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The Algebra of Operators for Regular Events

by

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Preface

The work presented in this dissertation has not been submitted for a degree or diploma or any other qualification at any other university, and is to the best of my knowledge and belief original, except where explicit mention is made to the contrary in the text.

I would like to express my deepest gratitude to Dr. J.H. Conway for his help and encouragement in the writing of this dissertation. I would also like to record my thanks to Professor J.C. Abbott for initially stimulating my interest in pure mathematics.

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Churchill College,

Cambridge. June 1970

Contents

The work of Kleene [1] and of Rabin and Scott [2] has provided the impetus for the current interest in the algebraic properties of the classes of languages corresponding to various classes of automata or accepting devices. As a result of the recent studies of Ginsburg and Greibach [3], a uniform method of procedure has been employed for research in this direction. They considered six basic operations, that of union (+), concatenation (.), Kleene star (*), regular event intersection, homomorphism, and inverse homomorphism, and defined a (full) "abstract family of languages" as a non-empty class of events closed under these "AFL" operations. The class of regular events then becomes the 'smallest' (full) AFL and the need for regularity is fundamental for the study of AFL's . If we let S be a collection of events (over a finite alphabet) closed under the regular operations of +,., and *, then the question of the closure of a class of events $\mathcal{X}$ in S under regular intersection, homomorphism, and inverse homomorphism becomes one of examining the $\mathcal{X}$ class-preserving operators for S. In addition, the algebraic structure of S induces

an algebraic structure on the class, $\mathcal{O}[S]$ say, of operators for S, that is, for Ω and Ψ in $\mathcal{O}[S]$, we define:

$$(\Omega + \Psi)[E] = \Omega[E] + \Psi[E],$$
$$\Omega\Psi[E] = \Omega[\Psi[E]],$$
$$\Omega^*[E] = E + \Omega[E] + \Omega.\Omega[E] + \ldots..$$

for an event E in S. Thus we are led to an investigation of the algebra of operators for S, and in particular, to a study of the regular algebras of class-preserving operators for S. Defining a <u>regulator</u> as an operator which maps regular events to regular events, our basic aim in this dissertation is to study the algebra of regulators and we show that the questions on the closure properties for various classes of events then find a natural setting in this context.

Chapter 1 is an introduction to the theory of operators. For arbitrary classes of events $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, we introduce the notion of a class of generalized transductions, $\left[\begin{smallmatrix}\mathcal{A}\\ \mathcal{B}\end{smallmatrix}\right]_{\mathcal{C}}$, the operators of which are $\mathcal{C}$-class functions over a finite number of ordered pairs of the form,

$[{A \atop B}] = \{[{w \atop v}] \mid w \in A$, A an event in $\mathcal{A}$ and $v \in B$, B an event in $\mathcal{B}\}$,

where the composition is component wise, and we establish some preliminary results on their properties. We then prove a general theorem on the Peirce product of two transductions which has many implications.

In Chapter 2, we show that the general theorem provides us with a large class of regulators, the operator class, $[{\mathcal{S} \atop \mathcal{R}}]_{\mathcal{R}}$, where $\mathcal{S}$ is the collection of arbitrary events, and $\mathcal{R}$ is the collection of regular events. This class includes the operations of homomorphism, inverse homomorphism, regular event intersection, regular substitution, inverse substitution, event derivation or quotient, regular event "shuffling", and many others. We also obtain several characterizations of full AFL's, and full AFL's closed under full substitution, in terms of generalized transductions.

Chapter 3 is an investigation of several classes of regulators which themselves form regular algebras. Of special interest is the class of <u>total regulators</u>, operators which map every event to a regular event, and it is shown that this class can be inserted into any regular algebra of regulators. We then consider the effect of the regulators on the context-free languages, and contrast the results with

the first part of the chapter.

In Chapter 4, the algebra of commutative regular events is studied. We establish its algebraic properties and provide a proof that the axiom scheme of Redko suffices to prove all tautologies in the algebra. We also point out the flaw in Redko's original proof.

Employing the structure developed in Chapter 4, the last chapter is a study of the regulators for commutative events. We first show that regular equations of a certain form have regular solutions, and obtain Parikh's theorem on the commutative image of a context-free language as a special case. We then examine the properties of the regulators for commutative events analogously to the non-commutative case of Chapters 1-3, and conclude with some conjectures about a very large class of regulators for the commutative events, the proof of which might have several interesting implications in the study of the context-free languages.

Preliminaries

Let Σ be an infinite set of symbols, called <u>letters</u>. A <u>word</u> is a finite formal sequence of letters in Σ, possibly empty, and an <u>event</u> is a set of words over a finite subset

of Σ, the alphabet V say. As usual, we do not distinguish between a letter and the corresponding word of length 1, nor between a word and the corresponding event of cardinal 1. We will denote letters as a, b, ..., words as w, v, ..., and events as E, F, ..., . We define:

$$E + F = E \cup F, \text{ the set union of E and F,}$$

$$E.F = \{ef \mid e,f \text{ words}, e\varepsilon E, f\varepsilon F\}, \text{ (E.F can be written EF)},$$

$$E^* = 1 + E + E^2 + \ldots = \sum_{n \geq 0} E^n,$$

where $E^0 = 1$, the empty word, $E^n = E.(E^{n-1})$,

$$\sum_{\alpha \varepsilon A} E_\alpha = \bigcup_{\alpha \varepsilon A} E_\alpha \text{ for an index set A,}$$

and we partially order events by defining

$$E \leq F \text{ if and only if } E + F = F.$$

A standard algebra, S, is a set with three operations Σ, . , *, defined on it, called the standard operations, with special elements 1, 0 (the empty event), such that

S1: $\sum_{\alpha \varepsilon A} E_\alpha = 0$ if A is the empty set,

S2: $\sum_{\alpha \varepsilon A} \sum_{\beta \varepsilon B_\alpha} E_\beta = \sum_{\beta \varepsilon \bigcup_{\alpha \varepsilon A} B_\alpha} E_\beta,$

S3: $E.1 = 1.E = E$,

S4: $(E.F).G = E.(F.G)$,

S5: $\sum_{\alpha\varepsilon A} E_\alpha \cdot \sum_{\beta\varepsilon B} E_\beta = \sum_{<\alpha,\beta>\varepsilon A\times B} (E_\alpha . F_\beta)$,

S6: $E^* = \sum_{n\varepsilon N} E^n$ where $N = \{0,1,2,...\}$.

Now, n×n matrices over an S-algebra form an S-algebra with the operations of Σ, $(M.N)_{ik} = \sum_j M_{ij}.N_{jk}$, 1 the n×n identity matrix, 0 the n×n zero matrix, and M* defined by S6. Also it is clear that in any standard algebra, E*G is the least F satisfying F = G + EF, and this enables one to prove:

<u>Theorem</u> (Conway [4]). If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a matrix over an S-algebra which is partitioned so that A,D are square, then

$$M^* = \begin{bmatrix} (A^*BD^*C)^*A^* & (A^*BD^*C)^*A^*BD^* \\ (D^*CA^*B)^*D^*CA^* & (D^*CA^*B)^*D^* \end{bmatrix} .$$

We call arbitrary events <u>standard events</u>, (in the class $\mathcal{S}$), and an event is said to be <u>regular</u> if and only if it can be obtained from 0,1, and the events, a, (in some finite

alphabet V) by repeated applications of +,., and *. In particular, the theorem above implies that the star of a matrix with regular events as entries is again a matrix of regular events.

A <u>regular algebra</u>, R, is a set with special elements 0 and 1, and operations +,.,*, (the <u>regular operations</u>) which satisfy all the formal laws which +,.,*,0,1 satisfy in every S-algebra.

Let $\mathcal{R}$ denote the class of regular events; then it is clear that the regular events (over a finite alphabet) form a regular algebra. The work on regular events has been extensive, and we list here some results for historical reasons and for the sake of completeness.

We define a <u>finite machine</u>, M, as a 5-tuple, $\{S,V,T,s_0,F\}$, where,

(i) S is a finite set of <u>states</u>,

(ii) V is a finite set of <u>input letters</u>,

(iii) T is a function, $T: S\times V \to S$, the <u>transition function</u>,

(iv) s_0 is a state in S, the <u>initial state</u>,

(v) F is a subset of S, the <u>final states</u>.

For a word, $v = a_{i_1} \ldots a_{i_n}$, $a_i \varepsilon V$, we define

$$T(s_o,v) = T(\ldots(T(T(s_o,a_{i_1}),a_{i_2}),\ldots,a_{i_n}).$$

Then an event E is said to be <u>representable</u> if and only if there exists a finite machine, $M_E = \{S,V,T,s_o,F\}$, such that

$$v \varepsilon E \quad \text{if and only if } T(s_o,v) \ \varepsilon \ F.$$

Kleene's classic theorem is:

<u>Kleene Theorem</u>: An event E is representable if and only if it is regular.

Thus we have a finite machine characterization for the regular events. Rabin and Scott in a later paper [2] showed that if T above was a relation, that is, M was a finite automaton, the result was still valid, and in particular, that $\mathcal{R}$ was closed under event intersection, complement (with respect to a fixed alphabet), and word reversal, that is, for a regular event, E, the event $\{w^T | w \varepsilon E$, w^T the mirror image or transpose of $w\}$ was regular.

Brzozowski [5] introduced the notion of differentiation (we use here left differentiation), that is, for $a \varepsilon V$, and an event E,

$\delta_a[E] = \{w | aw \varepsilon E\}$, is called a <u>letter derivate</u>, and for a word, $v = a_{i_1} \ldots a_{i_n}, a_i \varepsilon V$,

$$\delta_v[E] = \delta_{a_{i_n}}[\delta_{a_{i_{n-1}}}[\ldots[\delta_{a_{i_1}}[E]]\ldots]],$$

is called a <u>word derivate</u>.

We can then define an <u>event derivate</u> as

$$\delta_F[E] = \sum_{f \varepsilon F} \delta_f[E] = \{w | fw \varepsilon E \text{ for some } f \varepsilon F\}.$$

<u>Theorem</u> (Brzozowski, Conway): The word (respectively, event) derivates of a regular event E are regular events, and E is a regular event if and only if E has a finite number of word (event) derivates.

We conclude this discussion of regular events by introducing the decomposition theory or factor theory for (regular) events, again due to Conway [4], which is basic for the proof of some of our results.

<u>Definition</u>: For an event E,

$F.G.\ldots.H.\ldots.J.K$ is a <u>subfactorization</u> of E if

$F.G\ldots H\ldots J.K \leq E$. (*)

$F_1.G_1\ldots H_1\ldots J_1.K_1$ <u>dominates</u> a subfactorization if

$F \leq F_1, \ldots, K \leq K_1$, and

$F_1.G_1\ldots H_1\ldots J_1.K_1 \leq E$.

A term H is <u>maximal</u> if it cannot be increased without violating the inequality (*). A <u>factorization</u> of E is a subfactorization in which each term is maximal, and a <u>factor</u> is any element which can appear in a factorization.

Now any subfactorization is dominated by a factorization (not necessarily unique) in which every maximal term is unchanged. This enables one to prove:

<u>Lemma</u>: Any factor is a central factor in some 3-term factorization.

We say that F is a <u>left</u> (respectively, <u>right</u>) factor if it can appear at the left (right) in a factorization, and, as in the lemma, any left factor (right factor) is the left factor (right factor) in some two term factorization. Hence the condition that LR be a factorization defines a (1-1) correspondence, $L \leftrightarrow R$, between left and right factors. We index left and right factors, L_i, R_i, $i \varepsilon N$, so that $L_i \leftrightarrow R_i$, and we define the event E_{ij} by the condition that $L_i E_{ij} R_j$ be a subfactorization in which E_{ij} is maximal. Thus, E_{ij} is a factor, and by the lemma, any factor H is central in some 3-term factorization, LHR,

so that $H = E_{ij}$ for some i,j, not necessarily unique. In addition, we have that 1.E is a subfactorization in which E is maximal, hence dominated by a factorization $L_\ell.E$ for some ℓ. So $E = R_\ell$, and $L_\ell \geq 1$. Further, for any i,

$$L_\ell . L_i . R_i \leq E.$$

As $L_i.R_i$ must be maximal in this, we have that $L_i = E_{\ell i}$ for each i, and hence that $E = L_r = E_{\ell r}$ (by the symmetric argument to the one above for $E.1 \leq E.R_r \leq E$).

<u>Theorem</u> (1) Each E_{ij} is a factor and each factor is an E_{ij}.

(2) There exist indices ℓ, r, such that

$$E = L_r = R_\ell = E_{\ell r}, \quad L_i = E_{\ell i}, \quad R_i = E_{ir}.$$

Thus the factors naturally form a square matrix, among the entries of which is E.

Now as right factors are the maximal events R such that $K.R \leq E$ for some K, we have that,

$$K.R \leq E \text{ iff } k.R \leq E \text{ for every } k \varepsilon K \text{ iff } R \leq \delta_k[E] \text{ for every } k \varepsilon K$$

$$\text{iff } R \leq \bigcap_{k \varepsilon K} \delta_k[E],$$

and as R is maximal,

$$R = \bigcap_{k \varepsilon K} \delta_k[E].$$

In view of the fact that E is regular if and only if E has finitely many word derivates, we have:

<u>Theorem</u>: E has finitely many factors if and only if E is regular.

Hence, for a regular event E, the factor matrix, $\lceil E \rceil$, is finite. It is easy to show that (i) $1 \leq E_{ii}$, (ii) $E_{ij}E_{jk} \leq E_{ik}$, and (iii) $A.B \leq E_{ik}$ if and only if there exists j such that $A \leq E_{ij}$ and $B \leq E_{jk}$ (hence, for a word $uv \varepsilon E$, there exist factors L_i and R_i such that $u \varepsilon L_i$, $v \varepsilon R_i$). As a result, we have in addition:

<u>Theorem</u>: Factors of factors are themselves factors, and for a regular event E, the factor matrix, $\lceil E \rceil$, is its own star, that is,

$$\lceil E \rceil^* = \lceil E \rceil .$$

We conclude our preliminary remarks with a grammatical characterization of some of the classes of events which we will study, and in particular, gain another characterization of $\mathcal{R}$ as the class of events generated by one-sided linear grammars. For the equivalence of the event-classes and the corresponding classes of automata or accepting devices, see [6].

Definition: A grammar, Γ, is a 4-tuple, $\{V_N, V_T, A_0, P\}$, where

(i) V_N is a finite alphabet, $\{A_0, A_1, \ldots, A_q\}$ say, the non-terminal alphabet.

(ii) V_T is a finite alphabet, $\{a_1, \ldots, a_p\}$ say, the terminal alphabet.

(iii) A_0 in V_N is the unique initial non-terminal letter.

(iv) P is a finite set of productions of the form $\pi \to \psi$ where π and ψ are words in $(V_N + V_T)^*$.

For words u, z, in $(V_N + V_T)^*$, write $u \to z$ if there exists v and w in $(V_N + V_T)^*$ and a production $\pi \to \psi$ in P such that $u = v\pi w$, and $z = v\psi w$. Write $u \to^* z$ if there exists a finite sequence of words such that

$$u \to v \to w \to \ldots \to y \to z.$$

For a word u in $(V_N+V_T)^*$, let $Im_\Gamma(u)$ be the set $\{z\varepsilon V_T^* | u \rightarrow^* z\}$. The language (or event), L_Γ, generated by Γ is then $Im_\Gamma(A_0)$.

Let $\mathcal{Y}$ denote the class of events generated by grammars. This is a very large class of events and it can be shown that it contains a coded form of every recursively enumerable set.

We say that a grammar is context-sensitive if all productions have the form $vAw \rightarrow v\psi w$ where A is in V_N and ψ is a non-trivial word in $(V_N+V_T)^*$. The class of events, $\mathcal{U}$, generated by the context sensitive grammars (and possibly adding the empty word) is the class of context-sensitive languages, which correspond to the events accepted by the linear bounded automata.

