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# INSTITVTIONVM CALCVLI INTEGRALIS

## VOLVMEN SECVNDVM

IN QVO METHODVS INVENIENDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE DIFFERENTIALIVM SECVNDI ALTIORISVE GRADVS PERTRACTATVR.

Editio altera et correctior.



*PETROPOLI*

Impensis Academiae Imperialis Scientiarum

1792.

## INDEX CAPITVM,

*in Volumine secundo contentorum.*

Sectio prima, de resolutione aequationum differentialium secundi gradus, duas tantum variabiles inuoluentium.



CAP.

CAL-

# CALCVLI INTEGRALIS
## LIBER PRIOR.

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### PARS SECVNDA,
SEV
### METHODVS INVENIENDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE DIFFERENTIALIVM SECVNDI ALTIORISVE GRADVS.

### SECTIO PRIOR,
DE
### RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM SECVNDI GRADVS DVAS TANTVM VARIABILES INVOLVENTIVM.

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# CAPVT I.

## DE INTEGRATIONE FORMVLARVM DIFFERENTIALIVM SECVNDI GRADVS SIMPLICIVM.

### Definitio.

706.

Positis binis variabilibus $x$ et $y$, si vocetur $\partial y = p \partial x$ et $\partial p = q \partial x$, aequatio quaecunque, relationem inter quantitates $x$, $y$, $p$ et $q$ definiens, vocatur aequatio differentialis secundi gradus inter binas variabiles $x$ et $y$.

### Corollarium 1.

707. Quemadmodum ergo littera $p$ implicat rationem differentialium primi gradus, dum est $p = \frac{\partial y}{\partial x}$, ita littera $q = \frac{\partial p}{\partial x}$ implicat rationem differentialium secundi gradus. Sumto enim vt vulgo fieri solet, elemento $\partial x$ constante, erit $\partial p = \frac{\partial \partial y}{\partial x}$, ideoque $q = \frac{\partial \partial y}{\partial x^2}$.

### Corollarium 2.

708. Quatenus ergo in aequatione proposita littera $q$ inest, eatenus ea est differentialis secundi gradus. Si enim $q$ abesset, ob solam $p$ esset tantum differentialis primi gradus; ac si neque $p$ neque $q$ inesset, aequatio foret inter $x$ et $y$, neque quicquam praeterea quaereretur.

### Corollarium 3.

709. Methodus ergo desideratur, proposita aequatione quacunque praeter binas variabiles $x$ et $y$, etiam quantita-

tes $p = \frac{\partial y}{\partial x}$ et $q = \frac{\partial p}{\partial x}$ inuoluente, inueniendi relationem inter ipsas $x$ et $y$, vnde pateat, qualis $y$ sit functio ipsius $x$, seu vicissim.

## Scholion 1.

710. Hoc modo litteram $q$ introducendo aequationes differentio-differentiales a conditione illa, qua quodpiam differentiale primi gradus pro constante assumi solet, liberantur. Cum enim ad meras quantitates finitas reuocentur, quae rationem differentialium primi gradus exprimunt, consideratio differentialis constantis ne locum quidem habere potest. Quando ergo aequationes differentio-differentiales more solito ita exhibentur, vt quodpiam differentiale constans sit assumtum, introducendo litteras $p = \frac{\partial y}{\partial x}$ et $q = \frac{\partial p}{\partial x}$, species differentialium penitus tollitur, dum aequatio tantum quantitates finitas complectitur. Atque etiam vicissim proposita aequatione inter quantitates finitas $x, y, p, q$, ea ad formam vulgarem infinitis modis reduci potest, prout aliud atque aliud differentiale pro constante assumitur, quae tamen omnes formae specie diuersae inter se perfecte conueniunt, quin etiam nullo differentiali constante assumto euolutio in formam solitam fieri potest.

## Scholion 2.

711. Primum igitur breuiter exponi conueniet, quomodo aequatio more solito per differentialia secundi gradus expressa ad formam nostram reduci queat, quodcunque differentiale constans fuerit assumtum. Sit $\partial s$ hoc differentiale pro constante sumtum, cuius ergo ratio ad $\partial x$, ob $\frac{\partial y}{\partial x} = p$, per $p$ et forte ipsas variabiles $x$ et $y$ datur; ponatur ergo $\partial s = v\,\partial x$, vt $v$ fiat quantitas finita. Iam cum in aequatione occurrant $\partial\partial x$ et $\partial\partial y$, vel alterutrum saltem, loco $\partial\partial x$ scribatur $\partial s\,\partial . \frac{\partial x}{\partial s}$, quia ob $\partial s$ constans fit vtique $\partial s . \partial . \frac{\partial x}{\partial s} = \partial\partial x$.

Erit

Erit ergo $\partial\partial x = \partial s \partial . \frac{1}{v} = - \frac{\partial s \partial v}{v v}$. Simili modo loco $\partial\partial y$ ſcribendo $\partial s . \partial . \frac{\partial y}{\partial s} = \partial s \partial . \frac{p}{v}$, fiet $\partial \partial y = \frac{\partial s (v \partial p - p \partial v)}{v v}$. Cum igitur $v$ per $p$, $x$ et $y$ detur, erit

$$\partial v = M \partial x + N \partial y + P \partial p = \partial x (M + N p + P q);$$

ob $\partial p = q \partial x$, ſicque fiet

$$\partial \partial x = - \frac{\partial x^2}{v} (M + N p + P q) \text{ et}$$

$$\partial \partial y = \frac{\partial x^2}{v} (q v - M p - N' p^2 - P p q),$$

hique valores loco $\partial\partial x$ et $\partial\partial y$ ſubſtituti in aequatione tantum differentialia primi gradus relinquent, quibus omnibus ad $\partial x$ reductis, aequatio per diuiſionem prorſus a differentialibus liberabitur. Deinde viciſſim huiusmodi aequatio inter $x$, $y$, $p$ et $q$ propoſita in formam ſolitam, ſumto quopiam elemento $\partial s$ conſtante, euoluetur, ſi primo pro $p$ vbique ſcribatur $\frac{\partial y}{\partial x}$, loco $q$ autem $\frac{1}{\partial x} \partial . \frac{\partial y}{\partial x} = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3}$, vbi quidem nullius adhuc elementi conſtantis ratio eſt habita. At ob $\partial s = v \partial x$ conſtans, inſuper erit

$$v \partial\partial x + \partial v \partial x = 0, \text{ ſeu ob } \partial v = M \partial x + N \partial y + P \partial . \frac{\partial y}{\partial x},$$

$$v \partial \partial x + M \partial x^2 + N \partial x \partial y + \frac{P (\partial x \partial \partial y - \partial y \partial \partial x)}{\partial x} = 0,$$

vnde pro lubitu vel $\partial \partial x$ vel $\partial \partial y$ elidi poteſt, neutro autem eliſo infinitae formae aequiualentes exhiberi poſſunt.

## Scholion 3.

712. Hinc ergo praeſtantia formae finitae, ad quam hic aequationes differentio-differentiales reuocamus, prae more ſolito eas exhibendi luculenter perſpicitur; cum eadem aequatio more ſolito infinitis modis, prout aliud atque aliud elementum conſtans aſſumitur, repraeſentari poſſit, dum noſtro more eadem aequatio ſemper ad vnicam formam reducitur. Quodſi ergo noſtro more aequationes prodeant diuerſae, certum

tum eſt iis quoque diuerſas relationes, inter variabiles $x$ et $y$ exprimi, cum contra ſolito more diuerſiſſimae aequationes differentio-differentiales eandem relationem indicare queant, ex quibus plerumque difficile eſt eam eligere, quae ad reſolutionem maxime ſit accommodata. Cum igitur hic eiusmodi methodus requiratur, cuius ope propoſita quacunque aequatione inter quaternas quantitates $x$, $y$, $p$ et $q$, relatio inter binas variabiles $x$ et $y$ definiri queat, quoniam haec quaeſtio vires humanas ſuperare videtur, a caſibus ſimpliciſſimis erit exordiendum. Caſus autem ſimpliciſſimi ſine dubio ſunt, quando in aequatione propoſita duae tantum inſunt quantitates, ſcilicet vel $x$ et $q$ tantum, vel $y$ et $q$, vel $p$ et $q$; hoc eſt ſi $q$ aequetur functioni vel ipſius $x$, vel ipſius $y$, vel ipſius $p$ tantum; quos caſus in hoc capite euoluere conſtituimus.

## Definitio.

713. Formula differentio-differentialis ſimplex eſt, quando poſito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, quantitas $q$ aequatur functioni vel ipſius $x$, vel ipſius $y$, vel ipſius $p$ tantum.

## Corollarium 1.

714. Triplices ergo habemus formulas differentio-differentiales ſimplices, quarum reſolutionem in hoc capite doceri conuenit, prout quantitas $q$ vel per functionem ipſius $p$, vel ipſius $x$, vel ipſius $y$ tantum determinatur.

## Corollarium 2.

715. Si ergo X denotet functionem ipſius $x$, Y ipſius $y$, et P ipſius $p$ tantum, terna genera harum formularum ſimplicium ſunt 1) $q = X$, 2) $q = Y$, 3) $q = P$; in quibus continetur caſus ſimpliciſſimus $q = \text{Conſt.}$

Corol-

## Corollarium 3.

716. Si has formulas more ſolito exprimere velimus, ob $q = \frac{\partial p}{\partial x} = \frac{1}{\partial x} \partial . \frac{\partial y}{\partial x}$, ſumto elemento $\partial x$ conſtante, erit $q = \frac{\partial \partial y}{\partial x^2}$; ſumto elemento $\partial y$ conſtante, erit $q = - \frac{\partial y \partial \partial x}{\partial x^3}$; nullo autem ſumto conſtante, erit $q = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3}$, quibus ſimplicitas earum formularum haud mediocriter offuſcatur.

## Corollarium 4.

717. Si elementum $\sqrt{(\partial x^2 + \partial y^2)}$, quod ſaepe fit, conſtans accipiatur, erit $\partial x \partial \partial x + \partial y \partial \partial y = 0$; vnde poſtremus valor ipſius $q$, vel ob $\partial \partial y = - \frac{\partial x \partial \partial x}{\partial y}$ abit in $q = - \frac{(\partial x^2 + \partial y^2) \partial \partial x}{\partial x^3 \partial y}$, vel ob $\partial \partial x = - \frac{\partial y \partial \partial y}{\partial x}$ abit in

$$q = \frac{(\partial x^2 + \partial y^2) \partial \partial y}{\partial x^4}.$$

## Scholion.

718. Repudiata ergo penitus vulgari ratione aequationes differentio-differentiales exprimendi, quippe qua formulae in ſe ſatis ſimplices vehementer complicatae euadere poſſent, ratione hic ſtabilita vtamur, indeque reſolutionem huiusmodi formularum ſimplicium doceamus.

## Problema 92.

719. Poſito $\partial y = p \partial x$ et $\partial p = q \partial x$, ſi $q$ aequetur functioni cuicunque ipſius $p$, inuenire relationem inter ipſas variabiles $x$ et $y$.

## Solutio.

Sit ergo $q = P$, denotante P functionem quamcunque ipſius $p$: quoniam igitur eſt $q = \frac{\partial p}{\partial x}$, erit $\partial p = P \partial x$, hincque

$$\partial x = \frac{\partial p}{P}, \text{ et } \partial y = p \partial x = \frac{p \partial p}{P}.$$

Ex

Ex quo consequimur integrando

$$x = a + \int \frac{\partial p}{P}, \text{ et } y = b + \int \frac{p \partial p}{P};$$

ita vt tam $x$, quam $y$ per eandem nouam variabilem $p$ determinetur. Atque cum duae nouae constantes $a$ et $b$ per duplicem integrationem sint introductae, hoc integrale pro completo erit habendum.

## Corollarium 1.

720. Aequatio $q = P$, cuius integrationem hic tradidimus, si in formam consuetam, sumto $\partial x$ constante, resoluatur, ob $q = \frac{\partial \partial y}{\partial x^2}$, transmutabitur in $\partial \partial y = \partial x^2 f : \frac{\partial y}{\partial x}$; quae est aequatio differentio-differentialis, in qua ipsae variabiles $x$ et $y$ non occurrunt.

## Corollarium 2.

721. Talis quoque forma prodit, si elementum $\partial y$ vel alia expressio differentialis, in quam ipsae $x$ et $y$ non ingrediuntur, veluti $\sqrt{(\partial x^2 + \partial y^2)}$ pro constante sumatur. Hoc ergo modo omnis aequatio differentio-differentialis in quam ipsae variabiles $x$ et $y$ non ingrediuntur, integrari poterit.

## Corollarium 3.

722. Sin autem huiusmodi elementum $y \partial x - x \partial y$ constans assumatur, vt $y \partial \partial x - x \partial \partial y = 0$, ob

$$q = \frac{1}{\partial x} \partial . \frac{\partial y}{\partial x} = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3}, \text{ fiet}$$

$$q = \frac{(y \partial x - x \partial y) \partial \partial x}{x \partial x^3} = \frac{(y \partial x - x \partial y) \partial \partial y}{y \partial x^3},$$

quae expressio, si aequetur functioni ipsius $p = \frac{\partial x}{\partial y}$, integrari poterit.

Corol-

## Corollarium 4.

723. Si fuerit P quantitas conſtans, vt ſit $q = f$, erit
$$x = a + \frac{p}{f} \text{ et } y = b + \frac{pp}{2f},$$
vnde fit
$$y = b + \tfrac{f}{2}(x-a)^2, \text{ ſeu } y = \tfrac{1}{2}fxx - afx + \tfrac{1}{2}aaf + b,$$
ſeu mutata forma conſtantium $y = \frac{1}{2}fxx + Cx + D$.

## Scholion.

724. Cum ſcilicet aequatio differentio-differentialis duplici integratione indigeat, ſi vtraque omni extenſione inſtituatur, duae nouae conſtantes arbitrariae introducuntur; in quo criterium, num huiusmodi integrale ſit completum, conſiſtit. Quemadmodum enim aequationum differentialium primi gradus integratio completa vnam conſtantem arbitrariam implicat, ita ſi aequatio differentialis fuerit ſecundi gradus, binae conſtantes nouae in integrale completum ingredientur, ternae autem ac plures, ſi aequatio differentialis fuerit tertii altiorisue gradus. Problemata autem, quorum reſolutio ad huiusmodi aequationes differentiales altiorum graduum deducunt, natura ſua ita ſunt comparata, vt ſolutionis determinatio totidem conſtantes requirat. Ita in aequatione $q = f$, ſeu ſumto $\partial x$ conſtante, $\partial\partial y = f\partial x^2$, aequatio integralis completa $y = \frac{1}{2}fxx + Cx + D$ duas conſtantes nouas C et D inuoluit, quod etiam in ſubiunctis exemplis patebit.

## Exemplum 1.

725. *Aequationis differentio-differentialis $a\partial\partial y = \partial x \partial y$, in qua elementum $\partial x$ conſtans eſt ſumtum, integrale completum inuenire.*

Poſito $\partial y = p\partial x$ et $\partial p = q\partial x$, erit $\partial\partial y = q\partial x^2$, hincque $aq = p$, et $P = \frac{p}{a}$. Quocirca integratio praebet

 $x =$

$$x = \int \frac{a \partial p}{p} = C + a\,l\,p \text{ et } y = \int a \partial p = D + a p.$$

Cum igitur fit

$$p = \frac{y - D}{a}, \text{ erit } x = C + a\,l\,\frac{y-D}{a},$$

quae est aequatio integralis completa binas constantes C et D inuoluens.

## Exemplum 2.

726. *Posito $\partial x$ constante, inuenire aequationem inter $x$ et $y$, vt fiat* $\frac{(\partial x^2 + \partial y^2)\sqrt{(\partial x^2 + \partial y^2)}}{-\partial x \partial\partial y} = a.$

Posito $\partial y = p\,\partial x$, ob $\partial x$ constans, erit $\partial\partial y = \partial p \partial x$, sicque nostra aequatio est $\frac{(1+pp)\sqrt{(1+pp)}}{-\partial p}\partial x = a$, vnde fit

$$\partial x = \frac{-a \partial p}{(1+pp)^{\frac{3}{2}}} \text{ et } \partial y = \frac{-a p \partial p}{(1+pp)^{\frac{3}{2}}}.$$

Per integrationem ergo nanciscimur

$$x = A - \frac{a p}{\sqrt{(1+pp)}}, \text{ et } y = B + \frac{a}{\sqrt{(1+pp)}};$$

vnde concludimus

$$(A - x)^2 + (y - B)^2 = a a.$$

## Corollarium.

727. Si $x$ et $y$ denotent coordinatas rectangulas lineae curuae, formula $\frac{(\partial x^2 + \partial y^2)\sqrt{(\partial x^2 + \partial y^2)}}{-\partial x \partial\partial y}$ exprimit eius radium osculi, qui ergo vt sit constans $= a$, aequatio integralis inuenta circulum radio $a$ describendum indicat.

## Exemplum 3.

728. *Posito $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ eoque sumto constante, inuenire aequationem inter $x$ et $y$, vt fiat* $\frac{\partial s \partial y}{\partial\partial x} = \frac{a \partial x}{\partial y}$.

Pona-

Ponatur $\partial y = p\,\partial x$, erit $\partial s = \partial x \sqrt{(1+pp)}$, et ob $\partial s$ conſtans

$$\partial\partial x \sqrt{(1+pp)} + \frac{p\,\partial x\,\partial p}{\sqrt{(1+pp)}} = 0, \text{ ſeu}$$

$$\partial\partial x = \frac{-p\,\partial x\,\partial p}{1+pp},$$

vnde aequatio propoſita abit in

$$\frac{p\,\partial x^2 \sqrt{(1+pp)}}{-p\,\partial x\,\partial p}(1+pp) = \frac{a}{p}, \text{ ſeu}$$

$$\partial x = \frac{-a\,\partial p}{p(1+pp)^{\frac{3}{2}}}, \text{ et } \partial y = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}}, \text{ ergo}$$

$$y = D - \frac{ap}{\sqrt{(1+pp)}}.$$

At pro illa formula ſtatuatur $p = \frac{1}{r}$, eritque

$$\partial x = \frac{arr\,\partial r}{(1+rr)^{\frac{3}{2}}} = \frac{a\,\partial r}{\sqrt{(1+rr)}} - \frac{a\,\partial r}{(1+rr)^{\frac{3}{2}}};$$

vnde fit integrando

$$x = C - \frac{ar}{\sqrt{(1+rr)}} + a\,l\,[r + \sqrt{(1+rr)}], \text{ ſeu}$$

$$x = C - \frac{a}{\sqrt{(1+pp)}} + a\,l\,\frac{1+\sqrt{(1+pp)}}{p}.$$

## Exemplum 4.

729. *Poſito $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$, hocque elemento ſumto conſtante, fieri oporteat* $\frac{\partial s\,\partial y}{\partial\partial x} = a$ Ang. tang. $\frac{\partial y}{\partial x}$.

Si fiat vt ante $\partial y = p\,\partial x$, orietur haec aequatio integranda

$$\frac{-\partial x(1+pp)^{\frac{3}{2}}}{\partial p} = a \text{ Ang. tang. } p, \text{ ſeu}$$

 $\partial x =$

$$\partial x = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}}\ \text{Ang. tang.}\ p, \text{ et}$$

$$\partial y = \frac{-ap\,\partial p}{(1+pp)^{\frac{3}{2}}}\ \text{Ang. tang.}\ p.$$

Cum nunc ſit $\partial.$ Ang. tang. $p = \frac{\partial p}{1+pp}$, erit

$$x = \frac{-ap}{\sqrt{(1+pp)}}\ \text{Ang. tang.}\ p - a\int\frac{p\,\partial p}{(1+pp)^{\frac{3}{2}}}, \text{ et}$$

$$y = \frac{a}{\sqrt{(1+pp)}}\ \text{Ang. tang.}\ p - a\int\frac{\partial p}{(1+pp)^{\frac{3}{2}}}.$$

Quamobrem colligimus

$$x = C - \frac{ap}{\sqrt{(1+pp)}} - \frac{ap}{\sqrt{(1+pp)}}\ \text{Ang. tang.}\ p, \text{ et}$$

$$y = D - \frac{a}{\sqrt{(1+pp)}} + \frac{a}{\sqrt{(1+pp)}}\ \text{Ang. tang.}\ p.$$

## Corollarium 1.

730. Si $x$ ſit abſciſſa et $y$ applicata curuae, radius oſculi proportionalis eſſe debet angulo, quem curuae tangens cum axe conſtituit; vnde patet hanc curuam fore quandam ſpiralem, circa originem abſciſſarum ſe euoluentem.

## Corollarium 2.

731. Si angulus ille, cuius tangens $= p$, ponatur $= \Phi$, erit $p =$ tang. $\Phi$, hincque

$$x = C - a\ \text{coſ.}\ \Phi - a\Phi\ \text{ſin.}\ \Phi, \text{ et}$$

$$y = D - a\ \text{ſin.}\ \Phi + a\Phi\ \text{coſ.}\ \Phi;$$

vnde colligitur

$$x\ \text{coſ.}\ \Phi + y\ \text{ſin.}\ \Phi = C\ \text{coſ.}\ \Phi + D\ \text{ſin.}\ \Phi - a.$$

Corol-

## Corollarium 3.

732. Vt fumto $\Phi = 0$, ambae $x$ et $y$ euanefcant, fumi debet $C = a$ et $D = 0$, eritque

$$x = a - a \operatorname{cof.} \Phi - a \Phi \operatorname{fin.} \Phi, \text{ et}$$

$$y = - a \operatorname{fin.} \Phi + a \Phi \operatorname{cof.} \Phi;$$

vnde quamdiu angulus $\Phi$ eft minimus, erit

$$x = - \tfrac{1}{2} a \Phi \Phi + \tfrac{1}{8} a \Phi^4 \text{ et } y = - \tfrac{1}{3} a \Phi^3 + \tfrac{1}{30} \Phi^5:$$

ideoque proxime

$$\frac{x^3}{yy} = - \tfrac{9}{8} a, \text{ feu } yy = - \frac{8 x^3}{9 a}.$$

## Problema 93.

733. Pofito $\partial y = p \partial x$ et $\partial p = q \partial x$, fi quantitas $q$ aequetur functioni ipfius $x$, quae fit $X$, definire relationem inter binas variabiles $x$ et $y$.

## Solutio.

Cum ergo fit $q = X$, erit $q \partial x = \partial p = X \partial x$, vnde integrando colligimus $p = \int X \partial x + C$, atque hinc ob $\partial y = p \partial x$ adipifcemur

$$y = \int \partial x \int X \partial x + C x + D.$$

At eft

$$\int \partial x \int X \partial x = x \int X \partial x - \int X x \partial x,$$

vti fumendis differentialibus fponte patet. Quare aequatio integralis completa relationem inter binas variabiles $x$ et $y$ continens eft

$$y = x \int X \partial x - \int X x \partial x + C x + D$$

duas conftantes arbitrarias $C$ et $D$ inuoluens. Quae ergo erit algebraica, fi ambae formulae differentiales $X \partial x$ et $X x \partial x$ integrationem admittant.

## Corollarium 1.

734. Quodſi ergo ſit $q = 0$, ſeu ſumto $\partial x$ conſtante, $\partial\partial y = 0$, vt ſit $X = 0$, erit aequatio integralis completa $y = Cx + D$.

## Corollarium 2.

735. Aequationes ergo differentio-differentiales, quas hoc modo integrare licet, ſumto $\partial x$ conſtante, in hac forma $\partial\partial y = X\partial x^2$ continentur, vnde prima integratio praebet $\partial y = \partial x \int X \partial x + C$, et altera $y = \int \partial x \int X \partial x + Cx + D$.

## Corollarium 3.

736. Sin autem differentiale $\partial y$ capiatur conſtans ob $p = \frac{\partial y}{\partial x}$, erit $\partial p = -\frac{\partial y \partial\partial x}{\partial x^2} = q\partial x$, et forma aequationum hoc modo integrandarum erit $-\partial y \partial\partial x = X \partial x^3$.

## Corollarium 4.

737. Quodſi elementum $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ ſit conſtans, ob $\partial x \partial\partial x + \partial y \partial\partial y = 0$, erit

$$\partial p = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^2} = \frac{-\partial s^2 \partial\partial x}{\partial x^2 \partial y} = q\partial x.$$

Hinc forma aequationum hoc modo integrandarum eſt

$$-\partial s^2 \partial\partial x = X\partial x^3 \partial y.$$

Vel cum etiam ſit

$$\partial p = q\partial x = +\frac{\partial s^2 \partial\partial y}{\partial x^3},$$

ea erit $\partial s^2 \partial\partial y = X\partial x^4$.

## Scholion.

738. Hic manifeſtum eſt, quantum interſit aequationes differentio-differentiales a forma ſolita, vbi elementum quodpiam conſtans eſt aſſumtum, ab hac conditione liberare

et

et ad formam hic stabilitam reducere. Si enim proponatur haec aequatio $\partial s^2 \partial \partial y = X \partial x^4$, in qua elementum $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constans sit assumtum, haud facile patet, quomodo eius integratio sit suscipienda. Nostra autem methodo, si ponamus $\partial y = p \partial x$, vt sit

$$\partial s = \partial x \sqrt{(1 + pp)} \text{ et } \partial \partial y = p \partial \partial x + \partial x \partial p,$$

induit ista aequatio hanc formam

$$\partial x^2 (1 + pp)(p \partial \partial x + \partial x \partial p) = X \partial x^4 \text{ seu}$$
$$(p \partial \partial x + \partial x \partial p)(1 + pp) = X \partial x^2.$$

At quia $\partial s$, ac proinde quoque $\partial s^2 = \partial x^2 (1 + pp)$ est constans, erit

$$\partial \partial x (1 + pp) + p \partial x \partial p = 0, \text{ seu } \partial \partial x = \frac{-p \partial x \partial p}{1 + pp},$$

ideoque

$$p \partial \partial x + \partial x \partial p = \frac{\partial x \partial p}{1 + pp},$$

ita vt fiat $\partial p = X \partial x$, quae aequatio iam facillime tractatur. Hic scilicet in subsidium vocari debent ea, quae supra de integratione formularum differentialium simplicium sunt tradita.

## Exemplum 1.

739. *Sumto $\partial x$ constante, si fuerit $\partial \partial y = a x^n \partial x^2$, integrale completum inuestigare.*

Cum sit $\frac{\partial \partial y}{\partial x} = a x^n \partial x$, ob $\partial x$ constans, erit integrando $\frac{\partial y}{\partial x} = \frac{a}{n+1} x^{n+1} + C$, hincque denuo integrando

$$y = \frac{a}{(n+1)(n+2)} x^{n+2} + Cx + D:$$

vbi casus $n = -1$ et $n = -2$ seorsim sunt euoluendi.

I. Ergo si $n = -1$, erit $\frac{\partial \partial y}{\partial x} = \frac{a \partial x}{x}$, hincque $\frac{\partial y}{\partial x} = a l x + C$, vnde cum sit $\partial y = a \partial x l x + C \partial x$, erit denuo inte-

integrando $y = \alpha x l x - \alpha x + C x + D$, seu loco $C - \alpha$ scribendo $C$, habebitur $y = \alpha x l x + C x + D$.

II. Si $n = -2$ et $\frac{\partial \partial y}{\partial x} = \frac{\alpha \partial x}{x^2}$, erit $\frac{\partial y}{\partial x} = \frac{-\alpha}{x} + C$, hincque $y = -\alpha l x + C x + D$.

## Exemplum 2.

740. *Posito* $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ *constante, si fuerit*

$$\frac{\partial s^2 \partial \partial y}{\partial x^4} = \frac{1}{a} \operatorname{cos.} \frac{x}{c},$$

*inuenire integrale completum.*

Ex superioribus constat fore $\frac{\partial s^2 \partial \partial y}{\partial x^4} = q$, ita vt proposita sit haec aequatio $q = \frac{1}{a} \operatorname{cos.} \frac{x}{c}$, vnde fit

$$q \partial x = \partial p = \frac{\partial x}{a} \operatorname{cos.} \frac{x}{c}$$

et integrando

$$p = \frac{c}{a} \operatorname{fin.} \frac{x}{c} + C = \frac{\partial y}{\partial x}.$$

Quare obtinebitur

$$y = -\frac{cc}{a} \operatorname{cos.} \frac{x}{c} + C x + D,$$

quae est aequatio integralis completa.

## Problema 94.

741. Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si quantitas $q$ aequetur functioni cuicunque ipsius $y$ tantum, quae sit $Y$, inuenire aequationem integralem completam inter $x$ et $y$.

## Solutio.

Cum sit $q = Y = \frac{\partial p}{\partial x}$, erit $\partial x = \frac{\partial p}{Y}$, hincque $p \partial x = \partial y = \frac{p \partial p}{Y}$; vnde conficitur haec aequatio inter $p$ et $y$ separata $p \partial p = Y \partial y$, quae integrata praebet

$$\tfrac{1}{2} p p = \int Y \partial y + \tfrac{1}{2} C \text{ et } p = \sqrt{(C + 2\int Y \partial y)} = \frac{\partial y}{\partial x}.$$

Hinc

Hinc ergo porro concluditur $x = \int \frac{\partial y}{\sqrt{(C + 2\int Y \partial y)}}$, quae integratio denuo constantem arbitrariam inducit, ita vt hoc modo aequatio integralis completa inter $x$ et $y$ obtineatur.

## Corollarium 1.

742. Cum sit $q = \frac{\partial p}{\partial x}$ et $\partial x = \frac{\partial y}{p}$, erit $q = \frac{p \partial p}{\partial y}$. Quare cum aequatio proposita sit $q = Y$, erit $\frac{p \partial p}{\partial y} = Y$, hincque $p \partial p = Y \partial y$, vnde praecedens integratio sponte deducitur.

## Corollarium 2.

743. Sumto elemento $\partial x$ constante, cum sit $q = \frac{\partial \partial y}{\partial x^2}$, aequationes hic integratae habebunt hanc formam $\partial \partial y = Y \partial x^2$, cuius integratio si per $\partial y$ multiplicetur, est manifesta, fit enim

$$\tfrac{1}{2} \partial y^2 = \partial x^2 \int Y \partial y + \tfrac{1}{2} C \partial x^2,$$

ob $\partial x$ constans, hincque $\partial x = \frac{\partial y}{\sqrt{(C + 2\int Y \partial y)}}$, vt ante.

## Scholion.

744. En ergo specimen aequationum differentialium, quae per idoneum multiplicatorem integrabiles redduntur, ex quo intelligitur hanc methodum etiam in his aequationibus vsum habere posse; deinceps autem locus erit hanc methodum vberius excolendi, cuius quippe vsus praecipue in aequationibus differentialibus altiorum graduum est insignis, vbi variabilium separatio nihil subsidii affert. Atque hanc ob caussam iam supra hanc methodum per multiplicatores integrandi commendauimus, alterique per separationem procedenti longe antetulimus.

## Exemplum 1.

745. *Posito $\partial x$ constante, si fuerit $a a \partial \partial y = y \partial x^2$ inuenire integrale completum.*

 Mul-

Multiplicetur aequatio propofita per $2\,\partial y$, vt prodeat

$$2aa\,\partial y\,\partial\partial y = 2y\,\partial y\,\partial x^2,$$

quae ob $\partial x$ conftans habebit integrale

$$aa\,\partial y^2 = yy\,\partial x^2 + C\,\partial x^2,$$

vnde colligitur

$$\partial x = \frac{a\,\partial y}{\sqrt{(yy+C)}},$$

quae denuo integrata dat

$$x = a\,l\,[y + \sqrt{(yy+C)}] - a\,l\,b,$$

vnde concludimus, fumto $e$ pro numero cuius logarithmus eft $= 1$, fore

$$b\,e^{\frac{x}{a}} = y + \sqrt{(yy+C)},$$

et irrationalitatem tollendo

$$bb\,e^{\frac{2x}{a}} - 2by\,e^{\frac{x}{a}} = C,$$

ita vt fit

$$y = \tfrac{1}{2}b\,e^{\frac{x}{a}} - \frac{C}{2b}\,e^{-\frac{x}{a}}:$$

at forma conftantium C et $b$ mutata, habebitur

$$y = C\,e^{\frac{x}{a}} + D\,e^{-\frac{x}{a}},$$

quae eft aequatio integralis completa.

## Exemplum 2.

746. *Pofito $\partial x$ conftante, fi fuerit $aa\,\partial\partial y + y\,\partial x^2 = 0$; inuenire integrale completum.*

Multiplicatione per $2\,\partial y$ facta, aequationis

$$2aa\,\partial y\,\partial\partial y + 2y\,\partial y\,\partial x^2 = 0,$$

integrale eft

aa

$$a a \partial y^2 + y y \partial x^2 = c c \partial x^2,$$

vnde deducimus

$$\partial x = \frac{a \partial y}{\sqrt{(cc - yy)}},$$

quae denuo integrata dat

$$x = a \text{ Ang. fin. } \frac{y}{c} + b.$$

Erit ergo

$$\frac{y}{c} = \text{fin.} \frac{x - b}{a} = \text{cof.} \frac{b}{a} \text{ fin.} \frac{x}{a} - \text{fin.} \frac{b}{a} \text{ cof.} \frac{x}{a};$$

vel mutatis conftantibus $b$ et $c$, ita vt fit

$$c \text{ cof.} \frac{b}{a} = \text{C} \text{ et } - c \text{ fin.} \frac{b}{a} = \text{D}, \text{ erit}$$

$$y = \text{C fin.} \frac{x}{a} + \text{D cof.} \frac{x}{a}.$$

Vel retenta prima forma, habemus

$$y = \text{C fin.} \left(\frac{x}{a} + \alpha\right).$$

## Corollarium.

747. Hoc exemplum ex praecedente refolui potuiffet, cum fit

$$e^{u\sqrt{-1}} = \text{cof.} u + \sqrt{-1}.\text{fin.} u \text{ et } e^{-u\sqrt{-1}} = \text{cof.} u - \sqrt{-1}.\text{fin.} u,$$

ac viciffim

$$\text{cof.} u \sqrt{-1} = \tfrac{1}{2} e^{u} + \tfrac{1}{2} e^{-u} \text{ et fin.} u \sqrt{-1} = \frac{1}{2\sqrt{-1}} e^{u} - \frac{1}{2\sqrt{-1}} e^{-u}.$$

## Exemplum 3.

748. *Pofito $\partial x$ conftante, fi fuerit $\partial \partial y \sqrt{a y} = \partial x^2$, integrale completum inuenire.*

Cum ergo fit $2 \partial y \partial \partial y = \frac{2 \partial y}{\sqrt{a y}} . \partial x^2$,

erit integrando

$$\partial y^2 = \frac{4 \partial x^2 \sqrt{y}}{\sqrt{a}} + 4 n \partial x^2 = \frac{4 \partial x^2 (\sqrt{y} + n \sqrt{a})}{\sqrt{a}},$$

 vnde

vnde colligimus

$$2\partial x = \frac{\partial y \sqrt[4]{a}}{\sqrt{(\sqrt{y} + n\sqrt{a})}}.$$

Sit commoditatis gratia $n\sqrt{a} = b$ et $\sqrt{y} = z$, vt fiat

$$\partial y = 2z\partial z \text{ et } \frac{\partial x \sqrt{n}}{\sqrt{b}} = \frac{z\partial z}{\sqrt{(b+z)}},$$

cuius integrale eſt

$$\frac{x\sqrt{n}}{\sqrt{b}} = \tfrac{2}{3}(z - 2b)\sqrt{(b+z)} + C,$$

ſeu reſtituendo

$$\frac{x^{?}}{\sqrt[4]{a}} = \tfrac{2}{3}(\sqrt{y} - 2\sqrt{c})\sqrt{(\sqrt{y} + \sqrt{c})} + C,$$

vbi $c$ et C ſunt binae conſtantes arbitrariae. Er. ergo

$$\frac{3(x+f)}{2\sqrt[4]{a}} = (\sqrt{y} - 2\sqrt{c})\sqrt{(\sqrt{y} + \sqrt{c})},$$

poſito $C = \frac{-f}{\sqrt[4]{a}}$, et ſumtis quadratis

$$\frac{9(x+f)^2}{4\sqrt{a}} = y\sqrt{y} - 3y\sqrt{c} + 4c\sqrt{c}$$

## Scholion.

749. Forma ergo vulgaris aequationum hoc modo integrandarum, ſumto elemento $\partial x$ conſtante, eſt $\partial\partial y = Y\partial x^2$, quae per $\partial y$ multiplicata manifeſto fit integrabilis. Sin autem elementum $\partial y$ capiatur conſtans, ob $q = \frac{\partial p}{\partial x}$ et $p = \frac{\partial y}{\partial x}$, erit $q = -\frac{\partial y \partial\partial x}{\partial x^3}$, hincque forma vulgaris $\partial y \partial\partial x = -Y\partial x^3$. Porro ſumto elemento $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ conſtante, vt ſit $\partial x \partial\partial x + \partial y \partial\partial y = 0$, ob $\partial p = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^2}$, erit vel $q = -\frac{\partial s^2 \partial\partial x}{\partial x^3 \partial y}$ vel $q = \frac{\partial s^2 \partial\partial y}{\partial x^4}$, vnde naſcitur haec forma

$$-\frac{\partial s^2 \partial\partial x}{\partial x^3 \partial y} = Y \text{ vel } \frac{\partial s^2 \partial\partial y}{\partial x^4} = Y,$$

quae etiam per $\partial y$ multiplicatae integrabiles euadunt, etiamſi

hoc

hoc iam minus pateat. Simili modo si elementum $y\,\partial x$ sumatur constans, vt sit $y\,\partial\partial x + \partial x\,\partial y = 0$ et $\partial\partial x = -\frac{\partial x\,\partial y}{y}$, ob $\partial p = \frac{\partial\partial y}{\partial x} + \frac{\partial y^2}{y\,\partial x}$, orietur haec forma $y\,\partial\partial y + \partial y^2 = Yy\,\partial x^2$, cuius membrum prius integrabile redditur, si per functionem quamcunque ipsarum $y\,\partial y$ et $y\,\partial x$ multiplicetur, ergo etiam per $\frac{y\,\partial y}{y\,y\,\partial x^2}$, quo multiplicatore simul alterum membrum $Yy\,\partial x^2$ redditur integrabile. His igitur casibus simplicissimis aequationum differentio-differentialium expeditis, qui ne vlla quidem difficultate laborant, ad difficiliores progrediamur, ac primo quidem ad eas aequationes, in quibus altera binarum variabilium $x$ et $y$ ipsa non inest; ita vt aequatio proposita ternas tantum contineat litteras $x$, $p$ et $q$ vel $y$, $p$ et $q$, vtriusque enim ratio fere perinde est comparata.

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# CAPVT II.

## DE AEQVATIONIBVS DIFFERENTIO-DIFFERENTIALIBVS IN QVIBVS ALTERA VARIABILIVM IPSA DEEST.

### Problema 95.

750.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si detur aequatio quaecunque inter tres quantitates $x$, $p$ et $q$, in quam altera variabilis $y$ non ingrediatur, inuestigare relationem inter ipsas variabiles $x$ et $y$.

### Solutio.

Cum aequatio proposita has tres quantitates $x$, $p$ et $q$ contineat, loco $q$ scribatur eius valor $\frac{\partial p}{\partial x}$, atque habebitur aequatio differentialis primi gradus duas tantum quantitates variabiles $x$ et $p$ inuoluens, quam secundum praecepta prioris partis tractari, eiusque integrale inuestigari oportet. Integrali autem inuento, quod si fuerit completum constantem arbitrariam complectetur, inde vel $p$ per $x$, vel $x$ per $p$ determinari poterit. Priori casu quo $p$ per $x$ definire licet, vt $p$ aequatur functioni cuidam ipsius $x$, quae sit $= X$, ob $p = X$ fiet $p \partial x = \partial y = X \partial x$, vnde reperitur $y = \int X \partial x +$ Const. quae aequatio relationem desideratam inter $x$ et $y$ definit. Posteriori casu quo $x$ per $p$ detur, et functioni cuidam P ipsius $p$ aequatur, vt sit $x = P$, erit $y = \int p \partial x = \int p \partial P$, seu $y = Pp - \int P \partial p$.

Sin

Sin autem neque $x$ per $p$, neque $p$ per $x$ definiri queat, videndum eſt, num vtramque per nouam variabilem $u$ exprimere liceat, vnde fiat $x = V$ et $p = U$; tum enim habebitur $y = \int U \partial V$.

## Corollarium 1.

751. Huiusmodi ergo aequationum differentio-differentialium reſolutio ita inſtituitur, vt reuocetur ad aequationem differentialem primi gradus inter binas variabiles $x$ et $p$; quae ſi integrari queat, ſimul illius aequationis integratio habebitur, accedente quadam noua conſtante.

## Corollarium 2.

752. Si aequatio inter $x$, $p$ et $q$ propoſita ita fuerit comparata, vt $q$ vnicam dimenſionem non excedat, vel ſi ad talem formam reduci patiatur, orietur aequatio differentialis ſimplex, differentialia vnius tantum dimenſionis inuoluens, vbi praecepta ante tradita in vſum ſunt vocanda.

## Corollarium 3.

753. Sin autem quantitas $q$ plures obtineat dimenſiones, vel adeo tranſcendenter ingrediatur, tentanda ſunt ea artificia, quae in fine ſuperioris partis circa reſolutionem huiusmodi aequationum ſunt tradita.

## Scholion.

754. Quando in aequatione inter $x$, $p$ et $q$ littera $q$ vnicam habet dimenſionem, indeque poſito $q = \frac{\partial p}{\partial x}$ aequatio differentialis ſimplex naſcitur, praecipui caſus, quibus integratio ſuccedit, ſunt: 1) ſi aequatio haec differentialis ſeparationem admittat, 2) ſi alterutra variabilium $p$ et $x$, differentialium quoque ratione habita, vnam dimenſionem non ſuperet,

ac

ac 3) si ambae variabiles $x$ et $p$ vbique eundem dimensionum numerum constituant, quo casu aequatio homogenea appellatur. Casus minus late patentes, cuiusmodi supra euoluimus, hic non commemoramus. Deinde si quantitas $q$ vel pluribus dimensionibus sit implicata, vel adeo transcendenter ingrediatur, casus praecipui resolutionem admittentes, quemadmodum supra docuimus, sunt: 1) si proponatur aequatio quaecunque inter $x$ et $q$ deficiente $p$, 2) si aequatio tantum $p$ et $q$ contineat, quos binos quidem casus iam capite praecedente tractauimus; 3) si in aequatione proposita binae variabiles $p$ et $x$ vbique eundem dimensionum numerum constituant, 4) si in aequatione inter $x$, $p$ et $q$ altera binarum litterarum $x$ vel $p$ vnicam dimensionem obtineat, denique 5) si aequatio ita fuerit comparata, vt posito $x = v^{\mu}$, $p = z^{\mu+\nu}$ et $q = t^{\nu}$, aequatio oriatur homogenea inter $v$, $z$ et $t$, quae scilicet vbique eundem dimensionum numerum constituant. Secundum hos ergo casus exempla proferamus.

## Exemplum 1.

755. *Inuestigare aequationem inter $x$ et $y$, vt posito $\partial x$ constante, haec formula* $\frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial\partial y}$ *aequetur datae functioni ipsius $x$, quae sit* $= X$.

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, erit

$$\frac{(1+pp)^{\frac{3}{2}}}{q} = X = \frac{(1+pp)^{\frac{3}{2}}\partial x}{\partial p}, \text{ ideoque}$$

$$\frac{\partial x}{X} = \frac{\partial p}{(1+pp)^{\frac{3}{2}}},$$

vbi

vbi cum variabiles $x$ et $p$ ſint a ſe inuicem ſeparatae, integratio dat

$$\frac{p}{\sqrt{(1+pp)}} = \int \frac{\partial x}{X}.$$

Ponatur $\int \frac{\partial x}{X} = V$, integrali completo ſumto, erit V functio ipſius $x$, hinc

$$p = V\sqrt{(1+pp)} \text{ et } p = \frac{V}{\sqrt{(1-VV)}}.$$

Quare

$$\partial y = p\,\partial x = \frac{V\,\partial x}{\sqrt{(1-VV)}},$$

vnde obtinetur

$$y = \int \frac{V\,\partial x}{\sqrt{(1-VV)}}.$$

Tum vero praeterea elicitur elementum

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial x\sqrt{(1+pp)} = \frac{\partial x}{\sqrt{(1-VV)}},$$

cuius integrale praebet

$$\int \partial x\sqrt{(1+pp)} = \int \frac{\partial x}{\sqrt{(1-VV)}}.$$

## Corollarium 1.

756. Si $x$ et $y$ ſint coordinatae orthogonales curuae, erit formula $\frac{(1+pp)^{\frac{3}{2}}}{q}$ eius radius curuedinis, vnde hinc curva definitur, cuius radius curuedinis aequetur functioni cuicunque abſciſſae $x$.

## Corollarium 2.

757. Si ergo radius curuedinis debeat eſſe reciproce proportionalis abſciſſae $x$, ſumatur $X = \frac{aa}{2x}$, eritque

$$V = \int \frac{2x\,\partial x}{aa} = \frac{xx+ab}{aa}, \text{ hinc}$$

$$y = \int \frac{(xx+ab)\,\partial x}{\sqrt{[a^4 - (xx+ab)^2]}},$$

quae conditio praebet curuas a lamina elaſtica formatas.

## Corollarium. 3.

758. Si ſit $V = x^n$, ſeu $X = \frac{1}{n x^{n-1}}$, neglecta conſtante addenda, oritur $y = \int \frac{x^n \partial x}{\sqrt{(1 - x^{2n})}}$, quod integrale algebraice exhiberi poteſt caſibus, quibus eſt vel $n = \frac{1}{2i+1}$ vel $n = \frac{-1}{2i}$, denotante $i$ numerum integrum poſitiuum.

## Exemplum 2.

759. *Si poſito $\partial x$ conſtante, oporteat eſſe*

$$\partial x (\partial x^2 + \partial y^2) + x \partial y \partial\partial y = a \partial\partial y \sqrt{(\partial x^2 + \partial y^2)};$$

*inuenire aequationem inter $x$ et $y$.*

Poſito $\partial y = p \partial x$, noſtra aequatio ob $\partial\partial y = \partial p \partial x$ induit hanc formam

$$\partial x (1 + pp) + x p \partial p = a \partial p \sqrt{(1 + pp)},$$

quae per $\sqrt{(1 + pp)}$ diuiſa fit integrabilis, oritur enim

$$x \sqrt{(1 + pp)} = a p + b \text{ ſeu } x = \frac{ap + b}{\sqrt{(1 + pp)}}.$$

Cum nunc ſit

$$y = \int p \partial x = p x - \int x \partial p, \text{ erit}$$

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - \int \frac{\partial p (ap + b)}{\sqrt{(1 + pp)}},$$

et integratione euoluta

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - a \sqrt{(1 + pp)} - b l \frac{p + \sqrt{(1 + pp)}}{n}, \text{ ſeu}$$

$$y = \frac{bp - a}{\sqrt{(1 + pp)}} - b l \frac{p + \sqrt{(1 + pp)}}{n};$$

ita vt ambae variabiles $x$ et $y$ per $p$ definiantur.

Cum igitur ex priori eliciatur

$$p = \frac{ab + x \sqrt{(aa + bb - xx)}}{xx - aa} \text{ et } \sqrt{(1 + pp)} = \frac{bx + a \sqrt{(aa + bb - xx)}}{xx - aa},$$

erit his valoribus ſubſtitutis

$y =$

$$y = \frac{a(aa+bb-xx)+bx\sqrt{(aa+bb-xx)}}{bx+a\sqrt{(aa+bb-xx)}} - b\,l\,\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)},$$ seu
$$y = \sqrt{(aa+bb-xx)} - b\,l\,\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}.$$

## Corollarium.

760. Si constans priori integratione ingressa $b$ euanescens sumatur, aequatio inter $x$ et $y$ fit algebraica, erit enim $y = \sqrt{(aa-xx)}$. Sin autem $b$ non euanescat, aequatio integralis est transcendens, et logarithmos inuoluit.

## Exemplum 3.

761. *Posito $\partial x$ constante, si debeat esse*
$$aa\,\partial\partial y\sqrt{(aa+xx)} + aa\,\partial x\,\partial y = xx\,\partial x^2,$$
*inuenire aequationem inter $x$ et $y$.*

Posito $\partial y = p\,\partial x$, habebimus hanc aequationem
$$aa\,\partial p\sqrt{(aa+xx)} + aap\,\partial x = xx\,\partial x,$$ seu
$$\partial p + \frac{p\,\partial x}{\sqrt{(aa+xx)}} = \frac{xx\,\partial x}{aa\sqrt{(aa+xx)}},$$
in qua variabilis $p$ vnam dimensionem non superat. Cum ergo sit
$$\int\frac{\partial x}{\sqrt{(aa+xx)}} = l\,[x+\sqrt{(aa+xx)}],$$
haec aequatio integrabilis redditur, si multiplicetur per $x + \sqrt{(aa+xx)}$, tum enim prodit
$$p\,[x+\sqrt{(aa+xx)}] = \int\frac{xx\,\partial x\,[x+\sqrt{(aa+xx)}]}{aa\sqrt{(aa+xx)}},$$ seu
$$p\,[x+\sqrt{(aa+xx)}] = \frac{1}{aa}\int\frac{x^3\,\partial x}{\sqrt{(aa+xx)}} + \frac{x^3}{3aa},$$ et
$$\int\frac{x^3\,\partial x}{\sqrt{(aa+xx)}} = \tfrac{1}{3}(xx-2aa)\sqrt{(aa+xx)} + C,$$
hinc
$$p\,[x+\sqrt{(aa+xx)}] = \frac{(xx-2aa)\sqrt{(aa+xx)}+x^3}{3aa} + C.$$
Haec multiplicetur per $\sqrt{(aa+xx)} - x$, vt prodeat,
$$aap = \frac{-xx-2aa+2x\sqrt{(aa+xx)}}{3} + C\sqrt{(aa+xx)} - Cx,$$

 et

et quia $\partial y = p\,\partial x$, erit integrando

$$a a y = -\tfrac{1}{9} x^3 - \tfrac{2}{3} a a x + \tfrac{2}{9}(a a + x x)\sqrt{(a a + x x)}$$
$$- \tfrac{1}{2} C x x + C \int \partial x \sqrt{(a a + x x)}.$$

Quodsi ergo constans C euanescat, aequatio inter $x$ et $y$ erit algebraica; scilicet

$$9 a a y + 6 a a x + x^3 = 2(a a + x x)\sqrt{(a a + x x)}.$$

## Exemplum 4.

762. *Posito $\partial x$ constante, inuenire integrale huius aequationis differentio-differentialis*

$$(a a \partial y^2 + x x \partial x^2)\,\partial\partial y = n x \partial x^3 \partial y.$$

Fiat $\partial y = p\,\partial x$, et ob $\partial\partial y = \partial p\,\partial x$ habebimus

$$(a a p p + x x)\,\partial p = n p x \partial x,$$

quae aequatio cum sit homogenea, statuamus $x = p u$, eritque

$$p p (a a + u u)\,\partial p = n p p u (p\,\partial u + u\,\partial p), \text{ seu}$$
$$\frac{\partial p}{p} = \frac{n u \partial u}{a a + (1-n) u u},$$

quae integrata dat

$$l p = \frac{n}{2(1-n)}\, l\,[a a + (1-n) u u] + \text{Const.}$$

Hinc colligitur

$$p = C\,[a a + (1-n) u u]^{\frac{n}{2(1-n)}}, \text{ atque}$$
$$x = C u\,[a a + (1-n) u u]^{\frac{n}{2(1-n)}}.$$

Cum nunc sit

$$y = p x - \int x \partial p \text{ et } \partial p = C n u \partial u\,[a a + (1-n) u u]^{\frac{3n-2}{2(1-n)}},$$

erit

$$y = C C u\,[a a + (1-n) u u]^{\frac{n}{1-n}} - n C C \int u u \partial u\,[q a + (1-n) u u]^{\frac{2n-1}{2(1-n)}}.$$

Casu

Casu autem $n = 1$ erit

$$l p = \frac{uu}{2aa} + C, \text{ et } u = a \sqrt{2 l \tfrac{p}{c}}, \text{ hinc}$$

$$x = a p \sqrt{2 l \tfrac{p}{c}}, \text{ et } y = a p p \sqrt{2 l \tfrac{p}{c}} - a \int p \partial p \sqrt{2 l \tfrac{p}{c}}.$$

## Corollarium.

763. Si fuerit $n = \frac{1}{2}$, erit

$$x = C u \sqrt{(a a + \tfrac{1}{2} u u)} \text{ et}$$

$$y = C C u (a a + \tfrac{1}{2} u u) - \frac{C C u^3}{6} + D = C C u (a a + \tfrac{1}{3} u u) + D;$$

sicque relatio inter $x$ et $y$ algebraice exprimitur, quod etiam fit, si $n = \frac{2}{3}$, vel $n = \frac{3}{4}$, vel $n = \frac{4}{5}$, etc.

## Exemplum 5.

764. *Posito $\partial x$ constante, integrare hanc aequationem differentio-differentialem*

$$a \partial x \partial y^2 + x x \partial x \partial \partial y = n x \partial y \sqrt{(\partial x^4 + a a \partial \partial y^2)},$$

Fiat $\partial y = p \partial x$ et $\partial p = q \partial x$, vt fit $\partial \partial y = q \partial x^2$, et nostra aequatio induet hanc formam

$$a p p + q x x = n p x \sqrt{(1 + a a q q)},$$

quae est homogenea inter $p$ et $x$. Statuatur ergo $p = u x$, fietque

$$a u u + q = n u \sqrt{(1 + a a q q)}.$$

Iam vero est

$$\partial p = q \partial x = u \partial x + x \partial u,$$

vnde fit $\frac{\partial x}{x} = \frac{\partial u}{q - u}$.

At ex illa aequatione inter $q$ et $u$ colligitur

$$q = \frac{a u u + n u \sqrt{(1 - n n a a u u + a^4 u^4)}}{n n a a u u - 1}, \text{ et}$$

$$q - u = \frac{u(1 + a u - n n a a u u) + n u \sqrt{(1 - n n a a u u + a^4 u^4)}}{n n a a u u - 1},$$

 sicque

ſicque

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{nnaauu - 1}{1 + au - nnaauu + n\sqrt{(1 - nnaauu + a^4u^4)}}.$$

Dabitur ergo $x$ per $u$, hincque etiam $p = ux$ per $u$; vnde deducitur $y = \int p\,\partial x = \int ux\,\partial x$.

## Corollarium 1.

765. Illa aequatio differentialis transformatur in hanc

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + au - nnaauu - n\sqrt{(1 - nnaauu + a^4u^4)}}{nn - 1 - 2au + (nn-1)aauu},$$

vnde ratio integrationis facilius perſpicitur.

## Corollarium 2.

766. Notatu dignus autem eſt caſus $nn = 2$, quo fit

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + au - 2aauu - (1 - aauu)\sqrt{2}}{(1-au)^2}, \text{ ſeu}$$

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + 2au - (1+au)\sqrt{2}}{1 - au} = \frac{\partial u(1-\sqrt{2})}{u} + \frac{a\,\partial u(3 - 2\sqrt{2})}{1 - au};$$

vnde colligitur

$$lx = (1-\sqrt{2})\,lu - (3 - 2\sqrt{2})\,l(1 - au) + \text{Conſt. ſeu}$$

$$x\,u^{\sqrt{2}-1}(1 - au)^{3-2\sqrt{2}} = C.$$

## Exemplum 6.

767. *Sumto elemento* $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ *conſtante, inuenire integrale huius aequationis*

$$\partial x^3\,\partial y - x\,\partial s^2\,\partial\partial y = a\,\partial x\,\partial s\,\sqrt{(\partial\partial x^2 + \partial\partial y^2)}.$$

Poſito $\partial y = p\,\partial x$, erit $\partial s = \partial x\sqrt{(1 + pp)}$, et ob $\partial\partial s = 0$ fit

$$\partial\partial x = -\frac{p\,\partial p\,\partial x}{1 + pp} = \frac{-pq\,\partial x^2}{1+pp},$$

exiſtente $\partial p = q\,\partial x$, tum vero

$$\partial\partial y = p\,\partial\partial x + \partial p\,\partial x = \frac{-ppq\,\partial x^2}{1+pp} + q\,\partial x^2 = \frac{q\,\partial x^2}{1+pp},$$

ideoque

$$\sqrt{(\partial\partial x^2 + \partial\partial y^2)} = \frac{q\,\partial x^2}{\sqrt{(1+pp)}},$$

quibus substitutis, aequatio nostra induit hanc formam

$p - qx = aq$, quae differentiata praebet $-x\,\partial q = a\,\partial q$, ideoque $\partial q = 0$ et $q = \frac{1}{c}$. Hinc $p = \int q\,\partial x = \frac{x+a}{c}$, qui idem valor ex aequatione $p = (x+a)q$ sine integratione obtinetur. Tum vero est $y = \int p\,\partial x = \frac{xx+2ax}{2c} + b$, quae est aequatio integralis completa binas constantes $b$ et $c$ inuoluens.

## Exemplum 7.

768. *Sumto elemento $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constante, inuenire integrale huius aequationis differentio-differentialis*

$$\partial x^3 \partial y - x\,\partial s^2\,\partial\partial y = \frac{b\,\partial x^4\,\partial s^2\,\partial\partial y}{\sqrt{(\partial x^4 + aa\,\partial s^4\,\partial\partial y^2)}}.$$

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, ob $\partial s = \partial x\sqrt{(1+pp)}$, et $\partial\partial s = 0$, erit

$$\partial\partial x = \frac{-pq\,\partial x^2}{1+pp} \text{ et } \partial\partial y = \frac{-\partial x\,\partial\partial x}{\partial y} = \frac{-\partial\partial x}{p} = \frac{q\,\partial x^2}{1+pp},$$

ergo $\partial s^2\,\partial\partial y = q\,\partial x^4$, vnde aequatio nostra fit

$$p - qx = \frac{bq}{\sqrt{(1+aaqq)}},$$

quae differentiata praebet

$$-x\,\partial q = \frac{b\,\partial q}{(1+aaqq)^{\frac{3}{2}}};$$

vnde concluditur vel $\partial q = 0$ vel $x = \dfrac{-b}{(1+aaqq)^{\frac{3}{2}}}$.

Priori casu est $q = \frac{1}{c}$, et $p = \frac{x}{c} + \frac{b}{\sqrt{(cc+aa)}}$, hincque

$$y = \int p\,\partial x = \frac{xx}{2c} + \frac{bx}{\sqrt{(cc+aa)}} + f.$$

Posteriori casu quo $x = \dfrac{-b}{(1+aaqq)^{\frac{3}{2}}}$ fit

$p =$

$$p = \frac{-bq}{(1+aaqq)^{\frac{3}{2}}} + \frac{bq}{\sqrt{(1+aaqq)}} = \frac{aabq^3}{(1+aaqq)^{\frac{3}{2}}}.$$

At eſt

$$\partial x = \frac{+3aabq\partial q}{(1+aaqq)^{\frac{5}{2}}},$$ hincque

$$\partial y = p\,\partial x = \frac{3a^4bbq^4\partial q}{(1+aaqq)^4},$$

et ope reductionum

$$y = \frac{-\frac{1}{2}bbq - aabbq^3}{(1+aaqq)^3} + \tfrac{1}{2}bb\int\frac{\partial q}{(1+aaqq)^3}.$$

Eſt vero

$$\int\frac{\partial q}{(1+aaqq)^{n+1}} = \frac{q}{2n(1+aaqq)^n} + \frac{2n-1}{2n}\int\frac{\partial q}{(1+aaqq)^n}.$$

Ergo

$$\int\frac{\partial q}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \tfrac{3}{4}\int\frac{\partial q}{(1+aaqq)^2}$$ et

$$\int\frac{\partial q}{(1+aaqq)^2} = \frac{q}{2(1+aaqq)} + \tfrac{1}{2}\int\frac{\partial q}{1+aaqq}$$

$$= \frac{q}{2(1+aaqq)} + \frac{1}{2a}\,\text{Ang. tang. } aq.$$

Hinc

$$\int\frac{\partial q}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \frac{3q}{8(1+aaqq)} + \frac{3}{8a}\,\text{Ang. tang. } aq,$$

ideoque

$$y = \frac{-bbq(1+2aaqq)}{2(1+aaqq)^3} + \frac{bbq}{8(1+aaqq)^2} + \frac{3bbq}{16(1+aaqq)}$$
$$+ \frac{3bb}{16a}\,\text{Ang. tang. } aq,$$

exiſtente

$$x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}},$$

vnde fit

1 +

$$1 + aaqq = \sqrt[3]{\frac{bb}{xx}},$$

ita vt hoc modo aequatio inter $x$ et $y$ exhiberi poſſit. Hoc autem integrale, vt ſupra vidimus, tantum eſt particulare.

## Problema 96.

769. Poſito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, ſi detur aequatio quaecunque inter $y$, $p$ et $q$, ita vt variabilis $x$ ipſa in ea deſit, inueſtigare aequationem integralem inter $x$ et $y$.

## Solutio.

Cum ſit $q = \frac{\partial p}{\partial x}$ et $\partial x = \frac{\partial y}{p}$, erit $q = \frac{p\,\partial p}{\partial y}$; in aequatione ergo inter $y$, $p$ et $q$ vbique loco $q$ ſubſtituatur iſte valor $\frac{p\,\partial p}{\partial y}$, atque habebitur aequatio differentialis primi gradus binas tantum variabiles $p$ et $y$ inuoluens, cuius reſolutionem per methodos ſupra expoſitas tentari oportet. Inuenta autem aequatione integrali inter $p$ et $y$, inde vel $p$ per $y$, vel $y$ per $p$ definiatur, quo facilius altera integratio inſtitui poſſit. Si $y$ per $p$ commode definiri queat, vt $y$ aequetur functioni cuipiam ipſius $p$, quae ſit $= \mathrm{P}$, vt ſit $y = \mathrm{P}$, erit $\partial x = \frac{\partial \mathrm{P}}{p}$, hincque $x = \int \frac{\partial \mathrm{P}}{p} = \frac{\mathrm{P}}{p} + \int \frac{\mathrm{P}\,\partial p}{pp}$. Sin autem commodius $p$ per $y$ definire liceat, vt ſit $p = \mathrm{Y}$ denotante Y functionem quampiam ipſius $y$, ob $\partial x = \frac{\partial y}{p}$, habebitur $x = \int \frac{\partial y}{\mathrm{Y}}$. At ſi neutrum ſuccedat, nouam variabilem $u$ introducendo, per eam vtraque quantitas $p$ et $y$ definiatur, vt fiat $p = \mathrm{U}$ et $y = \mathrm{V}$, exiſtentibus U et V functionibus ipſius $u$, atque hinc erit $\partial x = \frac{\partial \mathrm{V}}{\mathrm{U}}$, et $x = \int \frac{\partial \mathrm{V}}{\mathrm{U}}$; hocque modo per duplicem integrationem integrale completum obtinebitur.

## Corollarium 1.

770. Huiusmodi ergo aequationum differentio-differentialium reſolutio quoque reuocatur ad aequationem diffe-

rentialem primi gradus, cuius resolutio si fuerit in potestate; simul illius integrale exhiberi poterit.

## Corollarium 2.

771. Si aequatio inter $y$, $p$ et $q$ ita fuerit comparata, vt ex ea commode valor ipsius $q$ elici queat, hincque $q$ aequetur functioni ipsarum $y$ et $p$, quae sit T, erit $p\partial p = T\partial y$, quae est aequatio differentialis primi gradus simplex.

## Corollarium 3.

772. Sin autem huiusmodi euolutio non succedat, dum littera $q$ vel ad altiores potestates exsurgit, vel signis radicalibus inuoluitur, vel adeo transcendenter ingreditur, aequatio differentialis quidem erit primi gradus sed complicata, quae methodis supra expositis erit tractanda.

## Scholion 1.

773. Cum paucis casibus aequationes differentiales primi gradus integrari queant, eosdem etiam hic notasse et per exempla illustrasse iuuabit. Interim vero et reliquos casus quasi solutos spectari conuenit, quandoquidem in aequationibus differentialibus altiorum ordinum id potissimum desideratur vt earum resolutio ad ordinem inferiorem reducatur. Perpetuo enim in Analysi quae ordine tractationis praecedunt, tanquam penitus confecta spectari solent, etiamsi plurima adhuc desiderentur, vt hoc modo multitudo desideratorum diminuatur. Ita quamuis longe adhuc absit, quominus aequationes algebraicas omnium ordinum resoluere valeamus, dum adeo vires nostrae non vltra quartum extenduntur, tamen in Analysi sublimiori omnium istarum aequationum resolutionem pro cognita habemus. Quod etiam vsu non caret, cum in praxi resolutio per approximationem, quam quousque luberit, exten-

extendere licet, sufficere possit. Simili modo etiam, quoniam methodum tradidimus, aequationum differentialium primi gradus integralia proxime inueniendi, merito totum negotium, vt plane confectum, est censendum, si eo resolutionem aequationum differentialium altiorum graduum reducere potuerimus. Quare in hac secunda parte statim atque aequationem differentialem secundi gradus ad primum gradum perduxerimus, totum negotium pro confecto erit habendum.

## Scholion 2.

774. Aequationes ergo differentio-differentiales, quae hoc modo ad differentiales primi gradus reducuntur, ita sunt comparatae, vt posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, variabilis $x$ ipsa inde tollatur, et aequatio inter solas tres variabiles $y$, $p$ et $q$ oriatur. Casus ergo quibus talis aequatio resolutionem admittit, duplicis sunt generis, ad quorum prius referendi sunt ii, quibus $q$ vnicam obtinet dimensionem, vnde $q$ functioni cuipiam ipsarum $y$ et $p$ aequari potest. Cum igitur sit $q = \frac{p\,\partial p}{\partial y} = f:(y \text{ et } p)$ quam ponamus $= T$, resolutio succedet. 1) Si $T$ sit functio homogenea vnius dimensionis ipsarum $y$ et $p$. 2) Si fuerit $T = \frac{P}{y+Q}$, designantibus $P$ et $Q$ functiones quascunque ipsius $p$ tantum, hinc enim fit

$$P\,\partial y = y\,p\,\partial p + Q\,p\,\partial p;$$

quorsum etiam refertur casus,

$$T = \frac{P}{y + Q y^n}.$$

3) Si fuerit $T = p(Yp + Z)$, si quidem $Y$ et $Z$ sint functiones quaecunque ipsius $y$, quia tum aequatio

$$\partial p = Y\,p\,\partial y + Y\,\partial y,$$

ob vnicam dimensionem ipsius $p$ est integrabilis, quorsum etiam referendus est casus $T = p(Yp + Zp^n)$. Pro altero genere si quantitas $q$ plures habeat dimensiones, vel signis radicalibus sit implicata, vel adeo transcendenter ingrediatur, aequatio inter $y$, $p$ et $q$, resolutionem admittet. 1) Si posito $q = pu$, vt sit $u = \frac{\partial p}{\partial y}$, aequatio resultet homogenea inter $y$ et $p$, in qua scilicet $y$ et $p$ vbique eundem dimensionum numerum compleant, vtcunque caeterum $u$ in eam ingrediatur. 2) Si in aequatione post substitutionem $q = pu$ inter $y$, $p$ et $u$ orta, altera quantitas $y$ vel $p$ vnicam obtineat dimensionem. 3) Si posito $y = v^\mu$, $p = z^{\mu+\nu}$ et $u = t^\nu$ aequatio oriatur homogenea inter ternas quantitas $v$, $z$ et $t$, huiusmodi enim aequationes supra resoluere docuimus.

## Exemplum 1.

775. *Posito elemento $\partial x$ constante, si habeatur haec aequatio differentio-differentialis*

$$\partial\partial y + A\partial x\partial y + By\partial x^2 = 0,$$

*eius integrale completum invenire.*

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, aequatio nostra erit

$$q + Ap + By = 0, \text{ seu } p\partial p + Ap\partial y + By\partial y = 0,$$

quae cum sit homogenea, posito $p = vy$, abit in

$$vvy\partial y + vyy\partial v + Avy\partial y + By\partial y = 0,$$

vnde fit

$$\frac{\partial y}{y} + \frac{v\partial v}{vv + Av + B} = 0.$$

Sit $vv + Av + B = (v + \alpha)(v + \beta)$,
vt sit $\alpha + \beta = A$ et $\alpha\beta = B$ erit

$$\frac{\partial y}{y} + \frac{\alpha\partial v}{(\alpha - \beta)(v + \alpha)} - \frac{\beta\partial v}{(\alpha - \beta)(x + \beta)} = 0,$$

hincque integrando

ly

$$ly + \frac{\alpha}{\alpha-\beta} l(v+\alpha) - \frac{\beta}{\alpha-\beta} l(v+\beta) = C$$ seu

$$y = a(v+\beta)^{\frac{\beta}{\alpha-\beta}} (v+a)^{\frac{-\alpha}{\alpha-\beta}},$$ ideoque

$$p = vy = av(v+\beta)^{\frac{\beta}{\alpha-\beta}} (v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}.$$

Tum vero eſt

$$\partial x = \frac{\partial y}{p} = \frac{\partial y}{vy},$$ vnde ob

$$\frac{\partial y}{y} = \frac{-v\partial v}{vv+Av+B},$$ erit

$$\partial x = \frac{-\partial v}{vv+Av+B} = \frac{\partial v}{(\alpha-\beta)(v+\alpha)} - \frac{\partial v}{(\alpha-\beta)(v+\beta)},$$ et

$$x = \frac{1}{\alpha-\beta} l \frac{v+\alpha}{v+\beta} + \text{Conſt.}$$

Verum haec reſolutio fit facilior ſequenti modo: Cum ſit

$$\frac{\partial y}{y} = \frac{-v\partial v}{(v+\alpha)(v+\beta)}$$ et $\partial x = \frac{-\partial v}{(v+\alpha)(v+\beta)}$, erit

$$\frac{\partial y}{y} + \alpha\partial x = \frac{-\partial v}{v+\beta}$$ et $\frac{\partial y}{y} + \beta\partial x = \frac{-\partial v}{v+\alpha}$, hinc

$$ly + \alpha x = la - l(v+\beta)$$ et $ly + \beta x = lb - l(v+\alpha)$.

Ergo

$$v+\beta = \frac{a}{y} e^{-\alpha x}$$ et $v+\alpha = \frac{b}{y} e^{-\beta x}$,

vnde fit

$$\alpha - \beta = \frac{1}{y}(b e^{-\beta x} - a e^{-\alpha x}),$$

ideoque mutatis conſtantibus

$$y = \mathfrak{A} e^{-\alpha x} + \mathfrak{B} e^{-\beta x},$$

quae integratio locum habet, ſi $\alpha$ et $\beta$ ſint quantitates reales et inaequales. Cum igitur poſuerimus

$$vv + Av + B = (v+\alpha)(v+\beta)$$ erit

$$\alpha = \tfrac{1}{2}A + \sqrt{(\tfrac{1}{4}AA - B)}$$ et $\beta = \tfrac{1}{2}A - \sqrt{(\tfrac{1}{4}AA - B)}$,

hinc prout expreſſio $\frac{1}{4}AA - B$ fuerit vel poſitiua, vel negatiua, vel euaneſcens, tres habebimus caſus euoluendos:

1) Sit $\frac{1}{2}A = m$ et $\sqrt{(\frac{1}{4}AA - B)} = n$, erit aequationis propositae integrale completum

$$y = \mathfrak{A}e^{-(m+n)x} + \mathfrak{B}e^{-(m-n)x} = e^{-mx}(\mathfrak{A}e^{-nx} + \mathfrak{B}e^{nx}).$$

2) Sit $\frac{1}{2}A = m$ et $\sqrt{(\frac{1}{4}AA - B)} = n\sqrt{-1}$, ob

$$e^{nx\sqrt{-1}} = \text{cos}.\, nx + \sqrt{-1}.\, \text{sin}.\, nx \text{ et}$$
$$e^{-nx\sqrt{-1}} = \text{cos}.\, nx - \sqrt{-1}.\, \text{sin}.\, nx,$$

erit constantibus mutandis

$$y = e^{-mx}(\mathfrak{C}\,\text{cos}.\, nx + \mathfrak{D}\,\text{sin}.\, nx) = \mathfrak{E}e^{-mx}\text{cos}.\,(nx + \epsilon).$$

3) Sit $\frac{1}{2}A = m$ et $\sqrt{(\frac{1}{4}AA - B)} = 0$, seu in casu primo $n = 0$, ob $e^{-nx} = 1 - nx$ et $e^{nx} = 1 + nx$ fiet

$$y = e^{-mx}(\mathfrak{C} + \mathfrak{D}x).$$

## Corollarium 1.

776. Ad aequationis ergo propositae integrale inueniendum, aequationis $vv + Av + B = 0$ radices inuestigari oportet, quibus inuentis facile erit integrale completum assignare.

## Corollarium 2.

777. Haec autem aequatio quadratica $vv + Av + B = 0$ insignem habet analogiam cum ipsa aequatione proposita

$$\partial\partial y + A\partial y\partial x + By\partial x^2 = 0,$$

ex qua quippe oritur scribendo $1$, $v$, $v^2$ loco $y$, $\frac{\partial y}{\partial x}$ et $\frac{\partial\partial y}{\partial x^2}$.

## Corollarium 3.

778. Formata autem aequatione hac algebraica $vv + Av + B = 0$, si eius factor sit $v + \alpha$, ex eo statim integrale particulare deducitur $y = \mathfrak{A}e^{-\alpha x}$, similiterque alter factor $v + \beta$ integrale particulare dabit $y = \mathfrak{B}e^{-\beta x}$, quibus coniunctis obtinetur integrale completum $y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x}$.

Scho-

## Scholion.

779. Infra methodus facilior tradetur huiusmodi aequationes differentio-differentiales tractandi, quae adeo ad talem formam

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = 0$$

patet, vbi P et Q sint functiones quaecunque ipsius $x$, quae etiam extendetur ad formam

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = X \partial x^2,$$

sumendo pro X functionem quamcunque ipsius $x$. Methodus scilicet ea inde haurietur, quod in huiusmodi aequationibus variabilis $y$ cum suis differentialibus $\partial y$ et $\partial \partial y$ vbique vnicam dimensionem constituat vel etiam nullam; eiusque ope resolutio ad aequationem differentialem primi gradus reducetur, quo ipso negotium pro confecto erit habendum. Quando autem hoc modo aequatio differentio-differentialis ad aequationem differentialem primi gradus reducitur, probe cauendum est, ne haec reductio pro integratione habeatur, quippe ad quam tantum ope idoneae substitutionis est peruentum; nihilo enim minus duae adhuc integrationes supersunt absoluendae, quibus totidem constantes arbitrariae introducantur, si quidem integrale completum desideretur, quemadmodum in hoc exemplo et praecedentibus clare videmus.

## Exemplum 2.

780. *Proposita aequatione differentio-differentiali*

$$a b \partial \partial y = \partial x \sqrt{(y y \partial x^2 + a a \partial y^2)},$$

*eius integrale inuestigare.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, haec aequatio abit in hanc

*a b q*

$$abq = \sqrt{(yy + aapp)} = \frac{abp\partial p}{\partial y}, \text{ ob } q = \frac{p\partial p}{\partial y},$$

quae cum ſit homogenea ponatur $p = \frac{y}{u}$, erit

$$y\partial y\sqrt{(1 + \frac{aa}{uu})} = \frac{aby}{u^3}(u\partial y - y\partial u), \text{ ſeu}$$

$$uu\partial y\sqrt{(aa + uu)} = abu\partial y - aby\partial u, \text{ vnde ſit}$$

$$\frac{\partial y}{y} = \frac{ab\partial u}{abu - uu\sqrt{(aa + uu)}}.$$

Ponatur $\sqrt{(aa + uu)} = su$; erit $uu = \frac{aa}{ss - 1}$,

$$\frac{\partial u}{u} = \frac{-s\partial s}{ss - 1}, \text{ et } \frac{\partial y}{y} = \frac{-bs\partial s}{bss - as - b} = \frac{-s\partial s}{ss - 2ns - 1},$$

poſito $\frac{a}{b} = 2n$. Ergo

$$\frac{2\partial y\sqrt{(nn+1)}}{y} = \frac{-\partial s[n + \sqrt{(nn+1)}]}{s - n - \sqrt{(nn+1)}} + \frac{\partial s[n - \sqrt{(nn+1)}]}{s - n + \sqrt{(nn+1)}},$$

ideoque

$$y^{2\sqrt{(nn+1)}} = \frac{C[s - n + \sqrt{(nn+1)}]^{n - \sqrt{(nn+1)}}}{[s - n - \sqrt{(nn+1)}]^{n + \sqrt{(nn+1)}}}.$$

Datur igitur $y$ per $s$, vt ſit $y = S$, hincque

$$u = \frac{a}{\sqrt{(ss - 1)}} \text{ et } p = \frac{S\sqrt{(ss - 1)}}{a}, \text{ atque}$$

$$\partial x = \frac{a\partial S}{S\sqrt{(ss - 1)}}, \text{ ſeu } \partial x = \frac{-as\partial s}{(ss - 2ns - 1)\sqrt{(ss - 1)}},$$

quae formula ad rationalitatem perduci et per logarithmos ſeu arcus circulares integrari poteſt.

## Exemplum 3.

781. *Poſito $\partial y = p\partial x$ et $\partial p = q\partial x$, inuenire integrale huius aequationis* $\frac{(pp + yy)\sqrt{(pp + yy)}}{2pp + yy - qy} = ny$.

Cum ſit $q = \frac{p\partial p}{\partial y}$; erit

$$\partial y(pp + yy)\sqrt{(pp + yy)} = 2nppy\partial y + ny^3\partial y - nyyp\partial p,$$

ob cuius homogeneitatem ponatur $p = uy$, fietque

$$y^3\partial y(uu + 1)^{\frac{3}{2}} = 2nuuy^3\partial y + ny^3\partial y - nuuy^3\partial y - nuy^4\partial u$$

vnde colligitur

$\frac{\partial y}{y}$

$$\frac{\partial y}{y} = \frac{-nu\partial u}{(uu+1)\sqrt{(uu+1)}-nuu-n} = \frac{nu\partial u}{(uu+1)[n-\sqrt{(uu+1)}]},$$

et $y$ per $u$ definitur; ex quo erit $p = uy$ et

$$\partial x = \frac{\partial y}{uy} = \frac{n\partial u}{(uu+1)[n-\sqrt{(uu+1)}]}.$$

Casu quo $n = 1$ erit

$$\frac{\partial y}{y} = \frac{-u\partial u}{(uu+1)[\sqrt{(uu+1)}-1]} = \frac{-\partial u[1+\sqrt{(uu+1)}]}{u(uu+1)}, \text{ et}$$

$$\partial x = \frac{-\partial u[1+\sqrt{(uu+1)}]}{uu(uu+1)}. \quad \text{Est vero}$$

$$\int \frac{\partial u}{u(uu+1)} = l\frac{u}{\sqrt{(uu+1)}}, \int \frac{\partial u}{uu(uu+1)} = -\frac{1}{u} - \text{Ang. tang. } u,$$

$$\int \frac{\partial u}{u\sqrt{(uu+1)}} = l\frac{\sqrt{(uu+1)}-1}{u}, \int \frac{\partial u}{uu\sqrt{(uu+1)}} = -\frac{\sqrt{(uu+1)}}{u};$$

vnde colligitur

$$y = \frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1} = C\left(1 + \frac{1}{\sqrt{(uu+1)}-1}\right) \text{ et}$$

$$x = D + \frac{1+\sqrt{(uu+1)}}{u} + \text{Ang. tang. } u.$$

Inde est

$$\sqrt{(uu+1)} = \frac{y}{y-a}; \text{ et } u = \frac{\sqrt{(2ay-aa)}}{y-a},$$

ideoque

$$x = D + \sqrt{\frac{2y-a}{a}} + \text{Ang. cos. } \frac{y-a}{y},$$

quae formulae introducendo angulum $\Phi$ cuius cosinus est $\frac{y-a}{y}$, ita commodius exhibentur

$$y = \frac{a}{1-\cos.\Phi} \text{ et } x = \zeta + \Phi + \cot.\tfrac{1}{2}\Phi.$$

## Corollarium 1.

782. Ex aequatione separata primum inuenta solutio particularis eruitur, tribuendo ipsi $u$ eiusmodi valorem constantem, vt denominator euanescat, qui est $u = \sqrt{(nn-1)}$; hinc $p = y\sqrt{(nn-1)}$ et $\partial x\sqrt{(nn-1)} = \frac{\partial y}{y}$, vnde fit

$$ly = la + x\sqrt{(nn-1)}.$$

## Corollarium 2.

783. Casu quo $n = 1$, hic casus particularis praebet $y = a$, pro valore quocunque alterius variabilis; fit enim $u = 0$, ideoque et $p = 0$, ita vt ex aequatione $\partial y = p\,\partial x$ quantitas $x$ non determinetur.

## Scholion.

784. Si $y$ designet radium vectorem ex puncto fixo ad curuam quampiam ductum, et $x$ angulum, quem iste radius cum recta quadam positione data constituit, formula

$$\frac{(pp + yy)\sqrt{(pp + yy)}}{2pp + yy - qy}$$

exprimit huius curuae radium curuedinis. In exemplo ergo proposito eiusmodi quaeritur curua, cuius radius curuedinis aequetur ipsi $ny$, cui quaestioni casu $n = 1$ vtique satisfacit valor $y = a$, qui praebet circulum, qui etiam ex aequatione integrali colligitur $y = \frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)} - 1}$, sumendo constantem $C$ nihilo aequalem, tum enim necesse est sit $u = 0$ et $p = 0$, sicque angulus $x$ non determinatur. Praeter circulum autem infinitae aliae lineae curuae satisfaciunt. At si $n > 1$, solutio particularis $ly = la + x\sqrt{(nn - 1)}$ praebet spiralem logarithmicam, praeter quam autem etiam infinitae aliae curuae satisfaciunt; casibus autem $n < 1$ nulla huiusmodi solutio particularis locum habet, sed formulas pro $\frac{\partial y}{y}$ et $\partial x$ inuentas reuera integrari oportet.

## Exemplum 4.

785. *Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, inuenire relationem inter $x$ et $y$, vt fiat $\frac{(pp + yy)\sqrt{(pp + yy)}}{2pp + yy - qy} = a$.*

Cum sit $q = \frac{p\,\partial p}{\partial y}$, ponatur $pp + yy = zz$, ob $p\,\partial p = q\,\partial y$ erit $q\,\partial y + y\,\partial y = z\,\partial z$, seu $q + y = \frac{z\,\partial z}{\partial y}$. Aequatio

tio autem propofita induit hanc formam:

$$z^3 = a(2zz - yy - qy) = a\left(2zz - \frac{yz\partial z}{\partial y}\right),\ \text{feu}$$

$$zz\partial y = 2az\partial y - ay\partial z,$$

vnde fit

$$\frac{\partial y}{y} = \frac{a\partial z}{2az - zz},\ \text{feu}\ \frac{2\partial y}{y} = \frac{\partial z}{z} + \frac{\partial z}{2a - z},$$

quare integrando colligitur

$$yy = \frac{Cz}{2a - z},\ \text{et}\ pp = zz - \frac{Cz}{2a - z} = \frac{-Cz + 2azz - z^3}{2a - z}.$$

At eft $z = \frac{2ayy}{C + yy}$, ergo

$$pp = \frac{4aay^4}{(C+yy)^2} - yy = \frac{yy[4aayy - (C+yy)^2]}{(C+yy)^2}.$$

Hinc igitur oritur

$$\partial x = \frac{(C+yy)\partial y}{y\sqrt{[4aayy - (C+yy)^2]}},$$

fit $yy = u$, erit

$$\partial x = \frac{(C+u)\partial u}{2u\sqrt{[4aau - (C+u)^2]}}.$$

Haec aequatio tractabilior redditur ponendo

$$u = 2aa - C + 2a\,\text{cof.}\,\Phi\sqrt{(aa - C)},$$

fit enim

$$\partial x = \frac{-a\partial\Phi[a + \text{cof.}\,\Phi\sqrt{(aa-C)}]}{2aa - C + 2a\,\text{cof.}\,\Phi\sqrt{(aa-C)}},\ \text{feu}$$

$$2\partial x = -\partial\Phi - \frac{C\partial\Phi}{2aa - C + 2a\,\text{cof.}\,\Phi\sqrt{(aa-C)}},$$

quae integrata dat

$$2x = \zeta - \Phi - \text{Ang. cof.}\ \frac{m + \text{cof.}\,\Phi}{1 + m\,\text{cof.}\,\Phi},$$

pofito $m = \frac{2a\sqrt{(aa-C)}}{2aa - C}$, vt fit

$$C = \frac{2aa\sqrt{(1-mm)}}{1+\sqrt{(1-mm)}}\ \text{et}\ \sqrt{(aa-C)} = \frac{ma}{1+\sqrt{(1-mm)}},$$

hincque $yy = \frac{2aa(1 + m\,\text{cof.}\,\Phi)}{1+\sqrt{(1-mm)}}$,

vnde fit

$$\cos\Phi = \frac{yy[1+\sqrt{(1-mm)}]-2aa}{2maa} \text{ et}$$

$$\frac{m+\cos\Phi}{1+m\cos\Phi} = \frac{yy[1+\sqrt{(1-mm)}]-2aa(1-mm)}{myy[1+\sqrt{(1-mm)}]}.$$

## Corollarium 1.

786. Cum fit $yy = \frac{2aa(1+m\cos\Phi)}{1+\sqrt{(1-mm)}}$, erit

$$yy = aa + bb + 2ab\cos\Phi,$$

fi ponatur $b = \frac{a[1-\sqrt{(1-mm)}]}{m}$, vnde fit

$$m = \frac{2ab}{aa+bb} \text{ et } \sqrt{(1-mm)} = \frac{aa-bb}{aa+bb},$$

hincque

$$2x = \zeta - \Phi - \text{Ang. cof. } \frac{2ab+(aa+bb)\cos\Phi}{aa+bb+2ab\cos\Phi}, \text{ feu}$$

$$2x = \zeta - \Phi - \text{Ang. fin. } \frac{(aa-bb)\sin\Phi}{yy}.$$

## Corollarium 2.

787. Si vt fupra radius vector $y$ cum angulo $x$ referatur ad lineam curuam, hanc curuam circulum effe oportet radio $= a$ defcriptum. Fit autem $\partial x = \frac{\partial\Phi(aa-ab\cos\Phi)}{aa+bb-2ab\cos\Phi}$ fumto $yy = aa + bb - 2ab\cos\Phi$, hincque

$$x = \zeta + \text{Ang. tang. } \frac{a\sin\Phi}{a\cos\Phi - b},$$

cuius applicatio ad Geometriam rem facit perfpicuam.

## Exemplum 5.

788. *Sumto elemento $\partial x$ conftante, fi proponatur haec aequatio $\partial\partial y(y\partial y + a\partial x) = \partial y(\partial x^2 + \partial y^2)$, eius integrale inuenire.*

Pofito $\partial y = p\partial x$ et $\partial p = q\partial x$ habebimus

$$q(py+a) = p(1+pp), \text{ et ob } q = \frac{p\partial p}{\partial y},$$

$$\partial p(py+a) = \partial y(1+pp), \text{ fiue}$$

$\partial y$

$$\partial y - \frac{p y \partial p}{1+pp} = \frac{a \partial p}{1+pp},$$

quae integrata dat

$$\frac{y}{\sqrt{(1+pp)}} = \frac{ap}{\sqrt{(1+pp)}} + b, \text{ ideoque}$$
$$y = ap + b\sqrt{(1+pp)} \text{ et}$$
$$x = \int \frac{\partial y}{p} = a\,l\,p + b\,l\,[p + \sqrt{(1+pp)}] + c,$$

ita vt $x$ et $y$ per eandem variabilem $p$ exprimantur. Si conſtans $b$ ſumatur $= 0$, obtinetur integrale particulare

$$y = ap \text{ et } x = a\,l\,p + c = a\,l\,\frac{y}{a} + c,$$

ſeu in exponentialibus $y = c\,e^{x:a}$. Sin autem ſumatur $b = a$, ob

$$p + \sqrt{(1+pp)} = \frac{y}{a} \text{ et } p = \frac{yy-aa}{2ay}, \text{ erit}$$
$$x = a\,l\,\frac{yy-aa}{2aa} + c \text{ ſeu } yy = aa + c\,e^{x:a}.$$

## Exemplum 6.

789. *Sumto $\partial x$ conſtante, huius aequationis differentio-differentialis*

$$\partial y^2 - y\,\partial\partial y = n\sqrt{(\partial x^2\,\partial y^2 + aa\,\partial\partial y^2)},$$

*integrale inuenire.*

Poſito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, erit

$$pp - qy = n\sqrt{(pp + aaqq)},$$

quae facto $q = pu$, vt ſit $\frac{p\,\partial p}{\partial y} = pu$ ideoque $\partial p = u\,\partial y$, abit in

$$pp - puy = np\sqrt{(1+aauu)} \text{ ſeu}$$
$$p - uy = n\sqrt{(1+aauu)}.$$

Iam quia $\partial p = u\,\partial y$ differentietur haec aequatio, prodibitque

$$-y\,\partial u = \frac{naau\,\partial u}{\sqrt{(1+aauu)}},$$

hinc vel $\partial u = 0$ vel $y = \frac{-naau}{\sqrt{(1+aauu)}}$.

1) Caſu $\partial u = 0$ fit $u = \alpha$, $p = \alpha y + \beta$, et $\partial x = \frac{\partial y}{\alpha y + \beta}$, hinc $\alpha x = l(\alpha y + \beta) + C$.

2) Si $y = \frac{-naau}{\sqrt{(1 + aauu)}}$, erit

$$p = uy + n\sqrt{(1 + aauu)} = \frac{n}{\sqrt{(1 + aauu)}},$$

hincque

$$\partial x = \frac{\partial y}{p} = \frac{-aa\partial u}{1 + aauu} \text{ et } x = -a \text{ Ang. tang. } au + C,$$

vel ob $u = \frac{y}{a\sqrt{(nnaa - yy)}}$, aequatio inter $x$ et $y$ quaeſita erit

$$\frac{b - x}{a} = \text{Ang. tang. } \frac{y}{\sqrt{(nnaa - yy)}} = \text{Ang. ſin. } \frac{y}{na},$$

vnde fit $y = na \text{ ſin. } \frac{b - x}{a}$. Haec autem relatio tantum pro integrali particulari eſt habenda.

CAP.

# CAPVT III.

## DE AEQVATIONIBVS DIFFERENTIO-DIFFERENTIALIBVS HOMOGENEIS, ET QVAE AD EAM FORMAM REDVCI POSSVNT.

## Problema 97.

790.

Aequationum differentio-differentialium homogenearum naturam explicare, atque ad formam finitam ponendo $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$ accommodare.

## Solutio.

Sumto elemento $\partial x$ conſtante, aequatio differentio-differentialis vulgari modo expreſſa dicitur homogenea, ſi non ſolum ipſis variabilibus $x$ et $y$, ſed etiam earum differentialibus $\partial x$ et $\partial y$, itemque ipſi $\partial\partial y$, ſingulis vnam dimenſionem tribuendo, omnes aequationis termini eundem dimenſionum numerum contineant; veluti in hac aequatione

$$xx\,\partial\partial y + x\,\partial x^2 + y\,\partial y^2 = 0,$$

vbi in ſingulis terminis ternae inſunt dimenſiones. Quodſi ergo ponamus $\frac{\partial y}{\partial x} = p$, ac $\frac{\partial p}{\partial x} = \frac{\partial\partial y}{\partial x^2} = q$, littera $p$ nullam dimenſionem, littera vero $q$ vnam dimenſionem negatiuam continere erit cenſenda. Hinc aequatio differentio-differentialis ad formam hic receptam reducta, vt nonniſi quantitates finitas $x$, $y$, $p$ et $q$ contineat, erit homogenea, ſi litteris $x$ et $y$ vnam dimenſionem tribuendo, litterae $p$ vero nullam, at litterae $q$ vnam

vnam dimensionem negatiuam, in singulis aequationis terminis idem oriatur dimensionum numerus. Vicissim ergo quoties haec proprietas in aequatione inter quaternas quantitates $x$, $y$, $p$ et $q$ proposita deprehenditur, ea aequatio erit homogenea et forma vulgari expressa manifesto homogeneitatem prae se feret.

## Corollarium 1.

791. Si ergo in aequatione tali homogenea inter $x$, $y$, $p$ et $q$ statuatur $y = ux$ et $q = \frac{v}{x}$, omnes termini eandem potestatem ipsius $x$ continebunt, qua ergo per diuisionem sublata, aequatio prodibit tres tantum variabiles $u$, $v$ et $p$ involuens.

## Corollarium 2.

792. Criterium igitur aequationis homogeneae inter quatuor quantitates $x$, $y$, $p$ et $q$ propositae in hoc consistit, vt posito $y = ux$ et $q = \frac{v}{x}$, quantitas $x$ prorsus ex calculo exterminetur.

## Corollarium 3.

793. Facta itaque hac substitutione, qua obtinetur aequatio inter ternas quantitates $u$, $v$ et $p$, ex ea pro lubitu vel $p$ per $u$ et $v$, vel $v$ per $u$ et $p$, vel $u$ per $v$ et $p$ definiri poterit.

## Scholion.

794. Simili modo ideam homogeneitatis in aequationibus differentio-differentialibus constituimus, quo in aequationibus differentialibus primi gradus sumus vsi. In his quidem, cum differentialia sponte eundem dimensionum numerum constituere debeant, homogeneitas ex solis ipsis variabilibus $x$ et $y$ diiudicatur. At in aequationibus differentio-differentialibus praeter ipsas variabiles $x$ et $y$ etiam litterae $q$ ratio in computo

puto dimenſionum haberi debet, ita tamen vt ipſi vna dimenſio negatiua ſit tribuenda; littera autem $p$ in hunc computum plane non ingreditur, quae ergo vtcunque aequationi implicetur, homogeneitatem non turbat. Plurimum autem intereſt probe noſſe indolem differentio-differentialium aequationum homogenearum, cum earum reſolutio ad reſolutionem aequationum differentialium primi gradus reduci poſſit, ita vt ſi haec ſucceſſerit, etiam ipſarum aequationum differentio-differentialium integratio habeatur, id quod in ſequenti problemate luculentius oſtendemus.

## Problema 98.

795. Propoſita aequatione differentio-differentiali homogenea, eius reſolutionem ad integrationem aequationis differentialis primi gradus reducere.

## Solutio.

Reducta aequatione ponendo $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$ ad formam hic receptam, vt habeatur aequatio inter quatuor quantitates finitas $x$, $y$, $p$ et $q$, ponatur $y = u\,x$ et $q = \frac{v}{x}$, ac cum aequatio ſit homogenea, hoc modo quantitas $x$ penitus ex calculo elidetur, ita vt proditura ſit aequatio inter ternas quantitates $u$, $v$ et $p$, ex qua vnam per binas reliquas definire liceat. Nunc igitur cum ſit $\partial y = p\,\partial x$, erit $u\,\partial x + x\,\partial u = p\,\partial x$, hincque $\frac{\partial x}{x} = \frac{\partial u}{p-u}$. Deinde ob $\partial p = q\,\partial x$, erit $\partial p = \frac{v\,\partial x}{x}$, ideoque $\frac{\partial x}{x} = \frac{\partial p}{v}$; ex quo duplici ipſius $\frac{\partial x}{x}$ valore colligitur $\frac{\partial u}{p-u} = \frac{\partial p}{v}$, ſeu $v\,\partial u = p\,\partial p - u\,\partial p$. Quodſi ergo ex illa aequatione quantitas $v$ definiatur per binas $p$ et $u$, habebitur aequatio differentialis primi gradus inter binas variabiles $p$ et $u$, cuius integratio ſi fuerit in poteſtate, vt $p$ per $u$ innoteſcat, aequatio $\frac{\partial x}{x} = \frac{\partial u}{p-u}$, in qua variabiles $x$ et $u$ ſunt feparatae, integretur, ficque $x$ per $u$ definietur, vnde fit $y=ux$; feu ftatim in hoc integrali loco $u$ fcribatur $\frac{y}{x}$, et habebitur aequatio inter $x$ et $y$ quaefita.

## Corollarium 1.

796. Totum ergo negotium reducitur ad integrationem huius aequationis differentialis fimplicis $v\partial u = p\partial p - u\partial p$, quae fi ope regularum fupra traditarum expediri queat, fimul aequationis differentio-differentialis integratio habetur.

## Corollarium 2.

797. Simul autem patet refolutionem huiusmodi aequationum duplicem integrationem requirere, vnde duae quantitates arbitrariae conftantes ingredientur, quibus integrale completum conftituitur.

## Corollarium 3.

798. Etiamfi autem integratio $v\partial u = p\partial p - u\partial p$ non fuccedat, tamen ingens lucrum eft rem eo perduxiffe, cum fupra methodus generalis fit tradita integralia omnium aequationum differentialium primi gradus proxime affignandi.

## Scholion.

799. Operae igitur pretium erit eos cafus perpendere, quibus aequatio $v\partial u = p\partial p - u\partial p$ integrationem admittit; quamobrem examinemus, qualis functio $v$ debeat effe ipfarum $p$ et $u$, vt hoc eueniat. Primum autem patet hoc fieri, fi $v$ fuerit functio homogenea vnius dimenfionis ipfarum $p$ et $u$, quoniam tum ipfa haec aequatio fit homogenea ac per regulas fupra expofitas ad integrationem perduci poteft. Deinde etiam integratio fuccedit fi fuerit $v$ functio quaecunque ipfius $p$, quoniam tum altera variabilis $u$ vnam dimenfionem non fuperat,

rat, et aequationis $\partial u + \frac{u \partial p}{v} = \frac{p \partial p}{v}$ integrale est

$$e^{\int \frac{\partial p}{v}} u = \int e^{\frac{\partial p}{v}} \frac{p \partial p}{v}.$$

Tertio integrationem absoluere licebit, si $v$ fuerit functio quaecunque quantitatis $p - u$. Posito enim $p - u = s$, vt sit $v$ functio ipsius $s$, ob $p = s + u$, nostra aequatio erit $v \partial u = s \partial s + s \partial u$, ideoque $\partial u = \frac{s \partial s}{v - s}$ et $u = \int \frac{s \partial s}{v - s}$, quae integratio adeo ad formulas simplices est referenda. Quarto manente $s = p - u$, si P, Q, R denotent functiones quascunque ipsius $s$, aequatio nostra $v \partial u = s \partial s + s \partial u$ tractari poterit, si fuerit $v = s + \frac{P s}{Q u + R u^n}$, tum enim fit $P \partial u = Q u \partial s + R u^n \partial s$. Quinto etiam patet, si denotantibus V et U functiones quascunque ipsius $u$, fuerit $v = s + V s s + U s^n$, integrationem quoque fore in potestate, fit enim aequatio nostra $V s \partial u + U s^{n-1} \partial u = \partial s$. Atque in genere si aequatio differentialis $\partial s = Z \partial u$ fuerit integrabilis, existente Z functione binarum variabilium $s$ et $u$, cum nostra aequatio fit $s \partial s = (v - s) \partial u$, habebimus $v = s + Z s$ pro omnibus casibus integrationem admittentibus.

## Exemplum 1.

800. *Sumto elemento $\partial x$ constante, si proponatur haec aequatio $x x \partial \partial y = x \partial x \partial y + n y \partial x^2$, eius integrale inuenire.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, erit $q x x = p x + n y$, vnde facto $y = u x$, prodit $q x = p + n u = v$, ita vt $v$ sit functio vnius dimensionis ipsarum $p$ et $u$, et aequatio nostra $(p + n u) \partial u = p \partial p - u \partial p$ fiat homogenea. Cum ergo fit $n u \partial u + p \partial u + u \partial p = p \partial p$, erit integrando

$$C + n u u + 2 p u = p p \text{ et}$$
$$p = u + \sqrt{[C + (n + 1) u u]}.$$



Ha-

Habebimus ergo $\frac{\partial x}{x} = \frac{\partial u}{\surd[C + (n+1)uu]}$, quae denuo integrata dat

$$lx = \frac{1}{\surd(n+1)} l \frac{u\surd(n+1) + \surd[C + (n+1)uu]}{D}, \text{ seu}$$

$$D x^{\surd(n+1)} = u\surd(n+1) + \surd[C + (n+1)uu]$$

hincque

$$D^2 x^{2\surd(n+1)} - 2 D x^{\surd(n+1)} u \surd(n+1) = C:$$

Sit $D = f\surd(n+1)$ et $C = g(n+1)$

vt habeatur

$$ffx^{2\surd(n+1)} - 2fx^{\surd(n+1)} u = g, \text{ existente } u = \frac{y}{x}.$$

Casu quo $n = -1$, ob $\frac{\partial x}{x} = \frac{\partial u}{\alpha}$, erit $\alpha l \frac{x}{a} = u = \frac{y}{x}$, ideoque $y = \alpha x l \frac{x}{a}$. At si $n+1$ sit numerus negatiuus integratio etiam angulos implicabit.

## Corollarium 1.

801. Si sit $n = 0$, huius aequationis $xx\partial\partial y = x\partial x\partial y$ integrale completum erit $ffx^2 - 2fy = g$, qui casus per se est perspicuus. Cum ex $\frac{\partial\partial y}{\partial y} = \frac{\partial x}{x}$ fluat

$$\frac{\partial y}{\partial x} = fx \text{ et } 2y = fxx - \frac{g}{f}.$$

## Corollarium 2.

802. Si sit $n = 3$, aequationis $xx\partial\partial y = x\partial x\partial y + 3y\partial x^2$ integrale completum est $ffx^4 - 2fxy = g$. Idem euenit si loco $\surd(n+1)$ scribatur $-2$, fit enim

$$\frac{ff}{x^4} - \frac{2fy}{x^3} = g \text{ et } ff - 2fxy = gx^4,$$

vtraque redit ad $y = \frac{\alpha}{x} + \beta x^3$.

## Exemplum 2.

803. *Sumto elemento $\partial x$ constante, si proponatur haec aequatio $\frac{xx\partial\partial y}{\partial x} = \surd(mxx\partial y^2 + nyy\partial x^2)$, eius integrale completum inuenire.*

Ob

Ob $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, habebimus $qxx = \surd(mppxx + nyy)$, quae posito $y = ux$ abit in

$$qx = \surd(mpp + nuu) = v, \text{ ob } q = \frac{v}{x}.$$

Quia ergo est $\frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial p}{v}$, erit

$$\partial u \surd(mpp + nuu) = (p - u)\,\partial p,$$

quae est aequatio homogenea. Ponatur ergo $p = su$; et prodit

$$\partial u \surd(mss + n) = (s - 1)(s\,\partial u + u\,\partial s);$$

hincque

$$\frac{\partial u}{u} = \frac{(s-1)\,\partial s}{\surd(mss+n) - ss + s},$$

ex qua fit

$$\frac{\partial x}{x} = \frac{\partial u}{(s-1)u} = \frac{\partial s}{\surd(mss+n) - ss + s},$$

vnde tam $u = \frac{y}{x}$ quam $x$ per eandem variabilem $s$ determinatur.

## Exemplum 3.

804. *Sumto elemento $\partial x$ constante, si proponatur haec aequatio $nx^3\,\partial\partial y = (y\,\partial x - x\,\partial y)^2$, eius integrale inuenire.*

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, erit $nx^3 q = (y - px)^2 = nxxv$ ob $q = \frac{v}{x}$. Si iam statuatur $y = ux$, fiet

$$nv = (u - p)^2 \text{ et } \frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial p}{v} = \frac{n\,\partial p}{(p-u)^2},$$

vnde habetur $n\,\partial p = p\,\partial u - u\,\partial u$: quae facto $p - u = s$, abit in $n\,\partial u + n\,\partial s = s\,\partial u$ seu $\partial u = \frac{n\,\partial s}{s-n}$, hinc $u = nl\frac{s-n}{\alpha}$. Tum vero ob $p - u = s$, erit

$$\frac{\partial x}{x} = \frac{\partial u}{s} = \frac{n\,\partial s}{s(s-n)} \text{ et } lx = l\frac{s-n}{\beta s}, \text{ ideoque}$$

$$x = \frac{s-n}{\beta s}, \text{ at } y = nxl\frac{s-n}{\alpha}.$$

Cum ergo sit

$$s = \frac{n}{1-\beta x}, \text{ erit } y = nxl\frac{n\beta x}{\alpha(1-\beta x)}.$$



Co-

## Corollarium.

805. Aequatio $n x^3 q = (y - p x)^2$ facilius resoluitur, ponendo $y - p x = z$, vnde fit $- x \partial p = \partial z$; quare ob $q \partial x = \partial p$, erit

$$n x^3 \partial p = z z \partial x = - n x x \partial z,$$

ideoque

$$\frac{1}{x} = \frac{1}{a} - \frac{n}{z} \text{ seu } \frac{x-a}{ax} = \frac{n}{y-px}, \text{ ergo}$$

$$y - p x = \frac{n a x}{x-a} = \frac{y \partial x - x \partial y}{\partial x}.$$

Quare

$$\frac{y \partial x - x \partial y}{xx} = \frac{n a \partial x}{x(x-a)} \text{ et}$$

$$\frac{y}{x} = n\, l \frac{x}{x-a} + C \text{ vt ante.}$$

## Exemplum 4.

806. *Sumto $\partial x$ constante, si proponatur haec aequatio*

$$(\partial x^2 + \partial y^2) \surd (\partial x^2 + \partial y^2) = n \partial x \partial \partial y \surd (xx + yy),$$

*integrale inuenire.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, aequatio proposita hanc induit formam

$$(1 + pp) \surd (1 + pp) = n q \surd (xx + yy),$$

quae facto $y = u x$ et $q = \frac{v}{x}$ transit in hanc

$$(1 + pp) \surd (1 + pp) = n v \surd (1 + uu).$$

Cum igitur sit $\frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial p}{v}$, ob

$$v = \frac{(1+pp) \surd (1+pp)}{n \surd (1+uu)} \text{ erit}$$

$$(1 + pp)^{\frac{3}{2}} \partial u = n (p - u) \partial p \surd (1 + uu)$$

cuius resolutio non sponte patet. Calculum autem ad angulos reuocando, sit $p = \text{tang.}\, \Phi$ et $u = \text{tang.}\, \omega$, erit

$\partial p =$

$$\partial p = \frac{\partial \Phi}{cof. \Phi^2}, \quad \partial u = \frac{\partial \omega}{cof. \omega^2},$$

$$\sqrt{(1 + pp)} = \frac{1}{cof. \Phi}, \quad \sqrt{(1 + uu)} = \frac{1}{cof. \omega}, \text{ et}$$

$$p - u = \frac{fin. (\Phi - \omega)}{cof. \Phi \; cof. \omega}, \text{ hincque}$$

$$\frac{1}{cof. \Phi^3} \cdot \frac{\partial \omega}{cof. \omega^2} = \frac{n \, fin. (\Phi - \omega)}{cof. \Phi \; cof. \omega} \cdot \frac{1}{cof. \omega} \cdot \frac{\partial \Phi}{cof. \Phi^2} \text{ fiue}$$

$$\partial \omega = n \, \partial \Phi \, \text{fin.} (\Phi - \omega) = \partial \Phi - (\partial \Phi - \partial \omega) \text{ ergo}$$

$$\partial \Phi = \frac{\partial \Phi - \partial \omega}{1 - n \, fin. (\Phi - \omega)}.$$

Pofito ergo $\Phi - \omega = \psi$, erit

$$\Phi = \int \frac{\partial \psi}{1 - n \, fin. \psi} \text{ et } \omega = \int \frac{\partial \psi}{1 - n \, fin. \psi} - \psi;$$

hinc ob $p = \text{tang.} \Phi$, et $u = \text{tang.} \omega$, obtinetur

$$\frac{\partial x}{x} = \frac{\partial u \; cof. \Phi \; cof. \omega}{fin. \psi} = \frac{\partial \omega \; cof. \Phi}{fin. \psi \; cof. \omega} = \frac{n \, \partial \psi \; cof. \Phi}{cof. \omega (1 - n \, fin. \psi)}.$$

Cafu $n = 1$, fit

$$\partial \Phi = \frac{\partial \psi}{1 - fin. \psi} = \frac{\partial \psi (1 + fin. \psi)}{cof. \psi^2},$$

$$\Phi = \text{tang.} \psi + \frac{1}{cof. \psi} = \frac{1 + fin. \psi}{cof. \psi} + \alpha,$$

$$\omega = \frac{1 + fin. \psi}{cof. \psi} + \alpha - \psi, \text{ et}$$

$$\frac{\partial x}{x} = \frac{\partial \Phi \; cof. \Phi}{cof. \omega} = \frac{\partial \Phi \; cof. \Phi}{cof. \Phi \; cof. \psi + fin. \Phi \, fin. \psi}.$$

Cum autem fit $\Phi - \alpha = \sqrt{\frac{1 + fin. \psi}{1 - fin. \psi}}$, erit

$$\text{fin.} \psi = \frac{(\Phi - \alpha)^2 - 1}{(\Phi - \alpha)^2 + 1}, \quad \text{cof.} \psi = \frac{2 (\Phi - \alpha)}{(\Phi - \alpha)^2 + 1}. \quad \text{Ergo}$$

$$\frac{\partial x}{x} = \frac{\partial \Phi \; cof. \Phi \, [(\Phi - \alpha)^2 + 1]}{2 (\Phi - \alpha) \, cof. \Phi + (\Phi - \alpha)^2 fin. \Phi - fin. \Phi}.$$

## Corollarium 1.

807. Si cafu $n = 1$ fumatur conftans $\alpha$ infinita, erit fin. $\psi = 1$, hinc $\psi = 90°$ et $\omega = \Phi - 90°$, tum vero $\frac{\partial x}{x} = \frac{\partial \Phi \; cof. \Phi}{fin. \Phi}$, ideoque $x = a \, \text{fin.} \Phi$, et $u = - \text{cot.} \Phi$, ergo $y = - a \, \text{cof.} \Phi$, et $xx + yy = aa$.

Co-

## Corollarium 2.

808. Eodem autem caſu $n = 1$, quo conſtans $\alpha$ non ſumitur infinita, fractionis cui $\frac{\partial x}{x}$ aequatur, numerator commode eſt differentiale denominatoris; vnde fit

$$x = a\,[(\Phi - \alpha)^2 \text{ſin.}\,\Phi - \text{ſin.}\,\Phi + 2\,(\Phi - \alpha)\,\text{coſ.}\,\Phi].$$

Tum vero eſt

$$\omega = \Phi - \text{Ang. tang.}\,\frac{(\Phi - \alpha)^2 - 1}{2\,(\Phi - \alpha)},$$

ideoque

$$u = \frac{y}{x} = \text{tang.}\,\omega = \frac{\text{tang.}\,\Phi - \frac{(\Phi - \alpha)^2 + 1}{2\,(\Phi - \alpha)}}{1 + \frac{(\Phi - \alpha)^2 - 1}{2\,(\Phi - \alpha)}\,\text{tang.}\,\Phi}, \text{ ſeu}$$

$$\frac{y}{x} = \frac{2\,(\Phi - \alpha)\,\text{ſin.}\,\Phi - (\Phi - \alpha)^2\,\text{coſ.}\,\Phi + \text{coſ.}\,\Phi}{(\Phi - \alpha)^2\,\text{ſin.}\,\Phi - \text{ſin.}\,\Phi + 2\,(\Phi - \alpha)\,\text{coſ.}\,\Phi};$$

conſequenter

$$y = -a\,[\,(\Phi - \alpha)^2\,\text{coſ.}\,\Phi - \text{coſ.}\,\Phi - 2\,(\Phi - \alpha)\,\text{ſin.}\,\Phi] \text{ et}$$

$$\sqrt{(x\,x + y\,y)} = [\,(\Phi - \alpha)^2 + 1].$$

## Scholion 1.

809. In genere etiam integrationem abſoluere licet. Cum enim ſit

$$\partial\Phi = \frac{\partial\psi}{1 - n\,\text{ſin.}\,\psi} \text{ et } \frac{\partial x}{x} = \frac{n\,\partial\Phi\,\text{coſ.}\,\Phi}{\text{coſ.}\,\omega}, \text{ erit}$$

$$\Phi + \alpha = \frac{1}{\sqrt{(1 - n\,n)}}\,\text{Ang. coſ.}\,\frac{n - \text{ſin.}\,\psi}{1 - n\,\text{ſin.}\,\psi};$$

vnde poſito

$$(\Phi + \alpha)\sqrt{(1 - n\,n)} = \theta, \text{ erit coſ.}\,\theta = \frac{n - \text{ſin.}\,\psi}{1 - n\,\text{ſin.}\,\psi};$$

hincque

$$\text{ſin.}\,\psi = \frac{n - \text{coſ.}\,\theta}{1 - n\,\text{coſ.}\,\theta} \text{ et coſ.}\,\psi = \frac{\text{ſin.}\,\theta\sqrt{(1 - n\,n)}}{1 - n\,\text{coſ.}\,\theta}.$$

At ob $\omega = \Phi - \psi$, habebitur

$$\frac{\partial x}{x} = \frac{n\,\partial\Phi\,\text{coſ.}\,\Phi\,(1 - n\,\text{coſ.}\,\theta)}{\text{coſ.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1 - n\,n)} + \text{ſin.}\,\Phi\,(n - \text{coſ.}\,\theta)}.$$

Iam

Iam cum ſit $\partial\theta = \partial\Phi\sqrt{(1-nn)}$, differentiale huius denominatoris eſt

$$-\partial\Phi\,\text{ſin.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1-nn)}+\partial\Phi\,\text{coſ.}\,\Phi\,\text{coſ.}\,\theta(1-nn)$$
$$+n\partial\Phi\,\text{coſ.}\,\Phi-\partial\Phi\,\text{coſ.}\,\Phi\,\text{coſ.}\,\theta+\partial\Phi\,\text{ſin.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1-nn)},$$

quod redit ad $n\partial\Phi\,\text{coſ.}\,\Phi(1-n\,\text{coſ.}\,\theta)$ ipſum ſcilicet numeratorem. Ita vt ſit

$$x=a[\text{coſ.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1-nn)}+\text{ſin.}\,\Phi(n-\text{coſ.}\,\theta)]$$

ſeu $x = a\,\text{coſ.}\,\omega\,(1-n\,\text{coſ.}\,\theta)$, ideoque

$$y = ux = a\,\text{ſin.}\,\omega\,(1-n\,\text{coſ.}\,\theta).$$

Aſſumto ergo angulo $\theta$, quaeratur angulus $\psi$, vt ſit

$$\text{ſin.}\,\psi=\frac{n-\text{coſ.}\,\theta}{1-n\,\text{coſ.}\,\theta} \text{ et } \text{coſ.}\,\psi=\frac{\text{ſin.}\,\theta}{1-n\,\text{coſ.}\,\theta}\sqrt{(1-nn)},$$

tum vero fiat

$$\omega=\frac{\theta}{\sqrt{(1-nn)}}-\alpha-\psi,$$

eritque integrale completum

$$x=a(1-n\,\text{coſ.}\,\theta)\,\text{coſ.}\,\omega \text{ et } y=a(1-n\,\text{coſ.}\,\theta)\,\text{ſin.}\,\omega.$$

## Scholion 2.

810. At ſi numerus $n$ ſit vnitate maior, haec integratio fit imaginaria, quod incommodum vt tollatur, notandum eſt, aequationis $\partial\Phi=\frac{\partial\psi}{1-n\,\text{ſin.}\,\psi}$ integrale eſſe

$$\Phi+\alpha=\frac{1}{\sqrt{(nn-1)}}\,l\,\frac{\sqrt{(n-1)(1+\text{ſin.}\,\psi)}+\sqrt{(n+1)(1-\text{ſin.}\,\psi)}}{\sqrt{(n-1)(1+\text{ſin.}\,\psi)}-\sqrt{(n+1)(1-\text{ſin.}\,\psi)}}.$$

Quare ſi ponatur $(\Phi+\alpha)\sqrt{(nn-1)}=\theta$, vt ſit

$$\partial\theta=\partial\Phi\sqrt{(nn-1)} \text{ et } \omega=\Phi-\psi=\frac{\theta}{\sqrt{(nn-1)}}-\alpha-\psi,$$

erit

$$\frac{e^\theta+1}{e^\theta-1}=\frac{\sqrt{(n-1)(1+\text{ſin.}\,\psi)}}{\sqrt{(n+1)(1-\text{ſin.}\,\psi)}}=\frac{(n-1)(1+\text{ſin.}\,\psi)}{\text{coſ.}\,\psi\sqrt{(nn-1)}};$$

vnde reperitur

$$\sin.\,\psi = \frac{e^\theta + 2n + e^{-\theta}}{ne^\theta + 2 + ne^{-\theta}} \text{ et } \cos.\,\psi = \frac{(e^\theta - e^{-\theta})\sqrt{(nn-1)}}{ne^\theta + 2 + ne^{-\theta}},$$

ita vt ex angulo $\theta$ definiantur anguli $\psi$, $\Phi$ et $\omega$. Cum iam fit

$$\frac{\partial x}{x} = \frac{n\,\partial\Phi \cos.\,\Phi}{\cos.\,\omega} = \frac{n\,\partial\Phi \cos.\,\Phi}{\cos.\,\Phi \cos.\,\psi + \sin.\,\Phi \sin.\,\psi}, \text{ erit}$$

$$\frac{\partial x}{x} = \frac{n\partial\Phi \cos.\,\Phi(ne^\theta + 2 + ne^{-\theta})}{\cos.\,\Phi(e^\theta - e^{-\theta})\sqrt{(nn-1)} + \sin.\,\Phi(e^\theta + 2n + e^{-\theta})},$$

vbi iterum commode euenit, vt numerator fit ipsum differentiale denominatoris, quemadmodum differentiationem instituenti mox patebit. Hinc ergo erit

$$x = a[\cos.\,\Phi(e^\theta - e^{-\theta})\sqrt{(nn-1)} + \sin.\,\Phi(e^\theta + 2n + e^{-\theta})] \text{ seu}$$

$$x = a\cos.\,\omega(ne^\theta + 2 + ne^{-\theta}), \text{ et ob } u = \tfrac{y}{x} = \text{tang.}\,\omega \text{ fit}$$

$$y = a\sin.\,\omega(ne^\theta + 2 + ne^{-\theta}).$$

Quocirca ex angulo $\theta$ primo quaeratur angulus $\psi$, vt fit

$$\sin.\,\psi = \frac{e^\theta + 2n + e^{-\theta}}{ne^\theta + 2 + ne^{-\theta}} \text{ et } \cos.\,\psi = \frac{(e^\theta - e^{-\theta})\sqrt{(nn-1)}}{ne^\theta + 2 + ne^{-\theta}},$$

quo inuento capiatur angulus $\omega = \frac{\theta}{\sqrt{(nn-1)}} - \alpha - \psi$, ac formulae illae pro $x$ et $y$ inuentae dabunt integrale completum ob duas constantes $a$ et $\alpha$ introductas.

## Scholion 3.

811. Cum hic praecipua pars integrationis fortuito successisse videatur, operae praetium erit in eius caussam inquirere, num forte ratio integrandi clarius perspici queat. Cum igitur fit

$$\Phi = \psi + \omega \text{ et } \partial\Phi = \frac{\partial\psi}{1 - n\sin.\,\psi},$$

hincque

$$\partial\omega = \frac{n\,\partial\psi \sin.\,\psi}{1 - n\sin.\,\psi} = n\,\partial\Phi \sin.\,\psi,$$

aequa-

aequatio

$$\frac{\partial x}{x} = \frac{n \partial \Phi \cos. \Phi}{\cos. \omega}, \text{ ob } \cos.\Phi = \cos.\psi \cos.\omega - \sin.\psi \sin.\omega,$$

in hanc resoluitur

$$\frac{\partial x}{x} = n \partial \Phi \cos. \psi - \frac{n \partial \Phi \sin. \psi \sin. \omega}{\cos. \omega},$$

quae ob

$$\partial \Phi = \frac{\partial \psi}{1 - n \sin. \psi} \text{ et } n \partial \Phi \sin. \psi = \partial \omega,$$

induit hanc formam integrabilem

$$\frac{\partial x}{x} = \frac{n \partial \psi \cos. \psi}{1 - n \sin. \psi} - \frac{\partial \omega \sin. \omega}{\cos. \omega},$$

ex qua elicitur

$$x = \frac{a \cos. \omega}{1 - n \sin. \psi} \text{ et } y = \frac{a \sin. \omega}{1 - n \sin. \psi}, \text{ ob } y = ux = x \operatorname{tang}. \omega.$$

En ergo in genere aequationis noſtrae hanc integrationem. Anguli $\omega$ et $\psi$ hanc inter ſe teneant relationem, vt ſit

$$\partial \omega = \frac{n \partial \psi \sin. \psi}{1 - n \sin. \psi},$$

tum vero erit

$$x = \frac{a \cos. \omega}{1 - n \sin. \psi} \text{ et } y = \frac{a \sin. \omega}{1 - n \sin. \psi}.$$

Quodſi ergo ponamus $\sqrt{(xx + yy)} = z$, vt ſit $x = z \cos. \omega$ et $y = z \sin. \omega$, erit

$$z = \frac{a}{1 - n \sin. \psi} \text{ et } \sin. \psi = \frac{z - a}{nz}, \text{ hinc}$$

$$\partial \omega = \frac{(z - a) \partial \psi}{a}. \text{ At ſit}$$

$$\partial \psi = \frac{a \partial z}{z \sqrt{[nnzz - (z - a)^2]}} \text{ ergo}$$

$$\partial \omega = \frac{(z - a) \partial z}{z \sqrt{[nnzz - (z - a)^2]}}$$

vnde angulus $\omega$ per $z$ definitur. Ad irrationalitatem tollendam, ſi ponamus

$$\sqrt{[nnzz - (z - a)^2]} = s(nz + z - a), \text{ fit}$$

$$z = \frac{a(ss + 1)}{(n+1)ss - n + 1} \text{ et } \partial \omega = \frac{2n \partial s (ss - 1)}{(ss + 1)[(n+1)ss - (n-1)]}, \text{ ſeu}$$

H 2 $\partial \omega$

$$\partial \omega = \frac{2 \partial s}{ss+1} - \frac{2 \partial s}{(n+1)ss - n + 1}$$

quae integratio eſt manifeſta.

## Problema 99.

812. Si aequatio differentio-differentialis tum demum fiat homogenea, ſi alteri variabili $y$ tribuantur $n$ dimenſiones, eius integrationem ad aequationem differentialem primi gradus reducere.

