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REACTION-DIFFUSION FISHER'S EQUATIONS VIA DECOMPOSITION METHOD

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ABSTRACT. The effect of the source, initial or boundary conditions in the use of Adomian decomposition method (ADM) on nonlinear partial differential equation or nonlinear equation in general is enormous. Sometimes the equation in question result to continuous exact solution in series form, other times it result to discrete approximate analytical solutions. In this paper, we show that continuous exact solitons can be obtained on application of ADM to the Fisher's equation with the deployment Taylor theorem to the terms(s) in question. And, the resulting series is split into the integral equations during the solution process. Resulting to multivariate Taylor's series of the exact solitons with the help of Adomian polynomials of the nonlinear reaction term correctly calculated. More physical results are further depicted in 2D, 3D and contour plots.

1. INTRODUCTION

A lot of attention and consideration has been devoted and dedicated to the investigation of the Fisher's equation in its various forms. Some in the original form as earlier conceived by [12] or its variance as can be seen in [2, 5–11, 13, 15–18] and the literatures therein. The equation is a nonlinear, parabolic,

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nonhomogeneous, reaction-diffusion, partial differential equation represented by the dynamical system

$$u_t = \nabla \cdot (\Phi \nabla u) + f(u), \qquad u(\vec{w}, 0) = \psi(\vec{w}), \tag{1.1}$$

where $\vec{w} = w(x, y, z)$. It is used in physical and biological sciences to describe processes of interaction among diffusion and reactions experiences in mathematical analysis. Specifically, it serves as models to describe evolution of neutron population in a Nuclear reactor, prototype model for propagating flame, a viral mutant in an infinitely long habitat, a chemical reaction propagation of genes, etc. Details of more application and uses of equation (1.1) can be seen in [17], [16] and the literatures therein. Equation (1.1) is also called Kolmogorov Petrovsky-Piscounov equation [14], u_t is the temporal evolution term acting as rate of change of say, concentration, $\nabla \cdot (\Phi \nabla u)$ is the diffusion term, Φ is the diffusion coefficient and $f(u)$ account for all local reactions.

Large sways of literature currently exist on how to solve the Fisher's equation mathematically be it in closed form saddled with burgoes assumptions or approximately using discrete parameters. Numerically, attention has been focused on the use of finite difference and finite elements methods as can be seen in [2, 5–9, 11, 13, 17] and the literatures therein. Authors [10, 15] implemented homotopy perturbation method and [16, 18] used ADM. In details, author [8] presented the one-dimensional Fisher equation using semi-implicit finite difference scheme to obtain discrete numerical results. [9] proposed a coupled implicit finite difference schemes based on Crank-Nicolson method to study the Burger-Fisher equation numerically. Author [13] studied this class of equation with an algorithm also based on finite difference and wavelet Galerkin methods with discretised domain using Crank-Nicolson scheme. [11] similarly studied the Fisher's equation using finite difference method based on Du-Fort Frankel scheme that is derived from Leapfrog explicit scheme. [2] studied the Fisher's equation numerically via the non-standard finite difference and the forward in time central space scheme. [6] utilized finite difference schemes to study the linear, nonlinear and coupled Fisher's equation numerically. Author [18] studied the Fisher's equation numerically using Adomian decomposition method. [5] presented an implicit and explicit scheme to study the Fisher's equation numerically. [15]presented the coupling of Elzaki and homotopy perturbation method. [7] presented a study using perturbation iteration technique to obtain numerical result. [17] studied equation (1.1) using Petrov-Galerkin finite element method. [16] studied using Adomian's method to obtain approximate result. [10] presented a study using homotopy perturbation method.

In this paper, we present a two level approach to obtain continuous, closed form results to the Fisher's equation using Adomian decomposition method on finite domain. The first step is to identify the source term(s) and obtain an equivalent Taylor series of it, which is a function of three variables going by (1.1). Then split the series terms into each integral equation that comprises the solution series. Resulting cumulatively to the multivariate Taylor series solution to the exact result. This development is particularly helpful to ensure rapid convergence if the term(s) in question are exponential, trigonometric or hyperbolic functions. Similar study is contain in [4]. In the next sections we state the ADM theory with the twist, followed by illustrative examples and we conclude.