A grammar is said to be context-free if every production in P has the left-hand side a letter in V_N, that is, $\pi\varepsilon V_N$ for $\pi \rightarrow \psi$ in P. This class of grammars generates the context-free languages, the class $\mathcal{C}$, which corresponds to the class of events accepted by push-down automata. $\mathcal{C}$ also includes the class of linear languages, $\mathcal{C}_\lambda$, which are generated by context-free grammars in which the productions are of the form,

$A \to bCd$, where A, C, are in V_N, and b, d, in $V_T \cup \{1\}$, or $A \to b$.

A context-free grammar is said to be <u>one-sided linear</u> if the productions have the form, $A \to bC$ (alternatively, $A \to Cb$), or $A \to b$, for A, b, C, as above, and these grammars generate the class of regular events, $\mathcal{R}$, as the productions of the grammar are in fact just state transitions when we consider the finite automata with the set of states V_N. We also have as a subclass of $\mathcal{R}$, the class of finite events, $\mathcal{F}$, where F is in $\mathcal{F}$ if and only if F is a finite sum of non-trivial words, w.

We have the following relations between these classes of events,

$$\mathcal{S} \supset \mathcal{T} \supset \mathcal{U} \supset \mathcal{C} \supset \mathcal{C}_\lambda \supset \mathcal{R} \supset \mathcal{F}$$

where all the inclusions are proper.

Finally, we remark that when we define $\mathcal{X}$-class functions for an arbitrary class of events, $\mathcal{X}$ say, we shall understand that $\mathcal{X}$ is either the empty class or contains a non-empty word in some event. This precludes $\mathcal{X}$ from being the exceptional event class, $\mathbf{1}$, consisting of only the empty event and 1.

Chapter 1

Operator Theory and Generalized Transductions

In this chapter we introduce the algebra of operators for a standard algebra. After defining the class of generalized transductions, we prove a general theorem on the Peirce product of two such operators, a result which has far ranging implications for the study of event classes, and in particular, provides us with a large class of regulators.

Let S be a standard algebra over a finite alphabet V. Let $\mathcal{O}[S]$ be the set of maps of S into itself.

Definition: Let O, Δ, C, respectively, denote the operators in $\mathcal{O}[S]$ such that

$O[E] = O$, the empty event in S,

$\Delta[E] = E$,

$C[E] = E^c$, the complement of E in S.

For operators Ω, Ψ in $\mathcal{O}[S]$,

$$(\Omega+\Psi)[E] = \Omega[E] + \Psi[E],$$

$$\Omega.\Psi[E] = \Omega[\Psi[E]],$$

$$\Omega^*[E] = \sum_{n\geq 0} \Omega^n[E], \text{ where } \Omega^0 = \Delta,\ \Omega^n = \Omega.\Omega^{n-1},$$

and we partially order $\mathcal{O}[S]$ by

$\Omega \leq \Psi$ if and only if $\Omega[E] \leq \Psi[E]$ for all E in S.

Let $\hat{\mathcal{L}}[S] = \{\Omega \varepsilon \mathcal{O}[S] \mid \Omega[\Sigma E_i] \geq \Sigma\Omega[E_i]\}$, the <u>super-linear</u> operators, and we let $\mathcal{L}[S] = \{\Omega \varepsilon \mathcal{O}[S] \mid \Omega[\Sigma E_i] = \Sigma\Omega[E_i]\}$, the <u>linear</u> operators.

For Ω in $\mathcal{O}[S]$, we define the <u>dual operator</u> ∂_Ω by

$\partial_\Omega[E] = \{w \mid \Omega[w] \cap E \neq 0\}$ for events E in S.

<u>Lemma 1.1</u>: (1) For words w and v in S,

$w \varepsilon \partial_\Omega[v]$ if and only if $v \varepsilon \Omega[w]$.

(2) $\Omega \leq \Psi$ implies that $\partial_\Omega \leq \partial_\Psi$.

(3) For Ω in $\mathcal{O}[S]$, ∂_Ω is in $\mathcal{L}[S]$.

<u>Proof</u>: (1) and (2) are immediate from the definition of ∂_Ω. (3): From (1) we have that $w \varepsilon \partial_\Omega[E]$ iff $\Omega[w] \cap E \neq 0$. But there exists $v \varepsilon E$ such that $v \varepsilon \Omega[w]$, iff there exists $v \varepsilon E$ such that $w \varepsilon \partial_\Omega[v]$, iff $w \varepsilon \sum_{v \varepsilon E} \partial_\Omega[v]$, and hence, $\partial_\Omega \varepsilon \mathcal{L}[S]$.

<u>Theorem 1.2</u>: $\partial : \mathcal{O}[S] \to \mathcal{L}[S]$ is an anti-homomorphism mapping Ω to ∂_Ω such that:

(1) $\partial_\Omega[0] = 0$, $\partial_\Delta = \Delta$, $\partial_0 = 0$,

(2) $\partial_{\Omega+\Psi} = \partial_\Omega + \partial_\Psi$,

(3) $\partial_{\Omega.\Psi} = \partial_\Psi.\partial_\Omega$,

(4) $(\partial_\Omega)^* = \partial_{\Omega^*}$,

(5) $\partial_{\partial_\Omega} \leq \Omega$ for Ω in $\hat{\mathcal{L}}[S]$, and

$\partial_{\partial_\Omega} = \Omega$ if and only if Ω is in $\mathcal{L}[S]$.

Proof: ∂ is well-defined in view of 1.1 (2). The proof of the theorem is immediate with the exception of (5).

Consider $\partial_{\partial_\Omega}[E]$ for an event E in S. Then $w \varepsilon \partial_{\partial_\Omega}[E]$ iff there exists $v \varepsilon E$ such that $v \varepsilon \partial_\Omega[w]$, which by 1.1 (1) is equivalent to saying that $w \varepsilon \sum_{v\varepsilon E} \Omega[v]$. Hence if Ω is in $\hat{\mathcal{L}}[S]$, then $\partial_{\partial_\Omega} \leq \Omega$, and if Ω is in $\mathcal{L}[S]$, then $\partial_{\partial_\Omega} = \Omega$. Now 1.1 (3) implies the converse, that is, if $\partial_{\partial_\Omega} = \Omega$, then Ω is in $\mathcal{L}[S]$.

Definition: For words w and v in S, we define the operator $[{}^w_v]$ by

$$[{}^w_v]\ [u] = \begin{cases} v \text{ if } w = u \\ 0 \text{ otherwise.} \end{cases}$$

We extend the operator $\begin{bmatrix} w \\ v \end{bmatrix}$ linearly so that for an event E,

$$\begin{bmatrix} w \\ v \end{bmatrix} [E] = \sum_{u \varepsilon E} \begin{bmatrix} w \\ v \end{bmatrix} [u],$$

that is, $\begin{bmatrix} w \\ v \end{bmatrix}$ is an operator in $\mathcal{L}[S]$. We also observe that

$$\partial_{\begin{bmatrix} w \\ v \end{bmatrix}} = \begin{bmatrix} v \\ w \end{bmatrix}.$$

The composition of two operators of this form then becomes:

$$\begin{bmatrix} w \\ v \end{bmatrix} \cdot \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} z \\ v \end{bmatrix} \quad \text{if } w = u, \text{ and}$$

$$0 \text{ otherwise (\underline{the Peirce product})}.$$

<u>Theorem 1.3</u>: $\mathcal{L}[S]$ is generated as a standard algebra by operators of the form $\begin{bmatrix} w \\ v \end{bmatrix}$ for words w and v in S, with the operations of union, Peirce product, and star.

<u>Proof</u>: For Ω in $\mathcal{L}[S]$, let $\Omega' = \Sigma \begin{bmatrix} w \\ v \end{bmatrix}$, $(v \varepsilon \Omega[w])$. Then Ω' is a linear operator, and for a word w in S,

$$\Omega'[w] = \{v | v \varepsilon \Omega[w]\} = \Omega[w],$$

that is, $\Omega = \Omega'$. Note that for operators Ω, Ψ, and Φ in

$\mathcal{L}[S]$, we have the left distributivity required for a standard algebra, that is, $\Omega(\Psi + \Phi) = \Omega.\Psi + \Omega.\Phi$, a property lacking for $\hat{\mathcal{L}}[S]$ operators.

From now on, we identify each operator Ω of $\mathcal{L}[S]$ with the set $\{\begin{bmatrix} w \\ v \end{bmatrix} | v \varepsilon \Omega[w]\}$ of ordered pairs $\begin{bmatrix} w \\ v \end{bmatrix}$.

<u>Corollary 1.3.1</u>: For a linear operator Ω,

$$\partial_\Omega = \partial_{\{\begin{bmatrix} w \\ v \end{bmatrix} | v \varepsilon \Omega[w]\}} = \{\begin{bmatrix} v \\ w \end{bmatrix} | w \varepsilon \partial_\Omega[v]\}.$$

<u>Transductions</u>

Let E be an $\mathcal{X}$-class event in a standard algebra over the alphabet $\{a_1, \ldots, a_p\}$. Associated with E is a function, f, of n variables, $\alpha_1, \ldots, \alpha_p$, say, such that

$$f(\alpha_1, \ldots, \alpha_p) = \Sigma\ \alpha_{i_1} \cdots\cdots \alpha_{i_n}, \quad (a_{i_1} \cdots\cdots a_{i_n} \varepsilon E).$$

We call functions of this type <u>$\mathcal{X}$-class functions</u>.

For a class of events $\mathcal{Y}$, let $\mathcal{X}(\mathcal{Y})$ denote the class of events of the form $f(F_1, \ldots, F_m)$, where f is an $\mathcal{X}$-class function of m variables, $m \varepsilon N$, and $F_1, \ldots, F_m$, are $\mathcal{Y}$-class events.

$\mathcal{X}(\mathcal{X}) \subseteq \mathcal{X}$ is equivalent to saying that the class of events $\mathcal{X}$ is closed under "full substitution" in the sense of [3]. For example, it follows from our definition of regular events that $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$, as an event which is a regular function of regular events can also be obtained from 0, 1, and a finite alphabet by a finite number of applications of +,., and *. The fact that $\mathcal{C}(\mathcal{C}) \subseteq \mathcal{C}$ is also well known [6], and trivially we have that $\mathcal{S}(\mathcal{S}) \subseteq \mathcal{S}$.

Let $\mathcal{X}[S]$ represent the subclass of $\mathcal{O}[S]$ that preserves $\mathcal{X}$-class events, that is,

$$\mathcal{X}[S] = \{\Omega \varepsilon \mathcal{O}[S] \mid \text{for every } \underline{\mathcal{X}\text{-event}}\ E \text{ in } S,\ \Omega[E] \text{ is in } \mathcal{X}\}.$$

It is clear that $\mathcal{O}[S] = \mathcal{S}[S]$. It is the investigation of $\mathcal{R}[S]$ that motivates the following definitions.

For standard algebras S_1 and S_2, let $S_1 \times S_2$ denote the standard algebra of orderd word pairs (or relations) of the form $\begin{bmatrix} w \\ v \end{bmatrix}$ for words w in S_1 and v in S_2, with the operations:

$$\Omega + \Psi = \{\begin{bmatrix} w \\ v \end{bmatrix} | \begin{bmatrix} w \\ v \end{bmatrix} \text{ in } \Omega \cup \Psi\},$$

$$\Omega \times \Psi = \{\begin{bmatrix} w_1 w_2 \\ v_1 v_2 \end{bmatrix} \Bigg| \begin{bmatrix} w_1 \\ v_1 \end{bmatrix} \text{ in } \Omega, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \text{ in } \Psi\},$$

$$\Omega^{\dagger} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Omega + \Omega\times\Omega + \cdots\cdots \quad ,$$

$$\sum_{\alpha\varepsilon A} \Omega_\alpha = \bigcup_{\alpha\varepsilon A} \Omega_\alpha \quad .$$

If f(a,b,c,...) is an $\mathcal{Q}$-class function of its arguments, then we write $f_\times(\Omega,\Psi,\Phi,\ldots)$ for the operator obtained from the linear operators $\Omega,\Psi,\Phi,\ldots$, and the function f with the operations of $+,\times,^{\dagger}$, and Σ corresponding to $+,.,*$, and Σ respectively.

For an event E in S_1 and an event F in S_2, we let

$$\begin{bmatrix} E \\ F \end{bmatrix} = \{ \begin{bmatrix} w \\ v \end{bmatrix} | w\varepsilon E, v\varepsilon F\}.$$

For classes of events $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Q}$, let $E_1,\ldots,E_n$ be $\mathcal{X}$-class events, $F_1,\ldots,F_n$ be $\mathcal{Y}$-class events, and f an $\mathcal{Q}$-class function of n variables. Then,

$$\Omega = f_{\times}\left(\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}, \ldots, \begin{bmatrix} E_n \\ F_n \end{bmatrix}\right),$$

is said to be an $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}_{\mathcal{A}}$-transduction, or an operator in the class $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}_{\mathcal{A}}$, if we interpret Ω as a linear operator mapping some standard algebra S_1 into S_2. Note that $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}_{\mathcal{A}}$ is defined for all $\mathcal{X}$-, $\mathcal{Y}$-, and $\mathcal{A}$-class events, not necessarily in a fixed standard algebra. However, Ω is in some standard algebra $S_1 \times S_2$ as the events E_i, F_j above are over some finite alphabet. In our use of these operators, we assume without loss of generality that $S = S_1 = S_2$ is a standard algebra over some finite alphabet, $\{a_1,\ldots,a_p\}$ say, unless otherwise specified.

Of special interest is the operator class $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}_{\mathcal{R}}$, the biregular operators over $\mathcal{X}$ and $\mathcal{Y}$, where we consider regular functions with the operations of $+$, $\times$, and †. For the sake of notation, we usually write $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}_{\mathcal{R}}$ as $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$.

Corollary 1.3.2: For a standard algebra S,

$$\begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}_{\mathcal{S}} \cap \; \mathcal{O}[S] = \mathcal{L}[S].$$

Proof: For an operator Ω in $\mathcal{L}[S]$, 1.3 implies that Ω is of the form $\Sigma\begin{bmatrix} w \\ v \end{bmatrix}$, $v \varepsilon \Omega[w]$, which clearly is an $\begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}_{\mathcal{S}}$ operator. The converse is immediate.

Definition: For an event E, we define the intersection operator $\cap_E$ by

$$\cap_E[F] = E \cap F$$

for all events F. If $E = f(a,b,c,\ldots)$, then the operator $\cap_E$ is $f_\times(\begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix}, \begin{bmatrix} c \\ c \end{bmatrix}, \ldots)$. For a class of events $\mathcal{Y}$, we define the operator class $\cap_{\mathcal{Y}}$ as the set of operators $\{\cap_E | E$ an event in $\mathcal{Y}\}$. For a class of events $\mathcal{X}$, we denote the class $\cap_{\mathcal{Y}}[\mathcal{X}]$ as $\mathcal{X}_{\mathcal{Y}}$. (We allow this asymmetric notation in view of the fact that in our usage, $\mathcal{Y}$ will be the class of regular events $\mathcal{R}$.)

We now state the main theorem for the Peirce product of generalized transductions.

Theorem 1.10: For classes of events $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$, such that $1 \varepsilon \mathcal{A}$,

$$\begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix} \cdot \begin{bmatrix} \mathcal{B} \\ \mathcal{D} \end{bmatrix}_{\mathcal{E}} \subseteq \begin{bmatrix} \mathcal{B} \\ (\mathcal{D}(\mathcal{R}))_{\mathcal{R}}(\mathcal{A}) \end{bmatrix}_{(\mathcal{E}(\mathcal{F}))_{\mathcal{R}}}.$$

To this end we begin with some results which are needed for the proof of the theorem but also prove interesting in their own right.