## Solutio.

Poſito $\partial y = p \partial x$ et $\partial p = q \partial x$, vt oriatur aequatio inter quaternas quantitates finitas $x$, $y$, $p$ et $q$, quae quomodo ratione homogeneitatis futura ſit comparata, videamus. Primo ergo cum pro $x$ vnam dimenſionem numerando, variabilis $y$ habeat $n$ dimenſiones, quantitati $p = \frac{\partial y}{\partial x}$ tribuendae ſunt $n-1$ dimenſiones, quantitati $q = \frac{\partial p}{\partial x}$ vero $n-2$ dimenſiones. Quocirca ponamus

$$y = x^n u, \; p = x^{n-1} t, \text{ et } q = x^{n-2} v,$$

et ob $\partial y = p \partial x$ et $\partial p = q \partial x$, habebimus

$$x \partial u + n u \partial x = t \partial x \text{ et } x \partial t + (n-1) t \partial x = v \partial x,$$

vnde colligimus

$$\frac{\partial x}{x} = \frac{\partial u}{t - n u} = \frac{\partial t}{v - (n-1) t},$$

ideoque

$$\partial u [v - (n-1) t] = \partial t (t - n u).$$

At factis ſuperioribus ſubſtitutionibus in aequatione inter $x$, $y$, $p$ et $q$, per hypotheſin variabilis $x$ ex calculo extruditur, ita vt prodeat aequatio inter tres tantum variabiles $u$, $t$ et $v$, ex qua litteram $v$ per binas $t$ et $u$ definire licebit. Quo valore ſubſtituto habebitur aequatio differentialis primi gradus inter binas variabiles $u$ et $t$, ex qua $t$ per $u$ determinari queat; ope

ope aequationis $\frac{\partial x}{x} = \frac{\partial u}{t - nu}$ definietur $x$ per $u$, hincque ob $u = \frac{y}{x^n}$ obtinebitur aequatio integralis inter $x$ et $y$, eaque ob duplicem integrationem completa.

## Corollarium 1.

813. Aequationum ergo inter $x$, $y$, $p$ et $q$ hoc modo tractabilium hoc est criterium, vt posito $y = x^n u$, $p = x^{n-1} t$ et $q = x^{n-2} v$, exponens $n$ eiusmodi determinationem patiatur, vt variabilis $x$ prorsus ex calculo per diuisionem egrediatur.

## Corollarium 2.

814. Si sit $n = 0$, aequatio ita est comparata, vt tribuendo ipsi $y$ eiusque differentialibus nullam dimensionem fiat homogenea. Hoc scilicet casu sola variabilis $x$ cum suis differentialibus dimensiones constituere censetur.

## Corollarium 3.

815. Contra vero si dimensiones ex sola variabili $y$ aestimentur, ita vt ea cum suis differentialibus $\partial y$ et $\partial \partial y$ vbique eundem dimensionum numerum constituat, exponens $n$ fiet infinitus.

## Scholion.

816. Si sola variabilis $x$ cum suis differentialibus vbique eundem dimensionum numerum complet, ob $n = 0$ fit $u = y$, et in aequatione inter $x$, $y$, $p$ et $q$ statui conueniet $p = \frac{t}{x}$ et $q = \frac{v}{xx}$, quo facto variabilis $x$ ex calculo deturbabitur, prodibitque aequatio inter $y$, $t$ et $v$, cuius ope aequatio differentialis $\partial y (v + t) = t \partial t$ ad duas tantum variabiles reducetur, qua resoluta erit $\frac{\partial x}{x} = \frac{\partial y}{t}$, vbi cum $t$ detur per $y$, integratio nullam habet difficultatem. Verum altero casu, quo variabilis $y$

sola cum suis differentialibus pares vbique dimensiones habet, ideoque exponens $n$ infinitus capi deberet, resolutio alio modo institui debet, quem mox docebimus, nisi forte permutando variabiles $x$ et $y$ casum ad praecedentem reducere lubuerit.

## Exemplum I.

817. *Sumto elemento $\partial x$ constante, si proponatur haec aequatio $x x \partial \partial y = \alpha y \partial x^2 + \beta x \partial x \partial y$, eius integrale inuenire.*

Perpendatur hic ista conditio, qua sola variabilis $x$ cum suo differentiali $\partial x$ vbique duas constituit dimensiones, eritque $n = 0$. Cum ergo posito $\partial y = p \partial x$ et $\partial p = q \partial x$, habeamus $q x x = \alpha y + \beta p x$, statuamus $p = \frac{t}{x}$ et $q = \frac{v}{x x}$, fietque $v = \alpha y + \beta t$, vnde adipiscimur istam aequationem differentialem

$$\alpha y \partial y + (\beta + 1) t \partial y = t \partial t,$$

ob cuius homogeneitatem faciamus $t = y z$, eritque

$$\alpha \partial y + (\beta + 1) z \partial y = y z \partial z + z z \partial y \text{ seu}$$

$$\frac{\partial y}{y} = \frac{z \partial z}{\alpha + (\beta + 1) z - z z}.$$

Sit $\alpha + (\beta + 1) z - z z = (f + z)(g - z)$ vt sit

$$\alpha = f g \text{ et } \beta + 1 = g - f,$$

reperieturque

$$\frac{\partial y}{y} = \frac{\frac{-f}{f+g} \partial z}{f + z} + \frac{\frac{g}{f+g} \partial z}{g - z}:$$

vnde colligitur integrando

$$l y = C - \frac{f}{f+g} l (f + z) - \frac{g}{f+g} l (g - z), \text{ seu}$$

$$y (f + z)^{\frac{f}{f+g}} (g - z)^{\frac{g}{f+g}} = a.$$

Tum

Tum vero eſt

$$\frac{\partial x}{x} = \frac{\partial y}{yz} = \frac{\partial z}{(f+z)(g-z)}, \text{ ſeu}$$

$$\frac{\partial x}{x} = \frac{1}{f+g} \cdot \frac{\partial z}{f+z} + \frac{1}{f+g} \cdot \frac{\partial z}{g-z}, \text{ hinc}$$

$$x = b\left(\frac{f+z}{g-z}\right)^{\frac{1}{f+g}}, \text{ ſeu } \frac{f+z}{g-z} = \left(\frac{x}{b}\right)^{f+g}.$$

Vnde cum ſit $z = \frac{g\,x^{f+g} - f\,b^{f+g}}{b^{f+g} + x^{f+g}}$, erit hoc valore ibi ſubſtituto

$$(f+g)\, b^g\, x^f\, y = a\,(b^{f+g} + x^{f+g}),$$

ſeu poſito $\frac{a}{f+g} = c$

$$y = c\left(\frac{b^f}{x^f} + \frac{x^g}{b^g}\right):$$

eſt vero $g - f = \beta + 1$ et $g + f = \sqrt{((\beta+1)^2 + 4\alpha)}$.

## Corollarium.

818. Quoniam in aequatione propoſita etiam ambae variabiles $x$ et $y$ ſimul vbique totidem dimenſiones habent, eam etiam ſecundum praecepta praecedentis problematis tractare licet.

## Exemplum 2.

819. *Poſito $\partial x$ conſtante, ſi aequatio differentio-differentialis duobus tantum terminis conſtet, vt ſit huiusmodi*

$$\partial\partial y = c\, x^\alpha\, y^\beta\, \partial x^{2-\gamma}\, \partial y^\gamma,$$

*eius integrale inueſtigare.*

Poſito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, habebitur haec forma $q = c\, x^\alpha\, y^\beta\, p^\gamma$, vbi exponentem $n$ ita definire licet, vt poſito $y = x^n u$, $p = x^{n-1} t$ et $q = x^{n-2} v$, variabilis $x$ per diviſionem tolli poſſit, capi enim oportet

$\alpha + \beta$

$$\alpha + \beta + \gamma(n-1) - n + 2 = 0, \text{ seu } n = \frac{-\alpha+\gamma-2}{\beta \quad \gamma-1};$$

tumque erit $v = c u^\beta t^\gamma$. Aequatio ergo differentialis primi gradus resoluenda erit

$$c u^\beta t^\gamma \partial u - (n-1) t \partial u = t \partial t - n u \partial t,$$

ex qua cum variabilis $t$ per $u$ fuerit determinata, integrari oportet hanc formulam $\frac{\partial x}{x} = \frac{\partial u}{t - n u}$, quo facto ob $u = \frac{y}{x^n}$, obtinebitur aequatio integralis quaesita inter $x$ et $y$.

Casus tantum $\beta + \gamma = 1$, quo $n$ fit infinitus, peculiarem postulat tractationem infra exponendam, nisi forte simul sit $\gamma = \alpha + 2$, tum enim exponens $n$ prorsus arbitrio nostro relinquitur, at aequatio erit homogenea.

## Exemplum 3.

820. *Sumto elemento $\partial x$ constante, si proponatur haec aequatio*

$$x^4 \partial\partial y = x^3 \partial x \partial y + 2 x y \partial x \partial y - 4 y y \partial x^2,$$

*eius integrale inuenire.*

Hic euidens est, si ipsi $y$ eiusque differentialibus $\partial y$ et $\partial\partial y$ binae dimensiones, ipsi $x$ vero et $\partial x$ singulae tribuantur, in omnibus terminis obtineri sex dimensiones. Quare cum posito $\partial y = p \partial x$ et $\partial p = q \partial x$, habeamus hanc aequationem

$$x^4 q = x^3 p + 2 x y p - 4 y y, \text{ faciamus}$$
$$y = x^2 u, \; p = x t \text{ et } q = v, \text{ prodibitque}$$
$$v = t + 2 u t - 4 u u.$$

At ob $n = 2$, aequatio differentialis nostra erit

$$\partial u (v - t) = \partial t (t - 2 u),$$

quae

quae abit in

$$2u\,\partial u\,(t-2u)=\partial t\,(t-2u),$$

vnde deducimus vel $t=2u$ vel $t=uu+c$, quos binos casus seorsim euoluamus.

1) Si $t=2u$, ob $\frac{\partial x}{x}=\frac{\partial u}{t-2u}$ fit $\partial u=0$, ideoque $u=C$, ac propterea $y=Cxx$, quod est integrale particulare, aequationi propositae vtique satisfaciens.

2) Sit $t=uu+c$, erit $\frac{\partial x}{x}=\frac{\partial u}{uu-2u+c}$, vbi tres casus sunt considerandi:

Primo si constans $c=1$, erit

$$l\frac{x}{a}=\frac{1}{1-u}=\frac{xx}{xx-y}, \text{ seu } xx=(xx-y)\,l\frac{x}{a}.$$

Secundo si constans $c=1-ff$, erit $\frac{\partial x}{x}=\frac{\partial u}{(u-1)^2-ff}$, hincque

$$lx=\frac{-1}{2f}\,l\frac{f+u-1}{f-u+1}+C,$$

ergo ob $u=\frac{y}{xx}$, erit

$$x=a\left(\frac{(f+1)xx-y}{(f-1)xx+y}\right)^{\frac{1}{2f}}.$$

Tertio si constans $c=1+ff$, ideoque $\frac{\partial x}{x}=\frac{\partial u}{(u-1)^2+ff}$, quae integrata dat

$$l\frac{x}{a}=\frac{1}{f}\text{ Ang. tang. }\frac{u-1}{f}, \text{ seu } \frac{u-1}{f}=\frac{y-xx}{fxx}=\text{tang. } f\,l\frac{x}{a}.$$

Pro ratione ergo constantis arbitrariae $c$ integratio vel algebraice succedit, vel a logarithmis, vel ab angulis pendet, vnde forma generali exprimi nequit.

## Scholion.

821. Integrale autem particulare primo inuentum $y=Cxx$ in nulla harum formarum, quibus integrale com-

 pletum

pletum conſtituitur, contineri deprehenditur: nihilo vero minus ſatisfacit aequationi differentio-differentiali propoſitae. Hoc ergo exemplo magis illuſtrantur ea, quae ſupra circa hoc paradoxon ſumus commentati, quod interdum aequationi differentiali ſatisfaciat aequatio finita, quae in integrali completo minime contineatur. Videmus igitur hoc idem paradoxon etiam in aequationibus differentio-differentialibus locum habere. Vtrum autem illa aequatio $y = Cxx$ inter integralia ſit admittenda, alia eſt quaeſtio, quae nondum penitus videtur confecta; hic quidem ipſa aequatio propoſita quaſi factores habens eſt cenſenda, ex quorum altero illa aequatio $y = Cxx$ naſcatur, verum multum abeſt, vt in hac explicatione acquieſcere queamus. Quin potius ipſa illa quaeſtio, ſiue geometrica fuerit ſiue alius diſciplinae, cuius ſolutio ad huiusmodi aequationem perduxerit, accurate perpendi debere videtur; vbi plerumque haud difficulter iudicari poteſt, vtrum quicquid aequationi differentiali ſatisfaciat, id etiam ipſi quaeſtioni conueniat nec ne? Veluti ſi deſcenſus grauis ex altitudine $= a$ labentis definiri debeat, et altitudo qua iam a terra diſtat ſit $x$, erit ibi celeritas vt $\sqrt{(a - x)}$, et elementum temporis $\partial t = \frac{-\partial x}{\sqrt{(a-x)}}$. Hic quidem euidens eſt iſti aequationi differentiali ſatisfieri ponendo $x = a$, ita vt tempus $t$ maneat indefinitum, quod tamen quaeſtioni neutiquam conuenit, quae noniſi vero integrali $t = 2\sqrt{(a - x)}$ reſoluitur.

## Problema 100.

822. Si in aequatione differentio-differentiali variabilis $y$ cum ſuis differentialibus $\partial y$ et $\partial \partial y$ vbique eundem dimenſionum numerum adimpleat, eius integrationem ad aequationem differentialem primi gradus reducere.

### Solutio.

## Solutio.

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, aequatio ita erit comparata, vt in ea ternae variabiles $y$, $p$, $q$, vbique eundem dimensionum numerum obtineant, altera variabili $x$ in computum dimensionum prorsus non ingrediente. Quare si statuatur $p = uy$ et $q = vy$, in omnibus terminis inerit eadem ipsius $y$ potestas, qua per diuisionem sublata habebitur aequatio inter ternas tantum variabiles $x$, $u$ et $v$, ex qua vnam per binas reliquas definire licebit, ita vt $v$ aequetur functioni cuipiam ipsarum $x$ et $u$. Iam ob $p = uy$ erit $\partial y = uy\,\partial x$, et ob $\partial p = q\,\partial x$ fiet $u\,\partial y + y\,\partial u = vy\,\partial x$, vnde sequitur

$$\frac{\partial y}{y} = u\,\partial x \quad \text{et} \quad \frac{\partial y}{y} = \frac{v\,\partial x - \partial u}{u},$$

ideoque $\partial u + uu\,\partial x = v\,\partial x$, quae aequatio differentialis duas tantum variabiles $x$ et $u$ complectitnr. Quam ergo si integrare liceat, vt relatio inter $x$ et $u$ inde innotescat, superest, vt formulae $u\,\partial x$ integrale inuestigetur, quo inuento erit $ly = \int u\,\partial x$, sicque aequatio orietur integralis inter $x$ et $y$, quae ob duplicem integrationem peractam duas constantes arbitrarias inuoluet, ideoque integrale completum exhibebit.

## Corollarium 1.

823. Huiusmodi ergo aequationum integratio reducitur ad huiusmodi aequationem differentialem $\partial u + uu\,\partial x = v\,\partial x$, cuius resolutio si succedat, simul illarum integratio habetur, cum formulae $u\,\partial x$ integratio difficultate careat.

## Corollarium 2.

824. Cum sit $\frac{\partial y}{y} = u\,\partial x$, erit $y = e^{\int u\,\partial x}$, qua substitutione aequatio differentio-differentialis proposita statim reducitur ad aequationem differentialem primi gradus, erit enim,

$$\frac{\partial y}{\partial x} = p = e^{\int u \partial x} u, \text{ et } \frac{\partial p}{\partial x} = q = \frac{e^{\int u \partial x}(\partial u + u u \partial x)}{\partial x},$$

ac tum formula exponentialis ſponte ex aequatione egreditur.

## Corollarium 3.

825. Viciſſim etiam propoſita aequatione differentiali primi gradus $\partial u + u u \partial x = v \partial x$, in qua $v$ ſit functio quaecunque ipſarum $x$ et $u$, ea poſito $u = \frac{\partial y}{y \partial x}$, in eiusmodi aequationem differentio-differentialem transformatur, in qua variabilis $y$ cum ſuis differentialibus $\partial y$ et $\partial \partial y$ vbique eundem dimenſionum numerum conſtituat.

## Scholion 1.

826. Haec reductio aequationum differentialium primi gradus ad gradum ſecundum legibus analyſeos aduerſari videtur, interim tamen ſubinde vſu non caret; quodſi enim alia methodo huiusmodi aequationes differentio-differentiales tractare liceat, dum earum integralia vel per ſeries exhibentur vel finite, ſimul integralia aequationum differentialium primi gradus innoteſcunt, quorum ratio plerumque aliunde vix perſpicitur. In ſequentibus autem videbimus, eiusmodi aequationes differentio-differentiales in quibus variabilis altera $y$ vnam dimenſionem non ſuperat, per ſeries commode integrari poſſe, atque adeo interdum has ſeries abrumpi, ita vt integrale finita expreſſione exhibeatur. Caeterum propoſita huiusmodi aequatione differentiali primi gradus $\partial u + u u \partial x = v \partial x$, ſubſtitutio $u = \frac{\partial y}{y \partial x}$ eo magis eſt notatu digna, quod ſumto elemento $\partial x$ conſtante fiat

$$\partial u = \frac{\partial \partial y}{y \partial x} - \frac{\partial y^2}{y y \partial x}, \text{ ideoque}$$

$$\partial u + u u \partial x = \frac{\partial \partial y}{y \partial x},$$

ita vt duo termini hoc modo in vnum coaleſcant.

Scho-

## Scholion 2.

827. Casus hic imprimis notasse iuuabit, quibus aequatio $\partial u + uu \partial x = v \partial x$ integrationem admittit. Hunc in finem sit $\partial u = V \partial x$ forma generalis aequationum resolubilium, et V certa functio ipsarum $x$ et $u$, ac manifestum est si fuerit $v = uu + V$ integrationem succedere. Primum ergo hoc eueniet si sit $V = \frac{X}{U}$, denotante X functionem ipsius $x$ et U ipsius $u$. Secundo si V sit functio homogenea nullius dimensionis ipsarum $x$ et $u$. Tertio si denotantibus X et $\Xi$ functiones quascunque ipsius $x$, fuerit $V = Xu + \Xi u^n$. Quarto si denotantibus P et Q functiones quascunque ipsius $u$, fuerit $V = \frac{1}{Px + Qx^n}$. Similique modo ex aliis formis integrabilibus alii casus concludentur.

## Exemplum 1.

828. *Sumto elemento $\partial x$ constante, si proponatur haec aequatio*

$$\alpha y \partial\partial y + \beta \partial y^2 = \frac{y \partial x \partial y}{\sqrt{(aa + xx)}},$$

*eius integrale inuenire.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, prodit

$$\alpha y q + \beta pp = \frac{yp}{\sqrt{(aa + xx)}},$$

quae facto $p = uy$ et $q = vy$, abit in hanc

$$\alpha v + \beta uu = \frac{u}{\sqrt{(aa + xx)}}, \text{ seu } v = \frac{u}{\alpha\sqrt{(aa + xx)}} - \frac{\beta uu}{\alpha},$$

vnde hanc aequationem resolui oportet

$$\partial u + uu \partial x = \frac{u \partial x}{\alpha\sqrt{(aa + xx)}} - \frac{\beta uu \partial x}{\alpha}.$$

Statuatur $u = \frac{1}{s}$, fietque

$$+\partial s + \frac{s \partial x}{\alpha\sqrt{(aa + xx)}} = (1 + \frac{\beta}{\alpha}) \partial x,$$

 quae

quae per $[x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}}$ multiplicata et integrata dat

$$s[x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}}=(1+\tfrac{\beta}{\alpha})\int\partial x[x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}}.$$

Fiat $x+\sqrt{(aa+xx)}=t^{\alpha}$, erit

$$aa=t^{2\alpha}-2t^{\alpha}x,\ \text{hinc}$$

$$x=\frac{t^{2\alpha}-aa}{2t^{\alpha}}=\tfrac{1}{2}t^{\alpha}-\tfrac{1}{2}aat^{-\alpha}\ \text{et}$$

$$\partial x=\tfrac{\alpha}{2}\partial t(t^{\alpha-1}+aat^{-\alpha-1});$$

ita vt fit

$$st=(1+\tfrac{\beta}{\alpha})\int\tfrac{\alpha}{2}\partial t(t^{\alpha}+aat^{-\alpha})\ \text{feu}$$

$$st=C+\frac{\alpha+\beta}{2}\left(\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right).$$

Porro eft $\frac{\partial y}{y}=u\partial x=\frac{\partial x}{s}$; at ex aequatione differentiali eft

$$(1+\tfrac{\beta}{\alpha})\frac{\partial x}{s}=\frac{\partial s}{s}+\frac{\partial x}{\alpha\sqrt{(aa+xx)}},$$

hincque

$$(1+\tfrac{\beta}{\alpha})ly=ls+\tfrac{1}{\alpha}l[x+\sqrt{(aa+xx)}]=lst+D$$

ergo $y=B(st)^{\frac{\alpha}{\alpha+\beta}}$. Quare fumto $C=\frac{\alpha+\beta}{2}A$, habebitur

$$y=B\left(A+\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

exiftente

$$x=\tfrac{1}{2}(t^{\alpha}-aat^{-\alpha}),\ \text{vel}\ t=[x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}},$$

ita vt aequatio inter $x$ et $y$ fit

$$Cy^{\frac{\alpha+\beta}{\alpha}}=A+\tfrac{1}{\alpha+1}[x+\sqrt{(aa+xx)}]^{\frac{\alpha+1}{\alpha}}$$
$$+\tfrac{aa}{1-\alpha}[x+\sqrt{(aa+xx)}]^{\frac{1-\alpha}{\alpha}}.$$

Scho-

## Scholion.

829. Hoc idem exemplum ita eſt comparatum, vt alia ratione facillime reſolui poſſit; aequatio enim

$$\alpha y q + \beta p p = \frac{y p}{\surd(a a + x x)},$$

ſi per $\frac{\partial x}{y p}$ multiplicetur, ob $q \partial x = \partial p$ et $p \partial x = \partial y$, abit in

$$\frac{\alpha \partial p}{p} + \frac{\beta \partial y}{y} = \frac{\partial x}{\surd(a a + x x)},$$

cuius ſinguli termini ſunt integrabiles. Prodit ergo

$$p^{\alpha} y^{\beta} = C\;[x + \surd(a a + x x)], \text{ hincque}$$

$$y^{\frac{\beta}{\alpha}} \partial y = C \partial x\, [x + \surd(a a + x x)]^{\frac{1}{\alpha}},$$

quae aequatio denuo integrata praebet integrale ante inuentum. Forma ergo generalis aequationum hoc modo reſolubilium eſt $P \partial p + Y \partial y + X \partial x = 0$, exiſtente P functione ipſius $p$, Y ipſius $y$ et X ipſius $x$, quae noſtro more repraeſentatur per $P q + Y p + X = 0$. Hinc ergo perſpicitur quomodo etiam aequationes differentio-differentiales ope idonei multiplicatoris ad integrationem perduci queant; quae methodus, cum in aequationibus differentialibus primi gradus inſignem vſum praeſtiterit, eo magis excolenda videtur, quod etiam ad aequationes differentiales altiorum graduum pateat, quod argumentum infra fuſius pertractare conabimur.

## Exemplum 2.

830. *Sumto elemento $\partial x$ conſtante, ſi proponatur haec aequatio*

$$x y \partial \partial y = y \partial x \partial y + x \partial y^2 + \frac{b x \partial y^2}{\surd(a a - x x)},$$

*eius integrale inuenire.*

Poſito $\partial y = p \partial x$ et $\partial p = q \partial x$, erit

$$x y q = y p + x p p + \frac{b x p p}{\surd(a a - x x)},$$

quae

quae facto $p = u y$ et $q = v y$, abit in

$$x v = u + u u x + \frac{b u u x}{\sqrt{(a a - x x)}}:$$

vnde oritur haec aequatio differentialis

$$\partial u + u u \partial x = \frac{u \partial x}{x} + u u \partial x + \frac{b u u \partial x}{\sqrt{(a a - x x)}}, \text{ seu}$$

$$\frac{x \partial u - u \partial x}{u u} = \frac{b x \partial x}{\sqrt{(a a - x x)}},$$

cuius integrale

$$C - \frac{x}{u} = - b \sqrt{(a a - x x)}, \text{ seu}$$

$$u = \frac{x}{C + b \sqrt{(a a - x x)}}, \text{ ergo } \frac{\partial y}{y} = \frac{x \partial x}{C + b \sqrt{(a a - x x)}}.$$

Statuatur $\sqrt{(a a - x x)} = t$, vt fit $x \partial x = - t \partial t$, erit

$$\frac{\partial y}{y} = \frac{- t \partial t}{C + b t} = - \frac{\partial t}{b} + \frac{C \partial t}{b (C + b t)}, \text{ et}$$

$$l y = - \frac{t}{b} + \frac{C}{b b} l (C + b t) + l c.$$

Sit $C = n b b$, erit

$$l \frac{y}{c} = - \frac{\sqrt{(a a - x x)}}{b} + n l \frac{n b + \sqrt{(a a - x x)}}{b}$$

vbi $c$ et $n$ sunt constantes arbitrariae.

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CA-

# CAPVT IV.

## DE AEQVATIONIBVS DIFFERENTIO-DIFFERENTIALIBVS IN QVIBVS ALTERA VARIABILIS VNICAM HABET DIMENSIONEM.

### Problema 101.

831.

Sumto elemento $\partial x$ constante, si proponatur aequatio huius formae $\partial\partial y + P\,\partial x\,\partial y + Q\,y\,\partial x^2 = 0$, vbi P et Q sint functiones quaecunque ipsius $x$, eam ad aequationem differentialem primi gradus reuocare.

### Solutio.

Ponendo $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, aequatio proposita induit hanc formam $q + P\,p + Q\,y = 0$, in qua si methodo ante exposita statuamus $p = u\,y$ et $q = v\,y$, obtinebimus hanc inter $x$, $u$ et $v$ aequationem $v + P\,u + Q = 0$, hincque $v = - P\,u - Q$. Tum vero fit

$$\partial y = u\,y\,\partial x \text{ et } u\,\partial y + y\,\partial u = v\,y\,\partial x,$$

ita vt sit

$$\frac{\partial y}{y} = u\,\partial x = \frac{v\,\partial x - \partial u}{u},$$

ideoque

$$\partial u + u\,u\,\partial x + P\,u\,\partial x + Q\,\partial x = 0$$

substituto pro $v$ valore.

Qua aequatione resoluta erit $ly = \int u\,\partial x$. Vel sine his substitutionibus statim in ipsa aequatione proposita ponamus $y = e^{\int u \partial x}$, vnde fit

$$\partial y = e^{\int u \partial x} u \partial x, \text{ et } \partial\partial y = e^{\int u \partial x}(\partial u \partial x + uu \partial x^2),$$

et cum facta substitutione quantitas exponentialis $e^{\int u \partial x}$ ex calculo tollatur, obtinebitur praecedens aequatio differentialis primi gradus

$$\partial u + uu\,\partial x + P u\,\partial x + Q\,\partial x = 0,$$

a cuius resolutione integratio aequationis differentio-differentialis propositae pendet.

## Corollarium 1.

832. Haec aequatio differentialis primi gradus pluribus modis in alias formas sibi fere similes transmutari potest. Veluti si ponamus $u = Mz$, prodit

$$M\partial z + z(\partial M + PM\partial x) + MMzz\partial x + Q\partial x = 0;$$

vbi pro M eiusmodi functionem ipsius $x$ accipere licet, vt terminus ipsa litera $z$ affectus euanescat, quod fit si

$$\partial M + MP\,\partial x = 0 \text{ seu } M = C\,e^{-\int P \partial x}.$$

## Corollarium 2.

833. Similis forma prodit ponendo $u = \frac{K}{z}$, fit enim

$$-\frac{K\partial z}{zz} + \frac{\partial K}{z} + \frac{KK\partial x}{zz} + \frac{KP\partial x}{z} + Q\,\partial x = 0 \text{ seu}$$

$$K\partial z - z(\partial K + KP\partial x) - Qzz\partial x - KK\partial x = 0,$$

vbi secundus terminus, sumendo $K = C\,e^{-\int P \partial x}$, pariter euanescit.

## Corollarium 3.

834. Similis transformatio generalius instituitur, ponendo $u = K + Mz$, prodit enim

$\partial K +$

$$\partial K + M\partial z + z\partial M + KK\partial x + 2KMz\partial x + MMzz\partial x^2$$
$$+ KP\partial x + MPz\partial x + Q\partial x = 0,$$

quae ordinata praebet

$$M\partial z + z(\partial M + 2KM\partial x + MP\partial x) + MMzz\partial x^2$$
$$+ \partial K + KK\partial x + KP\partial x + Q\partial x = 0;$$

vnde fecundus terminus tollitur fumendo

$$M = C e^{-\int \partial x(2K+P)} \text{ feu } K = \frac{-\partial M - MP\partial x}{2M\partial x}.$$

## Corollarium 4.

835. Adhuc generalius fimilis forma oritur, fi ponatur $u = \frac{K+Mz}{L+Nz}$, vnde prodit

$$\partial z(LM - KN) + L\partial K - K\partial L + z(L\partial M - M\partial L + N\partial K - K\partial N)$$
$$+ zz(N\partial M - M\partial N) + (K+Mz)^2\partial x + P(K+Mz)(L+Nz)\partial x$$
$$+ Q(L+Nz)^2\partial x = 0,$$

quae reducitur ad hanc formam

$$0 = \partial z(LM - KN)$$
$$+ z\left\{\begin{matrix} L\partial M - M\partial L + N\partial K - K\partial N \\ + 2KM\partial x + P(KN+LM)\partial x + 2LNQ\partial x \end{matrix}\right\}$$
$$+ zz(N\partial M - M\partial N + MM\partial x + MNP\partial x + NNQ\partial x)$$
$$+ L\partial K - K\partial L + KK\partial x + KLP\partial x + LLQ\partial x:$$

vbi pro K, L, M et N functiones eiusmodi ipfius $x$ accipere licet, vt forma prodeat tractatu facillima.

## Scholion.

836. Quoniam huiusmodi aequationes differentio-differentiales, in quibus variabilis $y$ vnicam habet dimenfionem frequentiffime occurrere folent, merito geometrae tantopere in refoluenda aequatione

$$\partial u + uu\partial x + Pu\partial x + Q\partial x = 0$$

 ftuſtudium et operam collocarunt, quae etiam forma generaliori ita repraeſentari poteſt

$$\partial z + P z \partial x + R z z \partial x + Q \partial x = 0,$$

cuius quidem caſum eximium $\partial z + z z \partial x = a x^n \partial x$ olim Comes *Riccati* in haud ſpernendum Analyſeos incrementum propoſuerat. Inter transformationes autem huius caſus praecipue notari meretur poſitio $x = t^{\frac{1}{n+2}}$, quae dat

$$\partial z + \frac{1}{n+2} z z t^{\frac{-n-1}{n+2}} \partial t = \frac{a}{n+2} t^{\frac{n-1}{n+2}} \partial t;$$

vnde ponendo $z = C t^{\frac{n}{n+2}} v$, prodit

$$(n+2) C t^{\frac{n}{n+2}} \partial v + C n t^{\frac{-2}{n+2}} v \partial t$$
$$+ 2 C C t^{\frac{n}{n+2}} v v \partial t = 2 a t^{\frac{n}{n+2}} \partial t, \text{ ſeu}$$
$$(n+2) C \partial v + \frac{n C v \partial t}{t} + 2 C C v v \partial t = 2 a \partial t:$$

ita vt hic nulla poteſtas indefinita ipſius $t$ occurrat. Si hic porro ponatur $v = \frac{\alpha}{t} + s$, fiet

$$\begin{aligned} &-\frac{(n+2) C \alpha \partial t}{tt} + (n+2) C \partial s + \frac{n C s \partial t}{t} + 2 C C s s \partial t = 2 a \partial t, \\ &+\frac{n C \alpha \partial t}{tt} \qquad\qquad\qquad + \frac{4 C C \alpha s \partial t}{t} \\ &+\frac{2 C C \alpha \alpha \partial t}{tt} \end{aligned}$$

vbi ſi capiatur $\alpha = \frac{-n}{4C}$, orietur

$$(n+2) C \partial s + 2 C C s s \partial t = 2 a \partial t - \frac{n(n+4) \partial t}{8 t t}$$

quae forma ſimpliciſſima videtur.

## Theorema.

837. Sumto elemento $\partial x$ conſtante, ſi aequationi differentio-differentiali

$$\partial \partial y + P \partial x \partial y + Q y \partial x^2 = 0$$

ſatis-

ſatisfaciant integralia particularia $y = M$, et $y = N$, ita vt ratio $M : N$ non ſit conſtans, erit eius integrale completum $y = \alpha M + \beta N$.

## Demonſtratio.

Cum valores $y = M$ et $y = N$ ſatisfaciant aequationi propoſitae, erit

$$\partial\partial M + P\,\partial x\,\partial M + Q M\,\partial x^2 = 0 \text{ et}$$

$$\partial\partial N + P\,\partial x\,\partial N + Q N\,\partial x^2 = 0,$$

vnde patet, ponendo $y = \alpha M + \beta N$, aequationi quoque ſatisfieri, cum fiat

$$\left.\begin{aligned} &+\alpha(\partial\partial M + P\,\partial x\,\partial M + Q M\,\partial x^2) \\ &+\beta(\partial\partial N + P\,\partial x\,\partial N + Q N\,\partial x^2) \end{aligned}\right\} = 0.$$

Quoniam vero hoc integrale $y = \alpha M + \beta N$ duas conſtantes $\alpha$ et $\beta$ complectitur, quas pro lubitu definire licet, id completum ſit neceſſe eſt, niſi forte N ſit multiplum ipſius M.

## Corollarium 1.

838. Ex datis ergo duobus integralibus particularibus huiusmodi aequationis, eius integrale completum formari poteſt, ſiquidem illa duo integralia ſint inter ſe diuerſa.

## Corollarium 2.

839. Cum poſito $y = e^{\int u\,\partial x}$ ſeu $u = \frac{\partial y}{y\,\partial x}$, prodeat

$$\partial u + uu\,\partial x + P u\,\partial x + Q\,\partial x = 0.$$

Si huic aequationi ſatisfaciant valores $u = \frac{\partial M}{M\,\partial x}$ et $u = \frac{\partial N}{N\,\partial x}$, eidem quoque ſatisfaciet valor $u = \frac{\alpha\,\partial M + \beta\,\partial N}{(\alpha N + \beta M)\,\partial x}$.

## Corollarium 3.

840. Si ergo aequationis $\partial u + uu\,\partial x + P u\,\partial x + Q\,\partial x = 0$ habeantur duo integralia particularia $u = R$ et $u = S$, ob

 M =

ſtudium et operam collocarunt, quae etiam forma generaliori ita repraeſentari poteſt

$$\partial z + P z \partial x + R z z \partial x + Q \partial x = 0,$$

cuius quidem caſum eximium $\partial z + z z \partial x = a x^n \partial x$ olim Comes *Riccati* in haud ſpernendum Analyſeos incrementum propoſuerat. Inter transformationes autem huius caſus praecipue notari meretur poſitio $x = t^{\frac{2}{n+2}}$, quae dat

$$\partial z + \tfrac{2}{n+2} z z t^{\frac{-n}{n+2}} \partial t = \tfrac{2a}{n+2} t^{\frac{n}{n+2}} \partial t;$$

vnde ponendo $z = C t^{\frac{n}{n+1}}$, prodit

$$\begin{aligned} (n+2) C t^{\frac{n}{n+2}} \partial v + C n t^{\frac{-2}{n+2}} v \partial t & \\ + 2 C C t^{\frac{n}{n+2}} v v \partial t &= 2 a t^{\frac{n}{n+2}} \partial t, \text{ ſeu} \end{aligned}$$

$$(n+2) C \partial v + \tfrac{n C v \partial t}{t} + 2 C C v v \partial t = 2 a \partial t:$$

ita vt hic nulla poteſtas indefinita ipſius $t$ occurrat. Si hic porro ponatur $v = \frac{\alpha}{t} + s$, fiet

$$\begin{array}{l} -\frac{(n+2) C \alpha \partial t}{t t} + (n+2) C \partial s + \frac{n C s \partial t}{t} + 2 C C s s \partial t = 2 a \partial t, \\ +\frac{n C \alpha \partial t}{t t} \qquad\qquad\qquad\qquad\quad + \frac{4 C C \alpha s \partial t}{t} \\ +\frac{2 C C \alpha \alpha \partial t}{t t} \end{array}$$

vbi ſi capiatur $\alpha = \frac{-n}{4C}$, orietur

$$(n+2) C \partial s + 2 C C s s \partial t = 2 a \partial t - \tfrac{n(n+4) \partial t}{8 t t}$$

quae forma ſimpliciſſima videtur.

## Theorema.

837. Sumto elemento $\partial x$ conſtante, ſi aequationi differentio-differentiali

$$\partial \partial y + P \partial x \partial y + Q y \partial x^2 = 0$$

ſatis-

ſatisfaciant integralia particularia $y = M$, et $y = N$, ita vt ratio $M : N$ non ſit conſtans, erit eius integrale completum $y = \alpha M + \beta N$.

## Demonſtratio.

Cum valores $y = M$ et $y = N$ ſatisfaciant aequationi propoſitae, erit

$$\partial\partial M + P\partial x\partial M + QM\partial x^2 = 0 \text{ et}$$

$$\partial\partial N + P\partial x\partial N + QN\partial x^2 = 0,$$

vnde patet, ponendo $y = \alpha M + \beta N$, aequationi quoque ſatisfieri, cum fiat

$$\left.\begin{array}{l} +\alpha(\partial\partial M + P\partial x\partial M + QM\partial x^2) \\ +\beta(\partial\partial N + P\partial x\partial N + QN\partial x^2) \end{array}\right\} = 0.$$

Quoniam vero hoc integrale $y = \alpha M + \beta N$ duas conſtantes $\alpha$ et $\beta$ complectitur, quas pro lubitu definire licet, id completum ſit neceſſe eſt, niſi forte $N$ ſit multiplum ipſius $M$.

## Corollarium 1.

838. Ex datis ergo duobus integralibus particularibus huiusmodi aequationis, eius integrale completum formari poteſt, ſiquidem illa duo integralia ſint inter ſe diuerſa.

## Corollarium 2.

839. Cum poſito $y = e^{\int u\partial x}$ ſeu $u = \frac{\partial y}{y\partial x}$, prodeat

$$\partial u + uu\partial x + Pu\partial x + Q\partial x = 0.$$

Si huic aequationi ſatisfaciant valores $u = \frac{\partial M}{M\partial x}$ et $u = \frac{\partial N}{N\partial x}$, eidem quoque ſatiſfaciet valor $u = \frac{\alpha\partial M + \beta\partial N}{(\alpha N + \beta M)\partial x}$.

## Corollarium 3.

840. Si ergo aequationis $\partial u + uu\partial x + Pu\partial x + Q\partial x = 0$ habeantur duo integralia particularia $u = R$ et $u = S$, ob

 M =

$$M = e^{\int R \partial x} \text{ et } N = e^{\int S \partial x},$$

integrale completum erit

$$u = \frac{\alpha e^{\int R \partial x} R + \beta e^{\int S \partial x} S}{\alpha e^{\int R \partial x} + \beta e^{\int S \partial x}} \text{ siue}$$

$$u = R + \frac{\beta e^{\int S \partial x} (S - R)}{\alpha e^{\int R \partial x} + \beta e^{\int S \partial x}}.$$

## Scholion.

841. Maximi momenti eſt haec obſeruatio, quod in huiusmodi aequationibus ex cognitis binis integralibus particularibus integrale completum aſſignari poſſit. Plerumque autem cognito vno integrali particulari, in eo ſignum radicale ineſſe ſolet, ob cuius ambiguitatem duo ſimul integralia particularia innoteſcunt. Ita ſi aequationi

$$\partial u + u u \partial x + P u \partial x + Q \partial x = 0$$

ſatisfaciat valor $u = T + \sqrt{V}$, eidem ſatisfaciet $u = T - \sqrt{V}$, vnde integrale completum erit

$$u = T + \sqrt{V} - \frac{2 \beta \sqrt{V}}{\alpha e^{2\int \partial x \sqrt{V}} + \beta} \text{ ſeu}$$

$$u = T + \frac{\alpha e^{2\int \partial x \sqrt{V}} \sqrt{V} - \beta \sqrt{V}}{\alpha e^{2\int \partial x \sqrt{V}} + \beta}.$$

Ac ſi forte ſit $\sqrt{V}$ imaginarium, puta $\sqrt{V} = X \sqrt{-1}$, ob

$$e^{\pm \int \partial x \sqrt{V}} = \text{coſ.} \int X \partial x \pm \sqrt{-1} . \text{ſin.} \int X \partial x, \text{ erit}$$

$$u = T + \frac{(\alpha - \beta) \text{coſ} \int X \partial x + (\alpha + \beta) \text{ſin.} \int X \partial x . \sqrt{-1}}{(\alpha + \beta) \text{coſ.} \int X \partial x + (\alpha - \beta) \text{ſin.} \int X \partial x . \sqrt{-1}} X \sqrt{-1},$$

ſeu poſito

$$(\alpha - \beta) \sqrt{-1} = \gamma \text{ et } \alpha + \beta = \delta,$$

$$u = T + \frac{\gamma \text{ coſ.} \int X \partial x - \delta \text{ ſin.} \int X \partial x}{\delta \text{ coſ.} \int X \partial x + \gamma \text{ ſin.} \int X \partial x} . X, \text{ vel etiam}$$

$$u = T + X \text{ tang.} (\int X \partial x + \zeta).$$

Pro-

## Problema 102.

842. Sumto elemento $\partial x$ constante, inuenire integrale completum huius aequationis differentio-differentialis

$$\partial\partial y + A\,\partial y\,\partial x + B\,y\,\partial x^2 = 0.$$

## Solutio.

Posito $y = e^{\int u \partial x}$ prodit, haec aequatio

$$\partial u + u u \partial x + A u \partial x + B \partial x = 0, \text{ siue}$$

$$\partial x = \frac{-\partial u}{uu + Au + B},$$

cui satisfit tribuendo ipsi $u$ eiusmodi valorem constantem vt euadat $uu + Au + B = 0$, qui sunt

$$u = -\tfrac{1}{2}A \pm \sqrt{(\tfrac{1}{4}AA - B)}.$$

Hinc ergo cum habeantur bina integralia particularia $y = e^{\int u \partial x}$, posito $\sqrt{(\tfrac{1}{4}AA - B)} = n$, erit integrale completum

$$y = e^{-\frac{1}{2}Ax}(\alpha e^{nx} + \beta e^{-nx}),$$

ac si sit $n$ numerus imaginarius, puta $n = m\sqrt{-1}$, erit

$$y = e^{-\frac{1}{2}Ax}(\alpha \operatorname{cos}. mx + \beta \operatorname{fin}. mx) = C e^{-\frac{1}{2}Ax} \operatorname{fin}.(mx + \gamma).$$

Sin autem sit $n = 0$, prodibit

$$y = e^{-\frac{1}{2}Ax}(\alpha + \beta x).$$

## Corollarium 1.

843. Ad integrale ergo aequationis propositae inueniendum, resolui oportet aequationem algebraicam $uu + Au + B = 0$, quae oritur ex proposita

$$\partial\partial y + A\,\partial y\,\partial x + B\,y\,\partial x^2 = 0,$$

si loco $y$, $\partial y$, $\partial\partial y$, scribatur $u^0$, $u^1$, $u^2$, et elementum $\partial x$ reiiciatur, tum enim binae radices illius aequationis dabunt integrale completum.

Corol-

## Corollarium 2.

844. Scilicet si aequationis $uu + Au + B = 0$ factores sint $(u+f)(u+g)$, ob valores $u = -f$ et $u = -g$, integrale completum erit $y = \alpha e^{-fx} + \beta e^{-gx}$. At si sit $g = f$, erit $y = e^{-fx}(\alpha + \beta x)$.

## Corollarium 3.

845. Si aequatio $uu + Au + B = 0$ habeat factores imaginarios, quo casu huiusmodi formam habebit

$$uu + 2fu \operatorname{cos}.\zeta + ff = 0, \text{ erit } u = -f\operatorname{cos}.\zeta \pm f\sqrt{-1}.\operatorname{sin}.\zeta,$$

hincque integrale completum erit:

$$y = e^{-fx\operatorname{cos}.\zeta}(\alpha \operatorname{cos}.fx\operatorname{sin}.\zeta + \beta \operatorname{sin}.fx\operatorname{sin}.\zeta), \text{ siue}$$
$$y = Ce^{-fx\operatorname{cos}.\zeta}\operatorname{sin}.(fx\operatorname{sin}.\zeta + \gamma).$$

## Scholion.

846. Idem integrale completum reperitur methodo consueta ex aequatione $\partial x = \frac{-\partial u}{uu + Au + B}$: posito enim

$$uu + Au + B = (u+f)(u+g), \text{ erit}$$
$$(g-f)\partial x = \frac{\partial u}{u+g} - \frac{\partial u}{u+f}, \text{ et } Ce^{(g-f)x} = \frac{u+g}{u+f},$$

vnde fit

$$u = \frac{g - Cfe^{(g-f)x}}{Ce^{(g-f)x} - 1}, \text{ seu}$$
$$u = \frac{-\alpha f e^{gx} + \beta g e^{fx}}{\alpha e^{gx} - \beta e^{fx}}.$$

Tum vero fit

$$\int u\partial x = -\int \frac{\alpha f e^{(g-f)x} - \beta g}{\alpha e^{(g-f)x} - \beta}\partial x = -\int \frac{u\partial u}{(u+f)(u+g)},$$

ideoque

$$\int u\partial x = \frac{f}{g-f} l(u+f) - \frac{g}{g-f} l(u+g).$$

Hinc

Hinc

$$y = e^{\int u\,\partial x} = C\,(u+f)^{\frac{f}{g-f}}(u+g)^{\frac{-g}{g-f}}.$$

At eſt

$$u+f = \frac{\beta(g-f)\,e^{fx}}{\alpha e^{gx} - \beta e^{fx}} \text{ et } u+g = \frac{\alpha(g-f)\,e^{gx}}{\alpha e^{gx} - \beta e^{fx}},$$

vnde colligitur mutando conſtantem C

$$y = \frac{C\,e^{\frac{ffx}{g-f}}\,e^{\frac{-ggx}{g-f}}}{(\alpha e^{gx} - \beta\, e^{fx})^{\frac{f-g}{g-f}}} = C\,e^{-(f+g)x}\,(\alpha e^{gx} - \beta e^{fx}),$$

ſeu $y = \alpha e^{-fx} + \beta e^{-gx}$, vt ante. Hinc ergo patet, quantum ſubſidium afferat formatio integralis completi ex binis particularibus.

## Problema 103.

847. Sumto elemento $\partial x$ conſtante, ſi proponatur haec aequatio differentio-differentialis

$$\partial\partial y + \frac{A\,\partial y\,\partial x}{x} + \frac{B\,y\,\partial x^2}{xx} = 0,$$

eius integrale completum inuenire.

## Solutio.

Ponatur $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, vt habeamus

$$q + \frac{Ap}{x} + \frac{By}{xx} = 0, \text{ ſeu } q = \frac{-Ap}{x} - \frac{By}{xx}.$$

Sit nunc $p = \frac{uy}{x}$, erit $\partial y = \frac{uy\,\partial x}{x}$, et

$$\partial p = \frac{u\,\partial y + y\,\partial u}{x} - \frac{uy\,\partial x}{xx} = -\frac{A\,\partial y}{x} - \frac{By\,\partial x}{xx},$$

vnde colligimus

$$\frac{\partial y}{y} = \frac{u\,\partial x}{x} = \frac{u\,\partial x - B\,\partial x - x\,\partial u}{xu + Ax}, \text{ ſeu}$$

$$x\,\partial u + B\,\partial x + uu\,\partial x + (A-1)\,u\,\partial x = 0,$$

hincque $\frac{\partial x}{x} = \frac{-\partial u}{uu + (A - 1)u + B}$; cui particulariter satisfit ponendo $uu + (A - 1)u + B = 0$.

Sit primo $uu + (A - 1)u + B = (u + f)(u + g)$, erit particulariter $ly = -flx$ et $y = x^{-f}$, similique modo $y = x^{-g}$, vnde integrale completum erit

$$y = \alpha x^{-f} + \beta x^{-g}.$$

Si sit $g = f$, statuatur $g = f - \omega$ euanescente $\omega$, erit

$$x^{-g} = x^{-f} \cdot x^{\omega} = x^{-f}(1 + \omega l x),$$

ergo hoc casu fit

$$y = x^{-f}(\alpha + \beta l x).$$

Sit denique

$$uu + (A - 1)u + B = uu + 2fu\,\text{cos.}\,\zeta + ff, \text{ erit}$$

$$u = -f(\text{cos.}\,\zeta \pm \sqrt{-1}.\,\text{sin.}\,\zeta),$$

ergo particulariter

$$y = x^{-f\,\text{cos.}\,\zeta} \cdot x^{\pm f\sqrt{-1}.\,\text{sin.}\,\zeta} = x^{-f\,\text{cos.}\,\zeta}[\text{cos.}(f\,\text{sin.}\,\zeta . lx) \pm \sqrt{-1}.\,\text{sin.}(f\,\text{sin.}\,\zeta . lx)],$$

quare integrale completum erit

$$y = C\,x^{-f\,\text{cos.}\,\zeta}\,\text{sin.}\,(f\,\text{sin.}\,\zeta . lx + \gamma).$$

## Corollarium. 1.

848. Huius ergo aequationis

$$\partial\partial y + (f + g + 1)\frac{\partial y \partial x}{x} + \frac{fgy\partial x^2}{xx} = 0,$$

integrale completum est

$$y = \alpha x^{-f} + \beta x^{-g}.$$

Huius autem

$$\partial\partial y + (2f + 1)\frac{\partial y \partial x}{x} + \frac{ffy\partial x^2}{xx} = 0,$$

integrale completum est

$$y = x^{-f}(\alpha + \beta l x).$$

Corol-

## Corollarium 2.

849. At si aequatio proposita huiusmodi formam habuerit

$$\partial\partial y + (1 + 2f \operatorname{cos}. \zeta)\frac{\partial y \partial x}{x} + \frac{ffy\partial x}{xx} = 0,$$

tum eius integrale completum erit

$$y = C x^{-f \operatorname{cos}. \zeta} \operatorname{fin}. (f \operatorname{fin}. \zeta\, l\, x + \gamma).$$

## Scholion.

850. Similem resolutionem quoque admittit haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\partial y \partial x}{x} + A x^n \partial y \partial x + B x^{2n} y \partial x^2 = 0.$$

Ponatur enim $\partial y = x^n y u \partial x$, et cum sit

$$\partial\partial y = x^n y \partial x \partial u + n x^{n-1} y u \partial x^2 + x^{2n} y u^2 \partial x^2,$$

erit per $y$ diuidendo

$$x^n \partial x \partial u + n x^{n-1} u \partial x^2 + x^{2n} uu \partial x^2 - n x^{n-1} u \partial x^2$$
$$+ A x^{2n} u \partial x^2 + B x^{2n} \partial x^2 = 0,$$

hinc

$$\partial u + x^n uu \partial x + A x^n u \partial x + B x^n \partial x = 0,$$

ideoque

$$x^n \partial x = \frac{-\partial u}{uu + Au + B},$$

cui particulariter satisfit ponendo $uu + Au + B = 0$, vnde $u$ duplicem consequitur valorem constantem quorum alter sit $u = -f$ alter $u = -g$. Quocirca integralia particularia erunt

$$y = e^{\frac{-f x^{n+1}}{n+1}} \quad \text{et} \quad y = e^{\frac{-g x^{n+1}}{n+1}}.$$

Sit breuitatis gratia $\frac{x^{n+1}}{n+1} = t$, erit integrale completum



$y =$

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

pro casu scilicet

$$uu + Au + B = (u+f)(u+g).$$

At pro casu

$$uu + Au + B = (u+f)^2, \text{ erit } y = e^{-ft}(\alpha + \beta t).$$

Casu autem quo

$$uu + Au + B = uu + 2fu\,\text{cos.}\,\zeta + ff, \text{ erit}$$
$$y = C e^{-ft\,\text{cos.}\,\zeta}\,\text{sin.}\,(ft\,\text{sin.}\,\zeta + \gamma).$$

Haec integratio adeo ad hanc formam extendi potest, sumendo pro X functionem quamcunque ipsius $x$

$$\partial\partial y - \frac{\partial x\,\partial y}{X} + AX\,\partial y\,\partial x + BXXy\,\partial x^2 = 0.$$

Posito enim $\partial y = Xuy\,\partial x$ seu $\frac{\partial y}{y} = Xu\,\partial x$, fit

$$X\partial x = \frac{-\partial u}{uu + Au + B},$$

vnde posito $\int X\partial x = t$, integrale completum se habebit vt ante. Scilicet

1) si $A = f + g$ et $B = fg$, erit integrale
$$y = \alpha e^{-ft} + \beta e^{-gt},$$

2) si $A = 2f$ et $B = ff$, erit integrale
$$y = e^{-ft}(\alpha + \beta t),$$

3) si $A = 2f\,\text{cos.}\,\zeta$ et $B = ff$, erit integrale
$$y = Ce^{-ft\,\text{cos.}\,\zeta}\,\text{sin.}\,(ft\,\text{sin.}\,\zeta + \gamma).$$

## Problema 104.

851. Sumto elemento $\partial x$ constante, si P, Q et X denotent functiones quascunque ipsius $x$, integrationem huius aequationis differentio-differentialis

$$\partial\partial y + P\,\partial y\,\partial x + Qy\,\partial x^2 = X\,\partial x^2,$$

ad aequationem differentialem primi ordinis reducere.

Solu-

## Solutio.

Hic fingulari modo procedamus, loco $y$ binas nouas incognitas introducendo. Statuatur fcilicet $y = uv$, et cum fit

$$\partial y = u\partial v + v\partial u \text{ et } \partial\partial y = u\partial\partial v + 2\partial u\partial v + v\partial\partial u,$$

aequatio noftra induet hanc formam

$$u\partial\partial v + 2\partial u\partial v + v\partial\partial u + Pu\partial x\partial v + Pv\partial x\partial u$$
$$+ Quv\partial x^2 = X\partial x^2.$$

Iam altera $v$ ita determinetur, vt termini ipfa littera $u$ affecti deftruantur, quod fit fi

$$\partial\partial v + P\partial x\partial v + Qv\partial x^2 = 0,$$

vnde per fuperiora $v$ per $x$ determinetur, quo facto fupereft haec aequatio

$$2\partial u\partial v + v\partial\partial u + Pv\partial x\partial u = X\partial x^2,$$

vnde cum $v$ iam detur per $x$, quantitas $u$ definiri debet. Ponatur $\partial u = s\partial x$, eritque

$$v\partial s + 2s\partial v + Psv\partial x = X\partial x,$$

quae multiplicata per $ve^{\int P\partial x}$ integrabilis redditur; prodit enim

$$vvse^{\int P\partial x} = \int e^{\int P\partial x} Xv\partial x, \text{ ideoque}$$

$$s = \frac{e^{-\int P\partial x}}{vv}\int e^{\int P\partial x} Xv\partial x \text{ et}$$

$$u = \int \frac{e^{-\int P\partial x}\partial x}{vv}\int e^{\int P\partial x} Xv\partial x.$$

Quare cum incognita $v$ fuerit determinata aequatione

$$\partial\partial v + P\partial x\partial v + Qv\partial x^2 = 0,$$

integrale aequationis propofitae erit

$$y = v\int \frac{e^{-\int P\partial x}\partial x}{vv}\int e^{\int P\partial x} Xv\partial x.$$

## Corollarium 1.

852. Vt integratio ad aequationem differentialem primi ordinis reuocetur, ponatur $v = e^{\int t \partial x}$, et quantitas $t$ definietur per hanc aequationem

$$\partial t + t t \partial x + P t \partial x + Q \partial x = 0,$$

quo facto integrale quaesitum erit

$$y = e^{\int t \partial x} \int e^{-\int (P + 2t) \partial x} \partial x \int e^{\int (P + t) \partial x} X \partial x.$$

## Corollarium 2.

853. Cum sit $(P + t) \partial x = -\frac{\partial t}{t} - \frac{Q \partial x}{t}$, erit

$$e^{\int (P + t) \partial x} = \frac{1}{t} e^{-\int \frac{Q \partial y}{t}}, \text{ hincque}$$

$$y = e^{\int t \partial x} \int e^{\int \frac{Q \partial x}{t} - \int t \partial x} t \partial x \int e^{-\int \frac{Q \partial x}{t}} X \frac{\partial x}{t},$$

vbi duplex integratio ad integrale completum perducit.

## Scholion 1.

854. Alio modo qui propius ad ante vsitatum accedat, eadem integratio institui potest. Ponatur scilicet pro aequatione proposita $\partial y = t y \partial x + v \partial x$, vbi $v$ certam ipsius $x$ functionem designet ex functione X determinandam. Cum igitur sit

$$\partial \partial y = y \partial t \partial x + t \partial x (t y \partial x + v \partial x) + \partial v \partial x,$$

erit facta substitutione

$$\left. \begin{matrix} y \partial t \partial x + t t y \partial x^2 + P t y \partial x^2 + Q y \partial x^2 \\ + t v \partial x^2 + \partial v \partial x + P v \partial x^2 + X \partial x^2 \end{matrix} \right\} = 0,$$

cuius aequationis vtraque pars, tam ea quae per $y$ multiplicatur, quam altera ab $y$ libera, seorsim nihilo aequetur, vnde has duas aequationes nanciscimur

$\partial t$

$$\partial t + tt\partial x + Pt\partial x + Q\partial x = 0 \text{ et}$$
$$\partial v + tv\partial x + Pv\partial x = X\partial x,$$

ex quarum illa $t$ per $x$ vt ante definiri debet, tum vero erit ex ista

$$e^{\int (P+t)\partial x} v = \int e^{\int (P+t)\partial x} X\partial x.$$

Iam vero ex aequatione assumta $\partial y - ty\partial x = v\partial x$ colligitur

$$e^{-\int t\partial x} y = \int e^{-\int t\partial x} v\partial x,$$

vbi si loco $v$ valor modo inuentus substituatur, praecedens integralis forma obtinetur.

## Scholion 2.

855. Ex hac operatione sequi videtur, aequationis propositae

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

integrationem necessario pendere ab integratione huius

$$\partial\partial v + P\partial v\partial x + Qv\partial x^2 = 0,$$

quandoquidem hac concessa illa exhiberi potest. Minime tamen hinc vicissim colligere licet, si posterioris resolutio vires nostras superet, etiam priorem nullo modo integrari posse, quin potius facile est infinitos casus exhibere, quibus prior integrationem admittat, cum tamen posterior irresolubilis existat. Sit enim $P = 0$ et $Q = \alpha x$, atque certum est aequationem posteriorem $\partial\partial v + \alpha xv\partial x^2 = 0$, nulla adhuc methodo resolui posse, cum posito $v = e^{\int t\partial x}$, abeat in

$$\partial t + tt\partial x + \alpha x\partial x = 0;$$

neque tamen hinc sequitur aequationem priorem

$$\partial\partial y + \alpha xy\partial x^2 = X\partial x^2,$$

semper esse intractabilem. Infiniti enim casus pro X assignari pos-

possunt, quibus integratio succedat. Sumta enim pro $y$ functione quacunque ipsius $x$, reperietur pro X eiusmodi functio, vt aequationi valor pro $y$ assumtus satisfaciat. Veluti posito $y = \frac{\beta x}{\alpha}$, ob $\partial\partial y = 0$ fit $X = \beta x x$, atque aequationi

$$\partial\partial y + \alpha x y \partial x^2 = \beta x x \partial x^2$$

vtique satisfacit integrale $y = \frac{\beta x}{\alpha}$. Interim tamen hoc integrale tantum est particulare, ac dubium adhuc relinquitur, an etiam integrale completum exhiberi possit. At posito $y = \frac{\beta x}{\alpha} + z$, pro integrali completo inueniendo prodit $\partial\partial z + \alpha x z \partial x^2 = 0$, quae cum resolutionem respuat, euidens est integrale completum etiam in genere exhiberi non posse, nisi simul altera aequatio integrationem admittat.

## Problema 105.

856. Sumto elemento $\partial x$ constante, inuenire integrale completum huius aequationis differentio-differentialis

$$\partial\partial y + A \partial y \partial x + B y \partial x^2 = X \partial x^2$$

denotante X functionem quamcunque ipsius $x$.

## Solutio.

Posito $y = u v$, aequatio proposita in duas sequentes resoluitur

$$\partial\partial v + A \partial v \partial x + B v \partial x^2 = 0 \text{ et}$$

$$v \partial\partial u + 2 \partial v \partial u + A v \partial x \partial u = X \partial x^2.$$

Quodsi ergo ex priore valor ipsius $v$ per $x$ definiatur, integrale completum ex posteriore ita se habebit, vt ob $P = A$ sit

$$y = v \int \frac{e^{-Ax} \partial x}{v v} \int e^{Ax} X v \partial x,$$

vbi

vbi cum duplex integratio integrale completum producat, sufficiet pro $v$ integrale particulare prioris aequationis assumsisse, id quod etiam ex solutione generali patebit. Cum igitur pro resolutione prioris aequationis formanda sit haec aequatio quadratica $tt + At + B = 0$, pro eius indole tres casus euolui conueniet.

I. Si $tt + At + B = (t+f)(t+g)$, vt sit $A = f+g$ et $B = fg$, erit $v = \alpha e^{-fx} + \beta e^{-gx}$. Sumatur primo tantum integrale particulare $v = e^{-fx}$, et ob $A = f+g$, fiet

$$y = e^{-fx} \int e^{(f-g)x} \partial x \int e^{gx} X \partial x:$$

sit $e^{(f-g)x} \partial x = \partial R$, et $\int e^{gx} X \partial x = S$, vt fiat

$$y = e^{-fx} \int S \partial R = e^{-fx} (RS - \int R \partial S);$$

at est $R = \frac{1}{f-g} e^{(f-g)x}$, vnde colligitur

$$y = \frac{1}{f-g} e^{-gx} S - \frac{1}{f-g} e^{-fx} \int e^{fx} X \partial x, \text{ siue}$$

$$(f-g) y = e^{-gx} \int e^{gx} X \partial x - e^{-fx} \int e^{fx} X \partial x,$$

quod idem integrale prodiisset, si altero particulari $v = e^{-gx}$ vsi essemus.

In genere autem sumto $v = \alpha e^{-fx} + \beta e^{-gx}$, statuatur vt ante

$$\frac{e^{-(f+g)x} \partial x}{vv} = \partial R, \text{ et } \int e^{(f+g)x} X v \partial x = S,$$

vt sit pariter

$$y = v \int S \partial R = v (RS - \int R \partial S).$$

Fingamus

$$R = \frac{C e^{\lambda x}}{v}, \text{ et ob } \partial v = -\partial x (\alpha f e^{-fx} + \beta g e^{-gx});$$

 erit

erit

$$\partial R = \frac{C e^{\lambda x} \partial x (\alpha \lambda e^{-fx} + \beta \lambda e^{-gx} + \alpha f e^{-fx} + \beta g e^{-gx})}{vv}:$$

Sit iam $\lambda = -g$, et $C\alpha(f-g) = 1$, vt $\partial R$ datum adipiscatur valorem, ob $C = \frac{1}{\alpha(f-g)}$, erit

$$R = \frac{e^{-gx}}{\alpha(f-g)v}, \text{ et } R\partial S = \frac{1}{\alpha(f-g)} e^{fx} X \partial x;$$

tum vero

$$S = \alpha \int e^{gx} X \partial x + \beta \int e^{fx} X \partial x,$$

vnde conficitur

$$y = v\left(\frac{e^{-gx}}{(f-g)v} \int e^{gx} X \partial x + \frac{\beta e^{-gx}}{\alpha(f-g)v} \int e^{fx} X \partial x - \frac{1}{\alpha(f-g)} \int e^{fx} X \partial x\right),$$

seu prorsus vt ante

$$(f-g)y = e^{-gx} \int e^{gx} X \partial x - e^{-fx} \int e^{fx} X \partial x.$$

II. Si $tt + At + B = (t+f)^2$, seu $A = 2f$ et $B = ff$, erit ex priori aequatione $v = e^{-fx}(\alpha + \beta x)$. Ponatur vt ante

$$\frac{e^{-2fx} \partial x}{vv} = \partial R, \text{ et } \int e^{2fx} X v \partial x = S,$$

vt habeatur $y = v(RS - \int R \partial S)$. Cum ergo sit

$$\partial R = \frac{\partial x}{(\alpha + \beta x)^2}, \text{ erit } R = -\frac{1}{\beta(\alpha + \beta x)} = -\frac{e^{-fx}}{\beta v}, \text{ et}$$

$$S = \alpha \int e^{fx} X \partial x + \beta \int e^{fx} X x \partial x = \int e^{fx} X \partial x (\alpha + \beta x),$$

quare

$$vRS = -\tfrac{\alpha}{\beta} e^{-fx} \int e^{fx} X \partial x - e^{-fx} \int e^{fx} X x \partial x, \text{ et}$$

$$\int R \partial S = -\tfrac{1}{\beta} \int e^{fx} X \partial x,$$

vnde conficitur

$y =$

$$y = e^{-fx} x \int e^{fx} X \partial x - e^{-fx} \int e^{fx} X x \partial x;$$

seu cum sit

$$\partial . e^{fx} y = \partial x \int e^{fx} X \partial x,$$

erit succinctius

$$y = e^{-fx} \int \partial x \int e^{fx} X \partial x.$$

III. Si $tt + At + B = tt + 2ft \operatorname{cos}. \zeta + ff$, seu $A = 2f \operatorname{cos}. \zeta$ et $B = ff$, erit

$$v = e^{-fx \operatorname{cos}. \zeta} \operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma).$$

Ponatur

$$\frac{e^{-2fx \operatorname{cos}. \zeta} \partial x}{vv} = \frac{\partial x}{\operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma)^2} = \partial R \text{ et}$$

$$e^{2fx \operatorname{cos}. \zeta} X v \partial x = e^{fx \operatorname{cos}. \zeta} X \partial x \operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma) = \partial S,$$

vt obtineatur $y = vRS - v \int R \partial S$. At est

$$R = -\frac{1}{f \operatorname{sin}. \zeta} \cdot \frac{\operatorname{cos}. (fx \operatorname{sin}. \zeta + \gamma)}{\operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma)};$$

hincque

$$vRS = -\frac{1}{f \operatorname{sin}. \zeta} e^{-fx \operatorname{cos}. \zeta} \operatorname{cos}. (fx \operatorname{sin}. \zeta + \gamma) \int e^{fx \operatorname{cos}. \zeta} X \partial x \operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma)$$

et

$$\int R \partial S = -\frac{1}{f \operatorname{sin}. \zeta} \int e^{fx \operatorname{cos}. \zeta} X \partial x \operatorname{cos}. (fx \operatorname{sin}. \zeta + \gamma).$$

Quocirca obtinebitur

$$fy \operatorname{sin}. \zeta = + e^{-fx \operatorname{cos}. \zeta} \operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma) \int e^{fx \operatorname{cos}. \zeta} X \partial x \operatorname{cos}. (fx \operatorname{sin}. \zeta + \gamma)$$
$$- e^{-fx \operatorname{cos}. \zeta} \operatorname{cos}. (fx \operatorname{sin}. \zeta + \gamma) \int e^{fx \operatorname{cos}. \zeta} X \partial x \operatorname{sin}. (fx \operatorname{sin}. \zeta + \gamma).$$

## Corollarium 1.

857. In hoc postremo integrali si ponamus $fx \operatorname{sin}. \zeta = \Phi$, erit

$$fe^{fx \operatorname{cos}. \zeta} y \operatorname{sin}. \zeta =$$
$$(\operatorname{sin}. \gamma \operatorname{cos}. \Phi + \operatorname{cos}. \gamma \operatorname{sin}. \Phi) \int e^{fx \operatorname{cos}. \zeta} X \partial x (\operatorname{cos}. \gamma \operatorname{cos}. \Phi - \operatorname{sin}. \gamma \operatorname{sin}. \Phi)$$
$$+ (\operatorname{sin}. \gamma \operatorname{sin}. \Phi - \operatorname{cos}. \gamma \operatorname{cos}. \Phi) \int e^{fx \operatorname{cos}. \zeta} X \partial x (\operatorname{sin}. \gamma \operatorname{cos}. \Phi + \operatorname{cos}. \gamma \operatorname{sin}. \Phi),$$

 seu

feu

$$\int e^{\int x\, cof.\zeta} y \text{ fin. } \zeta =$$

$$+\text{fin.}\gamma\text{cof.}\gamma\text{cof.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{cof.}\Phi - \text{fin.}\gamma^2\text{cof.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{fin.}\Phi$$
$$+\text{cof.}\gamma^2\text{fin.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{cof.}\Phi - \text{fin.}\gamma\text{cof.}\gamma\text{fin.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{fin.}\Phi$$
$$+\text{fin.}\gamma^2\text{fin.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{cof.}\Phi + \text{fin.}\gamma\text{cof.}\gamma\text{fin.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{fin.}\Phi$$
$$-\text{fin.}\gamma\text{cof.}\gamma\text{cof.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{cof.}\Phi - \text{cof.}\gamma^2\text{cof.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{fin.}\Phi$$

vnde patet angulum $\gamma$ prorfus ex calculo excedere; fit enim

$$\int e^{\int x\, cof.\zeta} y \text{fin.}\zeta = \text{fin.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{ cof.}\Phi - \text{cof.}\Phi\int e^{\int x\, cof.\zeta} X\partial x \text{fin.}\Phi.$$

## Corollarium 2.

858. Cum igitur loco vnius aequationis duas formaverimus integrandas, vidimus fufficere, fi alterius integrale faltem particulare fuerit cognitum. In ambobus enim praecedentibus cafibus conftantes $\alpha$ et $\beta$ integrale completum praebentes ex calculo fponte euanuerunt, et cafu tertio conftans $\gamma$ itidem exceffit.

## Exemplum.

859. *Sumto elemento $\partial x$ conftante, inuenire integrale huius aequationis*

$$\partial\partial y + A\,\partial y\,\partial x + B y\,\partial x^2$$
$$= \partial x^2\,[n(n-1)\,x^{n-2} + n A\,x^{n-1} + B\,x^n].$$

Hoc exemplum ita eft comparatum, vt ei manifefto fatisfaciat valor $y = x^n$, qui eius integrale particulare conftituit. Ad completum ergo inueniendum, fit $A = f + g$ et $B = fg$; et cum fit

$$X = f g\,x^n + n(f+g)\,x^{n-1} + n(n-1)\,x^{n-2}, \text{ erit}$$
$$\int e^{gx} X\,\partial x = f\,e^{gx}\,x^n + n\,e^{gx}\,x^{n-1} + \alpha, \text{ et}$$
$$\int e^{fx} X\,\partial x = g\,e^{fx}\,x^n + n\,e^{fx}\,x^{n-1} + \beta,$$

vnde ex forma inuenta prodit integrale completum

$(f-g)$

$$(f-g)y = fx^n + nx^{n-1} + \alpha e^{-gx} - gx^n - nx^{n-1} - \beta e^{-fx},$$

seu $y = x^n + \frac{\alpha}{f-g} e^{-gx} - \frac{\beta}{f-g} e^{-fx}$,

vel mutata constantium forma

$$y = x^n + \alpha e^{-fx} + \beta e^{-gx}.$$

Si sit $g = f$, ponatur $g = f + \omega$, existente $\omega = 0$, et ob $e^{-gx} = e^{-fx} \cdot e^{-\omega x} = e^{-fx}(1 - \omega x)$, vnde pro $\alpha + \beta$ et $-\beta\omega$ scribendo $\alpha$ et $\beta$, erit

$$y = x^n + e^{-fx}(\alpha + \beta x).$$

Sin autem sit

$$f = a + b\sqrt{-1} \text{ et } g = a - b\sqrt{-1},$$

cum fiat

$$y = x^n + e^{-ax}(\alpha e^{-bx\sqrt{-1}} + \beta e^{bx\sqrt{-1}}) \text{ ob}$$
$$e^{\pm bx\sqrt{-1}} = \text{cof.}\, bx \pm \sqrt{-1} \,.\, \text{fin.}\, bx,$$

mutata forma constantium habebimus

$$y = x^n + e^{-ax}(\alpha\, \text{cof.}\, bx + \beta\, \text{fin.}\, bx).$$

## Scholion.

860. In genere autem si huiusmodi aequationis

$$\partial\partial y + A\partial y\partial x + By\partial x^2 = X\partial x^2,$$

constet integrale particulare, seu valor ipsi satisfaciens $y = t$, integrale completum facile reperitur ponendo $y = t + z$. Cum enim per hypothesin sit

$$\partial\partial t + A\partial t\partial x + Bt\partial x^2 = X\partial x^2,$$

facta hac substitutione orietur

$$\partial\partial z + A\partial z\partial x + Bz\partial x^2 = 0,$$

vnde si $A = f + g$ et $B = fg$, colligitur

$$z = \alpha e^{-fx} + \beta e^{-gx},$$

 sicque

ficque integrale completum erit

$$y = t + \alpha e^{-fx} + \beta e^{-gx},$$

quemadmodum etiam in exemplo allato inuenimus.

## Problema 106.

861. Sumto elemento $\partial x$ conftante, fi proponatur haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\partial y \partial x}{x} + A x^n \partial y \partial x + B x^{2n} y \partial x^2 = X \partial x^2,$$

exiftente X functione quacunque ipfius $x$, eius integrale completum inueftigare.

## Solutio.

Refolutio huius aequationis vt fupra ex pofitione $y = uv$ deriuari poffet, fed alia methodo hic vtentes leui fubftitutione eam ad formam Problematis praecedentis reducamus. Scilicet ponamus $x^n \partial x = \partial t$, vt fit $x^{n+1} = (n+1) t$, qua fubftitutione functio X abeat in T functionem quandam ipfius $t$. Ne autem affumtio elementi $\partial x$ conftantis turbet, hanc conditionem tollamus ponendo $\partial y = p \partial x$ et $\partial p = q \partial x$, habebimusque

$$q - \frac{np}{x} + A x^n p + B x^{2n} y = X.$$

Cum nunc fit $\partial x = \frac{\partial t}{x^n}$, erit $p = \frac{x^n \partial y}{\partial t}$, hincque fumto elemento $\partial t$ conftante

$$\partial p = \frac{n x^{n-1} \partial x \partial y}{\partial t} + \frac{x^n \partial\partial y}{\partial t} = q \partial x = \frac{q \partial t}{x^n},$$

ergo

$$q = \frac{n x^{n-1} \partial y}{\partial t} + \frac{x^{2n} \partial\partial y}{\partial t^2},$$

ficque noftra aequatio erit

$$\frac{x^{2n}\partial\partial y}{\partial t^2} = \frac{n x^{n-1}\partial y}{\partial t} - \frac{n x^{n-1}\partial y}{\partial t} + \frac{A x^{2n}\partial y}{\partial t} + B x^{2n} y = X.$$

Sit $X x^{-2n} = \Theta$, quae quantitas, posito $x^{n+1} = (n+1)t$, vt functio ipsius $t$ spectari potest, sicque fiet

$$\partial\partial y + A\partial y\partial t + B y\partial t^2 = \Theta\partial t^2,$$

in qua aequatione elementum $\partial t$ constans est assumtum, cuius ergo integrale per superiora datur.

I. Si $A = f+g$ et $B = fg$, erit integrale

$$(f-g)y = e^{-gt}\int e^{gt}\Theta\partial t - e^{-ft}\int e^{ft}\Theta\partial t,$$

vbi restitutis valoribus $\partial t = x^n\partial x$ et $\Theta = X x^{-2n}$, retento breuitatis gratia $t = \frac{1}{n+1}x^{n+1}$, valor ipsius $y$ ita per $x$ exprimetur

$$(f-g)y = e^{-gt}\int e^{gt} x^{-n} X\partial x - e^{-ft}\int e^{ft} x^{-n} X\partial x.$$

II. Si $A = 2f$ et $B = ff$, erit integrale

$$y = e^{-ft} t\int e^{ft}\Theta\partial t - e^{-ft}\int e^{ft}\Theta t\partial t, \text{ seu}$$

$$y = e^{-ft}\int\partial t\int e^{ft}\Theta\partial t,$$

quod ergo per $x$ ita exprimetur

$$y = e^{-ft}\int x^n\partial x\int e^{ft} x^{-n} X\partial x.$$

III. Si denique $A = 2f\,\text{cos}.\zeta$ et $B = ff$, erit integrale

$$fe^{ft\,\text{cos}.\zeta} y\,\text{sin}.\zeta = \text{sin}.\Phi\int e^{ft\,\text{cos}.\zeta}\Theta\partial t\,\text{cos}.\Phi - \text{cos}.\Phi\int e^{ft\,\text{cos}.\zeta}\Theta\partial t\,\text{sin}.\Phi,$$

existente $\Phi = ft\,\text{sin}.\zeta$, seu $\Phi = \frac{f\,\text{sin}.\zeta}{n+1}x^{n+1}$, ob $t = \frac{1}{n+1}x^{n+1}$. Quare aequationis propositae integrale erit

$$fe^{ft\,\text{cos}.\zeta} y\,\text{sin}.\zeta = \text{sin}.\Phi\int e^{ft\,\text{cos}.\zeta} x^{-n} X\partial x\,\text{cos}.\Phi - \text{cos}.\Phi\int e^{ft\,\text{cos}.\zeta} x^{-n} X\partial x\,\text{sin}.\Phi.$$

## Corollarium 1.

862. Si $n = 0$, aequatio proposita abit in eam ipsam, quam problemate praecedente tractauimus, fitque $t = x$, vnde etiam integrale eodem redit.

Corol-

## Corollarium 2.

863. Sin autem ſit $n = -1$, aequatio noſtra fit

$$\partial\partial y + (A + 1)\frac{\partial y \partial x}{x} + \frac{B y \partial x^2}{xx} = X \partial x^2,$$

vbi ergo erit $t = l x$ et $e^{\lambda t} = x^\lambda$, tum vero pro caſu tertio angulus $\Phi = f$ ſin. $\zeta . l x$.

## Scholion.

864. Methodus qua hic vſi ſumus, huiusmodi aequationes differentiales integrandi, haud ſatis naturalis videtur, cum ad has quaſi ſolas formas ſit adſtricta. Quoniam igitur in aequationibus differentialibus primi gradus inuentio factorum, quibus eae per ſe integrabiles reddantur, inſignem fructum polliceri videbatur, eius quoque vſum in aequationibus differentialibus ſecundi gradus oſtendere conemur. Hic quidem nihil tam abſolutum expectare licet, quod ad omnes omnino aequationum formas pateat, ſed quantillum etiam praeſtare potuerimus, id haud contemnendum Analyſeos incrementum ſpectari debebit. Hac autem methodo eas potiſſimum aequationes differentiales, in quibus altera variabilis $y$ cum ſuis differentialibus vnam dimenſionem nusquam transgreditur, ſatis commode tractare licet, hincque via perſpicietur, quomodo eam magis excoli oporteat.

CA-

# CAPVT V.

DE

## INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM SECVNDI GRADVS, IN QVIBVS ALTERA VARIABILIS VNAM DIMENSIONEM NON SVPERAT, PER FACTORES.

### Problema 107.

865.

Sumto elemento $\partial x$ conſtante, ſi proponatur haec aequatio
$$\partial \partial y + A \partial x \partial y + B y \partial x^2 = X \partial x^2,$$
vbi X denotat functionem quamcunque ipſius $x$, inuenire functionem ipſius $x$, per quam haec aequatio multiplicata fiat integrabilis.

### Solutio.

Ponatur $\partial y = p \partial x$, vt habeatur forma differentialis primi gradus
$$\partial p + A p \partial x + B y \partial x = X \partial x,$$
quae multiplicata per V, functionem quandam ipſius $x$, fiat integrabilis; ſcilicet
$$V \partial p + A V p \partial x + B V y \partial x = V X \partial x,$$
vbi cum poſterius membrum $V X \partial x$ ſit integrabile, idem in priori eueniat, neceſſe eſt. At primo perſpicuum eſt eius integralis partem fore $V p$, vnde id ponatur $V p + S$ vt ſit $V p + S = \int V X \partial x$, fietque
$$\partial S = - p \partial V + A V p \partial x + B V y \partial x, \text{ ſeu}$$

$$\partial S = \partial y \left(A V - \tfrac{\partial V}{\partial x}\right) + B V y \partial x,$$

quae forma integrabilis reddi poteſt ſumendo $V = e^{\lambda x}$, erit enim

$$\partial S = e^{\lambda x}[(A-\lambda)\partial y + B y \partial x] \text{ et } S = (A-\lambda) e^{\lambda x} y,$$

vbi $\lambda$ ita debet accipi, vt fiat

$$A\lambda - \lambda\lambda = B \text{ ſeu } \lambda\lambda - A\lambda + B = 0.$$

Tum ergo erit

$$e^{\lambda x} p + (A-\lambda) e^{\lambda x} y = \int e^{\lambda x} X \partial x, \text{ ſeu}$$

$$\partial y + (A-\lambda) y \partial x = e^{-\lambda x} \partial x \int e^{\lambda x} X \partial x,$$

quae iam per $e^{(A-\lambda)x}$ multiplicata denuo fit integrabilis, datque

$$e^{(A-\lambda)x} y = \int e^{(A-2\lambda)x} \partial x \int e^{\lambda x} X \partial x.$$

Cum $\lambda$ ſit vna radix aequationis $\lambda\lambda - A\lambda + B = 0$, ſi ambas eius radices ponamus $f$ et $g$, vt ſit $\lambda = f$, erit $A - \lambda = g$, et aequatio integralis

$$e^{gx} y = \int e^{(g-f)x} \partial x \int e^{fx} X \partial x, \text{ ſeu}$$

$$e^{gx} y = \frac{1}{g-f} e^{(g-f)x} \int e^{fx} X \partial x - \frac{1}{g-f} \int e^{gx} X \partial x,$$

quae abit in formam ſupra inuentam

$$y = \frac{1}{g-f} e^{-fx} \int e^{fx} X \partial x - \frac{1}{g-f} e^{-gx} \int e^{gx} X \partial x.$$

## Corollarium 1.

866. Aequatio ergo propoſita ſeu inde nata

$$\partial p + A p \partial x + B y \partial x = X \partial x$$

fit integrabilis ſi ducatur in $e^{\lambda x}$, exiſtente $\lambda\lambda - A\lambda + B = 0$, ſicque duplex habetur factor vel $e^{fx}$ vel $e^{gx}$.

## Corollarium 2.

867. Ea autem per factorem $e^{fx}$ multiplicata, eius integrale erit

$\partial y$

$$\partial y + g y \partial x = e^{-fx} \partial x \int e^{fx} X \partial x,$$

ficque per integrationem ad aequationem differentialem primi gradus reducitur, quae denuo integrabilis redditur fi per $e^{gx}$ multiplicetur.

## Scholion.

868. Multiplicatorem V ita determinari oportebat, vt formula $\partial y (A V - \frac{\partial V}{\partial x}) + B V y \partial x$ fieret per fe integrabilis. Tum autem cum V fit functio ipfius $x$, integrale erit $y (A V - \frac{\partial V}{\partial x})$, vnde fiat neceffe eft

$$A \partial V - \frac{\partial \partial V}{\partial x} = B V \partial x, \text{ feu } \partial \partial V - A \partial x \partial V + B V \partial x^2 = 0,$$

a cuius aequationis integratione pendet inuentio factoris quaefiti V. Sufficit autem eius integrale particulare fumfiffe, dummodo enim aequatio propofita integrabilis reddatur, conftans arbitraria pro integrali completo reddendo ipfa integratione introducitur.

## Problema 108.

869. Sumto elemento $\partial x$ conftante, fi proponatur haec aequatio

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = X \partial x^2,$$

vbi P, Q et X fint functiones quaecunque ipfius $x$, inuenire multiplicatorem V, qui fit functio ipfius $x$, quo illa aequatio integrabilis reddatur.

## Solutio.

Quia aequatio per V multiplicata

$$V \partial \partial y + V P \partial y \partial x + V Q y \partial x^2 = V X \partial x^2,$$

integrabilis exiftit, prioris partis integrale ponatur $V \partial y + S y \partial x$, aliam enim formam habere nequit, ac fieri oportet

N 2 VP

$$VP\partial y\partial x + VQy\partial x^2 = \partial y\partial V + S\partial y\partial x + y\partial S\partial x,$$

vbi cum $S$ ſit neceſſario functio ipſius $x$, erit

$$VP\partial x = \partial V + S\partial x \text{ et } VQ\partial x = \partial S.$$

Inde autem eſt $S = VP - \frac{\partial V}{\partial x}$, quare multiplicator V definiri debet ex hac aequatione

$$VQ\partial x = V\partial P + P\partial V - \frac{\partial\partial V}{\partial x}, \text{ ſeu}$$

$$\partial\partial V - P\partial V\partial x + V\partial x(Q\partial x - \partial P) = 0,$$

quae ergo ſi reſolui potuerit, vel ſi ſaltem eius integrale quodpiam particulare innoteſcat, vt habeatur multiplicator V, aequationis propoſitae integrale erit

$$V\partial y + y + y(VP\partial x - \partial V) = \partial x\int VX\partial x,$$

quae porro integrabilis redditur, ſi ducatur in $\frac{1}{VV}e^{\int P\partial x}$, obtinebitur enim integrale

$$\frac{y}{V}e^{\int P\partial x} = \int\frac{\partial x}{VV}e^{\int P\partial x}\int VX\partial x, \text{ ſeu}$$

$$y = e^{-\int P\partial x}V\int e^{\int P\partial x}\frac{\partial x}{VV}\int VX\partial x,$$

quo duplici ſigno integrali gemina conſtans arbitraria introducitur, integrale completum conſtituens.

## Corollarium 1.

870. Inuentio ergo multiplicatoris V pendet etiam a reſolutione aequationis differentio-differentialis, quae autem propoſita ſimplicior eſt aeſtimanda, quod functionem X non inuoluat, et quantitas V cum ſuis differentialibus $\partial V$ et $\partial\partial V$ vbique vnam dimenſionem conſtituat.

## Corollarium 2.

871. Quodſi ergo ponatur $V = e^{\int v\partial x}$, quantitas $v$ determinabitur per hanc aequationem differentialem primi gradus

$\partial v$

$$\partial v + v v \partial x - P v \partial x + Q \partial x - \partial P = 0,$$

cuius si saltem integrale particulare constet, integratio aequationis propositae absolui poterit.

## Corollarium 3.

872. Dato autem multiplicatore V vicissim ratio aequationis propositae definitur, vt hoc modo integrabilis euadat. Erit enim vel

$$Q = \frac{\partial P}{\partial x} + \frac{P \partial V}{V \partial x} - \frac{\partial \partial V}{V \partial x^2}, \text{ vel}$$

$$\partial P + \frac{P \partial V}{V} = Q \partial x - \frac{\partial \partial V}{V \partial x}, \text{ vel integrando}$$

$$P V = \frac{\partial V}{\partial x} + \int Q V \partial x, \text{ seu } P = \frac{\partial V}{V \partial x} + \frac{\int Q V \partial x}{V}.$$

## Exemplum 1.

873. *Definire formam aequationis differentio-differentialis*

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = X \partial x^2,$$

*vt multiplicata per* $e^{\lambda x}$ *integrabilis euadat.*

Posito multiplicatore $V = e^{\int v \partial x} = e^{\lambda x}$, erit $v = \lambda$, et satisfieri oportet huic aequationi

$$\lambda \lambda \partial x - \lambda P \partial x + Q \partial x - \partial P = 0,$$

vnde fit $Q = \lambda P - \lambda \lambda + \frac{\partial P}{\partial x}$. Primum ergo hoc euenit si fuerint P et Q constantes, puta $P = A$ et $Q = B$, ac tum $\lambda$ definiri oportet ex hac aequatione $\lambda \lambda - A \lambda + B = 0$, qui est casus supra tractatus. Praeterea vero qualiscunque functio P fuerit ipsius $x$, modo sit $Q = \lambda P - \lambda \lambda + \frac{\partial P}{\partial x}$, aequatio in $e^{\lambda x}$ ducta erit integrabilis, integrali existente

$$e^{\lambda x} [\partial y + y \partial x (P - \lambda)] = \partial x \int e^{\lambda x} X \partial x, \text{ seu}$$

$$\partial y + (P - \lambda) y \partial x = e^{-\lambda x} \partial x \int e^{\lambda x} X \partial x,$$

 quae

quae vlterius per $e^{\int P\partial x - \lambda x}$ multiplicata et integrata, dat

$$y = e^{-\int P\partial x + \lambda x}\int e^{\int P\partial x - 2\lambda x}\partial x\int e^{\lambda x}X\partial x.$$

## Corollarium.

874. Sit $P = A + \alpha x$ et $Q = B + \beta x$, erit

$$B + \beta x = A\lambda + \alpha\lambda x - \lambda\lambda + \alpha, \text{ ergo}$$

$$B = A\lambda - \lambda\lambda + \alpha \text{ et } \beta = \alpha\lambda,$$

vnde ob $\lambda = \frac{\beta}{\alpha}$, coefficientes A, B, $\alpha$, $\beta$, ita comparatos esse oportet, vt sit

$$B\alpha\alpha = A\alpha\beta - \beta\beta + \alpha^3, \text{ seu } B\alpha\alpha + \beta\beta = \alpha(A\beta + \alpha\alpha).$$

## Exemplum 2.

875. *Definire formam aequationis differentio-differentialis*

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

*vt per* $e^{\int v\partial x}$, *existente* $v = \frac{\lambda}{x} + \mu x^n$, *multiplicata fiat integrabilis.*

Cum esse debeat

$$\partial v + vv\partial x - Pv\partial x + Q\partial x - \partial P = 0, \text{ erit}$$

$$\left.\begin{matrix} -\frac{\lambda}{xx} + \mu n x^{n-1} - \frac{\lambda P}{x} - \mu P x^n \\ + \frac{\lambda\lambda}{xx} + 2\lambda\mu x^{n-1} + \mu\mu x^{2n} + Q - \frac{\partial P}{\partial x} \end{matrix}\right\} = 0,$$

ergo

$$Q = \frac{\lambda(1-\lambda)}{xx} - (2\lambda + n)\mu x^{n-1} - \mu\mu x^{2n} + \frac{\lambda P}{x} + \mu P x^n + \frac{\partial P}{\partial x}.$$

Ponamus $P = \frac{\alpha}{x} + \beta x^n$, erit

$$Q = \frac{1}{xx}(\lambda - \lambda\lambda + \alpha\lambda - \alpha) + x^{n-1}(\beta\lambda + \alpha\mu + \beta n - 2\lambda\mu - n\mu) + x^{2n}(\beta\mu - \mu\mu).$$

Sit $Q = \frac{\gamma}{xx} + \delta x^{n-1} + \varepsilon x^{2n}$, fierique oportet

$$\lambda\lambda - (\alpha + 1)\lambda + \alpha + \gamma = 0,$$

$$\beta(\lambda + n) + \mu(\alpha - 2\lambda - n) = \delta, \text{ et } \mu(\beta - \mu) = \varepsilon,$$

vnde

vnde non solum pro multiplicatore litterae $\lambda$ et $\mu$, sed etiam certa relatio inter litteras $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, definitur.

Veluti si sit $\gamma = 0$ et $\delta = 0$, erit $(\lambda - \alpha)(\lambda - 1) = 0$, vnde $\lambda = \alpha$; tum $(\beta - \mu)(\alpha + n) = 0$, ergo $\alpha = \lambda = -n$ et $\mu\mu - \beta\mu + \varepsilon = 0$. Scilicet aequatio

$$\partial\partial y + \partial x \partial y (\beta x^n - \tfrac{n}{x}) + \varepsilon x^{2n} y \partial x^2 = X \partial x^2$$

multiplicatorem recipit $e^{\int v \partial x}$, existente $v = -\frac{n}{x} + \mu x^n$, sumto $\mu$ ita vt sit $\mu\mu - \mu + \varepsilon = 0$. Erit ergo multiplicator

$$V = \frac{1}{x^n} e^{\frac{\mu}{n+1} x^{n+1}} \text{ et } e^{\int P \partial x} = \frac{1}{x^n} e^{\frac{\beta}{n+1} x^{n+1}}.$$

Quare si ponamus $\frac{1}{n+1} x^{n+1} = t$, erit

$$y = x^n e^{-\beta t} \frac{1}{x^n} e^{\mu t} \int e^{\beta t - 2\mu t} x^n \partial x \int \frac{e^{\mu t} X \partial x}{x^n}, \text{ seu}$$

$$y = e^{(\mu - \beta)t} \int e^{(\beta - 2\mu)t} x^n \partial t \int \frac{e^{\mu t} X \partial x}{x^n}.$$

## Corollarium 1.

876. Si sumatur $\gamma = 0$ et $\varepsilon = 0$, erit
$\mu = \beta$, $\beta(\alpha - \lambda) = \delta$ et $(\lambda - \alpha)(\lambda - 1) = 0$, hinc
$\lambda = 1$, et $\delta = (\alpha - 1)\beta$, ideoque
$P = \frac{\alpha}{x} + \beta x^n$, $Q = (\alpha - 1)\beta x^{n-1}$,

et aequationis

$$\partial\partial y + (\tfrac{\alpha}{x} + \beta x^n)\partial x \partial y + (\alpha - 1)\beta x^{n-1} y \partial x^2 = X \partial x^2,$$

multiplicator $V = e^{\int v \partial x}$, existente $v = \frac{1}{x} + \beta x^n$, ita vt sit

$$V = x e^{\frac{\beta}{n+1} x^{n+1}} \text{ et } e^{\int P \partial x} = x^\alpha e^{\frac{\beta}{n+1} x^{n+1}}.$$

Corol-

## Corollarium 2.

877. Hoc ergo casu, posito $\frac{1}{n+1}x^{n+1}=t$, erit integrale

$$y=x^{1-\alpha}\int x^{\alpha-2}e^{-\beta t}\partial x\int e^{\beta t}Xx\partial x,$$

quae forma simplicius exhiberi nequit, propterea quod in genere formula $e^{-\beta t}x^{\alpha-2}\partial x$ integrationem non admittit.

## Scholion.

878. Cum igitur inuentio multiplicatorum, qui huiusmodi aequationem

$$\partial\partial y+P\partial x\partial y+Qy\partial x^2=X\partial x^2,$$

integrabilem reddunt, resolutionem huius aequationis postulet

$$\partial\partial V-P\partial V\partial x+V\partial x(Q\partial x-\partial P)=0,$$

quae in hac forma continetur

$$\partial\partial y+P\partial x\partial y+Qy\partial x^2=0,$$

videndum est, quomodo hanc formam etiam per multiplicatores tractari oporteat. Cuius multiplicator si fingatur V functio quaedam ipsius $x$, iterum ad praecedentem formam

$$\partial\partial V-P\partial V\partial x+V\partial x(Q\partial x-\partial P)=0,$$

deuenitur, atque si huius multiplicator statuatur $=U$ functioni ipsius $x$, hic definietur per hanc aequationem

$$\partial\partial U+P\partial U\partial x+QU\partial x^2=0,$$

ita vt sufficiat alteram harum duarum aequationum resoluisse. Ac supra quidem, vbi $y=uv$ posuimus, ad hanc posteriorem aequationem peruenimus: at mirum non est harum duarum aequationum alteram ab altera pendere, cum prior ex posteriori nascatur ponendo $U=e^{-\int P\partial x}V$, posterior vero ex priori ponendo $V=e^{\int P\partial x}U$, vti tentanti facile patebit. Quoniam igitur hoc modo difficultatem, si quae occurrit, tollere

lere non licet, inuestigandum est, an forte eiusmodi multiplicator, qui vtramque variabilem $x$ et $y$ cum suis differentialibus $\partial x$ et $\partial y$ seu $p = \frac{\partial y}{\partial x}$ inuoluat, negotium conficiat. At vero facile perspicitur exclusis differentialibus hoc fieri non posse; nam si multiplicator esset V functio ipsarum $x$ et $y$, ex primo termino $\partial \partial y$ nascetur integralis pars $V \partial y$, quae autem differentiata ponendo $\partial V = M \partial x + N \partial y$ inuolueret in differentiale partem $N \partial y^2$, in aequatione non occurrentem, quae etiam per reliquas integralis partes tolli non posset. Quare rem tentemus eiusmodi multiplicatoribus, qui etiam rationem differentialium $p = \frac{\partial y}{\partial x}$ complectantur; et cum ipsius $y$ cum suis differentialibus vbique sit idem dimensionum numerus, eadem proprietas etiam in multiplicatore insit necesse est; si enim diuersae inessent, singulae seorsim negotium essent confecturae.

## Problema 109.

879. Sumto elemento $\partial x$ constante, definire conditiones, vt multiplicator huius formae $Mp + Ny$, existente $p = \frac{\partial y}{\partial x}$ et M et N functionibus ipsius $x$, integrabilem reddat hanc aequationem

$$\partial \partial y + P \partial x \partial y + Q y \partial x^2 = 0,$$

vbi P et Q sunt functiones ipsius $x$.

## Solutio.

Ob $\partial y = p \partial x$, nostra aequatio est

$$\partial p + P p \partial x + Q y \partial x = 0,$$

quae per $Mp + Ny$ multiplicata fit

$$\begin{aligned} Mp\partial p + Ny\partial p + MPp\partial y + NPy\partial y + NQyy\partial x = 0, \\ + MQy\partial y \qquad\qquad \end{aligned}$$

quam integrabilem esse oportet. Ob terminos differentiali $\partial p$

 affe-

affectos pars integrabilis erit $\frac{1}{2}Mpp + Nyp$, vnde integrale ipsum statuatur $= \frac{1}{2}Mpp + Nyp + S$. Cuius differentiale cum ipsam illam aequationem praebere debeat, habebimus

$$\begin{aligned}\partial S = {} & MPp\partial y + NPy\partial y + NQyy\partial x \\ & + MQy\partial y \\ & - \tfrac{1}{2}pp\partial M - yp\partial N \\ & - Np\partial y\end{aligned}$$

quam ergo formulam integrabilem esset oportet, quae cum tantum differentialia primi ordinis $\partial x$ et $\partial y$ complectatur, necesse est vt quantitas $p$ ex calculo egrediatur. Posito ergo $\partial M = M'\partial x$ et $\partial N = N'\partial x$, ob $p\partial x = \partial y$, primus terminus continens $p$ ad nihilum redigi debet, vt sit

$$MPp\partial y - \tfrac{1}{2}M'p\partial y - Np\partial y = 0, \text{ seu}$$

$$MP - \tfrac{1}{2}M' - N = 0, \text{ vel } N = MP - \frac{\partial M}{2\partial x}.$$

Tum vero erit

$$\partial S = y\partial y(NP + MQ - N') + NQyy\partial x,$$

cuius formulae integrale est

$$S = \tfrac{1}{2}yy(NP + MQ - N'), \text{ vel } S = yy\int NQ\partial x,$$

quas duas formas congruere oportet, vnde fit

$$NP + MQ - \frac{\partial N}{\partial x} = 2\int NQ\partial x, \text{ seu}$$

$$N\partial P + P\partial N + M\partial Q + Q\partial M - \frac{\partial\partial N}{\partial x} - 2NQ\partial x = 0,$$

quae aequatio cum illa $N = MP - \frac{\partial M}{2\partial x}$ iuncta, conditiones quaesitas determinat, proditque tum aequatio integralis

$$\tfrac{1}{2}Mpp + Nyp + \tfrac{1}{2}yy\left(NP + MQ - \frac{\partial N}{\partial x}\right) = C.$$

## Corollarium 1.

880. Si functiones P et Q dentur, indeque M et N definiri oporteat, ob $N = MP - \frac{\partial M}{2\partial x}$, erit

$\partial N$

$$\partial N = M \partial P + P \partial M - \frac{\partial \partial M}{2 \partial x},$$

et functio M definietur per hanc aequationem

$$\frac{\partial^3 M}{2 \partial x^2} - \frac{3 P \partial \partial M}{2 \partial x}$$
$$+ (PP - \frac{5 \partial P}{2 \partial x} + 2Q) \partial M + M(2P\partial P - \frac{\partial \partial P}{\partial x} - 2PQ\partial x + \partial Q) = 0,$$

quae ob differentialia tertii ordinis parum iuuat.

## Corollarium 2.

881. Sin autem multiplicator $Mp + Ny$ detur, ipsa aequatio ita definitur, vt sit primo $P = \frac{N}{M} + \frac{\partial M}{2 M \partial x}$, vnde ex altera, quae est

$$\partial Q + \frac{Q \partial M}{M} - \frac{2 N Q \partial x}{M} = \frac{\partial \partial N}{M \partial x} - \frac{\partial . P N}{M},$$

haecque per $M e^{-2\int \frac{N \partial x}{M}}$ multiplicata, integrale dat

$$M Q e^{-2\int \frac{N \partial x}{M}} = \int e^{-2\int \frac{N \partial x}{M}} (\frac{\partial \partial N}{\partial x} - \partial . P N).$$

## Corollarium 3.

882. Sit hoc integrale $= Z$ eritque

$$Z = e^{-2\int \frac{N \partial x}{M}} (\frac{\partial N}{\partial x} - PN) + \int e^{-2\int \frac{N \partial x}{M}} (\frac{2 N \partial N}{M} - \frac{2 PNN \partial x}{M}),$$

quod posterius membrum, pro P valore substituto, abit in

$$\int e^{-2\int \frac{N \partial x}{M}} (\frac{2 N \partial N}{M} - \frac{2 N^3 \partial x}{MM} - \frac{NN \partial M}{MM}),$$

cuius integrale est manifesto $e^{-2\int \frac{N \partial x}{M}} \frac{NN}{M}$, ita vt sit

$$Z = e^{-2\int \frac{N \partial x}{M}} (\frac{\partial N}{\partial x} - \frac{N \partial M}{2 M \partial x}) + C, \text{ ideoque}$$

$$Q = \frac{C}{M} e^{2\int \frac{N \partial x}{M}} + \frac{\partial N}{M \partial x} - \frac{N \partial M}{2 MM \partial x}.$$

## Corollarium 4.

883. Propofita ergo hac aequatione

$$\frac{\partial\partial y}{\partial x}+\left(\frac{N}{M}+\frac{\partial M}{2M\partial x}\right)\partial y+y\left(\frac{C\partial x}{M}e^{2\int\frac{N\partial x}{M}}+\frac{\partial N}{M}-\frac{N\partial M}{2MM}\right)=0,$$

eam per $\frac{M\partial y}{\partial x}+Ny$ multiplicando, integrale fit

$$\frac{M\partial y^2}{2\partial x^2}+\frac{Ny\partial y}{\partial x}+\tfrac{1}{2}yy\left(Ce^{2\int\frac{N\partial x}{M}}+\frac{NN}{M}\right)=\text{Conft.}$$

## Scholion.

884. Cum ergo pro M et N quascunque functiones ipfius $x$ accipere liceat, innumerabiles hinc nacti fumus aequationum differentio-differentialium formas, quas ope multiplicatoris $\frac{M\partial y}{\partial x}+Ny$ integrare poffumus. Forma fcilicet generalis, quae hoc multiplicatore integrabilis redditur, eft vt vidimus

$$\frac{\partial\partial y}{\partial x}+\frac{\partial y}{2M\partial x}(\partial M+2N\partial x)$$
$$+\frac{y}{2MM}\left(2M\partial N-N\partial M+2CMe^{2\int\frac{N\partial x}{M}}\partial x\right),$$

ipfo integrali exiftente

$$\frac{M\partial y^2}{2\partial x^2}+\frac{Ny\partial y}{\partial x}+\tfrac{1}{2}yy\left(\frac{NN}{M}+Ce^{2\int\frac{N\partial x}{M}}\right),$$

vbi perfpicuum eft partem exponentialem conftanti C affectam vtrinque omitti poffe, cum ea fola ifta proprietate fit praedita. Quodfi partem exponentialem ad algebraicam reducamus ponendo $e^{2\int\frac{N\partial x}{M}}=L$, erit $\frac{2N\partial x}{M}=\frac{\partial L}{L}$ et $N=\frac{M\partial L}{2L\partial x}$, hincque $\partial N=\frac{M\partial\partial L}{2L\partial x}+\frac{\partial L\partial M}{2L\partial x}-\frac{M\partial L^2}{2LL\partial x}$; vnde ifta forma

$$\frac{\partial\partial y}{\partial x}+\frac{\partial y}{2\partial x}\left(\frac{\partial M}{M}+\frac{\partial L}{L}\right)+\tfrac{1}{2}y\left(\frac{\partial\partial L}{L\partial x}+\frac{\partial L\partial M}{2LM\partial x}-\frac{\partial L^2}{LL\partial x}+\frac{2CL\partial x}{M}\right),$$

quae per $\frac{M\partial y}{\partial x}+\frac{My\partial L}{2L\partial x}$ multiplicata, integrale praebet

$$\frac{M\partial y^2}{2\partial x^2}+\frac{My\partial L\partial y}{2L\partial x^2}+\tfrac{1}{2}yy\left(\frac{M\partial L^2}{4LL\partial x^2}+CL\right).$$

Vel

Vel ſi ponamus $\frac{\partial M}{M} + \frac{\partial L}{L} = \frac{2\partial K}{K}$, vt ſit $M = \frac{KK}{L}$, erit noſtra aequatio differentio-differentialis

$$\frac{\partial\partial y}{\partial x} + \frac{\partial y}{\partial x}\cdot\frac{\partial K}{K} + \frac{1}{2}y\left(\partial.\frac{\partial L}{L\partial x} - \frac{\partial L^2}{2LL\partial x} + \frac{\partial K\partial L}{KL\partial x} + \frac{2CLL\partial x}{KK}\right) = 0,$$

quae per $\frac{KK}{L}\left(\frac{\partial y}{\partial x} + \frac{y\partial L}{2L\partial x}\right)$ multiplicata, dabit integrale

$$\frac{KK}{2L}\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial L\partial y}{L\partial x^2} + yy\left(\frac{\partial L^2}{4LL\partial x^2} + \frac{CLL}{KK}\right)\right] = \text{Conſt.}$$

## Exemplum 1.

885. Sit $K = x^m(a+x)^n$ et $L = x^\mu(a+x)^\nu$, erit

$$\frac{\partial K}{K\partial x} = \frac{m}{x} + \frac{n}{a+x} = \frac{ma+(m+n)x}{x(a+x)},\ \text{et}$$

$\frac{\partial L}{L\partial x} = \frac{\mu}{x} + \frac{\nu}{a+x}$; vnde coefficieus ipſius $\frac{1}{2}y\partial x$ erit

$$-\frac{\mu}{xx} - \frac{\nu}{(a+x)^2} - \frac{\mu\mu}{2xx} - \frac{\mu\nu}{x(a+x)} - \frac{\nu\nu}{2(a+x)^2} + \frac{m\mu}{xx} + \frac{m\nu+n\mu}{x(a+x)}$$
$$+ \frac{n\nu}{(a+x)^2} + 2Cx^{2\mu-2m}(a+x)^{2\nu-2n},\ \text{ſeu}$$
$$\frac{\mu(2m-\mu-2)}{2xx} + \frac{m\nu+n\mu-\mu\nu}{x(a+x)} + \frac{\nu(2n-\nu-2)}{2(a+x)^2}$$
$$+ 2Cx^{2\mu-2m}(a+x)^{2\nu-2n},$$

vbi ſequentes caſus notaſſe iuuabit.

I. Sit $m = \mu + 1$ et $n = \nu$, erit ipſius $\frac{1}{2}y\partial x$ coefficiens

$$\frac{\mu\mu+4C}{2xx} + \frac{\nu(\mu+1)}{x(a+x)} + \frac{\nu(\nu-2)}{2(a+x)^2}.$$

Hinc iſta aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y\left(\frac{\mu+1}{x} + \frac{\nu}{a+x}\right)$$
$$+ \frac{1}{4}y\partial x\left[\frac{\mu\mu+4C}{xx} + \frac{2\nu(\mu+1)}{x(a+x)} + \frac{\nu(\nu-2)}{(a+x)^2}\right] = 0$$

multiplicata per

$$x^{\mu+2}(a+x)^\nu\left[\frac{\partial y}{\partial x} + \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\frac{y}{2}\right],$$

integrale dat

$$\frac{1}{2}x^{\mu+2}(a+x)^\nu\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial y}{\partial x}\left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) +\right.$$
$$\left.\frac{1}{4}yy\left(\frac{\mu\mu+4C}{xx} + \frac{2\mu\nu}{x(a+x)} + \frac{\nu\nu}{(a+x)^2}\right)\right] = \text{Conſt.}$$

II. Sit $m = \mu + \frac{1}{2}$ et $n = \nu + \frac{1}{2}$, erit ipsius $\frac{1}{4} y \partial x$ coefficiens

$$\frac{\mu(\mu-1)}{2xx} + \frac{2\mu\nu+\mu+\nu+4C}{2x(a+x)} + \frac{\nu(\nu-1)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y \left(\frac{2\mu+1}{2x} + \frac{2\nu+1}{2(a+x)}\right)$$
$$+ \tfrac{1}{4} y \partial x \left(\frac{\mu(\mu-1)}{xx} + \frac{2\mu\nu+\mu+\nu+4C}{x(a+x)} + \frac{\nu(\nu-1)}{(a+x)^2}\right) = 0,$$

multiplicata per

$$x^{\mu+1}(a+x)^{\nu+1}\left[\frac{\partial y}{\partial x} + \tfrac{1}{2} y \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\right],$$

integrale dabit

$$\tfrac{1}{2} x^{\mu+1}(a+x)^{\nu+1}\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial y}{\partial x}\left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) +\right.$$
$$\left.\tfrac{1}{4} yy \left(\frac{\mu\mu}{xx} + \frac{2\mu\nu+4C}{x(a+x)} + \frac{\nu\nu}{(a+x)^2}\right)\right] = \text{Const.}$$

III. Sit $m = \mu$ et $n = \nu + 1$, erit ipsius $\frac{1}{4} y \partial x$ coefficiens

$$\frac{\mu(\mu-2)}{2xx} + \frac{\mu(\nu+1)}{x(a+x)} + \frac{\nu\nu+4C}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y \left(\frac{\mu}{x} + \frac{\nu+1}{a+x}\right)$$
$$+ \tfrac{1}{4} y \partial x \left(\frac{\mu(\mu-2)}{xx} + \frac{2\mu(\nu+1)}{x(a+x)} + \frac{\nu\nu+4C}{(a+x)^2}\right) = 0,$$

multiplicata per

$$x^{\mu}(a+x)^{\nu+2}\left[\frac{\partial y}{\partial x} + \tfrac{1}{2} y \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\right],$$

dabit integrale

$$\tfrac{1}{2} x^{\mu}(a+x)^{\nu+2}\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial y}{\partial x}\left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) +\right.$$
$$\left.\tfrac{1}{4} yy \left(\frac{\mu\mu}{xx} + \frac{2\mu\nu}{x(a+x)} + \frac{\nu\nu+4C}{(a+x)^2}\right)\right] = \text{Const.}$$

## Corollarium 1.

886. Sit casu primo $\nu = 2$, $C = -\frac{1}{4}\mu\mu$, et habebitur haec aequatio

$$\frac{\partial\partial y}{\partial x} + \frac{(\mu+1)a + (\mu+3)x}{x(a+x)} \partial y + \frac{(\mu+1) y \partial x}{x(a+x)} = 0,$$

quae

quae per

$$x^{\mu+2}(a+x)^2\left(\frac{\partial y}{\partial x}+\frac{\mu a+(\mu+1)x}{2x(a+x)}y\right)$$

multiplicata, praebet integrale

$$\tfrac{1}{2}x^{\mu+2}(a+x)^2\left[\frac{\partial y^2}{\partial x^2}+\frac{\mu a+(\mu+2)x}{x(a+x)}\cdot\frac{y\partial y}{\partial x}+yy\left(\frac{\mu}{x(a+x)}+\frac{1}{(a+x)^2}\right)\right]=\text{Const.}$$

## Corollarium 2.

887. Sit casu tertio $\mu=2$ et $4C=-\nu\nu$, habebitur ista aequatio

$$\frac{\partial\partial y}{\partial x}+\partial y\cdot\frac{2a+(\nu+3)x}{x(a+x)}+\frac{(\nu+1)y\partial x}{x(a+x)}=0,$$

quae multiplicata per $xx(a+x)^{\nu+2}\left[\frac{\partial y}{\partial x}+\frac{1}{2}y\left(\frac{2}{x}+\frac{\nu}{a+x}\right)\right]$, dabit integrale

$$\tfrac{1}{2}xx(a+x)^{\nu+2}\left[\frac{\partial y^2}{\partial x^2}+\frac{y\partial y}{\partial x}\left(\frac{2}{x}+\frac{\nu}{a+x}\right)+yy\left(\frac{1}{xx}+\frac{\nu}{x(a+x)}\right)\right]=\text{Const.}$$

## Exemplum 2.

888. Sit $K=x^m(aa+xx)^n$ et $L=x^\mu(aa+xx)^\nu$, erit $\frac{\partial K}{K\partial x}=\frac{m}{x}+\frac{2nx}{aa+xx}$ et $\frac{\partial L}{L\partial x}=\frac{\mu}{x}+\frac{2\nu x}{aa+xx}$, et aequatio differentio-differentialis hanc induet formam

$$\frac{\partial\partial y}{\partial x}+\partial y\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+$$

$$\tfrac{1}{2}y\partial x\left\{\begin{matrix}\frac{\mu(2m-\mu-2)}{2xx}+\frac{2n\mu+2\nu(m-\mu+1)}{aa+xx}+\frac{2\nu(2n-\nu-2)xx}{(aa+xx)^2}\\+\frac{2Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\end{matrix}\right\}=0,$$

cuius in $x^{2m-\mu}(aa+xx)^{2n-\nu}\left[\frac{\partial y}{\partial x}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)\right]$ ductae integrale erit

$$\tfrac{1}{2}x^{2m-\mu}(aa+xx)^{2n-\nu}\times\left\{\begin{matrix}\frac{\partial y^2}{\partial x^2}+\frac{y\partial y}{\partial x}\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)\\+\frac{1}{4}yy\left[\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)^2+\frac{4Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\right]\end{matrix}\right\}=\text{Const.}$$

Euoluamus hic casus, quibus aequatio differentio-differentialis hanc

hanc obtinet formam

$$\frac{\partial\partial y}{\partial x}+\partial y\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{2}y\,\partial x\left(D+\frac{E}{xx}+\frac{F}{aa+xx}+\frac{Gxx}{(aa+xx)^2}\right)=0.$$

I. Sumatur $\mu = m$ et $\nu = n$, eritque $D = 2C$, $E = \frac{1}{2}m(m-2)$, $F = 2n(m+1)$, et

$$G = 2n(n-2).$$

II. Sumatur $\mu = m-1$ et $\nu = n$, eritque $D = 0$, $E = 2C+\frac{1}{2}(m-1)^2$, $F = 2n(m+1)$, et

$$G = 2n(n-2).$$

III. Sumatur $\mu = m-1$ et $2n-2\nu = -1$, seu $\nu = n+\frac{1}{2}$, erit vltimus terminus $\frac{2C(aa+xx)}{xx} = 2C+\frac{2Caa}{xx}$. Ergo

$$D = 2C,\ E = 2Caa+\tfrac{1}{2}(m-1)^2,\ F = 2(mn+n+1),$$
$$G = \tfrac{1}{2}(2n+1)(2n-5).$$

IV. Sumatur $\mu = m$ et $2n-2\nu = 1$, seu $\nu = n-\frac{1}{2}$, erit vltimus terminus $\frac{2C}{aa+xx}$, ideoque

$$D = 0,\ E = \tfrac{1}{2}m(m-2),\ F = 2C+2mn+2n-1,$$
$$G = \tfrac{1}{2}(2n-1)(2n-3).$$

V. Sumatur $\mu = m+1$ et $\nu = n-\frac{1}{2}$, erit vltimus terminus $\frac{2Cxx}{aa+xx} = 2C-\frac{2Caa}{aa+xx}$, ideoque

$$D = 2C,\ E = \tfrac{1}{2}(m+1)(m-2),\ F = -2Caa+2n(m+1),$$
$$G = \tfrac{1}{2}(2n-1)(2n-3).$$

VI. Sit $\mu = m-1$ et $\nu = n-\frac{1}{2}$, erit vltimus terminus $\frac{2C}{xx(aa+xx)} = \frac{2C}{aaxx}-\frac{2C}{aa(aa+xx)}$, vnde fit

$$D = 0,\ E = \frac{2C}{aa}+\tfrac{1}{2}(m-1)^2,\ F = \frac{-2C}{aa}+2mn+2n-2,$$
$$G = \tfrac{1}{2}(2n-1)(2n-3).$$

VII. Sit $\mu = m+1$ et $2n-2\nu = 2$, seu $\nu = n-1$, erit ter-

terminus vltimus $\frac{2Cxx}{(aa+xx)^2}$, ideoque

$$D=0,\ E=\tfrac{1}{2}(m+1)(m-3),\ F=2n(m+1),$$
$$G=2C+2(n-1)^2.$$

VIII. Sit $\mu=m+2$ et $\nu=n-1$, erit terminus vltimus $\frac{2Cx^4}{(aa+xx)^2}=2C-\frac{2Caa}{aa+xx}-\frac{2Caaxx}{(aa+xx)^2}$, hincque

$$D=2C,\ E=\tfrac{1}{2}(m+2)(m-4),\ F=-2Caa+2mn+2n+2,$$
$$\text{et } G=-2Caa+2(n-1)^2.$$

IX. Sit $\mu=m$ et $\nu=n-1$, erit terminus vltimus

$$\frac{2C}{(aa+xx)^2}=\frac{2C}{aa(aa+xx)}-\frac{2Cxx}{a^2(aa+xx)^2},$$

hincque

$$D=0,\ E=\tfrac{1}{2}m(m-2),\ F=\frac{2C}{aa}+2mn+2n-2,$$
$$G=\frac{-2C}{aa}+2(n-1)^2.$$

X. Sit $\mu=m-1$ et $\nu=n-1$, erit terminus vltimus

$$\frac{2C}{xx(aa+xx)^2}=\frac{2C}{a^4xx}-\frac{4C}{a^4(aa+xx)}+\frac{2Cxx}{a^4(aa+xx)^2},$$

hincque

$$D=0,\ E=\frac{2C}{a^4}+\tfrac{1}{2}(m-1)^2,\ F=\frac{-4C}{a^4}+2mn+2n-4,$$
$$G=\frac{2C}{a^4}+2(n-1)^2.$$

## Problema 110.

889. Sumto elemento $\partial x$ conſtante, ſi K et L denotent functiones quascunque ipſius $x$, inuenire integrale completum huius aequationis differentio-differentialis

$$\frac{\partial\partial y}{\partial x}+\frac{\partial y}{\partial x}\cdot\frac{\partial K}{K}+\tfrac{1}{2}y\left(\partial.\frac{\partial L}{L\partial x}-\frac{\partial L^2}{2LL\partial x}+\frac{\partial K\partial L}{KL\partial x}+\frac{2CLL\partial x}{KK}\right)=0.$$

## Solutio.

Quoniam haec aequatio integrabilis redditur, ſi multiplicetur per $\frac{KK}{L}\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)$, eius integrale completum, vt ſupra vidimus, est

$$\frac{KK}{2L}\left[\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLL}{KK}yy\right]=\text{Const.}$$

quam aequationem differentialem primi gradus adhuc integrari oportet, quod cum ob constantem indefinitam maxime sit difficile, ea neglecta, primo saltem integrale particulare inuestigemus. Erit ergo ex aequatione

$$\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLL}{KK}yy=0$$

radicem extrahendo

$$\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}=\frac{Ly}{K}\sqrt{-C}, \text{ seu } \frac{\partial y}{y}+\frac{\partial L}{2L}=\frac{L\partial x}{K}\sqrt{-C},$$

vnde fit

$$y\sqrt{L}=\alpha e^{\int\frac{L\partial x}{K}\sqrt{-C}}.$$

Cum igitur aequationi differentio-differentiali propositae satisfaciant hi duo valores

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{L\partial x}{K}\sqrt{-C}} \text{ et } y=\frac{\beta}{\sqrt{L}}e^{-\int\frac{L\partial x}{K}\sqrt{-C}},$$

bini coniuncti etiam satisfacient, quibus quoniam duae constantes arbitrariae introducuntur, eius integrale completum erit

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{L\partial x}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{L\partial x}{K}\sqrt{-C}},$$

quae expressio valet, si $\sqrt{-C}$ fuerit quantitas realis, sin autem sit imaginaria, erit

$$y=\frac{\gamma}{\sqrt{L}}\,\text{sin.}\left(\int\frac{L\partial x}{K}\sqrt{C}+\zeta\right):$$

sicque habetur integrale completum aequationis differentio-differentialis propositae.

## Corollarium 1.

890. Hinc igitur aequationis differentialis primi gradus

$$\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLLy^2}{KK}=\frac{AL}{KK},$$

quae

quae per se satis est difficilis, integrale assignare valemus, quod est

$$y = \frac{\alpha}{\sqrt{L}} e^{\int \frac{L \partial x}{K} \sqrt{-C}} + \frac{\beta}{\sqrt{L}} e^{-\int \frac{L \partial x}{K} \sqrt{-C}},$$

si modo debita relatio constantium $\alpha$ et $\beta$ respectu constantis **A** definiatur.

## Corollarium 2.

891. Erit autem per $\sqrt{L}$ multiplicando et differentiando

$$\partial y \sqrt{L} + \frac{y \partial L}{2\sqrt{L}} = \frac{\alpha L \partial x}{K} \sqrt{-C} . e^{\int \frac{L \partial x}{K} \sqrt{-C}}$$
$$- \frac{\beta L \partial x}{K} \sqrt{-C} . e^{-\int \frac{L \partial x}{K} \sqrt{-C}}.$$

Hinc

$$\frac{\partial y}{\partial x} + \frac{y \partial L}{2 L \partial x} = \frac{\sqrt{-C L}}{K} (\alpha e^{\int \frac{L \partial x}{K} \sqrt{-C}} - \beta e^{-\int \frac{L \partial x}{K} \sqrt{-C}}),$$

vnde fit

$$\frac{A L}{K K} = \frac{-C L}{K K} (\alpha e^{\int \frac{L \partial x}{K K} \sqrt{-C}} - \beta e^{-\int \frac{L \partial x}{K} \sqrt{-C}})^2$$
$$+ \frac{C L}{K K} (\alpha e^{\int \frac{L \partial x}{K} \sqrt{-C}} + \beta e^{-\int \frac{L \partial x}{K} \sqrt{-C}})^2,$$

ideoque $A = 4 C \alpha \beta$ seu $\beta = \frac{A}{4 C \alpha}$.

