follows that the effect of slip varies inversely as the radius of the tube. The smallest tube practicable in the experiments to be presently described had a diameter of about a millimetre. It is easy to show that the result of the existence of a slipping coefficient of the magnitude given by HELMHOLTZ would be to produce an increase in the volume of liquid flowing through the tube in a given time, which could not only be detected, but would be of such importance that it could not easily be masked.

In HELMHOLTZ's notation equation (i.) is written

$$\frac{1}{8}\frac{\pi\ (p_1 - p_2)}{\mu l}\ \{r^4 + 4\lambda r^3\},$$

the density of water being taken as unity.

Putting $r = \cdot05$ and $\lambda = \cdot23534$, we get

$$\frac{1}{8}\frac{\pi\ (p_1 - p_2)}{\mu l}\ \times\ 117\cdot67\ \times\ 10^{-6};$$

whereas if there is no slip, so that λ vanishes, the flux becomes

$$\frac{1}{8}\frac{\pi\ (p_1 - p_2)}{\mu l}\ \times\ 6\cdot25\ \times\ 10^{-6}.$$

If we take HELMHOLTZ's coefficient to be correct, the flow through a polished gilt tube of a millimetre in diameter is nearly *twenty times* as fast as through a glass tube of the same size.

HELMHOLTZ refers to some experiments made by GIRARD with copper tubes* which make the flow some four times faster than do POISEUILLE's formulæ. I shall return to the consideration of these observations later. The value of λ which HELMHOLTZ deduces from them is $0\cdot3984$ mm.

The discrepancy between these results and the generally received opinion that no slip occurred with any material seemed worthy of further investigation.

It is evident that the alleged coefficient could be investigated with much greater advantage by observing the flow of liquid through a small tube than by any experiments on oscillating spheres. In order to avoid all absolute determinations while searching for the existence of such a slip, I decided to observe the time of flow of a

* 'Mémoires de l'Institut,' 1813–1815.

given volume of water through a glass tube, and then to deposit a coating of silver on the interior surface of the tube. If the time of flow was the same as before, allowing for the (usually very small) change in diameter, it would be conclusive evidence against the existence of the effect. Such evidence I have most satisfactorily obtained.
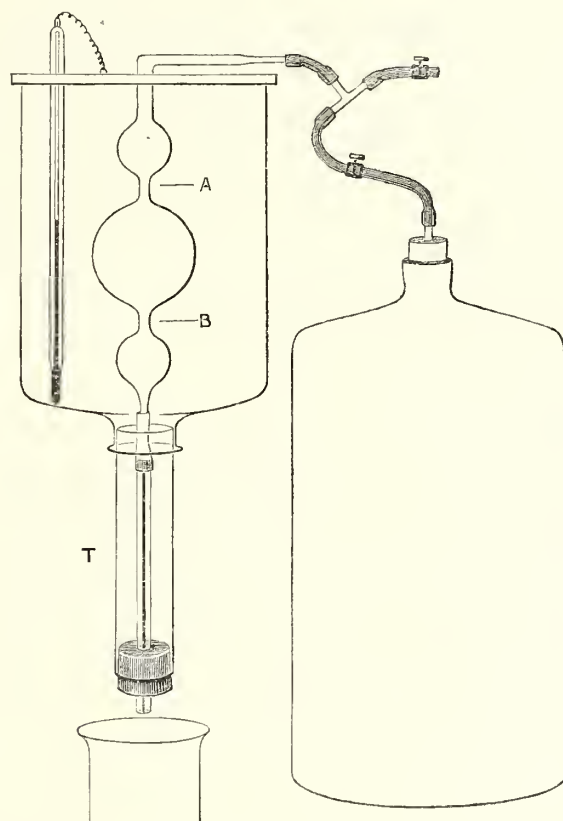
Since the experiments were to be merely comparative there was no object in attempting to keep the pressure constant throughout each observation, and the simplest apparatus could be used. The bulb tube AB was fixed in a large glass jar filled with water, and opening below into a wide tube T, also filled with water. The lower end of the bulb tube was attached to the capillary tube by an india-rubber joint, so that the ends of the two glass tubes should just meet. The top of the bulb tube could be put into connexion either with the air outside, or with an exhausted bottle which was used to fill the bulbs, as the apparatus was some distance from a pump. The temperature of the water in the glass jar and wide tube could be read off by a delicate thermometer.

The time taken by the upper surface of the water to fall from A to B was observed by a stop watch. Before silvering the tube the whole apparatus was repeatedly taken to pieces and set up again, and the time of flow shown to be unaltered.

The silvering solution was a modification of LIEBIG's, and was made according to a

recipe kindly given me by Dr. A. S. Lea. Each tube was dried and weighed, and the silvering solution then run through till a bright metallic mirror was deposited. It was washed out with a current of water, then with air, and finally dried and weighed again. The increase gave the weight of silver. At the end of the experiments with the tube thus silvered, it was again dried and weighed, and the silver dissolved off with nitric acid. The mean of these two results, which usually agreed to one or two tenths of a milligram, was taken to represent the weight of silver adhering to the tube during the experiments. Assuming the deposit to be uniform, this at once gave the change in diameter. The correction for temperature was calculated by Poiseuille's formula.

The first tube had a length of 37·57 cms., and an average radius (determined by filling with mercury) of 0·0451 cm.

The following series of experiments were made :—

Temperature.	Time of flow.
18·2	7 50·8
18·3	7 50·4
18·3	7 50·6
18·2	7 50·4
18·3	7 51·0
18·3	7 51·0
Means 18·26	7 50·7

The apparatus was then taken completely to pieces as it would be for silvering, and again set up after some hours, with the following results :—

Temperature.	Time of flow.
17·8	7 55·4
17·9	7 55·2

If we correct this to 18°·26 by Poiseuille's empirical formula we get 7′ 51″·0, a value identical with the above. The apparatus could thus be taken to pieces with safety.

Weight of tube when dry . . . . . = 16·3916 grams.
  „   with silver . . . . . = 16·3932   „

The apparatus was then again set up.

Temperature.	Time of flow.
$18°.4$	$7'\ 52''.6$
18.3	$7\ 52.2$
18.2	$7\ 52.4$
Means 18.30	$7\ 52.4$

The tube was then disconnected, dried, and weighed. Weight $= 16.3931$ grms., practically the same as before. The silver was then dissolved off, and the tube cleaned and dried. Weight $= 16.3916$.

The weight of silver deposited is, therefore, $.0015$ grm., and its thickness $.000014$ cm.

This gives a change in r^4 equivalent to 0.12 per cent. The time of flow for the unsilvered tube, corrected for change of radius, is $7'\ 51''.4$, while the observed time for the silvered tube is $7'\ 52''.4$. The difference of about 0.2 per cent. is probably due to slight irregularities in the thickness of the silver layer.

Two objections may be raised to this experiment. The first is that with such a thin film, the action between the water and the glass might still be effective, and prevent any slipping. When we remember, however, that the sphere of action of molecular forces is only about 10^{-8} cm., we see that no direct action can occur across a distance of 10^{-5} cm., and it is exceedingly unlikely that a layer of silver more than 1000 molecules thick, should be pervious to water, and thus allow of contact with the glass. In order, however, to entirely meet this objection, experiments were made with considerably thicker layers.

The second objection is that the silver might be deposited so irregularly that the choking effect might mask the quickening due to slip. It had been found, in a series of preliminary experiments, that if a tube was used whose diameter was much less than a millimetre, it was exceedingly difficult to get a uniform silver deposit, and the time of flow for the silvered tube was always much greater than for the unsilvered. To prevent this the tube had to be silvered in a vertical position, and various details, only to be learnt by experience, attended to. The time of flow for the silvered tube could, however, never be brought below that for the plain one, although as greater care was taken in the silvering it continually approximated to it, and this is conclusive against the existence of an effect at all comparable with that given by HELMHOLTZ for polished gold, or with that which he deduced from GIRARD's experiments on copper tubes. As the choking effect was naturally greater for small tubes, a series of experiments was next made on some of rather greater diameter.

Tube No. III.

Unsilvered.		Silvered.	
Temperature.	Time of flow.	Temperature.	Time of flow.
$20°·0$	1′ 2″·2	$20°·9$	1′ 1″·8
20·1	1 2·0	20·9	1 1·3
20·1	1 2·2	20·95	1 1·2
20·2	1 2·0	21·0	1 1·8
20·2	1 1·8	21·0	1 1·4
20·3	1 1·7	21·0	1 1·5
20·4	1 1·7	21·0	1 0·9
20·5	1 1·6	21·05	1 1·3
20·5	1 1·7	21·05	1 1·0
20·5	1 1·5	21·05	1 0·8
20·5	1 1·6		
20·6	1 1·5		
20·3	1 1·79	21·0	1 1·37

Thickness of film = ·0000237 cm.

Radius of tube = ·0776 cm.

Change in r^4 = 0·12 per cent.

Temperature correction = ·7 × 2·22 = 1·55 per cent.

Total correction 1·55 − ·12 = 1·43 per cent. = 0·87 second.

Time for glass tube, corrected   = 1′ 0″·92 ⎱
  „ observed for tube, silvered = 1′ 1″·37 ⎰

A change of + 0·7 per cent.

A rather smaller tube was then used, r = ·0642.

	Unsilvered.			Silvered.	
	Temperature.	Time.		Temperature.	Time.
1	$22°·0$	2′ 8″·4	1	$20°·2$	2′ 13″·0
2	22·0	2 8·2	2	20·2	2 13·1
3	22·05	2 8·4	3	20·2	2 12·9
4	22·05	2 8·3	4	20·2	2 13·4
5	22·1	2 8·4	5	20·25	2 13·0
6	22·15	2 8·0	6	20·3	2 13·0
7	22·2	2 8·0	7	20·4	2 12·8
8	22·2	2 8·2	8	20·4	2 12·6
9	22·2	2 8·0	9	20·45	2 12·4
10	22·25	2 7·8	10	20·5	2 12·4
			11	20·5	2 12·8
			12	20·5	2 12·4
Means	22·12	2 8·17	Means	20·34	2 12·82

Time for unsilvered tube, corrected for changes in

    temperature and radius. . . . . 2 13·59

Time observed for silvered tube. . . . . . . 2 12·82

A difference of − 0·6 per cent.

Another series was made with the same tube which gave as the mean of ten observations for each state :—

Unsilvered . . . . 2 18·78

    ,,   corrected . . . . 2 15·92

Silvered, observed . . . . 2 16·51

A difference of + 0·4 per cent. The silver was very thin, the change in r^4 being 0·10 per cent.

Thus, as the result of four series of observations with three different tubes, we have that the difference in the times of flow for the silvered and unsilvered tubes is never greater than 0·7 per cent. With both the first and second tubes, and in one series of observations with the third tube, the time of flow is slightly greater (by 0·2, 0·7, and 0·4 per cent.) for the silver surface, while in one case—the first series of observations with the third tube—the time is slightly less (by 0·6 per cent.).

These differences are all within the limits of experimental error, found by comparing the times of flow for the same tube in the same state on different occasions.

On the whole there is some evidence that the time is a little greater for the silvered surface, as would, of course, be the case if the deposit was not quite strictly uniform. In the one case, when the time was less, the temperature difference was largest, and errors likely to be most important.

These experiments may at any rate be considered conclusive against the existence of the large effect, for the existence of which I was searching.

A new series of observations was then undertaken to determine whether any slipping occurred in a silvered tube when the velocity of the water was greater than before, and the gradient of velocity was pushed near the limit beyond which the motion ceased to be linear.

This limit was calculated for each tube by means of a formula given by Professor OSBORNE REYNOLDS,[*] who found by experiment that in order to insure linear motion,

$$D v \rho / \mu \text{ must be } < 1400,$$

where D is the diameter of the tube, μ the coefficient of viscosity, ρ the density, and

[*] 'Phil. Trans.,' 1886.

v the velocity of the water calculated from the quantity which flows through in a given time.

It was advantageous to use as small a tube as possible, for two reasons: firstly, because any slipping effect is inversely proportional to the radius, and secondly, because the gradient of velocity can be pushed farther without exceeding the limits of linear motion, the smaller the tube.

A tube, whose diameter was ·084 cm., and whose length was 27·30 cms., was therefore taken, as it was the smallest which could conveniently be silvered, and the greatest allowable pressure calculated.

From the relation given above, we find that the pressure must not exceed $5600\mu^2l/r^3g = 303\cdot5$ cms. of water column.
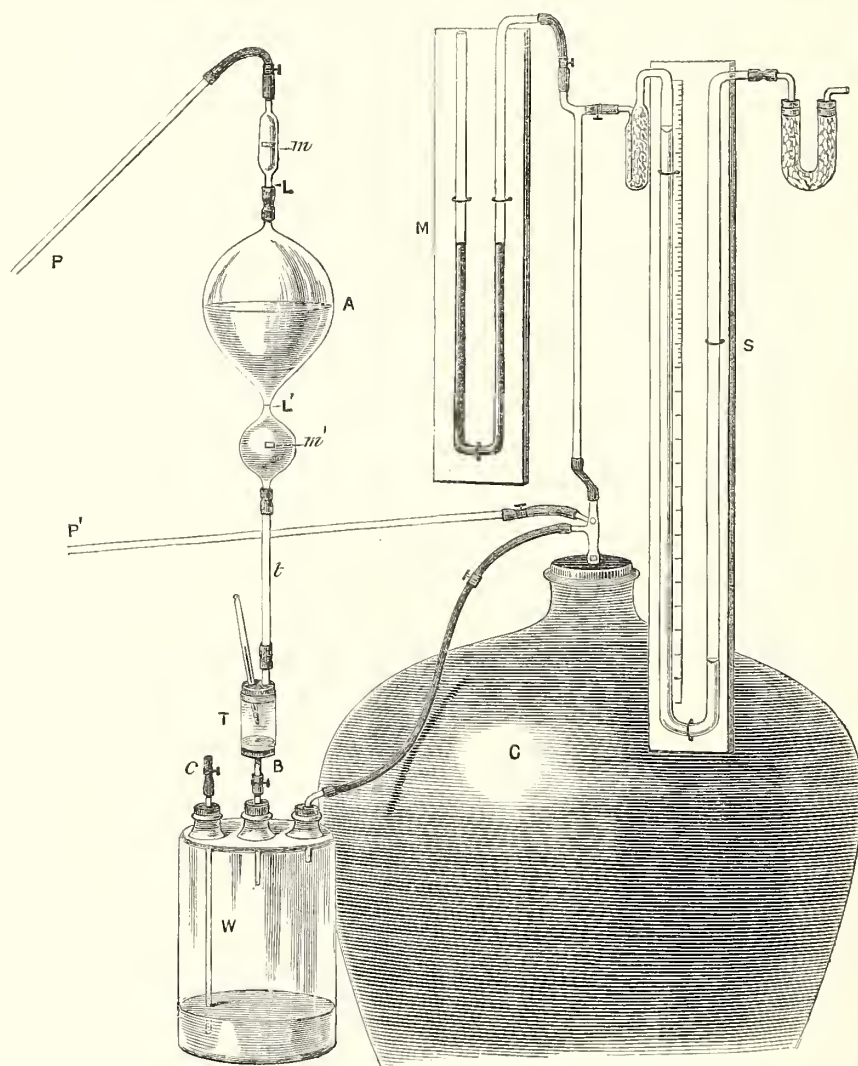
After many preliminary trials an apparatus was set up, which gave most excellent results.

The bulb had a capacity of about 850 c.c., and was fixed to a wooden frame to prevent breakage; this was screwed to the edge of a table. The water bath was eventually abolished, and the temperature of the water read off at intervals during each experiment as it passed through a broad tube, into which the capillary tube opened. The temperature of the water was thus read off immediately after it had passed the important place, and by taking readings at equal intervals while the bulb was emptying, a very accurate estimation of the average temperature could be obtained. Below the temperature tube was a three-necked WOLFF's bottle W; the second neck was connected to a large carboy C, and the third could be put into communication with the atmosphere. The carboy was connected to a three-way tube, the branches of which went, one to the WOLFF's bottle, one to an air pump P′, worked by the water supply, and one to the gauges S and M. The gauge S contained sulphuric acid, and M contained mercury. The latter was only used for the experiments in which higher pressures were employed.* The bulb was filled by disconnecting the joint B, and putting the lower orifice of the temperature tube in communication with c, the tube joining C and W being stopped. P was then put to the pump, and the water in W sucked up. When the bulb was filled, B was again connected, the carboy exhausted by putting P′ to the pump, and the tube from C to W opened. The pump was worked till the requisite pressure, as shown by the gauge, was reached.

The apparatus was then left till the temperature had become constant after the disturbances produced by exhausting, and the height of the gauge read off by a kathetometer. The pressure could be adjusted to within a millimetre or less by regulating the pumps, and small differences in corresponding experiments, were given by the readings of the kathetometer. The correction to be applied to the times of flow for a given small difference of pressure, was determined by observing the actual times of flow for pressures whose difference was considerably greater than that in the experiments to be compared, and keeping all other things unchanged. The small pressure correction could then be accurately estimated from the result of this auxiliary experiment. As before, the experiments were to be only comparative, and the pressure was allowed to fall during each observation. Any change introduced by this would affect the tube equally whether plain or silvered. In order to show the method of working and the degree of accuracy obtained, a complete account of an experiment is given in detail.

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* All permanent joints, corks, &c., were thickly covered with marine glue, and were quite air-tight.

Tube No. 3.

9. *Time.*—Start 0′ 0″; finish 15′ 23″·0 . . . 15′ 23″·0.

Pressure.—Gauge scale, adjusted at start to 22·00, reading at finish 21·47.

$$\text{Kathetometer readings—Start} \begin{Bmatrix} 22{\cdot}680 \\ 22{\cdot}583 \end{Bmatrix} 22{\cdot}682 \\ \left. \text{Finish} \begin{Bmatrix} 23{\cdot}229 \\ 23{\cdot}230 \end{Bmatrix} 23{\cdot}230 \right\} \quad . \quad . \quad 22{\cdot}956.$$

Temperature readings at intervals of two minutes—

13·00   12·88   12·86   12·90   12·92   12·93   12·95   12·98 . . 12°·93.

In order to make the pressure readings strictly comparable, the level of the water before each experiment was adjusted to a mark, *m*, in the top bulb, and after each to a mark *m′* in the lower bulb, and all pressure readings were taken while the water stood at these levels. The pressure was adjusted by the pump till the reading on the gauge scale was as nearly as possible 22·00, and the exact height then read off by a telescope in terms of kathetometer scale.

Tube No. 1.—$(r = {\cdot}042 \text{ cm.}; l = 27{\cdot}30 \text{ cms.})$

Total difference of pressure equivalent to about 250 cms. of water column. The limiting pressure is 305 cms.

	Pressure.	Temperature.	Time.
Unsilvered.	mm.		
1. Gauge reading	407·1	12°59	12′ 25″·7
2. ,, ,, ...	407·0	12·71	12 24·1
Silvered.			
3. Gauge reading	407·1	12·79	12 23·5

The temperature correction is 2·22 per cent. for 1° C., and in order to correct (2) to 12°·79, we must subtract 1″·4.

The pressure correction is for 0·1 mm. of sulphuric acid in a total pressure equivalent to about 1400 mm., *i.e.*, 1 in 14,000, and is therefore negligible.

The weight of silver deposited is 0·0008 grm., and the thickness 0·00001 cm., a change in r^4 of 0·1 per cent.

$$\left. \begin{array}{l} \text{The time for the unsilvered tube corrected} = 12' \ 23''{\cdot}4 \\ \text{ ,, \quad ,, \quad silvered \quad ,, \quad observed} = 12 \ 23{\cdot}5 \end{array} \right\}$$

At slightly different pressures—

	Pressure.	Temperature.	Time.
	mm.		
Silvered	406·4	12·52	12 29·7
Unsilvered	406·5	12·62	12 24·3
Unsilvered, corrected for temperature and change in r^4 . . .			12 26·4 ⎫
Silvered, observed time			12 29·7 ⎬

Tube No. 3.—($r = ·036$ cm. ; $l = 19·98$ cms.)

Total pressure at beginning of each observation 84·15 cms. of water + 21·028 cms. of mercury ; equivalent to 368·62 cms. of water. The critical pressure for this tube is 440 cms. of water. The tube was silvered with solutions of half strength very carefully.

	Mean reading of kathetometer.	Temperature.	Time.
Silvered	22·963	12·70	15 46·5
	22·956	12·93	15 23·0
	22·955	13·41	15 18·0
	22·966	12·76	15 34·0
	22·960	12·95	15 30·4
Unsilvered.	22·959	13·04	15 16·5
	23·004	13·69	15 7·3
	22·982	13·37	15 11·9 ⎫
Silvered, corrected for change in radius − 0·16 per cent., for temperature − 0·93 per cent., and for pressure + 0·14 per cent.			15 21·5 ⎬

In this very small tube there is thus a choking effect which increases the time by about 1 per cent.

The same tube was then re-coated with a very thin deposit which was just transparent to blue light.

Pressure.	Temperature.	Time.
mm.		
23·026	15·31	14 48·2
23·037	15·27	14 50·8
23·032	15·29	14 49·5

Time.

Unsilvered.—Pressure 23·005 mm. ; temperature 15°·64 .   .   .   14ʹ 43ʺ·2
   „   Corrected for temperature and pressure   14   51·7 ⎱
Silvered.—Observed time   14   49·5 ⎰

A decrease of 0·25 per cent.

The change in radius was inappreciable.

Thus the result of four series of experiments at these large differences of pressure is that in three cases the time for the silvered tube comes out slightly greater [by 0·014, 0·44, and 1·0 per cent.], and in one case slightly less [by 0·25 per cent.].

This agreement may be considered a quite satisfactory proof of identity.

In support of his view that there is a finite slipping coefficient HELMHOLTZ refers to some experiments of GIRARD,[*] who examined the time of flow of water through copper tubes, found that the motion was linear within limits which agree fairly well with those given by REYNOLDS' formula, but got times of flow much less than those observed by POISEUILLE for glass tubes. Thus with a tube whose diameter was 1·83 mm. and length 1790 mm., a quarter of a litre of water flowed through in 624·5 secs. under a pressure of 100 mm. of water and at a temperature of 0°·5, while POISEUILLE's formula gives 2949 secs.

As I could detect no error in GIRARD's account of his experiments I determined to repeat them. Messrs. ELLIOTT, of Sellyoak, Birmingham, most kindly undertook to manufacture some solid drawn copper tubes of the dimensions required, and they were entirely successful. The apparatus employed was essentially the same as that used by GIRARD, and was of the simplest possible nature. The results depended on the value obtained for the diameters of the tubes, as determined by weighing when empty and when full of water, and as no allowance could be made for irregularities or non-uniformities in the bore, calibration being impossible, it was useless to determine the time of flow or the temperature to any great degree of accuracy.

A glass jar was arranged in the manner shown in the figure, and the difference of level between the surface of the water in A and the orifice B determined by a kathetometer to a tenth of a millimetre. Below B was placed a 100 c.c. flask, and the time taken to fill this was observed with a stop watch to an accuracy of about 1 sec. in 400 or 600.

The temperature was observed in the jar A and also in B, and the two seldom differed by more than a tenth of a degree. In GIRARD's investigations the level of the water in A was allowed to fall during each experiment, and its mean value assumed to represent the effective driving pressure throughout. In order to determine whether such an arrangement was allowable for these small pressures, where the change was a large fraction of the whole, a comparison was made with a series of

[*] 'Mémoires de l'Institut,' 1813–1815.

experiments, during which the level was kept constant by allowing water to flow into A at the same rate as it flowed out.
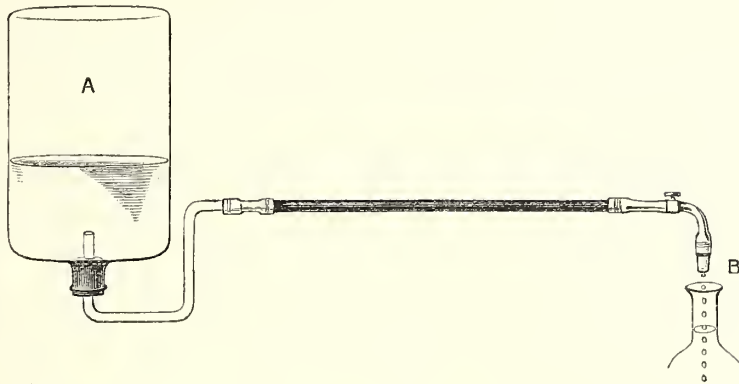
A glass tube was first used ($r = \cdot0568$ cm., $l = 22\cdot53$ cms.).

Pressure constant.

Level of reservoir, $19\cdot30$
„ orifice, $30\cdot23$ } . . . . $10\cdot93$ cms.

Temperature, $18°\cdot1$; $18°\cdot1$ . . . . $18°\cdot1$

Time : start, 6′ 0″ ; finish, 17′ 28″ . . . . 11′ 28″

Value of the coefficient of viscosity μ, deduced from
this by POISEUILLE's formula . . . . $\cdot01339$

Pressure varying.

Level of reservoir { Start, $18\cdot47$
Finish, $20\cdot22$ } $19\cdot35$
„ orifice . . . . . . $30\cdot24$ } . . . . $10\cdot59$ cms.

Temperature, $18°\cdot1$; $17°\cdot9$ . . . . $18°\cdot0$

Time : start, 8′ 0″ ; finish, 19′ 32″ . . . . 11′ 32″

Value of μ . . . . $\cdot01341$

A copper tube was then substituted.

Tube No. 5.—($r = \cdot0836$ cm., $l = 30\cdot86$ cms.)

Pressure constant.

Levels, $20\cdot05$; $26\cdot57$ . . . . $6\cdot52$ cms.

Temperature, $16°\cdot4$; $16°\cdot4$ . . . . $16°\cdot4$

Time : start, 0′ 0″ ; finish, 5′ 51″ . . . . 5′ 51″

Value of μ . . . . $\cdot01391$

Pressure varying.

Levels $\begin{Bmatrix} 18\cdot34 \\ 20\cdot06 \end{Bmatrix} 26\cdot57$ . . . . 7·37 cms.

Temperature, 16°·4 ; 16°·6 . . . . 16°·5

Time, 6′ 0″ ; 10′ 37″ . . . . 4′ 37″

Value of μ . . . . ·01241

Levels $\begin{Bmatrix} 20\cdot06 \\ 21\cdot78 \end{Bmatrix} 26\cdot56$ . . . . 5·64 cms.

Temperature, 16°·5 ; 16°·5 . . . . 16°·5

Time, 1′ 0″ ; 7′ 7″ . . . . 6′ 7″

Value of μ . . . . ·01258

Pressure constant.

Levels, 23·42 ; 26·56 . . . . 3·07 cms.

Temperature, 16°·5 ; 16°·6 . . . . 16°·6

Time, 7′ 0″ ; 18′ 33″ . . . . 11′ 33″

Value of μ . . . . ·01293

Levels, 21·56 ; 26·56 . . . . 5·00 cms.

Temperature, 16°·6 ; 16°·6 . . . . 16°·6

Time, 0′ 0″ ; 6′ 54″ . . . . 6′ 54″

Value of μ . . . . ·01258

Pressure varying.

Levels $\begin{Bmatrix} 21\cdot78 \\ 23\cdot48 \end{Bmatrix} 26\cdot56$ . . . . 3·93 cms.

Temperature, 16°·6 ; 16°·6 . . . . 16°·6

Time, 7′ 0″ ; 15′ 59″ . . . . 8′ 59″₄

Value of μ . . . . ·01288

Thus the experiments, both with the glass and with the copper tube, show that the time of flow is the same if the pressure be allowed to fall, as it is if the pressure be kept constant at the mean value of the falling pressure.

The results also show that the copper tube gives a value for μ practically identical with that given by the glass tube, and a little greater than that given by POISEUILLE's experiments, instead of about five times less.

The effect of modifying the interior surface was then investigated. Tubes were cleaned with acids and alkalies, polished with emery powder, coated with a film of oil, and amalgamated with mercury.

Tube No. 1.—($r = \cdot0803$ cm. ; $l = 30\cdot9$ cms.).

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms.	°	′ ″	
16·825	13·2	2 46·0	·01448
15·320	16·6	3 3·3	·01456

The tube was then cleaned with dilute nitric acid. The radius was re-determined, but was only changed by 0·08 per cent.

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms.	°	′ ″	
17·215	14·0	2 37·5	·01406
16·37	14·6	2 42·8	·01382

Cleaned with hydrochloric acid and potash.

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms.	°	′ ″	
17·86	13·7	2 32·5	·01412

Another tube was then taken and polished inside by working it along a stretched string, covered with fine emery powder.

Tube No. 2, polished with emery powder.—($r = \cdot08094$ cm. ; $l = 23\cdot30$ cms.)

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms.	°	′ ″	
16·43	13·0	2 9·5	·01510
14·69	13·0	2 23·6	·01497

The inside was then coated with a film of oil. The radius was re-determined, $r = \cdot08003$ cm.

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms.	°	′ ″	
16·56	13·6	2 13	·01494
15·07	13·6	2 27	·01502

Tube No. 5 was then cleaned with acid and amalgamated by leaving it for some time filled with mercury, and running a stream of mercury through several times. The radius was re-determined, $r = \cdot0833$ cm.

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms. 3·36 3·35 4·94	° 15·4 15·5 15·7	′ ″ 10 56 10 54·5 7 12	·01324 ·01317 ·01282

In the experiments with the same tube described above, when the surface was copper, the following results were obtained at similar pressures :—

Average pressure.	Temperature.	Time of flow.	Value of μ.
cms. 3·07 3·93 5·00	° 16·6 16·6 16·6	′ ″ 11 33 8 59 6 54	·01293 ·01288 ·01258

Thus in none of these experiments does the value of μ differ much from that given by POISEUILLE for glass tubes, but, like his, agrees with the formula deduced from the supposition that no slip occurs. In all cases it is slightly greater, which is readily explained by irregularities in the tubes, owing to the difficulty of drawing them. According to GIRARD's results, the value of μ should have about a quarter of the value given by POISEUILLE, but in none of the experiments described in this paper did it fall below POISEUILLE's value, and more decisive still, no change in the nature of the surface changed the rate of flow ; this is purely a comparative method, and seems much more reliable than the absolute method of GIRARD, which depends on accurate measurements of the radii of the tubes, differences in pressure, &c. GIRARD only used tubes of two sizes, and gives no account of the means he employed to estimate their radii. At the same time it should be noticed that his values for the two sizes agree fairly between themselves with the supposition that a slipping coefficient exists, whose value is about 0·4 mm. Any constant error in the estimation of the radius, would however be naturally of greater importance in the smaller tube, and may have led to the apparent agreement with the results of an effect, inversely proportional to the radius, and due to the existence of a finite slipping coefficient.

We must now return to the consideration of the experiments of HELMHOLTZ and PIOTROWSKI.

The discussion of the body of their paper I must leave to those with the requisite mathematical knowledge, merely observing in passing that it is remarkable that the value they deduce for the coefficient of viscosity of the liquid itself is considerably

greater (by about one-fourth) than that given by POISEUILLE's experiments. This seems to suggest that some slight modification in the application of the formulæ may be necessary, which will reduce the value deduced for the viscosity of the liquid, and increase that for its adhesion to the vessel to the value requisite for the condition of no slip.

By a preliminary series of experiments PIOTROWSKI claims to have shown that the friction on a body oscillating in contact with a liquid depends on the nature of the surface. He suspended a glass flask bifilarly, filled it with water, and observed the time of swing and the logarithmic decrement. He then silvered the inner surface and repeated his observations. The results are as follows :—

	Time of swing.	Logarithmic decrement.
Unsilvered	23·9333	0·0622182
	23·9333	0·0628467
	23·9333	0·0625325
Silvered	24·0088	0·0600305
	24·0076	0·0599622
	24·0082	0·0599964

As the result of these observations, PIOTROWSKI calculates that the ratio of the friction on glass to the friction on silver is as 1 : ·95645.

Independently of the fact that no account is given of any precautions to keep the temperature constant, or even to measure it, it is evident that the above determination is liable to errors due to changes in the suspension, which are very apt to occur, and that the agreement between the pairs of readings is not very close.

However, in order to test whether such an effect were appreciable, I undertook a series of experiments with an apparatus similar to that used by PIOTROWSKI.

A glass bulb was blown as nearly as possible spherical, and the neck drawn off
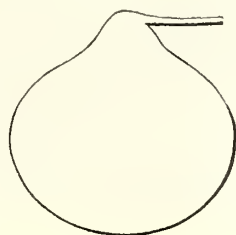
sideways into a fine tube. It was filled with water by means of an air pump, and always kept completely full; when left, a piece of india-rubber tubing filled with water was attached, so that if the temperature of the room sank, water, and not air was drawn in. During working the temperature was always slowly rising, and before

each observation the drop of water was removed by blotting paper. At any given temperature the apparatus was therefore in a definite state, and this was obtained at much less expenditure of time and patience than if the bulb had been always filled to a certain mark at a certain temperature, or the same mass of water always put in by adjusting the weight. A mirror was attached to the bulb by sealing wax, and the whole suspended bifilarly by a fine copper wire. The logarithmic decrement had to be reduced by suspending two brass balls at the end of a long bar magnet which was fixed to the bulb and by means of which the apparatus was set in oscillation.
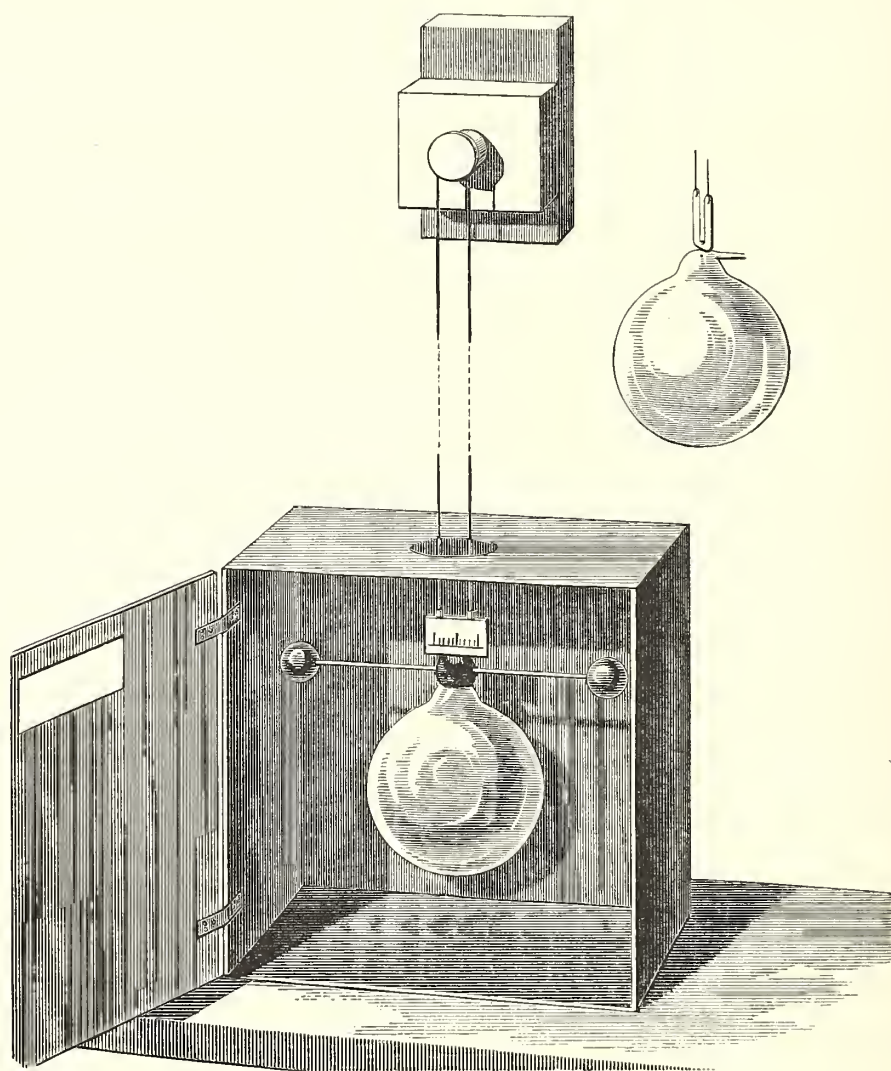
A series of preliminary observations gave as the ratio of the frictions 1 : 1·0001, an accuracy in the proof of identity which was not justified by the roughness of the observations, but which, at any rate, showed that the difference could not be very great.

After a week spent in preliminary investigation, the apparatus was set up in the

manner which had been found to answer best, and a series of observations taken, one of which is given in detail :—

TIMES OF TRANSIT.

$\overset{'}{3}2$ $\overset{''}{1}{\cdot}5$	$\overset{''}{10}{\cdot}5$	19	28	37	45·5	54·5	3·5	12	21	30	38·5
$\overset{'}{3}5$ $\overset{''}{6}{\cdot}5$	$\overset{''}{15}{\cdot}5$	24	32·5	41·5	50	59	8	16·5	25·5	34	43

∴ Time of 1 vibration = 8″·794.

Logarithmic decrement.—Zero 1·0 ; at end 1·0. Readings at one end of swing :—

42·2	36·7	32·0	27·9	24·3	21·2	18·5	16·16
14·17	12·40	10·90	9·58	8·44	7·44	6·60	5·85
5·20	4·63	4·16	3·74	3·37	3·06	2·77	

∴ Logarithmic decrement = ·14273.

The bulb was kept in a beaker of water whose temperature could be easily observed, till just before each observation, when it was rapidly dried with blotting paper, allowed to come to rest, and set oscillating by means of a strong bar magnet. The limit of each swing was read by means of a telescope mounted at the centre of a curved scale at about two metres distance, and the times of transit over the centre of the scale taken by a chronometer.

The following determinations were made :—

	Temperature.	Time of swing.	Logarithmic decrement.
Unsilvered	13·5	8·781″	·14224
	13·6	8·783	·14063
	13·6	8·787	·14115
	12·8	8·801	·14269
	12·9	8·796	·14328
		8·806	·14276
	13·0		·14204
			·14278
	13·1	8·798	·14305
			·14307
			·14289
	13·2	8·794	·14273
			·14192
			·14125
	13·8	8·732	·14066
	14·0	8·736	·14052
	14·0	8·740	·14094
	14·2	8·732	·14106
	14·2	8·750	·14093
			·14090
Silvered	12·6	8·764	·14338
			·14310
	12·6	8·768	·14274
			·14380
	12·9	8·837	·14304
	13·1	8·810	·14270
	14·1	8·829	·14232
	13·9	8·828	·14125
	13·8		·14191
	14·0	8·806	·14143
	14·1	8·808	·14115
			·14121
Unsilvered	14·0	8·823	·14167
	14·3	8·830	·14126
	14·4		·14101

If we take the means of these observations we get :—

	Temperature.	Time.	Logarithmic decrement.
Unsilvered	13·66°	8·779″	·141801
Silvered	13·40	8·806	·142335

From the series of observations with the unsilvered flask we find that the alteration in the logarithmic decrement for a change in temperature of 1° C. is about ·00160. For 0°·26 the change will be ·000416, and the logarithmic decrement of the unsilvered bulb corrected to a temperature of 13°·40 is ·142217.

By Helmholtz and Piotrowski's paper we see that the ratio of the friction on silver to the friction on glass is as $1 : \dfrac{8\cdot806 \times \cdot142217}{8\cdot779 \times \cdot142335} = 1 : 1\cdot0022.$

The change, if it exists at all, is according to these experiments less than $0\cdot3$ per cent.

A modification of Piotrowski's experiment was then tried. Instead of filling the oscillating flask with water, it was filled with sand, and oscillated as a rigid body in a large beaker of water. The temperature could then be accurately observed, and the ordinary investigation of the oscillations of a rigid body in a resisting medium, and acted on by a force proportional to the displacement, will hold.

Let k denote the frictional force proportional to the velocity,

M the moment of inertia,

λ the logarithmic decrement.

Then it is easily shown that

$$k = \frac{\lambda M}{\pi}\, p,$$

where

$$p = \sqrt{\left(\frac{\mu}{M} - \frac{k^2}{4M^2}\right)},$$

μ being the force of restitution for unit displacement.

$$\therefore\ k^2 = \frac{4\lambda^2 \mu M}{4\pi^2 + \lambda^2}.$$

Now in our case λ has a value of about $0\cdot2$, and λ^2 can therefore be neglected in comparison with $4\pi^2$,

$$\therefore\ k = \frac{\lambda}{\pi}\,\sqrt{(\mu M)}\quad \text{(approximately)}.$$

When the flask is silvered μ is unchanged, as the weight of silver is much too small to appreciably alter the bifilar couple, and, therefore, we get for the ratio of the frictions

$$\frac{k}{k'} = \frac{\lambda\sqrt{M}}{\lambda'\sqrt{M'}} = \frac{\lambda T}{\lambda' T'},$$

where T and T' are the respective times of vibration.

A thick platinum wire was attached to the bulb, and the bifilar arrangement fixed to this above the surface of the water.

At the conclusion of the experiments, the bulb filled with sand was oscillated in air and the logarithmic decrement found to be a very small fraction of that observed when the bulb was in water. This meets the objection that the chief resistance

might be due to the suspension, and a small change in that part due to the water be inappreciable.

Logarithmic decrement in air . . = ·00284.

Time of swing . . . . = 9·952 seconds.

The logarithmic decrement due to the air, suspension, &c., is only about 2 per cent. of that due to the water.

	Temperature.	Time of swing.	Logarithmic decrement.
Unsilvered	° 12·6 12·6 12·7	, 9·766 9·769 9·772	·19745 ·19882 (1) ·19830 ·19795
	12·9 13·0	9·741 9·722	·19694 ·19729 (2) ·19685
	13·0 13·7	9·737 9·686	·19392 ·19331 (3) ·19330
	13·8 13·7 13·8	9·725 9·739 9·731	·19374 ·19333 (4) ·19412
	13·18	9·739	·19579
Silvered	13·6 13·5	9·884 9·891	·19588 ·19717 (5) ·19622
	13·55	9·887	·19642
Resilvered	12·5 12·6	9·892 9·892	·19945 ·19943 (6) ·19943
	12·9 13·0	9·882 9·907	·19811 ·19793 (7) ·19775
	14·5 14·6	9·894 9·885	·19406 ·19363 (8) ·19331
	9·75 9·75	9·891 9·904	·20688 ·20695 (9) ·20772
Unsilvered	10·0 10·0	9·939 9·936	·20687 ·20721 (10) ·20636
	12·7 12·8	9·917 9·920	·19881 ·19862 (11) ·19860

Between each of the series of observations marked (1), (2) . . . the suspending wire was re-adjusted, but an inspection of the results shows that they agree well among themselves, and that therefore the bulb might safely be moved for silvering. After

the observations marked (4) the bulb was suspended in the silvering solution for about an hour, and then removed. It was then found that the top hemisphere was covered with a black sediment from the solution on top of the silver, while the under hemisphere was silvered as usual. Nevertheless a series of three determinations of the logarithmic decrement was taken. The mean of these shows an increase in both the logarithmic decrement and in the time of swing, even though the temperature was actually higher. It was thought that this might be due to the black sediment, which could not be removed alone; so the whole deposit was dissolved off and the bulb resilvered, the solution being kept stirred and frequently changed. This time the deposit was bright and uniform. The observations were, however, identical with the last. The only explanation of this (unless we suppose that the friction is about 3 per cent. greater for silver, instead of 4 per cent. less as PIOTROWSKI deduced) is to suppose that a change had occurred in the suspension. To test this, a series of observations (6) to (9) were taken with the silver on to get the temperature correction, and immediately after (9) the beaker of water was removed from under the bulb, and one of nitric acid of the same temperature put in its place without disturbing the suspension. As soon as the silver was dissolved the bulb was washed, and the beaker of water replaced.

A series of observations (10) were at once taken with the glass surface. Thus by comparing (9) with (10) we get a comparison of the friction on glass with the friction on silver, free from all possible errors due to change of suspension, and at temperatures whose difference is only $0°\cdot25$. The means are—

	Temperature.	Time of swing.	Logarithmic decrement.
(9) Silvered	$9°\cdot75$	$9\cdot898''$	$\cdot20718$
(10) Unsilvered	$10\cdot00$	$9\cdot938$	$\cdot20681$

The temperature correction for the logarithmic decrement is by (8) and (9) $\cdot002815$ for $1°$ C. or $\cdot00070$ for $0°\cdot25$.

Therefore the logarithmic decrement for the glass surface corrected to $9°\cdot75$ is $\cdot20751$.

The ratio of the frictions is $\dfrac{9\cdot938 \times \cdot20751}{9\cdot898 \times \cdot20718} : 1 = 1\cdot00564 : 1$.

Thus the effect is, if it exists at all, less than $0\cdot6$ per cent. instead of 4 per cent.

Another independent comparison can be taken between the series marked (6) and (7) and the series marked (11). The means are—

	Temperature.	Time of swing.	Logarithmic decrement.
(6) and (7)	12°75	9·893	·19868
(11)	12·75	9·922	·19867

The ratio of the frictions is 1·00288 : 1, the change being less than 0·3 per cent. greater for glass. Thus within the limits of experimental error the friction on silver is the same as the friction on glass, and this part of Piotrowski's paper is certainly misleading.

It is evident that the oscillation method is much inferior to that in which the flow of water through tubes is observed. The experiments described in the early part of this paper show that the difference in the time of flow for a glass and silver tube is less than *one-half per cent.*, in a case where the existence of a slipping coefficient of only one-half the magnitude of that deduced by Helmholtz for gold, would make the time of flow for the silver tube about *twelve times* less than the time of flow for the glass tube.

The arguments sometimes used in favour of the contact theory of electromotive force, based on the differences in friction of a liquid on different solid surfaces, must now be admitted to be without value. It is certain that no slip occurs, at any rate in the case of substances which are wetted by the liquid.

In conclusion I must offer my most sincere thanks to Professor J. J. Thomson and to Mr. Glazebrook for the help they have given me, and the many valuable suggestions they have made.

X. *A Determination of " v," the Ratio of the Electromagnetic Unit of Electricity to the Electrostatic Unit.*

By J. J. Thomson, *M.A., F.R.S., Cavendish Professor of Experimental Physics, Cambridge, and* G. F. C. Searle, *B.A., Peterhouse; Demonstrator at the Cavendish Laboratory, Cambridge.*

The experiments made by one of us in 1883 having given a value of " *v* " considerably smaller than the one found by several recent researches, it was thought desirable to repeat those experiments. The method used in 1883 was to find the electrostatic and electromagnetic measures of the capacity of a condenser; the electrostatic measure being calculated from the dimensions of the condenser, the electromagnetic measure determined by finding the resistance which would produce the same effect as that produced by the repeated charging of the condenser placed in one arm of a Wheatstone's Bridge. In the experiments of 1883 the condenser used in determining the electromagnetic measure of the capacity was not the same as the one for which the electrostatic measure had been calculated, but an auxiliary one, without a guard ring, the equality of the capacity of this condenser and that of the guard ring condenser being tested by the method given in Maxwell's 'Electricity and Magnetism,' vol. 1, p. 324.

In repeating the experiment we adopted at first the method used before, using, however, a key of different design for testing the equality of the capacity of the two condensers by Maxwell's method. We got very consistent results, practically identical with the previous ones. We may mention here, since it has been suggested that the capacity of the leads might account for the small values of " *v* " obtained, that this capacity is allowed for by the way the comparison between the capacities of the auxiliary and guard ring condensers is made, for the same leads are used both in this comparison and in the determination of the electromagnetic measure of the capacity of the auxiliary condenser; the capacity of the auxiliary condenser, plus that of its leads, is made equal to the capacity of the guard ring condenser, and it is the capacity of the auxiliary condenser, plus its leads, which is determined in electromagnetic measure. As the introduction of the auxiliary condenser introduced increased possibilities of error, we endeavoured to determine directly the electromagnetic measure of

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the capacity of the guard ring condenser, by using a complicated commutator which worked both the guard ring and the condenser. At first we tried one where the contacts were made by platinum styles attached to a tuning fork, but as the results were not so regular as we desired, we replaced the tuning fork commutator by a rotating one driven by a water motor. A stroboscopic arrangement was fixed to this commutator so that its speed might be kept regular and measured. With this arrangement, which worked perfectly, we got values for the electromagnetic measure of the capacity of the condenser distinctly less than those obtained by the old method. We then endeavoured to find out the cause of this difference, and after a good deal of trouble discovered that in the experiments by which the equality of the capacities of the guard ring and auxiliary condensers was tested by MAXWELL's method, the guard ring did not produce its full effect. When the guard ring of the standard condenser was taken off, and its capacity made equal by MAXWELL's method to the capacity of the auxiliary condenser, the two methods gave identical results; but the effect of adding the guard ring was less in the old method than in the new. We found also, by calculation, that the effect produced by the guard ring in the old method was distinctly too small, while that determined by the new method agreed well with its calculated value. As the new method was working perfectly satisfactorily, and as it possesses great advantages over the old one, inasmuch as we get rid entirely of the auxiliary condenser, and can also alter the speed of the rotating commutator with very much greater ease and considerably greater accuracy than in any arrangement where the speed is governed by a tuning fork, we discarded the old method and adopted the new one which we now proceed to describe, beginning by considering the errors to which this method is liable.

Advantages of the Method of Determining " v."

The best way of discussing the advantages of this method is to consider the quantities which have to be measured and the accuracy which can be obtained in their measurement. The investigation naturally divides into two parts (1) the determination of the capacity of a condenser in electrostatic measure; (2) the determination of the capacity of the same condenser in electromagnetic measure. Let us begin by considering the first part. The condenser consisted of two co-axial cylinders, the inner cylinder being provided with a guard ring. If the distribution of electricity on the middle part of the inner cylinder were the same as that on an equal length, l, of an infinite cylinder whose radius is a, surrounded by a co-axial infinite cylinder of radius b, the electrostatic measure of the capacity would be $\frac{1}{2} l/\log b/a$. The actual case may differ from this ideal one in some or all of the following ways. (1) The two cylinders may not be quite co-axial; this, however, is not important if we know the distance betwen the axes, as we can find the capacity of the system got by placing one cylinder anywhere inside another. (2) The cross sections of the cylinders may

not be accurately circles. The effect on the capacity of a slight departure from circularity is calculated below, so that this effect may be corrected. (3) The conductors may not be true cylinders but swell or contract slightly as we proceed along their lengths; we show, however, below, how to correct for an effect of this kind. (4) The existence of the air space between the guard ring and the middle cylinder will cause the distribution of electricity near the ends of this cylinder to be irregular, and there will also be some electricity on the cross section of the cylinder; we have, therefore, found the distribution of electricity in a case so nearly resembling this as to allow us to use the result as a correction. In the arrangement we used the potential of the guard ring differed slightly from that of the middle cylinder, the very small correction due to this is, however, easily calculated. Since we know the corrections, the capacity of the condenser can be calculated in terms of its dimensions, and the only errors to which we are liable are those which may be made in the determination of these dimensions. The lengths which have to be measured with great accuracy are the length of the middle cylinder, its radius and that of the outer cylinder, and the distance between the cylinders. The first three of these are long enough to be measured by the ordinary methods of measuring length, without danger of an error greater than one part in 3000, the fourth, however, is too small to be measured with so great an accuracy by these methods, it was determined, therefore, by finding v, the volume of water required to fill the space between the two cylinders, then d the distance between the cylinders is given by the formula

$$d = \frac{v}{\pi l (a + b)}$$

where l is the length of the middle cylinder and a and b the radii of the two cylinders. In this way the percentage error of d was not greater than those of a, b, and l. Since an accuracy of one part in 3000 can be obtained in the measurements of the dimensions of the cylinders, and since the electrostatic measure of the capacity is of the dimensions of a length, this measure of the capacity can be obtained correct to one part in 3000.

We now pass on to the determination of the capacity in electromagnetic measure. This was determined by balancing, in a Wheatstone's bridge, a discontinuous current produced by rapidly charging the condenser against a steady current derived from the battery which charged the condenser. In order to calculate the electromagnetic measure of the capacity it is necessary to know accurately the number of times per second the condenser is charged, and to keep this number constant. The charging and discharging of the condenser were effected by a commutator driven by a Thirlmere Water Motor, the water being obtained, not from the main, but from a cistern at the top of the Laboratory. The number of revolutions per second made by the commutator was compared by a stroboscopic arrangement with the frequency of an electrically driven tuning fork. The observer (G.F.C.S.) was able, after practice, to govern the

speed of the commutator so efficiently that when the condenser was in action the spot of light reflected from the mirror of the galvanometer did not move over more than half a millimetre.

The accuracy of measurement of the number of times the condenser was charged per second is thus practically the same as the accuracy of the determination of the frequency of the tuning-fork ; this frequency could be determined (see *infra*) to less than one part in 10,000.

The limit which is practically put on the determination of the electromagnetic measure of the capacity of the condenser is that imposed by the galvanometer. With the galvanometer we employed, which was one made in the laboratory, having about 30,000 turns and a resistance of 17,400 legal ohms, when the resistance of the variable arm of the Wheatstone's bridge was 2500 ohms, an alteration of 2 ohms could be detected ; thus the measurement of the resistance equivalent to the repeatedly charged condenser could be made to one part in 1250 : an error of this magnitude would cause an error of one part in 2500 in the value of " v," and as all the other measurements were more accurate than this, there seems no reason why this method should not give as accurate a value of " v " as that obtained for the ohm.

The electromagnetic way of measuring the capacity affords us the means of testing the accuracy of the corrections applied to the electrostatic measure of the capacity ; we availed ourselves of this in the case of the correction for the effect of the air space between the middle cylinder and the guard-ring ; we altered the thickness of this air space and found that the effect of this alteration was accurately represented by the correction we employed. One great advantage of the method is the ease with which the number of times per second the condenser is charged can be altered ; this affords a valuable means of detecting any leakage or any effect due to self-induction.

Calculation of the Electrostatic Measure of the Capacity of the Condenser.

Description of the Condenser.—The condenser, which was designed some years ago by Lord RAYLEIGH, is represented in section in fig. 1, and in plan in fig. 2. BHPD is a thick ebonite board, placed in an approximately horizontal position ; in this board two concentric circular grooves are cut. A cylindrical brass ring, HP, whose external diameter is about 23 cm., and whose height is about 10 cm., fits into the smaller of these grooves. Three pieces of ebonite carefully ground down to the same thickness (about 3 mm. in most of the experiments), with V-shaped grooves cut in them to increase the distance over which the electricity would have to leak are placed at equal ntervals on the top of this ring. On these the brass cylinder FGMN is placed ; this cylinder is of exactly the same diameter as the cylindrical ring HP, and is about 60 cm. long. The cylinders GFMN and HP are placed so that their axes are coincident. On the top of this cylinder three pieces of ebonite similar to those on HP are placed, and

upon the top of these a cylindrical ring EL, similar to the ring at the bottom. Another brass cylinder, ABCD, made in three pieces, two rings somewhat similar in height to the rings HP, EL, and a long middle piece of the same length as the cylinder FGMN, is then fitted over the other cylinders, the bottom ring fitting into the outer groove in the ebonite board; the internal diameter of this cylinder is about 25 cm.

Fig. 1.

Fig. 2.
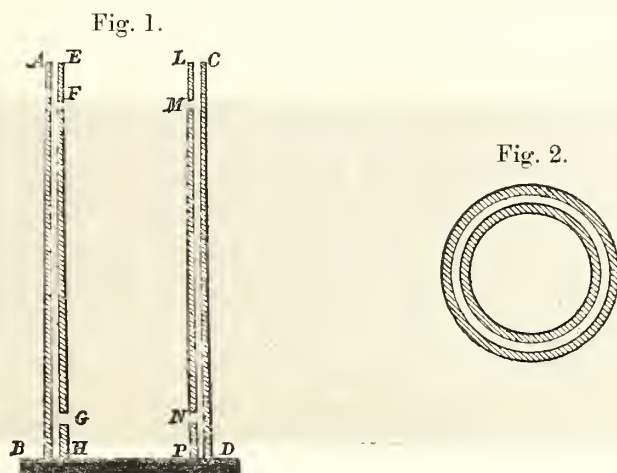
The cylinders are made co-axial by means of three pieces of ebonite worked down to the same thickness (the difference between the radii of the cylinders) pushed by rods attached to them down between the cylinders, the cylinders are adjusted until these three pieces of ebonite arranged symmetrically round the cylinder are each just in contact with the two cylinders; the rods were then removed. The insulation between the inner and outer cylinders and between the inner cylinder and its guard rings was tested by connecting one of these to earth, and the other to a charged gold leaf electroscope; the condenser was not used unless there was no appreciable loss of electricity shown by the electroscope in five minutes.

Calculation of the Capacity.—The capacity of the system regarded as two co-axial cylinders of circular section with a uniform distribution of electricity over them is

$$\tfrac{1}{2} \, l / \log \frac{a}{b},$$

where a is the radius of the outer cylinder, b that of the inner, and l the length of the cylinder FGMN.

Correction for want of coincidence between the Axes.—It is shown in a paper by J. J. Thomson, "On the Determination of the number of Electrostatic units in the Electromagnetic unit of Electricity" ('Phil. Trans.,' 1883, p. 714), that if c be the small distance between the axes of the cylinders, the capacity is

$$\tfrac{1}{2} \frac{l}{\log \frac{a}{b}} \left\{ 1 + \frac{c^2}{(a^2 - b^2) \log \frac{a}{b}} \right\}.$$

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Correction for the want of equality in the distribution produced by the air spaces between the inner cylinder and the guard rings.—To find this correction we shall find the distribution of electricity on the system represented in the figure, when AB is the section by the plane of the paper of an infinite horizontal metal plane, and CDE, FGH sections of conductors. CD and GH being horizontal and at the same distance from AD; and DE and FG vertical. Let h be the distance between the planes CD, and AB, and $2c$ the breadth of the slit DE FG.
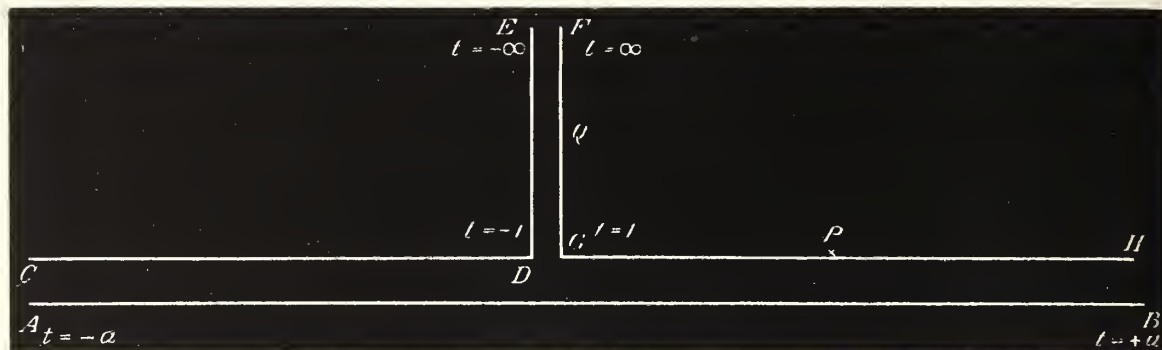
Let us take AB as the axis of x and the vertical line midway between ED and FG as the axis of y. Then writing z for $x + \iota y$, and supposing that ϕ and ψ are the stream and potential functions respectively, we find by using SCHWARZ's method that the solution of the problem is given by the equations

$$dz = - \mathrm{A} \frac{\{1 - t^2\}^{\frac{1}{2}}}{t^2 - a^2} dt. \quad a < 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\phi + \iota\psi = \mathrm{B} \log \frac{t - a}{t + a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

t being supposed to have all real values from $-\infty$ to $+\infty$.

For putting $t = \sin\theta$, $a = \sin\alpha$, and integrating (1) we find

$$z = \mathrm{A} \left(\theta - \tfrac{1}{2} \cot\alpha \log \frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)} \right)$$

or

$$x + \iota y = \mathrm{A} \left(\theta - \tfrac{1}{2} \cot\alpha \log \frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)} \right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

as θ goes from 0 to α. the right hand side of this equation is real so that $y = 0$ and x ranges from 0 to $+\infty$, this gives the positive half of the plane AB. As θ goes from α to $\tfrac{1}{2}\pi$

$$x + \iota y = \mathrm{A} \left(\theta - \tfrac{1}{2} \cot\alpha \log \frac{\sin(\theta - \alpha)}{\sin(\alpha + \theta)} + \tfrac{1}{2}\iota\pi \cot\alpha \right)$$

so that $y = \frac{1}{2} A\pi \cot \alpha$, and x ranges from ∞ to $\frac{1}{2} A\pi$, so if

$$h = \tfrac{1}{2} A\pi \cot \alpha \quad \ldots \ldots \ldots \ldots \quad (4)$$

$$c = \tfrac{1}{2} A\pi \quad \ldots \ldots \ldots \ldots \ldots \quad (5)$$

this will give the portion GH of the diagram. When $\sin \theta$ is greater than unity we may put

$$\theta = \tfrac{1}{2} \pi + \iota \vartheta,$$

the right hand side of (3) now equals

$$A\left\{ \frac{\pi}{2} + \iota \vartheta - \tfrac{1}{2} \cot \alpha \log \frac{\cos \alpha \, (\epsilon^\vartheta + \epsilon^{-\vartheta}) + \iota \sin \alpha \, (\epsilon^\vartheta - \epsilon^{-\vartheta})}{\cos \alpha \, (\epsilon^\vartheta + \epsilon^{-\vartheta}) - \iota \sin \alpha \, (\epsilon^\vartheta - \epsilon^{-\vartheta})} + \tfrac{1}{2} \iota \pi \cot \alpha \right\}$$

calling the quantity under the logarithm $P + \iota Q$, we see since

$$P + \iota Q = \frac{\cos \alpha \, (\epsilon^\vartheta + \epsilon^{-\vartheta}) + \iota \sin \alpha \, (\epsilon^\vartheta - \epsilon^{-\vartheta})}{\cos \alpha \, (\epsilon^\vartheta + \epsilon^{-\vartheta}) - \iota \sin \alpha \, (\epsilon^\vartheta - \epsilon^{-\vartheta})}$$

that

$$P - \iota Q = \frac{\cos \alpha \, (\epsilon^\vartheta + \epsilon^{-\vartheta}) - \iota \sin \alpha \, (\epsilon^\vartheta - \epsilon^{-\vartheta})}{\cos \alpha \, (\epsilon^\vartheta + \epsilon^{-}) + \iota \sin \alpha \, (\epsilon^\vartheta - \epsilon^{-\vartheta})};$$

multiplying these together we see that $P^2 + Q^2 = 1$. So that $\log (P + \iota Q)$ is wholly imaginary. Thus we see that as ϑ ranges from 0 to ∞, and t therefore from 1 to ∞, $x = \frac{1}{2} A\pi$ and y ranges from $\frac{1}{2} A\pi \cot \alpha$ to ∞, thus this range of values of t gives the portion GF of the figure. Since the real part of the right hand side of the equation (3) changes sign with θ or t, we see that the other portions of the figure are given by the negative values of t.

Since

$$\phi + \iota \psi = B \log \frac{t - a}{t + a}$$

we see that as long as t is between $- a$ and a, that is for the portion AB of the figure,

$$\phi + \iota \psi = \iota \pi B + \text{real quantities,}$$

so that $\psi = \pi B$, and, therefore, the potential is constant over AB; when t is not between these values, that is for the other portion of the figure $B \log \{(t - a)/(t + a)\}$ is real, and therefore $\psi = 0$.

Thus these equations give us the solution of the problem when AB is maintained at the potential πB and CDE, FGH are at zero potential.

The quantity of electricity on the conductor FGP when P is a point on GH

$$= \frac{1}{4\pi} (\phi_\mathrm{P} - \phi_\mathrm{F})$$

$$= \frac{\mathrm{B}}{4\pi} \log \frac{t - a}{t + a},$$

where t is the value of t at P. If we represent the increase in the quantity of electricity, due to the irregularity of the distribution, by supposing a strip of breadth d to be added to the conductor GH, and the distribution of electricity to be regular, and the same as if the air space were not present; the equation to find d is, if x is the value of x at P and V, the difference of potential between AB and GH.

$$\frac{\mathrm{V}}{4\pi h} \{x - c + d\} = - \frac{\mathrm{B}}{4\pi} \log \frac{t - a}{t + a},$$

substituting for x and t their values in terms of θ, and remembering that

$$\mathrm{V} = \pi\mathrm{B}, \qquad c = \tfrac{1}{2}\mathrm{A}\pi \qquad h = \tfrac{1}{2}\mathrm{A}\pi \cot \alpha$$

we get

$$\mathrm{A} \left(\theta - \tfrac{1}{2} \cot \alpha \log \frac{\sin (\theta - \alpha)}{\sin (\alpha + \theta)} \right) - c + d = - \tfrac{1}{2}\mathrm{A} \cot \alpha \log \frac{(\sin \theta - \sin \alpha)}{(\sin \alpha + \sin \theta)} ;$$

or

$$d = c \left\{ 1 - \frac{2}{\pi} \theta \right\} - \frac{h}{\pi} \log \frac{(\sin \theta - \sin \alpha) \sin (\alpha + \theta)}{(\sin \alpha + \sin \theta) \sin (\theta - \alpha)} .$$

Now if P be some distance from G we may put $\theta = \alpha$ and we get

$$d = c \left\{ 1 - \frac{2}{\pi} \alpha \right\} - \frac{h}{\pi} \log \cos^2 \alpha ,$$

from equations (4) and (5) we see that $\tan \alpha = c/h$, so that

$$d = c \left\{ 1 - \frac{2}{\pi} \tan^{-1} \frac{c}{h} \right\} + \frac{1}{\pi} h \log \left\{ 1 + \frac{c^2}{h^2} \right\}.$$

To deduce the corresponding solution for the cylinders from this we must multiply by the correction for curvature $1 + \tfrac{1}{4} h/a$, where a is here the radius of the inner cylinder, so that we have finally, if D be the whole breadth to be added for the two air spaces,

$$\mathrm{D} = 2 \left[c \left\{ 1 - \frac{2}{\pi} \tan^{-1} \frac{c}{h} \right\} + \frac{1}{\pi} h \log \left(1 + \frac{c^2}{h^2} \right) \right] \left(1 + \tfrac{1}{4} \frac{h}{a} \right).$$

Now in our condenser l was about 60, $2c = \cdot 3$, and $h = 1$, so that if we put D = 2c the value of the capacity will be correct to 1 part in 2000.

Correction for a small difference of potential between the guard ring and the middle cylinder.—The arrangement we used necessitates the existence of a small difference of potential between the cylinder and the guard ring, a slight modification of the preceding investigation will enable us to find the correction for this. If we put

$$\phi + \iota\psi = \frac{V}{\pi} \log \frac{t-a}{t+a} + \frac{\delta V}{\pi} \log (t+a),$$

then the potential over FGH $= 0$, that over EDC $= \delta V$, and over AB $= V$.

The breadth of the strip which must be added to compensate for the electricity on the portion QGH due to the difference of potential δV between CD and GH is

$$\frac{h}{\pi} \frac{\delta V}{V} \log \frac{(t_Q + a)}{(t_H + a)}.$$

Now, if QG is large,

$$GQ = A \log \frac{2t_Q}{\epsilon},$$

where ϵ is the base of the Napierian logarithms. Hence, since $t_H = a = \sin \alpha$, the breadth of the additional strip is

$$\frac{h}{\pi} \frac{\delta V}{V} \left\{ \frac{QG}{A} - \log \frac{4 \sin \alpha}{\epsilon} \right\};$$

but $A = c\, 2/\pi$ and $\sin \alpha = c/h$ approximately, hence the breadth of the strip for the two guard rings is

$$h \frac{\delta V}{V} \left\{ \frac{QG}{c} - \frac{2}{\pi} \log \frac{4c}{h\epsilon} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In our experiments $h/c = 6\cdot6$, QG $= 1$, so that the correction amounts to a strip whose breadth is about

$$7\cdot5 \frac{\delta V}{V}.$$

The value of $\delta V/V$ depended on the speed, the usual value was about $\frac{1}{183}$. In this case the breadth of the strip would be about $\frac{1}{30}$ of a centimetre, and, since the length of the cylinder was about 60 cm., the correction amounts to about 1 part in 1800.

To test the accuracy of these corrections, determinations of the capacity of the condenser were made when the top guard ring was separated from the middle cylinder (1) by pieces of ebonite ·504 cm. thick, (2) by pieces ·067 cm. thick. The capacity of the cylinders with the thick ebonite was greater than that with the thin by about 11 parts in 2760. According to the results we have obtained for these

corrections the effect of increasing the thickness of the ebonite would be to add a breadth ·218 cm. to the cylinder in consequence of the increased air space, and ·06 in consequence of the difference of potential, thus the two would add ·28 to the length of the cylinder, and would increase the capacity by $\frac{·28}{61} \times 2760$, or 12 parts in 2760. Thus the observed and calculated results agree well together.

Correction for ellipticity of the cross section.—Let us consider the case of a cylinder whose cross section is represented by the equation

$$r = b \{1 + e \cos 2\theta\},$$

placed inside one whose cross section is represented by

$$r = a \{1 + \alpha \cos 2\theta + \beta \sin 2\theta\},$$

where, since the measurements of the cylinder show that e, α, β are less than $\frac{1}{2000}$, we can neglect the squares of these quantities.

Let the potential between the cylinders be given by

$$V = A \log r + \frac{C \cos 2\theta}{r^2} + Dr^2 \cos 2\theta + \frac{E \sin 2\theta}{r^2} + Fr^2 \sin 2\theta.$$

Then, neglecting the squares of e, α, β, the difference of potential between the cylinders is

$$A \log \frac{a}{b},$$

and to the same approximation the charge per unit length is $\frac{1}{2}A$, thus the capacity per unit length is $\frac{1}{2} \log a/b$. Here a and b are the means of any two radii of the cylinders at right angles to one another. If we take these values as the radii of the cylinders the only correction required will be one of the order of one part in $(2000)^2$, which may be neglected.

Correction of Conicality.—We may see how to get rid of this correction by considering the electrical distribution on two infinite conductors, the one a plane perpendicular to the axis of y, the other a corrugated plane represented by the equation

$$y = h + \beta \sin \frac{2\pi x}{l},$$

the other plane being taken as the plane of xz. Let V the potential between the planes be given by

$$V = Ay + C \sin \frac{2\pi x}{l} \{\epsilon^{2\pi y/l} - \epsilon^{-2\pi y/l}\}$$

putting $y = h + \beta \sin 2\pi x/l$, and making the potential constant and equal to V_0

$$V_0 = Ah$$
$$C = -\frac{A\beta}{\epsilon^{2\pi h/l} - \epsilon^{-2\pi h/l}} \Bigg\} \text{ neglecting } \beta^2.$$

Thus σ, the surface density on the plane of xz,

$$= \frac{V_0}{4\pi h} \left\{ 1 - \frac{\beta \sin \frac{2\pi x}{l} \frac{2\pi}{l}}{\epsilon^{2\pi h/l} - \epsilon^{-2\pi h/l}} \right\}$$

$$= \frac{V_0}{4\pi h} \left\{ 1 - \frac{(y - h) \frac{2\pi}{l}}{\epsilon^{2\pi h/l} - \epsilon^{-2\pi h/l}} \right\}.$$

Thus, if we choose h so that it is the *mean* distance between the plates, for the breadth on which we wish to find the charge, the second term will vanish in our integration, and we get for Q the quantity of electricity on a breadth x

$$Q = \frac{V_0}{4\pi h} x.$$

Thus we can use the ordinary formula even when the plates are slightly inclined, provided h is the mean distance. Any correction to this will be the order of the square of the inclination at least, and in our case may be neglected.

Measurement of Dimensions of Condenser.

The dimensions are all referred to the standard metre of the Cavendish Laboratory which has been compared with the standard of the Board of Trade. The errors of the divisions are too small to affect the measurements given below. The comparison of the lengths with the standard metre was made by means of a pair of reading microscopes with micrometer screws. The pitch of the screws is accurately $\frac{1}{50}$th of an inch, and the head of the screw is divided into 100 parts, so that one division of the screw-head corresponds to ·0002 inch. The tenths of divisions are easily read and are recorded. The screws were tested by Mr. FITZPATRICK when working with Mr. GLAZEBROOK at the Specific Resistance of Mercury, and were found to be free from sensible error in either pitch or uniformity.

The standard metre is correct at $0°$ C., and its temperature coefficient is ·000017 per $1°$ C.

We require the dimensions of the condenser at $16°$ C. The metal of which the condenser is made is much the same as that of the standard metre, so that if we assume that the temperatures of the condenser and standard metre are the same at

the time of comparison we shall simply have to correct the metre to 16° C. The temperature of the room never differed from 16° by more than 2°, so that no appreciable error can be introduced on this account.

External Diameter of Inner Cylinder.

The sliding calipers of the laboratory were used to measure this. The bar of the calipers rested on the flat top of the cylinder, so that the calipers could be moved backwards and forwards along the top. The jaws are supposed to be at right angles to the bar along which the sliding one moves, but this was found not to be exactly the case. To obviate this difficulty a small piece of brass was fastened to the end of one jaw, so that the contacts were made at the ends of both jaws. The calipers were then placed under the microscope and two definite marks read off. The standard metre was then placed beneath the microscopes and treated in the same way. The distance between the marks when the jaws of the calipers were in contact was determined by the micrometer screw alone.

The readings of the screws are given in terms of $\frac{1}{5}$ inch.

Maximum Diameter of top end of Cylinder.

	Calipers.		23·8 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
1	1·1765	·6643	1·2308	·5970
2	1·1930	·6420	1·1939	·6280
3	1·8100	·5642	1·7980	·5648
4	1·8676	·5134	1·8793	·4965

The numbers in (4) are the mean of three readings.

These measurements gave as the distance between the marks—

$$
\begin{aligned}
23\cdot8 \text{ cm.} & \quad . \quad . \quad . \quad . \quad . \quad -\cdot00260 \text{ in.} \quad (1) \\
,, & \quad . \quad . \quad . \quad . \quad . \quad -\cdot00262 \text{ ,, } \quad (2) \\
,, & \quad . \quad . \quad . \quad . \quad . \quad -\cdot00220 \text{ ,, } \quad (3) \\
,, & \quad . \quad . \quad . \quad . \quad . \quad -\cdot00104 \text{ ,, } \quad (4) \\
\text{Mean} \quad 23\cdot8 \text{ cm} & \quad . \quad . \quad . \quad . \quad . \quad -\cdot00211 \text{ in.} \\
& \quad\quad\quad\quad\quad\quad\quad = 23\cdot7946 \text{ cm.}
\end{aligned}
$$

Distance between the marks with jaws of calipers closed. Read with left-hand screw—

(1.)	1·2779	(2.)	1·5778	(3.)	2·0561
	·7538		1·0430		1·5192
	·5241		·5348		·5369

(4.)	2·3553	(5.)	2·2183	(6.)	2·2187
	1·8136		1·6891		1·6895
	·5417		·5292		·5292

(7.)	2·3550	(8.)	2·5410	(9.)	2·5330
	1·8172		2·0096		1·9998
	·5378		·5314		·5332

The mean of these is

$$\frac{\cdot 53314}{5} \text{ in.} = \cdot 10662 \text{ in.} = \cdot 2708 \text{ cm.}$$

Thus we find that the maximum diameter at the top of the cylinder is

$$23 \cdot 7946 - \cdot 2708 = 23 \cdot 5238 \text{ cm.,}$$

referred to the standard at 16° C.

To reduce the standard to 0° C we must multiply by $(1 + 16 \times \cdot 000017)$ and we get as the true diameter

$$23 \cdot 5302 \text{ cm.}$$

MINIMUM diameter of top end of Cylinder.

	Calipers.		23·8 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
1. (Mean of 4 readings)	2·2158	0·4338	2·1822	0·4279
2. (Mean of 4 „)	1·9271	0·7133	1·8625	0·7406

Giving as minimum diameter of top end

(1.) 23·8 cm. — ·2708 cm. — ·00790 in.

(2.) 23·8 cm. — ·2708 cm. — ·00746 in.

Correcting for temperature, we find

Minimum diameter of top end = 23·5161 cm.

Bottom end of Cylinder.

MAXIMUM DIAMETER.

	Calipers.		23·8 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
1. (Mean of 4 readings)	1·4078	1·2042	1·4171	1·1950
2. (Mean of 4 ,,)	1·5469	1·0671	1·5910	1·0196

MINIMUM DIAMETER.

	Calipers.		23·8 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
1. (Mean of 4 readings)	1·6915	·9584	1·6814	0·9318
2. (Mean of 4 ,,)	1·6916	·9581	1·6495	0·9634

Thus, maximum diameter of bottom end equals

$$(1.) \quad 23\cdot8 \text{ cm.} - \cdot2708 \text{ cm.} + \cdot00002 \text{ in.}$$
$$(2.) \quad 23\cdot8 \text{ cm.} - \cdot2708 \text{ cm.} - \cdot00068 \text{ in.}$$

Correcting for temperature, we find

$$\text{Maximum diameter of bottom end} = 23\cdot5348 \text{ cm.}$$

The minimum diameter of bottom end equals

$$(1.) \quad 23\cdot8 \text{ cm.} - \cdot2708 \text{ cm.} - \cdot00734 \text{ in.}$$
$$(2.) \quad 23\cdot8 \text{ cm.} - \cdot2708 \text{ cm.} - \cdot00736 \text{ in.}$$

Correcting for temperature

$$\text{Minimum diameter of bottom end} = 23\cdot5169 \text{ cm.}$$

Collecting these results we have for the inner cylinder

Top end   .   Maximum diameter = 23·5302.
          Minimum diameter = 23·5161.
Bottom end   Maximum diameter = 23·5348.
          Minimum diameter = 23·5169.
  Mean of these . . . . . . = 23·5245.

The corresponding ends of the measured diameters were found to be almost exactly on the same generating line, so that though the cylinder is slightly elliptical and conical, it is free from anything of the nature of helicality.

Measurement of the Internal Diameter of the Outer Cylinder.

This was found to be a good deal more troublesome than the measurement of the external diameter of the inner cylinder, the plan finally adopted was to fix two pieces of hardened steel to the ends of the jaws of the sliding calipers, thus
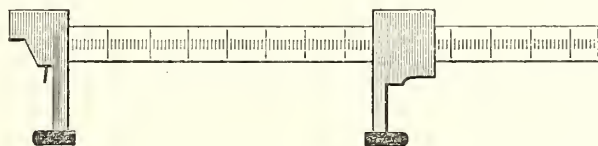
One side of each piece was polished, and the end was then ground and polished on a fine oilstone so as to form a good edge with the polished face. The shape of the end was semi-circular. In this way the edges made contact with the cylinder, and the cross wires (one of which was set carefully perpendicular to the line of travel of the microscopes) could be easily focussed on to the end of the steel.

It was not found practicable to determine exactly when contact was made in the same way as was done for the inner cylinder, since when the calipers were set to nearly the size of the cylinder scarcely any movement was possible. The sharp edges were also an impediment to the motion. We found, however, that by insulating one of the steel contact pieces we could determine accurately by the aid of a telephone when contact was made. As the cylinder was found to be nearly circular, and the formula for a slightly elliptical cylinder outside a circular one indicates that the lengths of two diameters at right angles to each other are required, two such diameters were measured. The following are the details of the measurements, each of the numbers being the mean of four observations :—

Top end.

Diameter A.

	Calipers.		25·4 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
(1)	1·3821	1·1721	1·3859	1·1793
(2)	1·3319	1·2060	1·3255	1·2324
(3)	1·1822	1·3445	1·1462	1·3993

DIAMETER B.

	Calipers.		25·4 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
(1)	1·1701	1·3843	1·1793	1·3787
(2)	1·1873	1·3762	1·1614	1·4032
(3)	1·1268	1·4415	1·1150	1·4485
(4)	1·8703	0·6891	1·8875	0·6620

BOTTOM end.

DIAMETER A.

	Calipers.		25·4 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
(1)	1·4352	1·0999	1·4289	1·1209
(2)	1·4734	1·0699	1·4918	1·0586

DIAMETER B.

	Calipers.		25·4 cm.	
	Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.
(1)	1·6109	0·9270	1·6131	0·9385
	1·6978	0·8427	1·7106	0·8371

Taking the mean of these and correcting for temperature we find

Top end   .   Diameter A = 25·4154 cm.
Diameter B = 25·4056 cm.
Bottom end   Diameter A = 25·4125 cm.
Diameter B = 25·4122 cm.
Mean of these .   .   .   .   = 25·4114 cm.

Measurement of the Length of the Cylinder.

The length of the cylinder was transferred from the cylinder to the reading micro-scopes by means of the beam compasses of the laboratory ; care being taken to keep the bar of the compasses parallel to the length of the cylinder while setting the compasses to the length of the cylinder.

On account of the length of the cylinder it was found difficult to ascertain by

moving the beam compasses just when contact was complete. A small piece of thin sheet steel (about ·03 cm. thick) was interposed between the end of the cylinder and the point of the beam compasses. The compasses were considered adjusted when a slight resistance to the motion of the feeling piece was perceived. The beam compasses were then placed under the microscopes, and the distance between two definite marks on their points determined. The points were then placed close together, so that the same resistance to the motion of the sliding piece was felt as in the former case. The distance between the marks was then ascertained by means of one of the microscopes and its screw. The distance being so small it seems unnecessary to compare it with the divisions of the standard metre.

DISTANCE between the Marks when the Compasses were Closed.

	Right-hand mark.	Left-hand mark.	
(1)	2·3156	1·7220	Mean of 4 observations.
(2)	2·1283	1·5308	,, 4 ,,
(3)	2·1287	1·5290	,, 5 ,,
(4)	2·9120	2·3154	,, 6 ,,

Giving, as the distance between the marks, ·11937 in., or ·3032 cm.

DISTANCE between the Marks when the Compasses were Open.

Calipers.		61·3 cm.		
Left-hand screw.	Right-hand screw.	Left-hand screw.	Right-hand screw.	
2·1297	·5698	2·0656	·5532	Mean of 4 observations.
2·4270	·2537	2·3819	·2415	,, ,,

Giving as the distance betwen the marks when open, 61·3 cm. − ·0138 in.

Hence the length of the cylinder when corrected for temperature equals 60·9784 cm.

The Distance between the Inner and Outer Cylinders.

Since the difference of the mean diameters is only about 1·09 cm., and since, on account of the difficulties of measurement and the irregularities in the shape of the cylinders, it is impossible to arrive at any satisfactory result by subtracting the mean diameter of one cylinder from that of the other, we had to apply some other method. We adopted that used in the experiment of 1883, which was to ascertain the amount of water required to fill the space between the two cylinders This amount was determined by weighing. The water employed was distilled, and was boiled a few hours previous to its use to enable it to absorb air bubbles more readily.

A 500 c.c. flask was filled with water and weighed, its contents were then transferred to the cylinders, and it was then weighed again. The difference in weight gives the weight of water transferred to the cylinder. This process was repeated until the space between the cylinders was full.

The weights of the full and empty flasks were determined to 1 centigrm.

The 500 grm. weight used to balance the water was compared with the standard 500 grm. weight of the Laboratory and found too heavy by ·055 grm. This has been allowed for. The equality of the arms of the balance was also tested.

The two cylinders were fastened down to a flat metal plate with a thin layer of cement so as to be quite water-tight. To get any accurate estimate of the volume of water required to fill the space it was necessary to provide some means to ascertain when the space was exactly full. The effects of capillarity, grease, &c., preclude any very accurate result being obtained when there is no top or cover fitted to the top of the cylinder, and as it was necessary to see whether any air bubbles were left inside a glass top had to be used. Two holes were bored through the glass, and tubes were fixed into these. The water was introduced through one tube, and the air escaped through the other. Although no difficulty was experienced in making the joint at the bottom quite water-tight with any of the cements employed, it took us several days to make a satisfactory joint at the top. The ease with which tightness at the bottom was secured was probably owing to the great weight of the cylinders.

The top gave us all the more trouble, because we could not tell whether it was watertight or not until we had almost completed the filling in of the water. If, then, the joint proved bad the whole of the time spent in weighing the water poured in was wasted.

We tried Prout's elastic glue, then gutta-percha dissolved in benzene, but both of these failed. The water seemed to loosen the hold of the glue upon the glass, so that although the system seemed air-tight it would not remain water-tight for more than a minute or two.

We finally tried some red wax which had been sent to the Laboratory by Professor THRELFALL, who obtained it in Germany, and this answered very well. It looks somewhat like a mixture of bees'-wax and sealing wax, and as it never gets quite hard it never cracks. It possessed another property which was also useful for our purpose, viz., that of melting at a comparatively low temperature. To apply the other cements in a satisfactory manner the glass had to be heated to a somewhat high temperature, and this frequently cracked it.

The wax when melted became very fluid, so that only an extremely thin layer was included between either the top or bottom plate and the cylinders.

At first the water was poured in through a funnel inserted into one of the tubes, but with this arrangement it was found impossible to get rid of the last air bubble, since fresh quantities of air were continually carried down the tube. After some experiments with different arrangements for filling, we finally adopted as the means

of introducing the water a tube with a funnel top provided with a tap. The funnel could always be kept partially full of water by means of the tap, so that no air was introduced. The cylinders were slightly tilted so that the exit tube was at the highest place. As the last few grammes were poured in the air was gradually swept along by the advancing water and driven up the exit tube. The small bubble which sometimes remained under the exit tube was easily removed by agitating it by a fine wire introduced through the exit tube.

The amount of water remaining in the tap and the amount which rose up the exit tube were ascertained afterwards by detaching the tap and tube, and finding the weight of water required to fill them to the same extent as when the cylinders were filled.

While the water was still about 3 cm. from the glass the air was exhausted from the cylinders by a water pump ; the pressure of the air was reduced to about 50 mm. of mercury. This had the effect of removing a good many bubbles, and we may hope that the experiment was free from error in this respect. The wax employed for fixing the top on to the cylinders stood this difference of pressure perfectly. A drying tube was placed between the cylinders and the tube leading to the pump in order to catch any water which might be carried off in the pumping. The increase in weight of the drying tube was found to be not more than ·02 or ·03 gramme, and this has been allowed for. The weighings have also been corrected to a vacuum.

The annexed sketch shows a section of the cylinders by a plane through their axis, and through the tap and exit tube.
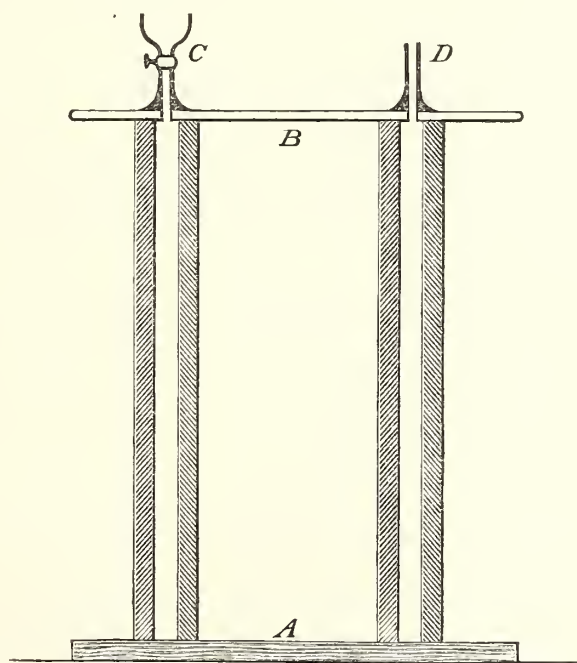
A is the flat metal plate.   C the funnel and tap.
B the glass plate.   D the exit tube.

The following are the results of the weighings on two separate days :—

	[1.] Temp. 17.	[2.] Temp. 15·3.
Weight of water put into funnel	4403·15	4403·53
Weight of water left in tap and tube	1·33	1·35
Volume of a piece of wax underneath the glass top	·70	0
Weight of water in space between the cylinders is therefore	4402·52	4402·18
Correction to vacuum	4·66	4·66
Correction for temperature	5·10	3·91
Correction for error in 500-grm. weight	·50	·50
Correction for inequality of arms	·09	·04
Volume between the cylinders is	4412·87	4411·29

The mean of these is

$$4412·08 \text{ c.c.}$$

which we take as the value of the volume between the cylinders.

If d is the mean distance between the cylinders, l the length of the inner cylinder, a and b the radii of the outer and inner cylinders respectively

$$d = \frac{4412·08}{\pi (a + b) l} ;$$

so that

$$d = ·94128 \text{ cm.}$$

$$\frac{a}{b} = 1 + \frac{a - b}{b} = 1·0800262 ;$$

$$2 \log \frac{a}{b} = ·15397063.$$

The thicknesses of the pieces of ebonite between the guard rings and the cylinder at the top and bottom were respectively ·2934 and ·288. Correcting for the air space the effective length of the cylinder is

$$60·9784 + ·2907 = 61·2691 \text{ cm.}$$

Hence the electrostatic measure of the capacity

$$= \frac{61·2691}{·15397063} ;$$

$$= \mathbf{397·927} \text{ cm.}$$

No correction is required for want of coincidence between the axes of the cylinders, for if c be the distance between the axes the correction is proportional to $c^2/(a^2 - b^2)$. In our experiments c was less than ·01, so that the correction only amounts to one part in more than 30,000.

The difference of potential between the guard ring and the cylinder depends upon the speed of the commutator, so that the correction on this account is made on the electromagnetic measure of the capacity.

The Electromagnetic Measure of the Capacity.

The arrangement employed in this measurement is represented in fig. 3.

Fig. 3.
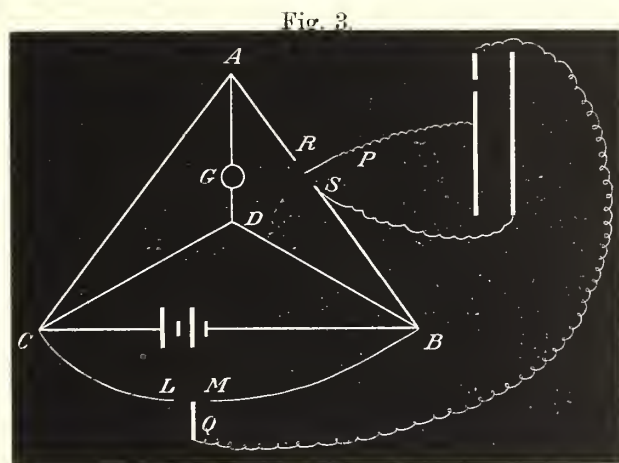
ABCD is a Wheatstone's bridge with the galvanometer at G, and the battery between B and C. The arm AB is broken at R and S, which are two poles of a commutator, which alternately come into contact with a spring P, connected with the middle part of the inner cylinder of the condenser. The outer cylinder is connected to S. The points C and B are connected respectively with L and M, the two poles of a commutator, which alternately come into contact with a spring Q, attached to the guard ring of the condenser. The system is arranged so that when the commutators are working the order of events is as follows :—

   I.   P on S. Condenser discharged.
          Q on M. Guard ring discharged.

  II. P on R. Condenser begins to charge.
          Q on M.

 III. P on R. Condenser completely charged to potential (A)–(B).
          Q on L. Guard ring charged to potential (C)–(B).

 IV. P on S. Condenser begins discharging.
          Q on L.

  V. P on S. Condenser discharged.
          Q on M. Guard ring discharged.

Thus, when the commutators are working, there will, owing to the flow of electricity to the condenser, be a succession of momentary currents through the galvanometer. The resistances are so adjusted that the effect of these momentary currents on the galvanometer just balances the effect due to the steady current, and there is no deflection of the galvanometer.

To investigate the relation between the resistances when this is the case, let us suppose that when the guard ring and condenser are charging

$$\dot{x} = \text{current through BC.}$$

$$\dot{y} = \text{current through AR.}$$

$$\dot{z} = \text{current through AD.}$$

$$\dot{w} = \text{currrent through CL.}$$

Thus, if a, b, α, β, γ are the resistances in the arms BC, AC, AD, BD, CD respectively, L the coefficient of self induction of the galvanometer, and E the electromotive force of the battery, we have

$$L\ddot{z} + (b + \gamma + \alpha)\dot{z} + (b + \gamma)\dot{y} + \gamma\dot{w} - \gamma\dot{x} = 0 \quad \ldots \ldots \quad (1)$$

$$(a + \gamma + \beta)\dot{x} - (\gamma + \beta)\dot{y} - \gamma\dot{z} - (\gamma + \beta)\dot{w} - E = 0 \quad \ldots \ldots \quad (2)$$

Now it is evident that the currents are expressed by equations of the following kind

$$\dot{x} = \dot{x}_1 + \dot{x}_2,$$

$$\dot{z} = \dot{z}_1 + \dot{z}_2,$$

where $\dot{x}_1$ and $\dot{z}_1$ express the steady currents when no electricity is flowing into the condenser, and $\dot{x}_2$, $\dot{z}_2$ are of the form $A\epsilon^{-\lambda t}$, $B\epsilon^{-\lambda t}$, and express the variable parts of the currents due to the charging of the condenser, $\dot{y}$ and $\dot{w}$ will be of the form $C\epsilon^{-\lambda t}$, $D\epsilon^{-\lambda t}$; t in all these equations is the time which has elapsed since the condenser commenced to charge.

Equations (1) and (2) will thus contain constant terms, and terms multiplied by $\epsilon^{-\lambda t}$, the latter must separately vanish, hence we have

$$L\ddot{z}_2 + (b + \gamma + \alpha)\dot{z}_2 + (b + \gamma)\dot{y} + \gamma\dot{w} - \gamma\dot{x}_2 = 0 \ldots \ldots \quad (3)$$

$$(a + \gamma + \beta)\dot{x}_2 - (\gamma + \beta)\dot{y} - \gamma\dot{z}_2 - (\gamma + \beta)\dot{w} = 0 \ldots \ldots \quad (4)$$

Let Z, X be the quantities of electricity which have passed through the galvanometer and battery respectively, in consequence of the charging of the condenser, and

Y and W the charges in the condenser and guard ring. Then integrating equations (3) and (4), over a time extending from just before the condenser began to charge until it is fully charged, remembering that at each of these times $\dot{z}_2 = 0$, we get

$$(b + \gamma + a)\, Z + (b + \gamma)\, Y + \gamma W - \gamma X = 0$$

$$(a + \gamma + \beta)\, X - (\gamma + \beta)\, Y - \gamma Z - (\gamma + \beta)\, W = 0,$$

hence eliminating X

$$Z\left(b + \gamma + a - \frac{\gamma^2}{a + \gamma + \beta}\right) + Y\left(b + \gamma - \frac{\gamma(\gamma + \beta)}{a + \gamma + \beta}\right) + W\gamma \frac{a}{a + \gamma + \beta} = 0.$$

In our experiments, the battery resistance is very small, being less than 1 ohm, while β is 500,000 ohms, b 200,000 ohms, and γ 3000 ohms, thus the third term is less than $\frac{1}{5,000,000}$th part of the second, and may be neglected, and we get, neglecting the battery resistance

$$Z = - \frac{b}{b + \gamma + a - \dfrac{\gamma^2}{\gamma + \beta}}\, Y.$$

If {A} {B} {D} denote the potentials of A, B, D when the condenser is fully charged, C the capacity of the condenser, then

$$Y = C\,[\{A\} - \{B\}].$$

But

$$\frac{\{A\} - \{B\}}{a + \beta \dfrac{(b + a + \gamma)}{\gamma}} = \frac{\{A\} - \{D\}}{a}.$$

The right-hand side of this equation is $\dot{z}_1$, the steady current through the galvanometer, so that

$$Y = - C\dot{z}_1\left(a + \beta\frac{(b + a + \gamma)}{\gamma}\right). \qquad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Z = - \dot{z}_1 bC\, \frac{\left\{a + \beta\dfrac{(b + a + \gamma)}{\gamma}\right\}}{b + \gamma + a - \dfrac{\gamma^2}{\gamma + \beta}}. \qquad . \quad . \quad . \quad . \quad . \quad (6)$$

If the condenser is charged n times per second, the quantity of electricity which passes in consequence through the galvanometer per second is nZ. If the galvanometer

needle remains undeflected, the quantity of electricity which passes through the galvanometer in unit time must be zero.   But this quantity is $nZ + \dot{z}_1$, so that

$$nZ + \dot{z}_1 = 0.$$

Substituting this relation in equation (6), we get

$$C = \frac{1}{n}\frac{\gamma}{b\beta}\frac{\left\{1 - \dfrac{\gamma^2}{(\gamma + \beta)(b + \gamma + \alpha)}\right\}}{1 + \dfrac{\gamma\alpha}{(b + \alpha + \gamma)\beta}}. \qquad \ldots \ldots \quad (7)$$

From this equation, if we know the resistances and the speed, we can calculate the capacity.

In order to apply the correction for the difference of potential between the guard ring and the inner cylinder, we require $\{A\} - \{C\}/\{A\} - \{B\}$, now

$$\frac{\{A\} - \{C\}}{\{A\} - \{B\}} = \frac{b}{\alpha} \cdot \frac{\{A\} - \{D\}}{\{A\} - \{B\}}$$

$$= \frac{b}{\alpha + \beta\dfrac{(b + \alpha + \gamma)}{\gamma}}.$$

Where n was equal to 64 the approximate values of the resistances were

$$b = 200{,}000 \text{ ohms.}$$
$$\alpha = 20{,}000 \quad \text{,,}$$
$$\beta = 500{,}000 \quad \text{,,}$$
$$\gamma = 3{,}000 \quad \text{,,}$$

substituting the values

$$\frac{\{A\} - \{C\}}{\{A\} - \{B\}} = \tfrac{1}{183}.$$

We shall now go on to discuss the details of the method whose theory we have just given.

The Commutator.

The general view of this is shown in figs. (4), (5), (6).   The framework is strongly made of cast iron, and somewhat resembles the headstock of a lathe, it is provided with two hardened steel centres, capable of adjustment, between which runs the axle of the commutator.   This was made of tool steel but not hardened.   The centres were very good, and the apparatus ran with very little friction and wear.

We will now describe the different pieces fixed to the axle beginning from the left of fig. 4.