<u>Definition</u>: For standard algebras S_1 with an alphabet V_1 and S_2 with an alphabet V_2, a <u>substitution</u>, Ψ, is a linear operator mapping S_1 to S_2 such that

$$\Psi[E.F] = \Psi[E].\Psi[F]$$

for all events E, F in V_{1*}. We say that Ψ is in the operator class <u>SUB</u>.

For a class of events $\mathcal{X}$, Ψ is said to be a <u>$\mathcal{X}$-substitution</u> (in the operator class <u>$\mathcal{X}$-SUB</u>) if $\Psi[a]$ is an $\mathcal{X}$-class event for a in $V_1 \cup \{1\}$. If in addition, we have that $\Psi[1] = 1$, Ψ is said to be a <u>unit substitution</u>.

A <u>homomorphism</u>, ϕ, is a unit substitution such that letters are mapped to words. ϕ is said to be in the operator class <u>HOM</u>. ϕ is a <u>letter homomorphism</u> if $\phi[a]$ is in $V_2 \cup \{1\}$ for $a \varepsilon V_1$, and is a <u>1-free homomorphism</u> if $\phi[a]$ is in $V_2{}^* \setminus 1$.

For a substitution Ψ (respectively, homomorphism ϕ), ∂_Ψ (respectively ∂_ϕ) is called a <u>dual substitution</u> or

inverse substitution (respectively, inverse homomorphism).

Lemma 1.4: SUB $\subseteq \begin{bmatrix} \mathcal{R} \\ \mathcal{S} \end{bmatrix}$.

Proof: For $\Psi \in$ SUB, it is sufficient to consider the effect of Ψ on the unit word and the letters of the alphabet in its domain. Then we have that

$$\Psi = \left(\begin{bmatrix} a_1 \\ \Psi[a_1] \end{bmatrix} + \ldots + \begin{bmatrix} a_p \\ \Psi[a_p] \end{bmatrix} + \begin{bmatrix} 1 \\ \Psi[1] \end{bmatrix} \right)^{\dagger} ,$$

an operator in $\begin{bmatrix} \mathcal{R} \\ \mathcal{S} \end{bmatrix}$.

Corollary 1.4.1: $\mathcal{A}$ -SUB $\subseteq \begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix}$ for a class of events $\mathcal{A}$.

Corollary 1.4.2: The operator classes HOM, ∂_{HOM}, $\mathcal{R}$-SUB, and $\partial_{\mathcal{R}\text{-SUB}}$ are subclasses of $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$.

We now prove a decomposition theorem for $\begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix}$ operators.

Theorem 1.5: For a class of events $\mathcal{A}$ such that $1 \varepsilon \mathcal{A}$, and an operator $\Omega \varepsilon \begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix}$,

$$\Omega = \theta \cdot \cap_E \cdot \partial_\tau ,$$

where θ is an $\mathcal{Q}$-substitution, E is a regular event, and τ is a letter homomorphism.

Proof: The theorem is proved for the case where $\Omega:S_1 \to S_2$ with the standard algebras over the alphabets V_1, $= \{a_1,\ldots,a_p\}$ say, and V_2. As $\mathcal{Q}(\mathcal{R}) \subseteq \mathcal{Q}$, we may assume that Ω is of the form

$$h_\times\left(\begin{bmatrix} a_1 \\ 1 \end{bmatrix},\ldots,\begin{bmatrix} a_p \\ 1 \end{bmatrix},\begin{bmatrix} 1 \\ A_1 \end{bmatrix},\ldots,\begin{bmatrix} 1 \\ A_q \end{bmatrix}\right),$$

where $A_1,\ldots,A_q$ are $\mathcal{Q}$-class events and h is a regular function.

Let (i) $\theta = \left(\begin{bmatrix} a_1 \\ 1 \end{bmatrix}+\ldots+\begin{bmatrix} a_p \\ 1 \end{bmatrix}+\begin{bmatrix} c_1 \\ A_1 \end{bmatrix}+\ldots+\begin{bmatrix} c_q \\ A_q \end{bmatrix}\right)^\dagger$

with $\{c_1,\ldots,c_q\}$ an alphabet of q letters distinct from V_1, and as $1 \in \mathcal{Q}$, θ is an $\mathcal{Q}$-substitution,

(ii) $\tau = \left(\begin{bmatrix} a_1 \\ a_1 \end{bmatrix}+\ldots+\begin{bmatrix} a_p \\ a_p \end{bmatrix}+\begin{bmatrix} c_1 \\ 1 \end{bmatrix}+\ldots+\begin{bmatrix} c_q \\ 1 \end{bmatrix}\right)^\dagger$,

a letter homomorphism,

(iii) $E = h(a_1,\ldots,a_p,c_1,\ldots,c_q)$ a regular event.

$$\text{Then } \theta.\cap_E.\partial_\tau = \theta.h_\times\left(\begin{bmatrix}a_1\\a_1\end{bmatrix},\ldots,\begin{bmatrix}a_p\\a_p\end{bmatrix},\begin{bmatrix}c_1\\c_1\end{bmatrix},\ldots,\begin{bmatrix}c_q\\c_q\end{bmatrix}\right).\partial_\tau$$

$$= h_\times\left(\begin{bmatrix}a_1\\1\end{bmatrix},\ldots,\begin{bmatrix}a_p\\1\end{bmatrix},\begin{bmatrix}c_1\\A_1\end{bmatrix},\ldots,\begin{bmatrix}c_q\\A_q\end{bmatrix}\right).\partial_\tau$$

$$= h_\times\left(\begin{bmatrix}a_1\\1\end{bmatrix},\ldots,\begin{bmatrix}a_p\\1\end{bmatrix},\begin{bmatrix}1\\A_1\end{bmatrix},\ldots,\begin{bmatrix}1\\A_q\end{bmatrix}\right)$$

as was to be shown.

<u>Corollary 1.5.1</u>: For $\Omega \in \begin{bmatrix}\mathcal{R}\\\mathcal{R}\end{bmatrix}$, $\Omega = \theta.\cap_E.\partial_\tau$ where θ and τ are letter homomorphisms and E a regular event.

<u>Proof</u>: As $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$, we may assume that Ω is of the form,

$$h_\times\left(\begin{bmatrix}a_1\\1\end{bmatrix},\ldots,\begin{bmatrix}a_p\\1\end{bmatrix},\begin{bmatrix}1\\a_1\end{bmatrix},\ldots,\begin{bmatrix}1\\a_q\end{bmatrix}\right),$$

and θ above becomes a letter homomorphism.

<u>Corollary 1.5.2</u>: For a class of events $\mathcal{A}$ such that $1\in\mathcal{A}$,

$$\begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix} \subseteq \mathcal{A}\text{-SUB.} \cap_{\mathcal{R}} . \mathcal{R}\text{-SUB.}$$

<u>Lemma 1.6</u>: For a letter homomorphism $\tau, \partial_\tau \in \mathcal{R}$-SUB.

<u>Proof</u>: Let τ be a letter homomorphism mapping an alphabet $V_1 = \{a_1, \ldots, a_p, c_1, \ldots, c_q\}$ say, to an alphabet $V_2 = \{b_1, \ldots, b_p\}$ say, such that

$$\tau[a_i] = b_i, \ i=1,\ldots,p, \text{ and}$$

$$\tau[c_j] = 1, \ \ j=1,\ldots,q.$$

Then, $\partial_\tau = \left(\begin{bmatrix} 1 \\ c_1+\ldots+c_q \end{bmatrix} + \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} + \ldots + \begin{bmatrix} b_p \\ a_p \end{bmatrix} \right)^{\dagger}$, is trivially a substitution, and for $i=1,\ldots,p$, $\partial_\tau[b_i]$ is a regular event of the form $(c_1+\ldots+c_q)^*(a_{i_1}+\ldots+a_{i_s})(c_1+\ldots+c_q)^*$ where $\tau[a_{i_t}] = b_i$, $t=1,\ldots,s$.

<u>Corollary 1.5.3</u>: $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix} \subseteq \mathcal{R}\text{-SUB.} \cap_{\mathcal{R}} . \mathcal{R}\text{-SUB.}$

<u>Definition</u>: For a $\mathcal{X}$-class function f of n variables, $x_1, \ldots, x_n$ say, and a $\mathcal{Y}$-class function g of m variables, $y_1, \ldots, y_m$ say, we define the <u>ordered pair function</u> $f \wedge g$ (of nm variables) as

$$f \wedge g(\langle x_1,y_1\rangle,\langle x_1,y_2\rangle,\ldots,\langle x_1,y_m\rangle,\ldots,\langle x_i,y_j\rangle,\ldots,\langle x_n,y_{m-1}\rangle,\langle x_n,y_m\rangle)$$

$$f(\{\sum_{j=1}^{m}\langle x_1,y_j\rangle\},\ldots,\{\sum_{j=1}^{m}\langle x_n,y_j\rangle\}) \bigcap g(\{\sum_{i=1}^{n}\langle x_i,y_1\rangle\},\ldots,\{\sum_{i=1}^{n}\langle x_i,y_m\rangle\}$$

so that $\langle x_{k_1},y_{k_1}\rangle \ldots \langle x_{k_p},y_{k_p}\rangle$ is in

$$f \wedge g(\langle x_1,y_1\rangle,\langle x_1,y_2\rangle,\ldots,\langle x_i,y_j\rangle,\ldots,\langle x_n,y_{m-1}\rangle,\langle x_n,y_m\rangle)$$

if and only if $x_{k_1} \ldots x_{k_p} \in f(x_1,\ldots,x_n)$ and

$y_{k_1} \ldots y_{k_p} \in g(y_1,\ldots,y_m)$.

Lemma 1.7: Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of events such that $\mathcal{X}(\mathcal{F}) \subseteq \mathcal{X}$ and $\mathcal{Y}(\mathcal{F}) \subseteq \mathcal{Y}$. Then for an $\mathcal{X}$-class function f and a $\mathcal{Y}$-class function g, $f \wedge g$ is an $\mathcal{X}_{\mathcal{Y}}$-class function.

Proof: The event intersection specified above is an $\mathcal{X}_{\mathcal{Y}}$-event (of ordered pairs of words), and thus an $\mathcal{X}_{\mathcal{Y}}$-class function.

Theorem 1.8: For classes of events $\mathcal{E}$ and $\mathcal{D}$, such that $\mathcal{E}(\mathcal{F}) \subseteq \mathcal{E}$,

$$(\mathcal{E}(\mathcal{D}))_{\mathcal{R}} \subseteq \mathcal{E}_{\mathcal{R}}(\mathcal{D}_{\mathcal{R}}) .$$

Proof: Let $g(D_1,\ldots,D_m)$ be an $\mathcal{E}(\mathcal{D})$ event where g is a $\mathcal{E}$-class function and $D_1,\ldots,D_m$ are events in $\mathcal{D}$. Let E be a regular event and note that $\mathcal{R}(\mathcal{F}) \subseteq \mathcal{R}$ as $\mathcal{F} \subseteq \mathcal{R}$.

Now let $\lceil E \rceil$ be the factor matrix for E and replace each factor E_{ij} by a variable e_{ij}, so that we transform $\lceil E \rceil$ to a matrix $\lceil e \rceil$ say. We then consider $\lceil e \rceil^*$, a matrix of regular events in the variables e_{ij}, and examine the $\ell r^{\underline{th}}$ entry where $E = E_{\ell r}$ in $\lceil E \rceil$. This is a regular function f of the e_{ij}, and as $\lceil E \rceil^* = \lceil E \rceil$, we may replace each variable in the function by its corresponding factor. Thus we obtain E as a regular function of its factors, that is, $E = f(E_1,\ldots,E_n)$, where we have listed the factors of E with single subscripts for the sake of notation.

Then for a word $d_{i_1}\ldots d_{i_p}$ in the event $g(D_1,\ldots,D_m)$, $d_{i_j} \in D_{i_j}$, to be a word in the event E, there must be an event $E_{k_1}\ldots E_{k_p}$ in $f(E_1,\ldots,E_n)$ such that $d_{i_j} \in E_{k_j}$, i and j as above. Then 1.7 provides the result as we now have that

$$f(E_1,\ldots,E_n) \cap g(D_1,\ldots,D_m) = f \wedge g(E_1 \cap D_1,\ldots,E_1 \cap D_m,\ldots,E_i \cap D_j,\ldots,E_n \cap D_m),$$

where $f \wedge g$ is an $\mathcal{E}_{\mathcal{R}}$-class function, and $E_i \cap D_j$ is an event in $\mathcal{D}_{\mathcal{R}}$.

<u>Theorem 1.9</u>: For classes of events $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$,

(*) $(\mathcal{A}(\mathcal{B}))(\mathcal{D}) \subseteq \mathcal{A}(\mathcal{B}(\mathcal{D}))$.

(*) is an equality if $\mathcal{B}$ is invariant under all permutations of Σ.

<u>Proof</u>: Any event in $(\mathcal{A}(\mathcal{B}))(\mathcal{D})$ or $\mathcal{A}(\mathcal{B}(\mathcal{D}))$ can be interpreted as an event of the form $\Omega.\Psi\,[f(x_1,\ldots,x_n)]$, where Ω is a $\mathcal{D}$-substitution, Ψ is a $\mathcal{B}$-substitution, and f is an $\mathcal{A}$-class function over the variables $x_1,\ldots,x_n$. (Note that $\Omega.\Psi$ is a $\mathcal{B}(\mathcal{D})$-substitution.)

<u>Proof of 1.10</u>: We consider the product of an operator Ω in $\begin{bmatrix}\mathcal{R}\\ \mathcal{A}\end{bmatrix}$ and an operator Ψ in $\begin{bmatrix}\mathcal{B}\\ \mathcal{D}\end{bmatrix}_{\mathcal{E}}$. In view of the fact that $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$, we may assume that Ω is of the form

$$h_x\left(\begin{bmatrix}a_1\\ 1\end{bmatrix},\ldots,\begin{bmatrix}a_p\\ 1\end{bmatrix},\begin{bmatrix}1\\ A_1\end{bmatrix},\ldots,\begin{bmatrix}1\\ A_q\end{bmatrix}\right),$$

where $A_1,\ldots,A_q$ are $\mathcal{A}$-class events and h is a regular function. Further, as in the proof of 1.5, $\Omega = \theta \cdot \cap_E \cdot \partial_\tau$ where

$$\text{(i)} \quad \theta = \left(\begin{bmatrix} a_1 \\ 1 \end{bmatrix} + \ldots + \begin{bmatrix} a_p \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ A_1 \end{bmatrix} + \ldots + \begin{bmatrix} c_q \\ A_q \end{bmatrix} \right)^{\dagger},$$

with $\{c_1,\ldots,c_q\}$ an alphabet of q letters distinct from $a_1,\ldots,a_p$,

($1 \varepsilon \mathcal{A}$ so that θ is an $\mathcal{A}$-substitution.)

(ii) $E = h(a_1..,a_p,c_1,..,c_q)$ is a regular event,

$$\text{(iii)} \quad \tau = \left(\begin{bmatrix} a_1 \\ a_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_p \\ a_p \end{bmatrix} + \begin{bmatrix} c_1 \\ 1 \end{bmatrix} + \ldots + \begin{bmatrix} c_q \\ 1 \end{bmatrix} \right)^{\dagger},$$

a letter homomorphism.