## Scholion 1.

892. Quamuis ergo aequationem propositam ope idonei multiplicatoris integrare licuerit, altera tamen integratio maximis difficultatibus premi videbatur. Interim tamen ope substitutionis aequatio illa differentialis primi gradus tractatu facilis redditur; posito enim $y = \frac{z}{\sqrt{L}}$, vt sit $\partial y \sqrt{L} + \frac{y \partial L}{2\sqrt{L}} = \partial z$, oritur $(\frac{\partial z}{\partial x \sqrt{L}})^2 + \frac{C L z z}{K K} = \frac{A L}{K K}$, hinc $\frac{\partial z}{\partial x} = \frac{L}{K} \sqrt{(A - C z z)}$, seu $\frac{\partial z}{\sqrt{(A - C z z)}} = \frac{L \partial x}{K}$, quae integrata dat

$$l[z\surd-C+\surd(A-Czz)]=\int\frac{L\partial x}{K}\surd-C+lB,$$

vnde praecedens integrale eruitur. Caeterum forma noſtrae aequationis differentio-differentialis aliquanto commodius exhiberi poteſt hoc modo: Si P et R ſint functiones quaecunque ipſius $x$, ſumaturque elementum $\partial x$ conſtans, huius aequationis

$$\partial\partial y-\partial y\left(\frac{\partial P}{P}+\frac{\partial R}{R}\right)-y\left(\partial\cdot\frac{\partial P}{P}-\frac{\partial P\partial R}{PR}+\frac{aaRR\partial x^2}{PP}\right)=0,$$

bis integratae integrale completum eſt

$$y=\alpha P e^{a\int\frac{R\partial x}{P}}+\beta P e^{-a\int\frac{R\partial x}{P}},$$

ſiquidem $a$ ſit quantitas realis. At ſi $a=0$, erit

$$y=P\left(\alpha+\beta\int\frac{R\partial x}{P}\right).$$

Sin autem ſit $aa=-cc$, erit

$$y=\alpha P\,\text{ſin.}\left(\beta+c\int\frac{R\partial x}{P}\right).$$

Tum vero illa aequatio integrabilis redditur, ſi multiplicetur per $\frac{1}{RR\partial x^2}\left(\partial y-\frac{y\partial P}{P}\right)$, eritque integrale primum

$$\frac{1}{2RR\partial x^2}\left[\left(\partial y-\frac{y\partial P}{P}\right)^2-\frac{aaRR}{PP}y^2\partial x^2\right]=\text{Conſt.}$$

Hinc patet in illa aequatione differentio-differentiali commode hanc ſubſtitutionem adhiberi $y=Pz$, qua ea transformatur in

$$\partial\partial z+\partial z\left(\frac{\partial P}{P}-\frac{\partial R}{R}\right)-\frac{aaRR}{PP}z\partial x^2=0,$$

quae per $\frac{PP\partial z}{RR\partial x^2}$ multiplicata, ſponte fit integrabilis. Quin etiam poſito $\frac{P}{R}=S$, vt habeatur

$$\partial\partial z+\frac{\partial S\partial z}{S}-\frac{aaz\partial x^2}{SS}=0,$$

multiplicator $\frac{SS\partial z}{\partial x^2}$, ſtatim dat integrale

$$\frac{SS\partial z^2}{2\partial x^2}-\tfrac{1}{2}aazz=\text{Conſt.}$$

Scho-

## Scholion 2.

893. Viciſſim ergo ex hac forma ſimpliciſſima

$$SS\partial\partial z + S\partial S\partial z - aaz\partial x^2 = 0,$$

quae per $\partial z$ multiplicata integrabilis redditur, formas magis complicatas deriuare potuiſſemus, ponendo $z = \frac{y}{P}$ et $S = \frac{P}{R}$. Quae quanquam in formis generalibus ſatis perſpicua, tamen in exemplis determinatis plerumque haec deriuatio nimis eſt occulta, quam vt menti occurrere poſſit. Veluti in caſibus §. 888. euolutis, ſi N°. IX. ſumamus $m=2$, et $C=(n-1)^2 aa$, fiet $D=0$, $E=0$, $F=2n(n+1)$, et $G=0$, vnde habetur haec aequatio

$$\frac{\partial\partial y}{\partial x} + 2\partial y\left(\frac{1}{x} + \frac{nx}{aa+xx}\right) + \frac{n(n+1)y\partial x}{aa+xx} = 0, \text{ ſeu}$$

$$\partial\partial y + \frac{2\partial x\partial y[aa+(n+1)xx]}{x(aa+xx)} + \frac{n(n+1)y\partial x^2}{aa+xx} = 0,$$

quae integrabilis redditur ope multiplicatoris

$$xx(aa+xx)^{n+1}\left(\frac{\partial y}{\partial x} + \frac{y(aa+nxx)}{x(aa+xx)}\right),$$

integrali exiſtente

$$\tfrac{1}{2}xx(aa+xx)^{n+1}\left[\left(\frac{\partial y}{\partial x} + \frac{y(aa+nxx)}{x(aa+xx)}\right)^2 + \frac{(n-1)^2 aayy}{(aa+xx)^2}\right] = \text{Conſt.}$$

Pro integrali ergo particulari erit

$$\frac{\partial y}{y} + \frac{\partial x}{x} + \frac{(n-1)x\partial x}{aa+xx} = \pm\frac{(n-1)a\partial x\sqrt{-1}}{aa+xx},$$

vnde colligitur

$$xy(aa+xx)^{\frac{n-1}{2}} = \alpha\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)^{\pm\frac{(n-1)}{2}}.$$

Ergo bina integralia particularia coniuncta dant

$$y = \frac{\alpha}{x}(a-x\sqrt{-1})^{-n+1} + \frac{\beta}{x}(a+x\sqrt{-1})^{-n+1},$$

integrale completum. Hoc autem caſu aequatio noſtra ad formam ſimpliciſſimam reducetur ope ſubſtitutionis

$$y = \frac{n}{x}(aa + xx)^{\frac{1-n}{2}},$$

cuius ratio et inuentio difficilius perspicitur.

## Problema III.

894. Sumto elemento $\partial x$ constante, inuestigare conditiones quibus aequatio differentio-differentialis

$$\partial\partial y + P\,\partial x\,\partial y + Q\,y\,\partial x^2 = 0,$$

integrabilis redditur, ope multiplicatoris huius formae

$$\frac{y\,\partial x^2}{L\,\partial y^2 + M\,y\,\partial y\,\partial x + N\,y\,y\,\partial x^2},$$

denotantibus litteris L, M, N, P, Q functiones ipsius $x$.

## Solutio.

Tribuatur denominatori huius fractionis talis forma

$$(\partial y + R\,y\,\partial x)(\partial y + S\,y\,\partial x),$$

ac leui attentione adhibita patet, integrale huiusmodi formam esse habiturum

$$V + l\,\frac{\partial y + R\,y\,\partial x}{\partial y + S\,y\,\partial x} = \text{Const.}$$

cuius ergo differentiale aequationem propositam producere debet. Dat autem differentiatio

$$\partial V + \frac{(S-R)\,y\,\partial x\,\partial\partial y + (R-S)\,\partial x\,\partial y^2 + y\,\partial x\,\partial y(\partial R - \partial S) + yy\,\partial x^2(S\,\partial R - R\,\partial S)}{(\partial y + R\,y\,\partial x)(\partial y + S\,y\,\partial x)} = 0,$$

quae ad communem denominatorem reducta, abit in

$$\left.\begin{array}{r} (S-R)y\partial x\partial\partial y + (R-S)\partial x\partial y^2 + y\partial x\partial y(\partial R - \partial S) + yy\partial x^2(S\partial R - R\partial S) \\ + \partial V\,\partial y^2 + (R+S)\,y\,\partial x\,\partial y\,\partial V + RS\,yy\,\partial x^2\,\partial V \end{array}\right\} = 0.$$

Statuatur $\partial V = (S - R)\,\partial x$, vt aequatio per $y$ diuisibilis euadat, sicque orietur haec aequatio

$$\left.\begin{array}{r} (S-R)\,\partial\partial y + \partial y(\partial R - \partial S) \quad + y\,\partial x(S\,\partial R - R\,\partial S) \\ + (SS - RR)\,\partial x\,\partial y + RS\,(S-R)\,y\,\partial x^2 \end{array}\right\} = 0,$$

quae vt cum forma proposita conueniat, fieri oportet

P =

$$P = (R + S) + \frac{\partial R - \partial S}{(S - R)\partial x}, \text{ et } Q = R\,S + \frac{S\,\partial R - R\,\partial S}{(S - R)\partial x},$$

quos valores si functiones P et Q habuerint, aequatio

$$\partial\partial y + P\,\partial x\,\partial y + Q\,y\,\partial x^2 = 0$$

per $\frac{(S - R)\,y\,\partial x}{(\partial y + R\,y\,\partial x)(\partial y + S\,y\,\partial x)}$ multiplicata, integrale dabit

$$\int (S - R)\,\partial x + l\,\frac{\partial y + R\,y\,\partial x}{\partial y + S\,y\,\partial x} = \text{Const.}$$

Si ponamus $S = M + N$ et $R = M - N$, erit

$$P = 2M - \frac{\partial N}{N\,\partial x} \text{ et } Q = M\,M - N\,N + \frac{\partial M}{\partial x} - \frac{M\,\partial N}{N\,\partial x}.$$

## Corollarium. 1.

895. Quaecunque ergo functiones ipsius $x$ loco M et N assumantur, indeque definiantur

$$P = 2M - \frac{\partial N}{N\,\partial x} \text{ et } Q = M\,M - N\,N + \frac{\partial M}{\partial x} - \frac{M\,\partial N}{N\,\partial x},$$

huius aequationis

$$\partial\partial y + P\,\partial x\,\partial y + Q\,y\,\partial x^2 = 0$$

integrale erit

$$2\int N\,\partial x + l\,\frac{\partial y + (M - N)\,y\,\partial x}{\partial y + (M + N)\,y\,\partial x} = \text{Const.}$$

## Corollarium 2.

896. Si ponatur $y = e^{\int z\,\partial x}$, fiet nostra aequatio differentialis primi gradus

$$\partial z + z\,z\,\partial x + P\,z\,\partial x + Q\,\partial x = 0,$$

cuius propterea integrale erit

$$2\int N\,\partial x + l\,\frac{z + (M - N)}{z + (M + N)} = \text{Const.}$$

## Corollarium 3.

897. Si velimus, vt sit $P = 0$ et aequatio habeatur huiusmodi $\partial\partial y + Q\,y\,\partial x^2 = 0$, capi debet $2\,M = \frac{\partial N}{N\,\partial x}$ eritque

que $Q = \frac{\partial M}{\partial x} - MM - NN$, eiusque aequatio integralis

$$2\int N \partial x + l \frac{\partial y + (M - N) y \partial x}{\partial y + (M + N) y \partial x} = \text{Const.}$$

## Corollarium 4.

898. In genere autem prout conftans capiatur vel $+\infty$ vel $-\infty$, obtinebitur integrale particulare vel

$$\partial y + (M + N) y \partial x = 0 \quad \text{vel}$$
$$\partial y + (M - N) y \partial x = 0,$$

vnde erit vel

$$y = \alpha e^{-\int (M+N) \partial x} \text{ vel } y = \beta e^{-\int (M-N) \partial x},$$

ex quibus noftrae aequationis colligitur integrale completum

$$y = e^{-\int M \partial x} (\alpha e^{-\int N \partial x} + \beta e^{\int N \partial x}).$$

## Exemplum 1.

899. Sit $M = \alpha$ et $N = \beta$, erit $P = 2\alpha$ et $Q = \alpha\alpha - \beta\beta$, vnde huius aequationis

$$\partial \partial y + 2 \alpha \partial x \partial y + (\alpha\alpha - \beta\beta) y \partial x^2 = 0$$

integrale eft

$$2 \beta x + l \frac{\partial y + (\alpha - \beta) y \partial x}{\partial y + (\alpha + \beta) y \partial x} = \text{Conft.}$$

In quantitatibus autem finitis integrale completum

$$y = e^{-\alpha x} (A e^{-\beta x} + B e^{\beta x}).$$

Cafu autem quo $\beta\beta = -\gamma\gamma$, aequatio

$$\partial \partial y + 2 \alpha \partial x \partial y + (\alpha\alpha + \gamma\gamma) y \partial x^2 = 0$$

bis integrata dat

$$y = A e^{-\alpha x} \text{fin.} (\gamma x + C).$$

At fi $\gamma = 0$, aequationis

$$\partial \partial y + 2 \alpha \partial x \partial y + \alpha\alpha y \partial x^2 = 0$$

inte-

integrale eſt

$$y = e^{-\alpha x}(\mathrm{A} + \mathrm{B}\,x).$$

## Exemplum 2.

900. Si $\mathrm{M} = \frac{\alpha}{x}$ et $\mathrm{N} = \beta x^n$, erit $\mathrm{P} = \frac{2\alpha - n}{x}$, et

$$\mathrm{Q} = \frac{\alpha\alpha}{xx} - \beta\beta x^{2n} - \frac{\alpha}{xx} - \frac{\alpha n}{xx} = \frac{\alpha(\alpha - n - 1)}{xx} - \beta\beta x^{2n}.$$

Ergo huius aequationis

$$\partial\partial y + \frac{(2\alpha - n)\partial x \partial y}{x} + \frac{\alpha(\alpha - n - 1) y \partial x^2}{xx} - \beta\beta x^{2n} y \partial x^2 = 0$$

integrale primum eſt

$$\frac{2\beta}{n+1} x^{n+1} + l\frac{x\partial y + (\alpha - \beta x^{n+1}) y\partial x}{x\partial y + (\alpha + \beta x^{n+1}) y\partial x} = \mathrm{Conſt.}$$

Integrale autem ſecundum

$$y = x^{-\alpha}\left(\mathrm{A}\, e^{\frac{-\beta x^{n+1}}{n+1}} + \mathrm{B}\, e^{\frac{\beta x^{n+1}}{n+1}}\right):$$

ſi $\beta = 0$, erit id

$$y = x^{-\alpha}(\mathrm{A} + \mathrm{B}\, x^{n+1}),$$

ſin autem $\beta\beta = -\gamma\gamma$, erit

$$y = \mathrm{A}\, x^{-\alpha} \text{ ſin. } \left(\frac{\gamma}{n+1} x^{n+1} + \mathrm{C}\right).$$

## Corollarium 1.

901. Sumto $n = 2\alpha$, vt habeatur haec aequatio

$$\partial\partial y - \frac{\alpha(\alpha+1) y \partial x^2}{xx} - \beta\beta x^{4\alpha} y \partial x^2 = 0,$$

erit eius integrale completum

$$y = x^{-\alpha}\left(\mathrm{A}\, e^{\frac{-\beta}{2\alpha+1} x^{2\alpha+1}} + \mathrm{B}\, e^{\frac{\beta}{2\alpha+1} x^{2\alpha+1}}\right),$$

ſi ſit $\beta = 0$, erit id

$$y = x^{-\alpha}(\mathrm{A} + \mathrm{B}\, x^{2\alpha+1}),$$

at si $\beta\beta = -\gamma\gamma$, erit hoc integrale

$$y = A x^{-\alpha} \text{ sin.} \left(\frac{\gamma}{2\alpha+1} x^{2\alpha+1} + C\right).$$

## Corollarium 2.

902. Ponamus $\alpha = -1$, vt habeamus hanc aequationem $\partial\partial y - \frac{\beta\beta y \partial x^2}{x^4} = 0$, cuius ergo integrale erit

$$y = x\left(A e^{\frac{\beta}{x}} + B e^{-\frac{\beta}{x}}\right),$$

vbi notandum, si sit $\beta\beta = -\gamma\gamma$, fore $y = A x \text{ sin.} \left(\frac{\gamma}{x} + C\right)$.

## Exemplum 3.

903. Ponatur $N = \frac{A x^m}{\alpha + \beta x^n}$, vt sit

$$\frac{\partial N}{N \partial x} = \frac{m}{x} - \frac{\beta n x^{n-1}}{\alpha + \beta x^n},$$

et sumatur

$$M = \frac{m}{2x} - \frac{\beta n x^{n-1}}{2(\alpha + \beta x^n)},$$

vt fiat $P = 0$ et

$$Q = \frac{-m}{2xx} - \frac{\beta n(n-1) x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta\beta n n x^{2n-2}}{2(\alpha + \beta x^n)^2}$$
$$- \frac{m m}{4 x x} + \frac{\beta m n x^{n-2}}{2(\alpha + \beta x^n)} - \frac{\beta\beta n n x^{2n-2}}{4(\alpha + \beta x^n)^2}$$
$$- \frac{A A x^{2m}}{(\alpha + \beta x^n)^2}$$

siue

$$Q = \frac{-m(m+2)}{4xx} + \frac{n(m-n+1)\beta x^{n-2}}{2(\alpha+\beta x^n)} + \frac{\beta\beta n n x^{2n-2} - 4AAx^{2m}}{4(\alpha+\beta x^n)^2},$$

et

et ob $\int M \partial x = \frac{1}{2} l N$, erit integrale

$$y = \frac{1}{\sqrt{N}} \left( C e^{-\int N \partial x} + D e^{\int N \partial x} \right)$$

huius aequationis $\partial \partial y + Q y \partial x^2 = 0$. Vt expreſſio ipſius Q fiat ſimplicior, hoc fieri poteſt pluribus modis, dum numerator partis poſtremae per $\alpha + \beta x^n$ diuiſibilis redditur.

I. Sit $m = n - 1$ et $AA = \frac{1}{4} \beta \beta n n$, eritque $Q = -\frac{(nn-1)}{4xx}$, tum vero

$$N = \frac{\frac{1}{2} \beta n x^{n-1}}{\alpha + \beta x^n} \text{ et } \int N \partial x = \frac{1}{2} l (\alpha + \beta x^n);$$

vnde aequationis $\partial \partial y - \frac{(nn-1) y \partial x^2}{4xx} = 0$ integrale eſt

$$y = \frac{1}{\sqrt{x^{n-1}}} (C + D \alpha + D \beta x^n).$$

II. Sit $2m = -2$ ſeu $m = -1$ et $4AA = \alpha \alpha n n$, erit

$$Q = \frac{1}{4xx} - \frac{nn\beta x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta nn x^{n-2} - \alpha nn x^{-2}}{4(\alpha + \beta x^n)}, \text{ ſeu}$$

$Q = \frac{1-nn}{4xx}$, vt ante.

III. Sit $2m = -n - 2$ ſeu

$$m = \frac{-n-2}{2} \text{ et } 4AA = -\frac{\alpha^3 nn}{\beta},$$

fietque

$$Q = \frac{-(nn-4)}{16xx} - \frac{3nn\beta x^{n-2}}{4(\alpha + \beta x^n)} + \frac{nn(\beta\beta x^{n-2} - \alpha\beta x^{-2} + \alpha\alpha x^{-n-2})}{4\beta(\alpha + \beta x^n)}$$

ſeu

$$Q = \frac{4 - nn}{16xx} + \frac{nn(\alpha\alpha x^{-n-2} - \alpha\beta x^{-2} - 2\beta\beta x^{n-2})}{4\beta(\alpha + \beta x^n)},$$

Q 2 quae

quae expreſſio abit in

$$Q=\frac{4-nn}{16xx}+\frac{nn}{4\beta}(\alpha x^{-n-2}-2\beta x^{-2})=\frac{4-9nn}{16xx}+\frac{nn\alpha}{4\beta x^{n+2}}.$$

Quare cum ſit

$$N=+\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}}\cdot\frac{x^{-\frac{n-2}{2}}}{\alpha+\beta x^n},\text{ erit}$$

$$\int N\partial x=\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}}\int\frac{\partial x}{(\alpha+\beta x^n)x^{\frac{n+2}{2}}};$$

ſumatur $n=\frac{2}{3}$, vt fiat $m=-\frac{4}{3}$, $Q=\frac{\alpha}{9\beta x^{\frac{8}{3}}}$, et

$$N=\frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\frac{x^{-\frac{4}{3}}}{\alpha+\beta x^{\frac{2}{3}}},\text{ hinc}$$

$$\int N\partial x=\frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{\partial x}{(\alpha+\beta x^{\frac{2}{3}})\,x^{\frac{4}{3}}};$$

ſicque aequationis

$$\partial\partial y+\frac{\alpha}{9\beta x^{\frac{8}{3}}}y\,\partial x^2=0$$

integrale erit

$$y=\frac{1}{\sqrt{N}}(Ce^{-\int N\partial x}+De^{\int N\partial x}).$$

Sin autem capiatur $n=-\frac{2}{3}$, vt fiat

$$m=-\frac{2}{3}\text{ et }Q=\frac{\alpha}{9\beta x^{\frac{4}{3}}},\text{ erit}$$

N =

$$N = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{2}{3}}}{\alpha+\beta x^{-\frac{2}{3}}} \text{ et}$$

$$\int N\partial x = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{x^{\frac{1}{3}}\partial x}{\alpha x+\beta x^{\frac{1}{3}}} = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{\partial x}{\alpha x^{\frac{2}{3}}+\beta},$$

vnde aequatio

$$\partial\partial y + \frac{\alpha y\partial x^2}{9\beta x^{\frac{4}{3}}} = 0$$

ſimili modo integratur.

## Scholion 1.

904. Aequationem ergo $\partial\partial y + A x^m y \partial x^2 = 0$ his caſibus integrare licuit, $m=0$, $m=-4$, $m=-\frac{4}{3}$, $m=-\frac{8}{3}$ et $m=-2$, ſeu $m=-2\pm\frac{2}{1}$, et $m=-2\pm\frac{2}{3}$. Quodſi vlterius ponamus $N = \frac{A x^\lambda}{\alpha+\beta x^n+\gamma x^{2n}}$, ſimili modo integrationem caſuum iſtius aequationis $m=-2\pm\frac{2}{5}$ impetrabimus, quibus quoque aequatio differentialis primi gradus

$$\partial z + zz\partial x + A x^m \partial x = 0$$

integrationem admittit. Haec autem caſuum integrabilium inveſtigatio nimis eſt operoſa, quam vt eam fuſius proſequamur, praeſertim cum infra methodus occurrat haec omnia commodius euoluendi.

## Scholion 2.

905. Ex his colligere licet, quantus fructus ex inventione multiplicatorum, quibus etiam aequationes differentio-differentiales integrabiles redduntur, expectari queat, etiamſi exempla hic tractata tantum leue huius methodi ſpecimen referant.

ferant. Aliquas autem saltem multiplicatorum formas hic sum contemplatus, neque vllum est dubium, quin plures aliae formae pari successu in vsum vocari queant. In hoc porro capite tantum eiusmodi aequationes differentio-differentiales tractavimus, in quibus altera variabilis $y$ cum suis differentialibus $\partial y$ et $\partial \partial y$ vbique vnicam obtinet dimensionem. Verum eadem methodus quoque ad alia huiusmodi aequationum genera extenditur, quae etsi parum adhuc est exculta, tamen vsu non carebit sequens applicatio, vbi integratio aliarum aequationum differentialium secundi gradus, quae aliis methodis tractatu difficillimae videntur, ope multiplicatorum docebitur.

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CA-

# CAPVT VI.

## DE

## INTEGRATIONE ALIARVM AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM PER IDONEOS MVLTIPLICATORES INSTITVENDA.

### Problema 112.

906.

Posito elemento $\partial x$ constante, si proposita sit huiusmodi aequatio

$$\partial\partial y + \frac{A y \partial x^2}{(Byy + C + 2Dx + Exx)^2} = 0,$$

inuenire multiplicatorem, quo ea integrabilis reddatur.

### Solutio.

Tentetur talis multiplicatoris forma $2P\partial y + 2Qy\partial x$, vbi P et Q sint functiones ipsius $x$, et producti

$$2\partial\partial y(P\partial y + Qy\partial x) + \frac{2Ay\partial x^2(P\partial y + Qy\partial x)}{(Byy + C + 2Dx + Exx)^2} = 0,$$

integrale statuatur

$$P\partial y^2 + 2Qy\partial x\partial y + V\partial x^2 = \text{Const.}\,\partial x^2,$$

vbi V sit functio binas variabiles $x$ et $y$ complectens. Erit ergo facta aequalitate

$$\begin{matrix} +\partial P\partial y^2 & +2y\partial x\partial Q\partial y \\ +2Q\partial x\partial y^2 + \partial x^2\partial V & \end{matrix} - \frac{2Ay\partial x^2(P\partial y + Qy\partial x)}{(Byy + C + 2Dx + Exx)^2} = 0,$$

quae per integrationem valorem ipsius V suppeditare nequit, nisi

nisi sit $\partial P + 2 Q \partial x = 0$, eritque tum

$$\partial V = \frac{A y (2 P \partial y - y \partial P)}{(B y y + C + 2 D x + E x x)^2} - 2 y \partial y . \frac{\partial Q}{\partial x},$$

cuius formulae, siquidem integrationem admittat, integrale ex variabilitate ipsius $y$ erit

$$\frac{- A P}{B (B y y + C + 2 D x + E x x)} - y y . \frac{\partial Q}{\partial x} = V.$$

Sumto ergo $y$ constante, necesse est sit

$$-\frac{A \partial P (B y y + C + 2 D x + E x x) + 2 A P \partial x (D + E x)}{B (B y y + C + 2 D x + E x x)^2} - y y . \frac{\partial \partial Q}{\partial x} = \frac{- A y y \partial P}{(B y y + C + 2 D x + E x x)^2},$$

cui satisfit, si $\frac{\partial \partial Q}{\partial x^2} = 0$ et

$$- \partial P (C + 2 D x + E x x) + 2 P \partial x (D + E x) = 0,$$

quae duae conditiones an simul consistere possint, videndum est. Posterior autem dat

$$\frac{\partial P}{P} = \frac{2 D \partial x + 2 E x \partial x}{C + 2 D x + E x x}, \text{ ideoque } P = C + 2 D x + E x x,$$

vnde fit

$$Q = - \frac{\partial P}{2 \partial x} = - D - E x, \text{ hinc } \frac{\partial Q}{\partial x} = - E \text{ et } \frac{\partial \partial Q}{\partial x^2} = 0.$$

Multiplicator ergo quaesitus est

$$2 \partial y (C + 2 D x + E x x) - 2 y \partial x (D + E x),$$

hincque obtinetur integrale

$$\frac{\partial y^2}{\partial x^2} (C + 2 D x + E x x) - \frac{2 y \partial y}{\partial x} (D + E x) - \frac{A (C + 2 D x + E x x)}{B (B y y + C + 2 D x + E x x)} + E y y = \text{Const.}$$

seu vtrinque addendo $\frac{A}{B}$,

$$\frac{\partial y^2}{\partial x^2} (C + 2 D x + E x x) - \frac{2 y \partial y}{\partial x} (D + E x) + \frac{A y y}{B y y + C + 2 D x + E x x} + E y y = \text{Const.}$$

## Corollarium 1.

907. Haec ergo aequatio $\partial \partial y + \frac{a a y \partial x^2}{(y y + x x)^2} = 0$, vbi $A = a a$, $B = 1$, $C = 0$, $D = 0$, et $E = 1$, integrabilis

bilis redditur multiplicatore $2xx\partial y - 2yx\partial x$, eiusque integrale erit

$$\frac{xx\partial y^2}{\partial x^2} - \frac{2xy\partial y}{\partial x} + yy + \frac{aayy}{yy+xx} = bb.$$

## Corollarium 2.

908. Si hic ponatur $y = ux$, ob $\partial y = u\partial x + x\partial u$, habebimus

$$\begin{aligned} &xxuu + \frac{2ux^3\partial u}{\partial x} + \frac{x^4\partial u^2}{\partial x^2} + \frac{aauu}{1+uu} = bb \\ &- 2xxuu - \frac{2ux^3\partial u}{\partial x} \\ &+ xxuu, \end{aligned}$$

fiue

$$\frac{x^4\partial u^2}{\partial x^2} = \frac{bb+(bb-aa)uu}{1+uu}, \text{ ergo } \frac{\partial x}{xx} = \frac{\partial u\sqrt{(1+uu)}}{\sqrt{[bb+(bb-aa)uu]}},$$

vnde tam $x$ quam $y$ per $u$ determinatur.

## Corollarium 3.

909. Simili etiam modo integratio in genere perfici poteſt. Sit enim breuitatis gratia

$$C + 2Dx + Exx = Bzz, \text{ erit } D + Ex = \frac{Bz\partial z}{\partial x},$$

et aequatio noſtra fiet

$$\frac{Bzz\partial y^2}{\partial x^2} - \frac{2By z\partial y\partial z}{\partial x^2} + Eyy + \frac{Ayy}{B(yy+zz)} = \frac{K}{B},$$

quae poſito $y = uz$, abit in

$$\frac{Bz^4\partial u^2}{\partial x^2} - \frac{Buuzz\partial z^2}{\partial x^2} + Euuzz + \frac{Auu}{B(1+uu)} = \frac{K}{B}.$$

At $\frac{zz\partial z^2}{\partial x^2} = \frac{(D+Ex)^2}{BB}$, vnde oritur

$$\frac{Bz^4\partial u^2}{\partial x^2} + \frac{CE-DD}{B}uu = \frac{K+(K-A)uu}{B(1+uu)}, \text{ ſeu}$$

$$\frac{BBz^4\partial u^2}{\partial x^2} = \frac{K+(K-A+DD-CE)uu+(DD-CE)u^4}{1+uu},$$

ita vt ſit, reſtituto valore ipſius $z$,

$$\frac{\partial x}{C+2Dx+Exx} = \frac{\partial u\sqrt{(1+uu)}}{\sqrt{[K+(K-A+DD-CE)uu+(DD-CE)u^4]}},$$

 ſicque

ſicque $x$ definitur per $u$, indeque etiam

$$y = uz = u\sqrt{\frac{C + 2Dx + Exx}{B}}.$$

## Scholion.

910. Hinc patet ſubſtitutio, qua tam ipſa aequatio differentio-differentialis propoſita, quam multiplicator ad formam commodiorem reduci debet. Poſito enim ad abbreuiandum $C + 2Dx + Exx = Bzz$, aequatio noſtra

$$\partial\partial y + \frac{Ay\partial x^2}{B^2(yy+zz)^2} = 0,$$

ope ſubſtitutionis $y = uz$ transformatur in

$$z\partial\partial u + 2\partial z\partial u + u\partial\partial z + \frac{Au\partial x^2}{BBz^3(1+uu)^2} = 0,$$

cuius multiplicator eſt

$$2B(zz\partial y - yz\partial z), \text{ ſeu } 2Bz^3\partial u,$$

vel ſimpliciter $z^3\partial u$. Cum autem ſit $\partial z = \frac{\partial x(D+Ex)}{Bz}$, erit

$$\partial\partial z = \frac{E\partial x^2}{Bz} - \frac{\partial x\partial z(D+Ex)}{Bzz} = \frac{E\partial x^2}{Bz} - \frac{\partial x^2(D+Ex)^2}{BBz^3} = \frac{(CE-DD)\partial x^2}{BBz^3},$$

ita vt ſit $z^3\partial\partial z = \frac{CE-DD}{BB}\partial x^2$, vnde aequatio noſtra per $z^3\partial u$ multiplicata induit hanc formam

$$z^4\partial u\partial\partial u + 2z^3\partial z\partial u^2 + \frac{CE-DD}{BB}u\partial u\partial x^2 + \frac{Au\partial u\partial x^2}{BB(1+uu)^2} = 0,$$

manifeſto integrabilem, integrali exiſtente

$$\tfrac{1}{2}z^4\partial u^2 + \frac{CE-DD}{2BB}uu\partial x^2 - \frac{A\partial x^2}{2BB(1+uu)} = \tfrac{1}{2}\text{Conſt.}\,\partial x^2,$$

cuius adeo noua integratio ob $z$ functionem ipſius $x$ mox in oculos incurrit, cum ſit

$$zz\partial u = \partial x\sqrt{\left(\text{Conſt.} + \frac{DD-CE}{BB}uu + \frac{A}{BB(1+uu)}\right)},$$

vbi variabiles $u$ et $x$ ſponte ſeparantur. Caeterum hic notetur, functionem pro $z$ aſſumtam ſatisfacere aequationi $z^3\partial\partial z = \alpha\partial x^2$, cum tamen eius ratio non ſit manifeſta. Multiplicando autem hanc aequationem per $\frac{2\partial z}{z^3}$ prodit, $2\partial z\partial\partial z = \frac{2\alpha\partial x^2\partial z}{z^3}$,

cuius

cuius integrale est $\partial z^2 = \beta \partial x^2 - \frac{\alpha \partial x^2}{zz}$, seu $\partial x = \frac{z \partial z}{\sqrt{(\beta zz - \alpha)}}$, vnde porro fit $\beta x + \gamma = \sqrt{(\beta zz - \alpha)}$, ideoque

$$\beta zz = \alpha + \gamma\gamma + 2\beta\gamma x + \beta\beta xx,$$

quae est ipsa nostra forma.

## Problema 113.

911. Sumto elemento $\partial x$ constante, inuenire formam generaliorem aequationum differentio-differentialium quae ope huiusmodi multiplicatoris $M y \partial x + N \partial y$ integrabiles reddantur.

## Solutio.

Quia multiplicator ope substitutionis $y = R u$ in formam simplicissimam $S \partial u$ transmutari potest, hac substitutione ipsa aequatio differentio-differentialis induat hanc formam

$$\partial\partial u + P \partial x \partial u + \frac{U \partial x^2}{S} = 0,$$

cuius postremum membrum per $S \partial u$ multiplicatum sponte est integrabile, si quidem U denotet functionem quamcunque ipsius $u$, dum R, S et P sint functiones ipsius $x$. Cum ergo aequatio

$$S \partial u \partial\partial u + P S \partial x \partial u^2 + U \partial x^2 \partial u = 0,$$

debeat esse integrabilis, posito integrali

$$\tfrac{1}{2} S \partial u^2 + \partial x^2 \int U \partial u = \tfrac{1}{2} C \partial x^2,$$

necesse est sit

$$\tfrac{1}{2} \partial S \partial u^2 = P S \partial x \partial u^2, \text{ seu } P \partial x = \frac{\partial S}{2 S}.$$

Quocirca haec forma generalis

$$\partial\partial u + \frac{\partial S \partial u}{2 S} + \frac{U \partial x^2}{S} = 0,$$

per $S \partial u$ multiplicata dabit integrale

$$S \partial u^2 = \partial x^2 (C - 2 \int U \partial u),$$

ſicque $x$ definitur per $u$, indeque etiam

$$y = uz = u\sqrt{\frac{C+2Dx+Exx}{B}}.$$

## Scholion.

910. Hinc patet ſubſtitutio, qua tam ipſa aequatio differentio-differentialis propoſita, quam multiplicator ad formam commodiorem reduci debet. Poſito enim ad abbreuiandum $C+2Dx+Exx = Bzz$, aequatio noſtra

$$\partial\partial y + \frac{Ay\partial x^2}{B^2(yy+zz)^2} = 0,$$

ope ſubſtitutionis $y = uz$ transformatur in

$$z\partial\partial u + 2\partial z\partial u + u\partial\partial z + \frac{Au\partial x^2}{BBz^3(1+uu)^2} = 0,$$

cuius multiplicator eſt

$$2B(zz\partial y - yz\partial z), \text{ ſeu } 2Bz^3\partial u,$$

vel ſimpliciter $z^3\partial u$. Cum autem ſit $\partial z = \frac{\partial x(D+Ex)}{Bz}$, erit

$$\partial\partial z = \frac{E\partial x^2}{Bz} - \frac{\partial x\partial z(D+Ex)}{Bzz} = \frac{E\partial x^2}{Bz} - \frac{\partial x^2(D+Ex)^2}{BBz^3} = \frac{(CE-DD)\partial x^2}{BBz^3},$$

ita vt ſit $z^3\partial\partial z = \frac{CE-DD}{BB}\partial x^2$, vnde aequatio noſtra per $z^3\partial u$ multiplicata induit hanc formam

$$z^4\partial u\partial\partial u + 2z^3\partial z\partial u^2 + \frac{CE-DD}{BB}u\partial u\partial x^2 + \frac{Au\partial u\partial x^2}{BB(1+uu)^2} = 0,$$

manifeſto integrabilem, integrali exiſtente

$$\tfrac{1}{2}z^4\partial u^2 + \frac{CE-DD}{2BB}uu\partial x^2 - \frac{A\partial x^2}{2BB(1+uu)} = \tfrac{1}{2}\,\text{Conſt.}\,\partial x^2,$$

cuius adeo noua integratio ob $z$ functionem ipſius $x$ mox in oculos incurrit, cum ſit

$$zz\partial u = \partial x\sqrt{\left(\text{Conſt.} + \frac{DD-CE}{BB}uu + \frac{A}{BB(1+uu)}\right)},$$

vbi variabiles $u$ et $x$ ſponte ſeparantur. Caeterum hic notetur, functionem pro $z$ aſſumtam ſatisfacere aequationi $z^3\partial\partial z = \alpha\partial x^2$, cum tamen eius ratio non ſit manifeſta. Multiplicando autem hanc aequationem per $\frac{2\partial z}{z^3}$ prodit, $2\partial z\partial\partial z = \frac{2\alpha\partial x^2\partial z}{z^3}$,

cuius

cuius integrale eſt $\partial z^2 = \beta \partial x^2 - \frac{\alpha \partial x^2}{zz}$, ſeu $\partial x = \frac{z \partial z}{\sqrt{(\beta zz - \alpha)}}$, vnde porro fit $\beta x + \gamma = \sqrt{(\beta zz - \alpha)}$, ideoque

$$\beta zz = \alpha + \gamma\gamma + 2\beta\gamma x + \beta\beta xx,$$

quae eſt ipſa noſtra forma.

## Problema 113.

911. Sumto elemento $\partial x$ conſtante, inuenire formam generaliorem aequationum differentio-differentialium quae ope huiusmodi multiplicatoris $My\partial x + N\partial y$ integrabiles reddantur.

## Solutio.

Quia multiplicator ope ſubſtitutionis $y = Ru$ in formam ſimpliciſſimam $S\partial u$ transmutari poteſt, hac ſubſtitutione ipſa aequatio differentio-differentialis induat hanc formam

$$\partial\partial u + P\partial x \partial u + \frac{U \partial x^2}{S} = 0,$$

cuius poſtremum membrum per $S\partial u$ multiplicatum ſponte eſt integrabile, ſi quidem U denotet functionem quamcunque ipſius $u$, dum R, S et P ſint functiones ipſius $x$. Cum ergo aequatio

$$S\partial u \partial\partial u + PS\partial x \partial u^2 + U\partial x^2 \partial u = 0,$$

debeat eſſe integrabilis, poſito integrali

$$\tfrac{1}{2} S\partial u^2 + \partial x^2 \int U\partial u = \tfrac{1}{2} C\partial x^2,$$

neceſſe eſt ſit

$$\tfrac{1}{2}\partial S \partial u^2 = PS\partial x \partial u^2, \text{ ſeu } P\partial x = \frac{\partial S}{2S}.$$

Quocirca haec forma generalis

$$\partial\partial u + \frac{\partial S \partial u}{2S} + \frac{U\partial x^2}{S} = 0,$$

per $S\partial u$ multiplicata dabit integrale

$$S\partial u^2 = \partial x^2 (C - 2\int U\partial u),$$

quod denuo integratum praebet

$$\int \frac{\partial x}{\sqrt{S}} = \int \frac{\partial u}{\sqrt{(1 - 2\int U \partial u)}}.$$

Cum igitur haec fint manifesta, ponendo $u = \frac{y}{R}$ ad formas magis intricatas regrediamur, ita vt iam fit $U =$ funct. $\frac{y}{R}$. Nunc vero eft

$$\partial u = \frac{\partial y}{R} - \frac{y \partial R}{RR} \text{ et } \partial \partial u = \frac{\partial \partial y}{R} - \frac{2 \partial R \partial y}{RR} - \frac{y \partial \partial R}{RR} + \frac{2 y \partial R^2}{R^3},$$

vnde aequatio noftra fit

$$\frac{\partial \partial y}{R} - \frac{2 \partial R \partial y}{RR} - \frac{y \partial \partial R}{RR} + \frac{2 y \partial R^2}{R^3} + \frac{U \partial x^2}{S} = 0,$$
$$+ \frac{\partial S \partial y}{2 R S} \qquad - \frac{y \partial R \partial S}{2 RRS},$$

quae per $\frac{S}{RR}(R \partial y - y \partial R)$ multiplicata integrabilis redditur. Vt igitur ad formam fupra propofitam accedamus, ftatuamus $S = \alpha R^4$, et aequatio

$$\frac{\partial \partial y}{R} - \frac{y \partial \partial R}{RR} + \frac{U \partial x^2}{\alpha R^4} = 0$$

per $\alpha RR(R \partial y - y \partial R)$ multiplicata integrabilis redditur. Seu haec aequatio

$$R \partial \partial y - y \partial \partial R + \frac{\partial x^2}{RR} f : \frac{y}{R} = 0$$

per $R \partial y - y \partial R$ multiplicata fit integrabilis.

Vt via ad integrationem perueniendi magis occultetur, ponatur $f : \frac{y}{R} = \frac{\alpha y}{R} + V$, vt V fit functio homogenea nullius dimenfionis ipfarum $y$ et R, ac ponatur $y \partial \partial R = \frac{\alpha y \partial x^2}{R^3}$, vt fiat

$$R \partial \partial y + \frac{V \partial x^2}{RR} = 0, \text{ feu } \partial \partial y + \frac{V \partial x^2}{R^3} = 0,$$

quae multiplicatore $R(R \partial y - y \partial R)$ redditur integrabilis. At cum fit $\partial \partial R = \frac{\alpha \partial x^2}{R^3}$, erit vt fupra vidimus

$$R = \sqrt{(\alpha + 2 \beta x + \gamma x x)},$$

vnde dum V fit functio homogenea nullius dimenfionis ipfarum

rum $y$ et $R = \sqrt{(\alpha + 2\beta x + \gamma x x)}$, aequatio

$$\partial \partial y + \frac{V \partial x^2}{(\alpha + 2\beta x + \gamma x x)^{\frac{3}{2}}} = 0$$

ope multiplicatoris

$$(\alpha + 2\beta x + \gamma x x)\partial y - (\beta + \gamma x) y \partial x,$$

integrabilis euadit

## Corollarium 1.

912. Posito autem $R = \sqrt{(\alpha + 2\beta x + \gamma x x)}$, aequatio nostra per $R R \partial y - R y \partial R$ multiplicata fit

$$R R \partial y \partial \partial y - R y \partial R \partial \partial y + \frac{V \partial x^2 (R \partial y - y \partial R)}{R R} = 0,$$

cuius integrale est

$$\tfrac{1}{2} R R \partial y^2 - R y \partial R \partial y + \int y \partial y (R \partial \partial R + \partial R^2)$$
$$+ \partial x^2 \int V \partial \tfrac{y}{R} = \text{Const.}\ \partial x^2,$$

vbi est

$$R \partial \partial R + \partial R^2 = \partial . R \partial R = \partial . (\beta + \gamma x) \partial x = V \partial x^2,$$

sicque integrale est

$$R R \partial y^2 - 2 R y \partial R \partial y + \gamma y y \partial x^2 \int V \partial . \tfrac{y}{R} = \text{Const.}\ \partial x^2.$$

## Corollarium 2.

913. Quia V est functio ipsius $\frac{y}{R}$, formulae $\int V \partial . \frac{y}{R}$ integrale habetur. Pro vlteriore vero integratione posito $y = R u$ et $\int V \partial u = U$, habebitur

$$R^4 \partial u^2 - R R u u \partial R^2 + \gamma R R u u \partial x^2 + 2 U \partial x^2 = G \partial x^2,$$

seu $R^4 \partial u^2 = \partial x^2 [G - 2U + (\beta\beta - \alpha\gamma) u u]$,

hincque

$$\frac{\partial x}{\alpha + 2\beta x + \gamma x x} = \frac{\partial u}{\sqrt{[G - 2U + (\beta\beta - \alpha\gamma) u u]}},$$

ac porro $y = u \sqrt{(\alpha + 2\beta x + \gamma x x)}$.

 Scho-

## Scholion.

914. Haec ergo aequatio $\partial\partial y + \frac{V\partial x^2}{R^3} = 0$, existente $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, multo latius patet ea quam in praecedente problemate tractauimus, propterea quod hic pro V accipere licet functionem quamcunque homogeneam nullius dimensionis ipsarum $y$ et R. Si enim sumatur $V = \frac{AF^3 y}{(\alpha yy - RR)^2}$, ipsa aequatio primum tractata oritur. Caeterum ex methodo, qua illam aequationem elicuimus apparet, eam per restrictionem ad hanc formam occultam esse perductam, cum ea aequatio, vnde est nata

$$R\partial\partial y - y\partial\partial R + \frac{\partial x^2}{RR} f : \frac{y}{R} = 0$$

perspicue integrationem admittat, si per $R\partial y - y\partial R$ multiplicetur. Est enim

$$R\partial\partial y - y\partial\partial R = \partial.(R\partial y - y\partial R) \text{ et } \frac{R\partial y - y\partial R}{RR} = \partial.\frac{y}{R},$$

vnde facta multiplicatione habebimus

$$(R\partial y - y\partial R)\partial.(R\partial y - y\partial R) + \partial x^2 f : \frac{y}{R}\ \partial.\frac{y}{R} = 0,$$

cuius aequationis vtrumque membrum per se est integrabile. In aequatione autem inde eruta integrabilitas minus perspicitur, multo magis integratio est abscondita in aequationibus sequentibus.

## Problema 114.

915. Sumto elemento $\partial x$ constante, integrationem huius aequationis

$$yy\partial\partial y + y\partial y^2 + Ax\partial x^2 = 0$$

ope multiplicatoris eam integrabilem reddentis perficere.

## Solutio.

Hic frustra tentatur multiplicator huius formae $L\partial y + M\partial x$; tentetur ergo haec forma

$$3L\partial y^2 + 2M\partial x\partial y + N\partial x^2$$

ac

ac ponatur producti integrale

$$Lyy\partial y^3 + Myy\partial x\partial y^2 + Nyy\partial x^2\partial y + V\partial x^3 = C\partial x^3$$

cuius differentiatio perducit ad hanc aequationem

$$\begin{aligned}\partial x^3\partial V = & 3Ly\partial y^4 + 2My\partial x\partial y^3 + Ny\partial x^2\partial y^2 + 2AMx\partial x^3\partial y + ANx\partial x^4 \\ & -2Ly\partial y^4 - yy\partial x\partial y^3(\tfrac{\partial L}{\partial x}) + 3ALx\partial x^2\partial y^2 - yy\partial x^3\partial y(\tfrac{\partial N}{\partial x}) \\ & -yy\partial y^4(\tfrac{\partial L}{\partial y}) - 2My\partial x\partial y^3 - yy\partial x^2\partial y^2(\tfrac{\partial M}{\partial x}) \\ & -yy\partial x\partial y^3(\tfrac{\partial M}{\partial y}) - 2Ny\partial x^2\partial y^2 \\ & -yy\partial x^2\partial y^2(\tfrac{\partial N}{\partial y})\end{aligned}$$

quae formula vt integrationem admittat, membra quae $\partial y^4$, $\partial y^3$ et $\partial y^2$ continent, euanescere debent: vnde primo colligitur $L - y(\frac{\partial L}{\partial y}) = 0$, vbi $(\frac{\partial L}{\partial y})$ nascitur ex differentiatione ipsius L posito $x$ constante. Consideretur ergo $x$ vt quantitas constans, eritque $\frac{\partial L}{L} = \frac{\partial y}{y}$, ideoque $L = yf:x$. Negligamus autem hanc functionem ipsius $x$, seu eius loco vnitatem sumamus, vt fit $L = y$ et $(\frac{\partial L}{\partial x}) = 0$: secundo ergo esse debet $(\frac{\partial M}{\partial y}) = 0$. Sumamus igitur $M = 0$, etiamsi M denotare posset functionem quamcunque ipsius $x$, quandoquidem videbimus hoc modo negotium confici posse. Tertio itaque habebimus

$$-Ny + 3Axy - yy(\tfrac{\partial N}{\partial y}) = 0;$$

sumto ergo $x$ constante, erit $3Ax\partial y = N\partial y + y\partial N$, ideoque $Ny = 3Axy$ seu $N = 3Ax$, vbi iterum functionem ipsius $x$, quae loco constantis ingrederetur, negligimus. Cum igitur hactenus inuenimus $L = y$, $M = 0$ et $N = 3Ax$, erit $\partial V = -3Ayy\partial y + 3AAxx\partial x$, quae formula cum sponte sit integrabilis scilicet $V = -Ay^3 + AAx^3$, multiplicator nostram aequationem integrabilem reddens erit $= 3y\partial y^2 + 3Ax\partial x^2$, et producti integrale habebitur

$$y^3\partial y^3 + 3Axyy\partial x^2\partial y - Ay^3\partial x^3 + AAx^3\partial x^3 = C\partial x^3$$

quod ob constantem C est integrale completum.

Corol-

## Corollarium 1.

916. Huius integralis membrum primum commode in tres factores resolui potest. Si ponantur formulae $z^3 - A$ factores $(z-\alpha)(z-\beta)(z-\gamma)$, vt sit

$$\alpha = \sqrt[3]{A},\ \beta = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A} \text{ et } \gamma = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A},$$

erit integrale inuentum

$$\left(\frac{y\partial y}{\partial x} - \alpha y + \alpha\alpha x\right)\left(\frac{y\partial y}{\partial x} - \beta y + \beta\beta x\right)\left(\frac{y\partial y}{\partial x} - \gamma y + \gamma\gamma x\right) = C,$$

existente

$$\alpha+\beta+\gamma=0,\ \alpha\beta+\alpha\gamma+\beta\gamma=0 \text{ et } \alpha\beta\gamma=1.$$

Posito enim $\frac{y\partial y}{\partial x} = z$, habetur haec forma

$$z^3 + 3Axyz - Ay^3 + AAx^3,$$

cuius factor si ponatur $z - p - q$, fit

$$z^3 - 3pqz - p^3 - q^3 = 0,$$

ideoque

$$p = y\sqrt[3]{A} \text{ et } q = -x\sqrt[3]{A^2}.$$

## Corollarium 2.

917. Sumto ergo constante $C=0$, tria obtinentur integralia particularia

$$y\partial y - \alpha y\partial x + \alpha\alpha x\partial x = 0,$$

et loco $\alpha$ scribendo $\beta$ et $\gamma$,

$$y\partial y - \beta y\partial x + \beta\beta x\partial x = 0 \text{ et}$$

$$y\partial y - \gamma y\partial x + \gamma\gamma x\partial x = 0,$$

quae posito $y = ux$ dant $\frac{\partial x}{x} = \frac{-u\partial u}{uu - \alpha u + \alpha\alpha}$, et porro integrando

$$lx = l\frac{\alpha}{\sqrt{(\alpha\alpha - \alpha u + uu)}} - \frac{1}{\sqrt{3}}\text{Ang. tang.}\frac{u\sqrt{3}}{2\alpha - u} + \text{Const.}$$

Scho-

## Scholion 1.

918. Aequationem autem differentialem primi ordinis inuentam difficile eſt denuo integrare. A poteſtatibus quidem differentialium, ponendo $\partial y = p\partial x$ et $y = ux$, vnde fit $\frac{\partial x}{x} = \frac{\partial u}{p-u}$, liberari poteſt, prodit enim

$$x^3(u^3p^3 + 3Auup - Au^3 + AA) = C,$$

quae ſumtis logarithmis, differentiata dat

$$\frac{\partial x}{x} + \frac{uu\partial p(upp+A) + u\partial u(up^3 + 2Ap - Au)}{u^3p^3 + 3Auup - Au^3 + AA} = 0,$$

quae loco $\frac{\partial x}{x}$ ſcripto $\frac{\partial u}{p-u}$, abit in

$$\partial u(upp+A)^2 + uu(p-u)\partial p(upp+A) = 0,$$

ac per $upp + A$ diuidendo, oritur

$$A\partial u + upp\partial u + puu\partial p - u^3\partial p = 0,$$

quae ponendo $p = \frac{q}{u}$ aliquanto fit ſimplicior, ſcilicet

$$A\partial u + q\partial q + qu\partial u - uu\partial q = 0,$$

cui autem poſito $A = m^3$, etſi particulariter ſatisfacit $q = mu - mm$, tamen inde integratio completa erui vix poſſe videtur. Caeterum eadem haec aequatio inter $p$ et $u$ immediate elicitur ex aequatione differentio-differentiali propoſita, quoniam in ea binae variabiles $x$ et $y$ vbique eundem dimenſionum numerum conſtituunt. Poſito enim $\partial y = p\partial x$ et $y = ux$, abit ea in

$$uux\partial p + upp\partial x + A\partial x = 0, \text{ ſeu } \frac{\partial x}{x} = \frac{-uu\partial p}{A+upp} = \frac{\partial u}{p-u},$$

quae eſt ipſa praecedens aequatio.

## Scholion 2.

919. Interim tamen aequatio propoſita complete integrari poteſt, indeque etiam eae, quas ex ea elicuimus. Hoc autem prorſus ſingulari ratione praeſtatur, aequationem illam adeo ad differentialia tertii ordinis euehendo. Cum enim ſit

$$y\,\partial.\frac{y\,\partial y}{\partial x}+A\,x\,\partial x=0,$$

statuatur $\frac{\partial x}{y}=\partial v$, vt fiat

$$y\,\partial.\frac{\partial y}{\partial v}+A\,x\,\partial x=0,\quad \text{seu}\quad \partial.\frac{\partial y}{\partial v}+A\,x\,\partial v=0,$$

quae sumto elemento $\partial v$ constante, denuo differentiata praebet

$$\frac{\partial^2 y}{\partial v}+A\,\partial x\,\partial v=0,\quad \text{seu}\quad \partial^3 y+A\,y\,\partial v^3=0:$$

quae forma ita est comparata, vt si ei particulariter satisfaciant $y=P$, $y=Q$, $y=R$, etiam satisfaciat $y=DP+EQ+FR$. Iam vero illi satisfacit $y=e^{-\alpha v}$, si fuerit $\alpha^3=A$; cum igitur in Coroll. 1. ternae litterae $\alpha$, $\beta$, $\gamma$ eadem conditione sint praeditae, habebitur integrale completum

$$y=D\,e^{-\alpha v}+E\,e^{-\beta v}+F\,e^{-\gamma v};$$

vnde ob $A\,x=-\frac{\partial\partial y}{\partial v^2}$, erit

$$x=\frac{-D\,\alpha\alpha\,e^{-\alpha v}-E\,\beta\beta\,e^{-\beta v}-F\,\gamma\gamma\,e^{-\gamma v}}{A}:$$

seu mutatis constantibus, ob $A=\alpha^3=\beta^3=\gamma^3$,

$$x=+\mathfrak{A}\,e^{-\alpha v}+\mathfrak{B}\,e^{-\beta v}+\mathfrak{C}\,e^{-\gamma v}$$

$$y=-\mathfrak{A}\,\alpha\,e^{-\alpha v}-\mathfrak{B}\,\beta\,e^{-\beta v}-\mathfrak{C}\,\gamma\,e^{-\gamma v}.$$

Hinc ergo aequationis

$$A\,\partial u+q\,\partial q+q\,u\,\partial u-u\,u\,\partial q=0$$

integrale completum his formulis continetur

$$u=\frac{-\mathfrak{A}\,\alpha\,e^{-\alpha v}-\mathfrak{B}\,\beta\,e^{-\beta v}-\mathfrak{C}\,\gamma\,e^{-\gamma v}}{\mathfrak{A}\,e^{-\alpha v}+\mathfrak{B}\,e^{-\beta v}+\mathfrak{C}\,e^{-\gamma v}}\quad \text{et}$$

$$q=\frac{\mathfrak{A}\,\alpha\,\alpha\,e^{-\alpha v}+\mathfrak{B}\,\beta\,\beta\,e^{-\beta v}+\mathfrak{C}\,\gamma\,\gamma\,e^{-\gamma v}}{\mathfrak{A}\,e^{-\alpha v}+\mathfrak{B}\,e^{-\beta v}+\mathfrak{C}\,e^{-\gamma v}},$$

ob $q=p\,u=\frac{y\,\partial y}{x\,\partial x}=\frac{\partial y}{x\,\partial v}$, quod insigne est specimen integrationis methodo directa vix perficiendae.

Pro-

## Problema 115.

920. Sumto elemento $\partial x$ conftante, fi proponatur haec aequatio $2y^3 \partial\partial y + yy\partial y^2 + X\partial x^2 = 0$, exiftente $X = \alpha + \beta x + \gamma xx$, inuenire multiplicatorem, qui eam integrabilem reddat.

## Solutio.

Hic fruftra tentantur multiplicatores formae

$$L\partial y + M\partial x \text{ et } L\partial y^2 + M\partial x\partial y + N\partial x^2;$$

fumamus ergo multiplicatorem huius formae

$$2L\partial y^3 + M\partial x^2\partial y + N\partial x^3,$$

et integrale ftatuatur

$$Ly^3\partial y^4 + My^3\partial x^2\partial y^2 + 2Ny^3\partial x^3\partial y + S\partial x^4 = 0,$$

vnde per differentiationem colligitur

$$\begin{aligned}\partial x^4\partial S = {} & 2Lyy\partial y^5 + Myy\partial x^2\partial y^3 + Nyy\partial x^3\partial y^2 + MX\partial x^4\partial y + NX\partial x^5 \\ & -3Lyy\partial y^5 + 2LX\partial x^2\partial y^3 - y^3\partial x^3\partial y^2\left(\frac{\partial M}{\partial x}\right) - 2y^3\partial x^4\partial y\left(\frac{\partial N}{\partial x}\right) \\ & -y^3\partial y^5\left(\frac{\partial L}{\partial y}\right) - 3Myy\partial x^2\partial y^3 - 6Nyy\partial x^3\partial y^2 \\ & -y^3\partial x^2\partial y^3\left(\frac{\partial M}{\partial y}\right) - 2y^3\partial x^3\partial y^2\left(\frac{\partial N}{\partial y}\right)\end{aligned}$$

vbi fumimus L effe functionem ipfius $y$ tantum. Vt ergo termini $\partial y^5$ continentes deftruantur, erit

$$-L - \frac{y\partial L}{\partial y} = 0 \text{ et } L = \frac{1}{y}.$$

Deinde pro deftructione terminorum per $\partial y^3$ affectorum erit

$$-2Myy + \frac{2X}{y} - y^3\left(\frac{\partial M}{\partial y}\right) = 0,$$

et fumto $x$ conftante

$$\partial M + \frac{2M\partial y}{y} = \frac{2X\partial y}{y^4},$$

quae per $yy$ multiplicata et integrata praebet

$$Myy = P - \frac{2X}{y} \text{ et } M = \frac{P}{yy} - \frac{2X}{y^3}$$

 deno-

denotante P functionem quamcunque ipsius $x$. Iam ad terminos $\partial y^2$ tollendos, erit

$$-5\,Nyy - y\frac{\partial P}{\partial x} + \frac{2\partial X}{\partial x} - 2y^3\left(\frac{\partial N}{\partial y}\right) = 0,$$

et sumto $x$ constante

$$2y^3\partial N + 5\,Nyy\,\partial y = \frac{2\partial X}{\partial x}\partial y - \frac{\partial P}{\partial x}.y\,\partial y,$$

quae per $\sqrt{y}$ diuisa et integrata dat

$$2\,N\,y^{\frac{5}{2}} = \frac{4\partial X}{\partial x}\sqrt{y} - \frac{2\partial P}{3\partial x}y\sqrt{y},$$

neglecta functione ipsius $x$ addenda, quoniam irrationalitas $\sqrt{y}$ in calculum non ingreditur. Erit ergo $N = \frac{2\partial X}{yy\partial x} - \frac{\partial P}{3y\partial x}$, ac propterea

$$\partial S = \partial y\left(\frac{PX}{yy} - \frac{2XX}{y^3} - \frac{4y\partial\partial X}{\partial x^2} + \frac{2yy\partial\partial P}{3\partial x^2}\right) + \frac{2X\partial X}{yy} - \frac{X\partial P}{3y},$$

vnde fit integrando

$$S = \frac{XX}{yy} - \frac{PX}{y} - \frac{2yy\partial\partial X}{\partial x^2} + \frac{2y^3\partial\partial P}{9\partial x^2}$$
$$+\int\left(\frac{P\partial X}{y} + \frac{2X\partial P}{3y} + \frac{2yy\partial^3 X}{\partial x^3} - \frac{2y^3\partial^3 P}{9\partial x^3}\right),$$

quae finite exprimetur si $P = 0$, cum ob $X = \alpha + \beta x + \gamma xx$ fit $\partial^3 X = 0$. Quocirca habemus

$$L = \frac{1}{y},\ M = -\frac{2X}{y^3},\ \text{et}\ N = \frac{2\partial X}{yy\partial x},$$

atque

$$S = \frac{XX}{yy} - \frac{2yy\partial\partial X}{\partial x^2} + \text{Const.}$$

vnde aequatio integralis est

$$y^2\partial y^4 - 2X\partial x^2\partial y^2 + 4y\partial X\partial x^2\partial y + \frac{XX\partial x^4}{yy} - 2yy\partial x^2\partial\partial X = C\partial x^4.$$

Aequatio ergo proposita

$$2y^3\partial\partial y + yy\,\partial y^2 + \partial x^2(\alpha + \beta x + \gamma xx) = 0,$$

integrabilis redditur multiplicata per

$$\frac{2\partial y^3}{y} - \frac{2(\alpha + \beta x + \gamma xx)\partial x^2\partial y}{y^3} + \frac{2\partial x^3(\beta + 2\gamma x)}{yy},$$

tum vero est integrale

$$y^2 \partial y^4 - 2 \partial x^2 \partial y^2 (\alpha + \beta x + \gamma x x) + 4 y \partial x^3 \partial y (\beta + 2 \gamma x) - 4 \gamma y y \partial x^4 + \frac{(\alpha + \beta x + \gamma x x)^2 \partial x^4}{y y} = C \partial x^4,$$

seu

$$[y y \partial y^2 - (\alpha + \beta x + \gamma x x) \partial x^2]^2 + 4 y^3 \partial x^3 \partial y (\beta + 2 \gamma x), - 4 \gamma y^4 \partial x^4 = C y y \partial x^4.$$

## Scholion 1.

921. Integrale hoc ita eſt intricatum, vt alia methodo vix inueniri potuiſſe videatur, verum etiam ita eſt comparatum, vt nulla pateat methodus id porro integrandi, vnde prima integratio parum lucri attuliſſe eſt iudicanda. Quemadmodum autem in praecedente problemate integrale completum ex alio fonte hauſimus, ita hic ſimili modo integrale eruere licet, quod eo magis eſt notatu dignum, cum aequatio propoſita in ſe ſpectata ſolutu ſit difficillima. Ponamus ſcilicet itidem $\partial x = y \partial v$, et cum ſit

$$\partial \partial y = \partial x \partial . \frac{\partial y}{\partial x} = y \partial v \partial . \frac{\partial y}{y \partial v},$$

erit ſumendo iam elementum $\partial v$ conſtans

$$\partial \partial y = y \partial v (\frac{\partial \partial y}{y \partial v} - \frac{\partial y^2}{y y \partial v}) = \partial \partial y - \frac{\partial y^2}{y}.$$

Hinc noſtra aequatio induit hanc formam

$$2 y^3 \partial \partial y - y y \partial y^2 + y y \partial v^2 (\alpha + \beta x + \gamma x x) = 0, \text{ ſeu}$$

$$2 y \partial \partial y - \partial y^2 + \partial v^2 (\alpha + \beta x + \gamma x x) = 0,$$

quae denuo differentiata praebet

$$2 y \partial^3 y + y \partial v^3 (\beta + 2 \gamma x) = 0, \text{ ſeu}$$

$$2 \partial^3 y + \partial v^3 (\beta + 2 \gamma x) = 0,$$

differentietur iterum, prodibitque

$$2 \partial^4 y + 2 \gamma y \partial v^4 = 0, \text{ ſeu } \partial^4 y + \gamma y \partial v^4 = 0,$$

quam aequationem ſi aliunde reſoluere, valoremque ipſius $y$

per $v$ exprimere liceat, erit $x = \int y \partial v$, seu sine integratione $x = -\frac{\partial^3 y}{\gamma \partial v^3} - \frac{\beta}{2\gamma}$. At manifestum est, isti aequationi differentiali quarti ordinis satisfacere $y = e^{\lambda v}$, si sit $\lambda^4 + \gamma = 0$. Ponamus ergo $\gamma = -n^4$, et quatuor ipsius $\lambda$ habebuntur valores $\lambda = \pm n$ et $\lambda = \pm n \sqrt{-1}$, vnde eius integrale completum est

$$y = A e^{nv} + B e^{-nv} + C \sin. (nv + \zeta),$$

hincque

$$x = + \frac{A}{n} e^{nv} - \frac{B}{n} e^{-nv} - \frac{C}{n} \cos. (nv + \zeta) + \frac{\beta}{2n^4},$$

qui ergo valores quoque satisfacient aequationi inter $x$ et $y$ propositae, dummodo constantes A, B, C, et $\zeta$ ita a se pendentes capiantur, vt quantitati quoque $\alpha$ conueniant. His nempe valoribus substitutis fieri debet

$$\alpha + \beta x - n^4 x x + \frac{2 y \partial\partial y - \partial y^2}{\partial v^2} = 0;$$

vbi tantum terminos constantes considerasse sufficit, quibus accenseri debent ii, qui quadratum sinus cosinusue anguli $nv + \zeta$ continent, quippe ex quorum combinatione quantitas constans exsurgit. Cum ergo sit

$$2y = 2 A e^{nv} + 2 B e^{-nv} + 2 C \sin. (nv + \zeta),$$

$$\frac{\partial\partial y}{\partial v^2} = nn A e^{nv} + nn B e^{-nv} - nn C \sin. (nv + \zeta),$$

$$\frac{\partial y}{\partial v} = n A e^{nv} - n B e^{-nv} + n C \cos. (nv + \zeta),$$

$$x = \frac{A}{n} e^{nv} - \frac{B}{n} e^{-nv} - \frac{C}{n} \sin. (nv + \zeta) + \frac{\beta}{2n^4},$$

erit sumtis terminis memoratis

$$\beta x = \frac{\beta\beta}{2n^4}$$

$$-n^4 x x = 2nn AB - nn CC \cos. (nv + \zeta)^2 - \frac{\beta\beta}{4n^4},$$

$$\frac{2y\partial\partial y}{\partial v^2} = 4nn AB - 2nn CC \sin. (nv + \zeta)^2,$$

$$-\frac{\partial y^2}{\partial v^2} = 2nn AB - nn CC \cos. (nv + \zeta)^2,$$

ergo

ergo

$$\alpha + 8nnAB - 2nnCC + \frac{\beta\beta}{4n^4} = 0, \text{ ideoque}$$

$$C = \surd\left(\frac{\alpha}{2nn} + \frac{\beta\beta}{8n^6} + 4AB\right), \text{ vel}$$

$$\alpha = 2nn(CC - 4AB) - \frac{\beta\beta}{4n^4} \text{ et}$$

$$\alpha + \beta x + \gamma xx = 2nn(CC - AB) - \left(\frac{\beta}{2nn} - nnx\right)^2.$$

Manent ergo tres constantes A, B, et $\zeta$ indeterminatae, ita vt nullum sit dubium, quin formulae pro $x$ et $y$ datae integrale completum exhibeant.

## Scholion 2.

922. Aequationes differentio-differentiales, quas in his duobus prob ematibus tractauimus, ad similem formam reduci possunt. Prior enim

$$y(y\partial\partial y + \partial y^2) + X\partial x^2 = 0,$$

existente $X = Ax$ vel $X = \alpha + \beta x$, si ponatur $y\partial y = \frac{1}{2}\partial z$ seu $yy = z$, induit hanc formam

$$\tfrac{1}{2}\partial\partial z\surd z + X\partial x^2 = 0,$$

quae ope multiplicatoris $\frac{3\partial z^2}{4\surd z} + 3X\partial x^2$ integrabilis redditur. Altera vero aequatio

$$yy(2y\partial\partial y + \partial y^2) + X\partial x^2 = 0,$$

existente $X = \alpha + \beta x + \gamma xx$, posito $y = z^{\frac{2}{3}}$, fit

$$\partial y = \tfrac{2}{3}z^{-\frac{1}{3}}\partial z \text{ et } \partial\partial y = \tfrac{2}{3}z^{-\frac{1}{3}}\partial\partial z - \tfrac{2}{9}z^{-\frac{4}{3}}\partial z^2,$$

hinc $2y\partial\partial y + \partial y^2 = \frac{4}{3}z^{\frac{1}{3}}\partial\partial z$, sicque aequatio hanc induit formam $\frac{4}{3}z^{\frac{5}{3}}\partial\partial z + X\partial x^2 = 0$, quae integrabilis redditur ope huius multiplicatoris

$$\frac{16\partial z^3}{27 z^{\frac{5}{3}}} - \frac{4X\partial x^2\partial z}{3 z^{\frac{7}{3}}} + \frac{2\partial X\partial x^2}{z^{\frac{4}{3}}}.$$

Hinc colligimus, pro aequatione $\partial\partial z + \frac{2X\partial x^2}{\sqrt{z}} = 0$ fore multiplicatorem $\partial z^2 + X\partial x^2\sqrt{z}$, pro aequatione autem

$$\partial\partial z + \frac{3X\partial x^2}{4z\sqrt[3]{zz}} = 0$$

multiplicatorem fore

$$\partial z^3 - \frac{9X\partial x^2\partial z}{4\sqrt[3]{zz}} + \tfrac{27}{8}\partial X\partial x^2\sqrt[3]{z},$$

seu sub vno conspectu

| pro aequatione | multiplicator erit |
|---|---|
| $\partial\partial z + \frac{X\partial x^2}{\sqrt{z}} = 0$ | $\partial z^2 + 2X\partial x^2\sqrt{z}$, |
| $\partial\partial z + \frac{X\partial x^2}{z\sqrt[3]{zz}} = 0$ | $\partial z^3 + \frac{3X\partial x^2\partial z}{\sqrt[3]{zz}} + \tfrac{9}{2}\partial X\partial x^2\sqrt[3]{z}$. |

Caeterum hae integrationes maxime sunt notatu dignae, cum ex aequationibus differentialibus altioribus perfici queant. Ita cum ex hac aequatione, vbi $\partial v$ constans

$$\partial^3 y + A\partial v\partial\partial y + B\partial v^2\partial y + Cy\partial v^3 = 0 \text{ fit}$$

$$y = \mathfrak{A}e^{\alpha v} + \mathfrak{B}e^{\beta v} + \mathfrak{C}e^{\gamma v},$$

si fuerint $\alpha$, $\beta$, $\gamma$, radices huius aequationis

$$r^3 + Ar^2 + Br + C = 0,$$

ponamus $\partial v = \frac{\partial x}{y}$, et cum sit

$$\partial\partial y = \partial v\partial\frac{\partial y}{\partial v} = \frac{\partial x}{y}\partial.\frac{y\partial y}{\partial x} \text{ et}$$

$$\partial^3 y = \partial v^2\partial.\frac{\partial\partial y}{\partial v^2} = \partial v^2\partial.\left(\frac{1}{\partial v}\partial.\frac{\partial y}{\partial v}\right) = \frac{\partial x^2}{yy}\partial.\left(\frac{y}{\partial x}\partial.\frac{y\partial y}{\partial x}\right),$$

si iam $\partial x$ constans sumamus, erit

$$\partial\partial y = \partial\partial y + \frac{\partial y^2}{y} \text{ et}$$

$\partial^3 y$

$$\partial^3 y = \frac{1}{yy}\,\partial . y\,(y\,\partial\partial y + \partial y^2) = \partial^3 y + \frac{4\,\partial y\,\partial\partial y}{y} + \frac{\partial y^3}{yy},$$

hincque per $yy$ multiplicando

$$yy\,\partial^3 y + 4y\,\partial y\,\partial\partial y + \partial y^3 + A\,\partial x\,(y\,\partial\partial y + \partial y^2)$$
$$+ B\,\partial x^2\,\partial y + C\,\partial x^3 = 0,$$

quae integrata dat

$$yy\,\partial\partial y + y\,\partial y^2 + Ay\,\partial x\,\partial y + By\,\partial x^2 + (Cx + D)\,\partial x^2 = 0,$$

quae ergo per superiora integrari potest.

## Problema 116.

923. Definire conditiones functionum P, Q, R et L, M, N, vt haec aequatio differentio-differentialis

$$\partial\partial y + P\,\partial y^2 + Q\,\partial x\,\partial y + R\,\partial x^2 = 0$$

integrabilis reddatur multiplicatore

$$3L\,\partial y^2 + 2M\,\partial x\,\partial y + N\,\partial x^2.$$

## Solutio.

Facta multiplicatione integratio terminorum per $\partial\partial y$ affectorum dat

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y,$$

quare ponatur integrale

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y + V\,\partial x^3 = C\,\partial x^3,$$

cuius differentiale aequari debet formulae propositae in multiplicatorem ductae, vnde oritur

$$\begin{array}{lllll}
\partial x^3\,\partial V = 3LP . \partial y^4 & + 3LQ . \partial x\,\partial y^3 & + 3LR . \partial x^2\,\partial y^2 & & \\
 & + 2MP & + 2MQ & + 2MR\,\partial x^3\,\partial y & \\
 - \left(\frac{\partial L}{\partial y}\right) & - \left(\frac{\partial L}{\partial x}\right) & + NP & + NQ & + NR\,\partial x^4 \\
 & - \left(\frac{\partial M}{\partial y}\right) & - \left(\frac{\partial M}{\partial x}\right) & - \left(\frac{\partial N}{\partial x}\right) & \\
 & & - \left(\frac{\partial N}{\partial y}\right). & &
\end{array}$$

 Hic

Hic ergo fieri oportet

$$3LP - \left(\frac{\partial L}{\partial y}\right) = 0,$$

$$3LQ + 2MP - \left(\frac{\partial L}{\partial x}\right) - \left(\frac{\partial M}{\partial y}\right) = 0,$$

$$3LR + 2MQ + NP - \left(\frac{\partial M}{\partial x}\right) - \left(\frac{\partial N}{\partial y}\right) = 0.$$

Tum vero erit

$$\partial V = \left[2MR + NQ - \left(\frac{\partial N}{\partial x}\right)\right]\partial y + NR\,\partial x,$$

quae formula integrabilis esse debet. Ex illis autem aequationibus colligitur

$$P = \frac{1}{3L}\left(\frac{\partial L}{\partial y}\right),\quad Q = \frac{1}{3L}\left(\frac{\partial L}{\partial x}\right) + \frac{1}{3L}\left(\frac{\partial M}{\partial y}\right) - \frac{2M}{9LL}\left(\frac{\partial L}{\partial y}\right) \text{ et}$$

$$R = \frac{1}{3L}\left(\frac{\partial M}{\partial x}\right) + \frac{1}{3L}\left(\frac{\partial N}{\partial y}\right) - \frac{N}{9LL}\left(\frac{\partial L}{\partial y}\right) - \frac{2M}{9LL}\left(\frac{\partial L}{\partial x}\right) - \frac{2M}{9LL}\left(\frac{\partial M}{\partial y}\right) + \frac{4MM}{27L^3}\left(\frac{\partial L}{\partial y}\right).$$

## Corollarium 1.

924. Si L, M, et N fuerint functiones ipsius $x$ tantum, erit $P = 0$, $Q = \frac{\partial L}{3L\partial x}$ et $R = \frac{\partial M}{3L\partial x} - \frac{2M\partial L}{9LL\partial x}$, hinc

$$\partial V = \left(\frac{2M\partial M}{3L\partial x} - \frac{4MM\partial L}{9LL\partial x} + \frac{N\partial L}{3L\partial x} - \frac{\partial N}{\partial x}\right)\partial y + \frac{N\partial M}{3L} - \frac{2MN\partial L}{9LL},$$

ac coefficiens ipsius $\partial y$ debet esse constans. Quare per $L^{\frac{1}{3}}$ diuidendo habebitur

$$\frac{C\partial x}{\sqrt[3]{L}} = \frac{2M\partial M}{3L\sqrt[3]{L}} - \frac{4MM\partial L}{9LL\sqrt[3]{L}} + \frac{N\partial L}{3L\sqrt[3]{L}} - \frac{\partial N}{\sqrt[3]{L}},$$

et integrando

$$C\int\frac{\partial x}{\sqrt[3]{L}} = \frac{MM}{3L\sqrt[3]{L}} - \frac{N}{\sqrt[3]{L}},\ \text{seu}\ N = \frac{MM}{3L} - CL^{\frac{1}{3}}\int\frac{\partial x}{\sqrt[3]{L}},$$

ergo

$$V = Cy + \int\left(\frac{\partial M}{3L} - \frac{2M\partial L}{9LL}\right)\left(\frac{MM}{3L} - CL^{\frac{1}{3}}\int\frac{\partial x}{\sqrt[3]{L}}\right).$$

Coro-

## Corollarium 2.

925. Sit $M = S \sqrt[3]{L^2}$, erit

$$\partial M = \partial S \sqrt[3]{L^2} + \frac{2 S \partial L}{3 \sqrt[3]{L}}, \text{ et}$$

$$V = Cy + \tfrac{1}{3} \int \frac{\partial S}{\sqrt[3]{L}} (\tfrac{1}{3} S S \sqrt{L} - C L^{\frac{1}{3}} \int \frac{\partial x}{\sqrt[3]{L}}), \text{ seu}$$

$$V = Cy + \tfrac{1}{27} S^3 - \tfrac{1}{3} C \int \partial S \int \frac{\partial x}{\sqrt[3]{L}},$$

tum vero

$$N = \tfrac{1}{3} S S \sqrt[3]{L} - C L^{\frac{1}{3}} \int \frac{\partial x}{\sqrt[3]{L}} = (\tfrac{1}{3} S S - C \int \frac{\partial x}{\sqrt[3]{L}}) \sqrt[3]{L},$$

atque $P = 0$, $Q = \frac{\partial L}{3 L \partial x}$ et $R = \frac{\partial S}{3 \partial x \sqrt[3]{L}}$. Quare haec aequatio

$$\partial \partial y + \frac{\partial L \partial y}{3 L} + \frac{\partial S \partial x}{3 \sqrt[3]{L}} = 0$$

integrabilis redditur multiplicatore

$$3 L \partial y^2 + 2 S \partial x \partial y \sqrt[3]{L^2} + \partial x^2 (\tfrac{1}{3} S S - C \int \frac{\partial x}{\sqrt[3]{L}}) \sqrt[3]{L};$$

et integrale est

$$L \partial y^3 + S \partial x \partial y^2 \sqrt[3]{L L} + \partial x^2 \partial y (\tfrac{1}{3} S S - C \int \frac{\partial x}{\sqrt[3]{L}}) \sqrt[3]{L} + C y \partial x^3$$
$$+ \tfrac{1}{27} S^3 \partial x^3 - \tfrac{1}{3} C \partial x^3 \int \partial S \int \frac{\partial x}{\sqrt[3]{L}} = 0.$$

## Corollarium 3.

926. Hic quidquid pro constante C assumatur, idem integrale prodire debet. Hinc si $C = 0$, aequationis

$$\partial \partial y + \frac{\partial L \partial y}{3 L} + \frac{\partial S \partial x}{3 \sqrt[3]{L}} = 0$$

multiplicator erit

$$3\,L\,\partial y^2 + 2\,S\,\partial x\,\partial y\,\sqrt[3]{L^2} + \tfrac{1}{3}\,S\,S\,\partial x^2\,\sqrt[3]{L},$$

et integrale

$$L\,\partial y^3 + S\,\partial x\,\partial y^2\,\sqrt[3]{LL} + \tfrac{1}{3}\,S\,S\,\partial x^2\,\partial y\,\sqrt[3]{L} + \tfrac{1}{27}\,S^3\,\partial x^3 = D\,\partial x^6,$$

seu $(\partial y\,\sqrt[3]{L} + \tfrac{1}{3}\,S\,\partial x)^3 = D\,\partial x^3$.

## Scholion 1.

927. Ex iisdem quoque conditionibus, si dentur functiones P, Q et R, definiri poterunt functiones L, M, N, quatenus quidem postrema conditio integrabilitatis patitur. Veluti si sit $P = \frac{n}{y}$, $Q = 0$ et R functio ipsius $x$ tantum, puta $R = X$, vt habeatur haec aequatio

$$\partial\partial y + \frac{n\,\partial y^2}{y} + X\,\partial x^2 = 0,$$

cuius multiplicator si sumatur

$$3\,L\,\partial y^2 + 2\,M\,\partial x\,\partial y + N\,\partial x^2,$$

vt integrale sit

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y + V\,\partial x^3 = C\,\partial x^3,$$

erit primo $\frac{3nL}{y} - \left(\frac{\partial L}{\partial y}\right) = 0$, et sumta $x$ constante $\frac{\partial L}{L} = \frac{3n\,\partial y}{y}$, hinc $L = S\,y^{3n}$, denotante S functionem ipsius $x$. Deinde est

$$\frac{2nM}{y} - y^{3n}\,\frac{\partial S}{\partial x} - \left(\frac{\partial M}{\partial y}\right) = 0,$$

et sumta $x$ constante

$$\partial M - \frac{2nM\,\partial y}{y} + \frac{\partial S}{\partial x}\,.\,y^{3n}\,\partial y = 0,$$

quae per $y^{-2n}$ multiplicata et integrata dat

$$y^{-2n}\,M + \frac{\partial S}{(n+1)\,\partial x}\,y^{n+1} = T \text{ funct. ipsius } x.$$

Ergo

$$M = T\,y^{2n} - \frac{\partial S}{(n+1)\,\partial x}\,y^{3n+1}.$$

Ter-

Tertio fieri debet

$$3\,S\,X y^{3n} + \frac{nN}{y} - \frac{\partial T}{\partial x}\,.\,y^{2n} + \frac{\partial\partial S}{(n+1)\partial x^2}\,.\,y^{3n+1} - \frac{\partial N}{\partial y} = 0,$$

vnde fumta $x$ conftante

$$\partial N + \frac{nN\partial y}{y} + \frac{\partial T}{\partial x}\,.\,y^{2n}\partial y - \frac{\partial\partial S}{(n+1)\partial x^2}\,.\,y^{3n+1}\partial y - 3\,S\,X y^{3n}\partial y = 0,$$

quae per $y^{-n}$ multiplicata et integrata dat

$$y^{-n}N + \frac{\partial T}{(n+1)\partial x} y^{n+1} - \frac{\partial\partial S}{2(n+1)^2\partial x^2} y^{2n+2} - \frac{3SX}{2n+1} y^{2n+1} = Uf:x,$$

feu

$$N = U y^n - \frac{\partial T}{(n+1)\partial x} y^{2n+1} + \frac{\partial\partial S}{2(n+1)^2\partial x^2} y^{3n+2} + \frac{3SX}{2n+1} y^{3n+1}.$$

Ex his autem fit

$$\partial V = \partial y \left\{ \begin{array}{l} 2\,T X y^{2n} - \frac{2X\partial S}{(n+1)\partial x} y^{3n+1} - \frac{\partial U}{\partial x}\,.\,y^n + \frac{\partial\partial T}{(n+1)\partial x^2} y^{2n+1} \\ \qquad - \frac{\partial^3 S}{2(n+1)^2\partial x^3} y^{3n+2} - \frac{3\partial.SX}{(2n+1)\partial x} y^{3n+1} \end{array} \right\}$$

$$+ X\partial x\left(U y^n - \frac{\partial T}{(n+1)\partial x} y^{2n+1} + \frac{\partial\partial S}{2(n+1)^2\partial x^2} y^{3n+2} + \frac{3SX}{2n+1} y^{3n+1}\right),$$

quae formula vt integrationem admittat, effe oportet

$$2y^{2n}\partial.TX - 2y^{3n+1}\,.\,\frac{\partial.X\partial S}{(n+1)\partial x} - y^n\,.\,\frac{\partial\partial U}{\partial x} + y^{2n+1}\,.\,\frac{\partial^3 T}{(n+1)\partial x^2}$$
$$- y^{3n+2}\,.\,\frac{\partial^4 S}{2(n+1)^3\partial x^3} - 3\,y^{3n+1}\,\frac{\partial\partial.SX}{(2n+1)\partial x} - n\,U X y^{n-1}\partial x$$
$$+ \frac{(2n+1)X\partial T}{(n+1)} y^{2n} - \frac{(3n+2)X\partial\partial S}{2(n+1)^2\partial x} y^{3n+1} - \frac{3(3n+1)SXX\partial x}{(2n+1)} y^{3n} = 0,$$

hic ergo fingulae poteftates ipfius $y$, quatenus funt inaequales, feorfim deftrui debent. Quare poteftas $y^{n-1}$ dat $U = 0$; vnde etiam poteftas $y^n$ ad nihilum redigitur. Poteftas $y^{2n}$ dat

$$(2n+2)\,T\,\partial X + (2n+2)\,X\,\partial T + (2n+1)\,X\,\partial T = 0,$$

feu $X^{2n+2}\,T^{4n+3} = A$; at poteftas $y^{2n+1}$ praebet $\partial^3 T = 0$, feu $T = \alpha + \beta x + \gamma x x$. Poteftas vero $y^{3n}$ poftulat $S = 0$, nifi fit $n = -\frac{1}{3}$; quo cafu etiam poteftates $y^{3n+1}$ et $y^{3n+2}$ fponte euanefcunt. Cum ergo fit $U = 0$, $S = 0$ et $T = \alpha + \beta x + \gamma x x$, hincque $X = B(\alpha + \beta x + \gamma x x)^{\frac{-4n-3}{2n+2}}$, haec aequatio

$$\partial \partial y + \frac{n \partial y^2}{y} + B(\alpha + \beta x + \gamma x x)^{\frac{-4n-3}{2n+2}} \partial x^2 = 0,$$

integrabilis redditur ope multiplicatoris

$$2(\alpha + \beta x + \gamma x x) y^{2n} \partial y - \frac{\partial x (\beta + 2 \gamma x)}{n+1} y^{2n+1}.$$

## Scholion 2.

928. Quanquam plurimum abeſt, quominus haec methodus ſatis adhuc ſit culta, tamen ſpecimina in hoc capite tradita abunde declarant, quanta incrementa inde expectare queamus, vnde eius cultura maxime Geometris commendanda videtur. Quoniam igitur methodi, quibus in reſolutione aequationum differentio-differentialium vti conuenit, ſatis luculenter ſunt expoſitae, ad ſequens caput progrediamur, vbi integrationem huiusmodi aequationum, quatenus quidem id commode fieri poteſt, per ſeries infinitas oſtendemus.

CA-

# CAPVT VII.

## DE RESOLVTIONE AEQVATIONIS $\partial y y + a x^n y \partial x^2 = 0$ PER SERIES INFINITAS.

### Problema 117.

929.

Sumto elemento $\partial x$ constante, aequationem differentio-differentialem $\partial \partial y + a x^n y \partial x^2 = 0$ per seriem infinitam integrare.

### Solutio.

Quaerimus hic seriem secundum potestates ipsius $x$ progredientem, quae valorem ipsius $y$ exprimat; et quia in altero aequationis nostrae termino quantitas $x$ cum suo differentiali $\partial x$ nullam, in altero vero $n+2$ dimensiones occupat, euidens est exponentes potestatum ipsius $x$ differentia $n+2$ ascendere vel descendere debere.

I. Ascendant primo exponentes, et fingatur series

$$y = A x^{\lambda} + B x^{\lambda+n+2} + C x^{\lambda+2n+4} + \text{etc.}$$

eritque

$$\frac{\partial \partial y}{\partial x^2} = \lambda(\lambda-1) A x^{\lambda-2} + (\lambda+n+2)(\lambda+n+1) B x^{\lambda+n} + \text{etc.}$$

$$a x^n y = \quad . \quad . \quad . \quad . \quad a A x^{\lambda+n}$$

vnde patet primum terminum solitarium euanescere debere, vt sit $\lambda(\lambda-1) = 0$. Quare capi oportet vel $\lambda = 0$ vel $\lambda = 1$, sic-ficque duplex feries obtinetur

$$y = A + B\,x^{n+2} + C\,x^{2n+4} + D\,x^{3n+6} + E\,x^{4n+8} + \text{etc.}$$
$$+ \mathfrak{A}x + \mathfrak{B}\,x^{n+3} + \mathfrak{C}\,x^{2n+5} + \mathfrak{D}\,x^{3n+7} + \mathfrak{E}\,x^{4n+9} + \text{etc.}$$

fubftitutione ergo facta fieri oportet

$$\begin{aligned} 0 = {} & (n+2)(n+1)Bx^n + (2n+4)(2n+3)Cx^{2n+2} + (3n+6)(3n+5)Dx^{3n+4} + \text{etc.} \\ & + aA \qquad\qquad\quad + aB \qquad\qquad\qquad\qquad + aC \end{aligned}$$

$$\begin{aligned} 0 = {} & (n+3)(n+2)\mathfrak{B}x^{n+1} + (2n+5)(2n+4)\mathfrak{C}x^{2n+3} + (3n+7)(3n+6)\mathfrak{D}x^{3n+5} + \text{etc.} \\ & + a\mathfrak{A} \qquad\qquad\quad + a\mathfrak{B} \qquad\qquad\qquad\qquad + a\mathfrak{C} \end{aligned}$$

vnde litteris A et $\mathfrak{A}$ arbitrio noftro relictis, reliquae per eas ita determinantur

$$B = \frac{-aA}{(n+1)(n+2)};\quad C = \frac{-aB}{2(2n+3)(n+2)};\quad D = \frac{-aC}{3(3n+5)(n+2)} \text{ etc.}$$
$$\mathfrak{B} = \frac{-a\mathfrak{A}}{(n+3)(n+2)};\quad \mathfrak{C} = \frac{-a\mathfrak{B}}{2(2n+5)(n+2)};\quad \mathfrak{D} = \frac{-a\mathfrak{C}}{3(3n+7)(n+2)} \text{ etc.}$$

ficque habebitur integrale completum ita expreffum

$$\begin{aligned} y = {} & A - \frac{aAx^{n+2}}{1(n+1).(n+2)} + \frac{a^2Ax^{2n+4}}{1.2(n+1)(2n+3).(n+2)^2} \\ & - \frac{a^3Ax^{3n+6}}{1.2.3(n+1)(2n+3)(3n+5).(n+2)^3} + \text{etc.} \\ & + \mathfrak{A}x - \frac{a\mathfrak{A}x^{n+3}}{1(n+3).(n+2)} + \frac{a^2\mathfrak{A}x^{2n+5}}{1.2(n+3)(2n+5).(n+2)^2} \\ & - \frac{a^3\mathfrak{A}x^{5n+7}}{1.2.3(n+3)(2n+5)(3n+7).(n+2)^3} + \text{etc.} \end{aligned}$$

II. Defcendant iam exponentes, et ficta ferie

$$y = A\,x^{\lambda} + B\,x^{\lambda-n-2} + C\,x^{\lambda-2n-4} + \text{etc.}$$

habebitur

$$\frac{\partial\partial y}{\partial x^2} = \lambda(\lambda-1)A\,x^{\lambda-2} + (\lambda-n-2)(\lambda-n-3)B\,x^{\lambda-n-4} + \text{etc.}$$
$$a\,x^n y = a\,A\,x^{\lambda+n} + a\,B\,x^{\lambda-2} + \text{etc.}$$

vbi

vbi cum terminus $x^{\lambda+n}$ ſui ſimilem non habeat, tolli nequit, ita vt hinc nulla aequationis reſolutio obtineatur.

## Corollarium 1.

930. Geminata ſeries pro $y$ inuenta, quoniam litterae A et $\mathfrak{A}$ arbitrio noſtro relinquuntur, integrale completum aequationis differentio-differentialis $\partial\partial y + a x^n y \partial x^2 = 0$ exhibet; tribuendo autem litteris A et $\mathfrak{A}$ datos valores, integralia particularia naſcentur.

## Corollarium 2.

931. Si ponamus $n + 2 = m$, ſeu $n = m - 2$, huius aequationis $\partial\partial y + a x^{m-2} y \partial x^2 = 0$ integrale completum ita commodius exprimetur

$$y = \begin{cases} A - \dfrac{aAx^m}{1(m-1).m} + \dfrac{a^2Ax^{2m}}{1.2(m-1)(2m-1).m^2} \\ - \dfrac{a^3Ax^{3m}}{1.2.3(m-1)(2m-1)(3m-1).m^3} + \text{etc.} \\ \mathfrak{A}x - \dfrac{a\mathfrak{A}x^{m+1}}{1(m+1).m} + \dfrac{a^2\mathfrak{A}x^{2m+1}}{1.2(m+1)(2m+1).m^2} \\ - \dfrac{a^3\mathfrak{A}x^{3m+1}}{1.2.3(m+1)(2m+1)(3m+1).m^3} + \text{etc.} \end{cases}$$

## Corollarium 3.

932. Si exponens $m$ fuerit poſitiuus et vnitate maior, hae ſeries eo magis conuergunt, quo minor valor quantitati $x$ tribuatur: aliis vero caſibus in praxi hae ſeries adhiberi nequeunt, niſi forte eae ipſae in alias conuergentes transformari poſſint.

## Scholion 1.

933. Dantur tamen casus, quibus hac series omni plane vsu destituuntur, quod euenit, si quispiam factorum denominatores constituentium euanescat, sicque omnes termini sequentes in infinitum excrescant, quibus casibus series in alias formas transmutari conuenit. Hic primo occurrit casus $m=0$ seu $n=-2$, quo vtriusque seriei omnes termini praeter primos fiunt infiniti, hoc vero casu aequatio, quae est $\partial\partial y+\frac{a y \partial x^2}{x^2}=0$, cum sit homogenea, singularem integrationem admittit: inueniri enim potest potestas ipsius $x$, quae pro $y$ substituta aequationi satisfacit. Ponatur scilicet $y=x^\lambda$, prodibitque

$$\lambda(\lambda-1)x^{\lambda-2}+a x^{\lambda-2}=0,\ \text{seu}\ \lambda\lambda-\lambda+a=0$$

vnde colligitur $\lambda=\frac{1}{2}\pm\sqrt{(\frac{1}{4}-a)}$, ob quem duplicem valorem est integrale completum

$$y=A x^{\frac{1}{2}+\sqrt{(\frac{1}{4}-a)}}+B x^{\frac{1}{2}-\sqrt{(\frac{1}{4}-a)}},$$

quae aequatio casu $a>\frac{1}{4}$ abit in hanc formam

$$y=A x^{\frac{1}{2}}\,\text{fin.}\,[(a-\tfrac{1}{4})^{\frac{1}{2}}\, l x+\alpha];$$

vnde patet, casu $a=\frac{1}{4}$ fore

$$y=(A+B\, l x)\sqrt{x}.$$

## Scholion 2.

934. Reliqui casus ad incommodum ducentes sunt, si vel $m=\frac{1}{i}$ vel $m=-\frac{1}{i}$ denotante $i$ numerum quemcunque integrum. Casu $m=\frac{1}{i}$ prior tantum series fit incongrua, casu vero $m=-\frac{1}{i}$ posterior tantum. Quare illo casu ponendo $A=0$ hoc vero $\mathfrak{A}=0$, series saltem vna idonea habetur, integrale particulare exhibens. Verum cognito integrali particulari quod sit $y=P$, inde aequationis

$$\partial\partial y+a x^{m-2} y\,\partial x^2=0$$

inte-

integrale completum eruitur ponendo $y = Pz$, vnde fit

$$P\partial\partial z + 2\partial P\partial z + z\partial\partial P + ax^{m-2}Pz\partial x^2 = 0,$$

at per hypothefin eft

$$\partial\partial P + ax^{m-2}P\partial x^2 = 0,$$

ergo prodit

$$P\partial\partial z + 2\partial P\partial z = 0, \text{ feu}$$

$$PP\partial z = C\partial x \text{ et } z = C\int\frac{\partial x}{PP}.$$

Cum autem P fit feries infinita, hinc valorem ipfius $z$ cognofcere haud licet. At cafibus illis memoratis pars integralis logarithmum ipfius $x$ inuoluit, quod vel inde intelligitur quod $\frac{x^0}{0}$ aequiualeat ipfi $lx$. Quare in aequatione

$$\partial\partial y + ax^{m-2}y\partial x^2 = 0,$$

ponendo $y = p + qlx$, ob $\partial y = \partial p + \frac{q\partial x}{x} + \partial q\, lx$, erit

$$\partial\partial p + \frac{2\partial x\partial q}{x} - \frac{q\partial x^2}{xx} + \partial\partial q\, lx + apx^{m-2}\partial x^2$$
$$+ aqx^{m-2}\partial x^2 lx = 0,$$

in qua partes $lx$ inuoluentes feorfim deftruantur neceffe eft; ita vt hae binae habeantur aequationes

$$\partial\partial q + aqx^{m-2}\partial x^2 = 0 \text{ et}$$
$$\partial\partial p + \frac{2\partial x\partial q}{x} - \frac{q\partial x^2}{xx} + apx^{m-2}\partial x^2 = 0,$$

vbi pro $q$ ea binarum fuperiorum ferierum accipi debet, quae cafu oblato incommodo caret, eaque conftituta ex pofteriori aequatione facile quantitas $p$ per feriem exprimetur. Huiusmodi cafus in fequentibus exemplis euoluamus; tantum notemus illam operationem perinde fe habere, etiamfi loco $lx$ fumatur $lx + \alpha$, ita vt inuentis $p$ et $q$ futurum fit

$$y = \alpha q + p + qlx, \text{ feu } y = p + ql\beta x.$$

## Exemplum 1.

935. *Posito* $m = 1$, *hanc aequationem*

$$\partial\partial y + \frac{a y \partial x^2}{x} = 0$$

*per series resoluere.*

Posito $y = p + q\,l\,x$, capi oportet

$$q = \mathfrak{A}x - \frac{a\mathfrak{A}x^2}{1.2} + \frac{a^2\mathfrak{A}x^3}{1.2.2.3} - \frac{a^3\mathfrak{A}x^4}{1.2.2.3.3.4} + \text{etc.}$$

pro qua ponamus breuitatis gratia

$$q = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$$

tum vero quaeratur $p$ ex hac aequatione

$$\frac{\partial\partial p}{\partial x^2} + \frac{2\partial q}{x\partial x} - \frac{q}{xx} + \frac{ap}{x} = 0:$$

fingamus ergo

$$p = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{llllll} & 2C + 6Dx & + 12Exx & + 20Fx^3 & + 30Gx^4 & \\ +2\mathfrak{A}\frac{1}{x} + 4\mathfrak{B} & + 6\mathfrak{C} & + 8\mathfrak{D} & + 10\mathfrak{E} & + 12\mathfrak{F} & \\ -\mathfrak{A} - \mathfrak{B} & - \mathfrak{C} & - \mathfrak{D} & - \mathfrak{E} & - \mathfrak{F} & \\ +aA + aB & + aC & + aD & + aE & + aF & \end{array}\right\} = 0$$

Cum iam dentur coefficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. erit $A = -\frac{\mathfrak{A}}{a}$; quantitas B non determinatur, tum vero

$$C = \frac{-3\mathfrak{B}}{2} - \frac{aB}{1.2} = \frac{+3a\mathfrak{A}}{1^3.2^2} - \frac{aB}{1.2},$$

$$D = \frac{-5\mathfrak{C}}{6} - \frac{aC}{2.3} = \frac{-5a^2\mathfrak{A}}{1^2.2^3.3^2} - \frac{aC}{2.3},$$

$$E = \frac{-7\mathfrak{D}}{12} - \frac{aD}{3.4} = \frac{+7a^3\mathfrak{A}}{1^2.2^2.3^3.4^2} - \frac{aD}{3.4},$$

$$F = \frac{-9\mathfrak{E}}{20} - \frac{aE}{4.5} = \frac{-9a^4\mathfrak{A}}{1^2.2^2.3^2.4^3.5^2} - \frac{aE}{4.5}, \text{ etc.}$$

vbi pro B scribere licet 0, quandoquidem in integrali $y = p + q\,l\,x$ addimus partem $\alpha q$, quae ex littera B oritur, ita vt sit

$$p = A + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

Hinc

Hinc erit

$$C = \frac{3a\mathfrak{A}}{1^3 \cdot 2^2}, \quad D = \frac{-14a^2\mathfrak{A}}{1^3 \cdot 2^3 \cdot 3^2}, \quad E = \frac{+70a^3\mathfrak{A}}{1^4 \cdot 2^4 \cdot 3^4 \cdot 4^2}, \quad F = \frac{-404a^4\mathfrak{A}}{1^5 \cdot 2^5 \cdot 3^5 \cdot 4^5 \cdot 5^2}, \text{ etc.}$$

vbi notetur esse

$$14 = 3 \cdot 3 + 5 \cdot 1, \quad 70 = 4 \cdot 14 + 7 \cdot 1 \cdot 2, \quad 404 = 5 \cdot 70 + 9 \cdot 1 \cdot 2 \cdot 3,$$

et pro sequente $2688 = 6 \cdot 404 + 11 \cdot 1 \cdot 2 \cdot 3 \cdot 4$; estque

$$y = p + \alpha q + q\, l\, x.$$

## Exemplum 2.

936. *Posito $m = -1$, hanc aequationem*

$$\partial\partial y + \frac{a y \partial x^2}{x^3} = 0,$$

*per series resoluere.*

Posito $y = p + \alpha q + q\, l\, x$, capi oportet

$$q = A - \frac{aA}{1 \cdot 2x} + \frac{a^2 A}{1 \cdot 2^2 \cdot 3x^2} - \frac{a^3 A}{1 \cdot 2^2 \cdot 3^2 \cdot 4x^3} + \text{ etc.}$$

pro qua ponatur breuitatis gratia

$$q = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \text{ etc.}$$

tum vero quantitas $p$ ex hac aequatione definiri debet

$$\partial\partial p + \frac{2\partial q \partial x}{x} - \frac{q \partial x^2}{xx} + \frac{a p \partial x^2}{x^3} = 0.$$

Fingamus ergo

$$p = \mathfrak{A}x + \mathfrak{B} + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{ etc.}$$

vnde facta substitutione prodit

$$\left.\begin{array}{l} \frac{a\mathfrak{A}}{xx} + \frac{a\mathfrak{B}}{x^3} + \frac{a\mathfrak{C}}{x^4} + \frac{a\mathfrak{D}}{x^5} + \frac{a\mathfrak{E}}{x^6} + \frac{a\mathfrak{F}}{x^7} + \text{etc.} \\ -A - B - C - D - E - F \\ \quad -2B - 4C - 6D - 8E - 10F \\ \quad +2\mathfrak{C} + 6\mathfrak{D} + 12\mathfrak{E} + 20\mathfrak{F} + 30\mathfrak{G} \end{array}\right\} = 0,$$

et coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, etc. ita determinantur, vt fit $\mathfrak{A} = \frac{A}{\alpha}$; fecundus $\mathfrak{B}$ non definitur, tum vero eft

$$\mathfrak{C} = \frac{3\,B}{1.2} - \frac{\alpha\,\mathfrak{B}}{1.2} = \frac{-3\,\alpha\,A}{1^3.2^2} - \frac{\alpha\,\mathfrak{B}}{1.2},$$
$$\mathfrak{D} = \frac{5\,C}{2.3} - \frac{\alpha\,\mathfrak{C}}{2.3} = \frac{5\,\alpha^2\,A}{2^3.3^2} - \frac{\alpha\,\mathfrak{C}}{2.3},$$
$$\mathfrak{E} = \frac{7\,D}{3.4} - \frac{\alpha\,\mathfrak{D}}{3.4} = \frac{-7\,\alpha^3\,A}{2^2.3^3.4^2} - \frac{\alpha\,\mathfrak{D}}{3.4},$$
$$\mathfrak{F} = \frac{9\,E}{4.5} - \frac{\alpha\,\mathfrak{E}}{4.5} = \frac{9\,\alpha^4\,A}{2^2.3^2.4^3.5^2} - \frac{\alpha\,\mathfrak{E}}{4.5}, \text{ etc.}$$

fi fumatur, id quod fine detrimento generalitatis fieri licet, $\mathfrak{B} = 0$, ita vt fit

$$p = \mathfrak{A}\,x + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc. erit}$$
$$\mathfrak{C} = \frac{-3\,\alpha\,A}{1^3.2^2},\ \mathfrak{D} = \frac{+14\,\alpha^2\,A}{1^3.2^3.3^2},\ \mathfrak{E} = \frac{-70\,\alpha^3\,A}{1^3.2^3.3^3.4^2},$$
$$\mathfrak{F} = \frac{+404\,\alpha^4\,A}{1^3.2^3.3^3.4^3.5^2}, \text{ etc.}$$

qui valores fimiles funt praecedentibus.

## Exemplum 3.

937. *Pofito* $m = \frac{1}{2}$, *hanc aequationem*

$$\partial\partial y + \frac{\alpha\,y\,\partial x^2}{x\sqrt{x}} = 0$$

*per feries refoluere.*

Pofito $y = p + \alpha\,q + q\,l\,x$, capi oportet

$$q = \mathfrak{A}\,x - \frac{4\,\alpha\,\mathfrak{A}}{1.3}\,x^{\frac{3}{2}} + \frac{16\,\alpha^2\,\mathfrak{A}}{1.2.3.4}\,x^2 - \frac{64\,\alpha^3\,\mathfrak{A}}{1.2.3.3.4.5}\,x^{\frac{5}{2}}$$
$$+ \frac{256\,\alpha^4\,\mathfrak{A}}{1.2.3.3.4.4.5.6}\,x^3 - \text{etc.}$$

pro qua breuitatis gratia fcribatur

$$q = \mathfrak{A}\,x + \mathfrak{B}\,x^{\frac{3}{2}} + \mathfrak{C}\,x^2 + \mathfrak{D}\,x^{\frac{5}{2}} + \mathfrak{E}\,x^3 + \mathfrak{F}\,x^{\frac{7}{2}} + \mathfrak{G}\,x^4 + \text{etc.}$$

tum vero quantitas $p$ ex hac aequatione definiri debet

$$\partial\partial p + \frac{2\,\partial x\,\partial q}{x} - \frac{q\,\partial x^2}{xx} + \frac{\alpha\,p\,\partial x^2}{x\sqrt{x}} = 0.$$

Fingamus ergo

$$p = \Delta + A\,x^{\frac{1}{2}} + B\,x + C\,x^{\frac{3}{2}} + D\,x^2 + E\,x^{\frac{5}{2}} + F\,x^3 + \text{etc.}$$

prodi-

prodibitque facta substitutione

$$\left.\begin{array}{l}\frac{a\Delta}{\alpha\sqrt{x}}+\frac{aA}{x}+\frac{aB}{\sqrt{x}}+aC+aD\sqrt{x}+aEx+aFx\sqrt{x}\\ \quad-\mathfrak{A}-\mathfrak{B}-\mathfrak{C}-\mathfrak{D}\quad-\mathfrak{E}\quad-\mathfrak{F}\\ \quad+2\mathfrak{A}+3\mathfrak{B}+4\mathfrak{C}+5\mathfrak{D}\quad+6\mathfrak{E}\quad+7\mathfrak{F}\\ -\frac{A}{4}+0\;+\frac{3}{4}C+2D+\frac{15}{4}E\quad+6F\quad+\frac{35}{4}G\end{array}\right\}=0.$$

Hinc colligitur fore

$$A=\frac{-\mathfrak{A}}{a},\ \Delta=\frac{-\mathfrak{A}}{4aa},$$

et B non determinatur, porro

$$C=\frac{-4aB}{1.3}-\frac{8\mathfrak{B}}{1.3}=\frac{-4aB}{1.3}+\frac{32a\mathfrak{A}}{1^2.3^2},$$

$$D=\frac{-4aC}{2.4}-\frac{12\mathfrak{C}}{2.4}=\frac{-4aC}{2.4}-\frac{12.16a^2\mathfrak{A}}{1.3.2^2.4^2},$$

$$E=\frac{-4aD}{3.5}-\frac{16\mathfrak{D}}{3.5}=\frac{-4aD}{3.5}+\frac{16.64a^3\mathfrak{A}}{1.3.2.4.3^2.5^2},$$

$$F=\frac{-4aE}{4.6}-\frac{20\mathfrak{E}}{4.6}=\frac{-4aE}{4.6}-\frac{20.256a^4\mathfrak{A}}{1.3.2.4.3.5.4^2.6^2},\ \text{etc.}$$

Quodsi iam ponatur $B=0$ vt sit

$$p=\frac{-\mathfrak{A}}{4aa}-\frac{\mathfrak{A}}{a}\sqrt{x}+*+Cx\sqrt{x}+Dx^2+Ex^2\sqrt{x}+Fx^3+\text{etc.}$$

erit

$$C=\frac{8.4a\mathfrak{A}}{1^2.3^2},\ D=\frac{-100.16a^2\mathfrak{A}}{1^2.3^2.2^2.4^2},\ E=\frac{1884.64a^3\mathfrak{A}}{1^2.3^2.2^2.4^2.3^2.5^2},$$

$$F=\frac{-52416.256a^4\mathfrak{A}}{1^2.3^2.2^2.4^2.3^2.5^2.4^2.6^2},\ \text{etc.}$$

vbi notetur esse

$$100=2.4.8+1.3.12,\ 1884=3.5.100+1.3.2.4.16,$$

$$52416=4.6.1884+1.3.2.4.3.5.20.$$

## Exemplum 4.

938. *Posito* $m=-\frac{1}{2}$, *hanc aequationem*

$$\partial\partial y+\frac{ay\partial x^2}{xx\sqrt{x}}=0$$

*per series resoluere.*

Posi-

Poſito $y = p + \alpha q + q\, l\, x$, capi oportet

$$q = A - \frac{4 a A}{1.3} x^{-\frac{1}{2}} + \frac{16 a^2 A}{1.3.2.4} x^{-1} - \frac{64 a^3 A}{1.3.2.4.3.5} x^{-\frac{3}{2}} + \text{etc.}$$

pro qua breuitatis cauſa ſcribamus

$$q = A + B x^{-\frac{1}{2}} + C x^{-1} + D x^{-\frac{3}{2}} + E x^{-2} + F x^{-\frac{5}{2}} + \text{etc.}$$

et littera $p$ ex hac aequatione definiri debet

$$\partial \partial p + \frac{2 \partial x \partial q}{x} - \frac{q \partial x^2}{x x} + \frac{a p \partial x^2}{x x \sqrt{x}} = 0.$$

Fingamus

$$p = \Delta x + \mathfrak{A} \sqrt{x} + \mathfrak{B} + \mathfrak{C} x^{-\frac{1}{2}} + \mathfrak{D} x^{-1} + \mathfrak{E} x^{-\frac{3}{2}} + \mathfrak{F} x^{-2} + \mathfrak{G} x^{-\frac{5}{2}} + \text{etc.}$$

et facta ſubſtitutione prodit

$$\left.\begin{array}{l} \frac{\mathfrak{A} \Delta}{x \sqrt{x}} + \frac{a \mathfrak{A}}{x x} + \frac{a \mathfrak{B}}{x x \sqrt{x}} + \frac{a \mathfrak{C}}{x^3} + \frac{a \mathfrak{D}}{x^3 \sqrt{x}} + \frac{a \mathfrak{E}}{x^4} + \frac{a \mathfrak{F}}{x^4 \sqrt{x}} \\ \quad - A \quad - B \quad - C \quad - D \quad - E \quad - F \\ \quad\quad - B \quad - 2C \quad - 3D \quad - 4E \quad - 5F \\ - \frac{1}{4} \mathfrak{A} \quad + \frac{3}{4} \mathfrak{C} \quad + 2 \mathfrak{D} \quad + \frac{15}{4} \mathfrak{E} \quad + 6 \mathfrak{F} \quad + \frac{35}{4} \mathfrak{G} \end{array}\right\} = 0,$$

vnde ſequentes determinationes colliguntur

$$\mathfrak{A} = \frac{A}{a} \text{ et } \Delta = \frac{\mathfrak{A}}{4 a} = \frac{A}{4 a a},$$

at $\mathfrak{B}$ non determinatur: porro

$$\mathfrak{C} = \frac{-4 a \mathfrak{B}}{1.3} + \frac{8 B}{1.3} = \frac{-4 a \mathfrak{B}}{1.3} - \frac{8.4 a A}{1^2.3^2},$$

$$\mathfrak{D} = \frac{-4 a \mathfrak{C}}{2.4} + \frac{12 C}{2.4} = \frac{-4 a \mathfrak{C}}{2.4} + \frac{12.16 a^2 A}{1.3.2^2.4^2},$$

$$\mathfrak{E} = \frac{-4 a \mathfrak{D}}{3.5} + \frac{16 D}{3.5} = \frac{-4 a \mathfrak{D}}{3.5} - \frac{16.64 a^3 A}{1.3.2.4.3^2.5^2},$$

$$\mathfrak{F} = \frac{-4 a \mathfrak{E}}{4.6} + \frac{20 E}{4.6} = \frac{-4 a \mathfrak{E}}{4.6} + \frac{20.256 a^4 A}{1.3.2.4.3.5.4^2.6^2}, \text{ etc.}$$

Quodſi iam ſumatur $\mathfrak{B} = 0$, erit

$$\mathfrak{C} = \frac{-8.4 a A}{1^2.3^2}, \quad \mathfrak{D} = \frac{+100.16 a^2 A}{1^2.3^2.2^2.4^2}, \quad \mathfrak{E} = \frac{-1884.64 a^3 A}{1^2.3^2.2^2.4^2.3^2.5^2}, \text{ etc.}$$

qui numeri vt ante progrediuntur.

Scho-

## Scholion.

939. Ex his exemplis perſpicitur, quomodo ſeries aequationem

$$\partial\partial y + a x^{m-2} y \partial x^2 = 0$$

reſoluentes in reliquis caſibus, quibus $m = \pm \frac{1}{i}$, inueniri oporteat; vbi obſeruetur, ſi ſit $m = + \frac{1}{i}$, pro $q$ hanc ſeriem accipi debere

$$q = \mathfrak{A} x + \mathfrak{B} x^{1+\frac{1}{i}} + \mathfrak{C} x^{1+\frac{2}{i}} + \mathfrak{D} x^{1+\frac{3}{i}} + \text{etc.}$$

tum vero formam ipſius $p$ tali ſerie exprimi

$$p = A + B x^{\frac{1}{i}} + C x^{\frac{2}{i}} + D x^{\frac{3}{i}} + \text{etc.}$$

cuius coefficientes ex ſuperioribus vt ante definiantur. Sin autem ſit $m = -\frac{1}{i}$, pro $q$ ſumatur ſeries

$$q = A + B x^{-\frac{1}{i}} + C x^{-\frac{2}{i}} + D x^{-\frac{3}{i}} + \text{etc.}$$

at pro $p$ huiusmodi formam accipi conueniet

$$p = \mathfrak{A} x + \mathfrak{B} x^{1-\frac{1}{i}} + \mathfrak{C} x^{1-\frac{2}{i}} + \mathfrak{D} x^{1-\frac{3}{i}} + \text{etc.}$$

vnde pariter ſingulos coefficientes vno excepto determinare licebit. Atque hoc artificium in genere eſt tenendum, quoties in reſolutione aequationis generalis ad ſeries peruenitur, cuius coefficientes certis caſibus in infinitum excreſcunt, quod plerumque indicio eſt, logarithmos eſſe introducendos. Verum etiam eadem aequatio $\partial\partial y + a x^n y \partial x^2 = 0$ aliis modis per ſeries reſolui poteſt, dum ea ante reſolutionem in aliam formam transmutatur, vbi cum euenire poſſit, vt ſeries certis caſibus abrumpatur, quibus adeo integrale reuera aſſignari poteſt, talem transformationem maxime notabilem hic explicemus.

## Problema 118.

940. Aequationem differentio-differentialem

$$\partial\partial y + a x^n y \partial x^2 = 0$$

in aliam formam transfundere, cuius resolutio per series infinitas commode institui possit.

## Solutio.

Vtamur substitutione $y = e^{\int p \partial x} z$, vbi $p$ sit certa functio ipsius $x$ aequationem commode resolubilem suppeditans. Erit ergo

$$\partial y = e^{\int p \partial x} (\partial z + p z \partial x) \text{ et}$$

$$\partial\partial y = e^{\int p \partial x} (\partial\partial z + 2 p \partial x \partial z + z \partial x \partial p + p p z \partial x^2),$$

vnde aequatio proposita abit in

$$\partial\partial z + 2 p \partial x \partial z + z \partial x \partial p + p p z \partial x^2 + a x^n z \partial x^2 = 0,$$

vbi $p$ ita capiatur, vt fiat

$$p p + a x^n = 0, \text{ seu } p = x^{\frac{n}{2}} \sqrt{-a}.$$

Ponamus ideo $a = -c c$ et $n = 2 m$, vt proposita sit haec aequatio

$$\partial\partial y - c c x^{2m} y \partial x^2 = 0,$$

quae posito

$$p = c x^m \text{ et } y = e^{\int p \partial x} z = e^{\frac{c}{m+1} x^{m+1}} z,$$

induet hanc formam

$$\partial\partial z + 2 c x^m \partial x \partial z + m c x^{m-1} z \partial x^2 = 0,$$

in qua, cum $x$ occupet vel nullam, vel $m + 1$ dimensiones, fingamus ipsius $z$ valorem

$$z = A x^\lambda + B x^{\lambda + m + 1} + C x^{\lambda + 2m + 2} + \text{etc.}$$

quo substituto fit

λ

$$\begin{array}{l} \lambda(\lambda-1)A x^{\lambda-2} + (\lambda+m+1)(\lambda+m)B x^{\lambda+m-1} = 0, \\ \qquad\qquad\qquad\quad + 2\lambda A c \\ \qquad\qquad\qquad\quad + m A c \end{array}$$

vnde perspicuum est sumi debere vel $\lambda = 0$ vel $\lambda = 1$. Consequimur ergo seriem duplicatam huiusmodi

$$z = A + B x^{m+1} + C x^{2m+2} + D x^{3m+3} + E x^{4m+4} + \text{etc.}$$
$$\mathfrak{A} x + \mathfrak{B} x^{m+2} + \mathfrak{C} x^{2m+3} + \mathfrak{D} x^{3m+4} + \mathfrak{E} x^{4m+5} + \text{etc.}$$

qua substituta fit

$$\left.\begin{array}{llll} (m+1)mBx^{m-1} & +2(m+1)(2m+1)Cx^{2m} & +3(m+1)(3m+2)Dx^{3m+1} \\ +mAc & +2(m+1)Bc & +4(m+1)Cc \\ & +mBc & +mCc \end{array}\right\} = 0$$

$$\left.\begin{array}{llll} +(m+1)(m+2)\mathfrak{B}x^{m} & +2(m+1)(2m+3)\mathfrak{C}x^{2m+1} & +3(m+1)(3m+4)\mathfrak{D}x^{3m+2} \\ +2\mathfrak{A}c & +2(m+2)\mathfrak{B}c & +2(2m+3)\mathfrak{C}c \\ +m\mathfrak{A}c & +m\mathfrak{B}c & +m\mathfrak{C}c \end{array}\right\} = 0$$

vnde vtrique coefficientes sequenti modo determinantur

$$\begin{array}{l|l} B = \frac{-mAc}{m(m+1)} & \mathfrak{B} = \frac{-(m+2)\mathfrak{A}c}{(m+2)(m+1)} \\ C = \frac{-(3m+2)Bc}{2(2m+1)(m+1)} & \mathfrak{C} = \frac{-(3m+4)\mathfrak{B}c}{2(2m+3)(m+1)} \\ D = \frac{-(5m+4)Cc}{3(3m+2)(m+1)} & \mathfrak{D} = \frac{-(5m+6)\mathfrak{C}c}{3(3m+4)(m+1)} \\ E = \frac{-(7m+6)Dc}{4(4m+3)(m+1)} & \mathfrak{E} = \frac{-(7m+8)\mathfrak{D}c}{4(4m+5)(m+1)} \\ \text{etc.} & \text{etc.} \end{array}$$

vbi bini coefficientes A et $\mathfrak{A}$ manent indeterminati, ita vt hoc integrale completum sit censendum.

*Aliter.* Sumta serie, in qua exponentes ipsius $x$ decrescant, fieri debet

$$2\lambda + m = 0 \text{ seu } m = -2\lambda,$$

vt aequatio nostra fit

$$\partial\partial y - cc x^{-4\lambda} y \partial x^2 = 0,$$

 quae

quae posito

$$p = c\,x^{-2\lambda} \text{ et } y = e^{\int p \partial x}\,z = e^{\frac{-c}{2\lambda-1}\,x^{-2\lambda+1}}\,z,$$

abit in

$$\partial\partial z + 2\,c\,x^{-2\lambda}\,\partial x\,\partial z - 2\,\lambda\,c\,x^{-2\lambda-1}\,z\,\partial x^2 = 0.$$

Ponamus ergo

$$z = A\,x^{\lambda} + B\,x^{3\lambda-1} + C\,x^{5\lambda-2} + D\,x^{7\lambda-3} + \text{etc.}$$

et substitutione facta prodit

$$\left.\begin{array}{llll} & \lambda(\lambda-1)Ax^{\lambda-2} + (3\lambda-1)(3\lambda-2)Bx^{3\lambda-3} + (5\lambda-2)(5\lambda-3)Cx^{5\lambda-4} & & \\ +2\lambda Acx^{-\lambda-1} & +2(3\lambda-1)Bc + 2(5\lambda-2)Cc & +2(7\lambda-3)Dc & \\ -2\lambda Ac & -2\lambda Bc \quad -2\lambda Cc & -2\lambda Dc & \end{array}\right\} = 0$$

vnde coefficientes ita determinantur

$$B = \frac{-\lambda(\lambda-1)A}{(4\lambda-2)c} = \frac{-\lambda(\lambda-1)A}{2(2\lambda-1)c}$$

$$C = \frac{-(3\lambda-1)(3\lambda-2)B}{(8\lambda-4)c} = \frac{-(3\lambda-1)(3\lambda-2)B}{4(2\lambda-1)c}$$

$$D = \frac{-(5\lambda-2)(5\lambda-3)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(5\lambda-3)C}{6(2\lambda-1)c}$$

$$E = \frac{-(7\lambda-3)(7\lambda-4)D}{(16\lambda-8)c} = \frac{-(7\lambda-3)(7\lambda-4)D}{8(2\lambda-1)c}$$

etc.

Hic vnius tantum litterae A valor arbitrio nostro relinquitur, ex quo haec series tantum integrale particulare exhibet

## Corollarium 1.

941. Ex solutione priori patet, alteram seriem terminari quoties

$$(2\,i + 1)\,m + 2\,i = 0, \text{ seu } m = \frac{-2i}{2i+1},$$

alteram vero quoties

$$(2\,i - 1)\,m + 2\,i = 0, \text{ seu } m = \frac{-2i}{2i-1},$$

denotante $i$ numerum integrum quemcunque. His ergo casibus integrale saltem particulare finite exprimi potest.

Corol-

## Corollarium 2.

942. Altera solutio praebet seriem finitam, quoties fuerit vel

$$(2i+1)\lambda - i = 0 \text{ vel } (2i-1)\lambda - i = 0,$$

hoc est

$$\lambda = \frac{i}{2i \pm 1} \text{ et } m = \frac{-4i}{2i \pm 1},$$

vt ante. Reliquis vero casibus haec series in infinitum excurrit.

## Corollarium 3.

943. Casus ergo, quibus haec aequatio

$$\partial\partial y - cc\, x^n \partial x^2 = 0$$

atque adeo posito $y = e^{\int u \partial x}$ etiam haec

$$\partial u + uu\partial x = cc\, x^n \partial x$$

integrationem saltem particularem admittit, sunt $n = \frac{-4i}{2i \pm 1}$, sumendo pro $i$ numerum integrum quemcunque.