Fig. 4. Fig 5.
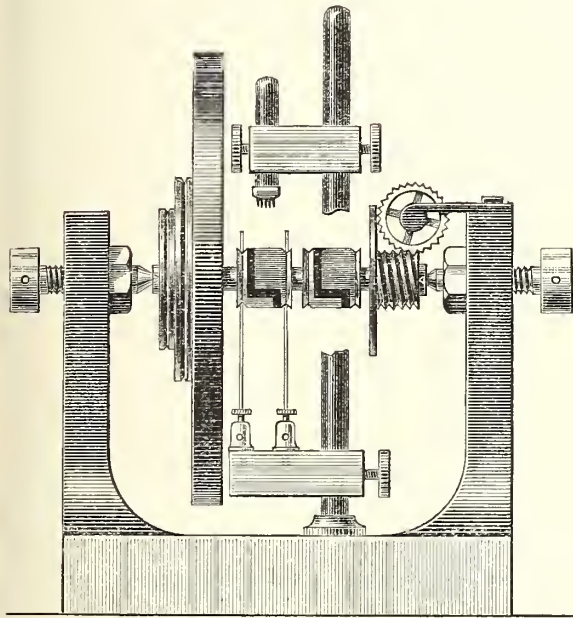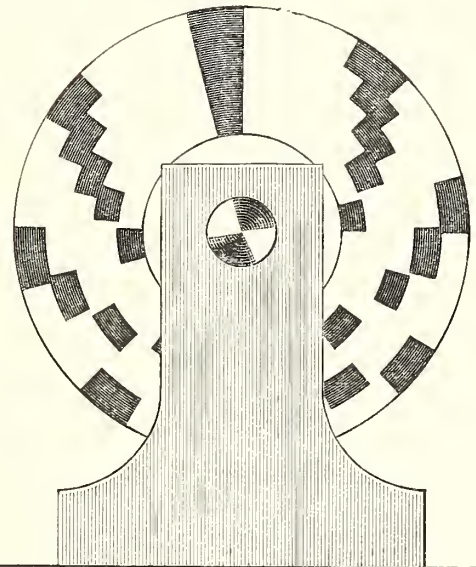
SCALE OF CENTIMETRES.

First there is a wooden disc with two grooves for the driving string, next comes the wooden disc on which the stroboscopic pattern is painted. The end view of this is shown in fig. 5.

Beyond this are the two commutators, which are exactly alike, each is made of two portions of brass tube fixed to an ebonite bush. Two grooves are turned in the continuous portion of the tube, in which rest two wires by means of which electrical connection is made with the parts of the commutator. Although the slits in the commutator were not more than about 2 mm. wide yet no trouble at all was experienced through want of insulation. The commutator could be easily cleaned by scraping the ebonite between the parts of the commutator with a sharp tool. There was also very good insulation between the commutator and the axle. The insulation was tested several times by means of the gold leaf electroscope, and was found practically perfect.

Following the two commutators is an ebonite disc to prevent oil from the centres getting to the commutators.

The last thing on the axle is an endless screw in which gears a worm-wheel of 30 teeth. This wheel is furnished with a pin which makes contact with a spring once in every revolution of the wheel, i.e., every 30 revolutions of the commutator. The spring is insulated from the framework, and the contact of the pin with it completes the circuit required for exciting one of the electromagnets of the recording apparatus. The spring is not shown in the figure.

The supports for the contact wires of the commutator and for the brushes were two pillars screwing into the base of the framework. One of these is shown in the figure, it is shown broken, and the two parts separated in order to show both parts of the commutator. Fig. 6 shows the arrangement as it would look in an end view if the

Fig. 6.
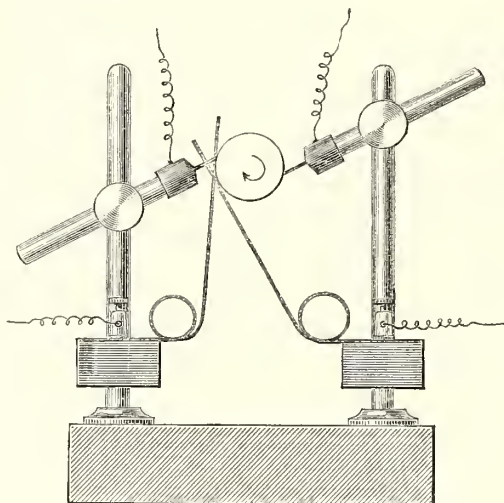
stroboscopic disc were absent. The wires for making contact with the commutator are clamped under the two binding screws and are made with a curl Ω so as to be very elastic and to ensure contact with the commutator. Before the curl was put into the wires it was found that the jarring of the apparatus gradually caused the wires (though made of hard drawn brass) to relax their pressure on the commutator and to make uncertain contacts. With the curled wires no trouble was found in this respect, and the pressure could be made much less, saving both friction and wear. The binding screws clamping the wires are fixed to a piece of ebonite which can slide on the pillar and be fixed in any position by a screw. The pillar also carries a support for the brush. The brush itself is made of very fine hard drawn brass wire, soldered into a brass piece of a suitable shape. The way in which the brush acted depended a good deal upon the regularity and straightness of its wires. We found it best not to have a thick bunch of wire, but a thin layer only a few wires thick. The brass piece into which the brush wires were soldered, fits on the end of an ebonite rod passing through the brush holder and capable of being fixed in its proper position by means of a screw.

The whole arrangement was clamped down to a thick iron slab, resting on a strong table. In this way the vibrations, which would otherwise have been set up, were avoided.

The tuning fork, by means of which the speed was observed and regulated, was placed on a separate table. At first the two were on the same table, but it was found that at the speed at which the commutator made one revolution for each

complete vibration of the fork, the vibrations set up by the fork were sufficient to make the contacts uncertain. As finally fixed, the apparatus worked extremely satisfactorily, and would run for several hours without either the brushes or the springs requiring any adjustment.

The commutator was driven from a water motor which was supplied from a tank at the top of the laboratory to secure a constant head of water. It was driven by a band of fine fishing line joined with a "long splice"; any rougher method of joining produced a joint, the effect of whose passage over the pulley of the commutator was plainly seen by the observer at the galvanometer. A second band went from a small pulley on the motor to a pulley fixed within easy reach of the observer stationed at the tuning fork. The regulation of the speed was done by letting the auxiliary band run through the fingers, and slightly pressing it. This was found to be a much better plan than regulating by the band driving the commutator. But, in spite of this, the speed of the commutator as judged by the steadiness of the pattern seen through the slits of the tuning fork was subject to incessant small agitations, and it required considerable vigilance on the part of the observer at the fork to keep the pattern quite at rest. A heavier disc on the commutator would no doubt have made this easier.

The supply of water was so adjusted that it was able to drive the commutator slightly faster then the speed required. The necessary fine adjustment was made by slightly pressing the regulating band.

Determination of the Speed of the Commutator.

To ascertain the speed at which the commutator was being driven its stroboscopic disc was observed through a pair of narrow slits fastened to the prongs of an electrically maintained fork. This fork made approximately 64 complete vibrations per second. The disc was provided with circles containing 4, 5, 6, 7, 8 spots at equal intervals, so that when a distinct pattern was observed through the slits on the fork the commutator made one of the following numbers of revolutions per second :—

$$16\cdot0, \quad 18\cdot3, \quad 21\cdot35, \quad 25\cdot6, \quad 32\cdot0, \quad 36\cdot6 \quad 42\cdot7, \quad 48\cdot0,$$
$$51\cdot2, \quad 54\cdot9, \quad 64, \quad 73\cdot2, \quad 76\cdot9, \quad 80\cdot1, \quad 85\cdot4.$$

These numbers are respectively $\frac{1}{4}$, $\frac{2}{7}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{4}{7}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{6}{7}$, 1, $\frac{8}{7}$, $\frac{6}{5}$, $\frac{5}{4}$, and $\frac{4}{3}$ of 64.

Any one of these speeds could be obtained by simply regulating the supply of water to the motor. Higher speeds could, of course, have been observed, but the motor would not drive the commutator much faster than 80 revolutions per second.

Experiments at most of these speeds will be found below.

It will be observed that this method gives a great choice of speeds, all of which can be determined with the same accuracy, and whose relation one to another is known with absolute accuracy.

The standard to which the speed was referred during the experiments was the

standard fork used by Lord RAYLEIGH. This makes about 128 complete vibrations per second. The electrically driven fork maintained another fork whose natural period is about half its own. This gave beats with the standard, and by counting the beats the speed of the fork through which the stroboscopic disc was observed could be determined in terms of that of the standard fork.

Determination of the Speed of the Standard Fork.

It was considered advisable to make a new determination of the standard fork. Lord RAYLEIGH ('Phil. Trans.,' 1883, p. 320) found that its speed at $t°$ C was

$$128 \cdot 140 \left\{ 1 - (t - 16) \times \cdot 00011 \right\}.$$

We have taken Lord RAYLEIGH's value of the temperature coefficient.

We determined the speed of the standard in the following way. At one end of the commutator an endless screw was fixed with a cog-wheel geared into it. The cog-wheel makes one revolution for 30 made by the commutator. A pin fixed to the wheel touches a spring once in every revolution, and completes an electric circuit
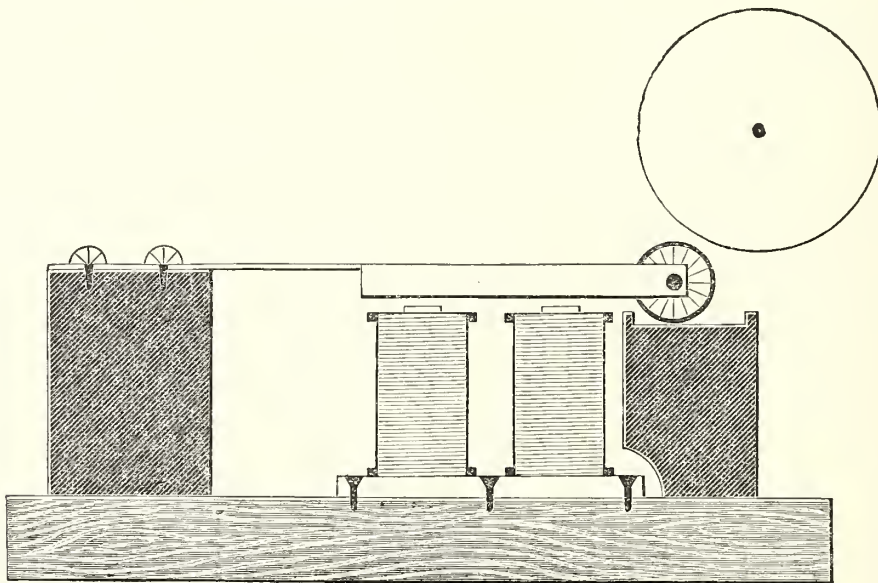
which causes a mark to be made on the tape of the recording apparatus. The laboratory clock is also arranged so as to complete a circuit once every second. A paper tape is pulled along at an approximately uniform rate between guides on a block of wood. Two electro-magnets are fixed at right angles to the guides. One of these is excited when contact is made by the cog-wheel of the commutator, the other when contact is made by the clock. The armatures are kept away from their magnets by springs fixed to one end.

The other end of the armature carries a fork in which a small disc can rotate. When the magnet is not excited the disc presses against a roller covered with printer's ink. When the magnet is excited the disc presses on the paper tape as it passes underneath it. The disc is kept inked by revolving the inking roller.

The general disposition of the apparatus will be seen from the figure.

As the tape is drawn along we get two series of marks, one due to the commutator, the other to the clock. By comparing the position of the marks made by the commutator with those made by the clock we get a very accurate method of timing the commutator. We can find the time at which say 15 commutator contacts are made, and then, after the lapse of 5 or 10 minutes, find the time at which another 15 commutator contacts are made. Subtracting the one set of times from the others we get 15 intervals which should, of course, be equal. Taking the average we get a very accurate value for the time occupied by (say) 600 revolutions of the cog wheel.

The paper tape was at first drawn along by a Morse receiver, but as this did not act very uniformly it was replaced by a small winding arrangement driven by the same motor as the commutator. This ran perfectly uniformly.

The method of experimenting was as follows : when the observing forks had been running for a few minutes, so that everything had got steady, the beats between the standard fork and the auxiliary fork were counted. The forks were so arranged that 20 beats occurred in about 65 seconds. The motor was then started and regulated so that the commutator could be kept at half the speed of the fork, i.e., about 32 revolutions per second.

The recording apparatus was then started and allowed to run for 5 or 10 minutes, the commutator being kept at a constant speed by observing the stroboscopic pattern through the fork. The apparatus was then stopped and the beats again counted.

Our reason for adopting this method was that we had originally intended to measure the speed of the commutator while we were making the electrical observations. This was, however, found to be too laborious, and had to be given up. The Laboratory clock was compared with the clock belonging to the Cambridge Philosophical Society, which is regularly rated from the Observatory.

The last three observations, when the apparatus was working very satisfactorily give for the rate of the standard fork at 16° C.

December 19 (tape running for 6 minutes) . . . 128·1081
February 14 (,, ,, 10 ,,) . . . 128·0909
,, 15 ,, ,, ,, ,, . . . 128·1146

Mean . . . . 128·1045

Thus according to our observations the fork is slightly slower than when used by Lord RAYLEIGH. This is what might be expected from the secular softening of the steel.

The Galvanometer.

This was one made in the Laboratory. It has two coils, each with about 16,000 turns of wire. The resistance of the coils when in series is 17,380 legal ohms. Great care was taken with the insulation, which, when tested by means of a gold leaf electroscope, was found to be practically perfect.

The Battery.

We used 36 very small storage cells, two sets of 18 being placed in parallel. The battery had thus an electromotive force of about 36 volts. The small size of the battery enabled us to insulate it with ease. The insulation was tested by a gold leaf electroscope, but no leak could be detected.

The Resistances.

The resistances used were contained in three boxes.

I. A Wheatstone's bridge box, No. 1256, ELLIOTT. Legal ohms, with coils ranging from 1 to 5000 ohms, and proportional arms, each containing 10, 100, 1000 ohms.

II. A box by ELLIOTT, containing 4 coils, 10,000, 20,000, 30,000, 40,000 B.A. units.

III. A box by MUIRHEAD, containing originally 10 coils, each 100,000 B.A. units.

The box I. was provided with an aperture for a thermometer. The other boxes had no such provision, but as they were always kept permanently on the same table as box I. we may hope that their temperature did not differ much from that of I.

The resistances of the coils in II. and III. were always ascertained by means of I. It was, therefore, necessary to obtain some definite knowledge of I. with reference to some coils whose values are accurately known.

The standard coils used were

No. 141, C.L.C. No. 102 . . .	10·00103 legal ohms at	16°·7 C.
No. 143, C.L.C. No. 104 . . .	99·9977 ,, ,,	16°·05 C.
No. 145, C.L.C. No. 106 . . .	1000·306 ,, ,,	17°·4 C.

The Legal ohm being assumed to be 1/·9889 B.A. unit.

See 'British Association Report,' 1885.

Mr. GLAZEBROOK informs us that the temperature coefficient of these coils may be taken as ·0003 per 1° C. We shall take the same temperature coefficient for the coils in the boxes.

Calling the coils in the "proportional arm" which joins the rest of the box 10_a, 100_a, 1000_a, those in the other arm 10_b, 100_b, 1000_b we found

$$\frac{1000_b}{1000_a} = 1 - \frac{\cdot 3}{10000}$$

$$\frac{1000_b}{100_a} = 10 \left\{ 1 - \frac{1\cdot 8}{10000} \right\}$$

$$\frac{1000_b}{10_a} = 100 \left\{ 1 + \frac{\cdot 76}{10000} \right\}.$$

Correcting for the inequalities of 1000_a and 1000_b we found by comparison with the Standard coils the following values at 16° C., for the various coils in box I.

		Legal ohms.
100_a	. . . .	100·144
100_β	. . . .	100·144
100	. . . .	100·113.

The last resistance was made up from all the coils from 1 to 50,

		Legal ohms.
1000_a	. . . .	1001·06
1000_β	. . . .	1001·01
1000	. . . .	1001·06.

The last was made up of all the coils from 1 to 500,

		Legal ohms.
2000	. . . .	2002·22
5000	. . . .	5004·88.

We see from this that the coils are very consistent, and on the average greater than legal ohms, in the ratio of 1·0011 to 1.

The box II. was measured as a whole, using the "proportional" coils 1000_b and 10_c. The apparent value was 98765. Hence the true value is

$$98765 \times 1\cdot 0011 \left(1 + \frac{\cdot 76}{10000} \right) = 98870 \text{ legal ohms at } 16° C.$$

The coils of box III. were tested in the same way and were found to be

	Resistance as measured by box I.	True resistance in legal ohms.
1.	98723	98878
2.	98617	98732
3.	98690	98805
4.	98727	98842
5.	98717	98832
6.	98768	98883

The box III. had one of its coils wrong when we began to use it, and during part of our work when we were attempting to use much greater battery power than we finally adopted, three more of the coils gave way. For this reason we could only use six out of the ten coils. These six coils, as well as box II., kept their resistances very constant during the whole of the investigation. They were tested several times during a whole year, and no perceptible change was detected.

The resistance boxes were placed on blocks of paraffin, and the insulation tested by a gold leaf electroscope.

Method of making the Observations.

The battery, resistance boxes, galvanometer, and condenser were connected as in the diagram (3), the commutator being placed so near to the condenser that a very short wire sufficed to make the connection. The insulation of the whole system when connected up was tested from time to time by a gold leaf electroscope.

The electrically driven fork was set going and the beats of the auxiliary fork with the standard fork observed. The water supply was then adjusted so that the motor drove the commutator at the required speed. One observer (G. F. C. S.) observed the stroboscopic disc of the commutator through the slits of the fork, and kept the speed steady by means of the controlling arrangement already described. The other observer (J. J. T.) observed the galvanometer. When the speed of the commutator had got steady, resistances were taken out of the Wheatstone's bridge box in the arm CD until no deflection was produced when the galvanometer circuit was broken. The sensitiveness of the galvanometer was such that the effect produced by altering the resistance in CD by 2 ohms in 3000 could be detected, and the speed of the commutator was kept so steady that the light reflected from the galvanometer mirror did not move over more than half a division; the deflection produced by the condenser when not balanced was more than 500 scale divisions.

The battery was then reversed and the operation repeated. The wire connecting the condenser to the commutator was then detached from the condenser, and the same operation repeated. In this way the capacity of the wire was determined. The temperatures of the fork and resistances were then read, and the beats of the auxiliary fork with the standard fork again determined.

The results of these observations are exhibited in the annexed Table.

1 Date.	2 Speed.	3 Beats per minute.	4 Temperature of fork.	5 b.	6 β.	7 Value of γ for condenser + wire.	8 Value of γ for wire.	9 Value of γ for condenser.	10 Reduced to 64.	11 Temperature of resistances.	12 Correction to 16° for resistances.	13 Correction to 21 beats per minute.	14 Correction to 16'' for fork.	15 Correction to $nC = \frac{\gamma}{b\beta}$	16 Correction to legal ohms.	17 Correction for difference of potential between g. r. and condenser.	18 Corrected value of γ.	19 Electromagnetic measure of capacity, × 10⁻²¹.
Dec. 17	64	21·4	17·5	197753	494089	2797·5	39·5	2758	2758	16·5	− ·41	+ ·14	+ ·45	− 1·37	+ 3·04	+ 1·72	2761·6	443·448
,,	32	21·4	17·5	,,	,,	1399·5	20	1379·5	2759	16·5	− ·41	+ ·14	+ ·45	− ·68	+ 3·04	+ ·86	2762·4	443·476
,,	48	21·5	17·5	,,	,,	2099	30	2069	2758·7	16·5	− ·41	+ ·18	+ ·45	− 1·07	+ 3·04	+ 1·29	2762·1	443·528
,,	80	21·5	17·5	,,	,,	3498	50·25	3447·75	2758·2	16·5		+ ·18	+ ·45	− 1·80	+ 3·04	+ 2·15	2761·8	443·480
,,	64	21·8	18	,,	,,	2798	40·5	2757·5	2757·5	17·2	− 1	+ ·29	+ ·60	− 1·37	+ 3·04	+ 1·72	2760·8	443·319
,,	55	21·8	18	,,	,,	2399	34·5	2364·5	2758·6	17·2	− 1	+ ·29	+ ·60	− 1·15	+ 3·04	+ 1·49	2761·9	443·496
,,	42	21·8	18	,,	,,	1866	27	1839	2758·5	17·2	− 1	+ ·29	+ ·60	− ·91	+ 3·04	+ 1·14	2761·6	443·448
Dec. 18	64	21·7	17·5	292585	39,257	3355	49	3306	3306	17·2	− 1·2	+ ·29	+ ·54	− 1·66	+ 3·64	+ 3·20	3310·8	443·112
,,	48	21·7	17·5	,,	,,	2517·5	37·5	2480	3306·7	17·2	− 1·2	+ ·29	+ ·54	− 1·32	+ 3·64	+ 2·40	3311·1	443·153
,,	32	21·7	17·5	,,	,,	1679·5	25	1654·5	3309	17·2	− 1·2	+ ·29	+ ·54	− ·84	+ 3·64	+ 1·60	3313·0	443·407
,,	80	21·7	17·5	,,	,,	4193	60·5	4132·5	3306	17·4	− 1·4	+ ·29	+ ·54	− 2·38	+ 3·64	+ 4·00	3310·2	443·032
,,	64	21·8	18·2	,,	,,	3359·5	49	3310·5	3310·5	17·2	− 1·2	+ ·35	+ ·79	− 1·66	+ 3·64	+ 3·20	3315·6	443·754
,,	55	21·8	18·2	,,	,,	2881·5	43	2838·5	3311·6	17·5	− 1·5	+ ·35	+ ·79	− 1·54	+ 3·64	+ 2·74	3315·7	443·768
,,	48	21·8	18·2	,,	,,	2520	38	2482	3309·3	18	− 2	+ ·35	+ ·79	− 1·32	+ 3·64	+ 2·40	3313·2	443·424
,,	32	21·6	18·2	,,	,,	1680·5	25	1655·5	3311	18	− 2	+ ·26	+ ·79	− ·84	+ 3·64	+ 1·60	3314·4	443·594
,,	16	21·6	18·2	,,	,,	841	13	828	3312	18	− 2	+ ·26	+ ·79	− ·41	+ 3·64	+ ·80	3315·0	443·675
,,	80	21·6	18·2	,,	,,	4196	61·5	4134·5	3307·6	18·1	− 2·1	+ ·26	+ ·79	− 2·38	+ 3·64	+ 4·00	3311·8	443·246
Dec. 20	64	21·6	16·6	197748	494094	2799	41·5	2757·5	2757·5	16·2	− ·16	+ ·22	+ ·18	− 1·37	+ 3·04	+ 1·72	2761·2	443·390
,,	32	21·8	16·8	,,	,,	1399·5	21	1378·5	2757	16·4	− ·33	+ ·29	+ ·24	− ·68	+ 3·04	+ ·86	2760·4	443·262
,,	48	21·8	16·8	,,	,,	2101	30·75	2070·25	2760·3	16·4	− ·33	+ ·29	+ ·24	− 1·07	+ 3·04	+ 1·29	2763·7	443·792
,,	80	21·8	16·6	,,	,,	3500	51·5	3448·5	2758·8	16·4	− ·33	+ ·22	+ ·18	− 1·80	+ 3·04	+ 2·15	2762·2	443·551
,,	55	21·6	16·6	,,	,,	2401·5	35·25	2366·25	2760·6	16·4	− ·33	+ ·22	+ ·18	− 1·15	+ 3·04	+ 1·49	2764·0	443·840
,,	64	21·6	16·6	,,	,,	2800·5	41·3	2759·2	2759·2	16·7	− ·57	+ ·22	+ ·18	− 1·37	+ 3·04	+ 1·72	2762·4	443·583

Explanation of Table of Results and Methods of Reduction.

Column 2 gives the approximate speed of the commutator.

Column 3 gives the beats per minute between the standard fork and the auxiliary fork driven by the electrically maintained fork. The auxiliary fork vibrated twice as fast as the driving fork, and was slightly slower than the standard.

Column 4. The temperature of the fork given by a thermometer hung a short distance from the fork.

Columns 5 and 6. The resistances of the arms AC and BD in legal ohms at 16° C.

Column 7 gives the value of the resistance in CD required for the balance when both the condenser and the connecting wire were joined to the key of the commutator. We shall denote this by γ_1. Each number is the mean of two observations made with the current from the battery flowing first in one direction and then in the opposite. The difference between the two readings very seldom amounted to more than 1 ohm.

Column 8 gives the value of the resistance in CD required for the balance when the wire alone is joined to the key of the commutator. We shall denote this by γ_2. Each is the mean of two readings corresponding to the two directions of the battery current.

Column 9 gives the difference between the last two columns. We must remark that, since the formula $n\mathrm{C} = \gamma/\beta b$ is not sufficiently accurate for our purpose, that this difference does not strictly represent the value of γ, which would be required to balance the condenser alone. With the resistances employed we may write the formula (7)

$$n\mathrm{C} = \frac{\gamma}{\beta b} \left\{ 1 - \frac{\gamma(\gamma + \alpha)}{(\alpha + b)\beta} \right\},$$

so that if, as in our case, we keep β and b constant, $n\mathrm{C}$ is not quite proportional to γ. Strictly, we should find the capacity of the combination of the condenser and wire, and then subtract the capacity of the wire. What we have done is to calculate as if $\gamma_1 - \gamma_2$ represented the capacity of the condenser. The difference amounts to writing in the small term $\gamma(\gamma + \alpha)/(\gamma + b)\beta$, $\gamma_1 - \gamma_2$ instead of γ_1, since the correcting term inside the bracket, in the case of the wire, is too small to be appreciable. Since, however, this term is already very small, and does not affect the result by much more than one part in 2000, and the change we have made only alters its value by 1 per cent. at most, it is evident that it will produce no appreciable effect on the value of the capacity.

Column 10 is headed "(reduced to 64)." Since the value of the capacity is given by

$$\mathrm{C} = \frac{\gamma}{n\beta b} \left\{ 1 - \frac{\gamma(\gamma + \alpha)}{(\alpha + b)\beta} \right\},$$

or with sufficient accuracy for our purpose by

$$C = \frac{\gamma}{nb\beta}\left\{1 - \frac{\gamma\alpha}{b\beta}\right\};$$

and if γ_a, γ_b are the values of γ corresponding to the speeds n_a, n_b, then

$$\frac{\gamma_a}{n_a}\left\{1 - \frac{\gamma_a\alpha}{b\beta}\right\} = \frac{\gamma_b}{n_b}\left\{1 - \frac{\gamma_b\alpha}{b\beta}\right\},$$

or

$$\gamma_a\left(1 - \frac{\gamma_a\alpha}{b\beta}\right) = \frac{n_a}{n_b}\gamma_b\left\{1 - \frac{\alpha}{b\beta}\gamma_b\right\}.$$

Hence we see that we may take as the corresponding uncorrected value of γ_a the value $\gamma_b n_a/n_b$, if we apply to this the correction $1 - \frac{\alpha}{b\beta}\gamma_b$ instead of the correction $\left(1 - \frac{\alpha}{b\beta}\gamma_a\right)$, this is what we have done in compiling this table, and this column contains the values of $\gamma_b n_a/n_b$.

Column 11 gives the temperature of the box from which the resistance was taken. The temperature of the other two boxes are assumed to be the same. Since they were always on the same table as the γ box, and the temperature of this box varied very little, this assumption will not lead to any serious error.

Column 12. " Correction to 16° for resistances" from the formula

$$nC = \frac{\gamma}{b\beta}$$

we see that if we take the temperature coefficients to be the same for the three resistances and equal to ·0003, then we may throw all the effect of the variation on γ if for each degree above 16 we *subtract* from γ ·0003 γ. In the first and third sets this amounts to ·82 ohm for each degree, in the second to 1 ohm per degree.

Column 13. " Correction to 21 beats per minute." The auxiliary fork always made rather more than 21 beats per minute with the standard. When making 21 beats the speed of the auxiliary fork is 127·7545 complete vibrations per second, and 1 additional beat per minute indicates a diminution in the speed in the ratio $1 - \frac{1}{128\times60}$ to 1. Since we wish to treat the observations as if the speed were constant we must for each additional beat increase γ in the ratio $1 + \frac{1}{128\times60}$ to 1, *i.e.*, in the first and third set observations we must add ·36 ohm to γ, and in the second set ·43 ohm.

Column 14. " Correction to 16° for fork." The fork has a temperature coefficient of − ·00011, so that if we regard the speed as constant we must increase γ in the proportion 1·00011 to 1 for each degree the temperature of the fork exceeds 16° C. In the first and third sets this amounts to ·30, and in the second set to ·36 ohm for each degree of excess of temperature above 16° C.

Column 15. " Correction to $nC = \gamma/b\beta$." We see from equation (7), page 606, that this is only an approximation to the correct value for nC, which is

$$nC = \frac{\gamma}{b\beta} \frac{\left\{1 - \dfrac{\gamma^2}{(\gamma + \beta)(b + \gamma + \alpha)}\right\}}{\left\{1 + \dfrac{\gamma\alpha}{(b + \alpha + \gamma)\beta}\right\}}$$

or with sufficient accuracy for our purpose

$$nC = \frac{\gamma}{b\beta}\left\{1 - \frac{\gamma(\gamma + \alpha)}{\beta(b + \alpha + \gamma)}\right\},$$

thus, if we wish to use the formula $nC = \gamma/b\beta$ we must diminish γ in the ratio $1 - \dfrac{\gamma(\gamma + \alpha)}{\beta(b + \alpha + \gamma)}$ to 1. The amount by which γ is to be diminished is given in column 15.

Column 16. "Correction to legal ohms." The values of b and β given in columns (5) and (6) are already expressed in terms of legal ohms, but the values of γ are expressed in terms of the resistances in the Wheatstone bridge box, which are greater than legal ohms in the proportion 1·0011 to 1. We have then to add 3·04 ohms to γ in the first and third sets of experiments, and 3·64 ohms in the second.

Column 17. "Correction for the difference of potential between the middle cylinder and the guard ring." By equation (6), page 9, if the difference of potential between the outer cylinder and the guard ring is less by δV than that between the outer cylinder and the inner cylinder, the capacity is greater than it would be if $\delta V = 0$ in the ratio

$$1 + \frac{\delta V}{V}\frac{h}{l}\left\{\frac{t}{c} - \frac{2}{\pi}\log\frac{4c}{h\epsilon}\right\} \text{ to 1.}$$

where V is the difference of potential between the cylinders, t the thickness of the guard ring, $2c$ the thickness of the pieces of ebonite, and h the distance between the cylinders.

Since $t = 1$, $h = 1$, $c = $ ·145 and $\delta V/V = -b/\left\{\alpha + \beta\dfrac{(b + \alpha + \gamma)}{\gamma}\right\}$ the capacity is less than it would be if the guard ring and the cylinder were at the same potential in the ratio of

$$1 - \frac{7\cdot5}{61}\frac{b\gamma}{\alpha\gamma + \beta(b + \alpha + \gamma)} \text{ to 1,}$$

so that in order to get the corresponding value of γ when the guard ring and cylinder are at the same potential, we must add to γ

$$\frac{7\cdot5}{61}\frac{b\gamma^2}{\alpha\gamma + \beta(b + \alpha + \gamma)},$$

and it is this correction which is given in column 17.

Column 18 contains the values of γ which result when all these corrections are made.

Column 19 contains the values of the electromagnetic measure of the capacity calculated from the formula

$$C = \frac{\gamma}{n\beta b}.$$

Since the auxiliary fork makes 21 beats per minute with the standard fork, which makes 128·1045 vibrations per second, and since the observing fork makes half the number of vibrations of the auxiliary fork

$$n = 63\cdot8773.$$

Assuming that the B.A. unit $= \cdot 9867 \times 10^9$ in absolute measure which corresponds to the legal ohm being $\cdot 9978 \times 10^9$, we have

$$C = \frac{\gamma \times 10^{-9}}{b\beta \times 63\cdot8773 \times \cdot 9977}$$

We have supposed that the resistance to the variable current which passes through the resistance coils during the charging of the condenser, is the same as the resistance to a steady current. This is justified by the following investigation. Though in our experiments the resistances were so large that the charging was not oscillatory, let us suppose that it is oscillatory and has the maximum frequency $1/\sqrt{LC}$. Then if r' is the resistance to this oscillatory current, r the resistance to a steady current

$$r' = r\{1 + kn^2\},$$

where k is a small numerical coefficient and

$$n = \frac{\pi}{\sqrt{LC}} \frac{a^2}{\sigma},$$

where a is the radius of the cross section of the wire, and σ its specific resistance. The coefficient of self induction of a galvanometer similar to the one we used was found some time ago to be 5×10^9, $\sigma = 2 \times 10^4$ and $a = \cdot 023$ for 32 B.W.G. ; substituting these numbers we find $n = 10^{-3}$ about, and

$$n^2 = 10^{-6}.$$

so that the correction will only amount to one part in a million, and may be neglected.

The mean of the first set of observations	$= 443\cdot471 \times 10^{-21}.$	
,, second ,,	$= 443\cdot417 \times 10^{-21}.$	
,, third ,,	$= 443\cdot569 \times 10^{-21}.$	
The mean of all the observations . .	$= 443\cdot486 \times 10^{-21}.$	

4 K 2

The means of the observations at different speeds are given in the following table.

Speed.	Electromagnetic measure of capacity $\times 10^{-21}$.
80	443·327
64	443·434
55	443·701
48	443·478
42	443·448
32	443·460
16	443·675

The means for the different speeds thus agree very well together, the greatest difference from the mean being about one part in 2000. There does not seem any indication of an effect depending on the number of times the condenser is charged per second, such as was very marked in Professor ROWLAND's experiments ('Phil. Mag.,' vol. 28, 1889).

Since the electrostatic measure of the capacity is 397·927, and the electromagnetic measure 443·454 $\times 10^{-21}$

$$\text{``} v \text{''} = \sqrt{\left(\frac{397\cdot927}{443\cdot486 \times 10^{-21}} \right)}$$

$$= 2 \textbf{·9955} \times \textbf{10}^{10} \text{ cm. sec}^{-1}.$$

The value of the B.A. unit is taken to be ·9867 $\times 10^9$ in absolute measure. This value of " v " agrees very nearly indeed with the value obtained by the most recent experiments for the velocity of light in air, these are

CORNU (1878)	. . . .	$3\cdot003 \times 10^{10}$ cm. sec^{-1}.
MICHELSON (1879)	. . .	$2\cdot9982 \times 10^{10}$.
MICHELSON (1882)	. . .	$2\cdot9976 \times 10^{10}$.
NEWCOMB (1885)	. . .	$2\cdot99615$
,, ,,	. . .	$2\cdot99682$ $\left. \right\} \times 10^{10}$.
,, ,,	. . .	$2\cdot99766$

In conclusion we desire to express our thanks to Mr. R. S. COLE, of Emmanuel College, who has given us valuable assistance on several occasions.

The following table taken from Mr. E. B. Rosa's paper on the Determination of "v" ('Phil. Mag.,' vol. 28, 1889, p. 315), gives the results of previous determinations of "v":—

1856. Weber and Kohlrausch	$3 \cdot 107$	$\times 10^{10}$.
1869. W. Thomson and King	$2 \cdot 808$	$\times 10^{10}$.
1868. Maxwell	$2 \cdot 842$	$\times 10^{10}$.
1872. M'Kichan	$2 \cdot 896$	$\times 10^{10}$.
1879. Ayrton and Perry	$2 \cdot 960$	$\times 10^{10}$.
1880. Shida	$2 \cdot 955$	$\times 10^{10}$.
1883. J. J. Thomson	$2 \cdot 963$	$\times 10^{10}$.
1884. Klemenčič	$3 \cdot 019$	$\times 10^{10}$.
1888. Himstedt	$3 \cdot 009$	$\times 10^{10}$.
1889. W. Thomson	$3 \cdot 004$	$\times 10^{10}$.
1889. E. B. Rosa	$2 \cdot 9993$	$\times 10^{10}$.

INDEX

TO THE

PHILOSOPHICAL TRANSACTIONS (A)

FOR THE YEAR 1890.

G.

I.

K.

L.

M.

P.

R.

S.

T.

V.

W.

LIST OF STATIONS.

No.	Name.	No.	Name.	No.	Name.
1	Aberdeen.	70	Carlisle.	139	Southend.
2	Arinagower (Coll).	71	Chesterfield.	140	Spalding.
3	L. Aylort.	72	Chichester.	141	Stoke-on-Trent.
4	Ayr.	73	Clenchwarton.	142	Sutton Bridge.
5	Ballater.	74	Clifton.	143	Swansea.
6	Banavie.	75	Clovelly.	144	Swindon.
7	Banff.	76	Coalville.	145	Taunton.
8	Berwick.	77	Colchester.	146	Thetford.
9	Boat of Garten.	78	Cromer.	147	Thirsk.
10	L. Boisdale.	79	Dover.	148	Tilney.
11	Bunnahabhain.	80	Falmouth.	149	Tunbridge Wells.
12	Callernish.	81	Gainsborough.	150	Wallingford.
13	Campbelton.	82	Giggleswick.	151	Weymouth.
14	Canna.	83	Gloucester.	152	Wheelock.
15	Carstairs.	84	Grantham.	153	Whitehaven.
16	Crianlarich.	85	Guernsey (L'Erée).	154	Windsor.
17	Crieff.	86	,, (Peter Port).	155	Wisbech.
18	Cumbrae.	87	Harwich.	156	Worthing.
19	Dalwhinnie.	88	Harpenden.	G	Greenwich.
20	Dumfries.	89	Haslemere.	S	Stonyhurst.
21	Dundee.	90	Holyhead.		
22	Edinburgh.	91	Horsham.		
23	Elgin.	92	Hull.	157	Armagh.
24	L. Eriboll.	93	Ilfracombe.	158	Athlone.
25	Fairlie.	94	Jersey (Grouville).	159	Bagnalstown.
26	Fort Augustus.	95	,, (S. Louis).	160	Ballina.
27	Gairloch.	96	,, (S. Owen).	161	Ballywilliam.
28	Glasgow.	97	Kenilworth.	162	Bangor.
29	Golspie.	98	Kettering.	163	Bantry.
30	Hawick.	99	Kew.	164	Carrick-on-Shannon.
31	L. Inver.	100A	King's Lynn.	165	Castlereagh.
32	Inverness.	100B	,, ,, (Gaywood).	166	Cavan.
33	Iona.	101	King's Sutton.	167	Charleville.
34	Kirkwall.	102	Lampeter.	168	Clifden.
35	Kyle Akin.	103	Leeds.	169	Coleraine.
36	Lairg.	104	Leicester.	170	Cookstown Junction.
37	Lerwick.	105	Lincoln.	171	Cork.
38	Lochgoilhead.	106	Llandudno.	172	Donegal.
39	L. Maddy.	107	Llangollen.	173	Drogheda.
40A	Oban.	108	Llanidloes.	174	Dublin.
40B	,, (Kerrera).	109	Loughborough.	175	Enniskillen.
41	Pitlochrie.	110	Lowestoft.	176	Galway.
42	Port Askaig.	111	Mablethorpe.	177	Gort.
43	Portree.	112A	Malvern (Colwall Green).	178	Greenore.
44	Row (Gairloch).	112B	,, (Great Malvern).	179	Kells.
45	Scarnish (Tiree).	112C	,, (Malvern Wells).	180	Kildare.
46	Soa.	112D	,, (Mathon).	181	Kilkenny.
47	Stirling.	113	Manchester (Old Trafford).	182	Killarney.
48A	Stornoway (Ard Point).	114	Manton.	183	Kilrush.
48B	,, (Castle).	115	March.	184	Leenane.
49	Strachur.	116	Melton Mowbray.	185	Limerick.
50	Stranraer.	117	Milford Haven.	186	Lisdoonvarna.
51	Stromness.	118	Newark.	187	Lismore.
52	E. L. Tarbert.	119	Newcastle.	188	Londonderry.
53	Thurso.	120	Northampton.	189	Oughterard.
54	Wick.	121	Nottingham.	190	Parsonstown.
		122	Oxford.	191	Sligo.
		123	Peterborough.	192	Strabane.
55	Aberystwith.	124	Plymouth.	193	Tipperary.
56	Alderney.	125	Port Erin.	194	Tralee.
57	Alnwick.	126	Preston.	195	Valentia.
58	Alresford.	127	Purfleet.	196	Waterfoot.
59	Appleby.	128	Pwllheli.	197	Waterford.
60	Barrow.	129	Ramsey.	198	Westport.
61	Bedford.	130	Ranmore.	199	Wexford.
62	Birkenhead.	131	Reading (Caversham).	200	Wicklow.
63	Birmingham.	132	Redcar.		
64	Braintree.	133	Ryde.		
65	Brecon.	134	St. Cyres (Exeter).	201	Chepstow.
66	Bude Haven.	135	St. Leonards.	202	Goodrich.
67	Cambridge.	136	Salisbury.	203	Hereford.
68	Cardiff.	137	Scarborough.	204	Ross.
69	Cardigan.	138	Shrewsbury.	205	Tintern.
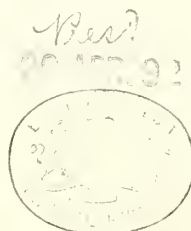
Plate I. Index Map.
showing position and
index number of
each station.

Plate II.
Terrestrial Isogonals 1886.
———— from formula.
----- District curves.
——— Isogonals for epoch 1872.

West, Newman lith

Plate III.
Terrestrial Isoclinals.
-------- from district curves.
———— from formula.
———— for epoch 1842·5.
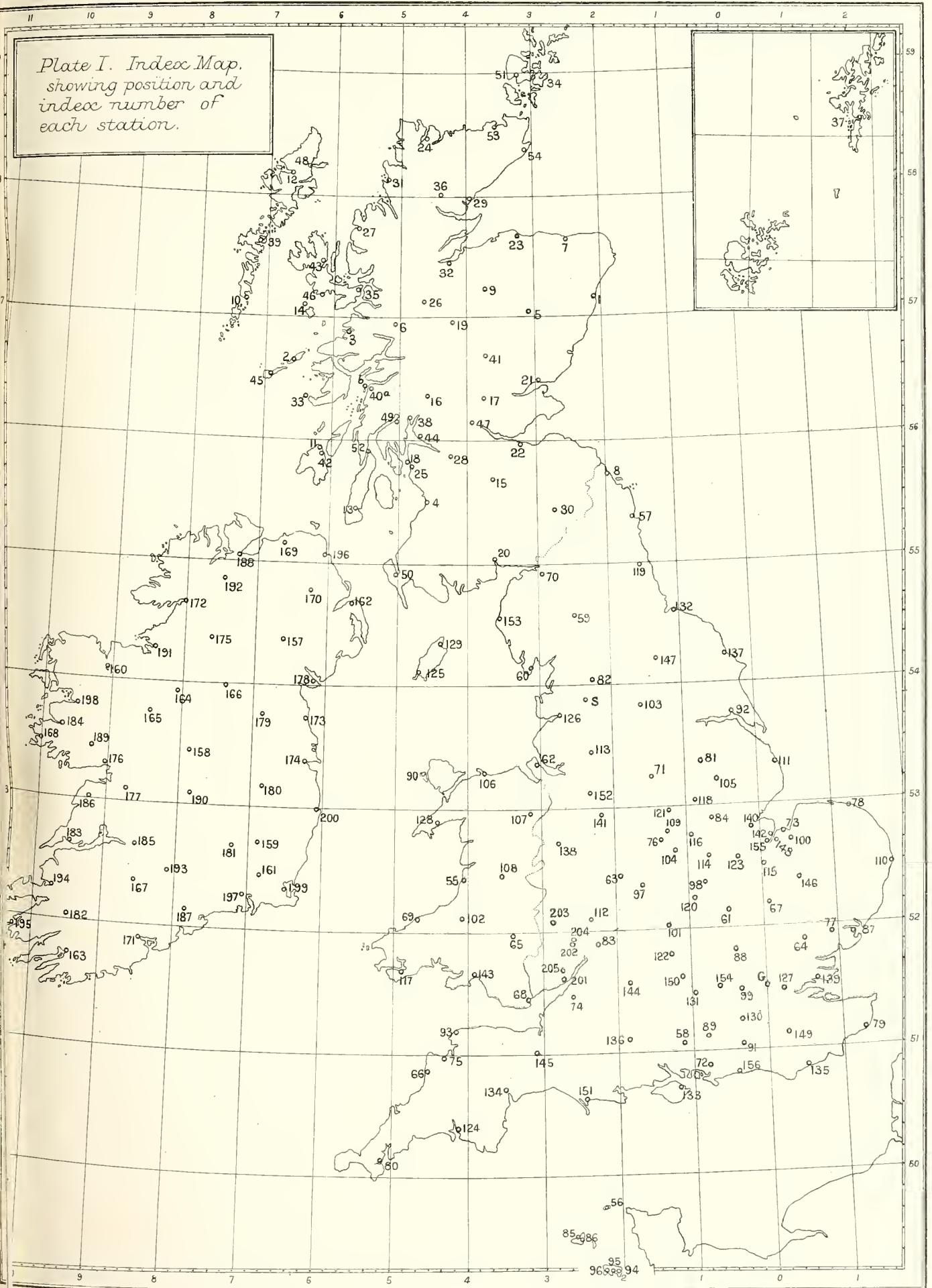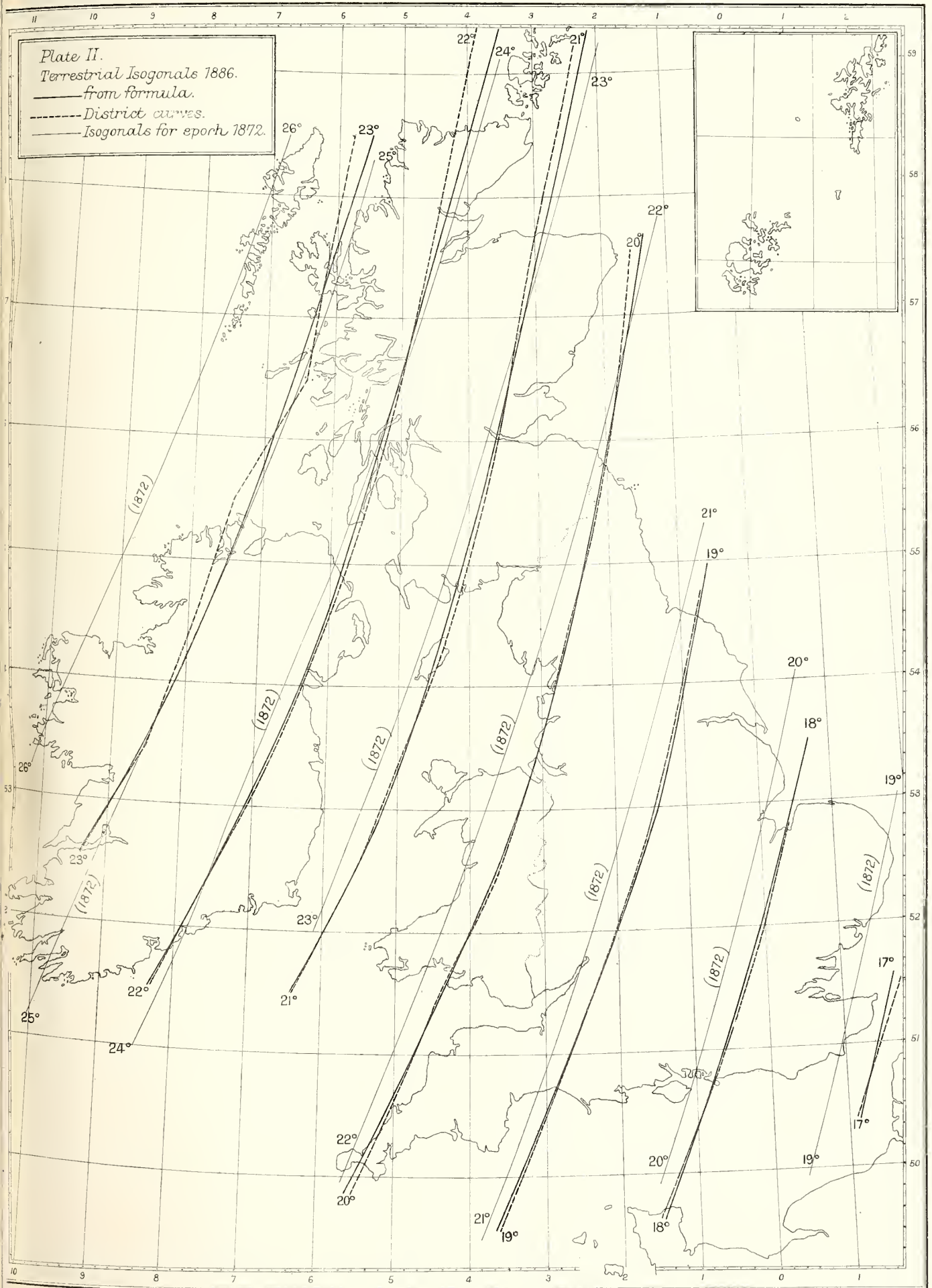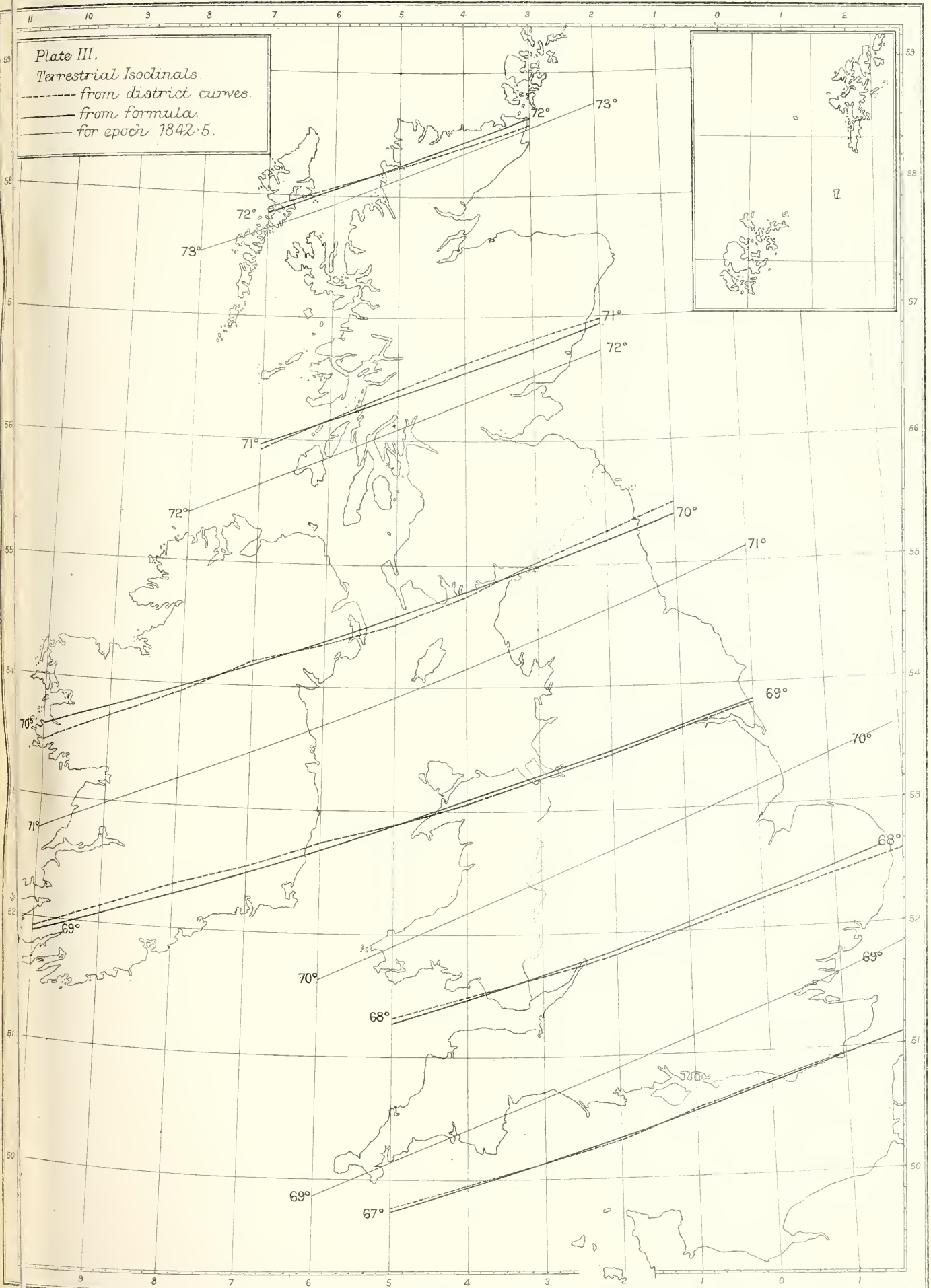
Plate IV.
Terrestrial Lines of Equal
Horizontal Force.
-------- from district lines.
———— from formula.

1·50

1·55

1·55

1·60

1·60

1·65

1·65

1·70

1·70

1·75

1·80

1·75

1·85

1·80

1·85
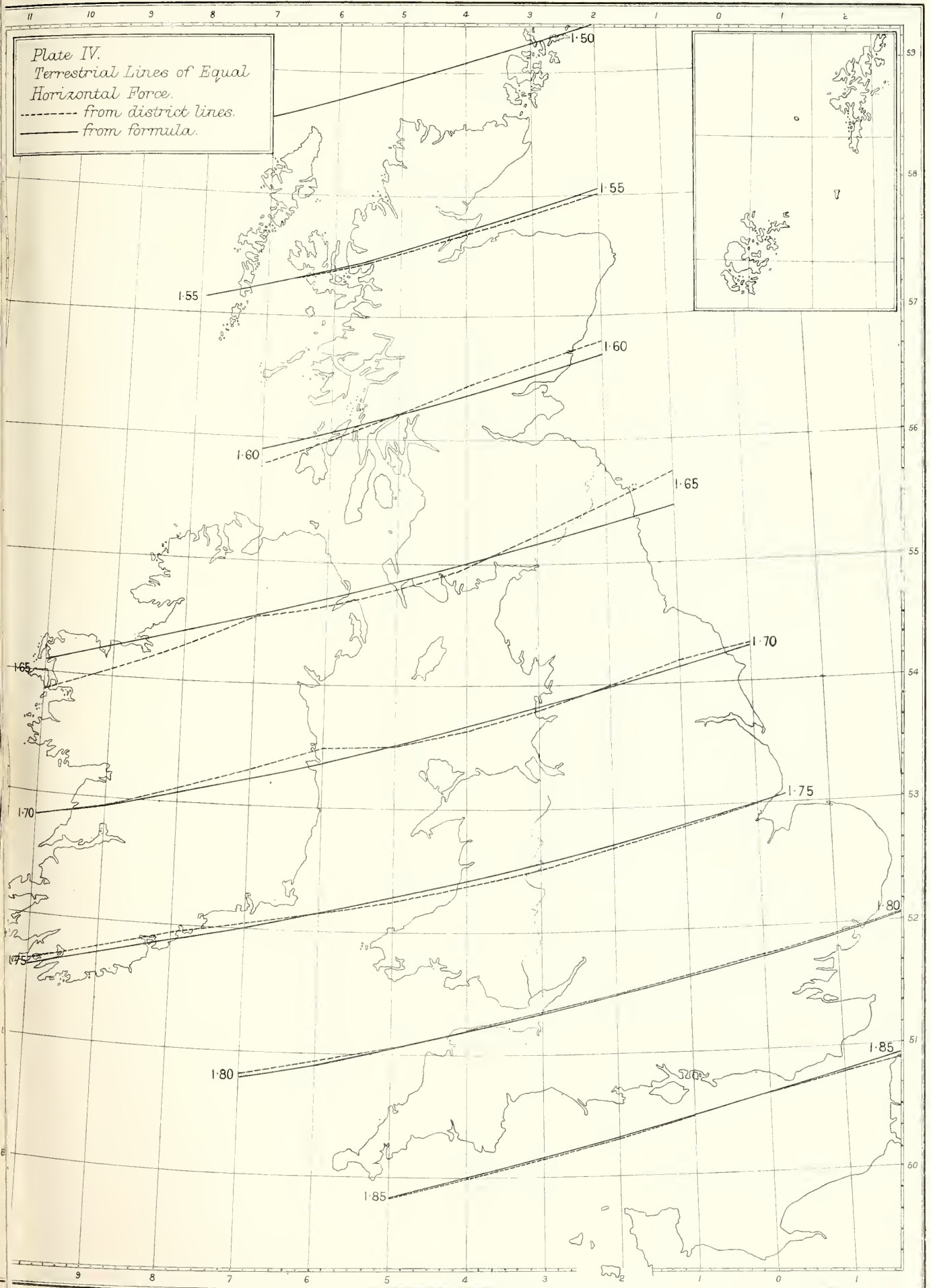
Plate V.
Declinations reduced
to Jan. 1. 1886. & True
Isogonals.
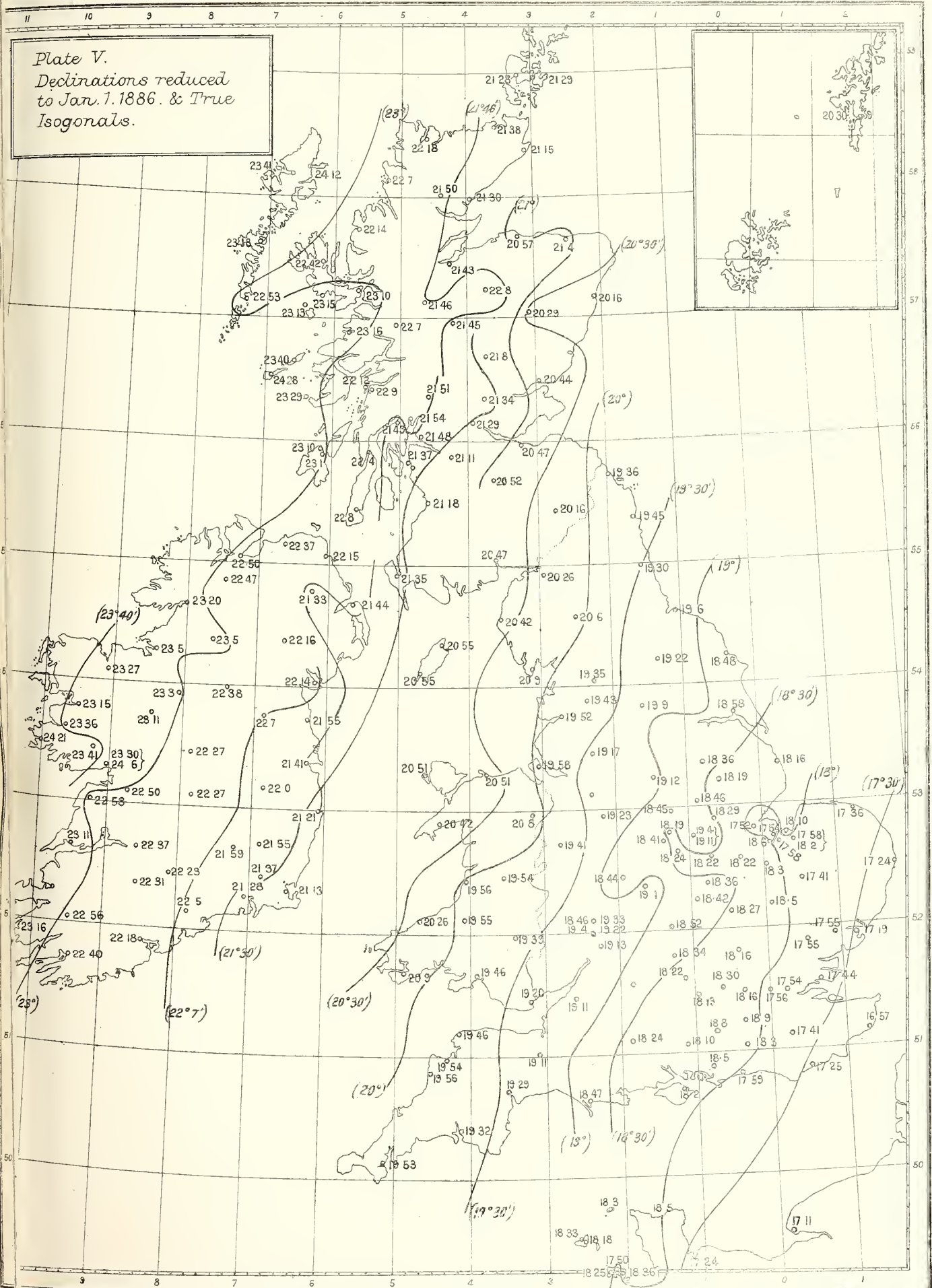
Plate VI.
Dips reduced to Jan.1.1886;
and True Isoclinals.
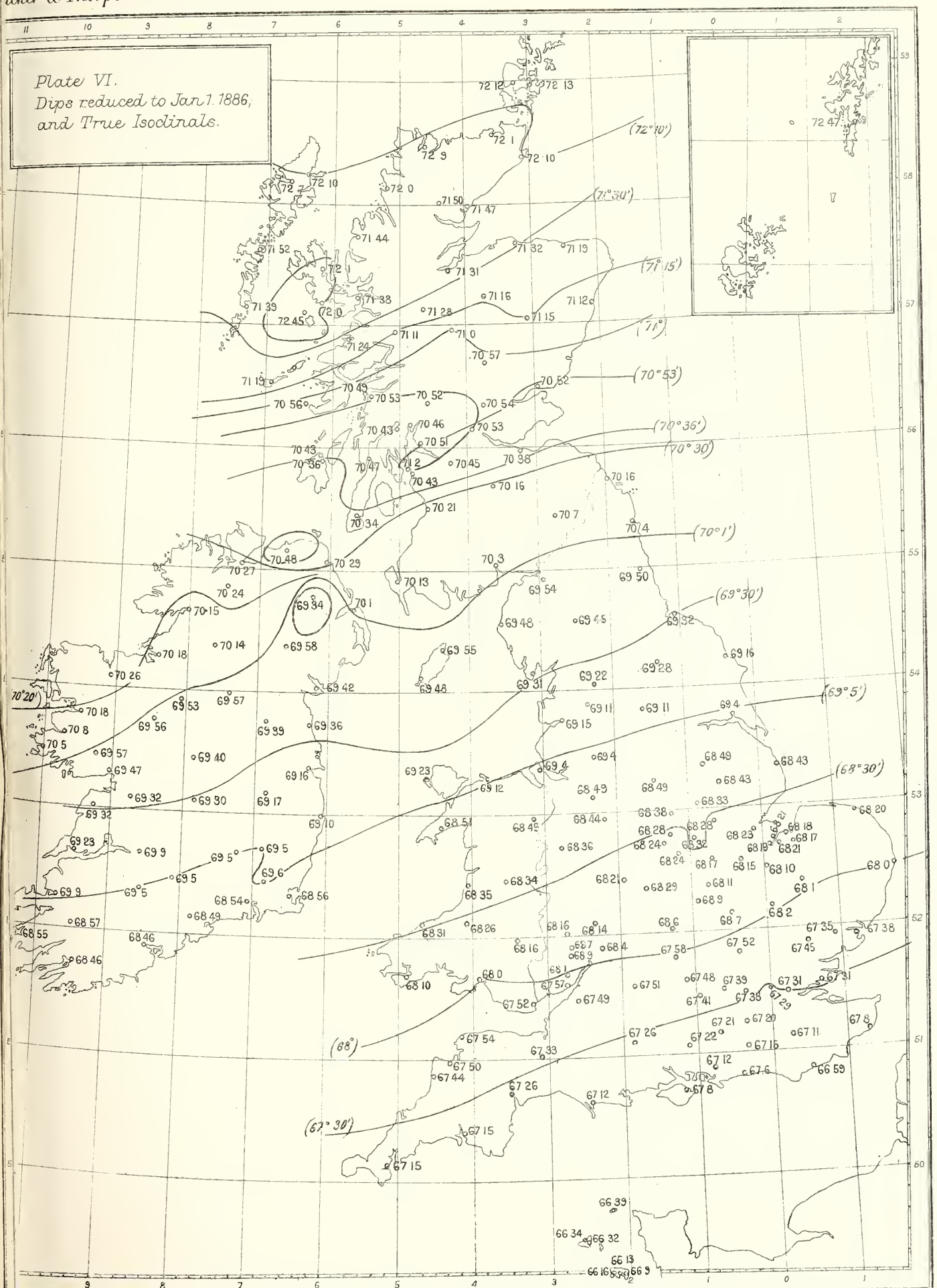
Plate VII.
Horizontal Forces reduced
to Jan. 1. 1886. and True
Lines of Equal Horizontal
Force.

Plate VIII.
Vertical Forces reduced
to Jan. 1. 1886. & True lines
of Equal Vertical Force.
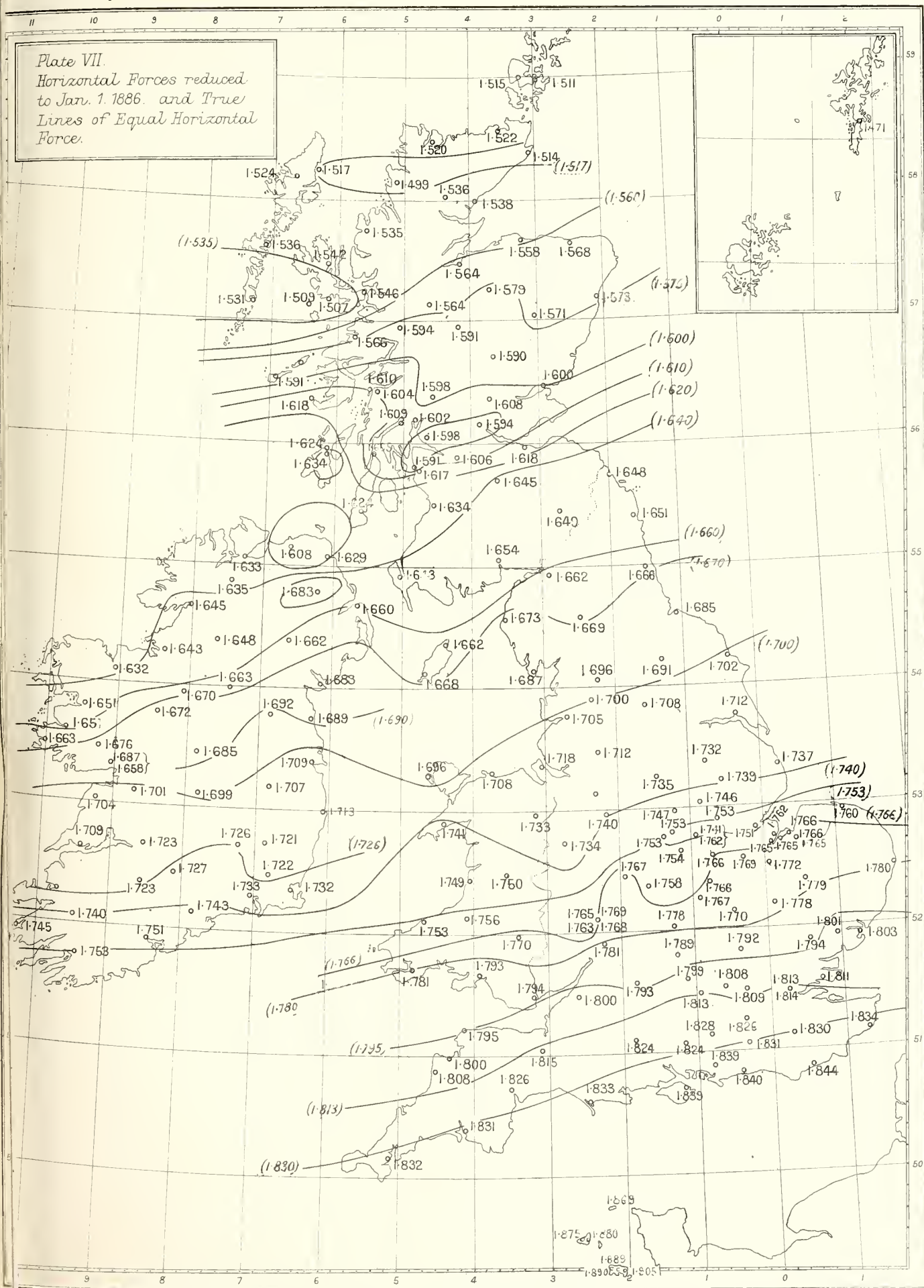
Plate IX. Declination Disturbances in minutes of arc & Lines of no Disturbance. The figures are underlined when the Disturbance is negative & the needle is deviated to the East.
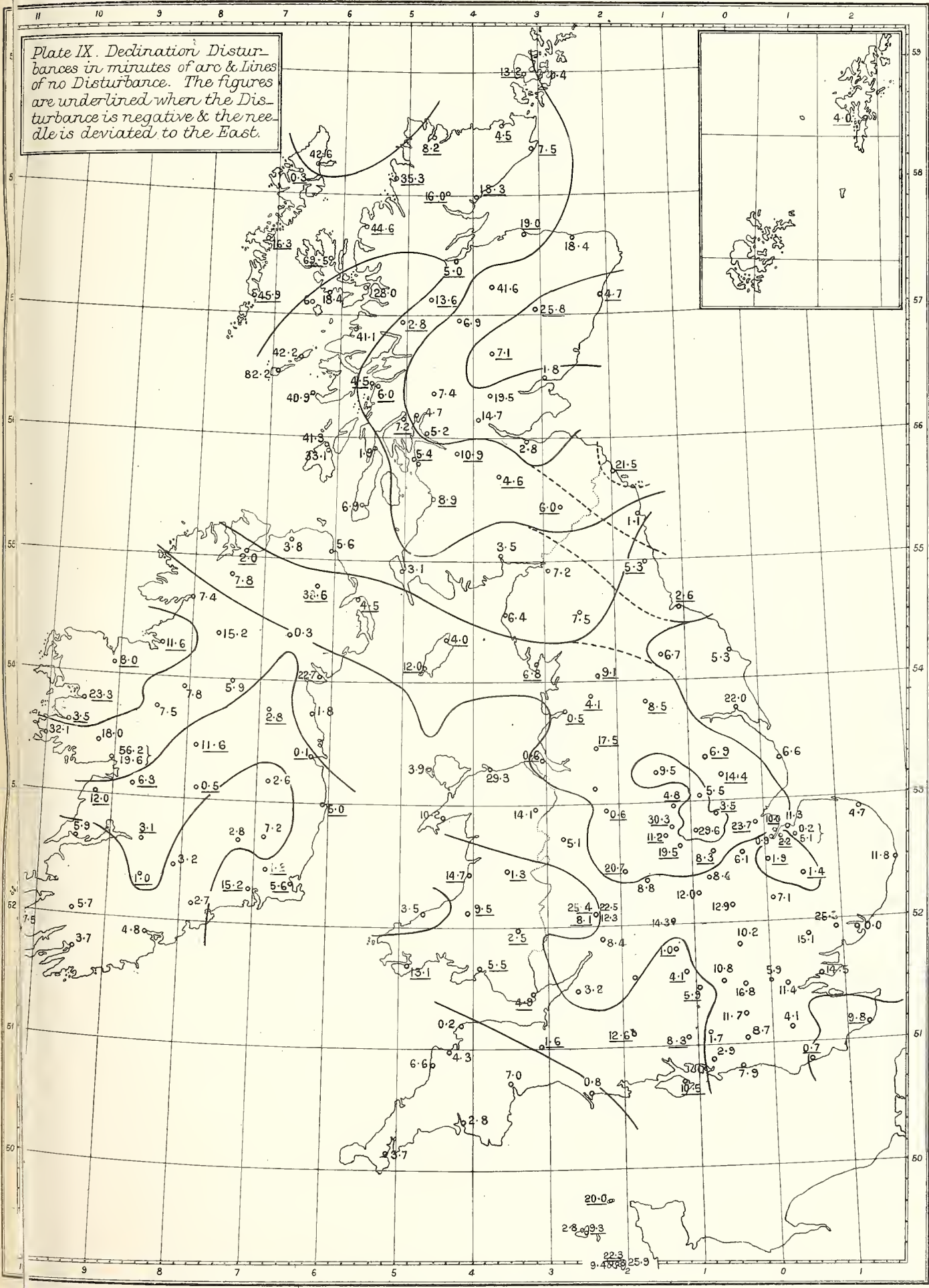
Plate X. Horizontal Force Dis_
turbances in terms of 0·0001
metric units, & lines of no
disturbance. The figures
are underlined when the force
is less than its calculated value.
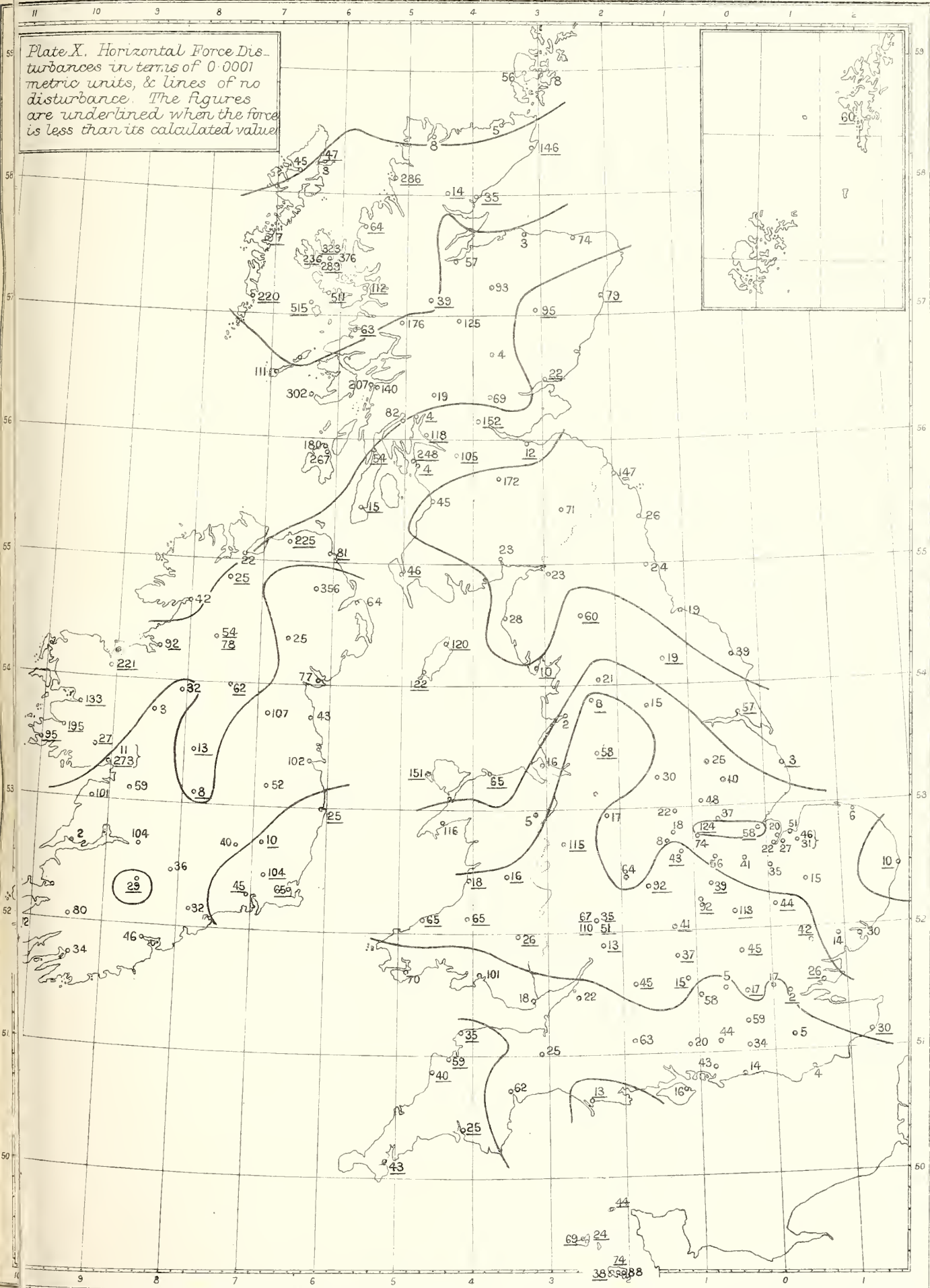
Plate XI. Vertical Force Dis_
turbances in terms of 0·0001
metric units & lines of no Dis_
turbance. The figures are un_
derlined when the Force is less
than its calculated value.
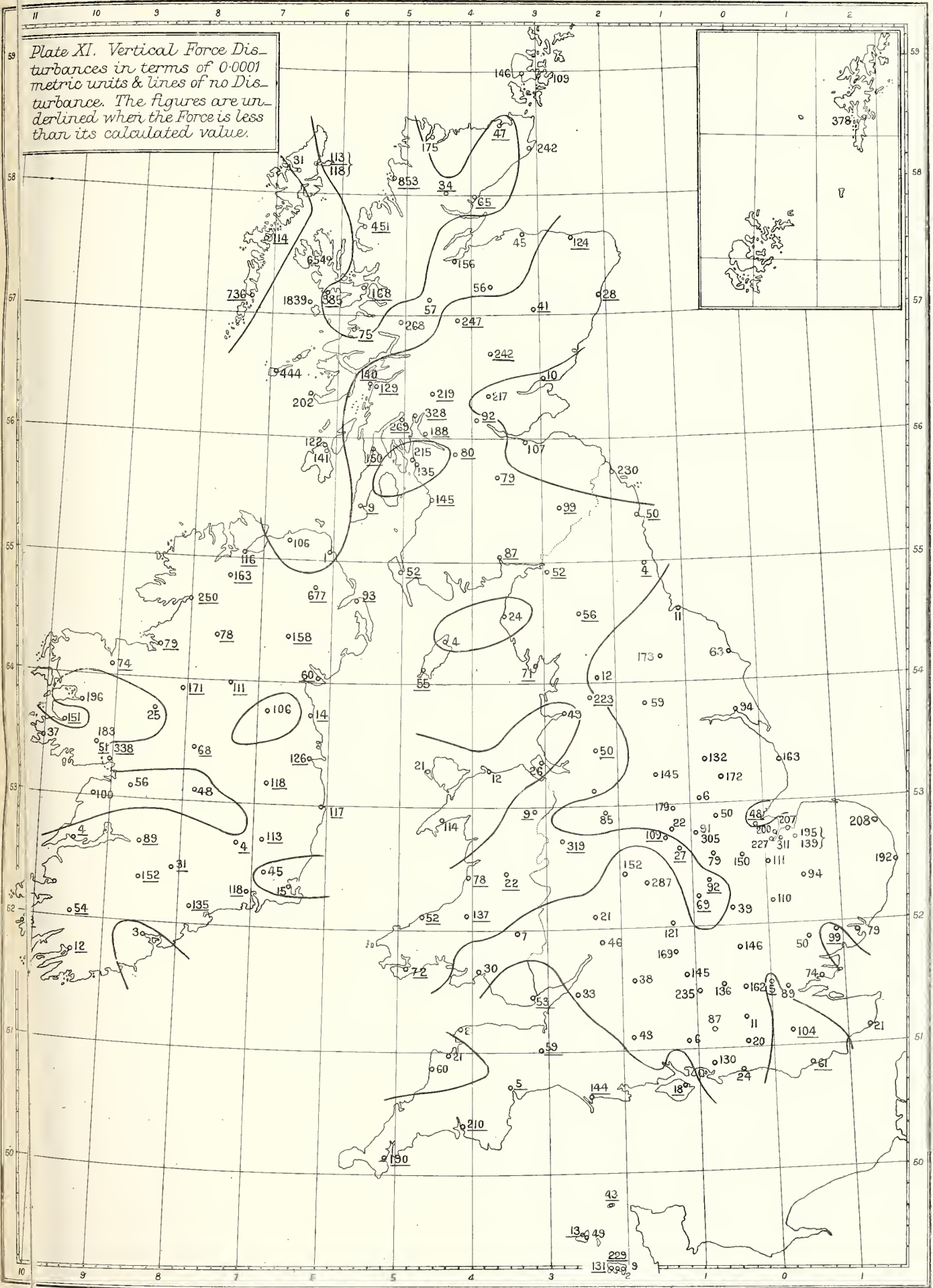
Plate XII. Dip Disturbances
in terms of minutes of arc.
The figures are underlined
when the Dip is less than
its calculated value.
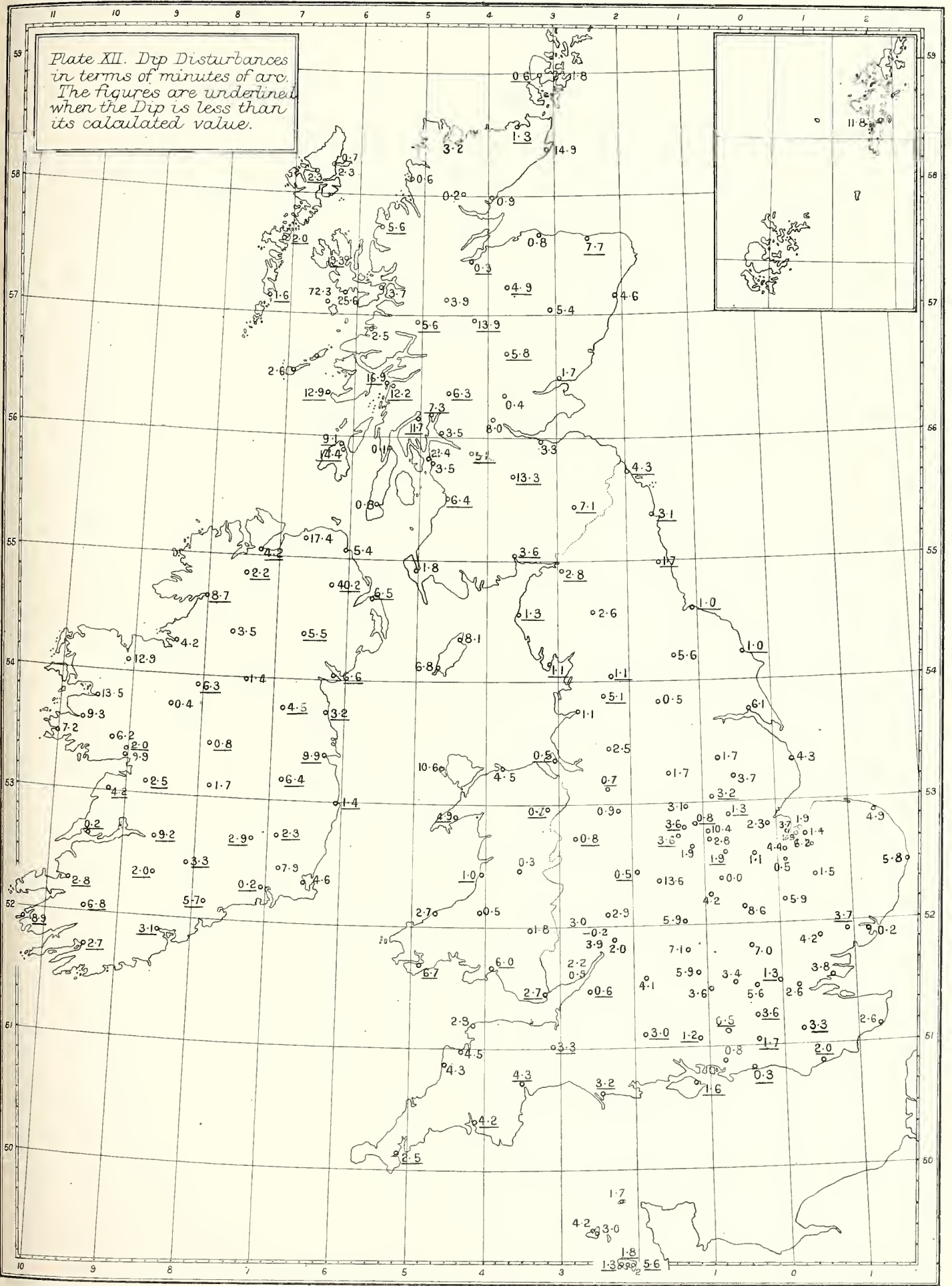
Plate XIII.

MAP OF MAGNETIC DISTURBING FORCES
IN THE UNITED KINGDOM.

The shaded parts are regions of positive
Vertical Force Disturbance.

The arrows represent the Horizontal Disturbing
Forces i: magnitude & direction.

Valley Line

Ridge Line

Ridge Line

Valley Line

Valley Line

Ridge Line

Valley Line

Ridge Line

Valley Line

Ridge Line

Valley Line

Valley Line

Ridge Line

Valley Line

Ridge Line
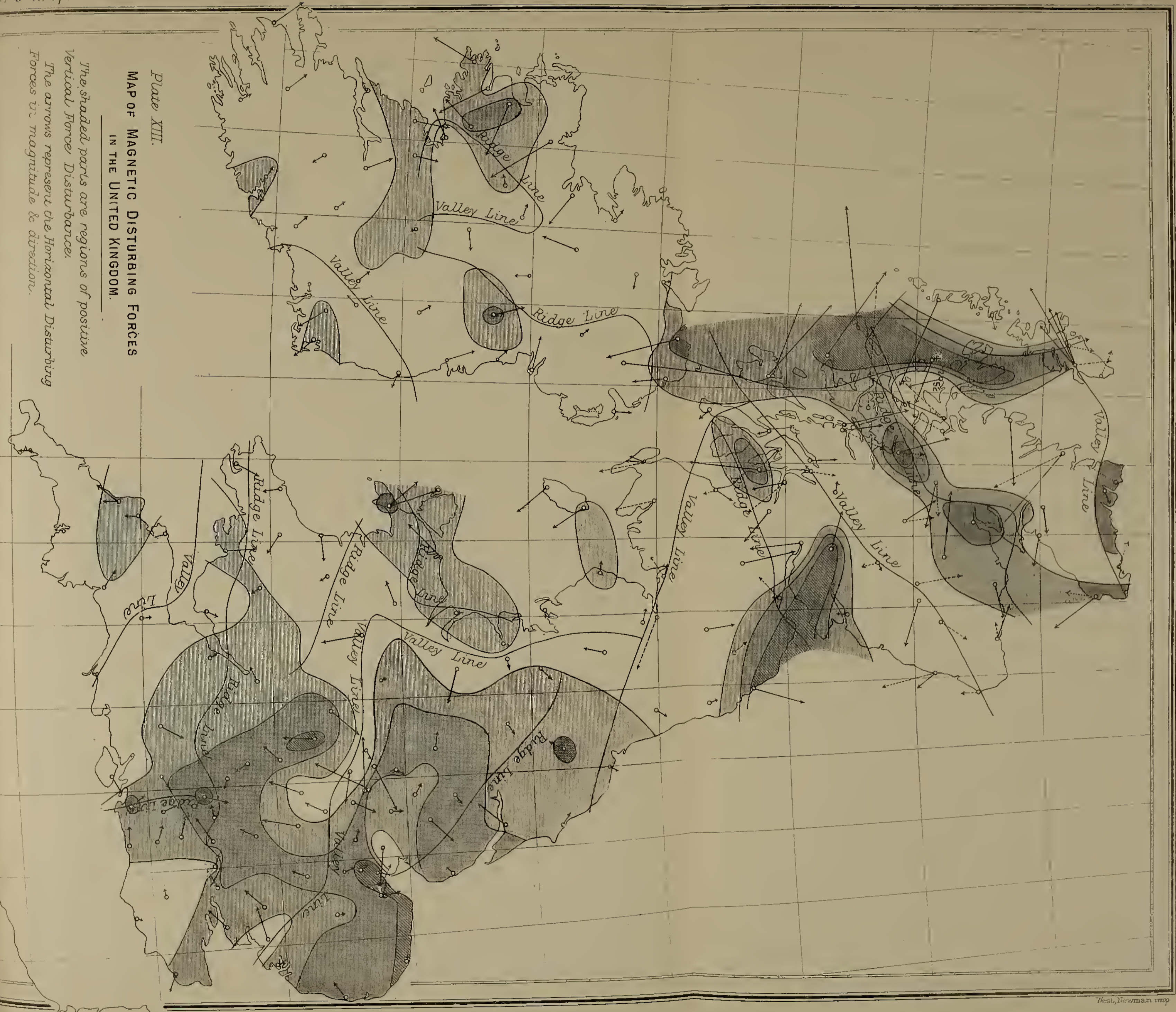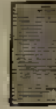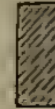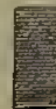
Plate XIV.

GEOLOGICAL MAP OF THE UNITED KINGDOM.

The districts shaded thus //// are the principal regions of Magnetic attraction.

The numerals are the index numbers of the stations.

● indicates a Peak or point to which the Horizontal Disturbing Forces converge.
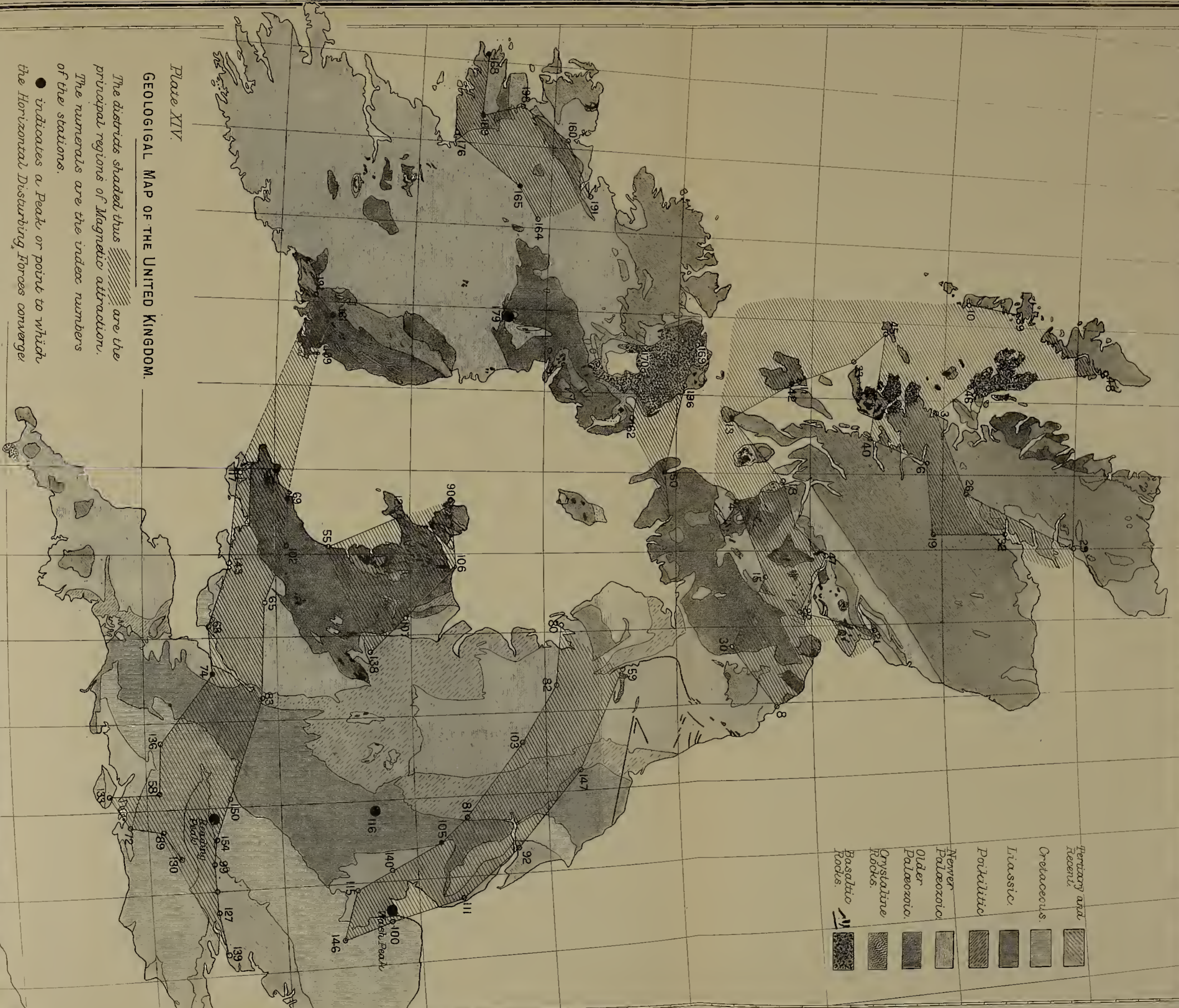
Tertiary and Recent.

Cretaceous.

Liassic.

Palaeozoic.

Newer Palaeozoic.

Older Palaeozoic.

Crystalline Rocks.

Basaltic Rocks.

Fig. 1.
Induced Magnetisation.

Fig. 3.
Coefficient of Induced Magnetisation.

Fig 5.
Increase of Induced Magnetisation
caused by first pressure.
Loss of Induced Magnetisation
when rod removed after
pressure cycles and replaced.

Fig. 7.
Cyclic effect of pressure on Magnetisation.
February experiments.
January experiments.
December experiments.

Fig. 2.
Induced Magnetisation.
Previous to pressures.
After Six pressure cycles.
With pressure on initially.

Fig. 4.
Coefficient of Induced Magnetisation.
Previous to pressures.
After pressure cycles.
With pressure on initially.

Fig 6.
Ratio of Increase of Magnetisation due to "Shock-effect" of first pressure
to Induced Magnetisation previous to pressure.

Fig. 8.
Ratio of Cyclic effect of pressure on Magnetisation
to Induced Magnetisation after pressure cycles.
February experiments.
January experiments.

Fig. 9.

Residual Magnetisation.

—•— No pressures during current

—×— Six pressure cycles, and current broken with pressure on.

Fig. 10.

Residual Magnetisation.

—+— No pressures during current and rod initially un-demagnetised

—•— No pressures during current.

—○······○— Six pressure cycles, and current broken when pressure "off."

—×——×— Six pressure cycles, and current broken when pressure "on".

Fig. 11.

Residual Magnetisation

—×——×— No pressures during current and rod initially un-demagnetised.

—•——•— No pressures during current.

—×······×— Six pressure cycles, and current broken when pressure "off."

—○——○— Six pressure cycles and current broken when pressure "on".

Fig. 14.

—•——•— Loss of Residual Magnetisation due to first pressure, when no pressures during current.

—×······×— Loss of Residual Magnetisation due to first pressure, when 6 pressure cycles during current.

—○——○— Loss of Residual Magnetisation due to the removal of pressure existing when current broken, when 6 pressure cycles during current.

Fig. 15.

—•——•— Ratio of "Shock-effect" of first pressure, in case —— of fig 12 to Initial Residual Magnetisation.

—×······×— Ratio of "Shock-effect" of first pressure, in case —— of fig.12 to Initial Residual Magnetisation.

—○——○— Ratio of "Shock-effect" of removal of pressure, in case —— of fig.12 to Initial Residual Magnetisation.

Fig. 13.

I Same as curve —— of fig. 12

II Fraction of Residual Magnetisation shaken out by pressure cycles.

Fig. 12.

Ratio of Residual Magnetisation just after current broken to Induced Magnetisation just before current broken.

—•——•— No pressures during current.

—×······×— Six pressure cycles, and current broken when pressure "off."

—○——○— Six pressure cycles, and current broken when pressure "on".

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY.

	£	s.	d.		£	s.	d.		£	s.	d.		£	s.	d.
1801. Part I...	0	17	0	1836. Part I...	1	10	0	1855. Part II...	1	6	0	1874. Part II...	3	0	0
—— Part II...	0	17	6	—— Part II...	2	0	0	1856. Part I...	2	0	0	1875. Part I...	3	0	0
1802. Part I...	0	11	0	1837. Part I...	1	8	0	—— Part II...	1	4	0	—— Part II. .	3	0	0
—— Part II...	0	17	6	—— Part II...	1	8	0	—— Part III...	1	4	0	1876. Part I...	2	8	0
1803. Part II...	0	13	6	1838. Part I...	0	13	0	1857. Part I...	1	8	0	—— Part II...	2	8	0
1804. Part I...	0	10	6	—— Part II...	1	8	0	—— Part II...	1	4	0	1877. Part I...	1	16	0
—— Part II...	0	12	6	1839. Part I...	0	18	0	—— Part III...	1	2	0	—— Part II...	2	5	0
1805. Part I...	0	10	0	—— Part II...	1	1	6	1858. Part I...	1	8	0	Vol. 168 (extra)	3	0	0
—— Part II...	0	11	6	1840. Part I...	0	18	0	—— Part II...	3	0	0	1878. Part I...	1	16	0
1806. Part I...	0	13	6	—— Part II...	2	5	0	1859. Part I...	2	10	0	—— Part II...	3	0	0
—— Part II...	0	17	6	1841. Part I...	0	10	0	—— Part II...	2	5	0	1879. Part I...	2	0	0
1807. Part I...	0	10	0	—— Part II...	1	10	0	1860. Part I...	0	16	0	—— Part II...	1	12	0
—— Part II...	0	15	6	1842. Part I...	0	16	0	—— Part II...	2	1	6	1880. Part I...	2	5	0
1812. Part I...	0	17	6	—— Part II...	1	2	0	1861. Part I...	1	3	0	—— Part II...	2	0	0
—— Part II...	0	17	6	1843. Part I...	0	10	0	—— Part II...	1	5	0	—— Part III...	1	1	0
1813. Part I...	0	14	0	—— Part II...	1	10	0	—— Part III...	1	7	6	1881. Part I...	2	10	0
—— Part II...	0	18	0	1844. Part I...	0	10	0	1862. Part I...	2	14	0	—— Part II...	1	10	0
1824. Part I...	0	12	6	—— Part II...	1	10	0	—— Part II...	3	0	0	—— Part III...	2	2	0
—— Part II...	1	0	0	1845. Part I...	0	16	0	1863. Part I...	1	14	0	1882. Part I...	1	14	0
—— Part III...	1	4	0	—— Part II...	1	0	0	—— Part II...	1	7	6	—— Part II...	2	0	0
1825. Part I...	1	4	0	1846. Part I...	0	7	6	1864. Part I.*..	0	11	0	—— Part III...	2	10	0
1826. Part I...	1	2	6	—— Part II...	1	12	0	—— Part II...	1	7	6	—— Part IV...	1	0	0
—— Part II...	0	12	6	—— Part III...	1	12	0	—— Part III...	1	10	0	1883. Part I...	1	10	0
—— Part III...	2	0	0	—— Part IV...	1	12	0	1865. Part I...	2	2	0	—— Part II...	2	10	0
—— Part IV...	1	2	6	1847. Part I...	0	14	0	—— Part II...	1	5	0	—— Part III...	1	12	0
1827. Part II...	0	18	0	—— Part II...	0	16	0	1866. Part I...	1	14	0	1884. Part I...	1	8	0
1828. Part I...	1	1	0	1848. Part I...	1	0	0	—— Part II...	2	7	6	—— Part II...	1	16	0
—— Part II...	0	10	0	—— Part II...	0	14	0	1867. Part I...	1	3	0	1885. Part I...	2	10	0
1829. Part I...	0	16	0	1849. Part I...	1	0	0	—— Part II...	1	15	0	—— Part II...	2	5	0
—— Part II...	0	14	0	—— Part II...	2	5	0	1868. Part I...	2	5	0	1886. Part I...	1	8	0
1830. Part I...	1	10	0	1850. Part I...	1	10	0	—— Part II ..	2	0	0	—— Part II...	1	15	0
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OF

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(B.)