Let $\Psi = g_x \left(\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}, \ldots, \begin{bmatrix} B_m \\ D_m \end{bmatrix} \right)$ where g is an $\mathcal{E}$-class function, the B_i, $\mathcal{B}$-class events, and the D_j, $\mathcal{D}$-class events. We consider

$$\theta \cdot \cap_E \cdot \partial_\tau \cdot \Psi ,$$

and we observe that

$$\partial_\tau \cdot \Psi = g_\times\left(\begin{bmatrix} B_1 \\ \partial_\tau[D_1] \end{bmatrix}, \ldots, \begin{bmatrix} B_m \\ \partial_\tau[D_m] \end{bmatrix}\right).$$

As in the proof of 1.8, there exists a regular function f such that $E = f(E_1,\ldots,E_n)$, the E_i the factors of E, and hence,

$$\cap_E \cdot \partial_\tau \cdot \Psi = f \wedge g_\times\left(\cap_{E_1} \cdot \begin{bmatrix} B_1 \\ \partial_\tau[D_1] \end{bmatrix}, \ldots, \cap_{E_i} \cdot \begin{bmatrix} B_j \\ \partial_\tau[D_j] \end{bmatrix}, \ldots, \cap_{E_n} \cdot \begin{bmatrix} B_m \\ \partial_\tau[D_m] \end{bmatrix}\right).$$

Composing this product with the $\mathcal{A}$-substitution θ, we obtain

$$\Omega \cdot \Psi = f \wedge g_\times\left(\begin{bmatrix} B_1 \\ \theta[\partial_\tau[D_1] \cap E_1] \end{bmatrix}, \ldots, \begin{bmatrix} B_j \\ \theta[\partial_\tau[D_j] \cap E_i] \end{bmatrix}, \ldots, \begin{bmatrix} B_m \\ \theta[\partial_\tau[D_m] \cap E_n] \end{bmatrix}\right).$$

1.9 implies that $(\mathcal{E}(\mathcal{F}))(\mathcal{F}) \subseteq \mathcal{E}(\mathcal{F})$ as $\mathcal{F}(\mathcal{F}) \subseteq \mathcal{F}$, so that 1.7 implies that $f \wedge g$ is an $(\mathcal{E}(\mathcal{F}))_{\mathcal{R}}$ class function. Then

$$\Omega \cdot \Psi \quad \varepsilon \quad \begin{bmatrix} \mathcal{B} \\ (\mathcal{D}(\mathcal{R}))_{\mathcal{R}}(\mathcal{A}) \end{bmatrix}_{(\mathcal{E}(\mathcal{F}))_{\mathcal{R}}}$$

as was to be shown.

Corollary 1.10.1: For classes of events $\mathcal{B}$ and $\mathcal{E}$ such that $\mathcal{E}(\mathcal{F}) \subseteq \mathcal{E}$,

$$\begin{bmatrix} \mathcal{S} \\ \mathcal{B} \end{bmatrix}_{\mathcal{E}} [\mathcal{R}] \subseteq \mathcal{E}_{\mathcal{R}}(\mathcal{B}) .$$

Proof: The dual result of 1.10 is that

$$\begin{bmatrix} \mathcal{D} \\ \mathcal{B} \end{bmatrix}_{\mathcal{E}} \cdot \begin{bmatrix} \mathcal{A} \\ \mathcal{R} \end{bmatrix} \subseteq \begin{bmatrix} (\mathcal{D}(\mathcal{R}))_{\mathcal{R}}(\mathcal{A}) \\ \mathcal{B} \end{bmatrix}_{\mathcal{E}_{\mathcal{R}}}$$

Letting $\mathcal{D} = \mathcal{S}$, $\mathcal{A} = \mathbb{1}$ the class consisting of the empty event and the empty word, E a regular event, and Ω an operator in $\begin{bmatrix} \mathcal{S} \\ \mathcal{B} \end{bmatrix}_{\mathcal{E}}$, we then have,

$$\Omega[E] = \Omega \cdot \begin{bmatrix} 1 \\ E \end{bmatrix} [1] = \begin{bmatrix} 1 \\ \Omega[E] \end{bmatrix} [1].$$

Now $\begin{bmatrix} 1 \\ \Omega[E] \end{bmatrix}$ is an operator in the class $\begin{bmatrix} (\mathcal{S}(\mathcal{R}))_{\mathcal{R}}(\mathbb{1}) \\ \mathcal{B} \end{bmatrix}_{\mathcal{E}_{\mathcal{R}}}$

which is actually the class $\begin{bmatrix} \mathbb{1} \\ \mathcal{B} \end{bmatrix}_{\mathcal{E}_{\mathcal{R}}}$ as the image of

any non-zero function of empty words is again the empty word. Thus $\Omega[E]$ is an event in $\mathcal{E}_{\mathcal{R}}(\mathcal{B})$ as was to be proved.

Corollary 1.10.2: For a class of events $\mathcal{X}$ such that $\mathcal{X}(\mathcal{X}) \subseteq \mathcal{X}$ and $\mathcal{X}_{\mathcal{R}} \subseteq \mathcal{X}$,

$$\begin{bmatrix} \mathcal{S} \\ \mathcal{X} \end{bmatrix}_{\mathcal{X}} [\mathcal{R}] \subseteq \mathcal{X}.$$

Corollary 1.10.3: For classes of events $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{E}$, such that $1 \varepsilon \mathcal{A}$ and $\mathcal{E}(\mathcal{F}) \subseteq \mathcal{E}$,

$$\begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix} [\mathcal{E}(\mathcal{D})] \subseteq \mathcal{E}_{\mathcal{R}}\left((\mathcal{D}(\mathcal{Q}))_{\mathcal{R}} (\mathcal{A}) \right).$$

Proof: The proof is similar to that of 1.10.1.

Corollary 1.10.4: For classes of events $\mathcal{A}$ and $\mathcal{X}$ such that $1 \varepsilon \mathcal{A}$,

$$\begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix} [\mathcal{X}] \subseteq (\mathcal{X}(\mathcal{R}))_{\mathcal{R}} (\mathcal{A})$$

One might hope for a more general result than 1.10 by replacing $\begin{bmatrix} \mathcal{R} \\ \mathcal{A} \end{bmatrix}$ by $\begin{bmatrix} \mathcal{X} \\ \mathcal{A} \end{bmatrix}_{\mathcal{Y}}$ for arbitrary classes of events $\mathcal{X}$ and $\mathcal{Y}$. However, we can argue, heuristically at best, that 1.10 is a 'best possible' result. For

$\Omega \varepsilon \begin{bmatrix} \mathcal{X} \\ \mathcal{A} \end{bmatrix}_{\mathcal{Y}}$, as above, we may consider the operator as the composition of an $\mathcal{A}$-substitution θ, an intersection operator $\cap_Y$ for a $\mathcal{Y}$-class event Y, and a dual $\mathcal{X}$-substitution ∂_τ. When we consider $\partial_\tau . \Psi$, we have no technique of determining the interplay between the $\mathcal{D}$- and $\mathcal{X}$-class events, unless we are considering letters as in the case $\mathcal{X} = \mathcal{R}$. Similarly, when we consider $\cap_Y . \Psi'$, it is only the fact that regular events have finitely many factors that enables us to prove a theorem of the nature of 1.8. Below we give further support for our contention when we examine the class of events $\mathcal{S}$ and the class of context-free events $\mathcal{C}$ which are the most natural classes in which to expect some generalization.

For a finite alphabet $V = \{a_1,\ldots,a_p\}$, let

$$\Delta_V = \left(\begin{bmatrix} a_1 \\ a_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_p \\ a_p \end{bmatrix} \right)^{\dagger},$$

the unit operator for $\mathcal{O}[S]$ and the identity $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operator over S. When the alphabet over which we are working is obvious, the subscript V will be suppressed.

Recalling the definitions of differentiation, for two events E and F, let

$\delta_{E^l}[F] = \{v|$ there exists $w \varepsilon E$ such that $wv \varepsilon F\}$ and

$\delta_{E^r}[F] = \{w|$ there exists $v \varepsilon E$ such that $wv \varepsilon F\}$.

The differential operators are discussed more fully in the third chapter where we consider their effect on regular events. Ginsburg and Spanier [7] have shown that differentiation of context-free events by context-free events does not preserve context-freeness, and for the sake of completeness, we describe their example.

Let L_1 be the context-free event generated by the productions

$$S_1 \rightarrow aS_1b^2 \quad S_1 \rightarrow bS_1a^3 \quad S_1 \rightarrow cS_1cba \quad S_1 \rightarrow d,$$

and L_2 the context-free event generated by the productions

$$S_2 \rightarrow aS_2a \quad S_2 \rightarrow bS_2b \quad S_2 \rightarrow cS_2c \quad S_2 \rightarrow d.$$

It is clear that $L_2 = \{wdw^T | w \varepsilon (a+b+c)^*\}$, and in [7] it is shown that

$$\delta_{L_2^\ell}[L_1] = \{ba, a^4, a^3b^2, a^2b^4, ab^6, b^8, b^7a^3, b^6a^6, .., a^{24}, ...\}.$$

But $\mathcal{C}$ is an AFL, so that $\mathcal{C}$ is closed under regular intersection. $\cap_{a^* \cdot \delta_{L_2}} [L_1] = \{a^n \mid n = 4.6^i,\ i \geq 0\}$ is a non-periodic event and thus not context-free (see Chapter 5 for a discussion of context-free events over a single letter), so $\delta_{L_2^\ell}[L_1]$ is not context-free. It is clear that $\delta_{L_2^\ell}$ is actually the operator $\begin{bmatrix} L_2 \\ 1 \end{bmatrix} \times \Delta_V$ where $V = \{a,b,c\}$. As this operator is in the class $\begin{bmatrix} \mathcal{C} \\ \mathcal{R} \end{bmatrix}$, we have that $\begin{bmatrix} \mathcal{C} \\ \mathcal{R} \end{bmatrix} [\mathcal{C}] \not\subseteq \mathcal{C}$.

In light of our method of proof for 1.10.1, we have the following:

<u>Proposition 1.11</u>: $\begin{bmatrix} \mathcal{C} \\ \mathcal{R} \end{bmatrix} \cdot \begin{bmatrix} \mathcal{R} \\ \mathcal{C} \end{bmatrix} \not\subseteq \begin{bmatrix} \mathcal{C} \\ \mathcal{C} \end{bmatrix}_{\mathcal{C}}$

<u>Corollary 1.11.1</u>: $\begin{bmatrix} \mathcal{C} \\ \mathcal{C} \end{bmatrix}_{\mathcal{C}} \cdot \begin{bmatrix} \mathcal{C} \\ \mathcal{C} \end{bmatrix}_{\mathcal{C}} \not\subseteq \begin{bmatrix} \mathcal{C} \\ \mathcal{C} \end{bmatrix}_{\mathcal{C}}$

When we consider the class of all events $\mathcal{S}$, we do have that

$$\begin{bmatrix} \mathcal{S} \\ \mathcal{R} \end{bmatrix} [\mathcal{S}] \subseteq \mathcal{S} ,$$

but the similar result of 1.11 for this class also follows.

Proposition 1.12: $\begin{bmatrix}\mathcal{S}\\ \mathcal{R}\end{bmatrix} \circ \begin{bmatrix}\mathcal{R}\\ \mathcal{S}\end{bmatrix} \not\subseteq \begin{bmatrix}\mathcal{S}\\ \mathcal{S}\end{bmatrix}$.

Proof: Let V be the finite alphabet {a,b,c,d} and X the event $\{wdw|\ w\varepsilon(a+b+c)^*\}$. Consider the operators

$$\Omega = \begin{bmatrix}X\\ 1\end{bmatrix} \times \begin{bmatrix}d\\ 1\end{bmatrix} \times \Delta_{\{a,b,c\}} \text{ and } \Psi = \Delta_{\{a,b,c\}} \times \begin{bmatrix}1\\ d\end{bmatrix} \times \begin{bmatrix}1\\ L_2\end{bmatrix}$$

where L_2 is the context-free language in the example above.

Ω and Ψ are operators in the classes $\begin{bmatrix}\mathcal{S}\\ \mathcal{R}\end{bmatrix}$ and $\begin{bmatrix}\mathcal{R}\\ \mathcal{S}\end{bmatrix}$ respectively, and $\Omega.\Psi = \{\begin{bmatrix}ududz\\ z\end{bmatrix} \circ \begin{bmatrix}w\\ wdvdv^T\end{bmatrix} \mid u,z,w,v\ (a+b+c)^*\}$

$$= \{\begin{bmatrix}w\\ w^T\end{bmatrix} \mid w\varepsilon(a+b+c)^*\}.$$ The following lemma provides the result.

Lemma 1.13: $\theta = \{\begin{bmatrix}w\\ w^T\end{bmatrix} \mid w\varepsilon(a+b+c)^*\}$ is not an $\begin{bmatrix}\mathcal{S}\\ \mathcal{S}\end{bmatrix}$ operator.

Proof: We first note that each biregular operator over $\mathcal{R}$ corresponds to a linear context-free event, with productions of the form

$$A \rightarrow \begin{bmatrix}b\\ d\end{bmatrix} \times C$$

corresponding to the linear productions of the form

$$A \rightarrow bCd,$$

where A and C are non-terminal letters and b,c are terminal. Hence $\{uv^T | \begin{bmatrix} u \\ v \end{bmatrix} \varepsilon\ \theta'\}$ for an operator θ' in $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ is a linear language (see Gruska [8] and Rosenberg [9] where this correspondence is fully studied.). If θ is an $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operator, then the set of words $L = \{ww | w\varepsilon(a+b+c)^*\}$ would be a context-free event, a contradiction as L is a context-sensitive, non context-free event [10].

Now, if θ were an $\begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}$ operator, there would exist a regular function f and events $S_1,\ldots,S_n$, $T_1,\ldots,T_n$, such that $S_i \neq 0$, $T_i \neq 0$, and

$$\theta = f_\times\left(\begin{bmatrix} S_1 \\ T_1 \end{bmatrix}, \ldots, \begin{bmatrix} S_n \\ T_n \end{bmatrix}\right),$$

and for some $j\varepsilon\{1,\ldots,n\}$, S_j or T_j is an infinite (in fact, non-regular) event. Without loss of generality, assume that S_j is such an event. Then for a word $x_{i_1}\ldots x_j\ldots x_{i_p}$ in the event $f(x_1,\ldots,x_n)$, we have that the corresponding event word $S_{i_1}\ldots S_j\ldots S_{i_p}$ is an infinite event. For a word

w in the event $T_{i_1} \ldots T_j \ldots T_{i_p}$, it follows that

$$\{ \begin{bmatrix} v \\ w \end{bmatrix} \mid v \varepsilon S_{i_1} \ldots S_j \ldots S_{i_p} \}$$

is an <u>infinite</u> operator event and a sub-operator of θ, another contradiction.

As $\mathcal{S}$ is the class of <u>all</u> events, we do not have the analogous result of 1.11.1 for this class. But we <u>do</u> have:

<u>Lemma 1.14</u>: $\begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}_{\mathcal{S}} \circ \begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}_{\mathcal{S}} \subseteq \begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}_{\mathcal{S}}$.

<u>Proof</u>: $\begin{bmatrix} \mathcal{S} \\ \mathcal{S} \end{bmatrix}_{\mathcal{S}}$ is the universal class of linear operators by 1.3.2.

Chapter 2

The Biregular Operators

Theorem 1.10 provides us with several important consequences in our study of regular events.

Theorem 2.1: $\begin{bmatrix}\mathcal{S}\\ \mathcal{R}\end{bmatrix} \cap \mathcal{O}[S] \subseteq \mathcal{R}[S]$ for any standard algebra S.

Proof: As $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$ and $\mathcal{R}_{\mathcal{R}} \subseteq \mathcal{R}$, 1.10.2 provides the result.

Corollary 2.1.1: Regular events are closed under inverse substitution.

Proof: $\mathrm{SUB} \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{S}\end{bmatrix}$ by 1.4.

Definition: For events E and F, we define the shuffle of E and F, $E \sqcup\!\sqcup F$, as the event $\{e_{i_1} f_{i_1} \ldots e_{i_n} f_{i_n} \mid e_{i_1} \ldots e_{i_n} \varepsilon E,\ f_{i_1} \ldots f_{i_n} \varepsilon F\}$, and the alternate shuffle of E and F, $E \sqcup\!\sqcup_{alt} F$, as the event

$$\{a_{i_1} b_{i_1} \ldots a_{i_n} b_{i_n} b_{i_{n+1}} \ldots b_{i_m} \mid a_{i_1} \ldots a_{i_n} \varepsilon E,\ b_{i_1} \ldots b_{i_m} \varepsilon F,$$

$$n \leq m,\ a_i,\ b_j \text{ letters}\}$$

$\bigcup \{a_{i_1} b_{i_1} \ldots a_{i_n} b_{i_n} a_{i_{n+1}} \ldots a_{i_m} \mid a_{i_1} \ldots a_{i_m} \varepsilon E,\ b_{i_1} \ldots b_{i_n} \varepsilon F,$

$n \leq m,\ a_i,\ b_j$ letters$\}$.