2. Decomposition Theory with a Twist

The decomposition method by G. Adomian [1] writes the Fisher type equation (1.1) as

$$L_t\left[u\left(\vec{w},t\right)\right]+R\left[u\left(\vec{w},t\right)\right]+N\left[u\left(\vec{w},t\right)\right]=\alpha\left(\vec{w},t\right), \tag{2.1}$$

where $L_t(u)$ is a partial derivative operator which is same as u_t in equation (1.1). Similarly, $R\left[u\left(\vec{w},t\right)\right]=\nabla\cdot\left(\Phi\nabla u\right)$ is the linear operator containing partial derivatives of $\vec{w}$, $N\left[u\left(\vec{w},t\right)\right]=f(u)$ is a nonlinear analytic operator and $\alpha\left(\vec{w},t\right)$ is a non-homogeneous arbitrary function which is zero in equation (1.1). By the dictates of [1] and widely reported in [3, 4] $L_t^{-1}(u)=\int_0^t(u)dt$ is applied to equation (2.1) which becomes

$$u\left(\vec{w},t\right)=\beta\left(\vec{w},t\right)+L_t^{-1}\left\{\alpha\left(\vec{w},t\right)-R\left[u\left(\vec{w},t\right)\right]-N\left[u\left(\vec{w},t\right)\right]\right\} \tag{2.2}$$

where

$$u\left(\vec{w},t\right)=\sum_{n=0}^{\infty}u_n\left(\vec{w},t\right), \tag{2.3}$$

(2.4) $$N\left[u\left(\vec{w},t\right)\right]=\sum_{n=0}^{\infty}A_n\left[u_0\left(\vec{w},t\right),u_1\left(\vec{w},t\right),u_2\left(\vec{w},t\right),\cdots,\right]$$

$n\in\{0\cup\mathbb{Z}^+\}$. $A_n\left[u_0\left(\vec{w},t\right),u_1\left(\vec{w},t\right),u_2\left(\vec{w},t\right),\cdots\right]$ are the Adomian polynomials computed as

(2.5) $$A_n\left[u_0,u_1,u_2,\cdots\right]=\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}N\left[\sum_{k=0}^{\infty}\lambda^k u_k(\vec{w},t)\right]_{\lambda=0}.$$

From equation (2.2) and as similarly been reported in [4], $\beta\left(\vec{w},t\right)=u\left(\vec{w},0\right)=\psi\left(\vec{w}\right)$ which we represent in a twist as

(2.6) $$\beta\left(\vec{w},t\right)=\sum_{k=0}^{\infty}\psi_n(\vec{w}).$$

Equation (2.6) is a Taylor series expansion of $\beta\left(\vec{w},t\right)$. On putting equations (2.3), (2.4), (2.5) and (2.6) in (2.2) and simplifying we obtain the following recurrence relation

$$\begin{aligned}
&u_0\left(\vec{w},t\right)=\psi_0(\vec{w})\\
&u_1\left(\vec{w},t\right)=\psi_1(\vec{w})+L_t^{-1}\left\{-R\left[u_0\left(\vec{w},t\right)\right]-A_0\left[u_0\left(\vec{w},t\right)\right]\right\}\\
&u_2\left(\vec{w},t\right)=\psi_2(\vec{w})+L_t^{-1}\left\{-R\left[u_1\left(\vec{w},t\right)\right]-A_1\left[u_0\left(\vec{w},t\right),u_1\left(\vec{w},t\right)\right]\right\}\\
&\qquad\cdots\\
&u_n\left(\vec{w},t\right)=\psi_n(\vec{w})+L_t^{-1}\left\{-R\left[u_{n-1}\left(\vec{w},t\right)\right]-A_{n-1}\left[u_\delta\right]\right\},
\end{aligned}$$

where $u_\delta=u_0\left(\vec{w},t\right),u_1\left(\vec{w},t\right),u_2\left(\vec{w},t\right),\cdots,u_{n-1}\left(\vec{w},t\right)$, resulting to

(2.7) $$u\left(\vec{w},t\right)=lim_{\eta\to\infty}\sum_{n=0}^{\eta}u_n\left(\vec{w},t\right)$$

3. Illustrative Results

In this section, we consider a one dimensional spatial variable case of equation (1.1) to illustrate exact solution procedure.

Example 1. *Consider equation* (1.1) *with* $\nabla\cdot(\Phi\nabla u)=u_{xx}$, $f(u)=6u(1-u)$ *and* $u(x,0)=(1+e^x)^{-2}$ *as contained in [5, 10, 11, 13, 15, 18] with exact solution* *therein as* $u(x,t) = (1 + e^{x-5t})^{-2}$*. Following the dictates of section 2 we have*

$$\psi_0(x) = \frac{1}{4}, \qquad A_0 = u_0^2$$
$$\psi_1(x) = -\frac{1}{4}x, \qquad A_1 = 2u_0u_1$$
$$\psi_2(x) = \frac{1}{16}x^2, \qquad A_2 = 2u_0u_2 + u_1^2$$
$$\psi_3(x) = \frac{1}{48}x^3, \qquad A_3 = 2u_0u_3 + 2u_1u_2 \cdots$$

Substituting accordingly, we get

$$u(x,t) = \frac{1}{4} - \frac{1}{4}x + \frac{5}{4}t + \frac{1}{16}x^2 - \frac{5}{8}xt + \frac{25}{16}t^2 - \frac{125}{48}t^3 + \cdots. \tag{3.1}$$

Equation (3.1) *is the multivariate Taylor series expansion of the exact solution*

$$u(x,t) = (1 + e^{x-5t})^{-2}.$$

More results representation are shown in figures 1 and 2.