FOR THE YEAR MDCCCXC.

VOL. 181.

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FOR THE YEAR MDCCCXC.

VOL. 181.

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The RUMFORD MEDAL to HEINRICH HERTZ, for his work in Electro-magnetic Radiation.

A ROYAL MEDAL to DAVID FERRIER, F.R.S., for his Researches on the Localisation of Cerebral Functions.

A ROYAL MEDAL to JOHN HOPKINSON, F.R.S., for his Researches in Magnetism and Electricity.

The DAVY MEDAL to EMIL FISCHER, for his Discoveries in Organic Chemistry, and especially for his Researches on the Carbo-hydrates.

The DARWIN MEDAL to ALFRED RUSSEL WALLACE, for his Independent Origination of the Theory of the Origin of Species by Natural Selection.

The Bakerian Lecture, "The Discharge of Electricity through Gases (Preliminary Communication)," was delivered by Professor A. SCHUSTER, F.R.S.

The Croonian Lecture, "On some Relations between Host and Parasite in certain Epidemic Diseases of Plants," was delivered by Professor H. M. WARD, F.R.S.

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(B.)

VOL. 181.

LIST OF ILLUSTRATIONS.

PHILOSOPHICAL TRANSACTIONS.

I. *A Chemical Inquiry into the Phenomena of Human Respiration.*

By WILLIAM MARCET, *M.D., F.R.S.*

Received June 3,—Read June 20, 1889.

[PLATES 1, 2.]

A LOVE of mountaineering is a natural inducement to inquire into the influence of altitude on the phenomena of respiration, and I began giving attention to that subject in 1875; my first results, however, were not communicated to the Royal Society till the month of March, 1878. In September, 1877, Dr. A. MERMOD, then Mr. MERMOD, had contributed to the " Bulletin de la Société Vaudoise des Sciences Naturelles," an interesting paper on this same subject, which he had treated in its chemical aspect, much in the same way as I had done. The title of his paper is " Nouvelles recherches physiologiques sur l'influence de la dépression atmosphérique sur l'habitant des montagnes." The following are the conclusions he arrives at :—

(1.) The uninterrupted and prolonged stay at increasing altitudes is attended with a greater frequency of the pulse.

(2.) The uninterrupted and prolonged stay at 1100 metres above the sea level is not attended with any acceleration of the respiratory movements.

From these two laws the author concludes that—

(3.) The mean fraction showing the relation between the respiratory frequency and the pulsations of the heart, becomes smaller and smaller on removing to stations increasing in altitude.

(4.) The temperature of the body does not appear markedly diminished by removing from a dwelling at 142 metres to another at 1100 metres.

(5.) Instead of an increase, there is rather a reduction in the weight of the air breathed by removing higher and higher above the sea level.

(6.) The absolute and relative quantity of carbonic acid exhaled from the lungs increases by removing into higher regions, although there be no change in the respiratory frequency ; at the same time the weight of the air breathed is reduced.

Dr. MERMOD's investigations were undertaken at two stations only, one of them being Strasburg, at an altitude of 142 metres, and the other Ste. Croix, at 1100 metres ; the difference of altitude amounting therefore to the small figure of 958 metres. Thirty-five experiments were made at Strasburg, and thirty-two at Ste. Croix, on the volumes of air breathed and weights of carbonic acid expired. The mean barometric pressure was 744·89 mm. at Strasburg, and 669·21 mm. at Ste. Croix, giving a difference of 75·68 mm. The extent of the inquiry appears hardly sufficient to determine the influence of atmospheric pressure on the chemical phenomena of respiration, but the work is evidently well done, and I feel called upon to discuss it in some of its bearings.

On one point I agree with Dr. MERMOD, that there is a reduction in the weight of air breathed within a given time at increasing altitudes, and depending on atmospheric pressure alone. My experiments, which extend to altitudes of 3300 and 4000 metres, some of them made under circumstances doing away with the influence of cold, certainly yield that result. I have expressed this law by stating that there is, at increasing altitudes, a reduction in the volume of air breathed, reduced to 0° and 760 mm.

Dr. MERMOD's sixth and last conclusion deals with the amount of carbonic acid exhaled from the lungs within a given time at different altitudes. At first I was disposed to conclude with him, that more carbonic acid is expired as the atmospheric pressure falls from increasing altitude, as I found that more carbonic acid was exhaled per minute in the mountains than in the plains; but it occurred to me that the increased cold experienced on ascending in the Alps was not unlikely to exert an influence on this phenomenon, and that the excess of carbonic acid expired in Alpine districts might be due to *cold* and not to *rarefied air*. My experiments, made on the peak of Tenerife, in order to clear up this point, showed conclusively that when no cold is experienced on reaching higher levels, there is no increase in the amount of carbonic acid expired within a given time.

Dr. MERMOD does not lose sight of the question of temperature, as he records the temperature of the atmosphere at the time of every experiment, and he finds that the mean of all the temperatures observed was the same at both stations. At his lower station, his mean temperature is 12°·65 C., and at his higher station, 12°·68 C. He thus disposes of any influence due to differences of temperature. The effect of food is done away with by making the experiments between 7 and 8 o'clock in the morning, and before breakfast. The barometer reading is recorded at the time of every experiment, but it is much to be regretted that the author quotes no hygrometric observations.

By considering together the temperatures of the air when the different experiments were made, and corresponding weights of carbonic acid expired, as stated in Dr. MERMOD's Tables, a very remarkable influence of the temperature at the higher station will be found. In order to show this, I have extracted from Dr. MERMOD's paper, and placed opposite each other in a tabular form, the mean weights of carbonic acid expired at both stations for each degree of temperature—from $9°$ to $10°$, $10°$ to $11°$, and so on :—

Temperatures at time of experiment.	Strasburg. CO_2 expired per minute.	Ste. Croix. CO_2 expired per minute.	Increase for Ste. Croix.	
			Per cent.	
9 to 10	0·392	0·387	0·0	
10 ,, 11	0·394	0·419	6·0	mean 3·1
11 ,, 12	0·388	0·401	3·2	
12 ,, 13	0·374	0·397	5·8	
13 ,, 14	0·373	0·393	5·1	mean 7·0
14 ,, 15	0·371	0·413	10·2	

First of all we observe that, as the temperature increases at the lower station, the weight of carbonic acid expired per minute as a rule falls, while at the higher station the weight of carbonic acid expired per minute is nearly the same at every temperature.

If, on the other hand, the weights of carbonic acid expired at Strasburg and Ste. Croix be compared with each other *for similar temperatures*, we find no increase of carbonic acid at Ste. Croix for the lowest temperature ($9°$ to $10°$), although a considerable increase, amounting to $10·2$ per cent., at the highest temperature. By collecting the temperatures and corresponding weights of carbonic acid into two groups, the group for the lowest temperatures will give an increase of carbonic acid of only $3·1$ per cent. for Ste. Croix (the highest station), while the group for the highest temperatures will show an excess of carbonic acid of $7·0$ per cent. for that same station.

The analysis of the tables in another form yields a similar conclusion. I have disposed as follows the figures for carbonic acid corresponding to the ten lowest and ten highest temperatures for each station :—

Strasburg.				Ste. Croix.			
Ten lowest temperatures.		Ten highest temperatures.		Ten lowest temperatures.		Ten highest temperatures.	
Temperatures.	CO_2.	Temperatures.	CO_2.	Temperatures.	CO_2.	Temperatures.	CO_2.
9·50	0·393	15·60	0·354	8·75	0·393	16·00	0·387
9·60	0·413	15·40	0·342	9·00	0·385	16·00	0·447
9·60	0·370	14·20	0·396	9·30	0·391	15·90	0·412
10·00	0·390	14·00	0·350	9·40	0·422	15·90	0·376
10·50	0·390	13·80	0·358	9·50	0·388	15·90	0·423
10·80	0·396	13·80	0·377	9·50	0·374	15·80	0·413
10·90	0·398	13·70	0·348	9·55	0·374	15·50	0·404
11·20	0·394	13·50	0·379	9·90	0·394	15·30	0·386
11·20	0·382	13·40	0·361	10·10	0·418	14·90	0·427
12·60	0·390	13·40	0·358	10·30	0·421	14·50	0·391
Mean 10·59	0·392	14·08	0·362	9·53	0·396	15·57	0·406

It follows from this Table that the experiments made at the ten lowest temperatures at Strasburg yield an increase of carbonic acid of 7·6 per cent. over the experiments made at the ten highest temperatures. This is just what might have been expected from the effect of the cold. At Ste. Croix, however, the conclusion is very different, as at that station there is a slight excess of 2·4 per cent. carbonic acid for the experiments made at the highest temperature over the mean carbonic acid found at the lowest temperatures at that same station. Moreover, if we compare with each other the mean weights of carbonic acid expired at the ten lowest temperatures for the two stations, they will be found nearly exactly the same (0·392 and 0·396), while for the ten highest temperatures there is much more carbonic acid expired at Ste. Croix than at Strasburg (0·406 and 0·362), the increase being by no less than 10·8 per cent. This appears at first sight quite unaccountable, although I think it can be satisfactorily explained.

There was clearly some influence on the experiments at Ste. Croix connected with the highest temperatures which was absent at Strasburg. That influence I believe to be a lower state of relative humidity of the air, which, by producing increased perspiration and increased evaporation at the lungs, exerted a cooling action on the body, thus necessitating an increase of animal combustion. It is true that the atmosphere was warmer, but this circumstance failed to compensate for the cold produced by increased perspiration and evaporation.

This explanation appears to me to account satisfactorily for the increase of carbonic acid observed at the higher station : it was not due to the fall of atmospheric pressure, but to increased dryness of the atmosphere, acting precisely in the same way as an accession of cold would have done.

Before entering any further upon this communication I must beg to acknowledge the valuable aid of my assistant, Mr. C. F. Townsend, F.C.S., to whose diligent, methodical, and careful work I am greatly indebted for the results obtained in the present inquiry. We made together the numerous calculations for the reductions, thus checking the figures in every possible way to insure accuracy.

Since my attention was first given to the chemical phenomena of respiration, I have had the honour of communicating to the Royal Society a succession of papers on the "Influence of Altitude on Respiration," which have appeared in volumes 27, 28, 29, and 31 of the 'Proceedings.' These inquiries show in the most conclusive manner that altitude exerts an action on respiration depending exclusively on the fall of atmospheric pressure, and the law can be expressed as follows :—" The volumes of air breathed, reduced to 0° and 760 mm., in order to produce a certain weight, say, 1 gramme of carbonic acid in the body, are smaller on mountains under diminished atmospheric pressures than they are in the plains under higher pressures." My earliest experiments on the Breithorn (13,685 feet—4171 metres), the Col St. Théodule (10,899 feet—3322 metres), the Riffel (8428 feet—2568 metres), the St. Bernard (8115 feet—2473 metres), and the Col du Géant (11,030 feet—3362 metres) were all attended with a fall of the atmospheric temperature on reaching into higher altitudes; this circumstance necessarily produced an increased combustion in the body in order to overcome the action of the cold, and introduced an element into the inquiry which was likely to interfere with the exclusive influence altitude might exert on the chemical phenomena of respiration. The only way of overcoming this difficulty was to resort to a mountain where it was possible to ascend to a considerable altitude without meeting with the cold air of northern Alpine districts. Thus I was induced to visit the Peak of Tenerife, taking with me the instruments required to repeat the experiments undertaken in the Swiss Alps. I spent three weeks on the Peak in the summer of 1878, engaged in these inquiries. The temperature, though varying somewhat at the three stations I had selected at various altitudes, was always hot in the daytime, and there was no cause for any increased formation of carbonic acid in the body towards the resistance of cold. The result was most striking. While in the cold Swiss Alps I had observed an increased expiration of carbonic acid when ascending on myself and others who submitted to the experiment, on the Peak of Tenerife there was no such increase. The mean weight of carbonic acid expired at the three stations by two different persons was, with one exception at one station only—in the case of a Chamonix guide—nearly the same for each of them respectively ; the volumes of air breathed were lessened,* so that the law remained unchanged : that at increasing alti-

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* There is one exception in the case of the Chamonix guide, the volume of air he expired (for 1 grm. CO_2) being somewhat greater at the higher than the intermediate station on the Peak ; but the mean volume of air expired at these two stations, taken collectively, was much less than the mean volume of air expired at the seaside.

tudes less air breathed, reduced to 0° and 760 mm., than at the sea level, is required to produce 1 grm. of carbonic acid in the body. The experiments on the Peak of Tenerife, by doing away entirely with the influence of cold, placed this fact beyond doubt.

As it is important towards a clear understanding of this paper that the results obtained formerly should be present to the reader, I beg to subjoin them in a tabular form.

EXPERIMENTS on the Influence of Altitude upon the Chemical Phenomena of Respiration (on Myself).

Stations.	Altitude.	Weight of CO_2 expired.	Litres of air expired reduced to 0° and 760 mm.
	feet.	grm.	
Near Geneva	1,230	1	13·6
St. Bernard	8,115	1	Mean at and above
Riffel	8,428	1	8115 feet up to
Col St. Théodule	10,899	1	13,685 feet.
Summit of Breithorn . . .	13,685	1	11·05

ON the Peak of Tenerife (on Myself).

Stations.	Altitude.	Weight of CO_2 expired.	Litres of air expired reduced to 0° and 760 mm.
	feet.	grm.	
Seaside	..	1	12·24
Guajara	7,090	1	11·9
Alta-Vista	10,700	1	10·7
Foot of terminal cone . . .	11,740	1	10·6

COL du Géant (on Myself).

Stations.	Altitude.	Weight of CO_2 expired.	Litres of air expired reduced to 0° and 760 mm.
	feet.	grm.	
Near Geneva	1,230	1	15·5
			Mean before and
			after ascent.
Courmayeur	3,945	1	14·35
Summit of Col du Géant . .	11,030	1	12·6

COL du Géant (on a Companion).

Stations.	Altitude.	Weight of CO_2 expired.	Litres of air expired reduced to 0° and 760 mm.
	feet.	grm.	
Geneva	1,230	1	13·7
			Mean before and
			after ascent.
Courmayeur	3,945	1	14·8
Summit of Col du Géant . .	11,030	1	12·6

The experiments were made by determining as carefully as possible the volume of air expired within a given time, and finding the amount of carbonic acid this air contained by means of PETTENKOFER's method carried out with the aid of an instrument I had constructed for that purpose. Then, knowing the weight of carbonic acid in a given volume of air, it was easy to find, by a very simple calculation, the volume of air (reduced to 0° and 760 mm. pressure) corresponding to, or holding, 1 grm. of carbonic acid. The method and instruments employed in this inquiry have been fully described in a previous communication.

Being desirous of ascertaining whether at such stations as those frequented by invalids the same effects were produced on the function of respiration as at greater elevations, I repeated the experiment in the year 1882 on the Rigi Mountain, in Switzerland, in the company of a young engineer, Mr. R. THURY. As the results obtained have never been published, beyond a short reference to them in a paper to the Alpine Club, I shall beg to include a tabular statement of these experiments in the present communication. The inquiry made on the Rigi confirms in every way the law ruling the chemical phenomena of respiration at different altitudes. Mr. THURY submitted alone to these experiments.

Fifteen experiments were made on the 3rd and 4th of September near Geneva, at an altitude of 1230 feet (375 metres), mean barometer pressure of 728 mm. and temperature of 15°·9 C. ; and eighteen experiments were made as soon as possible afterwards at the Rigi Staffel, at a mean pressure of 639 mm. and mean temperature of 7°·6 C. during the experiments. The Rigi Staffel is situated at an elevation of 5230 feet (1594 metres), giving a difference of altitude of 4000 feet, or 1219 metres, and of 89 mm. mean pressure, between the two stations. It might have been expected that more carbonic acid would be exhaled in a given time on the cold mountain station than in the valley of Geneva, and this turned out to be the case ; as, while a mean of 0·350 grm. CO_2 was expired near Geneva the lower and warmer station, this figure rose to a mean of 0·445 grm. at the higher or colder station—giving an excess of no less than 21 per cent. of carbonic acid for the Rigi. The amount of air expired, say breathed, for the expiration of 1 grm. of carbonic acid, reduced to 0° and 760 mm., was 10·78 litres in the Geneva valley, and only 9·45 litres on the Rigi station. Therefore, for a mean difference of 89 mm. atmospheric pressure (including the influence of reduced temperature), less air by 12 per cent. was breathed on the mountain to supply the oxygen required by the body to burn the same amount of carbon as in the valley.

EXPERIMENTS at Malagny, near Geneva, on Mr. R. THURY, 1883.   Barometer = 728 mm.

Number.	Date.	Time with reference to food taken.	Temperature of air.	Grammes of CO_2 expired per minute.	Litres of air expired per minute (reduced).	Litres of air expired (reduced) corresponding to 1 grm. CO_2.	Percentage volume of CO_2 in air expired.
		h. m.	° C.				
1	Sept. 3	2 10 after lunch . .	20·0	·334	3·65	10·93	4·66
2	,,	3 25 ,, ,, . .	18·5	·319	3·40	10·65	4·77
3	,,	3 55 ,, ,, . .	17·5	·303	3·29	10·88	4·67
4	,,	4 40 ,, ,, . .	16·5	·343	3·60	10·48	4·85
5	,,	5 20 ,, ,, . .	15·0	·294	3·17	10·77	4·72
6	Sept. 4	0 45 after breakfast .	16·0	·358	3·99	11·12	4·57
7	,,	1 30 ,, ,, .	16·5	·401	4·52	11·29	4·50
8	,,	2 25 ,, ,, .	16·0	·418	4·53	10·84	4·69
9	,,	3 0 ,, ,, .	15·0	·374	4·17	11·17	4·55
10	,,	0 45 after lunch . .	14·5	·410	4·40	10·75	4·73
11	,,	1 15 ,, ,, . .	15·0	·359	3·86	10·75	4·73
12	,,	2 10 ,, ,, . .	15·0	·340	3·49	10·28	4·94
13	,,	2 50 ,, ,, . .	14·5	·305	3·34	10·94	4·64
14	,,	4 0 ,, ,, . .	14·0	·366	3·82	10·42	4·88
15	,,	4 55 ,, ,, . .	14·0	·329	3·45	10·47	4·85
		2 52 mean time after a meal	15·9 mean	·350 mean	3·78 mean	10·78 mean	4·72 mean

EXPERIMENTS at the Rigi Staffel on Mr. THURY, 1883.   Barometer = 639 mm.

Number	Date.	Time with reference to food taken.	Temperature of air.	Grammes of CO_2 expired per minute.	Litres of air expired per minute (reduced).	Litres of air expired (reduced) corresponding to 1 grm. CO_2.	Percentage volume of CO_2 in air expired.
		h. m.	° C.				
1	Sept. 7	1 45 after lunch . .	9·0	·332	3·23	9·72	5·23
2	,,	2 30 ,, ,, . .	9·5	·326	3·24	9·94	5·11
3	,,	3 10 ,, ,, . .	10·0	·364	3·51	9·64	5·27
4	,,	3 55 ,, ,, . .	7·5	·379	3·61	9·54	5·32
5	,,	4 40 ,, ,, . .	7·5	·451	4·08	9·05	5·61
6	,,	5 15 ,, ,, . .	7·5	·416	3·93	9·44	5·38
7	Sept. 9	1 30 ,, ,, . .	8·0	·454	4·23	9·30	5·46
8	,,	2 22 ,, ,, .	9·0	·533	4·60	8·62	5·89
9	,,	3 5 ,, ,, .	6·0	·382	3·79	9·93	5·12
10	,,	3 50 ,, ,, .	6·0	·453	4·22	9·32	5·45
11	,,	4 27 ,, ,, . .	5·0	·488	4·44	9·11	5·58
12	Sept. 10	0 35 after breakfast .	6·5	·503	4·48	8·92	5·70
13	,,	1 8 ,, ,, .	7·0	·532	4·98	9·37	5·42
14	,,	1 48 ,, ,, .	8·0	·511	4·85	9·49	5·35
15	,,	2 33 ,, ,, .	6·0	·511	5·00	9·48	5·36
16	,,	3 30 ,, ,, .	7·5	·491	4·94	10·06	5·05
17	,,	4 25 no lunch . . .	8·5	·444	4·40	9·92	5·12
18	,,	5 30 ,, ,, . . .	8·5	·442	4·55	10·28	4·94
		3 7 mean time after a meal	7·6 mean	·445 mean	4·23 mean	9·45 mean	5·35 mean

The following investigation was carried out in a laboratory of the Physiological Department at University College, which Professor SCHÄFER has kindly placed at my disposal. Two persons—my assistant, Mr. C. F. TOWNSEND, and my laboratory attendant, WILLIAM ALDERWOOD, who has worked for the last five years in my laboratory, and whom I can thoroughly trust to carry out my instructions—both kindly submitted to experiment.

The inquiry has to be prefaced with a statement of the method of investigation adopted. The main objects were to collect and measure the air expired from the lungs within a given time, and then find out, as accurately as possible, the weight or volume of carbonic acid it contained. I had proposed at first to determine also the volume of air inspired, but met with so many difficulties in carrying this out satisfactorily that it was given up. There are two possible sources of error in a work of this kind, first on account of the resistance breathing has to overcome when the expired air is either collected for analysis or passed through absorbing media; and, secondly, from the attention of the person who is breathing being directed to the experiment, so that respiration becomes a voluntary act instead of being purely automatic. In EDWARD SMITH's experiments* the air was inspired through a dry gas meter, and expired through a succession of vessels which absorbed the water and carbonic acid, the latter being estimated by weighing. This method, although possessed of the advantage of allowing each experiment to be continued for an indefinite length of time, appeared to me hardly delicate enough for the object I had in view, especially because of the unavoidable resistance to free respiration, although it might be but slight. The air expired had therefore to be collected in a bell-jar, in such a way that the person under experiment might not be aware, at the time, that he was breathing with an object in view. I had two bell-jars constructed for this purpose by Messrs. W. PARKINSON and Co., engineers, by whom they were admirably made. Each bell-jar could hold 40 litres of air, and was graduated on a scale into litres and fractions of litres. They were made of japanned iron, and suspended over tanks filled with a solution of pure sodium chloride; the solution in the tank used for most of the experiments had a density of 1·074, corresponding to 10 per cent. of sodium chloride. The tanks had an annular form, so that the fluid should expose as small a surface as possible to the air contained in the jars. The bell-jars were counterpoised over a pulley fixed to a cycloid to which another regulating weight was suspended.

The cycloid was so constructed that the increased leverage of the regulating weight counterbalanced exactly the increase of weight of the bell-jar as it ascended from the tank and *vice versâ*. A gauge fixed to the dome of the bell-jar and charged with almond oil indicated the state of tension of the air in the jar. I found that by keeping the outside of the bell-jars carefully lubricated with almond oil their motion through the salt water was certainly facilitated, and after careful adjustment the bell-jars were so nicely suspended that air could be expired quite unconsciously into them through a

* 'Phil. Trans.,' 1859.

pipe about an inch in diameter, opening under the dome of the bell-jar. But this still proved insufficient, as it was necessary to devise some method enabling the person under experiment to breathe without his being aware that his breath was being collected for experiment. The difficulty was overcome as follows :—A double-way stop-cock, with passages not much less than an inch in diameter, was fixed to an upright stand screwed to the floor, and an india-rubber tubing of a corresponding diameter was connected with it, long enough to be used by a person sitting in a recumbent posture in a deck chair—there was a vulcanite mouthpiece in the distal end of the india-rubber tube. By means of a handle the double-way cock could be turned in such a position as to enable the person sitting in the deck chair to breathe either into the open air or into the bell-jar. A little silk flag placed inside a glass tube introduced in the course of the india-rubber tubing, flapped during the expiration and remained still while air was inspired, thus acting as a guide throughout the experiment. This arrangement answered the purpose in every way ; the person experimented upon was instructed to inspire through the nose and expire through the tube while in communication with the open air, and without paying any attention to the bell-jar, the little flag showing clearly the time when the respiration had become perfectly slow and natural. Then, without giving any notice and unknown to the person under experiment the handle of the double way cock was turned during an inspiration and the expired air directed towards the bell-jar ; this was so delicately suspended that no appreciable difference of sensation was experienced in the breathing, and it frequently happened that the bell-jar was half way up while the person in the chair still thought he was breathing into the outside air. In order to deviate in no way from natural respiration in these experiments, no valvular face-piece was used, the air, as stated above, being inspired through the nose and expired through the mouth. A little practice made it quite easy, but in order to ensure absolute accuracy the two persons who submitted to experiment closed their nose during expiration by a light pressure with the index finger of each hand held in such a position as to do this with as little effort as possible. After a time the movement became quite automatic, and it was too slight to cause the addition of any appreciable amount of carbonic acid in the expired air.

The experiment was made in the following way :—The person sat in a semi-recumbent posture in a comfortable deck chair, his body and legs being thoroughly supported and the feet resting on a stool at the same height as the legs, thus every muscular strain was done away with. Then he began inspiring through the nose and expiring from the mouth through the india-rubber tube and double-way cock into the open air ; at first the little flag in the track of the air expired was seen to fly about wildly, then by degrees its movements became more regular, and after from ten minutes to a quarter of an hour the respirations were observed to be quite quiet and steady. The scale of the bell-jar being at 0° the stop cock was turned during an inspiration, and the air expired directed into the bell-jar. The instant the bell-jar

commenced rising a stopwatch or chronograph was started, which registered minutes and seconds only. After about 35 litres of air had been collected the watch was stopped at the end of an expiration, and the stop-cock, turned during the next inspiration, the air expired being diverted into the external air. The volume of air collected was read off on the scale, exactly under atmospheric pressure. In some few instances the air was analysed at once (in which case care was taken to rinse out the bell-jar previously with expired air), but as a rule the bell-jar was filled twice or three times so as to obtain two experiments yielding as near as possible the same volume of air expired in a given time. In many cases the difference did not exceed two or three per cent., but it occasionally happened to be much greater. These occurrences were perplexing, and led to careful discussions before the figures were accepted. The air collected in the bell-jar was next submitted to analysis, but first of all its temperature was observed with a thermometer introduced *in situ* through the bell-jar, then the barometer was read.[*]

The determination of carbonic acid in expired air and atmospheric air has taken up my attention for many years, and after trying various forms of volumetric methods, in which volumes varying from 1 to 7 litres of air expired were submitted to analysis, I finally fell back upon PETTENKOFER's method with which I was well acquainted and which was known to yield reliable results, while simple and speedy in its manipulations. It was necessary to collect the air from the bell-jar into a vessel of the proper shape and size, and agitate this air with a solution of barium hydrate of known strength. The vessel is a cylinder made of strong glass of a capacity of 1 litre, it is closed at both ends by brass caps, screwed over a brass flange cemented to the ends of the cylinder, washers are interposed at the points of contact so as to make the cylinder absolutely air-tight, a brass tube with a stop-cock is soldered to each cap. One of the tubes is widened out and fitted with a thread, so as to screw to a brass piece fixed to a bottle of a capacity of a little over 100 c.c.; the connection being secured air-tight with a washer (a drawing of the instrument is given on Plate 2), the bottle is closed by a wide brass cap screwing upon it perfectly air-tight, so that the bottle can be readily washed out and dried with a towel. The air is drawn from the bell-jar into the dry glass cylinder by means of a pump, the cylinder being first exhausted and then filled with the air from the bell-jar. The operation is repeated eight or ten times, when a weight being placed on the bell-jar 10 or 12 litres more of expired air are driven through the cylinder, which is closed by turning the tap while the air is in transit. Next, the cylinder still attached by india-rubber tubing to the pipe leading from the bell-jar is immersed in water held in a cylindrical tin vessel

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[*] An experiment is quoted here in the abstract of this paper ('Proceedings,' vol. 46, p. 344), towards testing the accuracy of the method of breathing in the bell-jars. It was found subsequently, on repeating it a large number of times, that the results obtained varied to such an extent, from some cause connected exclusively with the experiment, as to make it useless with reference to the object in view. Hence it is omitted on the present occasion.

and the temperature of the water is recorded, corresponding to the temperature of the air to be analysed, the barometer is again read off. This, of course, is done after removing the weights from the bell-jar, and with the air in the bell-jar exactly under atmospheric pressure. Thus 1 litre of saturated air at a given temperature and under a known barometric pressure was obtained for analysis. The analysis was made by taking up 100 c.c. of the normal barium solution with a pipette and introducing it into the bottle, which was next screwed upon the cylinder and tightened with gas pliers. Then the stop-cock closing the cylinder next to the bottle was opened, and by turning the cylinder upside down the alkaline fluid was run into it from the bottle, when on shaking, the solution at once turned milky.

A number of experiments were made to find out how long the agitation of the cylinder had to be continued to ensure the complete combination of the carbonic acid. The experiments were performed both qualitatively and quantitatively. The qualitative test was made by shaking the air with the alkaline solution for a definite time, then aspiring this same air into a second similar cylinder in which a vacuum had been made, the displacement being effected over water. This air was tested for carbonic acid by shaking it with clear baryta water.

After agitating for 10 and 20 minutes the presence of carbonic acid in the second cylinder was still obvious, but after 30 minutes none was left, the solution of barium remaining perfectly clear after brisk agitation.

The quantitative experiment was made by determining the amount of carbonic acid absorbed by the barium hydrate from 1000 c.c. of the same stock of expired air after agitating the cylinder during 20 minutes, 30 minutes, or 40 minutes. The following results were obtained :—

After 20 minutes' agitation.   .   .   .   . CO_2 4·130 per cent. in air expired.
   „   30 minutes'   „   .   .   .   .   .   „   4·194   „   „
   „   40 minutes'   „   .   .   .   .   .   „   4·183   „   „

Therefore, after 20 minutes, there still remained a small portion of CO_2 unabsorbed; the very slight difference between the results obtained after 30 and 40 minutes is within the probable error of the analyses. As half an hour's agitation was objectionable from the length of time, experiments were made to ascertain whether this period could not be shortened by determining the CO_2 in the cylinder under a pressure exceeding that of the atmosphere.

For this purpose a cylinder was used into which a round graduated brass plunger was introduced—an instrument I had experimented with for some time while searching for a good volumetric method of estimating CO_2. The plunger moving through a close-fitting stuffing box was found to be quite air tight, and by depressing or raising it the pressure in the cylinder could be increased or decreased by the volume of the plunger introduced or withdrawn.

By this means the agitation could be made at will, either below the atmospheric pressure, at that pressure, or above it.

On depressing the plunger to 50 c.c. the air after complete combination was very nearly under atmospheric pressure. On depressing the plunger to 100 c.c. the pressure amounted to very nearly 38 mm. of mercury. On depressing it to 80 c.c. the pressure amounted to nearly 23 mm. of mercury.

It was found that the combination took place much more rapidly at a pressure exceeding that of the atmosphere than below or at the atmospheric pressure.

In ordinary circumstances the absorption of 50 c.c. in the cylinder, from expired air under analysis, would cause a vacuum or fall of pressure of about 38 mm. of mercury. This circumstance militates strongly against complete absorption, but by increasing the pressure as stated above to 23 mm. over and above the atmospheric pressure (after absorption), it was found that after agitating for 10 minutes no CO_2 was left uncombined, and with a pressure of 38 to 40 mm. the combination appeared complete after agitating for 5 minutes.

In the course of the present inquiry the use of the plunger was given up,[*] and the air in the cylinder was compressed by driving into it about 100 c.c. of common air, free from carbonic acid. This was done with a pear-shaped vulcanised india-rubber bag attached to the end of one of the terminal tubes, as shown in the accompanying drawing. The air was forced into the cylinder with the hand, the stopcock again closed and the bag removed. Thus, a volume of about 100 c.c. of air, free from CO_2 (passed through pumice stone moistened with potassium hydrate), was introduced into the cylinder, producing after combination a pressure of about 38 mm. of mercury. After shaking for a quarter of an hour, the bottle containing the milky alkaline solution was unscrewed from the cylinder and its contents were rapidly poured into a dry glass-stoppered bottle of a capacity of about 100 c.c., which was sealed with paraffin. The next day the precipitate had separated, and the clear fluid was titrated with a solution of oxalic acid of such a strength that 1 c.c. corresponded to 0·001 mgr. CO_2.

Fourteen experiments were undertaken to test the accuracy of this analytical method. In each experiment two analyses were made from the same stock of air in the bell-jar, and the accuracy of the results goes far to show how well the bell-jar retained its equilibrium with the atmospheric pressure. The results are tabulated as follows :—

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[*] Some determinations of CO_2, made with the plunger, by PETTENKOFER's method, have required a small correction owing to the plunger coming out slightly greasy from the stuffing box. A number of experiments showed the correction to amount to a mean of 1 per cent.

Found Carbonic Acid as expired per minute.

Gramme.	Difference.		Per cent.
0·3905 } 0·3881 }	0·0024	=	0·61
0·4581 } 0·4571 }	0·0010	=	0·22
0·4228 } 0·4203 }	0·0015	=	0·35
0·5108 } 0·5087 }	0·0021	=	0·41
0·4403 } 0·4393 }	0·0019	=	0·25
0·4625 } 0·4652 }	0·0027	=	0·58
0·4015 } 0·4007 }	0·0008	=	0·20
0·3630 } 0·3641 }	0·0011	=	0·30
0·4263 } 0·4243 }	0·0020	=	0·47
0·4348 } 0·4345 }	0·0003	=	0·07
0·4246 } 0·4234 }	0·0012	=	0·28
0·4011 } 0 4006 }	0·0005	=	0·12
0·4033 } 0·4046 }	0·0013	=	0·32
0·3766 } 0·3758 }	0·0008	=	0·21
Means . .	0·0013	=	0·31

The following precautions were taken in these analyses as necessary to ensure absolutely correct results by PETTENKOFER's method :—

1. The distilled water used for diluting the solution of barium hydrate (100 to 200 c.c.), was thoroughly boiled, and kept in glass stoppered Winchester quarts. Before using this water it was carefully tested and rejected if one drop of the dilute alkaline solution (25 c.c. normal barium solution in 100 c.c. aq.) failed to show a reaction in about 30 c.c. of the distilled water coloured with two or three drops of solution of turmeric.

2. The same burette was used in every experiment, and every titration was commenced at 0 of the burette scale, the burette being always refilled.

3. The air drawn into the burette by displacement of the fluid was filtered through a tube containing pumice stone moistened with a saturated solution of potassium hydrate in very dilute glycerine, and connected with the burette by a cork and india-rubber tubing. Thus none of the atmospheric carbonic acid could reach the solution in the burette.

4. During the titration air blown from a bellows over pumice stone, moistened with potassium hydrate in dilute glycerine, was driven through the wide-mouth bottle in which the titration was being made, so that atmospheric carbonic acid never came in

contact with the contents of the flask ; this precaution was found absolutely necessary to insure accurate results.

5. The analyses of the alkaline solution were made the day after the experiment except when a Sunday intervened. I have found it necessary not to delay the analyses of atmospheric air, a precaution perhaps less important when a large proportion of CO_2 is present, as occurs in air expired from the lungs.

6. The standard solution of oxalic acid used in PETTENKOFER's method usually undergoes a change after about a fortnight, when mould is formed in the fluid. In order to avoid any error from this cause, the solution was prepared very carefully by heating the dry flask in which it was to be made to a sufficient temperature to destroy the bacteria, and allowing it to cool with its opening plugged with cotton wool. The weighed oxalic acid was rapidly transferred to the flask, and the solution made with fresh boiled distilled water. The solution prepared in this way keeps for a much longer time unaltered in strength, than if these precautions are not taken.

There is practically I believe, only one source of error in PETTENKOFER's method of analysis—from the uncertainty of the acidity of the oxalic acid used, as equal weights of different samples do not always give exactly the same degree of acidity. This source of error can be easily obviated by substituting hydrochloric acid for oxalic acid. The analyses referred to in this paper have all been made with oxalic acid from the same stock kept in a glass stoppered bottle.

In order to make quite certain of the strength of the solution, a standard solution of pure hydrochloric acid was prepared by titrating it with a weighed quantity of anhydrous sodium carbonate. The alkalinity of the barium solution was then determined with the standard oxalic acid, and also with the standard hydrochloric acid, while a known volume of the same barium solution was precipitated with sulphuric acid, and determined by weight as barium sulphate. The results obtained were as follows :—

With oxalic acid 100 c.c. barium solution were found equivalent to . . . . 0·2683 grms. CO_2
With hydrochloric acid, 100 c.c. . . . . 0·2675 „
By weight, 100 c.c. . . . . 0·2679 „

It was therefore obvious that the oxalic acid used in these analyses was quite pure, and moreover that the barium hydrate was uncontaminated with any other alkali.

It has been long known that a certain proportion of the oxygen of the air inspired is not returned in the expired air as carbonic acid, and REGNAULT and REISET in their admirable experiments[*] have shown that a portion of the oxygen inhaled is occluded in the body. While experimenting on dogs they found the mean relation between the weight of the oxygen expired and the total weight of oxygen consumed

[*] 'Annales de Chimie et de Physique,' 1849, vol. 26.

to be 0·744. I have subjected human respiration to experiments of a similar description, and obtained results approximating very much to those arrived at by REGNAULT and REISET, but the inquiry was beset with difficulties, and the figures obtained for occluded air do not agree close enough to justify their being quoted on this occasion. We may so far accept an occlusion of from 1·8 to 2 per cent. of the air inspired; therefore, the volumes of air *expired*, as given in this paper, increased by that proportion, will yield very nearly the corresponding volumes of air inspired. Since, however, the air *expired* has only been taken into account in the present inquiry, the results may safely be accepted as correct *relatively to each other* without introducing any correction.

The calculations these experiments necessitated were simple, though somewhat laborious, until I had completed with the aid of Mr. TOWNSEND a table for reduction of volumes, which was found of the greatest use. As I am not aware of such a table of figures being in existence, and as it may be useful to others, it is included in the present communication.

The horizontal headings are Barometer readings from 740 to 780 mm., and the vertical headings are the temperatures from 10° C. to 25° C. inclusive. The reduction of one litre (1000 c.c.) to 0° C. and 760 mm. is given for every Barometer reading from 740 to 780 mm., and every temperature from 10° C. to 25° C.

The correction for fractions of millimetres is *additive* and obtained by multiplying the fraction by the constant 1·2 c.c., or perhaps more correctly 1·25 c.c. The correction for fractions of degrees Centigrade is *subtractive*, and obtained by multiplying the fraction by the constant 4·3. The table was made with the usual formula,

$$V_0 = \frac{Vt(p-f)}{760(1+\alpha t)}.$$

Every second or third column of reductions was calculated from the formula, and the intervening ones were calculated with constants. The figures obtained by the use of constants were tested in a number of cases by calculation from the formula and found to be correct within 0·1 per cent. With the assistance of these tables the calculation of a whole analysis could be made easily within a quarter of an hour, in some cases ten minutes were found sufficient for the purpose.

The carbonic acid in the laboratory air was taken into account, though not determined in direct connection with the present experiments. A number of analyses of this air made on previous occasions showing that the mean CO_2 present could be taken in clear weather at 5 parts in 10,000, and in dull weather at 7 parts in 10,000, the carbonic acid expired from the lungs was in every case corrected according to these data.

TABLE for the reduction to 0° C. and 760 mm. of 1 litre of air saturated with humidity, from 10° to 25° C., and 740 to 780 mm.

Barometer.

Tempera-ture.	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759	760
10	927·7	928·9	930·2	931·4	932·6	933·9	935·1	936·4	937·6	938·9	940·4	941·6	942·9	944·2	945·4	946·6	947·9	949·2	950·5	951·8	953·0
11	923·6	924·8	926·1	927·3	928·5	929·8	931·0	932·3	933·5	934·8	936·3	937·5	938·8	940·1	941·3	942·5	943·8	945·1	946·4	947·7	948·9
12	919·5	920·5	921·8	923·0	924·2	925·5	926·7	928·0	929·3	930·5	931·8	933·1	934·3	935·6	936·8	938·0	939·4	940·7	942·0	943·3	944·4
13	915·4	916·7	918·0	919·2	920·4	921·7	922·9	924·2	925·4	926·7	928·0	929·2	930·4	931·7	932·9	934·1	935·5	936·8	938·1	939·4	940·5
14	911·3	912·6	913·9	915·1	916·3	917·6	918·8	920·1	921·3	922·6	923·8	925·0	926·2	927·6	928·8	930·0	931·3	932·6	933·9	935·2	936·3
15	907·1	908·4	909·7	910·9	912·1	913·4	914·6	915·9	917·1	918·4	919·6	920·8	922·0	923·3	924·5	925·7	927·1	928·4	929·7	931·0	932·0
16	902·9	904·2	905·5	906·7	907·9	909·2	910·4	911·7	912·9	914·2	915·4	916·6	917·8	919·1	920·3	921·5	922·8	924·1	925·4	926·7	927·8
17	898·7	900·0	901·3	902·5	903·7	905·0	906·2	907·5	908·7	910·0	911·1	912·3	913·5	914·8	916·0	917·2	918·5	919·8	921·1	922·4	923·5
18	894·5	895·8	897·1	898·3	899·5	900·8	902·0	903·3	904·5	905·8	906·8	908·0	909·2	910·5	911·8	913·0	914·2	915·5	916·8	918·1	919·2
19	890·2	891·5	892·7	893·9	895·1	896·4	897·6	898·9	900·1	901·2	902·5	903·7	904·9	906·2	907·4	908·6	909·9	911·2	912·5	913·8	914·8
20	885·9	887·2	888·4	889·6	890·8	892·1	893·3	894·6	895·8	897·1	898·1	899·3	900·5	901·7	902·9	904·1	905·3	906·5	907·7	908·9	910·4
21	881·8	883·0	884·3	885·5	886·7	888·6	889·2	890·5	891·7	893·0	894·0	895·2	896·4	897·7	898·9	900·1	901·3	902·6	903·9	905·2	906·2
22	877·1	878·3	879·5	880·7	881·9	883·2	884·4	885·7	886·9	888·2	889·0	890·2	891·4	892·9	894·1	895·3	896·6	897·9	899·2	900·5	901·4
23	872·6	873·8	875·0	876·2	877·4	878·7	879·9	881·2	882·4	883·7	884·7	885·9	887·1	888·4	889·6	890·8	892·0	893·3	894·6	895·9	896·9
24	868·1	869·3	870·6	871·8	873·0	874·3	875·5	876·8	878·0	879·3	880·1	881·3	882·5	883·8	885·0	886·2	887·5	888·8	890·1	891·4	892·3
25	863·5	864·7	865·9	867·1	868·3	869·6	870·8	872·1	873·3	874·6	875·7	876·9	878·1	879·3	880·5	881·7	882·9	884·2	885·5	886·8	887·9

Barometer.

Tempera-ture.	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779	780	
10	954·3	955·6	956·8	958·0	959·3	960·6	961·8	963·1	964·4	965·7	967·0	968·3	969·6	970·8	972·1	973·3	974·6	975·9	977·1	978·4	—
11	950·2	951·5	952·7	953·9	955·2	956·5	957·7	959·0	960·3	961·6	962·8	964·1	965·4	966·6	967·9	969·1	970·4	971·7	972·9	974·2	—
12	945·7	947·0	948·2	949·4	950·7	951·9	953·1	954·4	955·7	957·0	958·2	959·5	960·8	962·0	963·3	964·5	965·8	967·1	968·3	969·6	—
13	941·8	943·1	944·3	945·5	946·8	948·1	949·3	950·6	951·9	953·1	954·3	955·6	956·9	958·1	959·4	960·6	961·9	963·2	964·4	965·7	—
14	937·6	938·9	940·1	941·3	942·6	943·8	945·0	946·3	947·6	948·8	950·1	951·3	952·6	953·8	955·1	956·3	957·5	958·8	960·0	961·3	—
15	933·3	934·6	935·8	937·0	938·2	939·5	940·7	942·0	943·2	944·4	945·8	947·0	948·2	949·6	950·8	952·0	953·2	954·5	955·7	957·0	—
16	929·1	930·4	931·6	932·8	934·0	935·2	936·4	937·7	938·9	940·1	941·4	942·6	943·9	945·2	946·4	947·6	948·8	950·1	951·3	952·6	—
17	924·7	926·0	927·2	928·5	929·7	930·9	932·1	933·4	934·6	935·8	937·1	938·3	939·6	940·9	942·1	943·3	944·5	945·8	947·0	948·3	—
18	920·4	921·7	922·9	924·2	925·4	926·6	927·8	929·1	930·3	931·5	932·7	933·9	935·2	936·5	937·7	938·9	940·1	941·4	942·6	943·9	—
19	916·0	917·2	918·4	919·7	920·9	922·2	923·4	924·7	925·9	927·1	928·3	929·5	930·8	932·0	933·2	934·4	935·6	936·9	938·1	939·4	—
20	911·6	912·8	914·0	915·2	916·5	917·7	918·9	920·2	921·4	922·6	923·8	925·0	926·3	927·5	928·8	930·0	931·2	932·5	933·7	935·0	—
21	907·4	908·6	909·9	911·1	912·3	913·5	914·7	916·0	917·2	918·4	919·6	920·8	922·1	923·3	924·5	925·7	926·9	928·2	929·4	930·7	—
22	902·6	903·8	905·0	906·3	907·5	908·7	909·9	911·2	912·4	913·6	914·8	916·0	917·2	918·4	919·7	920·9	922·1	923·4	924·6	925·7	—
23	898·0	899·2	900·4	901·7	902·9	904·1	905·3	906·6	907·8	909·0	910·2	911·4	912·6	913·8	915·1	916·3	917·5	918·8	920·0	921·3	—
24	893·4	894·6	895·8	897·1	898·3	899·5	900·7	902·0	903·2	904·4	905·6	906·8	908·0	909·2	910·4	911·6	912·8	914·0	915·2	916·5	—
25	888·9	890·1	891·3	892·6	893·8	895·0	896·2	897·4	898·6	899·8	900·9	902·1	903·3	904·5	905·7	906·9	908·1	909·3	910·5	911·7	—

The first series of experiments refers to WILLIAM ALDERWOOD, aged 23, a strong man in excellent health, with a well developed chest measuring 83 centims. in circumference at nipples; he weighs 8st. 1lb. without clothes. Seventy-four complete experiments were made in his case, but as they extended from the winter into the spring, the temperature of the laboratory (heated by hot water pipes) underwent rather great changes during that period; this circumstance was found to exert a decided influence on his respiration, the cold reducing the volumes of air breathed for 1 grm. CO_2 expired; hence it became necessary to divide the 74 experiments into two groups, one group including those made at a laboratory temperature varying from 14° to 17°·3 C., and the other group including those made at a temperature varying from 17°·3 to 21°·7 C.

The first group numbered 25 experiments, and the second 49 experiments. The results are given in the form of tables, the means of which have been made into two charts or series of curves (see Plate 1). Each chart shows three curves, one for the carbonic acid, one for the volumes of air expired, and one for the relation between the volumes and the corresponding weights of carbonic acid.

[In relatively cold weather (14° to 17°·3, see table, p. 19) it is difficult to define the precise hour during which the maximum carbonic acid is expired, as the proportion exhaled is relatively high from 0 to 1 hour, 1 to 2 hours, and 2 to 3 hours after food; the actual maximum is from 2 to 3 hours after food, after which time the fall is marked and continued. In relatively warm weather (17° to 21°·7, see table, p. 21) it is also difficult to assign the period for the maximum expiration of carbonic acid, the highest figure is between 1 and 2 hours after a meal, but the fall does not begin till the fourth hour. It may be said that with this person under experiment, the expiration of carbonic acid begins to lessen somewhat earlier after food in cold weather than in warmer weather; but it would be premature to apply this result in a general way.

The chart for the *comparatively cold weather* (WILLIAM ALDERWOOD under experiment) shows distinctly, from an inspection of the curved line marked CO_2, that the carbonic acid expired is high (with a slight fall from 1 to 2 hours after food) till from 2 to 3 hours after food, after which time the fall is rapid and regular till from 5 to 6 hours after food, when the experiments come to an end. In this same chart the curve for the volumes of air expired in comparatively cold weather (see the dotted line) coincides very closely with the curve for CO_2, from 2 to 3 hours till from 3 to 4 hours, and from 4 to 5 hours till from 5 to 6 hours, after food; then, while the line for CO_2 falls fast and regularly, that for volumes of air expired remains nearly horizontal for an hour (from 3 to 4 hours till from 4 to 5 hours after food); this means that the volumes of air are increasing in proportion to the weight of CO_2 they contain, hence the curve for relations shoots up rapidly; but in the following hour (from 4 to 5 till from 5 to 6) the curves for volume and CO_2 show again a marked tendency to coincide.

In the chart for the *relatively warm weather* (WILLIAM ALDERWOOD under experi-

ment) the curve for CO_2 exhibits a slight rise during the first hour after food, then a slight fall and again a slight rise, the maximum being between 1 and 2 hours after food; but the CO_2 remains high till between 3 and 4 hours after food, while by that time the curve was falling rapidly in colder weather. After a period, from 3 to 4 hours after food, the curve falls at somewhat the same speed as it does in colder weather. In this same chart the curves for the volumes of air and CO_2 expired in comparatively warm weather (see the dotted line) coincide very closely with each other from the time food is taken, till from 3 to 4 hours later; then, while the curve for CO_2 continues its regular fall, that for volumes of air expired diverges, crossing the line for CO_2. The effect on the curve for relations is to make it shoot up all but precisely to the same extent as in the corresponding curve for warmer weather. This circumstance certainly shows that the phenomenon is not accidental, but is actually the rule, at all events in the present case. A similar change of coincidence in the curves for CO_2 and volumes of air expired is observed in the other person who submitted to experiment.—*October* 23, 1889.]

WILLIAM ALDERWOOD under experiment.—Table showing (1st) Weight of Carbonic Acid expired per minute; (2nd) Volume of Air expired per minute reduced to 0° and 760 mm.; (3rd) Relation of Volume of Air expired (reduced) to Weight of CO_2. *Temperature of the Laboratory Air at the time of experiment from* 14° *to* 17°·3 *C.*

Weight of Carbonic Acid expired per minute in every hour up to six hours after a meal.

Hours after a meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
0·460	0·485	0·512	0·461	0·444	0·397
0·431	0·435	0·484	0·445	0·400	0·376
0·483	0·410	0·414	0·429	0·405	
		0·395	0·402	0·385	
		0·471	0·423	0·392	
		0·482			
		0·505			
Means 0·458	0·443	0·466	0·432	0·405	0·386

Volume of Air (reduced) expired per minute.

Hours after a meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
3·946	4·402	4·243	4·033	3·984	3·635
3·782	3·924	4·293	3·886	3·883	3·629
4·166	3·800	3·678	3·734	3·700	
		3·543	3·594	3·682	
		4·209	3·845	3·934	
		4·205			
		4·527			
Means 3·965	4·042	4·100	3·818	3·837	3·632

Relation between Volume of Air expired (reduced) and Weight of Carbonic Acid ; i.e., Volume of Air expired for 1 grm. CO_2.

Hours after a meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
8·58	9·08	8·29	8·75	8·97	9·16
8·78	9·02	8·87	8·73	9·71	9·65
8·63	9·27	8·88	8·70	9·13	
		8·97	8·94	9·56	
		8·94	9·09	10·04	
		8·72			
		8·96			
Means 8·66	9 12	8·80	8·84	9·48	9·40

WILLIAM ALDERWOOD under experiment.—Table showing (1st) Weight of Carbonic Acid expired per minute ; (2nd) Volume of Air expired per minute reduced to 0° and 760 mm. ; (3rd) Relation of Volume of Air expired (reduced) to Weight of CO_2. *Temperature of the Laboratory Air at the time of experiment from 17°·3 to 21°·7 C.*

Weight of Carbonic Acid expired per minute in every hour up to six hours after a meal.

Hours after a meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
0·421	0·458	0·467	0·465	0·397	0·388
0·456	0·469	0·436	0.489	0·385	0·358
0·472	0·456	0·420	0·438	0·387	0 384
0·430	0·448	0·411	0·417	0·386	0·376
0·424	0·438		0·424	0·370	0·380
	0·440		0·469	0·356	0·352
			0·400	0·381	0·384
				0·395	0·365
				0·442	0·390
				0·356	0·371
				0·383	0·344
				0·368	0·360
					0·349
					0·338
					0·378
Means 0·440	0·451	0·433	0·443	0·384	0·368

Volume of Air (reduced) expired per minute.

Hours after a meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
3·933	4·020	4·171	4·152	3·803	3·776
4·161	4·340	4·120	4·477	3·673	3·532
4·389	4·354	3·969	4·041	3·722	3·989
4·049	4·263	3·754	3·856	3·782	3·769
4·036	4·169		3·965	3·704	3·681
	4·115		4·269	3·477	3·378
			3·878	3·676	3·684
				3·817	3·486
				4·185	3·879
				3·484	3·796
				3·701	3·582
				3·704	3·434
					3·381
					3·191
					3·701
Means 4·114	4·210	4·003	4·091	3·727	3·617

Relation between Volume of Air expired (reduced) and Weight of Carbonic Acid; i.e., Volume of Air expired for 1 grm. CO_2.

Hours after a meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
9·34	8·78	8·93	8·94	9·58	9·73
9·12	9·25	9·45	9·16	9·54	9·87
9·29	9·55	9·45	9·23	9·62	10·39
9·45	9·52	9·13	9·25	9.80	10·03
9·51	9·52		9·35	10·01	9·69
	9·35		9·27	9·77	9·60
			9·69	9·65	9·55
				9·66	9·95
				9·47	10·23
				9·78	10·41
				9·66	9·54
				10·07	9·69
					9·44
					9·79
					9·60
Means 9·34	9·33	9·24	9·27	9·72	9·83

Influence of Atmospheric Pressure on Respiration.

[In order to make the influence of atmospheric pressure on respiration perfectly clear to the reader, the experiments have been disposed in the form of two tables. In the first table they are divided into two groups (under the form of means) of an equal number, the first group including the experiments attended with the highest barometer readings, the second including those attended with the lowest readings.   In the second table (pp. 23 and 24) each of the groups of experiments considered (under the form of means) in the preceding table, has been divided into two sub-groups of an equal number of experiments, each of the sub-groups for the *extreme* highest and *extreme* lowest pressures only being entered (under the form of means).— *October* 23, 1889.]

It is certainly unexpected to find that local changes of atmospheric pressure have a most positive and marked influence on respiration, the effect being of the same kind as that I had observed previously in my experiments at increasing altitudes.   In order to show the action of the variations of local pressures on respiration it was necessary to eliminate as much as possible the influence of food, and with this object in view the experiments made every *two hours* after a meal were grouped together, and the pressures and corresponding relations for each group placed in a tabular form.

TABLE showing that the Fall of the Barometer is attended by a Reduction in the Volume of air Expired (reduced) for 1 grm. CO_2 (temperatures 17°·3 C. to 21°·7 C.), WILLIAM ALDERWOOD under Experiment. The experiments are divided into two groups—those attended with the highest barometer readings and those with the lowest.

0 *to* 2 *Hours.*

	mm.		Litres.
1st group.—Mean highest barometer .	765·9	Relations (*i.e.*, vol. air expired per 1 grm. CO_2)	9·423
2nd „ Mean lowest barometer .	754·1	„	9·228
Difference	11·8	Fall for 11·8 mm. =	0·195

Number of Experiments, 11.

2 *to* 4 *Hours.*

	mm.		Litres.
1st group.—Mean highest barometer .	767·0	Relations	9·322
2nd „ Mean lowest barometer .	747·7	„	9·180
Difference	19·3	Fall for 19·3 mm. =	0·142

Number of Experiments, 11.

4 *to* 6 *Hours.*

	mm.		Litres.
1st group.—Mean highest barometer .	766·4	Relations	9·838
2nd „ Mean lowest barometer .	746·2	„	9·684
Difference	20·2	Fall for 20·2 mm. =	0·154

Number of Experiments, 27.

EXTREMES. (Temperatures, 17°·3 to 21°·7.)

0 *to* 2 *Hours after Food.*

The three Highest and three Lowest Pressures and corresponding Relations.

	mm.		Litres.
1st group.—Mean highest barometer .	769·5	Relations	9·517
2nd „ Mean lowest barometer .	751·5	„	9·080
Difference	18·0	Fall for 18 mm. =	0·437

2 *to* 4 *Hours after Food.*

The three Highest and three Lowest Pressures and corresponding Relations.

		mm.		Litres.
1st group.—Mean highest barometer .	768·8	Relations	9·270
2nd „ Mean lowest barometer .	741·7	„	9·260
Difference	27·1	Fall for 27·1 mm. =	0·010

4 *to* 6 *Hours after Food.*

The six Highest and six Lowest Pressures and corresponding Relations.

		mm.		Litres.
1st group.—Mean highest barometer .	771·7	Relations	10·097
2nd „ Mean lowest barometer .	738·0	„	9·665
Difference	33·7	Fall for 33·7 mm. =	0·432

Therefore, from the present enquiry with reference to this person under experiment, and a similar remark applies to the other, the relations are invariably found to be lowered with falling pressures; or, in other words, with a falling atmospheric pressure the volume of air breathed to expire 1 grm. CO_2 is lessened.   This is shown very clearly by a consideration of the ultimate results obtained from the present table, these are as follows :—

From the Collective Readings.

	Fall of barometric pressure.	Fall in the volume of air breathed for 1 grm. CO_2.
	mm.	Litres.
0 to 2 hours after food . .	11·9	0·195
2 to 4 hours after food . .	19·3	0·142
4 to 6 hours after food . .	20·2	0·154
Mean	17·1	0·164

Therefore for a mean fall of 17·1 mm. pressure there was a mean reduction of 0·164 litre in the air breathed to expire 1 grm. CO_2, amounting to a reduction of 0·0960 litre in the volume of air breathed for a fall of pressure of 10 mm.

The extreme readings for barometer and relations (extreme halves of each group) gave much greater differences except in one case, to which special reference will be made ; they are as follows :—

	Fall of barometric pressure.	Fall in the volumes of air expired for 1 grm. CO_2.
	mm.	Litres.
0 to 2 hours after food . .	18·0	0·437
2 to 4 hours after food . .	27·1	0·010
4 to 6 hours after food . .	33·7	0·432
Mean	26·3	0·293

Therefore, for a mean fall of 26·3 mm. pressure, there was a mean reduction of 0·293 litre in the air breathed to expire 1 grm. CO_2, amounting to a reduction of 0·111 litre for 10 mm. pressure. The small figure, 0·010, for the fall of relation from 2 to 4 hours after food, shows that the influence of food appears to interfere with the action of atmospheric pressure. It will now be seen that the fall in the volume of air breathed (for 1 grm. CO_2) is very nearly the same, whether the barometer readings (for every two hours) be taken collectively, or whether the extreme readings only be considered. If the barometer readings be taken collectively, a fall of volume of 0·0960 litre is observed for 10 mm. pressure; if the extreme barometric readings only be considered, the fall of volume will amount to 0·111 litre for every 10 mm. pressure.

The experiments made at a comparatively low atmospheric temperature give similar results as to the effect of barometric pressures, they are, however, too few in number to allow of their being submitted to a special consideration.

The experiments for barometer readings and corresponding volumes of air for the expiration of 1 grm. CO_2 for each person under experiment have been disposed under the form of curves. These curves show that there is a general tendency to a fall of relations with a subsidence of barometric pressure, but from 2 to 4 hours after a meal the influence of food appears to overcome to a certain extent the effect of the barometer, the axis of the curve for relations being nearer to a horizontal line.

The second series of experiments refers to C. F. TOWNSEND, aged 23 years, and weighing 8 stone without clothes; no necessity was found to divide them on account of temperatures; they amount to 56 in number as reported in detail in the accompanying Table (pp. 26 and 27). It will be seen that the maximum weight of carbonic acid (0·439 grm.) and maximum volume of air expired (4·488 litres) occur from 1 to 2 hours after a meal; [but the excess of CO_2 and air expired from 1 to 2 hours after a meal over that expired from 0 to 1 and 2 to 3 hours after food, is so small that the period for the maximum effect is very difficult to define. It has been stated in the abstract of this paper ('Proceedings,' vol. 46) as one of the conclusions arrived at, that these researches confirm " the known usual influence of food on the formation of carbonic acid in the body, the maximum expired occurring between two or three hours after a meal . . ." This result must be accepted in a general way only, as the effect of food upon the formation and elimination of CO_2 appears to vary

with the individual. Moreover, according to my experiments, the influence of digestion towards increasing the formation of CO_2 in the body, which shows itself within the first hour after food has been taken, may last during one or two hours, or even longer, and it is very difficult to assign a period for its maximum effect.— January 4, 1890.]

Three experiments were made before breakfast, yielding a mean of 0·343 grm. CO_2 per minute, and 3·157 litres of air expired.

The curves (Plate 2) for volumes and carbonic acid have not such a parallel course as in the case of WILLIAM ALDERWOOD. Throughout a period of two hours after food the curve for relations is nearly horizontal, showing that the volumes of air and weights of carbonic acid closely follow each other during that time. Then the curve shoots up gradually to the end of the chart. While the line for carbonic acid is seen to fall rather rapidly, the volumes of air expired remain nearly the same.

TABLE showing (1st) Weight of Carbonic Acid expired per minute ; (2nd) Volume of Air reduced to 0° and 760 mm. expired per minute ; (3rd) Relation of Volume of Air expired (reduced) to Weight of CO_2.—C. F. TOWNSEND under experiment.

Weight of Carbonic Acid expired per Minute in every Hour up to 6 Hours after a Meal.

Hours after a Meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
0·465	0·506	0·463	0·388	0·414	0·402
0·411	0·442	0·455	0·425	0·435	0·409
0·451	0·383	0·419	0·397	0·401	0·360
0·445	0·440	0·429	0·386	0·425	0·414
0·398	0·470	0·447	0·397	0·403	0·356
0·430	0·401	0·413	0·462	0·373	
0·401	0·428	0·434	0·405		
	0·406	0·439	0·413		
	0·450	0·440	0·350		
	0·450	0·420	0·440		
	0·454	0·454	0·416		
		0·396	0·394		
		0·372			
		0·410			
		0·431			
Means 0·429	0·439	0·428	0·404	0·410	0·388

Volume of Air reduced expired per Minute.

Hours after a Meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
5·080	5·305	4·853	3·937	4·725	4·397
4·175	4·570	4·864	4·583	4·626	4·784
4·572	3·926	4·454	4·449	4·204	4·239
4·424	4·441	4·673	3·994	4·515	4·397
4·163	4·861	4·688	3·603	4·224	4·121
4·197	4·185	4·234	5·069	4·083	
4·012	4·275	4·336	4·369		
	4·115	4·496	4·465		
	4·644	4·515	4·167		
	4·592	4·289	4·557		
	4·454	4·427	4·392		
		3·973	4·047		
		3·960			
		4·425			
		4·474			
Means 4·390	4·488	4·444	4·303	4·390	4·388

Relation between Volume of Air expired (reduced) and Weight of Carbonic Acid;
i.e., Volume of Air expired for 1 Grm. CO_2.

Hours after a Meal.					
0 to 1.	1 to 2.	2 to 3.	3 to 4.	4 to 5.	5 to 6.
10·92	10·48	10·48	10·15	11·41	10·94
10·14	10·35	10·68	10·65	10·63	11·69
10·13	10·27	10·63	11·20	10·48	11·77
9·94	10·09	10·90	10·38	10·62	10·62
10·45	10·24	10·48	9·56	10·48	11·57
9·76	10·43	10·25	11·18	10·95	
10·00	9·98	9·99	10·79		
	10·13	10·25	10·82		
	10·31	10·26	11·86		
	10·20	10·21	10·37		
	9·79	9·73	10·56		
		10·07	10·27		
		10·65			
		10·80			
		10·39			
Means 10·19	10·22	10·38	10·65	10·76	11·32

TABLE showing that the Fall of the Barometer is attended by a reduction in the Volume of Air expired (reduced) for 1 Grm. CO$_2$. (C. F. TOWNSEND under Experiment.)

BAROMETER READINGS AND RELATIONS FROM ALL THE EXPERIMENTS DIVIDED INTO TWO GROUPS.

0 *to* 2 *Hours after Food.*

Litres.

Mean highest barometer . . . . . 746·5   Relations (*i.e.*, vol. air expired for
                                                            1 grm. CO$_2$) . . . . 10·273
Mean lowest barometer . . . . . 750·4   ,, ... 10·139
    Difference . . . . . 14·1   Fall for 14·1 mm. = 0·134
Number of Experiments, 18.

2 *to* 4 *Hours after Food.*

Litres.

Mean highest barometer . . . . . 766·7   Relations 10 533
Mean lowest barometer . . . . . 753·9   ,, ... 10·412
    Difference . . . . . 12·8   Fall for 12·8 mm. = 0·121
Number of Experiments, 27.

4 *to* 6 *Hours after Food.*

Litres.

Mean highest barometer . . . . . 761·2   Relations . . . . 11·157
Mean lowest barometer . . . . . 752·0   ,, ... 10·767
    Difference . . . . . 9·2   Fall for 9·2 mm. = 0·390
Number of Experiments, 11.

EXTREMES.

0 *to* 2 *Hours after Food.*

The Five Highest and Five Lowest Barometer Readings and Corresponding Relations.

Litres.

Mean highest barometer . . . . . 768·6   Relations . . . . 10·424
Mean lowest barometer . . . . . 747·9   ,, ... 10·118
    Difference . . . . . 20·7   Fall for 20·7 mm. = 0·306

2 *to* 4 *Hours after Food.*

The Seven Highest and Seven Lowest Barometer Readings and Corresponding Relations.

Litres.

Mean highest barometer . . . . . 770·9   Relations . . . . 10·503
Mean lowest barometer . . . . . 750·8   ,, ... 10·301
    Difference . . . . . 20·1   Fall for 20·1 mm. = 0·202

4 to 6 Hours after Food.

The Three Highest and Three Lowest Barometer Readings and Corresponding Relations.

		Litres.
Mean highest barometer . . . . . 763·4	Relations	11·544
Mean lowest barometer . . . . . 747·6	,, ...	10·957
Difference . . . . . 15·8	Fall for 15·8 mm. =	0·587

This Table shows a result corresponding with that obtained from similar experiments on WILLIAM ALDERWOOD. The fall of Barometric readings and corresponding relations taken collectively give the following figures :—

TAKEN COLLECTIVELY.

	Differences of barometric pressure.	Fall in the volumes of air expired for 1 grm. CO₂.
	mm.	Litres.
Fall of from 0 to 2 hours after food . .	14·1	0·134
,, from 2 to 4 hours after food . .	12·8	0·121
,, from 4 to 6 hours after food . .	9·2	0·390
Mean	12·0	0·215

Therefore, for a mean fall of 12 mm. pressure, there was a mean reduction of 0·215 litre of air breathed for 1 grm. CO_2 expired, amounting to a reduction of 0·179 for 10 mm.

The Table for extreme readings gives :—

	Differences of barometric pressure.	Fall in the volumes of air expired for 1 grm. CO₂.
	mm.	litres.
Fall of from 0 to 2 hours after food . . .	20·7	0·306
,, from 2 to 4 hours after food . . .	20·1	0·202
,, from 4 to 6 hours after food . . .	15·8	0·587
Mean.	18·9	0·365

Therefore, with the extreme readings, for a mean fall of 18·9 mm. of the barometer, there was a mean fall of 0·365 litre in the air breathed to expire 1 grm. carbonic acid, amounting to a reduction of 0·194 for 10 mm. This is very near to 0·179, the corresponding figure obtained for the barometer readings and relations taken collectively. Consequently, with one person under experiment, the mean reduction

for the experiments taken collectively amounted to 0·0942 litre for a fall of 10 mm.; and with another it amounted to 0·179 litre. It follows that the effect of falling atmospheric pressures varies in degree with different individuals. The curve shows distinctly, in the case of C. F. TOWNSEND, the same influence of a falling barometer on respiration as the curve for W. ALDERWOOD. From 0 to 2 hours after food, and from 4 to 6 hours after food, there is a distinct fall; from 2 to 4 hours after food the fall is still observed though less marked, showing again that food appears to lessen the influence of the atmospheric pressure on respiration.

The results obtained from the present inquiry are as follows :—

1. The phenomenon is confirmed that less air, reduced to 0° C. and 760 mm., is breathed at high than low altitudes, for the formation in the body of a given weight of carbonic acid.

[2. The experiments on W. ALDERWOOD certainly show that a local fall of temperature is attended with a slight reduction in the volumes of air breathed for the formation and expiration of 1 grm. CO_2, a phenomenon similar to that resulting from a fall of atmospheric pressure.—25th February, 1890.]

3. The influence of food, during digestion, on the formation of carbonic acid, commences within the first hour after a meal has been taken, and lasts for two or three hours, the period for the maximum amount varying throughout that time.

4. The influence of food on the *relation* between the volumes of air breathed and the corresponding weights of carbonic acid expired is clearly shown, the volumes following, more or less, the fluctuations in the carbonic acid; but the CO_2 expired has a marked tendency to fall more rapidly than the corresponding volumes of air as time elapses after a meal. The harmony of the tracings appears to recover itself, however, over night, and the lines are again nearly parallel before the first morning meal.

5. The variations of local atmospheric pressures have a marked influence on respiration, less air being taken into the lungs for the formation and emission of a given weight of carbonic acid under *lower* atmospheric pressures than under *higher* pressures; but this influence varies in degree according to different persons. In the present inquiry with two subjects experimented upon; in one case, for a fall of pressure of 10 mm. (0·394 inch), there was a mean reduction of 0·094 litre in the volume of air breathed for 1 grm. CO_2 expired; in the other case, the reduction was greater, and amounted to 0·179 litre.

6. The influence of atmospheric pressures on the volume of air breathed is apparently not the same throughout the whole day, being somewhat less marked from 2 to 4 hours after a meal, when the action of food may be considered at its maximum. Thus, digestion apparently reduces more or less, the effects of any local change of pressure on respiration.

The present investigation shows beyond doubt that different individuals breathe different volumes of air to burn in the body and expire a given weight of carbonic acid. The two persons experimented upon on the present occasion yielded respec-

tively a mean of 9·29 and 10·51 litres, while another expired a mean of 11·30 litres for 1 grm. carbonic acid. No doubt, the less the volume of air inspired for the combustion of a certain weight of carbon the more readily the oxygen taken into the lungs finds its way into the blood, and consequently the more perfect the action of the respiratory organs. In the present case, one of the subjects submitted to experiment was 60 years of age, and he breathed 11·30 litres, against 10·51 litres and 9·29 litres for young men both 23 years of age; although of the same weight, one stronger physically and with a more fully-developed chest breathed (expired) 9·29 litres, against 10·51 for the other, to burn the same weight of carbon in his body.

[33   ]

II. *On Out-lying Nerve-cells in the Mammalian Spinal-Cord.*

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[PLATES 3, 4.]

To describe the ganglion-cells of the Mammalian spinal cord as confined to the grey substance of the cord is not quite strictly correct. BEISSO* was the first to draw attention to the fact, that apart from axis-cylinder processes which pass into the ventral roots from cells of the ventral cornu, there project also from those cells of the cornu which lie next the white column other branches to mingle with the fibres of the bundles of the ventral nerve-roots. The ganglion-cells of the grey matter often, by one or more of their processes, jut partially into the white matter. The descriptions of BEISSO,† PICK,‡ and SCHIEFFERDECKER§ have further shown that in certain situations in the anterior and lateral columns, ganglion-cells lie outside the grey substance in the surrounding white matter.

Since GASKELL,‖ in 1885, drew attention to the ganglion-cells in the cord of Alligator, lying at the periphery of the antero-lateral column, and, of course, quite removed from the central grey matter, I have often searched in the cord of the Mammalia for evidence of similarly situated cells; always, however, without success. The search has, however, persuaded me that isolated ganglion-cells are no infrequent constituents of the white columns. The cords examined by me have been chiefly those of Man, the Monkey (Bonnet, Jew, and Rhesus), and the Dog. A number of sections have also been prepared from the Cat, Lion, Calf, Rat, Mouse, Rabbit, and Guinea-pig. The out-lying ganglion-cells in the white matter may conveniently be considered in three sections, according as their situation is within the anterior (ventral), the lateral, or the posterior (dorsal) white column respectively.

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* TORQUATO BEISSO, 'Del Midoll. Spinal.,' p. 37, 1873.
† *Loc. cit.*
‡ 'Archiv für Psychiatrie,' vol. 4.
§ 'Archiv Mikrosk. Anat.,' vol. 10.
‖ 'Proceedings of the Physiol. Soc.,' Dec., 1885.

Out-lying Cells in the Anterior Column.

Among the bundles of transverse fibres slanting from the mesial edge of the ventral horn toward the depth of the ventral fissure and the white commissure can, not infrequently, be found multipolar ganglion-cells. The cells somewhat resemble in form the cells of the ventral cornu itself, but I have not found them quite so large as are the latter. They are, in my specimens, obviously triradiate. The largest I have measured is 38 μ in diameter, from the middle of one side across the cell-body to the base of the opposite cell-process. Fig. 1, Plate 3, shows the position and form of one of these out-lying individuals, from the cervical cord of the Dog between the points of exit of the first and second nerves. The section is in the transverse plane of the cord. Fig. 2, Plate 4, gives an example of another such cell from the lower part of the lumbar enlargement of the Human cord. The cell does not lie far out from the limiting edge of the grey cornu. It is much smaller than a neighbouring cell belonging to the superficial region of the cornu. The cornual cell can be seen to give a branch into the same bundle of transverse fibres as harbours the smaller and out-lying cell. Similar cells I have observed to occur in both the cervical and in the lumbar regions of the cord, and have seen examples in the Monkey and Cat, as well as in Man and in the Dog. They have somewhat the appearance of being aberrant members of a group of cells which occurs scattered in more or less broken fashion along the deeper part of the mesial edge of the ventral grey horn, and best seen just above the lumbar enlargement. This group was called attention to in the Human cord by PICK,[*] and is characterised, according to him, by the fact that the cells give off large processes into the fibre-bundles of the white commissure.

The references to out-lying cells in the anterior column which I have been able to discover are three. The earliest by TORQUATO BEISSO in 1873 ; he figures a specimen, which was obtained from the cord of the Ox. He writes,[†] " La figura dimostra come una grossa cellula bipolare contenuta nell' intreciamento della commissura mandi un prolungamento che ripiegando si perde fra le fibre più mediane della sostanza grigia." A second reference is by SCHIEFFERDECKER.[‡] In speaking of the structure of the cord in the lumbar region of, apparently, the Canine cord, he writes that he has found occasional cells in the white commissure, sometimes bipolar, but more often " shaped like arrowheads." He was, it would seem, not aware of the observation by BEISSO. A third reference is by PICK,[§] who, after describing the cells of the ventral cornu which give off processes from the mesial edge of the cornu, goes on to speak of the occasional occurrence in the white commissure of the Human cord of cells similar to those noted by SCHIEFFERDECKER in the lumbar cord of the Dog. He concludes by

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[*] *Loc. cit.*
[†] *Loc. cit.*, p. 37.
[‡] VIRCHOW's ' Archiv,' vol. 67, p. 598.
[§] *Loc. cit.*

adding that such cells probably exist not merely in the lumbar region but in other regions as well. My own preparations confirm the description of Schiefferdecker. In them the cells are like arrow-heads in shape; one of the cell-processes, occasionally two, disappears into fibre-bundles in a direction toward the adjacent grey horn, and one process is directed toward the bottom of the ventral fissure, as if to pass with medullated nerve-fibres into the white commissure. One cell process may project, as, for instance, in fig. 2, in a median direction, neither toward the grey cornu nor the white commissure. This may pass into a fibre-bundle lying further ventrally than that which contains the parent cell, and in that may turn either toward the grey cornu, or, and more probably, toward the white commissure. It is to be remembered, however, that the direction at first taken by it is the same as that of numerous bundles of fine medullated fibres which radiate into the anterior white column from the ventral part of the mesial border of the ventral cornu, as if to reach the angle of white matter forming the lip of the ventral fissure. I have found the cells in the cervical as well in the lumbar region of the cord. It is difficult not to think that these isolated cells as in the anterior column are connected with the fibres among which they lie. Most of these appear certainly to pass between the mesial portion of the ventral horn and the opposite side of the cord in the white commissure. In bundles starting from the cornu further ventrally, it is not usual in one and the same section to see that the bundles pass actually into the commissure, although they slant in the required direction. It is possible that they take a longitudinal course within the white column. The view of Bidder[*] and his pupils[†] that the fibres of the white commissure run to nerve-cells in the ventral cornu, although opposed by Stilling[‡] and untrue in the sense in which Gerlach[§] advocated it, has much in its favour. In the white commissure are collected together fibres from and for manifold end-stations, and that some of these fibres are of the kind described by Bidder is rendered all the more probable by the presence of the outlying cells above-mentioned.

Out-lying Ganglion-Cells in the Lateral Column.

The remarkable group of nerve-cells discovered by Berger[||] in the cord of Alligator and allied forms still remains, so far as I am aware, an unexplained fact. I have, as already stated, never seen any unequivocal trace of it in the Mammalian cord. It may be that a vestige of it does really exist in the shape of a thickened rib in the sub-pial

* In a letter to R. Wagner in 1854. See Wagner's 'Neurolog. Untersuchungen,' 1854.

† Kupffer, 'De Medullae Spinalis Textura in Ranis,' 1854, p. 30; Owsjannikow, 'Disq. Microsc. de Med. Spin. textur. imprimis in Piscibus fact.,' 1854, p. 36.

‡ 'Neue Untersuch. ü. d. Bau des Rückenmarks,' 1859; also Frommann, 'Untersuch. ü. d. Normal. Anat. des Rückenmarks,' 1864 and 1867.

§ Stricker's 'Handbuch,' vol. 2.

|| "Ueber ein eigenthümliches Rückenmarks band einiger Reptilien und Amphibien." 'Sitzungsb. der Mat. Class. der Kaiserlich. Akademie zu Wien,' Feb., 1878.

membrane over the lateral column at a point which in the cervical cord lies nearer the exit of the ventral than that of the dorsal nerve-root, and in the lumbar and lower thoracic cord, is nearer to the dorsal than to the ventral nerve-root. This thickening occupies a shallow fissure, and is better seen in young animals than in adult. It is particularly evident in the lower thoracic and upper lumbar regions. In fully-grown Dogs, in the Monkey, and, to a less extent, in the Human adult, a small septum, more marked than any of the neighbouring, projects into the lateral column at the level of the ventral extremity of the area of the crossed pyramidal tract; it has repeatedly seemed to me that this septum is identical with the thickening of the sub-pial membrane found in the still imperfectly developed cord of the young animal. In cords in which the crossed pyramidal tract comes to form a large portion of the lateral column, it would seem that with the growth of the tract the shallow bay containing tissue continuous with the sub-pial membrane becomes, so to say, unfolded and more shallow, even to disappearance. But a septum thrusting itself into the lateral column at the spot records the site of its existence. That the septum is of real morphological significance, I believe, from the fact that it divides from one another two masses of nerve-fibres, which offer a strong contrast in their appearance. On the dorsal aspect of the septum lie coarse fibres of fairly equable size (cerebellar); on the ventral exists an admixture composed largely of fine fibres (ascending antero-lateral). In the Monkey, in which animal, as I have indicated,[*] a portion of the pyramidal tract lies outside the cerebellar along the surface of the lateral column, the fibres of the pyramidal tract, as marked out by the degeneration method, generally, when followed along the surface in the ventral direction, cease abruptly at the little septum in question. In some instances, especially in the upper cervical region, a few scattered fibres belonging to the pyramidal tract may be found on the ventral side of the septum; but this appears to be quite exceptional for the cord as a whole. But I have never found nerve-cells either in the septum or in the sub-pial band.

Into relation with the deeper region of the lateral column come cells of the lateral cornu, especially where that cornu is well developed. The small fusiform cells of the lateral horn are for the most part situate in that part of the horn which abuts upon the white substance of the column. Cells can very frequently be found not far within, but yet distinctly within,[†] the limits of the white matter surrounding the horn, most frequently on the dorsal aspect of the horn. There seems no room for doubt that these are out-lying members of the lateral cornu group. A certain number of nerve-cells, evidently, I think, also out-lying members of the same group, lie further out still separated by a much greater distance from the grey matter. They are in every case placed upon fine strands of connective tissue which cross the white column (fig. 3, Plate 3), and in these strands are to be found fine medullated nerve-

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[*] 'Journ. of Physiol.,' vol. 10, p. 429.

[†] These are seen in CLARKE'S fig. 2, from thoracic cord of Ox, 'Phil. Trans.,' 1859.

fibres. That these distant cells are to be considered as belonging to the group of the lateral horn is to be inferred from the similarity of them to those in shape and size. They are of oval figure, with the long axis in the transverse plane of the cord. They are of the same size as, or slightly larger than, the lateral horn-cells. Five of them, taken without selection, measured respectively 32, 30, 34, 33, and 29 μ, giving the average of 31·6 μ. Five cells taken from the lateral cornu in the same preparations measured respectively 30, 35, 26, 28, and 31 μ, giving an average for the five of 30 μ. They are, moreover, arranged with their long axes upon lines which radiate backwards and outwards from the lateral cornu, suggesting a common origin from that point (fig. 4, Plate 3). They appear to be more numerous in the Human cord than in the other of the cords examined. The vertical fibres of the white matter in which they lie embedded belong to the area of the lateral column, called by FLECHSIG the lateral limiting layer; but it may well be that those lying furthest out are beyond this layer, and within the area of the crossed pyramidal tract. The arrangement of the cells in the transverse plane, and the coincidence of the longer axis of the cell-body with the direction of the medullated fibres running horizontally in the same connective tissue septa as contain the cells, suggest, however, that the cells are connected with the horizontal fibres radiating between the grey matter and the white column, rather than with the vertical fibres of the column. But the horizontal fibres must be in turn continuous with certain of the vertical; with which of these, however, is a question that at present there is no possibility of deciding.

To the dorsal side of the lateral horn, and often continuous with it, e.g., in the lower part of the cervical enlargement, is the processus reticularis. It merges dorsally in the lateral portion of the cervix and caput cornu dorsalis. In the bars of its reticulum made up of many interlacing horizontally and vertically running bands of fibres, numerous ganglion cells exist. For the most part, the cells seem attached to those bands of the interlacement which are horizontal. They present more variety of form than do the previously mentioned groups, and they seem to be equally numerous upon frontally running, and upon sagitally running bands of the formation. Many are small and fusiform, and many are somewhat larger and triangular. The triangular cells resemble closely the cells above-described in the bands of the white commissure, and likened by SCHIEFFERDECKER* to arrow-heads in their shape. The cells of the reticularis may, one must imagine, be related to the numerous small ganglion cells, chiefly fusiform, but often triangular, which occupy the grey matter bordering on the processus. These, as CLARKE stated,† lie for the most part with their length parallel to the nearest portion of free edge of the grey substance, except that the deeper lying of them are mostly directed outward, as if belonging to continuations of the cross-running transverse fibres of the commissures. These cells point into the bars of extension of the grey matter which form the roots of the processus reticularis, and

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* *Loc. cit.*

† *Loc. cit.*

the frontally directed cells in the reticularis may be considered to belong to part of the same system as do they.

Besides the smaller nerve cells in the processus reticularis are some much larger, indeed, quite to be reckoned among the large ganglion cells of the cord. These are multipolar, and never, so far as I have seen, of bipolar form. They may measure 50 μ across, but it is the size of the cell-processes rather than of the cell-body, which is characteristic. On this account they contrast more strikingly with such cells as compose, for instance, the posterior vesicular column of CLARKE, than do even the multipolar cells of the ventral cornu. Lying well out in the reticularis the appearance they offer invites the idea that in them may be provided a nodal junction for the fibres that are there arriving along various planes. But it is difficult to adduce evidence for such a view. SCHIEFFERDECKER,* who has furnished what appears to be the only definite reference to the cells of the processus reticularis, considers the cells typical representatives of his "conducting ganglion-cells" (Leiter-zellen) as distinguished from "reflex ganglion-cells" (Reflex-zellen), the latter being such, for instance, as the multipolar cells grouped in the ventral grey horn. He does not advert to the existence of any large multipolar cells in the reticularis, and although the smaller spindle-shaped and triangular elements would obviously agree with the characters he describes for his "conducting-cells," it is not so obvious that he could attach the same meaning to the large-sized multipolar individuals that one not rarely finds. Examples of these appear, however, in the reticularis of all regions of the cord.

Where the beams of the processus reticularis are thrust backwards towards the dorsal angle of the lateral column, and enclose a finer meshwork than they do more ventrally, ganglion-cells elongated in the direction of the length of the beams exist within them. Many are fusiform, and occur near or on bundles of medullated fibres which sweep laterally round the caput cornu dorsalis toward the base of the cornu. In the sacral cord, of the Monkey and Dog at least, nerve-cells are to be found lying considerably distant from the caput cornu within the dorsal part of the lateral white column, as represented in fig. 5, Plate 4. The arrangement of these tends to show a curvilinear grouping parallel with the outer edge of the lateral limb of the substantia gelatinosa. Those that are within the strands continuous with the processes of the reticularis may, it would seem reasonable to conjecture, be connected with medullated nerve-fibres from the dorsal spinal nerve-roots, which curve round the dorsal horn on its lateral side. The individuals lying further out it is difficult to suggest connections for. Their extension in the transverse plane of the cord must, one would think, mean that they communicate mainly with fibres that are taking a horizontal course in the lateral column at their level, but there is no evidence of more than sparsely scattered horizontal fibres in the neighbourhood.

* VIRCHOW'S 'Archiv,' *loc. cit.*

Out-lying Ganglion-Cells in the Posterior Column.

As in the antero-lateral, so also in the posterior column of the cord out-lying gang-lion-cells exist. And it would seem that the anatomy of the out-lying cells in the posterior column is less complex than that of the former classes. In the first place, the cells appear to be confined to one portion of the posterior column, namely, the extero-posterior division of it, and in that division to one particular area, namely, the radicular zone. In the second place, one has obtained definite evidence of their existence from a certain region only of the length of the cord, namely, from that which is especially characterised by the large dimensions assumed by the posterior vesicular column of CLARKE.

The out-lying cells appear to be situated in the root-zone exclusively upon certain of the horizontal radiations of the root-fibres from the white column into the grey cornu. These radiations are among those which in transverse sections through the cord can be seen to pass from the dorso-median limit of the root-zone in a ventral and lateral direction to enter the grey substance in the neighbourhood of CLARKE's column, often describing as they do so bold curves with the convexity turned toward the postero-median fissure. They are radiations consisting of root-fibres, which belong to the median division in KRAUSE's[*] classification of the bundles of the dorsal roots. They enter the grey matter without previously passing through any part of the mesial limb of the gelatinous substance. BECHTEREW[†] has shown that they are composed of fibres which already are possessed of the medullary sheath in the human fœtus of 25 cm. length, and are therefore among the earliest of the fibres of the cord to be developed. It is also to be remembered that their fibres are among those of largest calibre in the whole of the dorsal spinal nerve-root.

The form of the out-lying cells found in these radiations of the dorsal root may be described as oval (figs. 6, 7, 8, Plate 4); one end is usually considerably the less pointed. A nucleus, which is large and contains a very obvious nucleolus, is fairly centrally placed within the cell, though often somewhat nearer the blunter of the two extremities. The cells have not been observed isolated in teased preparations, but they may really be of a more symmetrically bipolar figure than the foregoing description would suggest. Although they are certainly chiefly extended in the plane at right angles to the long axis of the cord, a very slight inclination in such a plane, either of themselves or of the section in which they lie, would suffice to truncate one end of the spindle-shaped cell-body. In such a case the nucleus, if centrally placed, would appear nearer to the more rounded pole. That this does actually happen and frequently explains the somewhat unipolar appearance of the cell, is suggested by the fact that the more pointed end, continued into an obvious cell-process, is sometimes turned with the fibres of the dorsal root in the direction of the grey substance of the

* ' Allgemeine u. Mikroskop. Anatomie,' 1876, p. 389.
† ' Archiv f. Anat. u. Physiol., Anat. Abth.,' 1887, p. 126.

cornu, and sometimes reversely, the blunter end, devoid of any obvious cell process, is directed centrally, while the tapering extremity mingles with root-fibres on the distal side, and is turned away from the grey matter of the cornu.

In rare instances is observable a deviation from the usual shape, which is less easy to reconcile with a belief that the cells are all of fairly regular bipolar form. Such an instance is represented in fig. 8, Plate 4. The impression is given of possession by the cell of two processes issuing from the cell-body in somewhat close juxtaposition, and both turned in a direction absolutely away from the grey matter of the dorsal horn. A little eminence rising from the ventrally-directed edge of the cell, is all that can suggest the giving off of a process toward the grey cornu. This very slight suggestion is perhaps strengthened by the fact that nerve-fibres (of the roots) in their centripetal path sweep by the little eminence in question, and the direction in which the eminence is set is certainly the same as theirs. As will have been gathered, the outlying cells strongly resemble in form the cells of CLARKE'S column. This may be easily seen by a comparison of the two in one and the same section, especially in the lowest portions of the column, where, in Man, as pointed out by MOTT,* the arrangement of the component cells so as to have their longer axes parallel to the long axis of the cord is not so marked as is the case higher up. There, where cells of CLARKE'S column may thus be viewed lengthwise in the same transverse section as contains the out-lying cells, the similarity in shape between them is too obvious to be overlooked.

There is also a correspondence between the two in size. Measurements made of ten of the out-lying cells, taken at random, gave them an average dimension of $68 \cdot 7 \mu$. This is larger than the average diameter assigned by some authorities (GERLACH, HENLE†) to cells of CLARKE'S column, but agrees fairly with the measurements given by SCHWALBE‡ and MOTT :§ and the average size obtained by measurements of ten cells of CLARKE'S column from the same sections as furnished the out-lying cells measured, is 73μ.|| The actual measurements run as follows :—

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* 'Journ. of Anat. and Physiol.,' 1888. "Microsc. Exam. of CLARKE'S Column in Man, the Monkey, and the Dog," by FRED. MOTT, M.D., B.S. Lond., M.R.C.P., Lect. on Physiol. and Med. Regist., Charing Cross Hospital.

† 'Handb. der Nervenlehre,' 1879.

‡ 'Lehrb. der Neurologie,' 1881.

§ 'Journ. of Anat. and Physiol.,' 1888.

|| [GASKELL has recently (' Journ. of Physiol.,' vol. 10, 1889, p. 157) referred to the cells of CLARKE'S column as "divisible into two groups, of different-sized cells," and on that ground exception might be taken to the method of measurements for comparison employed above. I must confess, however, I have not been able to convince myself of the division of the cells into two groups, nor do I interpret the observations of MOTT (' Journ. of Anat. and Physiol.,' loc. cit.) to be in support of the division of the cells into two groups.—June, 1890.]

	Out-lying Cell.	Clarke's Cell.
	μ	μ
One specimen, at level of 1st lumbar nerve-root	74	68
,, ,, ,, ,, ,, ...	68	79
,, ,, ,, ,, ,, ...	65	72
,, ,, between 1st lumbar and 12th thoracic nerve-roots ..	60	68
,, ,, ,, ,, ,, ,, ,, ..	76	69
,, ,, at level of 12th thoracic nerve-root	61	63
,, ,, ,, ,, ,, ,, ...	61	77
,, ,, between 12th and 11th thoracic nerve-roots	59	73
,, ,, at level of 8th thoracic nerve-root	55	61
,, ,, ,, ,, ,, ,, ...	63	55

Another point of similarity between these cells and those of CLARKE's column is the considerable resistance of both, in comparison with the cells of the ventral cornu, to shrinkage and other damage incident during the ordinary hardening processes. The out-lying cells are well preserved in ordinary bichromate preparations. The degree of depth of tint assumed by them, in hæmatoxylin preparations, varies in various preparations, but not in the same preparation. The differences observable in respect to depth of stain between cells of the ventral cornu on the one hand, and, on the other, the cells of CLARKE's column, are often extremely striking. When large series of specimens are searched through, these differences are seen to be of capricious nature, and, I think, of little real significance. Although it will often happen that the cells of CLARKE's column, as has been claimed for them by v. LENHOSSEK* and others, become stained more deeply than, and with an apparently different kind of sepia tint (in, for instance, " WEIGERT " preparations), to the cells of other groups, notably of those of the ventral cornu, yet the reverse is quite frequently the case even in sections from the same levels of the cord. Without, therefore, laying more stress on the fact than the above limitation leaves, it is, perhaps, worthy of note that the out-lying cells in the radicular zone are found tinted in the same degree and manner as are the cells of the vesicular column.

Also, as connecting them with CLARKE's column is to be mentioned the peculiar distribution of the out-lying cells in the length of the cord. I have not observed them lower than the level of exit of the 3rd lumbar nerve-root nor higher than the level of the exit of the 5th thoracic. They appear, therefore, best seen where CLARKE's column is best developed. And where, as above stated,† the cells of CLARKE's column are placed, many of them, at least, with their length in the transverse plane of the cord, so are the out-lying cells themselves. This suggests that perhaps in upper thoracic and other levels where CLARKE's cells are more segmentally distributed and lie parallel with the length of the cord, out-lying ganglion-cells, if

* ' Archiv Mikrosk. Anat.,' vol. 33, 1889.
† And cf. MOTT, loc. cit.

related to them, might also be so arranged and a little difficult of discovery. A cross section of a bipolar cell, imbedded in the fibres of the white column, even when the plane of the section passed through it allowing recognition of the nucleus, might be somewhat hard to detect. In longitudinal sections they should be clearly discoverable. I have not, however, obtained specimens which prove that the cells exist beyond the limits mentioned above. I have not found them in the region of the sacral nucleus of CLARKE* and STILLING, nor in the upper cervical cord where a cell-group, in some respects resembling the posterior vesicular column, is obvious.

As to the particular position in the radicular zone occupied by the out-lying cells, they may be far removed from the limits of the grey matter (*cf.* figs. 9, 10†), in the middle of the extero-posterior column, indeed, near the edge of that column where it abuts upon the, in this region of the cord, ill-defined confines of the column of GOLL. The cells far distant from the grey matter appear in my specimens more numerously in the lumbar than in the thoracic cord, although there are many in the latter. The distant cells are not so numerous as others, quite resembling them, which lie in the portion of the root-zone that is nearer to (fig. 11, Plate 3), and close outside (figs. 12, 13, Plate 3), the grey matter of CLARKE's column. Occasionally the cells are situated far ventrally, near the dorsal commissure. Even from that district of the extero-posterior column, not a few bundles of fibres proceed towards CLARKE's group; it is presumable that these are root bundles, if so, the radicular zone extends ventrally sufficiently far to include them and the out-lying cells in question. The position of the isolated ganglion-cells upon these bundles favours the supposition.

A point to be mentioned is that just as CLARKE's group at its lower end is placed well backward in the base of the dorsal horn, considerably behind the *niveau* of the dorsal commissure, and when traced upwards is found to rapidly exchange this position for a more ventral one, so do the out-lying cells in the posterior columns experience a shift of their general position in the same sense. In the lower levels of their distribution they lie nearer the dorsal part of the periphery of the cord than in levels situated higher. A comparison of fig. 9, from near the exit of the 2nd lumbar nerve-root with fig. 10, from the level of the 8th thoracic, shows this difference in general position of the cells within the posterior column.

The proportion of the number of out-lying cells to the number of the cells of CLARKE's column, varies in different sections from the same level, and at different levels. Two out-lying cells to eight of CLARKE's cells is a high ratio. In the middle of the thoracic region, several successive sections may not reveal a single out-lying cell, even if the sections be made as thick as is compatible with a satisfactory examination of them, and as many as eight or nine of CLARKE's cells are to be found in the section of the vesicular column.

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* CLARKE latterly did not consider that this nucleus corresponded to his posterior vesicular column of the lumbo-thoracic region.

† The positions of the cells are marked by crosses.