<u>Corollary 2.1.2</u>: Regular events are closed under shuffling and alternate shuffling.

<u>Proof</u>: Let $E = f(a_1, \ldots, a_p)$ be a regular event and F a regular event over the alphabet $V = \{b_1, \ldots, b_q\}$. Consider the $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operator

$$\Omega = f_\times(\Delta_V \times \begin{bmatrix} 1 \\ a_1 \end{bmatrix} \times \Delta_V, \ldots, \Delta_V \times \begin{bmatrix} 1 \\ a_p \end{bmatrix} \times \Delta_V)$$

$$= \{ \begin{bmatrix} w_1 \ldots w_n \\ v_1 w_1 \ldots v_n w_n \end{bmatrix} \mid w_1 \ldots w_n \varepsilon V^*,\ v_1 \ldots v_n \varepsilon E\},$$

It is clear that $\Omega[F] = E \sqcup\!\sqcup F$, and as $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix} [\mathcal{R}] \subseteq \mathcal{R}$, $E \sqcup\!\sqcup F$ is a regular event.

Now let $V' = \{b'_i \mid b_i \varepsilon V\}$ be an alphabet disjoint from $\{a_1, \ldots, a_p\}$, and let θ, τ be the letter homomorphisms

$$\left(\sum_{i=1}^{q}\begin{bmatrix} b'_i \\ b_i \end{bmatrix} + \sum_{j=1}^{p}\begin{bmatrix} a_j \\ a_j \end{bmatrix}\right)^{\dagger}, \quad \left(\sum_{i=1}^{q}\begin{bmatrix} b_i \\ b'_i \end{bmatrix}\right)^{\dagger} \text{ respectively.}$$

Then $\Psi = \theta.\bigcap_G.f_\times\left(\Delta_{V'}\times\begin{bmatrix} 1 \\ a_1 \end{bmatrix}\times\Delta_{V'},\ldots,\Delta_{V'}\times\begin{bmatrix} 1 \\ a_p \end{bmatrix}\times\Delta_{V'}\right).\tau$,

where G is the regular event $(\sum_{i,j}^{q,p} a_j b'_i)^*.((a_1+..+a_p)^*+V'^*)$, is an $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operator as $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}\circ\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$. It follows that $\Psi[F] = \theta.\bigcap_G[E \sqcup F']$, where $F' = F(b'_1,\ldots,b'_q)$, is the event $E \underset{alt}{\sqcup} F$, hence is regular.

There are many more operators in Chapter 3 which we can show are regulators, that is, preserve regularity, by interpreting them as $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operators. Thus $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ is a large class of regulators closed under + and composition. It is not closed under star.

<u>Proposition 2.2</u>: $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ is not closed under *, and thus $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operators over a standard algebra S do not form a regular algebra of regulators for S.

<u>Proof</u>: Consider $\Omega = \begin{bmatrix} a \\ a^2 \end{bmatrix}^{\dagger}$ in $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$. Then,

$\Omega^*[a] = \{a^{2^n} \mid n \geq 0\}$ which is not a regular event. By

1.10.2, Ω^* is not in $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$.

In fact, through a suitable Godel numbering of the words in V^*, (where V is a finite alphabet with at least two distinct letters), we can effectively generate all coded recursively enumerable sets as the images of words under the stars of certain $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$ operators, in the sense of "normal" systems [11].

<u>Theorem 2.3</u>: Let τ be a semi-Thue system with alphabet $V = \{a_1, \ldots, a_p\}$, $p \geq 2$, and axiom word u. Then there exists an operator, Ω, in $\begin{bmatrix}R\\ \mathcal{R}\end{bmatrix}$ such that the 'theorems' of τ are exactly the words of $\Delta_V \cdot \Omega^*[u]$.

<u>Proof</u>: Let $V' = \{a'_1, \ldots, a'_p\}$ be a disjoint primed alphabet corresponding to V, and $\overline{V} = V + V'$. We mimic the normal system obtained from τ as follows:

(1) for productions of the form $a_i v \to v a'_i$ with v a word in V^*, let $\Omega_1 = \sum_{i=1}^{p} \begin{bmatrix} a_i \\ 1 \end{bmatrix} \times \Delta_{\overline{V}} \times \begin{bmatrix} 1 \\ a'_i \end{bmatrix}$,

(2) for productions of the form $a'_j v \to v a_j$, let

$$\Omega_2 = \sum_{j=1}^{p} \begin{bmatrix} a'_j \\ 1 \end{bmatrix} \times \Delta_{\overline{V}} \times \begin{bmatrix} 1 \\ a_j \end{bmatrix},$$

(3) For words w_k, $k = 1,2,..,m$, over V, and the corresponding words w'_k over V' (that is, if $w_k = a_{k_1}..a_{k_n}$ then $w'_k = a'_{k_m}...a'_{k_1}$), let

$$\Omega_3 = \sum_{k=1}^{m} \begin{bmatrix} w_k \\ 1 \end{bmatrix} \times \Delta_V \times \begin{bmatrix} 1 \\ w'_k \end{bmatrix}$$

correspond to the set of productions $w_k v \to v w'_k$.

Then for any semi-Thue system τ, the theorems of τ are the unprimed words obtained by a finite number of applications of the productions of the above form to the axiom word u. It is clear that this set of theorems is equivalent to the event

$\Delta_V \cdot (\Omega_1 + \Omega_2 + \Omega_3)^*[u]$, and as $\Omega_1 + \Omega_2 + \Omega_3$ is in $\begin{bmatrix} \mathcal{Q} \\ \mathcal{Q} \end{bmatrix}$,

the theorem follows.

<u>Definition</u>: For linear operators Ω, Ψ, we define $\Omega \cap \Psi$ as the set $\{\begin{bmatrix} v \\ w \end{bmatrix} | \begin{bmatrix} v \\ w \end{bmatrix} \varepsilon\ \Omega, \begin{bmatrix} v \\ w \end{bmatrix} \varepsilon\ \Psi\}$. Note that $\Omega \cap \Psi$ is <u>not</u> the same operator which maps E to $\Omega[E] \cap \Psi[E]$.

<u>Proposition 2.4</u>: $\begin{bmatrix} \mathcal{Q} \\ \mathcal{Q} \end{bmatrix}$ is not closed under intersection.

<u>Proof</u>: Let $\Omega = \begin{bmatrix} a \\ a \end{bmatrix}^{\dagger} \times \begin{bmatrix} 1 \\ ba^* \end{bmatrix}$ and $\Psi = \begin{bmatrix} 1 \\ a^*b \end{bmatrix} \times \begin{bmatrix} a \\ a \end{bmatrix}^{\dagger}$. Both these operators are in $\begin{bmatrix} \mathcal{Q} \\ \mathcal{Q} \end{bmatrix}$ but their intersection,

$$\Omega \cap \Psi = \{ \begin{bmatrix} a^n \\ a^n b a^n \end{bmatrix} \mid n \geq 0 \} ,$$

is not, as $\Omega \cap \Psi[a^*] = \{a^n b a^n \mid n \geq 0\}$ is not a regular event.

Corollary 2.4.1: $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ is not closed under complement (with respect to a fixed alphabet).

Elgot and Mezei [12] have considered the class of 'binary transductions' obtained from the component-wise regular closure of the ordered pairs $<a_1,1>,\ldots,<a_p,1>$, $<1,a_1>,\ldots,<1,a_p>$, for a finite alphabet $\{a_1,\ldots,a_p\}$. It is clear that the class of binary transductions is equivalent to the operator class $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$. They have also shown that

Theorem A: $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix} \cdot \begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$,

a result which follows from 1.10.

Fischer and Rosenberg [13] have investigated the classes of events, $\mathcal{N}_n$, $n \geq 0$, accepted by n-tape non-deterministic finite automata in the sense of Rabin and Scott, and the resultant decision problems for these classes of events. They have shown that these classes

correspond to "n-regular" events in a natural sense, and in particular, for n = 2, to the binary transductions of Elgot and Mezei, and thus to $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$. For the sake of completeness, we list here the results of their work.

Theorem B: The following decision problems for $\mathcal{T}_n$ ($n \geq 2$) are recursively insoluble:

(1) the disjointness problem,

(2) the containment problem,

(3) the universe problem,

(4) the cofiniteness problem,

(5) the equivalence problem.

Biregular Operators and AFL's:

We now show that the biregular operators and $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$ operators in particular play an important role in the theory of AFL's.

Definition: Given an infinite set of symbols, Σ, an abstract family of languages (AFL) is a family $\mathcal{L}$ of events of Σ^* with the following properties:

(1) For each X in $\mathcal{L}$, there is a finite set $V \subseteq \Sigma$ such that $X \subseteq V^*$.

(2) $\mathcal{X}$ contains a non-empty event.

(3) $\mathcal{X}$ is closed under the operations of +, ., $^{*+1}$, inverse homomorphism, 1-free homomorphism, and intersection with regular events. A full AFL is an AFL closed under arbitrary homomorphism.

Lemma 2.5: If a class $\mathcal{X}$ containing a non-empty event is closed under +, $^{*+1}$, and homomorphism, then $\mathcal{X}$ is closed under *.

Proof: For an $\mathcal{X}$-class event E over the finite alphabet $\{a_1,\ldots,a_p\}$,

$$E^* = E^{*+1} + \left(\begin{bmatrix} a_1 \\ 1 \end{bmatrix} + \ldots + \begin{bmatrix} a_p \\ 1 \end{bmatrix}\right)^{\dagger} [F], \quad (F \neq 0,\ F \varepsilon \mathcal{X}).$$

Theorem 2.6: A class of events $\mathcal{X}$ is a full AFL if and only if $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$ and $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$.

Proof: As homomorphism, inverse homomorphism, and regular intersection are $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operators, if $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$ and $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$, then $\mathcal{X}$ is a full AFL. Note that the non-emptiness of $\mathcal{X}$ is implied by the fact that $\mathcal{X}$ is closed under *, hence the empty word is in some $\mathcal{X}$-event, and $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$.

The fact that every $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operator can be represented in the form $\theta.\cap_E.\partial_\tau$ where θ and τ are (letter) homomorphisms and E is a regular event implies the converse.

<u>Corollary 2.6.1</u>: Full AFL's are closed under regular event shuffling, regular alternate shuffling, regular substitution, and regular event differentiation.

<u>Proof</u>: All of these operations can be interpreted as $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operators.

<u>Corollary 2.6.2</u>: $\mathcal{R}$ is a full AFL, and if $\mathcal{X}$ is a full AFL, then $\mathcal{X} \supseteq \mathcal{R}$.

<u>Proof</u>: The fact that $\mathcal{R}$ is a full AFL follows from 2.1 and $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$. As $\mathcal{X}$ is a non-empty class, there is a word w in some event X in $\mathcal{X}$. As $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$, we have that $\begin{bmatrix} w \\ E \end{bmatrix}[X] = E$ is a $\mathcal{X}$-class event for every regular event E.

The theorem above and its proof suggest the following definition and corollary.

<u>Definition</u>: For a finite alphabet V, the map $EXP_x: V^* \to (V+x)^*$ is an <u>expansion</u> if $EXP_x[a_{i_1} \ldots a_{i_n}] = x^*a_{i_1}x^* \ldots x^*a_{i_n}x^*$ for letters a_i in V. The map

$CON_x : (V+x)^* \to (V \setminus x)^*$ is a <u>contraction</u> if

$$CON_x[x^{p_o} a_{i_1} x^{p_1} \dots x^{p_{n-1}} a_{i_n} x^{p_n}] = a_{i_1} \dots a_{i_n} .$$

<u>Lemma 2.7</u>: $\Omega \in \left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right]$ if and only if Ω is a finite composition of expansions, contractions, and intersection operators in $\bigcap_{\mathcal{R}}$.

<u>Proof</u>: The letter homomorphisms ϕ and τ in 1.5 may be replaced by iterated products of contractions (and then ∂_τ is an iterated expansion).

<u>Corollary 2.6.3</u>: A class of events $\mathcal{X}$ is a full AFL if and only if $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$ and $\mathcal{X}$ is closed under expansions, contractions, and intersection with regular events.

The choice of nomenclature of [3] is unfortunate from the viewpoint of $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right]$ operators as an AFL might better be called an "unfull" AFL, rather than adding the adjective "full" when we remove the restriction on the type of homomorphism allowed. We may define a subclass $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right]_1$ of $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right]$ to cater for this and remark that our characterizations of full AFL's carry through for AFL's and $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right]_1$ with the obvious restrictions.

Definition: Ω is said to be an $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}_1$ operator if and only if there exists a regular function, f, such that

$$\Omega = f_x\left(\begin{bmatrix}u_1\\ w_1\end{bmatrix}, \ldots, \begin{bmatrix}u_n\\ w_n\end{bmatrix}\right)$$

where $u_i \varepsilon V_1{}^*$, $w_i \varepsilon V_2{}^* \setminus 1$.

Theorem 2.7: For an arbitrary non-empty class of events $\mathcal{X}$,

$$\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right) \qquad \text{is a full AFL.}$$

Proof: We are required to show

$$\text{(i)}\ \mathcal{R}\left(\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right)\right) \subseteq \mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right) \text{ and}$$

$$\text{(ii)}\ \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}\left[\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right)\right] \subseteq \mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right)$$

Now (i) follows trivially from 1.9 and the fact that $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$. 1.10.3 implies that

$$\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{R}(\mathcal{Y})] \subseteq \mathcal{R}_{\mathcal{R}}\left((\mathcal{Y}(\mathcal{R}))_{\mathcal{R}}(\mathcal{R})\right) \subseteq \mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{Y}]\right)$$

for an arbitrary class of events $\mathcal{Y}$, and 1.10 implies that $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix} \cdot \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$, whence (ii).

Corollary 2.7.1: For a class of events $\mathcal{X}$ such that

$\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{R}(\mathcal{X})$, we have

(i) $\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right) \subseteq \mathcal{R}(\mathcal{X})$, in other words,

$\mathcal{R}$-closure preserves $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$-closure,

(ii) $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{R}(\mathcal{X})] \subseteq \mathcal{R}(\mathcal{X})$,

(iii) $\mathcal{R}(\mathcal{X})$ is a full AFL.

Proof: Immediate.

Corollary 2.7.2: For a class of events $\mathcal{X}$ such that

$\mathcal{X}_{\mathcal{R}} \subseteq \mathcal{X}$ and $\mathcal{X}(\mathcal{F}) \subseteq \mathcal{X}$, then

$\mathcal{R}(\mathcal{X}) \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]$ implies

(i) $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{R}(\mathcal{X})] \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]$,

(ii) $\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right) \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]$,

(iii) $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]$ is a full AFL.

Proof: We verify (ii). $\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right) = \mathcal{R}\left((\mathcal{X}(\mathcal{R}))_{\mathcal{R}}(\mathcal{R})\right)$ which is a subclass of $\mathcal{R}(\mathcal{X}(\mathcal{R}))$ by 1.8, 1.9, and the fact that $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$. This class is equivalent to the

class $\mathcal{R}$-SUB $[\mathcal{R}(\mathcal{X})]$ which is $\subseteq \left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right][\mathcal{X}] \subseteq \left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right][\mathcal{X}]$ by 1.10 and the fact that $\mathcal{R}$-SUB is a subclass of $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{R}\end{smallmatrix}\right]$.

In a similar vein, we can also show that $\mathcal{R}$-closure preserves, individually, closure under substitution or regular intersection.