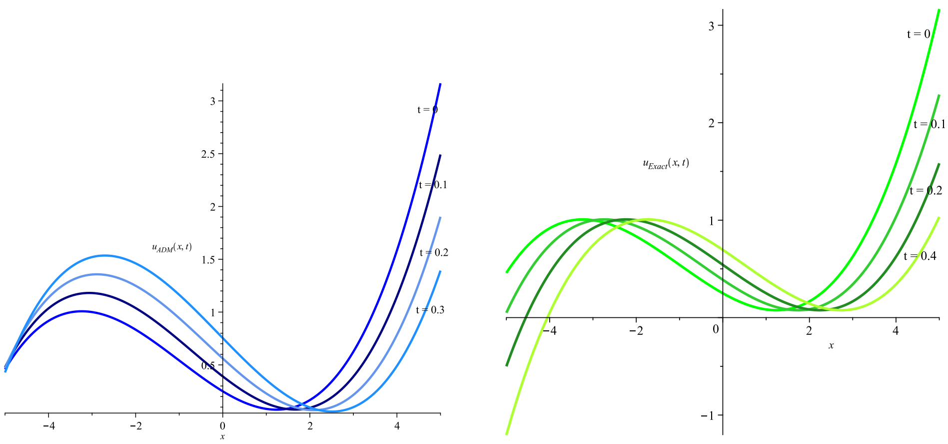

FIGURE 1. 2D plots of Example 1

Example 2. *Consider equation* (1.1) *with* $\nabla \cdot (\Phi \nabla u) = u_{xx}$*,* $f(u) = u(1 - u^6)$ *and* $u(x,0) = \left(1 + e^{\frac{3}{2}x}\right)^{-\frac{1}{3}}$ *as contained in [10, 13, 18] with exact solution therein as*

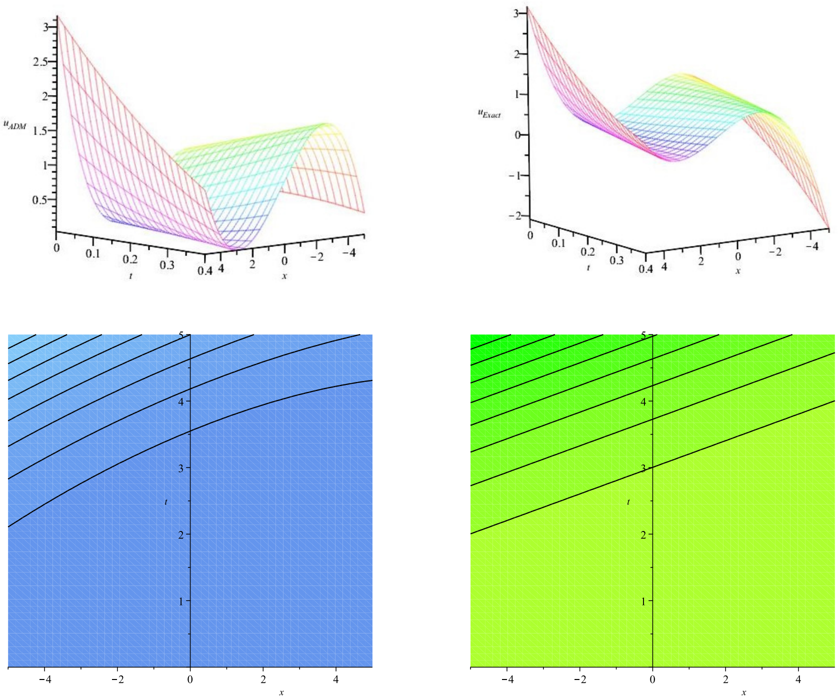

FIGURE 2. 3D and contour plots of Example 1

$u(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{3}{4}\left(x - \frac{5}{2}t\right)\right)\right]^{\frac{1}{3}}$. *Similarly, from dictates of section 2 we have*

$$
\begin{aligned}
\psi_0(x) &= \frac{1}{2}\eta, & A_0 &= u_0^7\\
\psi_1(x) &= -\frac{1}{8}\eta x, & A_1 &= 7u_0^6 u_1\\
\psi_2(x) &= \frac{1}{32}\eta x^2, & A_2 &= 7u_0^6 u_2 + 21u_0^5 u_1^2\\
\psi_3(x) &= \frac{1}{96}\eta x^3, & A_3 &= 7u_0^6 u_3 + 42u_0^5 u_1 u_2 + 35u_0^4 u_1^3\\
&\cdots
\end{aligned}
$$