Altogether, it appears to me most probable that the cells are to be considered outstanding members of CLARKE's group; and it appears to me that they suggest a connection between the cells of CLARKE's column and certain other cells of the dorsal horn. SCHWALBE* creates a class, which he terms "the solitary cells of the posterior horn," to include all the cells in the grey matter of the dorsal cornu, after deduction from it of the substantia gelatinosa and the vesicular column of CLARKE. He groups, therefore, under one title, cells differing greatly from one another in appearance, and of relationships probably widely different. He, further, in his explanatory figure,† indicates as the chief of these "solitary cells of the posterior horn" the comparatively closely-arranged smallish cells at the base of the processus reticularis, which CLARKE described as part of his tractus intermedio lateralis. These cells are certainly anything rather than "solitary" in their distribution (cf. fig. 4, Plate 3), and it seems to me that CLARKE was justified in not considering them within the confines of the dorsal cornu at all. But among ganglion-cells, situated undoubtedly within the dorsal cornu, are certain that might more fittingly be termed "solitary" cells, and seem to form a division apart. These resemble strikingly, in some features, the out-lying cells of the radicular zone. If in sections of, for instance, the lower lumbar cord, the fibre-bundles of the dorsal roots be examined where they plunge at intervals through the median limb of the gelatinosa, certain sparse ganglion-cells can be found (figs. 16, 17, 18, 19.) Large and spindle-shaped, they occur in the sections for the most part singly; but also, though rarely, two or even three together. The region of the dorsal horn that extends between the substantia gelatinosa and the longitudinally running bundles of the dorsal root, is very scantily occupied by ganglion-cells. It is in this region, and especially in that part of it adjoining the gelatinosa, that the large ganglion-cells chiefly appear. The longer axis of the cell is, in most instances, set parallel with the lines taken by the bundles of root-fibres that pierce the gelatinosa horizontally. The shape of the cell is that of a broad spindle, tapering to processes at the ends. The constancy with which these cells are seen, lying near or upon the root-bundles, and their direction with the bundles, makes a connection between the cells and fibres of the bundles highly probable. If one of these cells, with the adjacent root bundle, were situated in the white matter of the exteroposterior column, I doubt whether it would be possible to discriminate between it and the out-lying cells, described in this paper as so resembling the cells of CLARKE's column. May not these cells, therefore, be equivalent to the cells of CLARKE's column? The more so, that in many cases they lie barely more within the grey substance than do the out-lying cells of the radicular zone, for they are seen on the mesial aspect of the gelatinosa, between it and the white matter of the posterior column. There, also, they lie near to and parallel with certain of the bundles of the

* 'Lehrbuch d. Neurologie,' p. 350.
† Letter ε in fig. 221, p. 347.

coarser fibres of the dorsal nerve-root. In this position CLARKE has figured them,[*] in his fig. 22, Pl. XXIV., Phil. Trans., 1859.[†]

The chief interest attaching to ganglion-cells lying in the white columns is that it may well be supposed they are connected with the nerve-fibres among which they are placed, and that in this way some knowledge may be gained as to the anatomy of the cells themselves, and of the cell-group of which they may be out-lying members, and of the fibre-bundles containing them.

When the out-lying cell is isolated in position, it is not easy to conceive what becomes of the branched processes of the cell (GERLACH's protoplasmic processes). The axis-cylinder process, if one exist, may be continuous with a fibre of the adjacent fibre-bundle. Are the cells possessed of more than one axis-cylinder process? BEISSO, SCHIEFFERDECKER, and FLECHSIG have at various times urged that cells of such a kind exist. In this connection arises a slight difficulty if the out-lying cells of the radicular zone are considered individuals belonging to CLARKE's column. In the case of the out-lying cells, it is not easy to see that there is evidence of their communicating with any representative of the fine-fibred plexus that constitutes so striking a feature of the vesicular column itself, and with which it is customary to believe that the cells of the column do communicate.

If the out-lying cells of the dorsal root-zone are, as seems most probable, out-lying cells of CLARKE's column, certain suggestions as to the anatomy of that group become obvious. These are : that the cells of that group are connected *directly* with certain of the fibres of the mesial division of the sensory or posterior nerve-root, which, after an upward course in BURDACH's column, plunge into the grey matter of the base of the dorsal cornu : that some, at least, of the cells of the group are interpolated, more or less immediately, into the course of medullated nerve-fibres of large calibre : that other outstanding individual cells to be reckoned as belonging to the group may be present as solitary cells in the near neighbourhood of the substantia gelatinosa of the dorsal horn.

Further, it seems likely that the cells in the root-zone may in the Mammalian cord represent the cells described by FREUD[‡] in the cord of *Petromyzon Planeri* as

* From the conus medullaris of the Ox.

† They can be seen also in a figure in VAN DER KOLK.

‡ 'Sitzungsberichte d. Kais. Akad. zu Wien,' vol. 75, III. Abth., 1877, also following year; and Lit. Einleitung, No. 12. See also KUTSCHIN, 'Ueb. d. Bau d. Rückenm. des Neunauges,' 1863, abstracted in SCHULTZE's *Archiv*, 1887, vol. 2 ; also KLAUSNER on Proteus. [It may be recalled here that STILLING ('Neue Untersuchungen über den Bau des Rückenmarks,' 1859) suggested that the "grosse runde" cells in the spinal cord of *Petromyzon*, which lie each side of, and dorsal to, the central canal, represent in *Petromyzon* the group of cells described by CLARKE in the thoracic region of the Mammalian cord, and now generally known as CLARKE's vesicular column. STILLING asserted that these cells in *Petromyzon* possess each a cell-process directly continuous with a nerve-fibre of the dorsal nerve-root. KUTSCHIN, in 1863 (*loc. cit.*), confirmed STILLING's description; he found certain fibres of the dorsal root in *Ammocœtes* and *Petromyzon* directly traceable to the large cells, which he spoke of as the inner central cell-

directly connected with fibres of the posterior roots, and named by him " Hinter-zellen." It is conceivable that the out-lying cells are really equivalent to cells of the dorsal root-ganglion, although placed in an intra-spinal position. If so, they may help to explain the observation of JOSEPH* that not all the nerve-fibres of the dorsal root degenerate on being cut off from the root-ganglion. JOSEPH's observations were, however, made on an upper cervical nerve-root.

If, as according to the view of HENSEN† and of SCHENK,‡ the cells of the dorsal root-ganglia are nothing else than nerve-cells that originally were a part of the cord itself, then the out-lying cells in the root-zone suggest that the posterior vesicular column of CLARKE may be composed of cells equivalent to those composing the root-ganglia, that have, however, retained their original position as a constituent of the cord.§ The occurrence of CLARKE's column in the thoracic region would then well agree with other features which show that region to present more primitive characters than, for instance, the regions of the enlargements.

group. FREUD's papers on the subject appeared in 1877 and 1878. He designated the large cells into which he traced the posterior root-fibres *innere Hinterzellen*. The name *äussere Hinterzellen* he reserved for certain cells, similar, he considered, to the *innere Hinterzellen*, which he found placed upon fibres of the posterior root, between the root-ganglion and the posterior horn of grey matter of the cord. According to FREUD, some of the *äussere Hinterzellen* lie at the surface of the cord, upon fibres of the posterior rootlets which run for a distance upward along the surface of the spinal cord. Other indi-viduals of the *äussere Hinterzellen* lie within the cord, upon nerve-fibres of the posterior roots in their intra-spinal course. If this nomenclature be transferred to the Mammalian cord, then the cells of the posterior vesicular column are the *innere Hinterzellen* of *Petromyzon*, and the out-lying cells of the posterior root-zone are the intra-spinal set of the *äussere Hinterzellen*. As judged of from FREUD's figures of *Ammocœtes'* cord, the *innere Hinterzellen* do certainly in their position, and fairly in their form also, justify STILLING's view of their identity with CLARKE's vesicular column of the Mammalian cord. The appearance of them in some sections of *Ammocœtes'* cord, which I have been permitted to examine through the kindness of Dr. GASKELL, also bears out in this respect the impression obtained from the figures given by FREUD. That the cells of CLARKE's column are connected on their proximal side with the fibres of the cerebellar tract admits of little doubt; as to their distal connections, it may be said that two views are current. The one, which has recently been ably supported by MOTT (*loc. cit.*), is that the cells are connected with afferent fibres, fibres of the posterior root. The other (GASKELL, HILL) is that CLARKE's column is connected with the efferent fibres of anterior roots. Of these views, the former, I think, derives support from the facts observed with regard to out-lying cells in the posterior root-zone of the cord; but it is difficult to believe that these out-lying cells, although they appear equivalent to members of CLARKE's column, can be at all closely connected with any fibres in the anterior root. Indeed, it appears more likely that not only the vesicular group of CLARKE, and the out-lying cells of the external posterior column, but also the cells described above (p. 43) as occurring in and near the gelatinosa, all belong to the afferent system entering by the posterior root.—June 12, 1890.]

* 'Archiv für Physiologie,' DU BOIS-REYMOND, 1888.

† " Beob. über d. Befruchtung u. Entwick. des Kaninch. u. Meerschw." 'Ztschr. f. Anat. u. Entwickl., vol. 1, 1876.

‡ " Die Entwick. der Gangl. u. des Lobus electricus." 'Sitz. d. Kais. Akad. zu Wien,' vol. 74.

§ Interesting in this connection are the small ganglia intercalaria which, as HYRTL was the first to note, occasionally occur on the posterior roots of the human cord, between the main root-ganglia and the cord itself. *Cf.* HYRTL.

Note.

Mention of any out-lying cells in the posterior column is not to be found in the classical monographs by CLARKE, STILLING, FROMMANN, DEITERS and MAX SCHÜLTZE, KRAUSE, &c., nor in the recent admirable papers by LISSAUER, BECHTEREW, DARK-SCHEWITZ, MOTT, and v. LENHOSSEK. One reference, however, does occur in a second and very recent paper by M. v. LENHOSSEK, in the 'Archiv f. mikrosk. Anatomie' of November (vol. 34, II., 1889). At the end of a paragraph, in which the writer there contends for a direct connection between CLARKE's column and the posterior roots, he clearly mentions the fact of the existence of isolated cells in the posterior column. To quote his words : " in der Regel findet man frei gewordenen Zellen an der Grenze zwischen grauer und weisser Substanz (he is speaking of CLARKE's column), doch begegnet man denselben zuweilen in der Einstrahlungszone der BURDACH'SCHEN Stränge ; ja, sie rücken hin und wieder fast bis zur Eintrittstelle der Hinterwurzeln, halten sich in Lage, auch Richtung ihrer Achse, an den Hinterwurzelfasern." My description above was written before I had read this sentence in LENHOSSEK's admirable and rather lengthy article.

Addendum.

(June 14, 1890.)

Out-lying nerve-cells, similar to those above described existing along the root-fibre bundles in the external posterior column of the lumbar and thoracic regions of the cord, occur in the region of the bulb imbedded in the funiculus cuneatus. In this situation they lie sometimes two or three together, with the long axis of the cell in the frontal plane, and close to or upon bundles of nerve-fibres of large calibre (posterior root-fibres), which pass through the funiculus on their way to reach the grey matter of the nucleus. The similarity of these out-lying cells to the out-lying cells of the posterior root-zone of the cord is too striking to escape attention. The funiculus cuneatus is, of course, equivalent in the bulb to the external posterior column in the cord ; it undoubtedly contains an area which corresponds to the posterior root-zone of the spinal posterior column. The out-lying cells in the funiculus cuneatus appear, in point of position, to hold the same relation to the inner mass of the nucleus cuneatus as is held by the out-lying cells in the external posterior column of the lumbo-thoracic cord to the vesicular column of CLARKE. This seems to indicate that in the bulb the homologue of the vesicular column of CLARKE is to be found in the cuneate nucleus.

It will, perhaps, be urged against this that the upper end of the vesicular column of CLARKE is the vagus nucleus (ROSS, HILL, GASKELL). But although a group of cells

in the upper cervical region appears to represent there the vesicular column of CLARKE, even with this group as a guide to the probable position of CLARKE's column, I have never been able to satisfy myself that in the redistribution and dislocation of grey matter, that can be followed out in a series of sections passing upward into the bulb, the vagus nucleus is continuous with the upper part of CLARKE's column, or that its position is that which the upper extremity of CLARKE's column would assume. Neither does the histological appearance of the vagus nucleus altogether favour, in my opinion, the view that it represents CLARKE's column. Its cells appear to resemble more closely, in form and arrangement, the smaller fusiform cells of the lateral horn of the cord—such as, for instance, are seen in fig. 4, Plate 3. On the other hand, the cells of the cuneate nucleus do much resemble, in form and arrangement, the cells of the vesicular column of the cord.

EXPLANATION OF PLATES 3, 4.

Figure 1, Plate 3. Portion of the anterior column adjacent to the mesial edge of the anterior grey cornu. Tri-radiate nerve-cell lying between the fasciculi of white matter. From a section, stained with carmine, and prepared from the cord of the Dog, between the 1st and 2nd cervical nerve-roots. Camera lucida and apochromatic immersion (ZEISS), with ocular 4.

Figure 2, Plate 4. Portion of the anterior column adjacent to the mesial edge of the grey cornu. Tri-radiate nerve-cell lying between fasciculi of white matter. A larger nerve-cell belonging to the inner group of the grey cornu is seen. WEIGERT hæmatoxylin preparation, from the cord of a full-term human fœtus, at the level of the 5th lumbar nerve-root. Camera lucida and ZEISS apochromatic immersion, with ocular 4.

Figure 3, Plate 3. Out-lying cells in the lateral column, near the cell-group of the lateral horn. At level of 8th thoracic nerve-root of Macacus; stained with aniline blue-black. Camera lucida and ZEISS apochromatic immersion, with ocular 4.

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Figures 7 and 8, Plate 4.   Out-lying cells in the posterior column, showing their relation to strands of connective tissue in which are medullated nerve-fibres. These cells are those the position of which is shown by two crosses in the posterior column of figure 9.   Camera lucida and ZEISS apochromatic immersion, with ocular 4.

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Figure 10, Plate 3.   Transverse section of human cord at level of 8th thoracic nerve-root, showing posterior horn and CLARKE's column.   The number and position of the cells of CLARKE's column are shown by crosses.   The position of an out-lying cell is indicated by the cross in the posterior column.   Camera lucida and ZEISS objective a, with ocular 4.

Figures 11, 12, and 13, Plate 3.   Outlines, taken with the camera from preparations from the human cord at the level of the 3rd lumbar, 11th thoracic, and 9th thoracic nerve-roots respectively, to show the position of out-lying cells relatively to the cells of CLARKE's column and to bundles of medullated fibres passing through the radicular zone towards CLARKE's column.   From WEIGERT hæmatoxylin specimens.   ZEISS A, ocular 4.

Figure 14, Plate 4.   The mesial limb of the substantia gelatinosa, showing bundles of root-fibres passing through it, and close to one of the bundles a solitary bi-polar nerve-cell.   Human cord, at level of 4th lumbar nerve-root.   WEIGERT hæmatoxylin.   ZEISS objective A, ocular 2.

Figure 15, Plate 4.   The mesial limb of the substantia gelatinosa, with bundles of root-fibres passing through it ; two large bi-polar nerve-cells in close relation to the bundles.   Cord of Puppy.   WEIGERT hæmatoxylin.   ZEISS objective A, ocular 2.

Figure 16, Plate 4.   Posterior horn, with medullated root-fibres sweeping through the mesial limb of the substantia gelatinosa.   In close relation to a bundle of fibres is a large nerve-cell.   At level of 6th thoracic nerve-root.   Lion cub. WEIGERT hæmatoxylin.   ZEISS a, ocular 4.

III. *An Experimental Investigation into the Arrangement of the Excitable Fibres of the Internal Capsule of the Bonnet Monkey* (Macacus Sinicus).*

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[Plates 5–11.]

Contents.

Introduction.

In the following paper we propose to give the results of a research on which we have been engaged nearly three years, and by which we hoped to elucidate the arrangement of the motor fibres in the internal capsule.

The fibres which connect the excitable areas in the cortex cerebri with the bulbo-spinal grey matter in the medulla oblongata and spinal cord are commonly spoken of as forming the *pyramidal tract.*

These fibres course downwards between the caudate and lenticular portions of the corpus striatum and the optic thalamus. In passing through this region they enter into the composition of the bundles of white fibres, which received from the older anatomists the unfortunately misleading title of the *internal capsule.* It is obvious

that, when the destination and function of these fibres is fully known, this term will be abandoned with great advantage for a specific topographical nomenclature. At present, however, we must use the old expression in default of a better one.

In previous investigations ('Phil. Trans., B., 1887, 1888) we have described the foci* of representation in the cortex cerebri of the Macacque Monkey (*Macacus sinicus*), of the movements of the chief divisions of the body.

It remained for us to see whether there were issuing from such foci bundles of fibres conveying the functional impulses originating in those foci, and whether such fibres were arranged, as has been suggested by one of us ('Lancet,' July, 1884), in an order similar to that which prevails on the surface of the cortex.

From what has been already done in this matter it is evident that the problem can be attacked in two different ways, either by stimulation of the motor fibres of the internal capsule, or by removing different parts of the so-called motor cortex, and tracing downwards the degeneration produced in the motor fibres through the internal capsule and crus cerebri.

In this present paper, we have employed only the former of these two methods, viz., electrical stimulation of the fibres of the internal capsule, as exposed in horizontal section of the hemisphere. Some knowledge of the arrangement of the fibres in the internal capsule has already been gained by experimental and anatomical research, and the facts as known at the present time we will now summarise.

HISTORICAL INTRODUCTION.

Previous writers, on this question, have investigated the subject by several methods in different kinds of animals, viz., Rodents, Carnivora, Monkeys, and Man. The methods employed were for the most part anatomical, or consisted in tracing paths of degeneration. Very rarely has excitation been resorted to. Considering the varied mode in which movement is represented in the cortex of the brain in different animals, we have thought it best to arrange the facts we have deduced from the above researches under the headings of the various species of the animal examined, in the order already given.

Birds.

PITRES† found that in Pigeons and Chickens ablation of a hemisphere produced no descending degeneration.

Mammals.

Rodents.

PITRES† found that in the Guinea Pig and Rabbit destruction of the excitable parts

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* By the term "focus" we mean the point where the movements of any given segment are most intensely represented.

† 'Comptes Rendus de l'Académie des Sciences,' vol. 99, July, 1884, p. 89.

of the cortex produced descending degeneration, which cannot be traced beyond the lower limits of the bulb—*i.e.*, beyond the anterior pyramid of the same side.

NOTHNAGEL* never observed motor paralysis to follow either unilateral or bilateral division of the internal capsule in the Rabbit.

MONAKOW† removed in the Rabbit the anterior portion of the hemisphere, including part of the excitable zone, and found that this was followed by atrophy of the anterior part of the internal capsule. Removal of the middle zone of the cortex produced atrophy of the third fifth of the capsule, while destruction of the hinder third of the hemisphere (*i.e.*, occipital portion) caused atrophy of the hinder third of the capsule. He further found‡ that removal of FERRIER's centres 6 and 9 produced respectively atrophy of the hinder and dorsal part, and of the posterior part of the capsule. Finally, the corpus striatum was always normal.

GUDDEN§ figures (fig. 4, Taf. VIII.) a section of the hemisphere of a Rabbit, the upper third of which, including a thin slice of the corpus striatum, had been removed. The specimen shows marked atrophy of the internal capsule. The same author‖ also removed in young Rabbits the frontal lobe ("stirnhirn") and found that while the corpus striatum remained normal, the "medial part of the peduncle" and the corresponding pyramid and spinal cord were atrophied.

Carnivora.

Several authors have made experiments on the Cat and Dog.

J. BURDON-SANDERSON¶ excited electrically in the Cat the fibres of the corona radiata, and subsequently the corpus striatum, producing the same movements as he had previously obtained by excitation of the cortex.

CARVILLE and DURET.** These authors divided the anterior two-thirds of the internal capsule in the Dog, and thus produced hemiplegia. In a second experiment they found that division of the posterior three-fourths produced also hemiplegia, but accompanied by hemianæsthesia.

VULPIAN,†† having extirpated the right sigmoid gyrus 183 days previously, the right internal capsule was found, *post mortem*, to be "much thinner" than that of the left side.

* 'Archiv für Pathologische Anatomie und Physiologie und für Klinische Medicin,' vol. 71, 1877, p. 280.

† "Weitere Mittheilungen über die Extirpation circumscripter Hirnrindenregionen, &c." 'Archiv für Psychiatrie und Nervenkrankheiten,' vol. 12, 1882, p. 535.

‡ *Loc. cit.*, p. 141.

§ 'Archiv für Psychiatrie,' &c., vol. 2, 1870, p. 693.

‖ 'Correspondenzblatt für Schweizer Aertzte,' vol. 2, p. 79.

¶ Note on the "Excitation of the Surface of the Cerebral Hemispheres by Induced Currents," 'Roy. Soc. Proc.,' vol. 22, p. 368, 1874.

** "Sur les Fonctions des Hemisphères Cérébraux," 'Archives de Physiologie, vol. 2, 1875, p. 465.

†† 'Archives de Physiologie,' vol. 3, 1876, p. 814.

PITRES* found marked descending degeneration in the Dog and Cat, after ablation of the excitable cortex, around the crucial sulcus. The major part of the degeneration ceased at the lower limit of the bulb.

GUDDEN† found that extirpation of the frontal region (stirnhirn) in young Dogs produced atrophy of the "medial part of the peduncle," corresponding pyramid and spinal cord.

LÖWENTHAL‡ found that in the Dog, after removal by Professor HERZEN of the left sigmoid gyrus, some of the lenticulo-striate fibres were atrophied just in front of the genu, and that, of the posterior limb, the fibres composing its anterior third were also atrophied. In a section taken lower down, just above the level of the anterior commissure, the portion of the posterior limb of the internal capsule found to be degenerated proved to be the second and third fifths from before back.

A section through the upper part of the crus showed the inner two-thirds to be degenerated ; while in a section lower down—i.e., opposite the exit of the third nerve —the middle third only was degenerated.

FRANCK and PITRES§ obtained, on excitation of the internal capsule in the Dog, the following results :—

Dividing the capsule into motor fasciculi, they found the arrangement from before backwards to be as follows :—

> Opening of eyelids and dilatation of pupils ;
> Movement of opposite fore foot only ;
> Associated movements of both opposite limbs ;
> Movement of opposite hind limb only ;
> Closing eyelids ;
> Pricking of the opposite ear.

VEYSSIÈRE,‖ by a special instrument, produced hemianæsthesia by a lesion of the posterior third of the hinder limb of the internal capsule.

Quadrumana.

SCHÄFER¶ noted the following facts in a Monkey previously operated on by Professors FERRIER and YEO :—

The parts removed were the ascending frontal gyrus, except a small portion at the

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* 'Comptes Rendus de l'Académie des Sciences,' vol. 99, July, 1884, p. 90.

† *Loc. cit.*

‡ "La Région pyramidale de la Capsule Interne chez le Chien, et la Constitution du Cordon antéro-latéral de la Moelle." 'Revue Médicale de la Suisse Romande,' vol. 6, p. 529, Plates VII. and VIII., 1886.

§ 'Leçons sur les Fonctions Motrices du Cerveau,' &c., par FRANCK. Paris, 1887, pp. 21, 22.

‖ 'Recherches Cliniques et Expérimentales sur l'Hemianæsthésie de Cause Cérébrale,' 1874.

¶ "Report on the Lesions, Primary and Secondary, in the Brain and Spinal Cord of the Macacque Monkey exhibited by Professors FERRIER and YEO." 'Journ. of Physiology,' vol. 4, Nos. 4 and 5, Dec., 1883, p. 323, Plate XII.

upper end next the longitudinal fissure, and except also its lower end, amounting nearly to one-third of the length of the whole gyrus; also the posterior third of the upper division of the anterior portion of the frontal lobe.

Further, the ascending parietal gyrus was removed, except 5 mm. at the upper and 6 mm. at the lower end, and a small piece of the parietal lobule; and, in addition, rather more than half of the ascending limb of the angular and the supra-marginal gyri.

The effects of this lesion were as follows :—

Internal capsule.—Sections of the internal capsule showed in its "central part" a number of bundles mostly degenerated, which were not all closely packed together, but were somewhat scattered, being here and there separated by others which presented a normal appearance. In many of the flattened degenerated bundles there was a thin layer forming a sort of cortical stratum, in which the fibres appear normal.

In a section of the crus cerebri a patch of degeneration was found to be situated in the central part of the crusta, localised near the ventral border as an oval area tapering mesially.

FERRIER[*] found in the Monkey that removal of the præfrontal regions was followed by degeneration of "the lowermost and most internal" fibres in "the transverse sections of the internal capsule and the most median bundles of the foot of the crus cerebri."

Examination of the photographs of the brains shews that in each case the cortical areas for turning of the head and eyes were encroached upon by the lesion.

Man.

TURCK[†] in his original memoir showed that in cases of long-standing hemiplegia the hinder and upper part of the capsule was degenerated, and also the middle segment of the crus.

FLECHSIG[‡] observed in a new-born infant, 44 cm. long, that the internal capsule was white only between the thalamus and lenticular nucleus. At the same time the laminæ medullares of the lenticular nucleus as well as the fibres surrounding their posterior extremities were brighter than the neighbouring parts. In the crus cerebri there was seen on the lower and outer side a sharply bounded bright band 3 mm. broad (pyramidal tract) which was continued up into the internal capsule close to the innermost zone of the lenticular nucleus. In a further paper[§] he shows reason for

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[*] "A record of experiments on the effects of Lesion of different regions of the Cerebral Hemispheres," with GERALD YEO, 'Phil. Trans.,' 1884.—Experiments 22 and 23, p. 528.

[†] 'Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften,' vol. 6, Wien, 1851, p. 300.

[‡] 'Die Leitungsbahnen im Gehirn und Rückenmark des Menschen,' Leipzig, 1876, pp. 27, 28, Taf. II., fig. 3.

[§] 'Archiv für Anat. und Phys. (Anat. Abth.),' 1881, p. 12.

believing that in a horizontal section through the upper border of the middle zone of the lenticular nucleus, the pyramidal tract, including the fibres to the seventh and twelfth nerves, occupies the second, third, and fourth sixths of the posterior limb of the internal capsule.

CHARCOT* sub-divided the capsule according to his well-known plan, as follows :—

The anterior limb he termed the lenticulo-striate bundle. The posterior limb he subdivided into thirds, the two anterior of which he regarded as belonging to the pyramidal tract, while the remaining third was, he believed, limited to the transmission of afferent impressions.

We may remark that, as will subsequently be seen, this description applies only to one sectional level of the capsule.

BRISSAUD† made the most numerous clinical observations on this point. According to him there exist in the crus cerebri four bundles, whose relations from behind forwards may be stated as follows :—

1. A posterior bundle for the transmission of sensory impressions (CHARCOT, MEYNERT).

2. A middle bundle for the innervation of the muscles of the limbs and trunk.

3. A bundle of small size, which he called the geniculate bundle on account of its connection with the genu of the internal capsule. This bundle includes motor fibres, and is distributed to the bulbar nuclei for the movements of the muscles of the face, the tongue (perhaps the soft palate) ; in fact to all parts of the head and face which can be acted on voluntarily.

4. An internal bundle which ends in like manner in the bulb, and whose degeneration only seems to be coincident with intellectual derangements.

To these four bundles the four following divisions of the internal capsule correspond.

1. To the posterior bundle, the posterior third of the hinder limb corresponds (*i.e.*, the region of hemianæsthesia, CHARCOT, RAYMOND).

2. To the middle bundle, the anterior two-thirds of the hinder limb.

3. The bundle which he called geniculate, corresponds to the genu of the capsule.

4. Finally the whole of the anterior limb of the internal capsule corresponds to the internal bundle of the crus.

Contrary to what certain authors have stated, the crus cerebri cannot become completely degenerated ; in other words there is a limit which the degeneration cannot overstep.

This maximum degeneration is recorded by him in observation VII., where the

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* 'Leçons sur les Localisations dans les Maladies du Cerveau et de la Moëlle Épinière,' 1876–1880, pp. 221–224.

† 'Recherches sur la Contracture Permanente des Hemiplégiques,' Paris, 1880, p. 38.

degeneration affected the internal three-fourths of the lower level, and only the posterior fourth escaped. In this case all the frontal lobe, the whole of the parietal lobe and a great part of the occipito-sphenoidal lobe had been destroyed by an extensive superficial softening.

MEYNERT[*] divides the internal capsule into five different kinds of bundles, viz. :—

1. Fibres passing from the cortex, as laminæ medullares, between the zones of the lenticular nucleus.

2. A bundle passing from the frontal region to the optic thalamus, and also forming part of the stratum zonale.

3. Fibres from the nucleus caudatus to the crus cerebri.

4. Fibres from the nucleus lenticularis to crus cerebri.

5. A group of fibres (the most posterior) which pass to the tegmentum.

HUGHES BENNETT[†] and CAMPBELL published a remarkable case in which the left arm was completely paralysed by a hæmorrhage in the internal capsule.

The lesion was found post-mortem and located as nearly as possible to the junction of the most anterior and second fourth of the posterior limb.

DEJÉRINE[‡] reported a case in which a tubercular tumour of the size of a nut, "seated on the internal capsule, between the posterior part of the optic thalamus and the capsule itself," produced paralysis of the right upper limb, together with incomplete right hemianæsthesia.

OBERSTEINER[§] figures an arrangement of the fibres of the capsule and crus similar to that given by BRISSAUD and CHARCOT.

WERNICKE[||] states that the anterior limb of the internal capsule is composed of fibres, which pass partly into the subthalamic region, and partly into the mesial side of the crus, and that the fibres coming from the nucleus caudatus pass circuitously through the laminæ medullares round the two inner zones of the lenticular nucleus.

ANATOMY OF THE INTERNAL CAPSULE.

Definition of the term.—As before said the term *internal capsule* has been given to the descending and ascending fibres of the *corona radiata*, while passing between the basal ganglia ; consequently we assume that the term is only applicable to the fibres so long as they are passing between the two following levels.

The upper of these levels is shown in Plate 11, fig. 2, which is constructed from

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[*] 'Psychiatrie,' Wien, 1884, p. 77.

[†] " Case of Brachial Monoplegia due to Lesion of the Internal Capsule," 'Brain,' vol. 8, p. 78, April, 1885.

[‡] Quoted in article " Encéphale," 'Dictionnaire des Sciences Médicales,' 1887.

[§] 'Anleitung beim Studium des Baues der Nervösen Centralorgane im gesunden und kranken Zustande,' 1888, p. 255.

[||] 'Lehrbuch von den Gehirnkrankheiten,' Cassel, 1881, vol. 1, p. 85, fig. 44.

a photograph of a transverse vertical section of a Monkey's (*Macacus sinicus*) right hemisphere. This superior level (black dotted line) is seen to be a plane directed slightly obliquely downwards and outwards, and resting on the upper surfaces of the caudate and lenticular nuclei.

The lower level is scarcely a plane, owing to the configuration of the basal ganglia, as about to be described, but its general direction is roughly represented by a plane inclined upwards and backwards in a line with the upper surface of the optic tract and drawn from the optic chiasma backwards and upwards to the posterior extremity, or the pulvinar, of the optic thalamus.

Of course these levels are purely arbitrary divisions of the fibres of the pyramidal tract, separating a part of them from the centrum ovale above and the crus cerebri below.

The internal capsule may be described as consisting of bundles of fibres, arranged like the rays of a fan, of which the handle if broadened would represent the crus cerebri, while if the fan be considered to be held laterally in a plane parallel to the sagittal section of the hemisphere, the outside rays would correspond respectively with what we shall hereafter term the antero-inferior and postero-superior borders of the internal capsule, this is represented in fig. 3, which is a photograph of a sagittal section of a Monkey's brain (*Macacus sinicus*).

Viewed in horizontal section the capsule is limited by lines drawn at right angles to the anterior and posterior extremities of the lenticular nucleus.

Although this view of the fibres composing the internal capsule is useful to bear in mind, it nevertheless does not express the whole facts of the case. In the first place, we have to remember that extending down towards the base of the brain, *i.e.*, towards the basal ganglia and internal capsule, is a large number of fibres converging from all parts of the cortex mantle.

Arranged in order from before back, these fibres may be enumerated as follows, irrespective of the impulses which they are generally considered to convey, classified according to the region of the cortex with which they are apparently connected.

       I. Præfrontal.
      II. Excitable, Pyramidal, or Fronto-Parietal.
    III. Temporal.
    IV. Occipito-Temporal.
     V. Occipital.

These fibres are shown in their relation to the basal ganglia, anterior commissure, &c., in the accompanying figs. 4, 5, 6. These are drawn from a dissection of a hardened Monkey's (*M. sinicus*) brain in which the large majority of the hemisphere has been cut away, leaving uninjured the nuclei of the corpus striatum. In fig. 4 the ventricular aspect of the mass thus obtained, and also in fig. 6, is seen a dotted line in front, showing how much of the grey matter forming the anterior perforated spot,

and part of the amygdaloid nucleus had to be cut away to reveal the anterior commissure (*a.c.*) running under the corpus striatum.

The convergence of the fibres of the capsule is seen in fig. 6, in which the whole of the corona radiata, outer capsule and claustrum, have been removed. It shows the caudate nucleus terminating posteriorly in the surcingle, which ends in the grey matter under the lenticular nucleus.

It is clear that these fibres, when they approach the basal ganglia, either end in these ganglia or pass between them to and from the mesencephalon.

(1) Do any of the fibres descending from the cortex end in the basal ganglia ?

This is a question which will recur, and as its elucidation does not immediately concern us in considering the present point, it may be postponed, see pp. 68, 72.

(2) Do the fibres pass between the basal ganglia to the mesencephalon, and if so which of the classes above enumerated ?

The answer to this question is easier. In the first place the important fibres known as the pyramidal tract pass downwards from the cortex, through the internal capsule into the crus and to the pyramidal region of the mesencephalon. These fibres are universally recognised to be the efferent or motor fibres of the cortex system, and it is the result of the excitation of these fibres which forms the subject of the present communication. In order to make the arrangement of these fibres— as we believe them to exist—more intelligible, we must first refer to the præ-frontal class.

I. *Præfrontal.*—The fibres coming from the præfrontal region unquestionably appear in the anterior limb of the internal capsule as horizontal bundles, and compose its antero-inferior border.

Excitation of the præfrontal region of the cortex, as is well-known, has produced no muscular movement except in the hands of MUNK.[*] Ablation of the same region has been stated by Professor FERRIER[†] to have been followed by descending degeneration in the fibres of the anterior limb of the internal capsule. His results, however, judging from the figures he gives of the brains thus operated upon, seem to have included portions of the excitable area for the representation of the movements of the head and eyes.

Anatomical investigation goes to show that the fibres of the anterior limb of the capsule pass to the mesial side of the crus and also to the subthalamic region. Of such fibres our method gave only negative evidence, since in Groups III., VI., VII. (see fig. 1) the anterior limb was practically inexcitable, it is therefore clear that these præfrontal fibres at any rate do not possess efferent motor function.

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[*] MUNK, "Ueber die Stirnlappen des Grosshirns;" 'Sitzungsberichte der k. Preussischen Akademie,' 1882.

[†] 'Functions of the Brain,' 2nd edition, pp. 398, 399.

Whatever be their function it is evident that the fibres entering the anterior limb are arranged much as is indicated in fig. 3, and that, regarding the hemisphere as fixed with its greatest length in a horizontal plane, we may group these fibres, according to their vertical relations to such a plane, into :—

       a. Superior and Descending.
       b. Horizontal.
       c. Inferior and Ascending.   See also figs. 3, 5, 6.

But further, on account of the manner in which the grey cortical mantle of the frontal lobe presents, roughly speaking, the form of a cap or segment of a globe, we must, in addition, sub-divide these fibres according to their relative position about the horizontal plane of the hemisphere. Hence we must speak of a middle bundle, viz., those from the tip and anterior part of the lobe ; of an internal group, viz., those coming from the mesial surface, *i.e.*, of the anterior third of the marginal and neighbouring gyri ; and of an external group, *i.e.*, those coming from the outer part of lower frontal gyri (of course excitable).

Many of these bundles will be referred to under the next heading, and the destination of the remainder, *i.e.*, of the truly præfrontal, we have already discussed.

To sum up then, it is evident that we have excitable fibres coming into the anterior limb from the so-called motor cortex lying on the external aspect of the hemisphere in the opercular region, and these we regard as motor. The mass, however, of the anterior limb is composed of the bundles of fibres coming from the most anterior and mesial surfaces of the frontal lobe, these being non-excitable.

Looked at in their sagittal arrangement, we find that the excitability diminishes from above down, or, in other words, the pyramidal portion decreases and the præfrontal and non-excitable increases.

II. *Excitable or Pyramidal, Fronto-Parietal.*—This is the collection of fibres commonly known as the pyramidal tract, and which on being stimulated produce movement in a distinctive manner. Strictly speaking, the term pyramidal tract should apply only to those fibres contained in the pyramids, *i.e.*, those for the muscles of the limbs and trunk. In fact, many authors divide the " motor " capsular fibres into two groups, one devoted to the cerebral nerves, and the other termed the pyramidal fibres. This distinction we consider fair, but the term pyramidal tract has been so extended as to include these two groups of so-called motor fibres. But, while we agree in general to this distinction, it must not be forgotten that these two sets of fibres are not in utterly distinct compartments, but that the area of each overlaps.

We would prefer, therefore, to employ the term excitable to express the whole number of fibres possessing " efferent " or " motor " function.

These fibres we presume come from most of the excitable portions of the cortex.

We say *most* of the excitable region, because the recent investigations of SCHÄFER[*] show that on excitation of the occipital and angular gyri certain movements of the eyes are evoked, which, by reason of their greater latency of production, seem to be a result of secondary stimulation of the area for such movements situated in the frontal region.[†]   Consequently, although this point is still greatly *sub judice*, it seems probable that the excitable fibres of the internal capsule only come from the fronto-parietal region.   The course they take to reach the various points in the capsule will be determined by the anatomical relations of the cortical mantle to the basal ganglia, the exact lines of such courses, though not yet determined anatomically, may be theoretically imagined.

We have ascertained the anterior and posterior limits respectively of the excitable fibres in each experiment, and have constructed the following outlines, see fig. 1 (twice the natural size), showing the delimitation of these fibres in each group, the position of each point being determined by the averages given in Table I., see next page.

III., IV., V. *The Temporal, Occipito-Temporal, and Occipital Fibres.*—The delimitation of these fibres we have only in the present research ascertained by the process of exclusion, *i.e.*, by their non-excitability.   We believe that they have for the most part to do with the general and special senses, but the method of the present research does not throw any fresh light on the subject.

It is, however, worthy of note that even in the narcotised state, when the scalpel divided these fibres at the hinder end of the capsule, there frequently occurred a marked general reflex movement.   Excitation of the peripheral ends of these cut fibres produced, of course, no effect.

[*] ' Roy. Soc. Proc.,' vol. 43, 1888, p. 411.
[†] *Cf.* also SHERRINGTON's very valuable anatomical confirmation of SCHÄFER's statements.   ' Proceedings of the Physiological Society,' March, 1889.

TABLE I.*

Group I.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Tongue protruded	7	..	24	15
Mouth opens	13	..	51	32
Eyes turn	22	..	60	41
Opening eyes	24	..	55	39
Head and eyes turn (simultaneously) .	41	..	54	47
Head turns	42	..	50	64
Retract angle of mouth	58	..	64	82
Shoulder	61	..	71	66
Elbow	62	..	87	74
Fingers	67	..	80	73
Abdomen	70	..	88	79
Wrist	75	..	81	78
Thumb	76	..	83	79
Hip	76	..	94	85
Small toes	78	81	98	88
Hallux	80	..	96	88
Knee	81	..	93	87
Ankle	86	..	95	90

Length of capsule, 100.

Group II.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Eyes turn	25	50	68	47
Opening eyes	37	..	61	49
Head turns	42	62	71	56
Mouth opens	43	..	57	..
Retract angle of mouth	48	55	83	65
Shoulder	55	..	88	71
Tongue protruded	57	..	61	59
Fingers	58	75	86	72
Elbow	58	81	87	72
Wrist	60	..	71	65
Abdomen	60	..	80	70
Thumb	73	81	90	81
Hip	75	..	86	80
Knee	79	..	85	82
Small toes	82	88	91	86
Ankle	84	..	96	90
Head and eyes turn	..	84	..	84
Hallux	92	93	100	96

Length of capsule, 100.

* See also p. 71.

Group III.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Opening eyes	41	54	62	50
Eyes turn	48	49	60	54
Head and eyes turn (simultaneously)	53	. .	60	56
Head turns	56	. .	70	63
Mouth opens	57	. .	71	64
Retract angle of mouth	58	71	79	68
Tongue protruded	62	. .	72	67
Elbow	66	73	88	77
Thumb	69	. .	84	77
Fingers	69	73	84	77
Shoulder	69	70	84	77
Wrist	70	74	88	79
Hip	79	. .	84	82
Abdomen	80	.	86	83
Knee	80	. .	86	83
Ankle	83	. .	89	86
Hallux	84	. .	91	87
Small toes	85	. .	91	87

Length of Capsule, 100.

Group IV.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Eyes turn	23	. .	57	40
Opening eyes	30	45	60	45
Head turns	34	55	65	55
Mouth opens	40	. .	62	51
Shoulder	50	. .	65	57
Retract angle of mouth	50	65	85	65
Fingers	52	. .	80	66
Head and eyes turn (simultaneously)	55	. .	. .	55
Wrist	55	. .	75	65
Elbow	55	. .	80	67
Tongue	55	. .	82	72
Thumb	62	. .	82	72
Ankle	72	. .	85	78
Abdomen	75	. .	85	80
Hip	75	. .	85	80
Knee	75	. .	85	80
Small toes	75	. .	90	82
Hallux	75	. .	90	82

Length of Capsule, 100.

Group V.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Eyes open	38	47	54	46
Eyes turn	38	48	59	48
Head turns.	41	..	58	49
Mouth opens	44	..	48	46
Head and eyes turn (simultaneously) .	53	..	60	56
Angle of mouth retracted	54	76	80	67
Tongue protruded	55	..	76	65
Elbow	57	..	82	69
Fingers	57	..	84	70
Shoulder	58	..	81	69
Abdomen	58	66	85	71
Wrist	62	80	82	72
Thumb	64	..	79	71
Hip	68	..	80	74
Knee.	74	..	77	75
Hallux	77	..	84	80
Small toes	79	..	85	82
Ankle	80	..	88	84

Length of capsule, 100.

Group VI.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Eyes open	40	..	64	52
Eyes turn	42	..	60	51
Head turns	47	61	66	56
Elbow	48	..	75	61
Angle of mouth retracted	51	..	78	64
Tongue protruded	55	..	..	55
Mouth open	55	..	63	59
Shoulder	57	..	69	63
Fingers	59	..	77	68
Thumb	60	..	72	66
Wrist	62	..	72	67
Hip	69	..	77	73
Ankle	71	72	82	79
Knee	76	..	82	79
Abdomen	77	..	83	80
Small toes	78	..	82	80
Hallux	78	..	82	80

Length of capsule, 100.

Group VII.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Eyes open	5	. .	51	28
Eyes turn	5	. .	51	28
Head turns	22	. .	45	33
Head and eyes turn	29	. .	35	32
Tongue protruded	35	. .	52	43
Abdomen	41	52	58	49
Angle of mouth retracted	45	55	65	55
Shoulder	45	. .	65	55
Wrist	48	. .	62	55
Elbow	51	. .	65	58
Thumb	55	. .	61	58
Hip	56	.	65	60
Fingers	57	. .	62	59
Hallux	61	. .	72	66
Ankle	64	. .	. .	64
Knee	66	. .	. .	66
Small toes	66	. .	72	69

Length of capsule, 100.

Group VIII.

Character of movement.	Anterior limit.	Point of observed maximal intensity.	Posterior limit.	Average.
Eyes turn	8	. .	41	24
Eyes open	16	. .	41	28
Angle of mouth retracted	16	28	50	33
Elbow	16	41	62	39
Shoulder	20	58	66	43
Head turns	33	41	66	48
Fingers	37	. .	62	49
Wrist	41	. .	62	51
Small toes	49	. .	66	57
Ankle	49	. .	75	62
Abdomen	50	. .	. .	50
Hallux	53	. .	66	59
Thumb	48	. .	66	62

Length of capsule, 100.

The Configuration of the Basal Ganglia.

It is now necessary to consider in detail the shapes and relative positions of the three chief divisions of basal grey matter, to the fibres of the internal capsule.

We think we may best accomplish this by describing in succession from above down the sections typical of each group of experiments (see p. 65). The position of the genu, which, as will be seen in fig. 1, gradually moves backwards and inwards, as we pass down the capsule to near the base, *i.e.*, until the posterior limb, as seen in these sections, changes from an oblique antero-posterior direction to an almost transverse position. This is strikingly exhibited in Table II., calculated from the observed dimensions of the capsule in each experiment.

TABLE II.

Group.	Average total length of capsule.	Length of Anterior Limb.	Length of Posterior Limb.
	m.m.	m.m.	m.m.
I.	18·2	7	11·2
II.	17	6	11
III.	18	7	11
IV.	20	9	11
V.	21	10	11
VI.	20	9	11
VII.	17	2	15
VIII.	12	0	12

The genu is of course the point of junction of the two limbs, and it will be seen at once that while the length of the posterior limb remains remarkably constant until its axis finally changes direction, the anterior limb on the other hand increases in size from above down, lying in the notch between the caudate and lenticular nuclei (see figs. 1, 2, and 5), until the section approaching the grey matter of the base of the brain, it rapidly disappears (as in Group VIII.), see fig. 1.

It will be convenient next to consider two angles; firstly that which the anterior limb makes with the posterior limb in forming the genu, and secondly the angle of inclination of the long axis of the posterior limb to the middle line. By subtraction of these two from 180° we naturally obtain at once the inclination of the anterior limb to the mesial plane.

Since the angle of the genu is correlated with that of the inclination of the anterior limb, we set both out in the following Table III. (The angles we measured with a goniometer on photographs of the specimens made just after each experiment.)

Each of the angles typical of the several groups was found by taking the average of all the angles of the capsules contained in each group.

TABLE III.

Group.	Angle of genu.	Angle of inclination of axis of posterior limb to mesial line.	Angle of inclination of axis of anterior limb to mesial line.
	°	°	°
I.	134·5	15·8	29·7
II.	123·3	20·3	36·4
III.	118·9	30·8	30·3
IV.	105·1	36·2	38·7
V.	97·1	44·8	38·1
VI.	87·9	55	37·1
VII.	Not ascertainable	40*	..
VIII.	No genu	57·5	..

From this Table it is clear how, while the posterior limb is constantly altering its inclination to the mesial plane, the anterior limb moves comparatively slightly from the same plane in the middle three-fifths of the depth of the capsule. In other words, the caudate wall of the groove separating the anterior ends of the striate nuclei is practically vertical, though slanting outwards rapidly above, more slowly below, as is seen to be the case, when a frontal section is made of the anterior limb. (See fig. 2.)

It is necessary now to give what we have regarded as the defining points of difference between the several groups of capsules experimented on. (See fig. 1.)

Group I. (the highest).—Genu forms a widely open angle (134·5°) so that the capsule is almost a segment of a circle. Of the lenticular nucleus only the outer zone or putamen appears, and it has a bi-convex outline. (See typical photograph, Plate 10, Experiment 35 of our series, and fig. 1.)

Group II.—The genu is distinct and less obtuse, being 123·3°. The second zone of the lenticular nucleus has not yet appeared, though the caudate reaches backward to a point a little posterior to the genu (the anterior part of the surcingle). (See typical photograph, Experiment 67.)

The outline of the lenticular is now becoming triangular.

Group III.—The angle of the genu is becoming still smaller, averaging 118·9°. The second zone of the lenticular nucleus is not yet visible though nearly exposed. The caudate nucleus reaches posteriorly to the genu. The lenticular nucleus is becoming more triangular. In the large majority of sections the pulvinar is evidently free in the ventricle. (See typical photograph, Experiment 42.)

Group IV.—This group is easily marked, since in it we have with the main features given under Group III., the first appearance of the second or middle zone of the lenticular nucleus. (See typical photograph, Experiment 31.)

Group V.—The second zone is more marked, the white interzonal or lamellar fibres

* Probably this figure is unexpectedly low, owing to (1) the crescentic shape of the capsule, (2) the number of observations being few, viz., two.

separating this zone from the putamen are in this section extended backwards almost to the end of the lenticular nucleus. The angle of the genu is nearly a right angle, and the limbs of the capsule are approximately equal in length. (See typical photograph, Experiment 69.)

Group VI.—This group is the same as Group V., from which, as the individual capsules show, it can only be separated by the smaller angle of the genu and by the axis of the posterior limb, becoming more transverse to the longitudinal axis of the hemisphere. (See typical photograph, Experiment 50.)

Group VII.—This group is very definite since it shows sections of all three zones of the lenticular nucleus, and since the level is below the capsular groove between the two nuclei in the corpus striatum, it passes through the basal grey matter of the floor of the lateral ventricle. The anterior commissure appears near the middle line; the part seen corresponds to the highest point of the arch it forms. Since this section is cut along the line of the Fissure of Sylvius, it necessarily is wholly below the frontal lobes. As will be seen, there is a slight anterior limb in this section, see fig. 1. (See typical photograph, Plate 11, Experiment 81.)

Group VIII.—This section just above the crus closely resembles Group VII., save that the anterior limb is no longer represented, and the two innermost zones of the lenticular nucleus cannot now be separated. (See typical photograph, Experiment 79.)

Notes on Groups VII. and VIII.—In these groups the basal ganglia and capsule, as divided by these sections, present a very constant figure; thus the posterior limb of the internal capsule makes an angle with the middle line of about 60°, its posterior border being formed by the optic thalamus, while anteriorly there are ranged along its front the respective ends of the three grey zones of the lenticular nucleus. If the section pass just above the Fissure of Sylvius the three zones will be seen, if lower, the two innermost zones are found to be confused into one grey mass and cannot be separated.

In the *highest sections* (*i.e.*, those in which the three zones are distinct) of these groups, there are the remains of the anterior limb of the capsule lying to the inner side of the innermost zone of the lenticular nucleus. This remnant of the anterior limb is (in the highest sections), about 3 mm. long in the antero-posterior direction. This bundle of white fibres is not in immediate contact with the anterior commissure, but is separated from it by some of the grey matter of the lenticular nucleus. This delimitation of what we consequently call the anterior limb can be easily seen, and it is from this point that in this group of sections we have commenced the measurement of the capsule, continuing the same as before along its outer margin, *i.e.*, contiguous to the lenticular nucleus, as far as the hinder end of the outer zone of that nucleus (see fig. 1).

The remnant of the anterior limb above described is bounded on its inner aspect,

i.e., along the middle line of the cerebrum by a narrow (1–2 mm. wide) strip of inexcitable tissue.   (? Septum lucidum and anterior perforated spot, grey matter.)

Summary and Application of the Anatomical Facts.

It remains to connect these anatomical data with the facts observed upon excitation.

In the first place discrimination must be made between the effects of exciting fibres in their continuity and on their transverse section respectively, since it is plain that the identity of result might otherwise lead to an undesirably extended view of the limits of the efferent tract.

Reference to fig. 1 and to Table I. shows that some of the efferent excitable fibres which partly compose the anterior limb in Group I. are rapidly concentrated in the posterior limb as we descend to Group III. These fibres are those for the movements of opening the mouth and protrusion of the tongue, as is shown by a glance at fig. 1, C and F.

At Group IV., fig. 1, it is evident that a further change now occurs   Apparently owing to the appearance of the second or middle zone of the lenticular nucleus, that part of the anterior limb which is opposite this second zone is found to be excitable, whereas the rest, which really exactly corresponds with the whole anterior limb of Group III., is inexcitable as in that Group.

It is to be observed (fig. 1) that this excitability of the posterior part of the anterior limb again diminishes until we arrive at Group VII., where, upon the further appearance of the third or innermost zone, the hinder end of this limb is seen to be excitable.

To speak next of the inexcitable fibres of the anterior limb, we have in the foregoing pages referred to the views of those who have treated the subject, and we only would express our belief that these fibres of the anterior limb are for the most part horizontal, and that they pass backwards through the genu to the thalamic region.

A point of equal interest is the arrangement of excitable and inexcitable fibres in the posterior limb of the capsule.

In Groups I. and II., fig. 1, the posterior limb is excitable in its whole length, but, in Group III., the hindermost part (*i.e.*, for the last $\frac{9}{100}$ths) is inexcitable. This inexcitable portion steadily increases as we pass downwards (see fig. 1), so that in Groups VI., VII., and VIII., the excitable part terminates posteriorly at the hinder limit of the middle zone.

The classical dictum,[*] that the sensory fibres are contained in the posterior *third* of the hinder limb of the capsule only holds good for the lowest levels. The gradual character of the development of this inexcitable (afferent) part of the capsule is interesting by way of contrast with the variable arrangement of the excitable fibres in the anterior limb just discussed.

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[*] CARVILLE and DURET, CHARCOT, *loc. cit.*

Genu.—Hitherto we have only indirectly spoken of the genu and its relation to the surrounding parts. To determine its position exactly at the different levels of the capsule, we first, by means of successive horizontal sections parallel to the longest antero-posterior diameter of the hemisphere, exposed the level of our Group I. We then inserted (vertically) a long needle at right angles to the plane of the section just at the genu immediately in front of the most anterior extremity of the optic thalamus. Each successive horizontal (parallel) section of the hemisphere contained, consequently, the position of the genu as in the highest section (the upper level, Group I.).

From these data we found that the position of the genu in its antero-posterior relations varied extraordinarily little, but that the position of the entering angle of the inner border was gradually transferred to that of the outer border, and, finally, in the last three groups, the puncture revealed the fact that the position of the genu, as seen in the highest section and now projected upon the lowest, had moved about 2 mm. in front of, and outside the angle of the outer border of the capsule, or 4 mm. from its original site. In other words, the genu had moved backwards and inwards, and this change of position was more rapidly developed in the lower than the upper sections.

Of course, the inward and backward direction of the efferent tracts is also seen (especially the former) in frontal sections of the capsule.

Laminæ Medullares.—The white fibres, which in horizontal sections of the hemisphere, appear as lamellæ to separate the zones of the lenticular nucleus, but which, in vertical frontal sections, appear to run into and connect the zones, we have repeatedly excited without any result. We, therefore, are led to conclude that these fibres, whatever be their function, do not belong to the great excitable cortico-efferent system.

OBERSTEINER* gives a schematic diagram, in which he figures fibres passing down from the cortex through the laminæ medullares to end in the subthalamic region.

WERNICKE† suggests that these interzonular, or lamellary fibres, are in part derived from the nucleus caudatus, and are passing down to the crus.

MEYNERT‡ believes that such fibres are derived from the cortex rather than from the nucleus caudatus.

METHOD OF INVESTIGATION IN THE PRESENT RESEARCH.

Animal Employed.—The Bonnet Monkey (*Macacus sinicus*), *i.e.*, the smaller variety of that species of Monkey, was the animal which we have invariably used.

In all we have performed 45 experiments, and in every case the animal was com-

* *Loc. cit.*, p. 334
† *Loc. cit.*
‡ *Loc. cit.*

pletely under the influence of ether, and was killed before it recovered from the anæsthetic.

With few exceptions the left hemisphere was the one used for the examination, and, as a rule, the cortex had just before been minutely explored.   It was found most convenient to remove the vault of the cranium for about 1 cm. to the other side of the middle line, and laterally as low as the zygoma in front, and behind to a corresponding level.

The dura mater was reflected over the area thus exposed, care being taken not to injure the longitudinal sinus ; the hemisphere was then drawn towards the middle line so as to allow of the middle cerebral artery being ligatured soon after its origin, this was easily done by passing a fine thread beneath the vessel through the contiguous portions of the orbital and temporo-sphenoidal lobes.

A horizontal incision was then made with a long sharp scalpel through the substance of the hemisphere, from the outer surface towards the middle line, but stopping short 1 or 2 mm. of the mesial plane.   In this way section of all arteries, save the branches of the ligatured middle cerebral was avoided, and, hence, hæmorrhage was in a great measure prevented.   The only vessel which gave much trouble was the lenticulo-striate artery running vertically through the outer zone of the lenticular nucleus.   In the first two experiments, we sought to arrest this hæmorrhage by a fine point of an actual cautery, but we found, to our surprise, that although the area touched was extremely small, yet the cut ends of the fibres of the adjacent internal capsule were so damaged (by the radiant heat of the cautery) as to be inexcitable ; moreover, we speedily found that all hæmorrhage could be easily arrested by placing on the bleeding point a small cube of amadou.

When the hæmorrhage had thus been controlled the upper portion of the hemisphere which, as described above, had not been completely separated, but merely raised like the lid of a box, was replaced, and preparations were made for recording the results of the stimulation.

The plan of recording which appeared to us likely to afford the most accurate results, and, at the same time, not to involve so much delay in exposing the capsule as to cause the death of the fibres, was as follows :—

Upon paper, on which fine lines were engraved by an engine with mathematical accuracy so as to cover the surface with squares of 1 mm. side, we drew by means of compasses the exact outline of the basal ganglia as exposed by the section.

In this way we obtained at once the cut surface of the internal capsule correctly projected on paper, divided into squares of 1 mm. ; these squares we then numbered from the front of the anterior limb of the capsule to the posterior end of the lenticular nucleus.   Finally, we stimulated each of these bundles of fibres thus obtained of 1 square mm. area, and recorded the effect produced.

Mode of Stimulation.—The electrodes used were two fine platinum points, 1 mm. apart, so that the excitation should be exactly limited to each square excited.   The

exciting current was an interrupted induced current obtained from the secondary coil of a Du Bois Reymond inductorium, supplied by a 1 litre bichromate cell. The distance of the secondary coil from the zero point of completely covering the primary was such as to give a minimal stimulus, and this was usually about 10 cm.

After the experiment was completed the animal was killed by excess of chloroform, the hemisphere removed, washed in salt solution and the surface photographed.

Upon the photograph thus obtained, the numbers representing the different bundles of fibres (1 mm. square) were transferred by compasses from the plan originally drawn on the ruled paper.

Mode of Enumeration and Recording Results.—We have just said that we numbered the millimetre squares of the internal capsule in order from before back ; we adopted this plan for the reason that as the anterior limit of the excitable portion of the capsule varies in position, *i.e.*, retreats as the section descends (*cf.* fig. 1.), it was necessary to express the position of the fibres having the same function by some means which should bear an absolute value in relation to the rest of the capsule. By this means we have been enabled to trace the bundles having the same function through their various changes of direction as they pass down through the different sections of the capsule to reach the crus.

Inasmuch as the capsule is divided into millimetre squares by the above method, each figure denoting any bundle of fibres represents at the same time the distance of that bundle from the front end of the anterior limb.

In plotting out the plan of the capsule during an experiment we arranged it so that the more important posterior limb should be drawn parallel to one direction of the rows of squares on the ruled paper ; consequently, as the anterior limb forms an obtuse angle with the posterior limb, the squares dividing it were necessarily échelonned. This, however, afforded no real difficulty, and the more so since in the most important sections very little of the anterior limb is excitable. We, therefore, expressed each square or bundle of fibres by a fraction, the numerator of which denoted the distance that it was situated from the anterior end of the capsule, while the denominator gave the total length of the capsule in that particular section ; in this wise one fraction, or, in other words, the position of one bundle of fibres in one section, is strictly comparable with that in another section.

In order to bring all the fractions thus obtained together, and to find the average position of the representation of any given movement in each group, we converted all the fractions into decimals to two places. We took two places as adequately accurate, since any error beyond would only amount to $\frac{1}{1000}$th of the length of the capsule, and as this actually amounts, on the average, to not more than ·02 mm., it is a length which is far too small to be considered among the errors of the experimental method.

The decimals thus obtained, denoting the squares indicating the localisation of each movement elicited, were next added together in the several groups, and the average taken.

Since these decimals were taken to two places, each capsule in every group becomes divided into a hundred parts, and if we drop the decimal point, we may regard the figure as a whole number, taking the whole capsule to be 100. For example, the movement of "protrusion of the tongue" was observed in Group II. to occur in two experiments, the length of the capsule being respectively in each case 12 mm. and 17 mm., the average therefore being 14 mm. The movement was observed to follow excitation of the first millimetre in each case, and to end at the third millimetre in the one case and at the fourth in the other. Consequently the movement began at $\frac{1}{12}$th and $\frac{1}{17}$th of the actual length of the respective capsules, and was last obtained at $\frac{3}{12}$ths and $\frac{4}{17}$ths.

On converting these fractions into decimals they become respectively ·08 and ·06 for the anterior limit and ·25 and ·23 for the posterior limit, and on taking the averages of these we obtain ·07 and ·24.

Now that we have the situations of the fibres expressed in terms of the average length of the capsules of each group, we need not keep the decimal point if we regard the capsule as divided into a hundred parts, and then take the figure as a whole number, so that in Table I. the limits of the representation of the protrusion of the tongue become 7 and 24, i.e., $\frac{7}{100}$ and $\frac{24}{100}$.

This mode of treating the figures has this great advantage, that whatever the length of the different capsules may be in any group, we can compare the function of the fibres in similar parts.

At the same time this mode of extending the differentiation of the capsule has this disadvantage, viz., that, although taken absolutely, as is represented by the fraction $\frac{1}{14}$, we obtained the movement from the most anterior millimetre of the capsule, yet in converting $\frac{1}{14}$ into a decimal to two places, we necessarily obtain $\frac{1}{14} = ·07$, and then, in regarding the capsule for purposes of relative comparison as divided into 100 parts (dropping the decimal point), we thus obtain the figure 7, meaning the first seven-hundredths, as representing the first portion at which the movement occurred.

In fact, in attempting to analyse and to compare the results of each experiment, i.e., the fractions, we found it impossible to employ the method of a common denominator, and therefore selected, at Mr. GOTCH's suggestion, the two places of decimals. We were obliged to take two places of decimals, since, as a glance at Table I.[*] shows, one place of decimals would not have given the differentiation we have discovered to exist, though at the same time it was clear to us that two places would so extend our subdivisions of the capsule as to make it difficult to convey the absolute expression of the amount (i.e., extent) of representation.

These averages are set out in the last column in Table I., in which the movements are also arranged in their order from before back.

In Table I. the first column gives the names of the movements observed, and, as just stated, in the order in which they occur from before back. The first column of figures denotes the distance from the anterior extremity of the capsule of the first

* See p. 60.

bundle of fibres, excitation of which produced the movement. The third column gives the most posterior point or bundle of fibres at which the movement of a part could be elicited, so that the distance between the figures in these two columns gives the total or absolute extent of the fibres from which that movement of a part could be obtained.

The intermediate or second column gives, in those cases where it was most marked, the point of maximal representation, *i.e.*, of greatest intensity.

Further, it is seen at once that the distribution of the fibres for any one part is not allocated to one small point in the capsule, but spread over it for a certain distance, so that the overlapping seen on the cortex occurs here, though to a larger extent.

To further elucdiate the arrangement of the fibres and to facilitate future reference, we have constructed a series of diagrams (fig. 1, A–R.) on exactly the same scale as fig. 1 (*i.e.*, twice the natural size), on which we have marked the extent of representation in the different groups of sections of each segment observed. In each figure the shaded part represents the fibres, excitation of which produces movement of the given segment. The limits of these fibres were obtained from the numbers of Table I. As we have already drawn attention in describing fig. 1, we would remark on the extraordinary diminution of the fibres before the successive appearance of the zones of the lenticular nucleus, and the equally striking increase of the fibres when these zones become visible, *e.g.*, Groups IV. and VII.

Good examples of this are to be seen in the diagrams for the turning of the eyes, the small toes and hallux (fig. 1, B, Q, R).

Physiological Relations of the Basal Ganglia to the Internal Capsule.

Although the ventricular surface of the caudate nucleus has been asserted to be excitable by Carville and Duret,[*] Ferrier,[†] and Minor,[‡] we have failed to obtain the slightest movement by exciting the sectional surfaces of either of the basal ganglia. We do not advance these observations as of special value, considering that we removed the ventricular surface, but the contrast between the absence of any result when the ganglia were excited and the marked spasm which immediately followed the application of the electrodes to the capsule is in harmony with the observations of Franck and Pitres,[§] and with whose general deductions on this point we fully agree.

To finally elucidate this important point, we have also excited the intra-ventricular surfaces of the basal ganglia, but without any positive result.

Consequently on review of the experimental evidence, it seems to us most likely that the basal ganglia are really inexcitable.

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[*] *Loc. cit.*
[†] Ferrier, *loc. cit.*, p. 264.
[‡] 'Neurologisches Centralblatt,' June, 1883.
[§] *Loc. cit.*

Analysis of Results.

On pp. 60–63 we give, in Table I., the order of arrangement of the representation of different parts of the body. We cannot well separate the consideration of this arrangement from that of the order in which the different characters of the movements are placed, hence we will reserve to p. 79 the discussion of the facts set out in Table I.

We will, however, proceed to discuss certain points of general interest upon which a clear opinion must be formed before the main results of this research can be given at length. Of such points we take first, as the most important, the phenomenon of bilateral movement.

Movements of Muscles of the same side as that stimulated.

The phenomenon of bilateral movement occurring upon excitation of one hemisphere has often been approached from its theoretical, experimental, and clinical aspects. It has been assumed by Broadbent[*] that this is effected by the impulses passing from the sound hemisphere across commissural fibres postulated to exist between the lower (*i.e.*, bulbo-spinal) centres of the two sides, and he explained the common mode of recovery from hæmorrhagic compression and destruction of the internal capsule by this theory. This theory has been favourably received by neurologists as an explanation of the phenomenon of bilateral movement, but we submit that the subject can only profitably be treated upon experimental facts as a basis.

Before any satisfactory conclusions can be arrived at, we must obviously first determine exactly how much bilateral representation of movement exists in the cortex of a normal uninjured nervous system, and, in this case, of the Bonnet Monkey. Further we must understand clearly what we mean by bilateral representation.

Beginning with this last, we mean that the same movement of symmetrical parts of both sides of the body is represented in one point of the cortex of both hemispheres, *i.e.*, that excitation of such a point evokes simultaneous and identical movement in the same part of both sides of the body.

In attempting to classify "bilateral movements" it is necessary to point out first that movements such as conjugate deviation of the eyes are not, strictly speaking, examples of bilateral representation. For while it is true both eyeballs move, nevertheless one is rotated out, the other rotated in, and since this conjugate deviation occurs as the result of a single idea, the movement of both eyeballs really corresponds to that of one limb, and this is more especially the case since, as just said, the muscles employed on each side are different. A final confirmation of this view lies in the fact that excitation of one hemisphere produces conjugate deviation in one direction only, while excitation of the other evokes an exactly opposite movement.

* 'British Medical Journal,' vol. 1, 1876, pp. 333, 371, 401.

To this point we shall subsequently return. We may now mention the movements which are credited with being bilaterally represented, and we shall arrange them in classes, the justification for which will subsequently appear.

Credited as Bilateral.

> Movements of Trunk Muscles (*i.e.*, Rectus, abdominis, &c.)
> Movements of Tongue
> Conjugate deviation of Eyes
> Turning of Head

Class I. Not bilateral, *vid. inf.*

> Retraction of Angle of Mouth
> Pursing of Lips
> Opening of Eyelids } Identical move-
> Closing of Eyelids }   ment

Class II. Imperfectly bilateral.

> Pouting of Lips
> Mastication
> Swallowing (*i.e.*, movements of soft palate)
> Adduction of vocal cords

Class III. Truly bilateral.

Class—I. *Movements asserted to be bilateral, but not actually so.*

Movements of the Trunk Muscles.—The muscles of the trunk have above all others in the body been commonly supposed to be bilaterally associated in action. This is, however, quite contrary to observed fact. We have never seen any movement of both recti or oblique abdominis follow excitation of the fibres for the trunk muscles in the capsule, although we invariably saw movement restricted to the muscles of the opposite side, notably the rectus. We have observed this to be the case clinically, and believe that the statements made in the opposite direction rest upon imperfect examination. Further confirmation is to hand in the description given by Professor Schäfer* and one of us, of the results of excitation and ablation respectively of the " trunk areas " in the marginal gyrus, from which it appears that the same unilateral representation was discovered to exist.

Movements of the Tongue.—As we are specially investigating the cortical representation of these movements, we desire to defer the detailed observation of the capsular fibres subserving the same, until the former is completed.

Conjugate Deviation of the Eyes and Turning of the Head.—We have already mentioned the movements of the eyeballs, and we now intend to strengthen what is said on p. 73.

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* Horsley and Schäfer, 'Phil. Trans.,' B, vol. 179, 1888, pp. 10 and 14.

We there showed that the deviation of the eyes was a combined movement of opposed muscles (*internal* rectus with *external* rectus) of both sides of the body to produce one effect, and that the opposite hemisphere, when excited, produced exactly the opposite effect.

Further, considering that the visual field is hemiopic, and that the two eyes always have their axes parallel (when not accommodated for near objects), we may look upon the two eyes as acting like one, and we may also regard their movements to be of the same kind as those of the head.

Stimulation of the left cortex produces movement of the head and also of the eyes to the right, and, conversely, when the right cortex is irritated these parts move to the left. Movement to the same side is never observed, hence these movements we regard as distinctly unilateral.

Class II.—Imperfectly Bilateral Movements.

(a.) *Opening of Eyelids.*—In 'Phil. Trans.,' B., 1888, p. 241, we describe how we observed in one Monkey that there were two millimetre-squares on the cortex, excitation of which caused opening of the opposite eyelids to occur before those of the same side. In the present research we have obtained the same result from the internal capsule in one case (Monkey No. 69), at points 50–55 (whole capsule, 100), the movement being also stronger.

The extreme rarity of this phenomenon—occurring, as it does, only twice in about seventy different experiments in the same variety of Macacus—makes it clear that the movement of *opening* the eyelids is practically bilaterally represented. However, the phenomenon is not without interest in discussing the evolution of function in the " motor" cortex.

(b.) *Closure of Eyelids.*—In contradistinction to what we find with regard to opening of one eye before the other, we have obtained closure of the opposite eyelids, not only taking place before those of the same side, but even independently of the eyelids of the same side.

We drew attention for the first time to this differentiation of cortical representation of the eyelids in our second paper just referred to, where we described the opening of one eye before the other. Since writing that paper, we have been engaged in investigating the cortical representation of closure of the eyes, and we have found that there is a nearly horizontal zonular strip of the cortex just below and slightly overlapping the lower border of the upper limb area of representation, in which the single movement of closure of the opposite eyelids is represented.

In some cases this is imperfect, *i.e.*, the eyelids of the side stimulated closed as well as those of the opposite side. It is clear, therefore, that this movement is only imperfectly bilateral, a fact of the greatest importance, as is evidenced by the fact that we have been able with this observation to correctly diagnose that the homologous portion of the cortex in man was the seat of an irritative lesion.

Hence we were not surprised to find that in the internal capsule there were fibres, excitation of which produced closure of the opposite eyelids only.

(c.) *Pursing of the Lips.*—This is another imperfectly bilateral movement, it being most marked always on the opposite side.

(d.) *Retraction of the Angle of the Mouth.*—Rarely have we observed truly bilateral, *i.e.*, symmetrical, retraction of the angle of the mouth, this being the exception to the rule of retraction to the opposite side.

Class III.

(a.) *Pouting of the Lips.*—This eversion of the lips appears to be truly a bilateral movement.

(b.) *Mastication.*—So far as we have been able to observe, this complex movement is mainly bilateral, but we are compelled to postpone its detailed consideration until our analysis of its cortical representation is complete.

(c.) *Swallowing, including movements of the Soft Palate.*—This in our experience of the capsule has always been bilateral on excitation.

(d.) *Adduction of the Vocal Cords* has been shown [*] to be perfectly bilaterally represented.

(e.) *Bilateral Movement of both Elbows.*—This (as flexion) we have found to occur four times altogether in our experiments, when the internal capsule of one side has been excited. The occasions, however, on which this was observed were so few, and the movement so slight and limited that we cannot regard it to be a proof of bilateral representation of this segment of the upper limb in one hemisphere. Moreover, during a long series of experiments upon the cortex this was never observed.

The explanation of this phenomenon we prefer to postpone until further knowledge makes its discussion profitable.

It occurred at point 57, and only in the central [†] Groups, Nos. IV., V., VI.

To sum up, therefore, we consider the following movements to be alone truly bilaterally represented.

Pouting of Lips.
Mastication.
Swallowing.
Adduction of the Vocal Cords.

It will now be expedient to describe the general results obtained by the various groups of experiments.

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[*] KRAUSE, in the Dog; SEMON and HORSLEY, in the Monkey. This is denied by MASINI, and by GAREL and DOR.

[†] *I.e.*, opposite the middle level of the capsule.

Lateral Arrangement of the Excitable Fibres in the Internal Capsule.

We now introduce a new branch of this subject, namely, the relations in lateral juxtaposition of the efferent fibres.

So far we have described the antero-posterior arrangement of these fibres, but we have postponed the consideration of the results obtained by exciting the capsule in individual points of its breadth.

As the capsule varied in width from 1 to 3 mm., and as the part of the sections involving the inner zones of the lenticular nucleus was also constituted of white fibres in and around those zones, we have sometimes found five rows of excitable points (each 1 mm. square). The result obtained from these points we shall now analyse. Altogether we were able to obtain two or more rows of 1 mm. squares in 26 experiments.

Anatomically speaking, in horizontal transverse section we may regard the fibres of the internal capsule as falling into three groups, viz. :—(a) those in the mid line ; (b) those next the lenticular nucleus, and (c) those on the inner side next the optic thalamus, &c.

Of the 26 cases above-mentioned, in seven the excitability appeared to be equally distributed throughout the breadth of the capsule. However, besides these seven, we found that in 15 the most excitable part was that next the lenticular nucleus, this gradually decreasing towards the optic thalamus, but in the remaining four the capsule was most excitable alongside of the thalamus. Considering the anatomical arrangement of the cortex mantle to the position of the basal ganglia and internal capsule, it is not at all improbable that these results may be referable to the fibres entering the capsule from above in an oblique direction, and hence in a horizontal section, they will offer for excitation not only a transversely cut end alongside the lenticular nucleus, but also longitudinal surfaces, which are progressively less and less exposed to excitation as we move across the capsule from the lenticular nucleus to the optic thalamus.

If correct, this view would postulate the theory that the fibres forming the part of the capsule nearest the middle line are inexcitable. This, at any rate, is evidently strongly suggested by the experiments. Moreover, in a frontal microscopic section of the region, the fibres which emerge from the gyrus fornicatus on reaching the capsule, are seen to occupy the most internal position therein, and excitation of the gyrus fornicatus has always failed to elicit movement.

To sum up this question, it is self-evident that, the lateral arrangement of the fibres is infinitely less important than that of the antero-posterior direction, and principally because the function of each point in any given transverse row of squares is the same in character though diminishing in intensity, as we pass from the outer to the inner side of the capsule.

TABLE IV.

Groups.																		
I.	Tongue	Mouth opens	Eyes turn	Eyes open	Head and Eyes turn	Head turns	Angle of Mouth	Shoulder	Fingers	Elbow	Wrist	Trunk	Thumb	Hip	Knee	Small toes	Hallux	Ankle
II.	Eyes turn	Eyes open	Mouth opens	Head turns	Tongue	Wrist	Angle of Mouth	Trunk	Shoulder	Fingers	Elbow	Hip	Thumb	Knee	Small toes	Head and Eyes turn	Ankle	Hallux
III.	Eyes open	Eyes turn	Head turns	Mouth opens	Tongue	Angle of Mouth	Elbow	Shoulder	Thumb	Fingers	Wrist	Hip	Trunk	Knee	Ankle	Hallux	Small toes	
IV.	Eyes turn	Eyes open	Mouth opens	Head turns	Head and Eyes turn	Shoulder	Angle of Mouth	Wrist	Fingers	Elbow	Tongue	Thumb	Ankle	Trunk	Hip	Knee	Hallux	Small toes
V.	Eyes open	Mouth opens	Eyes turn	Head turns	Head and Eyes turn	Tongue	Angle of Mouth	Elbow	Shoulder	Fingers	Trunk	Thumb	Wrist	Hip	Knee	Hallux	Small toes	Ankle
VI.	Eyes turn	Eyes open	Tongue	Head turns	Mouth opens	Elbow	Shoulder	Thumb	Wrist	Fingers	Hip	Aukle	Knee	Trunk	Hallux	Small toes		
VII.	Eyes turn	Eyes open	Head and Eyes turn	Head turns	Tongue	Trunk	Angle of Mouth	Shoulder	Wrist	Elbow	Thumb	Fingers	Hip	Ankle	Hallux	Knee	Small toes	
VIII.	Eyes turn	Eyes open	Angle of Mouth	Elbow	Shoulder	Head turns	Fingers	Trunk	Wrist	Small toes	Hallux	Ankle	Thumb					

Character of the Movements of the Parts Represented in the Fibres of the Internal Capsule.

We have given at pp. 60–63 the antero-posterior arrangement in the capsule of the movements of the different parts of the body (see Table), we now furnish two tables (see Tables IV. and V.) in the latter of which again taking the whole capsule as 100, the extent of representation of each kind of movement of the several parts of the body is given.

We wish first to review the Table IV., given on p. 78, in detail, to see how far the order of arrangement of representation of parts, *i.e.*, independently of the character of the movement of such parts in the capsule, agrees or disagrees with the arrangement which we have shown to exist in the cortex.

The accompanying diagram (fig. 7) the details of which we have partly published in the ' Philosophical Transactions,' 1887–88, the remainder being in course of publication, gives in bare outline the relative positions of the points of chiefest representation as we have found them, by minimal excitation, to exist in the cortex of the Bonnet Monkey (*Macacus sinicus*).

With this outline before us we would suggest that the main axes of the fibres of the internal capsule correspond as far as antero-posterior arrangement goes with lines drawn at right angles to the direction of the upper two-thirds of the fissure of ROLANDO. It also remains to be seen whether there is any relation between the arrangement in the cortex in the horizontal antero-posterior division and the order of fibres in the capsule.

Table IV. contains the antero-posterior arrangement of the average position of the representation in the capsule of the various parts of the body, set out horizontally. Although each observation is placed vertically under each other, this is not necessarily true of their anatomical position, as we have already shown.

It substantiates our view that the oblique lines over the " motor " cortex described above give truly the order of representation in the capsule. For while the movements of opening the eyelids and of the eyes are seen to occur in front in both cortex and capsule, those of the toes are equally the most posterior. Hence placed in order from before back we have eye movements, head movements, upper limb movements, trunk and lower limb movements.

Or stating in further detail the order of the representation as given in the foregoing table in the average position of each segment in each part of the body, the following general order of representation is arrived at, viz. :—

Summary of averages :—

44	Eyes open ⎫	
42	Eyes turn ⎭	Eye movements
50	Mouth opens	
50	Head and eyes turn ⎫	
52	Head turns ⎬	Head movements
53	Tongue	
60	Angle of mouth	
62	Shoulder ⎫	
64	Elbow	
66	Wrist ⎬	Upper limb movements
66	Fingers	
70	Thumb ⎭	
70	Trunk	
76	Hip ⎫	
77	Ankle	
78	Knee ⎬	Lower limb movements
78	Hallux	
80	Toes ⎭	

Further, if we state for each of these their relative positions to the whole of the capsule, we obtain figures which give the relative positions absolutely as above shown by the numbers placed at the left hand side of the column.

We draw attention by brackets to the association of parts in movement as already shown to occur in the cortex.

Nothing, we believe, shows more distinctly the truth of our position that the arrangement in the capsule is but an imitation of that on the cortex, which in its turn is but a peripheric projection of the order of the metameres of the whole body.

The accuracy of this general view is borne out by an analysis of the representation of the segments of one, the lower limb. In both the cortex and capsule the hip is most represented in front, the small toes hindermost (*i.e.*, posterior to the hallux), and the other joints intermediately. Similarly, in accordance with the view of SCHÄFER and HORSLEY[*] that the trunk muscles are represented mediately in the "motor" region, we find that in the capsule the representation of this part is, as a rule, between the upper limb and lower limb regions.

For the upper limb the arrangement in the capsule is analogous to that for the lower limb, namely, the largest joint, the shoulder, is most anterior, and the smallest, the most specialised, the thumb, is most posterior. By dropping verticals in fig. 7, on p. 81, this is evidently the cortical arrangement projected below.

* 'Phil. Trans.,' B, 1888, p. 9; 'Roy. Soc. Proc.,' vol. 36, 1884, p. 437.

Fig. 7.
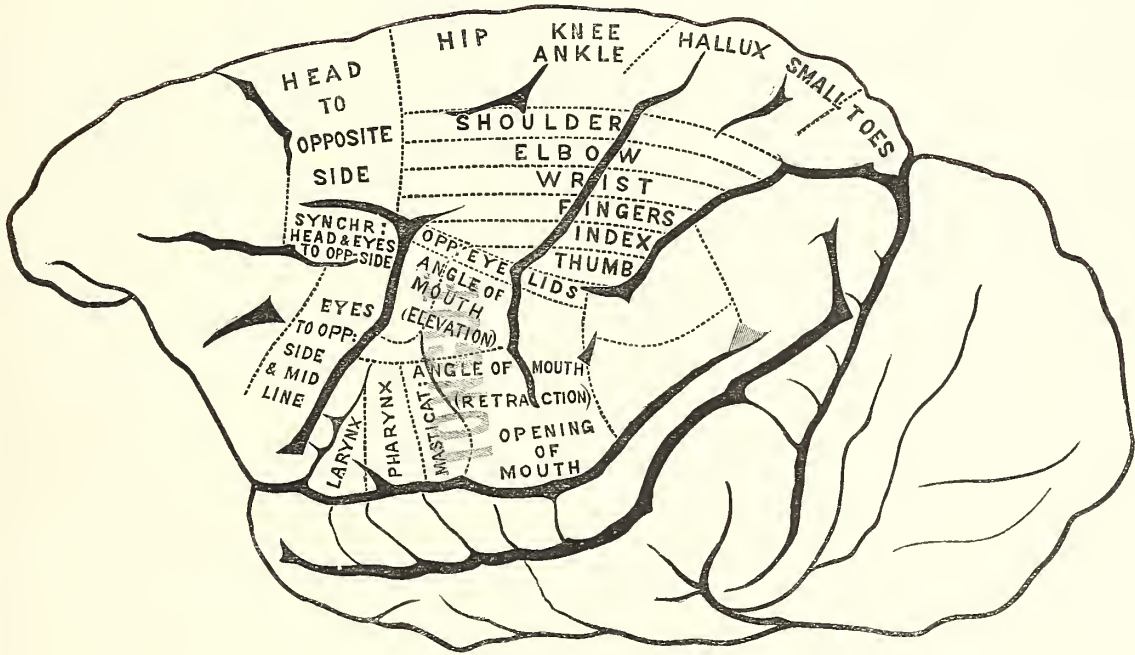
Arrangement of excitable fibres in the internal capsule.

Drawn from a photograph (magnified twice) of the outer surface of a Monkey's (*Macacus sinicus*) left hemisphere.

TABLE V.—(Length of Capsule = 100.)

Kind of Movement.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.
Eyelids, both sides open	24–25	37–61	41–62	30–60	38–54	40–64	5–51	16–41
„ opposite side more than same side	..	..	..	..	50–55	..	33	
„ both sides close	..	..	..	66–75	100 (1 observ.)	..		
„ opposite side alone closes			67–77	..	60–65	64	60–62	75 (1 observ.)
Eyeballs become parallel								
„ turn to opposite side	22–60	25–68	48–60	49–51	40	42–60	33	8–41
„ turn to opposite side and downwards	8–16	..	..	23–57	38–59	..	5–51	
„ turn up and to opposite side	..	..	..	..	25–60	..	5	
Pupils dilate		..	..	42	45	..	..	16–25
Head and eyes turn synchronously to opposite side	41–54	..	53–60	55	53–60	..	29–35	
Head turns to opposite side	42–64	42–71	56–70	34–65	41–58	47–66	22–45	33–66
„ turns, muzzle up	52	..	75	..	..	..	..	
„ retracted	..	..	..	65–90 (1 observ.)	..	..	..	
Lips, depression of opposite lower	..	..	..	..	54–68	..	..	41–50
„ pouting of both	..	56	..	..	..	..	..	
„ pursing of both, but especially on opposite side	..	..	..	65	..	..	..	
Angle of mouth,* opposite retracted	58–82	48–83	58–79	50–85	54–80	51–78	45–65	16–50
„ both sides retracted	..	..	58	..	..	63	..	
Mouth opens	13–51	43–57	57–71	40–62	44–48	..	..	
„ shuts	..	25–37	73 (1 observ.)	..	..	..	..	
„ opens and shuts (as in mastication)	14–69	44–47	77	45–47	61–63	88–94 (1 observ.)	45	
Tongue protruded straight	7–24	57–61	47	52–69	64–65	55	35–47	
„ to opposite side	..	..	..	66–83	57–62	..	..	
„ to same side	..			..	..	..	..	
„ to same side (concave to opposite side)		..	73		..	..	..	
„ narrowed	..	..	78	..	75–80	..	..	
„ retracted, heaped up posteriorly	..	..	52	63	76	..	..	

* We have not in this paper differentiated between elevation and retraction of the angles of the mouth.

Table V.—continued.

Kind of Movement.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.
Tongue retracted, tip retracted and protruded	..	..	..	77	64–69	..	52	
„ (1) to same side	..	50	..					
„ (2) to opposite side	..	50	77					
„ heaped behind, rolled, and protruded to same side	..	..	68	61–72	65–72			
Swallowing	..	..	..	65				
Barking	..	..	.36					
Shoulder, advancing	62–76	..	74–82	62–63	66–79			
„ retract	..	..	75	80	59	55		
„ adduction	..	68–86	75–83	60–83	66–80	65–89	45–62	49–66
„ abduction	..	..	77	40				16–25
„ rotation in	..	56	..	77	65	..	66	41–58
„ rotation out	67–70	63–70	67	59	55–71	..	51–54	16–50
Elbow, extension	70	58–62	66–73	60	57–59	55–61	41	17
„ confusion	75–91	68–74	75–80	58	70–76	57–67		
„ flexion	60–91	69–86	70–87	57–83	66–79	57–77	51–65	33–62
Wrist, extension	75–100	63–73	75–90	58–74	65–76	68–72	62	
„ flexion	..	..	64–73	60	78	..	70	62
„ pronation	80	..	89–100	45–72	67–71	65–69	41–64	41–50
„ supination	..	..	..	55–63	68	..	..	50
„ ulnar adduction	..	..	..	..	..	69		
„ confusion	..	..	..	..	..	60		
Index finger, flexion	..	..	..	..	..	60–69		
„ flexion, extension	..	..	..	..	..	65		
Fingers, extension	66	61–72	65	62–68	67–71	57–61	50	
„ extension, flexion	91–100 (1 observ.)	..	83	..				
„ interosseal flexion		..	84–90	..	50			
„ flexion	65–77	65–86	75–82	62–79	63–85	68–82	62	37–62
„ confusion	..	77	..					
Thumb, extension	..	61	..	70	79	65	55	
„ extend, flexion	..	..	94 (1 observ.)	(1 observ.)				

M 2

TABLE V.—continued.

Kind of Movement.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.
Thumb, interosseal flexion	83–87	79–88	82–88	..	76	62–73	61	62
" flexion	71–76	..	78–87	65–81	68–80	65		
" flexion, extension	..	..	..	..	..			
" abduction	..	..		45–50	..			
Trunk, opisthotonus	70–76	61–72	60	..				
" rotation to opposite side	..	..						
" arching to opposite side	..							
Abdomen, opposite rectus	70–88	60–80	80–86	80	58–85	77–83	41–58	
Tail, adduction or flexion to opposite side	..	..	87	..	..	77–88		
" extension to opposite side	..	..	84	..				
Hip, extension	100	83	88	..	52–71	68–79	65	
" confusion	62–71	77	81–87	..				
" flexion	..	85–90	74–85	74–91	72–76			
" adduction	85–89	75	88	85–90	85–90			
" rotation out	..	..	..	8 3	68–70			
" rotation in	..	68–81	..	..	75–85	69		
Knee, extension	100	..	78–82	..	75			
" confusion	..	77	84		..			
" flexion	79–83	81–93	84–86	76–88	74–77	76–82	66	
" rotation in	..	..	78		..			
Ankle, dorso flexion	78–81	92–96	83–89	73–82	83–88	72–85	64	58–78
" plantar extension	100	83	88–90	66	73	83	..	41
" eversion	88–94	..	77–94	85–90	78–82	69		
" inversion	..	87–93	78	..	71	77		
Small Toes, extension	91–95	84–94	86–92	76–94	82–86	82–88	61–72	58–66
" interosseal flexion	71	..	88–100		..			
" flexion	70–76	83	86	72–80	82	69		41–66
" flexion, extension					..			
Hallux, extension	82–88	92–98	84–91	79–92	72–83	82–88	..	66
" flexion	73	..	86–95	72–86	78–86	69	..	41–66
" flexion, extension	70–76							

We are now in a position to point out the chief facts of interest to be noted in Table V., and which illustrate the relation between the localisation of motor or efferent function in the cortex and that in the capsule. Under the drawing, fig. 7, we have inscribed in their accurate relative positions the arrangement of the capsular fibres. If lines be laid obliquely as stated on p. 79, the exact correspondence of the cortical capsular representation will be readily seen.

Movements of Eyelids.

The most striking among these is that of closure of the opposite eyelids only (or in an incompletely evolved cortex accompanied by slight movement of the same side). The cortical localisation of this movement we give in fig. 7, and a line drawn vertically shows it to be relatively far behind the area for the other movements of the lids.

So in the capsule it is found fully one-tenth to one-fifth of the length of the capsule further back, and thus comes into relation with the representation of the upper limb and angle of the mouth. No closer correspondence between the representation in the cortex and that in the capsule could well be imagined.

In passing on to the next feature of special interest we would mention that we only observed the minuter changes in the movements of the lips, mouth, tongue, and jaws towards the end of this work, since our analysis of the facial region (not yet published) has only been carried on during the last fifteen months. Hence our account of these movements is necessarily brief.

Upper Limb Movements.

Elbow.—In describing[*] the analysis of the upper limb cortical representation we have shown that extension of the elbow is represented most anteriorly and superiorly, and flexion most inferiorly and posteriorly, while confusion of these movements is localised between their foci of most intense representation. This is most clearly found to be also the case in the capsule. The elbow, being a simple hinge joint and allowing only of extension, flexion and confusion, affords a ready means for this comparison between the cortex and capsule, and therefore it is interesting to note that in every group of sections this close similarity of representation was established.

Fingers.—So also the hinge joints of the fingers show that extension is represented anteriorly and flexion posteriorly in the capsule, this difference being, as seen in Table V., present in every group.

Thumb.—The same also prevails for the thumb.

Lower Limb Movements.

Since the cortical representation of the movements of the lower limb was shown by our analysis to be relatively imperfect, it was to be expected that with the mingling

[*] 'Phil. Trans.,' B., 1887–88.

of fibres in the capsules, still less differentiation would be found.   This is obviously shown in Table V., but on inspection, the representation of even flexion of the *small toes* is consistently (every group) in front of that of extension.   To a minor degree and with less consistency (4 of 6 groups) the same holds for the hallux.

Relative Amount or Degree of Representation of Extension compared to Flexion.

This subject may not be without interest to those who have, from excitation experiments, attempted to gauge the relative amount of functional activity in the cortex.

The facts bearing upon this point obtained from the records of hinge joint movements are so clear and unmistakable that they are of value.

The following are the proportional rates of frequency as observed in the capsule.

	Flexion.	Extension.	Remarks.
Elbow	38	16	} Joints whose usual movement is powerful flexion
Fingers . . .	38	17	
Thumb . . .	22	9	Joint the flexion of which is usually carefully regulated by extension
Small toes . .	15	26	Joints very frequently spread in extension for standing
Hallux . . .	16	17	Joint used as a thumb as well as for support

The bearing of these facts obviously requires no further comment, and the more so as they are but confirmatory of the results we obtained in our analysis of the cortical representation of these parts.

CONCLUSIONS.

As the foregoing paper is but a collection of detailed facts we can only offer as conclusions a few bare statements.

1. The fibres of the excitable portion of the internal capsule are arranged from before back, in the same order as the foci of representation in the excitable part of the cortex, taken along lines drawn at right angles to the direction of the fissure of ROLANDO.

2. This applies to the great divisions of the body, *i.e.*, head, upper limb, &c.

3. The arrangement of the fibres for the different segments of each great division is likewise that which prevails in the cortex.

4. Similarly the character of each movement is represented in the capsule as in the cortex.

DESCRIPTION OF PLATES.

PLATES 5.–9.

Fig. 1.   This figure consists of a number of drawings of the internal capsule and basal ganglia of the left side in horizontal section, corresponding to the eight different levels into which we have grouped our observations, and enlarged twice the natural size.

  a. = Anterior commissure.
  C. = Caudate nucleus.
  F. = The point of fusion of the lenticular and caudate nuclei in the basal grey matter.
  i. = Inner zone of the lenticular nucleus.
  l. = Middle zone of lenticular nucleus.
  L. = Outer zone of lenticular nucleus or putamen.
  O. = Optic thalamus.

Figs. 1, A.–R., give on tracings of the outlines of fig. 1, the limitation of the fibres, excitation of which produced a particular movement, the character of which is indicated in the margin.

Figs. 4, 5, 6 (on Plate 6).   Views of a dissection of the hardened brain of a *Macacus sinicus*, in which almost the whole of the hemisphere has been cut away, leaving only the basal ganglia to show their relations to each other and to the internal capsule.   Fig. 4 is viewed from the inner or ventricular side, fig. 5 from below, and fig. 6 from the outer side.

  A.a. = Anterior and ascending division of præfrontal fibres, lying in the deep groove between the caudate and lenticular nuclei.
  a.c. = Anterior commissure.
  Ch. = Optic chiasma.
  Cb. = Cerebellum.
  H. = Horizontal fibres of præfrontal portion of capsule.
  Hi. = Hippocampus major.
  Gy.Hi. = Gyrus hippocampi.
  C.q. = Corpus quadrigeminum.
  l.m. = Outermost lamina medullaris.
  s.d. = Superior and descending division of præfrontal fibres.
  Ca. = Caudate nucleus.
  Pu. = Putamen or outer zone of lenticular nucleus.
  P.F. = Excitable ("motor") or fronto-parietal fibres.
  L. = Posterior limit of excitable fibres.

Note change of direction at this point opposite middle zone (*m*) of lenticular nucleus, the fibres forming the internal capsule behind and below (*m*) are respectively occipital, occipito-temporal, and temporal.

M.C. = Middle commissure.

III., V., and VI. denote the third, fifth, and sixth cranial nerves.

PLATE 10.

Groups I.–VI.   These are photographs typical of each level at which we have excited the capsule, corresponding to the first six groups of our observations.

Ca. = Caudate nucleus.

Le. = Lenticular nucleus.

O.Th. = Optic thalamus.

PLATE 11.

Groups VII. and VIII.   These are photographs of our last two levels and follow on Plate X.

NOTE.—Group VIII. has been reversed in the reproduction.

The section is taken from the left hemisphere, except in Groups IV. and VIII., which are taken from the right side.

Fig. 2.   A photograph of a transverse vertical section of the right hemisphere of *Macacus sinicus*, showing the limits of the capsule and the direction of its anterior limb in frontal section.

Ca. = Caudate nucleus.

C.c. = Corpus callosum.

Le. = Lenticular nucleus.

S.l. = Septum lucidum.

Fig. 3.   This figure is a photograph of a sagittal section of the hemisphere of a *Macacus sinicus* stained by PAL's method and mounted in balsam. The medullated fibres are consequently stained black, the grey matter being of a lighter greyish tint.

A.a. = Anterior or ascending fibres of præfrontal division of capsule.

H. = Horizontal fibres of the same.

S.d. = Superior or descending fibres of the same.

P.f. = Pyramidal fibres (excitable).

P. = Fibres entering the posterior limb of the capsule.

Ca. = Caudate nucleus.

Le. = Lenticular nucleus.

O.th. = Optic thalamus.

A.c. = Anterior commissure.

IV. *On the Organisation of the Fossil Plants of the Coal-Measures.*—Part XVII.

By Wm. C. Williamson, *LL.D., F.R.S., Professor of Botany in the Owens College, Manchester.*

[Plates 12–15.]

Lyginodendron Oldhamium and *Rachiopteris aspera.*

In the fourth of this series of Memoirs ('Phil. Trans.,' 1873, p. 377, *et seq.*) I described a remarkable plant under the name of *Dictyoxylon Oldhamium*; I also gave reasons for substituting the late Mr. Gourlie's generic name of *Lyginodendron* for that of *Dictyoxylon.* In the same Memoir (p. 403) I referred to some petioles, to which I proposed to assign the name of *Edraxylon*; but later researches demonstrated the necessity for abandoning this as a generic term and substituting for it the more comprehensive one of *Rachiopteris.* In my Memoir, Part VI. ('Phil. Trans.,' 1874, Plate 2, p. 679, *et seq.*), I described this proposed *Edraxylon* under the name of *Rachiopteris aspera.* Certain similar features exhibited by the above two plants led me to remark in Memoir IV., p. 403, after showing that the *Rachiopteris aspera* was obviously the petiole of a Fern, " I think it far from impossible that these may prove to belong to *Dictyoxylon* (*Lyginodendron*) *Oldhamium*; but since I have not yet succeeded in correlating them with any certainty, I shall add no more respecting them at present."

Since 1873 I have accumulated a vast amount of material illustrative of the structure and relations of these two plants, and am now in a position to demonstrate that they respectively represent the stem and petiole of the same organism which proves to be a Fern. I was long under the conviction that the remarkable exogenous development of the stems of many of the Carboniferous Cryptogams, which I have so continuously demonstrated to exist, and which is now so universally recognised by Palæontologists, had no existence amongst Ferns. I have now to show that this development did exist amongst Ferns as well as amongst the arborescent Lycopods and Calamites, in which it is so conspicuous. Fig. 1 (Plate 12) is part of a transverse section of a stem or branch of *Lyginodendron Oldhamium*, in which *a* represents the medulla; *b*, the exogenous xylem zone; *c*, the place of the inner cortex, wanting in this specimen; *d*, one of the pairs of vascular bundles, so characteristic of the cortex

of this plant ; e, the outermost cortex, composed, in transverse sections, of radiating bands of sclerenchyma, g, alternating with parenchymatous areas, f.   At k, k we find two bundles of tracheids, like those at d, forming the centre of the cortical structures of a petiole of *Rachiopteris aspera*, i, i, which petiole is organically united to the cortex e of the *Lyginodendron*.   The two bundles k, k are assuming the oblique relative positions seen in the similar bundles of the free petiole of *R. aspera*, represented in fig. 2.   Other sections in my cabinet, similar to fig. 1, demonstrate the same facts, viz., that the pairs of bundles, fig. 1, d, which form so characteristic a feature of transverse sections of the middle cortex of *Lyginodendron Oldhamium*, pass outwards, through the outer cortex, to become the tracheæal* bundles of the petioles of the plant, and which petioles I had previously designated *Rachiopteris aspera*.   I may state that my friend GRAF SOLMS-LAUBACH, who has obtained numerous specimens of the *Lyginodendron* associated with others of *Rachiopteris aspera* from a locality on the continent, agrees with me in the conclusion at which I have arrived respecting their unity.   The more perfect specimens of the *Lyginodendron* obtained during the last seventeen years have thrown yet further light upon those figured in 1873.   In the latter, as at fig. 1, c, no traces of the middle bark were preserved ; but examples from Halifax, for which I am indebted to my friends Mr. CASH and Mr. SPENCER, of Halifax, have supplied what was wanting.   Fig. 3 is a transverse section in which this inner cortex, c, is shown to consist of a zone of extremely delicate, thin-walled parenchymatous cells, scattered throughout which are numerous gum-canals, l.   Three of these canals are represented, enlarged 250 diameters, in figs. 4 and 5, embedded in the thin-walled cells, c, c, of the cortex.