<u>Lemma 2.8</u>: For arbitrary classes of events $\mathcal{X}$ and $\mathcal{Y}$,

(1) $\mathcal{X}(\mathcal{Y}) \subseteq \mathcal{R}(\mathcal{X}) \Rightarrow (\mathcal{R}(\mathcal{X}))(\mathcal{Y}) \subseteq \mathcal{R}(\mathcal{X})$

(2) $\mathcal{X}_{\mathcal{R}} \subseteq \mathcal{R}(\mathcal{X}) \Rightarrow (\mathcal{R}(\mathcal{X}))_{\mathcal{R}} \subseteq \mathcal{R}(\mathcal{X})$

<u>Proof</u>: (1) 1.9 and $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$.

(2) 1.8 and $\mathcal{R}_{\mathcal{R}} \subseteq \mathcal{R}$.

<u>Corollary 2.8.1</u>: $\mathcal{X}(\mathcal{R}) \subseteq \mathcal{R}(\mathcal{X}) \Rightarrow (\mathcal{R}(\mathcal{X}))(\mathcal{R}) \subseteq \mathcal{R}(\mathcal{X})$

<u>Corollary 2.8.2</u>: For classes of events $\mathcal{X}$ and $\mathcal{Y}$ such that $1 \varepsilon \mathcal{Y}$,

$$\left[\begin{matrix}\mathcal{R}\\ \mathcal{Y}\end{matrix}\right][\mathcal{X}] \subseteq \mathcal{R}(\mathcal{X}) \Rightarrow \left[\begin{matrix}\mathcal{R}\\ \mathcal{Y}\end{matrix}\right][\mathcal{R}(\mathcal{X})] \subseteq \mathcal{R}(\mathcal{X})$$

<u>Proof</u>: 1.10.3, 1.9, and $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$.

To conclude this section, we prove:

Theorem 2.9: For an arbitrary class of events $\mathcal{X}$,

$$\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{R}(\mathcal{X}) \iff \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}\cdot\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}.$$

Proof: First we note that the closure of $\mathcal{R}$ under regular substitution implies that $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} = \begin{bmatrix}\mathcal{R}\\ \mathcal{R}(\mathcal{X})\end{bmatrix}$. Then by 1.10, $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}\circ\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{R}\\ (\mathcal{X}(\mathcal{R}))_{\mathcal{R}}(\mathcal{R})\end{bmatrix}$, and this operator class trivially reduces to $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}(\mathcal{X})\end{bmatrix} = \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$.

Conversely, let Ω be an operator in $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$, and X an event in $\mathcal{X}$. As in the proof of 1.10.1,

$\Omega[X] = (\Omega.\begin{bmatrix}1\\ X\end{bmatrix})\ [1] = \begin{bmatrix}1\\ \Omega[X]\end{bmatrix}\ [1]$ where $\begin{bmatrix}1\\ \Omega[X]\end{bmatrix}$ is an operator in $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$. Hence $\Omega[X]$ is an $\mathcal{R}(\mathcal{X})$ event.

We remark that for any finite alphabet $V \subseteq \Sigma$, $\Delta_V \in \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$ so that $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}\cdot\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} = \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$ above.

Corollary 2.9.1: For a class of events $\mathcal{X}$ such that $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}\cdot\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$, $\mathcal{R}(\mathcal{X})$ is a full AFL.

Proof: 2.7.1.

Corollary 2.9.2: For an arbitrary class of events $\mathcal{X}$ such that $\mathcal{X} \supseteq \mathcal{R}$, $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{R}(\mathcal{X})$ if and only if $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \cdot \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$.

Proof: The same as in 2.9, but note that we require $1 \varepsilon \mathcal{X}$ to insure that we are dealing with $\mathcal{X}$-substitutions when decomposing the $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$-operators.

Full AFL's and closure under substitution:

Lemma 2.10: For a class of events $\mathcal{X}$ such that $\mathcal{X} \supseteq \mathcal{R}$,

$$\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X} \quad \text{implies that} \quad \mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$$

Proof: Let X_1 and X_2 be arbitrary events in $\mathcal{X}$ over the finite alphabets V_1 and V_2 respectively. Then $(\Delta_{V_1} + \begin{bmatrix}V_1{}^*\\ X_2\end{bmatrix})\ [X_1] = X_1 + X_2$ is an event in $\mathcal{X}$ as the operator is in $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$. Similarly, we have that $(\begin{bmatrix}1\\ X_2\end{bmatrix} \times \Delta_{V_1})\ [X_1] = X_2 . X_1$ and $\begin{bmatrix}1\\ X_1\end{bmatrix}^{\dagger}\ [1] = X_1{}^*$ are $\mathcal{X}$-events.

We are now in a position to present some characterizations for full AFL's in terms of the biregular operators.

Theorem 2.11: The following are equivalent for a class of events $\mathcal{X}$:

(1) $\mathcal{X}$ is a full AFL closed under $\mathcal{X}$-substitution (that is, with the "full substitution property" in the sense of [3]).

(2) $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$ and $\mathcal{X} \supseteq \mathcal{R}$.

(3) $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \cdot \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix} \subseteq \begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$, $\mathcal{X} \supseteq \mathcal{R}$, and $\mathcal{R}(\mathcal{X}) = \mathcal{X}$.

(4) $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$ and $\mathcal{X}$-SUB $[\mathcal{X}] \subseteq \mathcal{X}$.

Proof: We show in order that 1→2→3→4→1.

1→2: 2.6.2 shows that $\mathcal{X} \supseteq \mathcal{R}$ for a full AFL $\mathcal{X}$, and 2.6 along with the fact that every $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$ operator is the composition of a $\mathcal{X}$-substitution and an $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$-operator, implies that $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$.

2→3: As $\mathcal{X} \supseteq \mathcal{R}$, 2.10 implies that $\mathcal{R}(\mathcal{X}) = \mathcal{X}$, and 2.9.2 provides the result for the Peirce product of $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$ operators.

3→4: Again we have that $\mathcal{X} \supseteq \mathcal{R}$. Then 2.9.2 shows that $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X} = \mathcal{R}(\mathcal{X})$, which implies that $\mathcal{X}$ is

closed under full substitution. As $\mathcal{X} \supseteq \mathcal{R}$, we also have that $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$.

4→1: $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$ implies that $\mathcal{X} \supseteq \mathcal{R}$. Closure under $\mathcal{X}$-substitution implies that $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$. Hence, by 2.10, $\mathcal{R}(\mathcal{X}) = \mathcal{X}$.

Two other results in this direction seem worth mentioning.

<u>Theorem 2.12</u>: For a full AFL $\mathcal{X}$, closure under $\cap_{\mathcal{X}}$ implies that $\mathcal{X}$ is closed under full substitution. (The converse is false, for example, the context-free languages.)

<u>Proof</u>: As $\mathcal{X}$ is a full AFL, then $\mathcal{X}(\mathcal{R}) \subseteq \mathcal{X}$. Then for $\Omega \in \begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}_{\mathcal{X}}$, we may assume that Ω is of the form

$$f_x(\begin{bmatrix}a_1\\ 1\end{bmatrix}, \ldots, \begin{bmatrix}a_p\\ 1\end{bmatrix}, \begin{bmatrix}1\\ a_1\end{bmatrix}, \ldots, \begin{bmatrix}1\\ a_p\end{bmatrix}),$$

where f is an $\mathcal{X}$-class function. As $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}_{\mathcal{X}} \subseteq \mathcal{R}\text{-SUB}.\cap_{\mathcal{X}}.\mathcal{R}\text{-SUB}$, $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}_{\mathcal{X}}$ is a class of operators preserving $\mathcal{X}$.

Further, $\mathcal{X} \supseteq \mathcal{R}$ and $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$, so that $(X+a_1X_1+\ldots+a_mX_m)^*$ is a $\mathcal{X}$-class event, and hence a $\mathcal{X}$-class function, over the $\mathcal{X}$-class events $X, X_1, \ldots, X_m$,

and $\{a_1,\ldots,a_m\}$. Now, any $\mathcal{X}$-substitution is of the form $\Psi = \left(\begin{bmatrix}1\\ x\end{bmatrix} + \begin{bmatrix}a_1\\ x_1\end{bmatrix} +\ldots+ \begin{bmatrix}a_m\\ x_m\end{bmatrix}\right)^{\dagger}$, and thus an operator in $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}_{\mathcal{X}}$, so that Ψ preserves $\mathcal{X}$-class events.

<u>Theorem 2.13</u>: For full AFL's $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X}(\mathcal{Y})$ is a full AFL.

<u>Proof</u>: By 1.9, $\mathcal{R}(\mathcal{X}(\mathcal{Y})) = (\mathcal{R}(\mathcal{X}))\ (\mathcal{Y}) = \mathcal{X}(\mathcal{Y})$, so that $\mathcal{X}(\mathcal{Y})$ is closed under regular functions. The closure of $\mathcal{X}(\mathcal{Y})$ under regular intersection follows from 1.8, that is

$$\cap_{\mathcal{R}}[\mathcal{X}(\mathcal{Y})] = (\mathcal{X}(\mathcal{Y}))_{\mathcal{R}} \subseteq \mathcal{X}_{\mathcal{R}}(\mathcal{Y}_{\mathcal{R}}) = \mathcal{X}(\mathcal{Y}),$$

and the closure of $\mathcal{X}(\mathcal{Y})$ under regular substitution follows from 1.9 as $\mathcal{R}\text{-SUB}[\mathcal{X}(\mathcal{Y})] = \mathcal{X}(\mathcal{Y}(\mathcal{R})) = \mathcal{X}(\mathcal{Y})$.
Hence $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}(\mathcal{Y})] = \mathcal{X}(\mathcal{Y})$.

In 2.7, it was shown that for <u>any</u> class of events $\mathcal{X}$, $\mathcal{R}\left(\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}]\right)$ was a full AFL, but due to the problems involved with iterated substitution, the similar result for full AFL's closed under full substitution does not seem to be forthcoming.

We conclude the chapter with some examples to show that the closures of an event-class under regular intersection, regular substitutions, and regular functions over the class are in fact independent.

Proposition 2.18: Let $\mathcal{X}$ be an arbitrary class of events. Then

(1) $\cap_{\mathcal{R}}[\mathcal{X}] \subseteq \mathcal{X}$ and $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X} \not\Rightarrow \mathcal{R}\text{-SUB}[\mathcal{X}] \subseteq \mathcal{X}$,

(2) $\mathcal{R}\text{-SUB}[\mathcal{X}] \subseteq \mathcal{X}$ and $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X} \not\Rightarrow \cap_{\mathcal{R}}[\mathcal{X}] \subseteq \mathcal{X}$,

(3) $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X} \not\Rightarrow \mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$.

Proof: (1) The class of context-sensitive events, $\mathcal{U}$, is closed under regular intersection and +, ., *. However, it is not closed under regular substitution; it can be shown that HOM($\mathcal{U}$) is the class of events $\mathcal{T}$ which properly contains $\mathcal{U}$. [6]

(2) The class of events consisting of the empty event and all events containing 1 is closed under regular substitution, and the regular operations of +, ., *. It is not closed under regular intersection.

(3) The class of linear events $\mathcal{C}_\lambda$ is closed under the $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$-operators but not under $\mathcal{R}$-closure.

From the theory of $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$ operators, it is also quite easy to show some of the results of Greibach and Hopcroft [14], that is, for a class of events $\mathcal{X}$ such that $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}[\mathcal{X}] \subseteq \mathcal{X}$,

(i) closure under composition implies closure under +,

(ii) closure under +, * implies closure under composition.

Chapter 3

The Class of Linear Regulators

For a standard algebra S, $\mathcal{L}[S]$ forms a regular algebra in a natural way. The aim of this chapter is to investigate subclasses of $\mathcal{L}[S] \cap \mathcal{R}[S]$ which form regular algebras. As any finite sum of (linear) regulators is a (linear) regulator, the problem involved is one of investigating the effect of starring a regulator. In Chapter 2 it was shown that starring an $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{L}\end{smallmatrix}\right]$ operator led to the generation of all recursively enumerable events so that $\mathcal{L}[S] \cap \mathcal{R}[S]$ does not form a regular algebra itself. However, it is closed under the biregular operations.

Theorem 3.1: For a standard algebra S, $\mathcal{L}[S] \cap \mathcal{R}[S]$ is closed under regular functions of the operations of $+$, $\times$, and †.

Proof: Let Ω and Ψ be linear regulators and E a regular event. It is clear that $\Omega+\Psi$ is a linear regulator as $(\Omega+\Psi)[E] = \Omega[E] + \Psi[E]$. When we consider $\Omega\times\Psi$, the fact that E is regular implies that E can be represented as a

finite sum $\sum_{i=1}^{m} L_i R_i$ where L_i (respectively, R_i) is a left (respectively, right) factor of E, and for $uv \varepsilon E$, there exist L_i, R_i such that $u \varepsilon L_i$ and $v \varepsilon R_i$. Then $\Omega\times\Psi[E] = \Sigma\Omega\times\Psi[uv]$, for $uv\varepsilon E$, and hence equal to $\Sigma\Omega[u].\Psi[v] = \sum_{i=1}^{m} \Omega[L_i].\Psi[R_i]$, so that $\Omega\times\Psi$ is a linear regulator if Ω and Ψ are. Now, $\Omega^{\dagger}[E] = \begin{bmatrix}1\\1\end{bmatrix}[E]+\Omega.\Omega^{\dagger}[E] = \begin{bmatrix}1\\1\end{bmatrix}[E] + \{\Omega[e_1].\Omega[e_2]. \; .. \; .\Omega[e_n] \mid e_1 e_2 ... e_n \varepsilon E\}$, and as each subword of a word of E is a word of a factor of E, $\Omega.\Omega^{\dagger}[E]$ is then the $\ell r^{\underline{th}}$ entry in the star of the matrix $\lceil \Omega\ [E] \rceil$ where $\lceil E \rceil$ is the factor matrix of E and $E = E_{\ell r}$. That is, $\Omega.\Omega^{\dagger}[E]$ is the $\ell r^{\underline{th}}$ entry in

$$\begin{bmatrix} \Omega[E_{11}] & . & . & . & \Omega[E_{1m}] \\ . & & & & . \\ . & & & & . \\ \Omega[E_{m1}] & . & . & . & \Omega[E_{mm}] \end{bmatrix}^{*}$$

and as Ω is a linear regulator, the Conway Theorem on the star of a matrix provides the result.

The proof of 3.1 also implies the following:

Corollary 3.1.1: For a standard algebra S and an arbitrary class of events $\mathcal{X}$, the biregular closure of the operators in $\mathcal{L}[S]$ that map regular events to $\mathcal{X}$-class events is an operator class which maps regular events to $\mathcal{R}(\mathcal{X})$-events.

Proof: Let Ω and Ψ be operators as above and E a regular event in S. Then, as in the proof of 3.1, $(\Omega+\Psi)[E]$, $\Omega\times\Psi[E]$, and $\Omega^{\dagger}[E]$ are $\mathcal{R}(\mathcal{X})$-class events if $\Omega[E]$ and $\Psi[E]$ are.

The algebra of open convex operators:

Definition: For standard algebras S_1 and S_2, a convex operator Ω is a linear operator mapping S_1 to S_2 such that

$$\Omega[E.F] \leq \Omega[E].\Omega[F]$$

for all events E,F in S_1. Ω is said to be in the operator class CVX and it is clear that CVX $\supseteq$ SUB $\supseteq$ UNIT SUB $\supseteq$ HOM.

A linear operator Ω is said to be open (in the operator class OPN) if

$$\Omega[E.F] \geq E.\Omega[F] + \Omega[E].F \quad .$$

This condition is equivalent to

$$\Omega[E.F.G] \geq E.\Omega[F].G$$

which implies that OPN is the class of linear operators (for some standard algebra S over an alphabet V) such that $\Omega = \Delta_V \times \Omega \times \Delta_V$. For a standard algebra S, let $\Delta[S]$ be the class of open operators in $\mathcal{O}[S]$.