## Scholion.

944. Sufficit autem integrale particulare inuenisse, cum ex eo facile integrale completum erui possit. Cum enim in integrali insit littera $c$, dum aequatio differentialis tantum quadratum $cc$ continet, perinde est siue in integrali sumatur $+c$ siue $-c$. Hinc si integrale particulare sit $y = P + cQ$, erit etiam $y = P - cQ$ integrale particulare, vnde integrale completum erit

$$y = \alpha(P + cQ) + \beta(P - cQ), \text{ seu } y = \alpha P + \beta c Q.$$

Quo haec clarius explicentur, ad solutionem alteram aequationis $\partial\partial y - cc\, x^{-4\lambda} y \partial x^2 = 0$ accommodentur, pro qua ponendo breuitatis gratia $\frac{1}{1-2\lambda} x^{1-2\lambda} = t$, fecimus $y = e^{ct} z$, et inuenimus

$$z = A x^{\lambda} - \frac{\lambda(\lambda-1)A}{2(2\lambda-1)c} x^{3\lambda-1} + \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)A}{2.4(2\lambda-1)^2 cc} x^{5\lambda-2}$$
$$- \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)(5\lambda-2)(5\lambda-3)A}{2.4.6(2\lambda-1)^3 c^3} x^{7\lambda-3} + \text{etc.}$$

Pro qua expressione distinguendo terminos per potestates pares ipsius $c$ diuisos ab iis, qui per potestates impares sunt diuisi, scribamus $z = P - cQ$, ita vt iam P et Q tantum potestates pares ipsius $c$ contineant, eritque integrale particulare vnum

$$y = e^{ct}(P - cQ)$$

et alterum

$$y = e^{-ct}(P + cQ),$$

vnde completum erit

$$y = \tfrac{1}{2}P(\alpha e^{ct} + \beta e^{-ct}) - \tfrac{1}{2}cQ(\alpha e^{ct} - \beta e^{-ct}).$$

Hinc si $c$ sit numerus imaginarius seu $cc = -bb$, vt aequatio sit

$$\partial\partial y + bbx^{-4\lambda}y\partial x^2 = 0, \text{ erit}$$
$$z = P - bQ\sqrt{-1} \text{ et}$$
$$e^{ct} = e^{bt\sqrt{-1}} = \text{cof.}\, bt + \sqrt{-1}.\,\text{fin.}\, bt, \text{ ergo}$$
$$y = P\left(\tfrac{\alpha+\beta}{2}\,\text{cof.}\, bt + \tfrac{\alpha-\beta}{2}\sqrt{-1}.\,\text{fin.}\, bt\right)$$
$$- bQ\left(\tfrac{\alpha-\beta}{2}\,\text{cof.}\, bt + \tfrac{\alpha+\beta}{2}\sqrt{-1}.\,\text{fin.}\, bt\right)\sqrt{-1}.$$

Sit $\frac{\alpha+\beta}{2} = \gamma$ et $\frac{\alpha-\beta}{2}\sqrt{-1} = \delta$,

atque integrale completum hoc casu ita exprimetur

$$y = P(\gamma\,\text{cof.}\, bt + \delta\,\text{fin.}\, bt) - bQ(\delta\,\text{cof.}\, bt - \gamma\,\text{fin.}\, bt), \text{ seu}$$
$$y = (\gamma P - \delta bQ)\,\text{cof.}\, bt + (\delta P + \gamma bQ)\,\text{fin.}\, bt.$$

Casus ergo hoc modo integrabiles euoluamus.

## Exemplum 1.

945. *Integrale aequationis* $\partial\partial y - ccy\partial x^2 = 0$ *inuenire.*

Hic

Hic eſt $\lambda = 0$ et $z = A$, atque $t = x$, vnde ob $P = A$ et $Q = 0$, erit integrale completum

$$y = \alpha e^{cx} + \beta e^{-cx}.$$

Caſu autem $cc = -bb$, aequationis $\partial\partial y + bby\partial x^2 = 0$ integrale completum erit

$$y = \gamma \operatorname{cof.} bx + \delta \operatorname{fin.} bx.$$

## Exemplum 2.

946. *Integrale aequationis* $\partial\partial y - ccx^{-4}y\partial x^2 = 0$ *inuenire.*

Hic ob $\lambda = 1$ eſt $z = Ax$, et $t = -\frac{1}{x}$, vnde ob $P = x$ et $Q = 0$, fit

$$y = (\alpha e^{ct} + \beta e^{-ct})x.$$

Caſu autem $cc = -bb$ aequationis

$$\partial\partial y + bbx^{-4}y\partial x^2 = 0$$

integrale eſt

$$y = (\alpha \operatorname{cof.} bt + \beta \operatorname{fin.} bt)x, \text{ exiſtente } t = -\frac{1}{x}.$$

## Exemplum 3.

947. *Integrale aequationis* $\partial\partial y - ccx^{-\frac{4}{3}}y\partial x^2 = 0$ *inuenire.*

Ob $\lambda = \frac{1}{3}$, fit $B = -\frac{A}{3c}$, et $z = Ax^{\frac{1}{3}} - \frac{A}{3c}$, atque $t = 3x^{\frac{1}{3}}$, vnde $P = x^{\frac{1}{3}}$ et $Q = \frac{1}{3cc}$. Integrale ergo erit

$$y = (\alpha e^{ct} + \beta e^{-ct})x^{\frac{1}{3}} - (\alpha e^{ct} - \beta e^{-ct})\frac{1}{3c}.$$

Caſu autem $cc = -bb$, aequationis

$$\partial\partial y + bbx^{-\frac{4}{3}}y\partial x^2 = 0$$

inte-

integrale eſt

$$y = (\alpha \operatorname{cof.} bt + \beta \operatorname{fin.} bt)x^{\frac{1}{3}} + \tfrac{1}{3b}(\beta \operatorname{cof.} bt - \alpha \operatorname{fin.} bt).$$

## Exemplum 4.

948. *Integrale aequationis* $\partial\partial y - cc x^{-\frac{8}{3}} y \partial x^2 = 0$ *inuenire.*

Ob $\lambda = \frac{2}{3}$, fit $B = \frac{A}{3c}$ et $z = A x^{\frac{2}{3}} + \frac{A}{3c} x$, ita vt ſit $P = x^{\frac{2}{3}}$ et $Q = \frac{-x}{3cc}$. Poſito ergo $t = -3 x^{-\frac{1}{3}}$, integrale ita exprimitur

$$y = x^{\frac{2}{3}}(\alpha e^{ct} + \beta e^{-ct}) + \tfrac{x}{3c}(\alpha e^{ct} - \beta e^{-ct}).$$

Caſu autem $cc = -bb$ aequationis

$$\partial\partial y + bb x^{-\frac{8}{3}} y \partial x^2 = 0$$

integrale eſt

$$y = x^{\frac{2}{3}}(\alpha \operatorname{cof.} bt + \beta \operatorname{fin.} bt) - \tfrac{x}{3b}(\beta \operatorname{cof.} bt - \alpha \operatorname{fin.} bt).$$

## Exemplum 5.

949. *Integrale aequationis* $\partial\partial y - cc x^{-\frac{8}{5}} y \partial x^2 = 0$ *inuenire.*

Ob $\lambda = \frac{2}{5}$, eſt $B = \frac{-3A}{5c}$ et $C = \frac{+3A}{5^2 cc}$, hinc

$$z = A x^{\frac{2}{5}} - \tfrac{3A}{5c} x^{\frac{1}{5}} + \tfrac{3A}{5^2 cc}, \text{ ideoque}$$

$$P = x^{\frac{2}{5}} + \tfrac{3}{5^2 cc} \text{ et } Q = \tfrac{3}{5cc} x^{\frac{1}{5}}.$$

Poſito ergo $t = 5 x^{\frac{1}{5}}$, integrale erit

$$y = (x^{\frac{2}{5}} + \tfrac{3}{5^2 cc})(\alpha e^{ct} + \beta e^{-ct}) - \tfrac{3}{5c} x^{\frac{1}{5}}(\alpha e^{ct} - \beta e^{-ct})$$

caſu

casu autem $cc = -bb$, aequationis

$$\partial\partial y + bb\,x^{-\frac{8}{3}} y\,\partial x^2 = 0$$

integrale est

$$y = (x^{\frac{1}{3}} - \tfrac{3}{t^2 bb})(\alpha \operatorname{cof.} bt + \beta \operatorname{fin.} bt) + \tfrac{3}{tb}\, x^{\frac{1}{3}} (\beta \operatorname{cof.} bt - \alpha \operatorname{fin.} bt).$$

## Problema 119.

950. Aequationis differentio-differentialis

$$\partial\partial y + cc\,x^{\frac{-4i}{2i-1}} y\,\partial x^2 = 0$$

integrale completum assignare, denotante $i$ numerum integrum quemcunque.

## Solutio.

Sit breuitatis gratia $t = -(2i-1)\,x^{\frac{-1}{2i-1}}$ vnde fit $x^{\frac{1}{2i-1}} = -\frac{(2i-1)}{t}$, ac posito $y = e^{ct} z$, pro valore ipsius $z$ per seriem inuento

$$z = A\,x^{\frac{i}{2i-1}} + B\,x^{\frac{i+1}{2i-1}} + C\,x^{\frac{i+2}{2i-1}} + D\,x^{\frac{i+3}{2i-1}} + \text{etc.}$$

ob $\lambda = \frac{+i}{2i-1}$, hi coefficientes ita determinantur

$$B = \frac{i(i-1)A}{2(2i-1)c},\; C = \frac{(i+1)(i-2)B}{4(2i-1)c},\; D = \frac{(i+2)(i-3)C}{6(2i-1)c} \text{ etc.}$$

quibus substitutis, et introducto valore $x^{\frac{1}{2i-1}} = \frac{-(2i-1)}{t}$, erit

$$z = A\,x^{\frac{i}{2i-1}} \left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2.4\,cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^3 t^3} + \text{etc.}\right.$$

siue hoc modo

$$z = \frac{A}{t^i}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2.4\,cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6.\,c^3 t^3} + \text{etc.}\right).$$

Atque hinc aequationis propofitae integrale completum ita exprimetur

$$y = t^{-i}\left(1 + \frac{i(ii-1)(i-2)}{2.4\,cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$

$$- t^{-i}\left(\frac{i(i-1)}{2ct} + \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^3t^3} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct})$$

vbi in vtraque progreffione lex formationis fingulorum terminorum eft manifefta.

## Corollarium 1.

951. Hinc quoque illius aequationis

$$\partial\partial y + bb\,x^{\frac{-4i}{2i-1}}\,y\,\partial x^2 = 0$$

integrale completum, manente $t = -(2i-1)\,x^{\frac{-1}{2i-1}}$, eft

$$y = t^{-i}\left(1 - \frac{i(ii-1)(i-2)}{2.4\,bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,b^4t^4} - \text{etc.}\right)(\alpha \operatorname{cof}.bt + \beta \operatorname{fin}.bt)$$

$$+ t^{-i}\left(\frac{i(i-1)}{2bt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,b^5t^5} - \text{etc.}\right)(\beta \operatorname{cof}.bt - \alpha \operatorname{fin}.bt).$$

## Corollarium 2.

952. Si $i$ fit numerus negatiuus, haec integratio perinde fuccedit, aequationis enim

$$\partial\partial y - cc\,x^{\frac{-4i}{2i+1}}\,y\,\partial x^2 = 0$$

pofito $t = (2i+1)\,x^{\frac{1}{2i+1}}$, integrale erit

$$y = t^{i}\left(1 + \frac{i(ii-1)(i+2)}{2.4\,cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$

$$- t^{i}\left(\frac{i(i+1)}{2ct} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^5t^5} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct}).$$

## Corollarium 3.

953. Simili modo huius aequationis

$$\partial\partial y + bb\,x\,b^{\frac{-4i}{2i+1}}\,y\,\partial x^2 = 0,$$

pofito

posito $t = (2i+1)\, x^{\frac{1}{2i+1}}$, integrale completum erit

$$y = t^i \left(1 - \frac{i(ii-1)(i+2)}{2.4\, bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8.\, b^4 t^4} - \text{etc.}\right)(\alpha \operatorname{cos}.\, bt + \beta \operatorname{sin}.\, bt)$$

$$+ t^i \left(\frac{i(i+1)}{2bt} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\, b^3 t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\, b^5 t^5} - \text{etc.}\right)(\beta \operatorname{cos}.\, bt - \alpha \operatorname{sin}.\, bt).$$

## Corollarium 4.

954. In formulis finus et cofinus continentibus fi ponatur

$$\alpha = C \operatorname{fin}.\, \zeta \quad \text{et} \quad \beta = C \operatorname{cof}.\, \zeta,$$

expreffiones noftrae ita contrahuntur, vt fiat

$$\alpha \operatorname{cof}.\, bt + \beta \operatorname{fin}.\, bt = C \operatorname{fin}.\, (bt + \zeta) \quad \text{et}$$

$$\beta \operatorname{cof}.\, bt - \alpha \operatorname{fin}.\, bt = C \operatorname{cof}.\, (bt + \zeta),$$

vt iam hic C et $\zeta$ fint conftantes arbitrariae integrale completum reddentes.

## Scholion.

955. Hinc egregium adipifcimur adminiculum ad cafus integrabilitatis huius aequationis differentialis primi gradus

$$\partial u + uu\, \partial x + a x^n \partial x = 0$$

agnofcendos, fimulque integralia completa definienda; nafcitur enim haec aequatio ex ifta

$$\partial \partial y + a x^n y\, \partial x^2 = 0, \text{ ponendo } y = e^{\int u \partial x},$$

vnde ex illa viciffim haec oritur, ponendo $u = \frac{\partial y}{y \partial x}$. Cum igitur iftius integrale affignare licuerit cafibus quibus exponens $n = \frac{-4i}{2i+1}$, iisdem cafibus integrale aequationis differentialis primi gradus affignare licebit, vbi quidem duos cafus euolui conuenit, prout $a$ fuerit vel numerus negatiuus $a = -cc$ vel pofitiuus $a = +bb$. Hos igitur duos cafus pertractaffe operae erit pretium.

## Problema 120.

956. Denotante $i$ numerum integrum fiue pofitiuum fiue negatiuum quemcunque, inuenire integrale huius aequationis

$$\partial u + uu\,\partial x - cc\,x^{\frac{-4i}{2i+1}}\,\partial x = 0.$$

### Solutio.

Pofito $u = \frac{\partial y}{y\,\partial x}$, haec aequatio transformatur in iftam

$$\partial\partial y - cc\,x^{\frac{-4i}{2i+1}}\,y\,\partial x^2 = 0,$$

fumto elemento $\partial x$ conftante, cuius integrale affignauimus.

Pofito fcilicet $t = (2i+1)\,x^{\frac{1}{2i+1}}$, eft

$$y = (\alpha e^{ct} + \beta e^{-ct})\left(t^i + \frac{i(ii-1)(i+2)}{2.4\,cc}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^4}t^{i-4} + \text{etc.}\right)$$
$$-(\alpha e^{ct} - \beta e^{-ct})\left(\frac{i(i+1)}{2c}t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^5}t^{i-5} + \text{etc.}\right)$$

Ponamus breuitatis gratia

$$y = (\alpha e^{ct} + \beta e^{-ct})\,P - (\alpha e^{ct} - \beta e^{-ct})\,Q,$$

et cum fit

$$\partial t = x^{\frac{-2i}{2i+1}}\,\partial x, \text{ feu } \partial x = x^{\frac{2i}{2i+1}}\,\partial t,$$

erit

$$\frac{\partial y}{\partial x} = \frac{(\alpha e^{ct} + \beta e^{-ct})(\partial P - cQ\partial t) + (\alpha e^{ct} - \beta e^{-ct})(cP\partial t - \partial Q)}{x^{\frac{2i}{2i+1}}\,\partial t},$$

fiue

$$\frac{\partial y}{\partial x} = \frac{\alpha e^{ct}(\partial P + cP\partial t - \partial Q - cQ\partial t) + \beta e^{-ct}(\partial P - cP\partial t + \partial Q - cQ\partial t)}{x^{\frac{2i}{2i+1}}\,\partial t}.$$

At vero eft

$$\frac{\partial P}{\partial t} = i\, t^{i-1} + \frac{i(ii-1)(ii-4)}{2.\,4\, c^2}\, t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)}{2.\,4.\,6.\,8.\,c^4}\, t^{i-5} + \text{etc.}$$

$$cQ = \frac{i(i+1)}{2}\, t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.\,4.\,6\, c^2}\, t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.\,4.\,6.\,8.\,10\, c^4}\, t^{i-5} + \text{etc.}$$

$$cP = c\, t^i + \frac{i(ii-1)(i+2)}{2.\,4\, c}\, t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.\,4.\,6.\,8\, c^3}\, t^{i-4} + \text{etc.}$$

$$\frac{\partial Q}{\partial t} = \frac{i(ii-1)}{2c}\, t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)}{2.\,4.\,6\, c^3}\, t^{i-4} + \text{etc.}$$

vnde colligitur

$$\frac{\partial P - cQ\,\partial t}{\partial t} = \frac{-i(i-1)}{2}\, t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.\,4.\,6\, c^4}\, t^{i-3} - \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.\,4.\,6.\,8.\,10\, c^4}\, t^{i-5} - \text{etc.}$$

$$\frac{cP\partial t - \partial Q}{\partial t} = c\, t^i + \frac{i(ii-1)(i-2)}{2.\,4\, c}\, t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.\,4.\,6.\,8\, c^3}\, t^{i-4} + \text{etc.}$$

Ponamus ad abbreuiandum

$$P = t^i + \frac{i(ii-1)(i+2)}{2.\,4\, cc}\, t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.\,4.\,6.\,8\, c^4}\, t^{i-4} + \text{etc.}$$

$$Q = \frac{i(i+1)}{2c}\, t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.\,4.\,6\, c^3}\, t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.\,4.\,6.\,8.\,10\, c^5}\, t^{i-5} + \text{etc.}$$

$$R = t^i + \frac{i(ii-1)(i-2)}{2.\,4\, c\, c}\, t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.\,4.\,6.\,8\, c^4}\, t^{i-4} + \text{etc.}$$

$$S = \frac{i(i-1)}{2c}\, t^{i-1} + \frac{i(ii-1)(ii-4)(i-3)}{2.\,4.\,6\, c^3}\, t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.\,4.\,6.\,8.\,10\, c^5}\, t^{i-5} + \text{etc.}$$

vt fit

$$\frac{\partial y}{\partial x} = \frac{(\alpha e^{ct} + \beta e^{-ct})(-cS) + (\alpha e^{ct} - \beta e^{-ct})(cR)}{x^{\frac{2i}{2i+1}}}.$$

Quare cum fit $u = \frac{\partial y}{y\,\partial x}$, erit nostrae aequationis integrale completum

$$\tfrac{1}{c}\, x^{\frac{2i}{2i+1}}\, u = \frac{(\alpha e^{ct} - \beta e^{-ct})\, R - (\alpha e^{ct} + \beta e^{-ct})\, S}{(\alpha e^{ct} + \beta e^{-ct})\, P - (\alpha e^{ct} - \beta e^{-ct})\, Q},\ \text{fiue}$$

$$\tfrac{1}{c}\, x^{\frac{2i}{2i+1}}\, u = \frac{\alpha e^{ct}(R - S) - \beta e^{-ct}(R + S)}{\alpha e^{ct}(P - Q) + \beta e^{-ct}(P + Q)},$$

quod ob rationem constantium $\alpha : \beta$ arbitrariam, est completum.

## Corollarium 1.

957. Quaternae formulae P, Q, R, S, quae singulae casibus, quibus $i$ est numerus integer, abrumpuntur, ita a se inuicem pendent, vt sit primo

$$R = P - \frac{\partial Q}{c\,\partial t} \text{ et } S = Q - \frac{\partial P}{c\,\partial t};$$

tum vero

$$\partial P + \partial R = \frac{+2iR\,\partial t}{t} \text{ et } \partial Q + \partial S = \frac{2iS\,\partial t}{t}.$$

## Corollarium 2.

958. Posito ergo vel $\alpha = 0$ vel $\beta = 0$, integralia particularia algebraica aequationis

$$\partial u + uu\,\partial x - cc\,x^{\frac{-4i}{2i+1}}\,\partial x = 0$$

exhiberi possunt, quae sunt

$$\frac{1}{c}\,x^{\frac{2i}{2i+1}}\,u = \frac{R-S}{P-Q} \text{ et } \frac{1}{c}\,x^{\frac{2i}{2i+1}}\,u = \frac{-S-R}{P+Q},$$

ideoque hac vna formula comprehendi possunt

$$\frac{1}{c}\,x^{\frac{2i}{2i+1}}\,u = \frac{-S \pm R}{P \mp Q}.$$

## Scholion 1.

959. Pro variis ergo valoribus numeri $i$ tam quantitas $t$ quam litterae P, Q, R, S sequenti modo se habebunt: Primo scilicet si $i = 0$, erit $t = x$, atque $P = 1$, $Q = 0$, $R = 1$ et $S = 0$; reliquos casus in sequenti tabella repraesentemus

| | |
|---|---|
| $i = -1,\ t = -\frac{1}{x}$ | $i = 1,\ t = 3x^{\frac{1}{3}}$ |
| $P = \frac{1}{t},\quad Q = 0$ | $P = t,\ Q = \frac{1}{c}$ |
| $R = \frac{1}{t},\quad S = \frac{1}{ctt}$ | $R = t,\ S = 0$ |

$i =$

| | |
|---|---|
| $i = -2,\ t = -\frac{3}{x^{\frac{1}{3}}}$<br>$P = \frac{1}{t^2},\ Q = \frac{1}{c t^3}$<br>$R = \frac{1}{t^2} + \frac{3}{c c t^4},\ S = \frac{3}{c t^3}.$ | $i = 2,\ t = 5 x^{\frac{1}{5}}$<br>$P = t^2 + \frac{3}{c c},\ Q = \frac{3}{c} t.$<br>$R = t^2,\ S = \frac{1}{c} t.$ |
| $i = -3,\ t = \frac{-5}{x^{\frac{1}{5}}}$<br>$P = \frac{1}{t^3} + \frac{1.3}{c c t^5}$<br>$Q = \frac{3}{c t^4}$<br>$R = \frac{1}{t^3} + \frac{3.5}{c c t^5}$<br>$S = \frac{6}{c t^4} + \frac{3.5}{c^3 t^6}$ | $i = 3,\ t = 7 x^{\frac{1}{7}}$<br>$P = t^3 + \frac{3.5}{c c} t$<br>$Q = \frac{6}{c} t t + \frac{3.5}{c^3}$<br>$R = t^3 + \frac{1.3}{c c} t$<br>$S = \frac{3}{c} t t$ |
| $i = -4,\ t = -\frac{7}{x^{\frac{1}{7}}}$<br>$P = \frac{1}{t^4} + \frac{3.5}{c c t^6}$<br>$Q = \frac{6}{c t^5} + \frac{3.5}{c^3 t^7}$<br>$R = \frac{1}{t^4} + \frac{3.3.5}{c c t^6} + \frac{3.5.7}{c^4 t^8}$<br>$S = \frac{10}{c t^5} + \frac{3.5.7}{c^3 t^7}$ | $i = 4,\ t = 9 x^{\frac{1}{9}}$<br>$P = t^4 + \frac{3.3.5}{c c} t^2 + \frac{3.5.7}{c^4}$<br>$Q = \frac{10}{c} t^3 + \frac{3.5.7}{c^3} t$<br>$R = t^4 + \frac{3.5}{c c} t^2$<br>$S = \frac{6}{c} t^3 + \frac{3.5}{c^3} t$ |
| $i = -5,\ t = \frac{-9}{x^{\frac{1}{9}}}$<br>$P = \frac{1}{t^5} + \frac{3.3.5}{c c t^7} + \frac{3.5.7}{c^4 t^9}$<br>$Q = \frac{10}{c t^6} + \frac{3.5.7}{c^3 t^8}$<br>$R = \frac{1}{t^5} + \frac{3.5.7}{c c t^7} + \frac{3.5.7.9}{c^4 t^9}$<br>$S = \frac{15}{c t^6} + \frac{4.3.5.7}{c^3 t^8} + \frac{3.5.7.9}{c^5 t^{10}}$ | $i = 5,\ t = 11 x^{\frac{1}{11}}$<br>$P = t^5 + \frac{3.5.7}{c c} t^3 + \frac{3.5.7.9}{c^4} t$<br>$Q = \frac{15}{c} t^4 + \frac{4.3.5.7}{c^3} t^2 + \frac{3.5.7.9}{c^5}$<br>$R = t^5 + \frac{3.3.5}{c c} t^3 + \frac{3.5.7}{c^4} t$<br>$S = \frac{10}{c} t^4 + \frac{3.5.7}{c^3} t^2.$ |

$i =$

$$i = -6,\ t = \frac{-11}{x^{\frac{1}{11}}}$$

$$P = \frac{1}{t^6} + \frac{3.5.7}{cct^8} + \frac{3.5.7.9}{c^4 t^{10}}$$

$$Q = \frac{15}{ct^7} + \frac{4.3.5.7}{c^3 t^9} + \frac{3.5.7.9}{c^5 t^{11}}$$

$$R = \frac{1}{t^6} + \frac{2.3.5.7}{cct^8} + \frac{5.3.5.7.9}{c^4 t^{10}} + \frac{3.5.7.9.11}{c^6 t^{12}}$$

$$S = \frac{21}{ct^7} + \frac{4.5.7.9}{c^3 t^9} + \frac{3.5.7.9.11}{c^5 t^{11}}$$

---

$$i = 6,\ t = 13\, x^{\frac{1}{13}}$$

$$P = t^6 + \frac{2.3.5.7}{cc} t^4 + \frac{5.3.5.7.9}{c^4} t^2 + \frac{3.5.7.9.11}{c^6}$$

$$Q = \frac{21}{c} t^5 + \frac{4.5.7.9}{c^3} t^3 + \frac{3.5.7.9.11}{c^4} t$$

$$R = t^6 + \frac{3.5.7}{cc} t^4 + \frac{3.5.7.9}{c^4} t^2$$

$$S = \frac{15}{c} t^5 + \frac{4.3.5.7}{c^3} t^3 + \frac{3.5.7.9}{c^5} t.$$

## Scholion 2.

960. Dum hae formulae diligentius confiderantur, noua fe prodit relatio inter valores litterarum P, Q, R, S, quae in hoc confiftit, vt perpetuo fit $PR - QS = t^{2i}$, cuius veritas primo quidem per inductionem deprehenditur, tum vero etiam per relationes fupra datas demonftrari poteft. Si enim valores

$$R = P - \frac{\partial Q}{c\,\partial t} \text{ et } S = Q - \frac{\partial P}{c\,\partial t}$$

in aequationibus

$$\partial P + \partial R = \frac{2iR\,\partial t}{t} \text{ et } \partial Q + \partial S = \frac{2iS\,\partial t}{t}$$

fubftituantur, oriuntur hae duae aequationes

$$2\,\partial P - \frac{\partial\partial Q}{c\,\partial t} = \frac{2iP\,\partial t}{t} - \frac{2i\,\partial Q}{ct} \text{ et}$$
$$2\,\partial Q - \frac{\partial\partial P}{c\,\partial t} = \frac{2iQ\,\partial t}{t} - \frac{2i\,\partial P}{ct},$$

qua-

quarum illa per P, haec vero per $-Q$ multiplicata iunctim dant

$$2P\partial P - 2Q\partial Q + \frac{Q\partial\partial P - P\partial\partial Q}{c\partial t} = \frac{2i\partial t}{t}(PP - QQ) + \frac{2i}{ct}(Q\partial P - P\partial Q).$$

Ponatur

$$PP - QQ = M \text{ et } \frac{Q\partial P - P\partial Q}{c\partial t} = N, \text{ erit}$$

$$\partial M + \partial N = \frac{2i\partial t}{t}(M + N), \text{ seu } \frac{\partial M + \partial N}{M + N} = \frac{2i\partial t}{t},$$

hincque integrando $M + N = C\,t^{2i}$. At est

$$M + N = P\left(P - \frac{\partial Q}{c\partial t}\right) - Q\left(Q - \frac{\partial P}{c\partial t}\right) = PR - QS;$$

euidens autem est pro constante C vnitatem accipi debere.

## Problema 121.

961. Denotante $i$ numerum integrum siue positiuum siue negatiuum quemcunque, inuenire integrale completum huius aequationis

$$\partial u + uu\partial x + bbx^{\frac{-4i}{2i+1}}\partial x = 0.$$

## Solutio.

Posito $u = \frac{\partial y}{y\partial x}$, haec aequatio transformatur in istam

$$\partial\partial y + bbx^{\frac{-4i}{2i+1}}y\partial x^2 = 0,$$

sumto elemento $\partial x$ constante, cuius integrale supra est assignatum. Scilicet posito $t = (2i + 1)x^{\frac{1}{2i+1}}$, inuenimus (953, 954)

$$y = C\left(t^i - \frac{i(ii-1)(i+2)}{2.4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8b^4}t^{i-4} - \text{etc.}\right)\sin.(bt + \zeta)$$
$$+ C\left(\frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6b^3}t^{i-3} + \text{etc.}\right)\cos.(bt + \zeta),$$

cuius loco breuitatis gratia scribamus

 $y =$

$$y = CP\,\text{fin.}\,(bt+\zeta) + CQ\,\text{cof.}\,(bt+\zeta).$$

Hinc ob

$$\partial t = x^{\frac{-2i}{2i+1}}\partial x \quad \text{feu} \quad \partial x = x^{\frac{2i}{2i+1}}\partial t,$$

erit

$$\frac{\partial y}{\partial x} = \frac{C(\partial P - bQ\partial t)\,\text{fin.}\,(bt+\zeta) + C(\partial Q + bP\partial t)\,\text{cof.}\,(bt+\zeta)}{x^{\frac{2i}{2i+1}}\partial t}:$$

vnde cum fit $u = \frac{\partial y}{y\partial x}$, erit

$$u = \frac{(\partial P - bQ\partial t)\,\text{fin.}\,(bt+\zeta) + (\partial Q + bP\partial t)\,\text{cof.}\,(bt+\zeta)}{x^{\frac{2i}{2i+1}}\partial t\,[P\,\text{fin.}\,(bt+\zeta) + Q\,\text{cof.}\,(bt+\zeta)]}.$$

Ponamus

$$P + \frac{\partial Q}{b\partial t} = R \quad \text{et} \quad Q - \frac{\partial P}{b\partial t} = S,$$

vt fit

$$\frac{1}{b}x^{\frac{2i}{2i+1}}u = \frac{R\,\text{cof.}\,(bt+\zeta) - S\,\text{fin.}\,(bt+\zeta)}{P\,\text{fin.}\,(bt+\zeta) + Q\,\text{cof.}\,(bt+\zeta)},$$

erit

$$P = t^i - \frac{i(ii-1)(i+2)}{2.4\,bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8.10\,b^4}t^{i-4} - \text{etc.}$$

$$Q = \frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,b^5}t^{i-5} - \text{etc.}$$

$$R = t^i - \frac{i(ii-1)(i-2)}{2.4.\,bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,b^4}t^{i-4} - \text{etc.}$$

$$S = \frac{i(i-1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,b^5}t^{i-5} - \text{etc.}$$

atque ob angulum $\zeta$ introductum hoc integrale erit completum.

## Corollarium. 1.

962. Quaternarum ergo litterarum P, Q, R, S valores ita a fe inuicem pendent, vt fit primo

$$R = P + \frac{\partial Q}{b\partial t} \quad \text{et} \quad S = Q - \frac{\partial P}{b\partial t},$$

tum

tum vero etiam patet fore

$$\partial P + \partial R = \frac{2iR\partial t}{t} \text{ et } \partial Q + \partial S = \frac{2iS\partial t}{t}.$$

## Corollarium 2.

963. Deinde etiam colligitur fore $PR + QS = t^{2i}$ quae aequalitas ex praecedentis problematis formulis deducitur, ſumto $cc = -bb$, vbi $Q$ et $S$ abeunt in $Q\sqrt{-1}$ et $S\sqrt{-1}$.

## Corollarium 3.

964. Hic caſus a praecedente etiam hoc differt, quod hic nulla dentur integralia particularia algebraica. Quicunque enim valor angulo conſtanti $\zeta$ tribuatur, integrale ſemper ſinum et coſinum cuiusdam anguli inuoluit.

## Scholion 1.

965. Cum igitur aequationis

$$\partial u + uu\partial x + bb\, x^{\frac{-4i}{2i+1}} \partial x = 0$$

integrale completum, poſito $t = (2i+1)\, x^{\frac{1}{2i+1}}$, ſit

$$\frac{1}{b} x^{\frac{2i}{2i+1}} u = \frac{R \operatorname{cof.} (bt + \zeta) - S \operatorname{fin.} (bt + \zeta)}{P \operatorname{fin.} (bt + \zeta) + Q \operatorname{cof.} (bt + \zeta)},$$

pro ſingulis valoribus numeri $i$ quantitas $t$ cum litteris P, Q, R, S ita ſe habebit. Primo ſi $i = 0$, erit $P = 1$, $Q = 0$, $R = 1$ et $S = 0$, item $t = x$, ita vt integrale ſit $\frac{1}{b} u = \frac{\operatorname{cof.} (bx + \zeta)}{\operatorname{fin.} (bx + \zeta)}$; reliquos caſus ſequens tabella exhibet:

| | |
|---|---|
| $i = -1;\ t = -\frac{1}{x}$ | $i = 1;\ t = 3x^{\frac{1}{3}}$ |
| $P = \frac{1}{t};\ Q = 0$ | $P = t;\ Q = \frac{1}{b}$ |
| $R = \frac{1}{t};\ S = \frac{1}{btt}$ | $R = t;\ S = 0$ |



$i =$

| | |
|---|---|
| $i = -2;\ t = \frac{-3}{x^{\frac{1}{3}}}$<br>$P = \frac{1}{tt};\ Q = \frac{1}{bt^3}$<br>$R = \frac{1}{tt} - \frac{3}{bbt^4};\ S = \frac{3}{bt^3}$ | $i = 2;\ t = 5\,x^{\frac{1}{5}}$<br>$P = tt - \frac{3}{bb};\ Q = \frac{3}{b}\,t$<br>$R = tt;\ S = \frac{1}{b}\,t$ |
| $i = -3;\ t = \frac{-5}{x^{\frac{1}{5}}}$<br>$P = \frac{1}{t^3} - \frac{1.3}{bbt^5}$<br>$Q = \frac{3}{bt^4};$<br>$R = \frac{1}{t^3} - \frac{3.5}{bbt^5}$<br>$S = \frac{6}{bt^4} - \frac{3.5}{b^3 t^6}$ | $i = 3;\ t = 7\,x^{\frac{1}{7}}$<br>$P = t^3 - \frac{3.5}{bb}\,t$<br>$Q = \frac{6}{b}\,tt - \frac{3.5}{b^3}$<br>$R = t^3 - \frac{1.3}{bb}\,t$<br>$S = \frac{3}{b}\,tt$ |
| $i = -4;\ t = \frac{-7}{x^{\frac{1}{7}}}$<br>$P = \frac{1}{t^4} - \frac{3.5}{bbt^6}$<br>$Q = \frac{6}{bt^5} - \frac{3.5}{b^3 t^7}$<br>$R = \frac{1}{t^4} - \frac{3.3.5}{bbt^6} + \frac{3.5.7}{b^4 t^8}$<br>$S = \frac{10}{bt^5} - \frac{3.5.7}{b^3 t^7}$ | $i = 4;\ t = 9\,x^{\frac{1}{9}}$<br>$P = t^4 - \frac{3.3.5}{bb}\,tt + \frac{3.5.7}{b^4}$<br>$Q = \frac{10}{b}\,t^3 - \frac{3.5.7}{b^3}\,t$<br>$R = t^4 - \frac{3.5}{bb}\,tt$<br>$S = \frac{6}{b}\,t^3 - \frac{3.5}{b^3}\,t$ |
| $i = -5;\ t = \frac{-9}{x^{\frac{1}{9}}}$<br>$P = \frac{1}{t^5} - \frac{3.3.5}{bbt^7} + \frac{3.5.7}{b^4 t^9}$<br>$Q = \frac{10}{bt^6} - \frac{3.5.7}{b^3 t^8}$<br>$R = \frac{1}{t^5} - \frac{3.5.7}{bbt^7} + \frac{3.5.7.9}{b^4 t^9}$<br>$S = \frac{15}{bt^6} - \frac{4.3.5.7}{b^3 t^8} + \frac{3.5.7.9}{b^5 t^{10}}$ | $i = 5;\ t = 11\,x^{\frac{1}{11}}$<br>$P = t^5 - \frac{3.5.7}{bb}\,t^3 + \frac{3.5.7.9}{b^4}\,t$<br>$Q = \frac{15}{b}\,t^4 - \frac{4.3.5.7}{b^3}\,t^2 + \frac{3.5.7.9}{b^5}$<br>$R = t^5 - \frac{3.3.5}{bb}\,t^3 + \frac{3.5.7}{b^4}\,t$<br>$S = \frac{10}{b}\,t^4 - \frac{3.5.7}{b^3}\,t^2$ |

$$i = -6;\quad t = \frac{-11}{x^{\frac{1}{11}}}$$

$$P = \frac{1}{t^6} - \frac{3.5.7}{bbt^8} + \frac{3.5.7.9}{b^4 t^{10}}$$

$$Q = \frac{15}{bt^7} - \frac{4.3.5.7}{b^3 t^9} + \frac{3.5.7.9}{b^5 t^{11}}$$

$$R = \frac{1}{t^6} - \frac{2.3.5.7}{bbt^8} + \frac{5.3.5.7.9}{b^4 t^{10}} - \frac{3.5.7.9.11}{b^6 t^{12}}$$

$$S = \frac{21}{bt^7} - \frac{4.5.7.9}{b^3 t^9} + \frac{3.5.7.9.11}{b^5 t^{11}}$$

---

$$i = 6;\quad t = 13\, x^{\frac{1}{13}}$$

$$P = t^6 - \frac{2.3.5.7}{bb} t^4 + \frac{5.3.5.7.9}{b^4} tt - \frac{3.5.7.9.11}{b^6}$$

$$Q = \frac{21}{b} t^5 - \frac{4.5.7.9}{b^3} t^3 + \frac{3.5.7.9.11}{b^5} t$$

$$R = t^6 - \frac{3.5.7}{bb} t^4 + \frac{3.5.7.9}{b^4} tt$$

$$S = \frac{15}{b} t^5 - \frac{4.3.5.7}{b^3} t^3 + \frac{3.5.7.9}{b^5} t.$$

## Scholion 2.

966. Forma integralis inuenta modum fuppeditat, aequationem propofitam

$$\partial u + uu\,\partial x + A\, x^{\frac{-4i}{2i+1}} \partial x = 0$$

in fpeciem fimpliciorem transformandi. Primo enim ponatur

$$x^{\frac{2i}{2i+1}} u = v,\ \text{feu}\ u = x^{\frac{-2i}{2i+1}} v,$$

ac prodibit

$$x^{\frac{-2i}{2i+1}} \partial v - \frac{2i}{2i+1} x^{\frac{-4i-1}{2i+1}} v\,\partial x + x^{\frac{-4i}{2i+1}} vv\,\partial x$$
$$+ A\, x^{\frac{-4i}{2i+1}} \partial x = 0,\ \text{feu}$$

$$\partial v - \frac{2i}{2i+1} \cdot \frac{v\,\partial x}{x} + x^{\frac{-2i}{2i+1}} vv\,\partial x + A\, x^{\frac{-2i}{2i+1}} \partial x = 0.$$

Ponatur porro $t = (2i+1)x^{\frac{1}{2i+1}}$, erit

$$\partial t = x^{\frac{-2i}{2i+1}}\partial x, \text{ et } \frac{\partial t}{t} = \frac{1}{2i+1}\cdot\frac{\partial x}{x},$$

vnde fit

$$\partial v - \frac{2iv\partial t}{t} + vv\partial t + A\partial t = 0.$$

Sit insuper $v = \frac{i}{t} + z$, vt prodeat

$$-\frac{i\partial t}{tt} + \partial z - \frac{2ii\partial t}{tt} - \frac{2iz\partial t}{t} + \frac{ii\partial t}{tt} + \frac{2iz\partial t}{t} + zz\partial t + A\partial t = 0$$

seu

$$\partial z + zz\partial t - \frac{i(i+1)\partial t}{tt} + A\partial t = 0,$$

quae ergo quoties $i$ est numerus integer, est integrabilis. Simili modo haec aequatio

$$\partial u + uu\partial x + Ax^n\partial x = 0$$

generalius ita transformari potest: posito $u = x^\lambda v$ et $v = z - \frac{1}{2}\lambda x^{-\lambda-1}$, obtinetur

$$\partial z + x^\lambda zz\partial x + \tfrac{1}{4}\lambda(\lambda+2)x^{-\lambda-2}\partial x + Ax^{n-\lambda}\partial x = 0,$$

quae porro posito $x^\lambda\partial x = \partial t$, seu $x^{\lambda+1} = (\lambda+1)t$, abit in

$$\partial z + zz\partial t + \frac{\lambda(\lambda+2)\partial t}{4(\lambda+1)^2tt} + A(\lambda+1)^{\frac{n-2\lambda}{\lambda+1}}t^{\frac{n-2\lambda}{\lambda+1}}\partial t = 0,$$

quae aequatio est integrabilis, quoties $n = \frac{-4i}{2i+1}$, vnde numerum $\lambda$ pro lubitu assumendo, innumerabiles formae exhiberi possunt. Si capiatur $\lambda = -1$, fit $t = lx$ et

$$\partial z + zz\partial t - \tfrac{1}{4}\partial t + Ae^{(n+2)t}\partial t = 0.$$

CA-

# CAPVT VIII.

## DE ALIARVM AEQVATIONVM DIFFERENTIO - DIFFERENTIALIVM RESOLVTIONE PER SERIES INFINITAS.

### Problema 122.

967.

Formam generalem aequationum differentio - differentialium, quas commode per feries refoluere licet, exhibere, earumque integralia inueftigare.

### Solutio.

Primo alias aequationes commode per feries refoluere non licet, nifi in quibus altera variabilis $y$ cum fuis differentialibus $\partial y$ et $\partial \partial y$ nusqnam plus vna dimenfione obtinet; quoniam pro $y$ feriem infinitam fubftituendo in calculos nimis moleftos incideremus, fi vsquam plures dimenfiones ingrederentur. Huiusmodi ergo aequationes in hac forma

$$\partial \partial y + M \partial x \partial y + N y \partial x^2 = X \partial x^2$$

continentur. Tum vero vt feriei pro $y$ affumtae quilibet terminus per folum praecedentem determinetur, qui eft cafus refolutionis maxime notabilis, duplicis tantum generis terminos ratione alterius variabilis $x$ ineffe oportet, fiquidem ad dimenfiones, quas ipfa $x$ cum fuo differentiali $\partial x$ conftituit, refpiciamus. Vnde primo quidem, reiecto termino $X \partial x^2$, aequationes hoc modo refolubiles in hac forma continentur

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0.$$

Pro cuius resolutione fingamus

$$y=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

et facta substitutione, sequens serierum summa ad nihilum redigi debet

$$\begin{array}{llll}
\lambda(\lambda-1)Aax^\lambda & +(\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} & +\text{etc.}\\
 & +\lambda(\lambda-1)Ab & +(\lambda+n)(\lambda+n-1)Bb & \\
+\lambda Ac & +(\lambda+n)Bc & +(\lambda+2n)Cc & \\
 & +\lambda Ae & +(\lambda+n)Be & \\
+Af & +Bf & +Cf & \\
 & +Ag & +Bg. &
\end{array}$$

Hic ergo primo exponens $\lambda$ ita accipi debet, vt sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero pro reliquis fieri oportet

$$[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f]B=-[\lambda(\lambda-1)b+\lambda e+g]A,$$

$$[(\lambda+2n)(\lambda+2n-1)a+(\lambda+2n)c+f]C= -[(\lambda+n)(\lambda+n-1)b+(\lambda+n)e+g]B,$$

$$[(\lambda+3n)(\lambda+3n-1)a+(\lambda+3n)c+f]D= -[(\lambda+2n)(\lambda+2n-1)b+(\lambda+2n)e+g)C,$$

etc.

Cum igitur sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

si ponamus breuitatis causa

$$\lambda(\lambda-1)b+\lambda e+g=h,$$

erit

$$[n(n+2\lambda-1)a+nc]B=-hA$$

$$[2n(2n+2\lambda-1)a+2nc]C=-[n(n+2\lambda-1)b+ne+h]B$$

$$[3n(3n+2\lambda-1)a+3nc]D=-[2n(2n+2\lambda-1)b+2ne+h]C,$$

etc. Quia

Quia ergo nisi $a = 0$, pro $\lambda$ gemini inueniuntur valores, scilicet

$$\lambda = \frac{a - c \pm \sqrt{[(a-c)^2 - 4af]}}{2a},$$

binae series pro $y$ inueniuntur, quae vtcunque combinatae integrale completum aequationis propositae praebent.

## Aliter.

Proposita aequatione hac

$$xx(a+bx^n)\partial\partial y + x(c+ex^n)\partial x\partial y + (f+gx^n)y\partial x^2 = 0,$$

series quoque ordine retrogrado fingi potest

$$y = Ax^\lambda + Bx^{\lambda-n} + Cx^{\lambda-2n} + Dx^{\lambda-3n} + \text{etc.}$$

vnde oritur ad nihilum reducendum

| | | | |
|---|---|---|---|
| $+\lambda(\lambda-1)Abx^{\lambda+n}$ | $+(\lambda-n)(\lambda-n-1)Bbx^\lambda$ | $+(\lambda-2n)(\lambda-2n-1)Cbx^{\lambda-n}$ | + etc. |
| | $+\lambda(\lambda-1)Aa$ | $+(\lambda-n)(\lambda-n-1)Ba$ | |
| $+\lambda Ae$ | $+(\lambda-n)Be$ | $+(\lambda-2n)Ce$ | |
| | $+\lambda Ac$ | $+(\lambda-n)Bc$ | |
| $+Ag$ | $+Bg$ | $+Cg$ | |
| | $+Af$ | $+Bf$ | |

Hic ergo exponentem $\lambda$ ita accipi oportet, vt fiat

$$\lambda(\lambda-1)b + \lambda e + g = 0.$$

Tum vero si ponamus

$$\lambda(\lambda-1)a + \lambda c + f = h,$$

determinatio coefficientium ita se habebit

$$n[(n-2\lambda+1)b-e]B = -hA,$$

$$2n[(2n-2\lambda+1)b-e]C = -[n(n-2\lambda+1)a-nc+h]B,$$

$$3n[(3n-2\lambda+1)b-e]D = -[2n(2n-2\lambda+1)a-2nc+h]C,$$

etc.



Corol-

## Corollarium 1.

968. Ex priore folutione, fi $i$ denotet numerum integrum pofitiuum, feries affumta alicubi abrumpetur, fi fuerit

$$in(in+2\lambda-1)b+ine+b=0, \text{ vel}$$

$$(\lambda+in)(\lambda+in-1)b+(\lambda+in)e+g=0,$$

hoc eft

$$\left.\begin{matrix}\lambda\lambda b+\lambda(2in-1)b+in(in-1)b\\ +\lambda e \qquad\qquad +ine+g\end{matrix}\right\}=0.$$

## Corollarium 2.

969. Aequatio ergo noftra integrationem admittit, fi litterae $f$ et $g$ ita fuerint comparatae, vt fit

$$f=-\lambda(\lambda-1)a-\lambda c \text{ et}$$

$$g=-(\lambda+in)(\lambda+in-1)b-(\lambda+in)e.$$

Vel fumtis duobus numeris $\mu$ et $\nu$, vt fit $\nu-\mu$ diuifibile per exponentem $n$, fi fuerit

$$f=-\mu(\mu-1)a-\mu c \text{ et } g=-\nu(\nu-1)b-\nu e.$$

## Corollarium 3.

970. Cum hinc fit

$$\mu=\frac{a-c\pm\sqrt{[(a-c)^2-4af]}}{2a} \text{ et } \nu=\frac{b-e\pm\sqrt{[(b-e)^2-4bg]}}{2b},$$

aequatio habebit integrale algebraicum, fi fuerit $\nu-\mu=in$, denotante $i$ numerum integrum pofitiuum: hoc eft fi fit

$$in=\frac{c}{2a}-\frac{e}{2b}\pm\frac{\sqrt{[(b-e)^2-4bg]}}{2b}\mp\frac{\sqrt{[(a-c)^2-4af]}}{2a}.$$

## Corollarium 4.

971. Pro ferie autem inueuienda fi eueniat, vt exponens $\lambda$ fiat imaginarius, notari conuenit effe

$$x^{\alpha+\beta\sqrt{-1}}=x^{\alpha}.e^{\beta\sqrt{-1}.lx}=x^{\alpha}(\text{cof.}\beta lx+\sqrt{-1}.\text{fin.}\beta lx),$$

vnde

vnde binae feries ita combinari poterunt, vt integrale confequatur formam realem.

## Scholion.

972. Vtraque folutio generatim fpectata duplicem feriem pro variabili $y$ fuppeditat, pro gemino exponentis $\lambda$ valore, quarum combinatio integrale completum exhibet. Solutio fcilicet prior pro exponente $\lambda$ hos duos praebet valores

$$\lambda = \frac{a - c \pm \sqrt{[(a-c)^2 - 4af]}}{2a},$$

folutio vero pofterior

$$\lambda = \frac{b - e \pm \sqrt{[(b-e)^2 - 4bg]}}{4b},$$

ita vt hoc modo integrale completum duplici modo exprimi poffit; quae binae formae etiamfi maxime diuerfae, atque adeo interdum altera per exponentes imaginarios progrediatur, dum altera habet reales, tamen fibi aequipollentes effe debent. Quin etiam euenire poteft, vt altera folutio vel etiam vtraque ad integrale completum exhibendum fit inepta, dum vnicam feriem fuppeditet. Incommodum hoc pro vtraque folutione duplici modo accidere poteft; pro priori nempe folutione, vbi exponentem $\lambda$ ex hac aequatione

$$\lambda(\lambda - 1)a + \lambda c + f = 0,$$

definiri oportet, vnicus inde pro $\lambda$ eruitur valor, fi fuerit vel $a = 0$, vel $4af = (a-c)^2$, priori cafu tantum fit $\lambda = -\frac{f}{c}$, altero ipfius $\lambda$ valore quafi in infinitum abeunte. Pofteriori cafu vero ambo ipfius $\lambda$ valores fiunt inter fe aequales, fcilicet $\lambda = \frac{a-c}{2a}$. Idem incommodum in altera folutione locum habet, fi fuerit vel $b = 0$, vel $4bg = (b-e)^2$: vnde patet fieri poffe, vt altera folutio huiusmodi incommodo laboret, dum altera eo careat, quin etiam vt vtraque eodem inquinetur. Quocirca oftendi conueniet, quemadmodum etiam his

 caficasibus integrale completum inuestigari debeat; quorsum etiam casum referamus, quo ambo ipsius $\lambda$ valores fiunt imaginarii, quandoquidem ad imaginariam speciem tollendam singulari artificio est opus. Denique vero etiam binae series pro $y$ exhibendae difficultate premuntur, quoties bini valores ipsius $\lambda$ differentiam habent per exponentem $n$ diuisibilem, quorum casuum euolutio etiam explicari meretur.

## Problema 123.

973. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

si eueniat, vt binae series ascendentes pro $y$ assumtae vel in vnam coalescant, vel altera fiat impossibilis, integrale completum per series exprimere.

## Solutio.

Assumta serie

$$y=Ax^\lambda+Bx^{\lambda+n}+C^{\lambda+2n}+D^{\lambda+3n}+\text{etc.}$$

si eueniat vt bini valores ipsius $\lambda$ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0$$

vel fiant aequales, vel differentiam per $n$ diuisibilem obtineant, valor ipsius $y$ praeter potestates ipsius $x$ etiam logarithmum ipsius $x$ inuoluet. Quare pro aequationis resolutione statim ponamus $y=u+vlkx$, vt sit $y=u+vlx+\alpha v$, denotante $\alpha$ quantitatem constantem quamcunque. Hinc erit

$$\partial y=\partial u+\frac{v\partial x}{x}+\partial vlkx,$$

$$\partial\partial y=\partial\partial u+\frac{2\partial x\partial v}{x}-\frac{v\partial x^2}{xx}+\partial\partial vlkx,$$

quibus valoribus substitutis, aequatio nostra hanc induet formam

$$\left.\begin{array}{r} xx(a+bx^n)\partial\partial u + 2x(a+bx^n)\partial x\,\partial v - (a+bx^n)v\,\partial x^2 \\ +\,x(c+ex^n)\,\partial x\,\partial u + (c+ex^n)\,v\,\partial x^2 \\ +\,(f+gx^n)\,u\,\partial x^2 \\ +[xx(a+bx^n)\,\partial\partial v + x(c+ex^n)\,\partial x\,\partial v + (f+gx^n)v\partial x^2]\,lkx \end{array}\right\} = 0,$$

vbi partem postremam logarithmo affectam seorsim nihilo aequari oportet. Quare posito

$$v = A\,x^\lambda + B\,x^{\lambda+n} + C\,x^{\lambda+2n} + D\,x^{\lambda+3n} + \text{etc.}$$

exponenti $\lambda$ ex aequatione

$$\lambda(\lambda-1)\,a + \lambda c + f = 0,$$

is valor tribuatur, qui nulli incommodo est obnoxius; eritque pro reliquis coefficientibus,

$$\text{ponendo } \lambda(\lambda-1)\,b + \lambda e + g = nb,$$

vt sequitur

$$[(n+2\lambda-1)a+c]B + bA = 0,$$

$$2[(2n+2\lambda-1)a+c]C + [(n+2\lambda-1)b+e]B + bB = 0,$$

$$3[(3n+2\lambda-1)a+c]D + 2[(2n+2\lambda-1)b+e]C + bC = 0,$$

$$4[(4n+2\lambda-1)a+c]E + 3[(3n+2\lambda-1)b+e]D + bD = 0,$$

etc.

His coefficientibus ita definitis, quorum primus A arbitrio nostro relinquitur, ponamus

$$u = \Delta + \mathfrak{A}\,x^\lambda + \mathfrak{B}\,x^{\lambda+n} + \mathfrak{C}\,x^{\lambda+2n} + \mathfrak{D}\,x^{\lambda+3n} + \text{etc.}$$

qui valor si in priori aequatione cum serie pro $v$ inuenta substituatur, sequentes series ad nihilum reduci oportet

$$\begin{array}{llll}
xx(a+bx^n)\frac{\partial\partial\Delta}{\partial x^2} & +\lambda(\lambda-1)\mathfrak{A}ax^{\lambda} & +(\lambda+n)(\lambda+n-1)\mathfrak{B}ax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)\mathfrak{C}ax^{\lambda+2n}+\text{etc} \\
+x(c+ex^n)\frac{\partial\Delta}{\partial x} & & +\lambda(\lambda-1)\mathfrak{A}b & +(\lambda+n)(\lambda+n-1)\mathfrak{B}b \\
 & +\lambda\mathfrak{A}c & +(\lambda+n)\mathfrak{B}c & +(\lambda+2n)\mathfrak{C}c \\
+(f+gx^n)\Delta & & +\lambda\mathfrak{A}e & +(\lambda+n)\mathfrak{B}e \\
 & +\mathfrak{A}f & +\mathfrak{B}f & +\mathfrak{C}f \\
 & & +\mathfrak{A}g & +\mathfrak{B}g \\
 & +2\lambda Aa & +2(\lambda+n)Ba & +2(\lambda+2n)Ca \\
 & & +2\lambda Ab & +2(\lambda+n)Bb \\
 & & +B(c-a) & +C(c-a) \\
 & +A(c-a) & +A(e-b) & +B(e-b)
\end{array}$$

Cum autem fit

$$\lambda(\lambda-1)a+\lambda c+f=0 \text{ et } \lambda(\lambda-1)b+\lambda e+g=nb,$$

expreffio haec transmutabitur in hanc formam

$$xx(a+bx^n)\frac{\partial\partial\Delta}{\partial x^2}+x(c+ex^n)\frac{\partial\Delta}{\partial x}+(f+gx^n)\Delta$$

$$\begin{array}{lll}
[(2\lambda-1)a+c]Ax^{\lambda} & +[(2n+2\lambda-1)a+c]Bx^{\lambda+n} & +[(4n+2\lambda-1)a+c]Cx^{\lambda+2n}+\text{etc.} \\
 & +[(2\lambda-1)b+e]A & +[(2n+2\lambda-1)b+e]B \\
 & +n[(n+2\lambda-1)a+c]\mathfrak{B} & +2n[(2n+2\lambda-1)a+c]\mathfrak{C} \\
 & & +n[(n+2\lambda-1)b+e]\mathfrak{B} \\
 & +nb\mathfrak{A} & +nb\mathfrak{B}
\end{array}$$

vbi $\Delta$ denotat quosdam terminos feriei

$$\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\text{ etc.}$$

praemittendos, ita vt ordine retrogrado fit

$$\Delta=\mathfrak{a}x^{\lambda-n}+\mathfrak{b}x^{\lambda-2n}+\mathfrak{c}x^{\lambda-3n}+\ldots+\mathfrak{i}x^{\lambda-in}.$$

Quod principium quomodo quouis cafu fit conftituendum, fequentia funt obferuanda.

I. Prin-

I. Principium hoc locum habere nequit, nisi fuerit

$$(\lambda - in)(\lambda - in - 1)a + (\lambda - in)c + f = 0,$$

cum igitur sit

$$\lambda(\lambda - 1)a + \lambda c + f = 0,$$

inde erit

$$\lambda = in + \frac{a - c - \sqrt{[(a-c)^2 - 4af]}}{2a},$$

hinc vero

$$\lambda = \frac{a - c + \sqrt{[(a-c)^2 - 4af]}}{2a},$$

quandoquidem hi duo valores conuenire nequeunt, nisi ibi signum radicale negatiue, hic vero positiue accipiatur. Aequatis autem his valoribus fit

$$in = \tfrac{1}{a}\sqrt{[(a-c)^2 - 4af]}, \text{ seu}$$
$$iinnaa = (a-c)^2 - 4af, \text{ hincque}$$
$$f = \frac{(a-c)^2}{4a} - \tfrac{1}{4}iinna,$$

vnde fit

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in.$$

Quare si aequatio proposita ita fuerit comparata, vt sit

$$ina = \sqrt{[(a-c)^2 - 4af]},$$

tum sumto

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in,$$

ac pro $v$ sumta serie

$$v = Ax^{\lambda} + Bx^{\lambda+n} + \text{etc.}$$

alteram seriem $u$ ita constitui conuenit

$$u = \mathfrak{j}\,x^{\lambda-in} + \ldots\ldots + \mathfrak{a}\,x^{\lambda-n} + \mathfrak{A}\,x^{\lambda} + \mathfrak{B}\,x^{\lambda+n}$$
$$+ \mathfrak{C}\,x^{\lambda+2n} + \text{etc.}$$

Hic est casus, quo bini valores ipsius $\lambda$ ex aequatione

λ

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

differentiam habent per $n$ diuifibilem, vbi notandum, feriem $v$ a maiore valore ipfius $\lambda$, feriem vero $u$ a minore inchoari debere.

II. Principium $\Delta$ omitti nequit, nifi fuerit

$$(2\lambda-1)a+c=0;$$

quo cafu fit $\lambda=\frac{a-c}{2}$: atque hic eft cafus, quo aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0$$

binae radices fiunt inter fe aequales, ideoque $f=\frac{(a-c)^2}{4a}$. Continetur ergo hic cafus in praecedente, fumendo ibi $i=0$. Quare hoc modo refoluentur cafus, quibus bini valores ipfius $\lambda$ vel funt inter fe aequales, vel differentiam habent per exponentem $n$ diuifibilem. Sicque reperitur integrale completum per duas feries afcendentes $v$ et $u$ expreffum, quarum illa $v$ per $lx$ multiplicatur.

## Corollarium 1.

974. Quando ergo in aequatione propofita coefficientes $a$, $c$ et $f$ ita funt comparati, vt aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0$$

radices fint $\lambda=\mu$ et $\lambda=\mu-in$, denotante $i$ numerum integrum pofitiuum, integrale completum huiusmodi habebit formam $y=u+\alpha v+vlx$.

## Corollarium 2.

975. Hic autem binae quantitates $v$ et $u$ ex his aequationibus

$$\text{I.}\quad xx(a+bx^n)\partial\partial v + x(c+ex^n)\partial x\partial v + (f+gx^n)v\partial x^2 = 0,$$

$$\text{II.}\quad \left.\begin{aligned} xx(a+bx^n)\partial\partial u + x(c+ex^n)\partial x\partial u + (f+gx^n)u\partial x^2 \\ +2x(a+bx^n)\partial x\partial v - (a+bx^n)v\partial x^2 \\ +(c+ex^n)v\partial x^2 \end{aligned}\right\} = 0,$$

ita determinari poterunt, vt ponatur

$$v = \mathrm{A}x^\mu + \mathrm{B}x^{\mu+n} + \mathrm{C}x^{\mu+2n} + \mathrm{D}x^{\mu+3n} + \text{etc.}$$

$$u = \mathfrak{A}x^{\mu-in} + \mathfrak{B}x^{\mu-in+n} + \mathfrak{C}x^{\mu-in+2n} + \mathfrak{D}x^{\mu-in+3n} + \text{etc.}$$

Has ſcilicet ſeries ſubſtituendo omnes coefficientes ex vno definire licebit.

## Scholion.

976. Logarithmo ergo ipſius $x$ in ſubſidium vocato, his caſibus, quos commemorauimus, integrale completum aequationis propoſitae per ſeries aſcendentes exhiberi poteſt, dum ſine hoc artificio integrale tantum particulare inuenitur. Quando enim aequatio $\lambda(\lambda-1)a+\lambda c+f=0$ duas radices habet, quarum differentia per exponentem $n$ eſt diuiſibilis, puta $\lambda=\mu$ et $\lambda=\mu-in$, priore methodo ſola ſeries, quae incipit a poteſtate $x^\mu$, determinari poteſt; ſi enim altera a poteſtate $x^{\mu-in}$ incipiens pro $y$ aſſumeretur, coefficiens cuiusdam termini reperiretur infinitus, vnde ſequentes omnes forent quoque infiniti, quod incommodum introducendo logarithmus ipſius $x$ feliciter tollitur. Hunc igitur vſum iſtius reſolutionis aliquot exemplis illuſtraſſe iuuabit.

## Exemplum 1.

977. *Aequationis differentio - differentialis*

$$x\partial\partial y + \partial x\partial y + gx^{n-1}y\partial x^2 = 0$$

*integrale completum per ſeries aſcendentes exhibere.*

Hanc aequationem ad noſtram formam reducendo habebimus

$$xx\partial\partial y + x\partial x\partial y + gx^n y\partial x^2 = 0,$$

vbi ergo est $a = 1$, $b = 0$, $c = 1$, $e = 0$ et $f = 0$. Hinc $\lambda(\lambda - 1) + \lambda = 0$, seu $\lambda\lambda = 0$, ita vt bini valores ipsius $\lambda$ sint aequales et $= 0$. Quare posito $y = u + \alpha v + vlx$, resolui oportet has aequationes

$$\text{I. } xx\partial\partial v + x\partial x\partial v + gx^n v\partial x^2 = 0 \text{ et}$$

$$\text{II. } xx\partial\partial u + x\partial x\partial u + gx^n u\partial x^2 = 0$$
$$+ 2x\partial x\partial v.$$

Statuamus ergo

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc. et}$$
$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

ac prior aequatio praebet

$$\left.\begin{array}{llll} n(n-1)Bx^n & + 2n(2n-1)Cx^{2n} & + 3n(3n-1)Dx^{3n} & + \text{etc.} \\ + nB & + 2nC & + 3nD & \\ + Ag & + Bg & + Cg & \end{array}\right\} = 0,$$

vnde fit

$$B = \frac{-Ag}{nn},\ C = \frac{-Bg}{4nn},\ D = \frac{-Cg}{9nn},\ E = \frac{-Dg}{16nn},\ \text{etc.}$$

Tum vero altera aequatio dat

$$\left.\begin{array}{llll} nn\mathfrak{B}x^n & + 4nn\mathfrak{C}x^{2n} & + 9nn\mathfrak{D}x^{3n} & + \text{etc.} \\ + \mathfrak{A}g & + \mathfrak{B}g & + \mathfrak{C}g & \\ + 2nB & + 4nC & + 6nD & \end{array}\right\} = 0,$$

vnde colligitur

$$\mathfrak{B} = \frac{-\mathfrak{A}g}{nn} - \frac{2B}{n},\ \mathfrak{C} = \frac{-\mathfrak{B}g}{4nn} - \frac{2C}{2n},\ \mathfrak{D} = \frac{-\mathfrak{C}g}{9nn} - \frac{2D}{3n},\ \text{etc.}$$

Hic autem tuto assumere licet $\mathfrak{A} = 0$, quoniam termini ex $\mathfrak{A}$ oriundi, continentur in membro $\alpha v$. Cum igitur sit

$$B = \frac{-Ag}{n^2},\ C = \frac{+Ag^2}{1.4n^4},\ D = \frac{-Ag^3}{1.4.9n^6},\ E = \frac{+Ag^4}{1.4.9.16n^8},\ \text{etc.}$$

erit vt sequitur

$\mathfrak{B} =$

$$\mathfrak{B} = \frac{2Ag}{n^3},\ \mathfrak{C} = \frac{-2Agg}{4n^5} - \frac{2Agg}{2.1.4n^5} = \frac{-6Agg}{2.1.4n^5},$$

$$\mathfrak{D} = \frac{6Ag^3}{2.1.4.9n^7} + \frac{2Ag^3}{3.1.4.9n^7} = \frac{22Ag^3}{2.3.1.4.9n^7},$$

$$\mathfrak{E} = \frac{-22Ag^4}{2.3.1.4.9.16n^9} - \frac{2Ag^4}{4.1.4.9.16n^9} = \frac{-100Ag^4}{2.3.4.1.4.9.16n^9},$$

$$\mathfrak{F} = \frac{100Ag^5}{2.3.4.1.4.9.16.25n^{11}} + \frac{2Ag^5}{5.1.4.9.16.25n^{11}} = \frac{548Ag^5}{2.3.4.5.1.4.9.16.25n^{11}}$$

ſicque obtinentur ſequentes valores

$$\mathfrak{B} = \frac{2Ag}{n^3},\ \mathfrak{C} = \frac{-6Ag^2}{1.8n^5},\ \mathfrak{D} = \frac{22Ag^3}{1.8.27.n^7},\ \mathfrak{E} = \frac{-100Ag^4}{1.8.27.64n^9},$$

$$\mathfrak{F} = \frac{548Ag^5}{1.8.27.64.125n^{11}},\ \mathfrak{G} = \frac{-3528Ag^6}{1.8.27.64.125.216n^{13}},\ \text{etc.}$$

vbi numeratores 2, 6, 22, 100, 548, 3528, etc. ſinguli ita per binos praecedentes definiuntur

$$6 = 3.2 - 1.0,\ 22 = 5.6 - 4.2,\ 100 = 7.22 - 9.6,$$

$$548 = 9.100 - 16.22,\ 3528 = 11.548 - 25.100\ \text{etc.}$$

Conſequenter integrale ita exprimetur

$$y = \frac{2Ag}{n^3}x^n - \frac{6Ag^2}{1.8n^5}x^{2n} + \frac{22Ag^3}{1.8.27n^7}x^{3n} - \frac{100Ag^4}{1.8.27.64n^9}x^{4n} + \text{etc.}$$

$$+ A\left(1 - \frac{g}{nn}x^n + \frac{g^2}{1.4n^4}x^{2n} - \frac{g^3}{1.4.9n^6}x^{3n} + \frac{g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}\right)lx$$

$$+ \alpha - \frac{\alpha g}{nn}x^n + \frac{\alpha g^2}{1.4n^4}x^{2n} - \frac{\alpha g^3}{1.4.9n^6}x^{3n} + \frac{\alpha g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}$$

vbi A et $\alpha$ ſunt binae conſtantes arbitrariae.

## Exemplum 2.

978. *Aequationis differentio-differentialis*

$$x(1 - xx)\partial\partial y - (1 + xx)\partial x \partial y + xy\partial x^2 = 0,$$

*integrale completum per ſeries aſcendentes aſſignare.*

Ad formam noſtram reducta eſt

$$xx(1 - xx)\partial\partial y - x(1 + xx)\partial x\partial y + xxy\partial x^2 = 0,$$

ita vt ſit $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$, $f = 0$ et $g = 1$, vnde aequationis $\lambda(\lambda - 1) - \lambda = 0$ radices ſint $\lambda = 0$ et $\lambda = 2$, quarum differentia per $n = 2$ diuiſa dat 1.


Poſito

Pofito ergo $y = u + \alpha v + v\,l\,x$, ftatui debet

$$v = A\,x^2 + B\,x^4 + C\,x^6 + D\,x^8 + \text{etc. et}$$

$$u = \mathfrak{A} \quad + \mathfrak{B}\,x^2 + \mathfrak{C}\,x^4 + \mathfrak{D}\,x^6 + \mathfrak{E}\,x^8 + \text{etc.}$$

quae feries ex fequentibus aequationibus determinari debent

$$\text{I.}\quad x\,x\,(1 - x\,x)\,\partial\partial v - x\,(1 + x\,x)\,\partial x\,\partial v + x\,x\,v\,\partial x^2 = 0,$$

$$\text{II.}\quad \left.\begin{array}{r} x\,x\,(1 - x\,x)\,\partial\partial u - x\,(1 + x\,x)\,\partial x\,\partial u + x\,x\,u\,\partial x^2 \\ + 2\,x\,(1 - x\,x)\,\partial x\,\partial v - 2\,v\,\partial x^2 \end{array}\right\} = 0.$$

Hinc pro prioris determinatione fit

$$\left.\begin{array}{rrrrl} & 2\,A\,x^2 & + 12\,B\,x^4 & + 30\,C\,x^6 & + 56\,D\,x^8 + \text{etc.} \\ & & - 2\,A & - 12\,B & - 30\,C \\ -2\,A & & - 4\,B & - 6\,C & - 8\,D \\ & & - 2\,A & - 4\,B & - 6\,C \\ & & + A & + B & + C \end{array}\right\} = 0,$$

ideoque

$$2.\,4\,B = 1.\,3\,A,\quad 4.\,6\,C = 3.\,5\,B,\quad 6.\,8\,D = 5.\,7\,C,\ \text{etc.}$$

feu

$$B = \frac{1.\,3}{2.\,4}A,\quad C = \frac{1.\,3.\,3.\,5}{2.\,4.\,4.\,6}A,\quad D = \frac{1.\,3.\,3.\,5.\,5.\,7}{2.\,4.\,4.\,6.\,6.\,8}A,\ \text{etc.}$$

Ex altera vero aequatione reperitur

$$\left.\begin{array}{rrrrl} & 2\,\mathfrak{B}\,x^2 & + 12\,\mathfrak{C}\,x^4 & + 30\,\mathfrak{D}\,x^6 & + 56\,\mathfrak{E}\,x^8 + \text{etc.} \\ & & - 2\,\mathfrak{B} & - 12\,\mathfrak{C} & - 30\,\mathfrak{D} \\ -2\,\mathfrak{B} & & - 4\,\mathfrak{C} & - 6\,\mathfrak{D} & - 8\,\mathfrak{E} \\ & & - 2\,\mathfrak{B} & - 4\,\mathfrak{C} & - 6\,\mathfrak{D} \\ + \mathfrak{A} & & + \mathfrak{B} & + \mathfrak{C} & + \mathfrak{D} \\ + 4\,A & & + 8\,B & + 12\,C & + 16\,D \\ & & - 4\,A & - 8\,B & - 12\,C \\ - 2\,A & & - 2\,B & - 2\,C & - 2\,D \end{array}\right\} = 0.$$

vnde fieri oportet

$\mathfrak{A}$ +

$$\mathfrak{A}+2A=0,\quad 2.4\,\mathfrak{C}-1.3\,\mathfrak{B}+6B-4A=0,$$
$$4.6\,\mathfrak{D}-3.5\,\mathfrak{C}+10C-8B=0,$$
$$6.8\,\mathfrak{E}-5.7\,\mathfrak{D}+14D-12C=0,\ \text{etc.}$$

ſeu cum ſit

$$B=\tfrac{1.3}{2.4}A,\quad C=\tfrac{3.5}{4.6}B,\quad D=\tfrac{5.7}{6.8}C,\ \text{etc.}$$

erit $\mathfrak{A}=-2A$, tum vero

$$2.\quad 4\,\mathfrak{C}-1.3\,\mathfrak{B}-\tfrac{2.7}{2.4}A=0,\quad \mathfrak{C}=\tfrac{1.3}{2.4}\mathfrak{B}+\tfrac{2.7}{2^2.4^2}A;$$
$$4.\quad 6\,\mathfrak{D}-3.5\,\mathfrak{C}-\tfrac{2.21}{4.6}B=0,\quad \mathfrak{D}=\tfrac{3.5}{4.6}\mathfrak{C}+\tfrac{2.21}{4^2.6^2}B,$$
$$6.\quad 8\,\mathfrak{E}-5.7\,\mathfrak{D}-\tfrac{2.43}{6.8}C=0,\quad \mathfrak{E}=\tfrac{5.7}{6.8}\mathfrak{D}+\tfrac{2.43}{6^2.8^2}C,$$
$$8.\quad 10\,\mathfrak{F}-7.9\,\mathfrak{E}-\tfrac{2.73}{8.10}D=0,\quad \mathfrak{F}=\tfrac{7.9}{8.10}\mathfrak{E}+\tfrac{2.73}{8^2.10^2}D.$$

Dum ergo capiatur $\mathfrak{A}=-2A$, littera $\mathfrak{B}$ pro lubitu accipi poteſt; nihilque impedit, quo minus nihilo aequalis ſtatuatur, ſiquidem conſtans $\alpha$ ſupra eſt inducta.

## Exemplum 3.

979. *Aequationis differentio-differentialis*

$$xx(1+bxx)\partial\partial y+x(-5+exx)\partial x\partial y+(5+gxx)y\partial x^2=0$$

*integrale completum per ſeries aſcendentes exhibere.*

Quia hic eſt $a=1$, $c=-5$ et $f=5$, aequatio $\lambda(\lambda-1)-5\lambda+5=0$, ſeu $\lambda\lambda-6\lambda+5=0$, radices habet $\lambda=1$ et $\lambda=5$, quarum differentia 4 per $n=2$ diuidi poteſt. Poſito ergo $y=u+\alpha v+vlx$ ſtatuatur

$$v=Ax^5+Bx^7+Cx^9+Dx^{11}+Ex^{13}+\text{etc.}$$
$$u=\mathfrak{A}x+\mathfrak{B}x^3+\mathfrak{C}x^5+\mathfrak{D}x^7+\text{etc. et}$$

aequationes reſoluendae erunt

$$\text{I.}\quad xx(1+bxx)\partial\partial v+x(-5+exx)\partial x\partial v+(5+gxx)v\partial x^2=0\ \text{et}$$

$$\text{II.}\quad \left.\begin{aligned}&xx(1+bxx)\partial\partial u+x(-5+exx)\partial x\partial u+(5+gxx)u\partial x^2\\&+2x(1+bxx)\partial x\partial v-(1+bxx)v\partial x^2\\&+(-5+exx)v\partial x^2\end{aligned}\right\}=0,$$

 vbi

vbi prior ducit ad

$$\left.\begin{array}{llll}
5.4\,Ax^5 & +7.6\,Bx^7 & +9.8\,Cx^9 & +11.10\,Dx^{11}+\text{etc.}\\
-5.5\,A & -5.7\,B & -5.9\,C & -5.11\,D\\
+5\,A & +5\,B & +5\,C & +5\,D\\
 & +5.4\,Ab & +7.6\,Bb & +9.8\,Cb\\
 & +5\,Ae & +7\,Be & +9\,Ce\\
 & +Ag & +Bg & +Cg
\end{array}\right\}=0,$$

posterior vero ad

$$\left.\begin{array}{lllll}
 & +2.3\,\mathfrak{B}x^3 & +4.5\,\mathfrak{C}x^5 & +6.7\,\mathfrak{D}x^7 & +8.9\,\mathfrak{E}x^9+\text{etc.}\\
-5\,\mathfrak{A}x & -5.3\,\mathfrak{B} & -5.5\,\mathfrak{C} & -5.7\,\mathfrak{D} & -5.9\,\mathfrak{E}\\
+5\,\mathfrak{A} & +5\,\mathfrak{B} & +5\,\mathfrak{C} & +5\,\mathfrak{D} & +5\,\mathfrak{E}\\
 & & +2.3\,\mathfrak{B}b & +4.5\,\mathfrak{C}b & +6.7\,\mathfrak{D}b\\
 & +\mathfrak{A}e & +3\,\mathfrak{B}e & +5\,\mathfrak{C}e & +7\,\mathfrak{D}e\\
 & +\mathfrak{A}g & +\mathfrak{B}g & +\mathfrak{C}g & +\mathfrak{D}g\\
 & & +2.5\,A & +2.7\,B & +2.9\,C\\
 & & -6\,A & -6\,B & -6\,C\\
 & & & +2.5\,Ab & +2.7\,Bb\\
 & & & -Ab & -Bb\\
 & & & +Ae & +Be
\end{array}\right\}=0,$$

| Inde fit | seu |
|---|---|
| $12B+A(20b+5e+g)=0$ | $2.\ 6B+A(4.5b+5e+g)=0,$ |
| $32C+B(42b+7e+g)=0$ | $4.\ 8C+B(6.7b+7e+g)=0,$ |
| $60D+C(72b+9e+g)=0$ | $6.10D+C(8.9b+9e+g)=0,$ |
| etc. | etc. |

Hinc autem

$-4\mathfrak{B}$

$$-4\mathfrak{B}+\mathfrak{A}(e+g)=0,$$
$$0\mathfrak{C}+\mathfrak{B}(2.3b+3e+g)+4A=0,$$
$$2.6\mathfrak{D}+\mathfrak{C}(4.5b+5e+g)+8B+A(9b+e)=0,$$
$$4.8\mathfrak{E}+\mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0,$$
etc.

Ex prioribus formulis litterae B, C, D, etc. per A determinantur, ex posteriorum vero secunda fit $\mathfrak{B}=\frac{-4A}{2.3b+3e+g}$, ex prima autem $\mathfrak{A}=\frac{4\mathfrak{B}}{e+g}$, tum vero $\mathfrak{C}$ pro lubitu assumi potest, indeque reliqui coefficientes $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. definiuntur.

## Scholion.

980. Exemplum hoc occasionem nobis suppeditat phaenomena quaedam singularia obseruandi. Scilicet etiamsi integrale completum in genere $lx$ inuoluat, tamen id a logarithmo liberum prodit certis casibus. Primo nempe si sit $g=-e$, fit $\mathfrak{B}=0$, manente $\mathfrak{A}$ indefinito, tum vero ob $\mathfrak{B}=0$ capi oportet $A=0$, $B=0$, $C=0$, etc. ideoque $v=0$. Porro vero erit

$$2.\ 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0,$$
$$4.\ 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0,$$
$$6.\ 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0,$$
etc.

vbi $\mathfrak{C}$ altera est constans arbitraria, eritque aequationis

$$xx(1+bxx)\partial\partial y+x(-5+exx)\partial x\partial y+(5-exx)y\partial x^2=0,$$

integrale completum

$$y=\mathfrak{A}x+{}_*+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}+\text{etc.}$$

quod adeo finite exprimitur si $e=-(2i+5)b$, pro $i$ sumendo numeros 0, 1, 2, 3, 4, etc.

Secundo si fit

2. 3 b

2. $3b + 3e + g = 0$, seu $g = -6b - 3e$,
fit $\mathfrak{B} = -\frac{1}{2}\mathfrak{A}(3b+e)$, tum vero $A = 0$, $B = 0$, $C = 0$, ergo $v = 0$. Porro vero reperitur
etc. $\mathfrak{D} = -\frac{1}{6}\mathfrak{C}(7b+e)$, $\mathfrak{E} = -\frac{1}{8}\mathfrak{D}(9b+e)$, $\mathfrak{F} = -\frac{1}{10}\mathfrak{E}(11b+e)$ etc.
hincque

$$y = \mathfrak{A}x - \tfrac{1}{2}\mathfrak{A}(3b+e)x^3 + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{etc.}$$

vbi $\mathfrak{A}$ et $\mathfrak{C}$ arbitrio nostro reliquuntur.

Tertio si sit

4. $5b + 5e + g = 0$, seu $g = -20b - 5e$,

primo fit $B = 0$, $C = 0$, $D = 0$, etc. ideoque $v = Ax^5$, tum vero
$\mathfrak{B} = -\mathfrak{A}(5b+e)$, $-\mathfrak{B}(14b+2e) + 4A = 0$, seu $\mathfrak{B} = \frac{2A}{7b+e}$,
hincque $A = -\frac{1}{4}\mathfrak{A}(5b+e)(7b+e)$, porro

2. $6\mathfrak{D} + A(9b+e) = 0$,
4. $8\mathfrak{E} + 2\mathfrak{D}(11b+e) = 0$,
6. $10\mathfrak{F} + 2\mathfrak{E}(13b+e) = 0$,
etc.

Per $\mathfrak{A}$ ergo definiuntur coefficientes $\mathfrak{B}$, A, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. ac $\mathfrak{C}$ quoque arbitrio nostro relinquitur, vnde integrale completum hoc casu erit

$$y = Ax^5\,lx + \mathfrak{C}x^5 + \mathfrak{A}x + \mathfrak{B}x^3 + * + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{etc.}$$

quae expressio fit finita quoties $(2i+5)b + e = 0$.

## Exemplum 4.

981. *Si in priori exemplo sit* $e = -7b$ *et* $g = 15b$, *aequationis*

$$xx(1+bxx)\partial\partial y - x(5+7bxx)\partial x\partial y + 5(1+3bxx)y\partial x^2 = 0$$

*integrale completum algebraice exhibere.*

Erit

Erit ergo $\mathfrak{B} = + 2 \mathfrak{A} b$, $A = 0$, $\mathfrak{D} = 0$, $\mathfrak{E} = 0$, ideoque $v = 0$ et $u = \mathfrak{A} x + 2 \mathfrak{A} b x^3 + \mathfrak{C} x^5$, vnde pro $\mathfrak{A}$ et $\mathfrak{C}$ fumendo conftantes quascunque, erit integrale completum

$$y = \mathfrak{A} x (1 + 2 b x x) + \mathfrak{C} x^5.$$

Integralia ergo particularia erunt

$$y = \alpha x (1 + 2 b x x), \; y = \alpha x^5, \; y = \alpha x (1 + b x x)^2.$$

## Corollarium 1.

982. Pofito $y = e^{\int z \partial x}$, vt fit $z = \frac{\partial y}{y \partial x}$, aequationis huius differentialis primi gradus

$$x x (1 + b x x) \partial z + x x (1 + b x x) z z \partial x - x (5 + 7 b x x) z \partial x$$
$$+ 5 (1 + 3 b x x) \partial x = 0,$$

integrale completum eft $z = \frac{\mathfrak{A}(1 + 6 b x x) + 5 \mathfrak{C} x^4}{\mathfrak{A} x (1 + 2 b x x) + \mathfrak{C} x^5}$.

## Corollarium 2.

983. Aequatio autem differentio-differentialis integrabilis redditur, fi diuidatur per $x x (1 + b x x)^2$, eritque integrale

$$\frac{x \partial y - 5 y \partial x}{x (1 + b x x)} = C \partial x, \text{ feu } \partial y - \frac{5 y \partial x}{x} = C \partial x (1 + b x x),$$

quae per $x^5$ diuifa integrale praebet

$$\frac{y}{x^5} = \frac{-C}{4 x^4} - \frac{b C}{2 x^2} + D \text{ feu}$$
$$y = -\tfrac{1}{4} C x (1 + 2 b x x) + D x^5,$$

vt ante.

## Scholion.

984. Deficit autem adhuc integratio completa noftrae aequationis generalis per feries afcendentes, cafu quo $a = 0$, ideoque $\lambda c + f = 0$, vnde vnicus pro exponente $\lambda$ valor definitur $\lambda = \frac{-f}{c}$, qui tantum integrale particulare fuppeditat, atque hoc etiam tollitur, fi fuerit $c = 0$. Quia autem his ca-

 fibus

ſibus $a = 0$, coefficiens $b$ certo adſit neceſſe eſt, ex quo integrale completum per ſeries deſcendentes exhiberi poterit, cum aequatio $\lambda(\lambda - 1) b + \lambda e + g = 0$ duas ſemper contineat radices, ex quibus duplex ſeries obtinetur. Simile autem hic incommodum vſu venire poteſt, quando binae radices ipſius $\lambda$ vel prodeunt aequales, vel differentiam habent per exponentem $n$ diuiſibilem. Verum huic incommodo, ſeriem per $l\,x$ multiplicatam introducendo, ſimili methodo medela affertur, qua in hoc problemate ſumus vſi, ac ſuperfluum foret iſtam euolutionem hic repetere. Quodſi autem binae radices ipſius $\lambda$ tam pro ſeriebus aſcendentibus quam deſcendentibus fiant imaginariae, oſtendendum reſtat, quomodo integrale completum per ſeries infinitas exprimi oporteat.

## Problema 124.

985. Propoſita aequatione differentio-differentiali

$$xx(a+bx^n)\,\partial\partial y + x(c+ex^n)\,\partial x\,\partial y + (f+gx^n)\,y\,\partial x^2 = 0,$$

ſi eueniat vt aequatio

$$\lambda(\lambda - 1)\,a + \lambda c + f = 0$$

radices habeat imaginarias, eius integrale completum per ſeries aſcendentes exhibere.

## Solutio.

Ex ſupra allatis (971) colligitur hoc caſu ſtatui debere

$$y = v\,\text{ſin.}\,\mu\,l\,x + u\,\text{coſ.}\,\mu\,l\,x, \text{ vnde fit}$$

$$\partial y = \left(\partial v - \frac{\mu u\,\partial x}{x}\right)\text{ſin.}\,\mu\,l\,x + \left(\frac{\mu v\,\partial x}{x} + \partial u\right)\text{coſ.}\,\mu\,l\,x, \text{ et}$$

$$\partial\partial y = \left(\partial\partial v - \frac{2\mu\,\partial x\,\partial u}{x} + \frac{\mu u\,\partial x^2}{xx} - \frac{\mu\mu v\,\partial x^2}{xx}\right)\text{ſin.}\,\mu\,l\,x$$

$$+ \left(\partial\partial u + \frac{2\mu\,\partial x\,\partial v}{x} - \frac{\mu v\,\partial x^2}{xx} - \frac{\mu\mu u\,\partial x^2}{xx}\right)\text{coſ.}\,\mu\,l\,x,$$

qua facta ſubſtitutione, ſi terminos tam $\text{ſin.}\,\mu\,l\,x$ quam $\text{coſ.}\,\mu\,l\,x$ affectos ſeorſim ad nihilum redigamus, obtinebimus duas ſequenquentes aequationes

$$\text{I.}\quad \left.\begin{aligned} xx(a+bx^n)\partial\partial v + x(c+ex^n)\partial x\partial v + (f+gx^n)v\partial x^2 \\ -2\mu x(a+bx^n)\partial x\partial u - \mu\mu(a+bx^n)v\partial x^2 \\ +\mu(a+bx^n)u\partial x^2 \\ -\mu(c+ex^n)u\partial x^2 \end{aligned}\right\} = 0$$

$$\text{II.}\quad \left.\begin{aligned} xx(a+bx^n)\partial\partial u + x(c+ex^n)\partial x\partial u + (f+gx^n)u\partial x^2 \\ +2\mu x(a+bx^n)\partial x\partial v - \mu\mu(a+bx^n)u\partial x^2 \\ -\mu(a+bx^n)v\partial x^2 \\ +\mu(c+ex^n)v\partial x^2 \end{aligned}\right\} = 0$$

Iam pro $v$ et $u$ assumamus has series ascendentes

$$v = \mathrm{A}x^\lambda + \mathrm{B}x^{\lambda+n} + \mathrm{C}x^{\lambda+2n} + \mathrm{D}x^{\lambda+3n} + \text{etc.}$$

$$u = \mathfrak{A}x^\lambda + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} + \mathfrak{D}x^{\lambda+3n} + \text{etc.}$$

iisque substitutis, prior aequatio abit in hanc

| $\lambda(\lambda-1)\mathrm{A}ax^\lambda$ | $+(\lambda+n)(\lambda+n-1)\mathrm{B}ax^{\lambda+n}$ | $+(\lambda+2n)(\lambda+2n-1)\mathrm{C}ax^{\lambda+2n}$ |
|---|---|---|
| | $+\lambda(\lambda-1)\mathrm{A}b$ | $+(\lambda+n)(\lambda+n-1)\mathrm{B}b$ |
| $+\lambda\mathrm{A}c$ | $+(\lambda+n)\mathrm{B}c$ | $+(\lambda+2n)\mathrm{C}c$ |
| | $+\lambda\mathrm{A}e$ | $+(\lambda+n)\mathrm{B}e$ |
| $+\mathrm{A}f$ | $+\mathrm{B}f$ | $+\mathrm{C}f$ |
| | $+\mathrm{A}g$ | $+\mathrm{B}g$ |
| $-2\mu\lambda\mathfrak{A}a$ | $-2\mu(\lambda+n)\mathfrak{B}a$ | $-2\mu(\lambda+2n)\mathfrak{C}a$ |
| | $-2\mu\lambda\mathfrak{A}b$ | $-2\mu(\lambda+n)\mathfrak{B}b$ |
| $-\mu\mu\mathrm{A}a$ | $-\mu\mu\mathrm{B}a$ | $-\mu\mu\mathrm{C}a$ |
| | $-\mu\mu\mathrm{A}b$ | $-\mu\mu\mathrm{B}b$ |
| $+\mu\mathfrak{A}a$ | $+\mu\mathfrak{B}a$ | $+\mu\mathfrak{C}a$ |
| | $+\mu\mathfrak{A}b$ | $+\mu\mathfrak{B}b$ |
| $-\mu\mathfrak{A}c$ | $-\mu\mathfrak{B}c$ | $-\mu\mathfrak{C}c$ |
| | $-\mu\mathfrak{A}e$ | $-\mu\mathfrak{B}e.$ |

Hinc altera aequatio facile formatur permutandis litteris latinis et germanicis atque insuper signum numeri $\mu$ mutando.

Vtrinque ergo potestas prima $x^\lambda$ exigit has aequationes

$$A[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]-\mu\mathfrak{A}(2\lambda a-a+c)=0$$

$$\mathfrak{A}[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]+\mu A(2\lambda a-a+c)=0,$$

vnde necesse est vt sit

$$\text{tam } 2\lambda a-a+c=0$$

$$\text{quam } \lambda(\lambda-1)a+\lambda c+f-\mu\mu a=0.$$

Inde fit $\lambda=\frac{1}{2}-\frac{c}{2a}$, qui valor hic substitutus dat

$$-a\left(\frac{1}{4}-\frac{cc}{4aa}\right)+\frac{c}{2}-\frac{cc}{2a}+f=\mu\mu a=-\frac{a}{4}+\frac{c}{2}-\frac{cc}{4a}+f, \text{ seu}$$

$$\mu\mu a=\frac{4af-(a-c)^2}{4a}, \text{ ideoque}$$

$$\mu=\frac{\sqrt{[4af-(a-c)^2]}}{2a} \text{ et } \lambda=\frac{a-c}{2a}.$$

Vnde patet hanc solutionem locum habere si $4af>(a-c)^2$, quo ipso casu praecedens solutio fiebat imaginaria. Hic autem quantitates A et $\mathfrak{A}$ arbitrio nostro relinquuntur.

Terminus vero $x^{\lambda+n}$ vtrinque postulat has aequationes

$$B[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+A[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b]$$
$$-\mu\mathfrak{B}[2(\lambda+n)a-a+c]-\mu\mathfrak{A}(2\lambda b-b+e)=0 \text{ et}$$

$$\mathfrak{B}[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+\mathfrak{A}[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b]$$
$$+\mu B[2(\lambda+n)a-a+c]+\mu A(2\lambda b-b+e)=0.$$

Sit breuitatis gratia

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=nna=\alpha$$

$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta$$

$$2(\lambda+n)a-a+c=2na=\gamma$$

$$2\lambda b-b+e=\delta,$$

vt

vt habeamus

$$\mathrm{B}\alpha + \mathrm{A}\beta - \mu\mathfrak{B}\gamma - \mu\mathfrak{A}\delta = 0 \text{ et}$$

$$\mathfrak{B}\alpha + \mathfrak{A}\beta + \mu\mathrm{B}\gamma + \mu\mathrm{A}\delta = 0,$$

vnde colligitur

$$\mathrm{B} = \frac{-\mathrm{A}(\alpha\beta + \mu\mu\gamma\delta) + \mu\mathfrak{A}(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma} \text{ et}$$

$$\mathfrak{B} = \frac{-\mathfrak{A}(\alpha\beta + \mu\mu\gamma\delta) - \mu\mathrm{A}(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma}.$$

At vero ex valoribus aſſumtis eſt

$$\alpha = nna, \; \beta = \frac{(ae-bc)(a-c)}{2aa} - \frac{bf}{a} + g, \; \gamma = 2na, \; \delta = \frac{ae-bc}{a},$$

vnde ex aſſumtis A et $\mathfrak{A}$ definiuntur B et $\mathfrak{B}$, hincque porro C, $\mathfrak{C}$, D, $\mathfrak{D}$ etc.

## Exemplum 1.

986. *Sit $c = a$ et $f = a$, vt fiat $\mu = 1$, et inueſtigetur integrale huius aequationis*

$$xx(a+bx^n)\partial\partial y + x(a+ex^n)\partial x\partial y + (a+gx^n)y\partial x^2 = 0.$$

Hic ergo erit $\lambda = 0$ et $\mu = 1$, vnde poſito

$$y = v \text{ ſin. } lx + u \text{ coſ. } lx,$$

ac pro $v$ et $u$ ſumtis ſeriebus

$$v = \mathrm{A} + \mathrm{B}x^n + \mathrm{C}x^{2n} + \mathrm{D}x^{3n} + \text{etc.}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

coefficientes A et $\mathfrak{A}$ pro lubitu accipi poſſunt. Ex iis primo, ob $\alpha = nna$, $\beta = g - b$, $\gamma = 2na$ et $\delta = e - b$, erit

$$\mathrm{B} = \frac{-\mathrm{A}[nna(g-b) + 2na(e-b)] + \mathfrak{A}[nna(e-b) - 2na(g-b)]}{n^4aa + 4nnaa} \text{ ſeu}$$

$$\mathrm{B} = \frac{-\mathrm{A}[n(g-b) + 2(e-b)] + \mathfrak{A}[n(e-b) - 2(g-b)]}{na(nn+4)} \text{ et}$$

$$\mathfrak{B} = \frac{-\mathfrak{A}[n(g-b) + 2(e-b)] - \mathrm{A}[n(e-b) - 2(g-b)]}{na(nn+4)}.$$

Pro ſequentibus coefficientibus habebimus

$$C[2n(2n-1)a+2na+a-a]+B[n(n-1)b+ne+g-b]$$
$$-\mathfrak{C}(4na-a+a)-\mathfrak{B}(2nb-b+e)=0, \text{ seu}$$

$$4nnCa+B[(nn-n-1)b+ne+g]-4n\mathfrak{C}a$$
$$-\mathfrak{B}[(2n-1)b+e]=0, \text{ et}$$

$$4nn\mathfrak{C}a+\mathfrak{B}[(nn-n-1)b+ne+g]+4nCa$$
$$+B[(2n-1)b+e]=0,$$

quarum illa per $n$ multiplicata huic addatur, vt prodeat

$$4n(nn+1)Ca+B[(n^3-nn+n-1)b+(nn+1)e+ng]$$
$$+\mathfrak{B}[-(nn+1)b+g]=0, \text{ hinc}$$

$$C=\frac{-B[(n-1)(nn+1)b+(nn+1)e+ng]+\mathfrak{B}[(nn+1)b-g]}{4na(nn+1)} \text{ et}$$

$$\mathfrak{C}=\frac{-\mathfrak{B}[(n-1)(nn+1)b+(nn+1)e+ng]-B[(nn+1)b-g]}{4na(nn+1)}.$$

Porro erit

$$9nnDa+C[(4nn-2n-1)b+2ne+g]-6n\mathfrak{D}a$$
$$-\mathfrak{C}[(4n-1)b+e]=0$$

$$9nn\mathfrak{D}a+\mathfrak{C}[(4nn-2n-1)b+2ne+g]+6nDa$$
$$+C[(4n-1)b+e]=0,$$

quarum illa per $3n$, haec vero per $2$ multiplicata iunctim dant

$$3n(9nn+4)Da+C[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]$$
$$+\mathfrak{C}[(-4nn-n-2)b+ne+2g]=0;$$

vnde sequitur

$$D=\frac{-C[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]+\mathfrak{C}[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}$$

$$\mathfrak{D}=\frac{-\mathfrak{C}[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]-C[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}.$$

In genere autem ex coefficientibus quibuscunque M et $\mathfrak{M}$ sequentes N et $\mathfrak{N}$ definiuntur per has formulas

in

$in(iinn+4)Na$
$+M[[i(i-1)^2n^2-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$
$-\mathfrak{M}[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$
$in(iinn+4)\mathfrak{N}a$
$+\mathfrak{M}[[i(i-1)^2n^2-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$
$+M[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$

## Corollarium 1.

987. Si quantitates $b$, $e$, $g$ ita ſint comparatae, vt binae litterae ſibi reſpondentes N et $\mathfrak{N}$ euaneſcant, ſequentes omnes euaneſcent, et integrale completum forma finita exprimetur. Ita vt B et $\mathfrak{B}$ euaneſcant, fieri debet

$$2(g-b)=n(e-b) \text{ et } n(g-b)=-2(e-b);$$

vnde fit $g=e=b$, et ipſa aequatio propoſita factorem habebit $a+bx^n$.

## Corollarium 2.

988. In genere autem integrale finite exprimetur, ſi denotante $i$ numerum integrum quemcunque poſitiuum ſit

$$g=[(i-1)nn+\tfrac{1}{2}(i-2)n+1]b-\tfrac{1}{2}(i-2)ne,$$

tum vero

$$e=-[2(i-1)n-1]b,$$

vnde fit

$$g=[(i-1)^2nn+1]b.$$

## Exemplum 2.

989. *Sumto $n=1$, ſi ſit $e=-b$ et $g=2b$, huius aequationis*

$$xx(a+bx)\partial\partial y+x(a-bx)\partial x\partial y+(a+2bx)y\partial x^2=0$$

*integrale completum aſſignare.*

Ex

vbi prior ducit ad

$$\left.\begin{array}{rrrrl}
5\cdot 4\,A x^5 & +\ 7\cdot 6\,B x^7 & +\ 9\cdot 8\,C x^9 & +\ 11\cdot 10\,D x^{11} & +\ \text{etc.} \\
-5\cdot 5\,A & -5\cdot 7\,B & -5\cdot 9\,C & -\ 5\cdot 11\,D & \\
+\ 5\,A & +\ 5\,B & +\ 5\,C & +\ 5\,D & \\
 & +\ 5\cdot 4\,Ab & +\ 7\cdot 6\,Bb & +\ 9\cdot 8\,Cb & \\
 & +\ 5\,Ae & +\ 7\,Be & +\ 9\,Ce & \\
 & +\ Ag & +\ Bg & +\ Cg &
\end{array}\right\} = 0,$$

posterior vero ad

$$\left.\begin{array}{rrrrrl}
 & +\ 2\cdot 3\,\mathfrak{B} x^3 & +\ 4\cdot 5\,\mathfrak{C} x^5 & +\ 6\cdot 7\,\mathfrak{D} x^7 & +\ 8\cdot 9\,\mathfrak{E} x^9 & +\ \text{etc.} \\
-5\,\mathfrak{A}x & -\ 5\cdot 3\,\mathfrak{B} & -\ 5\cdot 5\,\mathfrak{C} & -\ 5\cdot 7\,\mathfrak{D} & -\ 5\cdot 9\,\mathfrak{E} & \\
+5\,\mathfrak{A} & +\ 5\,\mathfrak{B} & +\ 5\,\mathfrak{C} & +\ 5\,\mathfrak{D} & +\ 5\,\mathfrak{E} & \\
 & & +\ 2\cdot 3\,\mathfrak{B}b & +\ 4\cdot 5\,\mathfrak{C}b & +\ 6\cdot 7\,\mathfrak{D}b & \\
 & +\ \mathfrak{A}e & +\ 3\,\mathfrak{B}e & +\ 5\,\mathfrak{C}e & +\ 7\,\mathfrak{D}e & \\
 & +\ \mathfrak{A}g & +\ \mathfrak{B}g & +\ \mathfrak{C}g & +\ \mathfrak{D}g & \\
 & & +\ 2\cdot 5\,A & +\ 2\cdot 7\,B & +\ 2\cdot 9\,C & \\
 & & -\ 6\,A & -\ 6\,B & -\ 6\,C & \\
 & & & +\ 2\cdot 5\,Ab & +\ 2\cdot 7\,Bb & \\
 & & & -\ Ab & -\ Bb & \\
 & & & +\ Ae & +\ Be &
\end{array}\right\} = 0,$$

| Inde fit | seu |
| --- | --- |
| $12\,B + A(20b + 5e + g) = 0$ | $2\cdot 6\,B + A(4\cdot 5\,b + 5e + g) = 0,$ |
| $32\,C + B(42b + 7e + g) = 0$ | $4\cdot 8\,C + B(6\cdot 7\,b + 7e + g) = 0,$ |
| $60\,D + C(72b + 9e + g) = 0$ | $6\cdot 10\,D + C(8\cdot 9\,b + 9e + g) = 0,$ |
| etc. | etc. |

Hinc autem

$-4\,\mathfrak{B}$

$$-4\mathfrak{B}+\mathfrak{A}(e+g)=0,$$
$$0\mathfrak{C}+\mathfrak{B}(2.3b+3e+g)+4A=0,$$
$$2.6\mathfrak{D}+\mathfrak{C}(4.5b+5e+g)+8B+A(9b+e)=0,$$
$$4.8\mathfrak{E}+\mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0,$$
etc.

Ex prioribus formulis litterae B, C, D, etc. per A determinantur, ex posteriorum vero secunda fit $\mathfrak{B}=\frac{-4A}{2.3b+3e+g}$, ex prima autem $\mathfrak{A}=\frac{4\mathfrak{B}}{e+g}$, tum vero $\mathfrak{C}$ pro lubitu assumi potest, indeque reliqui coefficientes $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. definiuntur.

## Scholion.

980. Exemplum hoc occasionem nobis suppeditat phaenomena quaedam singularia obseruandi. Scilicet etiamsi integrale completum in genere $lx$ inuoluat, tamen id a logarithmo liberum prodit certis casibus. Primo nempe si sit $g=-e$, fit $\mathfrak{B}=0$, manente $\mathfrak{A}$ indefinito, tum vero ob $\mathfrak{B}=0$ capi oportet $A=0$, $B=0$, $C=0$, etc. ideoque $v=0$. Porro vero erit

$$2.\ 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0,$$
$$4.\ 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0,$$
$$6.\ 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0,$$
etc.

vbi $\mathfrak{C}$ altera est constans arbitraria, eritque aequationis

$$xx(1+bxx)\partial\partial y+x(-5+exx)\partial x\partial y+(5-exx)y\partial x^2=0,$$

integrale completum

$$y=\mathfrak{A}x+*+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}+\text{ etc.}$$

quod adeo finite exprimitur si $e=-(2i+5)b$, pro $i$ sumendo numeros 0, 1, 2, 3, 4, etc.

Secundo si fit

$$2.\ 3b + 3e + g = 0, \text{ seu } g = -6b - 3e,$$

fit $\mathfrak{B} = -\frac{1}{2}\mathfrak{A}(3b + e)$, tum vero $A = 0$, $B = 0$, $C = 0$, etc. ergo $v = 0$. Porro vero reperitur

$$\mathfrak{D} = -\frac{1}{6}\mathfrak{C}(7b + e),\ \mathfrak{E} = -\frac{1}{8}\mathfrak{D}(9b + e),\ \mathfrak{F} = -\frac{1}{10}\mathfrak{E}(11b + e) \text{ etc.}$$

hincque

$$y = \mathfrak{A}x - \frac{1}{2}\mathfrak{A}(3b + e)x^3 + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{ etc.}$$

vbi $\mathfrak{A}$ et $\mathfrak{C}$ arbitrio noſtro reliquuntur.

Tertio ſi ſit

$$4.\ 5b + 5e + g = 0, \text{ seu } g = -20b - 5e,$$

primo fit $B = 0$, $C = 0$, $D = 0$, etc. ideoque $v = Ax^5$, tum vero

$\mathfrak{B} = -\mathfrak{A}(5b + e)$, $-\mathfrak{B}(14b + 2e) + 4A = 0$, ſeu $\mathfrak{B} = \frac{2A}{7b+e}$, hincque $A = -\frac{1}{2}\mathfrak{A}(5b + e)(7b + e)$, porro

$$2.\ 6\mathfrak{D} + A(9b + e) = 0,$$
$$4.\ 8\mathfrak{E} + 2\mathfrak{D}(11b + e) = 0,$$
$$6.\ 10\mathfrak{F} + 2\mathfrak{E}(13b + e) = 0,$$

etc.

Per $\mathfrak{A}$ ergo definiuntur coefficientes $\mathfrak{B}$, $A$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. ac $\mathfrak{C}$ quoque arbitrio noſtro relinquitur, vnde integrale completum hoc caſu erit

$$y = Ax^5 lx + \mathfrak{C}x^5 + \mathfrak{A}x + \mathfrak{B}x^3 + * + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{ etc.}$$

quae expreſſio fit finita quoties $(2i + 5)b + e = 0$.

## Exemplum 4.

981. *Si in priori exemplo ſit* $e = -7b$ *et* $g = 15b$, *aequationis*

$$xx(1 + bxx)\partial\partial y - x(5 + 7bxx)\partial x\partial y + 5(1 + 3bxx)y\partial x^2 = 0$$

*integrale completum algebraice exhibere.*

Erit

Erit ergo $\mathfrak{B} = + 2 \mathfrak{A} b$, $A = 0$, $\mathfrak{D} = 0$, $\mathfrak{E} = 0$, ideoque $v = 0$ et $u = \mathfrak{A} x + 2 \mathfrak{A} b x^3 + \mathfrak{C} x^5$, vnde pro $\mathfrak{A}$ et $\mathfrak{C}$ ſumendo conſtantes quascunque, erit integrale completum

$$y = \mathfrak{A} x (1 + 2 b x x) + \mathfrak{C} x^5.$$

Integralia ergo particularia erunt

$$y = \alpha x (1 + 2 b x x),\ y = \alpha x^5,\ y = \alpha x (1 + b x x)^2.$$

## Corollarium 1.

982. Poſito $y = e^{\int z \partial x}$, vt ſit $z = \frac{\partial y}{y \partial x}$, aequationis huius differentialis primi gradus

$$x x (1 + b x x) \partial z + x x (1 + b x x) z z \partial x - x (5 + 7 b x x) z \partial x + 5 (1 + 3 b x x) \partial x = 0,$$

integrale completum eſt $z = \frac{\mathfrak{A}(1 + 6 b x x) + 5 \mathfrak{C} x^4}{\mathfrak{A} x (1 + 2 b x x) + \mathfrak{C} x^5}$.

## Corollarium 2.

983. Aequatio autem differentio-differentialis integrabilis redditur, ſi diuidatur per $x x (1 + b x x)^2$, eritque integrale

$$\frac{x \partial y - 5 y \partial x}{x (1 + b x x)} = C \partial x, \text{ ſeu } \partial y - \frac{5 y \partial x}{x} = C \partial x (1 + b x x),$$

quae per $x^5$ diuiſa integrale praebet

$$\frac{y}{x^5} = \frac{-C}{4 x^4} - \frac{bC}{2 x^2} + D \text{ ſeu}$$

$$y = -\tfrac{1}{4} C x (1 + 2 b x x) + D x^5,$$

vt ante.

## Scholion.

984. Deficit autem adhuc integratio completa noſtrae aequationis generalis per ſeries aſcendentes, caſu quo $a = 0$, ideoque $\lambda c + f = 0$, vnde vnicus pro exponente $\lambda$ valor definitur $\lambda = \frac{-f}{c}$, qui tantum integrale particulare ſuppeditat, atque hoc etiam tollitur, ſi fuerit $c = 0$. Quia autem his caſibus

ſibus $a = 0$, coefficiens $b$ certo adſit neceſſe eſt, ex quo integrale completum per ſeries deſcendentes exhiberi poterit, cum aequatio $\lambda(\lambda - 1)b + \lambda e + g = 0$ duas ſemper contineat radices, ex quibus duplex ſeries obtinetur. Simile autem hic incommodum vſu venire poteſt, quando binae radices ipſius $\lambda$ vel prodeunt aequales, vel differentiam habent per exponentem $n$ diuiſibilem. Verum huic incommodo, ſeriem per $lx$ multiplicatam introducendo, ſimili methodo medela affertur, qua in hoc problemate ſumus vſi, ac ſuperfluum foret iſtam euolutionem hic repetere. Quodſi autem binae radices ipſius $\lambda$ tam pro ſeriebus aſcendentibus quam deſcendentibus fiant imaginariae, oſtendendum reſtat, quomodo integrale completum per ſeries infinitas exprimi oporteat.

## Problema 124.

985. Propoſita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y + x(c+ex^n)\partial x\partial y + (f+gx^n)y\partial x^2 = 0,$$

ſi eueniat vt aequatio

$$\lambda(\lambda - 1)a + \lambda c + f = 0$$

radices habeat imaginarias, eius integrale completum per ſeries aſcendentes exhibere.

## Solutio.

Ex ſupra allatis (971) colligitur hoc caſu ſtatui debere

$$y = v\,\text{ſin.}\,\mu lx + u\,\text{coſ.}\,\mu lx,\ \text{vnde fit}$$

$$\partial y = \left(\partial v - \frac{\mu u\partial x}{x}\right)\text{ſin.}\,\mu lx + \left(\frac{\mu v\partial x}{x} + \partial u\right)\text{coſ.}\,\mu lx,\ \text{et}$$

$$\partial\partial y = \left(\partial\partial v - \frac{2\mu\partial x\partial u}{x} + \frac{\mu u\partial x^2}{xx} - \frac{\mu\mu v\partial x^2}{xx}\right)\text{ſin.}\,\mu lx$$

$$+ \left(\partial\partial u + \frac{2\mu\partial x\partial v}{x} - \frac{\mu v\partial x^2}{xx} - \frac{\mu\mu u\partial x^2}{xx}\right)\text{coſ.}\,\mu lx,$$

qua facta ſubſtitutione, ſi terminos tam ſin. $\mu lx$ quam coſ. $\mu lx$ affectos ſeorſim ad nihilum redigamus, obtinebimus duas ſequentes aequationes

$$\text{I.}\quad \left.\begin{array}{r} xx(a+bx^n)\partial\partial v + x(c+ex^n)\partial x\partial v + (f+gx^n)v\partial x^2 \\ -2\mu x(a+bx^n)\partial x\partial u - \mu\mu(a+bx^n)v\partial x^2 \\ +\mu(a+bx^n)u\partial x^2 \\ -\mu(c+ex^n)u\partial x^2 \end{array}\right\} = 0$$

$$\text{II.}\quad \left.\begin{array}{r} xx(a+bx^n)\partial\partial u + x(c+ex^n)\partial x\partial u + (f+gx^n)u\partial x^2 \\ +2\mu x(a+bx^n)\partial x\partial v - \mu\mu(a+bx^n)u\partial x^2 \\ -\mu(a+bx^n)v\partial x^2 \\ +\mu(c+ex^n)v\partial x^2 \end{array}\right\} = 0$$

Iam pro $v$ et $u$ assumamus has series ascendentes

$$v = \mathrm{A}x^\lambda + \mathrm{B}x^{\lambda+n} + \mathrm{C}x^{\lambda+2n} + \mathrm{D}x^{\lambda+3n} + \text{etc.}$$

$$u = \mathfrak{A}x^\lambda + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} + \mathfrak{D}x^{\lambda+3n} + \text{etc.}$$

iisque substitutis, prior aequatio abit in hanc

$$\begin{array}{lll}
\lambda(\lambda-1)\mathrm{A}ax^\lambda + & (\lambda+n)(\lambda+n-1)\mathrm{B}ax^{\lambda+n} + & (\lambda+2n)(\lambda+2n-1)\mathrm{C}ax^{\lambda+2n} \\
 & +\lambda(\lambda-1)\mathrm{A}b & +(\lambda+n)(\lambda+n-1)\mathrm{B}b \\
+\lambda\mathrm{A}c & +(\lambda+n)\mathrm{B}c & +(\lambda+2n)\mathrm{C}c \\
 & +\lambda\mathrm{A}e & +(\lambda+n)\mathrm{B}e \\
+\mathrm{A}f & +\mathrm{B}f & +\mathrm{C}f \\
 & +\mathrm{A}g & +\mathrm{B}g \\
-2\mu\lambda\mathfrak{A}a & -2\mu(\lambda+n)\mathfrak{B}a & -2\mu(\lambda+2n)\mathfrak{C}a \\
 & -2\mu\lambda\mathfrak{A}b & -2\mu(\lambda+n)\mathfrak{B}b \\
-\mu\mu\mathrm{A}a & -\mu\mu\mathrm{B}a & -\mu\mu\mathrm{C}a \\
 & -\mu\mu\mathrm{A}b & -\mu\mu\mathrm{B}b \\
+\mu\mathfrak{A}a & +\mu\mathfrak{B}a & +\mu\mathfrak{C}a \\
 & +\mu\mathfrak{A}b & +\mu\mathfrak{B}b \\
-\mu\mathfrak{A}c & -\mu\mathfrak{B}c & -\mu\mathfrak{C}c \\
 & -\mu\mathfrak{A}e & -\mu\mathfrak{B}e.
\end{array}$$

 Hinc

Hinc altera aequatio facile formatur permutandis litteris latinis et germanicis atque insuper signum numeri $\mu$ mutando.

Vtrinque ergo potestas prima $x^\lambda$ exigit has aequationes

$$A[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]-\mu\mathfrak{A}(2\lambda a-a+c)=0$$
$$\mathfrak{A}[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]+\mu A(2\lambda a-a+c)=0,$$

vnde necesse est vt sit

$$\text{tam } 2\lambda a-a+c=0$$
$$\text{quam } \lambda(\lambda-1)a+\lambda c+f-\mu\mu a=0.$$

Inde fit $\lambda=\frac{1}{2}-\frac{c}{2a}$, qui valor hic substitutus dat

$$-a\left(\tfrac{1}{4}-\tfrac{cc}{4aa}\right)+\tfrac{c}{2}-\tfrac{cc}{2a}+f=\mu\mu a=-\tfrac{a}{4}+\tfrac{c}{2}-\tfrac{cc}{4a}+f, \text{ seu}$$
$$\mu\mu a=\frac{4af-(a-c)^2}{4a}, \text{ ideoque}$$
$$\mu=\frac{\sqrt{[4af-(a-c)^2]}}{2a} \text{ et } \lambda=\frac{a-c}{2a}.$$

Vnde patet hanc solutionem locum habere si $4af>(a-c)^2$, quo ipso casu praecedens solutio fiebat imaginaria. Hic autem quantitates A et $\mathfrak{A}$ arbitrio nostro relinquuntur.

Terminus vero $x^{\lambda+n}$ vtrinque postulat has aequationes

$$B[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+A[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b]$$
$$-\mu\mathfrak{B}[2(\lambda+n)a-a+c]-\mu\mathfrak{A}(2\lambda b-b+e)=0 \text{ et}$$
$$\mathfrak{B}[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+\mathfrak{A}[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b]$$
$$+\mu B[2(\lambda+n)a-a+c]+\mu A(2\lambda b-b+e)=0.$$

Sit breuitatis gratia

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=nna=\alpha$$
$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta$$
$$2(\lambda+n)a-a+c=2na=\gamma$$
$$2\lambda b-b+e=\delta,$$

vt

vt habeamus

$$B\alpha + A\beta - \mu\mathfrak{B}\gamma - \mu\mathfrak{A}\delta = 0 \text{ et}$$

$$\mathfrak{B}\alpha + \mathfrak{A}\beta + \mu B\gamma + \mu A\delta = 0,$$

vnde colligitur

$$B = \frac{-A(\alpha\beta + \mu\mu\gamma\delta) + \mu\mathfrak{A}(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma} \text{ et}$$

$$\mathfrak{B} = \frac{-\mathfrak{A}(\alpha\beta + \mu\mu\gamma\delta) - \mu A(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma}.$$

At vero ex valoribus aſſumtis eſt

$$\alpha = nna,\ \beta = \frac{(ae-bc)(a-c)}{2aa} - \frac{bf}{a} + g,\ \gamma = 2na,\ \delta = \frac{ae-bc}{a},$$

vnde ex aſſumtis $A$ et $\mathfrak{A}$ definiuntur $B$ et $\mathfrak{B}$, hincque porro $C$, $\mathfrak{C}$, $D$, $\mathfrak{D}$ etc.

## Exemplum 1.

986. *Sit $c = a$ et $f = a$, vt fiat $\mu = 1$, et inueſtigetur integrale huius aequationis*

$$xx(a + bx^n)\partial\partial y + x(a + ex^n)\partial x\partial y + (a + gx^n)y\partial x^2 = 0.$$

Hic ergo erit $\lambda = 0$ et $\mu = 1$, vnde poſito

$$y = v \text{ ſin. } lx + u \text{ coſ. } lx,$$

ac pro $v$ et $u$ ſumtis ſeriebus

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

coefficientes $A$ et $\mathfrak{A}$ pro lubitu accipi poſſunt. Ex iis primo, ob $\alpha = nna$, $\beta = g - b$, $\gamma = 2na$ et $\delta = e - b$, erit

$$B = \frac{-A[nna(g-b) + 2na(e-b)] + \mathfrak{A}[nna(e-b) - 2na(g-b)]}{n^4aa + 4nnaa} \text{ ſeu}$$

$$B = \frac{-A[n(g-b) + 2(e-b)] + \mathfrak{A}[n(e-b) - 2(g-b)]}{na(nn+4)} \text{ et}$$

$$\mathfrak{B} = \frac{-\mathfrak{A}[n(g-b) + 2(e-b)] - A[n(e-b) - 2(g-b)]}{na(nn+4)}.$$

Pro ſequentibus coefficientibus habebimus

$$C[2n(2n-1)a+2na+a-a]+B[n(n-1)b+ne+g-b]$$
$$-\mathfrak{C}(4na-a+a)-\mathfrak{B}(2nb-b+e)=0, \text{ seu}$$

$$4nnCa+B[(nn-n-1)b+ne+g]-4n\mathfrak{C}a$$
$$-\mathfrak{B}[(2n-1)b+e]=0, \text{ et}$$

$$4nn\mathfrak{C}a+\mathfrak{B}[(nn-n-1)b+ne+g]+4nCa$$
$$+B[(2n-1)b+e]=0,$$

quarum illa per $n$ multiplicata huic addatur, vt prodeat

$$4n(nn+1)Ca+B[(n^3-nn+n-1)b+(nn+1)e+ng]$$
$$+\mathfrak{B}[-(nn+1)b+g]=0, \text{ hinc}$$

$$C=\frac{-B[(n-1)(nn+1)b+(nn+1)e+ng]+\mathfrak{B}[(nn+1)b-g]}{4na(nn+1)} \text{ et}$$

$$\mathfrak{C}=\frac{-\mathfrak{B}[(n-1)(nn+1)b+(nn+1)e+ng]-B[(nn+1)b-g]}{4na(nn+1)}.$$

Porro erit

$$9nnDa+C[(4nn-2n-1)b+2ne+g]-6n\mathfrak{D}a$$
$$-\mathfrak{C}[(4n-1)b+e]=0$$

$$9nn\mathfrak{D}a+\mathfrak{C}[(4nn-2n-1)b+2ne+g]+6nDa$$
$$+C[(4n-1)b+e]=0,$$

quarum illa per $3n$, haec vero per $2$ multiplicata iunctim dant

$$3n(9nn+4)Da+C[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]$$
$$+\mathfrak{C}[(-4nn-n-2)b+ne+2g]=0;$$

vnde sequitur

$$D=\frac{-C[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]+\mathfrak{C}[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}$$

$$\mathfrak{D}=\frac{-\mathfrak{C}[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]-C[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}.$$

In genere autem ex coefficientibus quibuscunque M et $\mathfrak{M}$ sequentes N et $\mathfrak{N}$ definiuntur per has formulas

in

$in(iinn+4)Na$

$$+M[[i(i-1)^2n^2-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$$
$$-\mathfrak{M}[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$$

$in(iinn+4)\mathfrak{N}a$

$$+\mathfrak{M}[[i(i-1)^2n^2-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$$
$$+M[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$$

## Corollarium 1.

987. Si quantitates $b$, $e$, $g$ ita fint comparatae, vt binae litterae fibi refpondentes N et $\mathfrak{N}$ euanefcant, fequentes omnes euanefcent, et integrale completum forma finita exprimetur. Ita vt B et $\mathfrak{B}$ euanefcant, fieri debet

$$2(g-b)=n(e-b) \text{ et } n(g-b)=-2(e-b);$$

vnde fit $g=e=b$, et ipfa aequatio propofita factorem habebit $a+bx^n$.

## Corollarium 2.

988. In genere autem integrale finite exprimetur, fi denotante $i$ numerum integrum quemcunque pofitiuum fit

$$g=[(i-1)nn+\tfrac{1}{2}(i-2)n+1]b-\tfrac{1}{2}(i-2)ne,$$

tum vero

$$e=-[2(i-1)n-1]b,$$

vnde fit

$$g=[(i-1)^2nn+1]b.$$

## Exemplum 2.

989. *Sumto* $n=1$, *fi fit* $e=-b$ *et* $g=2b$, *huius aequationis*

$$xx(a+bx)\partial\partial y+x(a-bx)\partial x\partial y+(a+2bx)y\partial x^2=0$$

*integrale completum affignare.*

Ex

Ex formulis modo inuentis colligimus

$$B = \frac{-A(g+2e-3b)+\mathfrak{A}(e+b-2g)}{5a} = \frac{3Ab-4\mathfrak{A}b}{5a} \text{ et}$$

$$\mathfrak{B} = \frac{3\mathfrak{A}b+4Ab}{5a};$$

tum vero

$$C = \frac{-B(2e+g)+\mathfrak{B}(2b-g)}{5a} = 0 \text{ et } \mathfrak{C} = 0.$$

Quocirca habebimus

$$v = A + \frac{(3A-4\mathfrak{A})b}{5a}x, \text{ et } u = \mathfrak{A} + \frac{(3\mathfrak{A}+4A)b}{5a}x;$$

hincque integrale completum elicitur

$$y = A\,\text{fin.}\,lx + \mathfrak{A}\,\text{cof.}\,lx + \frac{bx}{5a}[(3A-4\mathfrak{A})\,\text{fin.}\,lx + (3\mathfrak{A}+4A)\,\text{cof.}\,lx].$$

## Corollarium 1.

990. Sumto $\mathfrak{A} = 0$, habebitur integrale particulare

$$y = A\left(\text{fin.}\,lx + \frac{3bx}{5a}\,\text{fin.}\,lx + \frac{4bx}{5a}\,\text{cof.}\,lx\right).$$

Sin autem fit $A = 0$, aliud habebitur

$$y = \mathfrak{A}\left(\text{cof.}\,lx - \frac{4bx}{5a}\,\text{fin.}\,lx + \frac{3bx}{5a}\,\text{cof.}\,lx\right).$$

## Corollarium 2.

991. Pofito $y = e^{\int s\partial x}$, aequatio noftra reducitur ad hanc

$$xx(a+bx)\partial s + xx(a+bx)ss\partial x + x(a-bx)s\partial x$$
$$+ (a+2bx)\partial x = 0,$$

cuius integrale habetur $s = \frac{\partial y}{y\partial x}$ inde definiendum, quae aequatio in plures alias formas transfundi poteft.