These canals are obviously formed by a schizogenetic separation of the cortical cells, c, and display no signs of having possessed the linings of epithelial cells so frequently seen in the similar canals of living plants.   Each canal is lined by a thin carbonaceous layer, a, within which is a free carbonaceous rod, b, with a round or oval section.   I presume that both these black elements are the carbonised remains of gums or resins with which the canals were primarily filled.

Fig. 3, b' b', shows an unusual occurrence of laminæ of tracheids growing inwardly into the medulla, to be referred to later.

Another feature characterising equally the outer cortex of *Lyginodendron* and that of its petioles is well shown in fig. 6, which represents a superficial tangential section of the cortex of *Lyginodendron Oldhamium*.   At its lower part this section repre-sents the structure already described in Memoir IV.   The fibrous bands g and the cellular areas f correspond to those indicated by the same letters in figs. 1 and 3, but

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* I have, throughout this Memoir, used the term *tracheæ* in the sense in which it is used by DE BARY as comprehending both the shorter, thickened, but elongated cells, *tracheids*, and the longer *tracheæ* or vessels.   DE BARY does so "specially in those cases where it is not certainly decided whether a tube belongs to the one or to the other subdivision," which is almost always the case amongst these coal plants.   (See DE BARY ' Comparative Anatomy of Phanerogams and Ferns,' p. 155.)

h, h′, and *h″* represent various sections of numerous peculiar peripheral appendages. Though so conspicuous when present, examples of this cortex are common in which no traces of these organs can be found. When perfect, each one is bottle-shaped, having a length of ·05 of an inch and a maximum diameter of ·02. The superficial layer of each of these appendages is prosenchymatous, the cells being especially elongated and narrow near and at the terminal neck of the bottle. Internally the appendage consists of a very regular parenchyma, as is shown by the longitudinal and transverse sections *h″, h″.* Owing to the slight obliquity of the entire section fig. 6, at *h′ h′* only the isolated projecting tips of these organs are seen. Many of my transverse sections of the *Lyginodendron* exhibit similar appendages ; thus, they are seen in fig. 1, *h,* and the basal portion of one is intersected at fig. 3, *h.* This latter, like many other similar ones, illustrates the nature of these structures. They are emergences, not hairs. Their internal parenchyma, *h,* is seen in fig. 3, to be an extension of the subjacent parenchyma, *f′,* of the outer cortex, whilst the prosenchymatous superficial coat is derived from the two fibrous bands, *g′, g′,* by which *f′* is bounded laterally. This explanation is yet better demonstrated by fig. 9, in which the two portions, *h, h′,* of the emergence are seen to be respectively developed from the fibrous laminæ, *g, g,* and from the intermediate parenchyma, *f,* of the outer cortex.

The very characteristic emergences, *h,* of figs. 1–3 and 6, reappear in the petioles hitherto described under the name of *Rachiopteris aspera.* In my Memoir VI., I pointed out* the existence, on the periphery of these petioles, of numerous " abortive hairs" or " tubercles," but I was not then aware of their full significance.

Fig. 7 represents a characteristic example of a transverse section of a young *Rachiopteris aspera,* from the periphery of which a number of these emergences, *h, h,* are given off. The origin of the superficial layer and of the internal parenchyma of each emergence is seen to be identical with that demonstrated in my description of fig. 9. Fig. 8 represents a longitudinal section of a similar petiole, in which these emergences are again seen at *h, h,* variously intersected.

The above facts combined make it clear that *Rachiopteris aspera* is merely a petiole of *Lyginodendron Oldhamium* ; hence the former name must disappear from our lists. It also follows that *Lyginodendron Oldhamium* is a true Fern, most probably belonging to some Sphenopterid type. This determination carries along with it the further one that the stems of some, at least, of the Ferns of the Carboniferous age developed their xylem or vascular structures exogenously, through the instrumentality of a meristemic zone of the innermost cortex, which practically must be regarded as a cambium layer. But the history of this interesting plant is not yet ended.

In various preceding memoirs, but notably in Part XVI., I have provoked a strong opposition amongst some botanists whose studies have been chiefly limited to living forms of vegetation, by revealing the existence, amongst the Lycopodiaceous plants of the Coal Measures, of a mode of development of a medulla which differs so widely from

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* Plate 52, figs. 8, 9, and 13, *k, k″,* pp. 681–2.

what occurs amongst living plants that the sceptics are unable to accept my account of it.   My late friend Professor DE BARY, when his students occasionally declared that certain suggested explanations were "impossible," usually replied, "in scientific research nothing is 'impossible.'"   I utilise this philosophic reply, and commend it to some of my new opponents.*

I now advance further and proceed to show that the process of medullary development which, as demonstrated in my Memoir XVI., occurs so commonly amongst the Lepidodendroid stems and branches, takes place, if possible, in even a more striking manner in the young growths of *Lyginodendron Oldhamium*.

In Plate 25 of my Memoir IV., fig. 16, I represented a young lateral branch, bursting outwards through the cortex of an older stem.   The development of these branches was demonstrated in several other figures in the same memoir.   I have through many past years accumulated numerous additional examples of these young branches, which now enable me to speak decisively respecting their structure and development.

Plate 13, fig. 10, exhibits a transverse section of one of these young branches as it emerges through the cortex of the parent stem, and before it has become invested by an independent cortex of its own.   Its centre, *a*, appears to consist of a solid cluster of barred tracheæ, of variable diameters, grouped in no special order.   Springing directly from the periphery of this central mass, a considerable number of tracheæal laminæ, *b*, radiate outwards, in regular order, constituting the xylem zone; these laminæ are separated by numerous conspicuous medullary rays—both being obviously the result of a process of exogenous growth.   Fig. 11 represents a second specimen, similar in most respects to fig. 10, but differs, first, in being of larger dimensions (both being drawn to the same scale), but, secondly, in a change which is taking place in its vascular centre, the tracheæ composing which axis are now becoming dissociated, producing an irregular cavity, *a*, which is forming in their midst; by this change the tracheæ are separated into four or five irregular groups, *a' a'*, each of which adheres closely to the inner border of the exogenous xylem cylinder, *b*.   Such examples show no trace of the proper bark of the branch; but in another specimen in my cabinet (Cabinet number 1885 D), the growth of which has advanced slightly beyond that of fig. 11, and in which the cavity, *a*, has increased yet more in size, the characteristic cortex of *Lyginodendron* is fully developed, showing that this branch had completely emerged from the outer cortex of the parent stem.

In fig. 12 we have a third similar, but yet larger, section, in which the central cavity, *a*, has not only undergone a further increase of area, but is now occupied by a

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* Some of our older botanists scarcely realise the progress made during the last few years in the study of what may be designated "anomalous" features in the tissues of living plants.   Such "anomalies" are less rare, even now, than many botanists are aware of.   In the primæval plant-world they abounded.   (See D. H. SCOTT, "On some Recent Progress in our Knowledge of the Anatomy of Plants." 'Annals of Botany,' vol. 4, No. 13, p. 147, November, 1889.)

perfectly developed parenchymatous medulla ; whilst the four fragments, a' a', of the primary central tracheæal axis of fig. 10 are still further separated from one another.

Fig. 13 is an outline sketch of the central part of a *Lyginodendron Oldhamium*, very advanced in growth, but only enlarged less than eight diameters instead of thirty, as is the case with figs. 10, 11, and 12.

The line b represents the inner border of the exogenous cylinder. The five points, a', a', are the five dissociated segments of the originally solid tracheæal bundle, corresponding to those a' a' of figs. 11 and 12. On making accurate measurements, I find that whilst the outer surfaces of these five bundles continue to be in the closest contact with seventy-one parts of the circumferential line b, the intervening spaces, a'' a'', non-existent in the young state of the branch, occupy 164 parts: so that whilst, in the first instance, the circumference of the circle b and that of the aggregate of the united bundles, a' a', were absolutely coincident, the circle has enlarged, at the least, to more than three times its original circumference. Such an enlargement was necessarily accompanied by a corresponding increase in either the number, or size, or both, of the tracheæal laminæ and their intervening medullary rays, constituting the exogenous xylem zone.

Some of my botanical friends have endeavoured to explain the corresponding phenomena amongst the Lepidodendra, by supposing that, where such differences exist in the size of the medullæ, they did not represent different stages of growth of similar branches, but were characteristics of different kinds of branches ; but no such improbable hypothesis is for a moment admissible in the case of this *Lyginodendron*. The effects of growth, and of growth alone, as described above, are seen in every one of the innumerable specimens of *Lyginodendron* that have yet been discovered. In this case, that these are the effects of growth, is seen, first, in the breaking up, by dissociation, of the solid axial bundle of tracheæ ; secondly, in the formation of an area in the centre of this bundle, which became occupied by a steadily expanding parenchymatous medulla, and, lastly, in the secondary effects of these central expansions on the tracheæal xylem and the cortical zones which enclose the medulla. To dispute this is, in my opinion, alike unscientific and futile.

When we note that the medulla of the largest stem of *Lyginodendron* which I have yet obtained has advanced from little more than a point to above an inch in mean diameter, it is clear that such an enlargement must have been accompanied by considerable changes in the relations of the more external tissues, however apparently permanent they might be.

In order to obtain some fairly accurate data in reference to this strain put upon these more external structures, I have counted as accurately as the subject admitted of, the number of the radiating laminæ of the tracheæ comprising the xylem zone which started from the inner margin of that zone, and also the number of those which reach its outer or cortical zone. The results of these observations made on sixteen

sections of stems and branches of various ages and sizes, are given in the following table :—

Cabinet numbers of the slides containing the specimens examined.	Number of the vascular laminæ in the		
	Inner margin.	Outer margin.	
Slide A. 1885 C.	44	54	Central axial bundle dissociating
,, B. 1885 A.	46	66	Central axis solid, fig. 10
,, C. 1883 A.	58	93	Central axis solid
,, D. 1885 A.	61	96	Central axis solid
,, E. 1883	62	95	Central axis solid. Branch not yet emerged from the parent stem
,, F. 1885 D.	65	89	Specimen a corticated independent branch
,, G. 1141	71	85	Specimen with a central medulla, fig. 12
,, H. 1885 H.	74	97	Central axis dissociating, fig. 11
,, I. 1885	78	99	Central axis solid
,, K. 1884	84	139	Independent corticated branch, medulla much enlarged
,, L. 1885 F.	95	150	Independent corticated branch, with a medulla
,, M. 1138	116	200	Independent branch, medulla large
,, N. 1885	135	233	Independent branch on stem
,, O. 1885	272	499	
,, P. 1128	364	642	The specimen represented in the outline, fig. 13
,, Q. 131	1120	5460	

The above table shows that as each stem or branch increased in size, an increase also took place in the number of the radiating laminæ that originated close to the medullary border of the exogenous zone, as well as of those which reached its cortical margin. We find that the eight very young sections, A, B, C, D, E, G, H, I, had a mean number of 69 such protoxyloid laminæ, whilst the seven larger branches, F, K, L, M, N, O, P, all of which had become independent or corticated branches, completely emerged from the parent cortex, had a mean of 161 such laminæ, the example P having 364. The very large stem Q shows even a much further increase, having 1120 of these primary laminæ. That the second column should show a large number of secondary laminæ intercalated between the periphery of the medulla and the innermost cortex was to be expected where an exogenous mode of growth had obviously existed.

It is clear that many of the laminæ thus indisputably added even to the *inner* border of the vascular zone, must have been intercalated chiefly in the intervals corresponding to a'', a'', of fig. 13 ; and it is perfectly clear that wherever they were developed along this expanding medullary margin they must have pushed apart the older lamellæ between which they were so intercalated. Since every one of the larger and older branches in the lower half of my table must once have been small and young, like those of the upper half, it seems indisputable that we have here a phenomenon of growth and expansion which has no parallel, that I am aware of, amongst living

plants. That 44 laminæ of my example A can be represented by 1120 similarly arranged ones in the section Q shows a remarkable expansion of the medullary border that was produced not by the mere enlargement of the existing tissues, but by the intercalation of new ones. That such must have been the case is, as I have already observed, demonstrated by the dissociation of the primarily contiguous tracheæ of the central axis a of fig. 10, already described. We realise the extent of this separation when we compare the intervals separating the bundles a' of fig. 13 with the corresponding ones a' of fig. 12. In estimating the sizes of those intervals we must remember that fig. 12 is enlarged more than fig. 13 by at least four times.

I repeat, therefore, that however brought about, we have two separate and independent proofs of the origination and development of the medullary area. First in the separation of the clusters of tracheæ which primarily formed one united cluster, and secondly in the enormous increase in the number of the vascular laminæ, the inner extremities of which, though commencing their growth at different periods of the plant's life, alike start from the medullary border of the vascular zone, and extend to its periphery.*

The origin of the cellular medulla in the young branches of *Lyginodendron* presents no real difficulties. Plate 26, fig. 24, of my Memoir, Part IV., shows that these branches were derived from the exogenous xylems of the parent stems. The medullary rays of those woody zones are large and multicellular (Plate 13, fig. 3, b''), hence we can easily understand how a few small examples of these cells could be enclosed amongst the tracheæ of a central axis like fig. 10, a, and yet escape detection. Personally, I have no doubt that such was their origin. There is no reason for supposing that the area a of examples like fig. 11 was not normally occupied by the young medulla. The cells in such examples have merely failed to be preserved. In one of my slides (Cabinet number 1883) is an obliquely transverse section of a very young branch, in which the tracheæ are readily distinguishable by the conspicuous reticulations of their walls; running through the central axis of this branch is a very thin line, consisting of a small number of minute, but distinct cells. These germs would amply suffice for commencing a meristemic action, which would produce such a medulla as I have described above. But another question is less easily answered. What has occasioned the expansion of the medullary area and its surrounding tissues? Has it been due to the centrifugal pressure of the growing medulla alone, or have other forces taken part in the process?

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* It is scarcely necessary to remind any botanist that though, as recent exogenous stems grow, an enormous increase takes place in the number of the radial laminæ of the wedges composing their woody zone, little or no increase is seen in the number of those laminæ, of which the inner extremities reach the central medulla, or the point which that medulla originally occupied. All the additions are intercalated more or less peripherally.

Heterangium Grievii.

In the Memoir, Part IV., already referred to, I also described under the above name a plant from the Cliff of Petticur, near Burntisland, in Fifeshire. Some months ago my young friend, Mr. Lomax, of Ratcliffe, found, for the first time, specimens of the same plant from Dulesgate, in Lancashire. But along with these he brought from the same locality a series of sections of what at first seemed to be a different species of *Heterangium*. But long and careful comparison of all the examples of that genus in my cabinet convinced me that the supposed new forms were merely the *H. Grievii* in a younger state of growth. The specimens previously described were characterised by the existence of a central mass composed of irregular clusters of tracheæ, imbedded amongst numerous parenchymatous cells (*loc. cit.*, Plate 28, fig. 30, *a*), surrounded by an exogenously developed zone of tracheæ (*ibid.*, *b*). This plant was also invested by a well-defined zone of sclerous prosenchyma, which, in the transverse sections, was seen to be more or less subdivided radially into cubical masses (*loc. cit.*, fig. 30, *k*, and Plate 29, fig. 35, *k*).

The central part of the transverse section (Plate 14, fig. 14, *a*) resembles the Petticur plant in consisting of numerous clusters of tracheæ aggregated into a large axial mass, into the composition of which many parenchymatous cells enter. That these cells may have become carbonised is probable from the fact that, when examined under a higher power, the appearances shown in fig. 15 are observed. Two of the clusters of tracheæ, enlarged 75 diameters, are here represented. In the centre of each cluster we have a number of larger tracheæ, *a, a*, surrounded by still more numerous smaller ones, intermingled with some cells, *d, d*. The investing zones, *c, c*, seem to me to be the carbonised remains of the inter-tracheæal cells which occupy similar positions in the Petticur plant. At fig. 14, *a'*, a tracheæal bundle is passing outwards from the central mass to some external appendage.

A thick cortex, fig. 14, *b, b*, invests these central tissues ; it is fundamentally parenchymatous, but the forms and arrangement of its component cells vary much in different portions of it, as will be shown when describing longitudinal sections of this tissue. The outer cortical zone is a layer of sclerous fibres, *e, e*. The aspect presented by these fibres, as seen in transverse sections, is shown in fig. 16. The lignified wall and central lumen of each cell are very distinctly seen. These fibres reappear at *e* in fig. 17, which represents a longitudinal section through the centre of a stem like fig. 14. At *a* we have the central vasculo-medullary axis, consisting of intermingled barred tracheæ and vertical rows of cells, the latter having chiefly rectangular transverse septa. Immediately external to this vascular axis we find an innermost layer, *b, b*, of the cortex, the cells of which are arranged in rather regular perpendicular lines, but more externally these cells become larger and more irregular in size, form, and arrangement, though still retaining a tendency to dispose themselves in vertical rows. This zone of the cortex is bounded externally by an irregularly undulating, ill-defined line, *b', b'*. Outside this

line the cortex, *c, d,* assumes a very different aspect. Its cells still arrange themselves in vertical but undulating lines of a peculiar character. At the points *c′, c′,* we find transversely disposed bands of very strongly defined cells, alternating with intermediate areas, *d, d,* through which the same lines of cells continue their vertical course but with altered flexures. The cells themselves, also within these areas, have very much thinner walls. The cell-clusters, *c′, c′,* as seen in these vertical sections, are parts of numerous masses of more or less definite lenticular form, which abound in this part of the cortex ; on making broad tangential sections through the same we note that, whilst many of these masses are arranged in horizontal transverse lines, many of them incline at very various oblique angles. Fig. 18 represents a small portion of a very superficial tangential section, in which *c* is the part of the cortex under consideration, and *e, e,* the fibrous investing layer. The section of one of the lenticular clusters of cells *c′* illustrates the peculiar symmetrical arrangement so frequently observable within these clusters. The longitudinal section, fig. 17, shows that at their inner margins these lenticular cell-masses compress the inner cortex *b,* those margins projecting into the concave flexures of the line *b′.* Similar clusters to those seen at fig. 18, *c′,* in the transverse sections of these stems. The sclerous cells of the zone *e* of fig. 14 are seen in figs. 17 and 18 to be narrow fibres of very great length in proportion to their diameters. This zone cannot fail to remind the histologist familiar with the structures of the rhizomes of recent Ferns of the very similar hypodermal layer seen in some species of *Nephrolepis.*

The true affinities of this plant are as yet undetermined, but I have strong conviction that it will be eventually proved to be a true Fern.

Bowmanites.

In volume 5 of the third series of the Transactions of the Literary and Philosophical Society of Manchester, I published, in 1871, a description of a fruit to which I gave the name of *Volkmannia Dawsoni,* and further illustrations of the same fruit were given in my Memoir, Part V., of this series.* At that date the verticillate-leaved plants of the Coal Measures were ill understood, and much confusion has resulted from that imperfect knowledge ; a confusion which a few words of explanation may help to remove.

Palæobotanists have for more than a century been familiar with the existence, in the Carboniferous strata, of a number of plants, the leaves of which were arranged in verticils. In his ' Prodrome,' published in 1828, BRONGNIART threw a considerable number of these plants into two groups to which he gave the generic names of *Sphenophyllum* and *Asterophyllites.* The former genus he defined, among other characters, as " feuilles verticillées, au nombre de six à douze, distinctes jusqu'à leur

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* ' Phil. Trans.,' 1874, p. 54, Plate 5, figs. 28–30.

base, cunéiformes, entières, ou émarginées, ou même bifides, à lobes plus ou moins profondément laciniés."

His genus, *Asterophyllites*, he defined as having " feuilles planes, plus ou moins linéaires, aiguës, traversées par une nervure moyenne simple, libres jusqu'à la base."[*] In his ' Tableau des Genres de Végétaux Fossiles,' published in 1849, he again defines the two above genera in almost identical words, adding, however, that *Sphenophyllum* differs from *Asterophyllites*, " par le nombre beaucoup moindre ce ces organes à chaque verticille, 6 à 8 ou 10, et par leur forme qui est triangulaire, tronquée au sommet, ou dentée et lobée, quelquefois très profondément " (*loc. cit.*, p. 52).

In 1864 Messrs. COEMANS and KIKX published their " Monographie des Sphenophyllum d'Europe." In it they define the leaves as " cuneatis, sessilibus, verticillatis, nervo medio destitutis; nervulis autem æqualibus, dichotomis" (*loc. cit.*, p. 414).

In my Memoir, Part V.,[†] I described a plant in which each foliar verticil consisted of 18 *linear uninerved leaves,* which characters identified it unquestionably with *Asterophyllites.* That these leaves have the characteristics of *Asterophyllites* and not of *Sphenophyllum*, is demonstrated by the numerous long and transverse sections of them in figs. 5, 14, 15, 16, and 17 of the above memoir, in which also the remarkable structure and development of the stem of the plant is described.

In May, 1870, M. RENAULT had presented to the French Academy a memoir, " Recherches sur l'Organisation des *Sphenophyllum* et des *Annularia*," but owing to the siege of Paris, during which the drawings were lost, the plates were only published in the ' Annales des Sciences Naturelles, Botanique,' 5ᵉ série, tome 18, 1873.[‡]

In this memoir, M. RENAULT described the internal structure of *Sphenophyllum Stephanense*, which structure proved to be identical, in most respects, with that of my Asterophyllitean stem. On the strength of this resemblance, M. RENAULT and H. GRAF ZU SOLMS-LAUBACH, have contended that I ought to have designated my plant a *Sphenophyllum*, and not an *Asterophyllites.* I, on the other hand, still insist that, in accordance with the definition of every author who has written on the subject, my uninerved leaves prove my plant not to be a *Sphenophyllum*, but rather demonstrate the very intimate relationship, not to say identity, of the two genera, at least, so far as some forms of *Asterophyllites* are concerned. Some others are unquestionably the leaves of Calamites.

The position of the *Volkmannia Dawsoni* referred to above, is affected by this controversy. In M. RENAULT's memoir, p. 9, there appears a note by M. BRONGNIART, *apropos* of my memoir of 1871 in the Manchester Transactions. M. BRONGNIART says, " Ce travail s'accorde dans plusieurs points importants avec les résultats obtenus un

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[*] ' Prodrome,' pp. 68 and 159.

[†] ' Phil. Trans.,' 1874, p. 41, *et seq.*, but delivered to the Royal Society on May 17, 1873.

[‡] By a coincidence M. RENAULT's memoir, as now constituted, was only delivered to the Academy on May 13, 1873, four days before mine was delivered to the Royal Society.

an auparavant par M. Renault, sans que M. Williamson eût de son côté connaissance de l'article des *Comptes Rendus* du 30 Mai, 1870.

"La plante fossile étudiée par M. Williamson, et nommée par lui *Volkmannia Dawsoni*, diffère cependant sans aucun doute, au moins spécifiquement, de celle décrite par M. Renault, par la forme du faisceau vasculaire central et par l'absence des zones de cellules quadrangulaires qui l'entourent dans les échantillons de France, cellules qui, par suite de l'epaisseur de leurs parois, ne doivent pas se détruire facilement."

My plant to which M. Brongniart refers, is a strobilus, not an ordinary stem. Herr Ch. E. Weiss, of Berlin, believes it to belong to the genus *Bowmanites* established by the late Mr. Binney.* No remains of the specimens described by Mr. Binney appear to be now discoverable, but since the important little sketch, Plate 12, fig. 3, of Binney's memoir, was made by Mr. Bowman, who was an accomplished botanist—made too when all the fragments of the original specimen were in his own possession—I presume that it may be accepted as representing a true form. If so, it certainly approaches nearer to what we find in my plant than any other hitherto described. If so, however, the objects which Mr. Binney designated macrospores, must be regarded as sporangia, which the rows of similar interbracteal spheres in my *Volkmannia Dawsoni* certainly are.

Notwithstanding the above facts, this subject would have been left in a very unsatisfactory position had not Professor Ch. E. Weiss fortunately obtained in Germany a specimen, which he has named *Bowmanites Germanicus*.† These figures correspond so closely with those of Mr. Bowman reproduced in Mr. Binney's memoir, that the genus *Bowmanites* can now be accepted as representing an extremely distinct type of Calamarian fructification. But until my memoir on *B. Dawsoni* was published, nothing was known of the internal organisation of this fruit ;‡ and until now, nothing was known of the structure of its vegetative organs.

But my friend, Mr. George Wild, of Bardsley, near Ashton-under-Lyne, whose name has so often been recorded in these Memoirs, brought me the section represented in Plate 15, fig. 19, and which is beyond all doubt a transverse section of a stem of *Bowmanites Dawsoni*. In its general features the structure of this section is identical with similar ones of my *Asterophyllites*, and of M. Renault's *Sphenophyllum Stephanense*, which latter differs materially from the Autun *Sphenophyllum* described by the same author. All these plants agree in having a central vascular

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* "Observations on the Structure of Fossil Plants found in the Carboniferous Strata," by E. W. Binney, F.R.S., F.G.S.—Part 2; "Lepidostrobus and some Allied Cones." (Palæontographical Society, 1870, p. 59.)

† 'Atlas von neunundzwanzig Lichtdrucktafeln zu der Abhandlung: Beiträge zur Fossilen-Flora, III. Steinkohlen-Calamarien, II.' Ch. E. Weiss. Berlin, 1884. Tafel XXI., figs. 12, 12 a, 12 b.

‡ [I have recently obtained important specimens throwing additional light on the organisation of this interesting fruit, which will be described in my next Memoir.—September 9th, 1890.]

axis, the transverse section of which is a triangle with each of the three angles more or less prolonged in equal measures. But in my fructification of *Bowmanites* the transverse section of this axis is very broadly robust, instead of being drawn out into three very long and slender radii. Not only so, but the extremity of each short arm is very broadly truncated ; a form that hitherto has only been seen in my *Bowmanites Dawsoni*. In fig. 19 the central triangle *a* is composed of a cluster of tracheæ grouped in no special order, except that the central ones are larger than those occupying the periphery of the triangle. In my fructification this central truncated triangle constitutes the sole vascular axis, a fact easy of explanation. Referring to my Memoir V., Plate 1, I have shown that in the very young twig of my *Asterophyllites*, fig. 1, this triangle constituted the only tracheæal bundle, but as the vegetative twig grew in age and size, it developed exogenously zones of tracheæ which were successively added, investing the triangular centre, as in figs. 2, 3, and 4, resulting in the conditions seen in Plate 2, figs. 9, 10, and 11 of the same memoir. The present figure 19 is as nearly as possible in the condition of the figure 9 just referred to. Its exogenously added tracheæ are arranged in equally regular radial and concentric lines; the sizes of the individual tracheæ enlarging as they were successively superimposed upon those previously developed. Portions of a layer of a cork-like cortex, *c*, are still preserved. A similar layer is seen, not in the *young* twigs of my *Asterophyllites* 1 and 2, but in the older ones 9, 10, and 11. It also appears in the middle of the cortex, enclosed, between an inner and an outer parenchymatous zone in M. RENAULT's Autun *Sphenophyllum* (*loc. cit.*, Plate 1, figs. 4 and 5, *g*), as well as in my fig. 16. The French savant designates it a " partie subéreuse."[*]

That this stem of *Bowmanites Dawsoni* is constructed on the same plan as that of my *Asterophyllites*, and of M. RENAULT's *Sphenophyllum Stephanense* is obvious enough, but to place all these three plants in the genus *Sphenophyllum*, because of this vegetative resemblance, as my friends M. RENAULT and GRAF SOLMS-LAUBACH wish to do, would be wrong. I have already shown that the leaves of my *Asterophyllites* are *not* those characteristic of *Sphenophyllum* ; and the fructification of my *Bowmanites Dawsoni* is altogether distinct from the strobili of *Sphenophyllum*. It must further be remembered that evidence is now being obtained from various quarters that there are Carboniferous plants, the branches of which bear both *Asterophyllitean* and *Sphenophylloid* leaves.

I therefore conclude once more that we must unite *Sphenophyllum* and some forms of *Asterophyllites* in one and the same genus ; it is equally clear that *Bowmanites*, though its peculiar fructification demonstrates that it constitutes a perfectly

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[*] The facts here referred to illustrate conditions so frequently seen amongst these Carboniferous plants, in which the exogenous development of a xylem zone as well as the differentiation of separate zones of the cortex mark advance of age. The fructifications, being young and temporary organs, seem invariably to have a central axis, the internal structure of which is identical with that of the youngest vegetative twigs of the parent plant. The *Bowmanites* described above is a good illustration of this truth.

distinct genus, has strongly marked features of affinity in the structure of its stem to the Sphenophylloid type. Hence all these plants may be recognised as constituting one of the several groups which collectively form the great family of the *Calamariæ*, and of which family the modern genus *Equisetum*, instead of being the central *type*, is but a poor, feeble, and degraded member.

Calamites.

My attention has long been drawn to the fact that, whilst sandstone casts of considerable dimensions, moulded in the interiors of the fistular medullæ of Calamitean *stems* are notoriously common, similar casts of the interiors of the smaller branches are extremely rare. Fig. 20, which represents (natural size) the interior cast of a single internode of such a branch is the smallest one preserving the common features of cylindrical form, longitudinal ridges and furrows, and, at the two ends, the two nodal transverse constrictions, that I have met with during fifty years of research. I have thought it desirable to demonstrate the cause of this state of things. The example here figured is ·5 of an inch in diameter. Comparing a medulla of this size with those of other decorticated specimens in my cabinets, in which the exogenous woody zone is preserved, I have no doubt that fig. 20 belonged to a branch or stem that was at least an inch in diameter, independently of its thick cortical zone. Seeing that I possess sections of decorticated Calamitean twigs not more than ·03 in diameter, myriads of similar ones must exist, but which are never represented by these sandstone casts. I possess a very fine leaf-bearing branch in sandstone, for which I am indebted to my old friend Sir WILLIAM DAWSON, of which the diameter, when uncompressed, has been about ·16 of an inch, including its cortex. It displays longitudinal internodal ridges, but these are but the irregular products of the shrivelling of a semi-herbaceous tissue, and have no affinity with the regular ridges and furrows of fig. 20.

I have brought together in my cabinet a graduated series of specimens which illustrate and explain the conditions referred to. Two of these specimens are represented in figs. 21 and 22. Fig. 21 is the centre of a decorticated branch, the entire diameter of which is ·25 of an inch, whilst that of its fistular medullary cavity, *a*, is about ·05. This cavity is filled with inorganic matter, whilst at *a'* a zone of medullary parenchyma is still preserved, which is about ·005 in. in thickness. This medulla intervenes between the circumferential border of the inorganic cast, *a*, and the inner angles of the vascular wedges, *b, b*. This zone of undisturbed medullary tissue protects the periphery of the once plastic cast, *a*, from the pressure of the elongated inner angles of the woody wedges, *b*. It is to be remembered that in the *very* young Calamitean twigs this medulla is *entire*—not fistular; but that at a very early stage of growth the centre of this entire medulla becomes ruptured. The incipient cavity thus formed increases in diameter as the branch grows, partly in consequence of the absorption of the medullary cells, and

partly through the transverse enlargement of the entire stem. The result is that so long as these changes do not extend beyond the state represented by fig. 21, the exterior of the cast, a, neither can, nor does, become longitudinally ridged and furrowed. But in the specimen represented in fig. 22, all these conditions are altered. The entire diameter of this specimen is about ·55 of an inch, whilst the maximum of its medullary cavity reaches ·4. It will be observed that not only all the medullary cells have disappeared from this specimen, but even the larger and less dense cells composing the medullary extremities of the primary medullary rays, c, c, have also been absorbed; the inner angles of the various tracheæal woody wedges, b, b, have alike resisted pressure and absorption, and have thus given to the longitudinal ridges and furrows so characteristic of the exteriors of these inorganic casts their rounded undulating contours. This specimen thus answers a question that has been put to me by some of my correspondents, viz., " If the ridges of an ordinary Calamite were formed in accordance with your hypothesis of internal casts, how is it that the ridges have a rounded outline instead of projecting as radiating, vertical plates, corresponding to the entire forms of the primary medullary rays to which you say those ridges owed their existence ?" The reply is, the process of absorption only reached the innermost extremities of those primary rays. As the latter organs passed outwards through the xylem zone, their component cells became smaller and more capable alike of resisting pressure and decay. I have not seen a single instance in which these more external cells had been absorbed during the lifetime of the plant. The fistular cavity thus formed has ordinarily been filled with sand or mud, but in the present example, as is so frequently the case with the plants preserved in our Lancashire and Yorkshire nodules from the Gannister Coals, this cavity is occupied by calcareous matter, which has been in a state of solution, and which has filtered through the tissues of the plant. The first deposits from this solution have formed, in fig. 22, stalagmitic layers, a', a', lining the walls of the fistular cavity ; but in the central part, a, the intruded material has assumed a crystalline condition. These secondary details, however, in no way affect the general conclusion, viz., that the ridges and furrows marking the exteriors of transverse sections of these internal casts exhibit rounded contours, and not stellate ones. The above specimen was obtained from above the Upper Foot Coal at Sholver Lane. My cabinet contains various other specimens which lead to the same conclusions.

The section, part of which is represented in fig. 21, is one of a series of nine, made from the same stem or branch. Eight of these are transverse sections, which show a gradual increase of from 14 to 24–25 in the number of the tracheæal wedges of the xylem as we ascend from below. I am indebted to my old auxiliary, Mr. GEORGE WILD, not only for this series, but also for the section fig. 22. The *specimen* from which the latter section was cut was collected by another of my valued coadjutors, Mr. JAMES NIELD, of Oldham.

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Plate.	Figure.		Page at which reference is made to the figure.
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V. *The Nitrifying Process and its Specific Ferment.*—Part I.

By Percy F. Frankland, *Ph.D., B.Sc. (Lond.), A.R.S.M., F.I.C., Professor of Chemistry in University College, Dundee, and* Grace C. Frankland.

Communicated by Professor Thorpe, *F.R.S.*

The spontaneous oxidation of nitrogen in nature is a process which has long attracted attention in consequence of the great practical importance of the products to which it gives rise. Indeed, so great is the demand for potassium nitrate, nitre, or saltpetre, the principal derivative of the oxidised nitrogen, that the process of nitrification is often artificially stimulated by placing nitrogen, in the form of refuse animal matter, under the most favourable conditions for undergoing this change. The process of nitrification has thus been carried on for ages as a regular industry in India, and even in some European countries, thus especially in France during the Great Blockade. For a number of years past, however, the principal source of nitric acid and its derivatives has been the enormous deposits of nitrate of soda occurring in South America, which deposits themselves may, however, possibly be the product of a vast nitrification-process in a former period of the Earth's history.

But although nitrification had thus been practically studied for centuries, it was only in 1878 that the process was shown by Schlœsing and Müntz ('Compt. Rend.,' vol. 84, p. 301; 85, p. 1018) to be a fermentation change, and entirely dependent upon the presence of certain minute forms of life or micro-organisms. But although this connection was thus first experimentally demonstrated in 1878, it had with characteristic foresight been already surmised by Pasteur in 1862.

In the communication referred to above, these investigators clearly show that the process only takes place under conditions compatible with the life and growth of micro-organisms, and is immediately arrested by conditions, such as the application of strong heat or antiseptics, which are fatal to these forms of life. They further showed, moreover, that the nitrification-process could be induced by the introduction into suitable materials of the minutest quantity of matter from a medium already undergoing nitrification.

In the same year these remarkable results were fully confirmed by Warington ('Chem. Soc. Journ.,' 1878, p. 44), who further greatly extended our knowledge of

the conditions affecting the process of nitrification, in a very elaborate paper communicated to the Chemical Society in the following year (1879, pp. 429–456).

In subsequent publications ('Compt. Rend.,' vol. 89, pp. 891 and 1074), SCHLŒSING and MÜNTZ claim to have separated the organism causing nitrification, and they briefly describe the general appearance which nitrifying media present when examined with high microscopic powers.

It is, however, almost needless to say that we have no guarantee that they actually accomplished the isolation of this important organism, as the methods of bacteriological research at that time in general use were of a very imperfect character. In any case, the subject demanded re-investigation with the aid of more modern methods.

The results of a number of other researches on the subject of nitrification have been published from time to time, amongst which we may specially mention an extremely interesting paper by MUNRO ('Chem. Soc. Journ.,' 1886, pp. 632–680), in which the nature of the nitrogenous substances capable of undergoing nitrification is very fully discussed. In none of these communications, however, is any material advance made towards the identification and isolation of the active organism of nitrification.

We append, however, the following list of the principal contributions which have been made to our knowledge of the subject :—

"Note on the Ferment-Theory of Nitrification." F. H. STORER ('Chem. Soc. Journ.,' Abst., 1878, p. 932); in this it is shown that ammoniacal salts undergo nitrification in contact with peat, if air, filtered through cotton wool, is drawn through the mixture, whilst if the same mixture is first sterilised by heat, then the current of filtered air fails to bring about any nitrification.

"Nitrification," E. W. DAVY ('Pharm. Journ. Trans.,' [3], vol. 10, pp. 1–3; 'Chem. Soc. Journ.,' Abstr., 1879, p. 1047), discusses the formation of nitrates and nitrites in natural waters, and contends that light is not inimical to the change, which takes place very readily between 70° and 80° Fahr. It is further pointed out that, in the presence of organic matter, nitrites are formed.

WARINGTON, in "Alterations in the Properties of the Nitric Ferment by Cultivation" ('Chemical News,' vol. 44, p. 217; 'Chem. Soc. Journ.,' Abst., 1882, p. 79), states that when ammoniacal solutions are seeded from old nitrifying solutions only nitrites and no nitrates are formed, and he attributes this phenomenon to diminished energy of the nitrifying organism.

In a subsequent paper entitled "Nitrification, Part III." ('Chem. Soc. Journ.,' 1884, p. 637), WARINGTON describes a very extensive series of experiments, throwing light upon the special conditions which further nitrification.

PICHARD, in "Comparative Nitrifying Effect of Various Salts" ('Compt. Rend., vol. 98, p. 1289; 'Chem. Soc. Journ.', Abst., 1884, p. 924) shows that the presence of calcium sulphate greatly promotes nitrification, and institutes a comparison between the relative effect produced by the following salts; thus—

Calcium sulphate	100·00
Sodium sulphate	47·91
Potassium sulphate	35·78
Calcium carbonate	13·32
Magnesium carbonate	12·52

PICHARD's results as regards the beneficial effect of calcium sulphate are confirmed by WARINGTON in a paper " On the Action of Gypsum in Promoting Nitrification " ('Chem. Soc. Journ.,' 1885, p. 758), and again, in a further communication " On the Distribution of the Nitrifying Organism in the Soil " ('Chem. Soc. Journ.,' 1887, p. 119), he shows that by adding calcium sulphate to the ammoniacal solutions the latter may be made to nitrify by seeding with soil taken from greater depths than had hitherto been found capable of causing nitrification.

MUNRO, in " Formation of Nitrites during Nitrification of Ammoniacal Solutions" ('Chemical News,' vol. 56, pp. 62–64 ; 'Chem. Soc. Journ.,' Abst., 1888, p. 82), contends, in opposition to GAYON and DUPETIT, that the nitrite found during nitrification is actually the result of the direct oxidation of the ammoniacal nitrogen, and not due to a reduction of nitrate first formed.

In " Distribution of the Nitric Ferment and its Function in the Disintegration of Rocks " ('Annalen Chem. Phys.,' [6], vol. 11, pp. 136–144 ; 'Chem. Soc. Journ.,' Abst., 1887, p. 1135), MÜNTZ states that the nitrifying ferment is to be found at the highest elevations of the Alps, and ascribes to it the power of causing the disintegration of rocks. He also observes that organisms which appear to be identical with the nitrifying ones are capable of reducing nitrates when air is excluded.

It is not a little surprising that after the overwhelming evidence of such numerous investigators, the vital nature of the nitrification process should have been again disputed. This has been done, however, by FRANK ('Forschungen auf dem Gebiete der Agriculturphysik,' vol. 10, p. 56), whose contention has however, received no support, and has been rebutted by PLATH (' BIEDERMANN'S Centralblatt,' 1888, pp. 6–8 ; 'Chem. Soc. Journ.,' Abst., 1888, p. 521), and by LANDOLT (' BIEDERMANN's Centralblatt,' 1888, p. 577).

The modern methods of bacteriological study were first brought to bear upon the subject of nitrification by HERÆUS (' Zeitsch. f. Hygiene,' 1886, p. 193). This investigator, after inducing nitrification in an ammoniacal solution by means of a small quantity of garden soil, endeavoured to isolate the organisms present in the nitrifying solution by means of the method of gelatine-plate cultivation. In this manner he obtained four different organisms in a state of purity, and these he distinguished by the signs ρ, σ, ϕ and χ respectively. These pure cultures he inoculated into ammoniacal solutions of the following composition :—

	I.	II.
	grm.	grm.
Potassium phosphate	·05	·05
Magnesium sulphate	·01	·01
Calcium chloride	·05	·05
Ammonium carbonate	1·00	1·00
Grape sugar	1·00	none
	in 1000 c.c. of distilled water.	in 1000 c.c. of distilled water.

After remaining three days at 30° C. the solutions exhibited traces of nitrous acid as indicated by the iodide of starch reaction. After six days this reaction was decidedly stronger, but still the quantity of nitrous acid was not sufficient to admit of quantitative estimation. The experiments were repeated again with slight modifications, but again the amount of nitrous acid was insufficient for quantitative purposes. Upon this evidence, unsupported as it is by a single instance of nitrification on such a scale as to admit of the quantitative determination of the nitrous and nitric acids formed, HERÆUS claims for the organisms ρ, σ, ϕ, and χ the nitrifying power. This is the more remarkable, as nitrous and nitric acids are so eminently fitted for quantitative estimation, even when present in only the minutest traces. He even goes further and claims similar oxidizing properties for the well-known forms *Micrococcus prodigiosus*, *B. ramosus*, *the cheese Spirilla*, FINKLER'S *Spirilla, the typhoid bacillus of* KOCH-GAFFKY, *B. anthracis*, and *Staphylococcus citreus*, having obtained nitrous acid reactions when these organisms were respectively grown in sterilised and diluted urine. So positively, indeed, is it asserted in the above paper, that the organisms in question are endowed with nitrifying properties, that a casual reader will almost infallibly carry away the impression that the conclusions drawn are altogether beyond dispute. It is in this way, no doubt, that HERÆUS's statements have been generally incorporated without question in the ordinary text-book literature of the day.

HERÆUS's investigations were obviously calculated to be of the greatest importance to all interested in the question of nitrification, but his conclusions concerning the nitrifying power of the bacteria designated by ρ, σ, ϕ, and χ did not admit of immediate criticism, and could not be controverted or verified without a lengthy experimental inquiry. On the other hand, his assertions concerning the nitrifying properties of the well-known organisms mentioned above could be easily put to the test by any one possessing pure cultures of the same forms. Now, in the course of an investigation on " The Action of some Specific Micro-organisms on Nitric Acid " (' Chem. Soc. Journ.,' 1888, p. 373), one of us had occasion to experiment with two of the same organisms, viz., *M. prodigiosus* and *B. ramosus*, but in the case of neither could any evidence of nitrifying power be obtained, whilst, on the contrary, both organisms were found to be capable of reducing nitric to nitrous acid. This observation at once

indicated that a very different interpretation should be put upon HERÆUS's results to that which he had himself suggested.

The indications of nitrous acid which he had obtained on growing these organisms in diluted urine were obviously due to reduction of the small quantity of nitrates almost invariably present in normal urine, and not to any oxidation of the ammonia at all. This was pointed out by one of us in the paper in question, and, of course, casts a suspicion also upon the supposed nitrification obtained by HERÆUS with the pure cultures ρ, σ, ϕ, and χ, to which we had no access.

In further endeavours to obtain nitrification by means of pure cultures, one of us experimented with no less than 33 different forms obtained by us from air and water,* but in every case the results were negative. Such a number of negative results obtained when working under precisely similar conditions to those indicated by HERÆUS naturally tended to shake our belief in his having ever obtained more than traces of nitrous or nitric acids in any of his experiments with pure cultures.

NOTE.—Of other investigations professedly made with pure growths the following may be mentioned :—

" Nitrification," A. CELLI and F. MARINO-ZUCO (' Gazz. Chim. Ital.,' vol. 17, pp. 99–103 ; ' Chem. Soc. Journ.,' Abst., 1887, p. 858), in which the authors claim to have found a micrococcus (*M. cereus*) which behaved as a very efficient nitrifying agent. They further mention preliminary experiments purporting to show that certain organisms (*B. saprogenus*, *B. fluidificans*, and *M. luteus*) destroy nitrates when taken from gelatine, but produce nitrates when taken from potato-cultures.

" Changes induced in Water by the Development of Bacteria," T. LEONE (' Gazz. Chim. Ital.,' vol. 16, p. 505 ; ' Chem. Soc Journ.,' Abst., 1887, p. 615), in which the author asserts that the same organisms may successively produce nitrification and reduction of nitrates according to circumstances.

" The Chemical Action of some Micro-organisms," R. WARINGTON (' Chem. Soc. Journ.,' 1888, p. 751). In this paper the author records his failure to produce nitrification with pure growths obtained by gelatine-plate culture from nitrifying solutions ; he reviews the present position of the question of nitrification, and concludes with the words " an organism which nitrifies as soil nitrifies has yet to be isolated."

Failing to obtain nitrification with any of the pure cultivations of organisms in our collection, we determined to approach the matter from a different side, viz., to induce nitrification by a mixture of organisms, and then from this nitrifying mixture to attempt to isolate the particular organism or organisms responsible for the process.

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* " Studies on some New Micro-organisms obtained from Air," by GRACE C. FRANKLAND and PERCY F. FRANKLAND, ' Phil. Trans.,' 1887, p. 257 ; also " Ueber einige typische Mikro-organismen im Wasser und im Boden," by the same, ' Zeitsch. f. Hygiene,' vol. 6, 1889, p. 373.

The readiest means of inducing nitrification in suitable ammoniacal solutions is, as is well known, by the introduction of a minute trace of ordinary garden soil. The solution we employed for the purpose had the following composition :—

$$\left.\begin{array}{ll}\text{Salt solution* } \quad . \quad . \quad . \quad . & \text{100 c.c.} \\ \text{Ammonium chloride} \quad . \quad . & \text{·5 grm.} \\ \text{Carbonate of lime (pure) .} & \text{5·0 grms.}\end{array}\right\} \text{Diluted to 1000 c.c.}$$

The solution was thus entirely free from organic matter, as the results of previous investigators had clearly indicated that nitrification can take place in purely mineral solutions, whilst we hoped, through the absence of organic matter, to eliminate such other forms as require the same for their growth and multiplication.

The solution was invariably placed in small 6 oz. medicine bottles, which were filled to a depth of 2–3 inches, the bottles being previously plugged with cotton-wool, and sterilised at 150° C., whilst after introducing the liquid, the whole was steam-sterilised for upwards of an hour on three successive days.

Five bottles of this description were inoculated with a minute quantity of garden soil on May 9th, 1887, and placed in an incubator at 30° C.

On examination after eleven days, the bottles were all of them turbid, exhibited a slight skin on the surface and yielded the characteristic reactions for nitrous and nitric acids. Three of these solutions, which had nitrified, were submitted to gelatine plate cultivation; the plates developed numerous colonies, many of which caused liquefaction of the gelatine. A number of different colonies were picked out with the sterile platinum needle, and individually inoculated into separate bottles containing the sterile ammoniacal solution, but not one of these yielded any nitrous or nitric acid on subsequent examination after suitable incubation.

From one of the original bottles which had nitrified after addition of garden soil, a second bottle of ammoniacal solution was inoculated with a platinum needle; when this bottle had undergone nitrification, a fresh bottle was similarly inoculated from it, and so on in a long series of generations. The dates of inoculation of these successive generations, with other particulars, are recorded in the following table :—

---

* This salt solution was prepared by dissolving 1 grm. of potassium phosphate, ·2 grm. of crystallized magnesium sulphate, and ·1 grm. fused calcium chloride in 1000 c.c. of water.

Generation.	Date of Inoculation.	Quantity taken for Inoculation.		Date when Nitrification first observed.	Remarks.
I.	9. 5.1887	Original garden soil . . .		20. 5.1887	* Incubator at 30° C.
II.	25. 6.1887	3 needle-loops from	I.	30. 6.1887	* ,, ,,
III.	1. 7.1887	,, ,,	II.	7. 7.1887	,, ,,
IV.	14. 7.1887	,, ,,	III.	23. 7.1887	,, ,,
V.	25. 7.1887	,, ,,	IV.	17. 8.1887	* ,, ,,
VI.	26. 8.1887	,, ,,	V.	1.10.1887	,, ,,
VII.	3.10.1887	1 ,, ,,	VI.	7.10.1887	* ,, ,,
VIII.	7.10.1887	1 needle-point from	VII.	17.10.1887	,, ,,
IX.	17.10 1887	,, ,,	VIII.	29.10.1887	,, ,,
X.	7.11.1887	,, ,,	IX.	30.11.1887	At first in incubator then at 15–20° C.
XI.	1.12.1887	,, ,,	X.	15.12.1887	15–20° C.
XII.	16.12.1887	,, ,,	XI.	13. 1.1888	,,
XIII.	28. 1.1888	,, ,,	XII.	20. 2.1888	* ,,
XIV.	29. 2.1888	,, ,,	XIII.	5. 4.1888	,,
XV.	7. 4.1888	,, ,,	XIV.	27. 4.1888	* ,,
XVI.	30. 4.1888	,, ,,	XV.	10. 5.1888	,,
XVII.	12. 5.1888	,, ,,	XVI.	26. 5.1888	,,
XVIII.	19. 7.1888	,, ,,	XVII.	3. 9.1888	,,
XIX.	3. 9.1888	,, ,,	XVIII.	1.10.1888	,,
XX.	11.10.1888	,, ,,	XIX.	20.11.1888	,,
XXI.	24.11.1888	,, ,,	XX.	26. 2.1889	,,
XXII.	26. 2.1889	,, ,,	XXI.	4. 5.1889	,,
XXIII.	28. 6.1889	,, ,,	XXII.	18.10.1889	,,
XXIV.	4.11.1889	,, ,,	XXIII.	17.12.1889	,,

In the case of those generations marked with a * plates were poured and numerous bottles inoculated with isolated colonies,† but in no case could nitrification be induced. In the plates poured from the later generations the colonies had the appearance of being all of the same kind, and thus differed markedly from the plates obtained from the earlier generations in which there was a great variety of colonies, including liquefying ones which were entirely absent in the later plates.

Not only were bottles inoculated with single colonies, but, thinking that nitrification might be due to the combined action of two or more distinct organisms, bottles were sometimes inoculated with two or more different kinds of colonies from the same plate, and also in some cases with a large number of colonies taken from an overcrowded plate. In no single instance, however, have we succeeded in obtaining nitrification from a plate-cultivation, either from a single colony or from a number of colonies.

In addition to the series of experiments with the line of direct generations referred to above, several series of collateral experiments were being simultaneously carried on.

These collateral experiments were principally directed towards the separation and isolation of the nitrifying organism by the method of attenuation or dilution.

† These bottles were maintained in some cases at 30° C., in others at 15–20° C.

Dilution Experiments, Series I., commenced October 20, 1887.

50 c.c. of sterilised distilled water were placed in a sterile stoppered bottle and then inoculated with a few drops taken from the "sixth generation" (see p. 113), and the mixture was then violently shaken for some time to thoroughly disintegrate any coherent masses of organisms that might have been introduced. From the attenuation thus prepared a number of test-tubes containing sterile ammoniacal solution were respectively inoculated with one loop of a platinum needle, another series of test-tubes with two loops, and a single test-tube received as much as twelve drops from the above attenuation.

The several tubes thus inoculated on October 20, 1887, were examined on November 30, 1887, when the one which had received twelve drops yielded strong reactions both with diphenylamine and sulphanilic acid, whilst the tubes inoculated with one and two loops respectively gave no reactions, nor did they when subsequently examined on December 23, 1887. Thus only in the case of the tube which had received twelve drops had any nitrifying organisms been introduced, whilst the smaller quantities (one and two loops) employed for the other tubes must have been quite free from these organisms.

The solution which had been nitrified with the twelve drops added as above was reasonably to be regarded as purer than the "sixth generation" from which it had been obtained by large dilution. This "twelve-drop attenuation" as it may be called, therefore, was cultivated further in successive generations, each giving rise to well-marked nitrification. The third generation of this series was plate-cultivated, and inoculations made with a number of the colonies, but in no case did nitrification result.

The second generation of this twelve-drop attenuation was employed as the starting point for the

Dilution Experiments, Series II., commenced March 5, 1888.

The solution used for dilution in these experiments had been inoculated on December 1, 1887, it was found to have nitrified on December 23, 1887, but on re-examining on March 5, 1888, it not only gave reactions with diphenylamine and sulphanilic acid, but also with NESSLER's solution, and it might, therefore, be taken that the nitrifying organism was still in full activity. Two drops of this solution were added to about 50 c.c. of sterilised water, well shaken, and then five bottles containing sterile ammoniacal solution were inoculated as follows :—

No. 1 bottle with 9 drops.
,, 2   ,,   ,, 7   ,,
,, 3   ,,   ,, 5   ,,
,, 4   ,,   ,, 3   ,,
,, 5   ,,   ,, 1 drop.

These bottles were kept in the dark at the ordinary temperature of the laboratory, and when examined on April 5, 1888, they were all found to give strong reactions both with diphenylamine and sulphanilic acid.

In this series the dilution employed had, therefore, been insufficient, but No. 5 bottle, which had received only one drop, would obviously be likely to be the purest of the series.

Plates were poured from this No. 5 bottle as the purest, and from the resulting colonies obtained a number of bottles containing sterile ammoniacal solution were inoculated, but in no case was nitrification induced.

Dilution Experiments, Series III., commenced April 7, 1888.

No. 5 bottle, as the purest of the last series of experiments, was taken as the starting-point for this series. 1 drop from this bottle was diluted with 50 c.c. of sterilised distilled water, and from this *first dilution* 3 bottles (Nos. 1, 2, and 3), containing sterile ammoniacal solution, were inoculated with 1 drop each. Each of these bottles thus received about $\frac{1}{1000}$ drop of the original solution (bottle No. 5 above).[*]

Further 5 c.c. of the above "first dilution" were diluted to 50 c.c. with sterile distilled water, and from this *second dilution* a bottle No. 4 was inoculated with ·5 c.c. (or $\frac{1}{1000}$ drop of original solution), a bottle No. 5 with 1 c.c. (or $\frac{1}{500}$ drop of original solution), a bottle No. 6 with 2 c.c. (or $\frac{1}{250}$ drop of original solution), and a bottle No. 7 with 3 c.c. (or $\frac{1}{167}$ drop of original solution).

The whole series thus consisted of—

No. 1 bottle containing about $\frac{1}{1000}$ drop of nitrifying solution.

,, 2	,,	,,	,,	,,	,,
,, 3	,,	,,	,,	,,	,,
,, 4	,,	,,	,,	,,	,,
,, 5	,,	,,	$\frac{1}{500}$	,,	,,
,, 6	,,	,,	$\frac{1}{250}$	,,	,,
,, 7	,,	,,	$\frac{1}{167}$	,,	,,

All these bottles were kept at 15–20° C. and were found to have nitrified when examined on May 10, 1888.

Thus, again, the dilution had been inadequate for assured isolation, but bottles Nos. 1–4 would obviously be likely to contain the nitrifying organism in a greater state of purity than the remainder Nos. 5–7. One of these purest bottles was, therefore, made the starting-point for the next series of experiments.

Dilution Experiments, Series IV., commenced May 12, 1888.

One of the presumably purest bottles of the previous series was made the point of departure for the renewed attempt at isolation.

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[*] In this and subsequent calculations it is assumed that 1 c.c. consists of about 20 drops.

After a preliminary microscopic examination with a view to roughly estimating the number of micro-organisms in a given volume of the liquid, the following process of dilution was carried out :—

First Dilution (1 drop in 50 c.c. of sterile water).

> No. 1 bottle inoculated with 1 drop ($= \frac{1}{1000}$ drop of original nitrifying solution).
> No. 2 bottle   ditto.

Second Dilution (3 drops of First Dilution in 50 c.c. of sterile water).

> No. 3 bottle inoculated with 1 drop ($= \frac{3}{1,000,000}$ drop of original).
> No. 4 bottle   ditto.
> No. 5 bottle   ditto.
>
> No. 6 bottle inoculated with 3 drops ($= \frac{9}{1,000,000}$ drop of original).
> No. 7 bottle   ditto.
> No. 8 bottle   ditto.
>
> No. 9 bottle inoculated with 5 drops ($= \frac{15}{1,000,000}$ drop of original).
> No. 10 bottle   ditto.
> No. 11 bottle   ditto.
>
> No. 12 bottle inoculated with 10 drops ($= \frac{30}{1,000,000}$ drop of original).
> No. 13 bottle   ditto.
> No. 14 bottle   ditto.
>
> No. 15 bottle inoculated with 2 c.c. ($= \frac{120}{1,000,000}$ drop of original).
> No. 16 bottle   ditto.
> No. 17 bottle   ditto.
>
> No. 18 bottle inoculated with 5 c.c. ($= \frac{300}{1,000,000}$ drop of original).
>
> No. 19 bottle inoculated with 7 c.c. ($= \frac{420}{1,000,000}$ drop of original).
>
> No. 20 bottle inoculated with 10 c.c. ($= \frac{600}{1,000,000}$ drop of original).

The object of this wide range of attenuation was to endeavour to secure nitrification in only a part of the series, in which case it would be probable that the most diluted bottle in which nitrification still took place had received only one or, at any rate, only few individuals of the nitrifying organism.

These bottles were examined on the 7th July, 1888, when it was found that of the three most highly diluted solutions, contained in bottles Nos. 3, 4, and 5, only bottle No. 4 had nitrified, whilst bottles Nos. 3 and 5 gave no reactions either with diphenylamine or sulphanilic acid. This, of course, suggested the possibility that bottle No. 4 had been nitrified by a pure growth. Two gelatine-tubes * were inoculated

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* In all cases in which inoculations were made into gelatine-peptone, the needle was both stabbed into the material and also streaked along its surface, with a view to obtaining the depth and surface characters of any resulting growth.

from this bottle No. 4 on July 7, 1888, and on July 12 a very delicate growth in the depth and on the surface of the gelatine was observed. On microscopic examination, This was seen to be due to very characteristic long and slender bacilli. From one of these gelatine-tubes exhibiting this growth two bottles containing sterile ammoniacal solution were inoculated on July 12, 1888, and both of these bottles subsequently (examined on September 3 and October 1, 1888) nitrified.

This might at first sight be supposed to afford evidence that the nitrification had been caused by a pure growth taken from the gelatine-tubes, but on taking the following circumstances into consideration this was seen to bear another interpretation.

Thus, on July 7th, 1888, two bottles containing sterile ammoniacal solution were inoculated from the nitrified bottle No. 4 in the above series ; already, on July 12th, these bottles were found to be distinctly opalescent, and one of them was tested for nitrous and nitric acids but with negative results ; on microscopic examination, the bottle appeared to contain two different organisms, one a small fat bacillus, and the other a long bacillus, much resembling that observed in the gelatine-culture mentioned above. Bottles 3 and 5 of the above series, and which had not nitrified, were also microscopically examined and found to contain apparently pure growths of the short fat bacilli. Bottle 3 was inoculated into gelatine to compare the growth with that obtained from bottle 4, with the result that the same growth was formed, and this, on microscopical examination, was seen to consist of the same long bacillus. Somewhat later (July 15, 1888) the gelatine tube, inoculated from bottle 4, began developing a second and less conspicuous surface growth than that which first appeared, and this second growth, on microscopic examination, was found to consist of the short fat bacillus. Thus, both the long and short bacilli were obtained from bottle 4, which had nitrified, they were also both obtained from bottle 3, which had not nitrified ; and, therefore, the only probable conclusion was that neither of these organisms were connected with the nitrifying process.

The nitrification in bottle 4 must, therefore, have been caused by some third organism which had not grown in the two gelatine-tubes inoculated therefrom, and the nitrification induced by inoculating these gelatine-tube growths, as described above, must have been due to the transference of some of this third organism along with the visible growth. This is the more likely as the inoculation from the gelatine-tubes into the bottles was made only five days after the inoculation (July 7, 1888) of the gelatine-tubes themselves from bottle No. 4 ; and, on subsequently (September 3, 1888) endeavouring to cause nitrification by inoculating from these gelatine-tubes, the attempt failed, probably because the growth had increased, and there was, therefore, less probability of carrying any of the original matter introduced into the tube with the inoculating needle.

Although the nitrification occasioned by inoculation from the gelatine-tubes referred to above, could certainly not have been caused by a pure growth, still there was every reason to believe that the bottles which had thus nitrified, would contain a less admix-

ture of other organisms than anything we had previously dealt with, and these were, therefore, made the basis for further attempts to isolate the nitrifying organisms.

One of these bottles, which had been inoculated from the gelatine-tube as above, and which had duly nitrified, was transmitted through four generations of culture in bottles containing sterile ammoniacal solution. Thus from the gelatine-tube inoculated on July 7, 1888 :—

I. Bottle containing ammoniacal solution was inoculated from above July 12, 1888 ; examined and found reactions with diphenylamine and sulphanilic acid, September 3, 1888.

II. Bottle inoculated from I. bottle, September 3, 1888 ; examined and found reactions for nitrification, October 1, 1888.

III. Bottle inoculated from II. bottle, November 11, 1888 ; examined and found reactions for nitrification, February 26, 1889.

This III. bottle was made the starting point for the following experiments :—

Dilution Experiments, Series V., commenced March 20, 1889.

One drop from the nitrifying solution in III. bottle referred to above, was diluted with 50 c.c. of sterile distilled water, giving the

First Dilution . 1 drop $= \frac{1}{1000}$ of original drop.

Second Dilution (1 drop of First Dilution mixed with 50 c.c. sterile water).

$\left\{ \begin{array}{l} \text{1 drop} = \frac{1}{1,000,000} \text{ of original drop.} \\ \text{No. 1 bottle inoculated with 1 drop } (= \frac{1}{1,000,000} \text{ of original drop).} \\ \text{No. 2 bottle} \qquad \text{ditto.} \\ \text{No. 3 bottle} \qquad \text{ditto.} \\ \\ \text{No. 4 bottle inoculated with 2 drops } (= \frac{2}{1,000,000} \text{ of original drop).} \\ \text{No. 5 bottle} \qquad \text{ditto.} \\ \text{No. 6 bottle} \qquad \text{ditto.} \\ \\ \text{No. 7 bottle inoculated with 3 drops } (= \frac{3}{1,000,000} \text{ of original drop).} \\ \text{No. 8 bottle} \qquad \text{ditto.} \\ \text{No. 9 bottle} \qquad \text{ditto.} \\ \\ \text{No. 10 bottle inoculated with 10 drops } (= \frac{10}{1,000,000} \text{ of original drop).} \\ \text{No. 11 bottle} \qquad \text{ditto.} \\ \text{No. 12 bottle} \qquad \text{ditto.} \end{array} \right.$

1 drop $= \frac{1}{50,000,000}$ of original drop.

No. 13 bottle inoculated with 1 drop ($= \frac{1}{50,000,000}$ of original drop).

No. 14 bottle   ditto.

No. 15 bottle   ditto.

No. 16 bottle   ditto.

No. 17 bottle inoculated with 3 drops ($= \frac{3}{50,000,000}$ of original drop).

No. 18 bottle   ditto.

No. 19 bottle   ditto.

No. 20 bottle   ditto.

Third Dilution (1 c.c. of Second Dilution mixed with 50 c.c. sterile water).

No. 21 bottle inoculated with 5 drops ($= \frac{5}{50,000,000}$ of original drop).

No. 22 bottle   ditto.

No. 23 bottle   ditto.

No. 24 bottle   ditto.

No. 25 bottle inoculated with 10 drops ($= \frac{10}{50,000,000}$ of original drop).

No. 26 bottle   ditto.

No. 27 bottle   ditto.

No. 28 bottle   ditto.

No. 29 bottle inoculated with 2 c.c. ($= \frac{40}{50,000,000}$ of original drop).

No. 30 bottle   ditto.

No. 31 bottle   ditto.

No. 32 bottle   ditto.

A few of the above bottles were examined for nitrous and nitric acids on May 4, 1889, or about six weeks after their inoculation, when the following results were obtained :—

Bottle No. 10 (dilution $\frac{10}{1,000,000}$) gave strong reactions with both diphenylamine and sulphanilic acid.

Bottle No. 1 (dilution $\frac{1}{1,000,000}$) gave distinct reactions.

Further bottles were similarly examined on May 7th, 1889 :—

Bottle No. 13 (dilution $\frac{1}{50,000,000}$) gave no reactions more than a blank bottle similarly preserved.

Bottle No. 14 (dilution $\frac{1}{50,000,000}$)   ditto.

Bottle No. 15 (dilution $\frac{1}{50,000,000}$)   ditto.

Bottle No. 2 (dilution $\frac{1}{1,000,000}$)   ditto.

Bottle No. 11 (dilution $\frac{10}{1,000,000}$) gave slightly stronger reactions than the blank bottle.

Bottle No. 12 (dilution $\frac{10}{1,000,000}$) gave very strong reactions with both reagents.

Further examinations, both chemical and microscopical, were made on the following day, May 8th, 1889, with the following results :—

Bottles Nos. 1 and 3 (dilution $\frac{1}{1,000,000}$) yielded strong reactions both with diphenylamine and with sulphanilic acid, and both on microscopic examination were found to contain *a large number of very short almost micrococcus-like bacilli.*

Bottle No. 2 (dilution $\frac{1}{1,000,000}$), on the other hand, gave no reactions with either of the above reagents.

Bottle No. 4 (dilution $\frac{2}{1,000,000}$) gave no reactions.

Bottle No. 5 (dilution $\frac{2}{1,000,000}$) (examined on May 28, 1889) gave strong reactions with both reagents.

Bottle No. 6 (dilution $\frac{2}{1,000,000}$) (examined May 28, 1889) gave no reactions with the two reagents.

Bottles Nos. 7 *and* 9 (dilution $\frac{3}{1,000,000}$) gave strong reactions with both reagents.

Bottle No. 8 (dilution $\frac{3}{1,000,000}$), on the other hand, gave no reactions.

On May 28th, 1889, some of the bottles containing liquid still more dilute than $\frac{1}{1,000,000}$ were examined, but in every case with negative results. Thus :—

Bottles Nos. 29, 30, 31, *and* 32 (dilution $\frac{40}{50,000,000}$), all yielded negative results with both reagents.

Thus of a number of bottles inoculated with about $\frac{1}{1,000,000}$ of the original, some nitrified, whilst others did not, or, in other words, into some a nitrifying organism was introduced, whilst in others the inoculation failed to convey such an organism.

On the other hand, all the bottles containing less than $\frac{1}{1,000,000}$ of the original failed to nitrify, no nitrifying organisms having been conveyed in these more highly attenuated inoculations.

The most highly attenuated bottles which underwent nitrification being those which had received the dilution $\frac{1}{1,000,000}$, or bottles Nos. 1, 2, and 3, it was in these that the nitrifying organism in the greatest state of purity was to be expected.

It was upon these bottles, therefore, that our further attention was concentrated.

Of these bottles Nos. 1, 2, and 3, as already mentioned, Nos. 1 and 3 had nitrified strongly, and contained micro-organisms visible with the microscope, whilst No. 2 had not nitrified.

The contents of these three bottles were now submitted to investigation by means of cultivation-experiments. The results of these cultivation-experiments were highly remarkable.

Thus on inoculating from each of these bottles into tubes of gelatine-peptone—

(1) The tube from bottle No. 1, which had nitrified, never developed any growth whatever, although preserved for many months, indeed until the gelatine had become quite dried up.

(2) The tube from bottle No. 2, which had not nitrified, exhibited already on the third day a growth, consisting of a white surface expansion and beaded in the depth, and causing no liquefaction of the gelatine. Under the microscope the growth was seen to consist of chains of small bacilli, with bright shining spores.

(3) The tube from bottle No. 3, which had nitrified, also exhibited a growth.

These results clearly showed that in these three bottles we had two distinct

organisms to deal with; the one causing nitrification, but refusing to grow in the gelatine, the other growing in the gelatine, but incapable of producing nitrification. On this hypothesis bottle No. 1 contained the nitrifying organism in a state of purity, bottle No. 2 the non-nitrifying organism also in a state of purity, whilst bottle No. 3 contained a mixture of the two organisms. This supposition was the more probable as the less highly attenuated, and therefore presumably less pure, bottle No. 5 (dilution $\frac{2}{1,000,000}$), and bottles Nos. 10 and 12 (dilution $\frac{10}{1,000,000}$), all of which had nitrified, yielded growths on being inoculated into gelatine-tubes.

These observations were confirmed both by repetition and by the pouring of gelatine-plates from bottles Nos. 1 and 5, the plates from No. 1 yielded no colonies whatever, whilst those from No. 5 showed numerous small colonies.

It was thus established, beyond doubt, that the bottle No. 1 had nitrified, that it contained numerous micro-organisms of a very short bacillar form, but that this organism could not be cultivated on gelatine-peptone.

We next endeavoured to continue the cultivation of this organism in the ammoniacal solution, in order to ascertain whether it retained its nitrifying power, as well as its property of refusing to grow in gelatine.

Accordingly, on June 28, 1889, five bottles containing sterile ammoniacal solution were inoculated with a needle from bottle No. 1. These bottles, which may be described as No. 1 (2nd generation), were kept in the dark, at the temperature of the air, during the long vacation, and examined on October 18, 1889. *They were all five found to have strongly nitrified, but on inoculating them respectively into five different gelatine-tubes, in no single case was a growth developed.*

By way of control we had on the same day inoculated a gelatine-tube from the 23rd generation of the direct series experiments referred to on page 113, and this tube had already on the third day developed a visible growth.

On microscropically examining these five bottles (No. 1, 2nd generation), we again found in each case the characteristic small, almost micrococcus-like bacillus, to which we have had occasion to refer so frequently before, and which we may now fitly describe more in detail, having by the above experiments clearly established its causal connection with the process of nitrification.

Characterisation of the Bacillus of Nitrification.

(1) We have noticed that the solutions which had undergone nitrification by the organism in question remained perfectly clear, whilst the solutions which suffered nitrification by the mixture of organisms present in the direct series of experiments (see p. 113), generally exhibited a thin surface-film, and sometimes slight opalescence.

(2) One of the most remarkable features of the organism is its capacity of apparently indefinite growth in a medium practically destitute of organic matter,

although this property does not serve to distinguish it from the other organisms which accompanied it until the final separation by the process of dilution was accomplished.

(3) On microscopical examination, the solutions nitrified by the pure culture exhibited numerous small bacillar forms, which stain with somewhat more difficulty than most micro-organisms. The bacilli are about 0.8μ in length, and hardly longer than broad, in fact, their form is intermediate between that of a bacillus and micrococcus, so that the term "bacillococcus" may not inappropriately be used to designate them. They occur both isolated, in pairs, and in small irregular groups.

The accompanying figure exhibits their appearance as taken from bottle No. 1 itself.

Fig. 1.

Thus their appearance, although not so characteristic as to admit of their ready detection when present in small numbers along with other organisms, is still sufficiently definite to enable them to be distinguished from many other forms.

The organisms in the living state exhibit vibratory movement only.

Quantitative Determination of Nitrification.

We have already pointed out how necessary it is in speaking of nitrification, to indicate quantitatively the extent to which the process has taken place, to insure that the slight traces of nitrous and nitric acids so frequently exhibited by water which has been in contact with the air have not been mistaken for true nitrification.

We have, therefore, quantitatively determined the condition of the mineral nitrogen in several of the solutions in question. In these quantitative determinations we have employed the methods previously described by one of us ('Chem. Soc. Journ.,' vol. 53, p. 368).

Thus the ammonia was determined by means of NESSLER's reagent, and comparison of the tint produced with that obtained with a standard solution of ammonium chloride. In determining the nitric acid, the nitrite was first destroyed by evaporating the solution with an excess of ammonium chloride, and then decomposing the residual nitrate with sulphuric acid and mercury, the nitric oxide evolved being carefully measured in a gas-analysis apparatus.

In the determination of the nitrous acid, on the other hand, the solution was evaporated to dryness, with a little pure caustic soda, to prevent the loss of nitrite by decomposition with ammonia present; the residue was then decomposed with urea and dilute sulphuric acid, the gas evolved (consisting of nitrogen and carbonic anhydride) was freed from carbonic anhydride by means of strong caustic soda, and the residual nitrogen was then carefully measured in the gas-analysis apparatus.

Bottle No. 1 (2nd Generation), inoculated on June 28, 1889, was thus examined on November 21, 1889, and found to contain :—

Ammoniacal nitrogen = 5·76 parts per 100,000
Nitrous   ,,   = 6·43   ,,
Nitric   ,,   = 0   ,,

22nd Generation of Direct Series from Garden Soil (see p. 113), inoculated on February 2, 1889, was examined on December 20, 1889, and found to contain :—

Ammoniacal nitrogen = 0   parts per 100,000
Nitrous   ,,   = 10·71   ,,
Nitric   ,,   = 0   ,,

23rd Generation of Direct Series from Garden Soil, inoculated on May 20, 1889, was examined on November 28, 1889, and found to contain :—

Ammoniacal nitrogen, not determined
Nitrous   ,,   = 8·92 parts per 100,000
Nitric   ,,   = 0

The Ammoniacal Solution used in these experiments was found to contain on November 21, 1889 :—

Ammoniacal nitrogen = 11·12 parts per 100,000.

From the above results it appears, therefore, that the pure growth had converted more than one-half of the ammoniacal nitrogen into nitrous nitrogen in less than five months, whilst no nitric nitrogen had been formed. The impure growths contained in the 22nd and 23rd generations had also only given rise to nitrous without any nitric nitrogen, even in the course of ten and a half months. In fact, in the case of the 22nd generation, the whole of the ammonia had been almost quantitatively converted into nitrous nitrogen.

That in the case of " Bottle No. 1 (2nd generation)," the sum of the ammoniacal and nitrous nitrogen is in slight excess of the ammoniacal nitrogen present in the original solution employed is probably due to the bottle in which this 2nd generation had undergone nitrification having been considerably larger than the others, and consequently more concentration by evaporation through the cotton-wool stopper must doubtless have taken place.

In order to put this point to the test, we, later on (February 15, 1890), again took a similar large bottle, in fact, a duplicate of No. 1 above, and found

N as NH_3 = 4·53 parts per 100,000
N as N_2O_3 = 8·00   ,,
Total N = 12·53   ,,

R 2

thus clearly showing that the excess of total nitrogen in these large bottles was due to the greater amount of concentration by evaporation through the cotton-wool stoppers.

The exclusive production of nitrous nitrogen in these experiments is of special interest, as it raises the possibility of there being distinct organisms concerned in the formation of nitrites and nitrates respectively; whilst, on the other hand, it is by no means impossible that the formation of nitrate may yet take place in these solutions, or that the same organism may produce nitrates under other conditions. These are points which will require considerably more time to determine.*

The principal results of our investigation may be summarised as follows :—

1. The isolation, by the method of fractional dilution, of a micro-organism present in ammoniacal solutions undergoing nitrification originally induced by a minute quantity of garden soil.

2. The organism in question is possessed of a characteristic form, being a very short bacillus, about $\cdot 8\mu$ long, hardly longer than broad, and exhibiting only vibratory motion.

3. The organism can be cultivated in suitable ammoniacal solutions to which no organic matter whatsoever has been added. In such solutions we have cultivated it for nearly three years.

4. In these solutions the growth of the micro-organism is accompanied by the gradual transformation of the ammoniacal into nitrous nitrogen, whilst hitherto we have not observed the formation of any nitric nitrogen in solutions inoculated with the pure growth.

5. The solutions thus nitrified remain perfectly transparent and pellucid.

6. The solutions nitrified by inoculating the organism in question, after its purification by the process of fractional dilution, have in every case yielded absolutely negative results when introduced into gelatine-peptone, the organism, as taken from such nitrifying solutions being apparently incapable of growth in this solid medium.

On the other hand, in the process of purification by dilution referred to above, the less diluted, and, therefore, presumably less pure portions, invariably yielded growths on being introduced into gelatine-peptone. The refusal to grow in gelatine thus serves as an invaluable guide in ascertaining the purity of the organism.

We are at present engaged in the further investigation of this interesting organism, but in consequence of the large amount of time which these observations involve, owing to the slowness of the process of nitrification, we have deemed it advisable no longer to delay the publication of this first part of our enquiry.

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* In his study of the phenomena of nitrification, WARINGTON ('Chemical News,' vol. 44, p. 217) found that ammoniacal solutions seeded from old nitrified solutions generally only yielded nitrites and no nitrates.

Appendix.

Although in the experiments recorded above we have uniformly found that the pure nitrifying solutions yielded no growth in gelatine-peptone, we have by no means abandoned the hope of cultivating the nitrifying organism on this and possibly on other solid media.

Thus, in our most recent experiments we find that on inoculating from the pure nitrifying solutions into broth, the latter develops after a considerable time (about twenty days at the ordinary temperature), a very characteristic growth, the liquid becomes turbid, a very thin whitish pellicle forms on the surface, and afterwards a considerable amount of glutinous deposit collects on the bottom, the whole liquid in fact becomes highly viscous and adheres to a needle in long strings.

On microscopic examination, this growth is seen to consist of small bacilli, about $1\cdot5\mu$ in length, and about $\cdot5\mu$ in breadth, sometimes, however, forming threads ($5\cdot7\mu$ in length) in which the divisions are generally not apparent, but sometimes the divisions were sufficiently discernible to show that these threads were really made up of four or five individuals hanging together, end to end.

The accompanying drawing shows the appearance of the organism taken from a broth-cultivation.

Fig. 2.

On inoculating from such a broth-tube into a second, it is found that the growth makes its appearance more rapidly than in the first, thus whilst tubes inoculated from the nitrifying solutions required about ten to twenty days before any conspicuous growth was developed, the broth-tubes inoculated from these exhibited the characteristic growth in about six to ten days.

Although the microscopic appearance of the organisms in the broth-cultures thus somewhat departs from their appearance in the nitrifying solutions (compare figs. 1 and 2), such slight divergence in form is by no means uncommon in the case of one and the same organism when growing in different media, and in previous communications on the morphological characters of micro-organisms we have frequently had to call attention to such differences of shape, which are in fact well recognised by all who are acquainted with the nature of these low forms of life. We are, however, able to place beyond all dispute that the organisms growing in the nitrifying solutions (fig. 1), and those growing in the broth (fig. 2) are one and the same, for on inoculating the broth-growth into an ammoniacal solution, the latter (which, as we shall point out below, actually nitrified), again yielded the same characteristic forms as were represented in fig. 1. Thus the accompanying drawing shows the microscopic appearance

of the organisms taken from the ammoniacal and nitrifying solution inoculated from a broth-tube, which had itself been inoculated from a previous nitrifying solution :—

Fig. 3.

Not only, however, is the growth in broth much more rapid when the inoculation is made from a previous broth-tube, instead of from the nitrifying solution itself, but on inoculating from the broth into gelatine-peptone, a very slow growth actually makes its appearance in the latter.

This growth in the gelatine appeared after about three weeks in the first inoculation from broth, but on subsequently inoculating from this gelatine-culture in gelatine again, the growth was more rapid, appearing in from ten to twelve days.

This gelatine-growth appears as a transparent, smooth, shining, thin greyish expansion on the surface, which slowly increases in thickness, and causes a depression with gradual liquefaction.   On microscopic examination, the growth was found to consist of small bacilli, often hanging together in pairs, in fact their appearance was intermediate between that of the bacilli from broth on the one hand (fig. 2) and from the nitrifying solutions (figs. 1 and 3) on the other.   Fig. 4 represents the appearance of the organism taken from a gelatine cultivation.

Fig. 4.

Our nitrification experiments with these broth and gelatine cultures have not yet proceeded far ; we have, however, already obtained undoubted conversion of ammonia into nitrous acid by means of two of the broth-cultures.   Thus a broth-tube was inoculated from one of the pure nitrifying solutions on October 21, 1889 ; on November 2, 1889 (*i.e.*, after twelve days), the broth had become slightly turbid ; on November 4, 1889, two bottles containing ammoniacal solution were inoculated from the broth-tube; one of these bottles was placed in the incubator at 25° to 27° C., whilst the other was left at the ordinary temperature of the air.   On December 17, 1889 (or after forty-three days), the incubator-bottle gave undoubted reactions with both diphenylamine and sulphanilic acid.   On January 10, 1890, these reactions had become very strong, and on February 12, 1890, the mineral nitrogen was quantitatively determined according to the methods described on page 122, with the following results :—

Ammoniacal nitrogen = 10·70 parts per 100,000.
Nitrous   ,,   = 1·10 ,,   ,,
                                   ———
                Total .   .   .   11·80 ,,   ,,

Nitric nitrogen was presumably absent; it could not, however, be tested for, owing to the insufficiency of the liquid available.

As the ammoniacal solution originally contained about 11 parts of ammoniacal nitrogen (see analysis, page 123), and as some concentration of the liquid had taken place whilst in the incubator, it is obvious that the above results indicate marked conversion of ammonia into nitrous acid.

In the same way a bottle containing ammoniacal solution was inoculated from a similar broth cultivation on November 19, 1889, and placed in the incubator at 24° C. On January 10, 1890, this yielded strong reactions with diphenylamine, and also with sulphanilic acid.

The duplicate bottle inoculated on November 4, 1889, but kept at the ordinary temperature (15–20° C.) has not yet nitrified. Nor has any nitrification yet taken place in a bottle inoculated from a gelatine culture on December 10, 1889.

There is, however, nothing surprising in the retarded nitrification, which appears thus, at best, to take place from broth or gelatine cultures, for there are many other instances of the physiological properties of organisms being modified by change of soil. It has, indeed, been frequently observed by one of us in the case of other fermentations, that the fermentative power of a particular organism towards a particular substance may be greatly modified by growing it in different media. This appears to have been already recognised by FITZ in his later work ('Berlin, Chem. Ges. Berichte,' 1882, p. 878) in which the following very suggestive, but hitherto little recognised passage occurs :—

" Die Fähigkeit, Gährung zu erregen, wird ausser durch hohe Temperatur auch aufgehoben, wenn dem Spaltpilz sehr reichlich und andauernd Sauerstoff dargeboten wird.

" Wenn man, z. B., eine einzige Zelle in eine verhältnissmässig grosse Menge Kulturflüssigkeit aussät, so wird die Gährfähigkeit beträchtlich herabgestimmt und oft auch ganz aufgehoben.

" Ebenso verhält es sich wenn man den Spaltpilz bei sehr reichlichem Luftzutritt viele Generationen hindurch in einer Kulturflüssigkeit, in welcher er keine Gährung verursachen kann, kultivirt."

BRIEGER ('Zeitsch. f. Physiol. Chemie', vol. 8, pp. 306–311) points out that the "pneumococcus" of FRIEDLÄNDER loses its pathogenic properties when cultivated in sugar solutions, and only regains them when again cultivated in the ordinary solid media.

Another very remarkable instance of the chemical properties of a micro-organism being altered by the medium in which it is cultivated, is exhibited by the well-known *B. prodigiosus*, which on long-continued culture on gelatine or agar-agar loses its pigment-producing power, which can, however, be restored by cultivation on potatoes.

In the present instance the nitrifying organism had been cultivated, as already mentioned, for nearly three years, in the dilute ammoniacal solutions practically

destitute of organic matter, to which we have referred, so that its sudden transference to rich, nutritive media, like broth and gelatine-peptone, might not unnaturally be expected to produce a profound change in its organisation.

We have already shown how the capacity of growing in gelatine was entirely lost through this long residence in the dilute ammoniacal medium, which capacity was only restored by a preliminary cultivation in broth, similarly, we presume that cultivation in broth and gelatine may greatly diminish, and in some cases possibly extinguish, the nitrifying power of the organism.

It is particularly noteworthy that the ammoniacal solutions inoculated from the broth cultures, *on inoculation into gelatine, yielded no growth whatever, but gave the characteristic tardy growth in broth,* in this respect, as in the microscopic form of the organism they contained, precisely resembling the pure nitrifying solutions from which these broth-cultures were originally obtained.

VI. *A Record of the Results obtained by Electrical Excitation of the so-called Motor Cortex and Internal Capsule in an Orang-Outang* (Simia satyrus).

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[Plates 16–21.]

Contents.

Introduction.

Having been occupied with the study of the minute representation of motor function in the cortex of the Bonnet Monkey (*Macacus sinicus*), it seemed to us extremely necessary to investigate the character of such representation in the cortex of an anthropoid Ape, in order that we might form a more correct estimate of the mode of localisation in Man. A comparative study of the brains and habits of the more easily obtainable anthropoids showed clearly that for our object the Orang was more suitable than the Chimpanzee, in being likely to afford results nearer to those presumed to exist in Man. We therefore procured a young Orang which, in the opinion of Professor D. J. Cunningham, of Trinity College, Dublin, was about $2\frac{1}{2}$ years old.

In addition to excitation of the cortex, we also investigated the movements obtained by stimulating the fibres of the internal capsule.

We shall therefore arrange the facts as follows :—

    Anatomy.

    Method of Investigation.

Results of Excitation—
    A. Cortex.
    B. Internal Capsule.
Summary and Review of the Character of Localisation in this Animal.

ANATOMY.

We say nothing here respecting the general anatomy of the animal, having deposited the carcass in the Hunterian Museum.

The area of cortex which we explored comprises the so-called motor region. This area is bounded below by the Sylvian fissure, above by the longitudinal fissure, posteriorly by the intra-parietal sulcus, and anteriorly by a line drawn vertically through about the middle of the three frontal convolutions. It includes the fissure of ROLANDO, the posterior half of the superior frontal sulcus (corresponding to the sulcus x in the Bonnet Monkey : *vide* 'Phil. Trans.,' B, 1887), the præcentral sulcus, and the postcentral sulcus in the parietal lobule (corresponding to sulcus z in the Bonnet Monkey). Of the two small sulci which are so commonly seen in the Bonnet Monkey and in Man at the foot of the ascending or central convolutions, the one which is usually situated at the foot of the ascending frontal (v in the Bonnet Monkey*) was absent, while that at the foot of the ascending parietal was not only strongly developed, but there was also another small indentation in front of it, in the foot of the gyrus.

We will now proceed to make a few remarks in detail on these fissures and sulci, as we believe that there are certain features in their conformation which are of practical importance for the purposes of localisation.

Fissure of ROLANDO.

We have already drawn attention to the shape of this fissure in the Bonnet Monkey ('Phil. Trans.,' *loc. cit.*), and we have shown that in that animal it presents a well marked genu at the junction of its middle and lower thirds, and we have further shown that this genu indicates the lower border of the upper limb area of representation.

In the Orang this genu is extremely well marked, so that in tracing the fissure from below upwards it makes almost a right angle as it turns backward, and then, as a necessary consequence to reach the upper margin of the hemisphere, it has to turn again upwards almost at a right angle, see figs. 1, 2, 3, Plates 16, 17. It follows from this that the principal axes of the ascending or central convolutions being parallel to the Fissure of ROLANDO are very flexuous, instead of being merely oblique as in the Bonnet Monkey. This extreme angularity seems to be a very characteristic feature of the anthropoid Apes, as contrasted with the brain of Man, in whom, however, it is sometimes seen.

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* See forthcoming paper by the authors on the facial region of representation.

It will now be important to note the position of the genu of this fissure in the Orang and Man, as compared with the Bonnet Monkey. In the last, it is situated at the junction of the middle and lower thirds of the fissure, in the Orang and in Man it is found at about the mid-point between the upper margin of the hemisphere and the Fissure of SYLVIUS, or higher. We would suggest that this is due to the greater development in the higher animals of the convolutions about the lower end of the Fissure of ROLANDO, including the third frontal gyrus. We think this is borne out by the fact that it is the lower part of the Fissure of ROLANDO, *i.e.*, that part below the genu, which is so much lengthened.

Of further curvation, the Fissure of ROLANDO often exhibits at its upper end a slight bend forwards which, in man, is surmounted by a small curve in the opposite direction.

The Præcentral Sulcus

Consists of a vertical arm, which springs from close above the Fissure of SYLVIUS, and ascends till it bifurcates just below the horizontal level of the genu of the Fissure of ROLANDO, into a posterior ascending branch and an anterior horizontal. Where this latter joins the vertical limb, it forms a right angle anteriorly, which, as will be subsequently seen, is just as in the Bonnet Monkey, the focus for the representation of the movement of turning the head and eyes to the opposite side.

In this angle, formed by the limbs of the præcentral sulcus, there is another sulcus, directed downwards and forwards, and which we believe to be the homologue of the sulcus which we have temporarily called "*w*" in the Bonnet Monkey.* In the anthropoids, its anterior end seems to extend down to the orbital surface of the lobe, and to wind round in front of the anterior extremity of the Fissure of SYLVIUS.

(We believe that this sulcus corresponds to the inferior frontal sulcus of Man.)

We have pointed out in a previous communication ('Phil. Trans.,' B, 1887) that the sulcus in the Bonnet Monkey, which Professor SCHÄFER provisionally termed *x*, is the superior frontal sulcus of human nomenclature. We also showed that it consisted in its simplest condition of a vertical and a longitudinal limb. Now it has been proposed by JENSEN to call the vertical limb the superior præcentral sulcus; without giving our support to this step, we wish to emphasize the correctness of the description, and to point out that in the Orang this limb is well marked. As a rule, we believe that it will be found that this limb, or its secondary continuation, reaches down to the horizontal level of the genu of the Fissure of ROLANDO. For localising purposes, however, as we have shown in the Bonnet, the important feature of the superior frontal sulcus is not so much the vertical as the horizontal limb; for, on continuing the direction of the latter limb backwards across the ascending frontal and parietal convolutions it constitutes, according to our observation, the boundary

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* 'Phil. Trans.,' B, 1888.

between the representation of the upper and the lower limbs respectively ; this we similarly find to be the case in the Orang (see fig. 4, p. 150).

The Intraparietal Sulcus

In the present brain runs a characteristic course, viz., from near the lower end of the ascending parietal gyrus (where it bifurcates) upwards and backwards.   At the bifurcation one limb runs forwards and upwards towards the genu of the Fissure of Rolando, and the other downwards towards the Fissure of Sylvius.   As it courses across the parietal lobe it follows in a fairly parallel manner the direction of the Fissure of Rolando.

The post-central sulcus (Ramus Superior of the intraparietal sulcus, according to Cunningham, ' Journ. of Anat. and Physiol.,' 1889) is well marked, and begins near the longitudinal fissure at the upper margin of the hemisphere, and follows close round, strictly parallel to the Fissure of Rolando, reaching down the ascending parietal convolution almost as far as the genu of the Fissure of Rolando, *vide* figs. 2 and 3.

Convolutions.

The ascending frontal and parietal convolutions have the same position and relations as in Man and are bound by the same fissures and sulci.

The *three frontal convolutions* present more difficulty, as their boundaries are not so evident.   The division between the superior and middle convolutions is sufficiently indicated by the superior frontal sulcus ; but the line of separation between the middle and inferior convolutions is not so evident ; we think, however, that the sulcus " *w*," *i.e.*, the inferior frontal sulcus, marks the boundary between the two. Around the posterior end of this sulcus, as will be subsequently seen, is situated the focus of the representation of the movement of turning the eyes to the opposite side ; and since this particular focus of representation in the Bonnet surrounds the posterior end of the sulcus *w*, we think we are justified in regarding the sulcus marked I.F.S., in figs. 1, 2, and 3, as homologous to the *w* or inferior frontal sulcus of the Bonnet, and, consequently, as the dividing line between the middle and inferior *frontal convolutions.*

For topographical purposes, as was shown many years ago, the sulci are of very much more importance than the convolutions which they fashion, consequently, we need not prolong the description of the latter.   We must, however, recall attention to the determination by the sulci of representation or its absence, a point which we first observed in the Bonnet.   To repeat, representation of movement in the cortex is found only on the summits of the gyri of the convoluted surface, and the approach of a sulcus dimpling the cortex at all makes it at once inexcitable.   This, true of the Bonnet, is still more marked in the Orang, where, in addition, as will be seen, we have islands of smooth cortex, even between foci, which are inexcitable.

In the yet more highly divided human cortex this principle naturally must be held to prevail with still more force, as seems, in fact, to be the case, see p. 152, *et seq.*

METHOD OF INVESTIGATION.

In accordance with our present mode of recording the results of excitation of the cortex cerebri, we proceeded to expose and explore the surface of the so-called motor region in the left hemisphere in the following manner :—

1. *Anæsthetisation.*—The animal was thoroughly narcotised with ether. It was interesting to note that, in harmony with the greater size and the more highly evolved intelligence of the animal, it required, relatively, as well as of course absolutely, more ether than the lower Macacque Monkeys do.

2. *Operation.*—The brain was exposed by the method described elsewhere ('Brit. Med. Journ.,' 1887). The large flap of scalp and segments of the dura mater were carefully replaced after each measurement of the surface (*vide infra*) or excitation, so as to entirely preclude the possibility of the cortex becoming damaged by drying and cooling. For the special prevention of the two latter sources of error, the temperature of the room was maintained at 75°–80° F., the animal was placed upon a hot tin (temperature about 100° F.) and, finally, the surface of the brain was occasionally irrigated with warm ·75 per cent. salt solution.

We found that these precautionary measures preserved the excitability of the cortex at its normal level for many hours.

3. *Method of Recording.*—The positions of the sulci of the area exposed were then transferred by means of fine compasses to paper ruled mathematically with squares, whose sides measured 2 mm. We thus obtained a projection of the configuration of the cortex, the surface of which was already divided into squares of 4 square mm.* in extent. The actual drawing made at the time of the experiment is seen in fig. 3.

Each of these squares was then excited, the strength of the minimal stimulus noted, and also the resulting movement evoked.

4. *Mode of Excitation.*—The excitation was provided by the current from the secondary coil of a Du Bois-Reymond inductorium, furnished with a 1 litre bichromate cell. No side wire was employed (*vide* strength required *infra*). The electrodes were fine platinum points, 2 mm. apart, and they were applied horizontally, *i.e.*, in the long axis of the brain, each electrode resting on the mid point of the two sides of each square.

The strength of current required to evoke any movement was notably in excess of that which we have found necessary to use in the Macacque. With the Orang a minimal stimulus (*i.e.*, the weakest which would produce a movement) was obtained only when the secondary coil was moved to 9·5 centims. from the zero point of completely covering the primary coil. This current was such as to cause discomfort bordering on pain when applied to the tip of the tongue of the experimenter. This (minimal) stimulus was used throughout.

The method of investigating the internal capsule is given further on; see p. 147.

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* Throughout this paper we shall refer to these units of localisation as "squares."

Having now described the methods of investigation, we will now give in detail the results obtained by stimulating (A) the cortex, (B) the internal capsule.

In the description of the representation of movements obtained in the cortex we have thought it best to arrange the facts in the order in which the cortical areas were stimulated. This arrangement was similar to what we employed in the Bonnet Monkey, beginning with the foci for the smaller movements before proceeding to those for the larger, as we found by experience that the former were the first to lose their excitability. We shall therefore describe the different movements in the following order :—

Movements of the Eyes and Eyelids.
Movements of the Face and Mouth.
Movements of the Upper Limb.
Movements of the Lower Limb.

RESULTS OF EXCITATION.

(A.) *Cortex.—Left Hemisphere.*

Movements of Eyes and Eyelids.

(1) *Both Eyes move conjugately to the opposite side.*—The representation of this movement we found to extend in front of the præcentral sulcus, *i.e.*, to a slight extent just in front of its vertical limb, but lying principally in advance of the sulcus *w*.

The focus of the area is situated in the centre of the gyrus, at the points marked 2, 2, 2, in fig. 3. The whole representation extends as indicated by the stippling in fig. 1, Plate 18. The movement was also elicited from the centre of the gyrus, as above stated, at squares 3, 4, 5, 6 (fig. 3), but it was very slight, and at the angle formed by the two limbs of the sulcus there was no excitable point.

As compared to the same movement obtained from the Bonnet Monkey, *Macacus sinicus* (see 'Phil. Trans.,' B, 1888), this result from the Orang was characterised as being much slower in development and execution—in fact, resembling rather the movement* obtained from the angular gyrus in the Bonnet (*vide* Professor SCHÄFER, 'Roy. Soc. Proc.,' and confirmed by ourselves).

As indicated in the title of this section, no other movement was observed than that of both eyeballs to the opposite side, unaccompanied by the commonly-associated actions of opening the eyelids, dilatation of pupils, and turning of the head.

(2) *Combined Movement of Eyelids opening briskly, Eyeballs turning to the opposite side, and Head turning to the opposite side.*—This association of movements was obtained at squares 24, 25, 26, 27, 111, 112, 113, 114, 115, 116, 117, 118, 119, and appeared to be equably represented over the (dotted) area (see fig. 2, Plate 18), which is seen to be situated in the ascending frontal convolution. The sequence was that

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* First discovered by Professor FERRIER, 'Functions of the Brain,' 2nd ed., p. 243.

indicated in the title of this section, and more closely resembled the character of the corresponding movement obtained from the angle in front of the junction of the limbs of the præcentral sulcus in the Bonnet Monkey. The head and eyes moved in the horizontal plane.