For arbitrary classes of operators Π, Π', let

$\Pi_{\uparrow} = \{\Omega\varepsilon\Pi \mid \Omega\varepsilon\mathcal{O}[S]$ for some standard algebra S over an alphabet V such that $\Omega \geq \Delta_V\}$, the <u>increasing</u> Π-operators,

$\Pi_{OPN} = \{\Omega \mid \Omega\varepsilon\Pi, \Omega\varepsilon OPN\}$, the <u>open</u> Π-operators,

and for a standard algebra S,

$\Pi_{\Delta[S]} = \{\Omega \mid \Omega\varepsilon\Pi, \Omega\varepsilon\Delta[S]\}$,

$\Pi^* = \{\Omega^* \mid \Omega\varepsilon\Pi\}$,

$\Pi \cap \Pi' = \{\Omega \mid \Omega\varepsilon\Pi, \Omega\varepsilon\Pi'\}$,

$\mathcal{R}(\Pi) = \{f(\Omega_1,\ldots,\Omega_n) \mid f$ a regular function of n variables, $n\varepsilon N$, $\Omega_1,\ldots,\Omega_n\varepsilon\Pi\}$.

$\mathcal{R}(\quad)$ is a closure operator for operator classes as

(i) $\Pi \subseteq \mathcal{R}(\Pi)$,

(ii) $\mathcal{R}(\Pi) \subseteq \mathcal{R}(\mathcal{R}(\Pi))$, and

(iii) $\Pi \subseteq \Pi' \Rightarrow \mathcal{R}(\Pi) \subseteq \mathcal{R}(\Pi')$.

The aim of this section is to examine the regular closure of the operator class $CVX_{\Delta[S]}$ for a fixed standard algebra S, and to show that under certain restrictions, the dual operators of this class form a regular algebra of regulators.

<u>Lemma 3.2</u>: $\mathcal{R}(CVX \cap \mathcal{O}[S]) = CVX \cap \mathcal{O}[S]$.

<u>Proof</u>: $CVX \cap \mathcal{O}[S]$ is trivially closed under + and for convex operators Ω, Ψ, events E, F in S,

$$\Omega.\Psi[E.F] \quad = \quad \Omega[\Psi[E.F]] \ .$$

The linearity of CVX operators implies that this is

$$\leq \Omega[\Psi[E].\Psi[F]] \leq (\Omega.\Psi)[E].(\Omega.\Psi)[E].$$

The conclusion for $(CVX \cap \mathcal{O}[S])^*$ follows from the fact that

$$\Omega^*[E.F] = \sum_{n\geq 0} \Omega^n[E.F] \leq \sum_{n\geq 0} \Omega^n[E].\Omega^n[F] \leq \Omega^*[E].\Omega^*[F].$$

Lemma 3.3: $\mathcal{R}(\Delta[S]) = \Delta[S]$.

Proof: Again $\Delta[S]$ is trivially closed under + and for open Ω, Ψ, events E, F in S,

$$(\Omega.\Psi)[EF] \geq \Omega[E.\Psi[F]+\Psi[E].F] = \Omega[E.\Psi[F]]+\Omega[\Psi[E].F]$$

$$\geq E.(\Omega.\Psi)[F]+\Omega[E].\Psi[F]+\Psi[E].\Omega[F]+(\Omega.\Psi)[E].F,$$

so that $\Omega.\Psi$ is open. The conclusion for $(\Delta[S])^*$ operators follows from the lemma below.

Lemma 3.4: For an open operator Ω in $\Delta[S]$ and events E and F,

$$\Omega^*[E.F] \geq \Omega^*[E].\Omega^*[F].$$

Proof: $\Omega^*[E.F] = \sum_{n\geq 0} \Omega^n[E.F] = \sum_{p\geq 0} \sum_{n+m=p} \Omega^n.\Omega^m[E.F]$

$$= \sum_{p\geq 0} \sum_{n+m=p} \Omega^n\times\Delta.\Delta\times\Omega^m[E.F] \geq \sum_{p\geq 0} \sum_{n+m=p} \Omega^n\times\Delta[E.\Omega^m[F]]$$

$$\geq \sum_{p\geq 0} \sum_{n+m=p} \Omega^n[E].\Omega^m[F] = \Omega^*[E].\Omega^*[F].$$

Theorem 3.5: $\mathcal{R}(CVX_{\Delta[S]}) = CVX_{\Delta[S]}$.

Proof: The intersection of two closed classes is closed.

Corollary 3.5.1: $(CVX_{\Delta[S]})^* \subseteq SUB_{\Delta[S]} \subseteq CVX_{\Delta[S]}$.

Proof: For $\Omega \in CVX_{\Delta[S]}$, the proof of 3.2 shows that $\Omega^*[E.F] \leq \Omega^*[E].\Omega^*[F]$ and the lemma implies that this is an equality. Hence Ω^* is a substitution, and by 3.3, an operator in $SUB_{\Delta[S]}$. The right hand inclusion follows trivially. Note that the 'openness' of the operators is needed for this result. For example, $SUB^* \not\subseteq SUB$. (Consider the substitution $\begin{bmatrix} a \\ a^2 \end{bmatrix}^\dagger$ and then $a^6 \not\in \Omega^*[a^2]$, $a^6 \in \Omega^*[a].\Omega^*[a]$.)

We now prove the main result for the open convex operators.

Theorem 3.6: For a class of operators $\Pi[S]$ over a standard algebra S, let $\partial_{\Pi[S]}$ be the operator class $\{\partial_\Omega | \Omega \in \Pi[S]\}$. Then

$$\mathcal{R}(CVX_{\Delta[S]} \cap \partial_{\mathcal{Q}[S]}) = CVX_{\Delta[S]} \cap \partial_{\mathcal{Q}[S]} ,$$

in other words, the class of regulators whose duals are open convex substitutions form a regular algebra of regulators.

Proof: We show that every operator in the regular closure is a finite sum of $CVX_{\Delta[S]} \cap \partial\mathcal{R}[S]$ operators. As $\mathcal{R}[S]$ is closed under union and composition, 3.5 implies that it suffices to consider the star of such an operator sum, $\Omega_1+\ldots+\Omega_n$ say. Then $(\Omega_1+\ldots+\Omega_n)^* = (\Omega_1{}^*.\ \ldots\ .\Omega_n{}^*)^*$ is an open substitution (3.5.1 and the fact that $SUB_{\Delta[S]}$ is closed under composition) and as

$$SUB_{\Delta[S]} \subseteq \left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{S}\end{smallmatrix}\right] \cap \mathcal{O}[S] \subseteq \partial\mathcal{R}[S]\ ,$$

by 1.4 and 2.1, the result for star follows.

Corollary 3.6.1: $\mathcal{R}(\partial_{CVX_{\Delta[S]}} \cap \left[\begin{smallmatrix}\mathcal{S}\\ \mathcal{R}\end{smallmatrix}\right]) \subseteq \left[\begin{smallmatrix}\mathcal{S}\\ \mathcal{R}\end{smallmatrix}\right]$.

Corollary 3.6.2: $\mathcal{R}(\partial_{SUB_{\uparrow}} \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S]$, that is, the dual class of increasing substitutions forms a regular algebra of regulators.

Proof: As $SUB_{\Delta[S]} \subseteq CVX_{\Delta[S]} \cap \partial\mathcal{R}[S]$, it suffices to show that the operator classes $SUB_{\uparrow} \cap \mathcal{O}[S]$ and $SUB_{\Delta[S]}$ are equivalent, for then it follows that every regular function of increasing substitutions is equivalent to a finite sum of increasing substitutions.

Given $\Omega \varepsilon SUB_{\Delta[S]}$, $\Omega[1] \geq 1$ so that

$\Omega = \Delta\times\Omega\times\Delta \geq \Delta\times\begin{bmatrix}1\\1\end{bmatrix}\times\Delta = \Delta$. Hence $SUB_{\Delta[S]} \subseteq SUB_{\uparrow} \cap \mathcal{O}[S]$.

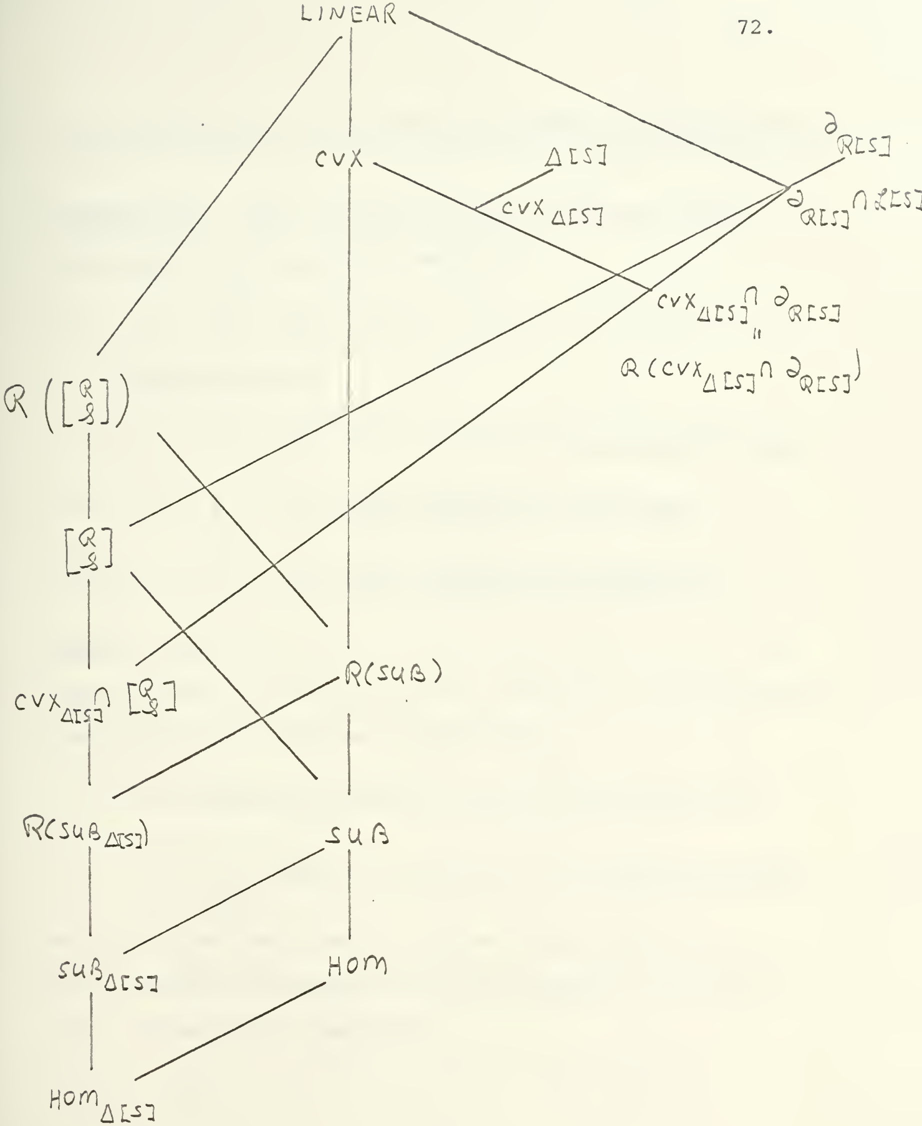
Conversely, let Ψ be an increasing substitution in $\mathcal{O}[S]$. Then $\Psi \geq \Delta$ and $\Psi = \Psi^{\dagger}$, both of which imply that

$$\Delta\times\Psi\times\Delta \geq \Psi = (\Delta+\Psi)^{\dagger} \geq \Delta\times\Psi\times\Delta,$$

and thus $\Psi \varepsilon SUB_{\Delta[S]}$.

Before concluding this section, we remark that $CVX_{\uparrow}$ properly contains CVX_{OPN} as the operator $\begin{bmatrix}a\\A\end{bmatrix} + \Delta$, for some non-empty event A in a standard algebra S, is a CVX operator which is not open.

The results above are schematically represented in the diagram below.


LINEAR
CVX
Δ[S]
CVX_Δ[S]
∂_R[S]
∂_R[S] ∩ L[S
CVX_Δ[S] ∩ ∂_R[S]
R(CVX_Δ[S] ∩ ∂_R[S])
R([R S])
[R S]
CVX_Δ[S] ∩ [R S]
R(SUB)
R(SUB_Δ[S])
SUB
SUB_Δ[S]
HOM
HOM_Δ[S]

<u>The Multiplier and Differential Calculus for Regular Events</u>

<u>Theorem 3.7</u>: For a standard algebra S, there exist embeddings of S into $\mathcal{L}[S]$ defined by

(1) $\ell: E \to E^{\ell}$, where $E^{\ell}[F] = E.F$ and E^{ℓ} is said to be a <u>left multiplier</u>,

(2) $r: E \to E^{r}$, where $E^{r}[F] = F.E$, the <u>right multipliers</u>,

(3) $\delta_{\ell}: E \to \delta_{E^{\ell}}$, the <u>left differential operators</u>,

(4) $\delta_{r}: E \to \delta_{E^{r}}$, the <u>right differential operators</u>.

<u>Proof</u>: Note that $E^{\ell}.F^{\ell} = (E.F)^{\ell}$ and $E^{r}.F^{r} = (F.E)^{r}$ so that the maps of (2) and (3) are actually 'anti-embeddings'. The proof of the theorem is immediate.

For a class of events $\mathcal{X}$, let $\mathcal{X}^{\ell}$ (respectively $\mathcal{X}^{r}$, $\delta_{\mathcal{X}^{\ell}}$, $\delta_{\mathcal{X}^{r}}$) denote the class of $\mathcal{X}$-<u>left multipliers</u> obtained from the events in $\mathcal{X}$ as above (respectively, the $\mathcal{X}$-right multipliers, left differential operators, right differential operators).

Corollary 3.7.1: For a class of events $\mathcal{X}$ in a standard algebra S, the regular closure of the operator class $\mathcal{X}^{\ell}$ (respectively $\mathcal{X}^{r}$, $\delta_{\mathcal{X}^{\ell}}$, $\delta_{\mathcal{X}^{r}}$) is the operator class $(\mathcal{R}(\mathcal{X}))^{\ell}$ (respectively $(\mathcal{R}(\mathcal{X}))^{r}$, $\delta_{(\mathcal{R}(\mathcal{X}))^{\ell}}$, $\delta_{(\mathcal{R}(\mathcal{X}))^{r}}$).

Corollary 3.7.2: For a standard algebra S, the operator classes $\mathcal{R}^{\ell} \cap \mathcal{O}[S]$, $\mathcal{R}^{r} \cap \mathcal{O}[S]$, $\delta_{\mathcal{R}^{\ell}} \cap \mathcal{O}[S]$, $\delta_{\mathcal{R}^{r}} \cap \mathcal{O}[S]$ form regular algebras of regulators for S.

Proof: As $\mathcal{R}(\mathcal{R}) \subseteq \mathcal{R}$, the proof follows immediately from the fact that regular events are closed under composition and differentiation.

Lemma 3.8: Let $\mathcal{X}$ be a class of events in a standard algebra S over a finite alphabet V such that $a \varepsilon \mathcal{X}$ for $a \varepsilon V$. Then $\mathcal{X}^{\ell}$ and $\mathcal{X}^{r}$ are subclasses of $\left[\begin{smallmatrix}\mathcal{R}\\ \mathcal{X}\end{smallmatrix}\right]$ and $\delta_{\mathcal{X}^{\ell}}$ and $\delta_{\mathcal{X}^{r}}$ are subclasses of $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{R}\end{smallmatrix}\right]$.