## Scholion.

992. Simili modo integratio per feries defcendentes inftituitur, fi exponentes fingulorum terminorum prodeant imagina-

ginarii; quod seorsim exposuisse ne opus quidem erit. Atque haec sufficiunt, vt pateat, quibusnam cautelis in resolutione aequationum per series infinitas sit vtendum. Summus autem vsus istarum euolutionum in hoc consistit, vt aequationes differentio-differentiales exhiberi queant, quarum saltem integrale particulare algebraicum assignare liceat, quos casus supra §. 969. indicauimus. Similis porro integratio per series infinitas pari modo extendi potest ad huiusmodi aequationes

$$xx(a+bx^n+\beta x^{2n})\partial\partial y+x(c+ex^n+\varepsilon x^{2n})\partial x\partial y$$
$$+(f+gx^n+\gamma x^{2n})y\partial x^2=0$$

tum autem seriei quaesitae quilibet terminus per duos praecedentes determinatur, ita vt si bini contigui euanescant, sequentes omnes in nihilum sint abituri. Quodsi autem terminus ab $y$ vacuus affuerit, resolutio in series fit facilior, cui propterea non immorandum censeo. Veluti si proponatur haec aequatio

$$xx\partial\partial y-x\partial x\partial y+ax^ny\partial x^2=bx^m\partial x^2,$$

series a potestate $x^m$ est inchoanda, ponendo

$$y=Ax^m+Bx^{m+n}+Cx^{m+2n}+Dx^{m+3n}+\text{etc.}$$

vnde fit

$$\begin{array}{lll} m(m-1)Ax^m+(m+n)(m+n-1)Bx^{m+n}+(m+2n)(m+2n-1)Cx^{m+2n}+\text{etc.}\\ -mA \qquad -(m+n)B \qquad\qquad -(m+n)C\\ -b \qquad\quad +Aa \qquad\qquad\qquad +Ba \end{array}$$

hincque

$$A=\frac{b}{m(m-2)},\ B=\frac{-Aa}{(m+n)(m+n-2)},\ C=\frac{-Ba}{(m+2n)(m+2n-2)},\ \text{etc.}$$

vbi quidem multa obseruanda occurrunt, quae per praecepta supra data expedire licet. Imprimis autem in hoc negotio iuuat, aequationem propositam ope substitutionis in alias transformasse, quarum resolutio per series fiat simplicior, quod cum pluribus modis fieri possit, hoc argumentum sequenti capite diligentius pertractare visum est, idque pro forma aequationum

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

quandoquidem pro aliis formis huiusmodi transformatio raro locum inuenit.

CAPVT

# CAPVT IX.

## DE TRANSFORMATIONE AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM.

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0.$$

### Problema 125.

993.

Aequationem differentio-differentialem

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

in qua L, M, N ſunt functiones quaecunque ipſius $x$, ſumto elemento $\partial x$ conſtante, ope ſubſtitutionis $y = e^{\int P\partial x} z$ in aliam formam transmutare.

### Solutio.

Cum hinc ſit $\frac{\partial y}{y} = P\partial x + \frac{\partial z}{z}$, erit differentiando

$$\frac{\partial\partial y}{y} - \frac{\partial y^2}{y^2} = \partial x\partial P + \frac{\partial\partial z}{z} - \frac{\partial z^2}{zz}, \text{ ergo}$$

$$\frac{\partial\partial y}{y} = \frac{\partial\partial z}{z} + \frac{2P\partial x\partial z}{z} + \partial x\partial P + PP\partial x^2.$$

Quare cum aequatio noſtra ſit

$$\frac{L\partial\partial y}{y} + \frac{M\partial x\partial y}{y} + N\partial x^2 = 0,$$

erit facta ſubſtitutione

$$\frac{L\partial\partial z}{z} + \frac{2LP\partial x\partial z}{z} + L\partial x\partial P + LPP\partial x^2$$
$$+ \frac{M\partial x\partial z}{z} + MP\partial x^2 + N\partial x^2 = 0,$$

ſeu per $z$ multiplicando

 L∂∂z

$$L\partial\partial z + (2LP + M)\partial x\partial z$$
$$+ z\partial x(L\partial P + LPP\partial x + MP\partial x + N\partial x) = 0,$$

vbi pro P functionem quamcunque ipsius $x$ accipere licet, vnde innumerabiles aequationes inter binas variabiles $x$ et $z$ obtinentur.

## Corollarium 1.

994. Quodsi ergo hanc aequationem transformatam integrare vel per seriem resoluere liceat, ex inuento valore ipsius $z$ habebitur $y = e^{\int P\partial x} z$.

## Corollarium 2.

995. Aequatio transformata similis est propositae, propterea quod in ea variabilis $z$ cum suis differentialibus $\partial z$ et $\partial\partial z$ vbique vnicam dimensionem occupat, perinde ac $y$ in aequatione proposita.

## Corollarium 3.

996. Si eueniat, vt ambae aequationes, proposita ac transformata, aeque commode per series resolui possint, hoc modo plures resolutiones eiusdem aequationis exhiberi possunt.

## Scholion 1.

997. Cum aequationes commode per series resolubiles in hac forma contineantur

$$xx(a+bz^n)\partial\partial x + x(c+ex^n)\partial x\partial y + (f+gx^n)y\partial x^2 = 0,$$

vbi est

$$L = xx(a+bx^n),\quad M = x(c+ex^n),\quad N = f+gx^n,$$

vt transformata similem obtineat formam, fieri oportet $LP = x(\mu+\nu x^n)$, ideoque $P = \frac{\mu+\nu x^n}{x(a+bx^n)}$. Hinc erit

$\partial P$

$$\partial P = \frac{-\mu a - \mu (n+1) b x^n + \nu (n-1) a x^n - \nu b x^{2n}}{x x (a + b x^n)^2} \partial x$$

ideoque

$$L \partial P + L P P \partial x + M P \partial x =$$

$$\left.\begin{matrix} -\mu a - (n+1) \mu b x^n + (n-1) \nu a x^n - \nu b x^{2n} \\ + \mu \mu + 2 \mu \nu x^n \qquad\qquad + \nu \nu x^{2n} \\ + \mu c + \mu e x^n + \nu c x^n \qquad\qquad + \nu e x^{2n} \end{matrix}\right\} : a + b x^n$$

vbi diuiſio per $a + b x^n$ ſuccedere debet. Statuatur quotus $= \mu b + \nu k x^n$, fietque

$$\mu = a - c + a b, \quad \nu = b - e + a k,$$

ac praeterea

$$2 \mu \nu - (n+1) \mu b + (n-1) \nu a + \mu e + \nu c = \mu b b + \nu a k,$$

vbi priores valores ſubſtituti praebent

$$(b - k + n)(b c - a e) = n a b (b - k) + a b (b - k)^2;$$

vnde fit vel

$$b - k = \frac{b c - a e}{a b}, \text{ vel } b - k = -n.$$

Litterarum ergo $b$ et $k$ altera arbitrio noſtro relinquitur, fitque aequatio transformata

$$x x (a + b x^n) \partial \partial z + x [2 \mu + c + (2 \nu + e) x^n] \partial x \partial z$$
$$+ [f + \mu b + (g + \nu k) x^n] z \partial x^2 = 0.$$

Huius autem reſolutio tam per ſeries aſcendentes, quam deſcendentes ſimiles ipſius $x$ poſtulat poteſtates: Subſtitutio autem ipſa fit

$$y = x^{\frac{a-c}{a} + b} (a + b x^n)^{\frac{b c - a e}{n a b} - \frac{(b-k)}{n}} z,$$

vbi ne ſola poteſtas ipſius $x$ ingrediatur, ſumi debet $b - k = -n$. Nihil intereſt quomodo hic $b$ accipiatur, ſumto ergo $b = 0$,

 fit

fit $k = n$, et fubftitutio

$$y = x^{\frac{a-c}{a}} (a + b x^n)^{\frac{bc-ae}{nab}+1} z,$$

quae ducit ad hanc aequationem

$$x x (a + b x^n) \partial\partial z + x [2a - c + (2(n+1) b - e) x^n] \partial x \partial z$$
$$+ [f + (n(n+1) b - ne + g) x^n] z \partial x^2 = 0.$$

## Scholion 2.

998. Supra §. 970. vidimus, aequationem propofitam inter $x$ et $y$ algebraicum admittere integrale, fi fuerit

$$\frac{a}{2a} - \frac{e}{2b} \cdot \cdot \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \cdot \cdot \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} = i n,$$

quae fi transformata fimili modo tractetur, integrale algebraicum affignari poterit, fi fuerit

$$-\frac{c}{2a} + \frac{e}{2b} \cdot \cdot \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \cdot \cdot \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} - n = i n,$$

quibus conditionibus coniunctis concludere licet, integrale algebraicum fatisfacere, dummodo haec formula

$$\frac{bc - ae}{2ab} \cdot \cdot \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \cdot \cdot \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b}:$$

diuifibilis extiterit per exponentem $n$, hic fignum . . ad ambiguitatem pofitiui ac negatiui defignandam adhibui. Quare fi ponamus

$$f = \frac{(a-c)^2 - bb}{4a} \text{ et } g = \frac{(b-e)^2 - kk}{4b},$$

integrabilitas locum habet, quoties haec expreffio $\frac{bc - ae \pm bh \pm ak}{2nab}$ fuerit numerus integer, fiue pofitiuus fiue negatiuus.

## Exemplum.

999. *Propofita aequatione*

$$x x (1 - x x) \partial\partial y + x (1 + 2 m x x) \partial x \partial y -$$
$$- m (m+1) x x y \partial x^2 = 0,$$

*inue-*

*inuenire casus, quibus integrale algebraicum saltem particulare assignari potest.*

Hic est $a = 1$, $b = -1$, $c = 1$, $e = 2m$, $f = 0$, $g = -m(m+1)$ et $n = 2$. Hinc deducimus

$$b = \sqrt{[(a-c)^2 - 4af]} = 0, \text{ et}$$

$$k = \sqrt{[(b-e)^2 - 4bg]} = \sqrt{[(2m+1)^2 - 4m(m+1)]},$$

hoc est $k = \pm 1$. Formula ergo numero integro aequalis est $\frac{-1-2m\pm 1}{-4}$, vnde geminos pro $m$ casus nanciscimur

vel $2m + 2 = \pm 4i$, vel $2m = \pm 4i$, hoc est

vel $m = \pm 2i - 1$, vel $m = \pm 2i$,

dummodo ergo $m$ sit numerus integer siue positiuus siue negatiuus, integrale particulare algebraicum exhiberi potest. Substitutio autem aequationem transformatam praebens est

$$y = (1-xx)^{\frac{-1-2m}{-2}+1} z = (1-xx)^{\frac{2m+3}{2}} z,$$

ipsa vero aequatio transformata

$$xx(1-xx)\partial\partial z + x[1 - 2(m+3)xx]\partial x \partial z$$
$$-(m+2)(m+3)xxz\partial x^2 = 0,$$

quam ex illa oriri manifestum est, si loco $m$ scribatur $-m-3$. Ipsa autem haec integralia reperiuntur, ob $\lambda\lambda = 0$, ponendo

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \text{etc.}$$

vnde fit

$$\left.\begin{array}{llll} 2Bxx + 12Cx^4 & +30Dx^6 & +56Ex^8 + \text{etc.} \\ \quad -2B & -12C & -30D \\ +2B + 4C & +6D & +8E \\ \quad +4mB & +8mC & +12mD \\ -m(m+1)A - m(m+1)B & -m(m+1)C & -m(m+1)D \end{array}\right\} = 0.$$

Ergo determinatio coefficientium ita se habet

B =

$$B = \frac{m(m+1)}{4} A, \quad C = \frac{(m-1)(m-2)}{16} B, \quad D = \frac{(m-3)(m-4)}{36} C, \text{ etc.}$$

Ac si ponatur

$$z = \mathfrak{A} + \mathfrak{B} x^2 + \mathfrak{C} x^4 + \mathfrak{D} x^6 + \mathfrak{E} x^8 + \text{etc. erit}$$

$$\mathfrak{B} = \frac{(m+2)(m+3)}{4} \mathfrak{A}, \quad \mathfrak{C} = \frac{(m+4)(m+5)}{16} \mathfrak{B}, \quad \mathfrak{D} = \frac{(m+6)(m+7)}{36} \mathfrak{E}, \text{ etc.}$$

## Problema 126.

1000. Aequationem differentio-differentialem

$$L \partial\partial y + M \partial x \partial y + N y \partial x^2 = 0,$$

ope substitutionis $\frac{\partial y}{y} = \frac{P z \partial x^2}{\partial z}$, in aliam eiusdem formae transmutare.

## Solutio

Hic scilicet quaeritur, qualem functionem ipsius $x$ pro P accipi oporteat, vt facta substitutione variabilis $z$ cum suis differentialibus $\partial z$ et $\partial\partial z$ vbique vnicam dimensionem obtineat. Cum igitur sit $\frac{\partial y}{y} = \frac{P z \partial x^2}{\partial z}$, erit differentiando

$$\frac{\partial\partial y}{y} - \frac{\partial y^2}{y^2} = \frac{-P z \partial x^2 \partial\partial z}{\partial z^2} + \frac{z \partial x^2 \partial P}{\partial z} + P \partial x^2 \text{ et}$$

$$\frac{\partial\partial y}{y} = \frac{-P z \partial x^2 \partial\partial z}{\partial z^2} + \frac{z \partial x^2 \partial P}{\partial z} + \frac{P P z z \partial x^4}{\partial z^2} + P \partial x^2,$$

quibus valoribus substitutis fit

$$\frac{-L P z \partial x^2 \partial\partial z}{\partial z^2} + \frac{L z \partial x^2 \partial P}{\partial z} + L P \partial x^2 + \frac{L P P z z \partial x^4}{\partial z^2}$$

$$+ \frac{M P z \partial x^3}{\partial z} + N \partial x^2 = 0.$$

Sumamus ergo $L P + N = 0$, seu $P = \frac{-N}{L}$, et multiplicando per $\frac{-\partial z^2}{P z \partial x^2}$, nanciscemur

$$L \partial\partial z - \frac{L \partial P \partial z}{P} - L P z \partial x^2 - M \partial x \partial z = 0, \text{ seu}$$

$$L \partial\partial z - M \partial x \partial z - \frac{L \partial N \partial z}{N} + \partial L \partial z + N z \partial x^2 = 0.$$

Aequatio ergo proposita ope substitutionis $\frac{\partial y}{y} = \frac{-N z \partial x^2}{L \partial x}$ transformatur in hanc

L ∂∂z

$$L\,\partial\partial z + \left(\frac{\partial L}{\partial x} - M - \frac{L\,\partial N}{N\,\partial x}\right)\partial x\,\partial z + N z\,\partial x^2 = 0.$$

Quodſi ergo hinc valor ipſius $z$ erui poſſit, habebitur etiam valor ipſius $y$ per $x$ expreſſus.

## Corollarium 1.

1001. Si in hac aequatione transformata viciſſim ponatur $\frac{\partial z}{z} = \frac{-N z\,\partial x^2}{L\,\partial y}$, ipſa aequatio propoſita exoritur, vnde hae duae aequationes ita inter ſe cohaerent, vt altera ex altera per ſimilem ſubſtitutionem producatur.

## Corollarium 2.

1002. Si in aequatione transformata ſecundum ſubſtitionem priorem ponatur $\frac{\partial z}{z} = Q\,\partial x + \frac{\partial v}{v}$, obtinebitur haec noua transformata

$$L\,\partial\partial v + \left(2LQ + \frac{\partial L}{\partial x} - M - \frac{L\,\partial N}{N\,\partial x}\right)\partial x\,\partial v$$
$$+ v\,\partial x\left(L\partial Q + LQQ\,\partial x + Q\,\partial L - MQ\,\partial x - \frac{LQ\,\partial N}{N} + N\partial x\right) = 0,$$

quae ergo ex ipſa propoſita deducitur ponendo

$$\frac{\partial y}{y} = \frac{-N v\,\partial x^2}{L(\partial v + Q v\,\partial x)}.$$

## Scholion 1.

1003. Hinc combinando ambas ſubſtitutiones, quibus in binis praecedentibus problematibus ſumus vſi, ſubſtitutionem huiusmodi generalem adipiſcimur

$$\frac{\partial y}{y} = \frac{P\,\partial z + Q z\,\partial x}{R\,\partial z + S z\,\partial x}\cdot\partial x,$$

quae ſi in aequatione propoſita

$$L\,\partial\partial y + M\,\partial x\,\partial y + N y\,\partial x^2 = 0$$

ſubſtituatur, functiones P, Q, R, S ita definiri debent, vt in aequatione reſultante variabilis $z$ cum ſuis differentialibus nusquam plus vna dimenſione teneat. Oriuntur autem termini quadrato $\partial z^2$ affecti, ad quos destruendos fieri oportet

$$L\partial x(PP+QR-PS)+L(R\partial P-P\partial R)+MPR\partial x+NRR\partial x=0,$$

seu $Q=\frac{PS}{R}-\frac{PP}{R}-\frac{\partial P}{\partial x}+\frac{P\partial R}{R\partial x}-\frac{MP}{L}-\frac{NR}{L}$,

tum vero peruenitur ad hanc aequationem

$$\left.\begin{aligned}&L\partial\partial z(PS-QR)+L\partial z(R\partial Q-Q\partial R+S\partial P-P\partial S)\\&+\partial x\partial z[2LPQ+M(QR+PS)+2NRS]\\&+Lz\partial x(S\partial Q-Q\partial S+QQ\partial x)+Sz\partial x^2(MQ+NS)\end{aligned}\right\}=0.$$

Verum facilius ad hanc aequationem generalem peruenitur, si ambae substitutiones alternatim in vsum vocentur.

## Scholion 2.

1004. Transformatio autem hic exposita eo magis est notatu digna, quod etiamsi aequatio transformata resolutionem admittat, inde tamen non nisi difficulter ipsa proposita resoluatur. Cum enim reperta fuerit functio ipsius $x$, quae loco $z$ substituta aequationi transformatae satisfaciat, pro valore ipsius $y$ inueniendo, insuper integrale huius aequationis $\frac{\partial y}{y}=\frac{-Nz\partial x}{L\partial z}$ inuestigari oportet, vbi etsi variabiles $x$ et $y$ a se inuicem sunt separatae, tamen difficultates insignes in ipsa integratione se exerere possunt. Fieri ergo poterit, vt ope huius substitutionis, eiusmodi aequationum integralia exhiberi queant, quae directa via vix inuestigare liceat. Scilicet si eueniat, vt integrale aequationis transformatae vel ope methodi cuiusdam supra expositae inueniri, vel per seriem abruptam exprimi possit, tum etiam ipsius aequationis propositae integrale habebitur. Etsi enim casu posteriori integrale tantum particulare innotescit, tamen ex eo semper in hoc aequationum genere integrale completum elici potest. Namque si aequationi

$$L\partial\partial y+M\partial x\partial y+Ny\partial x^2=0$$

par-

particulariter satisfaciat valor $y = X$, ponatur $y = Xv$, fietque

$$\left.\begin{array}{r} LX\partial\partial v + 2L\partial X\partial v + Lv\partial\partial X \\ + MX\partial x\partial v + Mv\partial x\partial X \\ + NXv\partial x^2 \end{array}\right\} = 0.$$

At quia $X = y$ per hypothesin aequationi satisfacit, erit

$$L\partial\partial X + M\partial x\partial X + NX\partial x^2 = 0 \text{ et}$$

$$LX\partial\partial v + (2L\partial X + MX\partial x)\partial v = 0, \text{ seu}$$

$$\frac{\partial\partial v}{\partial v} + \frac{2\partial X}{X} + \frac{M\partial x}{L} = 0,$$

vnde integrando oritur

$$XX\partial v = Ce^{-\int\frac{M\partial x}{L}}\partial x, \text{ porroque}$$

$$v = \int\frac{C\partial x}{XX}e^{-\int\frac{M\partial x}{L}};$$

ita vt integrale completum sit,

$$y = CX\int\frac{\partial x}{XX}e^{-\int\frac{M\partial x}{L}},$$

quod ergo ex quolibet integrali particulari $y = X$ elici potest.

## Exemplum.

1005. *Aequationem differentio-differentialem*

$$xx(a + bx^n)\partial\partial y + x(c + ex^n)\partial x\partial y + fy\partial x^2 = 0$$

*transformare ac per seriem integrare.*

Cum hic sit $L = xx(a + bx^n)$, $M = x(c + ex^n)$ et $N = f$, vtendum est hac substitutione

$$\frac{\partial y}{y} = \frac{-fz\partial x^2}{xx(a + bx^n)\partial z},$$

qua nostra aequatio reducitur ad hanc formam

$$xx(a + bx^n)\partial\partial z + x(2a - c + [(n+2)b - e]x^n)\partial x\partial z + fz\partial x^2 = 0;$$

pro cuius resolutione si ponatur

$$z = A x^{\lambda} + B x^{\lambda+n} + C x^{\lambda+2n} + \text{ etc.}$$

fieri debet

$$\lambda(\lambda - 1)a + \lambda(2a - c) + f = 0, \text{ seu}$$

$$\lambda\lambda a + \lambda(a-c) + f = 0, \text{ ergo } \lambda = \frac{-a+c \pm \sqrt{[(a-c)^2 - 4af]}}{2a}.$$

Series autem abrumpetur per 970, si haec expressio

$$-\frac{c}{2a} + \frac{e}{2b} - \frac{n}{2} \pm \left(\frac{e}{2b} - \frac{n}{2} - \frac{1}{2}\right) \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in,$$

denotante $i$ numerum integrum positiuum, hoc est

$$\text{vel } -\frac{c}{2a} + \frac{e}{b} - \frac{1}{2} - n \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in,$$

$$\text{vel } -\frac{c}{2a} + \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in.$$

Sin autem ipsa aequatio proposita hoc modo in seriem resolvatur, haec abrumpetur, si fuerit

$$\frac{c}{2a} - \frac{e}{2b} \pm \frac{(b-e)}{2b} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in,$$

hoc est

$$\text{vel } \frac{c}{2a} - \frac{e}{b} + \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in$$

$$\text{vel } \frac{c}{2a} - \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = in.$$

Vnde intelligitur, integrale finitum exhiberi posse, siue numerus integer $i$ sit positiuus siue negatiuus. Ad hanc vero duplicitatem iam prior substitutio perduxerat (998.), ita vt haec noua substitutio nullos nouos casus integrabiles suppeditet.

## Scholion 1.

1006. Vt tamen pateat, quomodo ex valore finito ipsius $z$ valor finitus ipsius $y$ elici queat, contemplemur casum

$$xx(a+bx^2)\partial\partial y + x(3a+ex^2)\partial x \partial y - 24ay\partial x^2 = 0,$$

vbi $n = 2$, $c = 3a$ et $f = -24a$, quae facta substitutione $\frac{\partial y}{y} = \frac{24az\partial x^2}{xx(a+bx^2)\partial z}$ abit in hanc

xx

$$x\,x\,(a+b\,x^2)\,\partial\partial z + x\,[-a+(4\,b-e)\,x\,x]\,\partial x\,\partial z$$
$$-24\,a\,z\,\partial x^2 = 0,$$

vbi pro ſerie aſcendente fit

$$\lambda\lambda - 2\,\lambda - 24 = 0 \text{ vel } (\lambda-6)(\lambda+4) = 0.$$

Statuatur

$$z = A\,x^{-4} + B\,x^{-2} + C + D\,x^2 + \text{etc.}$$

erit

$$\left.\begin{array}{lllll} 20\,A\,a\,x^{-4} & +\ 6\,B\,a\,x^{-2} & * & +\ 2Dax^2 + \text{etc.} \\ & +\,20\,A\,b & +\,6\,B\,b & * \\ +\ 4\,A\,a & +\ 2\,B\,a & * & -\ 2\,D\,a \\ & -\ 4\,A\,(4b-e) & -\ 2\,B\,(4\,b-e) & * \\ -\,24\,A\,a & -\,24\,B\,a & -\,24\,C\,a & -\,24\,D\,a \end{array}\right\} = 0.$$

Cum ergo fit $D = 0$, ſequentes termini omnes tolluntur. Tum vero eſt

$$16\,B\,a = 4\,A\,(b+e),\ 24\,C\,a = -2\,B\,b + 2\,B\,e,$$

ergo

$$B = \tfrac{b+e}{4a},\ C = \tfrac{e-b}{12\,a}\,B = \tfrac{e\,e-b\,b}{48\,a\,a}\,A,$$

hincque

$$z = A\,\left(\tfrac{1}{x^4} + \tfrac{b+e}{4\,a\,x^2} + \tfrac{e\,e-b\,b}{48\,a\,a}\right) = \tfrac{A\,[48\,a\,a + 12\,a\,(b+e)\,x\,x + (e\,e-b\,b)\,x^4]}{48\,a\,a\,x^4},$$

vnde ſequitur

$$\partial z = A\,\partial x\,\left(\tfrac{-4}{x^5} - \tfrac{b-e}{2\,a\,x^3}\right) = \tfrac{-A\,\partial x}{2\,a\,x^5}\,[8\,a + (b+e)\,x\,x].$$

Ergo

$$\tfrac{\partial y}{y} = \tfrac{-[48\,a\,a + 12\,a\,(b+e)\,x\,x + (e\,e-b\,b)\,x^4]}{x\,(a+b\,x^2)\,[8\,a + (b+e)\,x\,x]}\,\partial x,$$

ſeu reſoluendo

$$\tfrac{\partial y}{y} = \tfrac{-6\,\partial x}{x} + \tfrac{(5\,b-e)\,x\,\partial x}{a+b\,x\,x} + \tfrac{2\,(b+e)\,x\,\partial x}{8\,a+(b+e)\,x\,x},$$

hincque integrando

$$y = \tfrac{A}{x^6}\,(a+b\,x\,x)^{\frac{5\,b-e}{2\,b}}\,[8\,a + (b+e)\,x\,x].$$

 Scho-

## Scholion 2.

1007. Quod hic casu fortuito euenisse videtur, vt ex valore ipsius $z$ inuento quantitas $y$ commode definiri potuerit, idem perpetuo euenire oportere, sequenti modo in genere ostendi potest. Cum enim aequatio proposita

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

ope substitutionis $\frac{\partial y}{y} = -\frac{Nz\partial x^2}{L\partial z}$ in hanc fit transformata

$$L\partial\partial z - M\partial x\partial z - \frac{L\partial N\partial z}{N} + \partial L\partial z + Nz\partial x^2 = 0,$$

si haec per $L\partial z$ diuidatur, prodit

$$\frac{\partial\partial z}{\partial z} - \frac{M\partial x}{L} - \frac{\partial N}{N} + \frac{\partial L}{L} = -\frac{Nz\partial x^2}{L\partial z} = \frac{\partial y}{y},$$

ex qua integrando elicitur

$$y = \frac{\alpha L\partial z}{N\partial x}e^{-\int\frac{M\partial x}{L}},$$

quae inuento valore ipsius $z$, statim sine vlteriori integratione praebet valorem ipsius $y$.

Cum porro sit

$$\partial y = \frac{-Nyz\partial x^2}{L\partial z}, \text{ erit } \partial y = -\alpha z\partial x . e^{-\int\frac{M\partial x}{L}},$$

hincque

$$y\partial y = \frac{-\alpha\alpha Lz\partial z}{N}e^{-2\int\frac{M\partial x}{L}},$$

atque hae relationes eo magis sunt notatu dignae, quod ex iis aequatio proposita nonnisi per plures ambages ad transformatam reduci possit. Ipsa enim formula pro $y$ substituta perducit ad aequationem differentialem tertii gradus, quae autem manifesto integrationem admittit, ipsamque aequationem hic inuentam suppeditat. Hinc igitur occasionem adipiscimur eiusmodi substitutiones inuestigandi, quae quidem ad differentialia tertii

tertii gradus ascendant, verum tamen per integrationem ad differentialia secunda redigi se patiantur.

## Problema 127.

1008. Aequationem differentio - differentialem

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

ope huiusmodi substitutionis $y = \frac{P\partial z}{\partial x}$ in aliam aequationem pariter differentio - differentialem transformare.

## Solutio.

Ob $y = \frac{P\partial z}{\partial x}$, fit

$$\partial y = \frac{P\partial\partial z + \partial P\partial z}{\partial x} \text{ et } \partial\partial y = \frac{P\partial^3 z + 2\partial P\partial\partial z + \partial z\partial\partial P}{\partial x},$$

quibus formulis substitutis oritur haec aequatio differentialis tertii gradus

$$LP\partial^3 z + 2L\partial P\partial\partial z + L\partial z\partial\partial P + MP\partial x\partial\partial z$$
$$+ M\partial x\partial P\partial z + NP\partial x^2\partial z = 0,$$

quam ita comparatam assumamus, vt per functionem ipsius $x$, quae sit Q, multiplicata integrabilis euadat. Integrabilis ergo sit haec forma

$$LPQ\partial^3 z + 2LQ\partial P\partial\partial z + MPQ\partial x\partial\partial z + LQ\partial z\partial\partial P$$
$$+ MQ\partial x\partial P\partial z + NPQ\partial x^2\partial z = 0,$$

cuius integrale sit

$$LPQ\partial\partial z + S\partial x\partial z + Tz\partial x^2 = C\partial x^2,$$

vnde colligitur

$$\partial\partial z(2LQ\partial P + MPQ\partial x) = \partial\partial z(\partial.LPQ + S\partial x);$$

$$\partial z(LQ\partial\partial P + MQ\partial x\partial P + NPQ\partial x^2) = \partial z(\partial x\partial S + T\partial x^2),$$

et $z\partial x^2\partial T = 0$, ideoque T quantitas constans.

Inde autem fit

$$S\partial x = LQ\partial P - LP\partial Q - PQ\partial L + MPQ\partial x;$$

ex

ex quo per alteram conditionem elicitur

$$T\partial x^2 = LQ\partial\partial P + MQ\partial x\partial P + NPQ\partial x^2 - LQ\partial\partial P - L\partial P\partial Q - Q\partial P\partial L$$
$$+ LP\partial\partial Q + L\partial P\partial Q + P\partial Q\partial L + PQ\partial\partial L + P\partial Q\partial L + Q\partial P\partial L$$
$$- MP\partial x\partial Q - MQ\partial x\partial P - PQ\partial x\partial M,$$ fiue

$$T\partial x^2 = P\partial\partial . LQ - P\partial x\partial . MQ + PNQ\partial x^2.$$

Quare cum T fit quantitas conftans, ponatur $T = \alpha$, atque hinc commode functio P definitur, fcilicet

$$P = \frac{\alpha\partial x^2}{\partial\partial . LQ - \partial x\partial . MQ + NQ\partial x^2},$$

hocque valore pro P affumto, aequatio propofita ope fubftitutionis $y = \frac{P\partial z}{\partial x}$ transformatur in hanc

$$LPQ\partial\partial z + \partial z(LQ\partial P - LP\partial Q - PQ\partial L + MPQ\partial x)$$
$$+ \alpha z\partial x^2 = C\partial x^2,$$

vbi cum $z$ conftante quantitate augere liceat, conftans C omitti poteft. Diuidatur ergo haec aequatio per P Q et prodibit

$$L\partial\partial z + \partial z\left(\frac{L\partial P}{P} - \frac{L\partial Q}{Q} - \partial L + M\partial x\right) + \frac{\alpha z\partial x^2}{PQ} = 0,$$

feu in poftremo termino valorem ipfius P fubftituendo

$$L\partial\partial z + \partial z\left(\frac{L\partial P}{P} - \frac{\partial . LQ}{Q} + M\partial x\right)$$
$$+ \frac{z}{Q}(\partial\partial . LQ - \partial x\partial . MQ + NQ\partial x^2) = 0,$$

atque hic pro Q functionem quamcunque ipfius $x$ accipere licet.

## Corollarium 1.

1009. Hinc praecedens fubftitutio deriuatur ponendo

$$\partial\partial . LQ - \partial x\partial . MQ = 0,$$ ideoque
$$\partial . LQ - MQ\partial x = C\partial x,$$ feu
$$e^{-\int\frac{M\partial x}{L}} . LQ = C\int e^{-\int\frac{M\partial x}{L}}\partial x + D.$$

Namque fi hic capiatur $C = 0$, erit

Q =

$Q = \frac{D}{L} e^{+\int \frac{M \partial x}{L}}$, et $P = \frac{\alpha \partial x^2}{N Q \partial x^2} = \frac{\alpha}{N Q}$, seu

$P = \frac{\alpha L}{N} e^{-\int \frac{M \partial x}{L}}$, vt ante.

## Corollarium 2.

1010. Sin autem ponamus

$\partial \partial . L Q - \partial x \partial . M Q = \partial X \partial x$, vt sit

$P = \frac{\alpha \partial x}{\partial X + N Q \partial x}$, erit

$\partial . L Q - M Q \partial x = X \partial x + A \partial x$,

porroque integrando

$$e^{-\int \frac{M \partial x}{L}} L Q = \int e^{-\int \frac{M \partial x}{L}} \partial x (X + A) + B \text{ et}$$

$$Q = \frac{1}{L} e^{\int \frac{M \partial x}{L}} \int e^{-\int \frac{M \partial x}{L}} \partial x (X + A) + \frac{B}{L} e^{\int \frac{M \partial x}{L}}.$$

## Corollarium 3.

1011. Ponatur $\int e^{-\int \frac{M \partial x}{L}} X \partial x = e^{-\int \frac{M \partial x}{L}} V$, et $A = 0$, $B = 0$, erit $X = \frac{\partial V}{\partial x} - \frac{M V}{L}$ et $Q = \frac{V}{L}$, ideoque

$$P = \frac{\alpha \partial x}{\frac{\partial \partial V}{\partial x} - \frac{M}{L} \partial V - V \partial . \frac{M}{L} + \frac{N}{L} V \partial x}.$$

Si igitur sit $V = \alpha$, erit $Q = \frac{\alpha}{L}$,

$$P = \frac{L L \partial x}{L N \partial x - L \partial M + M \partial L},$$

et aequatio resultans

$$L \partial \partial z + \partial z \left(\frac{L \partial P}{P} + M \partial x\right) + \frac{z \partial x (L N \partial x - L \partial M + M \partial L)}{L} = 0.$$

## Scholion.

1012. Haec autem nimis sunt generalia, quam vt inde quicquam ad vsum communem concludi possit. Vtcunque autem transformatio instituatur, et aequatio transformata in seriem resoluatur, haec nullis aliis casibus abrumpi videtur, nisi iis quibus ipsa aequatio proposita, et inde per primam substitutionem transformata, ad seriem alicubi terminatam reducitur. Ex quo perspicuum est ope huiusmodi transformationum vix vnquam nouos casus integrabiles erui posse. Verum dum hactenus loco variabilis $y$ aliam $z$ per substitutionem introduximus, altera $x$, ex cuius potestatibus series formabantur, retenta, nunc etiam paucis exploremus, quomodo loco ipsius $x$ aliam variabilem $t$ introducendo, transformationem perfici oporteat; vbi imprimis notetur necesse est, cum ante elementum $\partial x$ assumtum fuerit constans, iam in transformata elementum $\partial t$ constans accipi debere. Hic igitur $t$ scribetur loco certae cuiuspiam functionis ipsius $x$, quam autem ita comparatam esse decet, vt aequatio resultans ne nimis fiat complicata.

## Problema 128.

1013. Proposita aequatione differentio-differentiali

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

loco quantitatis $x$ aliam $t$ introducere, quae functioni cuipiam ipsius $x$ aequetur.

## Solutio.

Diuisa aequatione per $\partial x$, repraesentetur aequatio ita

$$L\partial.\frac{\partial y}{\partial x} + M\partial y + Ny\partial x = 0,$$

vt iam consideratio elementi $\partial x$, quod constans erat assumtum, sit exclusa. Cum $t$ aequetur functioni cuipiam ipsius $x$, fiat inde $\partial t = P\partial x$, seu $\partial x = \frac{\partial t}{P}$, vnde nacifcimur

$$L\partial.\frac{P\partial y}{\partial t} + M\partial y + \frac{Ny\partial t}{P} = 0,$$

ac sumto elemento $\partial t$ constante

$$LP\partial\partial y + L\partial P\partial y + M\partial t\partial y + \frac{Ny\partial t^2}{P} = 0,$$

vbi

vbi tantum supereſt, vt in quantitatibus finitis, quae adhuc variabilem $x$ complectuntur, eius loco altera $t$ introducatur.

## Exemplum.

1014. *Propoſita ſit haec aequatio*

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

*in quam loco formulae* $b+kx^n$ *introducatur* $t$.

Cum ergo ſit

$$t=b+kx^n,\text{ erit }\partial t=nkx^{n-1}\partial x,$$

ideoque

$$P=nkx^{n-1}\text{ et }\partial P=n(n-1)kx^{n-2}\partial x=\frac{(n-1)\partial t}{x}.$$

Quare habebimus

$$nkx^{n+1}(a+bx^n)\partial\partial y+(n-1)x\partial t\partial y(a+bx^n)+x(c+ex^n)\partial t\partial y$$
$$+\frac{(f+gx^n)y\partial t^2}{nkx^{n-1}}=0$$

ſiue

$$nk(a+bx^n)\partial\partial y+\frac{(n-1)\partial t\partial y(a+bx^n)+\partial t\partial y(c+ex^n)}{x^n}$$
$$+\frac{(f+gx^n)y\partial t^2}{nkx^{2n}}=0.$$

Nunc vero eſt $x^n=\frac{t-b}{k}$, qui valor ſubſtitutus praebet

$$n(ak-bb+bt)\partial\partial y+\frac{(n-1)\partial t\partial y(ak-bb+bt)+\partial t\partial y(ck-eb+et)}{t-b}$$
$$+\frac{(fk-gb+gt)y\partial t^2}{n(t-b)^2}=0.$$

Verum hic ita vbique $t-b$ occurrit, vt aequatio ſimplicior euadat loco $t-b$ ſcribendo $u$, tum autem perinde eſt, ac ſi

 loco

loco poteſtatis $x^n$ ſcripſiſſemus quantitatem $u$: neque ergo hinc quicquam lucri pro nouis ſeriebus eruendis redundat.

## Corollarium.

1015. Si in aequatione generali loco $x^m$ ſcribere velimus $t$, erit

$$\partial t = m x^{m-1} \partial x \text{ et } P = m x^{m-1},$$

et aequatio reſultabit, ob

$$\partial P = m(m-1) x^{m-2} \partial x = \frac{(m-1)\partial t}{x},$$

iſta

$$m L x^{m-1} \partial\partial y + \frac{(m-1) L \partial t \partial y}{x} + M \partial t \partial y + \frac{N y \partial t^2}{m x^{m-1}} = 0$$

ſeu

$$m L \partial\partial y + \frac{(m-1) L \partial t \partial y}{t} + \frac{M x \partial t \partial y}{t} + \frac{N x x y \partial t^2}{m t t} = 0.$$

## Scholion.

1016. Plura de huiusmodi aequationum transformationibus tradere haud neceſſe videtur, cum ex his fontibus haud difficulter omnes transformationes ad vſum idoneae deriuari queant. Datur autem alia methodus prorſus ſingularis huiusmodi aequationum differentio-differentialium integralia exprimendi, quae per formulas integrales binas variabiles inuoluentes expeditur, dum altera in integratione vt conſtans tractatur. Ita ſi $P$ fuerit functio quaecunque binarum variabilium $x$ et $u$, ac ponatur $y = \int P \partial x$, conſiderando $u$ in integratione vt conſtantem, integrale hoc $\int P \partial x$ erit functio ipſarum $x$ et $u$, quod ita determinatum, vt euaneſcat poſito $x = 0$, ſi deinceps ſtatuatur $x = a$, obtinebitur functio ipſius $u$ ipſi $y$ aequalis, quae ſi ſatisfaciat aequationi cuipiam differentiali inter $u$ et $y$ propoſitae, haec aequatio reſoluetur formula $y = \int P \partial x$, quae

quae vt eius integrale ſpectari poterit. Atque hoc modo innumerabilium aequationum differentio-differentialium integralia exhiberi poſſunt, quae aliis methodis prorſus intractabiles videntur. Quanquam autem formula $\int P \partial x$, ſpectata quantitate $u$ vt conſtante, actu integrari nequit, tamen eius integrale in hoc negotio pro cognito accipi poteſt, quia eius valor ſaltem per approximationes aſſignari poteſt. Scilicet dum ſumta $x$ pro abſciſſa, ſi P denotet applicatam orthogonalem ei conuenientem, formula $\int P \partial x$ exprimet aream eiusdem curuae abciſſae $x$ inſiſtentem, ac poſito $x = a$, area habetur determinata valori $y = \int P \partial x$, prouti cum modo definiuimus, aequalis, quae ergo, vti loqui ſolent, per quadraturas curuarum aſſignari poteſt, ex quo haec integrandi ratio commode appellatur conſtructio per quadraturas. Hic autem imprimis ad eam rationem, qua integralia in particularia et completa diſtinximus, attendi conueniet; vnde ſollicite eſt cauendum, ne integralia hoc modo inuenta pro completis habeantur, niſi quatenus binas conſtantes arbitrarias inuoluant. Cum igitur eidem aequationi differentiali infinita integralia particularia conueniant, mirandum non eſt, ſi hoc modo pro eadem aequatione propoſita plura integralia diuerſa inueniamus. Hoc autem argumentum fere prorſus eſt nouum, neque a quoquam adhuc pertractatum, ſiquidem nonnulla ſpecimina, quae equidem iam dudum dedi, excipiantur; ex quo dubitare non licet, quin iſta methodus, ſi diligentius excolatur, aliquando forte praeclara incrementa in Analyſin ſit allatura.

# CAPVT X.

## DE
## CONSTRVCTIONE AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM PER QVADRATVRAS CVRVARVM.

### Problema 129.

1017.

Si fuerit $y = \int V \partial x$, denotante V functionem quamcunque binarum quantitatum $x$ et $u$, quarum autem haec $u$ in integratione vt constans spectatur, post integrationem vero statuatur $x = a$, vt $y$ aequetur functioni cuidam ipsius $u$; quodsi iam $u$ variabilis sumatur, inuestigare valorem ipsius $\frac{\partial y}{\partial u}$.

### Solutio.

Cum $\int V \partial x$ exhibeat functionem quandam binarum quantitatum $x$ et $u$, cuius differentiale sumta $u$ constante est $= V \partial x$, si tam $u$ quam $x$ vt variabiles tractentur, differentiale aequationis $y = \int V \partial x$ talem habebit formam, $\partial y = V \partial x + U \partial u$, quae quia est differentiale verum, necesse est sit $(\frac{\partial V}{\partial u}) = (\frac{\partial U}{\partial x})$. At cum V sit functio data ipsarum $x$ et $u$, ponatur $\partial V = P \partial x + Q \partial u$, eritque $(\frac{\partial V}{\partial u}) = Q$, ideoque $(\frac{\partial U}{\partial x}) = Q$. Hinc considerata iterum $u$ vt constante, erit $\partial U = Q \partial x$, et $U = \int Q \partial x$, in qua integratione sola $x$ pro variabili habetur. Quocirca si hunc valorem $\int Q \partial x$ vt cognitum spectemus, quippe quem per quadraturas assignare licet, erit

erit $\partial y = V \partial x + \partial u \int Q \partial x$. Quaerimus autem id ipsius $y$ differentiale, quod ex variabilitate ipsius $u$ tantum nascitur; quod cum sit $\partial y = \partial u \int Q \partial x$ erit valor quaesitus $\frac{\partial y}{\partial u} = \int Q \partial x$, si nempe post integrationem itidem ponatur $x = a$.

## Corollarium. 1.

1018. Cum sit $y = \int V \partial x$ functio ipsarum $x$ et $u$, per integrationem autem formulae $V \partial x$, in qua $u$ constans spectatur, functio quaecunque ipsius $u$ loco constantis accedere possit, functio $y$ per se erit indeterminata, determinabitur autem statim, atque integrale $\int V \partial x$ ita accipiatur, vt euanescat posito $x = 0$.

## Corollarium 2.

1019. Hac conditione obseruata euanescet $y$ posito $x = 0$, quicunque valor alteri quantitati $u$ tribuatur, erit ergo etiam $y + \partial u \left(\frac{\partial y}{\partial u}\right) = 0$ facto $x = 0$, ergo etiam $\left(\frac{\partial y}{\partial u}\right) = 0$. Vnde patet $\int Q \partial x = \frac{\partial y}{\partial u}$ ita quoque accipi debere, vt posito $x = 0$ euanescat.

## Corollarium 3.

1020. Cum $y = \int V \partial x$ erit $\left(\frac{\partial y}{\partial x}\right) = V$, hinc

$$\left(\frac{\partial \partial y}{\partial u \partial x}\right) = \left(\frac{\partial V}{\partial u}\right).$$

At si ponatur $\left(\frac{\partial y}{\partial u}\right) = Z$, erit quoque

$$\left(\frac{\partial \partial y}{\partial u \partial x}\right) = \left(\frac{\partial Z}{\partial x}\right), \text{ ergo } \left(\frac{\partial Z}{\partial x}\right) = \left(\frac{\partial V}{\partial u}\right).$$

Quare spectata $u$ vt constante, erit

$$\partial Z = \partial x \left(\frac{\partial V}{\partial u}\right), \text{ et } Z = \int \partial x \left(\frac{\partial V}{\partial u}\right)$$

ideoque

$$\left(\frac{\partial y}{\partial u}\right) = \int \partial x \left(\frac{\partial V}{\partial u}\right).$$

Co-

## Corollarium 4.

1021. Quodsi ergo post integrationes ita absolutas, vt integralia euanescant posito $x = 0$, ponatur $x = a$, tam valor $y = \int V \partial x$ quam $\frac{\partial y}{\partial u} = \int \partial x \left(\frac{\partial V}{\partial x}\right)$ erit functio determinata ipsius $u$.

## Corollarium 5.

1022. Simili modo vlterius progrediendo erit

$$\frac{\partial \partial y}{\partial u^2} = \int \partial x \left(\frac{\partial \partial V}{\partial u^2}\right).$$

Quare si L, M et N denotent functiones quascunque ipsius $u$, erit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int \partial x \left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right]$$

totumque negotium huc redit, vt ista formula integrationem admittat.

## Scholion.

1023. Datis scilicet ipsius $u$ functionibus L, M, N, quaeri debet functio V binarum variabilium $x$ et $u$, ita vt spectata $u$ constante formula

$$\left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right] \partial x$$

absolute fiat integrabilis, cuius integrale, vt sit determinatum ita capiatur, vt posito $x = 0$, euanescat. Tum vero statuatur $x = a$, ac si illud integrale etiam hoc casu euanescat, erit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = 0,$$

hincque aequationi satisfacit valor $y = \int V \partial x$, lege indicata sumtus. Problema autem, datis functionibus L, M et N, inuestigandi functionem V maxime est indeterminatum, neque methodis adhuc cognitis in genere resolui potest; ex quo conueniet id inuerso modo tractari, vt sumta functione V, alterae L, M et N indagentur. Hinc aequationes differentio-diffe-

differentiales consequemur, quarum integralia hoc modo assignare valebimus, quae si aliis methodis tractari nequeant, insigne lucrum suppeditant. Quodsi integrale illud

$$\int [L (\tfrac{\partial \partial V}{\partial u^2}) + M (\tfrac{\partial V}{\partial u}) + N V] \partial x$$

posito $x = a$ non euanescat, sed datam ipsius $u$ functionem U exhibeat, valor $y = \int V \partial x$ conueniet huic aequationi

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = U,$$

quae cum infinitis modis in alias formas transmutari possit, etiam harum integralia innotescent, vbi simul hoc commode euenit, vt etiamsi integrale tantum particulare obtineatur, inde tamen plerumque integrale completum haud difficulter colligi queat.

## Problema 130.

1024. Inuenire aequationes differentio-differentiales formae $\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = U$, vt L, M, N et U sint functiones ipsius $u$, cuius elementum $\partial u$ hic pro constante accipitur, quarum integrale ope constructionis per quadraturas exhiberi possit.

## Solutio.

Sumatur functio quaecunque binarum variabilium $u$ et $x$, quae sit V, capiaturque integrale $\int V \partial x$ spectata quantitate $u$ vt constante, ita vt posito $x = 0$, euanescat, tum vero fiat $x = a$, denotante $a$ quantitatem quamcunque constantem, vt iam $\int V \partial x$ exprimat functionem quandam ipsius $u$ tantum, cui quantitas $y$ aequetur, vt sit $y = \int V \partial x$. Cum iam sit

$$\frac{\partial y}{\partial u} = \int \partial x (\tfrac{\partial V}{\partial u}), \text{ et } \frac{\partial \partial y}{\partial u^2} = \int \partial x (\tfrac{\partial \partial V}{\partial u^2}),$$

his integralibus pariter ita sumtis, vt posito $x = 0$ euanescant, tum vero statuatur $x = a$, quaerantur functiones L, M, N ipsius $u$, vt haec formula

 $\int \partial x$

$$\int \partial x \left[L\left(\frac{\partial\partial V}{\partial u^2}\right) + M\left(\frac{\partial V}{\partial u}\right) + N V\right]$$

fiat abſolute integrabilis, eiusque integrale ita determinetur, vt poſito $x = a$, fiat id $= U$. Quod ſi fuerit praeſtitum, euidens eſt, aequationi differentio-differentiali

$$\frac{L\,\partial\partial y}{\partial u^2} + \frac{M\,\partial y}{\partial u} + N y = U$$

ſatisfacere formulam aſſumtam $y = \int V\,\partial x$.

## Corollarium 1.

1025. Aſſumtio ergo functionis V non penitus arbitrio noſtro permittitur, ſed ad hoc potiſſimum eſt ſpectandum, vt talis forma

$$\int \partial x \left[L\left(\frac{\partial\partial V}{\partial u^2}\right) + M\left(\frac{\partial V}{\partial u}\right) + N V\right],$$

per ſe fiat integrabilis.

## Corollarium 2.

1026. Infinitae ergo hinc ſtatim excluduntur formae ad hunc ſcopum ineptae, cuiusmodi ſunt $V = U P$, exiſtente U functione ipſius $u$ et P ipſius $x$ tantum; quia tum foret

$$y = U \int P\,\partial x,\ \frac{\partial y}{\partial u} = \frac{\partial U}{\partial u} \int P\,\partial x \text{ et } \frac{\partial\partial y}{\partial u^2} = \frac{\partial\partial U}{\partial u^2} \int P\,\partial x,$$

quippe quae idem integrale complectuntur, ita vt ex earum coniunctione formula abſolute integrabilis confici nequeat.

## Exemplum 1.

1027. *Sit* $V = x^n \sqrt{\frac{uu+xx}{cc-xx}}$ *et* $y = \int x^n\,\partial x \sqrt{\frac{uu+xx}{cc-xx}}$ *integrali euaneſcente poſito* $x = 0$, *tum vero facto* $x = a$.

Erit ergo

$$\left(\frac{\partial V}{\partial u}\right) = x^n \frac{u}{\sqrt{(uu+xx)(cc-xx)}} \text{ et } \left(\frac{\partial\partial V}{\partial u^2}\right) = x^n \frac{xx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}},$$

et

et integrabilis reddi debet haec formula

$$x^n \partial x \left( \frac{Lxx}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} + \frac{Mu}{\sqrt{(uu+xx)(cc-xx)}} + N \sqrt{\frac{uu+xx}{cc-xx}} \right),$$

seu

$$\frac{x^n \partial x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} [Lxx + Mu(uu+xx) + N(uu+xx)^2].$$

Statuatur integrale $= \frac{x^{n+1} \sqrt{(cc-xx)}}{\sqrt{(uu+xx)}}$, cuius differentiale cum sit

$$\frac{(n+1)x^n (cc-xx)(uu+xx) - x^{n+2}(uu+xx) - x^{n+2}(cc-xx)}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} . \partial x,$$

seu

$$\frac{x^n \partial x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} \left\{ \begin{matrix} (n+1)ccuu + (n+1)ccxx - (n+1)uuxx - (n+1)x^4 \\ -ccxx \qquad -uuxx \end{matrix} \right\},$$

cum qua si proposita comparetur, fiet

$$Mu^3 + Nu^4 = (n+1)ccuu,$$

$$L + Mu + 2Nuu = ncc - (n+2)uu \text{ et}$$

$$N = -(n+1).$$

Hinc elicitur

$$Mu = (n+1)(cc+uu), \text{ seu } M = \tfrac{(n+1)(cc+uu)}{u}, \text{ et}$$

$$L = -(n+1)(cc+uu) + 2(n+1)uu + ncc - (n+2)uu,$$

seu $L = -cc - uu$.

Quamobrem habebimus

$$\frac{(cc+uu)\partial\partial y}{\partial u^2} + \frac{(n+1)(cc+uu)\partial y}{u \partial u} - (n+1)y = \frac{a^{n+1}\sqrt{(cc-aa)}}{\sqrt{(aa+uu)}},$$

 cui

cui aequationi satisfacit $y = \int x^n \partial x \sqrt{\frac{uu+xx}{cc-xx}}$, integratione absoluta vt est indicatum.

## Corollarium 1.

1028. Sumto ergo $a = c$, formula integralis

$$y = \int x^n \partial x \sqrt{\frac{uu+xx}{cc-xx}}$$

posito post integrationem $x = c$, exhibebit integrale huius aequationis

$$u(cc+uu)\partial\partial y - (n+1)(cc+uu)\partial u \partial y + (n+1) u y \partial u^2 = 0,$$

seu

$$\partial\partial y - \frac{(n+1)\partial u \partial y}{u} + \frac{(n+1) y \partial u^2}{cc+uu} = 0.$$

## Corollarium 2.

1029. Si sit $n = 1$, per integrationem inuenitur

$$\int x \partial x \sqrt{\frac{uu+xx}{cc-xx}} = \tfrac{1}{4}(cc+uu)\,\text{Ang. sin.}\,\frac{2xx-cc+uu}{cc+uu}$$
$$- \tfrac{1}{2}\sqrt{(ccuu+ccxx-uuxx-x^4)},$$
$$- \tfrac{1}{4}(cc+uu)\,\text{Ang. sin.}\,\frac{-cc+uu}{cc+uu} + \tfrac{1}{2}cu,$$

et posito $x = c$ fit

$$y = \tfrac{1}{4}(cc+uu)\,\text{Ang. cos.}\,\frac{uu-cc}{cc+uu} + \tfrac{1}{2}cu, \text{ hincque}$$
$$\frac{\partial y}{\partial u} = \tfrac{1}{2}u\,\text{Ang. cos.}\,\frac{uu-cc}{cc+uu} \text{ et}$$
$$\frac{\partial\partial y}{\partial u^2} = \tfrac{1}{2}\,\text{Ang. cos.}\,\frac{uu-cc}{cc+uu} - \frac{cu}{cc+uu},$$

quae formulae euidenter satisfaciunt aequationi

$$\partial\partial y - \frac{2\partial u \partial y}{u} + \frac{2y\partial u^2}{cc+uu} = 0.$$

## Corollarium 3.

1030. Hoc casu integrale etiam hoc modo exprimi potest

$$y = \tfrac{1}{4}(cc+uu)\,\text{Ang. sin.}\,\frac{2cu}{cc+uu} + \tfrac{1}{2}cu,$$

seu

seu cum eius multiplum quoduis aeque satisfaciat

$$y = (cc + uu) \text{ Ang. sin. } \frac{2cu}{cc+uu} + 2cu,$$

satisfacit vero etiam $y = cc + uu$, vnde integrale completum est

$$y = \alpha(cc+uu) \text{ Ang. sin. } \frac{2cu}{cc+uu} + 2\alpha cu + \beta(cc+uu).$$

## Scholion.

1031. Quod valor $y = cc + uu$ satisfaciat, ex integrali inuento concludere licet, quia enim Ang. sin. $\frac{2cu}{cc+uu}$ est functio multiplex et termino $2\pi$ augeri potest, integrale ipsum augeri potest termino $2\pi(cc+uu)$. At in genere differentia binorum integralium quoque satisfacit, ergo etiam satisfacere debet $y = 2\pi(cc+uu)$ et generatim $y = \beta(cc+uu)$. Ex hoc casu facilius perspicitur, quomodo valor assumtus aequationi generali satisfaciat, etiamsi is per integrationem euolui nequeat. Patet autem $n+1$ esse debere numerum positiuum, quia alioquin conditio integralis, vt posito $x = 0$ euanescat, impleri nequit.

## Exemplum 2.

1032. *Sumatur*

$$V = x^{n-1}(uu+xx)^{\mu}(cc-xx)^{\nu}, \text{ erit}$$

$$\left(\frac{\partial V}{\partial u}\right) = 2\mu u x^{n-1}(uu+xx)^{\mu-1}(cc-xx)^{\nu} \text{ et}$$

$$\left(\frac{\partial\partial V}{\partial u^2}\right) = 2\mu x^{n-1}(cc-xx)^{\nu}[(uu+xx)^{\mu-1} + 2(\mu-1)uu(uu+xx)^{\mu-2}]$$

seu $= 2\mu x^{n-1}(cc-xx)^{\nu}(uu+xx)^{\mu-2}[(2\mu-1)uu+xx]$.

Integrabilis igitur reddi debet absolute haec formula

$$\int x^{n-1}\partial x(cc-xx)^{\nu}(uu+xx)^{\mu-2} \times$$
$$(2\mu[(2\mu-1)uu+xx]L + 2\mu u(uu+xx)M + (uu+xx)^2 N)$$

seu

 $\int x^{n-1}$

$$\int x^{n-1}\partial x\,(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left\{\begin{matrix}2\mu(2\mu-1)Luu+2\mu Lxx & +Nx^4\\ +2\mu Mu^3 & +2\mu Muxx\\ +Nu^4 & +2Nuuxx\end{matrix}\right\}.$$

Statuatur integrale $x^n(uu+xx)^{\mu-1}(cc-xx)^{\nu+1}$, cuius differentiale cum fit

$$x^{n-1}\partial x\,(uu+xx)^{\mu-2}(cc-xx)^{\nu}\times$$
$$[n(uu+xx)(cc-xx)+2(\mu-1)xx(cc-xx)-2(\nu+1)xx(uu+xx)],$$

erit

$$2\mu(2\mu-1)Luu+2\mu Mu^3+Nu^4=nccuu,$$
$$2\mu L+2\mu Mu+2Nuu=ncc-nuu+2(\mu-1)cc-2(\nu+1)uu,$$
$$N=-n-2(\mu-1)-2(\nu+1)=-n-2\mu-2\nu.$$

At prima

$$2\mu(2\mu-1)L+2\mu Mu+Nuu=ncc$$

demta fecunda dat

$$4\mu(\mu-1)L-Nuu=(n+2\nu+2)uu-2(\mu-1)cc,\ \text{feu}$$
$$4\mu(\mu-1)L=-2(\mu-1)(uu+cc),\ \text{hinc}\ L=\frac{-cc-uu}{2\mu},$$

qui valor in prima fuftitutus dat

$$-(2\mu-1)(cc+uu)+2\mu Mu-(n+2\mu+2\nu)uu=ncc,$$

feu

$$2\mu Mu=(n+2\mu-1)cc+(n+4\mu+2\nu-1)uu.$$

Ergo

$$M=\frac{(n+2\mu-1)(cc+uu)}{2\mu u}+\frac{(\mu+\nu)}{\mu}u.$$

Si $n>0$, fuperius integrale euanefcit pofito $x=0$, quare fi ponamus $x=a$, orietur haec aequatio

$$-\frac{(cc+uu)\partial\partial y}{2\mu\partial u^2}+\frac{(n+2\mu-1)(cc+uu)\partial y}{2\mu u\partial u}+\frac{(\mu+\nu)u\partial y}{\mu\partial u}$$
$$-(n+2\mu+2\nu)y=a^n(aa+uu)^{\mu-1}(cc-aa)^{\nu+1},$$

cuius integrale eft

$$y=\int x^{n-1}\partial x\,(uu+xx)^{\mu}(cc-xx)^{\nu},$$

integrali hoc ita fumto vt euanefcat pofito $x=0$, tum vero pofito $x=a$.

Corol-

## Corollarium 1.

1033. Si capiatur $a = c$, vt poſtrema pars fiat $= 0$, ſiquidem exponens $\nu + 1$ ſit nihilo maior, formula

$$y = \int x^{n-1} \partial x (uu + xx)^{\mu} (cc - xx)^{\nu}$$

poſito $x = c$, poſt integrationem ita peractam vt caſu $x = 0$ fiat $y = 0$, erit integrale huius aequationis

$$u(cc+uu)\partial\partial y - (n+2\mu-1)(cc+uu)\partial u\partial y - 2(\mu+\nu)uu\partial u\partial y$$
$$+ 2\mu(n+2\mu+2\nu)uy\partial u^2 = 0.$$

## Corollarium 2.

1034. Sit $n + 2\mu - 1 = \alpha$, et $n + 4\mu + 2\nu - 1 = \beta$, fiet $2\mu = \alpha + 1 - n$ et $2\nu = \beta + 1 - n - 2\alpha - 2 + 2n = \beta - 1 + n - 2\alpha$, et aequationis

$$u(cc+uu)\partial\partial y - (\alpha cc + \beta uu)\partial u\partial y + (\alpha+1-n)(\beta-\alpha+n)uy\partial u^2 = 0$$

integrale erit

$$y = \int x^{n-1}\partial x (uu + xx)^{\frac{\alpha+1-n}{2}} (cc - xx)^{\frac{\beta-1+n-2\alpha}{2}},$$

poſito $x = c$, ſi modo ſit $n > 0$ et $\beta - 1 + n > 2\alpha$.

## Scholion.

1035. Haec conſtructio latiſſime ad hanc aequationem patet

$$xx(a+bx^n)\partial\partial z + x(c+ex^n)\partial x\partial z + (f+gx^n)z\partial x^2 = 0,$$

primo enim hic ſine detrimento amplitudinis ſumi poteſt $n = 2$, ponendo $x^n = uu$. Tum vero vti ſupra §. 997. vidimus, ponendo

$$z = x^{\frac{a-c}{a}+b}(a+bx)^{\frac{bc-ae}{nab}+1}y,$$

aequatio abit in hanc

$$xx(a+bx^n)\partial\partial y + x[2a-c+2ab+(2b-e+2nb+2bb)x^n]\partial x\partial y$$
$$+[f+ab-cb+abb+(g+(b-e+nb+bb)(n+b))x^n]y\partial x^2 = 0,$$

vbi

vbi si $b$ ita accipiatur, vt sit $abb + (a - c)b + f = 0$, prodit aequatio formae, cuius constructionem dedimus. In casibus autem specialibus difficultates occurrere possunt, quibus superandis sequentia exempla inseruiunt.

## Exemplum 3.

1036. *Sit* $V = e^{mux} x^n (c - x)^\nu$, *erit*
$\left(\frac{\partial V}{\partial u}\right) = m e^{mux} x^{n+1} (c-x)^\nu$ et $\left(\frac{\partial\partial V}{\partial u^2}\right) = mm e^{mux} x^{n+2} (c-x)^\nu$.

Integrabilem ergo reddi oportet hanc formulam

$$e^{mux} x^n \partial x (c - x)^\nu (mm L xx + m M x + N),$$

cuius integrale ponatur $= e^{mux} x^{n+1} (c - x)^{\nu+1}$, cuius propterea differentiale illi formulae aequari debet: quod cum sit

$$e^{mux} x^n \partial x (c-x)^\nu [mux(c-x) + (n+1)(c-x) - (\nu+1)x],$$

erit

$$N = (n+1)c, \quad mM = mcu - (n+\nu+2), \quad mmL = -mu.$$

Statuatur nunc $x = a$, et formula

$$y = \int e^{mux} x^n \partial x (c - x)^\nu$$

erit integrale huius aequationis

$$-\frac{u\partial\partial y}{m\partial u^2} + \frac{cu\partial y}{\partial u} - \frac{(n+\nu+2)\partial y}{m\partial u} + (n+1)cy = e^{mau} a^{n+1} (c-a)^{\nu+1}.$$

Hic poni potest $m = 1$, ac sumto $c = a$, aequationis

$$u\partial\partial y - au\partial u\partial y + (n+\nu+2)\partial u\partial y - (n+1)ay\partial u^2 = 0,$$

integrale est $y = \int e^{ux} x^n \partial x (a - x)^\nu$, posito post integrationem $x = a$, dum sit $\nu + 1 > 0$, et $n + 1 > 0$, vt integrale euanescens reddi possit posito $x = 0$.

## Corollarium 1.

1037. Si hic ponatur $y = e^{\int z\partial u}$, erit

$$u\partial z + uzz\partial u - auz\partial u + (n+\nu+2)z\partial u - (n+1)a\partial u = 0,$$

cuius

cuius integrale eſt

$$z = \frac{\partial r}{y\,\partial u} = \frac{\int e^{ux} x^{n+1}\,\partial x\,(a-x)^{\nu}}{\int e^{ux} x^{n}\,\partial x\,(a-x)^{\nu}}.$$

Illa aequatio autem poſito $z = \frac{1}{2}a + v$, transmutatur in hanc

$$u\,\partial v + uvv\,\partial u + (n+\nu+2)v\,\partial u - \tfrac{1}{4}aau\,\partial u - \tfrac{1}{2}(n-\nu)a\,\partial u = 0,$$

quae ponendo $v = u^{-n-\nu-2}s$ abit in hanc

$$u^{-n-\nu-1}\,\partial s + u^{-2n-2\nu-3}ss\,\partial u - \tfrac{1}{4}aau\,\partial u - \tfrac{1}{2}(n-\nu)a\,\partial u = 0.$$

## Corollarium 2.

1038. Sit porro

$$u^{-n-\nu-2}\,\partial u = \partial t \text{ ſeu } u^{-n-\nu-1} = -(n+\nu+1)t,$$

vt fiat

$$\partial s + ss\,\partial t - \tfrac{1}{4}aau^{2n+2\nu+4}\,\partial t - \tfrac{1}{2}(n-\nu)u^{2n+2\nu+2}\,\partial t = 0,$$

quae ergo aequatio etiam conſtrui poteſt. Vel ſit

$$-(n+\nu+1)t = r, \text{ erit}$$

$$\partial s - \frac{ss\,\partial r}{n+\nu+1} + \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}}\,\partial r}{4(n+\nu+1)} + \frac{(n-\nu)r^{\frac{-2n-2\nu-3}{n+\nu+1}}\,\partial r}{2(n+\nu+1)} = 0,$$

quae poſito $s = -(n+\nu+1)q$ abit in

$$\partial q + qq\,\partial r - \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}}\,\partial r - 2(n-\nu)r^{\frac{-2n-2\nu-3}{n+\nu+1}}\,\partial r}{4(n+\nu+1)^2} = 0,$$

hicque eſt

$$u = r^{\frac{-1}{n+\nu+1}} \text{ et } z = \tfrac{1}{2}a - (n+\nu+1)r^{\frac{n+\nu+2}{n+\nu+1}}q.$$

## Scholion.

1039. Cum aequationis differentio-differentialis

$$\frac{\partial\partial y}{\partial u} - a\,\partial y + \frac{(n+\nu+2)\partial u}{u} - \frac{(n+1)ay\,\partial u}{u} = 0,$$



inte-

integrale sit $y = \int e^{ux} x^n \partial x (a-x)^\nu$, videamus, quomodo haec ipsa aequatio in alias formas transfundi possit. Sit primo $u = \alpha t^\lambda$, ideoque $\partial u = \alpha\lambda t^{\lambda-1}\partial t$, vnde fit

$$\frac{1}{\alpha\lambda}\partial\cdot\frac{\partial y}{t^{\lambda-1}\partial t} - a\partial y + \frac{(n+\nu+2)\partial y}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\partial t}{t} = 0:$$

sumatur iam elementum $\partial t$ constans, eritque

$$\frac{\partial\partial y}{\alpha\lambda t^{\lambda-1}\partial t} - \frac{(\lambda-1)\partial y}{\alpha\lambda t^\lambda} - a\partial y + \frac{(n+\nu+2)\partial y}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\partial t}{t} = 0$$

seu

$$\partial\partial y - \alpha\lambda a t^{\lambda-1}\partial t\partial y + \frac{(\lambda n+\lambda\nu+\lambda+1)\partial t\partial y}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}y\partial t^2 = 0,$$

cuius integrale est

$$y = \int e^{\alpha t^\lambda x} x^n\partial x(a-x)^\nu. \qquad \text{I}$$

Ponatur porro

$$\frac{\partial y}{y} = P\partial t + \frac{\partial z}{z},$$ vt sit $z = e^{-\int P\partial t} y$, erit

$$\partial\partial z + 2P\partial t\partial z - \alpha\lambda a t^{\lambda-1}\partial t\partial z + (\lambda n+\lambda\nu+\lambda+1)\frac{\partial t\partial z}{t} + z\partial t\partial P$$
$$+ z\partial t^2\left[PP - \alpha\lambda a t^{\lambda-1}P + \frac{(\lambda n+\lambda\nu+\lambda+1)P}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}\right] = 0.$$

Ad terminos elemento $\partial z$ adfectos tollendos statuatur

$$P = \tfrac{1}{2}\alpha\lambda a t^{\lambda-1} - \frac{(\lambda n+\lambda\nu+\lambda+1)}{2t},$$

fietque ac prodibit haec aequatio

$$\partial\partial z - z\partial t^2\left[\frac{(\lambda n+\lambda\nu+\lambda)^2-1}{4tt} + \tfrac{1}{2}\alpha\lambda\lambda(n-\nu)at^{\lambda-2} + \tfrac{1}{4}\alpha^2\lambda^2a^2t^{2\lambda-1}\right] = 0,$$

cuius propterea integrale est

$$z = e^{-\frac{1}{2}\alpha a t^\lambda}\, t^{\frac{\lambda n+\lambda\nu+\lambda+1}{2}} \int e^{\alpha t^\lambda x} x^n\partial x(a-x)^\nu.$$

Quodsi ergo sit $\nu = n$, $\lambda\lambda(2n+1)^2 - 1 = 0$, seu

$\lambda = \frac{\pm 1}{2n+1}$, et $\alpha = \pm\frac{2}{\lambda} = \pm 2(2n+1)$,

habebitur haec aequatio

$\partial\partial z$

$$\partial\partial z - aazt^{\frac{\pm 2}{2n+1}-2}\partial t^2 = 0,$$

cuius integrale eſt

$$z = e^{\pm(2n+1)at^{\frac{\pm 1}{2n+1}}}\, t^{\pm\frac{1}{2}+\frac{1}{2}} \int e^{\pm 2(2n+1)t^{\frac{\pm 1}{2n+1}}x}\, x^n \partial x (a-x)^n.$$

Vel huius aequationis

$$\partial\partial z - aat^{2\lambda-2} z\partial t^2 = 0$$

integrale eſt

$$z = e^{-\frac{a}{\lambda}t^\lambda}\, t \int e^{\frac{2x}{\lambda}t^\lambda}\, x^{\pm\frac{1}{2\lambda}-\frac{1}{2}} \partial x (a-x)^{\pm\frac{1}{2\lambda}-\frac{1}{2}}:$$

vnde occaſionem arripimus huiusmodi integrationes generalius inueſtigandi.

## Exemplum 4.

1040. *Si ſint P et Q functiones quaecunque ipſius u, et capiatur*

$$y = P\int e^{Qx} x^{n-1}\partial x (a-x)^{\nu-1},$$

*poſito ſcilicet poſt integrationem* $x = a$, *erit hic valor ipſius y integrale cuiuspiam aequationis differentio-differentialis*

$$\frac{L\partial\partial y}{\partial u^2} + \frac{M\partial y}{\partial u} + Ny = 0$$

*quae quaeritur.*

Ad calculum contrahendum ponamus $\partial P = P'\partial u$ et $\partial P' = P''\partial u$, item $\partial Q = Q'\partial u$ et $\partial Q' = Q''\partial u$. Hinc erit

$$\frac{\partial y}{\partial u} = P'\int e^{Qx} x^{n-1}\partial x (a-x)^{\nu-1} + PQ'\int e^{Qx} x^n \partial x (a-x)^{\nu-1} \text{ et}$$

$$\frac{\partial\partial y}{\partial u^2} = P''\int e^{Qx} x^{n-1}\partial x (a-x)^{\nu-1} + 2P'Q'\int e^{Qx} x^n \partial x (a-x)^{\nu-1}$$
$$+ PQ''\int e^{Qx} x^n \partial x (a-x)^{\nu-1} + PQ'Q'\int e^{Qx} x^{n+1}\partial x (a-x)^{\nu-1}$$

 vnde

vnde colligitur

$$\frac{L\partial\partial y}{\partial u^2}+\frac{M\partial y}{\partial u}+Ny=$$

$$\int e^{\mathfrak{Q}x}x^{n-1}\partial x(a-x)^{\nu-1}\left\{\begin{matrix}LP''+2LP'Q'x+LPQ''x+LPQ'Q'xx\\+MP'+MPQ'x+NP\end{matrix}\right\}$$

quod integrale ſtatuatur $= e^{\mathfrak{Q}x}x^n(a-x)^\nu$, ita vt euaneſcat poſito $x=a$, dum ſit $\nu>0$, vti etiam euaneſcit caſu $x=0$ ſi modo $n>0$. Cum igitur huius formulae differentiale ſit

$$e^{\mathfrak{Q}x}x^{n-1}\partial x(a-x)^{\nu-1}[Qx(a-x)+na-(n+\nu)x],$$

eius comparatio cum forma inuenta praebet

$$LP''+MP'+NP=na,$$

$$2LP'Q'+LPQ''+MPQ'=aQ-(n+\nu), \text{ et}$$

$$LPQ'Q'=-Q, \text{ ergo } L=\frac{-Q}{PQ'Q'}, \text{ hinc}$$

$$M=\frac{aQ}{PQ'}-\frac{(n+\nu)}{PQ'}+\frac{2P'Q}{PPQ'Q'}+\frac{QQ''}{PQ'Q'Q'} \text{ et}$$

$$N=\frac{na}{P}+\frac{P''Q}{PPQ'Q'}-\frac{MP'}{P}:$$

ſicque aequatio differentio-differentialis erit cognita.

## Corollarium 1.

1041. Si velimus vt ſit $M=0$, erit

$$aQ-(n+\nu)+\frac{2P'Q}{PQ'}+\frac{QQ''}{Q'Q'}=0,$$

quae per $\frac{Q'\partial u}{Q}$ multiplicata abit in hanc

$$\frac{2\partial P}{P}+a\partial Q-\frac{(n+\nu)\partial Q}{Q}+\frac{\partial Q'}{Q'}=0,$$

cuius integrale eſt

$$\frac{e^{aQ}P^2Q'}{Q^{n+\nu}}=\text{Conſt. ſiue}$$

$$P=Ce^{-\frac{1}{2}aQ}Q^{\frac{n+\nu}{2}}\sqrt{\frac{\partial u}{\partial Q}}.$$

## Corollarium 2.

1042. Sit $Q = 2\alpha u^{\lambda}$, erit $Q' = 2\alpha\lambda u^{\lambda-1}$, et

$$P = C e^{-\alpha a u^{\lambda}} u^{\frac{\lambda(n+\nu-1)+1}{2}},$$ hinc

$$L = -\frac{1}{2\alpha\lambda\lambda} e^{\alpha a u^{\lambda}} u^{\frac{-\lambda(n+\nu+1)+3}{2}},$$ et

$$N = \frac{n a}{P} + \frac{Q \partial \partial P}{P P \partial Q^2},$$ at eſt

$$\frac{Q}{\partial Q^2} = \frac{u^{-\lambda+2}}{2\alpha\lambda\lambda\partial u^2},$$ et ob

$$\frac{\partial P}{P} = -\alpha\lambda a u^{\lambda-1}\partial u + \frac{\lambda(n+\nu-1)+1}{2}\cdot\frac{\partial u}{u},$$ erit

$$\frac{\partial\partial P}{P} = -\alpha\lambda(\lambda-1) a u^{\lambda-2}\partial u^2 - \frac{\lambda(n+\nu-1)-1}{2}\cdot\frac{\partial u^2}{u u} + \alpha\alpha\lambda\lambda a a u^{2\lambda-2}\partial u^2$$
$$-\alpha\lambda a[\lambda(n+\nu-1)+1] + \frac{[\lambda(n+\nu-1)+1]^2}{4}$$

ſeu

$$\frac{\partial\partial P}{P} = \alpha\alpha\lambda\lambda a a u^{2\lambda-2}\partial u^2 - \alpha\lambda\lambda(n+\nu) a u^{\lambda-2}\partial u^2 + \frac{\lambda\lambda(n+\nu-1)^2-1}{4}\cdot\frac{\partial u^2}{u u},$$

hinc

$$n a + \frac{Q}{\partial Q^2}\cdot\frac{\partial\partial P}{P} = \tfrac{1}{2}\alpha a a u^{\lambda} + \tfrac{1}{2}(n-\nu) a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda},$$ et

$$N = e^{\alpha a u^{\lambda}} u^{\frac{-\lambda(n+\nu-1)-1}{2}} [\tfrac{1}{2}\alpha a a u^{\lambda} + \tfrac{1}{2}(n-\nu) a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda}].$$

## Corollarium 3.

1043. Hinc erit

$$\frac{N}{L} = -2\alpha\lambda\lambda u^{\lambda-2}[\tfrac{1}{2}\alpha a a u^{\lambda} + \tfrac{1}{2}(n-\nu) a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda}],$$

et huius aequationis

$$\frac{\partial\partial y}{\partial u^2} = y[\alpha\alpha\lambda\lambda a a u^{2\lambda-2} + \alpha\lambda\lambda(n-\nu) a u^{\lambda-2} + \frac{\lambda\lambda(n+\nu-1)^2-1}{4 u u}]$$

integrale eſt

$$y = e^{-\alpha a u^{\lambda}} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{2\alpha u^{\lambda} x} x^{n-1}\partial x (a-x)^{\nu-1}.$$

Ponamus $\alpha = \frac{1}{\lambda}$,

$$\lambda(n-\nu) = f, \text{ seu } \nu = n - \frac{f}{\lambda}, \text{ et } \frac{\lambda\lambda(n+\nu-1)^2-1}{4} = g,$$

vnde fit

$$n = \frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda} \text{ et } \nu = \frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

et huius aequationis

$$\partial\partial y = y\,\partial u^2 (aa u^{2\lambda-2} + af u^{\lambda-2} + g u^{-2})$$

integrale est

$$y = e^{\frac{-a}{\lambda}u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{\frac{2x}{\lambda}u^\lambda} x^{n-1}\partial x (a-x)^{\nu-1}, \text{ seu}$$

$$y = e^{\frac{-a}{\lambda}u^\lambda} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{2x}{\lambda}u^\lambda} x^{\frac{f-\lambda+\sqrt{(1+4g)}}{2\lambda}} \partial x (a-x)^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}}.$$

## Corollarium 4.

1044. Si ponamus $\alpha = \frac{-1}{\lambda}$,

$$\lambda(n-\nu) = -f, \text{ et } \frac{\lambda\lambda(n+\nu-1)^2-1}{4} = g \text{ erit}$$

$$n = \frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda} \text{ et } \nu = \frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

vnde huius aequationis, quae cum praecedente conuenit,

$$\partial\partial y = y\,\partial u^2 (aa u^{2\lambda-2} + af u^{\lambda-2} + g u^{-2})$$

integrale erit

$$y = e^{\frac{a}{\lambda}u^\lambda} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{-2x}{\lambda}u^\lambda} x^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}} \partial x (a-x)^{\frac{+f-\lambda+\sqrt{(1+4g)}}{2\lambda}}$$

vbi necesse est sit $n > 0$ et $\nu > 0$.

## Exemplum 5.

1045. *Si ponamus* $y = \int \partial x (aa - xx)^{\nu-1} \operatorname{cos.} au^\lambda x$, *posito post integrationem* $x = a$, *vt* $y$ *aequetur certae functioni ipsius* $u$, *inuenire aequationem differentio-differentialem, cui ea satisfaciat.*

Cum

Cum fit

$$\frac{\partial y}{\partial u} = -\alpha\lambda u^{\lambda-1}\int x\,\partial x\,(aa-xx)^{\nu-1}\,\text{fin.}\,\alpha u^{\lambda}x, \text{ et}$$

$$\frac{\partial\partial y}{\partial u^2} = \int x\,\partial x\,(aa-xx)^{\nu-1}[-\alpha\lambda(\lambda-1)u^{\lambda-2}\,\text{fin.}\,\alpha u^{\lambda}x - \alpha\alpha\lambda\lambda u^{2\lambda-2}x\,\text{cof.}\,\alpha u^{\lambda}x].$$

Hinc erit $\frac{L\,\partial\partial y}{\partial u^2} + \frac{M\,\partial y}{\partial u} + Ny =$

$$\int(aa-xx)^{\nu-1}\partial x\left\{\begin{matrix}N\,\text{cof.}\,\alpha u^{\lambda}x - \alpha\lambda M u^{\lambda-1}x\,\text{fin.}\,\alpha u^{\lambda}x - \alpha\lambda(\lambda-1)Lu^{\lambda-2}x\,\text{fin.}\,\alpha u^{\lambda}x \\ -\alpha\alpha\lambda\lambda L u^{2\lambda-2}xx\,\text{cof.}\,\alpha u^{\lambda}x.\end{matrix}\right\}$$

Fingatur integrale $= (aa-xx)^{\nu}\,\text{fin.}\,\alpha u^{\lambda}x$, quod euanefcit pofito tam $x=0$ quam $x=a$, reperiturque comparatione inftituta

$$L = \frac{u^{-\lambda+2}}{\alpha\lambda\lambda},\quad M = \frac{2\lambda\nu-\lambda+1}{\alpha\lambda\lambda}u^{-\lambda+1},\quad N = \alpha aau^{\lambda}.$$

Quare huius aequationis

$$\frac{\partial\partial y}{\partial u^2} + (2\lambda\nu-\lambda+1)\frac{\partial y}{u\,\partial u} + \alpha\alpha\lambda\lambda aau^{2\lambda-2}y = 0$$

integrale eft

$$y = \int\partial x\,(aa-xx)^{\nu-1}\,\text{cof.}\,\alpha u^{\lambda}x.$$

## Corollarium 1.

1046. Si ergo fit $\nu = \frac{\lambda-1}{2\lambda}$ et $\alpha = \frac{1}{\lambda}$, huius aequationis

$$\frac{\partial\partial y}{\partial u^2} + aau^{2\lambda-2}y = 0$$

integrale eft

$$y = \int\partial x\,(aa-xx)^{\frac{-\lambda-1}{2\lambda}}\,\text{cof.}\,\tfrac{1}{\lambda}u^{\lambda}x,$$

fi quidem poft integrationem ftatuatur $x=a$, integrali ita fumto vt euanefcat pofito $x=0$.

## Corollarium 2.

1047. Si igitur fit $\frac{-\lambda-1}{2\lambda} = i$ numero integro, feu $\lambda = \frac{-1}{2i+1}$, huius aequationis

∂∂y

$$\partial \partial y + a a u^{\frac{-4i-4}{2i+1}} y \partial u^2 = 0$$

integrale eſt

$$y = \int \partial x (a a - x x)^i \cos. \tfrac{1}{\lambda} u^\lambda x$$

quod reuera exhiberi poteſt. Prodeunt ſcilicet caſus integrabiles ſupra indicati.

## Scholion.

1048. Cum poſuerimus $y = \int V \partial x$, exiſtente V functione quacunque ipſarum $u$ et $x$, quarum autem in hac integratione ſola $x$ vt variabilis tractatur, non opus eſt abſolute integrale ita determinari, vt euaneſcat poſito $x = 0$, ſed ſufficit vt certo quodam caſu $x = b$ euaneſcat, quo facto ſi porro ponatur $x = a$, vt $y$ aequetur functioni cuipiam ipſius $u$, quam per quadraturas aſſignare licet, quandoquidem hic integrationem formularum ſimplicium nobis concedi iure poſtulamus. Atque hic valor ipſius $y$ per $u$ expreſſus integrale exhibet cuiusdam aequationis differentio-differentialis

$$L \partial \partial y + M \partial u \partial y + N y \partial u^2 = U \partial u^2,$$

vbi autem neceſſe eſt, vt haec formula

$$\int \partial x \left[L \left(\tfrac{\partial \partial V}{\partial u^2}\right) + M \left(\tfrac{\partial V}{\partial u}\right) + N V\right]$$

integrari actu poſſit, quod integrale itidem ita eſt capiendum, vt euaneſcat poſito $x = b$, tum vero poſito $x = a$, id fiat $= U$.

## Problema 131.

1049. Si fuerint P et Q functiones ipſius $x$, at K functio ipſius $u$, ac ponatur

$$y = \int P \partial x (K + Q)^n,$$

integrali ita ſumto vt euaneſcat caſu $x = b$, tum vero ſtatuatur $x = a$, vt pro $y$ prodeat functio ipſius $u$, inuenire aequatiotionem differentio - differentialem inter $y$ et $u$, cui ille valor ipsius $y$ satisfaciat.

## Solutio.

Sit $\partial K = K' \partial u$ et $\partial K' = K'' \partial u$, et ob

$y = \int (K+Q)^n P \partial x$, erit $\frac{\partial y}{\partial u} = \int n K' (K+Q)^{n-1} P \partial x$,

ac denuo differentiando

$$\frac{\partial\partial y}{\partial u^2} = \int [n K'' (K+Q)^{n-1} + n(n-1) K'K' (K+Q)^{n-2}] P \partial x,$$

vnde si L, M, N denotent functiones ipsius $u$, erit haec expressio $\frac{L \partial\partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int P \partial x (K+Q)^{n-2} \times$

$$[N(K+Q)^2 + nMK'(K+Q) + nLK''(K+Q) + n(n-1)LK'K']$$

$$= \int P\partial x (K+Q)^{n-2} \begin{Bmatrix} NKK + nMKK' + nLKK'' + n(n-1)LK'K' \\ +2NKQ + nMK'Q + nLK''Q + NQQ \end{Bmatrix},$$

quae cum debeat esse integrabilis, statuatur integrale

$$= R(K+Q)^{n-1} + \text{Const.}$$

ita vt euanescat posito vt ante $x = b$, vbi R sit functio ipsius $x$ tantum. Cuius formae differentiale quia est

$$(K+Q)^{n-2} [K \partial R + Q \partial R + (n-1) R \partial Q],$$

oportet sit

$$[NKK + nMKK' + nLKK'' + n(n-1)LK'K'] P \partial x$$
$$+ (2NK + nMK' + nLK'') PQ\partial x + NPQQ\partial x$$
$$= K\partial R + Q \partial R + (n-1) R \partial Q.$$

Hic ergo duplicis generis termini adesse debent, alii ab $u$ plane liberi, alii vero functione K affecti, quos deinceps seorsim aequari conueniet. Hunc in finem ponamus

$$NKK + nMKK' + nLKK'' + n(n-1)LK'K' = A + \alpha K,$$
$$2NK + nMK' + nLK'' = B + \beta K, \text{ et}$$
$$N = C + \gamma K.$$

Ex binis prioribus elidendo M colligitur

$$-NKK+n(n-1)LK'K'=A+\alpha K-BK-\beta KK,$$

vnde ob $N=C+\gamma K$, concluditur

$$L=\frac{A+(\alpha-B)K-(\beta-C)KK+\gamma K^3}{n(n-1)K'K'},$$

hincque

$$M=\frac{B+\beta K-2NK-nLK''}{nK'},$$

ita vt ex functione K litterae L, M et N, determinantur, dum A, $\alpha$, B, $\beta$, C, $\gamma$ constantes quascunque denotant. Nunc autem superest vt efficiatur

$$(A+\alpha K)P\partial x+(B+\beta K)PQ\partial x+(C+\gamma K)PQQ\partial x$$
$$=K\partial R+Q\partial R+(n-1)R\partial Q,$$

vnde duplicis generis terminos seorsim aequando, fit

$$P\partial x(A+BQ+CQQ)=Q\partial R+(n-1)R\partial Q$$
$$P\partial x(\alpha+\beta Q+\gamma QQ)=\partial R,$$

ideoque

$$\frac{A+BQ+CQQ}{\alpha+\beta Q+\gamma QQ}=Q+\frac{(n-1)R\partial Q}{\partial R}, \text{ seu}$$
$$\frac{(n-1)R\partial Q}{\partial R}=\frac{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}{\alpha+\beta Q+\gamma QQ}, \text{ ergo}$$
$$\frac{\partial R}{R}=\frac{(n-1)\partial Q(\alpha+\beta Q+\gamma QQ)}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3},$$

vnde ex functione Q functio R definitur: tum vero erit

$$P\partial x=\frac{(n-1)R\partial Q}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}.$$

Abeat iam integrale illud $R(K+Q)^{n-1}+$ Const. in functionem U, posito $x=a$, ac valor initio assumtus

$$y=\int\frac{(n-1)R\partial Q(K+Q)^n}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3},$$

erit integrale huius aequationis differentio-differentialis

$$L\partial\partial y+M\partial u\partial y+Ny\partial u^2=U\partial u^2.$$

Corol-

## Corollarium 1.

1050. Cum pro Q functio quaecunque ipsius $x$ accipi possit, nihil impedit, quo minus sumamus $Q = x$. Tum igitur quaeri oportet R ex hac aequatione

$$\frac{\partial K}{K} = \frac{(n-1)\partial x(\alpha + \beta x + \gamma x x)}{A + (B-\alpha)x + (C-\beta)xx - \gamma x^3},$$

eritque pro K functione quacunque ipsius $u$ assumta

$$y = (n-1)\int \frac{R\,\partial x(K+x)^n}{A + (B-\alpha)x + (C-\beta)xx - \gamma x^3},$$

in quo, integrali ita sumto vt posito $x = b$ euanescat, deinceps statui debet $x = a$.

## Corollarium 2.

1051. Ex functione autem K aequatio differentio-differentialis ita formatur, vt sit

$$L = \frac{A - (B-\alpha)K + (C-\beta)KK + \gamma K^3}{1(n-1)\partial K^2}\partial u^2,$$

$$M = \frac{B - (2C-\beta)K - 2\gamma K^2}{n\,\partial K}\partial u - \frac{L\,\partial\partial K}{\partial u\,\partial K}, \text{ et } N = C + \gamma K.$$

Deinde in expressione $R(K+x)^{n-1}$ + Const. ita constituta, vt posito $x = b$ euanescat, ponatur $x = a$, et functio ipsius $u$ inde resultans vocetur U, eritque aequatio differentio-differentialis

$$L\,\partial\partial y + M\,\partial u\,\partial y + N y\,\partial u^2 = U\,\partial u^2.$$

## Corollarium 3.

1052. Si expressio $R(K+x)^{n-1}$ + Const. ita sit comparata, vt vtroque casu $x = b$ et $x = a$ euanescat, seu potius hi termini integrationis ita constituantur, vt hoc eueniat, formula pro $y$ assumta satisfaciet huic aequationi

$$L\,\partial\partial y + M\,\partial u\,\partial y + N y\,\partial u^2 = 0,$$

quae si deinceps in alias formas transmutetur, earum quoque integralia assignari poterunt.

 Pro-

## Problema 132.

1053. Si fuerint P, Q functiones ipsius $x$, at K functio ipsius $u$, ac ponatur $y = \int e^{KQ} P \partial x$, integrali ita sumto vt euanescat casu $x = b$, tum vero ponatur $x = a$, et $y$ aequabitur functioni ipsius $u$, quae satisfaciet cuipiam aequationi differentio-differentiali, quam inuenire oportet.

## Solutio.

Cum sit $y = \int e^{KQ} P \partial x$, erit

$$\frac{\partial y}{\partial u} = \int e^{KQ} K' P Q \partial x, \text{ et } \frac{\partial \partial y}{\partial u^2} = \int e^{NQ} P \partial x (K''Q + K'K'QQ);$$

vnde fit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int e^{KQ} P \partial x (N + MK'Q + LK''Q + LK'K'QQ);$$

cuius integrale statuatur $e^{KQ} R +$ Const. quae expressio euanescat posito $x = b$, fierique oportet

$$\partial R + K R \partial Q = P \partial x [N + (MK' + LK'')Q + LK'K'QQ],$$

et ob rationes ante allegatas faciamus

$$LK'K' = A + \alpha K, \; MK' + LK'' = B + \beta K, \;\; N = C + \gamma K,$$

eritque

$$L = \frac{A + \alpha K}{K'K'} \text{ et } M = \frac{B + \beta K}{K'} - \frac{LK''}{K'},$$

atque obtinebimus has aequationes

$$\partial R = P \partial x (C + BQ + AQQ),$$
$$R \partial Q = P \partial x (\gamma + \beta Q + \alpha QQ),$$

vnde colligitur

$$\frac{\partial R}{R} = \frac{\partial Q (C + BQ + AQQ)}{\gamma + \beta Q + \alpha QQ},$$

inuentaque functione R, erit

$$P \partial x = \frac{R \partial Q}{\gamma + \beta Q + \alpha QQ},$$

ita vt sit

$y =$

$$y = \int e^{KQ} \frac{R \partial Q}{\gamma + \beta Q + \alpha QQ}.$$

Si iam expreſſio $e^{KQ} R +$ Conſt. poſito $x = a$ abeat in functionem U, aequatio differentio-differentialis, cui hoc integrale conuenit, erit

$$L \partial \partial y + M \partial u \partial y + N y \partial u^2 = U \partial u^2.$$

## Corollarium 1.

1054. Hic pro Q ſcribere licet $x$ vt ante, vnde fit $\frac{\partial R}{R} = \frac{\partial x (C + Bx + Axx)}{\gamma + \beta x + \alpha xx}$, et $y = \int e^{Kx} \cdot \frac{R \partial x}{\gamma + \beta x + \alpha xx}$, et U oritur ex forma $e^{Kx} R +$ Conſt. poſito $x = a$. Valor autem ipſius R, pro ratione coefficientium $\alpha$, $\beta$, $\gamma$ varias formas induere poteſt.

## Corollarium 2.

1055. Pro K autem quaecunque functio ipſius $u$ accipi poteſt, a cuius indole aequatio differentio-differentialis pendet. Erit autem

$$L = \frac{A + \alpha K}{\partial K^2} \partial u^2,$$

$$M = \frac{B + \beta K}{\partial K} \partial u - \frac{(A + \alpha K) \partial u \partial \partial K}{\partial K^3}, \text{ et } N = C + \gamma K,$$

vnde aequatio differentio-differentialis eſt

$$\frac{(A + \alpha K) \partial \partial y}{\partial K^2} + \frac{(B + \beta K) \partial y}{\partial K} - \frac{(A + \alpha K) \partial \partial K \partial y}{\partial K^3} + (C + \gamma K) y = U.$$

## Corollarium 3.

1056. Cum hic etiam $u$ ex calculo excedat, perinde eſt cuiusmodi functio eius pro K aſſumatur, quin etiam ſine detrimento amplitudinis poni poteſt $K = u$, dummodo ratio elementi, quod conſtans aſſumitur, habeatur.

## Scholion 1.

1057. Si ergo ſumatur $K = u$, atque elementum $\partial u$ ſumatur conſtans, vt fiat $\partial \partial K = 0$, hinc iſta aequatio con-

 ſtrui

ſtrui poteſt

$$\frac{(A+\alpha u)\partial\partial y}{\partial u^2}+\frac{(B+\beta u)\partial y}{\partial u}+(C+\gamma u)y=U,$$

exiſtente U eiusmodi functione ipſius $u$, quam deſcripſimus. Simili autem modo ex praecedente problemate conſtrui poteſt haec aequatio

$$[A-(B-\alpha)u+(C-\beta)uu+\gamma u^3]\frac{\partial\partial y}{\partial u^2}+(n-1)[B-(2C-\beta)u-2\gamma uu]\frac{\partial y}{\partial u}$$
$$+n(n-1)(C+\gamma u)y=U,$$

quae aeque late patere eſt cenſenda, ac ſi functionem quamcunque ipſius $u$ loco K ſcripſiſſemus. Hinc enim loco $u$ ſcribendo functionem quamcunque ipſius $t$, ac $\partial t$ pro conſtante ſumendo, omnes illae formae deriuari poſſunt. Ex quo haec aequatio multo latius patet illa, quam ſupra in genere per ſeries infinitas reſoluimus. Plerumque autem hae aequationes ita ſunt comparatae, vt earum integratio aliis methodis expediri haud poſſit, quocirca haec methodus omnino digna videtur, ad quam vlterius excolendam Geometrae omnes vires intendant.

## Scholion 2.

1058. In inueſtigatione huiusmodi conſtructionum ita ſum verſatus, vt primo quaſi per coniecturam formulam quandam differentialem $\int V\partial x=y$, in qua V erat certa functio ipſarum $u$ et $x$, vbi autem $u$ vt conſtans ſpectabatur, aſſumſeram, indeque dato ipſi $x$ valore tributo pertigerim ad aequationem differentio-differentialem inter $u$ et $y$, cui formula illa aſſumta ſatisfaceret. Hic autem obſeruandum eſt illam formulam integralem non prorſus ab arbitrio noſtro pendere, ſed certa quadam indole praeditam eſſe debere, vt euolutione facta res perducatur ad aequationem differentialem ſecundi gradus. Quamdiu autem hanc electionem ſoli coniecturae permittimus, perpaucae huiusmodi formulae menti ſe offerunt,

quae

quae ad ſcopum propoſitum perducunt: multoque minus ſperare licet, vt hoc modo vnquam ad datam aequationem differentio-differentialem perueniamus, caſuique potiſſimum tribuendae videntur conſtructiones, quas hic tradidimus. Cum igitur longiſſime adhuc ſimus remoti a ſolutione problematis, quo propoſita quadam aequatione differentio-differentiali quaeritur formula illa eius integrationem ſuppeditans, quod problema, an vnquam ſolutionem ſit nacturum, admodum incertum videtur; eo magis opera eſt adhibenda, vt ſaltem pro caſibus particularibus inueſtigationem formulae integrantis ex indole aequationis propoſitae deriuare conemur, ſicque quodam modo viam ad ſolutionem directam paremus. Ad hoc autem ſeries infinitae, per quas huiusmodi aequationes ſupra reſoluere docuimus, vtiliter adhiberi poſſunt; vnde in ſequenti capite methodum exponam ex ſerie infinita ſolutionem cuiuspiam aequationis differentio-differentialis continente, formulam illam integralem inueſtigandi.

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CA-

# CAPVT XI.

## DE CONSTRVCTIONE AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM EX EARVM RESOLVTIONE PER SERIES INFINITAS PETITA.

### Problema 133.

1059.

Propofita ferie infinita

$$A + Bs + Cs^2 + \ldots\ldots + Ms^{i-1} + Ns^i + \text{etc.}$$

in qua fit $B = \frac{0m+b}{1n+k}A$, $C = \frac{1m+b}{2n+k}B$, $D = \frac{2m+b}{3n+k}C$, et in genere $N = \frac{(i-1)m+b}{in+k}M$, eius fummam per formulam integralem exprimere.

### Solutio.

Ponatur fumma quaefita $= z$, ita vt fit

$$z = A + Bs + Cs^2 + Ds^3 + \ldots\ldots + Ms^{i-1} + Ns^i + \text{etc.}$$

eritque differentiando

$$\frac{s\partial z}{\partial s} = 0A + 1Bs + 2Cs^2 + 3Ds^3 + \ldots\ldots + (i-1)Ms^{i-1} + iNs^i + \text{etc.}$$

ex cuius combinatione cum praecedente oritur

$$\frac{ms\partial z}{\partial s} + bz = bA + (m+b)Bs + (2m+b)Cs^2 + \ldots\ldots$$
$$+ [(i-1)m+b]Ms^{i-1} + (im+b)Ns^i + \text{etc.}$$

Deinde vero etiam fimili modo eft

$$\frac{ns\partial z}{\partial s} + kz = kA + (n+k)Bs + (2n+k)Cs^2 + \ldots\ldots$$
$$+ [(i-1)n+k]Ms^{i-1} + (in+k)Ns^i + \text{etc.}$$

ergo

ergo ob

$$(n+k)B=bA, \ (2n+k)C=(m+b)B, \text{ etc. erit}$$
$$\frac{ns\partial z}{\partial s}+kz=kA+bAs+(m+b)Bs^2+(2m+b)Cs^3+\text{etc.}$$

vnde manifesto conficitur

$$\frac{ns\partial z}{\partial s}+kz=kA+\frac{mss\partial z}{\partial s}+bsz, \text{ seu}$$
$$s\partial z(n-ms)+z\partial s(k-bs)=kA\partial s$$

hoc est

$$\partial z+\frac{z\partial s(k-bs)}{s(n-ms)}=\frac{kA\partial s}{s(n-ms)}.$$

Cum nunc sit

$$\frac{\partial s(k-bs)}{s(n-ms)}=\frac{k\partial s}{ns}+\frac{(mk-nb)\partial s}{n(n-ms)},$$

aequatio ista integrabilis fit multiplicata per

$$s^{\frac{k}{n}}(n-ms)^{\frac{nb-mk}{mn}}, \text{ proditque}$$
$$(n-ms)^{\frac{nb-mk}{ma}}s^{\frac{k}{n}}z=Ak\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1},$$

quod integrale ita capi oportet, vt posito $s=0$ fiat $z=A$, quo obseruato habebimus

$$z=Aks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1}.$$

## Corollarium 1.

1060. Peculiari solutione eget casus $m=0$, quo fit

$$\partial z+\frac{z\partial s(k-bs)}{ns}=\frac{Ak\partial s}{ns},$$

quae per $s^{\frac{k}{n}}e^{\frac{-bs}{n}}$ multiplicata praebet

$$e^{\frac{-bs}{n}}s^{\frac{k}{n}}z=\frac{Ak}{n}\int e^{\frac{-bs}{n}}s^{\frac{k}{n}-1}\partial s$$

ideoque

$$z = \frac{A\,k}{n}\, e^{\frac{b s}{n}}\, s^{\frac{-k}{n}} \int e^{\frac{-b s}{n}}\, s^{\frac{k}{n} - 1}\, \partial s,$$

integrali ita ſumto, vt fiat $z = A$ poſito $s = 0$.

## Corollarium 2.

1061. Caſus etiam $n = 0$ ſeorſim reſolui debet, aequatio enim

$$\partial z + z\,\partial s\left(\frac{b s - k}{m s s}\right) = -\frac{A\,k\,\partial s}{m s s}$$

multiplicari debet per $s^{\frac{b}{m}} e^{\frac{k}{m s}}$, et inuenitur integrale

$$e^{\frac{k}{m s}}\, s^{\frac{b}{m}}\, z = -\frac{A\,k}{m} \int e^{\frac{k}{m s}}\, s^{\frac{b}{m} - 2}\, \partial s,$$

ideoque

$$z = -\frac{A\,k}{m}\, e^{\frac{-k}{m s}}\, s^{\frac{-b}{m}} \int e^{\frac{k}{m s}}\, s^{\frac{b}{m} - 2}\, \partial s.$$

## Corollarium 3.

1062. Si fuerit et $m = 0$ et $n = 0$, ob $N = \frac{b}{k} M$, ſeries noſtra erit geometrica, aequatio vero noſtra erit

$$z\,\partial s\,(k - b s) = A\,k\,\partial s \text{ ſeu } z = \frac{A\,k}{k - b s},$$

vti natura rei manifeſto poſtulat.

## Scholion.

1063. Imprimis hic caſus notari meretur, quo eſt $k = 0$, et ſumma $z$ ſine ſigno integrali exprimi poteſt; erit namque

$$(n - m s)^{\frac{b}{m}}\, z = \text{Conſt.}$$

et quia ſi $s = 0$, fieri debet $z = A$, erit Conſt. $= A\, n^{\frac{b}{m}}$, ideoque

$$z = A\, n^{\frac{b}{m}}\, (n - m s)^{\frac{-b}{m}}, \text{ ſeu } z = A\left(\frac{n}{n - m s}\right)^{\frac{b}{m}},$$

vel

vel etiam

$$z = A\left(1 - \frac{ms}{n}\right)^{\frac{-b}{m}}.$$

At vero integratio etiam ſuccedit caſu quo $k = n$, erit enim

$$(n - ms)^{\frac{b}{m} - 1} sz = An\int \partial s\,(n - ms)^{\frac{b}{m} - 2},$$

quod integrale eſt $=$ Conſt. $- \frac{An(n - ms)^{\frac{b}{m} - 1}}{b - m}$, et quia poſito $s = 0$, fit $z = A$, erit $0 =$ Conſt. $- \frac{A}{b - m} \cdot n^{\frac{b}{m}}$, hincque

$$z = \frac{An}{(b - m)s}\left[\left(\frac{n}{n - ms}\right)^{\frac{b}{m} - 1} - 1\right] = \frac{An}{(b - m)s}\left[\left(1 - \frac{ms}{n}\right)^{1 - \frac{b}{m}} - 1\right].$$

Porro perſpicuum eſt, integrationem expediri poſſe caſu $k = 2n$, quo cum ſit

$$(n - ms)^{\frac{b}{m} - 2} ssz = 2An\int s\,\partial s\,(n - ms)^{\frac{b}{m} - 3},$$

erit hoc integrale

$$= \text{Conſt.} - \frac{2Ans}{b - 2m}(n - ms)^{\frac{b}{m} - 2} + \frac{2An}{b - 2m}\int \partial s\,(n - ms)^{\frac{b}{m} - 2} \text{ vel}$$

$$= \text{Conſt.} - \frac{2Ans}{b - 2m}(n - ms)^{\frac{b}{m} - 2} - \frac{2An(n - ms)^{\frac{b}{m} - 1}}{(b - m)(b - 2m)};$$

vbi Conſt. $= \frac{2An^{\frac{b}{m}}}{(b - m)(b - 2m)}$, ideoque

$$z = \frac{2Ann}{(b - m)(b - 2m)ss}\left[\left(\frac{n}{n - ms}\right)^{\frac{b}{m} - 2} - 1 - \frac{(b - 2m)s}{n}\right],$$

ſimilique modo etiam integratio caſibus $k = 3n$, $k = 4n$ etc. abſoluetur.

## Problema 134.

1064. Propoſita huiusmodi ſerie infinita

$$A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + D\mathfrak{D}u^3 + \ldots + M\mathfrak{M}u^{i-1} + \mathfrak{N}Nu^i + \text{etc.}$$

 coeffi-

coefficientium lege exiſtente

$$\mathrm{B} = \frac{0m+b}{1n-k}\mathrm{A},\ \mathrm{C} = \frac{1m+b}{2n+k}\mathrm{B},\ \mathrm{D} = \frac{2m+b}{3n+k}\mathrm{C} \ldots \mathrm{N} = \frac{(i-1)m+b}{in+k}\mathrm{M},$$

$$\mathfrak{B} = \frac{0\mu+\eta}{1\nu+\theta}\mathfrak{A},\ \mathfrak{C} = \frac{1\mu+\eta}{2\nu+\theta}\mathfrak{B},\ \mathfrak{D} = \frac{2\mu+\eta}{3\nu+\theta}\mathfrak{C} \ldots \mathfrak{N} = \frac{(i-1)\mu+\eta}{i\nu+\theta}\mathfrak{M},$$

eius ſummam per formulam integralem exprimere.

## Solutio.

Poſita ſumma

$$y = \mathrm{A}\mathfrak{A} + \mathrm{B}\mathfrak{B}u + \mathrm{C}\mathfrak{C}u^2 + \mathrm{D}\mathfrak{D}u^3 + \mathrm{E}\mathfrak{E}u^4 + \text{etc.}$$

conſideretur ſeries hoc modo formata

$$z = \mathrm{A} + \mathrm{B}ux + \mathrm{C}u^2x^2 + \mathrm{D}u^3x^3 + \mathrm{E}u^4x^4 + \text{etc.}$$

cuius ſumma poſito $ux = s$ eſt, vt modo inuenimus

$$z = \mathrm{A}ks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1},$$

integrali ita determinato, vt poſito $s = 0$ fiat $z = \mathrm{A}$.

Formetur huiusmodi formula integralis

$$\mathrm{V} = \int \mathrm{P}z\partial x = \int \mathrm{P}\partial x(\mathrm{A} + \mathrm{B}ux + \mathrm{C}u^2x^2 + \mathrm{D}u^3x^3 + \text{etc.})$$

in qua $u$ ſpectetur vt conſtans, pro P autem eiusmodi functio ipſius $x$ accipiatur, vt fiat

$$\int \mathrm{P}x\partial x = \frac{\mathfrak{B}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^2\partial x = \frac{\mathfrak{C}}{\mathfrak{B}}\int \mathrm{P}x\partial x,\ \int \mathrm{P}x^3\partial x = \frac{\mathfrak{D}}{\mathfrak{C}}\int \mathrm{P}x^2\partial x, \text{etc.}$$

poſtquam ſcilicet in his integralibus data lege ſumtis variabili $x$ datus quidem valor fuerit tributus. Cum igitur hinc ſit

$$\int \mathrm{P}x\partial x = \frac{\mathfrak{B}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^2\partial x = \frac{\mathfrak{C}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^3\partial x = \frac{\mathfrak{D}}{\mathfrak{A}}\int \mathrm{P}\partial x, \text{etc.}$$

erit

$$\mathrm{V} = (\mathrm{A} + \frac{\mathrm{B}\mathfrak{B}}{\mathfrak{A}}u + \frac{\mathrm{C}\mathfrak{C}}{\mathfrak{A}}u^2 + \frac{\mathrm{D}\mathfrak{D}}{\mathfrak{A}}u^3 + \text{etc.})\int \mathrm{P}\partial x,$$

vnde patet fore

$$y = \frac{\mathfrak{A}\mathrm{V}}{\int \mathrm{P}\partial x} = \frac{\mathfrak{A}\int \mathrm{P}z\partial x}{\int \mathrm{P}\partial x}.$$

Quare cum valor ipſius $z$ ſit cognitus, tantum ſupereſt, vt functio P ipſius $x$ conditionibus memoratis praedita inueſtigetur.

tur. In genere autem esse oportet

$$\int P x^i \partial x = \frac{\mathfrak{N}}{\mathfrak{M}} \int P x^{i-1} \partial x = \frac{(i-1)\mu+\eta}{i\nu+\theta} \int P x^{i-1} \partial x,$$

quae aequalitas cum sufficiat, vt tantum certo quodam casu, quo ipsi $x$ datus tribuitur valor, subsistat, ponamus in genere esse

$$(i\nu+\theta)\int P x^i \partial x = [(i-1)\mu+\eta]\int P x^{i-1} \partial x + x^i Q,$$

ita vt pro terminis integralibus sit $Q=0$. Differentiando ergo habebimus

$$(i\nu+\theta) P x^i \partial x = (i\mu-\mu+\eta) P x^{i-1} \partial x + x^i \partial Q + i x^{i-1} Q \partial x,$$

seu per $x^{i-1}$ diuidendo

$$(i\nu+\theta) P x \partial x = (i\mu-\mu+\eta) P \partial x + x \partial Q + i Q \partial x,$$

quae aequalitas cum pro omnibus valoribus ipsius $i$ aeque subsistere debeat, hinc duas adipiscimur aequationes

$$\nu P x \partial x = \mu P \partial x + Q \partial x, \text{ et } \theta P x \partial x = (\eta-\mu) P \partial x + x \partial Q;$$

vnde duplici modo colligimus

$$P \partial x = \frac{Q \partial x}{\nu x - \mu}, \text{ et } P \partial x = \frac{x \partial Q}{\theta x - (\eta-\mu)},$$

sicque alterum valorem per alterum diuidendo

$$\frac{x \partial Q}{Q \partial x} = \frac{\theta x + \mu - \eta}{\nu x - \mu}, \text{ seu } \frac{\partial Q}{Q} = \frac{\partial x(\theta x + \mu - \eta)}{x(\nu x - \mu)},$$

quae euoluitur in

$$\frac{\partial Q}{Q} = \frac{\eta-\mu}{\mu x} \partial x + \frac{\mu\theta + \mu\nu - \eta\nu}{\mu(\nu x - \mu)} \partial x:$$

hinc integrando elicitur

$$Q = x^{\frac{\eta}{\mu} - 1} (\nu x - \mu)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} \text{ per Const. mult.}$$

seu

$$Q = - x^{\frac{\eta}{\mu} - 1} (\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1},$$

vnde fit

$$P\,\partial x = x^{\frac{\eta}{\mu}-1}\partial x\,(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}.$$

Cum igitur ſit

$$(i\nu+\theta)\int P\,x^{i}\,\partial x = [(i-1)\mu+\eta]\int P\,x^{i-1}\,\partial x - x^{i+\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1},$$

ſi haec integralia ita capiantur, vt euaneſcant poſito $x=0$, tum vero ſtatuatur $x=\frac{\mu}{\nu}$, fiet vti hypotheſis noſtra poſtulat

$$\int P\,x^{i}\,\partial x = \frac{(i-1)\mu+\eta}{i\nu+\theta}\int P\,x^{i-1}\,\partial x;$$

at vero in hunc finem neceſſe eſt, vt ſit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0.$$

Quae conditio ſi locum habeat, ſeriei propoſitae ſumma ita exprimetur, vt ſit

$$y\int x^{\frac{\eta}{\mu}-1}\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}} = \mathfrak{A}\int x^{\frac{\eta}{\mu}-1}z\,\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

exiſtente

$$z = A k s^{\frac{-k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1},$$

integrali hoc ita ſumto, vt fiat $z=A$ poſito $s=0$. Hoc autem integrali inuento, pro $s$ ſcribatur $ux$, et hoc valore ipſius $z$ in illa formula ſubſtituto, quantitatem $u$ tanquam conſtantem tractari oportet, quoad illae integrationes lege praeſcripta fuerint abſolutae, tum enim pro $y$ prodibit functio ipſius $u$, ſummam ſeriei propoſitae exprimens.

## Corollarium. 1.

1065. Quia in geminatis coefficientibus noſtrae ſeriei ſimilis lex progreſſionis aſſumitur, ſingulas ſeries A, B, C, D etc. et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. inter ſe permutare licet, vnde hac methodo duplex formula ſummam ſeriei exprimens obtinetur.

Corol-

## Corollarium 2.

1066. Etsi functio Q non in calculum ingreditur, eam tamen nosse oportet, quoniam ex eius indole termini integrationis constitui debent, ita vt pro vtroque fiat $Q = 0$. Hi scilicet termini sunt $x = 0$ et $x = \frac{\mu}{\nu}$, dum fuerit

$$i + \frac{\eta}{\mu} - 1 > 0 \text{ et } \frac{\theta\mu - \eta\nu}{\mu\nu} + 1 > 0,$$

vbi $i$ est numerus integer positiuus.

## Corollarium 3.

1067. Pro functione Q casus quo vel $\mu = 0$ vel $\nu = 0$ seorsim sunt euoluendi. Priore quo $\mu = 0$, est

$$\frac{\partial Q}{Q} = \frac{\partial x(\theta x - \eta)}{\nu x x} = \frac{\theta \partial x}{\nu x} - \frac{\eta \partial x}{\nu x x}, \text{ vnde fit}$$

$$Q = e^{\frac{\eta}{\nu x}} x^{\frac{\theta}{\nu}}.$$

Posteriori quo $\nu = 0$, est

$$\frac{\partial Q}{Q} = \frac{\partial x(\theta x + \mu - \eta)}{-\mu x} = -\frac{\theta \partial x}{\mu} + \frac{\eta - \mu}{\mu} \cdot \frac{\partial x}{x},$$

ideoque

$$Q = e^{\frac{-\theta x}{\mu}} x^{\frac{\eta}{\mu} - 1}.$$

## Scholion.

1068. Constructiones hoc modo adornandae prorsus similes sunt iis, quas capite praecedente tradidimus, cum res etiam ad formulam integralem huiusmodi $\int V \partial x$ reducatur, in qua V est functio binarum variabilium $u$ et $x$, quarum illa autem $u$ in ipsa integratione constans reputatur, post integrationem vero ipsi $x$ datus quidam valor assignatur. Verum tamen haec constructio ad casus in superiori methodo non contentos extenditur, quandoquidem fieri potest, vt quantitas $z$ functiones maxime transcendentes inuoluat. Viciſſim autem vidimus, methodum praecedentem ad eiusmodi aequationes appli-

applicari posse, quae per series, quales hic tractamus, euolui nequeant, vnde in Analysin ex hoc fonte haud contemnenda incrementa hauriri posse videntur.

## Problema 135.

1069. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y+(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

valorem ipsius $y$ per formulam integralem construere.