Substituting accordingly, we get

$$u(x,t) = \frac{1}{2} - \frac{1}{8}\eta x + \frac{5}{16}\eta t - \frac{1}{32}\eta x^2 + \frac{5}{32}\eta x t - \frac{25}{128}\eta t^2 + \cdots, \tag{3.2}$$

where $\eta = 2^{\frac{2}{3}} = 4^{\frac{1}{3}}$. *Equation* (3.2) *is the multivariate Taylor series expansion of the exact solution*

$$u(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{3}{4}\left(x - \frac{5}{2}t\right)\right)\right]^{\frac{1}{3}}.$$

Further representation of our result are shown in figures 3 and 4.

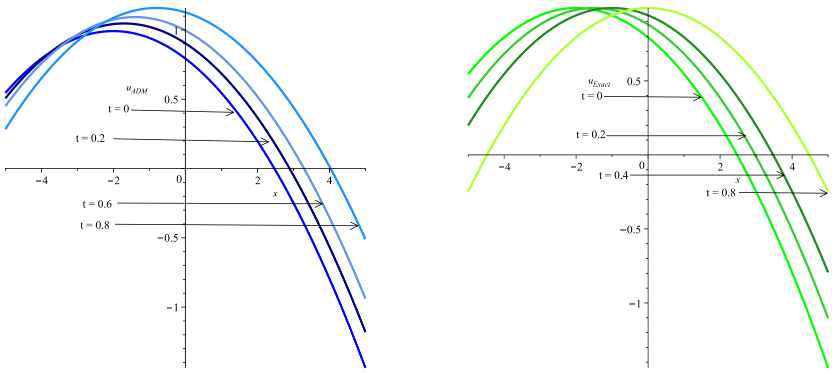

FIGURE 3. 2D plots of Example 2

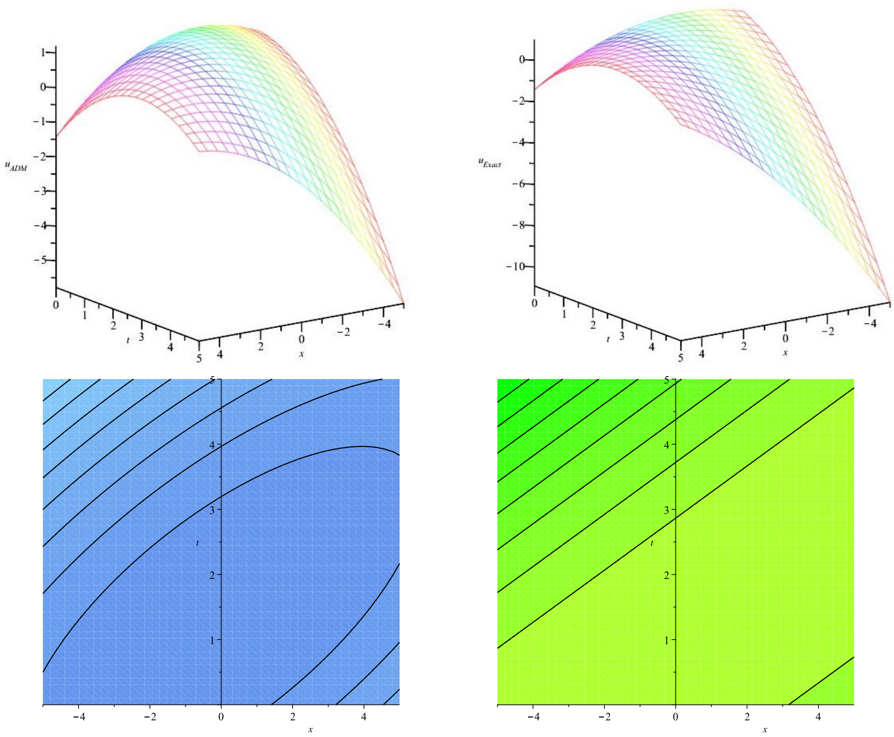

FIGURE 4. 3D and contour plots of Example 2

4. Conclusion

This work has been able to show that ADM does not only give approximate discrete solitons, as it has been currently portrayed in literatures, to the Fisher's equation but exact continuous solitary waves as it is evidently apparent here. However, correct calculation of Adomian polynomials is ultimate in obtaining the desires result which varies from one nonlinear situation to another.

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