(3) *Bilateral Closure of the Eyelids, but those of the opposite side more strongly than those of the same side.*—This movement, which we have found definitely marked in the Bonnet Monkey,* situated just between the lower end of the intra-parietal sulcus and the Fissure of ROLANDO, was here in the Orang observed to be just in front of the latter fissure at a homologous level, *i.e.*, at squares 120, 121, 122, 123, 124, 125, 126. (See fig. 3, Plate 18.) The lids of both eyes closed, but those of the same side far less completely than those of the opposite side.

Movements of the Face.

We shall next consider the movements of the lower part of the face, including the lips, mouth, and tongue, but before doing so, we must call special attention to the great development and mobility of the lips in the Orang, especially in the young animal (see p. 136).

The movements of the mouth and lips are very varied, and we have arranged them according to the following list.

Elevation of upper lip, opposite side
Retraction of angle of mouth, opposite side } Unilateral.
Eversion of lip, opposite side

Rolling in of both lips
Pursing of lips
Pouting of lips } Bilateral.
Closing of mouth

(4) *Elevation of upper lip of opposite side.*—This movement was obtained at squares 95, 96, 97, 121, 127; it was most marked at square 97; thus at 96 the latent period was long,† and at 127 the movement was very slightly marked. The representation of this movement is thus found as a narrow band in the ascending parietal gyrus close behind the Fissure of ROLANDO, and above the horizontal level of the lower end of the intraparietal sulcus (see fig. 4, Plate 18).

(5) *Retraction (horizontal) of the opposite angle of the mouth.*—Retraction (*i.e.*, with no elevation) of the angle of the mouth was elicited at squares 100, 101, 102, 103, 7, 8, 98, 99, 9, 10, 11, 12, 13, 14, 96.

The position of representation of this movement thus covers the middle third of that portion of the ascending frontal gyrus, which extends between the Fissure of SYLVIUS and the well-marked genu which we have in previous communications shown

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* Not yet published.   † *Cf.* next article on "Retraction," square 96.

to exist in the Fissure of ROLANDO.   The focus, or point of most intense representation, is situated in the line of squares 9, 10, 11, *i.e.*, just above the middle of the part now being considered.   See fig. 5, Plate 18.

All the squares for this movement are, with the exception of No. 96, situated in front of the Fissure of ROLANDO.   Excitation of No. 96 producing a very interesting result, viz., the mouth was first retracted and *subsequently* elevated.   Thus this square 96 forms a " border " square between the movements of retraction and elevation respectively, but the former predominates.

(6) *Eversion of the opposite lower lip*.—We obtained the representation of this movement at only one spot, viz., square 104.   See fig. 6, Plate 18.

(7) *Bilateral.   Rolling in of both lips, especially of the lower and of the opposite side*.—This movement consists in an incurving or inversion of the red margin of the lips over the teeth, so that the mouth being shut the skin of both lips comes into contact.   It involved the whole length of the oral fissure, but the opposite half more powerfully, and the movement of the lower lip was more extensive than that of the upper.   It was elicited from squares 15, 16, 17, 20, 21, 22, 23, being most represented at Nos. 22 and 23.   See fig. 7, Plate 18.

Before proceeding to describe the representation of pursing and pouting of the lips, it will be well that we should explain distinctly our means of recognising the difference between these two movements.

Mr. DARWIN[*] refers to the compression of the lips together in the performance of a delicate manual act by a young Orang ; he says, *loc. cit.*, " at each attempt the lips were firmly compressed, and at the same time slightly protruded."   We have expressed this combination of drawing together of the lips with some protrusion by the term pursing.   According to Professor THANE[†] the facial portion of the orbicularis oris acting alone projects the lips, whilst the two parts of the muscle acting as a whole, draw the lips together, the result being an action similar to that we have described as pursing.

When, on the other hand, we have eversion of the lips combined with extreme protrusion in anthropoids and children, a movement is produced which Mr. DARWIN (*loc. cit.*, p. 140) describes and figures as pouting, and which he considered to be an expression of disgust and disappointment, although, as he also states, a similar expression is used to denote the antithesis.   We may incidentally remark that we observed the same pouting when the administration of the ether in the present research was begun.

(8) *Pursing of both lips*.—This bilateral movement was obtained from two squares only, viz., 85 and 86, and was more marked at the former of these.   See fig. 8, Plate 18.

(9) *Pouting of both lips*.—Pouting was represented much more extensively than pursing, and as we consider it to be a further development of the latter by the super-position of eversion of the lips by means of the levator menti (THANE, *loc. cit.*, p. 281),

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[*] ' Expression of the Emotions,' 1872, p. 142.

[†] QUAIN's ' Anatomy,' 9th Edition, vol. 1, 1882, p. 280.

it is most interesting to observe that its representation was nearer than that of pursing to the Fissure of ROLANDO, passing round its lower end to reach towards the square in which eversion is most represented.   Pouting was observed at squares :— (*a*) In front of Fissure of ROLANDO, 87, 88, 89, 90.   (*b*) Behind Fissure of ROLANDO, 91, 92, 93, 94.

The focus was situated at No. 90, and the representation decreased from this point to No. 87, and as far as to No. 94 behind the Fissure of ROLANDO. See fig. 9, Plate 18.

(10) *Twitching of the opposite half of the upper part of the Orbicularis Oris.*— This remarkably specialised movement was restricted to the small triangular area lying between the upper end of the secondary sulcus in the foot of the ascending parietal gyrus, and reaching nearly to the genu of the Fissure of ROLANDO, viz., at squares 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, and was most marked at the vertical line passing through the middle of this region, viz., at squares 130, 134, 136. See Plate 19, fig. 10.

In passing to the next group of movements a prefatory explanation is needed of our use of the term, movements of the mouth.   By such term we wish to imply the opening and closure of the oral cavity, *i.e.*, lowering and raising of the lower jaw. Although we shall show that the lower jaw may be occasionally depressed within the closed lips, but that, if combined with compression of the lips, the mouth may remain shut, it is nevertheless more convenient and in accordance with general usage that the mandibular movements and the opening or closing of the labial fissure should be considered together.

Contrary to that which we had observed in the Bonnet (*Macacus sinicus*), in the Orang we never obtained evidence that the movement of mastication was represented in the cortex.   We, of course, only credit this fact with the value generally attached to negative evidence.   The various combinations of movements associated with the opening of the mouth we will consider in the chapter on the " Marches " observed, see p. 146.

(11) *Opening of the Mouth.*—This fundamentally important movement we found to be represented, exactly as in the Bonnet Monkey, at the foot of the ascending frontal and parietal gyri, around the lower end of the Fissure of ROLANDO, viz., at squares (*a*) in front of Fissure of ROLANDO, 20, 21, 22, 84, (*b*) behind the Fissure of ROLANDO, 91, 92.

It is universally well marked in the squares in front of the Fissure of ROLANDO, but only moderately in those which lie behind it (at 91, 92).   See Plate 19, fig. 11.

Movements of the Tongue.

We have found in the Bonnet Monkey (in some experiments shortly to be published) that the movements of the tongue are highly differentiated in cortical representation, constantly localised in the same position, and arranged in a definite order of grouping.   The movements of the tongue that we have found to be dis-

tinctly evoked by stimulating the cortex in the Orang we shall arrange in the order in which the varieties of representation are successively grouped in the cortex from above down.

Div. 1. Protrusion of whole organ.	Tongue flattened posteriorly, tip directed to opposite side.
	Tongue heaped up on the same side, tip directed to opposite side.
Div. 2. Rolling over of whole organ to side.	Tongue rolled over to opposite side.
Div. 3. Retraction of whole organ.	Tongue heaped up posteriorly, rolled over to opposite cheek.
	Tongue flattened throughout, the tip of the opposite side being retracted horizontally.

On summing up this table it will be seen that the movements are consecutively arranged in the cortex, so as to pass gradually one into the other from protrusion to retraction.

(12) *Tongue protruded, flattened posteriorly, tip directed to opposite side.*—This occurred at squares 7, 8, 98, 99, 9, 10, 11, 11'. This movement, shown in fig. 12, is the one upon the loss of which most stress has hitherto been laid in cases of destruction of the internal capsule and cortex in Man, but in our opinion it is but one and not the most important of a series of movements differing one from another, as will subsequently be seen. See Plate 19, fig. 12.

(13) *Tongue protruded, heaped up on the same side, and tip directed to the opposite side.*—This movement is represented next below the one just described, No. 12, and was noted at squares 12, 13, 14. See fig. 13.

It was seen that by " heaping up " to one side (for instance, the left) the tongue was narrowed transversely, the left side being thickened in the vertical diameter, so that that side and border bulged upwards.

(14) *Tongue neither protruded nor retracted, but rolled over to opposite side.*—This we found to occur at the next level below, viz., at squares 15, 16, 17.

It is a very important movement which has not hitherto received attention. In its execution the same side of the tongue rose and then the whole organ was tilted up and rolled over on its longitudinal axis, so that the previously horizontal dorsum was inclined towards the opposite cheek. See fig. 14.

(15) *Tongue retracted, heaped up posteriorly and rolled over to the opposite cheek.*—It is clear that this movement is a further extension of the last one (No. 14) with the addition of retraction. It was observed at squares 20, 21, 22. See fig. 15.

We need only add that the raising of the posterior part of the tongue was bilateral and occurred before the rolling over began. See " Marches," p. 146.

(16) *Tongue retracted slightly, flattened throughout, and the tip of the opposite side retracted horizontally.*—This rather singular movement we have never seen before; it was found at the lowest point of the representation of the tongue at squares 23, 84, 85, 86, 87, 88, 89, 90.

In this movement the tongue appeared to be adjusted horizontally to the floor of the mouth, while the upper surface of the opposite half was slightly hollowed (see fig. 16) and its tip retracted from the teeth.

Movements of the Upper Limb.

Proceeding now to investigate the mode of representation of the limbs, we will commence with that of the upper limb, taking the segments in order from below upwards, beginning with the thumb. But we cannot forbear from pointing out that it will be seen, especially now, how extremely integrated is the representation in this animal, and in the case of the upper limb we actually found it quite exceptional, with a minimal stimulus, for more than one segment to be represented at any given point.

Further, owing to the exaggeration of the bend of the ascending gyri above the genu of the Fissure of Rolando, the foci of the segments of the upper limb are arranged along a curved line which is horizontal at its lowest point and becomes vertical where the limb areas are contiguous, whereas, in the Bonnet Monkey, as we have shown, the foci are arranged in a straight but oblique line, parallel to the Fissure of Rolando. The functional arrangement and sequence of the representation of the segments being the same in the two animals, we think it is clear that this difference in horizontal level is simply due to the extreme bending of the Fissure of Rolando in the Orang. In illustration of this we show, in fig. 4, p. 150, the curve of the Fissure of Rolando in the two animals, with words indicating the foci of the segmental representation.

(17) *Movements of the Thumb.*—We observed the thumb to execute the following simple movements at the points indicated.

Action.	Square.
Flexion	38
Adduction, flexion	35, 36, 36'
Adduction	39, 83
Extension	37, 37', 138

We now have to point out the arrangement of the respective order of representation of these movements, and we shall compare that order with the simpler order, as seen in the Bonnet Monkey.

Thus, to begin :—Extension of the thumb we see is represented at 37 and 37', *i.e.*, in the foremost part of the area of representation. Now this is exactly the localisation we have demonstrated to occur in the Bonnet Monkey.

With a stronger current (8·5 cm.) we once obtained extension of the thumb at 138, which is, of course at the opposite pole of the thumb area.

The movement represented next to extension is that of adduction-flexion at squares 35, 36, 36'.

The above movements are represented in the ascending frontal gyrus: the next to be described are in the ascending parietal gyrus.   See fig. 17, Plate 19.

Of these the first is flexion at square No. 38, the second, adduction at squares Nos. 39 and 83.

As before stated, these movements of the thumb were single and unaccompanied by movement of any other joint, although the electrodes were finally kept applied for 5 seconds, a time more than sufficient in the lower Apes to generate movements in all segments of that limb, and probably, these in addition, be followed by others in the limb of the same side.   See conclusions, p. 149.

Finally, just as we have seen, extension is represented most anteriorly and superiorly, so now by reference to fig. 3, Plate 17, it appears that flexion and adduction are placed posteriorly and inferiorly.

The remarkable individual clearness of these and other movements is discussed on pp. 151, 152.

(18) *Movements of the Index Finger.*—The index finger is always represented in the Bonnet Monkey in the cortex next to the thumb area, and its focus in that animal is found to be in the ascending parietal gyrus (see fig. 18).   It is therefore of great interest, and we may add importance to note that also in the Orang the principal (indeed, almost the only) representation of this segment is situated next behind and above the thumb area, and in the ascending parietal gyrus, viz., at centres 40, 41, 83.

The kind of movement produced at each square was the well-known purposive or voluntary movement of extension ; the subordinate or associated movement of the index is alluded to in the next paragraph.   The focus of movement appeared to be at 40, and the representation diminished towards 41 and 83.   At 83 there was also observed adduction of the thumb, but there was no abduction of the index by the *abductor indicis* as might have perhaps been expected, the finger moved only in extension, *i.e.*, in its most highly integrated character.

(19) *Movements of the Fingers.*—The fingers (all moving in association) were represented in the ascending frontal gyrus at squares 28, 29, 30, *i.e.*, above and behind the thumb area.   The movement evoked at each was invariably extension, and this of a remarkably deliberate character, so that the sequence of the segments in action was very clearly demonstrated.   Thus the first digit to move was the little finger, which completely extended, next the ring finger, and finally, less marked extension of the remaining middle and index fingers.   No flexion of the fingers was noted to occur.   See fig. 19.

(20) *Movements of the Wrist.*—We have previously ('Phil. Trans.,' 1887, 1888) urged that the wrist and elbow should be regarded as joints subordinate in primary importance to those of the digits and shoulder.   In harmony with this view, we found in the Orang that the former was represented only in association, *e.g.*, with extension of the fingers, and with the following movements :—

Pronation at square No. 29.

Ulnar adduction at square No. 30.

See also Table of " Marches," and fig. 20, Plate 19.

(21) *Movements of the Elbow.*—The elbow we only saw move at one spot, viz., on exciting square No. 31, *i.e.*, next above and behind the wrist and fingers. The movement was flexion, and of primary character, no other segment being affected or associated with it, and the representation, though limited in area, was thus of a very high order (see p. 152). See fig. 21, Plate 19.

(22) *Movements of the Shoulder.*—In accordance with what has just been said about the subordination of the elbow and wrist to the shoulder and digits, it is evident why the shoulder should be represented at so many squares as we found it to be, viz., at Nos. 32, 33, 42, 43, 34, 44, 45, 46, 47.

It was represented around and below the hinder end of the superior frontal sulcus, see fig. 22, Plate 20, as in the Bonnet Monkey, while, owing to the special configuration of the sulci in the Orang, the representation shaded off at squares Nos. 46 and 47, into the area for the lower limb. In character the movement was always adduction, and was never associated with that of any other segment.

Movements of the Lower Limb.—Probably, in all the higher animals the representation of the lower limb is less integrated than that of the upper limb, but it was especially necessary to see what was the condition in an anthropoid like the Orang, whose customary vertical posture places it in an intermediate position between the Macacques and Man.

In the Bonnet Monkey, the lower limb area extends over the hinder third of the superior frontal gyrus, into the paracentral lobule on the mesial surface and backward into the ascending frontal convolution and superior parietal lobule, *i.e.*, the upper end of the ascending parietal gyrus.

In the Orang we found that, except with currents of such intensity (7 cms.) that obvious escape resulted, no movement was obtained on exciting the upper third of the ascending parietal gyrus. It is, however, of noteworthy interest that what we regarded as the effect of escape consisted only in extension and flexion of the hallux, viz., at squares 139, 140, 141, and this occurred between the postcentral sulcus and the Fissure of ROLANDO. Consequently, the representation of the lower limb in this anthropoid was confined to the part in front of the Fissure of ROLANDO (extending also over the mesial margin into the paracentral lobule[*]) ; and we shall now consider this region in detail, under the headings of the segments of the limb.

(23) *Movements of the Hallux.*—Omitting the squares 139, 140, and 141 just referred to, we obtained movements of the hallux at squares 56, 63, 64, 69, 76, 77, 78. See fig. 23, Plate 20.

Of these, the hallux moved primarily at 56 (where it was associated with movement

* See "Marches" and Right Hemisphere, p. 143.

of the ankle only) and 77 (where it was associated with movement of the small toes as well as of the ankle).

Character of the Movement evoked.—Flexion was noted at 56, 77, 78, and extension at 63, 64, 69, 76; consequently, the representation of flexion is anterior to that of extension. It will be seen that the hallux is at the highest point (*i.e.*, nearest the middle line) of the area only associated with the ankle and other toes. (*Vide infra.*)

(24) *Movements of the Small Toes.*—The small toes were represented at 54, 63, 64, 69, 70, 71, 76, 77, 78, 109, 110.

At squares Nos. 109, 110, which were situated on the anterior frontier of the lower limb area, we obtained the unique movement of very slight extension of the second toe, which is homologous with the index finger.

The primary movement of the small toes occurred on exciting squares Nos. 70, 71, 54, 78, No. 70 being the point of intensest representation; in fact, at 70 and 71 the small toes were the only segments which moved at all. See fig. 24, Plate 20.

Character of Movement.—This was in all cases extension. See also "Marches," p. 143.

(25) *Movements of all Toes.*—All the toes moved in synchronous association at squares 63, 64, 69, 76, the movement being invariably extension with separation of the digits. See also "Marches," p. 143.

(26) *Movements of the Ankle.*—The total of absolute representation of movements of the ankle is situated at squares 54, 55, 56, 60, 61, 62, 63, 64, 76, 77.

The primary movement was obtained at squares Nos. 55, 60. At 60 it was the only movement, consequently this affords another instance of the unique character of the localisation in the Orang. See p. 152, and fig. 25, Plate 20.

Character of Movement.—The nature of the movements observed was as follows :—

Eversion obtained at squares No. 54.
Dorsal flexion   ,,   ,,   Nos. 55, 56, 76, 77.
Plantar extension .,   .,   ,, 60, 61, 62, 63, 64.
Inversion   ,,   .,   No. 77 (observed only once on a repeated trial).

(27) *Movements of the Knee.*—The representation of the knee was found at squares 54, 55, 61, 62, 63, 64, 68. See fig. 26.

No primary movement was obtained, as was evidently probable from the condition of representation in the Macacque of such a subordinate joint.

Character of the Movement.—The movement was extension, except at square No. 68, where it was flexion.

(28) *Movements of the Hip.*—The total or absolute representation of the hip was found at the squares 48, 49, 50, 51, 52, 53, 61, 62, 63, 64, 65, 66, 68.

Primary Movement.—Just as with the shoulder, so the cortical representation of the hip in the Orang is not only very extensive, see fig. 27, but also singularly

integrated; thus movement of the hip (only) occurred at no less than eight squares, viz., Nos. 48, 49, 50, 51, 52, 53, 65, and 66.

The very large majority of these will be found to be aggregated on the upper border of the upper limb region, close to where the shoulder is most represented. See " Marches," p. 144, and conclusions, p. 151.

Character of the Movements.—In contradistinction to what we found in the Bonnet Monkey (*loc. cit.*) the character of the movement was extension of the hip (and knee), indeed with but three exceptions of which only one was a pure flexion.   The explanation of this we offer on p. 151.   The movements observed may be thus arranged :—

    Extension observed at squares Nos. 48, 49, 50, 51, 52, 53, 61, 62, 63, 64.
    Adduction . . . .   52, 53.
    Flexion   . . . .   68.
    Rotation   . . . .   65, 66.

See also next chapter, " Marches."

(A.) *Cortex.—Right Hemisphere.*

After the excitation of the left hemisphere had been completed as far as possible (*i.e.*, until the fall in blood pressure began to affect the excitability) we proceeded to rapidly expose the right hemisphere and to explore it with the same stimulus.

We found that the so-called motor points were symmetrically situated in the two hemispheres, and in consequence that the inexcitable regions (*vide infra*) which we were careful to explore first, as soon as the brain was exposed, were also similarly situated on the two sides.   The only trifling difference we noted was that at squares 121, 125 on the right side we obtained marked closure (blinking) of the opposite eyelids.   This means, of course, simply that on this side the representation of this movement was rather more highly differentiated.

We made an observation on the paracentral lobule in vertical line with square 78, *i.e.*, 7–8 mm. in front of the upper end of the Fissure of ROLANDO of this hemisphere, and found that the lower limb was alone represented there, *i.e.*, without the abdomen. See " Marches."

Marches.

Since, as a rule, the effect obtained by exciting the cortex of the Orang was a single primary movement, a march of the spasm from one group of muscles to another was exceptional.   We, therefore, have deemed it best to collect together under one heading all such " Marches " as follows :—

Area.	Square.	Primary movement.	Secondary movement.	Tertiary movement.	Quaternary movement.
Lower face	20, 21, 22	Mouth open	Tongue retracted, heaped behind, rolled to right		
Upper face	91, 92	Lowering of lower jaw	Lips pouted		
	96	Retraction of angle of mouth	Elevation of upper lip		
	121	Closure of both eyelids, especially of opposite side	Elevation of upper lip, marked		
	24, 25, 26, 27, 111, 112, 113, 114, 115, 116, 117, 118, 119	Eyelids open	Both eyes turn to opposite side	Head turns to opposite side	
Upper limb	83	Adduct thumb	Slight extend index		
	28	Extend little finger	Extend ring finger	Extend other fingers	Pronate wrist
	29	" "	" "	" "	Ulnar adduction of wrist
	30	" "	" "	" "	
Lower limb— Hip	61, 62	Extend hip and knee	Plantar-extend ankle	Extend digits	
	63, 64	" "	" "		
	68	Flex hip and knee			
Ankle	55	Dorsoflex ankle	Extend knee		
Small toes	54	Extend small toes	Evert ankle	Extend knee	
	78	" "	Flex hallux		
All toes	76	Extend all toes	Dorsoflex ankle		
Hallux	56	Flex hallux	Dorsoflex ankle		
	77	"	Extend small toes	Dorsoflex ankle	

From an inspection of the foregoing table which, with the exception of the combination of facial and oro-lingual movements to be mentioned later, is an absolute record of all the " marches " noted to have occurred, the extreme paucity of such combinations shows more clearly than words to what a remarkable extent the representation of each individual segment is integrated in the cortex of the Orang.   Too much stress cannot be laid upon this point since it demonstrates incontestably the great advance in evolution of function of the Orang's cortex above that of the Macacque, for, in the latter animal, single primary* movements were only rarely obtained, and even then by only momentary application of the electrodes.

In complete harmony with this view we also see that at only two groups of squares was there a combination of segmental movements in the upper limb (viz., 83, and 28, 29, 30), and these only involved the digits and wrist, whereas in the lower limb no less than nine such combinations were observed, and those (owing to duplication) at eleven squares.

The limited extent of such combinations is also manifested by the fact that all the segments of the lower limb were thrown into action at only two squares (63 and 64), and at a large number of the remainder only two segments were involved.

We have now to consider four points in this connection, these are :—

(1.) Combination of facial and oro-lingual movements.
(2.) Juxtaposition of synergic segments.
(3.) Character of movements in each march.
(4.) Relative position of squares from which marches were obtained.

* *I.e.*, a single movement not followed by any other movements, which latter we have termed secondary, tertiary, &c., in the successive order of their appearance ('Phil. Trans.,' B, 1887, 1888).

(1) *Combination of Facial and Oro-lingual Movements.*

Square. (List of squares in which face or tongue were represented.)	Tongue.	Face.
7 8 98 99 9 10 11 11′	Flattened, protruded to opposite side .	Opposite angle of mouth retracted
12 13 14	Tip to opposite side, heaped on same side	,, ,, ,,
15 16 17	Rolled over to opposite side	Rolling in of both lips, especially of lower (bilateral)
20 21 22	Retracted, heaped posteriorly and rolled to opposite side	Rolling in of both lips followed by mouth opening
23 24 85 86 87 88 89 90	Retracted slightly, flattened throughout, tip of opposite side retracted horizontally	,, ,, ,, Opening of mouth Pursing of lips ,, ,, Ponting of lip ,, ,, ,, ,, ,, ,,

The foregoing list shows which facial and oral movements are related with each other in synchronous action, and we do not see that it can be profitably amplified by further discussion.

(2) *Juxtaposition of Synergic Segments.*—Without a single exception the march involved segments in series according to their successive anatomical juxtaposition. The most striking instance of this phenomenon, and shown more minutely than anything noted by us in the Bonnet, is seen at square 29, where the successive order of segments involved was little finger, ring finger, second and index, wrist.

(3) *Character of Movements in each March.*—We find that the character of the movement in the stages of the march is simply a repetition of the character of the primary movement of the segment involved.

(4) *Relative Position of Squares from which Marches were Obtained.*—We have thought it important to note the regions where combinations of movements more frequently occurred, to see whether there was any definite relation among the squares one with another which would reveal any explanation of the phenomena in question.

Face Region.—Combination appeared to be universally distributed over the region for the face.

Upper Limb.—Combination was only found in the squares for representation of the small muscles (at squares 83, 28, 29, 30), *i.e.*, at the highest part of the hand region.

Lower Limb.—Combination occurred most at the squares situated in the vertical line drawn towards the middle of the lower limb region.

These facts are simply examples of the principle which we have previously enunciated, viz., that the representation of function is most intense on the summits of the gyri, and shades off to the sulci, where the excitability becomes nil.

(B) *Internal Capsule.*

We next proceed to raise the upper half of the right hemisphere (after ligature of the middle cerebral artery deeply in the Sylvian fissure just in front of the anterior limb of that fissure) by a horizontal section carried through the brain, as shown by the dotted line in fig. 1, Plate 16, so as to expose the internal capsule. See fig. 5, Plate 17.

Description.—The section exposing the internal capsule of the right hemisphere was a horizontal cut passing through (from before backwards) the third frontal gyrus, the foot of the ascending frontal and parietal gyri respectively, and crossing the posterior limb of the Sylvian fissure just where it turns upwards, viz., at about the junction of the middle and posterior thirds. The cut surface, viewed from above, exhibited, as is shown in the accompanying photograph, fig. 5, taken in the fresh state after the excitation, the following points. In front is seen the upper part of the anterior commissure, only the middle portion of which appears, as it is soon lost between the outer and middle zones of the lenticular nucleus. Posteriorly, in the middle line, the splenium of the corpus callosum limits the field of white fibres. Only the two outer-most zones of the lenticular nucleus are visible, and the caudate nucleus is only represented by a small portion of its head separated by a few white fibres from the putamen or outer zone of the lenticular nucleus. Of the anterior limb of the capsule, besides these fibres, there remains only a bundle of fibres about 4 mm. long, running directly backwards from the posterior surface of the anterior commissure. The hinder limb of the capsule extends from the inner border of the second or middle zone of the lenticular nucleus, and on section it is seen that the bundles of transversely cut fibres terminate posteriorly opposite the lamina medullaris situated between the zones.

We mapped out by compasses the cut surfaces of the basal ganglia and capsule, upon paper ruled with squares of 1 mm. side, taking care that the posterior limb was drawn parallel to the vertical lines of the squares.

Arbitrary numbers were then applied to each square from which result was obtained on excitation, and we now furnish the records of stimulation.

No. of square.	Distance of secondary coil.	Result.
	cm.	
1	10	Both eyes turn to opposite side
2	10	Both eyes turn to opposite side (more markedly than at No. 1) and open at the end of excitation
3	10	Both pairs of eyelids blink, then both eyes turn to the opposite side
4	10	Both pairs of eyelids blink, the mouth is strongly retracted to opposite side, tongue flattened, protruded, tip turning to the opposite side. Next opposite wrist flexed, doubtful movement of fingers
5	10	Both eyelids blink, both eyes turn to opposite side Mouth strongly retracted to opposite side, tongue protruded, tip to opposite side, opposite fingers and thumb flexed
5'	10	Nil
6	10	Mouth strongly retracted to opposite side. Tongue flattened, protruded to opposite side. Flexion of fingers (strong), flexion of fingers (slight). Flexion of wrist, doubtful flexion of elbow
6'	10	Doubtful flexion of fingers
7	10	Flexion of fingers, of thumb, of wrist, later mouth retracted to opposite side
7'	10	Flexion of finger and thumb, later flexion of wrist.
8	10	Flexion of fingers and thumb (very strongly), flexion of wrist, abduction of shoulder
8'	10	Flexion of fingers (less than 8)
9	10	Flexion of fingers
9'	10	Flexion of fingers and wrist, later eversion of ankle
10	9	Flexion of all toes, plantar extension of ankle, later pronation of wrist
10'	9	Very slight adduction of shoulder. Slight flexion of toes

As will be seen on fig. 5, Plate 17, the length of the posterior limb of the internal capsule is 18 mm. It will now be convenient to show more plainly the extent of distribution of the fibres for each particular movement.

Movement.	Extends	
	From—	To—
	m.	m.
Eyes turn to opposite side	1 max. 2	5
Eyelids closing (blinking)	3	5
Retraction of angle of mouth to opposite side	4	7
Tongue flattened and tip directed to opposite side	4	6
Flexion of thumb	5	8
Flexion of fingers	5	9'
Flexion of wrist	4	10
Flexion of elbow	6	..
Abduction of shoulder	8	..
Adduction of shoulder	..	10'
Flexion of toes	10	10'
Eversion of ankle	9'	..
Plantar extension of ankle	10	..

That this order of arrangement of the fibres in the capsule is mainly correct we have no doubt, in the light of our experiments on the capsule of the Bonnet Monkey

(Paper read to the Royal Society, December 12, 1889), but whether we obtained complete results is questionable, since after we had observed the above-mentioned facts, and were proceeding to amplify and control them, the animal died suddenly, evidently from syncope.

SUMMARY AND REVIEW.

We are now in a position to review the facts detailed above, and to examine into their real value.

I. *Condition of the Animal.*

(*a*) *Health.*—It had been in England about two months, and appeared to be in perfect health, and although the day before the experiment its appetite had begun to fail a little, to all appearances it was vigorous and healthy. It moved about in the vertical posture, and all its acts were performed completely, and with the deliberation special to the Orang.

(*b*) *Age.*—The animal was estimated to be about $2\frac{1}{2}$ years old. Since the animal was a young one, we wish it to be understood clearly that the facts which we put forward concerning the limitation of representation are not necessarily typical, as it is possible that this may be more extensive in an older animal. Bearing on this point we may mention that we have found that, given the same species of Macacque Monkey, a more extensive representation occurs with a larger, *i.e.*, an older, brain, and the converse also holds, but that the chief details and general plan are the same under the two circumstances.

Future investigation can alone decide this point.

II. *Vertical arrangement of the Representation of each Segment in levels transverse to the Fissure of* ROLANDO.

In the Bonnet Monkey (*Macacus sinicus*) we have shown that the representation of the segments of the various parts of the body is arranged along the Fissure of ROLANDO in horizontal levels, and that the boundary lines of these pass across the fissure. In the Orang the same arrangement holds as is well seen in the photograph on Plate 21. This photograph is constructed by writing on an enlarged copy of fig. 1, Plate 18, the various foci of representation. If we place the segments in successive order as we see them in the Bonnet and the Orang, the relations are as follows :—

Fig. 4.

BONNET.   ORANG.
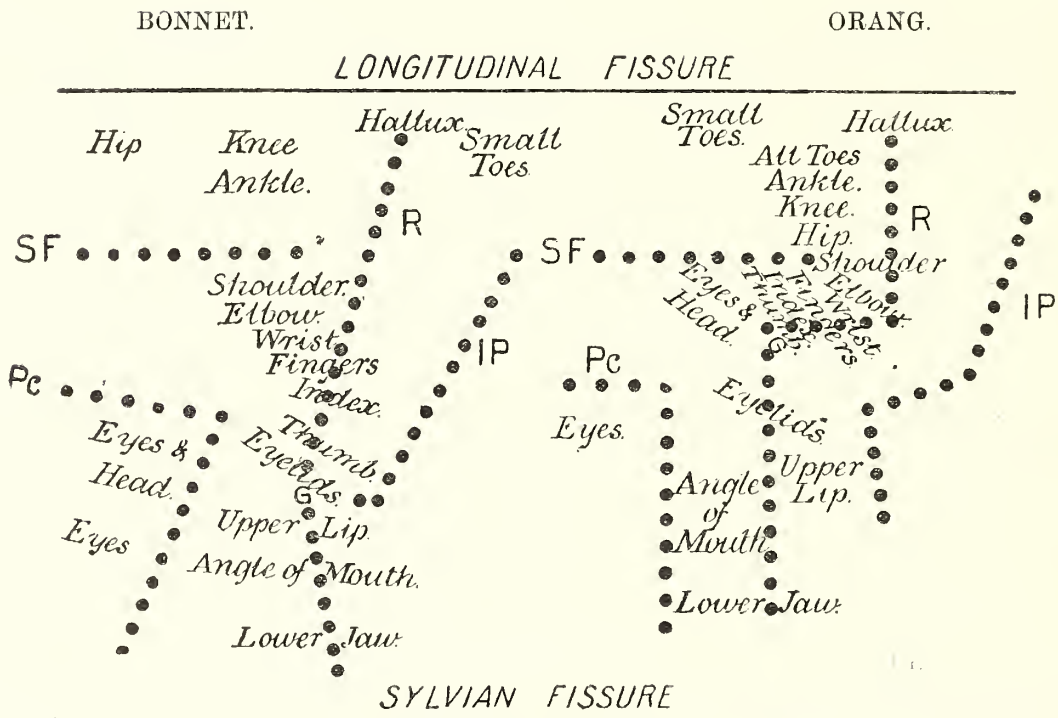
SF = Superior frontal sulcus.
R = Fissure of Rolando.
G = Genu of ditto.
Pc = Præcentral sulcus.
IP = Intraparietal sulcus.

We cannot but feel that the foregoing fig. 4 clearly shows that the general plan ot the arrangement of the representation of the segments in the two animals corresponds closely, and that the variations are rather due to the exaggeration of the sinuosities in the gyri of the Orang than to any special characters in that animal. Indeed, if the Fissure of ROLANDO in the Orang could be drawn out straight, the arrangement would be practically identical.

It is extremely interesting to see that the plan of the segments of the lower limb is truly horizontal in the Orang, i.e., transverse to the Fissure of ROLANDO, but antero-posteriorly placed in the Bonnet. We believe that this depends upon the following considerations, which we venture to suggest are not merely fanciful, and which we connect with the great representation in the Orang of extension of the hip as contrasted with the usual flexion of that joint in the Bonnet.

The habits of the two animals appear to afford an explanation of the difference of the mode of representation in their brains.

The Bonnet Monkey saves itself in flight by climbing, for which the first purposive movement of the lower limb is of necessity flexion of the hip. This, therefore, we find represented most anteriorly in the lower limb region, and therefore localised exactly

in the psychical order of events, *i.e.* (1), direction of the head towards the danger ; (2) initial movement of flight.

In the Orang the case is otherwise.   The vertical posture requires much purposive action.   An Orang placed under the same circumstances as the Bonnet rises on to its hind limbs by extension of the hip ; we, therefore, find that this movement is represented in this animal to great excess compared to that of flexion.

Further, in accordance with the now well recognised fact that as the foci of function are more spread over the convex surface of the hemisphere from the longitudinal towards the Sylvian fissure, so they are more highly, *i.e.*, purposively evolved, it is clear that in the Orang the hip possesses a far higher character of representation than it has in the Bonnet, and we believe that this subserves the vertical posture.   In addition, just as in the Bonnet the anatomical juxtaposition of the foci corresponds with the psychical order of events, in the Orang it is seen also that the focus of direction of the head towards the source of danger is situated *below* the focus for the hip— below because, as will be seen, what in the upper part of the hemisphere is horizontally directed in the Bonnet Monkey becomes vertically arranged in the Orang.

III. *Integration of Foci.*

The extraordinarily integrated character of representation of individual movements in the several foci thus observed is of such extreme importance in elucidating the mode of localisation of function in the brains of the highest animals, the main object of the present research, that we shall now examine their arrangement in successive detail from this standpoint.

The most important facts to notice in the comparison of the representation of the Anthropoids and Man with that in the Macacque and similar Monkeys, the highest animals on which experiments on the brain have hitherto been conducted, are that—

1. Instead of the excitable area of the cortex being continuous, it is (in the Orang) much interrupted by spaces from which no effect could be obtained even by the application of strong stimuli.

The inexcitable areas are indicated on fig. 3, and it will be noticed that they separate the areas of representations of the larger divisions of the body, and do not separate those for the segments of such divisions.

2. Reference to the figures shows also that a large part of the cortex generally considered to be excitable was not so in the Orang.   This is especially the case in the frontal lobe, the whole of which in front of the ascending frontal gyrus was inexcitable, with the exception of the area for the representation of turning the eyes to the opposite side, which is just in front of the præcentral sulcus.   See figs. 1 and 3.

Similarly, the ascending parietal gyrus gave relatively much fewer results than the same gyrus in the Bonnet Monkey, being practically wholly inexcitable in its upper third (the superior parietal lobule).

3. The chief fact in this connection, and indeed possibly the most important new point observed in this investigation, is the frequent limitation of movement to one segment when a point is excited, instead of, as in the Bonnet, a sequence of movements being thus produced. As an extreme instance of this singularity of function we may mention square 31, where flexion of the elbow and nothing else was obtained. The elbow in the Macacque, it will be remembered, was never unaccompanied by movements of other segments and rarely initiated a " March," whereas in the Orang we found it represented absolutely alone. (See " Marches," p. 144.)

It is well known that in the lower orders of animals the integration of representation becomes less perfect as we descend in the scale*—that, in fact, it is increasingly difficult to differentiate the areas for the limbs ; while, on the contrary, in ascending the scale, we have shown in our previous communications that it is easy to differentiate between the areas of representation of even the segments of the limbs ; and finally, when we now arrive at the Orang, the segments are not only differentiated in representation, but the nature of that representation is that of single movements.

There is now, from direct observation, reason to believe that in Man a similarly interrupted mode of representation exists.

The excitability to faradic stimulation of the human brain was established by the observations of BARTHOLOW[†] and SCIAMANNA.[‡]

The detailed results, however, are not sufficiently accurate to throw light upon the present question. Four observations, however, have been made under more favourable circumstances, although they cannot be fully accepted as deciding this question.

I. In October, 1886,[§] one of us (V. H.) explored (under anæsthesia, with morphia $\frac{1}{10}$ gr. and chloroform), for purposes of exact localisation, the right facial area in a boy, the subject of epileptic attacks, and found that excitation with an interrupted induced current of a strength just felt by the tongue only produced movements in the opposite (left) side of the face at points distant from each other, and not at intermediate points. In this case, however, the surface of the brain had been freely irrigated with 5 per cent. carbolic acid solution in water. The application of this lotion Professors SCHÄFER and HORSLEY had already found to soon depress the excitability of the cortex. This observation, therefore cannot stand alone.

II. In May, 1888, Dr. KEEN of Philadelphia[||] excited (under anæsthesia with morphia $\frac{1}{8}$ gr. and ether) with a faradic current of sufficient strength to cause contraction of the muscles of his own hand, the cortex in a man also the subject of epilepsy, and obtained movements as follows-- " The wrist (as observed by MORRIS J. LEWIS) moving in extension in the mid line and to the ulnar side at different

* FERRIER, ' Functions of the Brain.'

† BARTHOLOW, ' Amer. Journ. Med. Sciences,' April, 1874.

‡ ' Arch. di Psichiatria e Scienza Penale,' 1882.

§ ' Brit. Med. Journ.,' April 23, 1883.

|| ' International Journal of Medical Science,' 1888.

touches, and the fingers being extended and separated.   Above this centre were the shoulder and elbow centres and below, the face centre."

After Dr. KEEN had excised the focal representation of the wrist thus ascertained. he again excited the cortical surface above and below the point of removal and observed movement as follows :—"At the remaining part of the convolution at the upper margin of the excised portion, movements of the left elbow (flexion and extension) and shoulder, especially of the latter, which was raised and abducted, were noticed. Touching the part of this convolution remaining at the lower border of the excised portion produced an upward movement of the whole left face, no one muscle being noticeable in isolated contraction.   The platysma was not contracted nor was the angle of the mouth drawn downwards.   Touching the white matter at the bottom of the excision, produced again movements of the hand. . . ."

Immediately after the operation, the left hand was found to be paralysed as to all movements both of fingers and wrist.   The elbow was paretic, the shoulder and face perfectly unaffected.

Dr. KEEN's careful observation seems to us to bear out what we have said above as to the nature of localisation in the highest animals, including Man, and the more so since, although the current employed was evidently so very strong, it only produced movement of one or, at the most, two segments at each spot.

III. In June, 1888, Dr. LLOYD and Dr. DEAVER similarly explored* (under anæsthesia with ether preceded by chloroform and morphia ¼ gr.), the region of the middle of the Fissure of ROLANDO with a faradic current (strength not given).   Dr. LLOYD observed numerous and complex movements of the opposite upper limb and face.   As he says (loc. cit.) " the exact muscular movements which occur in the fit." Reference to his description makes it probable that he elicited epileptiform spasms.   Of immediate interest to us is his summary of the effects observed, as follows :—"I observed, especially in making these applications of faradism to the cortex. that considerable areas of it did not appear excitable at all to the strength of current employed, at least did not give muscular response anywhere, while the two comparatively narrow points above-mentioned reproduced almost exactly the muscular contractions of the epileptic seizures, and seemed to stand for more 'centres' than the diagrams of those who have experimented would allow to any such narrow areas."

IV. In a fourth case† published by Dr. NANCREDE, who operated for the relief of focal epilepsy, both the high integration of representation and, at the same time, the remarkable strength of current required to evoke the same were strikingly demonstrated.   Dr. NANCREDE found that excitation of " a spot in the ascending parietal convolution," corresponding to the point indicated by ourselves, " i.e., in the second fourth from below upwards, the thumb suddenly flexed," and an epileptic fit followed. And further, he found that " the shoulder, elbow, forearm, and facial centres were

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* ' International Journal of Medical Science,' November, 1888, p. 480.

† ' Medical News,' Philadelphia, 1888, p. 586.

readily picked out," and were situated as indicated by our previous observations.   In the discussion that followed the reading of Dr. NANCREDE's paper, Dr. MORRIS J. LEWIS, to whom had been intrusted the provision of the excitation, and, in part, observation of the effect, made the following important remarks.   He said, " No effect was produced by the application to the cortex until the current was so strong as to produce powerful and painful tetany of his first dorsal interosseous muscle," the skin being wet and metallic electrodes employed.   The battery was " one of FLEMMING's largest faradic batteries in perfect order, one half of the secondary coil being used with the graduating tube half withdrawn."

Finally, displacement of the electrodes laterally of the $\frac{1}{16}$th of an inch " caused the movement to cease," showing how extremely focal the effect produced was.

The objection to observation 1 does not apply in the same degree to observations 2, 3, and 4, since in 2 and 3 a weak solution of perchloride of mercury, and in 4 sterilised water was employed in irrigation of the surface of the brain.   On collating the observations, moreover, they all point in the same direction, viz., that the representation of movement in Man is very integrated, *i.e.*, in isolated areas, and requires powerful excitation for its demonstration.

Another example has recently occurred to one of us (V. H.) :—

V. R.L. ♂, age 39, operated upon for focal epilepsy at the request of Sir JAMES CRICHTON BROWNE, F.R.S.

The attacks began with a confused aura in the left hand and upper limb, which was then raised at the shoulder in extension, the motor spasm spreading rapidly to the lower limb and face of the left side, and subsequently to the right side, but the left was far more affected.   It was concluded that a previous diffused injury to the head had caused lesion of the right so-called motor area, and that the focus for the movements of the shoulder was the starting point of the attack.

The cortical area for the upper limb was therefore exposed.   The dura mater having been raised carefully, the cortex was excited with platinum electrodes, 2 mm. apart, a secondary current being obtained from a KRONECKER coil, furnished with a HELMHOLTZ side wire, and supplied by a weak GRENET (bichromate) cell.

The first excitation with the coil at 7000 gave no result, etherisation was then diminished, but as no result followed application of the electrodes to the point marked 1 in the table on p. 155, the strength of the coil was raised to 8000, and subsequently to 9000 ; this last, together with diminished etherisation, produced a single movement of the thumb, consequently this strength of current was retained.   Tested after the excitation was completed, this stimulus was found to be distinctly painful to the tongue and was, therefore, far in excess of that which would have sufficed to produce general epileptic convulsions in the Bonnet Monkey.   At the same time, it is interesting to note that the strength of current required to produce a movement from the cortex of the Orang was intermediate between these two extremes.

The cortical surface exposed included the two central quarters of the Fissure of

ROLANDO, the posterior extremity of the superior frontal sulcus (superior præcentral sulcus of JENSEN), and consequently a small piece of the ascending parietal, and a considerable part of the ascending frontal convolution, with very small portions of the superior and middle frontal convolutions. All these were explored with the current mentioned above, but only the ascending frontal gave any result.

No final conclusions can be ascribed to this fact, owing to reasons which have been already stated, although care was taken to irrigate with the mercurial lotion instead of carbolic. In the ascending frontal, the points indicated below by the numbers 1, 2, 3, 4, 5 from above downwards, were found to be excitable and produced movement as follows :—

The movements were observed by Sir JAMES CRICHTON BROWNE, Drs. BUXTON, WALTON, and RUSSELL.

ASCENDING Frontal Convolution.

Number.*	Character of movement.
1	Shoulder abducted, whole upper limb extended with pronation of wrist
2	Wrist, pronation ; elbow movement (? character)
3	Fingers all flexed. Wrist extended with (?) pronation
4	Index finger flexed
5	Thumb, slight abduction followed by flexion towards palm

We wish to point out that the order of representation in the above table is precisely the same as that observed in the Orang and also in the Bonnet Monkey. The movements were, for the most part, single, i.e., affecting only one segment of the limb, and were evidently primary in character (vide 'Phil. Trans.,' 1887). It was, of course, impossible to investigate these points more minutely as the object of the exploration had been attained, but it seemed as if the portions of cortex intervening between the above foci were either altogether inexcitable or much less excitable than the foci themselves.

We think that it will be now admitted that we are in a position to assert that the character of the motor function of the cortex, whether topographical or physiological, is the same in the higher animals, proceeding from the lower Apes through the Anthropoids to Man.

4. In complete accord with this we find that in the Orang no border centres of amalgamated function of contiguous areas of representation exist, such as, for instance, that in the Bonnet Monkey between the areas for the limbs and face.

5. We never observed epilepsy to follow excitation of any part, although, as has been already described (p. 140) the electrodes were occasionally kept in contact

* No. 1 was situated just below the level of the superior frontal sulcus, and No. 5 just above the middle of the genu, which was well marked.

for 5 seconds.   In view of the increasing readiness with which epilepsy follows cortical excitation as we pass from Apes to the Carnivora this observation is not without meaning.

Conclusion.

In conclusion we feel obliged to advert once more to the fact that the foregoing account is founded upon but one experiment.   In view of the difficulty and expense of obtaining material we have, however, thought it best to publish the results at once.

Description of Plates.

Plates 16, 17 (Figs. 1–5).

PLATE 16.

Fig. 1. View of left hemisphere (external surface) which was analysed by excitation, and photographed directly after the experiment, with the meninges intact. Natural size.   The photograph has been unavoidably reversed in printing.

        Sy.   Fissure of Sylvius.

        G.   Fissure of Rolando, placed just behind its genu.

        Pr.c.   Præcentral sulcus.

        S.F.S. Superior frontal sulcus.

        I.F.S. Inferior frontal sulcus.

        P.c.   Postcentral sulcus.

        I.P.   Intra-parietal sulcus.

Fig. 2. View of the same hemisphere after hardening in bichromate of ammonia and alcohol, and subsequently removing the meninges.   Natural size (shrunk in hardening).

        The letters are the same as in fig. 1.

PLATE 17.

Fig. 3. Photograph of the original drawing of the portion of the cortex examined, made at the time of the experiment.   The exact size, on paper ruled with 2 mm. squares, as described on p. 133.

        The letters are the same as in fig. 1, except that the Fissure of Rolando is called *R.*

        The numbers are referred to in the text.

Fig. 5. Horizontal section of the right hemisphere, showing the level where the internal capsule was excited (*vide* p. 147).

        i.c.   Internal capsule.

        pu.   Putamen, or outermost zone of lenticular nucleus.

        m.z.   Middle zone of lenticular nucleus.

        o.th. Optic thalamus.

        s.l.   Septum lucidum.

PLATES 18-20 (Figs. 1-27).

PLATE 18.

Fig. 1. Photograph of the external surface of the left hemisphere, which is tilted laterally, so that the genu of the Fissure of ROLANDO is opposite the centre of the lens.   Natural size.

The letters refer to the same parts as in fig. 1, Plate 16.

In figs. 1 to 27, the shading shows the area of representation of the various movements, the depth of shading showing the degree of representation.   The numbering of these figures corresponds to the description in the text (pp. 134-147).

Figs. 2-27 are photographs of the central convolutions corresponding to fig. 1.

Fig. 1. Both eyes move conjugately to the opposite side.

Fig. 2. Combined movement of eyelids opening briskly, eyeballs turning to the opposite side, and head turning to the opposite side.

Fig. 3. Bilateral closure of the eyelids, but those of the opposite side more strongly than those of the same side.

Fig. 4. Elevation of upper lip of opposite side.

Fig. 5. Retraction (horizontal) of the opposite angle of the mouth.

Fig. 6. Eversion of the opposite lower lip.

Fig. 7. Rolling in of both lips, especially of the lower and of the opposite side.

Fig. 8. Pursing of both lips.

Fig. 9. Pouting of both lips.

PLATE 19.

Fig. 10. Twitching of the opposite half of the upper part of the orbicularis oris.

Fig. 11. Opening of the mouth.

Fig. 12. Tongue protruded, flattened posteriorly, tip directed to opposite side.

Fig. 13. Tongue protruded, heaped up on the same side, and tip directed to the opposite side.

Fig. 14. Tongue neither protruded nor retracted, but rolled over to opposite side.

Fig. 15. Tongue retracted, heaped up posteriorly, and rolled over to opposite cheek.

Fig. 16. Tongue retracted slightly, flattened throughout, and the tip of the opposite side retracted horizontally.

Fig. 17. Movements of the thumb.

Fig. 18. Movements of the index finger.

Fig. 19. Extension of the fingers.

Fig. 20. Movements of the wrist.

Fig. 21. Flexion of the elbow.

PLATE 20.

Fig. 22. Adduction of the shoulder.
Fig. 23. Movements of the hallux.
Fig. 24. Extension of the small toes.
Fig. 25. Movements of the ankle.
Fig. 26. Movements of the knee.
Fig. 27. Movements of the hip.

PLATE 21.

This is a photograph (enlarged twice the size) of fig. 1, Plate 18, upon which are written the names of the foci of representation in their proper relative positions.

VII. *Development of the Sympathetic Nervous System in Mammals.*

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Communicated by A. Milnes Marshall, *F.R.S.*

[Plates 22–30].

I. Introduction.

The Mammalian sympathetic nervous system presented apparently insurmountable obstacles to both anatomists and physiologists till the publication of Gaskell's Monograph in 1885 (1). Our knowledge of its structure and functions is now, however, placed on a much firmer basis; and it is possible to enter seriously into a consideration of its morphology and development.

A. *Physiology.*

The *cells* composing the ganglia in the main sympathetic chain are shown by Gaskell to be trophic only. They are neither automatic nor reflex in their action, but merely nourish the fibres which pass from them centrally or peripherally. The gray *Rami communicantes* spring from the ganglia and are distributed as trophic fibres to the roots and trunk of each spinal nerve and their *meninges*, and the bodies of the vertebræ. In some cases (*e.g.* fore and hind limbs) the vasomotor fibres reach their destination through these Rami. The white *Rami communicantes* are only found in two regions. In the Dog between the tenth and twenty-fifth spinal nerves (second dorsal—second lumbar), and in the Rami of the second and third sacral nerves. This corresponds fairly accurately with their distribution in the human subject (2). In both regions they consist of very small medullated fibres ($1 \cdot 8\mu$ to $2 \cdot 7\mu$). In the anterior region the white Rami pass from the spinal nerves to the ganglia, and there separate into two groups; one set forms *vasomotor* fibres, which join the ganglia, become connected with the ganglion-cells, and are distributed peripherally as gray fibres, greatly increased in numbers. The other set does not join the ganglia, but, passing over them, forms the nerves distributed in the Splanchnics to the abdominal viscera as *viscero-inhibitory* fibres. In the posterior region

white Rami arise from the second and third (and in man fourth (2)) sacral nerves which pass over without joining the ganglia ; becoming connected with the hypogastric plexus, they are distributed as the *Nervi erigentes* (*vaso-inhibitory* fibres). They are possibly also *viscero-inhibitory*.

The only spinal nerves which come into direct communication with the sympathetic ganglia are the vasomotor fibres. In fact, so convinced is GASKELL of this that he maintains that " this chain might, therefore, be most appropriately called the chain of vasomotor ganglia, instead of the present meaningless title of main sympathetic chain." GASKELL shows, moreover, that these vasomotor fibres are conducted along the sympathetic chain from their source to their distribution. In the anterior thoracic region they pass forwards along the chain to be distributed to the carotid and vertebral arteries, to the arm, and to the heart. In the posterior part of the thorax they pass backwards to the aorta, abdominal vessels, and lower limbs.

The functions of the gangliated cord of the sympathetic thus appear to be :—
1. To nourish the vasomotor fibres after converting them into non-medullated fibres ; and 2. To act as " a guiding rod " to direct the course of the stream of vasomotor fibres emerging from the cerebro-spinal axis.

B. *Morphology.*

The prevailing notion among anatomists regarding the morphology of the gangliated cord of the sympathetic is one which springs from the analogy which it bears to the bilaterally gangliated chain in Invertebrates, and which has been elaborated by BALFOUR chiefly among embryologists (3). It is generally spoken of as being *segmental*, and the attempt is made to harmonise the arrangement of the ganglia with that of the vertebral somites. Instead, however, of presenting a distinctly metameric arrangement, there is anything but a correspondence with the segmentation of the body, as illustrated in the case of the spinal nerves or vertebræ. In the human subject, for example, in the neck there are only two, sometimes three, ganglia in the sympathetic cord. The first is elongated, connected with four spinal nerves, and sometimes irregularly constricted, and so " is favourable to the view that it may be regarded as consisting of several ganglia which have coalesced." (4). The last cervical ganglion is frequently fused with the first thoracic. In the gangliated cord in the other regions the same irregularity is seen. Of the thoracic ganglia there are usually twelve, but variations are formed occasionally by the " coalescence " of adjacent ganglia. In the loins there are generally four ganglia, but the number is often diminished. They are sometimes even fused into a single ganglion, much elongated, passing over several vertebræ, and connected at intervals with the lumbar nerves. " The sacral ganglia are usually four in number, but the variation both in number and size is more marked than in the thoracic or lumbar ganglia " (4).

The assertion of GASKELL (1), which I shall venture in the sequel to uphold from

embryological considerations, that the superior cervical ganglion is to be looked upon neither morphologically nor physiologically as a fundamental portion of the gangliated cord, but rather as a distal or collateral ganglion, makes it still more difficult to regard the gangliation of the chain as indicative of its metameric constitution.

It is noteworthy that the central connexions of the main chain form the only evidence of segmentation, which is not discredited by exceptional characters. These connexions are obviously truly segmental, being intimately related to the spinal nerves ; and the segmental hypothesis of the formation of the sympathetic is founded essentially on these anatomical relations. Hereafter I shall endeavour to demonstrate that the sympathetic is primarily unsegmented, and shall attempt to show adequate cause for the appearance of segmentation in the cord.

C. *Embryology.*

The recent memoirs bearing on the development of the sympathetic are comparatively few, and the excellent account given of the literature of the subject in Onodi's monograph (15), makes it unnecessary to repeat it here at length. Various steps in the process have been traced by previous observers ; but there is absolute divergence in their views regarding the most vital points. In the first place it cannot be said to be definitely proved from which of the embryonic layers the gangliated cord arises. On the one hand we have the weighty authority and historical researches of Remak (5, 6), supported by Bidder and Kupffer (7), Kölliker (8), Hensen (9), His ? (10), Götte (11), and Schwalbe (12), in defence of its origin from mesoblast. On the other hand Balfour's work on the development of Elasmobranchs (13), the memoir of Schenck and Birdsell (14), and the elaborate monograph of Onodi support the idea of its origin from epiblast. There is a third hypothesis to the effect that the sympathetic is originally derived from hypoblast. As far as I am aware, this theory has never been seriously supported by any observer, and, as it will not bear investigation, it is not necessary to discuss it further.

Those who, like Remak and his followers, uphold the mesodermal origin of the sympathetic, maintain that it springs from isolated segmental ganglia arising from the mesoblast in the neighbourhood of the protovertebræ, which ganglia obtain connexions with the periphery, with the spinal nerves, and antero-posteriorly with the contiguous ganglia.

Balfour (13) was the first to suggest the ectodermal origin of the sympathetic, in order to harmonise it with the rest of the nervous system. In his monograph on the development of Elasmobranchs, he makes some observations on the development of the ganglia. He says : " There may be seen short branches from the spinal nerves, which take a course towards the median line of the body, and terminate in small irregular cellular masses immediately dorsal to the cardinal veins. These form the first traces that have come under my notice of the sympathetic nervous system. In

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the youngest of my embryos, in which I have detected these, it has not been possible for me either definitely to determine the antero-posterior limits of the system, or to make certain whether the terminal masses of cells which form the ganglia are connected by a longitudinal commissure." There is here only one positive observation, namely the connexion of the ganglia with the spinal nerves, which might be obtained from an examination of the adult, and BALFOUR himself, while inferring that the ganglia spring from the spinal nerves, and therefore from ectoderm, admits that his investigations, " though they may naturally be interpreted in this way, do not definitely exclude a completely different method of development for the sympathetic system." In his 'Comparative Embryology' he states that " the sympathetic ganglia (in Elasmobranchs) are at first simply swellings on the main branches of the spinal nerves, some way below the ganglia. . . . I have been unable to find a longitudinal commissure connecting them in their early stages, and I presume they are at first independent, and become subsequently united into a continuous cord on each side." He also derives the suprarenal bodies of Elasmobranchs from the sympathetic. Other observers, BRUNN (16), KÖLLIKER (17) and BRAUN (18), have shown that the medullary parts of the suprarenals are derived from the sympathetic, the cortical parts from undifferentiated mesoderm.

SCHENCK and BIRDSELL (14), working with Chick, Dog, and Human embryos, came to the conclusion that the ganglia are derivatives of the central nervous system, and are pushed out from the spinal ganglia. Their paper is not, however, convincing ; the illustrations do not clearly express the meaning of the text ; and the facts adduced do not appear to justify the generalisations at which they arrive.

ONODI (15) has given the most complete account of the development of the sympathetic. His researches cover all the classes of vertebrates. In fishes he studied the question in *Scyllium canicula, Mustelus lœvis, Scymnus*, and *Torpedo*. In the first only he describes the earliest appearance of the sympathetic ganglia. as thickenings on the ventral ends of the spinal *ganglia*. These thickenings are composed of large cells, which stain deeply, are arranged in rows, and are sharply marked off from the pale mesoderm cells. By the further growth of this bulbous expansion the sympathetic ganglion is formed, remaining in connexion with the spinal nerve by a double row of cells. In the other examples, the commencement of the gangliated cord was not discovered ; in the youngest specimens examined the ganglia were fully formed, and connected to the spinal nerves (as BALFOUR described) by *Rami communicantes.* These *rami* ONODI describes as being first cellular, afterwards becoming fibrous. The main conclusion of his researches in Fishes agrees with that of BALFOUR, that the ganglia are segmental and ectodermal in origin. They disagree, however, in regard to the process of formation. BALFOUR regards the sympathetic ganglia as buds from the spinal nerves, the connecting stalk being gradually elongated into the *ramus communicans* as the ganglion grows. ONODI, on the other hand, asserts that ganglia

and rami are direct proliferations from the spinal ganglia. Nor in either case is the evidence clear regarding the fusion of the ganglia to form the connected chain.

In Birds and Mammals, ONODI did not see the earliest stages in the development of the sympathetic. In the Chick at 3 and 5 days the gangliated cord was present, and connected by rami to the spinal nerves. In the Rabbit (10 mm.) and Human embryos (18 mm.) the same conditions were found—ganglia already formed, with their rami communicantes. These observations do not differ markedly from those of other investigators—HIS, SCHENCK and BIRDSELL, BRAUN, &c. Yet, although in no case were the ganglia *seen* to spring from the spinal ganglia, ONODI expresses himself as convinced (from the appearance he describes in *Scyllium*) that that is the source from which they originally start.

Summarising the conclusions of previous observers discussed above,—the sympathetic gangliated cord in Mammals consists of vasomotor ganglia, trophic in their functions, and pseudo-metameric in their morphological arrangement. Two opposite views exist amongst embryologists regarding its development. In both views the segmental formation of the cord is upheld. According to one view it is mesodermal, and is formed *in situ*. According to the other view, it is ectodermal. But the observers who assert this view disagree as to its fundamental source. BALFOUR regards each ganglion as an offshoot from the spinal nerve ; ONODI and SCHENCK and BIRDSELL consider it as a direct proliferation from the spinal ganglion.

In the succeeding pages the attempt will be made to find the source of the sympathetic system in Mammals, by tracing its development from its first appearance to the date at which it can be fairly compared with the adult condition ; and thereafter, by a study of its morphology at different ages, and in different animals, to harmonise the conflicting views of anatomists and physiologists.

II. THE DEVELOPMENT OF THE SYMPATHETIC SYSTEM.*

The following researches were begun some years ago, during an investigation into the development of the spinal nervous system. It was then noted that the first apparent connexion between the sympathetic ganglia and the spinal nerves arose in the formation of the splanchnic or visceral branch of the latter, which passed into the splanchnic area and seemed to join the gangliated cord. Beginning at this point, and continuing the investigation into earlier and later stages, it has been possible to arrive at conclusions regarding the main features in the formation of the gangliated

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* Dealing exclusively with Mammals, in the following pages, it is necessary to explain the method used in determining the ages of the embryos employed. It is not possible, in my opinion, to be certain as to exact ages, as the time of impregnation after connexion varies in different cases ; and two embryos from the same uterus often differ in size and extent of development. From a comparison of a large series, however, it is possible to determine *approximately* the age of a given embryo. In the following description the probable age of the embryo is given, and when possible, its axial length also.

cord, its connexion with the spinal nerves, its terminations, cephalic and caudal, and its relations to the suprarenal bodies, and collateral ganglia. In the succeeding account the order followed will be that of development, as far as the subject under consideration allows, and as far as possible drawings from the actual specimens will be utilised to demonstrate the facts adduced.

1. *Formation of the Main Sympathetic Cord: Early Stages.*

The examination of very young Mammalian embryos leads only to negative conclusions regarding the origin of the sympathetic cord.

In Rat embryos about 5–6 days old, cut *in utero*, hardened in spirit and stained with aniline blue-black (BEVAN-LEWIS), the spinal cord is incompletely closed, the brain consists of its three primary vesicles, and the optic vesicles are in process of formation. In transverse sections through the thoracic region (Plate 22, fig. 1), the notochord (*No.*) is large and distinct, is surrounded by a sheath, and is in close contact with the spinal cord (*Sp.C.*). Below is the alimentary canal (*Al.*), in the ventral wall of which the first steps in the formation of the pulmonary passages are to be seen. On each side are the separate aortæ (*Ao.*) and muscle plates (*M.P.*), the inner layers of which are undergoing proliferation. The dorsal ganglia (*D.R.*) are beginning to form, as indistinct buds of fusiform cells (stained like those of the spinal cord more deeply than the surrounding mesoblast), springing from a hardly perceptible neural ridge on the summit of the spinal cord. There is no sign, as yet, of the ventral roots of the spinal nerves, or of the sympathetic cord. Between the muscle plate and aorta laterally, and the spinal cord, notochord, and alimentary canal in the middle line, is a clear space containing only some scattered embryonic cells with branching processes.

In Rabbit embryos aged about 7 days (axial length 5 mm.), hardened in picric acid, and stained with borax-carmine, the dorsal aortæ were fused into a single trunk, having a cardinal vein on each side; the lungs had begun to form, and the limbs were represented as rudimentary rounded buds. The spinal cord was almost entirely cellular, a very narrow belt of fibres being found only in the neighbourhood of the dorsal and ventral roots. The development of the spinal nerves was further advanced. In transverse sections through the thoracic region (Plate 22, fig. 2), the dorsal ganglion (*D.G.*) is distinctly seen: fusiform in shape, it is connected to the spinal cord (*Sp.C.*) by a bundle of fibres, among which are cells possessing large nuclei and fine tapering processes at either end. At its distal end, a bundle of fibres can be traced to join the ventral root. The latter is also distinct: it leaves the spinal cord as a number of separate fine fibres which combine with the dorsal root to form the nerve (*Sp.n.*). This can be traced outwards almost as far as the cardinal vein (*C.V.*), and to the level of the somato-splanchnic angle (*α*); or in the regions where the limbs are sprouting as far as the roots of the limbs, where in certain sections the fibres can be seen to separate

out slightly before their termination.   The nerves consist for the most part of fibres :
the cellular elements are plainly mesoblastic, and help in forming the nerve-sheaths.

There is no sign as yet of the formation of the superior primary division, or
splanchnic branch of the nerve.   The ventral root passes out entirely below the
dorsal ganglion, which is easily defined.   The interval before mentioned between the
aorta, cardinal vein, spinal nerve, and spinal cord, is now filled up by a mass of
mesoblastic cells (intermediate cell-mass) in process of vertebra formation.   The cells
are large, often with several nuclei, and are in an active state of division.   They all
stain equally with borax-carmine, except the nuclei of blood corpuscles, which stain
more deeply.

Sagittal sections of embryos from the same uterus show the spinal nerves in
transverse section ; but there is no evidence of the existence of the sympathetic cord
or ganglia.

In Mouse embryos of about 8 days, and in Rat embryos of a similar age (the axial
length of the latter being 7 mm.), the first indication of the formation of the sympa-
thetic cord has been found.   The general condition of development in both these
animals corresponds with that of the Chick at the end of the third and beginning of
the fourth day.   The spinal cord is cellular, a narrow belt of fibres being present on
each side in the neighbourhood of and between the nerve-roots.   The spinal nerve
has extended almost as far as the somato-splanchnic angle, and to the cardinal vein in
places where the latter is present ; and in some sections the superior primary division
is apparent.   The pulmonary diverticulum from the alimentary canal has become
divided into the two main bronchi.   The Wolffian duct and Wolffian body are
distinct.

The Mouse embryos (Plate 22, fig. 3) were hardened in picric acid and stained with
borax-carmine.   A change is now apparent among the cells of the mesoblast surrounding
the aorta (*Ao.*).   In the interval between the latter and the cardinal vein (*C.V.*), instead
of a uniformity in the cells composing the intermediate cell-mass, an irregular group of
cells is seen among them (*Sy.*), lying on the ventral side of the intercostal arteries, and
in close proximity to the aorta.   On close examination, this mass of cells is strikingly
different from the surrounding mesoblast.   The cells stain more deeply ; the nuclei are
larger, and often possess a considerable number of nucleoli ; and the cells form a
rounded clump closely massed together, only in some sections clearly demarcated from
the cells in which they are imbedded.   This mass is bilaterally symmetrical, and is
evident in continuous series of sections from the level of the cephalic border of the
fore-limb anteriorly, where it ends abruptly, to the middle of the trunk—the region of
the stomach—posteriorly.   It is comparatively large anteriorly, and gradually tapers
off and becomes indistinct posteriorly.   The mass in front is double the thickness
of that seen further back.   There is no connexion, fibrous or cellular, with the spinal
nerves or ganglia.

The Rat embryos of the same age were hardened in spirit, and stained with aniline

blue-black. They were in the same condition of development as the Mouse embryos. The spinal nerves (*Sp.N.*, Plate 22, fig. 4) were distinctly fibrous, terminating in a thickened brush-like ending in the neighbourhood of the cardinal or jugular vein (*V.*). The dorsal ganglia (*D.G.*) were well defined, and consisted of fusiform cells with large ovoid nuclei, and with long and fine processes directed towards the spinal cord and nerve. In transverse sections a more or less rounded mass of cells (*Sy.*) is seen lying between the aorta (*Art.*) or carotid artery, and the cardinal or jugular vein (*V.*). It is formed of a mass of cells closely packed together and sharply marked off from the ordinary mesoblast by a capsule composed of a single layer of flattened cells, which, like the blastema-cells in which the mass is imbedded, stain less deeply than the mass itself. The cells forming this group, besides being evident from their deeper staining, possess nuclei, which have a striking appearance in transverse sections. Some are large and prominent, with several nucleoli; others are very small; while some cells appear to be without nuclei. In transverse sections they are round or angular. These appearances are due to the fact that the cells are fusiform, and so are cut at different levels in transverse sections. The mass, as a whole, is in close proximity to the aorta or carotid artery. It is distinct and separate from the spinal nerves. It is traceable in transverse sections from a point slightly anterior to the root of the fore-limb to the middle of the trunk, being thickest in front, and gradually becoming attenuated and indistinct behind.

In longitudinal (sagittal) sections of embryos from the same uterus, prepared in the same way, one can make out by means of continuous series of sections that this apparently rounded mass is distinctly evident on either side of the middle line as a long rod or column of fusiform cells (*Sy.*, Plate 26, fig. 11), possessing large, deeply stained, ovoid nuclei, and fine thread-like processes. All the cells are directed antero-posteriorly, and bear a marked resemblance to the cells of the dorsal ganglia at this date (*D.G.*). The column lies in close proximity to the aorta and carotid vessels (*Art.*), and can be traced forwards as far as the level of the mouth (*Mo.*) and first vertebral segment (1*st V.S.*), where it ends abruptly in a pointed brush-like process composed of the filiform processes of the cells, and not apparently due to any admix-ture of fibres. It is marked off from the surrounding tissues by a narrow but distinctly evident interval. Behind the level of the lungs the column of cells becomes less distinct; it can, however, be followed to the anterior end of the kidneys (Wolffian bodies), where it becomes connected with a mass of mesodermal cells, which form one portion of the suprarenal bodies. Above and behind the kidneys the character of the column changes. It can be followed to the level of the hind limbs, but is more indefinite; its cells, while retaining their fusiform character and columnar arrange-ment, are less closely packed together, are less regularly arranged, and are less deeply stained than in the anterior part of the column. Traced backwards, it becomes more and more indistinct, and is finally lost in the region of the hind limbs. In its entire length I failed to find any connexion between this cellular column and the spinal

nerves. It is slightly thicker in front than behind, but is of almost uniform width throughout. It presents no evidence whatever of segmentation.

I have not the slightest hesitation in asserting that the cellular column above described is the precursor of the main sympathetic chain. It occupies the position of that chain; and there is no other structure to be found in the vicinity, in the form of isolated ganglia or otherwise, which might lead one to doubt its significance. It is mesodermal in origin; it is formed *in situ*; and it is unsegmented and unconnected with the spinal nerves. The next step in the investigation is to discover how this connexion takes place.

2. *Connexion of the Sympathetic Cord and Spinal Nerves: Formation of the* Rami communicantes.

In Rat embryos, aged between 8 and 9 days, axial length 8·5 mm., hardened in spirit, and stained with aniline blue-black, the above-mentioned cellular cord (*Sy.*, Plate 23, fig. 5) is slightly larger than before. In transverse sections it has the same close connexion with the aortic wall (*Ao.*); and in certain places can be traced into continuity with small bundles of similar cells, which form groups lying against the sides of the vessel, and tapering off in a ventral direction. These cells form the first sign of peripheral distribution from the cord. At this date the cord itself presents no sign of constriction or segmentation, and has the same antero-posterior limits as before.

It is in the spinal nerve that the most significant change is now seen. In previous sections the inferior primary division was traced as far as the cardinal vein, above the somato-splanchnic angle. It is now seen (*Sp.N.*) to divide at that spot (*a*) into two unequal parts, a larger external (*So.*) and a smaller internal branch (*Spl.*), both of which are very short and end abruptly in a fringe of nerve fibres lying between and against scattered mesoblastic cells. These two short branches form the first appearance of the somatic and splanchnic branches of the nerve. The former is directed towards the body wall, the latter inwards, behind the cardinal vein (*C.V.*) towards the median line.

In Mouse embryos of about 9 days, hardened in picric acid, and stained with borax-carmine, development has perceptibly advanced. Transverse sections (Plate 23, fig. 6) show the spinal cord (*Sp.C.*) with its cells arranged in two groups; a broad band of deeply stained cells surrounds the neural canal, while more externally a mass of paler and more scattered cells is found, the precursor of the proper gray matter of the cord. Outside the cells the belt of peripheral fibres has become distinctly thicker. The cellular cord (*Sy.*) presents the same characters as before; it is thick anteriorly, gradually tapering off posteriorly; and it possesses the same bundles of cells in connexion with it, passing ventrally, and embracing the sides of the aorta. In the spinal nerve the condition of development is advanced. The superior primary division is plainly seen; the somatic branch (*So.*) of the inferior

primary division (*I.D.*) is longer and larger, and in the regions of the limbs has entered their substance and has divided into dorsal and ventral branches. The splanchnic branch (*Spl.*) can now be traced to a point midway between the cellular sympathetic cord (*Sy*) and the origin of the branch from the spinal nerve. It is most evident in the thoracic region. In the neighbourhood of the origin of the trachea and anterior to that, it could not be found. There the cellular cord was very distinct ; the superior primary division and somatic branch were visible, but so far as I could discover, no splanchnic branch was present.

In transverse sections of Rat embryos at about 10 days, hardened in spirit and stained with aniline blue-black (Plate 23, fig. 7), the cellular cord of the sympathetic (*Sy.*) has much the same characters as in the last case. It still presents in continuous sections a uniform thickness. Cellular outgrowths from it (*Br.*) can be traced ventrally round the aorta, especially in the region of the kidney, and in front of it, to form the collateral ganglia, and to join the suprarenal bodies. Behind that point the cord becomes attenuated. The spinal nerve shews an advanced condition of development. The superior primary division (*S.D.*), though well formed, is not yet differentiated into its separate roots. The somatic part (*So.*) of the inferior primary division (*I.D.*) is seen divided into its dorsal and ventral branches (1.2.). The splanchnic part (*Spl.*) is very evident in the region in front of the kidney. Each branch is thick, and directed inwards above the cardinal vein (*C.V.*) ; and reaches almost, but not quite, up to the sympathetic cord. The interval between the two varied slightly in different nerves, but in no single case was the junction perfectly evident. Round the end of the splanchnic branch are large fusiform cells which obscure the ends of the nerve fibres, probably mesodermal cells, which are proceeding to form the sheaths of the nerves, possibly cells derived from the sympathetic cord and pursuing a central direction. The nerve fibres throughout the spinal nerves are larger than before, and appear in the form of wavy bands.

In transverse sections of Mouse embryos at about 11 days (Plate 24, fig. 8), hardened in picric acid, and stained with borax-carmine, development is further advanced. The spinal cord is more fibrous ; the anterior commissure is thicker, and the decussation of fibres within it more evident. The superior primary division of the spinal nerve (*S.D.*) is now traceable to two independent roots, one dorsal, the other ventral. The splanchnic branch (*Spl.*) has now joined the sympathetic cord (*Sy.*). The fibres on entering the mass present, in transverse sections, cut ends, and cannot be traced into direct communication with the cells. Surrounding the junction of the nerve with the cellular cord are groups of cells, some of them in continuity with, and apparently forming part of, the sympathetic cord, others resembling the blastema cells surrounding the splanchnic branch. Some of the cells of the sympathetic cord are ovoid, or occasionally stellate in transverse section, with one long process directed towards the nerve, and others radiating in different directions, chiefly towards the aorta. These processes can be seen joining others from neighbouring cells of the mass, in the midst of which

the fibres of the splanchnic branch can be seen coursing downwards and inwards. The splanchnic branch was not traced beyond the sympathetic cord. The latter was lost in the region of the neck in front, and in the region of the hind limb behind.

In transverse sections of Rat embryos, at about 12 days, hardened in picric acid, and stained with borax-carmine, general development has advanced a stage (Plate 24, fig. 9). The cells of the intermediate cell mass have grouped themselves closely round the notochord (*No.*), in process of vertebra formation. The cells of the spinal cord (*Sp.C.*) are more distinctly separated into two groups, and the fibrous belt is thicker, especially near the origins of the spinal nerve-roots. The ventral root-fibres are seen to be in direct continuity with the cells of the spinal cord; the dorsal ganglion has its cells more regularly arranged in vertical rows. The divisions of the spinal nerves have increased in length, and are more distinctly differentiated. The splanchnic branch (*Spl.*) can, in transverse sections, be traced into the sympathetic cord, with which it is directly connected. In the thoracic region its fibres can be traced proximally through the spinal nerve and into continuity with both dorsal and ventral roots. Distally, its fibres can be traced into direct connexion with the cells of the sympathetic cord. In some cases the fibres are traced into the mass, and their cut ends appear in transverse sections. In the anterior part of the thorax, the fibres cannot be traced beyond the sympathetic cord. In the region of the neck, and in front of the origin of the vertebral artery, no splanchnic branch at all could be found joining the sympathetic. In some situations—especially in the region of the stomach—the splanchnic branch is divisible into two bundles of fine wavy nerves, one of which joins the sympathetic, and either communicates by its fibres with the cells, or is directed forwards or backwards in the cellular column; the other bundle passes round the sympathetic cord, and can be followed in a ventral direction for some distance, in the interval between the aorta and cardinal vein. Here the wavy fibres are surrounded by large blastema cells, which lie pressed against their side; among them, also, are seen other cells, directly continuous by their processes with the cells of the sympathetic cord. In other places, even more distinct cellular branches proceed peripherally from the latter, round the aorta, to form plexuses, collateral ganglia, &c. The sympathetic itself has much the same constitution as before, the noteworthy points in its structure being found at the ends. Posteriorly it narrows, and finally disappears at the point of bifurcation of the aorta. Traced forwards, it becomes thicker, attaining its greatest size near the origin of the vertebral artery. It here gives off a slender fibro-cellular cord which accompanies that vessel, and itself suddenly narrows and becomes decidedly fibrous. It soon swells out again, and resumes its cellular characters, and in this form is traceable to the level of the top of the pharynx, along the common and internal carotid arteries. Lying near the ganglion of the vagus, it becomes gradually narrower, and can be followed only as a thin fibrous cord along the carotid artery, as it courses beneath the auditory capsule, where it finally disappears.

In Rat embryos, at about 13 days, hardened in spirit, and stained with aniline blue-black, the sympathetic cord can be followed backwards as far as the middle sacral artery, and slightly behind the bifurcation of the aorta. In coronal sections in the thoracic region, it shows no constriction whatever ; it is composed of fusiform cells arranged longitudinally, with nerve fibres coursing forwards and backwards among them, derived from the splanchnic branches of the spinal nerves, which can be seen to enter the cord obliquely, the fibres diverging and being traceable for some distance. The cellular branches from the sympathetic cord have now extended still further in a ventral direction, and the formation of the collateral ganglia and suprarenal bodies is more advanced. The splanchnic branches first become evident and join the sympathetic cord at the point where the dorsal aorta becomes single and median. In front of that point there was no evidence of a splanchnic branch joining the sympathetic cord. Again, behind the bifurcation of the aorta no splanchnic branches could be found. Between these points they were distinct and evident. In the region of the stomach and kidney the splanchnic branch was most clearly divisible into two parts, one joining the sympathetic, and the other passing round the outer side of it accompanied by a cellular branch from the cord, to join the semi-lunar ganglia, or solar plexus.

In Rat embryos about 14 days, stained with borax-carmine or aniline blue-black, the mesoderm has begun to show signs of cartilage formation. At this date the sympathetic cord still presents no sign of constriction, except at the anterior end. At the point of origin of the vertebral artery—the point to which it could originally be traced—it becomes narrow, its tenuity being due to the presence of nerve-fibres, and largely to the absence of cells. Having given off a large fibro-cellular cord, which accompanies the vertebral artery, it again swells out to form the "superior" cervical ganglion. This lies close to the ganglion of the vagus, with which it appears to communicate. It can be traced, as before, along the internal carotid artery to the auditory capsule, beneath which it becomes first fibrous, and then, gradually attenuating, finally disappears. The vertebral artery at its origin is completely surrounded by the cellular cord. There is no change at the caudal end.

The splanchnic branches in the thoracic and lumbar regions can now be traced proximally to the roots of the spinal nerves, and are found to receive fibres from both, especially the ventral roots. In the neck, in front of the origin of the vertebral artery, there is no sign of splanchnic branches joining the sympathetic. Branches, however, are seen springing from the ventral roots of the cervical nerves, which course horizontally inwards round the under surface of the vertebral artery, and become lost in the protovertebral mass around the notochord. These branches are situated at some distance from the sympathetic, on its dorsal aspect. As they are traced backwards successively towards the origin of the vertebral artery, they approach nearer to the cord, and finally, behind the origin of the vessel, the splanchnic branches proper appear, and partly join, partly pass over, the cord.

This latter arrangement is most obvious, however, in the posterior dorsal and

lumbar regions. Here the splanchnic branch (*spl.*, Plate 26, fig. 13) is distinctly formed of two parts. Each consists of fine nerve-fibres, among and around which are long fusiform mesodermal cells, forming a sheath for them. Reaching their extremities, the fibres of each portion separate into bundles of fine, wavy fibrils. These fibrils, in the case of that portion (*α*) which joins the sympathetic (*Sy.*), join the cells or separate out among them. The portion (*β*) which passes beyond the sympathetic cord, is accompanied by a cellular branch (*γ*) derived from the latter, and the two proceed in a ventral direction round the aorta. Around the cœliac axis, the cellular branches from both sides, coursing round the aorta, are now seen to unite together to complete the solar plexus. Both splanchnic branch and cellular branch from the sympathetic are larger and longer than before.

In transverse sections of a Human embryo about the end of the first month, hardened in spirit and stained with aniline blue-black, the sympathetic cord has very much the characters just described. The cord itself is large and uniform in width, widening out anteriorly to form the inferior cervical ganglion; beyond this it narrows, encloses the subclavian artery, and forms a fibrous cord; this again becomes cellular, and widens out into the " superior " cervical ganglion. No splanchnic branches join the cord in front of the level of the inferior cervical ganglion. In the thorax (Plate 28, fig. 16) the splanchnic branches are seen (*spl.*) arising from both roots of the spinal nerve (*I.D.*), and, as in the figure, terminating wholly in the sympathetic cord (*sy.*). Sometimes a small portion of a splanchnic branch can be traced round the ventral side of the cord, accompanied by a cellular branch from it. In the hinder thoracic region, a small part only of the splanchnic branch joins the cord, the greater part, along with cellular outgrowths from the sympathetic, passing onwards to form the solar plexus and semi-lunar ganglia, which are seen in process of formation on the ventral aspect of the aorta. A similar fibro-cellular bundle passes to join the suprarenal body. In the lumbar region the splanchnic branch can be seen for a considerable distance almost entirely unconnected with the sympathetic cord, and separated by an interval from it. The cord gradually narrows as it is followed backwards, and, becoming attenuated, disappears at the point of bifurcation of the aorta.

In Rat embryos at about 15 days, stained with borax-carmine or aniline blue-black, there is not much further change. The sympathetic cord has much the same extent as before (Plate 27, figs. 14 and 14A, *Sy.*); is uniform in width for the most part (except that it is considerably thickened in the situation where the fibro-cellular bundle arises to join the solar plexus, semilunar ganglion, and suprarenal body); and consists of cells which, especially anteriorly, are long, narrow, and fusiform, with deeply stained round or oval nuclei, and terminal filiform processes. In some cases the cells seem almost converted into fibres. Near the posterior end the cells are wider and shorter. The cell processes are seen to join the splanchnic branches directly. Careful examination of the anterior nerves strengthens my confidence that

the first splanchnic branch which joins the sympathetic does so just behind the origin of the vertebral artery.

In older embryos the sympathetic cord presents similar characters, modified by the collateral development of other parts and organs. In Mouse embryos of 17–18 days (Plate 25, fig. 10, and Plate 28, fig. 15), in which the cartilages have become evident, the cord is still uniform and unsegmented. The cervical part of the cord is unaltered; the "superior" cervical ganglion terminates as before, in front and behind. The cervical splanchnic branches do not join the sympathetic cord. The "inferior" cervical ganglion is joined by fine splanchnic branches, along which cellular branches can be seen coursing towards the ventral roots of the spinal nerves. In the thorax the cord is uniform, diminishing in diameter from before backwards. In transverse sections it is oval at the points where the intercostal arteries cross it, round in intermediate situations. The splanchnic branches arise from both spinal nerve roots, and in the anterior part of the thorax appear to join the sympathetic cord entirely; in the hinder part of the thorax they partly join, and partly pass, the cord. Cellular branches can be traced ventrally from the sympathetic, as well as others, which pass centrally along the splanchnic branches of the nerves. The whole thoracic portion appears much more minute than in younger embryos, on account of the great strides which other parts have taken in development. It narrows gradually to the level of the diaphragm, and as it passes through this it becomes very minute. Behind the diaphragm, and in relation to the liver, the cord is greatly increased in thickness, and gives off considerable bundles, partly fibrous, partly cellular, to form the solar plexus, suprarenal body, &c., and derived partly from the splanchnic branches directly, partly from the sympathetic cord. Behind that the cord again gradually narrows; the splanchnic branches in some sections wholly (as in fig. 10) join the cord; in others can be seen passing alongside it towards the mesentery, accompanied by a cellular stream from the sympathetic. In the lumbar region the splanchnic branches are very long and fine; they arise from both spinal nerve roots. They cease behind at the bifurcation of the aorta. None were found behind that point. Behind that level the cord gradually tapered off and lay in relation to the middle sacral artery, behind which, near its termination, a junction seemed to be effected between the cords of the two sides.

The connexions of the spinal nerves and sympathetic cord have now been sufficiently described. First a cellular, unsegmented column, the sympathetic, appears on each side of the middle line in the neighbourhood of the aorta. This is, secondarily, joined by branches from the inferior primary divisions of certain spinal nerves, which have a definite origin and distribution. They are fundamentally spinal nerves, deriving their origin from both dorsal and ventral roots, and being formed by the division of the inferior primary division into two branches, onesomatic, the other (the nerve in question) splanchnic. Coursing downwards and inwards, they reach the sympathetic cord. Here the splanchnic branch terminates in one of two ways. It

may end entirely in the sympathetic cord, as seems to occur in the anterior part of the thoracic region. [This mode of ending is not absolutely certain. It may be only apparent from the manner of section of the nerve trunks.] On the other hand, in the hinder dorsal and lumbar regions, the splanchnic branch on reaching the cord divides into two parts, one of which joins, the other passes beyond the cord. In either case the splanchnic fibres which join the cord become directly connected with the component cells. The connexion between the spinal nerves and sympathetic cord is, therefore, to be regarded as quite a secondary connexion.

In certain regions no such connexions can be made out. Behind the kidney and bifurcation of the aorta (*i.e.*, behind the lumbar region) the splanchnic branches cease. In front of the fore limbs and the origin of the vertebral arteries (*i.e.*, in the cervical region) the splanchnic branches do not join the sympathetic. In comparatively advanced embryos distinct nerves, analogous to the splanchnics, course inwards round the vertebral artery to the tissues lying round the growing vertebræ and beneath the spinal cord, but occupying a position altogether dorsal to the sympathetic cord and unconnected with it.

The regions, in short, where the sympathetic cord is joined by the splanchnic branches are those in which it lies in relation to the main vascular trunks—in the line of the embryonic aorta.

3. *Formation of the Sympathetic Ganglia.*

Up to the time of the formation of the cartilaginous vertebral *centra* there is no gangliation or constriction of the main sympathetic cord. Sagittal sections of embryos prior to this date have already been referred to and their characters noted. These are exemplified in Rat embryos of 8 days (Plate 26, fig. 11), of 10 days (fig. 12), of 15 days (Plate 27, figs. 14 and 14A), &c.

In sagittal sections of Mouse embryos of 17–18 days (Plate 28, fig. 15) the vertebral *centra* (*V.S.*) are distinctly cartilaginous, connected together by dense masses of cells, and plainly demarcated from the surrounding tissues. The segmentation of the body is also indicated by the intercostal vessels (*Int.*) lying opposite to the vertebral *centra*. Along the sympathetic cord there are seen at places, in successive series of sections and in relation to these segments, occasional bulgings. These are rare, however; sections through a considerable length of the cord at its thickest part show it as an unconstricted column (*Sy.*, fig. 15), composed of large fusiform cells with terminal processes, and with large, deeply-stained nuclei.

Sagittal sections of a Human embryo, about $\frac{3}{4}$ in. in length, hardened in spirit, and stained with aniline blue-black, show a slightly later condition (Plate 30, fig. 19). They are very instructive, as indicating the manner in which the gangliation of the sympathetic cord occurs. The several parts of the vertebræ are already moulded in cartilage, the intervals between the *centra* being filled up as before with embryonic cellular tissue. When serial sections of this embryo are examined, the first indication

of the presence of the sympathetic cord is found in isolated, segmentally-arranged masses, which in successive sagittal sections grow larger and larger, gradually fuse together, and give rise to a continuous fibro-cellular cord. The condition of the cord (*Sy.*), cut through in its thickest part, is indicated in fig. 12. It is mainly cellular, uniform in width except at certain points, and with a considerable admixture of nerve fibres on its ventral aspect. It is joined, dorsally, in the intervals between the vertebræ, by the splanchnic branches of the spinal nerves (*Spl.*), and at these "nodes" the cord has a greater width than in the intervals between them.

In this condition is seen the first commencement of gangliation or nodulation of the sympathetic cord. Two causes produce it : (1) the union of the splanchnic branches. This causes the cord at a given spot to attain a larger size, by the accession of a bundle of nerve fibres, some of which join the cells of the cord, and so insure their permanent occupation of that spot, as the cells of the ganglia ; and (2) the position of the cord in relation to the vertebral column. By the growth of the latter the cord is stretched, and the part lying against the vertebral somites is elongated and narrowed, and is indented by the growing vertebræ and intercostal vessels, the result being the formation of a longitudinal fibro-cellular commissure connecting the nodes or ganglia together. The ganglia themselves are permitted to increase in size, by their position, and by their connexion with the splanchnic branches of the spinal nerves. The commissures, at first cellular, become fibro-cellular, and afterwards fibrous, owing to the admixture of spinal fibres, and apparently also to the conversion into fibres of the original cells of the cord. By the formation and gradual constriction of the commissures the ganglia are made still more definite.

This process of ganglion formation, due primarily to these two causes, can be followed in older embryos ; but as the parts gradually attain their adult form the regularity of alternate swelling and constriction is not so evident. As the ganglia become defined in form, their position tends to become irregular. While one may lie in the interval between two vertebræ, the next may be seen opposite the vertebra itself. Again, two ganglia may be co-terminous, not by fusion, but from the fact that separation is incomplete. This arrangement is shown in Plate 29, fig. 18, taken from a sagittal section of a Rat embryo at about 22 days. The sympathetic cord (*Sy.*) is figured in the region of the diaphragm (*Di.*), just beyond the point at which the stream of cells arises for the formation of the semilunar ganglion (*S.G.*) and solar plexus (*Sol.*). The cord is constricted at intervals, so as to form ganglia and commissures, both of which are irregular, and bear, as may be seen, no definite relation to the vertebral segments (*V.S.*) or intercostal vessels (*Int.*). The commissures are, for the most part, fibro-cellular ; in some instances they are mainly fibrous ; in others there are groups of cells enclosed among the fibres : while in others again there are comparatively few fibres, so that the ganglia are practically fused together. The cord is here represented in its greatest thickness ; in sections on either side the irregularity of the ganglia (into which splanchnic branches can be traced) is much more marked.

From this point the gangliation of the cord gradually proceeds until the condition found in the adult is produced.

The truth of these observations has been tested by another method.   Fig. 21 (Plate 30) shows life-size drawings taken from dissections of Human embryos at various ages.   In the dissections made, individual variations were found in the two sides of the same body—differences in the state of gangliation of the cord, as well in the arrangement as of the spinal communications.

In an embryo in the third month (fig. 21, *A.*) the connecting branches from the spinal nerves were too fine to be seen with sufficient distinctness to warrant their being drawn.   In the upper half of the thoracic region, the sympathetic cord formed a wavy band which swelled out above and below.   Above, it joined the inferior cervical ganglion (*I.G.*), which was again connected by a narrower strip to the long, oval, superior cervical ganglion (*S.G.*), which occupied nearly the whole length of the neck. The vertebral (*V.P.*) and carotid plexuses (*C.P.*) were seen.   The lower swelling of the dorsal band marked the origin of the first root of the great Splanchnic nerve (*G.Spl.*).   In the lower dorsal region were two swellings, distinct, and connected to one another and the neighbouring parts of the cord, by short longitudinal commissures. Below these ganglia were two long oval masses separated by a slight constriction ; and below these, again, were two distinct ganglia joined by commissures.   The last lay free and unattached (to spinal fibres), on the cartilage representing the second sacral vertebra.   It is noteworthy that the diaphragm was complete in this embryo.

In an embryo of the fourth month (fig. 21, *B.*) the cord again forms a uniform band in the thoracic region, joined by the upper nine thoracic nerves.   Angular projections (ganglia ?) are seen at the junctions of the spinal branches, and also where the roots of the Splanchnic nerves arise, especially in the case of the great Splanchnic (*G.Spl.*). The thoracic band swells out above into the inferior cervical ganglion (*I.G.*), which is joined by branches from the eighth cervical and first dorsal nerves.   After giving off the vertebral plexus (*V.P.*) this forms a narrow commissure on which is a minute swelling, the middle cervical ganglion (*M.G.*), and finally broadens out into the superior cervical ganglion (*S.G.*), which ends above in the carotid plexus (*C.P.*).   In the lower thoracic region two constrictions separate off a small mass, which is joined by branches from the tenth, eleventh, and twelfth thoracic nerves.   Below the second constriction the cord forms a broad oval band, joined by the twelfth thoracic and all the lumbar nerves.   Lastly, a sacral portion is constricted off, which consists of three fairly definite ganglia and a terminal filament, the ganglia being joined by the first three sacral nerves.   I was not able to trace the existence of a connecting loop between the lower ends of the two cords.

In an embryo of the sixth month (fig. 21, *D.*) a very similar condition of things was found, so that it will be sufficient to note the differences from the last example.   The inferior cervical ganglion (*I.G.*) was much larger : was joined by branches from the

seventh, as well as the eighth cervical, and first dorsal nerves : and the commissure connecting it with the middle cervical ganglion was double.   The lumbar portion of the cord showed a faint indication of separation into four ganglia : was not joined by the last dorsal nerve : and was connected to the sacral portion by a double commissure.   The existence of a fourth sacral ganglion was noted as doubtful.

In an embryo of the fifth month (fig. 21, *C.*) the condition of the sympathetic cord, in some respects, more resembled the adult state than did the cord at the sixth month.   The inferior cervical ganglion (*I.G.*), after having given off the vertebral plexus and branches *to* the eighth cervical and first thoracic nerves, was continued up into the neck in the form of two incompletely separated swellings, which were joined respectively by the seventh and sixth spinal nerves.   The same ganglion was continued down into the thorax as far as the junction of the second thoracic nerve—receiving roots also from the eighth cervical and first thoracic nerves,—where it gave rise to a commissure connected below with an elongated mass, joined by the three succeeding nerves.   Below this were a series of co-terminous ganglia, joined on the one hand by the lower thoracic nerves, and on the other giving off the roots of the Splanchnic nerves.   The disposition of the cord in this portion of its extent appears to indicate the mode in which the individuality of the ganglia arises.   The eleventh thoracic nerve was connected with two small ganglia ; the twelfth was joined by three branches to two other ganglia.   In the loin a single oval mass was found, joined by long, fine branches from the lumbar nerves.   This was joined by a delicate commissure to the sacral ganglia, which were three in number, the first connected with two, the other two each with one sacral nerve.

These dissections could not, alone, be taken as proof of the original absence of segmentation of the sympathetic cord.   When, however, they are compared with one another, and with the disposition of the adult cord, and are considered also in the light of the foregoing investigations, they lend support to the view that at an early date gangliation of the cord has not occurred, but that it is due to later collateral changes in the cord itself, and in the surrounding parts.

The only observation I am acquainted with dealing with this question from the point of view of comparative anatomy is one by SWAN.   He figures the cord in the Snake as being imperfectly gangliated ; and he describes it as resembling a band rather than a chain of ganglia.   This observation I have had opportunities of repeating, with the result that I have found the cord precisely as he figures it.

4. *Cephalic and Caudal Terminations of the Sympathetic Cord.*

a. Cephalic termination ; superior cervical ganglion ; vertebral and carotid plexuses.

The embryonic cord when first formed can, in transverse sections, be traced forwards as far as the cephalic border of the fore-limb : in sagittal sections to the level of the mouth, or first vertebral segment (Rat, 8 days, Plate 26, fig. 11).   As growth continues

and the large vessels are better developed, the cervical part of the cord lies in distinct relationship to the large arteries. It very plainly divides into two parts at the point of origin of the vertebral artery. One fibro-cellular cord accompanies that vessel, having arisen from the main cord at the point where the artery crosses it. Growth still proceeding, this part of the cord becomes more and more fibrous, until at length it assumes a condition characteristic of the adult vertebral plexus, and is gradually lost on the anterior part of the vessel.

The other division, or main part of the sympathetic cord, accompanies the common and internal carotid arteries, at first only to the level of the mouth, where it forms a pointed extremity composed of cells, or a brushlike ending of cell processes. Growing larger along with the growth of the neck, this part of the cord becomes more and more constricted off from the part of the cord lying behind the level of the vertebral artery, by the formation of a fibrous commissure. A fibro-cellular mass is thus formed, which represents the "superior" cervical ganglion. The commissure connecting it posteriorly to the main cord is formed, partly by direct conversion of the cellular elements into fibres : partly by the passage into it of fibres derived from the anterior thoracic splanchnic branches. In this way the presence of a middle cervical ganglion may be explained, as representing a mass of cells derived from the elementary sympathetic cord which have been included among the fibres of the growing commissure.