## Solutio I.

Hanc aequationem supra (967.) ita in seriem euoluimus, vt posito

$$y=x^\lambda(\mathrm{A}+\mathrm{B}x^n+\mathrm{C}x^{2n}+\mathrm{D}x^{3n}+\mathrm{E}x^{4n}+\text{etc.}),$$

primo exponenti $\lambda$ tribui debeat radix huius aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero posito breuitatis gratia

$$\lambda(\lambda-1)b+\lambda e+g=b \text{ fit}$$

$$\mathrm{B}=\frac{-b}{n[na+(2\lambda-1)a+c]}\mathrm{A},$$

$$\mathrm{C}=\frac{-nnb-(2\lambda-1)nb-ne-b}{2n[2na+(2\lambda-1)a+c]}\mathrm{B},$$

$$\mathrm{D}=\frac{-4nnb-2(2\lambda-1)nb-2ne-b}{3n[3na+(2\lambda-1)a+c]}\mathrm{C},$$

ideoque, illius seriei positis binis terminis contiguis indefinite $\mathrm{M}x^{(i-1)n}+\mathrm{N}x^{in}$, generaliter

$$\mathrm{N}=\frac{-(i-1)^2nnb-(2\lambda-1)(i-1)nb-(i-1)ne-b}{in[ina+(2\lambda-1)a+c]}\mathrm{M},$$

vbi cum denominator iam habeat factores, quales ante assumsimus, numeratorem quoque in factores resoluamus, quo nihilo aequali posito inuenitur

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\frac{e}{2b}\pm\sqrt{\left[\tfrac{1}{4}(2\lambda-1)^2+\frac{(\lambda-1)e}{2b}+\frac{ee}{4bb}-\frac{b}{b}\right]},$$

seu

(i−1)

$$(i-1)n = -\tfrac{1}{2}(2\lambda-1) - \tfrac{e}{2b} \pm \tfrac{1}{2b}\sqrt{[(b-e)^2 - 4bg)}.$$

Ponatur breuitatis gratia

$$\sqrt{[(b-e)^2 - 4bg]} = q,$$

vt fit

$$(i-1)n = \frac{-(2\lambda-1)b - e \pm q}{2b},$$

et nostra relatio fiet

$$N = \frac{-[(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q][(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q]}{inb[ina + (2\lambda-1)a + c]}M.$$

Ponamus iam $x^n = u$, et seriem inuentam ita repraesentemus

$$\frac{y}{x^\lambda} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + \ldots\ldots + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

horumque duplicium coefficientium lex ita se habebit

$$N = \frac{(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q}{-inb}M \text{ et}$$

$$\mathfrak{N} = \frac{(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q}{ina + (2\lambda-1)a + c}\mathfrak{M}.$$

Cum igitur haec series similis sit ei, quam ante construximus, comparationem instituamus, et habebimus

$$m = nb,\ h = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q,$$
$$n = -nb, \text{ et } k = 0:$$
$$\mu = nb,\ \eta = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q,$$
$$\nu = na, \text{ et } \theta = (2\lambda-1)a + c.$$

Primum ergo quaeramus quantitatem $z$, et ne littera $x$ ambiguitatem creet, loco litterae $x$ in praecedente problemate vsurpatae vtamur littera $t$, sitque $ut = s$, et quia est $k = 0$, erit per §. 1063.

$$z = A(1+s)^{\frac{-(2\lambda-1)b-e+q}{2nb}} = A(1+ut)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

 Hoc

Hoc valore inuento tractetur in sequentibus integrationibus quantitas $u$ vt constans, et cum quod supra erat $y$ hic sit $\frac{y}{x^\lambda}$, et $t$ quod supra erat $x$, erit

$$\frac{y}{x^\lambda} \int t^{\frac{1}{\mu}-1} \partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}} = \mathfrak{A}\int t^{\frac{\eta}{\mu}-1} z\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

vbi cum sit $u=x^n$, hic valor statim pro $u$ scribi potest, vt fit

$$z = A(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}},$$

et in his integrationibus littera $x$ vt constans spectari debet. Quodsi autem fuerit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0,$$

haec integralia ita capi debent, vt euanescant posito $t=0$, quo facto ipsi $t$ tribui debet valor $t=\frac{\mu}{\nu}=\frac{b}{a}$. Cum vnitas sit minimus valor ipsius $i$, sufficit vt sit

$$\frac{(2\lambda-1)b+e-q}{2nb}>0, \text{ tum vero } \frac{(2\lambda+1)ab+2bc-ae-aq}{2nab}+1>0.$$

Nunc vero cum

$$\int t^{\frac{\eta}{\mu}-1} \partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

fiat quantitas constans, pro $y$ autem eius multiplum quoduis nostrae aequationi aeque satisfaciat, eius integrale ita exprimetur

$$y = C x^\lambda \int t^{\frac{\eta}{\mu}-1} z\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

existente

$$z = (1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

## Solutio II.

1071. Si coefficientes geminatos inter se permutemus, vt sit

$m =$

$$m = n b, \quad b = \tfrac{1}{2}(2\lambda - 1) b + \tfrac{1}{2} e + \tfrac{1}{2} q,$$
$$n = n a, \quad k = (2\lambda - 1) a + c,$$
$$\mu = n b, \quad \eta = \tfrac{1}{2}(2\lambda - 1) b + \tfrac{1}{2} e - \tfrac{1}{2} q,$$
$$\nu = - n b, \; \theta = 0,$$

ſumaturque $\lambda$ ex aequatione

$$\lambda(\lambda - 1) a + \lambda c + f = 0.$$

Primo ponatur $x^n t = s$, et quaeratur $z$ vt ſit

$$z = A k s^{\frac{-k}{n}} (n - m s)^{\frac{m k - n b}{m n}} \int s^{\frac{k}{n} - 1} \partial s (n - m s)^{\frac{n b - m k}{m n} - 1},$$

integrali ita definito, vt poſito $s = 0$ fiat $z = A$, qui quidem valor A eſt arbitrarius, tum ſpectata $x$ vt conſtante erit

$$y = C x^\lambda \int t^{\frac{\eta}{\mu} - 1} z \partial t (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

integrali ita ſumto vt euaneſcat poſito $t = 0$; tum vero facto $t = \frac{\mu}{\nu}$, ſi modo fuerit $\frac{\eta}{\mu} > 0$ et $1 - \frac{\eta}{\mu} > 0$, ob $\theta = 0$; vbi notetur $z$ per hanc aequationem differentialem definiri

$$\frac{\partial z}{\partial s} = \frac{A k - z(k - b s)}{s(n - m s)}.$$

## Solutio III.

1072. Per ſeriem deſcendentem aequationem propoſitam ita reſoluimus, vt poſito

$$y = x^\lambda (A + B x^{-n} + C x^{-2n} + D x^{-3n} + \text{etc.})$$

exponens $\lambda$ definiri debeat ex hac aequatione

$$\lambda(\lambda - 1) b + \lambda e + g = 0,$$

tum vero poſito $\lambda(\lambda - 1) a + \lambda c + f = h$, ſit

$$B = \frac{-h}{n[n b - (2\lambda - 1) b - e]} A,$$

$$C = \frac{-n n a - (2\lambda - 1) n a - n c - h}{2n[2n b - (2\lambda - 1) b - e]} B,$$

 et

et generatim

$$N = \frac{-(i-1)^2 nna + (2\lambda-1)(i-1)na + (i-1)nc - b}{in(inb-(2\lambda-1)b-e)} M,$$

quae aequalitas posito $\sqrt{[(a-c)^2 - 4af]} = p$, ita per factores exhibetur

$$N = \frac{-[(i-1)na - \frac{1}{2}(2\lambda-1)a - \frac{1}{2}c - \frac{1}{2}p][(i-1)na - \frac{1}{2}(2\lambda-1)a - \frac{1}{2}c + \frac{1}{2}p]}{ina(inb - (2\lambda-1)b - e)} M.$$

Quodsi iam ponamus $x^{-n} = u$, et talem seriem constituamus

$$\frac{y}{x^\lambda} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + \ldots\ldots + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

erit

$$N = \frac{(i-1)na - \frac{1}{2}(2\lambda-1)a - \frac{1}{2}c - \frac{1}{2}p}{-ina} M, \text{ et}$$

$$\mathfrak{N} = \frac{(i-1)na - \frac{1}{2}(2\lambda-1)a - \frac{1}{2}c + \frac{1}{2}p}{inb - (2\lambda-1)b - e} \mathfrak{M},$$

et habebimus comparatione instituta cum constructione generali

$$m = na, \quad b = -\tfrac{1}{2}(2\lambda-1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$

$$n = -na, \text{ et } k = 0,$$

$$\mu = na, \quad \eta = -\tfrac{1}{2}(2\lambda-1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$

$$\nu = nb, \text{ et } \theta = -(2\lambda-1)b - e.$$

Hinc posito $s = ut = x^{-n}t$, erit

$$z = A(1+s)^{\frac{-b}{m}} = A(1+x^{-n}t)^{\frac{-b}{m}},$$

quo valore inuento, si iam sola quantitas $t$ pro variabili habeatur, orietur haec constructio

$$y = Cx^\lambda \int t^{\frac{\eta}{\mu}-1} z\,\partial t\,(\mu - \nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

vbi termini integrationis ita sunt constituendi, vt vtroque fiat

$$t^{\frac{\eta}{\mu}}(\mu - \nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1} = 0.$$

Solu-

## Solutio IV.

1703. Hic etiam coefficientes geminati permutari possunt, vt sit

$$m = n a, \quad b = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$n = n b, \quad k = -(2\lambda - 1)b - e,$$
$$\mu = n a, \quad \eta = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$\nu = -n a, \text{ et } \theta = 0,$$

sumtoque vt ante $\lambda$ ex aequatione

$$\lambda(\lambda - 1)b + \lambda e + g = 0,$$

et posito $\sqrt{[(a-c)^2 - 4af)} = p$, fit $s = x^{-n} t$, et quaeratur $z$ ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{A k - z(k - b s)}{s(n - m s)},$$

ita vt posito $s = 0$ fiat $z = A$, vnde fit

$$z = A k s^{\frac{-k}{n}} (n - m s)^{\frac{mk - nb}{mn}} \int s^{\frac{k}{n} - 1} \partial s (n - m s)^{\frac{nb - mk}{mn} - 1},$$

tum vero spectata $x$ vt constante, erit

$$y = C x^{\lambda} \int t^{\frac{\eta}{\mu} - 1} z \partial t (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

vbi bini termini integrationis ita sumi debent, vt vtroque fiat

$$t^{\frac{\eta}{\mu}} (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} = 0.$$

## Scholion.

1074. Singulae hae constructiones plerumque pluribus modis exhiberi possunt, cum non solum $\lambda$ duplicem valorem habere queat, sed etiam formulae radicales $p$ et $q$ signo ambiguo gaudeant. At hae constructiones alio modo negotium conficiunt atque praecedentes, quod quo clarius appareat, consideremus aequationem

$$xx(1-xx)\partial\partial y - x(1+xx)\partial x\partial y + xxy\partial x^2 = 0,$$

ita vt fit $a=1$, $b=-1$, $c=-1$, $e=-1$; $f=0$ et $g=1$, atque $n=2$; vnde pro duabus prioribus constructionibus habetur $\lambda(\lambda-1)-\lambda=0$, ergo vel $\lambda=0$ vel $\lambda=2$, tum vero $q=\pm 2$. Constructio ergo prima dat $\lambda=0$, $m=-2$, $b=\mp 1$, $n=2$, $k=0$, $\mu=-2$, $\eta=\pm 1$, $\nu=2$, $\theta=-2$; vnde colligitur

$$z=(1+xxt)^{\mp\frac{1}{2}}, \text{ et } y=C\int t^{\mp\frac{1}{2}-1} z\partial t(1+t)^{\pm\frac{1}{2}-1}.$$

Valeant signa inferiora, vt fit

$$z=\surd(1+xxt) \text{ et } y=C\int\frac{z\partial t}{(1+t)\surd t(1+t)}.$$

Quae formulae quomodo satisfaciant, videamus. Sumta nempe sola $x$ variabili, fit

$$\frac{\partial z}{\partial x}=\frac{xt}{\surd(1+xxt)} \text{ et } \frac{\partial\partial z}{\partial x^2}=\frac{t}{(1+xxt)^{\frac{3}{2}}},$$

hinc

$$\frac{\partial y}{\partial x}=C\int\frac{xt\partial t}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{1}{2}}} \text{ et } \frac{\partial\partial y}{\partial x}=C\int\frac{t\partial t}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

vnde conficitur

$$xx(1-xx)\frac{\partial\partial y}{\partial x^2}-x(1+xx)\frac{\partial y}{\partial x}+xxy=C\int\frac{xx\partial t(1-xxtt)}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

cuius formulae integrale est $\frac{2Cxx\surd t}{\surd(1+t)(1+xxt)}$; quod cum euanescat tam casu $t=0$ quam casu $t=\infty$, cunstructio nostiae aequationis

$$y=C\int\frac{z\partial t}{(1+t)\surd t(1+t)}=C\int\frac{\partial t\surd(1+xxt)}{(1+t)\surd t(1+t)},$$

ita confici debet; sumto $x$ pro constante, integratio ita instituatur, vt integrale euanescat posito $t=0$, quo facto statuatur

tur $t = \infty$, et functio ipsius $x$, quae pro $y$ prodibit, satisfaciet aequationi propositae.

Sin autem secundam constructionem eligamus, sumta $\lambda = 0$, erit $m = -2$, $b = \pm 1$, $n = 2$, $k = -2$, $\mu = -2$, $\eta = \mp 1$, $\nu = 2$, $\theta = 0$, atque $z$ ita definiri debet ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{-2A + z(2+s)}{s(2+2s)},$$

existente $s = xxt$, vt posito $s = 0$ fiat $z = A$. Valeat signum superius, et cum sit

$$\partial z - \frac{z\partial s(2+s)}{2s(1+s)} = \frac{-A\partial s}{s(1+s)}:$$

multiplicetur per $\frac{\sqrt{(1+s)}}{s}$, eritque integrale

$$\frac{z\sqrt{(1+s)}}{s} = \text{Const.} - A\int\frac{\partial s}{ss\sqrt{(1+s)}}, \text{ seu}$$

$$z = A - \frac{As}{\sqrt{(1+s)}}\, l\, \frac{1+\sqrt{(1+s)}}{\sqrt{s}} + \frac{Bs}{\sqrt{(1+s)}},$$

quae posito $s = 0$ dat $z = A$, quicquid sit B. Deinde est

$$y = C\int t^{\frac{1}{2}-1} z\partial t(1+t)^{-\frac{1}{2}} \text{ seu } y = C\int\frac{z\partial t}{\sqrt{t(1+t)}},$$

qui valor quomodo satisfaciat haud facile ostendi potest; hocque magis ista methodus excoli meretur.

## Exemplum.

1075. *Constructiones aequationis differentio-differentialis*

$$xx(1-xx)\partial\partial y - x(1+xx)\partial x\partial y + xxy\partial x^2 = 0$$

*ex praecedente problemate oriundas exhibere.*

Ob $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$, $f = 0$, et $g = 1$, pro prima constructione habemus vel $\lambda = 0$ vel $\lambda = 2$, vnde obtinemus.

1.) Si $\lambda = 0$, vt modo inuenimus

$$m = -2,\ b = \mp 1,\ n = 2,\ k = 0,\ \mu = -2,\ \eta = \pm 1,$$
$$\nu = 2,\ \theta = -2, \text{ vnde fit}$$

$z =$

$$z=(1+xxt)^{\mp\frac{1}{2}} \text{ et } y=C\int t^{\mp\frac{1}{2}-1} z\partial t(1+t)^{\pm\frac{1}{2}-1},$$

ficque duplex oritur conftructio

altera $z=\sqrt{(1+xxt)}$ et $y=C\int\frac{z\partial t}{(1+xxt)\sqrt{t(1+t)}}$,

altera $z=\frac{1}{\sqrt{(1+xxt)}}$ et $y=C\int\frac{z\partial t}{t\sqrt{t(1+t)}}$.

2.) Si $\lambda=2$, erit ob $q=\pm 2$,

$$m=-2,\ b=-2\mp 1,\ n=2,\ k=0,\ \mu=-2,$$
$$\eta=-2\pm 1,\ \nu=2,\ \theta=2,$$

vnde fit

$$z=(1+xxt)^{\frac{-2\mp 1}{2}} \text{ et } y=Cx^2\int t^{+1\mp\frac{1}{2}-1} z\partial t(1+t)^{\pm\frac{1}{2}},$$

ficque duplex habetur conftructio

altera $z=(1+xxt)^{-\frac{3}{2}}$ et $y=Cx^2\int\frac{z\partial t\sqrt{(1+t)}}{\sqrt{t}}$,

altera $z=(1+xxt)^{-\frac{1}{2}}$ et $y=Cx^2\int\frac{z\partial t\sqrt{t}}{\sqrt{(1+t)}}$,

Pro fecunda conftructione generali habemus:

3.) Si $\lambda=0$, permutando illos indices,

$$m=-2,\ b=\pm 1,\ n=2,\ k=-2,\ \mu=-2,\ \eta=\mp 1,$$
$$\nu=2,\ \theta=0,$$

vnde pofito $xxt=s$, primo quaeratur $z$ ex hac aequatione $\frac{\partial z}{\partial s}=\frac{-2A+z(2\pm s)}{2s(1+s)}$, vt pofito $s=0$ fiat $z=A$; tum vero erit

$$y=C\int t^{\pm\frac{1}{2}-1} z\partial t(1+t)^{\mp\frac{1}{2}}.$$

Hinc ergo nafcitur duplex conftructio

altera $\frac{\partial z}{\partial s}=\frac{-2A+z(2+s)}{2s(1+s)}$ et $y=C\int\frac{z\partial t}{\sqrt{t(1+t)}}$,

altera $\frac{\partial z}{\partial s}=\frac{-2A+z(2-s)}{2s(1+s)}$ et $y=C\int\frac{z\partial t\sqrt{(1+t)}}{t\sqrt{t}}$.

4.) Si $\lambda=2$ habebimus

$$m=-2,\ b=-2\pm 1,\ n=2,\ k=2,\ \mu=-2,$$
$$\eta=-2\mp 1,\ \nu=2,\ \theta=0,$$

pofito-positoque $xxt = s$, vt ante, quaeratur $z$ ex aequatione

$$\frac{\partial z}{\partial s} = \frac{2A - z[2 + (2 \mp 1)s]}{2s(1+s)}, \text{ eritque}$$

$$y = Cx^2 \int t^{\pm \frac{1}{2}} z \partial t (1+t)^{\frac{-2 \mp 1}{2}};$$

ideoque etiam duplex constructio elicitur

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = Cx^2 \int \frac{z \partial t \sqrt{t}}{(1+t)\sqrt{(1+t)}}$

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = Cx^2 \int \frac{z \partial t}{\sqrt{t(1+t)}}$.

Ex solutione tertia colligimus primo

$$-\lambda(\lambda - 1) - \lambda + 1 = 0 \text{ seu } \lambda\lambda = 1,$$

ideoque

$$\lambda = \pm 1 \text{ et } p = \sqrt{4} = \pm 2; \text{ quare}$$

5.) si capiatur $\lambda = +1$, erit

$m = 2$, $b = \mp 1$, $n = -2$, $k = 0$, $\mu = 2$, $\eta = \pm 1$, $\nu = -2$, $\theta = 2$,

hincque

$$z = (1 + \tfrac{t}{xx})^{\pm \frac{1}{2}} \text{ et } y = Cx \int t^{\pm \frac{1}{2} - 1} z \partial t (1+t)^{-1 \mp \frac{1}{2}},$$

ita vt iterum duplex habeatur constructio

altera $z = \frac{1}{x}\sqrt{(xx+t)}$ et $y = Cx \int \frac{z \partial t}{(1+t)\sqrt{t(1+t)}}$

altera $z = \frac{x}{\sqrt{(xx+t)}}$ et $y = Cx \int \frac{z \partial t}{t\sqrt{t(1+t)}}$.

6.) Si capiatur $\lambda = -1$ erit

$m = 2$, $b = 2 \mp 1$, $n = -2$, $k = 0$, $\mu = 2$, $\eta = 2 \pm 1$, $\nu = -2$, $\theta = -2$,

hincque

$$z = (1 + \tfrac{t}{xx})^{-1 \pm \frac{1}{2}} \text{ et } y = \tfrac{C}{x} \int t^{\pm \frac{1}{2}} z \partial t (1+t)^{\mp \frac{1}{2}},$$

vnde binae constructiones fluunt

altera $z = \frac{x}{\sqrt{(xx+t)}}$ et $y = \frac{C}{x}\int \frac{z\,\partial t \sqrt{t}}{\sqrt{(1+t)}}$

altera $z = \frac{x^3}{(xx+t)^{\frac{3}{2}}}$ et $y = \frac{C}{x}\int \frac{z\,\partial t \sqrt{(1+t)}}{\sqrt{t}}$.

Ex solutione quarta denique concludimus

7.) Si $\lambda = +1$,

$m = 2,\ b = \pm 1,\ n = -2,\ k = 2,\ \mu = 2,\ \eta = \mp 1,$
$\nu = -2,\ \theta = 0.$

Posito nunc $s = \frac{t}{xx}$, quaeratur $z$ ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{-2A + z(2 \mp s)}{2s(1+s)},$$

vt posito $s = 0$ fiat $z = A$, tumque erit

$$y = Cx\int t^{\mp\frac{1}{2}-1} z\,\partial t(1+t)^{\pm\frac{1}{2}},$$

vnde duplex constructio

altera $\frac{\partial z}{\partial s} = \frac{-2A + z(2-s)}{2s(1+s)}$ et $y = Cx\int \frac{z\,\partial t\sqrt{(1+t)}}{t\sqrt{t}}$

altera $\frac{\partial z}{\partial s} = \frac{-2A + z(2+s)}{2s(1+s)}$ et $y = Cx\int \frac{z\,\partial t}{\sqrt{t(1+t)}}$.

8. Si $\lambda = -1$, habebitur

$m = 2,\ b = 2 \pm 1,\ n = -2,\ k = -2,\ \mu = 2,$
$\eta = 2 \mp 1,\ \nu = -2,\ \theta = 0,$

at posito $\frac{t}{zx} = s$, quaeri debet $z$ ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{+2A - z[2+(2\pm 1)s]}{2s(1+s)},$$

vt posito $s = 0$ fiat $z = A$, quo facto erit

$$y = \frac{C}{x}\int t^{\mp\frac{1}{2}} z\,\partial t(1+t)^{-1\pm\frac{1}{2}};$$

sicque duplex oritur constructio

altera

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = \frac{C}{x}\int \frac{z\,\partial t}{\sqrt{t(1+t)}}$

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = \frac{C}{x}\int \frac{z\,\partial t\,\sqrt{t}}{(1+t)\sqrt{(1+t)}}$.

Omnino ergo ſedecim conſtructiones ſumus conſecuti.

## Scholion.

1076. Periculum faciamus oſtendendi, quomodo hae conſtructiones, quae magis arduae videntur, aequationi propoſitae ſatisfaciant; atque in hunc finem eligamus conſtructionem poſteriorem No. 4. quae habet

$$\partial z + \frac{z\,\partial s(2+3s)}{2s(1+s)} = \frac{A\,\partial s}{s(1+s)},$$

haec per $s\sqrt{(1+s)}$ multiplicata praebet integrale

$$sz\sqrt{(1+s)} = A\int \frac{\partial s}{\sqrt{(1+s)}} = 2A\sqrt{(1+s)} + B, \text{ ſeu}$$

$$z = \frac{2A}{s} + \frac{B}{s\sqrt{(1+s)}}.$$

Iam vt poſito $s = 0$ fiat $z = A$, debet eſſe $B = -2A$, vt ſit

$$z = \frac{2A[\sqrt{(1+s)} - 1]}{s\sqrt{(1+s)}} = \frac{2A}{txx} - \frac{2A}{txx\sqrt{(1+txx)}}.$$

Hinc fit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{-4A}{tx^3} + \frac{2A(2+3txx)}{tx^3(1+txx)^{\frac{3}{2}}} \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{12A}{tx^4} - \frac{6A(2+5txx+4ttx^4)}{tx^4(1+txx)^{\frac{5}{2}}}.$$

Cum nunc ſit $y = C\int \frac{xxz\,\partial t}{\sqrt{t(1+t)}}$, erit

$$\left(\frac{\partial y}{\partial x}\right) = 2C\int \frac{xz\,\partial t}{\sqrt{t(1+t)}} + C\int \frac{xx\,\partial t}{\sqrt{t(1+t)}} \cdot \left(\frac{\partial z}{\partial x}\right) \text{ et}$$

$$\left(\frac{\partial\partial y}{\partial x^2}\right) = 2C\int \frac{z\,\partial t}{\sqrt{t(1+t)}} + 4C\int \frac{x\,\partial t}{\sqrt{t(1+t)}}\left(\frac{\partial z}{\partial x}\right) + C\int \frac{xx\,\partial t}{\sqrt{t(1+t)}}\left(\frac{\partial\partial z}{\partial x^2}\right),$$

hincque

$$x(1-xx)\frac{\partial\partial y}{\partial x^2}-x(1+xx)\frac{\partial y}{\partial x}+xxy=$$
$$C\int\frac{\partial t}{\sqrt{t(1+t)}}\cdot\left(\frac{2Axx}{t}-\frac{2Axx(1+4txx+3ttxx)}{t(1+txx)^{\frac{5}{2}}}\right),$$

quod integrale eſt

$$\frac{-4ACxx\sqrt{(1+t)}}{\sqrt{t}}+\frac{4ACxx\sqrt{(1+t)}}{(1+txx)^{\frac{3}{2}}\sqrt{t}}$$
$$=\frac{4ACxx\sqrt{(1+t)}}{\sqrt{t}}\left(\frac{1}{(1+txx)^{\frac{3}{2}}}-1\right),$$

et hac forma exprimi poteſt

$$-2Cx^4\left[3z+x\left(\tfrac{\partial z}{\partial x}\right)\right]\sqrt{t(1+t)}$$

vel etiam hoc modo

$$-2Cx^4\left(\tfrac{2A+z}{1+txx}\right)\sqrt{t(1+t)}.$$

Expreſſio autem iſta fit $=0$, primo ſi $t=-1$, deinde etiam ſi $t=0$, vnde valor pro $y$ inuentus

$$y=D\int\tfrac{\partial t}{t\sqrt{t(1+t)}}\left(1-\tfrac{1}{\sqrt{(1+txx)}}\right)$$

ita per integrationem definiri debet, vt euaneſcat poſito $t=0$; tum vero ponatur $t=-1$. Vel poſito $t=-v$ erit

$$y=D\int\tfrac{\partial v}{v\sqrt{v(1-v)}}\left(1-\tfrac{1}{\sqrt{(1-vxx)}}\right)$$

integrali ita ſumto vt euaneſcat poſito $v=0$, tum vero facto $v=1$.

Exemplum hoc ſufficit ad oſtendendum, quomodo conſtructiones exhibitae aequationi differentio-differentiali ſatisfaciant; interim vero ſi quantitas $z$ tranſcendenter, per logarithmos ſcilicet exprimitur, conſenſum nonniſi per calculos nimium taediosos declarare licet.

Pro-

# Problema 136.

1077. Posito $y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\partial t$, in qua integratione quantitas $x$ vt constans spectatur, integrali per terminos deinceps inuestigandos definito, vt $y$ aequetur certae functioni ipsius $x$, inuenire aequationes differentio-differentiales formae

$$Lxx.\frac{\partial\partial y}{\partial x^2} + Mx.\frac{\partial y}{\partial x} + Ny = 0,$$

cui ea functio satisfaciat.

## Solutio.

Cum sit ex principiis ante stabilitis

$$\frac{\partial y}{\partial x} = C\int \lambda t(1+t)^{\nu-1}(a+tx)^{\lambda-1}\partial t \text{ et}$$

$$\frac{\partial\partial y}{\partial x^2} = C\int \lambda(\lambda-1)tt(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t, \text{ erit}$$

$$Lxx.\frac{\partial\partial y}{\partial x^2}+Mx.\frac{\partial y}{\partial x}+Ny = C\int(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t \times$$
$$[\lambda(\lambda-1)Lttxx+\lambda Mtx(a+tx)+N(a+tx)^2] =$$
$$C\int(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t\begin{Bmatrix} Naa+2Natx+Nttxx \\ +\lambda Matx+\lambda Mttxx \\ +\lambda(\lambda-1)Lttxx \end{Bmatrix}$$

quae formula sumta $x$ constante absolute integrabilis esse debet. Ponatur ergo integrale

$$C(1+t)^{\nu}(a+tx)^{\lambda-1}(Paa+Qatx)$$

denotantibus P et Q functionibus quibuscunque ipsius $x$, erit eius differentiale

$$C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t \times$$
$$[\nu(a+tx)(Paa+Qatx)+(\lambda-1)x(1+t)(Paa+Qatx)+Qax(1+t)(a+tx)]$$

$$=C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t\begin{Bmatrix} +\nu Pa^3 & +\nu Qaatx & +\nu Qattxx \\ +(\lambda-1)Paax+\nu Paatx & & +(\lambda-1)Qattxx \\ +Qaax & +(\lambda-1)Paatx & +Qattxx \\ & +(\lambda-1)Qatxx & \\ & +Qaatx & \\ & +Qatxx & \end{Bmatrix}$$



qua

qua forma cum illa comparata, adipiſcimur

$$N = \nu P a + (\lambda - 1) P x + Q x$$

$$2 N + \lambda M = (\nu + 1) Q a + (\lambda + \nu - 1) P a + \lambda Q x$$

$$N + \lambda M + \lambda (\lambda - 1) L = (\lambda + \nu) Q a,$$

quarum aequationum extremae demta media praebent

$$\lambda(\lambda - 1) L = -(\lambda - 1) P a + (\lambda - 1) P x + (\lambda - 1) Q a - (\lambda - 1) Q x;$$

hincque

$$\lambda L = (a - x)(Q - P) \text{ ſeu } L = \tfrac{1}{\lambda}(a - x)(Q - P),$$

ſecunda autem demto primae duplo, dat

$$\lambda M = (\lambda - \nu - 1) P a - 2(\lambda - 1) P x + (\nu + 1) Q a + (\lambda - 2) Q x, \text{ ſeu}$$

$$\lambda M = [(\nu + 1) a + (\lambda - 2) x](Q - P) + \lambda (a - x) P.$$

Quare ſumtis pro P et Q functionibus quibuscunque ipſius $x$, ſi functiones L, M, N ita definiantur, vt ſit

$$L = \tfrac{1}{\lambda}(a - x)(Q - P)$$

$$M = \tfrac{1}{\lambda}[(\nu + 1) a + (\lambda - 2) x](Q - P) + (a - x) P$$

$$N = x(Q - P) + (\nu a + \lambda x) P,$$

aequationi differentio - differentiali

$$L x x \partial \partial y + M x \partial x \partial y + N y \partial x^2 = 0$$

ſatisfaciet formula integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t,$$

tractata $x$ vt conſtante, dummodo integrationis termini ita conſtituantur, vt vtroque haec expreſſio

$$(1 + t)^{\nu} (a + t x)^{\lambda - 1} (P a + Q t x) \text{ euaneſcat.}$$

Notari autem oportet, hos terminos non ab $x$ pendere debere. Primo autem patet hanc expreſſionem fieri $= 0$ caſu $t = -1$, ſi modo ſit $\nu > 0$. Deinde poſito $t = \infty$ etiam euaneſcet, ſi modo ſit $\nu + \lambda - 1 + 1$ numerus negatiuus, ſeu $\nu + \lambda < 0$. Quocirca ſi ſit $\nu > 0$ et $\nu + \lambda < 0$, integrale

$y =$

$$y = C\int (1+t)^{\nu-1} (a+tx)^{\lambda} \partial t$$

ita capi debet, vt posito $t = -1$ euanescat, tum vero statuatur $t = \infty$, functioque ipsius $x$ pro $y$ resultans satisfaciet aequationi propositae.

## Corollarium 1.

1078. Quoniam functiones P et Q in formulam integralem pro $y$ assumtam non ingrediuntur, manifestum est eandem formulam satisfacere omnibus aequationibus differentio-differentialibus, quicunque valores litteris P et Q tribuantur.

## Corollarium 2.

1079. Sumto ergo $Q = P$ eadem formula integralis

$$y = C\int (1+t)^{\nu-1} (a+t)^{\lambda} \partial t$$

satisfacit etiam huic aequationi differentiali primi gradus

$$(a-x)\,x\,\partial y + (\nu a + \lambda x)\,y\,\partial x = 0.$$

Huius vero integrale est

$$y = \frac{D\,(a-x)^{\lambda+\nu}}{x^{\nu}},$$

qui ergo valor quoque in genere nostrae aequationi differentio-differentiali satisfacit, id quod tentanti mox patebit.

## Corollarium 3.

1080. Hic ergo valor integralis

$$y = C\int (1+t)^{\nu-1} (a+tx)^{\lambda} \partial t$$

secundum terminos definitos sumtus, congruere debet cum formula algebraica $y = \frac{D\,(a-x)^{\lambda+\nu}}{x^{\nu}}$, si modo sit $\nu > 0$ et $\lambda + \nu < 0$.

Scho-

## Scholion.

1081. Parum ergo integratio hoc problemate exhibita habet in recessu. Verum reductio formulae integralis

$$y = C\int(1+t)^{\nu-1}(a+tx)^{\lambda}\partial t \text{ ad } y = \frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$$

eo magis est notatu digna, ad quam illa reducitur, si integrali ita sumto vt euanescat posito $t = -1$, ponatur $t = \infty$. Posito ergo $\lambda + \nu = -\mu$, vt $\mu$ et $\nu$ sint numeri positiui, erit

$$C\int\frac{(1+t)^{\nu-1}\partial t}{(a+tx)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}}.$$

Vel ponatur $1 + t = z$, erit

$$C\int\frac{z^{\nu-1}\partial z}{(a-x+xz)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}},$$

terminis illius integrationis existentibus $z = 0$ et $z = \infty$. Verum etiam haec obseruatio non magni est momenti, nam posito $a - x = ux$, fit

$$\frac{C}{x^{\mu+\nu}}\int\frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}} = \frac{D}{x^{\mu+\nu}u^{\mu}},$$

ideoque haec formula $\int\frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}}$ ita integrata vt euanescat posito $z = 0$, si tum ponatur $z = \infty$, hanc induet formam $\frac{A}{u^{\mu}}$, in qua A quantitatem constantem denotat ab $u$ non pendentem. Pendet autem ab exponentibus $\mu$ et $\nu$, lege ex casibus facile obseruanda. Scilicet posito

$$\int\frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}} = \frac{A}{u^{\mu}},$$

si

si sit $\nu = 1$, integrale illud praebet $-\frac{1}{\mu(u+z)^{\mu}} + \frac{1}{\mu u^{\mu}}$, et posito $z = \infty$, prodit $\frac{1}{\mu u^{\mu}}$, ita vt hoc casu sit $A = \frac{1}{\mu}$. Si sit $\nu = 2$, integratio quoque succedit, reperiturque $A = \frac{1}{\mu(\mu+1)}$ si $\nu = 3$ fit $A = \frac{1 \cdot 2}{\mu(\mu+1)(\mu+2)}$, et si $\nu = 4$ fit

$$A = \frac{1 \cdot 2 \cdot 3}{\mu(\mu+1)(\mu+2)(\mu+3)},$$

vnde in genere concludimus fore

$$A = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \ldots \ldots (\nu-1)}{\mu(\mu+1)(\mu+2) \ldots \ldots (\mu+\nu-1)}.$$

Quare integratione secundum regulam praescriptam instituta, erit

$$\frac{1}{\mu+1} \cdot \frac{2}{\mu+2} \cdot \frac{3}{\mu+3} \cdot \ldots \cdot \frac{\nu-1}{\mu+\nu-1} = \mu u^{\mu} \int \frac{z^{\nu-1} \partial z}{(u+z)^{\mu+\nu}}.$$

Quod si exponens $\nu$ non fuerit integer, valor ipsius A ope interpolationis huius formulae per factores procedentis definietur. Quadratura scilicet circuli ingredietur si exponens $\nu$ fractionem $\frac{1}{2}$ inuoluat, de huiusmodi autem interpolationibus alibi fusius egimus, neque hic locus est hoc argumentum vberius prosequendi. Restat vltimum huius sectionis caput, quo aequationum differentio-differentialium integratio per approximationes docebitur.

# CAPVT XII.

## DE
## AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM INTEGRATIONE PER APPROXIMATIONES.

### Problema 137.

1082.

Proposita aequatione differentio-differentiali quacunque, principia explicare, ex quibus integrationem per approximationes peti oportet.

### Solutio.

Versetur aequatio proposita inter binas variabiles $x$ et $y$, ac posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, dabitur aequatio inter quatuor quantitates $x$, $y$, $p$ et $q$, ex qua $q$ ita definire licebit, vt $q$ aequetur functioni cuidam trium quantitatum $x$, $y$ et $p$, qua vocata $= V$, sit $q = V$, seu $\partial p = V\,\partial x$. Hic primo obseruandum est, integrationem, vt sit determinata, duplicem determinationem requirere, seu duas conditiones quasi pro lubitu praescribi posse, quibus satisfaciat. Scilicet non sufficit, vt posito $x = a$ fiat $y = b$, quemadmodum in aequationibus differentialibus primi gradus vsu venire vidimus, sed aliam insuper conditionem adiicere licet, quae sit, vt posito $x = a$ fiat etiam $p = \frac{\partial y}{\partial x} = c$ quantitati datae. His ergo determinationibus constitutis, vt posito $x = a$, fiat $y = b$ et $p = c$, totum integrationis negotium huc reducitur, vt ipsi $x$ alium quemcunque valorem tribuendo, inuestigentur valores respondentes ipsius

ipſius $y$ et ipſius $p$; hoc enim ſi praeſtiterimus, aequationis propoſitae integrale perfecte definiuerimus, vt nihil praeterea deſiderari poſſit. Quod cum in genere fieri nequeat, approximationis ratio in hoc conſiſtit, vt ipſi $x$ valor quam minimum ab $a$ diſcrepans tribuatur, qui ſit $x = a + \omega$, et inquiratur, quantum valores quantitatum $y$ et $p$ a primitiuis $b$ et $c$ ſint diſcrepaturi. Hic pro principio aſſumimus, dum $x$ ab $a$ ad $a + \omega$ increſcit, etiam quantitatum $y$ et $p$ valores tam parum mutatum iri, vt inde functio V nullam variationem notabilem patiatur. Quare ſi ponamus, ſtatuendo $x = a$, $y = b$ et $p = c$, fieri $V = F$, eundem valorem F quantitas V retinere cenſebitur, dum $x$ ab $a$ vsque ad $a + \omega$ augetur. Cum igitur pro hoc interuallo minimo habeamus $\partial p = F \partial x$, erit integrando $p = F x + \text{Conſt.}$ Verum, quia poſito $x = a$, fieri debet $p = c$, erit $p = c + F x - F a$. Sit nunc $x = a + \omega$, atque habebimus $p = c + F \omega$, qui eſt valor ipſius $p$, valori $x = a + \omega$ reſpondens. Denique pro hoc minimo interuallo erit $\partial y = c \partial x$, ideoque $y = b + c x - a c$, et pro valore $x = a + \omega$ fit $y = b + c \omega$, qui eſt valor ipſius $y$ valori $x = a + \omega$ conueniens. Quocirca ſi valores primitiui ſint $x = a$, $y = b$ et $p = \frac{\partial y}{\partial x} = c$, ex iisque fiat $V = F$, ſequentes valores interuallo quam minimo ab illis remoti erunt

$$x = a + \omega,\ y = b + c\omega,\ p = c + F\omega,$$

qui ſi porro vt primitiui ſpectentur, ex iis ſimili modo per interuallum quam minimum progredi licet, ſicque tandem progreſſus per interuallum quantumuis magnum innoteſcet.

## Corollarium 1.

1083. Quo minora capiantur haec interualla, eo minus a vero aberrabitur, dummodo quantitates $c$ et F non ſint nimis magnae, ſin autem eae adeo in infinitum excreſcant, manifeftum eft, errorem in quantitatibus $y$ et $p$ infignem committum iri.

## Corollarium 2.

1084. Si quantitas $c$ vel F fiat vehementer magna, interuallum, quo $y$ vel $p$ crefcit, pro dato accipi poteft; ita pofito $c\omega = \psi$ erunt fequentes valores

$$x = a + \frac{\psi}{c},\ y = b + \psi \text{ et } p = c + \frac{F\psi}{c}.$$

At fi F prodeat quantitas permagna, valor ipfius $p$ interuallo minimo $\Phi$ augeri fumatur, vt fit $F\omega = \Phi$, eruntque valores fequentes $x = a + \frac{\Phi}{F}$, $y = b + \frac{c\Phi}{F}$ et $p = c + \Phi$.

## Corollarium 3.

1085. Si $b$ fit quantitas infinita, pro valore proximo ipfius $y$ expediet definiri

$$\frac{1}{y} = \frac{1}{b + c\omega} = \frac{1}{b} - \frac{c\omega}{bb} = -\frac{c\omega}{bb},$$

quae expreffio vt fit finita, etiam quantitas $c$ infinita fit oportet; alioquin valor ipfius $y$ refpondens non folum ipfi $x = a$ fed etiam ipfi $x = a + \omega$ maneret infinitus.

## Scholion 1.

1086. Quoties folutio alicuius problematis pendet ab integratione cuiuspiam aequationis differentialis fecundi gradus, toties conditiones problematis binas determinationes fuppeditare folent; quarum altera, dum ipfi $x$ certus quidam valor $x = a$ tribuitur, exigit vt $y$ datum valorem $y = b$, altera vero vt etiam ratio $\frac{dy}{dx} = p$ datum valorem $p = c$ confequatur. Quodfi ergo in genere aequationem quandam differentiodifferentialem integrare velimus, integrationem ita inftituere licet, vt pofito $x = a$, fiat $y = b$ et $p = c$, quantitatibus $a$,

$b$, $c$ ab arbitrio noſtro pendentibus. Interim tamen quandoque vſu venire poteſt, vt poſito $x = a$, valores ipſarum $y$ et $p$ non penitus ab arbitrio noſtro pendeant, ſed ex natura aequationis iam datos valores ſortiantur, quibus caſibus defectus determinationis aliis conditionibus compenſatur. Veluti ſi proponatur haec aequatio

$$x x (a - b x) \partial \partial y - 2 x (2 a - b x) \partial x \partial y$$
$$+ 2 (3 a - b x) y \partial x^2 = 6 a a \partial x^2,$$

quomodocunque ea per integrationem determinetur, poſito $x = 0$ neceſſario fit $y = a$ et $\frac{\partial y}{\partial x} = p = b$; ita vt pro caſu $x = 0$ valores quantitatum $y$ et $p$ minime arbitrio noſtro relinquantur. Integrale autem completum reperitur

$$y = a + b x + \frac{(A + B x) x x}{a - b x},$$

vbi etiamſi conſtantes A et B pro lubitu aſſumantur, tamen ſemper poſito $x = 0$ prodit $y = a$ et $p = b$. Huiusmodi ergo caſibus mirum non eſt, ſi pro dato ipſius $x$ valore, quantitatum $y$ et $p$ valores arbitrio noſtro haud permittantur.

## Scholion 2.

1087. Expoſita ratio aequationes differentio-differentiales per approximationes integrandi, dum per interualla minima progredimur, quemadmodum etiam in aequationibus differentialibus primi gradus fecimus, certis caſibus difficultatibus inuoluitur, vt niſi remedium afferatur, in vſum vocari nequeat. Primum hoc euenit, quando $c = \infty$, tum enim quantumvis exiguum accipiatur interuallum $\omega$, neque ipſius $y$ neque ipſius $p$ valorem cognoſcere licet. Simile quoque incommodum turbat, ſi poſitis $x = a$, $y = b$ et $p = c$, functio V fiat infinita, ideoque prodeat $F = \infty$, quo caſu valor ipſius $p$ non

 defi-

definitur. Deinde etiam casus quibus vel $c$ vel F euanescit; seorsim tractari conuenit: etsi enim tum valores ipsarum $y$ et $p$ satis accurate ostenduntur, tamen quia nullam mutationem patiuntur, dum mutatio altiore ipsius $\omega$ potestate exprimitur, ipsam mutationem inuestigari vtile est, quo in progressu minus a veritate aberretur. Sin autem quantitas $b$ euadat infinita, iam ante aduertimus, loco ipsius $y$ eius reciprocum $\frac{1}{y}$ explorari deberi. Quemadmodum ergo difficultatibus ante memoratis sit occurrendum, diligentius perpendamus.

## Problema 138.

1088. Si integrationem per interualla instituendo, pro initio cuiuspiam interualli, posito $x = a$, $y = b$ et $p = c$ eueniat, vt quantitas $c$ sit vel euanescens vel infinita, integrationem per hoc interuallum absoluere.

## Solutio.

Praecedens approximatio dederat $y = b + c(x - a)$, vnde si $c = 0$, incrementum ipsius $y$ altiore potestate ipsius $x - a$ exprimetur, scilicet $y = b + A(x - a)^{\lambda}$, existente $\lambda > 1$, reiectisque altioribus potestatibus, quae prae hac ob interuallum $x - a$ minimum recte contemnuntur. Sin autem sit $c = \infty$, valor ipsius $y$ simili modo repraesentari potest, $y = b + A(x - a)^{\lambda}$, existente $\lambda < 1$, ita tamen vt nihilo sit maior; vtroque ergo casu eadem inuestigatio est, vt ex aequatione proposita $\partial p = V \partial x$ tam coefficiens A quam exponens $\lambda$ definiatur. Iam ex illa aequatione deducimus

$$\frac{\partial y}{\partial x} = p = \lambda A (x - a)^{\lambda - 1} \text{ et}$$
$$\partial p = \lambda(\lambda - 1) A (x - a)^{\lambda - 2} \partial x,$$

ac necesse est, eandem expressionem resultare, si in formula

$V \partial x$

$V \partial x$ ftatuatur

$$y = b + A(x-a)^{\lambda} \text{ et } p = \lambda A (x-a)^{\lambda-1},$$

vnde euidens eft, fore

$$c = 0 \text{ fi } \lambda > 1, \text{ et } c = \infty \text{ fi } \lambda < 1.$$

Cum iam V fit functio ipfarum $x$, $y$ et $p$, ponatur vbique $x = a$, nifi quatenus formula $x - a$ ineft, quae relinquatur, tum vero $y = b$, nifi hinc prodeat $V = 0$ vel $V = \infty$; hoc enim fi eueniat, pro $y$ valor $b + A(x-a)^{\lambda}$ fcribatur, fimilique modo pro $p$ fcribatur $\lambda A (x-a)^{\lambda-1}$. Reiiciantur autem formulae $x - a$ poteftates altiores prae inferioribus, ficque pro V orietur expreffio huius formae $C(x-a)^{\mu}$, quae formulae $\lambda(\lambda-1) A (x-a)^{\lambda-2}$ aequari debet, vnde tam coëfficiens affumtus A quam exponens $\lambda$ definietur, ideoque vero proximi valores

$$y = b + A(x-a)^{\lambda} \text{ et } p = \lambda A (x-a)^{\lambda-1}$$

innotefcent, qui eo minus a veritate recedent, quo minor differentia inter $a$ et $x$ conftituitur. Cafus autem quo $\lambda = 1$ per fe eft perfpicuus, atque in praecedente problemate pertractatus, cum is fit folus quo quantitas $c$ finitum nancifcitur valorem.

## Corollarium 1.

1089. Si eueniat vt pofito $c = \infty$, quo cafu effe debebat $\lambda < 1$, functio V finitum obtineat valorem, cui formula $\lambda(\lambda-1) A (x-a)^{\lambda-2}$ aequari nequit, cafus per fe nihil habet difficultatis, et valor ipfius $y$ etiam pro minimo exceffu ipfius $x$ fuper $a$ reuera infinitus euadet.

Corol-

## Corollarium 2.

1090. Facilius hoc perſpicere licet ex exemplo $\partial p = 6 x \partial x$, vnde fit

$$p = c + 3xx - 3aa = \frac{\partial y}{\partial x}, \text{ hincque}$$
$$y = b + (c - 3aa)(x - a) + x^3 - a^3, \text{ ſeu}$$
$$y = b - ac + 2a^3 + (c - 3aa)x + x^3.$$

Si ergo conſtans $c$ ſumatur infinita, valor ipſius $y$ ſemper erit infinitus ſolo excepto caſu $x = a$.

## Corollarium 3.

1091. Sin autem ſumto $c = o$, quo caſu eſſe debet $\lambda > 1$, functio V finitum habeat valorem eumque adeo conſtantem, poſito $x = a$ et $y = b$, ei formula $\lambda(\lambda - 1)A(x - a)^{\lambda - 2}$ aequabitur ſumendo $\lambda = 2$, et $2A =$ illi valori conſtanti. Velut in praecedente exemplo fit $V = 6a = 2A$, hinc $A = 3a$, eritque proxime $y = b + 3a(x - a)^2$, quod etiam congruit cum integrali inuento, quod poſito $c = o$ eſt

$$y = b + 2a^3 - 3aax + x^3 = b + (x + 2a)(x - a)^2,$$

quae expreſſio facto $x = a$ in illam abit.

## Scholion 1.

1092. Poſito $c = o$ functio V, ſi in ea ſcribatur $x = a$, $y = b$ et $p = c = o$, valorem nanciſcetur vel infinite magnum, vel finitum, vel adeo euaneſcentem. Primo caſu, quo fit $V = \infty$, vt ei aequari poſſit $\lambda(\lambda - 1)A(x - a)^{\lambda - 2}$, ſumto $x = a$, neceſſe eſt ſit $\lambda < 2$, exiſtente $\lambda > 1$; vt autem hinc quantitates A et $\lambda$ definiantur, in functione V ſcribi oportet

$$y = b + A(x - a)^{\lambda} \text{ et } p = \lambda A(x - a)^{\lambda - 1},$$

itemque $x = a$, niſi vbi formula $x - a$ occurrit; hoc modo quia

quia per hypothesin casu $x = a$ fit $V = \infty$, iste valor prodibit $= C(x-a)^{-\alpha}$, quo collato cum $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ reperientur A et $\lambda$, dum ne prodeat $\lambda < 1$, qui casus cum $c = 0$ subsistere nequit. Secundo casu quo prodit $V =$ quantitati finitae, capi oportet $\lambda = 2$, sin autem tertio casu sit $V = 0$, sumi debet $\lambda > 2$, vt eius valor in formula

$$\lambda(\lambda-1)A(x-a)^{\lambda-2}$$

contineatur. At si debeat esse $c = \infty$, fieri nequit vt vidimus, vt functio V finitum obtineat valorem, multo minus euanescentem, nisi quidem casus incongruos, quibus $y$ perpetuo maneat infinita, admittere velimus. Tum igitur functio V necessario valorem infinitum induit cum formula

$$\lambda(\lambda-1)A(x-a)^{\lambda-2}$$

comparandum, ita vt sit $\lambda < 1$. Hinc igitur patet, determinationem quantitatis $c$ non semper arbitrio nostro relinqui, sed quandoque ex indole ipsius aequationis nobis praescribi. Veluti si proponatur haec aequatio $\partial\partial y = \frac{2\partial x^2}{(x-a)^3}$, erit

$$\frac{\partial y}{\partial x} = p = A - \frac{1}{(x-a)^2} \text{ et } y = B + Ax + \frac{1}{x-a},$$

vnde posito

$$x = a \text{ fit } c = A - \frac{1}{0^2} \text{ et } b = B + Aa + \frac{1}{0}, \text{ ergo}$$

$$A = c + \frac{1}{0^2}, \text{ et } B = b - ac - \frac{a}{0^2} + \frac{1}{0},$$

quare ne aequatio integralis omnino in infinitis versetur, litterae $b$ et $c$ non possunt non esse infinitae.

## Scholion 2.

1093. Non autem omnes ordines infinitorum et euanescentium in formula $(x-a)^\lambda$, casu $x = a$, contineri, iam obseruauimus; expressio scilicet $xx\,l\,x$, casu $x = 0$, infinities superat potestatem secundam $x^2$, interim tamen infinities minor eſt poteſtate $x^{1-\alpha}$, quantumuis etiam exigua fractio $\alpha$ accipiatur. Quare ſi in ſuperiori ſolutione formulas ita inſtruere velimus, vt ad omnes ordines tam infinitorum quam euaneſcentium pateant, ſtatui conueniet

$$y = b + A(x-a)^{\lambda}[l(x-a)]^{\mu},$$

vnde fit

$$\frac{\partial y}{\partial x} = p = \lambda A(x-a)^{\lambda-1}[l(x-a)]^{\mu} + \mu A(x-a)^{\lambda-1}[l(x-a)]^{\mu-1};$$

at poſito $x = a$, pars prior eſt ad poſteriorem vt $l(x-a)$ ad 1, hoc eſt vt $\infty : 1$, ex quo ſufficit ſumſiſſe

$$p = \lambda A(x-a)^{\lambda-1}[l(x-a)]^{\mu},$$

ex quo ſimili modo colligitur

$$\frac{\partial p}{\partial x} = \lambda(\lambda-1)A(x-a)^{\lambda-2}[l(x-a)]^{\mu},$$

quae expreſſio cum functione V, poſtquam in ea ſcripſerimus $x = a$, $y = b$ et $p = c$, ſeu potius

$$y = b + A(x-a)^{\lambda}[l(x-a)]^{\mu} \text{ et } p = \lambda A(x-a)^{\lambda-1}[l(x-a)]^{\mu},$$

comparari debet, vt inde tam conſtans A quam exponentes $\lambda$ et $\mu$ innoteſcant. Haec ergo tenenda ſunt in iſta inueſtigatione, quo ea ad plures caſus extendatur.

## Problema 139.

1094. Approximationem ante expoſitam accuratius perſequi, vt ſumtis interuallis etiam paulo maioribus minus a vero aberretur.

## Solutio.

Poſito $\partial y = p\,\partial x$, aequatio differentio-differentialis hac forma $\frac{\partial p}{\partial x} = V$ exhibeatur, ex qua ante $p$ ita definiuimus, quaſi V eſſet quantitas conſtans pro interuallo ſaltem vehementer paruo, vnde obtinuimus $p = c + V(x-a)$, poſtquam ſcilicet in V in V posuerimus $x = a$, $y = b$ et $p = c$, qui sunt valores primitiui per interuallum $x - a = \omega$ retinendi. Cum autem functio V interea non sit constans, quia $x$, $y$ et $p$ inuoluit, reuera erit

$$p = c + V(x - a) - \int (x - a) \partial V.$$

Ponamus igitur

$$\partial V = P \partial x + Q \partial y + R \partial p,$$

vt sit

$$\partial V = (P + Qp + RV) \partial x,$$

et nunc quantitatem $P + Qp + RV$ vt constantem spectemus, cuius valor prodeat ponendo $x = a$, $y = b$ et $p = c$, quo facto V in F abire supra sumsimus, eritque

$$p = c + F(x - a) - \tfrac{1}{2}(P + Qc + RF)(x - a)^2.$$

Hinc porro ob $\partial y = p \partial x$, fit

$$y = b + c(x - a) + \tfrac{1}{2}F(x - a)^2 - \tfrac{1}{6}(P + Qc + RF)(x - a)^3,$$

similique modo approximationem vlterius prosequi licet. Quando autem quantitates P, Q, R et V formulam $x - a$ eiusue potestates complectuntur, quam non amplius vt constantem spectare licet, eius ratio in integratione est habenda, qua fit vt in seriebus approximantibus formulae $x - a$ potestates non ordine ascendant. Tum igitur conueniet pro $p$ eiusmodi seriei initium assumi

$p = c + A(x - a)^\lambda$, vnde fit

$y = b + c(x - a) + \frac{A}{\lambda + 1}(x - a)^{\lambda + 1}$, et quia

$\frac{\partial p}{\partial x} = \lambda A (x - a)^{\lambda - 1}$,

huic formulae aequari debet functio V, postquam in ea pro $y$ et $p$ valores assumtos, et $a$ pro $x$ scripserimus, nisi formula $x - a$ ingrediatur, hoc modo tam exponens $\lambda$ quam coefficiens A determinabitur.

Si $c$ fit $= 0$ vel $= \infty$, eius ratio poteſt in calculum introduci, vt ponatur

$$p = f(x-a)^n + A(x-a)^\lambda,$$

vnde fit

$$y = b + \frac{f}{n+1}(x-a)^{n+1} + \frac{A}{\lambda+1}(x-a)^{\lambda+1},$$

qui valores ſi loco $x$ et $p$ ſubſtituantur in functione V, prodire debet

$$nf(x-a)^{n-1} + \lambda A(x-a)^{\lambda-1}.$$

## Corollarium 1.

1095. Hoc modo per interualla continuo vlterius progredi licet, dummodo ſingula non maiora accipiantur, quam vt errores commiſſi maneant inſenſibiles; atque hac quidem correctione errores illi diminuuntur, vt interualla etiam maiora ſtatui queant.

## Corollarium 2.

1096. Pro primo ſcilicet interuallo valores primitiui $x = a$, $y = b$ et $p = c$ pro lubitu aſſumuntur, et valores in fine interualli inuenti praebent valores initiales pro ſecundo interuallo, ex quibus calculus pro hoc interuallo perinde expeditur ac primo; ſicque continuo vlterius eſt progrediendum.

## Scholion.

1097. Huius problematis duplicem ſolutionem dedimus, quarum prior etſi latiſſime patere videtur, certis tamen caſibus in vſum vocari nequit; iis ergo altera ſolutione vti conueniet. Exiſtunt autem tantum plerumque pauciſſima eiusmodi interualla, quae poſteriorem methodum poſtulant, dum reliqua omnia ope prioris expedire licet. Euenit hoc, quando pro quopiam interuallo quantitates V et $c$ vel euaneſcunt vel in in infinitum excrefcunt; quin etiam fieri poteft, vt quantumvis exiguum interuallum accipiatur, quantitates $y$ et $p$ variationibus infinitis fint obnoxiae, quarum repraefentatio determinationem prorfus fingularem requirit. Veluti fi propoatur haec aequatio $\partial\partial y + \frac{y\partial x^2}{xx} = 0$, interuallum ab $x = 0$ vsque ad $x = \omega$, etiamfi $\omega$ quam minimum affumatur, infinitam mutationem in valoribus $y$ et $p$ indicat; id quod ex eius integrali completo perfpicitur, quod cum fit

$$y = A x^{\frac{1}{2}} \text{fin.} \left(\tfrac{\sqrt{3}}{2} l x + \alpha\right),$$

hincque

$$p = \tfrac{A}{2\sqrt{x}} \text{fin.} \left(\tfrac{\sqrt{3}}{2} l x + \alpha\right) + \tfrac{A\sqrt{3}}{2\sqrt{x}} \text{cof.} \left(\tfrac{\sqrt{3}}{2} l x + \alpha\right),$$

feu

$$p = \tfrac{A}{\sqrt{x}} \text{fin.} \left(\tfrac{\sqrt{3}}{2} l x + \alpha + 60^\circ\right),$$

euidens eft fi $x = 0$, fore quidem $y = 0$, fed ipfius $p$ valorem effe incertum. At ipfi $x$ valorem quam minimum tribuendo, $y$ quidem minimum retinebit valorem, fed qui pro minimo interuallo modo fit pofitiuus, modo euanefcens, modo negatiuus, ob maximam mutationem, quam $lx$ patitur, quantitas autem $p$ interea tranfit per omnes mutationes poffibiles. Idem luculentius perfpicitur ex hoc exemplo

$$\partial\partial y + \frac{2\partial x \partial y}{x} - \frac{ffy\partial x^2}{x^4} = 0,$$

cuius integrale eft $y = A\,\text{fin.} \left(\frac{f}{x} + \alpha\right)$; dum enim $x$ a $0$ ad $\omega$ crefcit, angulus $\frac{f}{x} + \alpha$ ab infinito ad finitum tranfibit, eiusque finus interea omnes mutationes ab $+1$ ad $-1$ infinities adeo fubibit. Quando ergo eiusmodi interualla occurrunt, mirum non eft, fi confuetae methodi approximandi deficiant, quippe quae hoc principio innituntur, quod mutationes per interualla minima, fint etiam valde paruae; his autem interuallis exceptis folutio praefcripta femper cum vfu adhiberi poteft.

## Exemplum 1.

1098. *Propofita aequatione*

$$\partial \partial y + \frac{y \partial x^2}{f x} = 0,$$

*eius integrationem per approximationem abfoluere.*

Cum ergo fit $\partial p = -\frac{y \partial x}{f x}$, erit $V = \frac{-y}{f x}$; quare fi pro initio interualli fit $x = a$, $y = b$ et $p = c$, inde tantillum progrediendo, per folutionem priorem, ob

$$P = \frac{y}{f x x} = \frac{b}{a a f},$$

$$Q = \frac{-1}{f x} = \frac{-1}{a f}, \text{ et } R = 0,$$

habebimus

$$p = c - \frac{b}{a f}(x - a) - \frac{1}{2}\left(\frac{b}{a a f} - \frac{c}{a f}\right)(x - a)^2 \text{ et}$$

$$y = b + c(x - a) - \frac{b}{2 a f}(x - a)^2.$$

Sumto ergo $x - a = \omega$, pro interuallo fequente erunt valores initiales

$$a' = a + \omega,$$

$$b' = b + c\,\omega - \frac{b\,\omega^2}{2 a f}, \text{ et } c' = c - \frac{b\,\omega}{a f} - \frac{(b - a c)\,\omega\,\omega}{2 a a f},$$

vnde fimili modo valores initiales pro interuallo fequente colliguntur. Verum fi pro quopiam interuallo fiat $a = 0$, operatio peculiari modo inftitui debet. Pofito fcilicet pro initio huius interualli $x = 0$, $y = b$ et $p = c$, ftatuatur

$$p = c + A x^\lambda \text{ et } y = b + c x + \frac{A x^{\lambda + 1}}{\lambda + 1}, \text{ erit}$$

$$\frac{\partial p}{\partial x} = \lambda A x^{\lambda - 1} = \frac{-y}{f x} = \frac{-b}{f x} - \frac{c}{f} - \frac{A x^\lambda}{(\lambda + 1) f},$$

cui nifi fit $b = 0$ fatisfieri nequit; prodiret enim $\lambda = 0$ et $A = \infty$, vnde concludimus poni debere $y = b + A x l x$, vt fit

fit $p = A\,l\,x + A$ et $\frac{\partial p}{\partial x} = \frac{A}{x} = \frac{-b - A\,x\,l\,x}{f\,x}$, hinc $A = -\frac{b}{f}$. Verum quo hinc $p$ accuratius cognoſcere liceat, ſtatuamus

$$y = b + A\,x\,l\,x + B\,x, \text{ erit}$$
$$p = A\,l\,x + A + B \text{ et } \frac{\partial p}{\partial x} = \frac{A}{x},$$

vnde concluditur vt ante $A = \frac{-b}{f}$, et B manet indeterminatum, ita vt ſit

$$y = b - \frac{b\,x}{f}\,l\,x + B\,x \text{ et } p = \frac{-b}{f}\,l\,x - \frac{b}{f} + B,$$

niſi ergo ſit $b = 0$, caſu $x = 0$, quantitas $c$ neceſſario eſt infinita. Quamobrem ſi interualli initio ſit $x = 0$, $y = b$ et $p = \infty$, pro eius fine et initio ſequentis erit

$$x = \omega,\; y = b - \frac{b\,\omega}{f}\,l\,\omega \text{ et } p = \frac{-b}{f}\,l\,\omega.$$

## Exemplum 2.

1099. *Propoſita ſit haec aequatio*

$$x\,x\,\partial\partial y - 2\,x\,\partial x\,\partial y + 2\,y\,\partial x^2 = \frac{x^2\,y\,\partial x^2}{f\,f},$$

*quam per approximationem integrari oporteat.*

Cum ſit

$$\frac{\partial p}{\partial x} = \frac{2\,p}{x} - \frac{2\,y}{x\,x} + \frac{y}{f\,f} = V, \text{ erit}$$
$$P = \frac{-2\,p}{x\,x} + \frac{4\,y}{x^3};$$
$$Q = \frac{-2}{x\,x} + \frac{1}{f\,f} \text{ et } R = \frac{2}{x}.$$

Hinc ſi pro cuiusque interualli initio ſit $x = a$, $y = b$, $p = c$, ob $F = \frac{2\,c}{a} - \frac{2\,b}{a\,a} + \frac{b}{f\,f}$, erit

$$p = c + \left(\frac{2\,c}{a} - \frac{2\,b}{a\,a} + \frac{b}{f\,f}\right)(x - a)$$
$$- \tfrac{1}{2}\left(\frac{c}{f\,f} + \frac{2\,b}{a\,f\,f}\right)(x - a)^2, \text{ et}$$
$$y = b + c\,(x - a) + \tfrac{1}{2}\left(\frac{2\,c}{a} - \frac{2\,b}{a\,a} + \frac{b}{f\,f}\right)(x - a)^2;$$

vnde calculus per interualla facile continuatur, dum ne ſit $a = 0$.

$a = 0$. Hoc autem casu quo $a = 0$, difficulter interuallum computo definitur, quia tum fieri nequit, vt quantitatibus $b$ et $c$ dati valores tribuantur, id quod inde facillime intelligitur, quod aequationis propositae integrale completum est

$$y = A e^{\frac{x}{f}} x + B e^{\frac{-x}{f}} x,$$

posito enim $x = 0$, necessario fit $y = 0$, nisi coefficientes A et B infinitos capere velimus. Sumto autem $b = 0$, approximatio est in promtu.

# CALCVLI INTEGRALIS
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# CAPVT I.

## DE
## INTEGRATIONE FORMVLARVM DIFFERENTIALIVM TERTII ALTIORISVE GRADVS SIMPLICIVM.

### Problema 140.

1100.

Sumto elemento $\partial x$ conſtante, inuenire integrale completum harum formularum $\partial^3 y = 0$, $\partial^4 y = 0$, $\partial^5 y = 0$ etc. atque in genere huius $\partial^n y = 0$.

### Solutio.

Cum $\partial x$ ſit conſtans, aequatio $\partial^3 y = 0$ per integrationem dat $\partial\partial y = \alpha\,\partial x^2$, hincque porro integrando $\partial y = \alpha x\,\partial x + \beta\,\partial x$, et tandem $y = \frac{1}{2}\alpha x^2 + \beta x + \gamma$.

Simili modo ex aequatione $\partial^4 y = 0$, per quadruplicem integrationem reperitur

1.) $\partial^3 y = \alpha\,\partial x^3$,

2.) $\partial\partial y = \alpha x\,\partial x^2 + \beta\,\partial x^2$,

3.) $\partial y = \frac{1}{2}\alpha x x\,\partial x + \beta x\,\partial x + \gamma\,\partial x$,

et tandem

4.) $y = \frac{1}{6}\alpha x^3 + \frac{1}{2}\beta x^2 + \gamma x + \delta$.

Ex aequatione autem $\partial^5 y = 0$, integratio quinquies repetita dat

$$y = \frac{1}{24}\alpha x^4 + \frac{1}{6}\beta x^3 + \frac{1}{2}\gamma x^2 + \delta x + \varepsilon.$$

At huius aequationis $\partial^6 y = 0$ integrale colligitur

$$y = \tfrac{1}{120}\alpha x^5 + \tfrac{1}{24}\beta x^4 + \tfrac{1}{6}\gamma x^3 + \tfrac{1}{2}\delta x^2 + \varepsilon x + \zeta,$$

ſicque ad huiusmodi formas $\partial^n y = 0$, quanticumque fuerint gradus, progredi licet, dummodo $n$ fuerit numerus integer poſitiuus.

## Corollarium 1.

1101. A ſimpliciſſima forma ergo incipiendo integralia ſequenti ordine procedunt

| Formularum | Integralia completa ſunt |
|---|---|
| $\partial y = 0$ | $y = \alpha$ |
| $\partial\partial y = 0$ | $y = \alpha x + \beta$ |
| $\partial^3 y = 0$ | $y = \tfrac{1}{2}\alpha x^2 + \beta x + \gamma$ |
| $\partial^4 y = 0$ | $y = \tfrac{1}{6}\alpha x^3 + \tfrac{1}{2}\beta x^2 + \gamma x + \delta$ |
| etc. | etc. |

## Corollarium 2.

1102. Quia conſtantes $\alpha$, $\beta$, $\gamma$ etc. ab arbritrio noſtro pendent, fractiones tuto reiicere licet, eritque

| Formularum | Integralia |
|---|---|
| $\partial y = 0$ | $y = \alpha$ |
| $\partial\partial y = 0$ | $y = \alpha x + \beta$ |
| $\partial^3 y = 0$ | $y = \alpha x^2 + \beta x + \gamma$ |
| $\partial^4 y = 0$ | $y = \alpha x^3 + \beta x^2 + \gamma x + \delta$ |
| $\partial^5 y = 0$ | $y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon$ |
| etc. | etc. |

## Corollarium 3.

1103. Quoti ergo ordinis eſt formula differentialis, totidem conſtantes arbitrarias eius integrale completum complectitur, quas pro quouis caſu oblato ſecundum conditiones praeſcriptas definiri oportet.

Scho-

## Scholion I.

1104. Ponendo $\partial y = p\,\partial x$, $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, $\partial r = s\,\partial x$ etc. omnes aequationes differentiales altiorum ordinum ad quantitates finitas reducuntur, in quibus nulla amplius ratio eius elementi, quod constans assumitur, habetur. Atque hinc formae omnium aequationum differentialium sequenti modo repraesentari possunt:

| Aequationum differentialium | Forma generalis |
|---|---|
| I. gradus | $p = f:(x \text{ et } y)$ |
| II. gradus | $q = f:(x, y, \text{ et } p)$ |
| III. gradus | $r = f:(x, y, p, \text{ et } q)$ |
| IV. gradus | $s = f:(x, y, p, q, \text{ et } r)$ |
| | etc. |

vbi quantitates $y$, $p$, $q$, $r$, $s$ etc. ita, excludendo $\partial x$, a se inuicem pendent, vt cum sit

$$\partial x = \frac{\partial y}{p} = \frac{\partial p}{q} = \frac{\partial q}{r} = \frac{\partial r}{s} = \frac{\partial s}{t} \text{ etc.}$$

sequentes relationes locum habeant

$$q\partial y = p\partial p,\ r\partial y = p\partial q,\ s\partial y = p\partial r,\ t\partial y = p\partial s \text{ etc.}$$
$$r\partial p = q\partial q,\ s\partial p = q\partial r,\ t\partial p = q\partial s \text{ etc.}$$
$$s\partial q = r\partial r,\ t\partial q = r\partial s \text{ etc.}$$
$$t\partial r = s\partial s \text{ etc.}$$

quarum formularum quaedam per se sunt integrabiles veluti

$$\int q\partial y = \tfrac{1}{2}pp,\ \int r\partial p = \tfrac{1}{2}qq,\ \int s\partial q = \tfrac{1}{2}rr,\ \int t\partial r = \tfrac{1}{2}ss \text{ etc.}$$

ex quibus porro ob $\int z\partial v = vz - \int v\partial z$ istae concluduntur

$$\int y\partial q = yq - \tfrac{1}{2}pp,\ \int p\partial r = pr - \tfrac{1}{2}qq,\ \int q\partial s = qs - \tfrac{1}{2}rr,\ \int r\partial t = rt - \tfrac{1}{2}ss, \text{etc.}$$

quarum ope ex praecedentibus deducitur

$$\int s\partial y = pr - \tfrac{1}{2}qq, \text{ hinc } \int y\partial s = ys - pr + \tfrac{1}{2}qq,$$

 $\int t\,\partial p$

$\int t\,\partial p = qs - \frac{1}{2}rr$, hinc $\int p\,\partial t = pt - qs + \frac{1}{2}rr$,

$\int u\,\partial q = rt - \frac{1}{2}ss$, hinc $\int q\,\partial u = qu - rt + \frac{1}{2}ss$.

Hinc porro definitur $\int y\,\partial u = yu - \int u\,\partial y$, at $\frac{\partial y}{p} = \frac{\partial t}{u}$, vnde

$\int r\,\partial u = yu - \int p\,\partial t = yu - pt + qs - \frac{1}{2}rr$.

Quare ſi differentialia iterum introducamus, obtinebimus ſequentes formulas integrales

$$\int y\,\partial y = \tfrac{1}{2}yy$$
$$\int y\,\partial^3 y = y\,\partial\partial y - \tfrac{1}{2}\partial y^2$$
$$\int y\,\partial^5 y = y\,\partial^4 y - \partial y\,\partial^3 y + \tfrac{1}{2}\partial\partial y^2$$
$$\int y\,\partial^7 y = y\,\partial^6 y - \partial y\,\partial^5 y + \partial\partial y\,\partial^4 y - \tfrac{1}{2}\partial^3 y^2$$

etc.

ita vt formula $\int y\,\partial^n y$ ſit integrabilis, quoties $n$ eſt numerus impar.

## Scholion 2.

1105. In aequationibus differentialibus ſecundi gradus formas ſimpliciores ita conſtituimus, vt $q$ aequetur functioni vel ipſius $x$ tantum, vel ipſius $y$, vel ipſius $p$, quas litteras maiusculas pro functionibus minuſcularum ſcribendo ita repraeſentare licet, vt ſit

vel $q = X$, vel $q = Y$, vel $q = P$.

Hinc ſimili modo pro aequationibus differentialibus tertii gradus formas ſimpliciores conſtituere poſſumus

$r = X$, $r = Y$, $r = P$, $r = Q$,

ita vt tantum binas quantitates variabiles inuoluant.

Pro quarto autem gradu eſſent formae ſimpliciores

$s = X$, $s = Y$, $s = P$, $s = Q$, $s = R$,

et pro quinto

$t = X$, $t = Y$, $t = P$, $t = Q$, $t = R$, $t = S$,

atque ita porro pro ſuperioribus.

Ve-

Verum hae formae non omnes aeque integrationem admittunt, dum aliae ne semel quidem, aliae semel tantum, aliae per omnes integrationes vsque ad relationem inter $x$ et $y$ perduci possunt, cuiusmodi sunt primae quaeque in quouis gradu. Semper autem id est propositum, vt relatio inter binas variabiles principales $x$ et $y$ eliciatur.

## Problema 141.

1106. Posito $\partial y = p\,\partial x$, $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, $\partial r = s\,\partial x$, $\partial s = t\,\partial x$, etc pro quouis differentialium gradu, si litterarum $p$, $q$, $r$, $s$, $t$, etc. quaepiam aequetur functioni ipsius $x$, quae sit X, inuenire relationem inter $x$ et $y$.

## Solutio.

Si primo sit $p = X$, per $\partial x$ multiplicando erit $p\,\partial x = \partial y = X\,\partial x$, hincque $y = \int X\,\partial x$, qui est casus formularum differentialium primi gradus simplicium.

Sit secundo $q = X$, erit $q\,\partial x = \partial p = X\,\partial x$, hinc $p = \int X\,\partial x$, et $p\,\partial x = \partial y = \partial x \int X\,\partial x$, ergo $y = \int \partial x \int X\,\partial x$, seu per simplicia integralia $y = x\int X\,\partial x - \int X x\,\partial x$.

Sit tertio $r = X$, ob $\partial q = r\,\partial x$ erit $q = \int X\,\partial x$, hincque

$$p = \int q\,\partial x = \int \partial x \int X\,\partial x = x\int X\,\partial x - \int X x\,\partial x,$$

ac tandem

$$y = \int p\,\partial x = \int \partial x \int \partial x \int X\,\partial x = \tfrac{1}{2} x x \int X\,\partial x - x\int X x\,\partial x + \tfrac{1}{2}\int X x x\,\partial x.$$

Sit quarto $s = X$, ac reperitur $y = \int \partial x \int \partial x \int \partial x \int X\,\partial x$, quae expressio euoluitur in hanc

$$y = \tfrac{1}{6} x^3 \int X\,\partial x - \tfrac{1}{2} x x \int X x\,\partial x + \tfrac{1}{2} x \int X x x\,\partial x - \tfrac{1}{6}\int X x^3\,\partial x.$$

Sit

Sit quinto $t = X$, erit $y = \int \partial x \int \partial x \int \partial x \int \partial x \int X \partial x$, seu

$$y = \tfrac{1}{24} x^4 \int X \partial x - \tfrac{1}{6} x^3 \int X x \partial x + \tfrac{1}{4} x^2 \int X x x \partial x$$
$$- \tfrac{1}{6} x \int X x^3 \partial x + \tfrac{1}{24} \int X x^4 \partial x;$$

vnde lex vlterius progrediendi est manifesta.

## Corollarium 1.

1107. Tot ergo habentur formulae integrales, quoti gradus aequatio fuerit differentialis, et quia quaelibet constantem arbitrariam assumit, totidem constantes in integrale ingrediuntur, quibus id completum redditur; quod idem ex priori forma, vbi totidem signa integralia implicantur, intelligitur.

## Corollarium 2.

1108. Sumto elemento $\partial x$ constante, sequentium formularum more consueto expressarum integralia completa ita se habebunt

I. Si $\partial y = X \partial x$ est $y = \int X \partial x$,

II. Si $\partial \partial y = X \partial x^2$ est $1 y = x \int X \partial x - \int X x \partial x$,

III. Si $\partial^3 y = X \partial x^3$ est
$$2 y = x^2 \int X \partial x - 2 x \int X x \partial x + \int X x^2 \partial x,$$

IV. Si $\partial^4 y = X \partial x^4$ est
$$6 y = x^3 \int X \partial x - 3 x^2 \int X x \partial x + 3 x \int X x^2 \partial x - \int X x^3 \partial x,$$

V. Si $\partial^5 y = X \partial x^5$ est
$$24 y = x^4 \int X \partial x - 4 x^3 \int X x \partial x + 6 x^2 \int X x^2 \partial x - 4 x \int X x^3 \partial x + \int X x^4 \partial x,$$

etc.

## Scholion.

1109. Formulas autem, quas supra secundo loco constituimus, functionem Y complectentes, post secundum gradum inte-

integrare non licet. Ex tertio enim ordine formula $r = Y$ etsi nouimus esse

$$r = \frac{p\,\partial q}{\partial y} = \frac{q\,\partial q}{\partial p} = \frac{\partial q}{\partial x},$$

nullo modo integrari potest; neque etiam hinc $q$ per $y$ determinari potest. Nam sumta forma $p\,\partial q = Y\,\partial y$, existente $p\,\partial p = q\,\partial y$ ob $p = \frac{Y\,\partial y}{\partial q}$, erit

$$\partial p = \frac{\partial y\,\partial Y}{\partial q} + Y\,\partial\,.\,\frac{\partial y}{\partial q},$$

hincque $p$ elidendo

$$\frac{Y\,\partial y^2\,\partial Y}{\partial q^2} + \frac{YY\,\partial y}{\partial q}\,\partial\,.\,\frac{\partial y}{\partial q} = q\,\partial y,$$

quae quidem aequatio est secundi gradus, sed neutiquam in genere resolutionem admittit. Ex quarto genere formula $s = Y$, ob $\int s\,\partial y = pr - \frac{1}{2}qq = \int Y\,\partial y$, semel integrari potest, sed hinc vlterius progredi non licet. Quas autem supra pro quouis gradu formulas simpliciores vltimo loco constituimus itemque penultimo, eae tractabiles deprehenduntur: earum ergo integrationem inuestigemus.

## Problema 142.

1110. Posito vt hactenus $\partial y = p\,\partial x$, $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, etc. litterae Y, P, Q, R, etc. denotent functiones cuiusque litterae minusculae cognominis, inuestigare integralia harum formularum simplicium

$$p = Y,\ q = P,\ r = Q,\ s = R,\ t = S,\ \text{etc.}$$

## Solutio.

Aequatio prima ob $p = \frac{\partial y}{\partial x}$ statim dat $\partial x = \frac{\partial y}{Y}$, ideoque $x = \int \frac{\partial y}{Y}$.

Aequatio secunda $q = P$ ob $q = \frac{\partial p}{\partial x}$ praebet $\partial x = \frac{\partial p}{P}$, et $\partial y = \frac{p\,\partial p}{P}$, vnde cum P sit functio ipsius $p$, vtraque variabilis $x$

 et

et $y$ per $p$ determinatur hoc modo

$$x=\int\frac{\partial p}{p} \text{ et } y=\int\frac{p\partial p}{p}.$$

Aequatio tertia $r=Q$ ob $r=\frac{\partial q}{\partial x}$ dat $\partial x=\frac{\partial q}{Q}$, hinc $q\partial x=\partial p=\frac{q\partial q}{Q}$, ita vt fit $x=\int\frac{\partial q}{Q}$ et $p=\int\frac{q\partial q}{Q}$; vnde colligimus $p\partial x=\partial y=\frac{\partial q}{Q}\int\frac{q\partial q}{Q}$, ergo $y=\int\frac{\partial q}{Q}\int\frac{q\partial q}{Q}$. Quare per eandem variabilem $q$ vtraque variabilis $x$ et $y$ ita determinatur, vt fit

$$x=\int\frac{\partial q}{Q} \text{ et } y=\int\frac{\partial q}{Q}\int\frac{q\partial q}{Q}.$$

Aequatio quarta $s=R=\frac{\partial r}{\partial x}$ dat $\partial x=\frac{\partial r}{R}$, vnde colligimus $r\partial x=\partial q=\frac{r\partial r}{R}$, ita vt fit $q=\int\frac{r\partial r}{R}$. Porro $q\partial x=\partial p$ dat $\partial p=\frac{\partial r}{R}\int\frac{r\partial r}{R}$, hincque $p=\int\frac{\partial r}{R}\int\frac{r\partial r}{R}$; et quia $p\partial x=\partial y$ habebimus $\partial y=\frac{\partial r}{R}\int\frac{\partial r}{R}\int\frac{r\partial r}{R}$, quare per $r$ ambae variabiles principales $x$ et $y$ ita definiuntur

$$x=\int\frac{\partial r}{R} \text{ et } y=\int\frac{\partial r}{R}\int\frac{\partial r}{R}\int\frac{r\partial r}{R}.$$

Aequatio quinta $t=S$ fimili modo tractata praebet

$$x=\int\frac{\partial s}{S} \text{ et } y=\int\frac{\partial s}{S}\int\frac{\partial s}{S}\int\frac{\partial s}{S}\int\frac{s\partial s}{S}$$

ficque facile vlterius progredi licet.

## Corollarium 1.

1111. Ex formula fecunda intelligitur, fi $x$ aequetur functioni ipfius $p$, vt fit $x=P$, fore $y=\int p\partial P=Pp-\int P\partial p$, quod quidem per fe eft manifeftum.

## Corollarium 2.

1112. Sin autem fit $x=Q$, ob $\partial x=\partial Q$ erit

$$q\partial x=\partial p=q\partial Q, \text{ et } p=\int q\partial Q,$$

hincque

$$y=\int\partial Q\int q\partial Q, \text{ feu } y=Q\int q\partial Q-\int qQ\partial Q.$$

Vel

Vel etiam cum sit

$$y = \int \partial Q (q Q - \int Q \partial q), \text{ erit } y = \tfrac{1}{2} q QQ + \tfrac{1}{2} \int QQ \partial q - Q \int Q \partial q,$$

siue hoc modo

$$2y = QQq - 2Q \int Q \partial q + \int QQ \partial q.$$

## Corollarium 3.

1113. Simili modo si $x = R$, erit

$$q = \int r \partial x = \int r \partial R \text{ et } p = \int q \partial x = \int \partial R \int r \partial R,$$

atque

$$y = \int p \partial x = \int \partial R \int \partial R \int r \partial R, \text{ seu}$$

$$2y = \int \partial R (RRr - 2R \int R \partial r + \int RR \partial r),$$

per praecedens Corollarium. Ergo per similes reductiones

$$6y = R^3 r - 3R^2 \int R \partial r + 3R \int RR \partial r - \int R^3 \partial r.$$

## Corollarium 4.

1114. At si fuerit $x = S$, reperietur per similes reductiones

$$24y = S^4 s - 4S^3 \int S \partial s + 6S^2 \int SS \partial s - 4S \int S^3 \partial s + \int S^4 \partial s:$$

hinc ergo per differentiationes retrogrediendo

$$24p \partial S = 4S^3 s \partial S - 12SS \partial S \int S \partial s + 12S \partial S \int SS \partial s - 4 \partial S \int S^3 \partial s,$$

seu

$$6p = S^3 s - 3SS \int S \partial s + 3S \int SS \partial s - \int S^3 \partial s, \text{ et}$$

$$2q = S^2 s - 2S \int S \partial s + \int SS \partial s,$$

tum $r = Ss - \int S \partial s$ et $s = s$.

## Problema 143.

1115. Iisdem manentibus denominationibus, quibus hactenus sumus vsi, inuestigare integralia harum formularum simpliciorum

$$q = Y, \; r = P, \; s = Q, \; t = R, \text{ etc.}$$

## Solutio.

Pro formula prima $q = Y$, cum ſit $q = \frac{p\,\partial p}{\partial y}$, erit $p\,\partial p = Y\,\partial y$ et $pp = 2\int Y\,\partial y$, hinc $p = \sqrt{2\int Y\,\partial y} = \frac{\partial y}{\partial x}$, vnde colligitur $x = \int \frac{\partial y}{\sqrt{2\int Y\,\partial y}}$, ſicque $x$ per $y$ determinatur.

Pro formula ſecunda $r = P$ ob $r = \frac{q\,\partial q}{\partial p}$, habebimus $q\,\partial q = P\,\partial p$ et $q = \sqrt{2\int P\,\partial p} = \frac{p\,\partial p}{\partial y} = \frac{\partial p}{\partial x}$, vnde concludimus

$$x = \int \frac{\partial p}{\sqrt{2\int P\,\partial p}}, \text{ et } y = \int \frac{p\,\partial p}{\sqrt{2\int P\,\partial p}}.$$

Pro formula tertia $s = Q$ ob $s = \frac{r\,\partial r}{\partial q}$, fiet

$$r = \sqrt{2\int Q\,\partial q} = \frac{q\,\partial q}{\partial p},$$

vnde ſequitur $p = \int \frac{q\,\partial q}{\sqrt{2\int Q\,\partial q}}$. Cum vero ſit $r = \frac{\partial q}{\partial x}$, erit $\partial x = \frac{\partial q}{\sqrt{2\int Q\,\partial q}}$, et ob $p\,\partial x = \partial y$ habebimus

$$x = \int \frac{\partial q}{\sqrt{2\int Q\,\partial q}}, \text{ et } y = \int \frac{\partial q}{\sqrt{2\int Q\,\partial q}} \int \frac{q\,\partial q}{\sqrt{2\int Q\,\partial q}}.$$

Pro formula quarta $t = R$ ob $t = \frac{s\,\partial s}{\partial r}$, nanciſcimur $s = \sqrt{2\int R\,\partial r}$. At eſt $s = \frac{\partial r}{\partial x}$, vnde fit $\partial x = \frac{\partial r}{\sqrt{2\int R\,\partial r}}$. Eſt vero etiam $s = \frac{r\,\partial r}{\partial q}$, ideoque $q = \int \frac{r\,\partial r}{\sqrt{2\int R\,\partial r}}$; ſed quoniam $p = \int q\,\partial x$, fit $p = \int \frac{\partial r}{\sqrt{2\int R\,\partial r}} \int \frac{r\,\partial r}{\sqrt{2\int R\,\partial r}}$, ex quo prodit $y = \int p\,\partial x$. Quocirca $x$ et $y$ ita per $r$ determinantur, vt fit

$$x = \int \frac{\partial r}{\sqrt{2\int R\,\partial r}}, \text{ et } y = \int \frac{\partial r}{\sqrt{2\int R\,\partial r}} \int \frac{\partial r}{\sqrt{2\int R\,\partial r}} \int \frac{r\,\partial r}{\sqrt{2\int R\,\partial r}}.$$

Pro formula quinta $u = S$, ob $u = \frac{t\,\partial t}{\partial s}$, adipiſcimur $t = \sqrt{2\int S\,\partial s} = \frac{\partial s}{\partial x}$, vt fit $\partial x = \frac{\partial s}{\sqrt{2\int S\,\partial s}}$. Eſt vero etiam $t = \frac{s\,\partial s}{\partial r}$, ergo $r = \int \frac{s\,\partial s}{\sqrt{2\int S\,\partial s}}$. Tum $q = \int r\,\partial x$, $p = \int q\,\partial x$ et $y = \int p\,\partial x$, ex quibus conficitur

$$x = \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \text{ et } y = \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \int \frac{s\,\partial s}{\sqrt{2\int S\,\partial s}}.$$

Scho-

## Scholion.

1116. Hi funt cafus, quibus formulas illas fimpliciores fupra recenfitas refoluere licet, neque methodus patet, qua reliquae tractari queant. Multo pauciores occurrunt cafus tractabiles in formis magis compofitis, vbi $\frac{\partial^n y}{\partial x^n}$ aequatur functioni binarum pluriumue quantitatum variabilium, ob quam penuriam parum admodum fuppetit, quod in hac fectione exponere queamus. Aequationum autem, quae per methodos adhuc inuentas tractari poffunt, haec fere eft forma generalis

$$Ay + B . \frac{\partial y}{\partial x} + C . \frac{\partial\partial y}{\partial x^2} + D . \frac{\partial^3 y}{\partial x^3} + \text{etc.} = 0,$$

fumto elemento $\partial x$ conftante, quae etiam ponendo

$$\partial y = p\,\partial x,\ \partial p = q\,\partial x,\ \partial q = r\,\partial x,\ \text{etc.}$$

ita repraefentari poteft

$$Ay + Bp + Cq + Dr + Es + \text{etc.} = 0.$$

Deinde vero etiam aequationes hac forma latius patente contentae refolutionem admittunt

$$Ay + Bp + Cq + Dr + Es + \text{etc.} = X,$$

denotante X functionem quamcunque ipfius $x$. Porro quoque fequentes formae, quae quidem ad illas reduci poffunt

$$Ay + \frac{Bp}{x} + \frac{Cq}{x^2} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = 0 \text{ et}$$

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = X,$$

quarum refolutio adeo fuccedit, ad quantumuis gradum etiam differentialitas affurgat; in harum ergo aequationum euolutione tractatio noftra verfabitur.

# CAPVT II.

## DE RESOLVTIONE AEQVATIONVM HVIVS FORMAE

$$Ay + B \cdot \frac{\partial y}{\partial x} + C \cdot \frac{\partial\partial y}{\partial x^2} + D \cdot \frac{\partial^3 y}{\partial x^3} + E \cdot \frac{\partial^4 y}{\partial x^4} + \text{etc.} = 0,$$

## SVMTO ELEMENTO $\partial x$ CONSTANTE.

### Problema 144.

1117.

Aequationis differentialis tertii gradus

$$Ay + \frac{B \partial y}{\partial x} + \frac{C \partial\partial y}{\partial x^2} + \frac{D \partial^3 y}{\partial x^3} = 0,$$

sumto elemento $\partial x$ constante, integrale completum inuenire.

### Solutio.

Cum A, B, C, D sint quantitates constantes, leui attentione adhibita patet, isti aequationi huiusmodi formam $y = e^{\lambda x}$ satisfacere; cum enim hinc sit

$$\frac{\partial y}{\partial x} = \lambda e^{\lambda x}, \quad \frac{\partial\partial y}{\partial x^2} = \lambda^2 e^{\lambda x}, \quad \frac{\partial^3 y}{\partial x^3} = \lambda^3 e^{\lambda x},$$

his substitutis et aequatione per $e^{\lambda x}$ diuisa, fit

$$A + \lambda B + \lambda^2 C + \lambda^3 D = 0,$$

vnde exponens $\lambda$ determinatur, qui cum tres valores sortiatur, qui sint $\alpha$, $\beta$, $\gamma$, habebimus tres formulas satisfacientes $y = e^{\alpha x}$, $y = e^{\beta x}$, $y = e^{\gamma x}$. Verum ex natura aequationis propositae perspicuum est, si ei satisfaciant valores $y = P$, $y = Q$, $y = R$, etiam his vtcunque coniungendis satisfacturum

$$y = \mathfrak{A} P + \mathfrak{B} Q + \mathfrak{C} R.$$

Quare

Quare ex ternis formulis inuentis nanciscimur hanc latissime patentem aeque satisfacientem

$$y = \mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x},$$

quae forma cum tres constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ complectatur, reuera erit integrale completum aequationis nostrae propositae.

## Corollarium 1.

1118. Integrale ergo completum tot constat partibus, quot radices habeat seu factores aequatio

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

cuius si factor fuerit $a + \lambda$, pars integralis erit $e^{-ax}$.

## Corollarium 2.

1119. Haec scilicet pars erit integrale huius aequationis $ay + \frac{\partial x}{\partial y} = 0$. Vnde si fit

$$A + B\lambda + C\lambda^2 + D\lambda^3 = (a + \lambda)(b + \lambda)(c + \lambda),$$

quaerantur valores ipsius $y$ ex his aequationibus simplicibus

$$ay + \frac{\partial y}{\partial x} = 0, \quad by + \frac{\partial y}{\partial x} = 0, \quad cy + \frac{\partial y}{\partial x} = 0,$$

qui si sint $y = P$, $y = Q$, $y = R$, integrale aequationis propositae erit

$$y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R.$$

## Corollarium 3.

1120. Si binae radices sint aequales, puta $\beta = \alpha$, consideretur differentia vt euanescens vel $\beta = \alpha + \omega$, et cum sit

$$e^{\beta x} = e^{\alpha x} \cdot e^{\omega x} = e^{\alpha x}(1 + \omega x),$$

euidens est loco $\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x}$ scribi debere

$$\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\alpha x} x = e^{\alpha x}(\mathfrak{A} + \mathfrak{B} x).$$

Ac

Ac ſi omnes tres radices fuerint aequales $\alpha = \beta = \gamma$, vt aequatio ſit

$$y + \frac{3\partial y}{a\partial x} + \frac{3\partial\partial y}{a^2\partial x^2} + \frac{\partial^3 y}{a^3\partial x^3} = 0.$$

ob $\alpha = \beta = \gamma = -a$, integrale completum erit

$$e^{-ax}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx).$$

## Corollarium 4.

1121. Si binae radices fiant imaginariae, vt ſit

$$\alpha = \mu + \nu\sqrt{-1} \text{ et } \beta = \mu - \nu\sqrt{-1},$$

loco $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$ ſcribi debet

$$e^{\mu x}(\mathfrak{A}e^{\nu x\sqrt{-1}} + \mathfrak{B}e^{-\nu x\sqrt{-1}}),$$

quae reducitur ad hanc formam

$$e^{\mu x}(\mathfrak{A}\,\mathrm{cof.}\,\nu x + \mathfrak{B}\,\mathrm{fin.}\,\nu x).$$

## Scholion 1.

1122. Quanquam aequatio propoſita triplicem integrationem poſtulat, antequam ad relationem finitam inter $x$ et $y$ perueniatur, hic tamen vna operatione, quae ne integrationi quidem eſt affinis, eo pertigimus. Coniectura ſcilicet formam collegimus aequationi particulariter ſatisfacientem, ſimulque tres huiusmodi formas ſumus conſecuti. Deinde ex ipſa aequationis indole intelleximus, ſi valores ſinguli $y = P$, $y = Q$, $y = R$ ſatisfaciant, etiam formam ex iis compoſitam $y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R$ ſatisfacere debere, quod niſi commode eueniſſet, ex illis ternis valoribus nihil amplius concludi potuiſſet. Ex eodem ergo principio in genere huiusmodi aequationes differentiales, quotcunque fuerint gradus, vno quaſi actu ita reſolui poterunt, vt adeo integrale completum aſſignetur.

Scho-

## Scholion 2.

1123. Quoniam aequationem differentialem tertii gradus

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} = 0,$$

in genere resoluere licuit, vt integrale completum esset

$$y = \mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x},$$

existentibus $\alpha$, $\beta$, $\gamma$ radicibus huius aequationis cubicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

hinc vsum non contemnendum pro aliis aequationibus, in quas illam transformare licet, percipiemus. Primo autem illam aequationem ad differentialem secundi gradus reuocare licet ope substitutionis $y = e^{\int u\partial x}$; vnde fit

$$\frac{\partial y}{\partial x} = e^{\int u\partial x}u,\ \frac{\partial\partial y}{\partial x^2} = e^{\int u\partial x}\left(\frac{\partial u}{\partial x} + uu\right),\ \text{et}$$

$$\frac{\partial^3 y}{\partial x^3} = e^{\int u\partial x}\left(\frac{\partial\partial u}{\partial x^2} + \frac{3u\partial u}{\partial x} + u^3\right),$$

ita vt aequatio transformata sit, diuisione per $e^{\int u\partial x}$ facta,

$$A + Bu + Cuu + Du^3 + C\frac{\partial u}{\partial x} + \frac{3Du\partial u}{\partial x} + \frac{D\partial\partial u}{\partial x^2} = 0,$$

cuius ergo ob $u = \frac{\partial y}{y\partial x}$ integrale est

$$u = \frac{\alpha\mathfrak{A}\, e^{\alpha x} + \beta\mathfrak{B}\, e^{\beta x} + \gamma\mathfrak{C}\, e^{\gamma x}}{\mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x}}.$$

Haec autem aequatio vlterius ad gradum primum reducitur ponendo $\partial x = \frac{\partial u}{t}$, cum enim elementum $\partial x$ sumtum sit constans, erit $t\partial\partial u - \partial t\partial u = 0$, seu $\partial\partial u = \frac{\partial t\partial u}{t}$, vnde fit $\frac{\partial u}{\partial x} = t$, et $\frac{\partial\partial u}{\partial x^2} = \frac{t\partial t}{\partial u}$, ita vt prodeat haec aequatio differentialis primi gradus

$$A + Bu + Cuu + Du^3 + t(C + 3Du) + \frac{Dt\partial t}{\partial u} = 0,$$

cuius ergo resolutio quoque est in potestate: vtriusque scilicet variabilis $u$ et $t$ valor per eandem variabilem $x$ exprimi potest.

ſt. Cum enim $y$ per $x$ detur, erit primo $u = \frac{\partial y}{y \partial x}$; tum vero $t + uu = \frac{\partial\partial y}{y \partial x^2}$, ob $\frac{\partial u}{\partial x} = t$. Loco $y$ ergo ſubſtituto valore ſupra inuento erit

$$u = \frac{\alpha \mathfrak{A} e^{\alpha x} + \beta \mathfrak{B} e^{\beta x} + \mathfrak{C} \gamma e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}}, \text{ et}$$

$$t + uu = \frac{\alpha \alpha \mathfrak{A} e^{\alpha x} + \beta \beta \mathfrak{B} e^{\beta x} + \gamma \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}},$$

dummodo $\alpha$, $\beta$, $\gamma$ ſint radices ex hac aequatione

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0.$$

Obſeruari autem conuenit, illam aequationem, poſito $t + uu = z$, abire in hanc formam

$$A + Bu + z(C + Du) + \frac{D \partial z}{\partial u}(z - uu) = 0,$$

quae latius patere videtur, quam illae eiusdem generis aequationes, quas ſupra tractauimus; cuius, quia ratio per methodos cognitas integrandi non conſtat, reſolutio facilime inſtituitur ponendo

$$u = \frac{\partial y}{y \partial x} \text{ et } z = \frac{\partial\partial y}{y \partial x^2},$$

vnde fit

$$\partial u = \frac{\partial\partial y}{y \partial x} - \frac{\partial y^2}{yy \partial x} \text{ et } \partial z = \frac{\partial^3 y}{y \partial x^2} - \frac{\partial y \partial\partial y}{yy \partial x^2},$$

ideoque

$$\frac{\partial z}{\partial u} = \frac{y \partial^3 z - \partial y \partial\partial y}{\partial x (y \partial\partial y - \partial y^2)} \text{ et } z - uu = \frac{y \partial\partial y - \partial y^2}{yy \partial x^2},$$

ſicque reſultat haec aequatio

$$A + \frac{B \partial y}{y \partial x} + \frac{C \partial\partial y}{y \partial x^2} + \frac{D \partial y \partial\partial y}{yy \partial x^3} + \frac{D y \partial^3 y - D \partial y \partial\partial y}{yy \partial x^3} = 0, \text{ ſeu}$$

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} = 0,$$

cuius reſolutio eſt oſtenſa.

Scho-

## Scholion 3.

1124. Aequatio illa differentialis primi gradus

$$D t \partial t + t \partial u (C + 3 D u) + \partial u (A + B u + C u^2 + D u^3) = 0,$$

cuius integrale inuenimus, diligentiore euolutione est digna. Ac primo quidem obseruo, eam integrabilem reddi, si diuidatur per hanc formam

$$D D t^3 + D t t (B + 2 C u + 3 D u u)$$
$$+ t (C + 3 D u)(A + B u + C u^2 + D u^3) + (A + B u + C u^2 + D u^3)^2,$$

vnde concludimus et hanc aequationem

$$D z \partial z - D u u \partial z + z \partial u (C + D u) + \partial u (A + B u) = 0$$

integrabilem fieri, si diuidatur per hanc formam

$$D D z^3 + D z z (B + 2 C u) + z [AC + (3AD + BC) u + (BD + CC) u u]$$
$$+ A A + 2 A B u + (A C + B B) u u + (A D + B C) u^3.$$

Vtrinque autem diuisor iste nihilo aequatus praebet integrale particulare, vnde cum $t$ vel $z$ ternos obtineant valores, singuli exhibebunt integralia particularia. Hinc operae pretium erit in genere aequationem

$$y \partial y + y P \partial x + Q \partial x = 0,$$

inuestigare, quae per formam

$$y^3 + L y y + M y + N,$$

diuisa integrabilis euadat. Per operationem autem supra explicatam inuenitur

$$\partial L = 2 P \partial x, \quad \partial M = P L \partial x + 3 Q \partial x,$$
$$\partial N = 2 Q L \partial x, \text{ et } P N - Q M = 0;$$

vnde colligitur

$$P \partial x = \tfrac{1}{2} \partial L, \quad Q \partial x = \tfrac{\partial N}{2 L},$$
$$\partial M = \tfrac{1}{2} L \partial L + \tfrac{3 \partial N}{2 N}, \text{ et } N \partial L = \tfrac{M \partial N}{L}, \text{ seu } M = \tfrac{N L \partial L}{\partial N},$$

 qui

qui valor ibi ſubſtitutus ſumto $\partial N$ conſtante dat

$$3\partial N^2 = LL\partial L\partial N + 2NLL\partial\partial L + 2NL\partial L^2,$$

quae per $\partial L$ multiplicata, tranſit in

$$3\partial L\partial N^2 = \partial . NL^2\partial L^2.$$

Verum commodius, ac ſingulari quidem modo, illae aequationes reſoluuntur, ſtatuendo

$$N = \alpha Z^2 \text{ et } L = \frac{\partial Z}{\partial z},$$

vnde ſumto elemento $\partial z$ conſtante deducitur $M = \frac{Z\partial\partial Z}{2\alpha z^2}$, hincque

$$\partial M = \frac{Z\partial^3 Z + \partial Z\partial\partial Z}{2\alpha z^4} \text{ et}$$

$$\tfrac{1}{2}L\partial L + \frac{3\partial N}{2L} = \frac{\partial Z\partial\partial Z}{2\partial z^2} + 3\alpha Z\partial z.$$

Ergo $\partial^3 Z = 6\alpha\partial z^3$, ideoque

$$Z = \alpha z^3 + \beta zz + \gamma z + \delta,$$

$$P\partial x = \frac{\partial\partial Z}{2\partial z} \text{ et } Q\partial x = \alpha Z\partial z.$$

Quocirca ſumto $Z = \alpha z^3 + \beta z^2 + \gamma z + \delta$, haec aequatio

$$y\partial y + y . \frac{\partial\partial Z}{2\partial z} + \alpha Z\partial z = 0$$

integrabilis redditur, diuiſa per hanc formam

$$y^3 + y^2 . \frac{\partial Z}{\partial z} + y . \frac{Z\partial\partial Z}{2\partial z^2} + \alpha ZZ.$$

Praeterea ſi $Z$ habeat factores, vt propoſita ſit haec aequatio

$$y\partial y + y\partial z(\alpha+\beta+\gamma+3z) + \partial z(\alpha+z)(\beta+z)(\gamma+z) = 0$$

diuiſor eam integrabilem reddens erit

$$[y+(\alpha+z)(\beta+z)][y+(\alpha+z)(\gamma+z)][y+(\beta+z)(\gamma+z)],$$

cuius ſinguli factores nihilo aequati praebent integrale particulare. Ex vnoquoque autem, more magis conſueto, integrale completum ita elicitur. Ponatur

$$y = v - (\alpha+z)(\beta+z),$$

ac

ac reperitur

$$v\partial v+v\partial z(\gamma+z)-\partial v(\alpha+z)(\beta+z)=0,$$

fit porro $\partial v=p\,\partial z$, eritque $v=\frac{p(\alpha+z)(\beta+z)}{p+\gamma+z}$, et differentiando locoque $\partial v$ ponendo $p\,\partial z$, orietur haec aequatio

$$\partial p(\alpha+z)(\beta+z)(\gamma+z)=\partial z[p^3+(2\gamma-\alpha-\beta)p^2+(\gamma-\alpha)(\gamma-\beta)p]$$

quae dat hanc feparatam

$$\frac{\partial z}{(\alpha+z)(\beta+z)(\gamma+z)}=\frac{\partial p}{p(p+\gamma-\alpha)(p+\gamma-\beta)}.$$

## Problema 145.

1125. Aequationis differentialis cuiuscunque gradus

$$Ay+B\frac{\partial y}{\partial x}+C\frac{\partial\partial y}{\partial x^2}+D\frac{\partial^3 y}{\partial x^3}+E\frac{\partial^4 y}{\partial x^4}+\text{etc.}=0$$

fumto elemento $\partial x$ conftante, integrale completum inuenire.

## Solutio.

Et huic aequationi euidens eft fatisfacere formulam $y=e^{\lambda x}$, cum enim hinc fit

$$\frac{\partial y}{\partial x}=\lambda e^{\lambda x},\ \frac{\partial\partial y}{\partial x^2}=\lambda^2 e^{\lambda x},\ \frac{\partial^3 y}{\partial x^3}=\lambda^3 e^{\lambda x},$$

et in genere $\frac{\partial^n y}{\partial x^n}=\lambda^n e^{\lambda x}$, facta fubftitutione peruenietur ad hanc aequationem, poftquam fcilicet per $e^{\lambda x}$ diuiferimus,

$$A+B\lambda+C\lambda^2+D\lambda^3+E\lambda^4+\text{etc.}=0,$$

ex qua valorem ipfius $\lambda$ definiri oportet. Hinc littera $\lambda$ totidem valores obtinebit, quoti fuerit ordinis aequatio differentialis propofita, quorum finguli aequationi aeque fatisfacient. Qui valores fi fint $\alpha$, $\beta$, $\gamma$, $\delta$, etc. integralia quidem particularia erunt

$$y=\mathfrak{A}e^{\alpha x},\ y=\mathfrak{B}e^{\beta x},\ y=\mathfrak{C}e^{\gamma x},\ \text{etc.}$$

Verum ex ipfa aequationis natura perfpicuum eft, aggregata quotcunque horum valorum, ideoque etiam omnium, perinde ſatisfacere. Cum igitur aggregatum omnium

$$y = \mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x} + \mathfrak{D}\, e^{\delta x} + \text{etc.}$$

tot complectatur conſtantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. quoti ordinis differentialis eſt aequatio propoſita, quin haec forma eius ſit integrale completum, dubitari nequit. Aſcendat aequatio differentialis ad gradum $n$, vt ſit

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n} = 0,$$

atque integrale completum ex $n$ partibus conſtabit, quas ex reſolutione huius aequationis algebraicae ordinis $n$, ſcilicet

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0,$$

definire oportet. Singuli nimirum eius factores ſimplices partes illas patefacient, ita ſi factor ſit $\alpha - \lambda$, ex eo integralis pars naſcitur $\mathfrak{A}\, e^{\alpha x}$, quae, vti manifeſtum eſt, ex integratione aequationis differentialis ſimplicis $\alpha y - \frac{\partial y}{\partial x} = 0$ naſcitur. Simili modo duo factores coniunctim

$$(\alpha - \lambda)(\beta - \lambda) = \alpha\beta - (\alpha + \beta)\lambda + \lambda\lambda$$

integralis portionem $\mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x}$ ſuppeditant, quae ſimul eſt integrale huius aequationis differentialis ſecundi gradus

$$\alpha\beta y - (\alpha + \beta)\frac{\partial y}{\partial x} + \frac{\partial\partial y}{\partial x^2} = 0.$$

Atque in genere ſi aequationis illius algebraicae factor ſit

$$a + b\lambda + c\lambda^2 + f\lambda^3 + \text{etc.} = 0,$$

ex hoc viciſſim formetur aequatio differentialis

$$ay + b\frac{\partial y}{\partial x} + c\frac{\partial\partial y}{\partial x^2} + f\frac{\partial^3 y}{\partial x^3} + \text{etc.} = 0,$$

cuius integrale completum ſi ſit $y = P$, id ſimul erit pars integralis aequationis propoſitae. Atque hoc modo ex ſingulis factoribus aequationis algebraicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0$$

de-

deriuabuntur singulae partes integralis quaesiti, quae iunctae eius integrale completum constituent, ita vt praecipuum negotium resolutioni huius aequationis innitatur.

## Corollarium 1.

1126. Si igitur istius aequationis algebraicae omnes factores simplices fuerint reales simulque inaequales, integratio nullam habet difficultatem. Si enim factor simplex sit $f+g\lambda$, integralis pars inde oriunda est $\mathfrak{A}\, e^{\frac{-fx}{g}}$.

## Corollarium 2.

1127. Si bini factores simplices sint aequales, seu factor fuerit $(f+g\lambda)^2$, pars integralis inde oriunda est $e^{\frac{-fx}{g}}(\mathfrak{A}+\mathfrak{B}x)$. Si factor sit cubus $(f+g\lambda)^3$, inde oritur pars integralis

$$e^{\frac{-fx}{g}}(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2),$$

et ex factore biquadrato $(f+g\lambda)^4$ huiusmodi pars

$$e^{\frac{-fx}{g}}(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3);$$

et ita porro pro quotcunque factoribus aequalibus, vti ex §. 1120. colligere licet.

## Corollarium 3.

1128. Si factores occurrant imaginarii, bini coniuncti exhibent factorem trinomium realem, cuius forma ita repraesentatur

$$ff+2fg\lambda\,\text{cos}.\,\zeta+gg\lambda\lambda,$$

vnde deducitur

$$\lambda=-\tfrac{f}{g}(\text{cos}.\,\zeta\pm\sqrt{-1}.\,\text{sin}.\,\zeta),$$

quo cum §. 1121. collato, fit $\mu=\frac{-f\,\text{cos}.\,\zeta}{g}$ et $\nu=\frac{f\,\text{sin}.\,\zeta}{g}$. Ex quo

quo pars integralis ex tali factore oriunda erit

$$e^{\frac{-fx\,\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$$

## Corollarium 4.

1129. Si huiusmodi formae quadratum inter factores occurrat

$$(ff+2fg\lambda\cos.\zeta+gg\lambda\lambda)^2,$$

seu duo huiusmodi factores sint aequales, considerentur quasi infinite parum discrepantes, vt in altero loco $\frac{f}{g}$ sit $\frac{f}{g}(1+\omega)$, et ob

$$e^{\frac{-fx\cos.\zeta}{g}(1+\omega)}=e^{\frac{-fx\cos.\zeta}{g}}\left(1-\frac{\omega fx}{g}\cos.\zeta\right),$$

$$\cos.\frac{fx\sin.\zeta}{g}(1+\omega)=\cos.\frac{fx\sin.\zeta}{g}-\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g},\text{ et}$$

$$\sin.\frac{fx\sin.\zeta}{g}(1+\omega)=\sin.\frac{fx\sin.\zeta}{g}+\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}:$$

ex hoc factore colligitur pars integralis

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}\\+\mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g}-\mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}+\mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

cui prior addi debet. Hunc in finem constantes ita contrahamus ponendo

$$\mathfrak{A}+\mathfrak{A}'=\mathfrak{E},\quad\frac{-\mathfrak{A}'\omega f\cos.\zeta}{g}+\frac{\mathfrak{B}'\omega f\sin.\zeta}{g}=\mathfrak{G},$$

$$\mathfrak{B}+\mathfrak{B}'=\mathfrak{F},\quad\frac{-\mathfrak{A}'\omega f\sin.\zeta}{g}-\frac{\mathfrak{B}'\omega f\cos.\zeta}{g}=\mathfrak{H},$$

vnde illae constantes vtique determinantur, eritque pars integralis respondens

$$e^{\frac{-fx\cos.\zeta}{g}}\left[(\mathfrak{E}+\mathfrak{G}x)\cos.\frac{fx\sin.\zeta}{g}+(\mathfrak{F}+\mathfrak{H}x)\sin.\frac{fx\sin.\zeta}{g}\right].$$

Scho-

## Scholion.

1130. En ergo vniuerſam methodum huiusmodi aequationum differentialium integralia inueniendi

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n} = 0$$

ita in compendium contractam. Procedatur vt iſte laterculus indicat

| loco | $y$ | $\frac{\partial y}{\partial x}$ | $\frac{\partial\partial y}{\partial x^2}$ | $\frac{\partial^3 y}{\partial x^3}$ | $\frac{\partial^4 y}{\partial x^4}$ | ..... | $\frac{\partial^n y}{\partial x^n}$ |
|---|---|---|---|---|---|---|---|
| ſcribatur | $1$ | $z$ | $z^2$ | $z^3$ | $z^4$ | ..... | $z^n$ |

vt oriatur haec aequatio algebraica

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots + Nz^n = 0,$$

cuius ſinguli factores reales, ſiue ſimplices ſiue duplicati, notentur, atque inſuper caſus quibus duo pluresue ſunt inter ſe aequales, probe obſeruentur. Tum cuiusmodi partes pro integrali quaeſito ex ſingulis factoribus oriantur, ex ſequente tabella intelligere licet:

| Factores | Partes integralis |
|---|---|
| $f + gz$ | $\mathfrak{A}\,e^{\frac{-fx}{g}}$ |
| $(f + gz)^2$ | $(\mathfrak{A} + \mathfrak{B}x)\,e^{\frac{-fx}{g}}$ |
| $(f + gz)^3$ | $(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2)\,e^{\frac{-fx}{g}}$ |
| $(f + gz)^4$ | $(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3)\,e^{\frac{-fx}{g}}$ |
| $ff + 2fgz\cos.\zeta + ggzz$ | $e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g} + \mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right)$ |
| $(ff + 2fgz\cos.\zeta + ggzz)^2$ | $e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{matrix}(\mathfrak{A} + \mathfrak{B}x)\cos.\frac{fx\sin.\zeta}{g}\\ (\mathfrak{a} + \mathfrak{b}x)\sin.\frac{fx\sin.\zeta}{g}\end{matrix}\right\}$ |
| $(ff + 2fgz\cos.\zeta + ggzz)^3$ | $e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{matrix}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx)\cos.\frac{fx\sin.\zeta}{g}\\ (\mathfrak{a} + \mathfrak{b}x + \mathfrak{c}xx)\sin.\frac{fx\sin.\zeta}{g}\end{matrix}\right\}$ |

etc.

Pro singulis autem factoribus diuersae litterae constantes scribi debent, vt integrale omnibus numeris completum obtineatur.

## Exemplum 1.

1131. *Aequationis differentialis quarti gradus*

$$y - \frac{2\partial y}{\partial x} + \frac{2\partial\partial y}{\partial x^2} - \frac{2\partial^3 y}{\partial x^3} + \frac{2\partial^4 y}{\partial x^4} = 0$$

*integrale completum assignare.*

Hinc oritur aequatio algebraica

$$1 - 2z + 2zz - 2z^3 + z^4 = 0,$$

quae in hos factores resoluitur $(1 - z)^2 (1 + zz)$, quorum prior ob $f = 1$ et $g = -1$ praebet hanc partem integralis $(\mathfrak{A} + \mathfrak{B} x) e^x$, alter vero factor ob $f = 1$, cos. $\zeta = 0$, $g = 1$, et sin. $\zeta = 1$, dat $\mathfrak{A}$ cos. $x + \mathfrak{B}$ sin. $x$. Quare integrale completum, quod quaeritur, erit

$$y = (\mathfrak{A} + \mathfrak{B} x) e^x + \mathfrak{C} \text{ cos. } x + \mathfrak{D} \text{ sin. } x$$

continens quatuor constantes arbitrarias. Quod si velimus, vt posito $x = 0$ fiat $y = 0$, fieri oportet $\mathfrak{A} + \mathfrak{C} = 0$, si etiam $\frac{\partial y}{\partial x}$ eodem casu euanescere debeat, ob

$$\frac{\partial y}{\partial x} = (\mathfrak{A} + \mathfrak{B} + \mathfrak{B} x) e^x - \mathfrak{C} \text{ sin. } x + \mathfrak{D} \text{ cos. } x$$

fieri debet $\mathfrak{A} + \mathfrak{B} + \mathfrak{D} = 0$. Si praeterea $\frac{\partial\partial y}{\partial x^2}$ euanescere debeat, ob

$$\frac{\partial\partial y}{\partial x^2} = (\mathfrak{A} + 2\mathfrak{B} + \mathfrak{B} x) e^x - \mathfrak{C} \text{ cos. } x - \mathfrak{D} \text{ sin. } x$$

fieri debet $\mathfrak{A} + 2\mathfrak{B} - \mathfrak{C} = 0$. Quare his tribus conditionibus satisfaciemus sumendo $\mathfrak{C} = -\mathfrak{A}$, $\mathfrak{B} = -\mathfrak{A}$ et $\mathfrak{D} = 0$, ita vt sit integrale

$$y = \mathfrak{A}(1 - x) e^x - \mathfrak{A} \text{ cos. } x.$$

Exem-

## Exemplum 2.

1132. *Aequationem differentialem quarti ordinis*

$$A y + C \frac{\partial\partial y}{\partial x^2} + E \frac{\partial^4 y}{\partial x^4} = 0,$$

*sumto elemento $\partial x$ constante, integrare.*

Aequatio algebraica ad integrationem perducens est

$$A + C z z + E z^4 = 0,$$

quae semper duos factores duplicatos reales habet, quorum forma duplex esse potest

vel $(aa + 2maz + nzz)(aa - 2maz + nzz)$

vel $(aa + mzz)(aa + nzz)$.

Ex priori est

$$A = a^4, \; C = 2naa - 4mmaa, \; E = nn,$$

ex posteriore vero

$$A = a^4, \; C = (m+n)aa, \; E = mn:$$

semper autem terminum primum A biquadrato $a^4$ repraesentare licet, et prior resolutio locum habet, si E sit numerus positiuus, et $2naa - C$ seu $2\sqrt{AE} - C$ quoque positiuus, ideoque $4AE > CC$. Posterior vero si $CC > 4AE$. Tum igitur videndum est, ad quamnam classem singuli factores pertineant, vnde sequentes casus occurrent.

I. Si omnes quatuor factores simplices sunt reales, erit

$$A + Czz + Ez^4 = (a+z)(a-z)(b+z)(b-z),$$

et haec habebitur aequatio

$$aabby - (aa+bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

cuius integrale completum est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-ax} + \mathfrak{C} e^{bx} + \mathfrak{D} e^{-bx},$$

 Ac

Ac si sit $b = a$, huius aequationis

$$a^4 y - \frac{2aa\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

integrale completum erit

$$y = (\mathfrak{A} + \mathfrak{B} x) e^{ax} + (\mathfrak{C} + \mathfrak{D} x) e^{-ax}.$$

II. Si duo factores simplices sint reales, duo vero imaginarii, erit

$$A + Czz + Ez^4 = (a + z)(a - z)(bb + zz),$$

et haec habebitur aequatio

$$aabby + (aa - bb)\frac{\partial\partial y}{\partial x^2} - \frac{\partial^4 y}{\partial x^4} = 0,$$

cuius integrale completum est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-ax} + \mathfrak{C} \operatorname{cos.} bx + \mathfrak{D} \operatorname{sin.} bx.$$

III. Si omnes factores simplices sint imaginarii, duo casus sunt euoluendi:

1) si $A + Czz + Ez^4 = (aa + zz)(bb + zz)$,

vnde huius aequationis

$$aabby + (aa + bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum erit

$$y = \mathfrak{A} \operatorname{cos.} ax + \mathfrak{B} \operatorname{sin.} ax + \mathfrak{C} \operatorname{cos.} bx + \mathfrak{D} \operatorname{sin.} bx:$$

2) si $A + Czz + Ez^4 = (aa + 2az\operatorname{cos.}\zeta + zz)(aa - 2az\operatorname{cos.}\zeta + zz)$,

vnde huius aequationis

$$a^4 y - \frac{2aa\partial\partial y}{\partial z^2}\operatorname{cos.} 2\zeta + \frac{\partial^4 y}{\partial x^4} = 0,$$

integrale completum est

$$y = e^{+ax\operatorname{cos.}\zeta}(\mathfrak{A} \operatorname{cos.} ax \operatorname{sin.}\zeta + \mathfrak{B} \operatorname{sin.} ax \operatorname{sin.}\zeta)$$
$$+ e^{-ax\operatorname{cos.}\zeta}(\mathfrak{C} \operatorname{cos.} ax \operatorname{sin.}\zeta + \mathfrak{D} \operatorname{sin.} ax \operatorname{sin.}\zeta).$$

At si sit priori casu $b = a$, seu posteriori $\operatorname{cos.}\zeta = 0$, huius aequationis

$a^4 y$

$$a^4 y + \frac{2aa\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum eſt

$$y = (\mathfrak{A} + \mathfrak{B}\, x) \operatorname{cof.} a\, x + (\mathfrak{C} + \mathfrak{D}\, x) \operatorname{fin.} a\, x.$$

## Scholion 1.

1133. Cum igitur aequationis

$$\mathrm{A} y + \frac{\mathrm{C}\partial\partial y}{\partial x^2} + \frac{\mathrm{E}\partial^4 y}{\partial x^4} = 0$$

integrale aſſignari poſſit, omnes aequationes quas inde deriuare licet, integrari poterunt. At haec aequatio per $2\partial y$ multiplicata per integrationem ad differentialem tertii ordinis reducitur

$$\mathrm{A} yy + \frac{\mathrm{C}\partial y^2}{\partial x^2} + \frac{2\mathrm{E}\partial y \partial^3 y - \mathrm{E}\partial\partial y^2}{\partial x^4} = \mathrm{Conſt.}$$

In integrali autem ante inuento conſtantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definire licet, vt haec Conſt. euaneſcat, ideoque huius aequationis

$$\mathrm{A} yy + \frac{\mathrm{C}\partial y^2}{\partial x^2} + \frac{2\mathrm{E}\partial y \partial^3 y - \mathrm{E}\partial\partial y^2}{\partial x^4} = 0$$

integrale completum erit in noſtra poteſtate. Nunc ponatur $y = e^{\int v \partial x}$, vt ſit $v = \frac{\partial y}{y \partial x}$, et ob $\frac{\partial y}{\partial x} = e^{\int v \partial x} v$,

$$\frac{\partial\partial y}{\partial x^2} = e^{\int v \partial x} \left(\frac{\partial v}{\partial x} + v\, v\right), \text{ atque}$$

$$\frac{\partial^3 y}{\partial x^3} = e^{\int v \partial x} \left(\frac{\partial\partial v}{\partial x^2} + \frac{3 v \partial v}{\partial x} + v^3\right),$$

aequatio noſtra hanc induit formam

$$\mathrm{A} + \mathrm{C}\, v\, v + \mathrm{E}\left(\frac{2 v \partial\partial v}{\partial x^2} + \frac{4 v v \partial v}{\partial x} + v^4 - \frac{\partial v^2}{\partial x^2}\right) = 0.$$

Sit porro $\partial x = \frac{\partial v}{s}$, vt ſit

$$s = \frac{\partial v}{\partial x} = \frac{\partial\partial y}{y \partial x^2} - \frac{\partial y^2}{y y \partial x^2}, \text{ erit}$$

$$\frac{\partial\partial v}{\partial x} = \partial s \text{ et } \frac{\partial\partial v}{\partial x^2} = \frac{s \partial s}{\partial v}:$$

vnde reſultat haec aequatio differentialis primi gradus

$$\mathrm{A} + \mathrm{C}\, v\, v + \mathrm{E}\left(\frac{2 v s \partial s}{\partial v} - s\, s + 4\, v\, v\, s + v^4\right) = 0,$$

cuius

cuius relatio inter $v$ et $s$ ita ex relatione inter $x$ et $y$ inuenta definitur, vt fit

$$v = \frac{\partial y}{y \partial x} \quad \text{et} \quad s = \frac{y \partial\partial y - \partial y^2}{y y \partial x^2}.$$

## Scholion 2.

1134. Retenta autem illa constante per integrationem ingressa, vt habeatur

$$A y y + \frac{C \partial y^2}{\partial x^2} + \frac{2E \partial y \partial^3 y - E \partial\partial y^2}{\partial x^4} = G,$$

in integrali completo, quo $y$ per $x$ exprimitur, constantes 𝔄, 𝔅, ℭ, 𝔇, quantitati huic G conformitur determinari poterunt. Nunc igitur ponatur $\partial x = \frac{\partial y}{u}$, vt fit $\frac{\partial y}{\partial x} = u$, erit

$$\frac{\partial\partial y}{\partial x^2} = \frac{u \partial u}{\partial y}, \quad \text{et} \quad \frac{\partial^3 y}{\partial x^2} = \partial . \frac{u \partial u}{\partial y},$$

ideoque $\frac{\partial^3 y}{\partial x^3} = \frac{u}{\partial y} \partial . \frac{u \partial u}{\partial y}$. Vnde obtinetur haec aequatio differentialis secundi gradus

$$A y y + C u u + E \left(\frac{2 u u}{\partial y} \partial . \frac{u \partial u}{\partial y} - \frac{u u \partial u^2}{\partial y^2}\right) = G,$$

vbi consideratio elementi pro constante assumti est exuta. Nihil ergo impedit, quo minus sumamus $\partial y$ pro constante, fietque

$$A y y + C u u + E \left(\frac{2 u^3 \partial\partial u}{\partial y^2} + \frac{u u \partial u^2}{\partial y^2}\right) = G,$$

quae ergo aequatio etiam integrari potest.

Vel si ponamus $y y = p$ et $u u = q$, sumto elemento $\partial p$ constante, prodibit haec aequatio

$$A p + C q + E \left(\frac{4 p q \partial\partial q + 2 q \partial p \partial q - p \partial q^2}{\partial p^2}\right) = G.$$

Vel si in illa aequatione ponatur $u = r^{\frac{2}{3}}$, erit

$$A y y + C r^{\frac{4}{3}} + \tfrac{4}{3} E r^{\frac{5}{3}} . \frac{\partial\partial r}{\partial y^2} = G.$$

Quarum formarum integratio sine hoc subsidio maxime ardua videtur.

Pro-

## Problema 146.

1135. Propofita aequatione differentiali ordinis cuiuscunque $a^n y \pm \frac{\partial^n y}{\partial x^n} = 0$, vbi elementum $\partial x$ conftans eft affumtum, eius integrale completum inueftigare.

## Solutio.

Aequatio algebraica folutioni inferuiens eft $a^n \pm z^n = 0$, pro cuius refolutione duos cafus confiderari conuenit, prout fignum vel fuperius vel inferius valeat.

I. Valeat fuperius, vt haec propofita fit aequatio

$$a^n y + \frac{\partial^n y}{\partial x^n} = 0,$$

et formulae $a^n + z^n$ factores reales funt

$$aa - 2az\cos.\tfrac{\pi}{n} + zz,\quad aa - 2az\cos.\tfrac{3\pi}{n} + zz,$$

$$aa - 2az\cos.\tfrac{5\pi}{n} + zz,\ \text{etc.}$$

quorum vltimus eft vel $aa - 2az\cos.\frac{n\pi}{n} + zz$ vel

$$aa - 2az\cos.\tfrac{(n-1)\pi}{n} + zz,$$

prout vel $n$ vel $n - 1$ fuerit numerus impar, atque illo quidem cafu loco factoris quadrati $aa + 2az + zz$ eius radix $a + z$ fumi debet.

Hinc iftius aequationis integrale completum eft

$$y = + e^{ax\cos.\frac{\pi}{n}} (\mathfrak{A}\cos.ax\sin.\tfrac{\pi}{n} + \mathfrak{B}\sin.ax\sin.\tfrac{\pi}{n})$$

$$+ e^{ax\cos.\frac{3\pi}{n}} (\mathfrak{C}\cos.ax\sin.\tfrac{3\pi}{n} + \mathfrak{D}\sin.ax\sin.\tfrac{3\pi}{n})$$

$$+ e^{ax\cos.\frac{5\pi}{n}} (\mathfrak{E}\cos.ax\sin.\tfrac{5\pi}{n} + \mathfrak{F}\sin.ax\sin.\tfrac{5\pi}{n}),$$

etc.

cuius

cuius expreſſionis, ſi $n$ ſit numerus impar, vltima pars ſit $\mathfrak{N}\, e^{-ax}$. Quod integrale etiam ita poteſt exhiberi

$$y = \mathfrak{A} e^{ax \cos.\frac{\pi}{n}} \cos.\left(ax \sin.\tfrac{\pi}{n} + \mathfrak{a}\right) + \mathfrak{B} e^{ax \cos.\frac{3\pi}{n}} \cos.\left(ax \sin.\tfrac{3\pi}{n} + \mathfrak{b}\right)$$
$$+ \mathfrak{C} e^{ax \cos.\frac{5\pi}{n}} \cos.\left(ax \sin.\tfrac{5\pi}{n} + \mathfrak{c}\right) + \mathfrak{D} e^{ax \cos.\frac{7\pi}{n}} \cos.\left(ax \sin.\tfrac{7\pi}{n} + \mathfrak{d}\right)$$
$$+ \text{etc.}$$

quae forma eousque continuari debet, quoad ſimiles termini recurrant.

II. Si valeat ſignum inferius, propoſitaque ſit haec aequatio

$$a^n y - \frac{\partial^n y}{\partial x^n} = 0,$$

formulae $a^n - z^n$ factores reales ſunt

$$a - z,\ aa - 2z \cos.\tfrac{2\pi}{n} + zz,\ aa - 2az \cos.\tfrac{4\pi}{n} + zz,$$
$$aa - 2az \cos.\tfrac{6\pi}{n} + zz, \text{ etc.}$$

quorum ſi $n$ numerus par, vltimus eſt $a + z$, ſin autem impar

$$aa - 2az \cos.\tfrac{(n-1)\pi}{n} + zz.$$

Quare aequationis huius integrale completum eſt

$$y = \mathfrak{A} e^{ax} + e^{ax \cos.\frac{2\pi}{n}} \left(\mathfrak{B} \cos.ax \sin.\tfrac{2\pi}{n} + \mathfrak{C} \sin.ax \sin.\tfrac{2\pi}{n}\right)$$
$$+ e^{ax \cos.\frac{4\pi}{n}} \left(\mathfrak{D} \cos.ax \sin.\tfrac{4\pi}{n} + \mathfrak{E} \sin.ax \sin.\tfrac{4\pi}{n}\right)$$
$$+ e^{ax \cos.\frac{6\pi}{n}} \left(\mathfrak{F} \cos.ax \sin.\tfrac{6\pi}{n} + \mathfrak{G} \sin.ax \sin.\tfrac{6\pi}{n}\right),$$
$$\text{etc.}$$

quod integrale etiam ita exprimi poteſt

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{ax \cos.\frac{2\pi}{n}} \cos.\left(ax \sin.\tfrac{2\pi}{n} + \mathfrak{b}\right)$$
$$+ \mathfrak{C} e^{ax \cos.\frac{4\pi}{n}} \cos.\left(ax \sin.\tfrac{4\pi}{n} + \mathfrak{c}\right) + \mathfrak{D} e^{ax \cos.\frac{6\pi}{n}} \cos.\left(ax \sin.\tfrac{6\pi}{n} + \mathfrak{d}\right),$$
$$\text{etc.}$$

quae

quae forma eousque est continuanda, quamdiu termini a prioribus diuersi prodeunt.

## Scholion 1.

1136. Pro variis ergo exponentis $n$ valoribus integralia sequenti modo se habebunt, ac primo quidem pro aequatione $a^n y + \frac{\partial^n y}{\partial x^n} = 0$.

I. Aequationis $a y + \frac{\partial y}{\partial x} = 0$ integrale est

$$y = \mathfrak{A} e^{-ax}.$$

II. Aequationis $a^2 y + \frac{\partial \partial y}{\partial x^2} = 0$ integrale est

$$y = \mathfrak{A} \operatorname{cos.} (a x + \mathfrak{a}).$$

III. Aequationis $a^3 y + \frac{\partial^3 x}{\partial x^3} = 0$ integrale est

$$y = \mathfrak{A} e^{\frac{1}{2} a x} \operatorname{cos.} (\tfrac{a x \sqrt{3}}{2} + \mathfrak{a}) + \mathfrak{B} e^{-ax}.$$

IV. Aequationis $a^4 y + \frac{\partial^4 y}{\partial x^4} = 0$ integrale est

$$y = \mathfrak{A} e^{\frac{a x}{\sqrt{2}}} \operatorname{cos.} (\tfrac{a x}{\sqrt{2}} + \mathfrak{a}) + \mathfrak{B} e^{\frac{-a x}{\sqrt{2}}} \operatorname{cos.} (\tfrac{a x}{\sqrt{2}} + \mathfrak{b}).$$

V. Aequationis $a^5 y + \frac{\partial^5 y}{\partial x^5} = 0$ integrale est

$$y = \mathfrak{A} e^{a x \operatorname{cos.} 36^\circ} \operatorname{cos.} (a x \operatorname{sin.} 36^\circ + \mathfrak{a})$$
$$+ \mathfrak{B} e^{-a x \operatorname{cos.} 72^\circ} \operatorname{cos.} (a x \operatorname{sin.} 72^\circ + \mathfrak{b}) + \mathfrak{C} e^{-ax}.$$

VI. Aequationis $a^6 y + \frac{\partial^6 y}{\partial x^6} = 0$ integrale est

$$y = \mathfrak{A} e^{\frac{a x \sqrt{3}}{2}} \operatorname{cos.} (\tfrac{1}{2} a x + \mathfrak{a})$$
$$+ \mathfrak{B} \operatorname{cos.} (a x + \mathfrak{b}) + \mathfrak{C} e^{\frac{-a x \sqrt{3}}{2}} \operatorname{cos.} (\tfrac{1}{2} a x + \mathfrak{c}),$$

etc.

Simili autem modo pro altera forma

$$a^n y - \frac{\partial^n y}{\partial x^n} = 0,$$

integrationes ad valores fimpliciores exponentis $n$ accommodatae ita fe habebunt

I. Aequationis $a y - \frac{\partial y}{\partial x} = 0$ integrale eft
$$y = \mathfrak{A}\, e^{a x}.$$

II. Aequationis $a^2 y - \frac{\partial \partial y}{\partial x^2} = 0$ integrale eft
$$y = \mathfrak{A}\, e^{a x} + \mathfrak{B}\, e^{-a x}.$$

III. Aequationis $a^3 y - \frac{\partial^3 y}{\partial x^3} = 0$ integrale eft
$$y = \mathfrak{A}\, e^{a x} + \mathfrak{B}\, e^{-\frac{1}{2} a x} \operatorname{cof.} \left(\frac{a x \surd 3}{2} + \mathfrak{b}\right).$$

IV. Aequationis $a^4 y - \frac{\partial^4 y}{\partial x^4} = 0$ integrale eft
$$y = \mathfrak{A}\, e^{a x} + \mathfrak{B} \operatorname{cof.} (a x + \mathfrak{b}) + \mathfrak{C}\, e^{-a x}.$$

V. Aequationis $a^5 y - \frac{\partial^5 y}{\partial x^5} = 0$ integrale eft
$$y = \mathfrak{A}\, e^{a x} + \mathfrak{B}\, e^{a x \operatorname{cof.} 72^\circ} \operatorname{cof.} (a x \operatorname{fin.} 72^\circ + \mathfrak{b})$$
$$+ \mathfrak{C}\, e^{-a x \operatorname{cof.} 36^\circ} \operatorname{cof.} (a x \operatorname{fin.} 36^\circ + \mathfrak{c}).$$

VI. Aequationis $a^6 y - \frac{\partial^6 y}{\partial x^6} = 0$ integrale eft
$$y = \mathfrak{A}\, e^{a x} + \mathfrak{B}\, e^{\frac{1}{2} a x} \operatorname{cof.} \left(\frac{a x \surd 3}{2} + \mathfrak{b}\right)$$
$$+ \mathfrak{C}\, e^{-\frac{1}{2} a x} \operatorname{cof.} \left(\frac{a x \surd 3}{2} + \mathfrak{c}\right) + \mathfrak{D}\, e^{-a x},$$

ficque quousque libuerit progredi licet

## Scholion 2.

1137. Quamuis methodus, qua hic fum vfus, expedite integralia aequationum in propofita forma contentarum fuppeditet, a principiis tamen integrationis omnino abhorret. Cum enim aequatio differentialis eft altioris gradus, leges integrationis poftulant, vt toties feorfim integretur, antequam ad relationem finitam inter binas variabiles perueniatur, et dum fin-ſingulae integrationes conſtantem arbitrariam recipiunt, hoc demum modo integrale completum obtinetur. Hactenus autem vna quaſi operatione integrale poſtremum eruimus, cum omnibus conſtantibus, quibus id completum redditur; reuera ſcilicet ſola coniectura vtentes plura integralia particularia ſumus adepti, atque natura aequationis commode permiſit, vt ex iis integrale completum formare liceret. Verum ſi leges integrandi ſtricte obſeruare velimus, propoſita verbi gratia aequatione differentiali quarti gradus, quadruplici integratione opus erit, quarum prima ea reducatur ad aequationem differentialem tertii gradus, tum vero iſta per nouam integrationem ad aequationem differentialem ſecundi gradus, quae denuo integrata ad gradum primum perducatur, haecque tandem iterum integrata relationem quaeſitam inter binas variabiles patefaciat. Atque hoc modo etiam aequationum hic tractatarum formam reſoluere licet, vt per continuas integrationes ad gradus ſimpliciores redigatur, quibus tandem eadem integralia, quae hic elicuimus, inueniantur. Cum autem haec methodus latius pateat, quam ad formas hic conſideratas, eiusque ope haec aequatio generalior integrari queat

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \text{etc.}$$

denotante X functionem quamcunque ipſius $x$, cui reſoluendae praecedens methodus minime ſufficit, nouam methodum ſtatim ad hanc formam generaliorem accommodabo.

# CAPVT III.

## DE

## INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM HVIVS FORMAE

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \text{etc.}$$

### Problema 147.

1138.

Proposita aequatione differentiali

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \ldots\ldots + \frac{N\partial^n y}{\partial x^n},$$

sumto elemento $\partial x$ constante, et significante X functionem quamcunque ipsius $x$, inuenire functionem ipsius $x$, per quam haec aequatio multiplicata integrabilis euadat.

### Solutio.

Sit $P\partial x$ iste multiplicator quem quaerimus, et cum prius membrum X eo integrabile reddatur, eius rationem ex altero membro definiri oportet. Facile autem intelligitur, formam huius multiplicatoris P eiusmodi fore $e^{\lambda x}$, ita vt quantitas $\lambda$ definiri debeat. Sit ergo $e^{\lambda x}\partial x$ multiplicator, atque hanc formam

$$e^{\lambda}\partial x\,(Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \ldots\ldots + \frac{N\partial^n y}{\partial x^n},$$

integrabilem esse oportet, cuius integrale propterea statuatur

$e^{\lambda x}$

$$e^{\lambda x}\left(A'y + \frac{B'\partial y}{\partial x} + \frac{C'\partial\partial y}{\partial x^2} + \ldots\ldots + \frac{M'\partial^{n-1}y}{\partial x^{n-1}}\right),$$

ita vt huius differentiale cum illa forma congruere debet, quod cum fit

$$e^{\lambda x}\partial x\left\{\begin{matrix}\lambda A'y + \frac{\lambda B'\partial y}{\partial x} + \frac{\lambda C'\partial\partial y}{\partial x^2} + \ldots\ldots + \frac{\lambda M'\partial^{n-1}y}{\partial x^n} \\ + \frac{A'\partial y}{\partial x} + \frac{B'\partial\partial y}{\partial x^2} + \ldots\ldots\ldots\ldots\ldots + \frac{M'\partial^n y}{\partial x^n}\end{matrix}\right\},$$

necesse est fit

$$A' = \frac{A}{\lambda},\ B' = \frac{B - A'}{\lambda},\ C' = \frac{C - B'}{\lambda},\ D' = \frac{D - C'}{\lambda};\ \ldots$$

$$\ldots M' = \frac{M - L'}{\lambda},$$ atque $M' = N$. Hinc erit

$$A' = \frac{A}{\lambda},$$

$$B' = \frac{B}{\lambda} - \frac{A}{\lambda^2},$$

$$C' = \frac{C}{\lambda} - \frac{B}{\lambda^2} + \frac{A}{\lambda^3},$$

$$D' = \frac{D}{\lambda} - \frac{C}{\lambda^2} + \frac{B}{\lambda^3} - \frac{A}{\lambda^4},$$

$$\vdots$$

$$M' = \frac{M}{\lambda} - \frac{L}{\lambda^2} + \frac{K}{\lambda^3} \ldots\ldots \pm \frac{A}{\lambda^n},$$ et

$$0 = \frac{N}{\lambda} - \frac{M}{\lambda^2} + \frac{L}{\lambda^3} \ldots\ldots \mp \frac{A}{\lambda^{n+1}},$$

vbi ex vltima aequatione quantitas $\lambda$ erui debet, quae aequatio induit hanc formam

$$A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 \ldots\ldots \pm N\lambda^n = 0,$$

vnde cum $\lambda$ sortiatur $n$ valores, totidem quoque multiplicatores inueniuntur.

 Videa-

Videamus, quomodo hae determinationes pro singulis valoribus exponentis $n$ se habeant.

I. Si $n = 1$; erit $A - B\lambda = 0$, tum vero

$$A' = \frac{A}{\lambda} = B.$$

II. Si $n = 2$; erit $A - B\lambda + C\lambda^2 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C.$$

III. Si $n = 3$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 = 0$ tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2,$$
$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda \text{ et}$$
$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D.$$

IV. Si $n = 4$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3,$$
$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2,$$
$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda,$$
$$D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E.$$

V. Si $n = 5$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 - F\lambda^5 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3 + F\lambda^4$$
$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2 - F\lambda^3$$
$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda + F\lambda^2$$
$$D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E - F\lambda$$
$$E' = \frac{E\lambda^4 - D\lambda^3 + C\lambda^2 - B\lambda + A}{\lambda^5} = F$$

atque ita porro.

In-

Inuento autem hoc multiplicatore $e^{\lambda x}\partial x$, prius aequationis membrum fit $\int e^{\lambda x} X \partial x$, et aequatio propofita, quae eft differentialis gradus $n$, per integrationem reducitur ad hanc vno gradu fimpliciorem

$$\int e^{\lambda x} X \partial x = e^{\lambda x}\left(A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + M'\frac{\partial^{n-1} y}{\partial x^{n-1}}\right).$$

## Corollarium 1.

1139. Integratione ergo hac prima inftituta, aequatio propofita vno gradu deprimitur, et definitis coefficientibus $A'$, $B'$, $C'$, etc. ex fuperioribus formulis, aequatio integralis hac forma exhiberi poteft

$$e^{-\lambda x}\int e^{\lambda x} X \partial x = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + M'\frac{\partial^{n-1} y}{\partial x^{n-1}}.$$

## Corollarium 2.

1140. Cum prius membrum $e^{-\lambda x}\int e^{\lambda x} X \partial x$ fit functio ipfius $x$ conftantem arbitrariam inuoluens, fi eius loco ponatur $X'$, haec aequatio fimilem formam habet atque ipfa propofita, ideoque eadem methodo iterum integrari et ad gradum differentialitatis $n - 2$ reduci poteft, quae huiusmodi formam habebit

$$X'' = A''y + B''\frac{\partial y}{\partial x} + C''\frac{\partial\partial y}{\partial x^2} + \ldots\ldots\ldots + L''\frac{\partial^{n-2} y}{\partial x^{n-2}}.$$

## Corollarium 3.

1141. Hoc modo vlterius progrediendo tandem ad aequationem differentialem primi gradus peruenietur

$$X^{(n-1)} = A^{(n-1)}y + B^{(n-1)}\frac{\partial y}{\partial x},$$

quae fimili modo ad aequationem finitam $X^{(n)} = A^{(n)}y$ reducitur, qua relatio inter ipfas variabiles $x$ et $y$ exprimitur.

Scho-

## Scholion.

1142. Haec igitur est methodus huiusmodi aequationes differentiales altiorum graduum successiue per gradus integrandi, vbi tot opus est integrationibus, quoti gradus differentialis fuerit ipsa aequatio proposita. Totum ergo negotium situm est in inuentione successiua coefficientium, quos ex praecedentibus ope multiplicatoris definiri oportet. In genere quidem lex, qua ii continuo ex antecedentibus determinantur, non ita est perspicua, vt inde forma integrabilis extremi perspici possit; verum quia ex capite superiori nouimus, casu quo primum membrum X euanescit, etiam vltimum integrale lege satis simplici contineri, idem hic vsu venire merito suspicamur, eamque legem facillime agnoscemus, si pedetentim a gradibus inferioribus ad altiores progrediamur. Ac primo quidem casu, quo aequatio est differentialis primi gradus $X = Ay + B\frac{\partial y}{\partial x}$, multiplicator erit $e^{\lambda x}\partial x$, posito $A - \lambda B = 0$, vt sit $\lambda = \frac{A}{B}$, et cum sit $A' = \frac{A}{\lambda} = B$, integrale erit

$$\int e^{\lambda x} X \partial x = B e^{\lambda x} y \text{ seu } e^{-\lambda x}\int e^{\lambda x} X \partial x = By.$$

Ad hanc similitudinem aequationes graduum altiorum euoluamus, ac formam integralis vltimi inuestigemus.

## Problema 148.

1143. Proposita aequatione differentiali secundi gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2},$$

per duplicem integrationem relationem inter $x$ et $y$ inuestigare.

## Solutio.

Sit $e^{\lambda x}\partial x$ multiplicator hanc aequationem per se integrabilem reddens, eritque $A - B\lambda + C\lambda^2 = 0$, tum sumatur

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C,$$

posi-

positoque

$$e^{-\lambda x}\int e^{\lambda x} X \partial x = X',$$

aequatio semel integrata est

$$X' = A' y + B' \frac{\partial y}{\partial x}.$$

Huius iam multiplicator sit $e^{\mu x} \partial x$, eritque $A' - B' \mu = 0$, ac statuatur $A'' = \frac{A'}{\mu} = B'$; positoque $e^{-\mu x} \int e^{\mu x} X' \partial x = X''$ habebimus $X'' = A'' y$, quae est aequatio bis integrata relationem quaesitam inter $x$ et $y$ exprimens.

Cum igitur hic sit $A'' = B'$ et $B' = C$, erit $A'' = C$. Deinde loco $A'$ et $B'$ substitutis valoribus, aequatio $A' - B' \mu = 0$ induit hanc formam

$$B - C\lambda - C\mu = 0, \text{ seu } B - C(\lambda + \mu) = 0,$$

ex qua cum sit $\lambda + \mu = \frac{B}{C}$, patet $\lambda + \mu$ aequari summae binarum radicum aequationis $A - B\lambda + C\lambda^2 = 0$. Quoniam igitur $\lambda$ eius vna est radix, $\mu$ necessario eius alteram radicem denotat. Quare si ex aequatione proposita, vti in capite praecedente fecimus, hanc formemus aequationem $A + Bz + Czz = 0$, eius radices erunt $z = -\lambda$ et $z = -\mu$. Seu si factores eius statuamus $C(\alpha + z)(\beta + z)$, litterae $\alpha$ et $\beta$ praebebunt valores $\lambda$ et $\mu$. Hinc cum sit

$$X' = e^{-\alpha x}\int e^{\alpha x} X \partial x \text{ erit}$$

$$X'' = e^{-\beta x}\int e^{(\beta - \alpha) x} \partial x \int e^{\alpha x} X \partial x. \text{ At}$$

$$\int e^{(\beta-\alpha)x}\partial x \int e^{\alpha x} X \partial x = \frac{1}{\beta - \alpha} e^{(\beta-\alpha)x}\int e^{\alpha x} X \partial x - \frac{1}{\beta-\alpha}\int e^{\beta x} X \partial x;$$

vnde concluditur

$$X'' = \frac{1}{\beta-\alpha} e^{-\alpha x}\int e^{\alpha x} X \partial x + \frac{1}{\alpha - \beta} e^{-\beta x}\int e^{\beta x} X \partial x.$$

Quocirca aequationis propositae integrale completum est

$$Cy = \frac{1}{\beta-\alpha} e^{-\alpha x}\int e^{\alpha x} X \partial x + \frac{1}{\alpha-\beta} e^{-\beta x}\int e^{\beta x} X \partial x$$

 vbi

vbi litterae $\alpha$ et $\beta$ ita ſunt capiendae, vt ſit

$$A + Bz + Czz = C(\alpha + z)(\beta + z).$$

## Corollarium 1.

1144. Si bini hi factores ſint aequales, ſeu $\beta = \alpha$, erit

$$X'' = e^{-\alpha x}\int \partial x \int e^{\alpha x} X \partial x = e^{-\alpha x} x \int e^{\alpha x} X \partial x - e^{-\alpha x}\int e^{\alpha x} X x \partial x,$$

ideoque caſu

$$A + Bz + Czz = C(\alpha + z)^2,$$

aequationis noſtrae integrale eſt

$$Cy = e^{-\alpha x}(x\int e^{\alpha x} X \partial x - \int e^{\alpha x} X x \partial x).$$

## Corollarium 2.

1145. Si bini factores ſint imaginarii, quod euenit ſi

$A + Bz + Czz = C(ff + 2fz \operatorname{cof.} \theta + zz)$, erit

$\alpha = f(\operatorname{cof.} \theta + \sqrt{-1}.\operatorname{fin.} \theta)$ et

$\beta = f(\operatorname{cof.} \theta - \sqrt{-1}.\operatorname{fin.} \theta)$ hinc

$e^{\alpha x} = e^{fx \operatorname{cof.} \theta}(\operatorname{cof.} fx \operatorname{fin.} \theta + \sqrt{-1}.\operatorname{fin.} fx \operatorname{fin.} \theta)$ et

$e^{\beta x} = e^{fx \operatorname{cof.} \theta}(\operatorname{cof.} fx \operatorname{fin.} \theta - \sqrt{-1}.\operatorname{fin.} fx \operatorname{fin.} \theta)$, atque

$\beta - \alpha = -2\sqrt{-1}.f \operatorname{fin.} \theta$.

## Corollarium 3.

1146. Quo haec facilius ſubſtituere queamus, ſit breuitatis gratia

$e^{fx \operatorname{cof.} \theta} = m$, $\operatorname{cof.} fx \operatorname{fin.} \theta = p$, et $\operatorname{fin.} fx \operatorname{fin.} \theta = q$,

vt ſit

$e^{\alpha x} = mp + mq\sqrt{-1}$, et $e^{\beta x} = mp - mq\sqrt{-1}$.

Hinc fit

$\int e^{\alpha x} X \partial x = \int mpX\partial x + \int mqX\partial x \sqrt{-1}$ et

$\int e^{\beta x} X \partial x = \int mpX\partial x - \int mqX\partial x \sqrt{-1}$.

Tum

Tum vero eſt

$$e^{-\alpha x} = \frac{p - q\sqrt{-1}}{m}, \text{ et } e^{-\beta x} = \frac{p + q\sqrt{-1}}{m}.$$

## Corollarium 4.

1147. Ex his colligimus

$$e^{-\alpha x}\int e^{\alpha x} X \partial x = \frac{p}{m}\int m p X \partial x - \frac{q\sqrt{-1}}{m}\int m p X \partial x$$
$$+ \frac{p\sqrt{-1}}{m}\int m q X \partial x + \frac{q}{m}\int m q X \partial x;$$

et ſumto $\sqrt{-1}$ negatiuo, prodit $e^{-\beta x}\int e^{\beta x} X \partial x$, quae forma inde ſubtracta relinquit

$$-\frac{2q\sqrt{-1}}{m}\int m p X \partial x + \frac{2p\sqrt{-1}}{m}\int m q X \partial x;$$

hocque reſiduum diuidi debet per

$$\beta - \alpha = -2\sqrt{-1}.f\,\text{ſin.}\,\theta.$$

Vnde integrale colligitur

$$Cy = \frac{q}{m f \text{ſin.}\theta}\int m p X \partial x - \frac{p}{m f \text{ſin.}\theta}\int m q X \partial x.$$

## Corollarium 5.

1148. Reſtituantur pro $m$, $p$, $q$ valores aſſumti, atque aequationis noſtrae, ſi fuerit

$$A + Bz + Czz = C(ff + 2fz\,\text{coſ.}\,\theta + zz)$$

integrale erit

$$Cy = e^{-fx\text{coſ.}\theta}\left(\frac{\text{ſin.}fx\,\text{ſin.}\theta}{f\text{ſin.}\theta}\int e^{fx\text{coſ.}\theta} X \partial x\,\text{coſ.}fx\,\text{ſin.}\theta - \frac{\text{coſ.}fx\,\text{ſin.}\theta}{f\text{ſin.}\theta}\int e^{fx\text{coſ.}\theta} X \partial x\,\text{ſin.}fx\,\text{ſin.}\theta\right),$$

quae ergo expreſſio aequiualet illi, ſi $\alpha$ et $\beta$ valores imaginarios obtineant.

## Problema 149.

1149. Propoſita aequatione differentiali tertii gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3}$$

per triplicem integrationem eius integrale completum inuenire.

## Solutio.

Posito multiplicatore $e^{\lambda x}\partial x$, debet esse

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0,$$

tum sumatur

$$A' = B - C\lambda + D\lambda^2,\ B' = C - D\lambda \text{ et } C' = D,$$

positoque

$$e^{-\lambda x}\int e^{\lambda x} X \partial x = X',$$

aequatio semel integrata praebet

$$X' = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2}.$$

Huius porro multiplicator statuatur $e^{\mu x}\partial x$, vt sit

$$A' - B'\mu + C'\mu^2 = 0,$$

sumaturque

$$A'' = B' - C'\mu \text{ et } B'' = C',$$

et posito

$$e^{-\mu x}\int e^{\mu x} X'\partial x = X'',$$

aequatio secunda integralis est

$$X'' = A''y + B''\frac{\partial y}{\partial x},$$

cuius multiplicator erit $e^{\nu x}\partial x$, sumendo $A'' - B''\nu = 0$, at posito $A''' = B''$, erit aequatio integralis tertia

$$e^{-\nu x}\int e^{\nu x} X''\partial x = A'''y = Dy,$$

quaeri ergo oportet quantitates $\lambda$, $\mu$, $\nu$. Est vero primo

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0, \text{ tum}$$

$$B - C(\lambda + \mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0,$$

et ob

$$A'' = C - D(\lambda + \mu) \text{ et } B'' = D,$$

erit tertio

$$C - D(\lambda + \mu + \nu) = 0;$$

ex

ex qua postrema aequalitate patet, $\lambda+\mu+\nu$ aequari summae radicum aequationis primae, cuius $\lambda$ est vna radix. Quod autem $\mu$ et $\nu$ sint reliquae radices, hoc modo ostenditur. Consideretur aequatio

$$A+Bz+Cz^2+Dz^3=0,$$

cuius si vna radix sit $z=-\lambda$, seu $\lambda+z$ vnus factor, diuidatur per eum aequatio, ac prodibit

$$Dz^2+(C-D\lambda)z+B-C\lambda+D\lambda\lambda=0,$$

quae est ipsa aequatio secunda $C'zz+B'z+A'=0$, cuius radices sunt $z=-\mu$ et $z=-\nu$, seu factores $(\mu+z)$ $(\nu+z)$, vti in problemate praecedente ostendimus. Quare si formulae

$$A+Bz+Cz^2+Dz^3,$$

factores sint

$$D(\alpha+z)(\beta+z)(\gamma+z),$$

pro integrali vltimo inueniendo ponatur

$$e^{-\alpha x}\int e^{\alpha x}X\partial x=X',\quad e^{-\beta x}\int e^{\beta x}X'\partial x=X'',\text{ et}$$
$$e^{-\gamma x}\int e^{\gamma x}X''\partial x=X''',$$

eritque $Dy=X'''$. Verum per reductionem integralium est, vti supra vidimus

$$X''=\frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}X\partial x+\frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}X\partial x,$$

hincque porro

$$\int e^{\gamma x}X''\partial x=\frac{1}{(\beta-\alpha)(\gamma-\alpha)}e^{(\gamma-\alpha)x}\int e^{\alpha x}X\partial x-\frac{1}{(\beta-\alpha)(\gamma-\alpha)}\int e^{\gamma x}X\partial x$$
$$+\frac{1}{(\alpha-\beta)(\gamma-\beta)}e^{(\gamma-\beta)x}\int e^{\beta x}X\partial x-\frac{1}{(\alpha-\beta)(\gamma-\beta)}\int e^{\gamma x}X\partial x,$$

vbi bini postremi termini contrahuntur in

$$\frac{1}{(\alpha-\gamma)(\gamma-\beta)}\int e^{\gamma x}X\partial x.$$

Quamobrem integrale quaesitum est

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x}X\partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x}X\partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x}X\partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

## Corollarium 1.

1150. Si formulae $A + Bz + Cz^2 + Dz^3$ duo factores fuerint aequales, puta $\gamma = \beta$, erit

$$\int e^{\beta x}X''\partial x = \frac{1}{(\beta-\alpha)^2}e^{(\beta-\alpha)x}\int e^{\alpha x}X\partial x - \frac{1}{(\beta-\alpha)^2}\int e^{\beta x}X\partial x$$
$$+ \frac{1}{\alpha-\beta}x\int e^{\beta x}X\partial x - \frac{1}{\alpha-\beta}\int e^{\beta x}Xx\partial x,$$

ideoque integrale hoc casu erit

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x}X\partial x - e^{-\beta x}\int e^{\beta x}X\partial x}{(\beta-\alpha)^2} + \frac{e^{-\beta x}x\int e^{\beta x}X\partial x - e^{-\beta x}\int e^{\beta x}Xx\partial x}{\alpha-\beta}.$$

## Corollarium 2.

1151. Si omnes tres factores sint aequales, seu $\alpha = \beta = \gamma$, erit

$$e^{\alpha x}X'' = \int\partial x\int e^{\alpha x}X\partial x = x\int e^{\alpha x}X\partial x - \int e^{\alpha x}Xx\partial x, \text{ et}$$
$$e^{\alpha x}X''' = \int e^{\alpha x}X''\partial x = \int\partial x\int\partial x\int e^{\alpha x}X\partial x, \text{ seu}$$
$$e^{\alpha x}X''' = \tfrac{1}{2}xx\int e^{\alpha x}X\partial x - x\int e^{\alpha x}Xx\partial x + \tfrac{1}{2}\int e^{\alpha x}Xxx\partial x,$$

vnde integrale hoc casu erit

$$Dy = \tfrac{1}{2}e^{-\alpha x}(xx\int e^{\alpha x}X\partial x - 2x\int e^{\alpha x}Xx\partial x + \int e^{\alpha x}Xxx\partial x),$$

seu

$$Dy = e^{-\alpha x}\int\partial x\int\partial x\int e^{\alpha x}X\partial x.$$

## Scholion.

1152. In genere etiam nulla integralium reductione adhibita integrale nostrae aequationis ita exprimi potest, vt sit

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}\partial x\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

posito

$$A + Bz + Cz^2 + Dz^3 = D(\alpha+z)(\beta+z)(\gamma+z),$$

vbi

vbi imprimis notatu dignum occurrit, quod ternas litteras $\alpha$, $\beta$, $\gamma$ quomodocunque inter se permutare licet, ita vt haec integralis expressio sex modis variari possit. In problemate etiam praecedente, vbi duo tantum factores occurrunt

$$C(\alpha + z)(\beta + z) = A + Bz + Cz^2,$$

aequationis

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2}$$

integrale completum ita exhiberi potest

$$Cy = e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

ac permutatis litteris $\alpha$ et $\beta$ etiam hoc modo

$$Cy = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{\beta x}X\partial x.$$

Quarum formularum aequalitas si fuerit perspecta, id quod tentanti facile patebit, praecedentium quoque variationem declarat. Sit enim $e^{-\alpha x}\int e^{\alpha x}X\partial x = X'$, erit pro superiori formula

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}\partial x\int e^{\beta x}X'\partial x,$$

cui cum aequalis sit ista

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}\partial x\int e^{\gamma x}X'\partial x,$$

erit etiam pro $X'$ valore restituto

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}\partial x\int e^{(\gamma-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

quae a prima hoc tantum differt, quod litterae $\beta$ et $\gamma$ sunt permutatae. Quod autem etiam litterae $\beta$ et $\gamma$ cum $\alpha$ permutari queant, hoc modo difficilius ostenditur, ex reductione autem in solutione adhibita, atque adeo ex ipsa solutionis indole per se est manifestum.

## Problema 150.

1153. Proposita aequatione differentiali quarti gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4},$$

sumto

ſumto elemento $\partial x$ conſtante, et denotante X functionem quamcunque ipſius $x$, eius integrale inueſtigare.

## Solutio.

In ſubſidium vocetur formula algebraica ex aequatione propoſita facile formanda

$$A + Bz + Czz + Dz^3 + Ez^4 = P,$$

quae in factores ſuos ſimplices reſoluatur, vt ſit

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

et multiplicator aequationem noſtram integrabilem reddens erit $e^{\lambda x}\partial x$, ſumendo $\lambda$ aequali vni litterarum $\alpha$, $\beta$, $\gamma$, $\delta$; ſumatur ergo $\lambda = \alpha$, vt ſit multiplicator $e^{\alpha x}\partial x$, atque poſito $e^{-\alpha x}\int e^{\alpha x} X \partial x = X'$, aequatio ſemel integrata erit

$$X' = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2} + D'\frac{\partial^3 y}{\partial x^3},$$

vbi A′, B′, C′, D′ ita determinantur, vt ſit

$$A' = \frac{A}{\alpha},\ B' = \frac{-A + B\alpha}{\alpha^2},\ C' = \frac{C\alpha^2 - B\alpha + A}{\alpha^3},\ D' = \frac{D\alpha^3 - C\alpha^2 + B\alpha - A}{\alpha^4},$$

ſeu

$$A' = \frac{A}{\alpha},\ B' = \frac{B - A'}{\alpha},\ C' = \frac{C - B'}{\alpha},\ D' = \frac{D - C'}{\alpha},$$

vel etiam

$$A = A'\alpha,\ B = B'\alpha + A',\ C = C'\alpha + B',\ D = D'\alpha + C'.$$

Ex quibus determinationibus liquet, ſi ponatur

$$A' + B'z + C'z^2 + D'z^3 = Q,$$

hanc formulam Q naſci ex formula P, ſi haec per $\alpha + z$ diuidatur, ita vt ſit

$$Q = \frac{P}{\alpha + z} = E(\beta + z)(\gamma + z)(\delta + z).$$

Eodem ergo modo ſecundam integrationem inſtituamus ope multiplicatoris $e^{\beta x}\partial x$, et poſito

$$e^{-\beta x}\int e^{\beta x} X'\partial x = X'',$$

erit

erit aequatio integralis

$$X'' = A'' y + B'' \frac{\partial y}{\partial x} + C'' \frac{\partial \partial y}{\partial x^2},$$

coefficientibus $A''$, $B''$, $C''$ ita ſumtis, vt ſit

$$A'' + B'' z + C'' z^2 = \frac{P}{(\alpha + z)(\beta + z)} = E(\gamma + z)(\delta + z).$$

Hinc porro ope multiplicatoris $e^{\gamma x} \partial x$ integrando, ſi ponamus $e^{-\gamma x} \int e^{\gamma x} X'' \partial x = X'''$, inueniemus

$$X''' = A''' y + B''' \frac{\partial y}{\partial x},$$

exiſtente

$$A''' + B''' z = \frac{P}{(\alpha + z)(\beta + z)(\gamma + z)} = E(\delta + z).$$

Ac tandem ope multiplicatoris $e^{\delta x} \partial x$, poſita forma

$$e^{-\delta x} \int e^{\delta x} X''' \partial x = X'''',$$

integrale vltimum reperitur

$$X'''' = A'''' y \text{ exiſtente } A'''' = E.$$

Haec igitur omnia colligendo, integrale quaeſitum erit

$$E y = e^{-\delta x} \int e^{(\delta - \gamma) x} \partial x \int e^{(\gamma - \beta) x} \partial x \int e^{(\beta - \alpha) x} \partial x \int e^{\alpha x} X \partial x,$$

quae expreſſio iam ſine vllis ambagibus ex reſolutione formae principalis

$$P = A + B z + C z^2 + D z^3 + E z^4,$$

in factores ſcilicet

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

confici poteſt, vbi notandum quomodocunque ordo litterarum $\alpha$, $\beta$, $\gamma$, $\delta$ permutetur, pro $E y$ ſemper eundem valorem prodire debere.

## Corollarium 1.

1154. Cum ſit $X' = e^{-\alpha x} \int e^{\alpha x} X \partial x$, erit vti iam vidimus

$$X'' = e^{-\beta x}\int e^{\beta x} X' \partial x = e^{-\beta x}\left(\frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x} X \partial x - \frac{1}{\beta-\alpha}\int e^{\beta x} X \partial x\right),$$

seu

$$X'' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{\beta-\alpha} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{\alpha-\beta}.$$

## Corollarium 2.

1155. Porro ob $X''' = e^{-\gamma x}\int e^{\gamma x} X'' \partial x$, erit simili modo reductionem instituendo

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\beta-\alpha)(\alpha-\gamma)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\beta)(\beta-\gamma)},$$

quae reducitur ad hanc formam

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

## Corollarium 3.

1156. Hinc simili modo euoluitur valor $X''''$, vbi quidem sufficeret primum membrum eruisse, quippe ex quo ob permutabilitatem reliqua sponte formantur. Hoc modo integrale nostrae aequationis reperietur hac forma expressum

$$Ey = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{e^{-\delta x}\int e^{\delta x} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

## Scholion.

1157. Si duae pluresue radices sint aequales vel imaginariae, integralia inuenta transformationem postulant, quam deinceps inuestigabimus. Atque haec postrema quidem forma magis

magis apta videtur, vnde transformationes repetantur. Ita pro factorum aequalitate si sit $\delta=\gamma$, bina postrema membra tantum reductionem postulant, ad quam inueniendam ponatur $\delta=\gamma-\omega$, et penultimum membrum erit $-\frac{e^{-\gamma x}\int e^{\gamma x}X\partial x}{\omega(\alpha-\gamma)(\beta-\gamma)}$; pro vltimo autem notandum est, esse

$$\frac{1}{\alpha-\delta}=\frac{1}{\alpha-\gamma+\omega}=\frac{1}{\alpha-\gamma}-\frac{\omega}{(\alpha-\gamma)^2},\text{ et }\frac{1}{\beta-\delta}=\frac{1}{\beta-\gamma}-\frac{\omega}{(\beta-\gamma)^2},$$

hincque

$$\frac{1}{(\alpha-\delta)(\beta-\delta)}=\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{\omega(2\gamma-\alpha-\beta)}{(\alpha-\gamma)^2(\beta-\gamma)^2},\text{ vnde}$$

$$\frac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}=\frac{1}{\omega(\alpha-\gamma)(\beta-\gamma)}+\frac{2\gamma-\alpha-\beta}{(\alpha-\gamma)^2(\beta-\gamma)^2}.$$

Tum vero pro numeratore erit

$$e^{-\delta x}=e^{-\gamma x}(1+\omega x),\text{ et }e^{\delta x}=e^{\gamma x}(1-\omega x),$$

ideoque

$$e^{-\delta x}\int e^{\delta x}X\partial x=e^{-\gamma x}\int e^{\gamma x}X\partial x+\omega e^{-\gamma x}x\int e^{\gamma x}X\partial x$$
$$-\omega e^{-\gamma x}\int e^{\gamma x}Xx\partial x,$$

atque hinc bina vltima membra ob terminos per $\omega$ diuisos se destruentes, abeunt in hanc formam

$$\frac{(2\gamma-\alpha-\beta)e^{-\gamma x}\int e^{\gamma x}X\partial x}{(\alpha-\gamma)^2(\beta-\gamma)^2}+\frac{e^{-\gamma x}x\int e^{\gamma x}X\partial x-e^{-\gamma x}\int e^{\gamma x}Xx\partial x}{(\alpha-\gamma)(\beta-\gamma)},$$

quae expressio etiam ex priori forma elicitur. Eodem modo problema in genere resolui potest.

## Problema 151.

1158. Proposita aequatione differentiali cuiuscunque gradus

$$X=Ay+B\frac{\partial y}{\partial x}+C\frac{\partial\partial y}{\partial x^2}+D\frac{\partial^3 y}{\partial x^3}+\dots+N\frac{\partial^n y}{\partial x^n},$$

sumto elemento $\partial x$ constante, et denotante $X$ functionem quamcunque ipsius $x$, eius integrale inuestigare.

## Solutio.

Formetur ex hac aequatione formula algebraica

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = P,$$

quae in factores simplices resoluatur, vt sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots\ldots (\nu + z),$$

quorum numerus est $n$. Quodsi iam simili modo per singulas integrationes continuo progrediamur, tandem ad hanc aequationem integralem extremam perueniemus

$$Ny = e^{-\nu x}\int e^{(\nu-\mu)x}\partial x \int e^{(\mu-\lambda)x}\partial x \int \ldots\ldots \int e^{(\beta-\alpha)x}\partial x \int e^{\alpha x} X \partial x,$$

seu cum factores inter se permutare liceat, erit etiam

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x \int e^{(\beta-\gamma)x}\partial x \int \ldots\ldots \int e^{(\mu-\nu)x}\partial x \int e^{\nu x} X \partial x.$$

Haec vero expressio per similes reductiones, quibus supra sumus vsi, in sequentes partes resolui potest, ad quas commodius repraesentandas sit breuitatis gratia

$$(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha) \ldots\ldots (\nu - \alpha) = \alpha',$$
$$(\alpha - \beta)(\gamma - \beta)(\delta - \beta) \ldots\ldots (\nu - \beta) = \beta',$$
$$(\alpha - \gamma)(\beta - \gamma)(\delta - \gamma) \ldots\ldots (\nu - \gamma) = \gamma',$$
$$\vdots$$
$$(\alpha - \nu)(\beta - \nu)(\gamma - \nu) \ldots\ldots (\mu - \nu) = \nu',$$

hincque erit

$$Ny = \frac{1}{\alpha'} e^{-\alpha x}\int e^{\alpha x} X \partial x + \frac{1}{\beta'} e^{-\beta x}\int e^{\beta x} X \partial x$$
$$+ \frac{1}{\gamma'} e^{-\gamma x}\int e^{\gamma x} X \partial x + \ldots\ldots + \frac{1}{\nu'} e^{-\nu x}\int e^{\nu x} X \partial x.$$

Ne autem opus sit ad valores $\alpha'$, $\beta'$, $\gamma'$, etc. inueniendos, tot factores in se inuicem multiplicare, cum sit

$$\frac{P}{N(\alpha + z)} = (\beta + z)(\gamma + z)(\delta + z) \ldots\ldots (\nu + z),$$

cui-euidens eſt, hanc formulam praebere valorem $\alpha'$, ſi in ea ſtatuatur $z = -\alpha$; hoc autem caſu fractionis $\frac{P}{N(\alpha + z)}$ tam numerator, quam denominator euaneſcit, ex quo eius valor erit $\frac{\partial P}{N \partial z}$. Quare cum ſit

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n, \text{ erit}$$

$$\frac{\partial P}{\partial z} = B + 2Cz + 3Dz^2 + \ldots\ldots + nNz^{n-1},$$

quae expreſſio vocetur Q, vnde patet fore

$$\alpha' = \frac{Q}{N} \text{ poſito } z = -\alpha,$$

$$\beta' = \frac{Q}{N} \text{ poſito } z = -\beta,$$

$$\gamma' = \frac{Q}{N} \text{ poſito } z = -\gamma,$$

etc.

Vel cum his valoribus ſubſtitutis aequatio integralis per N diuidi queat, ſequentes valores colligantur

$$B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots\ldots \pm nN\alpha^{n-1} = \mathfrak{A},$$

$$B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots\ldots \pm nN\beta^{n-1} = \mathfrak{B},$$

$$B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots\ldots \pm nN\gamma^{n-1} = \mathfrak{C},$$

$$\vdots$$

$$B - 2C\nu + 3D\nu^2 - 4E\nu^3 \ldots\ldots \pm nN\nu^{n-1} = \mathfrak{N},$$

quibus conſtitutis erit integrale quaeſitum

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X \partial x + \text{etc.}$$

quoad omnes factores fuerint in computum ducti.

## Corollarium 1.

1159. Cum ſit

$$\alpha' = \frac{\mathfrak{A}}{N}, \ \beta' = \frac{\mathfrak{B}}{N}, \ \gamma' = \frac{\mathfrak{C}}{N}, \text{ etc. erit}$$

$$\mathfrak{A} = N\alpha', \ \mathfrak{B} = N\beta', \ \mathfrak{C} = N\gamma', \text{ etc.}$$

 Hinc

Hinc ob

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = N(\alpha+z)(\beta+z)(\gamma+z)\ldots\ldots(\nu+z),$$

erit

$$\mathfrak{A} = \frac{P}{\alpha+z} \text{ posito } z = -\alpha,$$
$$\mathfrak{B} = \frac{P}{\beta+z} \text{ posito } z = -\beta,$$
$$\mathfrak{C} = \frac{P}{\gamma+z} \text{ posito } z = -\gamma,$$

et ita porro.

## Corollarium 2.

1160. Regula ergo huius aequationis propositae

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n},$$

integrale completum inueniendi ita se habet. Formetur inde formula algebraica haec

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = P,$$

cuius quaerantur omnes factores simplices, qui sint

$$\alpha + z,\ \beta + z,\ \gamma + z,\ \delta + z, \text{ etc.}$$

quorum multitudo numero $n$ est aequalis, tum pro singulis his factoribus sequentes quantitates constantes definiantur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. vt sit

$$\mathfrak{A} = \frac{P}{\alpha+z} \text{ posito } z = -\alpha, \text{ seu}$$
$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots\ldots \pm nN\alpha^{n-1},$$
$$\mathfrak{B} = \frac{P}{\beta+z} \text{ posito } z = -\beta, \text{ seu}$$
$$\mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots\ldots \pm nN\beta^{n-1},$$
$$\mathfrak{C} = \frac{P}{\gamma+z} \text{ posito } z = -\gamma, \text{ seu}$$
$$\mathfrak{C} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots\ldots \pm nN\gamma^{n-1},$$

his omnibus inuentis erit integrale quaesitum

$y =$

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X \partial x + \text{etc.}$$

quae forma tot conſtat partibus, quot fuerint factores ſimplices in formula P.

## Corollarium 3.

1161. Cum hoc modo integrale tot conſtet partibus, quoti ordinis eſt aequatio differentialis propoſita, et vna quaeque pars per integrationem vnam inuehat conſtantem arbitrariam; manifeſtum eſt integrale ope huius regulae inuentum fore completum.

## Scholion.

1162. Integratio ergo huiusmodi aequationum differentialium nulla amplius laborat difficultate, ſi modo formulae illius algebraicae P omnes factores ſimplices, ſeu quod eodem redit, huius aequationis algebraicae

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = 0,$$

omnes radices numero $n$ aſſignari queant. Hic vero duplicis generis caſus occurrunt, quibus haec integratio vehementer impeditur, quando ſcilicet vel duo pluresue eorum factorum ſimplicium inter ſe fiunt aequales, vel imaginarii, quo quidem poſteriori caſu hoc tantum incommodi accedit, quod partes quaepiam integralis inuenti imaginaria inuoluant, quae autem facta reductione ſe mutuo deſtruunt. Priori vero caſu partes ex factoribus aequalibus oriundae adeo fiunt infinitae, ſed ita diuerſis ſignis affectae, vt coniunctim nihilominus quantitatem finitam referant, cuius valorem nonniſi per plures ambages elicere licet, vbi probe notandum eſt, vtroque caſu inuentionem reliquarum integralis partium, quae factoribus inaequalibus conueniunt, neutiquam hinc turbari. Methodum autem huic fini accommodatam in ſequenti problemate explicabo.

Pro-

## Problema 152.

1163. Propofita aequatione differentiali cuiuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4} + \dots + N\frac{\partial^n y}{\partial x^n},$$

fi forma algebraica inde facta

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots + Nz^n$$

duos pluresue factores fimplices inter fe habeat aequales, partem integralis inde oriundam inueftigare.

## Solutio.

Sint primo duo factores $\alpha + z$ et $\beta + z$ inter fe aequales, feu $\beta = \alpha$, reliquus vero factor formae P fit $= Q$, vt habeatur

$$P = (\alpha + z)(\beta + z)Q = (\alpha + z)^2 Q,$$

pofito autem $z = -\alpha$, abeat Q in $\mathfrak{C}$. Iam initio faltem litterae $\alpha$ et $\beta$ vt diuerfae fpectentur, excepta quantitate $\mathfrak{C}$ quae vtrinque fit eadem, atque pro binis integralis partibus ex his binis factoribus oriundis habebimus

$$\mathfrak{A} = (\beta - \alpha)\mathfrak{C} \text{ et } \mathfrak{B} = (\alpha - \beta)\mathfrak{C}.$$

Partes autem integralis inde oriundae littera $v$ defignentur, vt fit

$$(\beta - \alpha)\mathfrak{C}v = e^{-\alpha x}\int e^{\alpha x}X\partial x - e^{-\beta x}\int e^{\beta x}X\partial x;$$

vnde differentiando colligimus

$$(\beta - \alpha)\mathfrak{C}\partial v = -\alpha e^{-\alpha x}\partial x\int e^{\alpha x}X\partial x + \beta e^{-\beta x}\partial x\int e^{\beta x}X\partial x,$$

ad hanc addatur prior per $\beta\partial x$ multiplicata, fietque

$$(\beta - \alpha)\mathfrak{C}\partial v + (\beta - \alpha)\mathfrak{C}\beta v\partial x = (\beta - \alpha)e^{-\alpha x}\partial x\int e^{\alpha x}X\partial x,$$

quae per $\beta - \alpha$ diuifa, et per $e^{-\alpha x}$ multiplicata, ob $\beta = \alpha$, integrale praebet

$$\mathfrak{C}e^{\alpha x}v = \int\partial x\int e^{\alpha x}X\partial x.$$

Quo-

Quocirca loco binarum partium ex factoribus aequalibus $\alpha+z$ et $\beta+z$ oriundarum scribi oportet hanc formulam

$$v = \tfrac{\mathfrak{C}}{1} e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x,$$

vbi $\mathfrak{C}$ oritur ex forma $\frac{P}{(\alpha+z)^2}$ posito $z = -\alpha$.

Ponamus iam formulam P tres habere factores simplices aequales, vt sit $\alpha+z = \beta+z = \gamma+z$, quosquidem initio vt diuersos spectemus.

Ponamus ergo $P = (\alpha+z)(\beta+z)(\gamma+z)Q$, abeatque Q in $\mathfrak{M}$, posito $z = -\alpha$, ac pro integralis partibus habebimus

$$\mathfrak{A} = (\beta-\alpha)(\gamma-\alpha)\mathfrak{M}, \quad \mathfrak{B} = (\alpha-\beta)(\gamma-\beta)\mathfrak{M},$$
$$\mathfrak{C} = (\alpha-\gamma)(\beta-\gamma)\mathfrak{M}.$$

Hinc si summam trium integralis partium, quam quaerimus, littera $v$ denotemus, erit

$$\mathfrak{M}v = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

Cum nunc sit

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)} + \frac{1}{(\alpha-\gamma)(\beta-\gamma)} = 0,$$

erit differentiando

$$\frac{\mathfrak{M}\partial v}{\partial x} = \frac{-\alpha e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\beta e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} - \frac{\gamma e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)},$$

ad quam si prima per $\alpha$ multiplicata addatur, fit

$$\mathfrak{M}\left(\frac{\partial v}{\partial x} + \alpha v\right) = \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{\gamma-\beta} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{\beta-\gamma}.$$

Haec aequatio denuo differentietur, vt prodeat

$$\mathfrak{M}\left(\frac{\partial\partial v}{\partial x^2} + \frac{\alpha \partial v}{\partial x}\right) = \frac{-\beta e^{-\beta x}\int e^{\beta x} X \partial x}{\gamma-\beta} - \frac{\gamma e^{-\gamma x}\int e^{\gamma x} X \partial x}{\beta-\gamma},$$

ad quam illa per $\beta = \alpha$ multiplicata, si addatur, oritur

$$\mathfrak{M}\left(\frac{\partial \partial v}{\partial x^2} + \frac{2\alpha \partial v}{\partial x} + \alpha\alpha v\right) = e^{-\gamma x}\int e^{\gamma x} X \partial x = e^{-\alpha x}\int e^{\alpha x} X \partial x;$$

vnde iam omnia incommoda sunt sublata. Multiplicetur nunc per $e^{\alpha x}\partial x$, et integratio dabit

$$\mathfrak{M}\, e^{\alpha x}\left(\frac{\partial v}{\partial x} + \alpha v\right) = \int \partial x \int e^{\alpha x} X \partial x,$$

quae per $\partial x$ multiplicata denuo fit integrabilis, proditque

$$\mathfrak{M}\, e^{\alpha x} v = \int \partial x \int \partial x \int e^{\alpha x} X \partial x.$$

Quocirca si forma P factorem habeat cubicum $(\alpha + z)^3$, quaeratur quantitas $\mathfrak{M}$, vt sit

$$\mathfrak{M} = \frac{P}{(\alpha + z)^3}, \text{ posito } z = -\alpha,$$

et integralis pars hinc oriunda erit

$$\frac{1}{\mathfrak{M}} e^{-\alpha x}\int \partial x \int \partial x \int e^{\alpha x} X \partial x.$$

Simili modo si formula P quatuor habeat factores aequales, vt sit $P = (\alpha + z)^4 Q$, capiatur $\mathfrak{M} = \frac{P}{(\alpha + z)^4}$ seu $\mathfrak{M} = Q$, posito $z = -\alpha$, et integralis pars inde nata erit

$$\frac{1}{\mathfrak{M}} e^{-\alpha x}\int \partial x \int \partial x \int \partial x \int e^{\alpha x} X \partial x:$$

sicque etiam casus, quibus formula P adhuc plures habet factores aequales, facile resoluentur.

*Nota.* Tota haec solutio est vitiosa, propterea quod licet quantitates $\alpha$, $\beta$, $\gamma$, etc. quae ponuntur aequales, vt diuersae spectentur, tamen pro singulis membris quantitas $\mathfrak{M}$ eundem valorem retinere assumitur. Quodsi enim litterae $\alpha$, $\beta$, $\gamma$, etc. infinite parum a se inuicem discrepare concipiantur, etiam in valoribus littera $\mathfrak{M}$ indicatis differentiam infinite paruam agnoscere oportet, vnde cum singulae partes integralis fiant infinitae, iisque euolutis membra infinita se mutuo tollant, ex differentiis infinite paruis litterae $\mathfrak{M}$ partes quoque finitae emergunt. Correctionem horum errorum petere licet ex seq. Probl. 154, dum factores aequales in aequationem peculiarem coniiciuntur. Malui autem hunc correctionis laborem industriae lectorum relinquere, quam hoc opus a tali errore liberare, saepe enim plus prodest errores, in quos etiam exercitatis incidere contingit, conseruari, quo melius harum rerum studiosi addiscant quanta circumspectione cauendum sit, ne in ratiocinando hallucinemur.

Corol-

## Corollarium 1.

1164. Notatu hic omnino eſt dignum, quod hae formulae

$$\partial v + \alpha v \partial x,\ \partial\partial v + 2\alpha \partial x \partial v + \alpha^2 v \partial x^2,$$
$$\partial^3 v + 3\alpha \partial x \partial\partial v + 3\alpha^2 \partial x^2 \partial v + \alpha^3 v^3 \partial x^3$$

et in genere haec

$$\partial^n v + \tfrac{n}{1}\alpha \partial x \partial^{n-1} v + \tfrac{n(n-1)}{1.2}\alpha^2 \partial x^2 \partial^{n-2} v + \tfrac{n(n-1)(n-2)}{1.2.3}\alpha^3 \partial x^3 \partial^{n-3} v + \text{etc.}$$

ſi ſemel per $e^{\alpha x}$ multiplicentur, ſucceſſiue toties integrationem admittant, quot vnitates continet index $n$, ita vt poſtremum integrale ſit $e^{\alpha x}$.

## Corollarium 2.

1165. Ratio autem huius phaenomeni inde eſt manifeſta, quod ſi formula $e^{\alpha x} v$ continuo differentietur, ſumto elemento $\partial x$ conſtante, formulae illae differentiales per $e^{\alpha x}$ multiplicatae prodeant, ita vt ſit

$$\partial^n . e^{\alpha x} v = e^{\alpha x}(\partial^n v + \tfrac{n}{1}\alpha \partial x \partial^{n-1} v + \tfrac{n(n-1)}{1.2}\alpha^2 \partial x^2 \partial^{n-2} v + \text{etc.})$$

## Corollarium 3.

1166. Aeque memoratu dignum eſt alterum phaenomenum, quod ſolutio iſta nobis offert; ſumtis ſcilicet numeris quibuscunque $\alpha$, $\beta$, $\gamma$, $\delta$, etc. ſequentes aequalitates ſemper locum habere, vt ſit

$$\frac{1}{\alpha-\beta} + \frac{1}{\beta-\alpha} = 0,$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = 0,$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = 0,$$
etc.

quotcunque numeri hoc modo capiantur.

## Corollarium 4.

1167. Si formula P in factores simplices resoluta ponatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z) \ldots\ldots (\mu+z)(\nu+z),$$

expressio integralis prius inuenta (1158.) quae erat

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{(\beta-\gamma)x}\partial x\int e^{(\gamma-\delta)x}\partial x \ldots\ldots \int e^{(\mu-\nu)x}\partial x\int e^{\nu x}X\partial x,$$

ob factores aequales nulla implicatur difficultate, forma autem posterior, qua integrale in partes ex singulis factoribus ortas distributum exhibetur, et quae ad vsum multo magis accommodata videtur, eo difficiliori egebat euolutione.

## Scholion.

1168. Phaenomenum Corollario 3. obseruatum eo maiorem attentionem meretur, quod etiam ad Arithmeticam vulgarem transferri potest, vbi vsu adeo insigni non cariturum videtur, praecipue quod eius demonstratio minime sit obuia, sed ex profundioribus Analyseos penetralibus repeti debeat, ex quo haud alienum fore arbitror, si huic insigni Theoremati arithmetico hic locum concedam, idque eo magis, quod solutio problematis hic exposita sine demonstratione istius Theorematis minime foret perfecta.

## Theorema Arithmeticum.

1169. Si habeantur numeri quotcunque $a$, $b$, $c$, $d$, etc. ex iisque dum a quolibet singuli reliqui subtrahantur, formentur sequentia producta

$$(a-b)(a-c)(a-d)(a-e) \text{ etc.} = \mathfrak{A}$$
$$(b-a)(b-c)(b-d)(b-e) \text{ etc.} = \mathfrak{B}$$
$$(c-a)(c-b)(c-d)(c-e) \text{ etc.} = \mathfrak{C}$$
$$(d-a)(d-b)(d-c)(d-e) \text{ etc.} = \mathfrak{D}$$

etc.

sem-

ſemper habebitur

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.} = 0.$$

## Demonſtratio.

Conſideretur ſecundum principia in **Introductione ad Analyſin** infinitorum tradita haec fractio

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d)\,\text{etc.}},$$

vbi Z denotet eiusmodi functionem rationalem integram ipſius $z$, in qua ſumma poteſtas ipſius $z$ minor ſit numero factorum denominatoris; haecque fractio reſolui poterit in has fractiones ſimplices, quibus ea iunctim ſumtis ſit aequalis, ſcilicet

$$\frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \frac{D}{z-d} + \text{etc.}$$

Ad quam reſolutionem ſumamus illum numeratorem $Z = z^n$, exiſtente $n$ numero integro minore quam denominator continet factores, atque hi numeratores ita definiuntur, vt ſit

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\,\text{etc.}}$$

$$B = \frac{b^n}{(b-a)(b-c)(b-d)\,\text{etc.}}$$

$$C = \frac{c^n}{(c-a)(c-b)(c-d)\,\text{etc.}}$$

etc.

Cum igitur iſtae fractiones negatiue ſumtae nempe

$$\frac{A}{a-z} + \frac{B}{b-z} + \frac{C}{c-z} + \frac{D}{d-z} + \text{etc.}$$

ad fractionem propoſitam adiectae in nihilum abeant, ſi $z$ ſit numerorum propoſitorum $a, b, c, d$, etc. vltimus, quorum adeo multitudo maior eſt quam $n+1$, ponatur

 $(a-b)$

$$
\begin{array}{l}
(a-b)(a-c)(a-d)\ldots\ldots(a-z)=\mathfrak{A}\\
(b-a)(b-c)(b-d)\ldots\ldots(b-z)=\mathfrak{B}\\
(c-a)(c-b)(c-d)\ldots\ldots(c-z)=\mathfrak{C}\\
(d-a)(d-b)(d-c)\ldots\ldots(d-z)=\mathfrak{D}\\
\qquad\qquad\text{etc.}\\
(z-a)(z-b)(z-c)\ldots\ldots(z-y)=\mathfrak{Z}
\end{array}
$$

vt fractio propofita fit $\frac{z^n}{\mathfrak{Z}}$. Atque hinc perfpicuum eft, fummam omnium harum fractionum effe

$$\frac{a^n}{\mathfrak{A}}+\frac{b^n}{\mathfrak{B}}+\frac{c^n}{\mathfrak{C}}+\frac{d^n}{\mathfrak{D}}+\ldots\ldots\ldots+\frac{z^n}{\mathfrak{Z}}=0$$

dum fit $n+1$ minor numero terminorum. Sumto ergo $n=0$ oritur cafus Theorematis.

## Corollarium 1.

1170. Haec fi transferantur ad numeros $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. fupra (1160.) definitos, vbi aliquod leue difcrimen in factorum conftitutione probe eft notandum, intelligetur effe

$$
\begin{array}{l}
\frac{1}{\mathfrak{A}}+\frac{1}{\mathfrak{B}}+\frac{1}{\mathfrak{C}}+\frac{1}{\mathfrak{D}}+\text{etc.}=0\\
-\frac{\alpha}{\mathfrak{A}}-\frac{\beta}{\mathfrak{B}}-\frac{\gamma}{\mathfrak{C}}-\frac{\delta}{\mathfrak{D}}-\text{etc.}=0\\
+\frac{\alpha^2}{\mathfrak{A}}+\frac{\beta^2}{\mathfrak{B}}+\frac{\gamma^2}{\mathfrak{C}}+\frac{\delta^2}{\mathfrak{D}}+\text{etc.}=0\\
-\frac{\alpha^3}{\mathfrak{A}}-\frac{\beta^3}{\mathfrak{B}}-\frac{\gamma^3}{\mathfrak{C}}-\frac{\delta^3}{\mathfrak{D}}-\text{etc.}=0\\
\qquad\text{etc.}
\end{array}
$$

donec perueniatur ad hanc formam

$$\pm\frac{\alpha^{n-1}}{\mathfrak{A}}\pm\frac{\beta^{n-1}}{\mathfrak{B}}\pm\frac{\gamma^{n-1}}{\mathfrak{C}}\pm\frac{\delta^{n-1}}{\mathfrak{D}}\pm\text{etc.}$$

cuius fumma non amplius eft euanefcens, fed aequalis fractioni $\frac{1}{\mathfrak{A}}$.

Co-

## Corollarium 2.

1171. Hoc etiam ex euolutione formae in Theoremate adhibitae colligere licet; etenim ſi ea ſtatuatur

$$\frac{z^{n-1}}{(z-a)(z-b)(z-c)\ldots\ldots(z-y)},$$

exiſtente omnium litterarum $a$, $b$, $c$, etc. numero $= n$, quia hic numerator $z^{n-1}$ tot habet dimenſiones, quot ſunt factores in denominatore, pars integra in hac fractione contenta eſt vnitas; quae etiam facta reſolutione conſeruatur, et in applicatione ad caſum memoratum abit in $\frac{1}{\mathfrak{K}}$.

## Scholion.

1172. Poſt huius Theorematis demonſtrationem demum clare a poſteriori oſtendi poteſt, quemadmodum integrale ſupra (1160.) exhibitum aequationi differentiali ibidem propoſitae ſatisfaciat. Notatis enim expreſſionibus §. 1170, cum ſupra inuenerimus integrale

$$y = \frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\partial x + \frac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\partial x + \text{etc.}$$

erit continuo differentiando

$$\frac{\partial y}{\partial x} = -\frac{\alpha}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\partial x - \frac{\beta}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\partial x - \text{etc.}$$

$$\frac{\partial\partial y}{\partial x^2} = \frac{\alpha^2}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\partial x + \frac{\beta^2}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\partial x + \text{etc.}$$

$$\frac{\partial^3 y}{\partial x^3} = -\frac{\alpha^3}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\partial x - \frac{\beta^3}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\partial x - \text{etc.}$$

etc.

vsque ad

$$\frac{\partial^{n-1} y}{\partial x^{n-1}} = \pm\frac{\alpha^{n-1}}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\partial x \pm \frac{\beta^{n-1}}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\partial x \pm \text{etc.}$$

vnde ſequens forma differentialis reſultat

$\partial^n y$

$$\frac{\partial^n y}{\partial x^n} = \mp \frac{\alpha^n}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x \mp \frac{\beta^n}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x \mp \text{etc.}$$

$$\mp \left( \frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \text{etc.} \right) X,$$

quod postremum membrum abit in $\frac{1}{N}$ X.

Si iam omnes hae formae singulatim multiplicentur per quantitates A, B, C, D . . . . . N, quoniam est

$$A - B\alpha + C\alpha^2 - D\alpha^3 + \ldots \mp N\alpha^n = 0,$$

$$A - B\beta + C\beta^2 - D\beta^3 + \ldots \mp N\beta^n = 0,$$

propterea quod $\alpha + z$, $\beta + z$, $\gamma + z$, etc. sunt factores formae

$$A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n,$$

manifesto obtinebimus

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots + N\frac{\partial^n y}{\partial x^n} = X,$$

quae est ipsa aequatio differentialis initio proposita.

## Problema 153.

1173. Proposita aequatione differentiali cuiuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots + N\frac{\partial^n y}{\partial x^n},$$

si expressio algebraica hinc formata

$$P = A + Bz + Cz^2 + Dz^3 \ldots \ldots + Nz^n$$

duos habeat factores simplices imaginarios factore duplici $ff + 2fz\cos.\theta + zz$ contentos, inuestigare partes integralis hinc oriundas.

## Solutio.

Sint $\alpha+z$ et $\beta+z$ hi duo factores imaginarii, vt sit $\alpha=f(\text{cos}.\theta+\sqrt{-1}.\text{sin}.\theta)$ et $\beta=f(\text{cos}.\theta-\sqrt{-1}.\text{sin}.\theta)$, ob

$$(\alpha+z)(\beta+z)=ff+2fz\,\text{cos}.\,\theta+zz,$$

ac statuatur $P=(ff+2fz\,\text{cos}.\,\theta+zz)\,Q$, existente

$$Q=A'+B'z+C'z^2+\ldots\ldots+N'z^{n-2}.$$

Cum igitur integralis partes ex binis illis factoribus simplicibus imaginariis ortae sint

$$\frac{\mathfrak{A}}{\mathfrak{a}}e^{-\alpha x}\int e^{\alpha x}X\,\partial x+\frac{\mathfrak{B}}{\mathfrak{b}}e^{-\beta x}\int e^{\beta x}X\,\partial x=v,$$

hos valores imaginarios ad realitatem perduci oportet. Erunt autem $\mathfrak{A}$ et $\mathfrak{B}$ quantitates imaginariae resultantes ex forma

$$(f\,\text{cos}.\,\theta\mp\sqrt{-1}.f\,\text{sin}.\,\theta+z)\,Q,$$

si loco $z$ scribatur

$$-f\,\text{cos}.\,\theta\mp\sqrt{-1}.f\,\text{sin}.\,\theta.$$

At facta hac substitutione fit

$$Q=A'-B'f\,\text{cos}.\,\theta+C'ff\,\text{cos}.\,2\theta-D'f^3\,\text{cos}.\,3\theta+\text{etc.}$$
$$\mp\sqrt{-1}.B'f\text{sin}.\theta\pm\sqrt{-1}.C'ff\text{sin}.2\theta\mp\sqrt{-1}.D'f^3\text{cos}.3\theta\pm\text{etc.}$$

Ponamus breuitatis gratia

$$A'-B'f\,\text{cos}.\theta+C'ff\,\text{cos}.\,2\theta-D'f^3\text{cos}.\,3\theta+\text{etc.}=\mathfrak{M}\text{ et}$$
$$-B'f\,\text{sin}.\theta+C'ff\,\text{sin}.\,2\theta-D'f^3\,\text{sin}.\,3\theta+\text{etc.}=\mathfrak{N},$$

vt sit $Q=\mathfrak{M}\pm\mathfrak{N}\sqrt{-1}$, vbi signorum ambiguorum superius valet pro litteris $\alpha$ et $\mathfrak{A}$, inferius pro litteris $\beta$ et $\mathfrak{B}$. Hinc ergo erit

$$\mathfrak{A}=-2\sqrt{-1}.f\,\text{sin}.\,\theta\,(\mathfrak{M}+\mathfrak{N}\sqrt{-1})\text{ et}$$
$$\mathfrak{B}=+2\sqrt{-1}.f\,\text{sin}.\,\theta\,(\mathfrak{M}-\mathfrak{N}\sqrt{-1}),$$

ideoque

$$2 v \sqrt{-1} . f \sin. \theta = \frac{-e^{-\alpha x} \int e^{\alpha x} X \partial x}{\mathfrak{M} + \mathfrak{N} \sqrt{-1}} + \frac{e^{-\beta x} \int e^{\beta x} X \partial x}{\mathfrak{M} - \mathfrak{N} \sqrt{-1}}.$$

Est vero

$$e^{\alpha x} = e^{f x \cos. \theta} [\cos. (f x \sin. \theta) + \sqrt{-1} . \sin. (f x \sin. \theta)] \text{ et}$$

$$e^{\beta x} = e^{f x \cos. \theta} [\cos. (f x \sin. \theta) - \sqrt{-1} . \sin. (f x \sin. \theta)].$$

Sit breuitatis gratia angulus $f x \sin. \theta = \Phi$ erit

$$2 \sqrt{-1} . v (\mathfrak{M}\mathfrak{M} + \mathfrak{N}\mathfrak{N}) f \sin. \theta =$$

$$-(\mathfrak{M}-\mathfrak{N}\sqrt{-1}) e^{-fx\cos.\theta}(\cos.\Phi-\sqrt{-1}.\sin.\Phi)\int e^{fx\cos.\theta} X\partial x(\cos.\Phi+\sqrt{-1}.\sin.\Phi)$$

$$+(\mathfrak{M}+\mathfrak{N}\sqrt{-1}) e^{-fx\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int e^{fx\cos.\theta} X\partial x(\cos.\Phi-\sqrt{-1}.\sin.\Phi)$$

$$= e^{-fx\cos.\theta} 2\sqrt{-1} . (\mathfrak{M} \sin.\Phi + \mathfrak{N} \cos.\Phi) \int e^{fx\cos.\theta} X \partial x \cos.\Phi$$

$$- e^{-fx\cos.\theta} 2\sqrt{-1} . (\mathfrak{M} \cos.\Phi - \mathfrak{N} \sin.\Phi) \int e^{fx\cos.\theta} X \partial x \sin.\Phi.$$

Quocirca habebimus integralis partem quaesitam

$$v = \frac{\left\{ \begin{array}{l} + e^{-fx\cos.\theta} (\mathfrak{M} \sin.\Phi + \mathfrak{N} \cos.\Phi) \int e^{fx\cos.\theta} X \partial x \cos.\Phi \\ - e^{-fx\cos.\theta} (\mathfrak{M} \cos.\Phi - \mathfrak{N} \sin.\Phi) \int e^{fx\cos.\theta} X \partial x \sin.\Phi \end{array} \right\}}{(\mathfrak{M}\mathfrak{M} + \mathfrak{N}\mathfrak{N}) f \sin.\theta}$$

existente $\Phi = f x \sin. \theta$.

## Corollarium 1.

1174. Praecipuum igitur opus hic consistit in inuentione formulae imaginariae $\mathfrak{M} + \mathfrak{N} \sqrt{-1}$, quae colligi debet ex quantitate Q, dum loco $z$ scribitur valor imaginarius

$$-f (\cos. \theta + \sqrt{-1} . \sin. \theta),$$

vnde hoc commodi nascitur, vt loco $z^n$ scribi oporteat

$$(-f)^n (\cos. n \theta + \sqrt{-1} . \sin. n \theta).$$

## Corollarium 2.

1175. Cum sit $Q = \frac{P}{ff + 2fz \cos. \theta + zz}$, etiam ex hac forma per eandem substitutionem formula imaginaria $\mathfrak{M} + \mathfrak{N} \sqrt{-1}$

 inue-

inueniri poteſt, vbi autem notandum eſt, hac ſubſtitutione tam numeratorem P quam denominatorem euaneſcere. Ex quo manifeſtum eſt, valorem iſtius formulae rite obtineri ex hac fractione

$$\frac{\partial P}{2(f\,\text{cof.}\,\theta + z)\,\partial z} = \frac{\partial P}{-2\sqrt{-1}.\,f\,\text{fin.}\,\theta\,\partial z}.$$

## Corollarium 3.

1176. Quoniam igitur eſt

$$\frac{\partial P}{\partial z} = B + 2Cz + 3Dz^2 + 4Ez^3 + \ldots\ldots nNz^{n-1},$$

ſi ſtatuamus

$$\mathfrak{P} = B - 2Cf\,\text{cof.}\,\theta + 3Df^2\,\text{cof.}\,2\theta - 4Ef^3\,\text{cof.}\,3\theta \ldots$$
$$\pm nNf^{n-1}\,\text{cof.}\,(n-1)\,\theta,$$
$$\mathfrak{Q} = -2Cf\,\text{fin.}\,\theta + 3Df^2\,\text{fin.}\,2\theta - 4Ef^3\,\text{fin.}\,3\theta \ldots$$
$$\pm nNf^{n-1}\,\text{fin.}\,(n-1)\,\theta,$$

vt facta ſubſtitutione fiat

$$\frac{\partial P}{\partial z} = \mathfrak{P} + \sqrt{-1}.\,\mathfrak{Q},$$

habebimus

$$\mathfrak{M} + \sqrt{-1}.\,\mathfrak{N} = \frac{\mathfrak{P} + \sqrt{-1}.\,\mathfrak{Q}}{-2\sqrt{-1}.\,f\,\text{fin.}\,\theta} = \frac{-\mathfrak{Q} + \sqrt{-1}.\,\mathfrak{P}}{2f\,\text{fin.}\,\theta},$$

ideoque

$$\mathfrak{M} = \frac{-\mathfrak{Q}}{2f\,\text{fin.}\,\theta} \text{ et } \mathfrak{N} = \frac{\mathfrak{P}}{2f\,\text{fin.}\,\theta}.$$

## Corollarium 4.

1177. Immediate ergo ex quantitate P indeque derivatis $\mathfrak{P}$ et $\mathfrak{Q}$, poſito $fx\,\text{fin.}\,\theta = \Phi$, integralis pars ex factore duplici $ff + 2fz\,\text{cof.}\,\theta + zz$ nata erit expreſſa

$$= \frac{2e^{-fx\,\text{cof.}\,\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} (\mathfrak{P}\,\text{cof.}\,\Phi - \mathfrak{Q}\,\text{fin.}\,\Phi) \int e^{fx\,\text{cof.}\,\theta}\, X\,\partial x\,\text{cof.}\,\Phi \\ \hline + (\mathfrak{P}\,\text{fin.}\,\Phi + \mathfrak{Q}\,\text{cof.}\,\Phi) \int e^{fx\,\text{cof.}\,\theta}\, X\,\partial x\,\text{fin.}\,\Phi \end{array} \right\}.$$

## Scholion.

1178. Quotcunque ergo forma

$$P = A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

habuerit factores duplices, pro singulis ope horum praeceptorum partes integralis facile definiuntur, et quia hinc inuentio partium, quae factoribus simplicibus conueniunt, siue ii sint inaequales siue aequales, non turbatur, omnibus partibus in vnam summam coniectis habebitur integrale completum aequationis differentialis propositae. Verum tamen haec praecepta non sufficiunt, si factorum duplicium bini pluresue inter se fuerint aequales; huiusmodi enim casus peculiarem exigunt euolutionem similem eius, qua pro casu duorum pluriumue factorum simplicium inter se aequalium sum vsus. Ne autem hanc tractationem nimis protraham, sufficiet casum pro duobus factoribus duplicibus inter se aequalibus euoluisse, cum inde methodus ad plures facile extendatur.

## Problema 154.

1179. Proposita aequatione differentiali cuiuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n},$$

si expressio algebraica inde formata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

habeat factorem duplicem quadratum

$$(ff + 2fz\cos.\theta + zz)^2,$$

partem integralis ei conuenientem inuestigare.

## Solutio.

Ponamus ergo $P = (ff + 2fz\cos.\theta + zz)^2 Q$, sitque

Q =

$$Q = A' + B'z + C'zz + \ldots + N'z^{n-4},$$

ac primo imaginaria non curantes statuamus

$$\alpha = f(\text{cos}.\,\theta + \sqrt{-1}.\,\text{sin}.\,\theta) \text{ et}$$

$$\beta = f(\text{cos}.\,\theta - \sqrt{-1}.\,\text{sin}.\,\theta),$$

vt sit

$$P = (\alpha + z)^2\,(\beta + z)^2\,Q.$$

Iam ex iis quae supra (1163.) de binis factoribus simplicibus aequalibus docuimus, ponamus formam

$$\frac{P}{(\alpha+z)^2} = (\beta + z)^2\,Q, \text{ posito } z = -\alpha, \text{ abire in } \mathfrak{A},$$

at hanc formam

$$\frac{P}{(\beta+z)^2} = (\alpha + z)^2\,Q, \text{ posito } z = -\beta, \text{ in } \mathfrak{B},$$

quibus quantitatibus $\mathfrak{A}$ et $\mathfrak{B}$ inuentis, ibi ostendi fore integralis partes hinc oriundas

$$\frac{1}{\mathfrak{A}}e^{-\alpha x}\int \partial x \int e^{\alpha x}X\,\partial x + \frac{1}{\mathfrak{B}}e^{-\beta x}\int \partial x \int e^{\beta x}X\,\partial x = v,$$

quas, cum iam imaginaria inuoluant, ad realitatem reduci oportet. Faciamus vt in problemate praecedente

$$\mathfrak{M} = A' - B'f\,\text{cos}.\,\theta + C'f^2\,\text{cos}.\,2\theta - D'f^3\,\text{cos}.\,3\theta + \text{etc.}$$

$$\mathfrak{N} = \quad - B'f\,\text{sin}.\,\theta + C'f^2\,\text{sin}.\,2\theta - D'f^3\,\text{sin}.\,3\theta + \text{etc.}$$

vt quantitas Q, posito $z = -\alpha = -f(\text{cos}.\,\theta + \sqrt{-1}.\,\text{sin}.\,\theta)$, abeat in $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, at posito

$$z = -\beta = -f(\text{cos}.\,\theta - \sqrt{1}.\,\text{sin}.\,\theta),$$

in $\mathfrak{M} - \mathfrak{N}\sqrt{-1}$.

Cum iam sit $(\beta - \alpha)^2 = (-2\sqrt{-1}.f\,\text{sin}.\,\theta)^2 = -4ff\,\text{sin}.\,\theta^2$, cui quoque $(\alpha-\beta)^2$ aequatur, erit

$$\mathfrak{A} = -4ff\,\text{sin}.\,\theta^2\,(\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et}$$

$$\mathfrak{B} = -4ff\,\text{sin}.\,\theta^2\,(\mathfrak{M} - \mathfrak{N}\sqrt{-1}),$$

 ideo-

ideoque

$$-4ff\,\text{fin.}\,\theta^2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})v = (\mathfrak{M}-\mathfrak{N}\sqrt{-1})e^{-\alpha x}\int\partial x\int e^{\alpha x}X\partial x$$
$$+(\mathfrak{M}+\mathfrak{N}\sqrt{-1})e^{-\beta x}\int\partial x\int e^{\beta x}X\partial x.$$

At posito $fx\,\text{fin.}\,\theta = \Phi$, est vt vidimus

$$e^{\alpha x} = e^{fx\,\text{cof.}\,\theta}(\text{cof.}\,\Phi+\sqrt{-1}.\,\text{fin.}\,\Phi),$$
$$e^{-\alpha x} = e^{-fx\,\text{cof.}\,\theta}(\text{cof.}\,\Phi-\sqrt{-1}.\,\text{fin.}\,\Phi),$$
$$e^{\beta x} = e^{fx\,\text{cof.}\,\theta}(\text{cof.}\,\Phi-\sqrt{-1}.\,\text{fin.}\,\Phi),$$
$$e^{-\beta x} = e^{-fx\,\text{cof.}\,\theta}(\text{cof.}\,\Phi+\sqrt{-1}.\,\text{fin.}\,\Phi),$$

vnde illius aequationis alterum membrum abit in

$$+e^{-fx\,\text{cof.}\,\theta}[\mathfrak{M}\,\text{cof.}\,\Phi-\mathfrak{N}\,\text{fin.}\,\Phi-\mathfrak{N}\sqrt{-1}.\,\text{cof.}\,\Phi-\mathfrak{M}\sqrt{-1}.\,\text{fin.}\,\Phi]$$
$$\times\int\partial x\int e^{fx\,\text{cof.}\,\theta}X\partial x(\text{cof.}\,\Phi+\sqrt{-1}.\,\text{fin.}\,\Phi)$$
$$+e^{-fx\,\text{cof.}\,\theta}[\mathfrak{M}\,\text{cof.}\,\Phi-\mathfrak{N}\,\text{fin.}\,\Phi+\mathfrak{N}\sqrt{-1}.\,\text{cof.}\,\Phi+\mathfrak{M}\sqrt{-1}.\,\text{fin.}\,\Phi]$$
$$\times\int\partial x\int e^{fx\,\text{cof.}\,\theta}X\partial x(\text{cof.}\,\Phi-\sqrt{-1}.\,\text{fin.}\,\Phi),$$

vbi partes imaginariae sponte se destruunt, ita vt obtineatur

$$v = \frac{\left\{\begin{array}{l}-e^{-fx\,\text{cof.}\,\theta}(\mathfrak{M}\,\text{cof.}\,\Phi-\mathfrak{N}\,\text{fin.}\,\Phi)\int\partial x\int e^{fx\,\text{cof.}\,\theta}X\partial x\,\text{cof.}\,\Phi\\-e^{-fx\,\text{cof.}\,\theta}(\mathfrak{N}\,\text{cof.}\,\Phi+\mathfrak{M}\,\text{fin.}\,\Phi)\int\partial x\int e^{fx\,\text{cof.}\,\theta}X\partial x\,\text{fin.}\,\Phi\end{array}\right\}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\,\text{fin.}\,\theta^2}$$

seu hoc modo

$$v = \frac{-e^{-fx\,\text{cof.}\,\theta}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\,\text{fin.}\,\theta^2}\left\{\begin{array}{l}+(\mathfrak{M}\,\text{cof.}\,\Phi-\mathfrak{N}\,\text{fin.}\,\Phi)\int\partial x\int e^{fx\,\text{cof.}\,\theta}X\partial x\,\text{cof.}\,\Phi\\+(\mathfrak{M}\,\text{fin.}\,\Phi+\mathfrak{N}\,\text{cof.}\,\Phi)\int\partial x\int e^{fx\,\text{cof.}\,\theta}X\partial x\,\text{fin.}\,\Phi\end{array}\right\},$$

quae expressio ab imaginariis penitus est liberata.

*Nota.* Etiam haec solutio insigni correctione indiget diligentiae lectorum relicta.

## Corollarium 1.

1180. Quoniam formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ nascitur ex quantitate Q, si loco $z$ scribatur

—f

$-f(\cos.\theta + \sqrt{-1}.\sin.\theta)$,

eadem positione quoque reperietur ex forma

$$\frac{P}{(ff + 2fz\cos.\theta + zz)^2},$$

verum hic tam numerator quam denominator prodit euanescens

## Corollarium 2.

1181. Orietur ergo quoque idem valor ex formula

$$\frac{\partial P}{4\partial z\,[f^3\cos.\theta + ffz(1 + 2\cos.\theta^2) + 3fzz\cos.\theta + z^3]},$$

vbi cum idem incommodum denuo recurrat, orietur quoque ex hac formula

$$\frac{\partial\partial P}{4\partial z^2\,[ff(1 + 2\cos.\theta^2) + 6fz\cos.\theta + 3zz]}.$$

## Corollarium 3.

1182. Statuatur hic primo in denominatore

$$z = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$$

fietque haec formulae

$$\frac{\partial\partial P}{8ff\partial z^2\sin.\theta^2}.$$

Deinde cum sit

$$\frac{\partial\partial P}{2\partial z^2} = C + 3Dz + 6Ezz + \dots + \frac{n(n-1)}{1.2}Nz^{n-2},$$

ponamus breuitatis gratia

$$\mathfrak{P} = C - 3Df\cos.\theta + 6Eff\cos.2\theta - \dots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\cos.(n-2)\theta,$$

$$\mathfrak{Q} = -3Df\sin.\theta + 6Eff\sin.2\theta - \dots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\sin.(n-2)\theta,$$

vt sit facta substitutione

$$\frac{\partial\partial P}{2\partial z^2} = \mathfrak{P} + \mathfrak{Q}\sqrt{-1},$$

ideoque

$$\mathfrak{M} + \mathfrak{N}\sqrt{-1} = \frac{-\mathfrak{P} - \mathfrak{Q}\sqrt{-1}}{4ff\sin.\theta^2},$$

et consequenter

$\mathfrak{M} =$

$$\mathfrak{M} = \frac{-\mathfrak{P}}{4ff\,\text{fin.}\,\theta^2} \text{ et } \mathfrak{N} = \frac{-\mathfrak{Q}}{4ff\,\text{fin.}\,\theta^2}.$$

Quos ergo valores in parte integralis inuenta fubftituere licet.

## Corollarium 4.

1183. Facta autem fubftitutione, factor duplex quadratus $(ff + 2fz\,\text{cof.}\,\theta + zz)^2$ hanc praebet integralis partem

$$v = \frac{2\,e^{-fx\,\text{cof.}\,\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} +(\mathfrak{P}\,\text{cof.}\,\Phi - \mathfrak{Q}\,\text{fin.}\,\Phi)\int \partial x \int e^{fx\,\text{cof.}\,\theta} X \partial x\,\text{cof.}\,\Phi \\ +(\mathfrak{P}\,\text{fin.}\,\Phi + \mathfrak{Q}\,\text{cof.}\,\Phi)\int \partial x \int e^{fx\,\text{cof.}\,\theta} X \partial x\,\text{fin.}\,\Phi \end{array} \right\},$$

vbi $\Phi$ denotat angulum $fx\,\text{fin.}\,\theta$.

## Scholion.

1184. Si hanc expreffionem cum ea, quam problemate praecedente inuenimus, comparemus, vix actuali fimili euolutione erit opus pro cafibus magis complicatis. Ita fi quantitas

$$P = A + Bz + Cz^2 + Dz^3 + \dots + Nz^n,$$

factorem habeat duplicem cubicum

$$(ff + 2fz\,\text{cof.}\,\theta + zz)^3,$$

quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ ita definiuntur, vt fit

$$\mathfrak{P} = D - 4Ef\,\text{cof.}\,\theta + 10Fff\,\text{cof.}\,2\theta - 20Gf^3\,\text{cof.}\,3\theta + \dots \pm \frac{n(n-1)(n-2)}{1.2.3} Nf^{n-3}\,\text{cof.}\,(n-3)\theta,$$

$$\mathfrak{Q} = -4Ef\,\text{fin.}\,\theta + 10Fff\,\text{fin.}\,2\theta - 20Gf^3\,\text{fin.}\,3\theta + \dots \pm \frac{n(n-1)(n-2)}{1.2.3} Nf^{n-3}\,\text{fin.}\,(n-3)\theta,$$

quibus inuentis, erit integralis pars hinc nata

$$\frac{2\,e^{-fx\,\text{cof.}\,\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} +(\mathfrak{P}\,\text{cof.}\,\Phi - \mathfrak{Q}\,\text{fin.}\,\Phi)\int \partial x \int \partial x \int e^{fx\,\text{cof.}\,\theta} X \partial x\,\text{cof.}\,\Phi \\ +(\mathfrak{P}\,\text{fin.}\,\Phi + \mathfrak{Q}\,\text{cof.}\,\Phi)\int \partial x \int \partial x \int e^{fx\,\text{cof.}\,\theta} X \partial x\,\text{fin.}\,\Phi \end{array} \right\},$$

neque iam vlterior progreffio vlli amplius difficultati eft obnoxia. Quocirca aequationis hoc capite propofitae refolutionem

nem ita concinne mihi equidem absoluisse videor, vt nihil amplius desiderari possit. Interim hoc argumentum maxime illustrabitur, si haec praecepta ad exempla particularia accommodabimus; cui instituto sequens caput est destinatum. Ante autem insignem proprietatem circa huiusmodi aequationes generales proponam, quae in Analysi ingentem vsum habitura videtur.

## Problema 155.

1185. Proposita aequatione differentiali cuiuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^{m+n} y}{\partial x^{m+n}},$$

si formula algebraica inde nata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^{m+n}$$

duobus factoribus constet $P = QR$, vt sit

$$Q = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \ldots\ldots + \mathfrak{N}z^m \text{ et}$$

$$R = a + bz + cz^2 + \ldots\ldots + nz^n,$$

integrationem illius aequationis ad integrationem binarum aequationum simpliciorum reuocare.

## Solutio.

Si formam integralem primo (1158.) perpendamus, haud difficulter inde colligimus, postquam hanc aequationem integrauerimus

$$X = \mathfrak{A}v + \mathfrak{B}\frac{\partial v}{\partial x} + \mathfrak{C}\frac{\partial\partial v}{\partial x^2} + \ldots\ldots + \mathfrak{N}\frac{\partial^m v}{\partial x^m},$$

indeque valorem ipsius $v$ per $x$ et $X$ definiuerimus, valorem ipsius $y$ pro aequatione proposita ex hac aequatione erutum iri

$$v = \mathfrak{a}y + \mathfrak{b}\frac{\partial y}{\partial x} + \mathfrak{c}\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + \mathfrak{n}\frac{\partial^n y}{\partial x^n},$$

cuius ratio adeo in promtu eſt poſita; dum ex hac aequatione valores pro $v$ eiusque differentialibus ſubſtituantur. Prodibit enim

$$\begin{aligned} X = \mathfrak{A}\mathfrak{a}y &+ \mathfrak{A}\mathfrak{b}.\tfrac{\partial y}{\partial x} + \mathfrak{A}\mathfrak{c}.\tfrac{\partial\partial y}{\partial x^2} + \mathfrak{A}\mathfrak{d}.\tfrac{\partial^3 y}{\partial x^3} + \text{etc.} \\ &+ \mathfrak{B}\mathfrak{a} \qquad + \mathfrak{B}\mathfrak{b} \qquad + \mathfrak{B}\mathfrak{c} \\ &\qquad\qquad + \mathfrak{C}\mathfrak{a} \qquad + \mathfrak{C}\mathfrak{b} \\ &\qquad\qquad\qquad\qquad + \mathfrak{D}\mathfrak{a}. \end{aligned}$$

Cum autem per hypotheſin ſit $P = QR$, ſeriebus $Q$ et $R$ in ſe multiplicatis, neceſſe eſt fieri

$$A = \mathfrak{A}\mathfrak{a},\ B = \mathfrak{A}\mathfrak{b} + \mathfrak{B}\mathfrak{a},\ C = \mathfrak{A}\mathfrak{c} + \mathfrak{B}\mathfrak{b} + \mathfrak{C}\mathfrak{a},\ \text{etc.}$$

ſicque haec poſtrema aequatio ad ipſam propoſitam reducitur.

## Corollarium 1.

1186. Si tantum ad factores ſimplices reſpiciamus, prioris aequationis integrale per huiusmodi terminos exprimitur

$$v = \Gamma e^{-\alpha x}\int e^{\alpha x}X\partial x \ \text{etc.}$$

poſterioris vero aequationis integrale per huiusmodi

$$y = \Delta e^{-\beta x}\int e^{\beta x}v\partial x \ \text{etc.}$$

## Corollarium 2.

1187. Quodſi iam in ſingulis terminis poſterioris integralis ſubſtituamus ſingulos prioris, fiet

$$y = \Gamma\Delta e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial x,$$

quae forma ad hanc reducitur

$$y = \frac{\Gamma\Delta}{\beta-\alpha}\left(e^{-\alpha x}\int e^{\alpha x}X\partial x - e^{-\beta x}\int e^{\beta x}X\partial x\right),$$

cuiusmodi termini per integrationem aequationis propoſitae immediate inueniuntur.

## Corollarium 3.

1188. Si hic fuiſſet $\beta = \alpha$, ſine vlla reductione ſtatim prodiiſſet forma

$$y = \Gamma \Delta e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x$$

ſupra pro caſu duorum factorum ſimplicium aequalium inuenta. Interim cum totum negotium ad reſolutionem in factores vel ſimplices vel duplices reales redeat, ipſa aequatio propoſita modo ante expoſito facillime expeditur.

# CAPVT IV.

## APPLICATIO METHODI INTEGRANDI IN CAPITE PRAECEDENTE TRADITAE AD EXEMPLA.

### Problema 156.

1189.

Propofita hac aequatione differentiali

$$X = a^n y + \frac{\partial^n y}{\partial x^n}$$

eius integrale completum inuenire.

### Solutio.

Hic ergo eft $P = a^n + z^n$, vbi primo obferuetur, fi $n$ fit numerus impar, factorem fimplicem effe $a + z$, ex quo nafcitur pars integralis

$$\mathfrak{A}\, e^{-ax} \int e^{ax} X \partial x,$$

exiftente $\mathfrak{A}$ valore ex forma $\frac{P}{a+z}$ emergente, fi ponatur $z = -a$, qui ergo valor cum fit etiam $\frac{\partial P}{\partial z} = n z^{n-1}$, ob $n - 1$ numerum parem, erit $\mathfrak{A} = n a^{n-1}$, ideoque haec integralis pars

$$\frac{1}{n a^{n-1}} e^{-ax} \int e^{ax} X \partial x.$$

Reliqui factores omnes in hac forma continentur

$$aa - 2az \cos. \theta + zz, \text{ exiftente } \theta = \frac{(2i+1)\pi}{n},$$

vbi $i$ denotat numerum integrum quemcunque et $\pi$ angulum duobus rectis aequalem. Comparata hac forma cum Probl.

153.

153. et Coroll. 1. fit $f = -a$, et ob $z = a(\cos.\theta + \sqrt{-1}.\sin.\theta)$, ex forma $\frac{\partial P}{\partial z}$ colligitur

$$\mathfrak{P} = n a^{n-1} \cos.(n-1)\theta \text{ et } \mathfrak{Q} = n a^{n-1} \sin.(n-1)\theta;$$

cum igitur fit

$$\cos. n\theta = -1 \text{ et } \sin. n\theta = 0, \text{ erit}$$

$$\mathfrak{P} = -n a^{n-1} \cos.\theta \text{ et } \mathfrak{Q} = n a^{n-1} \sin.\theta.$$

Quare posito $f x \sin.\theta = -a x \sin.\theta = \Phi$, integralis pars ex quolibet factore duplici oriunda est

$$\frac{2 e^{a x \cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} (-\cos.\theta \cos.\Phi - \sin.\theta \sin.\Phi) \int e^{-a x \cos.\theta} X \partial x \cos.\Phi \\ (-\cos.\theta \sin.\Phi + \sin.\theta \cos.\Phi) \int e^{-a x \cos.\theta} X \partial x \sin.\Phi \end{array} \right\}$$

seu

$$\frac{-2 e^{a x \cos.\theta}}{n a^{n-1}} [\cos.(\theta - \Phi) \int e^{-a x \cos.\theta} X \partial x \cos.\Phi - \sin.(\theta - \Phi) \int e^{-a x \cos.\theta} X \partial x \sin.\Phi]$$

et pro $\Phi$ valore restituto

$$\frac{-2 e^{a x \cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} \cos.(\theta + a x \sin.\theta) \int e^{-a x \cos.\theta} X \partial x \cos.(a x \sin.\theta) \\ + \sin.(\theta + a x \sin.\theta) \int e^{-a x \cos.\theta} X \partial x \sin.(a x \sin.\theta). \end{array} \right\}$$

Iam pro $\theta$ successiue substituantur anguli $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, $\frac{7\pi}{n}$, quamdiu ipso $\pi$ sunt minores, omnesque hae formae in vnam summam coniectae, quibus casu quo $n$ est numerus impar insuper addi oportet formam primo inuentam

$$\frac{1}{n a^{n-1}} e^{-a x} \int e^{a x} X \partial x,$$

dabunt integrale quaesitum.

## Corollarium 1.

1190. Casu quidem quo $n$ est numerus impar, vltimus valor ipsius $\theta$ foret $\pi$, quem autem hic omitti iussimus,

 inde

inde autem ob

$$a\,x\,\sin.\,\theta = 0 \quad\text{et}\quad \cos.\,\theta = -1,$$

prodiret vltima pars integralis

$$\frac{2\,e^{-a\,x}}{n\,a^{n-1}}\int e^{a\,x}\,X\,\partial x,$$

dupla eius quam capi conuenit; cuius ratio est, quod sumto $\theta = \pi$ formula $a\,a + 2\,a\,z + z\,z$ non amplius ipsa est factor, sed eius radix quadrata $a + z$, ex quo hunc casum seorsim erui necesse erat.

## Corollarium 2.

1191. Si est $X = 0$, formulae integrales abeunt in constantes arbitrarias, et ex factore

$$a\,a - 2\,a\,z\cos.\,\theta + z\,z$$

oritur haec pars integralis

$$\frac{-2\,e^{a\,x\cos.\,\theta}}{n\,a^{n-1}}\,[A\cos.\,(\theta + a\,x\sin.\,\theta) + \mathfrak{A}\sin.\,(\theta + a\,x\sin.\,\theta)],$$

quae reducitur ad hanc formam

$$A\,e^{a\,x\cos.\,\theta}\cos.\,(\zeta + a\,x\sin.\,\theta),$$

denotante $\zeta$ angulum constantem quemcunque, vti iam supra inuenimus.

## Problema 157.

1192. Proposita hac aequatione differentiali

$$X = a^n y - \frac{\partial^n y}{\partial x^n}$$

eius integrale completum inuenire.

## Solutio.

Forma algebraica hinc nata $P = a^n - z^n$ factorem semper habet $a - z$, vnde nascitur pars integralis $\frac{1}{n}e^{ax}\int e^{-ax}X\,\partial x$,

exi-

existente $\mathfrak{A} = \frac{P}{z-a}$ posito $z = a$. Cum ergo sit quoque

$$\mathfrak{A} = \frac{\partial P}{\partial z} = -n z^{n-1}, \text{ erit } \mathfrak{A} = -n a^{n-1},$$

ideoque haec pars integralis

$$\frac{-1}{n a^{n-1}} e^{ax} \int e^{-ax} X \partial x.$$

Deinde si $n$ sit numerus par, hincque $n-1$ impar, factor quoque erit $a+z$, qui praebet integralis partem

$$\frac{1}{n a^{n-1}} e^{-ax} \int e^{ax} X \partial x.$$

Reliqui factores omnes ipsius P sunt duplicis formae $aa - 2az \operatorname{cos}\theta + zz$, existente angulo $\theta = \frac{2i\pi}{n}$, qua cum generali supra vsurpata $ff + 2fz \operatorname{cos}\theta + zz$ comparata, fit $f = -a$, et ex forma $\frac{\partial P}{\partial z} = -n z^{n-1}$ quaeri oportet formulam $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, posito $z = a(\operatorname{cos}\theta + \sqrt{-1}.\operatorname{sin}\theta)$, vnde colligitur

$$\mathfrak{P} = -n a^{n-1} \operatorname{cos}(n-1)\theta \text{ et } \mathfrak{Q} = -n a^{n-1} \operatorname{sin}(n-1)\theta,$$

seu ob $\operatorname{cos} n\theta = 1$ et $\operatorname{sin} n\theta = 0$, fit

$$\mathfrak{P} = -n a^{n-1} \operatorname{cos}\theta \text{ et } \mathfrak{Q} = +n a^{n-1} \operatorname{sin}\theta.$$

Posito iam angulo $-ax \operatorname{sin}\theta = \Phi$, ex §. 1177. oritur pars integralis

$$\frac{2e^{ax\operatorname{cos}\theta}}{na^{n-1}}\left\{\begin{array}{l}(-\operatorname{cos}\theta\operatorname{cos}\Phi - \operatorname{sin}\theta\operatorname{sin}\Phi)\int e^{-ax\operatorname{cos}\theta} X\partial x \operatorname{cos}\Phi \\ +(-\operatorname{cos}\theta\operatorname{sin}\Phi + \operatorname{sin}\theta\operatorname{cos}\Phi)\int e^{-ax\operatorname{cos}\theta} X\partial x \operatorname{sin}\Phi\end{array}\right\},$$

quae vt ante reducitur ad hanc formam

$$\frac{-2e^{ax\operatorname{cos}\theta}}{na^{n-1}}\left\{\begin{array}{l}\operatorname{cos}(\theta + ax\operatorname{sin}\theta)\int e^{-ax\operatorname{cos}\theta} X\partial x \operatorname{cos}(ax\operatorname{sin}\theta) \\ +\operatorname{sin}(\theta + ax\operatorname{sin}\theta)\int e^{-ax\operatorname{cos}\theta} X\partial x \operatorname{sin}(ax\operatorname{sin}\theta)\end{array}\right\}.$$

Hic iam pro $\theta$ successiue scribantur anguli $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, etc. quamdiu sunt minores quam $\pi$, haeque partes omnes cum primum

mum inuenta atque etiam altera, si $n$ fuerit numerus par, in vnam summam collectae dabunt integrale quaesitum seu valorem ipsius $y$.

## Corollarium.

1193. Cum factor duplex generalis $aa - 2a\cos.\theta + zz$ casibus $\theta = 0$ et $\theta = \pi$ non praebeat ipsos factores simplices reales $a - z$ et $a + z$ sed eorum quadrata, haec ratio est, cur pars integralis inde eruta prodeat dupla eius, quam capi oportet.

## Problema 158.

1194. Proposita hac aequatione differentiali

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial\partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \ldots\ldots, + \frac{\partial^n y}{\partial x^n},$$

eius integrale completum inuestigare.

## Solutio.

Forma algebraica hinc nata est

$$P = 1 + z^2 + z^3 + z^4 + \ldots\ldots + z^n,$$

cuius omnes factores scrutari oportet. Cum igitur sit $P = \frac{1 - z^{n+1}}{1 - z}$, formae $1 - z^{n+1}$ factores capi conuenit, excluso $1 - z$; vnde primo patet, si fuerit $n + 1$ numerus par, factorem simplicem fore $1 + z$, ex quo nascitur pars integralis $\frac{1}{\mathfrak{A}} e^{-x} \int e^x X \partial x$, existente $\mathfrak{A} = \frac{P}{1+z} = \frac{1 - z^{n+1}}{1 + zz}$, posito $z = -1$. Erit ergo quoque $\mathfrak{A} = \frac{(n+1) z^n}{2z}$, ideoque $\mathfrak{A} = \frac{1}{2}(n + 1)$, vt haec pars integralis sit $\frac{2}{n+1} e^{-x} \int e^x X \partial x$.

Facto-

Factorum autem duplicium forma est $1 - 2z\cos.\theta + zz$, sumto angulo $\theta = \frac{2i\pi}{n+1}$, ita vt pro §. 1176. sit $f = -1$. Consideretur forma

$$\frac{\partial P}{\partial z} = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2},$$

quae posito $z = \cos.\theta + \sqrt{-1}.\sin.\theta$, abire sumitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, sicque erit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{1 - (n+1)\cos.n\theta + n\cos.(n+1)\theta - (n+1)\sqrt{-1}.\sin.n\theta + n\sqrt{-1}.\sin.(n+1)\theta}{1 - 2\cos.\theta + \cos.2\theta - 2\sqrt{-1}.\sin.\theta + \sqrt{-1}.\sin.2\theta}.$$

Cum vero sit

$$\sin.(n+1)\theta = 0 \text{ et } \cos.(n+1)\theta = 1, \text{ erit}$$

$$\sin.n\theta = -\sin.\theta \text{ et } \cos.n\theta = \cos.\theta, \text{ ideoque}$$

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{n + 1 - (n+1)\cos.\theta + (n+1)\sqrt{-1}.\sin.\theta}{-2\cos.\theta + 2\cos.\theta^2 - 2\sqrt{-1}.\sin.\theta(1-\cos.\theta)}, \text{ seu}$$

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{n+1}{2(1-\cos.\theta)} \cdot \frac{1 - \cos.\theta + \sqrt{-1}.\sin.\theta}{-\cos.\theta - \sqrt{-1}.\sin.\theta}:$$

multiplicetur huius fractionis numerator et denominator per $-\cos.\theta + \sqrt{-1}.\sin.\theta$ et prodibit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-(n+1)[1 + \cos.\theta - 2\cos.\theta^2 - \sqrt{-1}.\sin.\theta(1 - 2\cos.\theta)]}{2(1-\cos.\theta)},$$

ita vt sit

$$\mathfrak{P} = -\tfrac{1}{2}(n+1)(1 + 2\cos.\theta) \text{ et } \mathfrak{Q} = \tfrac{1}{2}(n+1)\frac{\sin.\theta(1 - 2\cos.\theta)}{1-\cos.\theta},$$

vnde fit

$$\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q} = \frac{(n+1)^2}{2(1-\cos.\theta)}.$$

Tum vero posito angulo $-x\sin.\theta = \Phi$, colligitur

$$\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi = \frac{-(n+1)[\cos.(\theta-\Phi) - \cos.(2\theta-\Phi)]}{2(1-\cos.\theta)},$$

$$\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi = \frac{+(n+1)[\sin.(\theta-\Phi) - \sin.(2\theta-\Phi)]}{2(1-\cos.\theta)},$$

cum autem sit

$$\cos.a - \cos.b = 2\sin.\frac{a+b}{2}\sin.\frac{b-a}{2}, \text{ et}$$

$$\sin.a - \sin.b = -2\sin.\frac{b-a}{2}\cos.\frac{a+b}{2},$$

 fit

fit hinc

$$\mathfrak{P}\,\text{cof.}\,\Phi - \mathfrak{Q}\,\text{fin.}\,\Phi = \frac{-(n+1)\,\text{fin.}\,\frac{1}{2}(3\,\theta - 2\,\Phi)}{2\,\text{fin.}\,\frac{1}{2}\,\theta} \text{ et}$$

$$\mathfrak{P}\,\text{fin.}\,\Phi + \mathfrak{Q}\,\text{cof.}\,\Phi = \frac{-(n+1)\,\text{cof.}\,\frac{1}{2}(3\,\theta - 2\,\Phi)}{2\,\text{fin.}\,\frac{1}{2}\,\theta},$$

ex quo integralis pars quaesita erit

$$\frac{-4}{n+1} e^{x\,\text{cof.}\,\theta}\,\text{fin.}\,\tfrac{1}{2}\theta \left\{ \begin{array}{l} \text{fin.}\,\frac{1}{2}(3\,\theta + 2\,x\,\text{fin.}\,\theta) \int e^{-x\,\text{cof.}\,\theta} X\,\partial x\,\text{cof.}\,(x\,\text{fin.}\,\theta) \\ -\text{cof.}\,\frac{1}{2}(3\,\theta + 2\,x\,\text{fin.}\,\theta) \int e^{-x\,\text{cof.}\,\theta} X\,\partial x\,\text{fin.}\,(x\,\text{fin.}\,\theta) \end{array} \right\}.$$

Pro $\theta$ ergo fucceffiue fubftituantur anguli

$$\frac{2\,\pi}{n+1},\ \frac{4\,\pi}{n+1},\ \frac{6\,\pi}{n+1},\ \text{etc.}$$

quamdiu funt minores quam $\pi$, haecque partes omnes in vnam fummam colligantur, cui fi $n+1$ fit numerus par, addatur infuper $\frac{2}{n+1} e^{-x} \int e^{x} X\,\partial x$, ficque obtinebitur valor ipfius $y$.

## Corollarium 1.

1195. Si aequatio propofita in infinitum progrediatur, vt fit $n$ numerus infinitus, anguli $\theta$ priores omnes funt infinite parui ideoque numero infiniti, quoad numerus par $2\,i$ ad $n+1$ rationem finitam habere incipiat, tum autem pro $\theta$ fequentur omnes anguli finiti in progreffione arithmetica increfcentes, cuius differentia eft $\frac{2\,\pi}{n+1}$, vsque ad $\pi$, quorum numerus itidem eft infinitus.

## Corollarium 2.

1196. Quamdiu angulus $\theta$ eft infinite paruus, integralis pars ex eo oriunda hanc induit formam

$$\frac{-\theta\,\theta\,e^{x}}{n+1}\left[(3+2\,x)\int e^{-x} X\,\partial x - \int e^{-x} X\,x\,\partial x\right],$$

quae

quae cum per cubum infiniti sit diuisa, etiam multitudo infinita huiusmodi formularum pro euanescente est habenda.

## Corollarium 3.

1197. Quodsi fuerit $X = 0$, vt huius aequationis

$$0 = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \dots\dots + \frac{\partial^n y}{\partial x^n}$$

integrale sit inuestigandum, erit eius pars quaecunque

$$e^{x\, cos.\, \theta}\,[A\, sin. \tfrac{1}{2}(3\theta + 2x\, sin.\, \theta) + \mathfrak{A}\, cos. \tfrac{1}{2}(3\theta + 2x\, sin.\, \theta)],$$

seu simplicius

$$A\, e^{x\, cos.\, \theta}\,(cos.\, \zeta + x\, sin.\, \theta).$$

Cum igitur si $n$ sit numerus infinitus, pro $\theta$ angulus quicunque accipi queat, erit istius aequationis integrale particulare quodcunque

$$y = A\, e^{x\, cos.\, \theta}\,(cos.\, x\, sin.\, \theta + \zeta),$$

sumendo pro $\zeta$ etiam angulum quemcunque.

## Scholion.

1198. Num autem huius aequationis differentialis in infinitum excurrentis

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} + \text{etc.}$$

denotante $X$ functionem quamcunque ipsius $x$, integrale commodius exprimi possit, quam per partium illarum innumerabilium euanescentium summam, quaestio est altioris indaginis, neque adhuc ad hunc scopum Analyseos fines satis videntur promoti. Casibus quidem, quibus $X$ est functio rationalis integra, puta

$$X = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.}$$

res nullam habet difficultatem, cum sumto

 $y =$

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.} + v,$$

hi coefficientes $\alpha$, $\beta$, $\gamma$, etc. semper ita definiri queant vt facta substitutione prodeat talis aequatio

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial \partial v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} + \text{etc.}$$

cui particulariter satisfacit valor

$$v = A e^{x \operatorname{cof.} \theta} \operatorname{cof.} (x \operatorname{fin.} \theta + \zeta),$$

sumtis pro $\zeta$ et $\theta$ angulis quibuscunque. Verum ex dato eiusmodi valore ipsius X inuenitur

$$\alpha = a - b,\ \beta = b - 2c,\ \gamma = c - 3d,\ \delta = d - 4e,\ \varepsilon = e - 5f,\ \text{etc.}$$

Verum in genere cum fiat

$$\frac{\partial X}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

euidens est semper, posito $y = X - \frac{\partial X}{\partial x} + v$, aequationem illam transformari in hanc

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial \partial v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} + \text{etc.}$$

## Corollarium.

1199. En ergo praeter expectationem integrationem completam huius aequationis differentialis in infinitum excurrentis

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} + \text{etc.}$$

pro qua iam nouimus esse

$$y = X - \frac{\partial X}{\partial x} + A e^{x \operatorname{cof.} \theta} \operatorname{cof.} (x \operatorname{fin.} \theta + \zeta),$$

quod postremum membrum ob angulos $\zeta$ et $\theta$ arbitrarios in infinitum multiplicari potest. Haecque forma maxime complicatae illi ex solutione oriundae aequiualere est censenda.

## Problema 159.

1200. Proposita hac aequatione differentiali

$$X = y + \frac{n \partial y}{a \partial x} + \frac{n(n-1) \partial \partial y}{1.2.a^2 \partial x^2} + \frac{n(n-1)(n-2) \partial^3 y}{1.\ 2.\ 3\ a^3 \partial x^3} + \text{etc.}$$

vbi

vbi quidem $n$ ſit numerus integer affirmatiuus, vt terminorum numerus ſit finitus, eius integrale completum inueſtigare.

## Solutio.

Formula algebraica hinc conſideranda ſit

$$P = 1 + \frac{n}{1} \cdot \frac{z}{a} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{z^2}{a^2} + \text{etc.} = (1 + \frac{z}{a})^n,$$

quae ergo meros habet factores ſimplices inter ſe aequales $z + a$. Cum igitur ſit $\frac{P}{(a+z)^n} = \frac{1}{a^n}$, ex §. 1163. ſtatim colligitur integrale quaeſitum

$$y = a^n e^{-ax} \int \partial x \int \partial x \int \partial x \ldots \ldots \int e^{ax} X \partial x,$$

quoad ſignorum integralium numerus aequetur exponenti $n$. Hanc autem formam ſequenti modo in integralia ſimplicia reſoluere licet, ope reductionis generalis qua eſſe nouimus

$$\int \partial x \int V \partial x = x \int V \partial x - \int V . x \partial x,$$

vnde fit

$$\int \partial x \int e^{ax} X \partial x = x \int e^{ax} X \partial x - \int e^{ax} X x \partial x,$$

$$\int \partial x \int \partial x \int e^{ax} X \partial x = \tfrac{1}{2} x^2 \int e^{ax} X \partial x - x \int e^{ax} X x \partial x + \tfrac{1}{2} \int e^{ax} X x^2 \partial x,$$

$$\int \partial x \int \partial x \int \partial x \int e^{ax} X \partial x = \frac{x^3 \int e^{ax} X \partial x - 3x^2 \int e^{ax} X x \partial x + 3x \int e^{ax} X x^2 \partial x - \int e^{ax} X x^3 \partial x}{1 . \quad 2 . \quad 3},$$

etc.

Cum igitur ſignorum integralium numerus ſit $= n$, concludimus fore

$$y = \frac{a^n e^{-x}}{1 . 2 \ldots \ldots (n-1)} [x^{n-1} \int e^{ax} X \partial x - \frac{(n-1)}{1} x^{n-2} \int e^{ax} X x \partial x$$
$$+ \frac{(n-1)(n-2)}{1 \cdot 2} x^{n-3} \int e^{ax} X x^2 \partial x - \text{etc.}],$$

vbi cum ſingula integralia conſtantem arbitrariam implicent, manifeſtum eſt, hoc integrale eſſe completum.

## Corollarium 1.

1201. Si ergo esset $X = 0$, aequationis differentialis propositae integrale completum foret

$$y = e^{-ax}(Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} \ldots + Mx + N),$$

vbi constantium arbitrariarum A, B, C, etc. numerus vtique est $= n$.

## Corollarium 2.

1202. Si numerus $n$ fuerit infinitus, simulque quantitas $a$ capiatur infinita, vt sit $a = nc$, aequatio integranda in infinitum excurret, eritque

$$X = y + \frac{\partial y}{c\,\partial x} + \frac{\partial\partial y}{1.2\,c^2\,\partial x^2} + \frac{\partial^3 y}{1.2.3\,c^3\,\partial x^3} + \text{etc.}$$

aequatio autem integralis ad hunc casum applicata nullam lucem foeneratur.

## Corollarium 3.

1203. Quaecunque autem $y$ functio fuerit ipsius $x$, constat si loco $x$ scribatur $x + \frac{1}{c}$, eam abire in

$$y + \frac{\partial y}{c\,\partial x} + \frac{\partial\partial y}{1.2\,c^2\,\partial x^2} + \frac{\partial^3 y}{1.2.3\,c^3\,\partial x^3} + \text{etc.}$$

quae cum esse debeat $= X$, viciſſim patet, $y$ aequari ei functioni ipsius $x$ quae nascitur ex X, si ibi loco $x$ scribatur $x - \frac{1}{c}$.

## Scholion 1.

1204. Quod quo facilius appareat obseruo, si proposita fuerit quaecunque eiusmodi aequatio

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

semper sine vlla integratione integrale particulare per approximationem hoc modo inueniri posse: statuatur

$$y = \alpha X + \beta\frac{\partial X}{\partial x} + \gamma\frac{\partial\partial X}{\partial x^2} + \delta\frac{\partial^3 X}{\partial x^3} + \text{etc.}$$

facta-

factaque fubftitutione habebitur

$$\begin{aligned} X = A\alpha X &+ A\beta \cdot \tfrac{\partial X}{\partial x} + A\gamma \cdot \tfrac{\partial \partial X}{\partial x^2} + A\delta \cdot \tfrac{\partial^3 X}{\partial x^3} + \text{etc.} \\ &+ B\alpha \quad\;\; + B\beta \quad\;\;\; + B\gamma \\ &\qquad\qquad + C\alpha \quad\;\;\; + C\beta \\ &\qquad\qquad\qquad\qquad\;\; + D\alpha \end{aligned}$$

ficque coefficientes $\alpha$, $\beta$, $\gamma$, $\delta$, etc. definiuntur, vt fit $\alpha = \frac{1}{A}$, reliqui vero

$$\beta = \frac{-B\alpha}{A} = \frac{-B}{A^2},$$
$$\gamma = \frac{-C\alpha - B\beta}{A} = \frac{-C}{A^2} + \frac{BB}{A^3},$$
$$\delta = \frac{-D\alpha - C\beta - B\gamma}{A} = \frac{-D}{A^2} + \frac{2BC}{A^3} - \frac{B^3}{A^4},$$
$$\epsilon = \frac{-E\alpha - D\beta - C\gamma - D\delta}{A} = \frac{-E}{A^2} + \frac{2BD + C^2}{A^3} - \frac{3BBC}{A^4} + \frac{B^4}{A^5},$$
etc.

quae fi accommodentur ad cafum problematis, fiet

$$y = X - \frac{n\,\partial X}{1\,a\,\partial x} + \frac{n(n+1)\,\partial\partial X}{1 \cdot 2\, a^2\, \partial x^2} - \frac{n(n+1)(n+2)\,\partial^3 X}{1 \cdot 2 \cdot 3\, a^3\, \partial x^3} + \text{etc.}$$

Hinc cafu quo $n = \infty$ et $a = nc$ colligitur

$$y = X - \frac{\partial X}{1\,c\,\partial x} + \frac{\partial\partial X}{1 \cdot 2\, c^2\, \partial x^2} - \frac{\partial^3 X}{1 \cdot 2 \cdot 3\, c^3\, \partial x^3} + \text{etc.}$$

quae expreffio etfi in infinitum excurrens manifefto definit eam ipfius $x$ functionem, quae nafcitur ex X, fi loco $x$ fcribatur $x - \frac{1}{c}$. Quodfi iam hanc nouam functionem figno $X'$ indicemus, ponamusque $y = X' + v$, aequatio Corollarii 2. abit in hanc

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial\partial v}{1 \cdot 2\, \partial x} + \frac{\partial^3 v}{1 \cdot 2 \cdot 3\, \partial x^3} + \text{etc.}$$

cuius integrale particulare quodcunque eft $v = A e^{-ncx} x^m$ exiftente $n$ numero infinito, et $m$ numero integro pofitiuo.

## Scholion 2.

1205. Haec me deducunt ad fequentem fpeculationem circa ferierum fummationem. Sit nempe feries quaecunque

$$\overset{1}{A},\ \overset{2}{B},\ \overset{3}{C},\ \overset{4}{D},\ \ldots\ldots\ \overset{x}{T},$$

cuius terminus indici $x$ respondens sit T functio quaecunque ipsius $x$. Statuatur summa omnium horum terminorum

$$A + B + C + D + \ldots\ldots + T = y,$$

at perspicuum est $y$ fore eiusmodi functionem ipsius $x$, vt si in ea loco $x$ scribatur $x - 1$, proditura sit eadem illa summa $y$ termino vltimo T mulctata, scilicet $y - T$. At loco $x$ scribendo $x - 1$, functio $y$ abit in

$$y - \frac{\partial y}{\partial x} + \frac{\partial\partial y}{1.2\,\partial x^2} - \frac{\partial^3 y}{1.2.3\,\partial x^3} + \text{etc.}$$

vnde oritur haec aequatio

$$T = \frac{\partial y}{\partial x} - \frac{\partial\partial y}{1.2\,\partial x^2} + \frac{\partial^3 y}{1.2.3\,\partial x^3} - \frac{\partial^4 y}{1.2.3.4\,\partial x^4} + \text{etc.}$$

quae semel integrata posito $\int T\,\partial x = X$, fit

$$X = y - \frac{\partial y}{1.2\,\partial x} + \frac{\partial\partial y}{1.2.3\,\partial x^2} - \frac{\partial^3 y}{1.2.3.4\,\partial x^3} + \text{etc.}$$

quam quomodo integrari conueniat videamus, dum eam aliquanto generaliorem reddemus.

## Problema 160.

1206. Proposita hac aequatione differentiali

$$X = \frac{ny}{a} - \frac{n(n-1)\partial y}{1.2\,a^2\,\partial x} + \frac{n(n-1)(n-2)\partial\partial y}{1.2.3\,a^3\,\partial x^2} - \text{etc.}$$

eius integrale completum inuestigare.