From the anterior end of the growing mass of the superior ganglion, a narrow bundle of fibres arises, which is closely applied to the internal carotid artery in its course beneath the auditory capsule. This bundle can be followed for a considerable distance, and is gradually lost upon the vessel, forming the carotid plexus. In fig. 17 (Plate 29), the anterior end of the cervical ganglion (*Sy.*) is shown in a sagittal section of a Mouse embryo of 17–18 days, along with this fibrous bundle accompanying the internal carotid artery (*Art.*) beneath the Eustachian tube (*Eust.*). In this embryo the stream of fibres forming the vertebral plexus has much the same appearance as that shown in the figure.

These parts—the vertebral and carotid plexuses, and superior cervical ganglion—I regard as pertaining to the collateral portion of the sympathetic system, for the following reasons :—(1), they are out-growths from the main cord (the ganglion, fundamentally a part of the original cord, undergoes extensive growth, and becomes soon constricted off from the main cord); and (2), no splanchnic branches enter this part of the cord. This agrees with GASKELL's conclusion regarding the ganglion, which from morphological considerations he regards as belonging "to the distal rather than to the proximal group of ganglia."

b. Caudal Termination.

In the youngest embryos in which the sympathetic cord was found (Rat, 8 days,

p. 166), behind the point at which the mass of cells arises for the formation of the suprarenal body, the cord is irregular in form, and composed of cells less closely packed together than in anterior regions. At a slightly later period it has extended further backwards (Rat, 12 days, Plate 27, fig. 14A), and becoming attenuated is lost at the point of bifurcation of the aorta. In still older embryos it passes further backwards on each side of the middle sacral artery, and on the dorsal aspect of the rectum, from which it is separated by cellular tissue.

In Rat embryos of 22 days the cord (Sy., Plate 30, fig. 20) can be traced backwards in successive transverse sections for a considerable distance, lying in the interval between the middle sacral artery (m.s.) and its accompanying veins (v.v.). The cords of the two sides approach one another as they pass backwards. Cellular branches (Br.) arise from them at intervals, and are directed downwards between the vessels. The cords themselves are more irregular in form than anteriorly, and are not joined by splanchnic branches. Near their posterior terminations, having approached very close to one another, they become connected here and there by transverse cellular commissures (Co.) of which one is represented in fig. 20. These are not in direct continuity with one another, but are separated by intervals, and beyond them the two cords taper off, and finally are lost in the undifferentiated mesoblast above the middle sacral artery. These commissures, which lie close together, and are formed by the fusion of outgrowths from the two cords, apparently give rise to the fibres connecting the adult coccygeal ganglia together, which form the loop containing the coccygeal ganglion, or *ganglion impar*. It is only in an advanced condition of development that the fusion of the two cords can be made out ; until they have reached their ultimate limits the fusion does not take place.

The posterior, like the anterior end of the sympathetic cord, may be regarded as an outgrowth from the original cord. It is formed by an extension of the cord backwards along the main vascular trunks, and it likewise receives no splanchnic branches from the spinal nerves.

5. *Collateral Distribution.*

(*a.*) The development of the superior cervical ganglia, the carotid and vertebral plexuses, has already been traced, and the opinion has been expressed that these structures properly belong to the collateral distribution of the sympathetic cord.

(*b.*) Regarding the formation of the peripheral branches, and collateral ganglia of the sympathetic, my investigations in the main support the conclusions arrived at by REMAK (5, 6). As already noted, cellular buds or branches arise from the sympathetic cord (figs. 7 and 13, Plates 23 and 26), and accompany the parts of the splanchnic branches of the spinal nerves which pass beyond it. These cellular outgrowths gradually increase in length and thickness. They consist of cells which resemble in every way those of the sympathetic cord itself. They communicate by

their processes with one another and with the cells of the cord ; and, especially in the hinder part of the thoracic region, form considerable masses traceable along the main vessels. In older embryos (Plate 29, fig. 18, Rat, 22 days) they form ganglia and nerves, in the way already described for the main cord ; and, in the region referred to, give rise to the Splanchnic nerves, semilunar ganglia, &c. These branches appear to follow in their development the same line of growth as the splanchnic branches of the spinal nerves ; although they are equally evident in places where the latter are not seen.

(c.) The mode of development of the gray *Rami communicantes* I have not made out with perfect satisfaction. They appear to take their origin from the sympathetic cord as cellular outgrowths, which in some cases are traceable along the splanchnic branches of the spinal nerves towards their roots ; but on this question I must withhold a definite opinion until supported by more conclusive evidence. I have not been able to satisfy myself about their formation in places where the splanchnic branches are absent, or are unconnected with the sympathetic cord.

(d.) The relation of the sympathetic cord to the development of the suprarenal bodies is very evident in younger embryos. A considerable column of cells (Plate 27, fig. 14A) can be traced in sagittal sections from the ventral aspect of the main cord, downwards and backwards, to join a mass of mesoblastic cells situated at the anterior end of the embryonic kidney (*K.*). Entering this mass the cellular column spreads out, so as to constitute the central portion or medulla of the suprarenal body (*Sr.C.*), while the tissue around forms the cortex. This agrees, in general terms, with the mode of development of the suprarenal bodies described by previous observers (13, 16, 17, 18). It must be borne in mind that according to this view the *whole* of the Mammalian suprarenal capsule is mesoblastic. The connecting stalk becomes converted into nerve fibres connecting the suprarenal body to the sympathetic cord.

III. Conclusions.

A. *The Development of the Sympathetic System.*

The following are the principal conclusions derived from the preceding investigations :—

1. The formation of the main sympathetic cord is the first event. It is developed as a cellular rod or column, uniform in outline, and without ganglia or constrictions. It is formed in, and derived from the mesoblastic tissue on either side of the embryonic aorta, and in front of the growing vertebral column. The cord appears after the formation of the roots and ganglia of the spinal nerves, and is entirely independent of them at first.

2. The connexion of the spinal nerves with the main sympathetic cord occurs secondarily. The inferior primary division of a (typical) spinal nerve divides, on reaching the junction of body wall and splanchnopleure, into a somatic and a splanchnic

branch. The splanchnic branch grows gradually in a mesial and ventral direction, towards the sympathetic cord, with which it finally becomes connected. It generally divides into two parts, of which one joins, the other passes beyond, the sympathetic cord. The fibres which join the cord are traceable into direct continuity with its cells, or can be followed forwards or backwards in its substance. The part of the splanchnic branch which does not join the sympathetic, can be traced beyond it into the splanchnic area.

This arrangement obtains for the spinal nerves in the posterior thoracic and lumbar regions. In the anterior part of the thorax the whole of the splanchnic branch appears to be joined to the sympathetic cord. Behind the level of the bifurcation of the aorta no splanchnic branches were found. In the neck, above the point of origin of the vertebral artery, they are unconnected with the sympathetic cord. Instead, they proceed from their origin in a ventral and mesial direction external to the vertebral artery, and round that vessel, to be distributed to the tissues of the neck lying on the dorsal aspect of the sympathetic cord.

These splanchnic branches correspond to the white *Rami communicantes*. Tracing them back to their origin they are found to be derived from both the dorsal and the ventral root of the spinal nerve, of which the latter usually contributes the greater number of fibres.

The gray *Rami communicantes* appear to arise from the sympathetic cord as cellular outgrowths, which find their way along the splanchnic branches towards their central connexions. They thus may be regarded as belonging to the group of collateral branches from the cord.

3. The formation of the ganglia on the main cord of the sympathetic is a subsequent event, and is subordinate to the connexion of the splanchnic branches of the spinal nerves with the cord. The causes leading to the formation of the ganglia are : mainly, the junction of the splanchnic branches, and the accession of a large number of nerve fibres at the point of entrance ; the consequent persistence of the cells of the cord, which are joined by these nerves, as ganglion cells ; and to a less extent, the anatomical relations of the cord to the bony segments, &c., over which it passes, which, in their growth, cause indentation of the cord at certain points.

This view is supported by the condition of affairs found in dissections of Human embryos at various periods of development (3–6 months), where the cord has the form of a band or strip, rather than a regularly nodulated chain : and by the evidence derived from the normal adult structure, where the "segmentation" of the cord is apparent, rather than real.

4. Cephalic termination. The parts derived from the sympathetic cord in the neck in front of the "inferior" cervical ganglion, may be regarded as belonging to the peripheral or collateral distribution of the nerve. A fibro-cellular bundle springs from the cord and accompanies the vertebral artery ; beyond this the original cord, which is, at first, limited anteriorly at the level of the mouth, becomes constricted by

the formation of a fibro-cellular commissure, separating off the "superior" cervical ganglion. This ends anteriorly in a fibrous bundle, which accompanies and is lost upon the internal carotid artery, beneath the auditory capsule. The "middle" cervical ganglion, when present, is to be regarded as formed of a group of cells which have been included in the commissure connecting the superior ganglion with the main cord, and which have persisted. The connexions of the sympathetic cord with the cranial nerves have not been thoroughly investigated, and reference to them has been purposely omitted, as they bear no direct relation to the development of the main cord.

The reasons for placing the parts named in the category of peripheral branches are that (1) they are essentially out-growths from the main sympathetic cord, and (2) no splanchnic branches join them from the spinal nerves.

5. Caudal termination. The sympathetic cord is at first ill-defined behind the region of the kidneys; it gradually extends further back, alongside the aorta and middle sacral artery, where the two cords become closely approximated. They become gradually more and more attenuated, and finally disappear. Near their termination they are joined together on the dorsal aspect of the middle sacral artery by cellular commissures, from which the connecting loop and *ganglion impar* are developed. No fusion of the two cords can be seen until they have reached their permanent posterior limit. The sympathetic cord behind the lumbar region may be regarded as belonging to the peripheral distribution of the cord for the same reasons as the cervical portion.

6. The peripheral branches from the sympathetic cord, including the collateral ganglia, as well as the medullary portions of the suprarenal bodies, the superior cervical ganglia, &c., are formed by out-growths from the cord, which are at first cellular. These give rise to ganglia, nerves and plexuses, and are accompanied by the parts of the splanchnic branches of the spinal nerves which do not join the ganglia. In this category are placed doubtfully the gray *Rami communicantes*.

B. *The Morphology of the Sympathetic System.*

1. *The Secondary Segmentation of the Sympathetic Cord.*

It has been shown that the main cord of the Mammalian sympathetic is primarily unsegmented, that in comparatively old Human embryos the differentiation of ganglia and longitudinal commissures is not definite, and that in the adult sympathetic cord there is no certain indication of true segmentation, except in the obviously metameric splanchnic nerves, which are connected with it. Now, in this respect, the sympathetic cord does not differ from, but agrees with, not only the other elements of the nervous system, both in vertebrates and invertebrates, but also with all other longitudinally placed structures in the body. It agrees, firstly, with the medullary tube, forming the cerebro-spinal axis, which, so far as present evidence goes, is formed without any sign of segmentation, the spinal nerves being secondary and subordinate formations.

Again, the gangliated cords of invertebrates, which are formed from epiblast, are at first developed as solid, uniform, and unconstricted columns, which only secondarily become constricted and gangliated. A second reason in support of the view here propounded is that we possess, in the gangliated cords of the sympathetic, bilaterally symmetrical structures, placed along the bony axis of the trunk, in relation to the main vascular and alimentary tubes. On the one hand, there is no evidence of strict segmentation in the branches distributed peripherally from these cords; and, on the other hand, the parts over which they preside—the heart, blood-vessels, lungs, intestinal tract, &c.—are all morphologically unsegmented, like other parts and organs in the Mammalian trunk which are longitudinally placed. The cerebro-spinal axis, the notochord, heart, vascular system, alimentary canal and diverticula, ureters, &c., present no signs of being developed in a segmental manner. In fact, it might be formulated as a morphological law, to which the sympathetic cord is no exception—*that longitudinally-placed structures, whether single and median, or laterally placed, are primarily and uniformly unsegmented.*[*]

2. From the morphological point of view, however, the most important conclusions derived from the investigations recorded above are undoubtedly the following :—

a. The independent origin of the sympathetic system, and its secondary connexion with the cerebro-spinal system.

b. Its development from the mesoblast.

Both these points are of interest, more particularly because, on the one hand, the sympathetic system has previously been regarded as a specialised portion of the general nervous system ; and, on the other hand, there is no fact in embryology more firmly established than this, that whenever a nervous system is present, it is developed from the epiblast.

As the foregoing observations are confined to Mammals—a highly specialised group of vertebrates—it may be objected that, until other and more primitive groups are investigated, it is unwise to draw conclusions of so general a nature. On the other hand, I would venture to urge the following considerations :—

(1) In maintaining the independent and mesoblastic origin of the sympathetic, one is supported by the investigations of a considerable number of embryologists (p. 161), including REMAK, who studied the subject in the Chick, the main difference between REMAK's view of the development of the Avian sympathetic system and the observations made in Mammals being in regard to the primary or secondary segmentation of the main sympathetic cord.

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[*] The longitudinal muscles of the trunk form an apparent exception to this rule, but an exception which in reality may be said to prove the rule. Formed from the muscle plates, and strictly segmental in origin, they retain this character in their fully-formed condition, being, as regards their deeper layers, closely related to the various body segments, in relation to which they are formed. With regard to the fusion of the segmental masses in their superficial parts to form elongated longitudinal muscles passing over several segments (and joints), they follow the same laws as are found to govern the development and evolution of muscles in other parts of the body.

(2) My own investigations reveal definitely the earliest appearance, and the complete history of the formation of the sympathetic system in Rodents (a comparatively primitive group of Mammals); and distinctly indicate the existence of a considerable period during which the sympathetic cord is entirely independent of the cerebro-spinal system.

(3) Investigators who have assigned an epiblastic origin to the sympathetic system differ materially as to the actual mode of development. As already stated (p. 162), BALFOUR regards the sympathetic ganglia (in Elasmobranchs) as outgrowths from the spinal nerves; while ONODI (working at the same group) maintains that they are direct proliferations from the spinal ganglia. Moreover, these authors have left the history of the development of the system in an incomplete state. The earliest steps in the process have not been described for Mammals; and the ultimate and peripheral branches and ganglia have not been traced back to the epiblast. The fragmentary nature of the observations, and the divergence of opinion regarding them, show that further investigations are necessary before the origin of the sympathetic system as an outgrowth from the cerebro-spinal system can be accepted as proved even for any group of vertebrates, still less for all.

(4) Besides the embryological facts recorded in the preceding pages, it may be further pointed out that there are strong grounds, histological and physiological, for drawing a sharp distinction between the sympathetic and the cerebro-spinal nervous systems.

The cells comprising the sympathetic ganglia differ from those of the spinal ganglia in histological character, the latter being rounded, the former angular. In birds, RAMAGE (25) describes the difference as being very evident, "que les cellules des ganglions sympathiques étaient nettement multipolaires, et qu'il était facile de les distinguer de celles du ganglion spinal, qui sont sphériques et unipolaires." The fibres of the sympathetic system are essentially different from those of the cerebro-spinal system, being non-medullated branching fibres.

Physiologically also, the sympathetic is clearly distinct from the cerebro-spinal system. Its functions appear to be to direct and regulate the distribution of certain nerves which emanate from the cerebro-spinal axis. The ganglia are not stated by physiologists to be either automatic or reflex, but only trophic, from which we may infer that they have a "vital influence" over the parts within their jurisdiction. The ganglia apparently neither augment, diminish, nor alter the character of the nerve impulses which permeate them.

From these considerations it may be regarded as very possible that under the term "sympathetic nervous system" are included two structures, entirely independent in nature, origin, and function, the *sympathetic* system, and the *nervous* system proper. Such a view would be in entire harmony with the mode of development described above, and would render more intelligible than has hitherto been possible certain developmental peculiarities, such as the conversion of part of the sympathetic system into the suprarenal bodies.

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EXPLANATION OF PLATES 22–30.

The figures of embryonic sections are in some cases compiled from several successive sections, as, for example, when a nerve had to be followed for some distance.

PLATE 22.

Fig. 1. Transverse section of Rat embryo (5–6 days) in thoracic region. *Sp.C.*, spinal cord ; *No.*, notochord ; *M.P.*, muscle plate ; *D.R.*, dorsal root of nerve ; *Al.*, alimentary canal ; *Ao.*, aorta.

Fig. 2. Transverse section of Rabbit embryo (7 days) in thoracic region. *D.G.*, dorsal ganglion and root ; *Sp.N.*, spinal nerve ; *C.V.*, cardinal vein ; *Co.*, cœlom ; *L.*, lung ; *A.V.*, allantoic vein ; *α*, somato-splanchnic angle. Other letters as before.

Fig. 3. Transverse section of Mouse embryo (8 days) in lower thoracic region. *Sp.N.*, *S.D.*, *I.D.*, superior and inferior primary divisions of spinal nerve ; *H.*, heart ; *W.D.*, Wolffian duct ; *W.B.*, Wolffian body ; *Sy.*, sympathetic cord.

Fig. 4. Transverse section of Rat embryo (8 days) in upper thoracic region. *L.*, trachea ; *Art.*, carotid artery ; *V.*, jugular vein ; *Sy.*, sympathetic cord.

PLATE 23.

Fig. 5. Transverse section of Rat embryo (8–9 days) in lumbar region. *Sp.N.*, spinal nerve, dividing at somato-splanchnic angle (*α*) into somatic (*So.*) and splanchnic (*Spl.*) branches ; *Me.*, mesentery.

Fig. 6. Transverse section of Mouse embryo (about 9 days) in lower thoracic region.

Fig. 7. Transverse section of Rat embryo (about 10 days) in lumbar region. *I.D.*, inferior primary division of nerve, dividing into somatic (*So.*) and splanchnic (*Spl.*) branches, somatic branch subdividing into dorsal or lateral (1) and ventral or inferior (2) branches ; *Sy.*, sympathetic cord, showing growth of cellular peripheral branch (*Br.*).

PLATE 24.

Fig. 8. Transverse section of Mouse embryo in region of fore-limb. *S.D.*, superior primary division, arising by two roots ; *Spl.*, splanchnic branch of spinal nerve joined to sympathetic cord (*Sy.*) ; *P.*, pleural cavity ; *L.*, lung ; *F.L.*, fore-limb.

Fig. 9. Transverse section of Rat embryo (12 days) in thoracic region.

PLATE 25.

Fig. 10. Transverse section of Mouse embryo (about 17–18 days) in lumbar region. *Sp.art.*, spinal artery ; *V.*, vertebra (cartilaginous) ; *Panc.*, pancreas ; *mes.*, mesentery ; *K.*, kidney ; *Spl.*, splanchnic branch, arising from two roots.

PLATE 26.

Fig. 11. Sagittal section of Rat embryo (about 8 days). *Sy.*, sympathetic cord; *Mo.*, mouth; *To.*, tongue; *Hy.*, hypoglossal nerve; *B.W.*, body-wall; *H.*, heart; *Per.C.*, pericardial cavity; *L.*, lung; *Pl.C.*, pleural cavity; *Art.*, carotid artery; *V.*, vein; *Va.*, vagus; *V.S.*, vertebral somites; *D.G.*, dorsal ganglia; *Br.*, brain.

Fig. 12. Sagittal section of Rat embryo (about 10 days).

Fig. 13. Transverse section of sympathetic cord (*Sy.*) in Rat embryo (about 14 days) in thoracic region. *Ao.*, aorta; *Spl.*, splanchnic branch of nerve, dividing into (α) a branch joining the cord, and (β) a branch passing beyond it—γ, a cellular branch arising from the cord, and accompanying (β).

PLATE 27.

Fig. 14. Sagittal section of Rat embryo (about 15 days), showing the anterior termination of the sympathetic (*Sy.*). *Sp.C.*, spinal cord; *Eust.*, Eustachian tube.

Fig. 14A. Sagittal section of same embryo (15 days), showing posterior termination of sympathetic (*Sy.*). *Sr.C.*, suprarenal body; *K.*, kidney; *Sto.*, stomach.

PLATE 28.

Fig. 15. Sagittal section of Mouse embryo (about 17–18 days). *V.S.*, vertebral somites (cartilaginous); *Int.*, intercostal vessels (*Cf.* fig. 10).

Fig. 16. Transverse section of Human embryo, aged about one month, in thoracic region.

PLATE 29.

Fig. 17. Sagittal section of Mouse embryo (about 17–18 days), showing anterior termination of superior cervical ganglion (*Sy.*). *Art.*, internal carotid artery.

Fig. 18. Sagittal section of Rat embryo (about 22 days) in lower thoracic and lumbar regions. *Lu.*, lung; *Di.*, diaphragm; *Li.*, liver; *Pl.C.*, pleural cavity; *Perit.C.*, peritoneal cavity; *S.G.*, semilunar ganglion; *Sol.*, solar plexus; *Sy.*, sympathetic cord.

PLATE 30.

Fig. 19. Sagittal section of Human embryo, about $\frac{3}{4}$ inch long. *Sy.*, sympathetic cord, joined by splanchnic branches (*Spl.*).

Fig. 20. Transverse section through sympathetic cords (*Sy.*), middle sacral artery M.S.A.), and veins (*V.V.*), with peripheral branches (*Br.*, *Br.*), and a connecting commissure (*Co.*). Rat embryo (about 22 days).

Fig. 21. The sympathetic cord, from dissections of Human embryos in the (*A.*) third, (*B.*) fourth, (*C.*) fifth, and (*D.*) sixth month. *S.G.*, *M.G.*, *I.G.*, superior, middle, and inferior cervical ganglia; *C.P.*, *V.P.*, carotid and vertebral plexuses; *G.Spl.*, great splanchnic nerve, with ganglion (*X.*). Connexions with various spinal nerves, numbered in order: *C.*, cervical, 1, 2, 3, 4, &c.; *D.*, thoracic; *L.*, lumbar; *S.*, sacral.

VIII. *An Experimental Investigation of the Central Motor Innervation of the Larynx.*
By Felix Semon, *M.D., F.R.C.P., and* Victor Horsley, *B.S., F.R.S.*
(From the Laboratory of the Brown Institution.)

[Plates 31, 32.]

Part I.—Excitation-experiments.

a. *Introduction.*

In the course of observations dealing with the physiology of the motor innervation of the larynx, the results of which were embodied in a paper recently presented by one of us (F. S.) to the Royal Society,[*] it was shown that the central innervation of the larynx played a much more important *rôle* in the function of respiration than had previously been accorded to it. More especially it was shown that certain nerve-centres were constantly at work in maintaining a reflex-tonus of the abductor muscles (posterior crico-arytænoids) indispensable for the mechanism of quiet respiration in Man.

In 1880, 1881 and 1883[†] it had been shown by one of us (F. S.) that these same muscles (the abductors) were more liable to degenerative changes in cases of organic disease of the motor nerves of the larynx from the medulla oblongata downwards, and at the same time that functional disorders of the laryngeal motor apparatus almost exclusively affected their antagonists, the adductors. The explanation of all these different phenomena presented great difficulties, and, although light was thrown upon them in very various and unlooked-for ways, it soon became evident that neither clinical and pathological observations in Man alone, nor experiments upon the peripheral nerve-mechanism of animals, would suffice to solve all the questions here involved.

Now, the fact just mentioned—viz., that in functional disorders (*e.g.*, hysterical aphonia) those motor laryngeal nerve-fibres only which subserve the volitionary function of the larynx (*i.e.*, phonation) ordinarily suffer, whilst, on the other hand, in

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[*] " On the Position of the Vocal Cords in Quiet Respiration in Man, and on the Reflex-tonus of their Abductor Muscles." 'Roy. Soc. Proc.,' 1890 (vol. 48, p. 403).

[†] (*a*) Foot note in German Edition of Morell Mackenzie's 'Diseases of the Throat and Nose,' vol. 1, p. 574.

    (*b*) 'Archives of Laryngology,' July, 1881

    (*c*) 'Berliner Klinische Wochenschrift,' No. 46, *et seq.*, 1883.

organic disease those fibres are usually primarily affected which are engaged in maintaining the automatic process of laryngeal respiration—seemed to indicate that there must be a central differentiation corresponding to these varied phenomena. Hence it was that the idea of the present research occurred to us.

In planning such an investigation we naturally turn first to the subject of the cortical representation of the larynx—a problem which, first touched upon by FERRIER and DURET, has been treated in detail by H. KRAUSE at the instigation of H. MUNK, and by MASINI.

From this point we might proceed to the consideration of the fibres leading from the cortex to the lower centres in the medulla, as they pass through the corona radiata and the internal capsule. The investigation of these fibres by means of the excitation-method has, so far as we know, not been undertaken by any previous observer, though some work by the ablation and degeneration method has been performed by KRAUSE. This latter method and its results will be fully dealt with by us in a subsequent paper.

Finally, we discuss the results of excitation of the central mechanism in the bulb, a branch of the subject which we commenced four years ago, and which also has not been treated, to our knowledge (confirmed by Professor GAD), by any previous observer.

Before passing to a complete historical retrospect of the question, and before describing the method of our own experiments, we may state that, since we have found certain important differences in the central structural arrangements in different species of animals, and also in animals of different ages belonging to the same species, it is essential that these should first be prominently and clearly expressed, in order that some discrepancies which exist in the statements of previous investigators and the classification of our own results might be properly understood.

To the same category belongs the question of the action of anæsthetics upon the neuro-muscular system, a question the importance of which for the present investigation can hardly be exaggerated ; since, for reasons which we will further discuss, the effect attained by exciting any given centre may be entirely modified by the mere degree of anæsthesia affecting the peripheral mechanism of nerve-endings and muscles.

In a previous communication* we have shown the importance of the action of ether upon the peripheral nervous system, a factor which must be properly discounted in considering the action of the anæsthetic upon the nervous mechanism as a whole.

These considerations of species, age, and anæsthesia will compel us to further subdivide the great groups into which we have arranged our facts.

In the following historical retrospect of the results obtained by previous investigators we have thought it best (to avoid misconception) to place each writer in the chronological order of the appearance of his publication.

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* " On an apparently peripheral and differential action of Ether upon the laryngeal muscles," 'British Medical Journal,' 1886.

b. *Historical Retrospect.*

Although various authors have observed the changes in thoracic respiration following excitation of the central nervous system, but few have studied the relationship of the latter to the larynx, either so far as respiration or as phonation is concerned.

As this relationship is the immediate subject of this paper, we shall confine ourselves in this retrospect to a brief *résumé* of the results of those authors who have directly investigated, either experimentally or clinically, the question here at issue.

On the experimental side undoubtedly FERRIER* was the first worker in the field of excitation. In the first edition of his well-known work he describes an instance in which excitation of a certain cortical area in the Dog elicited barking, and observes that similar observations had been made by him repeatedly.

Shortly afterwards DURET,[†] of Paris, referred to the same subject in connection with his cortical ablation and compression experiments, and reported similar results.

Although subsequent experimentation has shown that both these observers did not accurately localise the *focus* of the representation of phonation in the cortex, to them, undoubtedly, belongs the priority of the idea of a special representation of this function in the cortex.

Seven years after FERRIER's reference to the question the subject was, at the suggestion of H. MUNK, specially studied by H. KRAUSE,[‡] who was the first to localise accurately the phonatory area in the cortex of the Dog's brain. He made six excitation-experiments on Dogs, from which he localised the movement of closure of the glottis (always bilateral adduction) to be represented in the isthmus ("STIEL") of the præfrontal (præcrucial) gyrus. To his ablation and degeneration experiments we shall make reference in a future communication.

In his great work on the functions of the brain, FRANÇOIS FRANCK[§] states that excitation of the "zone motrice" of the cortex produces changes in the thoracic respiratory movements according to the duration and intensity of the excitation. These changes are acceleration, slowing, and variations of amplitude. He does not consider that respiratory centres can be considered to exist in the cortex cerebri, or that there is any differential representation of the larynx or other parts of the respiratory apparatus.

In the epileptic convulsion evoked by cortical excitation he observed what we have confirmed and extended, viz., that in the tonic stage the glottis is shut, whereas in

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* ' Functions of the Brain.'

† 'Traumatismes cérébraux,' 1878, p. 142.

‡ " Ueber die Beziehungen der Grosshirnrinde zu Kehlkopf und Rachen," 'Archiv für Anatomie und Physiologie, Physiol. Abth.,' 1884. This was preceded by a note under the same title, in the ' Sitzungsberichte der Kgl. Preuss. Akademie der Wissenschaften zu Berlin,' November, 1883.

§ FRANÇOIS FRANCK, ' Leçons sur les Fonctions Motrices du Cerveau,' Paris, 1887 pp. 146–8.

the clonic stage the cords execute movements synchronous with the twitchings of the thoracico-abdominal respiratory muscles.

MASINI* made in 1888 a series of both excitation and ablation experiments on the cortex of Dogs. In the former he found in four cases that excitation with weak currents produced earlier movements of the opposite vocal cord, followed by later and slower movements of the cord corresponding to the side of stimulation. He is the only experimenter, so far as we know, who has obtained such unilateral results. With stronger currents he obtained bilateral movements. According to his views the representation of the adductory movements of the vocal cords is spread over the whole motor area, though focussed in the centres for the soft palate and tongue.

ARONSOHN† mentions in a paper, published in the same year, briefly, that in his experiments he could not obtain any definite reply to the question whether the adductory movements of the vocal cords observed by him were actually due to the stimulation of the brain.

MOTT,‡ in repeating our observations, corroborated our results.

Two very important papers on the root-fibres of the motor nerves of the larynx, by GROSSMANN§ and GRABOWER‖ respectively, we only desire in this connection to mention by title, as the subject of their researches rather refers to the peripheral part of the nervous mechanism of the larynx.

The above short list comprises, so far as we know, all experimental contributions to this question, in which the intrinsic laryngeal movements have been actually observed or recorded, when the central nervous system has been stimulated.

The clinical evidence concerning the relationship of the larynx to the higher nervous centres is also very limited.

That lesions of the medulla oblongata may and do occasionally cause laryngeal paralysis, is, of course, admitted on all hands. In such cases it is, correctly speaking, rather a lesion of the vago-accessory nucleus and fibres in the medulla oblongata than a genuine central affection, to which the laryngeal paralysis must be attributed.

With regard to the higher nervous centres, however (cortex and internal capsule), matters are very different.

Although a certain number of cases have been published (e.g., by ANDRAL, GERHARDT, FRIEDREICH, FOVILLE, DUVAL, LÖRI, LEWIN, LIVIO RONCI, LUYS, CARTAZ, MASSEI, &c.¶), in which simultaneously with organic lesions of the higher centres

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* 'Archivi Italiani di Laryngologia,' Napoli, April, 1888, p. 45.

† "Zur Pathologie der Glottiserweiterung," 'Deutsche Med. Wochenschr.,' 1888.

‡ 'Brit. Med. Journal,' 1890.

§ "Ueber die Athembewegungen der motorischen Kehlkopfnerven, II. Theil," 'Sitzungsberichte der Kais. Akad. d. Wissensch. zu Wien,' November, 1889.

‖ "Das Wurzelgebiet der motorischen Kehlkopfnerven," 'Centralblatt für Physiologie,' January 4, 1890.

¶ The literature of these cases will be found in MASINI'S, GAREL and DOR'S, and in ROSSBACH'S papers hereafter referred to in the text. They are not quoted here at length, because these authors themselves admit that they cannot claim to be of a decisive character for the question at issue.

vocal disturbances or even laryngeal paralyses have been observed, no conclusions can be drawn from them for the decision of the present question, because all of them are in some way or other incomplete. In some of them the diagnosis of a cerebral lesion is not at all beyond doubt, in others multiple lesions existed, in a third category no laryngoscopic examination had been made, and again in others either no autopsy had taken place, or the descriptions of the latter, when made, are defective with regard to the condition of the medulla oblongata, the peripheral parts of the laryngeal nerves, the microscopic examination of the medulla, the nerve fibres, and the muscles themselves, &c.

In 1884 BRYSON DELAVAN* called attention to the possibility which existed in his opinion, viz., of localising from clinical observation the cortical motor centre for the larynx. Of the two cases, however, upon which he based his views, in one (SEGUIN's) no laryngoscopic examination had been made, whilst he himself frankly stated some years afterwards,† that in the second one, his own, in which he had attributed the laryngeal paralysis to a cortical lesion, the post-mortem examination had established the fact that in reality it was of bulbar nature.

In 1886 GAREL‡ recorded an observation, to which he added, in conjunction with M. DOR, another in 1890, which proved,§ in the opinion of these authors, that unilateral laryngeal paralysis could be caused by a unilateral lesion of the opposite hemisphere or internal capsule. We have elsewhere‖ taken exception to the interpretation given by these authors to the paralysis observed in their cases, because (a) the medulla oblongata was not microscopically examined at the post-mortem examination, and (b) the *total* paralysis (respiratory as well as phonatory) of the vocal cord observed in their cases is according to our ablation-experiments absolutely incompatible with the idea of its being due to a *cerebral* lesion.

Recently ROSSBACH¶ has published a very interesting paper on the subject, in which from a carefully observed case under his own care he draws the conclusion that unilateral paralysis of a vocal cord may be caused by a lesion of the opposite insula. Seeing however, that in his case there was atrophy of the corresponding side of the tongue due to an undoubted lesion of the ganglionic cells of the hypoglossal nucleus in the medulla oblongata, and that the integrity of the vago-accessory nucleus in the medulla was merely *surmised* from the fact, that the fibres of the nerve itself, and of the laryngeal muscles, had undergone no atrophy, while it was not at all shown that the

* 'Compte-rendu des Travaux de la Section de Laryngologie du VIII. Congrès International Périodique des Sciences Médicales.' Copenhague, 1884, p. 17.

† 'Transactions of the American Laryngological Association,' 1888, p. 195.

‡ 'Annales des Maladies de l'Oreille et du Larynx,' May, 1886.

§ *Ibidem*, April and May, 1890.

‖ 'Annales des Maladies de l'Oreille et du Larynx,' May and June, 1890.

¶ "Localisation des corticalen Stimmcentrums beim Menschen." 'Deutsches Archiv für klinische Medicin,' vol. 46, March, 1890.

laryngeal paralysis had existed so long that atrophy ought necessarily have followed, we cannot admit this case as actually proving such a relationship as that claimed by the author.

Finally, at this moment a paper on the subject is in course of publication from the pen of SCHEFF,[*] the outcome of which cannot yet be foretold.

The general observations of LANNOIS[†] bring nothing new, whilst GOTTSTEIN[‡] considers the question of the existence of a cortical centre for the larynx in man as still open.

This short list practically exhausts the clinical material in connection with the subject.

c. *Methods of Experimentation.*

The different points to be considered under this heading may conveniently be grouped in the following manner :—

1. Electrical methods.
2. Operative technique.

1. *Electrical methods.*—The excitation either of the cortex, corona radiata, or bulb was furnished by the secondary coil of a DU BOIS-REYMOND inductorium of the ordinary pattern, the distances of the secondary coil being marked in millimetres from a zero point where the primary coil was completely covered. In the later experiments we always employed a special inductorium with the HELMHOLTZ side wire, very kindly graduated for one of us by Professor KRONECKER, according to the manner he has published elsewhere, and by means of which alone is obtained an accurate knowledge of the increase in the strength of the stimulus when the secondary coil is pushed towards and over the primary. To facilitate the various references to the strength of the current in this paper we now give in parallel columns the divisions in millimetres according to the old method, and opposite them the divisions according to KRONECKER's method. The battery elements employed were either one Daniell cell, one 1 litre bichromate cell, (GRENET), or a thermoelectric element, 16 cm. high, 14 cm. diameter, of 19 metallic junctions.

Du BOIS-REYMOND inductorium.

cm.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.	25
Divisions of current	13,000	12,500	12,000	11,250	10,270	9,250	8,500	7,300	6,300	5,300	4,300	3,300	2,400	1,700	1,000	600	420	300	250	180	130	87	70	58	48	41

Fraction over

KRONECKER inductorium.

The electrodes employed were always fine platinum points which, in exciting the cortex, were fixed 2 mm. apart, but in exciting the corona radiata, internal capsule, or

* " Zur Pathologie d. Motilitätsstörungen d. Kehlkopfs." ' Allg. Wien. Med. Zeitung,' June, 1890.

† " Y a-t-il un Centre cortical du Larynx ? " ' Revue de Médecine et de Chirurgie,' August, 1885.

‡ ' Die Krankheiten des Kehlkopfs,' Dritte Auflage, 1890, p. 344.

bulb were placed 1 mm. apart. The stimulus was kept applied to the cortex for different periods of time according to varying circumstances. Thus if a very minimal stimulus was being employed it was necessary, in order to judge of its effect upon the cortex, to prolong the excitation during several respiratory movements of the vocal cords, *i.e.*, from 5 to 10 seconds. If, however, the stimulus was markedly adequate so that an unmistakeable effect was produced immediately and ceased directly the stimulation was discontinued, then the application of the electrodes was only maintained for from 1 to 2 seconds, so as not to exhaust the cortex.

2. *Operative technique.*—The first point to be considered of course is the anæsthesia, the importance of which has been already alluded to, and is considered in further detail on p. 195. We may say at once that we have almost invariably used ether, in a few cases also in part chloroform ; but inasmuch as the latter did not show any specific effect, differing much from that of ether, it is sufficient only to allude to it here as it may chance to be mentioned hereafter. The narcotisation was always conducted in the following manner :—The ether was first given by inhalation, and then complete unconsciousness having been produced tracheotomy was next performed and the anæsthetisation continued through a funnel and short* piece of rubber tubing, the funnel resting on cotton wool wet with ether and covered with a cloth. This method had the advantage that as it was usually found far more convenient to inspect the larynx from below all trouble connected with the inhalation of the narcotic was avoided, and as the trachea was always completely divided it was readily moved aside without twisting the larynx to either side, and consequently an uninterrupted view obtained. Furthermore, if too much anæsthetic was given by mistake, it was easy to restore the animal by inflation.

The tracheotomy having been done, as large a cannula as possible was inserted and fitted as above stated.

The animal was secured in a prone position and the operation proceeded with to expose the central parts of the brain. These may be referred to as cortex, corona radiata, internal capsule, and bulb. The second and the third of course consist only of fibres, whereas in the first and last case additional complications arise and extra precautions have to be taken owing to their being central apparatuses.

Cortex.—The anterior third of the skull, including the roof of the orbit, was then removed by first making a trephine opening and subsequently removing the bone piecemeal with fine bone forceps. All bleeding from the bone, which was always free, was immediately checked by wax. In this way the whole of the anterior extremity of the hemisphere, including the orbital surface, was completely exposed, *i.e.*, as far as the margin of the olfactory tract. For the better exposure of the same the tissues filling the orbit were drawn forward, the ocular contents having previously been removed from the eyeball through an incision in the sclerotic.† By means of constant irrigation

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* Purposely short to allow of proper change of the respiratory gases in ventilation of the lungs.

† Or the eyeball removed (HITZIG).

with warm water or normal salt solution shock was in very great measure avoided. This was also provided against in the usual way with hot vessels, coverings, &c.

In no case was the dura mater opened until immediately before the commencement of the excitation.

The animal was then fixed on its side so that when the anterior wall of the trachea had been removed a perfect view from below was obtained of the vocal cords in their whole length.

In all this preparation of the trachea, larynx, &c., special care was necessary to avoid injury of the recurrent laryngeal nerves. Between every period of the excitation of the cortex it was douched with warm water or saline solution, protected by the skin-flap from the air and kept warm with hot sponges.

Corona Radiata.—The investigation of the corona radiata of course enables one to decide many points concerning the amount of representation of various functions in the cortex. Naturally, however, the excitation of the few fibres descending from the comparatively extensive area of the cortex is not easy unless the electrodes be placed exactly upon the cut ends of the fibres. It is for this reason, no doubt, that a stronger stimulus* is usually required to successfully excite the fibres. The removal of the cortex was always effected by a single incision with a sharp scalpel at right angles to the supposed direction of the fibres.

Internal Capsule.—This was exposed by horizontal section through the hemisphere in the same way as that just described for the corona radiata. The line of the incision taken is referred to on p. 202, and the only point noteworthy here is the arrest of hæmorrhage, &c. Naturally bleeding comes almost entirely from the middle cerebral, the lenticulo-striate arteries and their branches. Bleeding from these vessels can be certainly arrested if a fragment of amadou be gently pressed on the open mouth of the vessel and left there. Usually the various bundles of fibres in the capsule were then explored with the electrodes, and finally a careful transcript of the surface of the section made with compasses upon paper ruled with squares of 1 mm. a side, and the corresponding bundles of fibres indicated by arbitrary numbers placed on the squares into which the capsule was by this means divided.

The Medulla Oblongata.—This was exposed for excitation by reflecting the muscles and periosteum by the usual T-shaped incision from the occipital bone and the dorsal surfaces of the arches of the first two vertebræ without injury to the vertebral vessels. These same arches and the neighbouring portion of the occipital bone were then cut away with bone forceps. The dura mater was then divided in the middle line and the middle lobe of the cerebellum slightly drawn upwards or a portion removed in order to expose the lower end of the fourth ventricle. A drawing was then made with compasses of the surfaces of the ventricle thus exposed on the 1 mm. ruled paper. Oozing from the cerebellar vessels was always readily controlled, even in the dog, by irrigation with hot water or saline solution. This had a very notable effect also in raising

* PUTNAM, FRANCK.

the excitability of the bulbar centres when these were depressed by shock, exposure, &c.

In every case a post-mortem examination was made, the drawings verified and the specimen preserved.

d. *Influence of the Anæsthetic and Strength of Stimulus.*

We have assumed that the construction of the whole neuro-muscular apparatus of the larynx comprises (1) an area of cortex popularly, though incorrectly, termed the cortical laryngeal centre, (2) a similar area in the bulb or medulla oblongata which may in like manner be spoken of as the bulbar laryngeal centre and is connected with the cortical apparatus by fibres running in the corona radiata and internal capsule, while finally the vago-accessory nerve roots and peripheral branches contain the fibres supplying the laryngeal muscles and mucous membrane.

It was first observed by HOOPER[*] that even when the recurrent nerves were cut inhalation of large doses of ether produced paralysis of the closers (the adductors) of the glottis before the openers (the abductors). We repeated[†] and extended in further detail this observation and showed that this remarkable peripheral and differential action of ether was, in all probability, due to the fact that the antagonistic muscles of the larynx belong to classes of muscles biologically different.

HOOPER's original observation was contested by DONALDSON,[‡] who found that in certain cases one of the two groups of muscles predominated according to the strength of the stimulus, that with a feeble current, opening, with a strong current, closing, of the glottis was produced. A further, later research by HOOPER[§] showed that the same effect is produced by varying the rate (*i.e.*, the intensity ?) of the stimuli, with a slow rate of interruption of a constant current abduction being obtained, with a quick rate adduction.

We have observed DONALDSON's result to be occasionally present, and from general considerations are ready to believe that HOOPER's second observation is correct. It is clear that the ether effect, and the intensity of stimulation effect mutually confirm each other. All these results apply in their entirety to Dogs only. Both, however, have to be kept in view when considering the results of exciting, as in the present research, the highest point of the efferent neuro-muscular track of the laryngeal apparatus in any animal, since it is obvious that the actual representation of movement (whether closure or opening of the glottis) might be masked by the condition of the peripheral mechanism; for instance, if this were poisoned with ether stimulation would give opening only of the glottis.

[*] 'Transactions of the American Laryngological Association,' vol. 7, 1886.

[†] 'Brit. Med. Journal,' 4–11 September, 1886. *Vide* also BIEDERMANN, 'Centralblatt für Physiologie,' and BOWDITCH, 'International Journal of the Medical Sciences,' April, 1887, p. 444.

[‡] 'American Journal of Medical Sciences,' July, 1886.

[§] 'Transactions of the American Laryngological Association,' 1888.

This combination of physiological relationship in the larynx has not, so far as we know, received much attention, we wish, therefore, to lay stress upon the fact that, until it has been further investigated, we shall not speak dogmatically upon the relative degree of representation of the two antagonistic movements in any one point of the cortex, but shall content ourselves with giving the results of our work.   The further consideration naturally arises, that possibly what applies to the peripheral mechanism *may* also be true of the central apparatus, that, in short, the activity of the cortical corpuscles in which the representation of adduction exists may be more easily extinguished by ether than that of the representation of abduction.   Considering, too, that this implies the earlier abolition of purposive function, it is not so fallacious to thus translate peripheral (*i.e.*, nerve ending?) conditions to a centre.   This, however, although a question of the utmost interest, does not appear to be capable of direct solution until it is possible to differentiate either anatomically or functionally the particular fibres in the pyramidal tracts in which closure or opening are respectively represented.

We hope that the results of excitation of the internal capsule, especially in the Cat, *vide infra*, will lend some aid in explanation of the difficulty as well as in the special examination of MASINI's results for the reasons before given.   We have carefully noted the effect of varying the strength of the excitation.

We have always employed the same rate of excitation (viz., 100 per second), since we believe that variation in rate is practically the same in effect as variation in intensity.

<div align="center">RESULTS.</div>

<div align="center">*Cortex.*</div>

General Statements.

We have directed our attention primarily to the investigation of the representation of phonatory laryngeal movements only, but we have not been unmindful of the concomitant representation of respiration.   This is a point which has already been the subject of research, and especially at the hands of FRANÇOIS FRANCK.   That author, as already mentioned in the Historical Retrospect, does not consider that a separate cortical apparatus exists for the phonatory and respiratory functions of the larynx respectively, but we believe that an extensive series of experiments will justify the course we have adopted.

We moreover think it right to discriminate between the representation of general respiration and that of the larynx, since we have, on the one hand, obtained laryngeal effects without any concomitant changes in general respiration,* and on the other hand, one excitation has often at the same moment induced changes of a

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* Since this was written KRAUSE has stated in the 'Berliner Klinische Wochenschrift,' No. 25, 1890, that he has observed the same differentiation of representation.

different type in the larynx and general respiration, *e.g.*, closure of the rima glottidis with general acceleration of respiration. We shall therefore proceed at once to describe the laryngeal effects only, and shall reserve what we may wish to state relating to the effects produced on general respiration for another communication. In arranging our results we divide them as indicated on p. 188, *i.e.*, first, according to the species of animal employed, and we shall commence with the highest order.

Monkey.

Species employed.—We have explored the cortex in examples of *Macacus sinicus*, *Macacus rhesus*, and *Macacus cynomolgus.*

Among these *Macacus sinicus* afforded the best results. The general conclusions we derive from our excitation experiments are as follows :—

Intrinsic Laryngeal Movements.—That in the foot of the ascending frontal gyrus, just behind the lower end of the præcentral sulcus (*vide* Plate 31, fig. 1),[*] there is a focus of representation of the movements of the vocal cords which are independent of movements of the pharynx, when the most anterior part of the focal area of representation is excited.

This focus is limited anteriorly by the præcrucial sulcus (*p.c.*, fig. 1) and a line continuing the direction of that sulcus to the Fissure of SYLVIUS, superiorly, by a line drawn horizontally (parallel to Fissure of SYLVIUS) through the upper extremity of the small secondary sulcus marked *v*, see fig. 1, inferiorly, by the Fissure of SYLVIUS, and posteriorly, by the sulcus *v*.

The purely intrinsic movement of adduction of the vocal cords seems to be represented in the front half (marked darkly F, in fig. 1) of this focal area. In the posterior half, *i.e.*, just in front of *v*, it is accompanied by pharyngeal movements especially. (BEEVOR and HORSLEY.)

In this focal area the character of the movement represented is invariably adduction (Monkey), and not only complete, but also perfectly bilateral. We have never observed abduction to follow excitation. Outside the focal area the intrinsic movements are also represented, but in a greatly diminished fashion. Thus, from the rest of the facial region, as high as the lower border of the upper limb region, as defined by BEEVOR and HORSLEY, and, posteriorly, as far as the secondary sulcus in the foot of the ascending parietal gyrus, a slight degree of adduction (as far as the cadaveric position) is obtained on excitation. The relative degree to which this exists is shown in fig. 1 by the depth of the stippling. It need hardly be added that, of course, the movements of the face, jaws, and tongue noted to be represented in this region (FERRIER, SCHÄFER, BEEVOR and HORSLEY) also occur in concomitant association.

This gradual decrease in representation from a focus or centre of most complete function is typical of all cortical motor representation, especially in the lower apes.

Extrinsic Laryngeal Movements.—The movements of the larynx as a whole,

* Confirmed by MOTT and SCHÄFER, ' Brit. Med. Journal,' May 17, 1890.

of course, possess considerable interest.   From anatomical reasons these resolve them-
selves under two headings—

(1) Upward Movement of the Larynx,
(2) Downward   ,,   ,,   ,,

(1) *Upward movement* of the larynx we noted to occur only in front of the Fissure
of ROLANDO, between it and the small secondary sulcus marked *v* in figure.

This movement is associated with swallowing and mastication, these actions
being (BEEVOR and HORSLEY) represented in the strip of cortex thus indicated.
Clearly, in this association, we have only the completion of the observed representation
of these complicated acts.

(2) *Downward movement* of the larynx we obtained by exciting the cortex just
behind the Fissure of ROLANDO between that sulcus and a line parallel to it, drawn
from the lower end of the intraparietal sulcus to the Fissure of SYLVIUS.

Once we observed confusion between the movement of the larynx upwards and
downwards to occur at this point, the downward direction predominating.

Epilepsy.—We have frequently observed epilepsy to follow excitation of the cortex
in this region.   At the commencement of the fit, the vocal cords are strongly
adducted, often with tremors, *i.e.*, confusion between adduction and respiratory (?)
abduction, this persisting during the tonic stage.   Then, in the subsequent clonic
stage, the movement of adduction alone occurs as single spasms and ceases, of course,
often suddenly.

Carnivora.—Dog.

Phonation.—The following account applies only to the adult animal ; we shall speak
later of the differences which are caused in the cortical representation according to the
age of the individual examined.   The nomenclature of the sulci employed will be that
given by LANGLEY in his admirable paper on this subject,* except that for brevity we
shall speak of the anterior limb of the sigmoid gyrus as the præcrucial gyrus, and
similarly of the posterior limb as the postcrucial gyrus.   In the original description
given by KRAUSE† of the cortical representation of the movements of the vocal cords,
that author localised the focus of the representation to be situated in the præcrucial
gyrus [or, as he called it, following MUNK, the gyrus præfrontalis (OWEN),] just where
that gyrus terminates below in a narrow pillar or isthmus connecting it with the
anterior composite gyrus.   At this point he found that excitation, as a rule, produced
bilateral adduction of the cords ; this we have completely confirmed, but we have also
found that while the spot indicated is a focus, nevertheless slight adduction can
be obtained by excitation of the whole of the lower two-thirds of the præcrucial
gyrus, and also a few millimetres of the upper extremity of the anterior composite
gyrus.

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* ' Journal of Physiology,' December, 1883, p. 248.
† ' Archiv f. Anatomie und Physiologie, Phys. Abth.,' 1884, p. 203.

This representation of the intrinsic movements of the larynx is not pure except very special care be taken to adjust the strength of the excitation sufficiently to prove a minimal stimulus to the point excited. As a rule, it is accompanied by movements of the tongue and of the pharynx, *e.g.*, swallowing.* Occasionally, however, we have obtained movements of the vocal cords without any concomitant action elsewhere. We have never, in the adult Dog, witnessed abduction, although we have explored the whole of the frontal lobe on its orbital and external nasal surfaces completely.

Respiration.—We now proceed to remark upon the representation of respiratory movements of the larynx also to be found in the cortex of the adult Dog. We have found that the following conditions may be produced with absolute certainty.

a. Acceleration of Respiratory Movements of the Vocal Cords.—This is chiefly represented just above the focus for the movements of the adduction of the cords, *i.e.*, just in the middle of the lower third of the præcrucial gyrus, but excitation of the whole of the præcrucial gyrus as a rule produces this effect in diminishing degree, as we move from the focal point just mentioned. In the performance of this act the abduction of the vocal cords is, as a rule, diminished in extent, as has already been mentioned in speaking of their phonatory representation in the same area, but they otherwise present the ordinary alternate rhythm of abduction and adduction.

Before passing to the next point observed we must here make a few remarks upon this extraordinary result, because it has not been, so far as we know, described in connexion with any function other than the respiratory.

Quickening of respiration has very properly been termed by RICHET[†] polypnœa, and has been proved by him to be the means whereby the thermo-taxic apparatus restores the equilibrium of the body by providing for extra loss or dissipation of heat.

Further, work of a memorable kind has been done upon this subject by OTT,[‡] who has directed special attention to this point, and has produced by excitation the same acceleration of respiration, but from excitation of the internal capsule. (*Vide* our remarks on the internal capsule.) We provisionally quite agree with the conclusions and explanations offered by these writers, and we would here remark, that so far as we can see, the results obtained by CHRISTIANI,[§] and probably those by MARTIN and BOOKER,[||] are to be explained as the results of excitation of the fibres leading down from this very extensive and cortical representation of the respiratory acceleration

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* [While this paper was passing through the press MUNK published, in the 2nd edition of his 'Gesammelte Mittheilungen über die Functionen der Grosshirnrinde,' a statement on p. 178 to the effect that we have confounded the focal representation of phonation with laryngeal movements when accompanied by associated movements of the mouth, pharynx, &c. The justice of this statement we leave to the opinion of our readers.—August, 1890.]

† 'La Chaleur Animale.'

‡ 'Laboratory Researches,' Easton, Pa.

§ 'Zur Physiologie des Gehirnes.' Berlin, 1885.

|| 'Journal of Physiology,' vol. 1, 1881.

that we have observed.   (*Vide* also François Franck, Lépine, Bochefontaine, Danilewsky, Goldstein, Sihler, Munk, &c., and our remarks on the internal capsule.)

b. Intensification of Respiratory Movements.—If the electrodes be moved over the surface of the sigmoid gyrus it will be noticed that as soon as they are placed opposite the lower or outer end of the crucial sulcus, there is then, together with the now very slight acceleration in rate, a distinctly more energetic action of the vocal cords.   Both the abduction and the adduction movements are more pronounced, and as the electrodes are carried back to the lower end of the postcrucial gyrus, *i.e.*, over the area of representation of the fore-limb, this intensification becomes still more marked, and is occasionally accompanied by slowing, and may in short be said to be there localised, as from this point upwards it gradually diminishes over the surface of the gyrus.

These observations are interesting in connexion with those made by Masini upon the distribution of the laryngeal represention in the Dog.   (*Vide* History.)

In reviewing the results just noted, we wish to point out that if the excitation of the cortex be first attempted with an extremely weak current and if then the stimulus be increased, the first effect in response to the weak current is the respiratory acceleration before noted, and the next effect is the intensification of the respiratory movements.   The purposive adduction obtained in the phonatory centre as a rule requires the strongest stimulus.

This has been found to be the case by most observers, the distance of the secondary coil from the zero point of the ordinary du Bois inductorium being required to be pushed, as Krause himself first stated, to 7·5 or even 4 cm.[*]   Although we have made a special investigation of this point with very weak as well as stronger currents, and although we have been very careful to preserve the vertical position of the larynx, and to record even the slightest action, we have never observed the unilateral movements described by Masini.

Epilepsy.—As in the Monkey, we have not infrequently observed epilepsy to follow the excitation of the cortex, this being more readily produced in the carnivorous animals than in the higher.   During the epileptic convulsions the vocal cords usually after preliminary adduction exhibit clonic adductory movements, in no wise differing from those previously described, save in their intermittent character.

Cat.

We have made a large number of experiments upon the cortex of the Cat because we found very early in our investigations, in 1886, that the movements of the larynx represented in the cortex of the Cat were not the same as those which we had obtained from the Monkey or even from the Dog.   In other words : contrary to what we have just seen, the movement of abduction was well represented in the cortex of the Cat, although we had never been able to find any genuine trace of it in the cortex of

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[*] Faradism, moreover, not Helmholtz modification.

the Dog. It will be necessary, perhaps, first to explain that the arrangement of the convolutions in the Cat as regards the frontal lobe is in its chief features the same as in the Dog, only in miniature. See Plate 32, figs. 2, 3, 4, 5. The coronal fissure bounds the sigmoid gyrus below, and running forward is separated from the supraorbital sulcus by the pillar or isthmus of the præcrucial gyrus joining the anterior composite gyrus as before described. In front of the supraorbital sulcus and between that sulcus and the rhinal fissure is situated, as in the Dog, the orbital lobe. The rhinal fissure, of course, separates the orbital lobe from the olfactory tract. We mention all these points because their special bearing will be seen directly.

Phonation.—The bilateral adduction of the vocal cords, as in phonation, we have observed in the Cat to be more frequently represented in the coronal gyrus at its anterior extremity, near the anterior composite gyrus, at a point which is in a line with the crucial sulcus and in the upper part of the gyrus, *i.e.*, close to the coronal fissure, at *d'*, figs. 2, 4, 5. We have further observed that representation of bilateral adduction is also to be found in the area corresponding to that discovered by KRAUSE in the Dog, *i.e.*, just above and in front of the anterior end of the coronal fissure, at *d*, figs. 2, 4, 5. This representation of adduction, whether in one place or the other, is very soon exhausted, although it is, when present—*i.e.*, in a full-grown old Cat— well marked. It requires a moderate stimulus for its production, *i.e.*, 7000 to 8000 of the KRONECKER coil.

Respiration, Abduction, Acceleration.—We have found that in the Cat there exists a point (at *b*, figs. 2, 3, 4, 5) in the lower border of the hemisphere, just above the rhinal fissure—*i.e.*, in the anterior composite gyrus—and just below the depression representing the antero-supra-sylvian sulcus, excitation of which produces well-marked—*i.e.*, as long as the excitation lasts—abduction of the vocal cords ; and that the same effect is produced, in a diminishing degree, round this point as a focus, and occasionally passes above the supra-sylvian sulcus on to the coronal and upper part of the anterior composite gyrus. This cortical representation of abduction in the Cat is very interesting, as probably indicating special respiratory requirements[*] in that animal, this deduction being confirmed by the experiments of LEGALLOIS[†] and others. Although we have found occasionally that in the Cat the præcrucial gyrus gave upon excitation, as in the Dog, acceleration of respiration, yet this was not so marked.

Intensification.—Similarly, intensification also was observed, but in no great degree.

Concomitant Movements.—Concomitant movements were observed (movements of the tongue, licking, mastication, and swallowing) when the anterior composite gyrus was excited, *i.e.*, when opening of the glottis (abduction) was produced. But we constantly observed the abduction to be completely maintained while thoracic respira-

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[*] In the discussion following a demonstration of some of our experiments at the 10th International Medical Congress, 1890, Prof. DU BOIS-REYMOND stated that he had observed phonation in the Cat and Cow to be of an inspiratory nature. Prof. EXNER stated the same with regard to the Pig.

[†] 'Sur le Principe de la Vie.'

tion went on unhindered.   On the other hand, when closure of the glottis was evoked, the respirations were distinctly slowed.

We cannot pass from this part of the subject without adverting to the remarkable way in which the laryngeal apparatus of the Cat is, as we have before remarked, extremely sensitive to the operation of the anæsthetic.   Especially was this the case in making these experiments upon the cortex, since we found that a moderately deep narcotisation was sufficient to change the effect observed from adduction to abduction. The discussion of this we resume further on.

Epilepsy.—We have occasionally observed epilepsy, and when it occurred the vocal cords were preliminarily brought together, and then moved in clonic adductory spasms.   We would now like to state that it is the phonatory—or, as we may say, the purposive—adductory representation of the laryngeal movements which is invariably brought out in the epileptic discharge of the cortex ; another proof, if proof be required, of our position that adduction is a movement characteristic of the highest cortical evolution as far as the larynx is concerned.

Rodents.—Rabbit.

Cortical representation of movement in the Rabbit is, of course, well known to be very limited indeed, both in extent and in degree, the cortical centres being soon exhausted and the effects produced being relatively insignificant.   We therefore only made two experiments upon this animal.   We obtained no focal representation of the larynx, but the intrinsic movement observed concurrently with the movements of other parts were, as a rule, adduction ; and, where this was most marked, the concomitant movement was that of swallowing.

Corona radiata.

We have, in a certain number of experiments, removed the cortex and excited the fibres beneath.   As might be expected, our results all point to one conclusion, viz., that excitation of the fibres which lead in a direct line from the particular part of the cortex removed to the internal capsule gives the same effects as did the cortex when that was excited.   Thus, in the Dog we have observed that the white fibres leading down from the præcrucial gyrus gave acceleration and those from the isthmus of that gyrus adduction, whereas the fibres coming from the lower end of the postcrucial gyrus gave intensification of the respiratory movements of the vocal cords.   Exactly the same is observed in the Cat, viz., that acceleration is coupled with adduction, and that, on excitation of the fibres coming from the abductor focus, we have as distinctly abduction.

In no case, in accordance with the previous observations of various authors, did we obtain epileptic movements of the laryngeal muscles, after the cortex had been excised, by stimulation of the fibres leading therefrom.

Internal Capsule.

It will be necessary to give a slight anatomical introduction before describing our results of the excitation of the internal capsule. In the first place we invariably have exposed the internal capsule by a horizontal section carried through the substance of the hemisphere from before back, after the usual preliminary ligature of the middle cerebral artery. In this manner the internal capsule in its whole length, basal ganglia, &c., were exposed, and readily distinguishable. The horizontal section before referred to passed in the Carnivora through the sub-prorean gyrus, the lower part of the orbital lobule just above the rhinal fissure, through about the middle of the Sylvian fissure, and so backwards through the lower end of the curved external convolutions and sulci, *i.e.*, ecto-sylvian, &c.

In the Monkey the section was carried horizontally backwards through the third frontal gyrus, the foot of the ascending frontal and parietal gyri respectively, across the Sylvian fissure, through the uppermost part of the temporo-sphenoidal lobe, and so through the occipital lobe. The anatomical features of the cerebral sections thus produced, although, of course, differing considerably in the various orders of animals, nevertheless are not so diverse as might be anticipated (see figs. 6, 7). In each case the section exposed both the caudate and lenticular nuclei of the corpus striatum as well as the optic thalamus. The sections in the Monkey thus correspond to about the level of Group 4, as described by BEEVOR and HORSLEY. These authors in their paper (*ante*, pp. 49–88), on the arrangement of the excitable fibres of the internal capsule of the Bonnet Monkey, just referred to, observed phonation to occur in one section of the capsule when these fibres were excited (*ibid.*, p. 83), but they did not directly observe the laryngeal movements.

In this connection it must be remarked that in the Carnivora the lenticular nucleus is relatively extremely narrow (see figs. 6, 7), so that the angle made by the two limbs of the capsule is a very open one, almost 150 to 160 degrees. Therefore, as will be seen directly, the general arrangement of the fibres, as far as the antero-posterior order goes, is the same in the two classes of animals, although the actual positions are not quite the same, owing to the non-development of the præfrontal region in the Carnivora. Since the irruption of præfrontal fibres which makes up the mass of the anterior limb in the Monkey and pushes the excitable fibres further backwards into the posterior limb does not occur in the brain of the Dog or Cat, in these animals it is the anterior limb which principally contains the excitable laryngeal fibres.

In all animals the most anteriorly placed laryngeal effect observed was the accelerated movements of the vocal cords connected with quick movements of the thorax in the polypnœa, which we have described to be produced by excitation of the præcrucial gyrus. In the Cat and in the Dog this acceleration of respiration is produced by excitation of a considerable number of fibres occupying the anterior half of the anterior limb, *i.e.*, opposite about the anterior half of the caudate nucleus as exposed in the

section before referred to. At the same time that acceleration was produced there was also obtained from the most posterior bundles of these fibres, in all animals, abduction, this combination being concentrated in the Monkey at the genu, where the fibres subserving the acceleration of respiration are chiefly grouped. In the Cat and in the Dog these abductor fibres were found just before the genu.

Taking the various species individually we find that, in (1) the Monkey, acceleration is first produced when the fibres immediately in front of the genu and exactly at the genu are stimulated; this is, of course, just opposite the sharp angle made by the middle zone of the lenticular nucleus. The fibres in the posterior limb immediately behind the genu when excited either give a mixed movement or simply produce on excitation diminution of abduction, and it is not until we get to the junction of the middle and posterior fourths of the capsular border of the middle zone of the lenticular nucleus that we get distinct adduction. There, however, the adduction is very accurately localised to a few bundles of fibres about 2 square millimetres in area. Just behind this point for a short distance diminished abduction was observed on stimulating the capsule in one case; it is clear, therefore, that in the Monkey the acceleration of respiration is only brought about by a few fibres at the situation of the genu, while the phonatory adduction of the cords is subserved by a few fibres in almost exactly the middle of the posterior limb of the capsule.

It may be interesting at this point to state that this is what might have been expected, inasmuch as it is just at this region that the observations before referred to* show that these fibres similarly subserve the movements of the tongue and pharynx.

(2). In the Dog the condition appears to prevail which is found in the Cat, about to be described, that is to say, that excitation of almost the whole of the anterior limb produces acceleration of respiration and that adduction begins to be produced at the genu. (*g*, fig. 7.) We will therefore proceed to describe the condition as it is seen in the Cat, in which animal we have investigated this point closely.

(3). In the Cat the most anterior effect is acceleration. This commences, according to the level of the section, either just in front of, or just opposite to, the anterior extremity of the caudate nucleus, and it persists throughout the front half of the anterior limb of the capsule. (1–3, fig 6.) At this centre point of the anterior limb the respirations become intensified, abduction much more notably marked, and in some cases persistent. (See fig. 6.)

Exactly at the genu, or in many cases just in front of it, *i.e.*, opposite the posterior extremity of the caudate nucleus (*vide* figs. 6 and 7), adduction is very sharply marked.

In most cases the bundle of fibres subserving this phonatory adducton is small, in others—possibly owing to slight difference in the obliquity of the section—

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* BEEVOR and HORSLEY, *loc. cit.*

adduction can be obtained from almost any point in the front half of the posterior limb, and in some cases there is, in addition, diminished abduction as far as the hinder end of the lenticular nucleus.

We are now in a position to discuss further the points postponed in speaking of the relative representation of respiration and phonation in the cortex.

It will be seen that in the capsule we have thus arranged antero-posteriorly the fibres subserving portions of the cortex which are placed rather transversely to the mesial plane than parallel to it, and that, as might be expected, the acceleration which is the highest and most anterior is similarly most anteriorly placed in the internal capsule. To return now to the polypnœa obtained by exciting these fibres in the internal capsule, OTT describes how by using needle electrodes he was enabled in the Rabbit and in the Cat to evoke exactly the same polypnœic condition that we produced in the above-mentioned circumstances. He was exciting the fibres, according to our view, of the capsule leading from the cortex. The very important results which he obtained, and the bearing which these facts have upon heat regulation and the pathology of fever will be found in his communication.

The next point is of course the relation which the laryngeal movements thus described bear to the other respiratory movements. Where we had to do simply with the acceleration of the laryngeal movements, of course such acceleration was exactly synchronous with acceleration of thoracic respiratory excursions.

But when the fibres posterior to this were excited, viz., those which produced persistent abductory movements of the vocal cords, then it was noticed that while the larynx remained wide open in persistent abduction as long as the excitation lasted, nevertheless the respiratory excursions of the thorax were greatly accelerated ; another instance of the differentiation of representation which we have before described. Next, when the vocal cords were brought together in adduction, as a rule there was some slowing of respiration and frequently movements of the opposite fore limb and swallowing.

Epilepsy.—After what has gone before it will be easily understood that we never saw epilepsy to occur from excitation of the internal capsule, and that all intrinsic laryngeal movements observed were strictly and completely bilateral.

Medulla oblongata.

We come now to the consideration of the laryngeal movements produced by direct excitation of the medulla oblongata. It will readily be conceded that here there is more than usual difficulty in discriminating between those laryngeal movements which subserve the process of respiration and those which have to do with phonation. We do not propose in the present case to do more than indicate what those movements are, and to leave the consideration of the shares which they individually possess in serving the two functions before mentioned to another time.

But we may observe incidentally that we are naturally inclined to regard that small area of the medulla from which adduction can be produced as a transit-station for the corresponding cortical, and as a centre for reflex phonatory impulses.   Whether this is really so the future will show.   What is, for the present, of more practical import in this connection is how far the bulbar apparatus for the larynx is bilateral or unilateral in its composition.   It will be seen directly that we have obtained from the one half of the medulla oblongata both bilateral and unilateral effects, that is, as far as they go the bilateral effect is for the most part complete, sometimes, *i.e.*, when the electrodes are moved outwards, with a preponderance of movements on the part of the cord on the same side, and the unilateral effect is of course movement of the cord on the same side as that stimulated.

We shall discuss the results then from these two standpoints, viz., (1) the bilateral effect, and (2) the unilateral effect.

We have only made observations on the Carnivora, and in this respect it is interesting and important to note that the results in the Cat are not different from those in the Dog.   In fig. 9 is given a photograph of the floor of the fourth ventricle in the Dog, the calamus scriptorius (*c.s.*, fig. 9) is well seen bounded laterally by the fasciculus gracilis passing up on each side towards the restiform body with the ala cinerea (*a.c.*, fig. 9) well marked.   The cerebellum has been divided and the right half removed, so as to expose the lower surface of the inferior peduncle completely, and only the upper part of the inferior vermiform process is seen to be in contact with the floor of the ventricle.

Within the area thus exposed are included all the movements of the vocal cords which may be reasonably attributed to phonation, and also almost all which are connected with respiration.   However, in accordance with what we have stated above, we shall now proceed to describe the movements of the vocal cords independently of the function they may be supposed to subserve.

1. *Adduction.*—Closure of the glottis is sharply and invariably produced when the upper margin of the calamus scriptorius, *i.e.*, the nucleus of the spinal accessory nerve, is stimulated, and the same thing is observed if the electrodes are carried outwards and upwards along the margin of the posterior pyramid for about 3 mm. from the middle line.   This action is distinctly bilateral.

Immediately above the calamus, in the situation of the ala cinerea, abduction is invariably produced.   This movement, like that of adduction, is also absolutely bilateral and persistent.   It extends upwards as high as the line drawn horizontally across the fourth ventricle at the transverse level of the upper end of the representation of the adduction.   (See fig. 9.)   Above this point, especially in the middle line as high as the level of the eighth nerve, excitation evokes movements of the cords about the cadaveric position (see dotted area, fig. 9), which may be looked upon as either a confusion or imperfect adduction.   This movement is strictly bilateral.

We have obtained unilateral adduction of the vocal cord on the same side by

excitation of the restiform body in a vertical direction beginning below, opposite and close to the upper end of the region just referred to as giving bilateral adduction, and proceeding along the outer part of the floor of the fourth ventricle as high as the centre of that cavity. (See fig. 9.)

Whether the adduction of the vocal cord of the same side obtained by excitation in the line thus drawn is due to excitation of the nucleus reserved for the movements of the vocal cord of the same side, or whether it is rather excitation of those fibres which run through the substance of the medulla, to appear at once as the roots of the vago-accessory nerves, remains doubtful, though we are naturally more inclined towards the latter view. However, we do not consider that any positive opinion can at present be expressed upon this point.

Abduction.—It has been mentioned above that the movement of abduction is obtained over the situation of the upper two-thirds of the ala cinerea and also from about the origin of the eighth nerve. This latter area is evidently more important and extends upwards to close to the debouchment of the aqueduct of SYLVIUS. (See fig. 9.) Further, this movement of abduction is persistent, and is better marked a short distance, about 2 mm. from the middle line, than in the middle line itself. We have never observed unilateral abduction similar to the unilateral adduction before mentioned.

It may be that fibres exist in the tegmentum or crusta which directly connect together the cortical and bulbar mechanisms now described, and that, as before suggested, the observations which would point to respiratory mechanisms existing in the basal parts of the cerebral hemisphere and in the mesencephalon have been the results of stimulation of such fibres by those looking for respiratory centres in these positions.

Before we had recognised completely the degree to which the etherisation must be carried we obtained some results upon the larynx when the dorsal surface of the spinal cord, *i.e.*, below the calamus scriptorius, was stimulated, but we saw from the first that the results obtained here were probably those of a reflex nature, and further investigation has confirmed this opinion. For the effects were only produced in their entirety as the excitation approached successively the posterior roots, and diminished as the electrodes were moved to intermediate parts of the spinal cord. Further, the effects were far more in the nature of acceleration of thoracic respiration than primary changes in the movements of the larynx.

We never observed any after effects, *e.g.*, clonus, &c., upon cessation of the stimulation of the medulla oblongata.

Influence of Age.

Those acquainted with the history of previous research on the subject of the innervation of the larynx know well that, from the first, experimental investigation has shown that there exists, according to the age, a great difference in the same

species of animal in the representation in the nervous system of respiration and phonation respectively.

The need for efficient respiratory action on the part both of the larynx and the thorax is much greater in the younger animal than in the adult, and is greater, as LEGALLOIS more especially has shown, in the Cat and the Horse than in any other animal, especially more than in the Dog. The present part of our paper will be devoted to summarising the facts that we have observed and the differences noted according to the age of the animal, and we shall group the results, as before, according to the species, beginning again with the highest.

(*a.*) *Monkey.*—In the Monkey we have only incomplete observations owing to the difficulty of obtaining many Monkeys, but all our observations went to show that phonatory adduction was less represented in the young animal than in the older adult.

(*b.*) *Dog.*—In the Dog these differences are also marked. In the young Dog even with strong currents we have frequently only obtained from the focus of laryngeal phonatory representation a high degree of acceleration of respiration with marked diminution of abduction, but no complete adduction ; in fact in the cortex of such an animal, while respiration is most efficiently represented, phonation is only imperfectly evolved.

(*c.*) *Cat.*—In the Cat, again, the same thing prevails in its entirety, with the additional point, viz., that this want of differentiation in the laryngeal movements was not confined to the larynx, but also prevailed in the centres for the movements of the limbs.

It is in accordance with points noticed by one of us (V. H.) in the representation of the limb-muscles in the cortex of the Monkey that a wider extent of cortex provides for a higher evolution or development of the motor representation, and probably the differences we have just been describing as produced by age are due to the greater or less provision of cortical material for subserving the particular function represented.

Conclusions.

The conclusions which we feel may legitimately be drawn from the foregoing experiments—and which, we may be permitted to add, are based not only upon the results of the work just described, but also upon much which we have not alluded to, and which will be described in subsequent communications—may best be put forward upon the plan according to which we believe the central innervation of the laryngeal movements is arranged.

In consequence of the laryngeal muscles serving, as particularly insisted upon by one of us (F.S.), in this connection, two distinct functions (1) phonation, (2) respiration, and in consequence of these two functions being, one connected with so-called purposive or volitionary acts, the other, on the contrary, related to those functions

which are commonly spoken of as belonging to organic life, it is easy to understand that our results bring us to the conclusion that the phonatory movements of the laryngeal muscles, *i.e.*, adduction, &c., are represented principally in the cortex, whereas the respiratory movements, *i.e.*, abduction, chiefly in inspiration, adduction less commonly and imperfectly in expiration, usually find their origin in impulses emanating from the bulb.

We have seen that the first of these functions, phonatory adduction, is represented most completely in the cortex the higher we ascend in the animal scale, and the more completely as the cortex is developed in the life of the individual, *i.e.*, by age, while *pari passu*, the function of respiration, although present in a minor degree, as evidenced by acceleration, is least represented where phonation is most completely developed. In further detail, adductory phonation is centred in the Monkey at the junction of the foot of the ascending frontal gyrus, and the base of the third frontal gyrus, in the Carnivora at the junction of the præcrucial, or præfrontal gyrus, with the anterior composite and coronal gyri. Respiration is represented in the higher animals only as acceleration, and that above and rather in front of the focus of phonation.

In the Cat only have we found a deliberate representation of abduction in the cortex.

Turning now to the lowest part of the central apparatus, the bulb, we see that there is therein situated a small but concentrated focus on each side of the middle line for bilateral closure of the glottis, and outside that a small area, excitation of which produces unilateral adduction of the vocal cord on the same side. Above this region of adduction there is also a representation of abduction, *i.e.*, inspiration.

Finally we have found that, connecting the cortical apparatus with the bulbar apparatus, there exist numerous connecting fibres which can be analysed as they pass down from the cortex through the corona radiata and internal capsule towards the bulb; that as far as they are concerned they simply subserve the functions which are proved to exist in the cortex; that the anatomico-functional arrangement of these fibres in the capsule enumerated from before backwards is :—acceleration of respiration, abduction, intensification, adduction ; and that their respective positions in the capsule are constant and strictly homologous in the different species of animals, according to the development of the cerebral hemispheres in the particular cases observed.

These results are in complete concord with the ideas concerning the whole central nervous mechanism which innervates the larynx, ideas which were put forward by one of us (F.S.) at the International Congress at Copenhagen in 1884. We are enabled to state this the more positively for the reason that we have in our hands the notes of a large number of ablation experiments which, by their control of excitation effects, prove to demonstration the correctness of the views and opinions we have enunciated.

DESCRIPTION OF PLATES.

PLATE 31.

Fig. 1. Photograph of right hemisphere of a Macacque's brain, *Macacus sinicus*.
    The total or absolute representation of the larynx is indicated by dotting,
the depth of the same showing the degree of representation.
       R. Fissure of ROLANDO.
     Pc. Præcentral sulcus.
      F. Focus of highest development of laryngeal representation, *i.e.,* pure
         intrinsic movement of adduction.
      v. The small secondary sulcus indicated by that letter.
    Sy. Fissure of SYLVIUS.

PLATE 32.

Fig. 2. Three-quarters view of Cat's brain, half natural size.
    c.s. Crucial sulcus.
     d. KRAUSE's centre of adduction.
    d'. Centre of adduction also noted by the authors.
     b. Focus of abduction.
Fig. 3. Base of Cat's brain, half natural size
Fig. 4. Side view of Cat's brain, natural size   }Lettering as above, fig. 2.
Fig. 5.   ,,   ,,   half natural size
Fig. 6. View of Cat's brain from above. The right hemisphere has been sliced
    horizontally, as described in the text, to show the internal capsule.
    olf. Olfactory lobe.
    p.f. Præfrontal fibres.
    Ca. Caudate nucleus.
     g. Genu.
   o.th. Optic thalamus.
    n.l. Lenticular nucleus (putamen).
    F.S. Fissure of SYLVIUS.
     1. Anterior limit of fibres excitation of which produces acceleration of
        respiration.
     2. Posterior limit of fibres representing acceleration and abduction
        and commencement of representation of adduction.
     3. Centre of anterior limb marking commencement of representation
        of abduction.

Fig. 7. Horizontal section of the right hemisphere of the Dog.   Lettering as in fig. 6.

Fig. 8. View of the bulb of a Cat, the cerebellum being split and drawn aside.

        C.q. Corpus quadrigeminum.

        cb.p. Posterior ditto and superior cerebellar peduncle.

        r. Restiform body.

        a.c. Ala cinerea.

        c.s. Calamus scriptorius.

Fig. 9. View of the bulb of a Dog, the cerebellum having been split and the right half removed.

        aq. Aqueduct of SYLVIUS.

        i.v.p. Inferior vermiform process.

     a.c., c.s. As above.

     The close oblique markings at the calamus and extending outwards indicate the focus of bilateral adduction.

     The vertical lines indicate the representation of bilateral abduction.

     The horizontal marks on the outer side indicate the area of unilateral adduction.

     The dotted area indicates the representation of incomplete adduction.

     All are photographs of the parts in the recent state.

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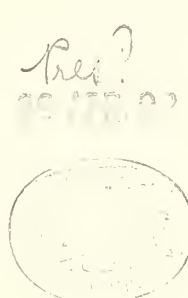
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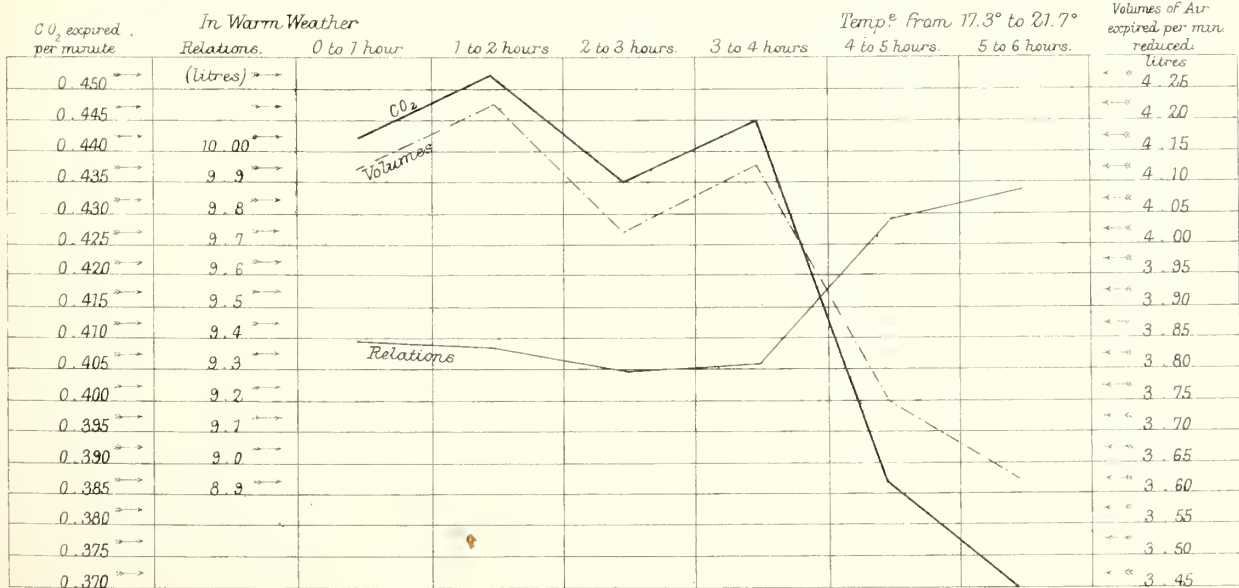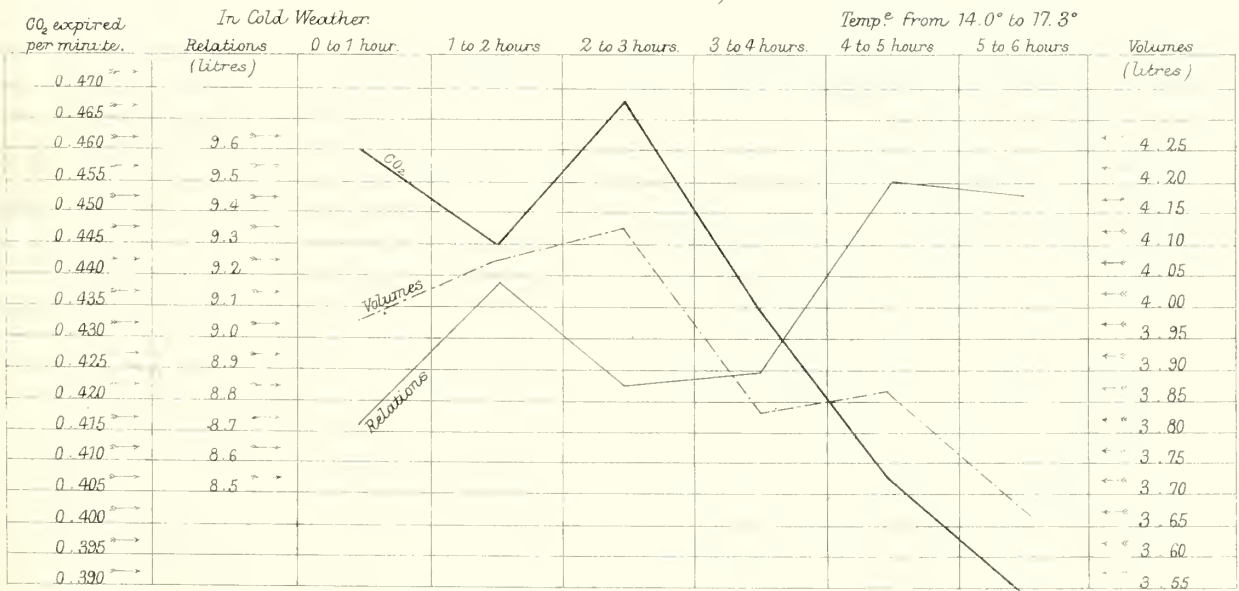
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India-Rubber Bag

Glass Tube

Glass Bottle

Instrument for the determination of Carbonic acid.

Chart showing that with a fall of the Barometer less Air has to be breathed to expire 1 gramme CO_2
(C. F. Townsend under experiment)

Bars.	772	769	766	763	760	757	754	751	748	745	Barometers

Relations

11.7
11.6
11.5
11.4
11.3
11.2
11.1
11.0
10.9
10.8
10.7
10.6
10.5
10.4
10.3
10.2
10.1
10.0

from 4 to 6 hours after food

2 to 4 hours

0 to 2 hours

Chart showing the weight of CO_2 & volumes of Air expired per minute together with their relations.
(C. F. Townsend)

Weight CO_2 Exp.d per minute	Relation of Weight CO_2 to Vol. air expired	Time after food	0 to 1 hour	1 to 2 hours	2 to 3 hours	3 to 4 hours	4 to 5 hours	5 to 6 hours	Volumes expired per minute
0.440	11.5	Before Breakfast							4.60
0.435	11.4								4.55
0.430	11.3		CO_2						4.50
0.425	11.2								4.45
0.420	11.1		Volumes						4.40
0.415	11.0								4.35
0.410	10.9								4.30
0.405	10.8								4.25
0.400	10.7								4.20
0.395	10.6								4.15
0.390	10.5								4.10
0.385	10.4								4.05
0.380	10.3								4.00
0.375	10.2			Relations					3.95
0.370	10.1								3.90
0.365	10.0								3.85
0.360									3.80
0.355									3.75
0.350									3.70
0.345									3.65
0.340									3.60

Fig 9.

Fig 1

Fig. 3

lat.cornu.

Fig 4

lat col
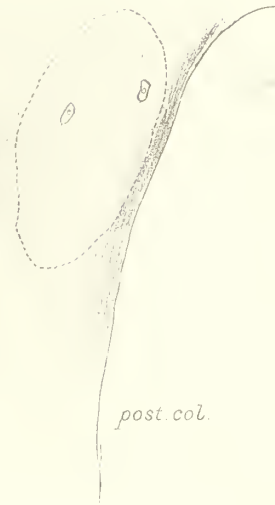
post.col.

Fig 13

Fig 11

Fig 12

Fig 10

Fig 7.

Fig 5.

Fig 2.

Fig 8.

Fig 6.

Fig 14.

Fig 16.

Fig 14.

Fig. 1.

oups.   I   II   III   IV   V   VI   VII   VIII

EYES
OPEN

EYES
TURN
TO
OPP. SIDE

MOUTH
OPENS
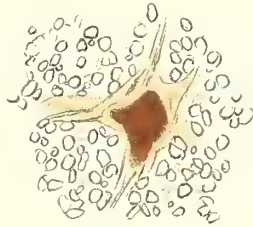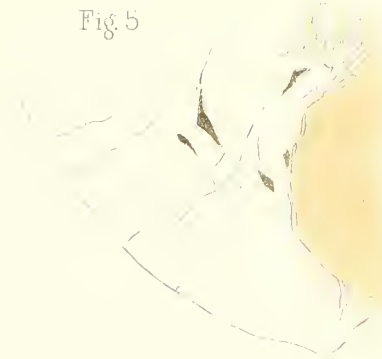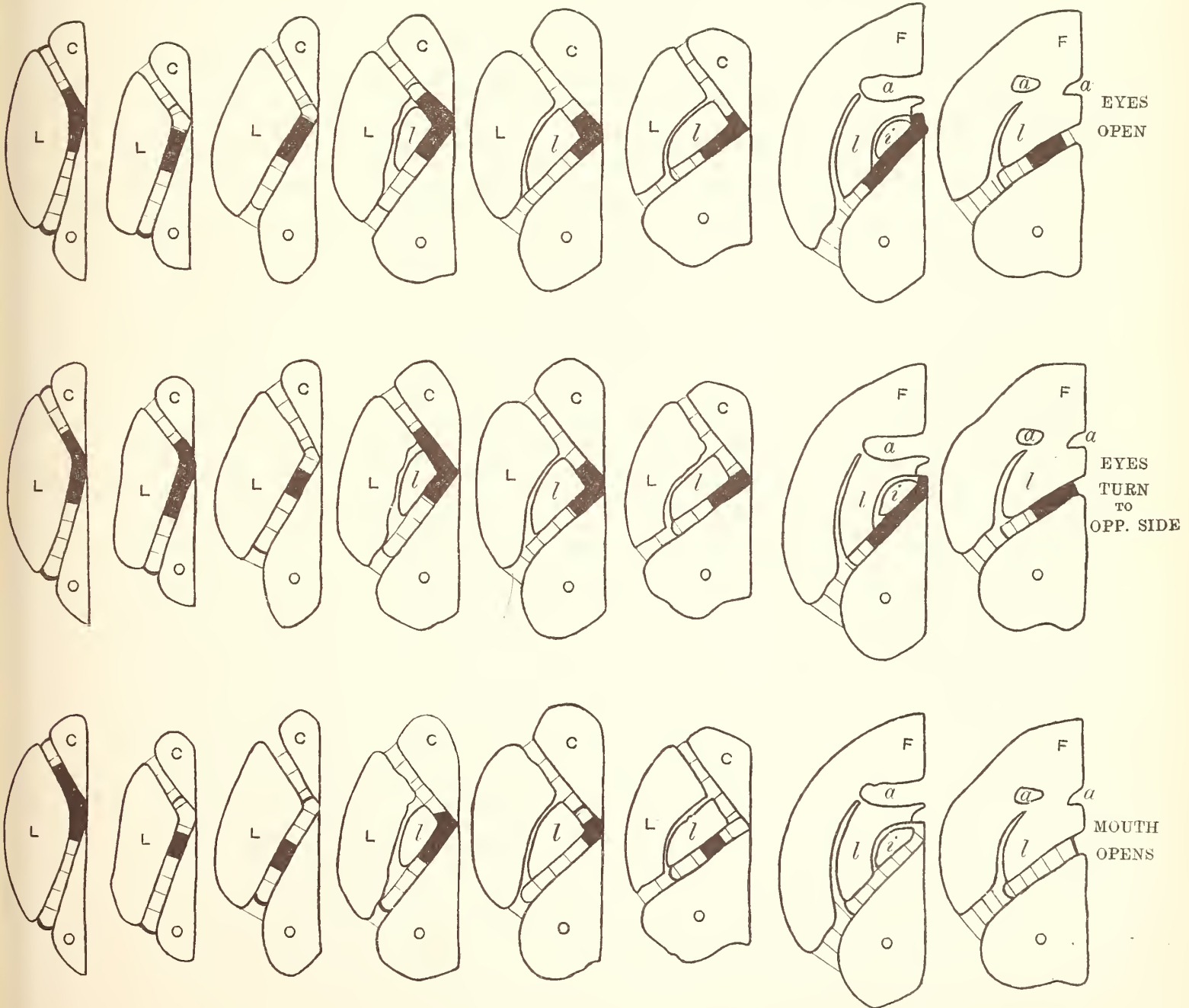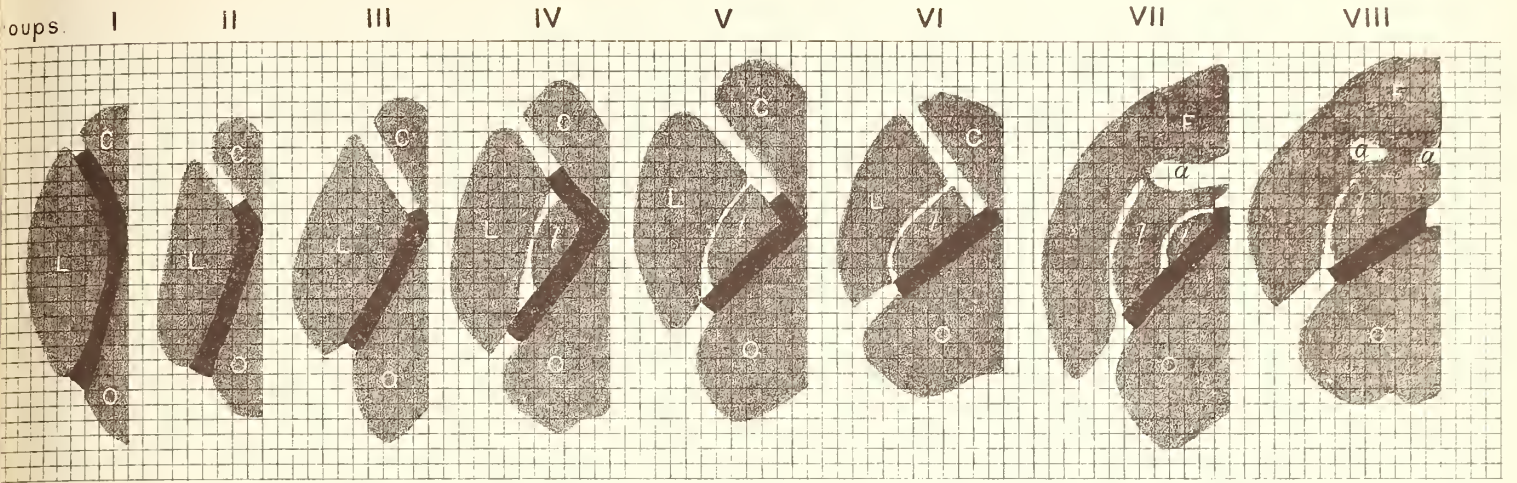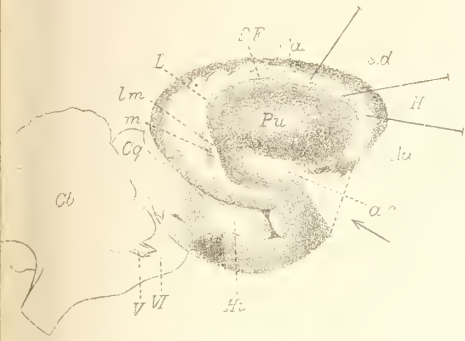
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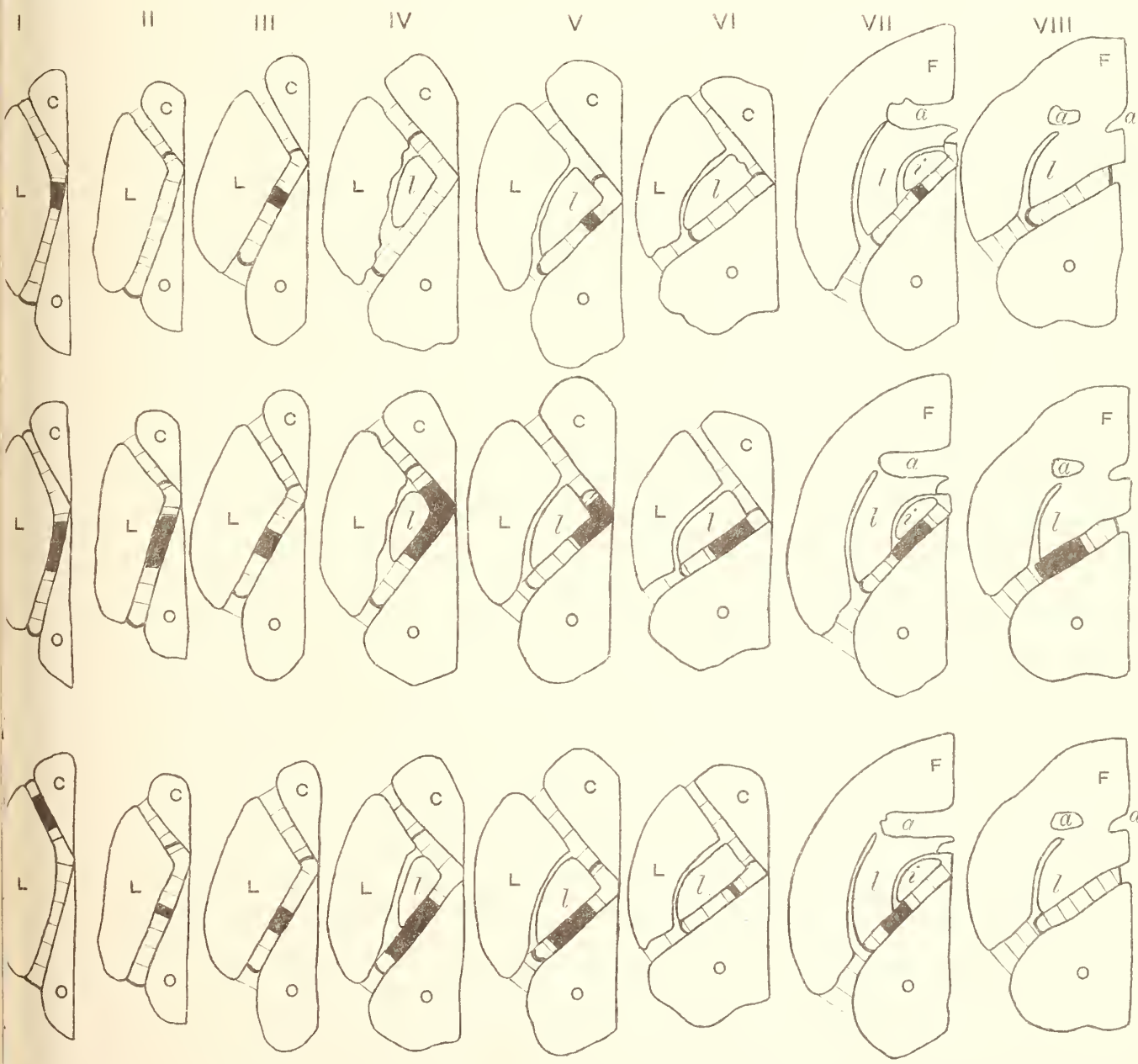
Fig. 5.

Fig. 4.