## Solutio.

Formetur inde haec quantitas algebraica

$$P = \frac{n}{a} - \frac{n(n-1)z}{1.\ 2\,a^2} + \frac{n(n-1)(n-2)zz}{1.\ 2.\ 3\,a^3} - \text{etc.} = \frac{1-(1-\frac{z}{a})^n}{z},$$

seu $P = \dfrac{a^n - (a-z)^n}{a^n z}$, cuius factor duplex quicunque hanc habe-

habebit formam

$$a\,a - 2\,a\,(a - z)\,\text{cof.}\,2\,\zeta + (a - z)^2,$$

exiftente angulo $2\,\zeta = \frac{2\,i\,\pi}{n}$. Abit autem haec forma in

$$2\,a\,a\,(1 - \text{cof.}\,2\,\zeta) - 2\,a\,z\,(1 - \text{cof.}\,2\,\zeta) + z\,z,$$

vel $4\,a\,a\,\text{fin.}\,\zeta^2 - 4\,a\,z\,\text{fin.}\,\zeta^2 + z\,z$,

quae cum generali $f\,f + 2\,f\,z\,\text{cof.}\,\theta + z\,z$ comparata dat

$$f = 2\,a\,\text{fin.}\,\zeta, \text{ et } \text{cof.}\,\theta = -\,\text{fin.}\,\zeta,$$

vnde

$$\theta = 90^\circ + \zeta, \text{ et } \text{fin.}\,\theta = \text{cof.}\,\zeta,$$

exiftente $\zeta = \frac{i\,\pi}{n}$. Iam ad partem integralis hinc ortam inueniendam confideretur forma

$$\frac{\partial\,P}{\partial\,z} = \frac{-\,a^n + [a + (n - 1)\,z]\,(a - z)^{n-1}}{a^n\,z\,z},$$

in qua pofito

$$z = -\,f\,(\text{cof.}\,\theta + \sqrt{-1}.\,\text{fin.}\,\theta), \text{ feu}$$

$$z = 2\,a\,\text{fin.}\,\zeta\,(\text{fin.}\,\zeta - \sqrt{-1}.\,\text{cof.}\,\zeta) = a\,(1 - \text{cof.}\,2\,\zeta - \sqrt{-1}.\,\text{fin.}\,2\,\zeta),$$

vt fit

$$a - z = a\,(\text{cof.}\,2\,\zeta + \sqrt{-1}.\,\text{fin.}\,2\,\zeta),$$

prodit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-1 + [n - (n-1)(\text{cof.}\,2\,\zeta + \sqrt{-1}.\,\text{fin.}\,2\,\zeta)]\,[\text{cof.}\,2\,(n-1)\,\zeta + \sqrt{-1}.\,\text{fin.}\,2\,(n-1)\,\zeta]}{-\,4\,a\,a\,\text{fin.}\,\zeta^2\,(\text{cof.}\,2\,\zeta + \sqrt{-1}.\,\text{fin.}\,2\,\zeta)}.$$

Cum autem fit

$$\text{cof.}\,2\,n\,\zeta = 1 \text{ et } \text{fin.}\,2\,n\,\zeta = 0, \text{ erit}$$

$$\text{cof.}\,2\,(n-1)\,\zeta = \text{cof.}\,2\,\zeta \text{ et } \text{fin.}\,2\,(n-1)\,\zeta = -\,\text{fin.}\,2\,\zeta,$$

ideoque

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-\,n + n\,(\text{cof.}\,2\,\zeta - \sqrt{-1}.\,\text{fin.}\,2\,\zeta)}{-\,4\,a\,a\,\text{fin.}\,\zeta^2\,(\text{cof.}\,2\,\zeta + \sqrt{-1}.\,\text{fin.}\,2\,\zeta)},$$

quae reducitur ad hanc formam

$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{4aa\sin.\zeta^2}(\cos.2\zeta-\sqrt{-1}.\sin.2\zeta-\cos.4\zeta+\sqrt{-1}\sin.4\zeta)$,
vnde concluditur

$$\mathfrak{P}=\frac{n}{4aa\sin.\zeta^2}(\cos.2\zeta-\cos.4\zeta)=\frac{n}{2aa\sin.\zeta}\sin.3\zeta,$$
$$\mathfrak{Q}=\frac{-n}{4aa\sin.\zeta^2}(\sin.2\zeta-\sin.4\zeta)=\frac{n}{2aa\sin.\zeta}\cos.3\zeta,$$

sicque est

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{nn}{4a^4\sin.\zeta^2},$$

et posito

$$\Phi=2ax\sin.\zeta\cos.\zeta=ax\sin.2\zeta,\ \text{fiet}$$
$$\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi=\frac{n}{2aa\sin.\zeta}\sin.(3\zeta-\Phi)\ \text{et}$$
$$\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi=\frac{n}{2aa\sin.\zeta}\cos.(3\zeta-\Phi).$$

Quocirca integralis pars hinc oriunda erit

$$\frac{4aa\sin.\zeta}{n}e^{2ax\sin.\zeta^2}\left\{\begin{array}{l}\sin.(3\zeta-\Phi)\int e^{-2ax\sin.\zeta^2}X\partial x\cos.\Phi\\+\cos.(3\zeta-\Phi)\int e^{-2ax\sin.\zeta^2}X\partial x\sin.\Phi\end{array}\right\},$$

vbi pro $\zeta$ successiue scribi debent hi anguli

$$\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \frac{4\pi}{n},\ \text{etc.}$$

quamdiu sunt angulo recto minores, at si $n$ sit numerus par ad has partes insuper addi oportet

$$-\frac{2aa}{n}e^{2ax}\int e^{-2ax}X\partial x,$$

sicque colligetur verus valor ipsius $y$.

## Corollarium 1.

1207. Si est $X=0$, pars integralis ex quolibet angulo $\zeta=\frac{i\pi}{n}$ nata induit hanc formam

$$e^{2ax\sin.\zeta^2}[A\sin.(3\zeta-ax\sin.2\zeta)+B\cos.(3\zeta-ax\sin.2\zeta)],$$

seu hanc

$$Ae^{2ax\sin.\zeta^2}\sin.(\alpha+ax\sin.2\zeta),$$

denotante $\alpha$ angulum quemcunque constantem.

Co-

## Corollarium 2.

1208. Inuento integrali particulari quocunque $y = V$ quod aequationi propositae satisfaciat, si ponamus deinceps $y = V + v$, orietur haec aequatio

$$0 = \frac{nv}{a} - \frac{n(n-1)\partial v}{1.2 a^2 \partial x} + \frac{n(n-1)(n-2)\partial\partial v}{1.2.3 a^3 \partial x^2} - \text{etc.}$$

ex quo integrale completum erit

$$y = V + A e^{2 a x \sin. \zeta^2} \sin. (\alpha + a x \sin. 2 \zeta),$$

vltima hac parte secundum omnes valores ipsius $\zeta$ multiplicata.

## Corollarium 3.

1209. Si sumamus $n = \infty$ et $a = n$, vt haec prodeat aequatio differentialis in infinitum excurrens

$$X = y - \frac{\partial y}{1.2 \partial x} + \frac{\partial\partial y}{1.2.3 \partial x^2} - \frac{\partial^3 y}{1.2.3.4 \partial x^3} + \frac{\partial^4 y}{1.2.3.4.5 \partial x^4} - \text{etc.}$$

erit $y$ terminus summatorius progressionis, cuius terminus generalis indici $x$ respondens est $T = \frac{\partial X}{\partial x}$. Quamdiu ergo angulus $\zeta = \frac{i\pi}{n}$ est infinite paruus, ob $\Phi = 2 i \pi x$, integralis pars quaelibet est

$$4 i \pi . e^{\frac{2 i i \pi\pi}{n} x} \left\{ \begin{array}{l} \sin. (\frac{3 i \pi}{n} - 2 i \pi x) \int e^{\frac{-2 i i \pi\pi}{n} x} X \partial x \operatorname{cos.} (2 i \pi x) \\ + \operatorname{cos.} (\frac{3 i \pi}{n} - 2 i \pi x) \int e^{\frac{-2 i i \pi\pi}{n} x} X \partial x \sin. (2 i \pi x) \end{array} \right\},$$

et omissis euanescentibus

$$4 i \pi [\operatorname{cos.}(2 i \pi x) \int X \partial x \sin.(2 i \pi x) - \sin.(2 i \pi x) \int X \partial x \operatorname{cos.}(2 i \pi x)],$$

si iam hic pro $i$ successiue omnes numeri integri 1, 2, 3, etc. substituantur, omnium formularum hoc modo resultantium summa dabit verum et completum valorem ipsius $y$.

## Scholion.

1210. Pro aequatione autem proposita methodo ante indicata integrale particulare per seriem differentialium inuenire licet, ponendo

$$y = AX + \frac{B\partial X}{\partial x} + \frac{C\partial\partial X}{\partial x^2} + \frac{D\partial^3 X}{\partial x^3} + \frac{E\partial^4 X}{\partial x^4} + \text{etc.}$$

facta enim substitutione reperitur

$$A = \frac{a}{n},\ B = \frac{n-1}{2n},\ C = \frac{nn-1}{12an},\ D = \frac{nn-1}{24aan},$$
$$E = \frac{-(nn-1)(nn-19)}{720a^3n},\ \text{etc.}$$

cuius quidem seriei difficile est legem progressionis in genere assignare. Verum pro casu $n = \infty$ et $a = n$, qui imprimis in doctrina progressionum est notatu dignus, hi coefficientes ita se habent.

$$A = 1,\ B = \tfrac{1}{2},\ C = \tfrac{1}{12},\ D = 0,\ E = \tfrac{-1}{720},\ \text{etc.}$$

vnde ea ipsa forma oritur, quam olim in genere pro termino summatorio dedi. Concesso autem hoc termino summatorio qui fit $= V$, probe notari conuenit, aequationem $y = V$ tantum esse integrale particulare aequationis propositae, completum vero facile exhiberi, si modo ad V addantur omnes huiusmodi formulae A sin. $(\alpha + 2i\pi x)$, pro $i$ scribendo successiue omnes numeros 1, 2, 3, 4, etc. vbi pro quolibet angulus $\alpha$ pro arbitrio assumi potest. Quod autem singuli hi valores aequationi

$$0 = v - \frac{\partial v}{2\partial x} + \frac{\partial\partial v}{6\partial x^2} - \frac{\partial^3 v}{24\partial x^3} + \frac{\partial^4 v}{120\partial x^4} - \frac{\partial^5 v}{720\partial x^5} + \text{etc.}$$

satisfaciant, ita facillime ostenditur. Posito breuitatis gratia $2i\pi = m$, vt sit $v = \text{fin.}(\alpha + mx)$, et facta substitutione fieri debet

$$0 = \begin{cases} \text{fin.}(\alpha + mx)(1 - \frac{mm}{6} + \frac{m^4}{120} - \text{etc.}) \\ \text{cof.}(\alpha + mx)(-\frac{m}{2} + \frac{m^3}{24} - \frac{m^5}{720} + \text{etc.}) \end{cases} = \begin{cases} \text{fin.}(\alpha + mx)\frac{1}{m}\,\text{fin.}\, m \\ \text{cof.}(\alpha + mx)\frac{1}{m}(\text{cof.}\, m - 1). \end{cases}$$

Cum autem sit $m = 2i\pi$, manifesto est tam fin. $m = 0$ quam cof. $m - 1 = 0$.

Pro-

## Problema 161.

1211. Propofita hac aequatione differentiali

$$X = y + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{\partial \partial y}{a^2 \partial x^2} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{\partial^4 y}{a^4 \partial x^4} + \text{etc.}$$

eius integrale completum inueftigare.

## Solutio.

Quantitas algebraica hinc formanda eft

$$P = 1 + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{zz}{aa} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{z^4}{a^4} + \text{etc.}$$

quae ad hanc formam manifefto reducitur

$$P = \tfrac{1}{2}(1 + \tfrac{z}{a})^n + \tfrac{1}{2} \cdot (1 - \tfrac{z}{a})^n = \frac{(a+z)^n + (a-z)^n}{2\, a^n},$$

cuius factor quicunque trinomialis eft

$$(a+z)^2 - 2(aa - zz)\,\text{cof.}\, 2\zeta + (a-z)^2,$$

fumendo

$$2\zeta = \frac{(2i+1)\pi}{n}, \text{ feu } \zeta = \frac{(2i+1)\pi}{2n}.$$

Haec autem forma abit in

$$2aa(1 - \text{cof.}\, 2\zeta) + 2zz(1 + \text{cof.}\, 2\zeta) = 4aa\,\text{fin.}\,\zeta^2 + 4zz\,\text{cof.}\,\zeta^2,$$

qui factor generalis repraefentetur hoc modo

$$aa\,\text{tang.}\,\zeta^2 + zz,$$

ficque comparatio cum forma generali

$$ff + 2fz\,\text{cof.}\,\theta + zz$$

praebet

$$f = -a\,\text{tang.}\,\zeta \text{ et } \theta = 90^\circ,$$

vnde fit

$$\Phi = -ax\,\text{tang.}\,\zeta, \; (1177.)$$

et valor pro $z$ fubftituendus

$$-f(\text{cof.}\,\theta + \sqrt{-1} \cdot \text{fin.}\,\theta) = a\,\text{tang.}\,\zeta \cdot \sqrt{-1};$$

 quo

quo pacto

$$\frac{\partial P}{\partial z} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2a^n}$$

abire ponitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$,

vnde fit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \tfrac{n}{2a}[(1+\text{tang.}\zeta.\sqrt{-1})^{n-1} - (1-\text{tang.}\zeta.\sqrt{-1})^{n-1}]$$

$$= \frac{n}{2a\,\text{cos.}\zeta^{n-1}}[\text{cos.}(n-1)\zeta + \sqrt{-1}.\text{sin.}(n-1)\zeta - \text{cos.}(n-1)\zeta$$
$$+\sqrt{-1}.\text{sin.}(n-1)\zeta],$$

ideoque $\mathfrak{P} = 0$ et $\mathfrak{Q} = \frac{n\,\text{sin.}(n-1)\zeta}{a\,\text{cos.}\zeta^{n-1}}$. At ob $n\zeta = \frac{2i+1}{2}\pi$, hincque cos. $n\zeta = 0$ et sin. $n\zeta = \pm 1$, prout $i$ fuerit numerus par vel impar, erit sin. $(n-1)\zeta = \pm$ cos. $\zeta$, ideoque $\mathfrak{Q} = \frac{\pm n}{a\,\text{cos.}\zeta^{n-2}}$. Quocirca ob cos. $\theta = 0$, integralis pars ex hoc factore oriunda est

$$\pm\frac{2a\,\text{cos.}\zeta^{n-2}}{n}(\text{cos.}\Phi\int X\partial x\,\text{sin.}\Phi - \text{sin.}\Phi\int X\partial x\,\text{cos.}\Phi),$$

seu ob $\Phi = -ax\,\text{tang.}\zeta$,

$$\pm\frac{2a\,\text{cos.}\zeta^{n-2}}{n}\left\{\begin{array}{l}\text{sin.}(ax\,\text{tang.}\zeta)\int X\partial x\,\text{cos.}(ax\,\text{tang.}\zeta)\\ -\text{cos.}(ax\,\text{tang.}\zeta)\int X\partial x\,\text{sin.}(ax\,\text{tang.}\zeta)\end{array}\right\};$$

vbi pro $\zeta$ successiue substituantur anguli

$$\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n},$$

quamdiu sunt recto minores, pro quibus ibi alternatim + et — scribi oportet; haeque partes omnes in vnam summam collectae dabunt valorem completum ipsius $y$, dummodo pro vltima

vltima parte ex angulo $\zeta = \frac{\pi}{2}$ oriunda, quod euenit si $n$ numerus impar, eius tantum semissis capiatur.

## Corollarium 1.

1212. Accommodemus haec statim ad casum $n = \infty$ et $a = nc$, vt proposita sit haec aequatio differentialis

$$X = y + \frac{\partial\partial y}{1.2 c^2 \partial x^2} + \frac{\partial^4 y}{1.2.3.4 c^4 \partial x^4} + \frac{\partial^6 y}{1 \ldots 6 c^6 \partial x^6} + \text{etc. in infinitum.}$$

Cum igitur hic valores ipsius $\zeta$ sint infinite parui, erit

$$\text{cos.}\,\zeta = 1 \text{ et tang.}\,\zeta = \zeta = \frac{(4i \pm 1)\pi}{2n},$$

hinc $ax \,\text{tang.}\,\zeta = (4i \pm 1) cx \cdot \frac{\pi}{2}$,
pro quo angulo scribamus $\omega$. Ergo pars integralis quaecunque

$$\pm 2c (\text{sin.}\,\omega \int X \partial x \,\text{cos.}\,\omega - \text{cos.}\,\omega \int X \partial x \,\text{sin.}\,\omega),$$

vbi signa ambigua sibi mutuo respondent.

## Corollarium 2.

1213. Si tantum angulus $\frac{\pi}{2} cx$ ponatur $= \Phi$, integrale vniuersum ita erit expressum

$$\begin{aligned} \frac{y}{2c} = &+ \text{sin.}\,\Phi \int X \partial x \,\text{cos.}\,\Phi - \text{cos.}\,\Phi \int X \partial x \,\text{sin.}\,\Phi \\ &- \text{sin.}\,3\Phi \int X \partial x \,\text{cos.}\,3\Phi + \text{cos.}\,3\Phi \int X \partial x \,\text{sin.}\,3\Phi \\ &+ \text{sin.}\,5\Phi \int X \partial x \,\text{cos.}\,5\Phi - \text{cos.}\,5\Phi \int X \partial x \,\text{sin.}\,5\Phi \\ &- \text{sin.}\,7\Phi \int X \partial x \,\text{cos.}\,7\Phi + \text{cos.}\,7\Phi \int X \partial x \,\text{sin.}\,7\Phi \\ &\text{etc.} \end{aligned}$$

quae formulae in infinitum sunt continuandae.

## Corollarium 3.

1214. Si ponamus $c = b\sqrt{-1}$, vt habetur haec aequatio infinita

$$X = y - \frac{\partial\partial y}{1.2 b^2 \partial x^2} + \frac{\partial^4 y}{1 \ldots 4 b^4 \partial x^4} - \frac{\partial^6 y}{1 \ldots 6 b^6 \partial x^6} + \text{etc.}$$

ac

... angulum $\frac{\pi}{2} b x$ vocemus $\psi$, erit integrale completum

$$z = + e^{-\psi} \int e^{\psi} X \partial x - e^{\psi} \int e^{-\psi} X \partial x$$
$$- e^{-3\psi} \int e^{3\psi} X \partial x + e^{3\psi} \int e^{-3\psi} X \partial x$$
$$+ e^{-5\psi} \int e^{5\psi} X \partial x - e^{5\psi} \int e^{-5\psi} X \partial x$$
$$\text{etc.}$$

## Scholion.

1215. Si pro aequatione Corollarii 1. methodo supra exposita quaeramus integrale particulare per differentialia ipsius X expressum, huncque in finem ponamus

$$y = A X - \frac{B \partial\partial X}{c^2 \partial x^2} + \frac{C \partial^4 X}{c^4 \partial x^4} - \frac{D \partial^6 X}{c^6 \partial x^6} + \frac{E \partial^8 X}{c^8 \partial x^8} - \text{etc.}$$

reperiemus hos coefficientium valores

$$A = 1,\ B = \frac{1}{1.2},\ C = \frac{5}{1.\ldots.4},\ D = \frac{61}{1.\ldots.6},$$
$$E = \frac{1385}{1.\ldots.8},\ F = \frac{50521}{1.\ldots.10},\ \text{etc.}$$

Hicque valor si ponatur $= V$, vocato angulo $\frac{\pi}{2} c x = \Phi$, erit integrale completum

$$y = V + A \sin.(\alpha + \Phi) + B \sin.(\beta + 3\Phi) + C \sin.(\gamma + 5\Phi)$$
$$+ D \sin.(\delta + 7\Phi) + \text{etc.}$$

## Problema 162.

1216. Proposita aequatione differentiali

$$X = y + \frac{n(n-1)}{1.2 a^2} \cdot \frac{\partial y}{\partial x} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4 a^4} \cdot \frac{\partial\partial y}{\partial x^2}$$
$$+ \frac{n \ldots (n-5)}{1.\ldots.6 a^6} \cdot \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

eius integrale completum inuestigare.

## Solutio.

Quantitas algebraica hinc formanda

$$P = 1 + \frac{n(n-1)}{1.2\,a^2} z + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4\,a^4} . zz + \text{etc.}$$
$$= \tfrac{1}{2}\left(1 + \tfrac{\sqrt{z}}{a}\right)^n + \tfrac{1}{2}\left(1 - \tfrac{\sqrt{z}}{a}\right)^n,$$

cum ex praecedente nafcatur, fi ibi loco $zz$ fcribatur $z$, fumto angulo $\zeta = \frac{2i+1}{2n}\pi$, factor quicunque erit $aa\,\mathrm{tang.}\,\zeta^2 + z$, ita vt huius formae omnes factores fimplices fint reales. Hoc ergo factore cum formula $\alpha + z$ comparato, erit $\alpha = aa\,\mathrm{tang.}\,\zeta^2$, et fumto $\mathfrak{A} = \frac{P}{\alpha + z}$ pofito $z = -\alpha$, erit integralis pars ex hoc factore oriunda $\mathfrak{A}\, e^{-\alpha x} \int e^{\alpha x} X \partial x$. Quia vero P euanefcit pofito $z = -\alpha$, erit quoque $\mathfrak{A} = \frac{\partial P}{\partial z}$; at eft differentiando

$$\frac{\partial P}{\partial z} = \frac{n}{4a\sqrt{z}}\left[\left(1 + \tfrac{\sqrt{z}}{a}\right)^{n-1} - \left(1 - \tfrac{\sqrt{z}}{a}\right)^{n-1}\right].$$

Quia igitur poni oportet $\frac{\sqrt{z}}{a} = \mathrm{tang.}\,\zeta . \sqrt{-1}$, erit

$$1 + \frac{\sqrt{z}}{a} = \frac{\mathrm{cof.}\,\zeta + \sqrt{-1}\,.\,\mathrm{fin.}\,\zeta}{\mathrm{cof.}\,\zeta} \quad \text{et} \quad 1 - \frac{\sqrt{z}}{a} = \frac{\mathrm{cof.}\,\zeta - \sqrt{-1}\,.\,\mathrm{fin.}\,\zeta}{\mathrm{cof.}\,\zeta}$$

hincque

$$\mathfrak{A} = \frac{n}{4a \cdot \mathrm{tang.}\,\zeta . \sqrt{-1}} \cdot \frac{2\sqrt{-1}\,.\,\mathrm{fin.}\,(n-1)\zeta}{\mathrm{cof.}\,\zeta^{n-1}} = \frac{n\,\mathrm{fin.}\,(n-1)\zeta}{2aa\,\mathrm{fin.}\,\zeta\,\mathrm{cof.}\,\zeta^{n-2}}.$$

Iam obferuetur effe $\mathrm{fin.}\, n\zeta = \mathrm{fin.}\,(2i+1)\frac{\pi}{2} = \pm 1$, (vbi fignum fuperius valet fi $i$ numerus par, inferius fi impar,) tum vero $\mathrm{cof.}\, n\zeta = 0$, vnde fit $\mathrm{fin.}\,(n-1)\zeta = \pm\,\mathrm{cof.}\,\zeta$, ex quo conficitur

$$\mathfrak{A} = \frac{\pm n}{2aa\,\mathrm{fin.}\,\zeta\,\mathrm{cof.}\,\zeta^{n-3}},$$

et pars integralis quaefita habebitur

$$\pm \frac{2aa\,\mathrm{fin.}\,\zeta\,\mathrm{cof.}\,\zeta^{n-3}}{n} e^{-aa\,\mathrm{tang.}\,\zeta^2 . x} \int e^{aa\,\mathrm{tang.}\,\zeta^2 . x} X \partial x.$$

Nunc igitur ipfi $\zeta$ fucceffiue tribuantur hi valores

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},\ \text{etc.}$$

quoad angulum rectum non fuperent, atque omnes iftae partes

tes in vnam fummam collectae, dabunt integrale completum feu valorem ipfius $y$.

## Corollarium 1.

1217. Si ponamus $n = \infty$ et $a = nc$, aequatio propofita in infinitum excurrit, eritque

$$X = y + \frac{\partial y}{1.2 c^2 \partial x} + \frac{\partial \partial y}{1.2.3.4 c^4 \partial x^2} + \frac{\partial^3 y}{1 \ldots 6 c^6 \partial x^3} + \text{etc.}$$

et forma algebraica inde nata

$$P = 1 + \frac{z}{1.2 c^2} + \frac{zz}{1.2.3.4 c^4} + \frac{z^3}{1 \ldots 6 c^6} + \text{etc.} = \tfrac{1}{2} e^{\frac{\sqrt{z}}{c}} + \tfrac{1}{2} e^{\frac{-\sqrt{z}}{c}},$$

quae omnes factores fimplices habet reales, et ob $\zeta$ infinite paruum erit tang. $\zeta = \zeta = \frac{2i+1}{2n}\pi$, indeque factorum forma generalis

$$z + \frac{(2i+1)^2}{4}\pi\pi cc, \text{ feu } 1 + \frac{4z}{(2i+1)^2 \pi\pi cc}.$$

## Corollarium 2.

1218. Ponatur breuitatis gratia angulus

$$\frac{2i+1}{2}\pi = \theta, \text{ erit}$$

$$aa \text{ tang.} \zeta^2 = \theta\theta cc, \text{ tum vero}$$

$$\text{cof.} \zeta = 1 \text{ et } \frac{a a \text{ fin.} \zeta}{n} = \theta cc,$$

ex quo integralis pars quaecunque erit

$$\pm 2\theta cc e^{-\theta\theta ccx} \int e^{\theta\theta ccx} X \partial x,$$

vbi pro $\theta$ fucceffiue omnes hos angulos fcribi oportet

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \text{ etc.}$$

## Corollarium 3.

1219. Perinde hic eft fiue $cc$ negatiue fiue pofitiue capiatur, hinc iftius aequationis differentialis infinitae

$$X = y + \frac{\partial y}{1.2 b \partial x} + \frac{\partial \partial y}{1.2.3.4 bb \partial x^2} + \frac{\partial^3 y}{1 \ldots 6 b^3 \partial x^3} + \text{etc.}$$

inte-

integrale erit

$$y = 2\,\theta\, b\, e^{-\theta\theta b x} \int e^{\theta\theta b x}\, X\, \partial x,$$

loco $\theta$ scribendo successiue omnes hos angulos, ambiguitate signi iam sublata

$$+\tfrac{\pi}{2},\ -\tfrac{3\pi}{2},\ +\tfrac{5\pi}{2},\ -\tfrac{7\pi}{2},\ +\ \text{etc.}$$

vnde si $X = 0$, integrale particulare quoduis est

$$y = A\, e^{-\theta\theta b x}.$$

## Problema 163.

1220. Proposita aequatione differentiali

$$X = \frac{n\,\partial y}{a\,\partial x} + \frac{n(n-1)(n-2)\,\partial^3 y}{1.\,2.\,3\,a^3\,\partial x^3} + \frac{n(n-1)(n-2)(n-3)(n-4)\,\partial^5 y}{1.\,2.\,3.\,4.\,5\,a^5\,\partial x^5} + \text{etc.}$$

eius integrale completum inuestigare.

## Solutio.

Etsi haec aequatio in $\partial x$ ducta sponte semel integratur, praestat tamen hanc formam retinere, vnde fit

$$P = \frac{nz}{a} + \frac{n(n-1)(n-2)\,zz}{1.\,2.\,3\,a^3} + \frac{n(n-1)(n-2)(n-3)(n-4)\,z^3}{1.\,2.\,3.\,4.\,5\,a^5} + \text{etc.}$$

quae manifesto ita exhiberi potest

$$P = \frac{\sqrt{z}}{2}\left[\left(1 + \frac{\sqrt{z}}{a}\right)^n - \left(1 - \frac{\sqrt{z}}{a}\right)^n\right],$$

cuius quidem statim vnus factor se offert $z$; reliqui vero in hac forma continentur

$$\left(1 + \frac{\sqrt{z}}{a}\right)^2 - 2\left(1 - \frac{z}{aa}\right)\cos.\, 2\,\zeta + \left(1 - \frac{\sqrt{z}}{a}\right)^2,$$

sumto angulo

$$2\,\zeta = \frac{2\,i\,\pi}{n} \text{ seu } \zeta = \frac{i\,\pi}{n},$$

haec vero forma abit in

$$2\,(1 - \cos.\, 2\,\zeta) + \frac{2z}{aa}\,(1 + \cos.\, 2\,\zeta),$$

vnde patet in genere factorem fore $aa\,\text{tang.}\,\zeta^2 + z$, quae etiam primum illum $z$ complectitur sumto $i = 0$. Hinc posito fito $aa\,\text{tang.}\,\zeta^2 = \alpha$, integralis pars huic factori respondens erit

$$\frac{1}{n} e^{-\alpha x} \int e^{\alpha x} X \partial x,$$

fi pofito

$$z = -aa\,\text{tang.}\,\zeta^2 \text{ feu } \sqrt{z} = a\,\text{tang.}\,\zeta \sqrt{-1},$$

capiatur

$$\mathfrak{A} = \frac{\partial P}{\partial z} = \frac{1}{4\sqrt{z}}\left[\left(1+\frac{\sqrt{z}}{a}\right)^n - \left(1-\frac{\sqrt{z}}{a}\right)^n\right]$$
$$+ \frac{n}{4a}\left[\left(1+\frac{\sqrt{z}}{a}\right)^{n-1} + \left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right].$$ At

$$\left(1+\frac{\sqrt{z}}{n}\right)^n = \frac{\text{cof.}\,n\zeta + \sqrt{-1}.\,\text{fin.}\,n\zeta}{\text{cof.}\,\zeta^n}, \text{ et } \left(1-\frac{\sqrt{z}}{a}\right)^n = \frac{\text{cof.}\,n\zeta - \sqrt{-1}.\,\text{fin.}\,n\zeta}{\text{cof.}\,\zeta^n},$$

quamobrem fiet

$$\mathfrak{A} = \frac{\text{fin.}\,n\zeta}{2a\,\text{tang.}\,\zeta\,\text{cof.}\,\zeta^n} + \frac{n\,\text{cof.}\,(n-1)\zeta}{2a\,\text{cof.}\,\zeta^{n-1}} = \frac{\pm n}{2a\,\text{cof.}\,\zeta^{n-1}},$$

ob $\text{fin.}\,n\zeta = 0$ et $\text{cof.}\,n\zeta = \pm 1$, prout numerus $i$ fuerit vel par vel impar. Quocirca integralis pars quaecunque ita erit expreffa

$$\pm \frac{2a\,\text{cof.}\,\zeta^{n-1}}{n} e^{-\alpha x} \int e^{\alpha x} X \partial x,$$

exiftente $\alpha = aa\,\text{tang.}\,\zeta^2$. Iam angulo $\zeta$ fucceffiue tribuantur hi valores

$$\frac{0\pi}{n},\ \frac{1\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},$$

quoad angulum rectum $\frac{\pi}{2}$ non excedant, haeque formulae omnes cum fuis fignis in vnam fummam coniectae dabunt valorem completum pro $y$.

## Corollarium 1.

1221. Prima igitur integralis pars nafcitur ex angulo $\zeta = 0$, vnde ea erit $+\frac{2a}{n}\int X \partial x$, cuius autem loco ob rationes

tiones supra allegatas circa factores simplices, eius tantum dimidium sumi debet, vt haec prima pars sit $= \frac{a}{n} \int X \partial x$, quod etiam inde patet, quod posito $z = 0$ fiat manifesto $\frac{P}{z} = \frac{n}{a}$.

## Corollarium 2.

1222. Idem tenendum esset de parte vltima, siquidem ex valore $\zeta = \frac{\pi}{2}$ nascatur, quod euenit si $n$ sit numerus par. Quia vero hoc casu fit cos. $\zeta = 0$, haec tota integralis pars per se euanescit.

## Corollarium 3.

1223. Si esset $X = 0$, quaelibet pars integralis foret $A\, e^{-a a \,tang.\, \zeta .^2 x}$, denotante A quantitatem constantem arbitrariam; foretque adeo haec aequatio

$$y = A\, e^{-a a \,tang.\, \zeta .^2 x}$$

integrale particulare aequationis, dummodo capiatur angulus $\zeta = \frac{i\pi}{n}$.

## Scholion.

1224. Hinc posito $n = \infty$ et $a = n\sqrt{b}$, integrari potest haec aequatio differentialis in infinitum excurrens

$$\frac{X}{\sqrt{b}} = \frac{\partial y}{1.b\,\partial x} + \frac{\partial\partial y}{1.2.3\,b\,b\,\partial x^2} + \frac{\partial^3 y}{1\ldots 5\,b^3\,\partial x^3} + \frac{\partial^4 y}{1\ldots\ldots 7\,b^4\,\partial x^4} + \text{etc.}$$

vel etiam haec per vnam integrationem ex ista nata

$$\sqrt{b}.\int X \partial x = y + \frac{\partial y}{1.2.3\,b\,\partial x} + \frac{\partial\partial y}{1\ldots 5\,b^2\,\partial x^2} + \frac{\partial^3 y}{1\ldots\ldots 7\,b^3\,\partial x^3} + \text{etc.}$$

Cum enim sit angulus $\zeta = \frac{i\pi}{n}$ infinite paruus, erit

$$\text{cos.}\,\zeta = 1, \text{ et } a \text{ tang.}\,\zeta = a\,\zeta = i\,\pi\sqrt{b},$$

ideoque

$$\alpha = a\,a \text{ tang.}\,\zeta^2 = i\,i\,\pi\,\pi\,b,$$

habebitur pars integralis quaecunque

$$\pm 2\sqrt{b}\,.\,e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}X\partial x,$$

vnde parte prima ex $i=0$ nata ad dimidium reducta, ob rationes supra allegatas, erit integrale completum

$$\frac{y}{\sqrt{b}}=\int X\partial x-2e^{-\pi\pi bx}\int e^{\pi\pi bx}X\partial x+2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}X\partial x$$
$$-2e^{-9\pi\pi bx}\int e^{9\pi\pi bx}X\partial x+2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}X\partial x-\text{etc.}$$

## Exemplum.

1225. Sit $n=6$ et $a=1$, vt integranda proponatur haec aequatio

$$X=\frac{6\partial y}{\partial x}+\frac{20\partial\partial y}{\partial x^2}+\frac{6\partial^3 y}{\partial x^3},\ \text{seu}$$
$$\int X\partial x=6y+\frac{20\partial y}{\partial x}+\frac{6\partial\partial y}{\partial x^2}.$$

Valores ergo pro angulo $\zeta$ et inde pendentes sunt

$$\zeta=0,\ 30^\circ,\ 60^\circ$$
$$\text{cos.}\,\zeta=1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},$$
$$\alpha=0,\ \frac{1}{3},\ 3,$$

ex quibus colligitur integrale quaesitum

$$y=\tfrac{1}{6}\int X\partial x-\tfrac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}X\partial x+\tfrac{1}{48}e^{-3x}\int e^{3x}X\partial x,$$

quod etiam aequationi satisfacere tentanti patebit.

CA-

# CAPVT V.

## DE INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM HVIVS FORMAE

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \text{etc.}$$

### Problema 164.

**1226.**

Propofita aequatione differentiali huius formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

definire functionem ipfius $x$, per quam ea multiplicata fiat integrabilis.

### Solutio.

Attendenti mox patebit, fimplicem poteftatem ipfius $x$ hoc praeftare. Sit igitur integrabilis haec aequatio

$$Xx^\lambda\partial x = Ax^\lambda y\partial x + Bx^{\lambda+1}\partial y + \frac{Cx^{\lambda+2}\partial\partial y}{\partial x} + \ldots + \frac{Nx^{\lambda+n}\partial^n y}{\partial x^{n-1}},$$

cuius integrale fit

$$\int Xx^\lambda\partial x = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}\partial y}{\partial x} + \frac{C'x^{\lambda+3}\partial\partial y}{\partial x^2} + \ldots + \frac{M'x^{\lambda+n}\partial^{n-1}y}{\partial x^{n-1}}.$$

Cum igitur huius differentiale illi debeat effe aequale, fequentes nancifcemur determinationes

A =

quo pacto

$$\frac{\partial P}{\partial z} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2a^n}$$

abire ponitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$,

vnde fit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \tfrac{n}{2a}[(1+\text{tang.}\zeta.\sqrt{-1})^{n-1} - (1-\text{tang.}\zeta.\sqrt{-1})^{n-1}]$$

$$= \frac{n}{2a\,\text{cof.}\zeta^{n-1}}[\text{cof.}(n-1)\zeta + \sqrt{-1}.\text{fin.}(n-1)\zeta - \text{cof.}(n-1)\zeta$$

$$+ \sqrt{-1}.\text{fin.}(n-1)\zeta],$$

ideoque $\mathfrak{P} = 0$ et $\mathfrak{Q} = \frac{n\,\text{fin.}(n-1)\zeta}{a\,\text{cof.}\zeta^{n-1}}$. At ob $n\zeta = \frac{2i+1}{2}\pi$, hincque cof. $n\zeta = 0$ et fin. $n\zeta = \pm 1$, prout $i$ fuerit numerus par vel impar, erit fin. $(n-1)\zeta = \pm$ cof. $\zeta$, ideoque $\mathfrak{Q} = \frac{+n}{a\,\text{col.}\zeta^{n-2}}$. Quocirca ob cof. $\theta = 0$, integralis pars ex hoc factore oriunda eft

$$\pm\frac{2a\,\text{cof.}\zeta^{n-2}}{n}(\text{cof.}\Phi\int X\partial x\,\text{fin.}\Phi - \text{fin.}\Phi\int X\partial x\,\text{cof.}\Phi),$$

feu ob $\Phi = -ax\,\text{tang.}\zeta$,

$$\pm\frac{2a\,\text{cof.}\zeta^{n-2}}{n}\left\{\begin{array}{l}\text{fin.}(ax\,\text{tang.}\zeta)\int X\partial x\,\text{cof.}(ax\,\text{tang.}\zeta)\\ -\text{cof.}(ax\,\text{tang.}\zeta)\int X\partial x\,\text{fin.}(ax\,\text{tang.}\zeta)\end{array}\right\};$$

vbi pro $\zeta$ fucceffiue fubftituantur anguli

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},$$

quamdiu funt recto minores, pro quibus ibi alternatim $+$ et $-$ fcribi oportet; haeque partes omnes in vnam fummam collectae dabunt valorem completum ipfius $y$, dummodo pro

vltima

vltima parte ex angulo $\zeta = \frac{\pi}{2}$ oriunda, quod euenit si $n$ numerus impar, eius tantum semissis capiatur.

## Corollarium 1.

1212. Accommodemus haec statim ad casum $n = \infty$ et $a = nc$, vt proposita sit haec aequatio differentialis

$$X = y + \frac{\partial\partial y}{1.2\, c^2 \partial x^2} + \frac{\partial^4 y}{1.2.3.4\, c^4 \partial x^4} + \frac{\partial^6 y}{1 \ldots 6\, c^6 \partial x^6} + \text{etc. in infinitum.}$$

Cum igitur hic valores ipsius $\zeta$ sint infinite parui, erit

$$\text{cos.}\,\zeta = 1 \text{ et tang.}\,\zeta = \zeta = \frac{(4i \pm 1)\pi}{2n},$$

hinc $a x \,\text{tang.}\,\zeta = (4i \pm 1) c x \,.\, \frac{\pi}{2}$,

pro quo angulo scribamus $\omega$. Ergo pars integralis quaecunque

$$\pm 2c\,(\text{sin.}\,\omega \int X \partial x \,\text{cos.}\,\omega - \text{cos.}\,\omega \int X \partial x \,\text{sin.}\,\omega),$$

vbi signa ambigua sibi mutuo respondent.

## Corollarium 2.

1213. Si tantum angulus $\frac{\pi}{2} c x$ ponatur $= \Phi$, integrale vniuersum ita erit expressum

$$\begin{aligned} \frac{y}{2c} = &+ \text{sin.}\,\Phi \int X \partial x \,\text{cos.}\,\Phi - \text{cos.}\,\Phi \int X \partial x \,\text{sin.}\,\Phi \\ &- \text{sin.}\,3\Phi \int X \partial x \,\text{cos.}\,3\Phi + \text{cos.}\,3\Phi \int X \partial x \,\text{sin.}\,3\Phi \\ &+ \text{sin.}\,5\Phi \int X \partial x \,\text{cos.}\,5\Phi - \text{cos.}\,5\Phi \int X \partial x \,\text{sin.}\,5\Phi \\ &- \text{sin.}\,7\Phi \int X \partial x \,\text{cos.}\,7\Phi + \text{cos.}\,7\Phi \int X \partial x \,\text{sin.}\,7\Phi \\ &\qquad \text{etc.} \end{aligned}$$

quae formulae in infinitum sunt continuandae.

## Corollarium 3.

1214. Si ponamus $c = b\sqrt{-1}$, vt habetur haec aequatio infinita

$$X = y - \frac{\partial\partial y}{1.2\, b^2 \partial x^2} + \frac{\partial^4 y}{1 \ldots 4\, b^4 \partial x^4} - \frac{\partial^6 y}{1 \ldots 6\, b^6 \partial x^6} + \text{etc.}$$

ac

ac iam angulum $\frac{\pi}{2} b x$ vocemus $\psi$, erit integrale completum

$$\frac{y}{b} = + e^{-\psi} \int e^{\psi} X \partial x - e^{\psi} \int e^{-\psi} X \partial x$$
$$- e^{-3\psi} \int e^{3\psi} X \partial x + e^{3\psi} \int e^{-3\psi} X \partial x$$
$$+ e^{-5\psi} \int e^{5\psi} X \partial x - e^{5\psi} \int e^{-5\psi} X \partial x$$
etc.

## Scholion.

1215. Si pro aequatione Corollarii 1. methodo supra exposita quaeramus integrale particulare per differentialia ipsius X expressum, huncque in finem ponamus

$$y = A X - \frac{B \partial\partial X}{c^2 \partial x^2} + \frac{C \partial^4 X}{c^4 \partial x^4} - \frac{D \partial^6 X}{c^6 \partial x^6} + \frac{E \partial^8 X}{c^8 \partial x^8} - \text{etc.}$$

reperiemus hos coefficientium valores

$$A = 1, \; B = \frac{1}{1.2}, \; C = \frac{5}{1.\ldots.4}, \; D = \frac{61}{1.\ldots.6},$$
$$E = \frac{1385}{1.\ldots.8}, \; F = \frac{50521}{1.\ldots.10}, \; \text{etc.}$$

Hicque valor si ponatur $= V$, vocato angulo $\frac{\pi}{2} c x = \Phi$, erit integrale completum

$$y = V + A \sin.(\alpha + \Phi) + B \sin.(\beta + 3\Phi) + C \sin.(\gamma + 5\Phi)$$
$$+ D \sin.(\delta + 7\Phi) + \text{etc.}$$

## Problema 162.

1216. Proposita aequatione differentiali

$$X = y + \frac{n(n-1)}{1.2 a^2} \cdot \frac{\partial y}{\partial x} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4 a^4} \cdot \frac{\partial\partial y}{\partial x^2}$$
$$+ \frac{n \ldots (n-5)}{1 \ldots 6 a^6} \cdot \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

eius integrale completum inuestigare.

## Solutio.

Quantitas algebraica hinc formanda

$$P = 1 + \frac{n(n-1)}{1.2\,a^2}z + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,3.\,4\,a^4}.zz + \text{etc.}$$
$$= \tfrac{1}{2}(1 + \tfrac{\sqrt{z}}{a})^n + \tfrac{1}{2}(1 - \tfrac{\sqrt{z}}{a})^n,$$

cum ex praecedente nafcatur, fi ibi loco $zz$ fcribatur $z$, fumto angulo $\zeta = \frac{2i+1}{2n}\pi$, factor quicunque erit $aa\,\text{tang.}\zeta^2 + z$, ita vt huius formae omnes factores fimplices fint reales. Hoc ergo factore cum formula $\alpha + z$ comparato, erit $\alpha = aa\,\text{tang.}\zeta^2$, et fumto $\mathfrak{A} = \frac{P}{\alpha + z}$ pofito $z = -\alpha$, erit integralis pars ex hoc factore oriunda $\mathfrak{A}\, e^{-\alpha x}\int e^{\alpha x} X \partial x$. Quia vero P euanefcit pofito $z = -\alpha$, erit quoque $\mathfrak{A} = \frac{\partial P}{\partial z}$; at eft differentiando

$$\frac{\partial P}{\partial z} = \frac{n}{4a\sqrt{z}}\left[(1 + \tfrac{\sqrt{z}}{a})^{n-1} - (1 - \tfrac{\sqrt{z}}{a})^{n-1}\right].$$

Quia igitur poni oportet $\frac{\sqrt{z}}{a} = \text{tang.}\,\zeta.\sqrt{-1}$, erit

$$1 + \frac{\sqrt{z}}{a} = \frac{\text{cof.}\,\zeta + \sqrt{-1}.\,\text{fin.}\,\zeta}{\text{cof.}\,\zeta} \text{ et } 1 - \frac{\sqrt{z}}{a} = \frac{\text{cof.}\,\zeta - \sqrt{-1}.\,\text{fin.}\,\zeta}{\text{cof.}\,\zeta}$$

hincque

$$\mathfrak{A} = \frac{n}{4a\cdot\text{tang.}\,\zeta.\sqrt{-1}}\cdot\frac{2\sqrt{-1}.\,\text{fin.}(n-1)\zeta}{\text{cof.}\,\zeta^{n-1}} = \frac{n\,\text{fin.}(n-1)\zeta}{2aa\,\text{fin.}\zeta\,\text{cof.}\zeta^{n-2}}.$$

Iam obferuetur effe $\text{fin.}\,n\zeta = \text{fin.}(2i+1)\frac{\pi}{2} = \pm 1$, (vbi fignum fuperius valet fi $i$ numerus par, inferius fi impar,) tum vero $\text{cof.}\,n\zeta = 0$, vnde fit $\text{fin.}(n-1)\zeta = \pm\,\text{cof.}\,\zeta$, ex quo conficitur

$$\mathfrak{A} = \frac{\pm n}{2aa\,\text{fin.}\,\zeta\,\text{cof.}\,\zeta^{n-3}},$$

et pars integralis quaefita habebitur

$$\pm\frac{2aa\,\text{fin.}\,\zeta\,\text{cof.}\,\zeta^{n-3}}{n}\,e^{-aa\,\text{tang.}\zeta^2.x}\int e^{aa\,\text{tang.}\zeta^2.x} X\,\partial x.$$

Nunc igitur ipfi $\zeta$ fucceffiue tribuantur hi valores

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},\ \text{etc.}$$

quoad angulum rectum non fuperent, atque omnes iftae partes in vnam summam collectae, dabunt integrale completum seu valorem ipsius $y$.

## Corollarium 1.

1217. Si ponamus $n = \infty$ et $a = nc$, aequatio proposita in infinitum excurrit, eritque

$$X = y + \frac{\partial y}{1.2 c^2 \partial x} + \frac{\partial \partial y}{1.2.3.4 c^4 \partial x^2} + \frac{\partial^3 y}{1 \ldots 6 c^6 \partial x^3} + \text{etc.}$$

et forma algebraica inde nata

$$P = 1 + \frac{z}{1.2 c^2} + \frac{zz}{1.2.3.4 c^4} + \frac{z^3}{1 \ldots 6 c^6} + \text{etc.} = \tfrac{1}{2} e^{\frac{\sqrt{z}}{c}} + \tfrac{1}{2} e^{\frac{-\sqrt{z}}{c}},$$

quae omnes factores simplices habet reales, et ob $\zeta$ infinite paruum erit $\operatorname{tang.} \zeta = \zeta = \frac{2i+1}{2n}\pi$, indeque factorum forma generalis

$$z + \frac{(2i+1)^2}{4}\pi\pi cc, \text{ seu } 1 + \frac{4z}{(2i+1)^2 \pi\pi cc}.$$

## Corollarium 2.

1218. Ponatur breuitatis gratia angulus

$\frac{2i+1}{2}\pi = \theta$, erit

$aa \operatorname{tang.} \zeta^2 = \theta\theta cc$, tum vero

$\operatorname{cos.} \zeta = 1$ et $\frac{a a \operatorname{sin.} \zeta}{n} = \theta cc$,

ex quo integralis pars quaecunque erit

$$\pm 2\theta cc e^{-\theta\theta ccx} \int e^{\theta\theta ccx} X \partial x,$$

vbi pro $\theta$ successiue omnes hos angulos scribi oportet

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \text{ etc.}$$

## Corollarium 3.

1219. Perinde hic est siue $cc$ negatiue siue positiue capiatur, hinc istius aequationis differentialis infinitae

$$X = y + \frac{\partial y}{1.2 b \partial x} + \frac{\partial \partial y}{1.2.3.4 bb \partial x^2} + \frac{\partial^3 y}{1 \ldots 6 b^3 \partial x^3} + \text{etc.}$$

inte-

integrale erit

$$y = 2\theta b e^{-\theta\theta b x} \int e^{\theta\theta b x} X \partial x,$$

loco $\theta$ scribendo successiue omnes hos angulos, ambiguitate signi iam sublata

$$+\tfrac{\pi}{2},\ -\tfrac{3\pi}{2},\ +\tfrac{5\pi}{2},\ -\tfrac{7\pi}{2},\ + \text{etc.}$$

vnde si $X = 0$, integrale particulare quoduis est

$$y = A e^{-\theta\theta b x}.$$

## Problema 163.

1220. Proposita aequatione differentiali

$$X = \frac{n\partial y}{a\partial x} + \frac{n(n-1)(n-2)\partial^3 y}{1.\ 2.\ 3\ a^3 \partial x^3} + \frac{n(n-1)(n-2)(n-3)(n-4)\partial^5 y}{1.\ 2.\ 3.\ 4.\ 5\ a^5 \partial x^5} + \text{etc.}$$

eius integrale completum inuestigare.

## Solutio.

Etsi haec aequatio in $\partial x$ ducta sponte semel integratur, praestat tamen hanc formam retinere, vnde fit

$$P = \frac{nz}{a} + \frac{n(n-1)(n-2)zz}{1.\ 2.\ 3\ a^3} + \frac{n(n-1)(n-2)(n-3)(n-4)z^3}{1.\ 2.\ 3.\ 4.\ 5\ a^5} + \text{etc.}$$

quae manifesto ita exhiberi potest

$$P = \frac{\sqrt{z}}{2}\left[\left(1 + \frac{\sqrt{z}}{a}\right)^n - \left(1 - \frac{\sqrt{z}}{a}\right)^n\right],$$

cuius quidem statim vnus factor se offert $z$; reliqui vero in hac forma continentur

$$\left(1 + \frac{\sqrt{z}}{a}\right)^2 - 2\left(1 - \frac{z}{aa}\right)\operatorname{cos.} 2\zeta + \left(1 - \frac{\sqrt{z}}{a}\right)^2,$$

sumto angulo

$$2\zeta = \frac{2i\pi}{n} \text{ seu } \zeta = \frac{i\pi}{n},$$

haec vero forma abit in

$$2(1 - \operatorname{cos.} 2\zeta) + \frac{2z}{aa}(1 + \operatorname{cos.} 2\zeta),$$

Vnde patet in genere factorem fore $aa\ \text{tang.}\ \zeta^2 + z$, quae etiam primum illum $z$ complectitur sumto $i = 0$. Hinc posito

 sito

sito $a\,a\,\text{tang.}\,\zeta^2 = \alpha$, integralis pars huic factori respondens erit

$$\tfrac{1}{n}\,e^{-\alpha x}\int e^{\alpha x}\,X\,\partial x,$$

si posito

$$z = -\,a\,a\,\text{tang.}\,\zeta^2 \text{ seu } \sqrt{z} = a\,\text{tang.}\,\zeta\,\sqrt{-1},$$

capiatur

$$\mathfrak{A} = \frac{\partial P}{\partial x} = \frac{1}{4\sqrt{z}}\left[\left(1+\tfrac{\sqrt{z}}{a}\right)^n - \left(1-\tfrac{\sqrt{z}}{a}\right)^n\right] + \frac{n}{4a}\left[\left(1+\tfrac{\sqrt{z}}{a}\right)^{n-1} + \left(1-\tfrac{\sqrt{z}}{a}\right)^{n-1}\right].$$ At

$$\left(1+\tfrac{\sqrt{z}}{a}\right)^n = \frac{\text{cos}.n\zeta + \sqrt{-1}.\,\text{sin}.n\zeta}{\text{cos}.\,\zeta^n}, \text{ et } \left(1-\tfrac{\sqrt{z}}{a}\right)^n = \frac{\text{cos}.n\zeta - \sqrt{-1}.\,\text{sin}.n\zeta}{\text{cos}.\,\zeta^n},$$

quamobrem fiet

$$\mathfrak{A} = \frac{\text{sin}.\,n\,\zeta}{2\,a\,\text{tang}.\,\zeta\,\text{cos}.\,\zeta^n} + \frac{n\,\text{cos}.(n-1)\zeta}{2\,a\,\text{cos}.\,\zeta^{n-1}} = \frac{\pm\,n}{2\,a\,\text{cos}.\,\zeta^{n-1}},$$

ob $\text{sin}.n\,\zeta = 0$ et $\text{cos}.\,n\,\zeta = \pm 1$, prout numerus $i$ fuerit vel par vel impar. Quocirca integralis pars quaecunque ita erit expressa

$$\pm\,\frac{2\,a\,\text{cos}.\,\zeta^{n-1}}{n}\,e^{-\alpha x}\int e^{\alpha x}\,X\,\partial x,$$

existente $\alpha = a\,a\,\text{tang}.\,\zeta^2$. Iam angulo $\zeta$ successiue tribuantur hi valores

$$\frac{0\pi}{n},\ \frac{1\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},$$

quoad angulum rectum $\frac{\pi}{2}$ non excedant, haeque formulae omnes cum suis signis in vnam summam coniectae dabunt valorem completum pro $y$.

## Corollarium 1.

1221. Prima igitur integralis pars nascitur ex angulo $\zeta = 0$, vnde ea erit $+\frac{2a}{n}\int X\,\partial x$, cuius autem loco ob rationes

tiones supra allegatas circa factores simplices, eius tantum dimidium sumi debet, vt haec prima pars sit $= \frac{a}{n} \int X \partial x$, quod etiam inde patet, quod posito $z = 0$ fiat manifesto $\frac{P}{z} = \frac{n}{a}$.

## Corollarium 2.

1222. Idem tenendum esset de parte vltima, siquidem ex valore $\zeta = \frac{\pi}{2}$ nascatur, quod euenit si $n$ sit numerus par. Quia vero hoc casu fit cos. $\zeta = 0$, haec tota integralis pars per se euanescit.

## Corollarium 3.

1223. Si esset $X = 0$, quaelibet pars integralis foret $A e^{-a a \, tang. \zeta^2 x}$, denotante A quantitatem constantem arbitrariam; foretque adeo haec aequatio

$$y = A e^{-a a \, tang. \zeta^2 x}$$

integrale particulare aequationis, dummodo capiatur angulus $\zeta = \frac{i\pi}{n}$.

## Scholion.

1224. Hinc posito $n = \infty$ et $a = n\sqrt{b}$, integrari potest haec aequatio differentialis in infinitum excurrens

$$\frac{X}{\sqrt{b}} = \frac{\partial y}{1.b\partial x} + \frac{\partial\partial y}{1.2.3 b b \partial x^2} + \frac{\partial^3 y}{1.\ldots.5 b^3 \partial x^3} + \frac{\partial^4 y}{1.\ldots.7 b^4 \partial x^4} + \text{etc.}$$

vel etiam haec per vnam integrationem ex ista nata

$$\sqrt{b}.\int X \partial x = \frac{y}{1} + \frac{\partial y}{1.2.3 b \partial x} + \frac{\partial\partial y}{1.\ldots.5 b^2 \partial x^2} + \frac{\partial^3 y}{1.\ldots.7 b^3 \partial x^3} + \text{etc.}$$

Cum enim sit angulus $\zeta = \frac{i\pi}{n}$ infinite paruus, erit

$$\text{cos.}\,\zeta = 1, \text{ et } a \, \text{tang.}\,\zeta = a\zeta = i\pi\sqrt{b},$$

ideoque

$$\alpha = a a \, \text{tang.}\,\zeta^2 = i i \pi\pi b,$$

habebitur pars integralis quaecunque

$$\pm 2\sqrt{b}\,.\,e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}X\partial x,$$

vnde parte prima ex $i=0$ nata ad dimidium reducta, ob rationes supra allegatas, erit integrale completum

$$\frac{y}{\sqrt{b}}=\int X\partial x-2e^{-\pi\pi bx}\int e^{\pi\pi bx}X\partial x+2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}X\partial x$$
$$-2e^{-9\pi\pi bx}\int e^{9\pi\pi bx}X\partial x+2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}X\partial x-\text{etc.}$$

## Exemplum.

1225. Sit $n=6$ et $a=1$, vt integranda proponatur haec aequatio

$$X=\frac{6\partial y}{\partial x}+\frac{20\partial\partial y}{\partial x^2}+\frac{6\partial^3 y}{\partial x^3},\ \text{seu}$$
$$\int X\partial x=6y+\frac{20\partial y}{\partial x}+\frac{6\partial\partial y}{\partial x^2}.$$

Valores ergo pro angulo $\zeta$ et inde pendentes sunt

$$\zeta=0,\ 30^\circ,\ 60^\circ$$
$$\text{cos}.\,\zeta=1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},$$
$$\alpha=0,\ \frac{1}{3},\ 3,$$

ex quibus colligitur integrale quaesitum

$$y=\tfrac{1}{6}\int X\partial x-\tfrac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}X\partial x+\tfrac{1}{48}e^{-3x}\int e^{3x}X\partial x,$$

quod etiam aequationi satisfacere tentanti patebit.

# CAPUT V.

## DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUIUS FORMAE

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \text{etc.}$$

### Problema 164.

**1226.**

Proposita aequatione differentiali huius formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \dots + \frac{Nx^n\partial^n y}{\partial x^n},$$

definire functionem ipsius $x$, per quam ea multiplicata fiat integrabilis.

### Solutio.

Attendenti mox patebit, simplicem potestatem ipsius $x$ hoc praestare. Sit igitur integrabilis haec aequatio

$$Xx^\lambda\partial x = Ax^\lambda y\partial x + Bx^{\lambda+1}\partial y + \frac{Cx^{\lambda+2}\partial\partial y}{\partial x^2} + \dots + \frac{Nx^{\lambda+n}\partial^n y}{\partial x^{n-1}},$$

cuius integrale sit

$$\int Xx^\lambda\partial x = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}\partial y}{\partial x} + \frac{C'x^{\lambda+3}\partial\partial y}{\partial x^2} + \dots + \frac{M'x^{\lambda+n}\partial^{n-1}y}{\partial x^{n-1}}.$$

Cum igitur huius differentiale illi debeat esse aequale, sequentes nanciscemur determinationes

A =

$A=(\lambda+1)A'$, hinc $(\lambda+1)A'=A$
$B=(\lambda+2)B'+A'$, $(\lambda+1)(\lambda+2)B'=(\lambda+1)B-A$
$C=(\lambda+3)C'+B'$, $(\lambda+1)(\lambda+2)(\lambda+3)C'=(\lambda+1)(\lambda+2)C-(\lambda+1)B+A$
$D=(\lambda+4)D'+C'$, $(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)D'=(\lambda+1)(\lambda+2)(\lambda+3)D$
$-(\lambda+1)(\lambda+2)C+(\lambda+1)B-A$
.
.
.
.
.
$N=M'$

integralis enim termini sequentes, qui inuoluerent differentialis gradum $\partial^n y$ altioresque, euanescere debent, quia alioquin integratio non successisset. Cum igitur in integrali littera $N'$ euanescat, peruenimus ad hanc aequationem

$$0=A-(\lambda+1)B+(\lambda+1)(\lambda+2)C-(\lambda+1)(\lambda+2)(\lambda+3)D\ldots\ldots\pm(\lambda+1)(\lambda+2)\ldots\ldots(\lambda+n)N$$

ex qua aequatione exponens $\lambda$ potestatis quaesitae $x^\lambda$ definiri debet. Formetur ergo talis expressio algebraica

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\ldots\ldots+N(z-1)(z-2)(z-3)\ldots\ldots(z-n),$$

huiusque quaerantur omnes factores simplices, vt sit

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \text{ etc.}$$

factorum horum numero existente $=n$. Iam ex quolibet factore $z+\alpha$ ad nihilum reducto, valor $z=-\alpha$ dabit potestatem $x^\alpha$, per quam proposita aequatio multiplicata integrabilis euadit, ita vt eius integrale sit futurum

$$x^{-\alpha-1}\int x^\alpha X\partial x=A'y+\frac{B'x\partial y}{\partial x}+\frac{C'x^2\partial\partial y}{\partial x^2}+\frac{D'x^3\partial^3 y}{\partial x^3}+\ldots+\frac{Nx^{n-1}\partial^{n-1}y}{\partial x^{n-1}}$$

vbi

vbi differentialium gradus vnitate est inferior. Ita autem haec aequatio integrata per propositam determinatur, vt sit

$$A = (\alpha + 1) A'$$
$$B = (\alpha + 2) B' + A'$$
$$C = (\alpha + 3) C' + B'$$
$$D = (\alpha + 4) D' + C'$$
etc.

donec perueniatur ad vltimum coefficientem N, qui vtrobique est idem.

## Corollarium 1.

1227. Quia aequatio integrata similis est ipsi propositae, ea per certam potestatem ipsius $x$ multiplicata denuo fiet integrabilis. Ad hanc enim potestatem inueniendam considerare oportet hanc formam algebraicam

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \ldots$$
$$\ldots\ldots + N(z-1)(z-2)\ldots\ldots(z-n+1),$$

cuius si fuerit factor simplex quicunque $z + \mu$, erit $x^{+\mu}$ illa potestas ipsius $x$, aequationem integrabilem reddens.

## Corollarium 2.

1228. Quodsi ipsa aequatio proposita per potestatem $x^{+\alpha}$ multiplicata reddita fuerit integrabilis, hic probe notari conuenit, quantitatem Q ex integrata formatam ita pendere a priori P ex ipsa proposita formata, vt sit $Q = \frac{P}{\alpha + z}$, quandoquidem per hypothesin $\alpha + z$ est factor ipsius P.

## Scholion 1.

1229. Ad hanc insignem proprietatem demonstrandam, quod scilicet sit $P = (\alpha + z) Q$, tantum opus est, vt quantitas Q per $\alpha + z$ multiplicetur; verum quo conclusio clarius in oculos occurrat, pro singulis terminis ipsius Q multiplicator bipartito est repraesentandus; ac pro primo quidem termino loco $\alpha+z$ scribatur $(\alpha+1)+(z-1)$, pro secundo $(\alpha+2)+(z-2)$, pro tertio $(\alpha+3)+(z-3)$, pro quarto $(\alpha+4)+(z-4)$ etc. ita vt cuiusque termini productum binis partibus exhibeatur, quam operationem hic totam apponam

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \text{etc.}$$

| Multipl. | | | |
|---|---|---|---|
| $\alpha+1$ | $\alpha+2$ | $\alpha+3$ | $\alpha+4$ |
| $z-1$ | $z-2$ | $z-3$ | $z-4$ |

$$\text{Prod.}\ (\alpha+1)A' + A'(z-1) \quad + B'(z-1)(z-2) \quad + C'(z-1)(z-2)(z-3)\ \text{etc.}$$
$$+(\alpha+2)B'(z-1) + (\alpha+3)C'(z-1)(z-2) + (\alpha+4)D'(z-1)(z-2)(z-3)$$

In solutione autem vidimus esse

$$(\alpha+1)A' = A,\ (\alpha+2)B' + A' = B,\ (\alpha+3)C' + B' = C,\ \text{etc.}$$

quocirca hoc productum hac forma exprimetur

$$A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

cui valor ipsius P est aequalis, sicque demonstrata est insignis illa proprietas memorata, quod sit $Q = \frac{P}{\alpha+z}$.

## Corollarium 3.

1230. Quodsi ergo valor ipsius P in factores simplices resolutus ita repraesentetur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)\ \text{etc.}$$

et ex factore $\alpha+z$ aequatio proposita per $x^\alpha$ multiplicata integretur, tum vero ex integrata simili modo valor Q formetur, erit

$$Q = N(\beta+z)(\gamma+z)(\delta+z)\ \text{etc.}$$

## Corollarium 4.

1231. Aequatio ergo integrata, postquam ad formam propositae fuerit perducta, vt posito

$$x^{-\alpha-1}\int x^{\alpha} X \partial x = X'$$

habeatur

$$X' = A'y + \frac{B'x\partial y}{\partial x} + \frac{C'x^2\partial\partial y}{\partial x^2} + \frac{D'x^3\partial^3 y}{\partial x^3} + \text{etc.}$$

haec denuo integrabilis reddetur, si multiplicetur per quampiam harum potestatum $x^{\beta}$, $x^{\gamma}$, $x^{\delta}$, etc. quae etiam ipsam propositam integrabilem reddidissent.

## Scholion 2.

1232. Antequam continuationem harum integrationum vlterius prosequar, conueniet eum casum formae propositae generalis seorsim euolui, quo prius aequationis membrum X in nihilum abit. Hoc enim casu hoc commodi vsu venit, vt statim sine integrationibus repetitis integrale completum exhiberi queat, idque simili modo quo supra in Capite II. sum vsus. Huic quidem casui quia iam multo facilius tractari potest, proprium caput assignare nolui, ne praecepta nimis multiplicari videantur.

## Problema 165.

1233. Proposita hac aequatione differentiali cuiuscunque ordinis

$$0 = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \text{etc.}$$

vbi variabilis $y$ cum suis differentialibus nusquam plus vna dimensione, altera vero $x$ adeo nullam obtineat, eius integrale completum inuenire.

## Solutio.

Particulariter huic aequationi satisfieri perspicuum est, si $y$ certae ipsius $x$ potestati aequetur, ponamus ergo esse $y = x^{\mu}$, et facta substitutione, cum vbique per $x^{\mu}$ diuiserimus, perueniemus ad hanc aequationem

$$0 = A + \mu B + \mu(\mu-1)C + \mu(\mu-1)(\mu-2)D + \text{etc.}$$

 vnde

vnde exponentem $\mu$ determinari oportet. Vel si secundum praeceptum praecedentis Problematis ex aequatione proposita hanc formemus formulam algebraicam

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

eamque in factores simplices resoluamus, vt fit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

euidens est posito $\mu = z - 1$, primae aequationi satisfieri sumendo

$$\mu = -\alpha - 1, \text{ vel } \mu = -\beta - 1, \text{ vel } \mu = -\gamma - 1, \text{ etc.}$$

ita vt quisque factor suppeditet integrale particulare. Cum igitur factorum numerus aequetur gradui differentialium summo, hinc colligetur integrale completum aequationis propositae

$$y = \mathfrak{A}\, x^{-\alpha-1} + \mathfrak{B}\, x^{-\beta-1} + \mathfrak{C}\, x^{-\gamma-1} + \mathfrak{D}\, x^{-\delta-1} + \text{etc.}$$

vbi tantum obseruari conuenit, si factorum illorum simplicium duo pluresue fuerint inter se aequales, integralis formam simili modo immutari debere, quo supra Capite II. sum vsus. Scilicet cum aequationes ibi tractatae ad praesentem formam reuocentur, si ibi loco $x$ scribatur $l\,x$, inde has regulas haurimus.

1.) Si forma P factorem habeat $(\alpha + z)^2$, inde nascitur pars integralis

$$x^{-\alpha-1}(\mathfrak{A} + \mathfrak{B}\, l\,x).$$

2.) Si forma P factorem habeat $(\alpha + z)^3$, pars integralis inde orta est

$$x^{-\alpha-1}[\mathfrak{A} + \mathfrak{B}\, l\,x + \mathfrak{C}\,(l\,x)^2].$$

3.) Si forma P factorem habeat $(\alpha + z)^4$, pars integralis inde orta erit

$$x^{-\alpha-1}[\mathfrak{A} + \mathfrak{B}\, l\,x + \mathfrak{C}\,(l\,x)^2 + \mathfrak{D}\,(l\,x)^3].$$

Si factores occurrant imaginarii, partes inde oriundae per solitam imaginariorum reductionem facile ad formam realem reuocabuntur, vti in corollariis docebo.

## Corollarium. 1.

1234. Si forma P duos habeat factores simplices imaginarios in formula $ff + 2fz\cos.\theta + zz$ contentos, hac cum producto $(\alpha + z)(\beta + z)$ comparata erit

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et } \beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

vnde fit

$$x^{-\alpha} = x^{-f\cos.\theta}\, x^{-f\sqrt{-1}.\sin.\theta} = x^{-f\cos.\theta}\, e^{-\sqrt{-1}.f\sin.\theta\, lx}.$$

Est vero

$$e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}.\sin.u,$$

ideoque habetur

$$x^{-\alpha-1} = x^{-f\cos.\theta}\left(\frac{\cos.(f\sin.\theta\, lx) - \sqrt{-1}.\sin.(f\sin.\theta\, lx)}{x}\right).$$

Quare cum $x^{-\beta-1}$ simili modo exprimatur, mutato signo ipsius $\sqrt{-1}$, ex factore duplici $ff + 2fz\cos.\theta + zz$ haec nascitur pars integralis

$$x^{-f\cos.\theta-1}[\mathfrak{A}\cos.(f\sin.\theta\, lx) + \mathfrak{B}\sin.(f\sin.\theta\, lx)],$$

quae etiam ita potest repraesentari

$$\mathfrak{A}\, x^{-f\cos.\theta-1}\cos.(\mathfrak{a} + f\sin.\theta\, lx),$$

denotante $\mathfrak{a}$ angulum constantem arbitrarium.

## Corollarium 2.

1235. Simili modo si factores aequales inuoluat, vt sit

$$(\alpha + z)^2(\beta + z)^2 = (ff + 2fz\cos.\theta + zz)^2,$$

litterae $\alpha$ et $\beta$ eosdem quos ante sortientur valores imaginarios, ex quorum reductione colligitur haec pars integralis inde oriunda

$$x^{-f\cos.\theta-1}[\mathfrak{A}\cos.(\mathfrak{a}+f\sin.\theta lx)+\mathfrak{B}lx.\cos.(\mathfrak{b}+f\sin.\theta lx)],$$

quatuor constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$ et $\mathfrak{b}$ continens.

## Corollarium 3.

1236. Hinc ergo euidens est, quomodo ex factoribus formae P, siue sint simplices siue duplices, siue inaequales siue aequales, singulas integralis partes assignari indeque totum integrale completum formari conueniat.

## Scholion.

1237. Totum ergo negotium huc redit, vt quantitas algebraica ex aequatione differentiali formata

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.}$$

in suos factores reales vel simplices vel duplices resoluetur, in quo plerumque maxima difficultas versatur, quoniam huiusmodi formae minus tractari sunt solitae. Cum vero haec resolutio isti aequationi cum generali, quam hoc capite euoluere suscepi, est communis, quicquid hic praestare licuerit, potius in aequatione generali ostendi conueniet, ad quam resoluendam propterea reuertor. Id tantum hic obseruasse necesse duco, quod si pro aequatione generali

$$X=Ay+\frac{Bx\partial y}{\partial x}+\frac{Cxx\partial\partial y}{\partial x^2}+\frac{Dx^3\partial^3 y}{\partial x^3}+\text{etc.}$$

vndecunque innotuerit integrale particulare, puta $y=V$ existente $V$ certa functione ipsius $x$, tum posito $y=V+v$ perueniri ad hanc aequationem

$$0=Av+\frac{Bx\partial v}{\partial x}+\frac{Cxx\partial\partial v}{\partial x^2}+\frac{Dx^3\partial^3 v}{\partial x^3}+\text{etc.}$$

cuius integrale completum per praecepta huius problematis inuentum si loco $v$ scribatur, habebitur integrale completum illius aequationis, quo pacto certe insigne calculi compendium obtinetur.

Proble-

## Problema 165.

1238. Proposita aequatione differentiali gradus cuiuscunque $n$ huius formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \dots + \frac{Nx^n\partial^n y}{\partial x^n},$$

eius integrale per integrationem $n$ vicibus repetitam inuenire.

## Solutio.

Ex hac aequatione formetur haec quantitas algebraica $P = A + B(z-1) + C(z-1)(z-2) + \dots + N(z-1)(z-2)\dots(z-n)$, cuius quaerantur omnes factores simplices, nullo habito respectu siue sint reales siue imaginarii, vt ea hoc modo exprimatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\dots\dots(\mu+z)(\nu+z),$$

factorum numero existente $= n$. Quo facto, initio huius capitis vidimus, quemlibet factorem puta $\alpha + z$ praebere potestatem $x^\alpha$, per quam nostra aequatio fiat integrabilis, atque adeo ostendimus, integrale inde ortum, si compendii gratia ponamus

$$x^{-\alpha-1}\int x^\alpha X\partial x = X',$$

fore

$$X' = A'y + \frac{B'x\partial y}{\partial x} + \frac{C'x^2\partial\partial y}{\partial x^2} + \dots + \frac{Nx^{n-1}\partial^{n-1}y}{\partial x^{n-1}},$$

ita vt sit $A' = \frac{A}{\alpha+1}$, caeterique coefficientes ita se habeant, vti ibidem docuimus; hic autem sufficiet ad primum potissimum respexisse. Absoluta iam prima integratione, si eadem lege ex aequatione semel integrata formemus quantitatem

$P' = A' + B'(z-1) + C'(z-1)(z-2) + \dots + N(z-1)(z-2)\dots(z-n+1)$, cuius resolutio in factores iam ex prima forma P constat, postquam §. 1229. demonstraui esse

P'

$$P' = N(\beta + z)(\gamma + z)(\delta + z) \ldots . (\mu + z)(\nu + z),$$

ita vt fit $P' = \frac{P}{z}$. Hinc ergo fimili modo factor $\beta + z$ fuppeditabit multiplicatorem $x^\beta$, quo haec aequatio integrabilis redditur, ac pofito

$$x^{-\beta - 1} \int x^\beta X' \partial x = X'',$$

vt fit

$$X'' = x^{-\beta - 1} \int e^{\beta - \alpha - 1} \partial x \int x^\alpha X \partial x,$$

integrale erit

$$X'' = A'' y + \frac{B'' x \partial y}{\partial x} + \frac{C'' x^2 \partial \partial y}{\partial x^2} + \ldots . + \frac{N x^{n-2} \partial^{n-2} y}{\partial x^{n-2}},$$

exiftente hic $A'' = \frac{A'}{\beta + 1} = \frac{A}{(\alpha + 1)(\beta + 1)}$. Quodfi hoc modo tot integrationes fucceffiue abfoluantur, quot vnitates continentur in indice $n$, ficque omnes factores fimplices formae P in vfum vocentur, tandem ad hanc peruenietur aequationem $X^{(n)} = A^{(n)} y$, quae eft ipfa integralis defiderata. Cum autem hic futurum fit

$$A^{(n)} = \frac{A}{(\alpha + 1)(\beta + 1)(\gamma + 1) \ldots . (\nu + 1)},$$

euidens eft denominatorem nafci ex forma $\frac{P}{N}$, fi loco $z$ fcribatur vnitas; tum autem fumto $z = 1$, prima forma manifefto dat $P = A$, ita vt denominator ifte fiat $= \frac{A}{N}$, ideoque $A^{(n)} = N$, quod etiam inde patet, quod omnium aequationum vltimi termini habeant eundem coefficientem N, quo ergo in poftrema integrali ipfe primus terminus $y$ debet effe affectus. Deinde vero eft

$$X^{(n)} = x^{-\nu - 1} \int x^{\nu - \mu - 1} \partial x \int x^{\mu - \lambda - 1} \partial x \ldots \ldots \int x^{\beta - \alpha - 1} \partial x \int x^\alpha X \partial x,$$

vbi cum numeri $\alpha$, $\beta$, $\gamma$, etc. vtcunque inter fe permutari poffunt, integrale quaefitum etiam hoc modo repraefentari poteft

$$N y = x^{-\alpha - 1} \int x^{\alpha - \beta - 1} \partial x \int x^{\beta - \gamma - 1} \partial x \ldots . \int x^{\mu - \nu - 1} \partial x \int x^\nu X \partial x.$$

Corol-

## Corollarium 1.

1239. Totum ergo negotium huc redit, vt forma algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

in suos factores simplices resoluatur, quibus inuentis vt sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots \ldots (\nu + z),$$

hinc integrale quaesitum facile exhibetur, et quidem pro factorum varia permutatione pluribus modis, qui autem omnes eundem valorem exprimunt, vti ex sequentibus clarius patebit.

## Corollarium 2.

1240. Cum haec forma integralis inuenta tot inuoluat integrationes, quoti gradus fuerit aequatio differentialis proposita, totidem quoque constantes arbitrariae ingerentur, quemadmodum indoles integrationis completae postulat.

## Scholion.

1241. Quoniam integrale inuentum pluribus integrationibus est inuolutum, ad vsum faciliorem conueniet hanc formam in partes resolui, quae singulae vnicum tantum signum integrale contineant. Hanc autem resolutionem simili modo instituere licet, quo supra sumus vsi, atque hic quidem totum negotium ad huiusmodi formulam reuocatur

$$\int x^{m-n-1} \partial x \int x^n X \partial x,$$

quae manifesto ita reducitur, vt sit

$$\frac{1}{m-n} x^{m-n} \int x^n X \partial x - \frac{1}{m-n} \int x^m X \partial x,$$

vbi tamen obseruandum est, si fuerit $m = n$, peculiari reductione opus esse, hocque casu fore

$$\int \frac{\partial x}{x} \int x^n X \partial x = lx . \int x^n X \partial x - \int x^n X \partial x \, lx.$$

Hac ergo regula vtemur in resolutione sequentium problematum, quibus successiue omnes gradus differentialium percurramus, praetermisso quidem gradu primo, cum aequationis

$$X = Ay + \frac{Nx\partial y}{\partial x}, \text{ ob}$$

$$P = A + N(z - 1) = N(\alpha + z),$$

integrale fit

$$Ny = x^{-\alpha-1}\int x^{\alpha} X \partial x,$$

quod nulla reductione indiget.

## Problema 167.

1242. Proposita hac aequatione differentiali secundi gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Nx^2\partial\partial y}{\partial x^2},$$

eius integrale per formulas integrales simplices euoluere.

## Solutio.

Cum sit

$$P = A + B(z-1) + N(z-1)(z-2),$$

statuatur

$$P = N(\alpha + z)(\beta + z),$$

eritque integrale per methodum praecedentem inuentum $Ny = X''$ existente

$$x^{\beta+1}X'' = \int x^{\beta-\alpha-1}\partial x \int x^{\alpha} X \partial x,$$

quae forma euoluitur in hanc

$$\frac{1}{\beta-\alpha} x^{\beta-\alpha}\int x^{\alpha} X\partial x - \frac{1}{\beta-\alpha}\int x^{\beta} X\partial x,$$

sicque erit

$$Ny = \frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha} X\partial x + \frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta} X\partial x.$$

Hinc

Hinc autem casum excipi oportet, quo $\beta = \alpha$, tum enim fit

$$x^{\alpha+1} X'' = \int \frac{\partial x}{x} \int x^\alpha X \partial x = l x \int x^\alpha X \partial x - \int x^\alpha X \partial x \, l x,$$

pro hoc igitur casu habebimus

$$N y = x^{-\alpha-1} \int \frac{\partial x}{x} \int x^\alpha X \partial x, \text{ seu}$$

$$N y = x^{-\alpha-1} (l x \int x^\alpha X \partial x - \int x^\alpha X \partial x \, l x),$$

vbi quidem prior forma praeferenda videtur.

## Corollarium 1.

1243. Si ambo factores simplices sint imaginarii, ponatur

$$(\alpha + z)(\beta + z) = ff + 2 f z \operatorname{cos.} \theta + zz,$$

eritque

$$\alpha = f(\operatorname{cos.} \theta + \sqrt{-1} . \operatorname{sin.} \theta) \text{ et}$$

$$\beta = f(\operatorname{cos.} \theta - \sqrt{-1} . \operatorname{sin.} \theta),$$

indeque

$$\beta - \alpha = -2 f \sqrt{-1} . \operatorname{sin.} \theta.$$

Tum vero

$$x^\alpha = x^{f \operatorname{cos.} \theta} [\operatorname{cos.} (f \operatorname{sin.} \theta . l x) + \sqrt{-1} . \operatorname{sin.} (f \operatorname{sin.} \theta . l x)],$$

$$x^{-\alpha} = x^{-f \operatorname{cos.} \theta} [\operatorname{cos.} (f \operatorname{sin.} \theta \, l x) - \sqrt{-1} . \operatorname{sin.} (f \operatorname{sin.} \theta \, l x)],$$

quae formulae mutato signo ipsius $\sqrt{-1}$ ad $x^\beta$ et $x^{-\beta}$ transferuntur.

## Corollarium 2.

1244. Ponatur breuitatis gratia angulus

$$f \operatorname{sin.} \theta \, l x = \Phi,$$

et facta substitutione habebimus

$$N x y = \frac{x^{-f \operatorname{cos.} \theta} (\operatorname{cos.} \Phi - \sqrt{-1} . \operatorname{sin.} \Phi) \int x^{f \operatorname{cos.} \theta} X \partial x (\operatorname{cos.} \Phi + \sqrt{-1} . \operatorname{sin.} \Phi)}{-2 f \sqrt{-1} . \operatorname{sin.} \theta}$$

$$+ \frac{x^{-f \operatorname{cos.} \theta} (\operatorname{cos.} \Phi + \sqrt{-1} . \operatorname{sin.} \Phi) \int x^{f \operatorname{cos.} \theta} X \partial x (\operatorname{cos.} \Phi - \sqrt{-1} . \operatorname{sin.} \Phi)}{2 f \sqrt{-1} . \operatorname{sin.} \theta}$$

 vbi

vbi partes imaginariae se sponte destruunt, fietque

$$N x y = \frac{x^{-f \cos.\theta}}{f \sin.\theta} (\sin.\Phi \int x^{f \cos.\theta} X \partial x \cos.\Phi - \cos.\Phi \int x^{f \cos.\theta} X \partial x \sin.\Phi).$$

## Corollarium 3.

1245. Forma ergo haec realis modo inuenta aequivalet illi imaginaria implicanti

$$N x y = \frac{x^{-\alpha}}{\beta - \alpha} \int x^{\alpha} X \partial x + \frac{x^{-\beta}}{\alpha - \beta} \int x^{\beta} X \partial x,$$

si fuerit

$$(\alpha + z)(\beta + z) = ff + 2 f z \cos.\theta + zz,$$

ponaturque $\Phi = f \sin.\theta\, l x$, quae reductio semel facta etiam in sequentibus vsum praestabit.

## Problema 168.

1246. Proposita hac aequatione differentiali tertii gradus

$$X = A y + \frac{B x \partial y}{\partial x} + \frac{C x x \partial\partial y}{\partial x^2} + \frac{N x^3 \partial^3 y}{\partial x^3},$$

eius integrale per formulas integrales simplices euoluere.

## Solutio.

Cum hic sit

$$P = A + B(z-1) + C(z-1)(z-2) + N(z-1)(z-2)(z-3),$$

ponatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z),$$

et cum per integrationem generalem prodeat $N y = X'''$, notetur esse $X''' = x^{-\gamma-1} \int x^{\gamma} X'' \partial x$, siquidem valorem ipsius $X''$ ex binis factoribus $\alpha + z$ et $\beta + z$ iam inuenimus; hinc enim per problema praecedens habetur

$$X'' = \frac{x^{-\alpha-1}}{\beta - \alpha} \int x^{\alpha} X \partial x + \frac{x^{-\beta-1}}{\alpha - \beta} \int x^{\beta} X \partial x:$$

vnde

vnde colligitur

$$\int x^\gamma X'' \partial x = \frac{x^{\gamma-\alpha}\int x^\alpha X \partial x}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\int x^\gamma X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{\gamma-\beta}\int x^\beta X \partial x}{(\alpha-\beta)(\gamma-\beta)} - \frac{\int x^\gamma X \partial x}{(\alpha-\beta)(\gamma-\beta)}.$$

Eſt vero

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)} = \frac{-1}{(\alpha-\gamma)(\beta-\gamma)}$$

quod quemadmodum cum per ſe liquet, tum vero ex Theoremate §. 1169. demonſtrato perſpicitur. Quocirca integrale quaeſitum ita obtinetur expreſſum

$$N x y = \frac{x^{-\alpha}\int x^\alpha X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{-\beta}\int x^\beta X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{x^{-\gamma}\int x^\gamma X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

## Corollarium 1.

1247. Si forma P duos habeat factores aequales, vt ſit $\beta = \alpha$, quia tum eſt

$$X'' = x^{-\alpha-1}\int \frac{\partial x}{x}\int x^\alpha X \partial x, \text{ erit}$$

$$x^\gamma X'' \partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int \frac{\partial x}{x}\int x^\alpha X \partial x - \frac{1}{\gamma-\alpha}\int x^{\gamma-\alpha-1}\partial x \int x^\alpha X \partial x, \text{ at}$$

$$\int x^{\gamma-\alpha-1}\partial x \int x^\alpha X \partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^\alpha X \partial x - \frac{1}{\gamma-\alpha}\int x^\gamma X \partial x;$$

vnde colligitur

$$N x y = \frac{x^{-\alpha}}{\gamma-\alpha}\int \frac{\partial x}{x}\int x^\alpha X \partial x - \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^\alpha X \partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^\gamma X \partial x.$$

## Corollarium 2.

1248. Similis forma oritur, ſi ſumatur $\gamma = \beta$, tum enim fit



$\int x^\gamma$

$$\int x^\gamma X''\partial x = \frac{x^{\beta-\alpha}}{(\beta-\alpha)^2}\int x^\alpha X\partial x - \frac{\int x^\beta X\partial x}{(\beta-\alpha)^2} + \frac{1}{\alpha-\beta}\int\frac{\partial x}{x}\int x^\beta X\partial x$$

ideoque

$$Nxy = \frac{x^{-\beta}}{\alpha-\beta}\int\frac{\partial x}{x}\int x^\beta X\partial x - \frac{x^{-\beta}\int x^\beta X\partial x}{(\beta-\alpha)^2} + \frac{x^{-\alpha}\int x^\alpha X\partial x}{(\beta-\alpha)^2}.$$

## Corollarium 3.

1249. Quodsi autem omnes tres factores inter se fuerint aequales $\alpha=\beta=\gamma$, erit

$$\int x^\gamma X''\partial x = \int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\alpha X\partial x;$$

ideoque hoc casu integrale ita succincte exprimitur

$$Nxy = x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\alpha X\partial x.$$

## Corollarium 4.

1250. Si duo factores sint imaginarii, scilicet

$(\alpha+z)(\beta+z)=ff+2fz\,\text{cos.}\,\theta+zz$, ob

$\alpha=f(\text{cos.}\,\theta+\sqrt{-1}.\,\text{sin.}\,\theta)$ et

$\beta=f(\text{cos.}\,\theta-\sqrt{-1}.\,\text{sin.}\,\theta)$,

postremum quidem nostri integralis membrum manet reale ob

$$(\alpha-\gamma)(\beta-\gamma)=\gamma\gamma-2\gamma f\,\text{cos.}\,\theta+ff,$$

at bina priora fient, posito $\Phi=f\,\text{sin.}\,\theta\, l\, x$,

$$\frac{x^{-f\text{cos.}\theta}(\text{cos.}\Phi-\sqrt{-1}.\text{sin}\,\Phi)\int x^{f\text{cos.}\theta}X\partial x(\text{cos.}\Phi+\sqrt{-1}.\text{sin.}\Phi)}{-2f\sqrt{-1}.\text{sin.}\theta[\gamma-f(\text{cos.}\theta+\sqrt{-1}.\text{sin.}\theta)]}$$

$$+\frac{x^{-f\text{cos.}\theta}(\text{cos.}\Phi+\sqrt{-1}.\text{sin.}\Phi)\int x^{f\text{cos.}\theta}X\partial x(\text{cos.}\Phi-\sqrt{-1}.\text{sin.}\Phi)}{2f\sqrt{-1}.\text{sin.}\theta[\gamma-f(\text{cos.}\theta-\sqrt{-1}.\text{sin.}\theta)]},$$

quae reducuntur ad hanc formam realem

$$\frac{x^{-f\text{cos}\theta}[\gamma\text{si.}\Phi-f\text{si.}(\theta+\Phi)]\int x^{f\text{cos.}\theta}X\partial x\,\text{cs.}\Phi-x^{-f\text{cos}\theta}[\gamma\text{cs.}\Phi-f\text{cs.}(\theta+\Phi)]\int x^{f\text{cos.}\theta}X\partial x\,\text{si.}\Phi}{f\,\text{sin.}\theta(\gamma\gamma-2\gamma f\text{cos.}\theta+ff)}.$$

Scho-

## Scholion.

1251. Quod ad factores imaginarios attinet, integralium inde natorum reductio facilius in genere instituetur, vnde in his differentialium gradibus determinatis, ei non amplius immorabor. Factores autem aequales hic data opera pro singulis gradibus accuratius persequi est visum, quia supra nimis cito ad euolutionem generalem properanti in insignem errorem illabi contigit, quem statim feliciter euitassem, si eadem methodo ibi essem vsus. Huiusmodi autem vitium circa factores imaginarios hic non est pertimescendum, cum in hoc negotio nihil sub specie infinite parui negligendum occurrat. Ex hoc autem fonte errores illi, quos supra commisi, sunt nati: quod vitium subtile quo clarius ob oculos ponatur, vna cum necessaria emendatione hic euoluam. Quaestio scilicet pro casu praesenti huc redit, vt valor harum duarum formularum

$$\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)(\gamma-\alpha)}+\frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)}$$

definiatur, casu quo $\beta=\alpha$ et ambo membra in infinitum excrescent; hunc in finem pono $\beta=\alpha+\omega$, existente $\omega$ particula euanescente, et cum sit

$$x^{\beta}=x^{\alpha}x^{\omega}=x^{\alpha}e^{\omega lx}=x^{\alpha}(1+\omega lx),$$

hincque

$$x^{-\beta}=x^{-\alpha}(1-\omega lx),$$

habebimus

$$\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{\omega(\gamma-\alpha)}-\frac{x^{-\alpha}(1-\omega lx)\int x^{\alpha}X\partial x(1+\omega lx)}{\omega(\gamma-\beta)}.$$

Quia nunc est

$$\frac{1}{\gamma-\alpha}=\frac{1}{\gamma-\beta+\omega}=\frac{1}{\gamma-\beta}-\frac{\omega}{(\gamma-\beta)^2},$$

prius membrum induit hanc formam

$x^{-\alpha}$

$$\frac{x^{-\alpha}\int x^{\alpha} X \partial x}{\omega(\gamma-\beta)} - \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\gamma-\beta)^2};$$

posterius vero euolutum istam

$$\frac{-x^{-\alpha}\int x^{\alpha} X \partial x}{\omega(\gamma-\beta)} + \frac{x^{-\alpha} l x \int x^{\alpha} X \partial x - x^{-\alpha}\int x^{\alpha} X \partial x\, l x}{\gamma-\beta},$$

sicque valor quaesitus casu $\beta = \alpha$ concluditur

$$\frac{x^{-\alpha}(l x \int x^{\alpha} X \partial x - \int x^{\alpha} X \partial x\, l x)}{\gamma-\alpha} - \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\gamma-\alpha)^2}, \text{ seu}$$

$$\frac{x^{-\alpha}}{\gamma-\alpha}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x - \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha} X \partial x,$$

cuius formulae posterius membrum, quod in vitiosa illa methodo erat omissum, inde resultat, quod hic ad discrimen inter expressiones $\gamma - \alpha$ et $\gamma - \beta$ respeximus, quam necessariam cautionem supra negleximus.

## Problema 169.

1252. Proposita hac aequatione differentiali quarti gradus

$$X = A y + \frac{B x \partial y}{\partial x} + \frac{C x x \partial\partial y}{\partial x^2} + \frac{D x^3 \partial^3 y}{\partial x^3} + \frac{N x^4 \partial^4 y}{\partial x^4},$$

eius integrale per formulas integrales simplices euoluere.

## Solutio.

Formata hinc expressione algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3)$$
$$+ N(z-1)(z-2)(z-3)(z-4),$$

statuatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

et

et per praecepta generalia est

$$Ny = X^{IV}, \text{ existente } X^{IV} = x^{-\delta-1}\int X''' x^{\delta} \partial x,$$

siquidem X''' ex tribus prioribus factoribus determinetur, quemadmodum in problemate praecedente est factum. Valorem scilicet ibi pro $Nxy$ inuentum hic per $x^{\delta-1}\partial x$ multiplicari oportet, vnde oritur

$$\int x^{\delta} X''' \partial x = + \frac{x^{\delta-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} - \frac{\int x^{\delta} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}$$
$$+ \frac{x^{\delta-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} - \frac{\int x^{\delta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{\delta-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} - \frac{\int x^{\delta} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)},$$

vbi ob rationes supra demonstratas tres postremi termini coalescunt in $+\frac{\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$, ita vt sit integrale quaesitum

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{x^{-\delta}\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$$

siquidem omnes factores sint inter se inaequales. Casus autem quibus duo pluresue sunt aequales, in corollariis explorabimus.

## Corollarium 1.

1253. Si fuerint duo factores aequales nempe $\delta = \gamma$; seu si sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)^2$$

ex eadem forma pro X''' ante inuenta oritur integrale

 Nxy

$$N x y = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta - \alpha)(\gamma - \alpha)^2} - \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\beta - \alpha)(\gamma - \alpha)^2}$$

$$+ \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\alpha - \beta)(\gamma - \beta)^2} - \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha - \beta)(\gamma - \beta)^2}$$

$$+ \frac{x^{-\gamma} \int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\alpha - \gamma)(\beta - \gamma)};$$

vbi membra negatiua ita repraesentari possunt

$$- \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha - \beta)} \left( \frac{1}{(\gamma - \alpha)^2} - \frac{1}{(\gamma - \beta)^2} \right).$$

## Corollarium 2.

1254. Si fuerint tres factores aequales, vt sit

$$P = N (\alpha + z)(\beta + z)^3,$$

ideoque $\delta = \gamma = \beta$, ex formula §. 1248. inuenta colligitur integrale

$$N x y = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta - \alpha)^3} - \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\beta - \alpha)^3}$$

$$- \frac{x^{-\beta} \int \frac{\partial x}{x} \int x^{\beta} X \partial x}{(\beta - \alpha)^2} + \frac{x^{-\beta} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\beta} X \partial x}{\alpha - \beta}.$$

## Corollarium 3.

1255. Si omnes quatuor factores fuerint aequales, vt sit

$$P = N (\alpha + z)^4, \text{ existente } \delta = \gamma = \beta = \alpha,$$

ex forma pro tribus aequalibus §. 1249. inuenta fit integrale

$$N x y = x^{-\alpha} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x.$$

Corol-

## Corollarium 4.

1256. Si habeatur $\beta = \alpha$ et $\delta = \gamma$, vt bini factores sint aequales scilicet $P = N(\alpha + z)^2 (\gamma + z)^2$, ex §. 1247. vbi factores erant $(\alpha + z)^2 (\gamma + z)$, colligitur integrale

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2} \int \frac{\partial x}{x} \int x^\alpha X \partial x - \frac{x^{-\gamma}}{(\gamma-\alpha)^2} \int x^{\gamma-\alpha-1} \partial x \int x^\alpha X \partial x$$

$$- \frac{x^{-\alpha}}{(\gamma-\alpha)^3} \int x^\alpha X \partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^3} \int x^\gamma X \partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2} \int \frac{\partial x}{x} \int x^\gamma X \partial x;$$

quae ob

$$\int x^{\gamma-\alpha-1} \partial x \int x^\alpha X \partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha} \int x^\alpha X \partial x - \frac{1}{\gamma-\alpha} \int x^\gamma X \partial x$$

contrahitur in hanc formam

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2} \int \frac{\partial x}{x} \int x^\alpha X \partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2} \int \frac{\partial x}{x} \int x^\gamma X \partial x$$

$$- \frac{2x^{-\alpha}}{(\gamma-\alpha)^3} \int x^\alpha X \partial x - \frac{2x^{-\gamma}}{(\alpha-\gamma)^3} \int x^\gamma X \partial x.$$

## Problema 170.

1257. Proposita hac aequatione differentiali quinti gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \frac{Nx^5\partial^5 y}{\partial x^5},$$

eius integrale per formulas integrales simplices euoluere.

## Solutio.

Cum hic sit quantitas algebraica formanda

$$P = A + B(z-1) + C(z-1)(z-2) + \dots$$
$$\dots + N(z-1)(z-2)(z-3)(z-4)(z-5)$$

 statua-

ſtatuatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)(\varepsilon + z);$$

ac ſi hi factores omnes ſint inter ſe inaequales, ex integrali praecedente nouam inſtituendo integrationem prodibit integrale quaeſitum

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)} + \frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma)} + \frac{x^{-\delta}\int x^{\delta}X\partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta)}$$
$$+ \frac{x^{-\varepsilon}\int x^{\varepsilon}X\partial x}{(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon)}.$$

Caſus quo duo pluresue factores ſunt aequales, in corollariis euoluemus.

## Corollarium 1.

1258. Si fuerint duo factores aequales, vt ſit

$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2$, ideoque $\varepsilon = \delta$

ex praecedente problemate colligitur integrale

$$Nxy = + \frac{x^{-\alpha}\int x^{\alpha}X\partial x - x^{-\delta}\int x^{\delta}X\partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}$$
$$+ \frac{x^{-\beta}\int x^{\beta}X\partial x - x^{-\delta}\int x^{\delta}X\partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma}X\partial x - x^{-\delta}\int x^{\delta}X\partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+ \frac{x^{-\delta}\int\frac{\partial x}{x}\int x^{\delta}X\partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

Corol-

## Corollarium 2.

1259. Si fuerint tres factores aequales, vt sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)^3, \text{ ideoque}$$

$$\varepsilon = \delta = \gamma$$

ex corollario 1. problematis praecedentis colligitur

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x - x^{-\gamma}\int x^{\gamma} X \partial x}{(\beta - \alpha)(\gamma - \alpha)^3} - \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\beta - \alpha)(\gamma - \alpha)^2}$$

$$+ \frac{x^{-\beta}\int x^{\beta} X \partial x - x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha - \beta)(\gamma - \beta)^3} - \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha - \beta)(\gamma - \beta)^2}$$

$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha - \gamma)(\beta - \gamma)}.$$

## Corollarium 3.

1260. Si quatuor factores sint aequales, vt sit

$$P = N(\alpha + z)(\beta + z)^4, \text{ ideoque}$$

$\varepsilon = \delta = \gamma = \beta$, erit per §. 1254

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta - \alpha)^4} - \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\beta - \alpha)^4} - \frac{x^{-\beta}\int \frac{\partial x}{x}\int x^{\beta} X \partial x}{(\beta - \alpha)^3}$$

$$- \frac{x^{-\beta}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\beta} X \partial x}{(\beta - \alpha)^2} + \frac{x^{-\beta}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha - \beta}.$$

Ac si omnes quinque sint inter se aequales seu

$$P = N(\alpha + z)^5, \text{ erit integrale}$$

$$Nxy = x^{-\alpha}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x.$$

## Corollarium 4.

1261. Si P habeat duos factores quadratos, vt sit

$P = N(\alpha + z)(\beta + z)^2(\gamma + z)^2$, ideoque $\delta = \gamma$ et $\varepsilon = \beta$,

 erit

erit ex §. 1253. integrale, reductione necessaria facta;

$$N x y = \frac{x^{-\alpha}\int x^{\alpha} X \partial x - x^{-\beta} x^{\beta} X \partial x}{(\beta-\alpha)^2 (\gamma-\alpha)^2} - \frac{x^{-\gamma} x^{\gamma} X \partial x + x^{-\beta} x^{\beta} X \partial x}{(\beta-\alpha)(\alpha-\gamma)^2 (\beta-\gamma)}$$

$$+ \frac{x^{-\beta}\int \frac{\partial x}{x} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^3} - \frac{x^{-\gamma}\int x^{\gamma} X \partial x + x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\beta-\gamma)^3}$$

$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma} X \partial x + x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\gamma)(\beta-\gamma)^3},$$

quae porro redigitur ad hanc formam

$$N x y = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)^2 (\gamma-\alpha)^2} + \frac{x^{-\beta}\int \frac{\partial x}{x}\int x^{\beta} X \partial x}{(\alpha-\beta)^2 (\gamma-\beta)^2} - \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)^2 (\gamma-\beta)^2} - \frac{2 x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^3}$$

$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^2 (\beta-\gamma)^2} - \frac{2 x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)^3}.$$

## Corollarium 5.

1262. Si P habeat et factorem quadratum et cubicum, vt sit

$$P = N(\alpha + z)^2 (\gamma + z)^3, \text{ ideoque } \beta = \alpha \text{ et } \epsilon = \delta = \gamma,$$

ex §. 1254. colligitur integrale

$$N x y = \frac{x^{-\alpha}\int \frac{\partial x}{x}\int x^{\alpha} X \partial x}{(\gamma-\alpha)^3} - \frac{3 x^{-\alpha}\int x^{\alpha} X \partial x}{(\gamma-\alpha)^4}$$

$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^2} - \frac{2 x^{-\gamma}\int \frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^3} - \frac{3 x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^4}.$$

## Scholion.

1263. Ex his formulis parum constat, quemadmodum eas vlterius pro maiori factorum numero continuari oportet, si quidem factorum aliquot inter se fuerint aequales; inte-

integralium enim partes, quae factoribus inaequalibus respondent, legem seruant manifestam. Quae autem partibus aequalibus respondent, adhibita certa reductione commodius exprimi possunt. Veluti pro casu corollarii 1. si breuitatis gratia ponatur $\alpha-\delta=p$, $\beta-\delta=q$ et $\gamma-\delta=r$, forma $x^{-\delta}\int x^{\delta}X\partial x$ ducta est in

$$\frac{1}{(p-q)(r-p)pp}+\frac{1}{(p-q)(q-r)qq}+\frac{1}{(r-p)(q-r)rr},\ \text{seu}$$
$$\frac{(q-r)qqrr+(r-p)pprr+(p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr},$$

cuius fractionis numerator est

$$-(p-q)(q-r)(r-p)(pq+pr+qr),$$

ita vt haec fractio reducatur ad istam

$$\frac{-pq-pr-pr}{ppqqrr}=-\frac{1}{pqr}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right).$$

Quando ergo est

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2,$$

integrale ita se habet

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}+\frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}+\frac{x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+\frac{x^{-\delta}\int\frac{\partial x}{x}\int x^{\delta}X\partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}-\frac{x^{-\delta}\int x^{\delta}X\partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta}+\frac{1}{\beta-\delta}+\frac{1}{\gamma-\delta}\right).$$

Pro casu autem $P=M(\alpha+z)(\beta+z)(\gamma+z)^3$ habebitur

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)(\gamma-\alpha)^3}+\frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)^3}$$
$$+\frac{x^{-\gamma}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)}-\frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma}+\frac{1}{\beta-\gamma}\right)$$
$$+\frac{x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{1}{(\beta-\gamma)^2}\right).$$

Tum

Tum vero pro casu $P = N(\alpha + z)(\beta + z)^4$ fit

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)^4} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha-\beta}$$
$$- \frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha-\beta}\cdot\frac{1}{\alpha-\beta} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^2}$$
$$- \frac{x^{-\beta}\int x^{\beta} X \partial x}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^3}.$$

At pro casu $P = (\alpha + z)(\beta + z)^2(\gamma + z)^2$ integrale ita se habet

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta-\alpha)^2(\gamma-\alpha)^2} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^2}$$
$$- \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^2}\left(\frac{1}{\alpha-\beta} + \frac{2}{\gamma-\beta}\right) + \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)^2}$$
$$- \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)^2}\left(\frac{1}{\alpha-\gamma} + \frac{2}{\beta-\gamma}\right).$$

Denique pro casu $P = N(\alpha + z)^2(\gamma + z)^3$ est

$$Nxy = \frac{x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha} X \partial x}{(\gamma-\alpha)^3} - \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\gamma-\alpha)^3}\cdot\frac{3}{\gamma-\alpha}$$
$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^2} - \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^2}\cdot\frac{2}{\alpha-\gamma}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha-\gamma)^2}\cdot\frac{3}{(\alpha-\gamma)^2},$$

vnde indoles harum formularum iam magis fit perspicua, simulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca iam problema generale aggredi licebit.

Pro-

## Problema 171.

1264. Proposita aequatione differentiali cuiuscunque gradus huius formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \ldots$$
$$\ldots + N(z-1)(z-2)\ldots\ldots(z-n),$$

omnes factores habeat inter se inaequales; valorem ipsius $y$ completum per formulas integrales simplices exhibere.

## Solutio.

Sint primo formae P omnes factores simplices reales

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ldots\ldots(\nu+z),$$

factorum numero existente $= n$, et ex antecedentibus patet, ex quolibet factore nasci integralis partem. Ad has partes inueniendas, eliciantur sequentes valores

1.) posito $z = -\alpha$ fit $\mathfrak{A} = \frac{P}{\alpha+z}$, seu $\mathfrak{A} = \frac{\partial P}{\partial z}$,

2.) posito $z = -\beta$ fit $\mathfrak{B} = \frac{P}{\beta+z}$, seu $\mathfrak{B} = \frac{\partial P}{\partial z}$,

3.) posito $z = -\gamma$ fit $\mathfrak{C} = \frac{P}{\gamma+z}$, seu $\mathfrak{C} = \frac{\partial P}{\partial z}$,

etc.

Cum igitur sit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots\ldots(\nu-\alpha) = \frac{\mathfrak{A}}{N},$$

littera N ex superioribus formis per diuisionem tolletur, fietque integrale quaesitum

$$xy = \frac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\partial x + \frac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\partial x + \frac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}X\partial x + \text{etc.}$$

quoad singuli factores fuerint exhausti.

Quodsi iam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio sequenti modo instituetur. Quoniam bini factores simplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha + z)(\beta + z) = ff + 2fz\cos.\theta + zz,$$

ita vt sit

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et } \beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

vnde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur, quarum cum vtraque deriuetur ex forma $\frac{\partial P}{\partial z}$, illa posito $z = -\alpha$ haec vero posito $z = -\beta$, in ipsa forma $\frac{\partial P}{\partial z}$ loco $z$ vbique scribatur

$$-f(\cos.\theta \pm \sqrt{-1}.\sin.\theta),$$

prodeatque $\mathfrak{P} \pm \mathfrak{Q}\sqrt{-1}$. Ac perspicuum est fore

$$\mathfrak{A} = \mathfrak{P} + \mathfrak{Q}\sqrt{-1} \text{ et } \mathfrak{B} = \mathfrak{P} - \mathfrak{Q}\sqrt{-1},$$

vbi notandum est, quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ esse reales. Deinde cum sit

$$x^{m+n\sqrt{-1}} = x^m e^{n\sqrt{-1}.lx} = x^m[\cos.(nlx) + \sqrt{-1}.\sin.(nlx)],$$

si breuitatis ergo ponamus angulum $f\sin.\theta\, lx = \Phi$, erit

$$x^\alpha = x^{f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi),$$
$$x^{-\alpha} = x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi),$$
$$x^\beta = x^{f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi),$$
$$x^{-\beta} = x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi).$$

Quare pro binis partibus

$$\frac{1}{\mathfrak{A}}x^{-\alpha}\int x^\alpha X\partial x + \frac{1}{\mathfrak{B}}x^{-\beta}\int x^\beta X\partial x,$$

ob $\mathfrak{A}\mathfrak{B} = \mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}$ habebimus

$$+\frac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{matrix}(\mathfrak{P}-\mathfrak{Q}\sqrt{-1})(\cos.\Phi-\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)X\partial x\\(\mathfrak{P}+\mathfrak{Q}\sqrt{-1})(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi-\sqrt{-1}.\sin.\Phi)X\partial x\end{matrix}\right\}$$

quae

quae forma ob partes imaginarias se tollentes reducitur ad hanc

$$\frac{\left\{\begin{array}{l} 2x^{-f\,cos.\theta}(\mathfrak{P}\,cos.\,\Phi - \mathfrak{Q}\,sin.\,\Phi)\int x^{f\,cos.\theta}\,X\,\partial x\,cos.\,\Phi \\ +\,2x^{-f\,cos.\theta}(\mathfrak{Q}\,cos.\,\Phi + \mathfrak{P}\,sin.\,\Phi)\int x^{f\,cos.\theta}\,X\,\partial x\,sin.\,\Phi \end{array}\right\}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}}.$$

Talisque forma ad integrale accedit, quoties forma P huiusmodi habet factorem duplicem $ff + 2fz\,cos.\,\theta + zz$.

## Corollarium 1.

1265. Etsi autem factorum simplicium ipsius P quidam sunt imaginarii, eorum qui sunt reales euolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

## Corollarium 2.

1266. Pars integralis ex binis factoribus imaginariis seu vno factore duplici oriunda aliquanto succinctius repraesentari potest, si ponatur

$$\mathfrak{P} = \mathfrak{O}\,cos.\,\zeta \text{ et } \mathfrak{Q} = \mathfrak{O}\,sin.\,\zeta,$$

sic enim ea fiet

$$\frac{2}{\mathfrak{O}}x^{-f\,cos.\theta}[cos.(\zeta+\Phi)\int x^{f\,cos.\theta}\,X\,\partial x\,cos.\,\Phi$$
$$+\,sin.(\zeta+\Phi)\int x^{f\,cos.\theta}\,X\,\partial x\,sin.\,\Phi],$$

vbi $\zeta$ et $\theta$ sunt anguli constantes, $\Phi$ vero variabilis ob $\Phi = f\,sin.\,\theta.\,l\,x$.

## Problema 172.

1267. Si pro aequatione differentiali in praecedente problemate proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam inuestigare.

## Solutio.

In forma ergo ante exhibita

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

ponamus esse $\beta = \alpha$, quoniam vero tum vtraque integralis pars oritur infinita, altera signo $+$, altera signo $-$ affecta, ita vt iunctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta = \alpha - \omega$, denotante $\omega$ quantitatem euanescentem, eritque

$$\mathfrak{A} = -N\omega(\gamma - \alpha)(\delta - \alpha)(\varepsilon - \alpha) \text{ etc. et}$$

$$\mathfrak{B} = +N\omega(\gamma - \beta)(\delta - \beta)(\varepsilon - \beta) \text{ etc.}$$

Ponatur iam

$$\frac{P}{(\alpha + z)(\beta + z)} = \frac{P}{(\alpha + z)^2} = Q,$$

vt sit

$$Q = N(\gamma + z)(\delta + z)(\varepsilon + z) \text{ etc.}$$

ac manifestum est fieri

$$\mathfrak{A} = -\omega Q, \text{ posito } z = -\alpha \text{ et}$$

$$\mathfrak{B} = \omega Q, \text{ posito } z = -\beta = -\alpha + \omega,$$

vnde intelligitur valorem ipsius $Q$ posteriorem excedere priorem suo differentiali $\partial Q$, si fiat

$$z = -\alpha \text{ et } \partial z = \omega,$$

ita vt sit

$$\mathfrak{B} = \omega\left(Q + \omega\frac{\partial Q}{\partial z}\right), \text{ posito } z = -\alpha,$$

hincque

$$\frac{1}{\mathfrak{B}} = \frac{1}{\omega Q} - \frac{\partial Q}{QQ\partial z} = \frac{1}{\omega Q} + \frac{1}{\partial z}\partial.\frac{1}{Q}, \text{ existente } \frac{1}{\mathfrak{A}} = -\frac{1}{\omega Q}:$$

tum vero cum sit

$$x^\beta = x^\alpha x^{-\omega} = x^\alpha(1 - \omega\, l\, x) \text{ et } x^{-\beta} = x^{-\alpha}(1 + \omega\, l\, x),$$

binae partes integralis quaesitae erunt

$$-\frac{1}{\omega Q} x^{-\alpha} \int x^{\alpha} X \partial x$$
$$+\left(\frac{1}{\omega Q}+\frac{1}{\partial z} \partial . \frac{1}{Q}\right) x^{-\alpha}(1+\omega l x) \int x^{\alpha} X \partial x(1-\omega l x),$$

vbi cum membra per $\omega$ diuisa se destruant, resultat

$$\frac{1}{Q} x^{-\alpha}\left(l x \int x^{\alpha} X \partial x-\int x^{\alpha} X \partial x l x\right)+\frac{1}{\partial z} \partial . \frac{1}{Q} . x^{-\alpha} \int x^{\alpha} X \partial x$$

seu

$$\frac{1}{Q} x^{-\alpha} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x+\frac{1}{\partial z} \partial . \frac{1}{Q} . x^{-\alpha} \int x^{\alpha} X \partial x,$$

siquidem tam in valore $\frac{1}{Q}$ quam in $\frac{1}{\partial z} \partial . \frac{1}{Q}$ vbique loco $z$ scribatur $-\alpha$. Cum vero sit $Q=\frac{P}{(\alpha+z)^2}$, hi valores inde facile inueniuntur.

## Corollarium 1.

1268. Quodsi ergo quantitas algebraica P ex aequatione differentiali formata factorem habeat quadratum $(\alpha+z)^2$, inde in integrale transferenda est haec portio

$$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x+\frac{1}{\partial z} \partial . \frac{(\alpha+z)^2}{P} . x^{-\alpha} \int x^{\alpha} X \partial x,$$

posito $z=-\alpha$; dum si hic factor $\alpha+z$ esset solitarius integralis pars inde oriunda foret

$$\frac{\alpha+z}{P} x^{-\alpha} \int x^{\alpha} X \partial x, \text{ posito } z=-\alpha.$$

## Corollarium 2.

1269. Cum sit $Q=\frac{P}{(\alpha+z)^2}$, casu $z=-\alpha$ fiet $Q=\frac{\partial \partial P}{2 \partial z^2}$; verum quia hic ipsi $z$ iam valor determinatus est tributus, hinc $\frac{\partial Q}{\partial z}$ colligere non licet, sed prima est vtendum qua fit $\frac{\partial Q}{\partial z}=\frac{(\alpha+z) \partial P-2 P \partial z}{(\alpha+z)^3 \partial z}$, cuius fractionis cum numerator et denominator casu $z=-\alpha$ euanescat, erit pro eodem casu

$$\frac{\partial Q}{\partial z}=\frac{(\alpha+z) \partial \partial P-\partial z \partial P}{3(\alpha+z)^2 \partial z^2}=\frac{(\alpha+z) \partial^3 P}{6(\alpha+z) \partial z^3}=\frac{\partial^3 P}{6 \partial z^3}.$$

## Corollarium 3.

1270. Hoc valore inuento, quia est eodem casu $z = -\alpha$, quantitas $Q = \frac{\partial\partial P}{2\partial z^2}$ erit

$$\frac{1}{\partial z}\partial.\frac{1}{Q} = -\frac{\partial Q}{QQ\partial z} = -\frac{2\partial z\partial^3 P}{3\partial\partial P^2}, \text{ seu}$$

$$\frac{1}{\partial z}\partial.\frac{1}{Q} = \frac{2\partial z}{3}\partial.\frac{1}{\partial\partial P},$$

ex quibus formulis, si factores ipsius P non sint euoluti, partes integralis facilius reperiuntur.

## Problema 173.

1271. Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum $(\alpha + z)^3$, integralis partem inde oriundam inuestigare.

## Solutio.

Ponamus ergo esse

$$P = (\alpha + z)^2(\gamma + z)R, \text{ existente } \gamma = \alpha - \omega,$$

vbi $\omega$ pro quantitate euanescente assumitur. Quod ergo ante erat Q, id hinc fit $Q = (\gamma + z)R$, et facto $z = -\alpha$, erit $Q = -\omega R$, si etiam in R ponatur $z = -\alpha$. Deinde cum sit

$$\frac{\partial Q}{\partial z} = R + \frac{(\gamma + z)\partial R}{\partial z} = R - \frac{\omega\partial R}{\partial z},$$

eodem casu erit

$$\frac{1}{\partial z}\partial.\frac{1}{Q} = -\frac{1}{\omega\omega R} + \frac{\partial R}{\omega\omega R\partial z} = -\frac{1}{\omega\omega}.\frac{1}{R} - \frac{1}{\omega\partial z}\partial.\frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha + z)^2$ per praecedens problema haec obtinetur integralis pars

$$-\frac{1}{\omega R}x^{-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\partial x - \left(\frac{1}{\omega\omega R} + \frac{1}{\omega\partial z}\partial.\frac{1}{R}\right)x^{-\alpha}\int x^\alpha X\partial x,$$

cuius ambo membra in infinitum excrescunt ob $\omega = 0$.

Adiiciamus autem partem ex tertio factore

$$\gamma + z = \alpha - \omega + z,$$

oriun-

oriundam, quae ob $\frac{P}{\gamma+z} = (\alpha+z)^2 R$ est

$$\frac{1}{(\alpha+z)^2 R} x^{-\gamma} \int x^{\gamma} X \partial x, \text{ posito } z = -\gamma = -\alpha + \omega.$$

Quod si iam R vt ante is fuerit valor, qui oritur posito $z = -\alpha$, augendo hunc valorem particula $\omega$, loco $\frac{1}{R}$ scribi debet

$$\frac{1}{R} + \frac{\omega}{\partial z} \partial . \frac{1}{R} + \frac{\omega^2}{1.2 \partial z^2} \partial \partial . \frac{1}{R} \text{ etc.}$$

si quidem valorem $z = -\alpha$ et hic retineamus: vnde haec integralis pars ob $\alpha + z = \omega$ erit

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega \partial z} \partial . \frac{1}{R} + \frac{1}{2 \partial z^2} \partial \partial . \frac{1}{R}\right) x^{-\alpha+\omega} \int x^{\alpha-\omega} X \partial x;$$

sicque manifestum est, illum ipsius $\frac{1}{R}$ valorem vsque ad secundam potestatem ipsius $\omega$ continuari debuisse, atque eadem lege hic alteram partem $x$ inuoluentem exprimi conueniet. Ad quod obseruo, si habeatur huiusmodi formula $x^{\omega} \int x^{-\omega} V \partial x$ secundum potestates ipsius $\omega$ euoluenda, id hac ratione commodissime fieri. Posito

$$v = x^{\omega} \int x^{-\omega} V \partial x, \text{ vt fit } x^{-\omega} v = \int x^{-\omega} V \partial x,$$

erit differentiando $\partial v - \frac{\omega v \partial x}{x} = V \partial x$, quare posito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' + \text{etc.}$$

habebitur, terminos secundum potestates ipsius $\omega$ disponendo,

$$\left.\begin{array}{l} \partial T + \omega \partial T' + \omega\omega \partial T'' + \omega^3 \partial T''' + \text{etc.} \\ -V \partial x - \omega T \frac{\partial x}{x} - \omega\omega T' \frac{\partial x}{x} - \omega^3 T'' \frac{\partial x}{x} - \text{etc.} \end{array}\right\} = 0$$

ideoque

$$T = \int V \partial x, \; T' = \int \frac{\partial x}{x} \int V \partial x, \; T'' = \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int V \partial x, \text{ etc.}$$

Consequenter cum in applicatione sit $V = x^{\alpha} X$, erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega \partial z} \partial . \frac{1}{R} + \frac{1}{2 \partial z^2} \partial \partial . \frac{1}{R}\right) x^{-\alpha} \Big(\int x^{\alpha} X \partial x + \omega \int \frac{\partial x}{x} \int x^{\alpha} X \partial x$$
$$+ \omega^2 \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x\Big),$$

qua

qua cum parte ex $(\alpha+z)^2$ nata iunctim sumta, omnia membra infinita se mutuo destruunt, et pro quantitatis $P=(\alpha+z)^3R$ factore cubico $(\alpha+z)^3$ in integrale ingreditur haec pars

$$\frac{1}{R}x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x+\frac{1}{\partial z}\partial.\frac{1}{R}.x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x$$
$$+\frac{1}{2\partial z^2}\partial\partial.\frac{1}{R}.x^{-\alpha}\int x^{\alpha}X\partial x,$$

si modo in quantitate $R=\frac{P}{(\alpha+z)^3}$ vbique scribatur $z=-\alpha$.

## Corollarium 1.

1272. Methodus in solutione huius problematis adhibita facile ad quotcunque factores aequales extendi potest. Si enim fuerit $(\alpha+z)^m$ factor quantitatis P, atque in hac fractione $\frac{(\alpha+z)^m}{P}$ suisque differentialibus, postquam fuerint euoluta, ponatur $z=-\alpha$, partes integralis inde natae ita se habebunt

| Factor quant. P | $\alpha+z$ | $(\alpha+z)^2$ | $(\alpha+z)^3$ |
|---|---|---|---|
| Pars integralis. | $\frac{\alpha+z}{P}x^{-\alpha}\int x^{\alpha}X\partial x$ | $\frac{(\alpha+z)^2}{P}x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x$<br>$\frac{1}{\partial z}\partial.\frac{(\alpha+z)^2}{P}.x^{-\alpha}\int x^{\alpha}X\partial x$ | $\frac{(\alpha+z)^3}{P}x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x$<br>$\frac{1}{\partial z}\partial.\frac{(\alpha+z)^3}{P}.x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x$<br>$\frac{1}{2\partial z^2}\partial\partial.\frac{(\alpha+z)^3}{P}.x^{-\alpha}\int x^{\alpha}X\partial x.$ |

## Corollarium 2.

1273. Si fuerint duo pluresue factores duplices inter se aequales, sumtis

$$\alpha=f(\cos.\theta+\sqrt{-1}.\sin.\theta)\text{ et}$$
$$\beta=f(\cos.\theta-\sqrt{-1}.\sin.\theta),$$

partes pro $(\alpha+z)^2$ et $(\beta+z)^2$ seorsim euolutae methodo supra

pra adhibita non difficulter coniungentur, et ad realitatem reducentur.

## Scholion.

1274. Simili methodo, qua hoc caput est pertractatum, in euolutione capitis III. huius sectionis vti oportebat, neque tum vllum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret, errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim. Quodsi enim in aequatione capitis III.

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \frac{E\partial^4 y}{\partial x^4} + \text{etc.}$$

statuatur $x = lv$, vt sit $\partial x = \frac{\partial v}{v}$, functio autem X abeat in functionem ipsius $v$ quae sit V, proueniet aequatio eius formae quam hic tractauimus. Dum autem ibi elementum $\partial x$ pro constanti est habitum, ad hanc conditionem exuendam ponamus

$$\partial y = p\,\partial x,\ \partial p = q\,\partial x,\ \partial q = r\,\partial x,\ \partial r = s\,\partial x,\ \text{etc.}$$

vt haec aequatio resultet

$$X = V = Ay + Bp + Cq + Dr + Es + Ft + \text{etc.}$$

Nunc autem ob $\partial x = \frac{\partial v}{v}$ adipiscimur, elemento $\partial v$ constante sumto

$$p = \frac{\partial y}{\partial x} = \frac{v\partial y}{\partial v},$$

$$q = \frac{\partial p}{\partial x} = \frac{vv\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},$$

$$r = \frac{\partial q}{\partial x} = \frac{v^3\partial^3 y}{\partial v^3} + \frac{3v^2\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},$$

$$s = \frac{\partial r}{\partial x} = \frac{v^4\partial^4 y}{\partial v^4} + \frac{6v^3\partial^3 y}{\partial v^3} + \frac{7vv\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v},$$

$$t = \frac{\partial s}{\partial x} = \frac{v^5\partial^5 y}{\partial v^5} + \frac{10v^4\partial^4 y}{\partial v^4} + \frac{25v^3\partial^3 y}{\partial v^3} + \frac{15vv\partial\partial y}{\partial v^2} + \frac{v\partial y}{\partial v}, \text{ etc.}$$

Quare aequatio inter $v$ et $y$ erit haec

 V =

$$\begin{array}{llllll} V = Ay + & \frac{Bv\partial z}{\partial v} + & \frac{Cvv\partial\partial y}{\partial v^2} + & \frac{Dv^3\partial^3 y}{\partial v^3} + & \frac{Ev^4\partial^4 y}{\partial v^4} + & \frac{Fv^5\partial^5 y}{\partial v^5} + \text{etc.} \\ & + C & + 3D & + 6E & + 10F & \\ & + D & + 7E & + 25F & & \\ & + E & + 15F & & & \\ & + F & & & & \end{array}$$

cuius integrationem hic docuimus. Imprimis autem notandum est quantitatem algebraicam P hinc formandam

$$\begin{aligned} P = & A + (B + C + D + E + F)(z-1) \\ & + (C + 3D + 7E + 15F)(z-1)(z-2) \\ & + (D + 6E + 25F)(z-1)(z-2)(z-3) + \text{etc.} \end{aligned}$$

ad hanc formam reduci

$$\begin{aligned} P = A + B(z-1) + C(z-1)^2 + D(z-1)^3 + E(z-1)^4 \\ + F(z-1)^5 + \text{etc.} \end{aligned}$$

quae quantitas algebraica ab illa, qua in capite III. ad integrationem sumus vsi, hoc tantum differt, quod ibi littera $z$ id quod hic formula $z-1$ expressimus; ex quo etiam ambarum integratio facillime altera ad alteram reducitur.

## Conclusio libri primi.

1275. Atque haec fere sunt, quae ad librum primum de calculo integrali pertinere sunt visa, vbi methodum tradere institui, functiones vnius variabilis ex data quacunque differentialium cuiusque ordinis relatione inuestigandi; quod opus mihi equidem ita pertractasse videor, vt vix quicquam eorum, quae adhuc de hoc argumento ab aliis sunt inuenta et in medium allata, sit praetermissum.

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