HEAD AND EYES SIMULTANEOUSLY TURN TO OPP. SIDE

HEAD TURNS TO OPP. SIDE

TONGUE

ANGLE
OF MOUTH

SHOULDER

ELBOW

WRIST
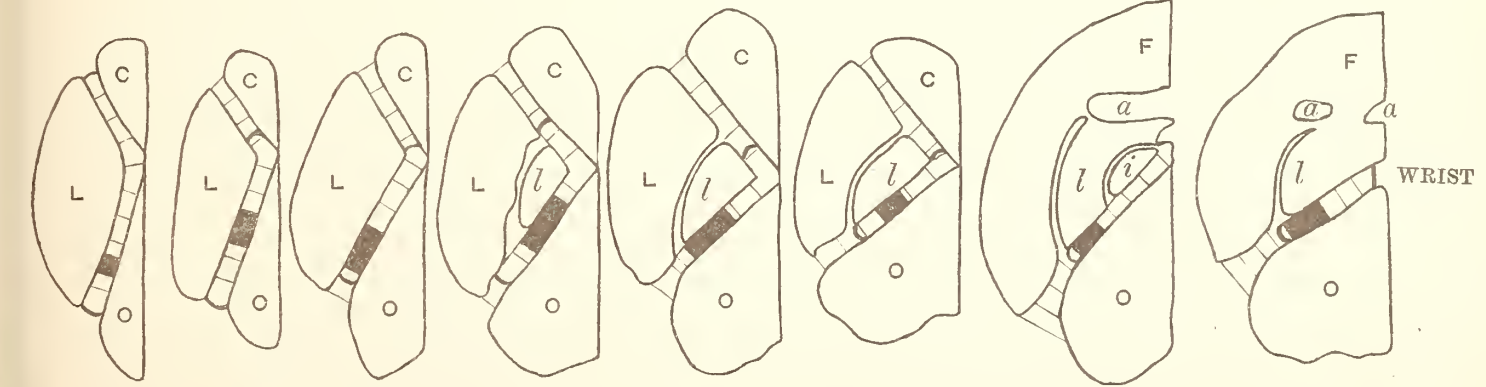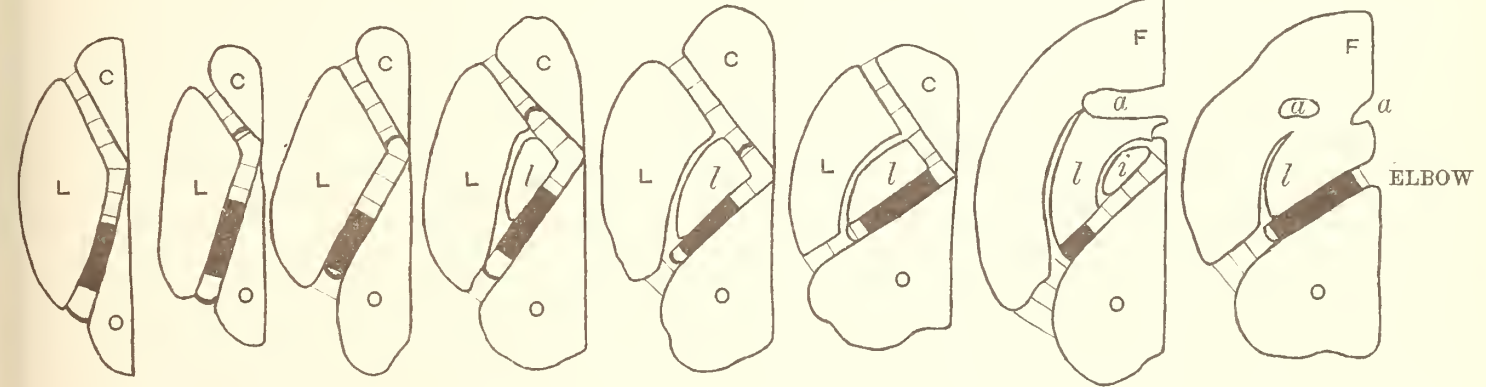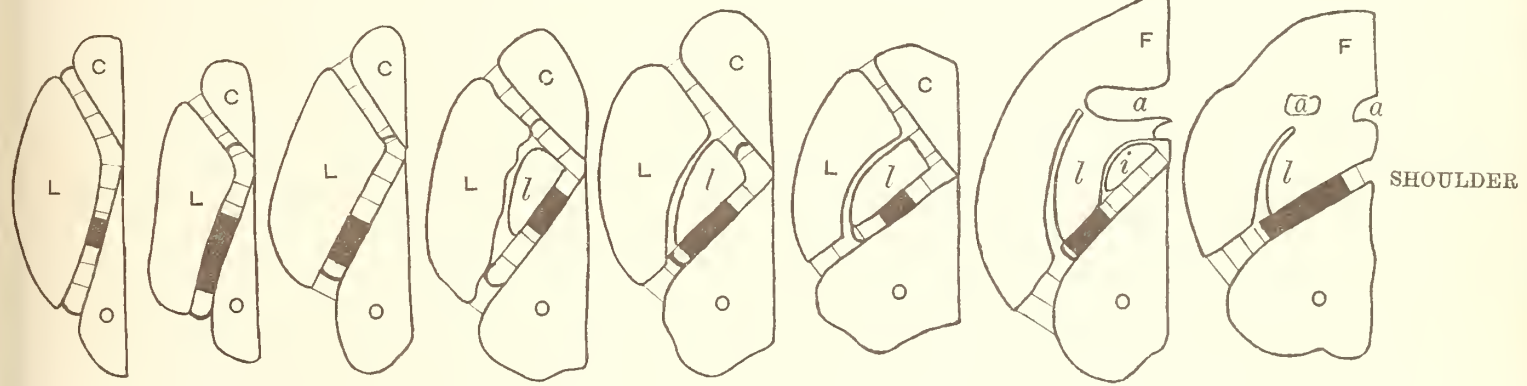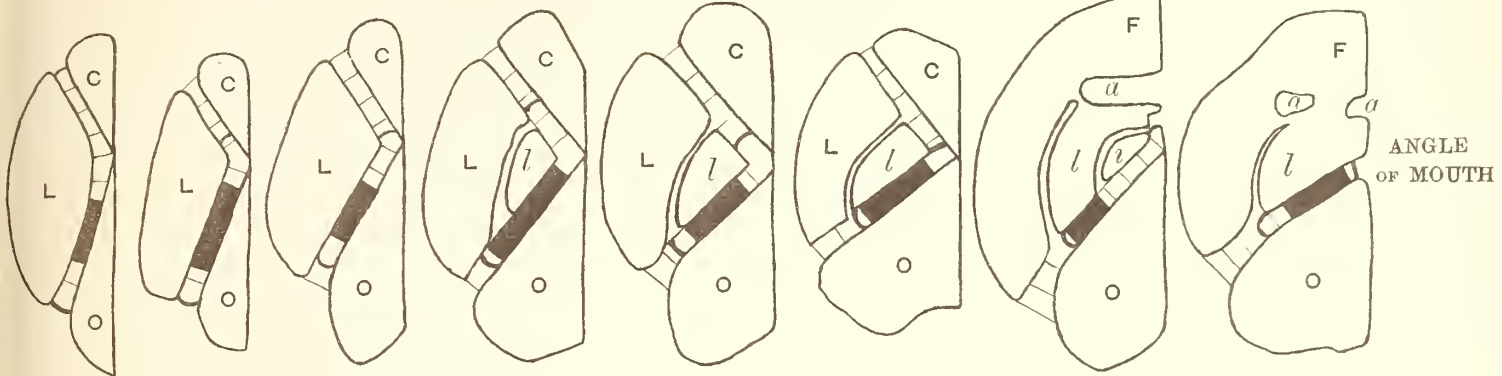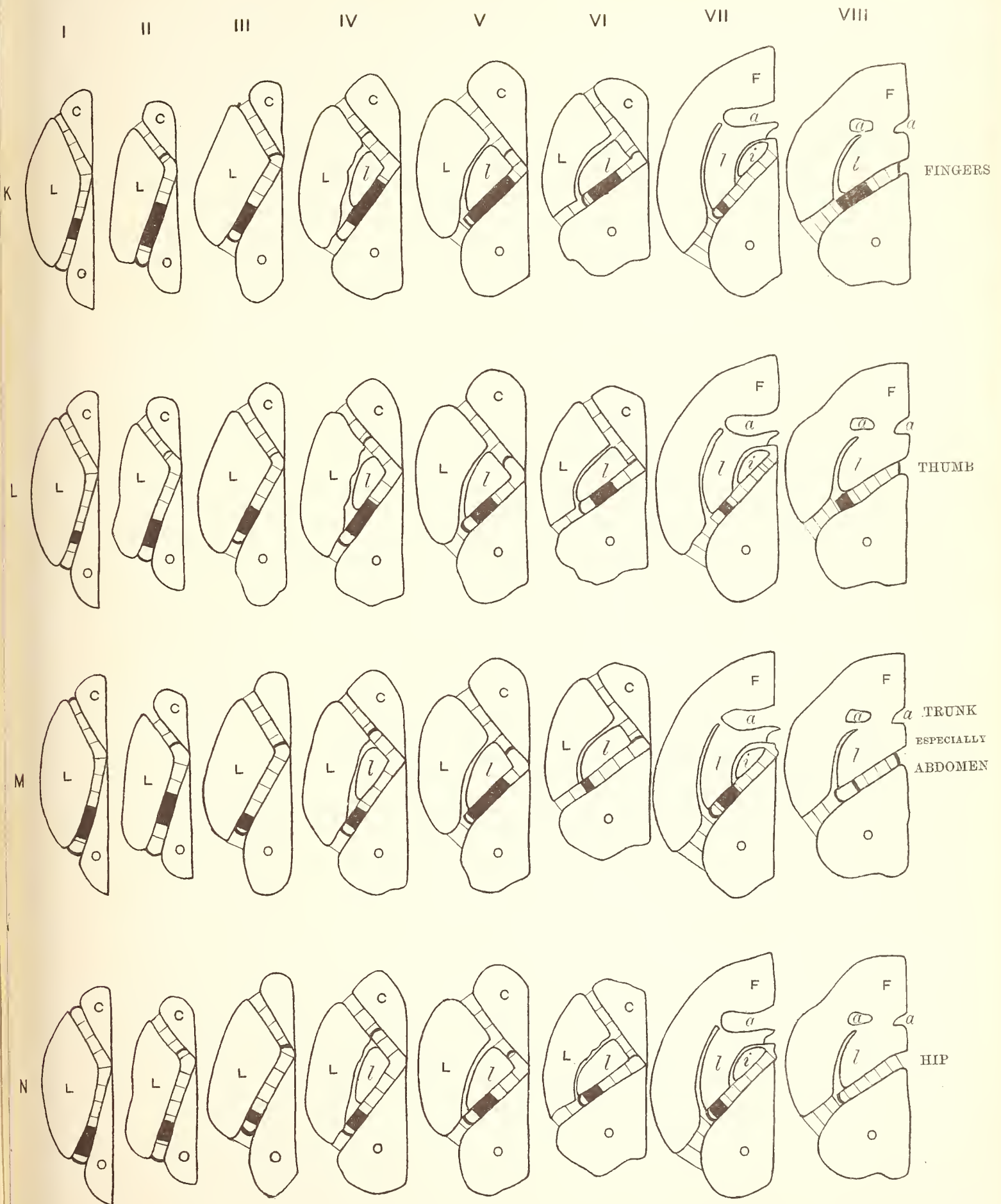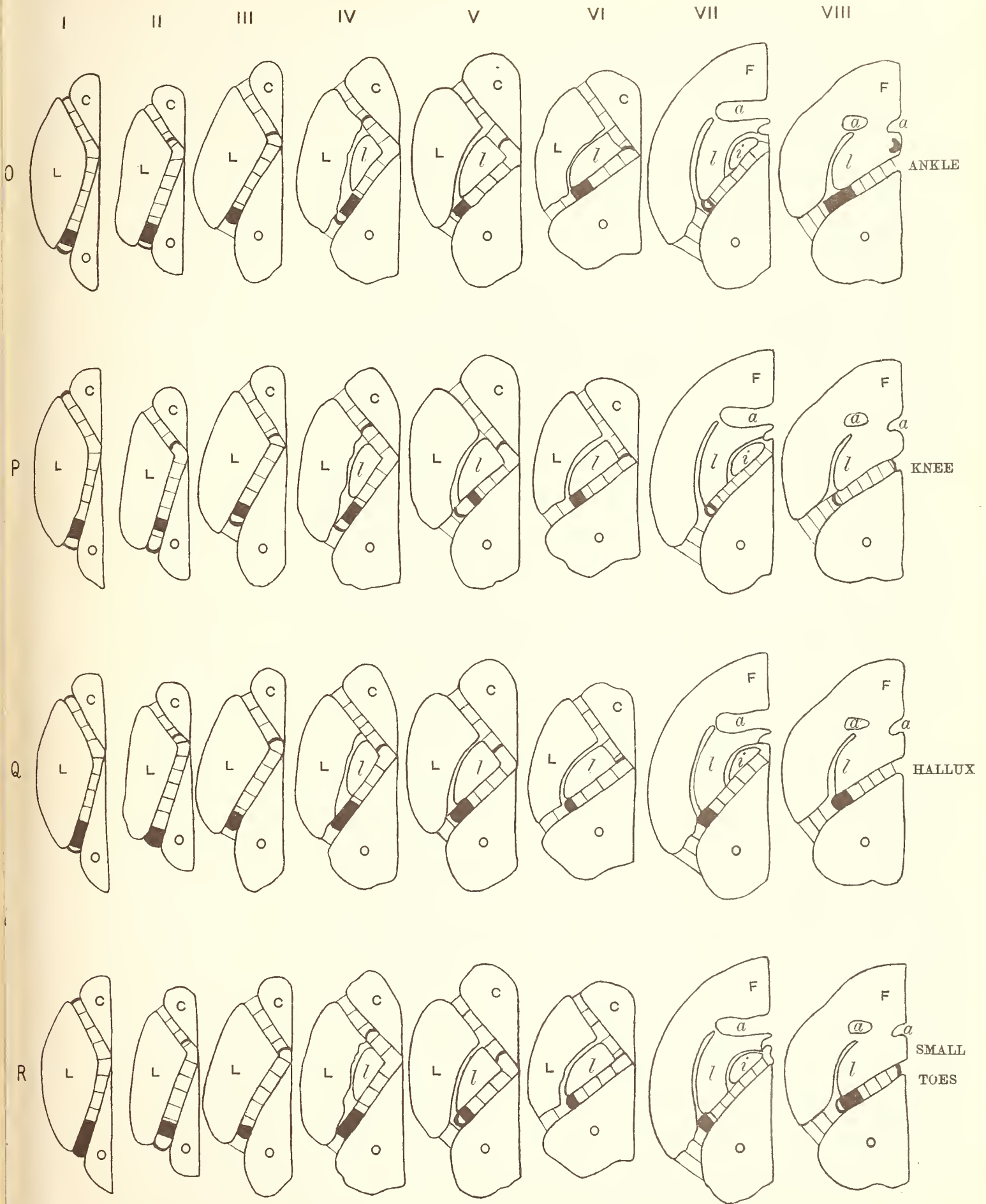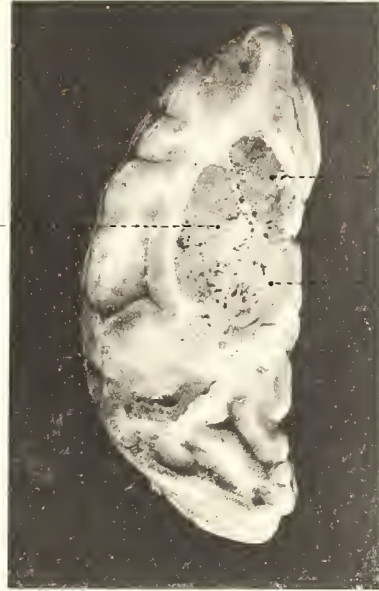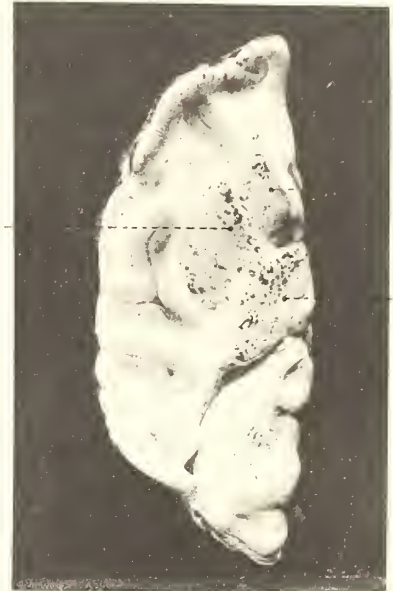
Group I.

Group II

Group III.

Group V.

Group IV.

Group VI.

P   Pf   Ca.

S.d

H

A.c
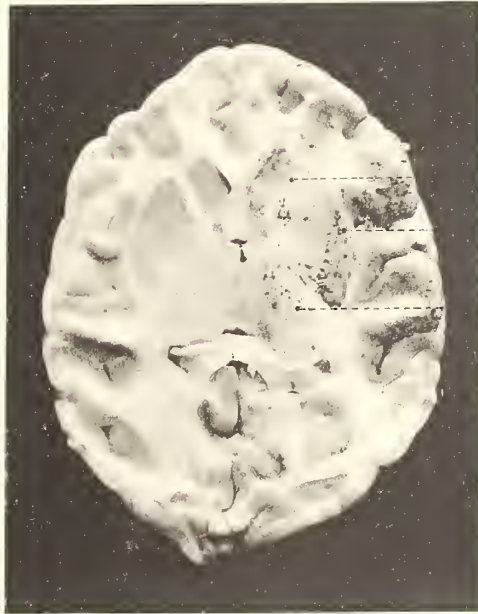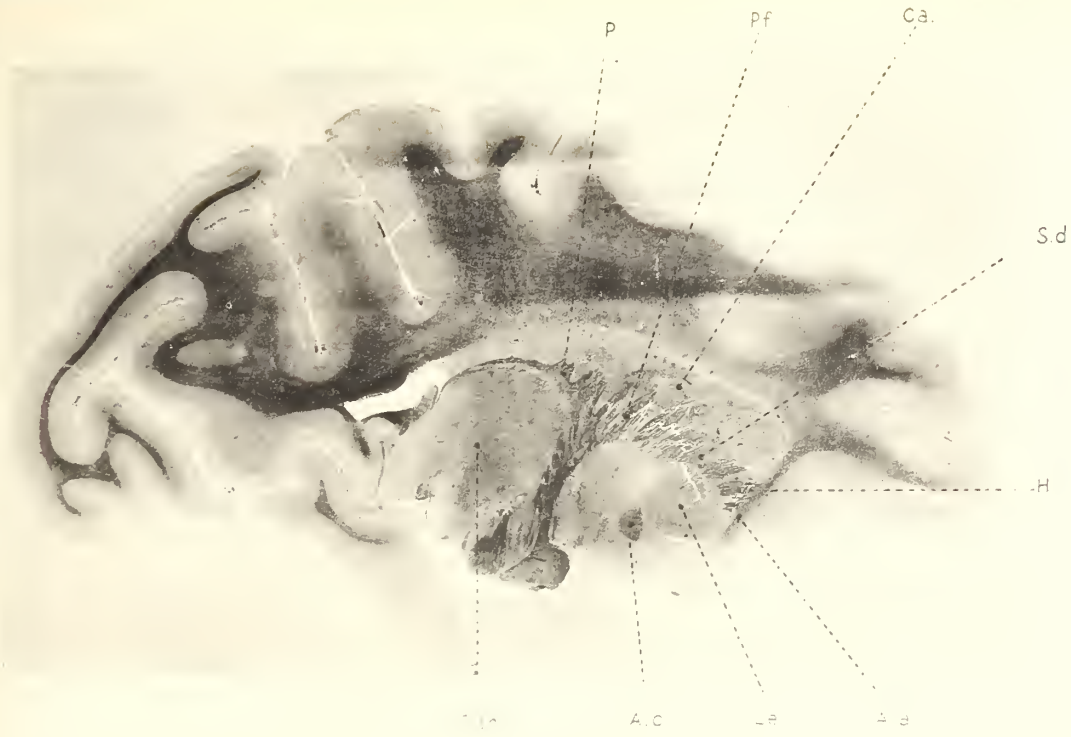
Fig 3.

Group VII.

Group VIII.

Le   Ca

Fig 2.

Le

Oth

Oth

Le

C c

S.l

Fig. 4.
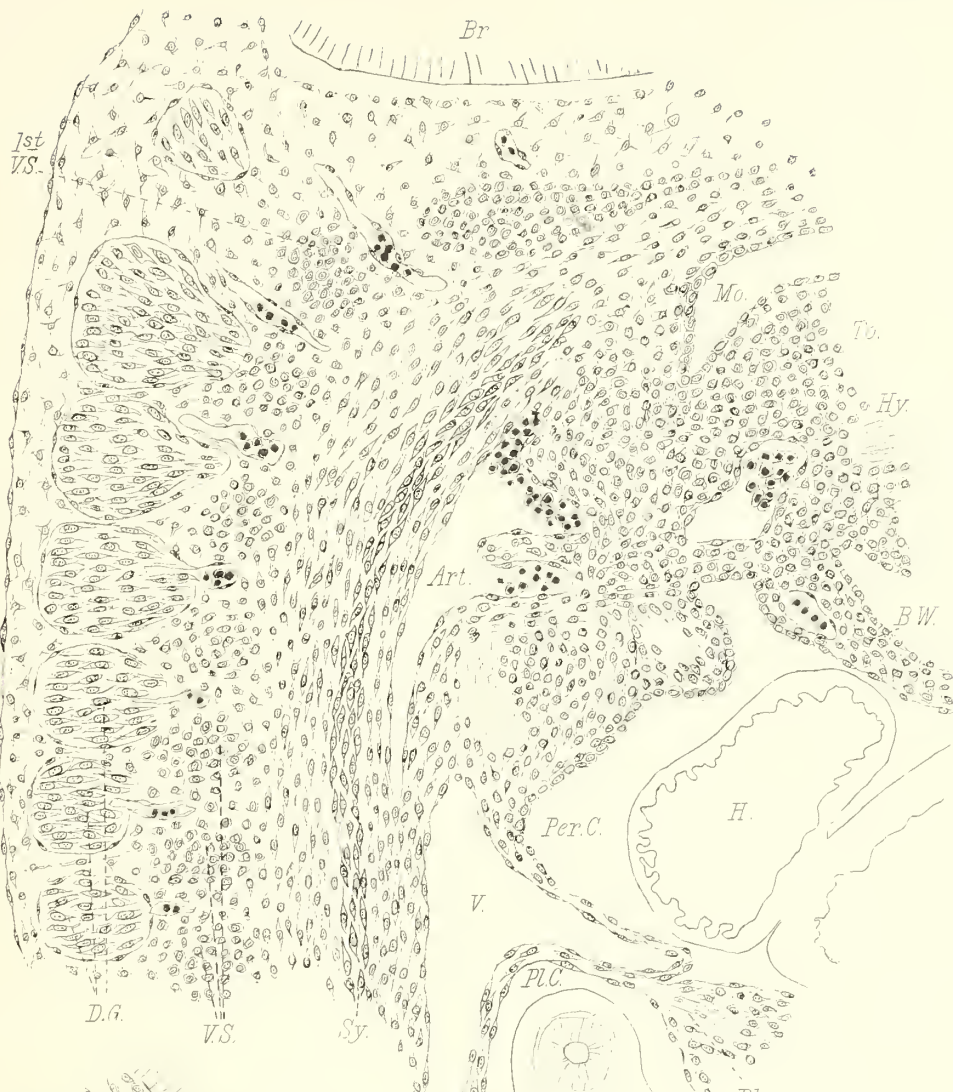
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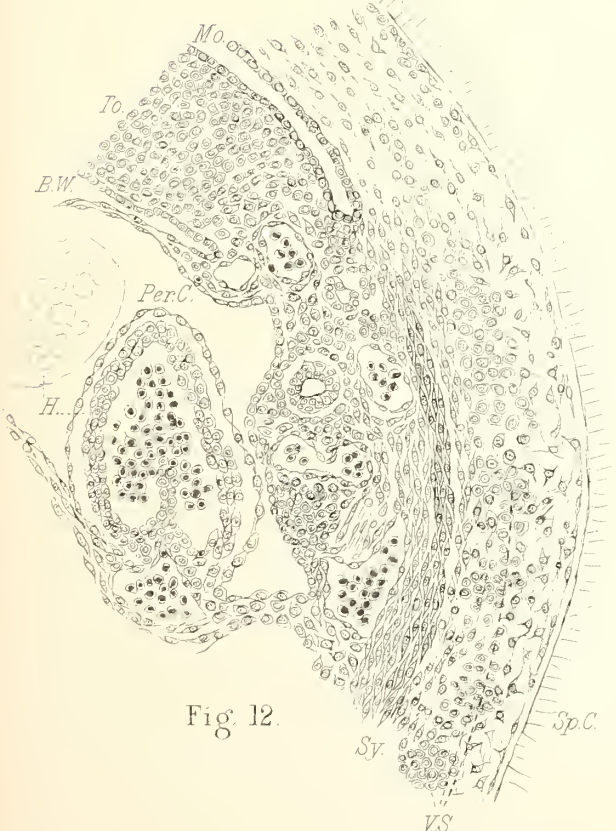
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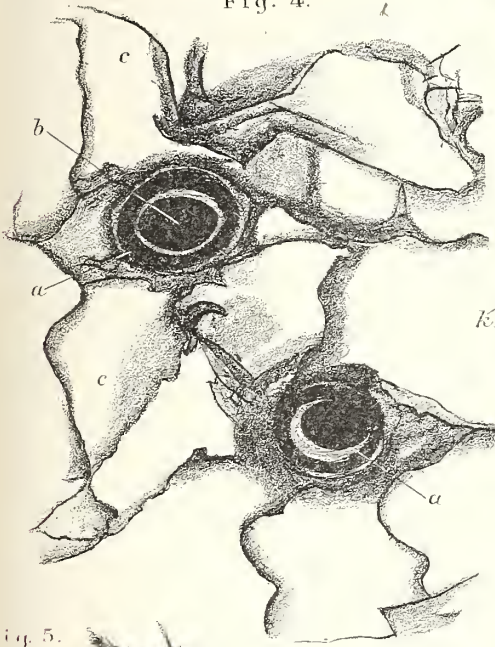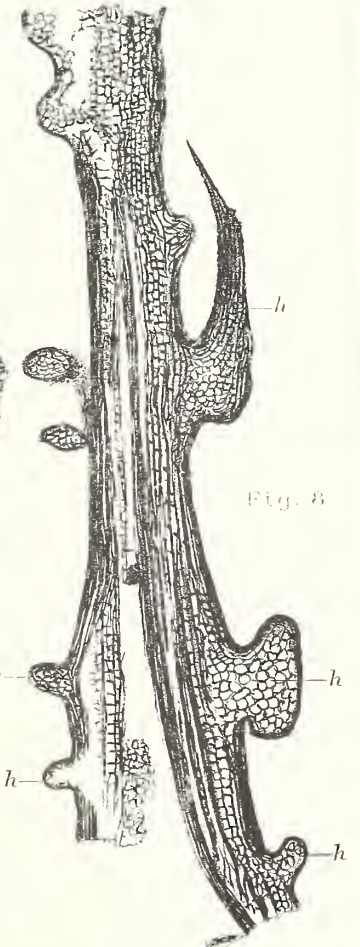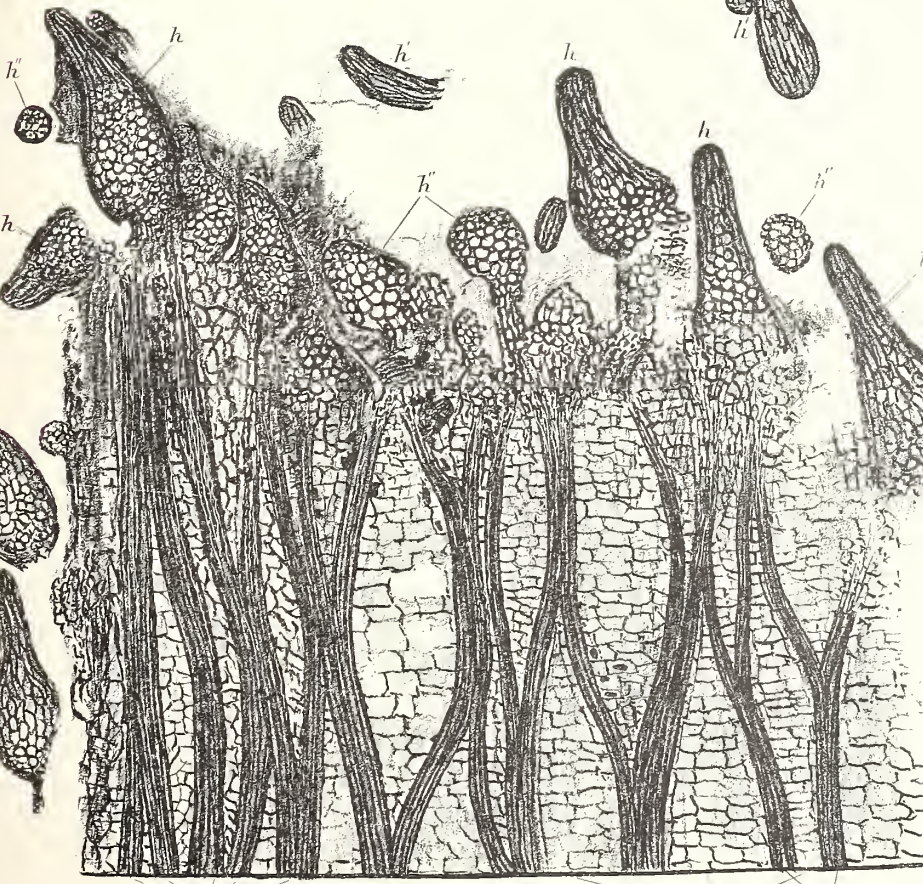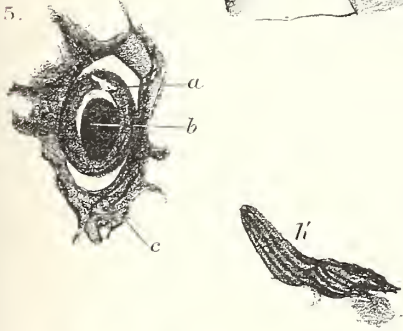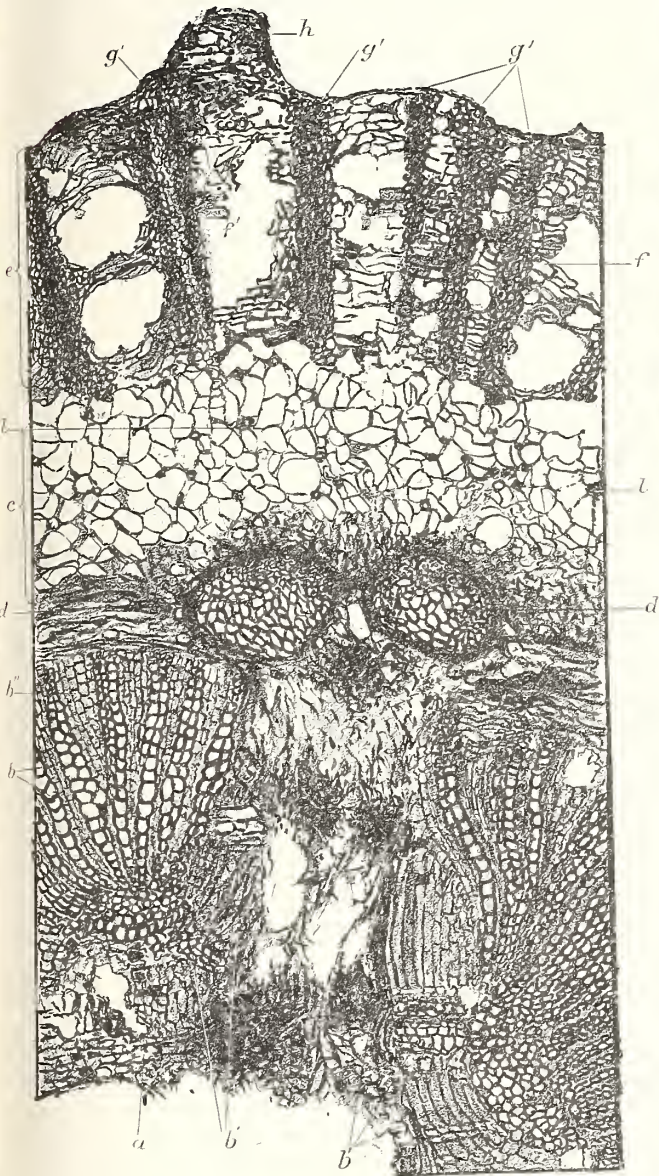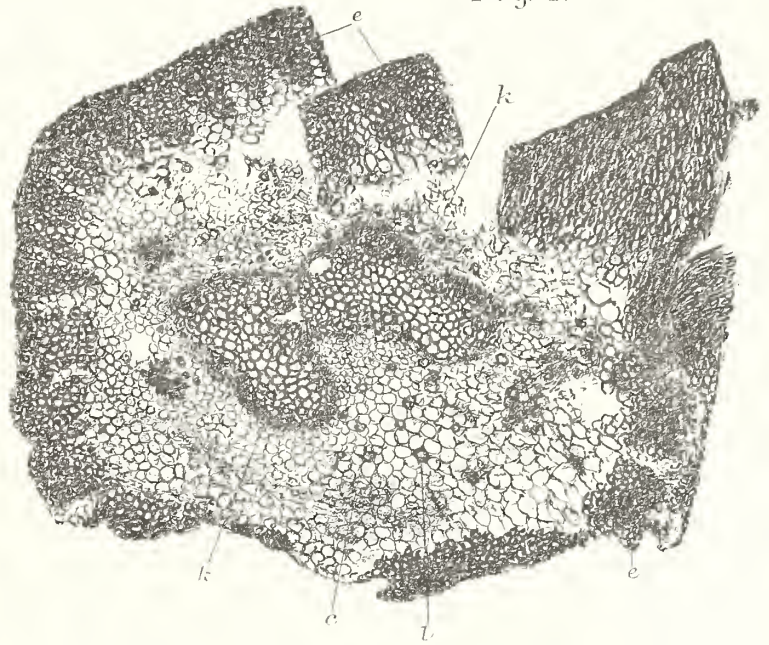
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Fig. 2.

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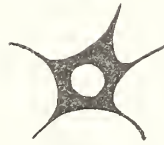
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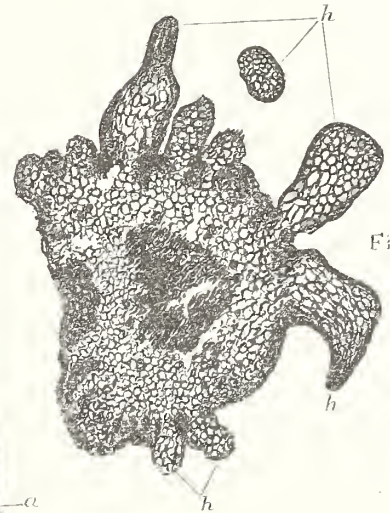
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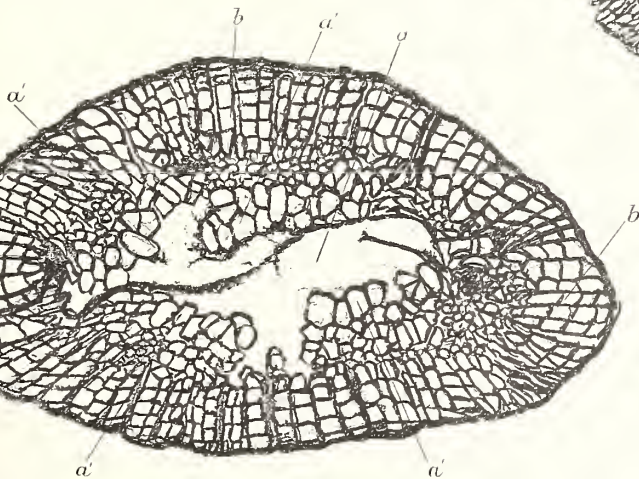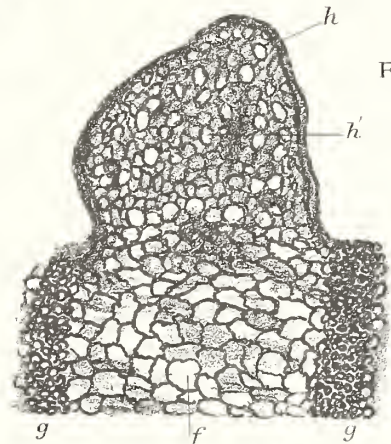
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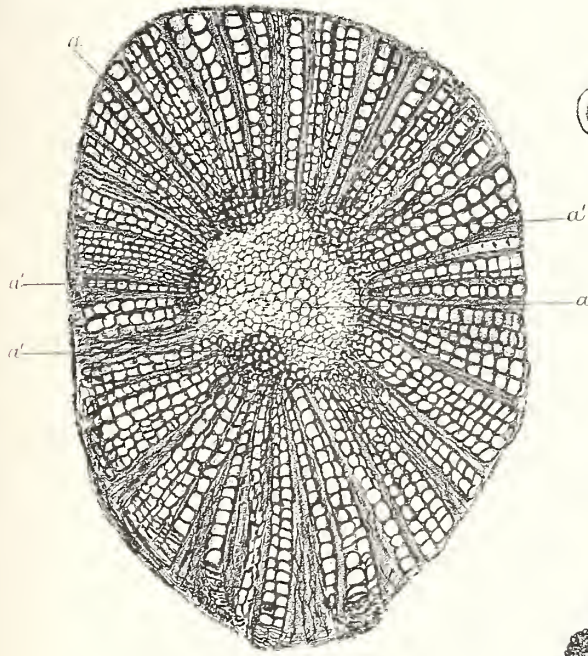
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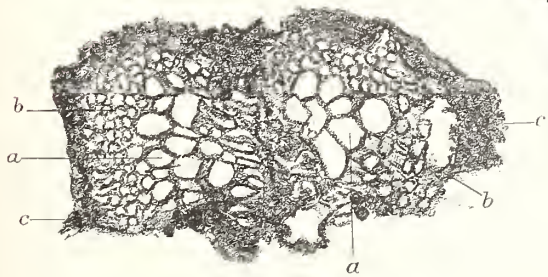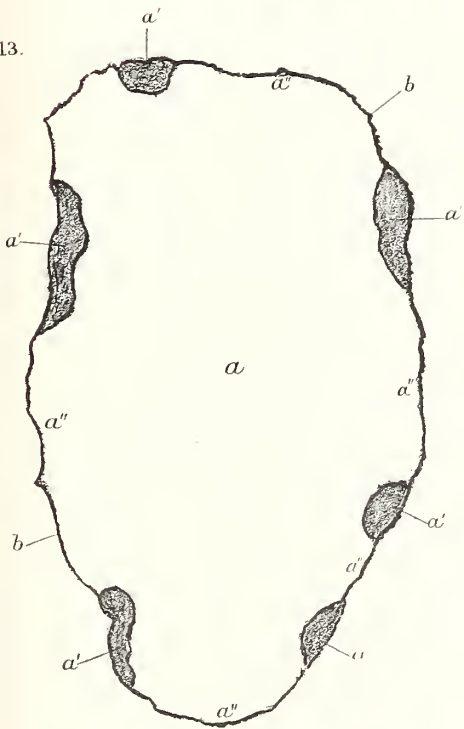
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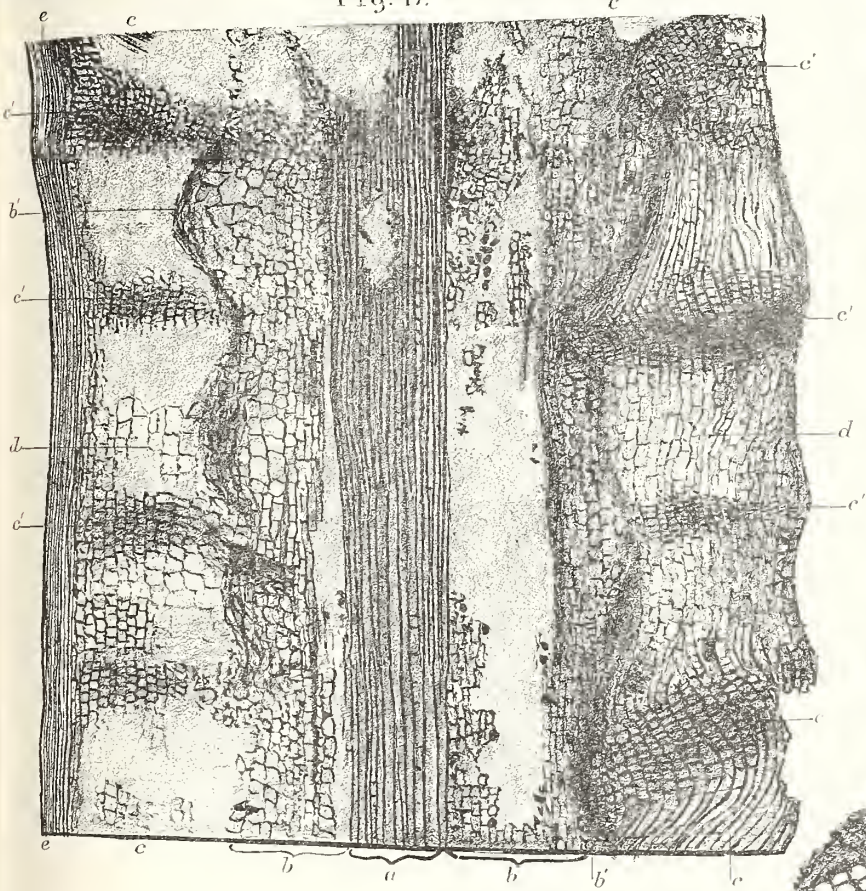
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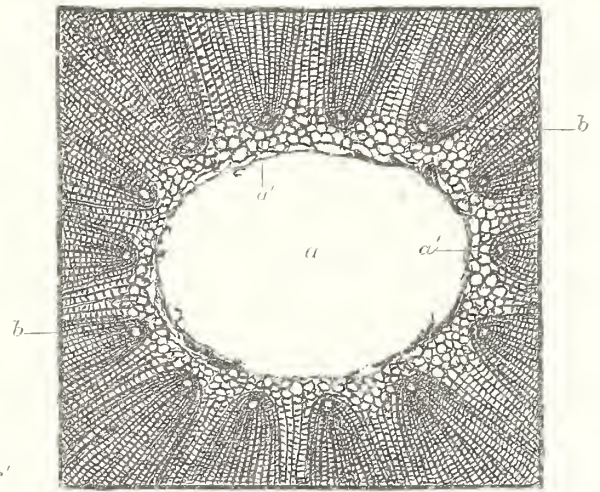
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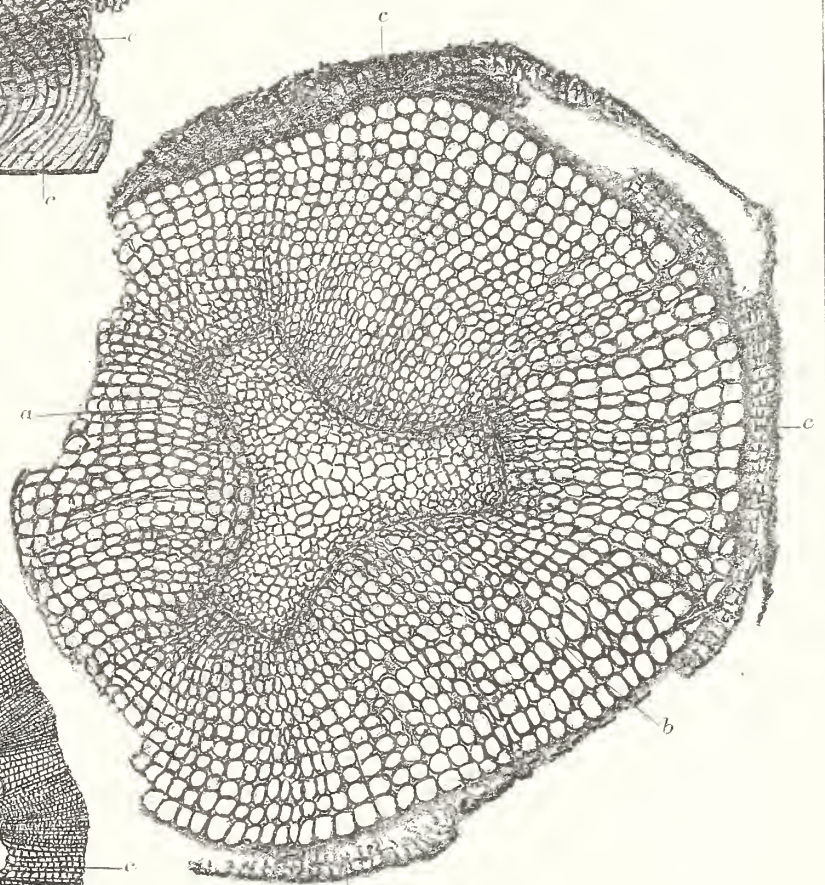
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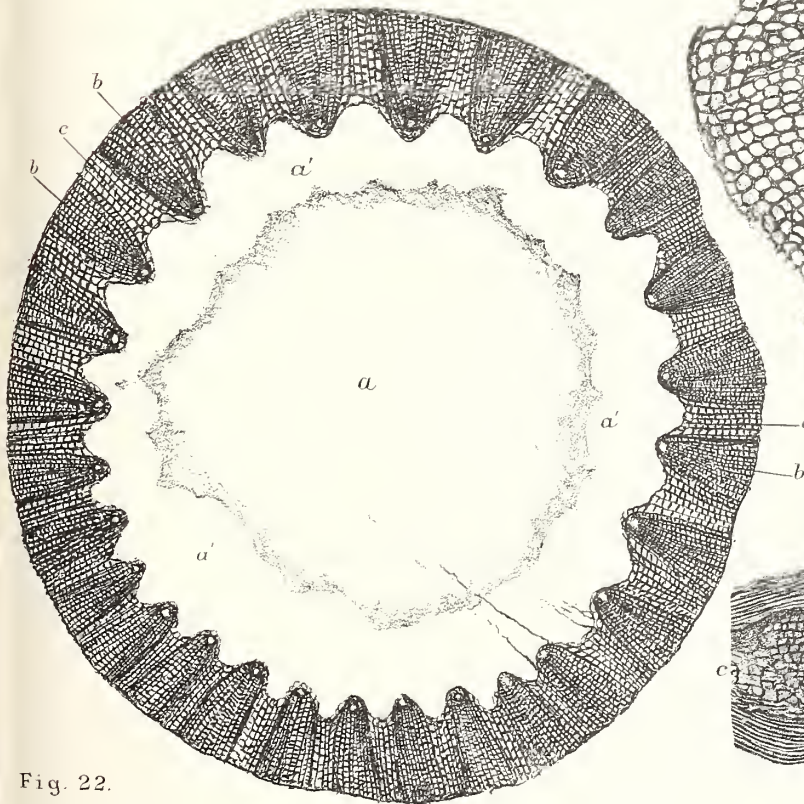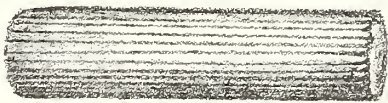
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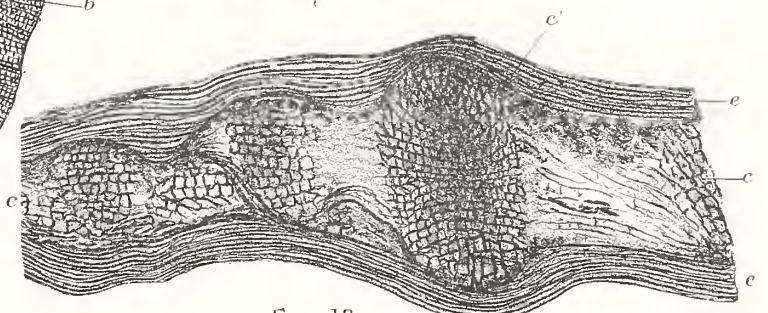
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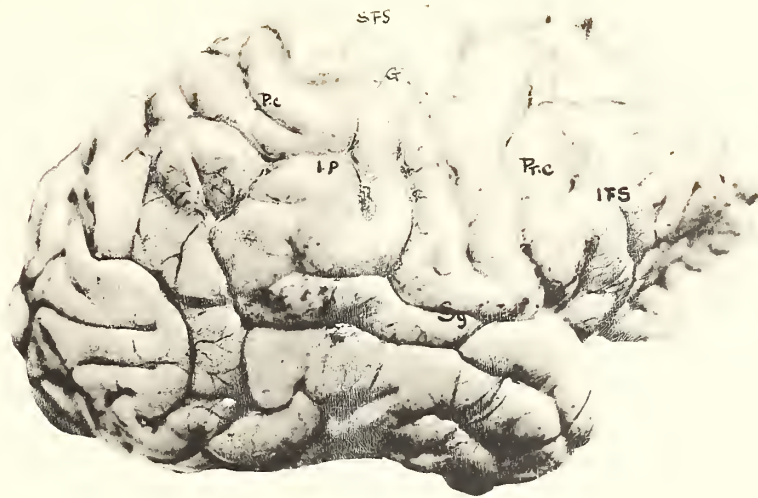
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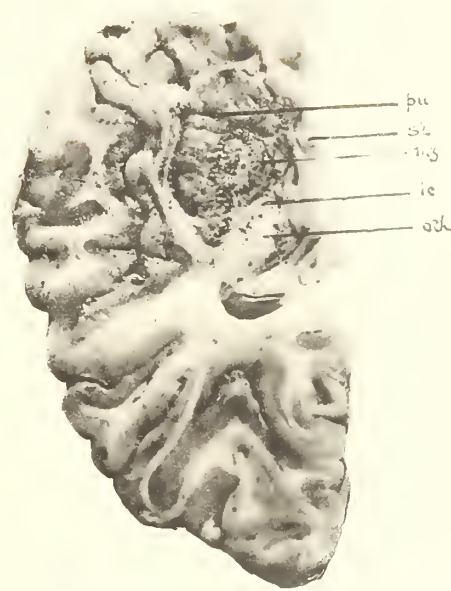
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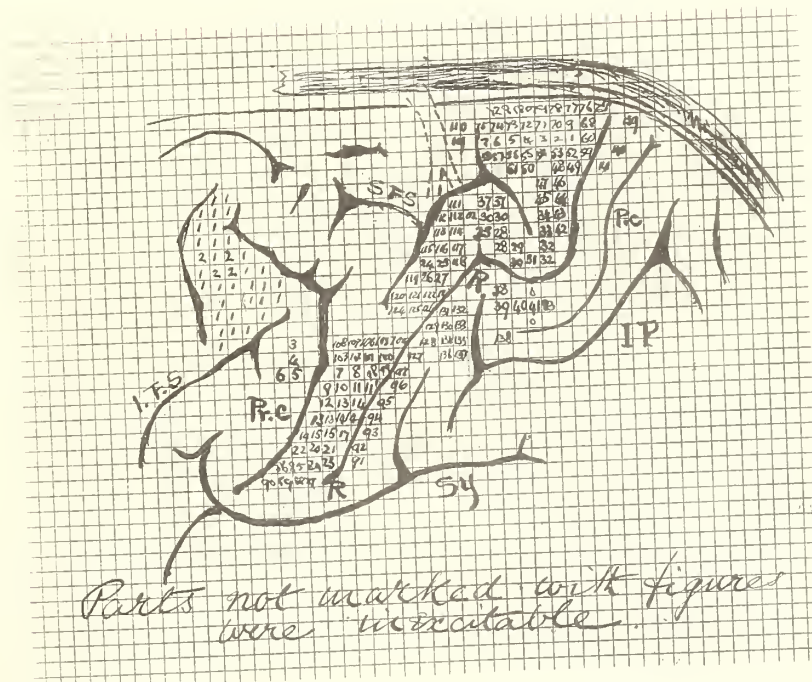
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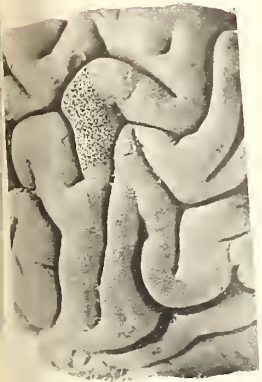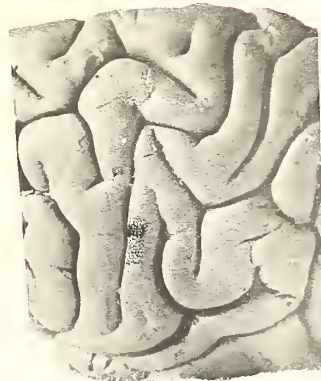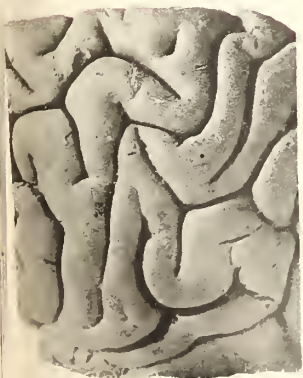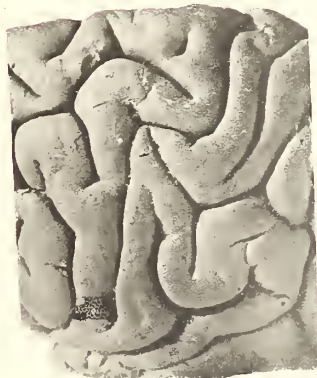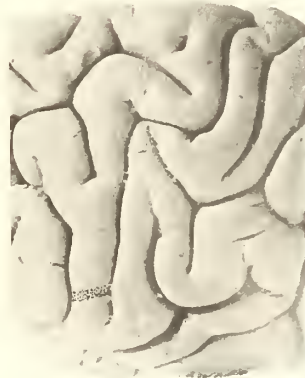
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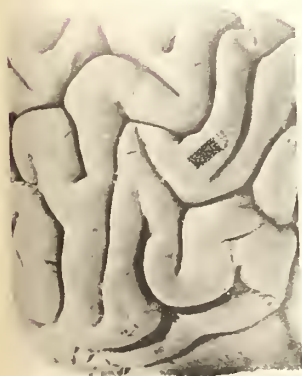
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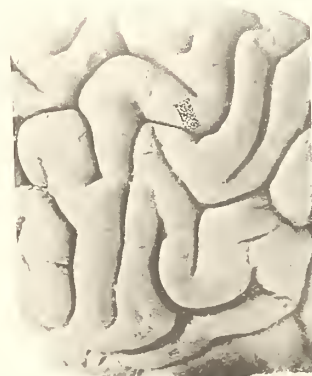
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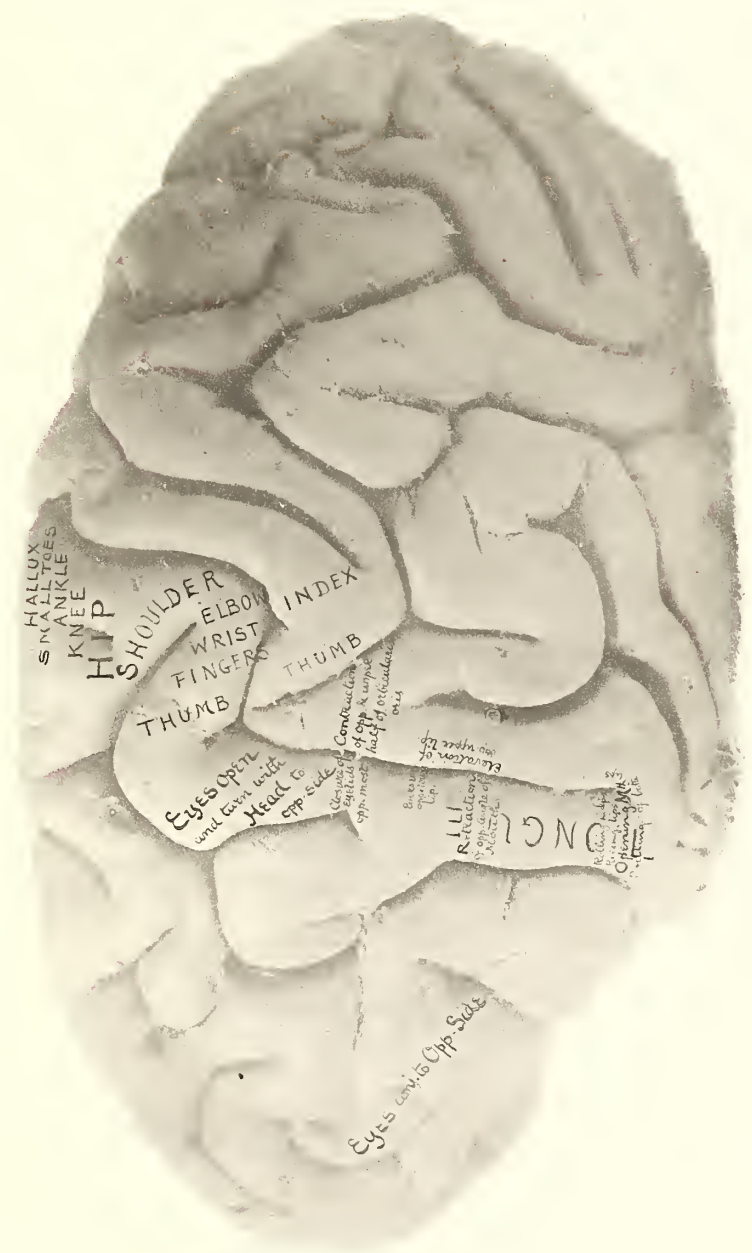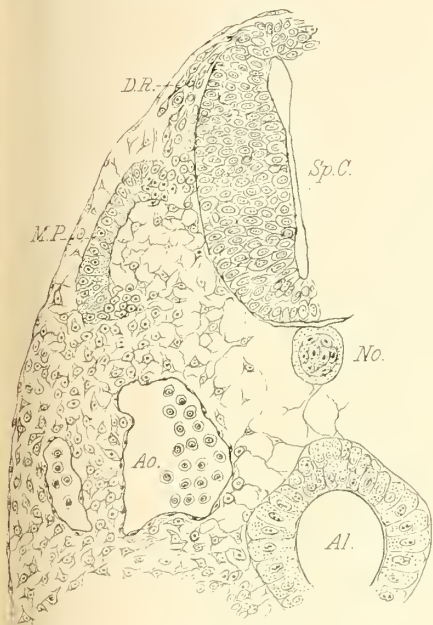
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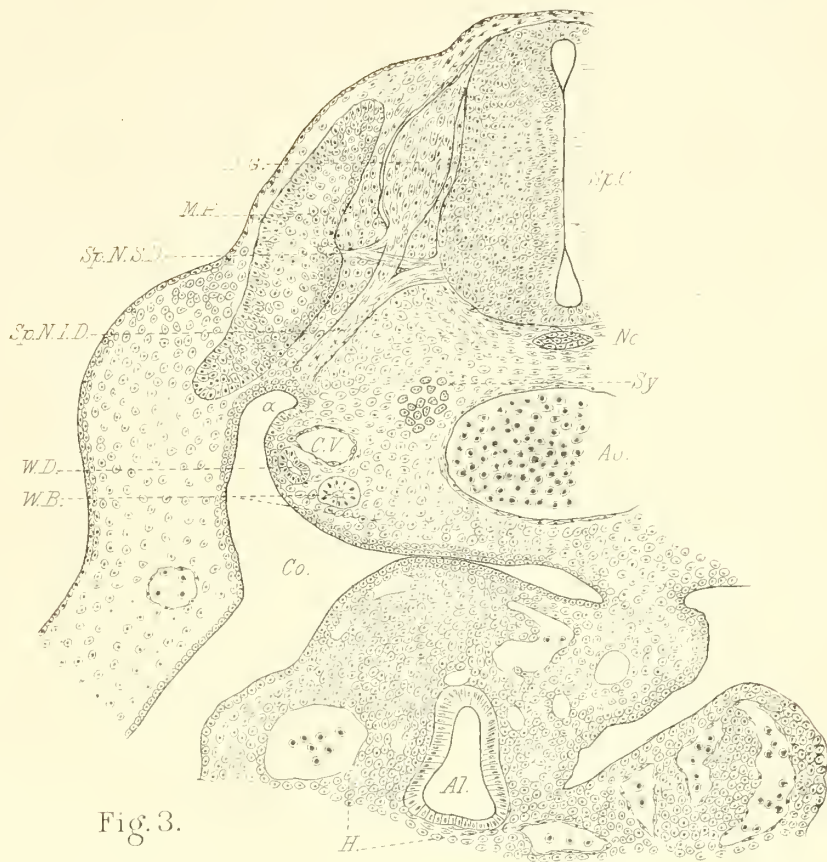
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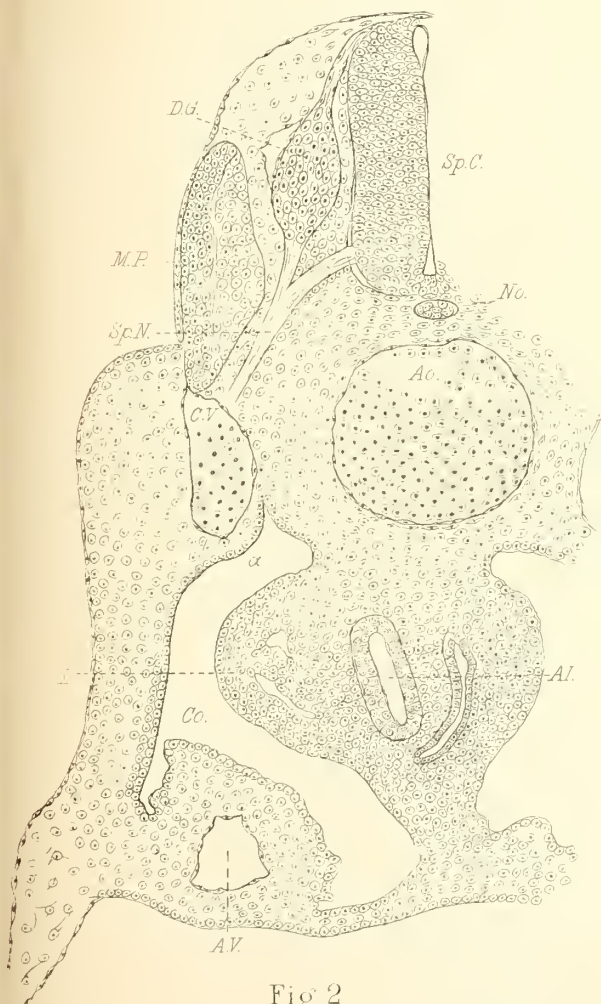
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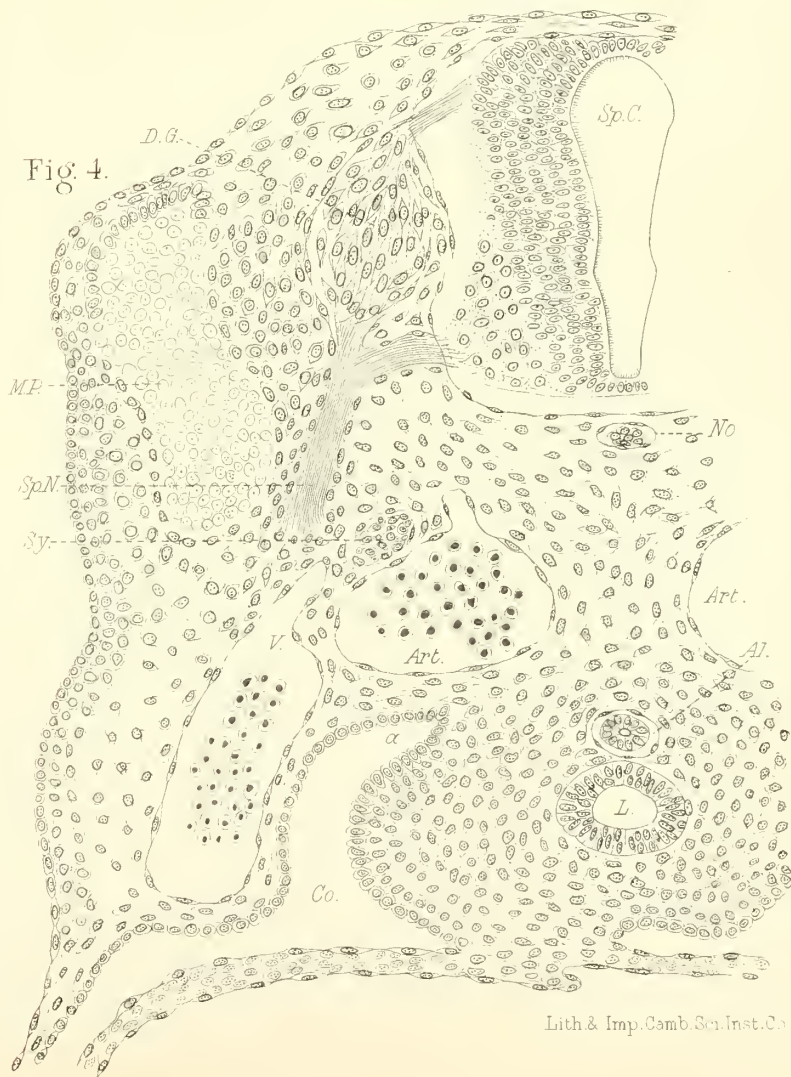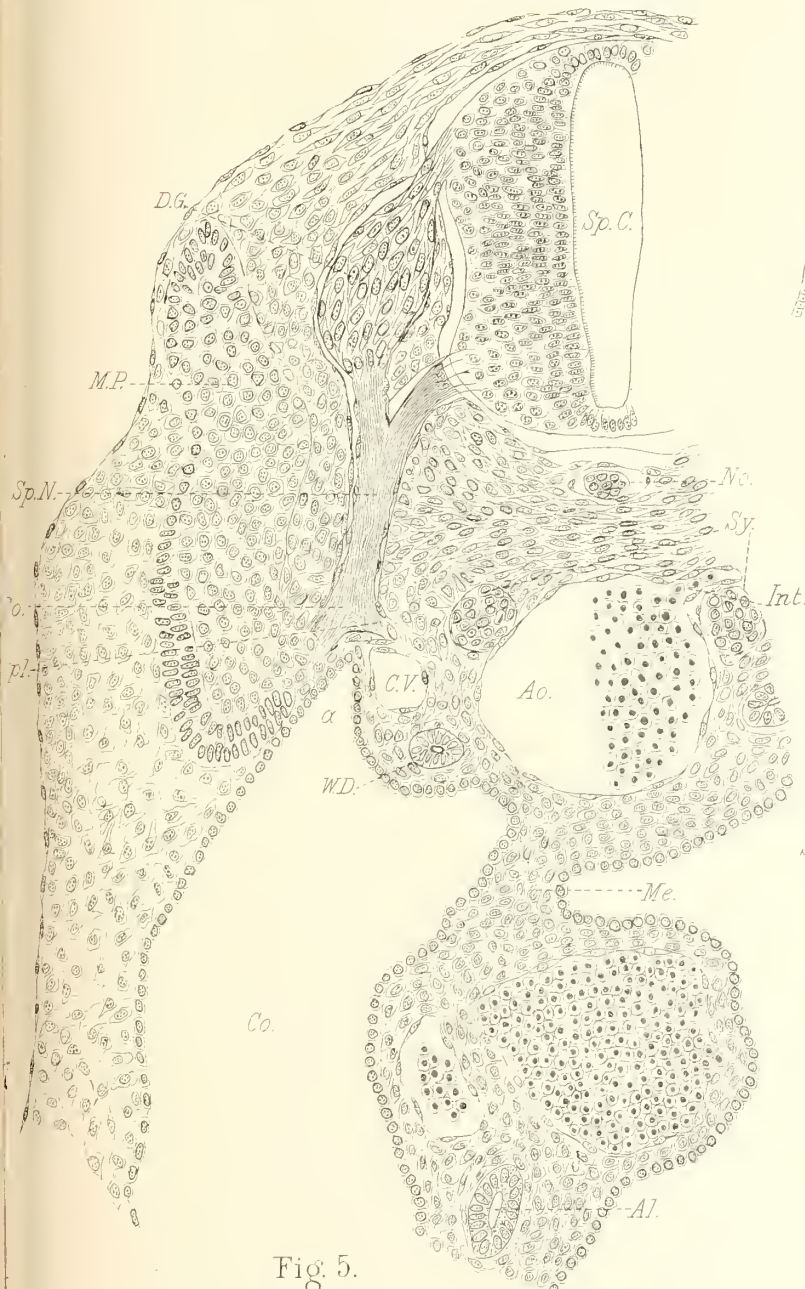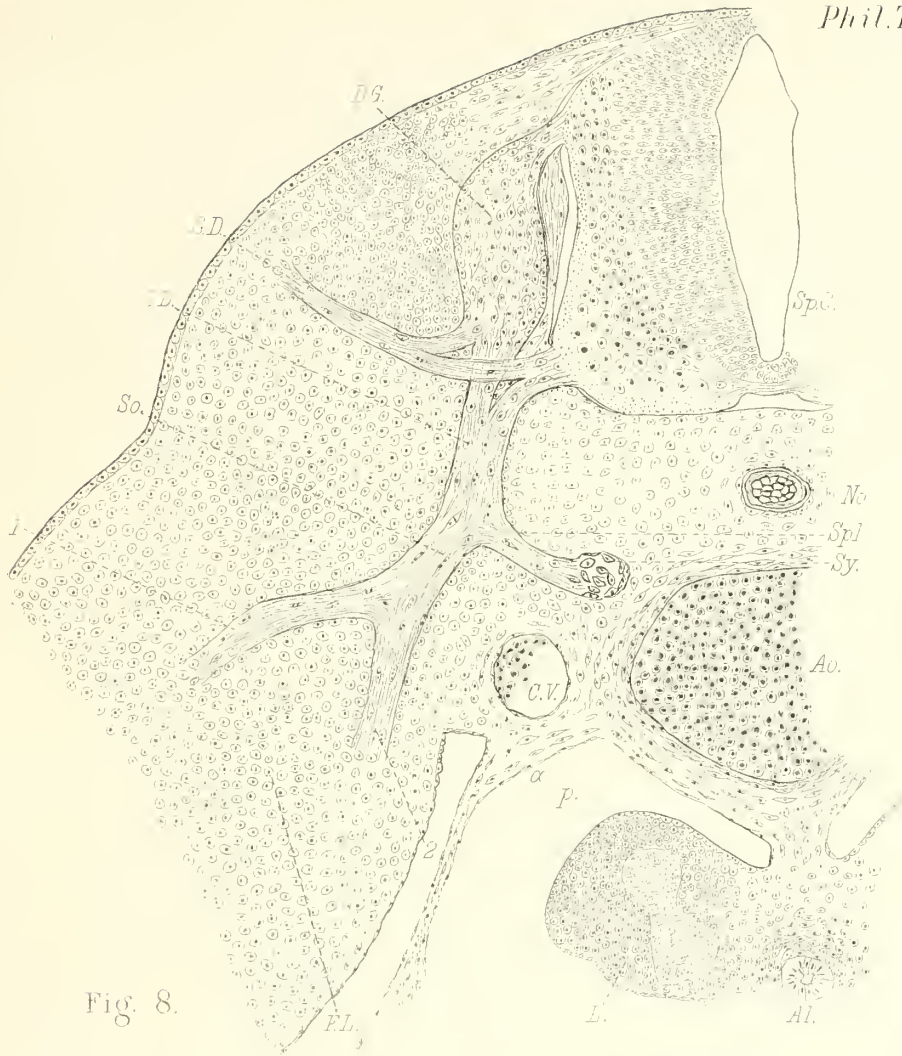
Fig 2

Fig. 5.

Fig. 6.

Fig. 7.

Fig. 8.

Fig 9.

Fig 10.

Fig. 11.

Fig. 12.

Fig. 13.

Sp.C.
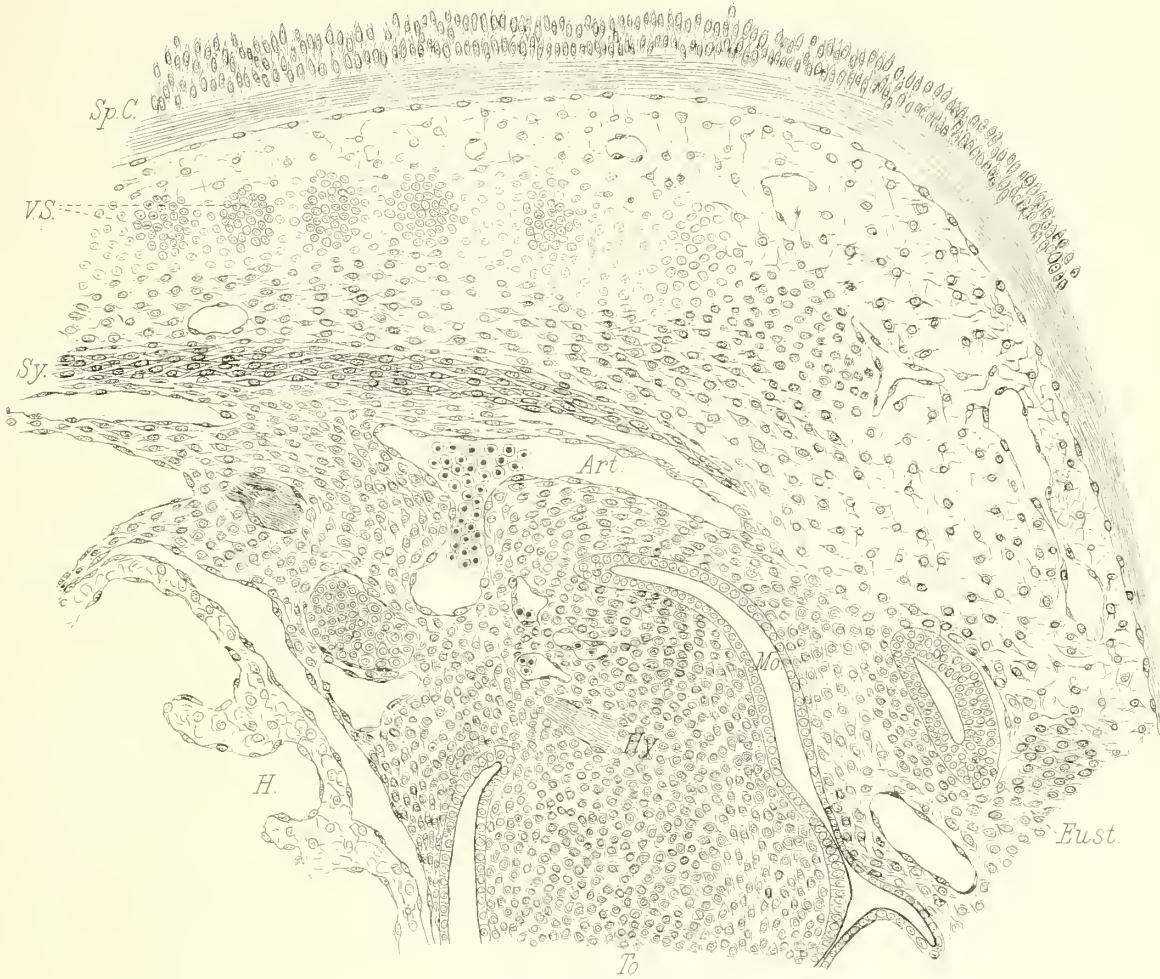
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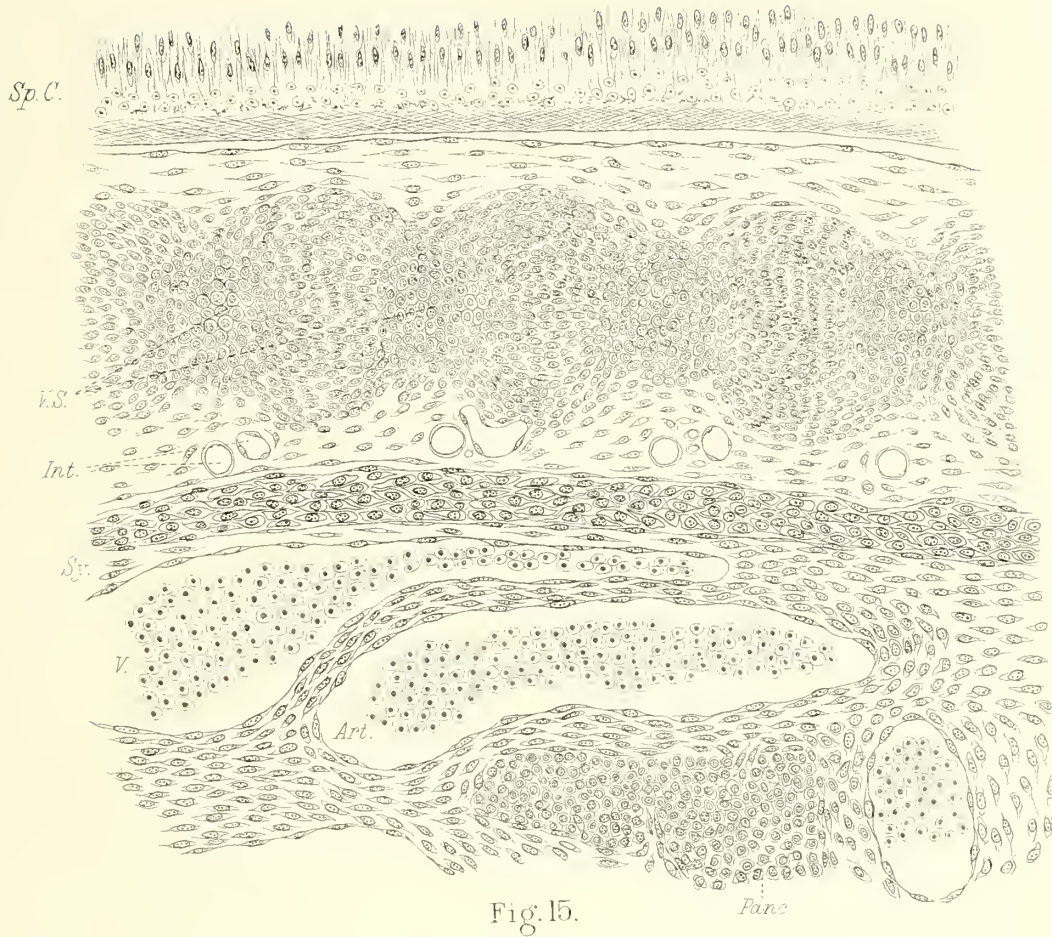
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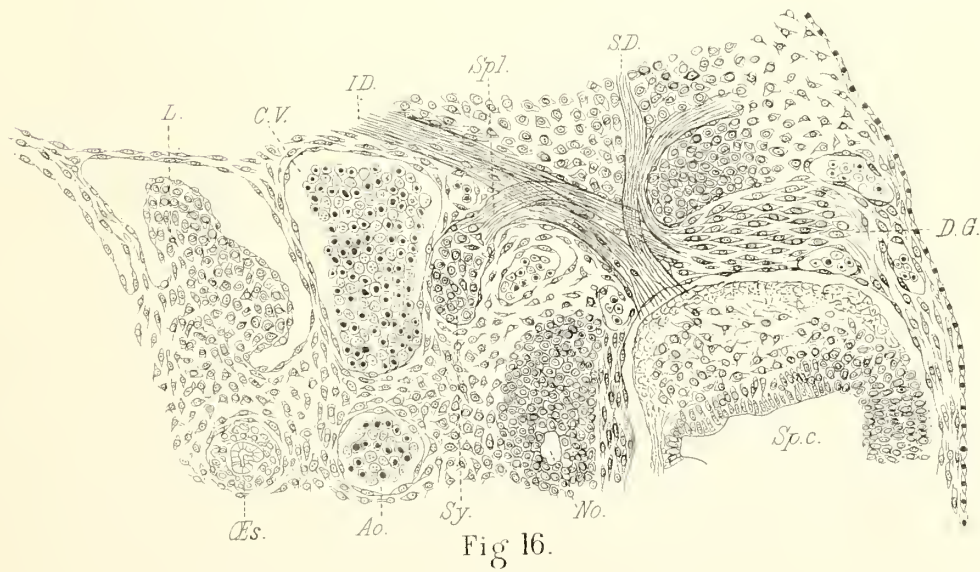Fig. 14.

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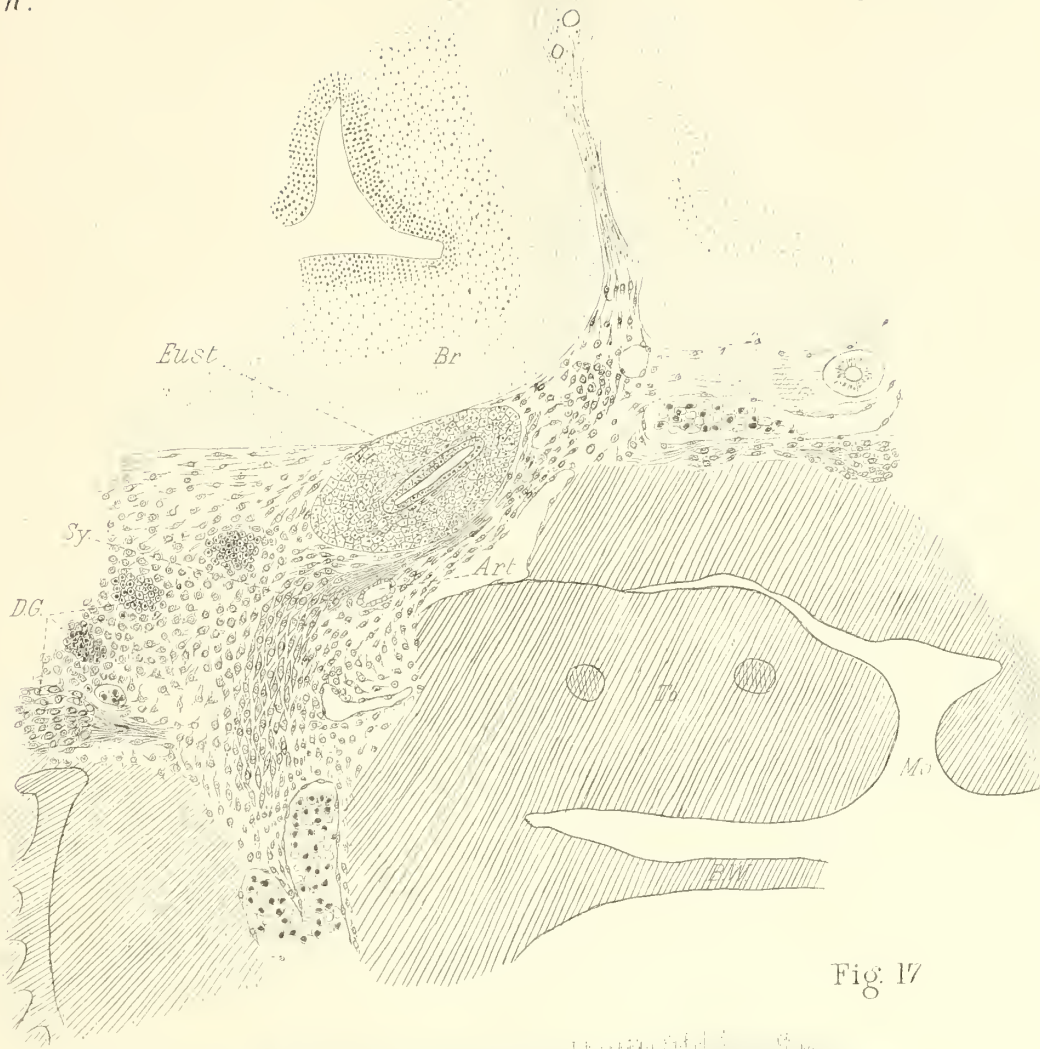
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Fig. 14a.

Fig. 15.

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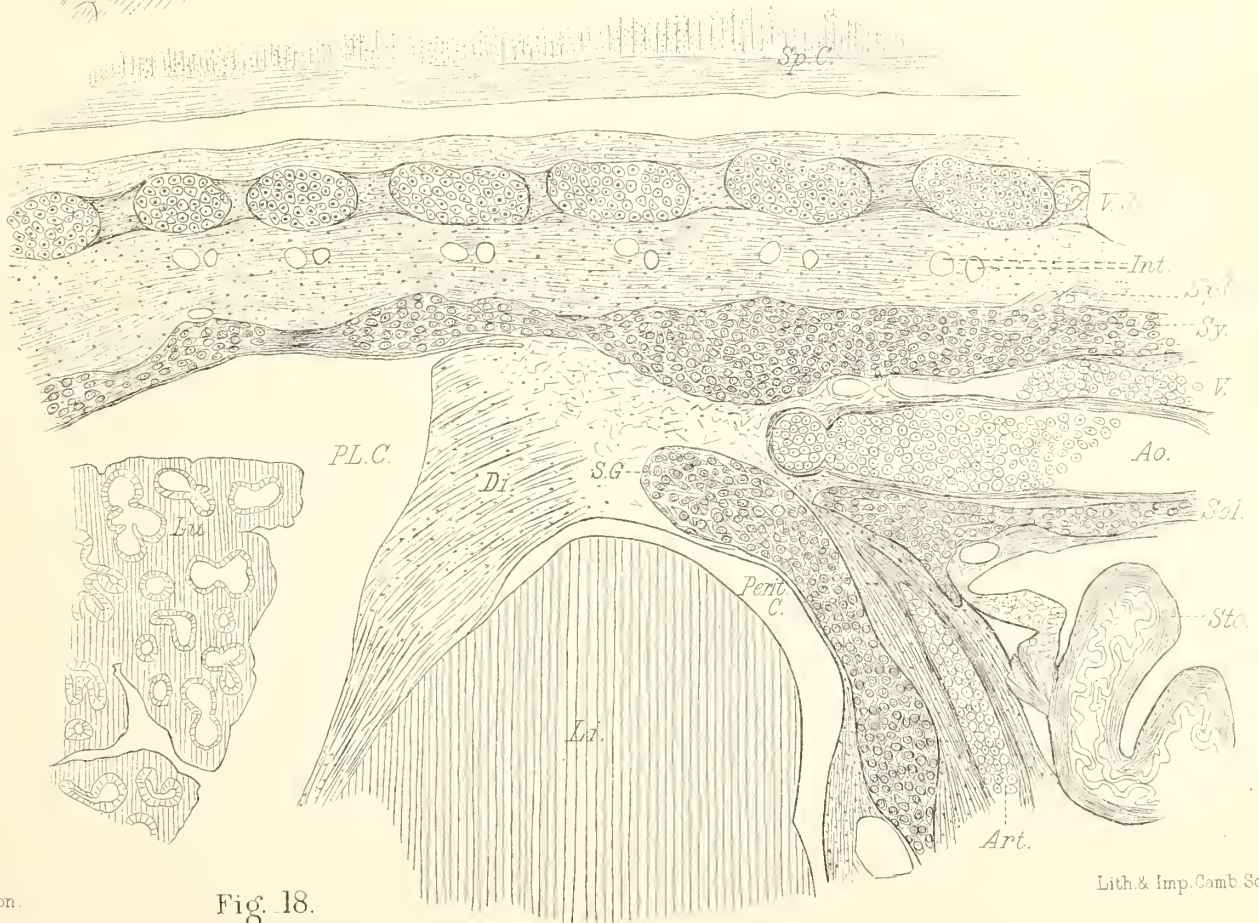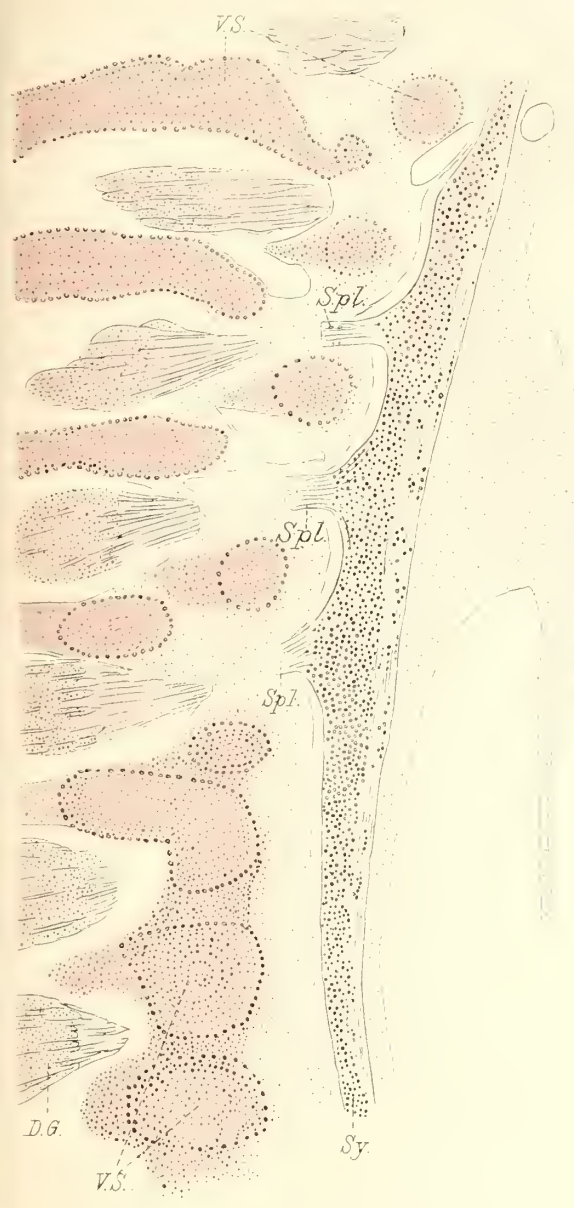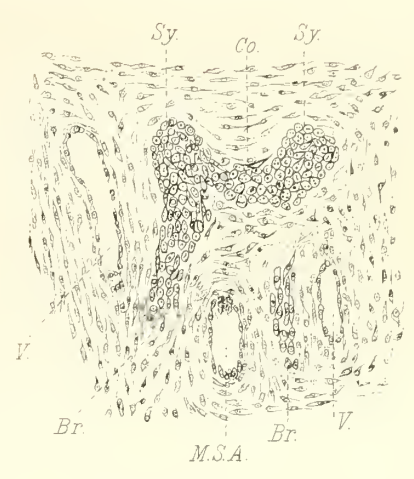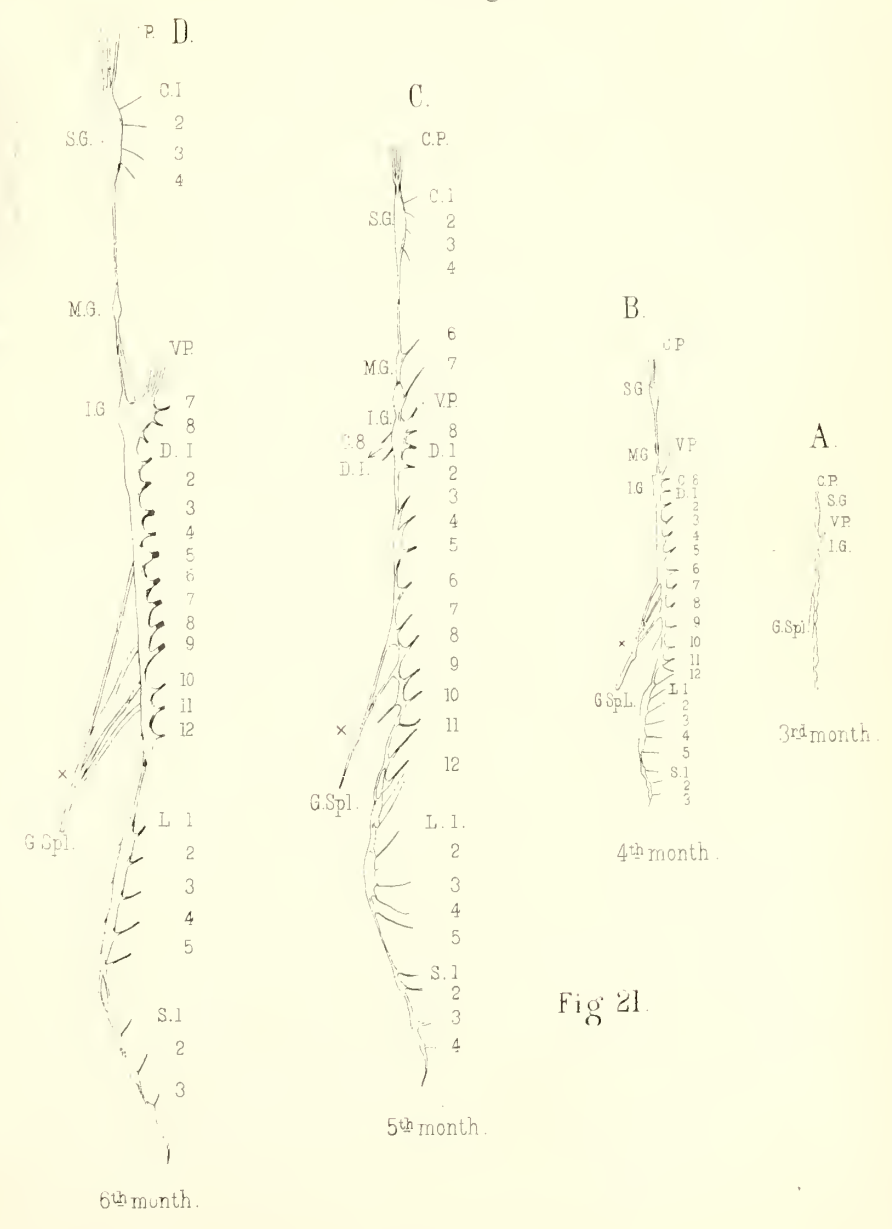Fig. 18.

Fig. 19.

Fig. 20.

Fig. 21.

3rd month.

4th month.

5th month.

6th month.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

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THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY.

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1801. Part I...	0	17	0	1836. Part I...	1	10	0	1855. Part II...	1	6	0	1874. Part II...	3	0	0
—— Part II...	0	17	6	—— Part II...	2	0	0	1856. Part I...	2	0	0	1875. Part I...	3	0	0
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1806. Part I...	0	13	6	—— Part II...	2	5	0	1859. Part I...	2	10	0	—— Part II...	3	0	0
—— Part II...	0	17	6	1841. Part I...	0	10	0	—— Part II...	2	5	0	1879. Part I...	2	0	0
1807. Part I...	0	10	0	—— Part II...	1	10	0	1860. Part I...	0	16	0	—— Part II...	1	12	0
—— Part II...	0	15	6	1842. Part I...	0	16	0	—— Part II...	2	1	6	1880. Part I...	2	5	0
1812. Part I...	0	17	6	—— Part II...	1	2	0	1861. Part I...	1	3	0	—— Part II...	2	0	0
—— Part II...	0	17	6	1843. Part I...	0	10	0	—— Part II...	1	5	0	—— Part III...	1	1	0
1813. Part I...	0	14	0	—— Part II...	1	10	0	—— Part III...	1	7	6	1881. Part I...	2	10	0
—— Part II...	0	18	0	1844. Part I...	0	10	0	1862. Part I...	2	14	0	—— Part II...	1	10	0
1824. Part I...	0	12	6	—— Part II...	1	10	0	—— Part II...	3	0	0	—— Part III...	2	2	0
—— Part II...	1	0	0	1845. Part I...	0	16	0	1863. Part I...	1	14	0	1882. Part I...	1	14	0
—— Part III...	1	4	0	—— Part II...	1	0	0	—— Part II...	1	7	6	—— Part II...	2	0	0
1825. Part I...	1	4	0	1846. Part I...	0	7	6	1864. Part I.*...	0	11	0	—— Part III...	2	10	0
1826. Part I...	1	2	6	—— Part II...	1	12	0	—— Part II...	1	7	6	—— Part IV...	1	0	0
—— Part II...	0	12	6	—— Part III...	1	12	0	—— Part III...	1	10	0	1883. Part I...	1	10	0
—— Part III...	2	0	0	—— Part IV...	1	12	0	1865. Part I...	2	2	0	—— Part II...	2	10	0
—— Part IV...	1	2	6	1847. Part I...	0	14	0	—— Part II...	1	5	0	—— Part III...	1	12	0
1827. Part II...	0	18	0	—— Part II...	0	16	0	1866. Part I...	1	14	0	1884. Part I...	1	8	0
1828. Part I...	1	1	0	1848. Part I...	1	0	0	—— Part II...	2	7	6	—— Part II...	1	16	0
—— Part II...	0	10	0	—— Part II...	0	14	0	1867. Part I...	1	3	0	1885. Part I...	2	10	0
1829. Part I...	0	16	0	1849. Part I...	1	0	0	—— Part II...	1	15	0	—— Part II...	2	5	0
—— Part II...	0	14	0	—— Part II...	2	5	0	1868. Part I...	2	5	0	1886. Part I...	1	8	0
1830. Part I...	1	10	0	1850. Part I...	1	10	0	—— Part II...	2	0	0	—— Part II...	1	15	0
—— Part II...	1	1	0	—— Part II...	3	5	0	1869. Part I...	2	10	0	1887. (A.)	1	3	0
1831. Part I...	1	10	0	1851. Part I...	2	10	0	—— Part II...	3	3	0	—— (B)	1	16	0
—— Part II...	1	12	0	—— Part II...	2	10	0	1870. Part I...	1	10	0	1888. (A.)	1	10	0
1832. Part I...	1	1	0	1852. Part I...	1	0	0	—— Part II...	1	18	0	—— (B.)	2	17	6
—— Part II...	2	0	0	—— Part II...	2	5	0	1871. Part I...	1	10	0	1889. (A.)	1	18	0
1833. Part I...	1	1	0	1853. Part I...	0	18	0	—— Part II...	2	5	0	—— (B.)	1	14	0
—— Part II...	2	18	0	—— Part II...	0	12	0	1872. Part I...	1	12	0	1890. (A.)	1	16	6
1834. Part I...	0	17	0	—— Part III...	1	2	0	—— Part II...	2	8	0	—— (B.)	1	5	0
—— Part II...	2	2	0	1854. Part I...	0	12	0	1873. Part I...	2	10	0				
1835. Part I...	1	2	0	—— Part II...	0	16	0	—— Part II...	1	5	0				
—— Part II...	0	14	0	1855. Part I...	0	16	0	1874. Part I...	2	8	0				

* This part is not sold separately.

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