Proof: For events E and F in S, let

$$\left(\begin{matrix}E\\F\end{matrix}\right] = \Delta_V \times \left[\begin{matrix}E\\F\end{matrix}\right],\ \left[\begin{matrix}E\\F\end{matrix}\right) = \left[\begin{matrix}E\\F\end{matrix}\right] \times \Delta_V,\ \left(\begin{matrix}E\\F\end{matrix}\right) = \Delta_V \times \left[\begin{matrix}E\\F\end{matrix}\right] \times \Delta_V.$$

Then $E^{\ell} = \left[\begin{matrix}1\\E\end{matrix}\right)$, $E^{r} = \left(\begin{matrix}1\\E\end{matrix}\right]$, $\delta_{E^{\ell}} = \left[\begin{matrix}E\\1\end{matrix}\right)$, $\delta_{E^{r}} = \left(\begin{matrix}E\\1\end{matrix}\right]$,

are $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$ and $\begin{bmatrix}\mathcal{X}\\ \mathcal{R}\end{bmatrix}$ operators respectively. Note that we require $a \varepsilon \mathcal{X}$ for $a \varepsilon V$ to ensure that Δ_V is an $\begin{bmatrix}\mathcal{R}\\ \mathcal{X}\end{bmatrix}$ operator (or $\begin{bmatrix}\mathcal{X}\\ \mathcal{R}\end{bmatrix}$ operator).

Corollary 3.8.1: For a class of events $\mathcal{X}$ in a standard algebra S,

$$^{\partial}\mathcal{X}^{\ell} = {}^{\delta}\mathcal{X}^{\ell} \qquad \text{and} \qquad {}^{\partial}\mathcal{X}^{r} = {}^{\delta}\mathcal{X}^{r},$$

in other words, the dual class of $\mathcal{X}$-left (respectively, right) multipliers is the class of $\mathcal{X}$-left (respectively, right) differential operators.

Proof: Immediate from the form of the operators in 3.8.

Corollary 3.8.2: Full AFL's are preserved under regular left and right multiplication and regular left and right differentiation.

Proof: These are $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$ operations and the result follows from 2.6.

The natural question to ask at this point is to what extent the regulators of 3.7.2 may be combined and still obtain a regular algebra of regulators. We examine the pairwise closure of these classes and obtain the following

results for a standard algebra S over an alphabet V.

Theorem 3.9: If V has cardinality 1, then

$$\mathcal{R}((\mathcal{R}^{\ell} \cup \mathcal{R}^{r}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S].$$

If V consists of two or more letters, then

$$\mathcal{R}((\mathcal{R}^{\ell} \cup \mathcal{R}^{r}) \cap \mathcal{O}[S]) \not\subseteq \mathcal{R}[S].$$

Theorem 3.10: $\mathcal{R}((\delta_{\mathcal{R}^{\ell}} \cup \delta_{\mathcal{R}^{r}}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S].$

Theorem 3.11: $\mathcal{R}((\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{\ell}}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S]$, and similarly,

$$\mathcal{R}((\mathcal{R}^{r} \cup \delta_{\mathcal{R}^{r}}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S].$$

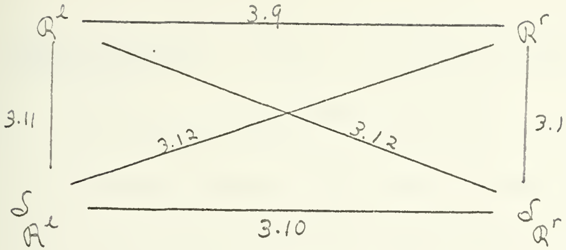
Theorem 3.12: If V has cardinality 1, then

$$\mathcal{R}((\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{r}}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S].$$

If V has two or more letters, then

$$\mathcal{R}((\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{r}}) \cap \mathcal{O}[S]) \not\subseteq \mathcal{R}[S].$$

Similarly for $\mathcal{R}((\mathcal{R}^{r} \cup \delta_{\mathcal{R}^{\ell}}) \cap \mathcal{O}[S])$.



<u>Proof of 3.9</u>: For an alphabet V of one letter, a say, we have the operator identity $a^\ell = a^r$ as S is a commutative algebra. The result for the single letter case follows from 3.7.2.

In the general case, let $a,b \varepsilon V$ and then $(a^\ell b^r)^*[1] = \{a^n b^n | n \geq 0\}$ is a non-regular event and the result follows. $\mathcal{Q}(\mathcal{R}^\ell \cup \mathcal{R}^r)$ actually corresponds to the regular algebra of Gruska [8] over ordered pairs of words $\langle w,v \rangle$ with the operations of union, composition defined by $\langle w_1,v_1 \rangle \odot \langle w_2,v_2 \rangle = \langle w_1 w_2, v_2 v_1 \rangle$, and star defined by the iterated composition. By defining $\langle w,v \rangle \otimes [E] = w.E.v$ for an event E, Gruska has shown that every linear language (over a finite alphabet) can be obtained as the image of an event in his pair algebra operating on the empty word and that context-free languages are preserved under his $\otimes$ operation.

Proof of 3.10: We prove a stronger result, that is,

$$\mathcal{R}((\delta_{\mathcal{S}^\ell} \cup \delta_{\mathcal{S}^r}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S].$$

For all words w, $v \varepsilon V^*$, we have the operator identity $w^\ell v^r = v^r w^\ell$, and hence for an operator Ω in the above class, Ω is of the form

$$\sum_{\alpha \varepsilon A} \delta_{E_\alpha{}^\ell} \cdot \delta_{F_\alpha{}^r}$$

for some index set A, events E_α, F_α in S. As any regular event G has only finitely many left and right event derivates, which themselves are regular events, there are only finitely many events of the form $\delta_{E^\ell} \cdot \delta_{F^r}[G]$ for events E and F in S. Hence $\Omega[G]$ is a regular event and is a union of some of these finitely many double derivates of G.

Corollary 3.10.1: For a standard algebra S, $\mathcal{R}((\delta_{\mathcal{S}^\ell} \cup \delta_{\mathcal{S}^r}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S]$, that is, the differential operators form a regular algebra of regulators.

Proof of 3.11: $\mathcal{R}((\mathcal{R}^\ell \cup \delta_{\mathcal{R}^\ell}) \cap \mathcal{O}[S])$ is an operator algebra generated by the operator 'alphabet' $\{a^\ell, \delta_{a^\ell} | a \varepsilon V\}$, where $V = \{a_1, \ldots, a_p\}$ say, satisfying

the relations

$$(*) \quad \delta_{a_i^{\ell}} \cdot a_j^{\ell} \quad = \quad 0 \qquad \text{if } i \neq j, \text{ and}$$

$$(**) \qquad\qquad\qquad = \quad 1 \qquad \text{if } i = j.$$

(Similar relations hold for the right operators and hence the proof for these operators is essentially the same.)

Let S' be the <u>free</u> standard algebra generated by the (disjoint) alphabet $V' = \{b_1,\ldots,b_p\} \cup \{c_1,\ldots,c_p\}$. Now for an operator Ω in the class above, there is a regular function f such that

$\Omega = f(a_1^{\ell},\ldots,a_p^{\ell}, \delta_{a_1^{\ell}},\ldots,\delta_{a_p^{\ell}})$ and we let

$E' = f(b_1,\ldots,b_p, c_1,\ldots,c_p)$ be the corresponding event in S'. Let $F' = c_1b_1+\ldots+c_pb_p$ and then

$(\Delta_{V'} \times \begin{bmatrix} 1 \\ F' \end{bmatrix} \times \Delta_{V'})^*[1] = G'$ is an event in S'. It is clear that $\binom{F'}{1}^* = \binom{G'}{1}^\dagger$, and that this is a regulator as it is an $\begin{bmatrix} \mathcal{L} \\ \mathcal{R} \end{bmatrix}$ operator. Then

$$H'=g(b_1,..,b_p,c_1,..,c_p)=\bigcap\nolimits_{(b_1+..+b_p)^*(c_1+..+c_p)^*} \cdot \binom{G'}{1}^\dagger[E']$$

is the regular event obtained from E' by imposing the relations $c_i.b_j = 0$ if $i \neq j$ and $c_i.b_j = 1$ if $i = j$. As these relations correspond to (*) and (**) above respectively, it follows that

$$\Omega = g(a_1^{\ell},\ldots,a_p^{\ell}, \delta_{a_1^{\ell}},\ldots,\delta_{a_p^{\ell}}).$$

Further, H' is a regular event in $(b_1+\ldots+b_p)^*(c_1+\ldots+c_p)^*$ and hence is a finite sum of regular events of the form B'.C' where $B' = L_i \cap (b_i+\ldots+b_p)^*$, $C' = R_i \cap (c_1+\ldots+c_p)^*$, L_iR_i one of the finitely many two term factorizations of H'. So Ω is a finite sum of operators of the form $\Psi.\Phi$ where Ψ is a regular left multiplier and Φ a regular left differential operator, hence Ω is a regulator.

<u>Corollary 3.11.1</u>: Any $\mathcal{R}(\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{\ell}})$ operator, over a standard algebra S, can be put into a "normal form",

$E_1^{\ell}.\delta_{F_1^{\ell}}+\ldots+E_n^{\ell}.\delta_{F_n^{\ell}}$, where the E_i and F_i are regular events in S, $n \varepsilon N$.

<u>Corollary 3.11.2</u>: For a standard algebra S, ∂ is an anti-isomorphism of $\mathcal{R}(\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{\ell}})$ (respectively,

$\mathcal{R}(\mathcal{R}^r \cup \delta_{\mathcal{R}^r})$ mapping $E^\ell.\delta_{F^\ell}$ to $F^\ell.\delta_{E^\ell}$ (respectively $E^r.\delta_{F^r}$ to $F^r.\delta_{E^r}$).

In view of 3.10.1, we conjecture that $\mathcal{R}(\mathcal{R}^\ell \cup \delta_{\mathcal{S}^\ell})$ is a regular algebra of regulators for a standard algebra S. The method of proof in 3.11 fails in that we cannot assert that H' is a regular function of the letters $b_1,\ldots,b_p$ and suitable standard events, $C_1,\ldots,C_p$ say.

Proof of 3.12: The single letter case follows from 3.11 as the commutivity of S implies the operator identity $a^\ell = a^r$.

When the alphabet consists of two or more letters, as in the proof of 2.3, the productions of a normal system correspond to $\mathcal{R}(\mathcal{R}^r \cup \delta_{\mathcal{R}^\ell})$ operators, and hence this class of operators maps words in S to arbitrary (coded) recursively enumerable sets. The similar result holds for $\mathcal{R}(\mathcal{R}^\ell \cup \delta_{\mathcal{R}^r})$ operators.

Before concluding this section, we prove a result which indicates to some extent that the algebras of regulators which we have been considering may be combined to form larger regular algebras of regulators.

Theorem 3.13: Let $\mathcal{S}_1$ denote the class of events containing the empty word. For a standard algebra S,

$$\mathcal{R}((\delta_{\mathcal{S}_1^{\ell}} \cup \delta_{\mathcal{S}_1^{r}} \cup \delta_{SUB_{\uparrow}}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S].$$

Proof: We consider the dual class of operators of the above form. As in the proof of 3.10, $\mathcal{S}_1^{\ell}$ and $\mathcal{S}_1^{r}$ operators commute, and for events E,F and a substitution, the operator identity

$$(*) \quad \Omega.E^{\ell}.F^{r} = (\Omega[E])^{\ell}.(\Omega[F])^{r}.\Omega$$

implies that it is sufficient to consider only the regular closure of $(\mathcal{S}_1^{\ell} \cup SUB_{\uparrow}) \cap \mathcal{O}[S]$. We omit the superscript $^{\ell}$ for the multiplier operators.

Lemma: Let Ω and Ψ be increasing substitutions and E an $\mathcal{S}_1$-class event in S. Then

(1) $\Omega.\Psi \in SUB_{\uparrow}$,

(2) $\Omega^* \in SUB_{\uparrow}$,

(3) $\mathcal{S}_1$ operators are closed under regular functions,

(4) $(E.\Omega)^* = \Omega^*[E^*].\Omega^*$.

Proof: (1) is immediate, (2) follows from 3.5.1, and (3) from 3.7.1. We prove (4).

$$(E.\Omega)^* = \Delta + E.\Omega + \ldots + (E.\Omega)^n + \ldots$$

$$= \Delta + E.\Omega + \ldots + E.\Omega[E].\Omega^2[E].\ \ldots\ .\Omega^n[E].\Omega^n + \ldots$$

As $\Delta \leq \Omega$, $\Omega^n \leq \Omega.(\Omega^n)$ so that

$$(E.\Omega)^* \leq \Delta + \Omega[E].\Omega + \ldots + \Omega^n[E].\Omega^n[E].\ \ldots\ .\Omega^n[E].\Omega^n + \ldots$$

$$= \Delta + \Omega[E].\Omega + \ldots + \Omega^n[E^n].\Omega^n + \ldots$$

$$\leq \Omega^* [E^*].\Omega^* .$$

Conversely, as the event E contains the empty word, the operator E contains Δ so that $E.\Omega \geq E+\Omega$ and $(E.\Omega)^* \geq \Omega^*.E^* = \Omega^*[E^*].\Omega^*$ as was to be shown.

We now assert that every operator in the class is a finite sum of operators of the form $E.\Omega$, for E,Ω as in the lemma, and (1), (2), (3), (*) imply that it suffices to consider the star of such an operator. Let $\Omega_1,\ldots,\Omega_n$ be increasing substitutions, and $E_1,\ldots,E_n$ $\mathcal{S}_1^{\ell}$ operators. Then

$$(E_1.\Omega_1+\ldots+E_n.\Omega_n)^* = ((E_1.\Omega_1)^*.(E_2.\Omega_2)^*.\ \ldots\ .(E_n.\Omega_n)^*)^*$$

$$= (\Omega_1^*[E_1^*].\Omega_1^*.\Omega_2^*[E_2^*].\Omega_2^*.\ \ldots\ .\Omega_n^*[E_n^*].\Omega_n^*)^*.$$

(1), (2) and (*) imply that this operator is equivalent to $(F.\Psi)^*$ for an $\mathcal{S}_1$-class event F and an increasing substitution Ψ. Another use of the lemma proves our assertion. Hence every operator in the dual class $\mathcal{R}((\delta_{\mathcal{S}_1}\ell \cup \partial_{SUB_\uparrow}) \cap \mathcal{O}[S])$ is a finite sum of regulators and hence a regulator.

The requirement that the differential operators contain the identity operator appears to be artificial, although no proof or counterexample of the result for $\mathcal{S}$ vice $\mathcal{S}_1$ has been found.

General Algebras of Regulators:

Our discussion above has been limited to specific classes of operators and their effects on regular events. In this section, we examine two operator classes which can be inserted into any regular algebra of regulators. The first to be considered is the class of total regulators.

<u>Definition</u>: For a standard algebra S, an operator Ω in $\mathcal{L}[S]$ is said to be a <u>total regulator</u> (in the operator class $\mathcal{R}_T[S]$) if, for any event E in S, $\Omega[E]$ is a regular event.

<u>Lemma 3.14</u>: For a regular function of p+k variables, there exists a finite number of regular functions h, f'_{i_j}, g_{i_j}, such that for p+k letters $b_1,\ldots,b_k,a_1,\ldots,a_p$,

$$f(b_1,..,b_k,a_1..,a_p)=h(a_1,..,a_p) +$$
$$+\sum_{j=1}^{k}\sum_{i_j=1}^{q_j} f'_{i_j}(a_1,..,a_p).b_j.g_{i_j}(b_1,..,b_k,a_1,\ldots,a_p) .$$

<u>Proof</u>: Let $E = f(b_1,\ldots,b_k,a_1,\ldots,a_p)$ and the aim is to decompose E into a finite sum of regular events of the above form. As E is regular, there are finitely many two term factorizations, L_iR_i, such that $E = \sum_{i=1}^{q} L_iR_i$, and we consider the intersection of E with

$$(a_1+..+a_p)^* + \sum_{j=1}^{k}(a_1+..+a_p)^*.b_j.(b_1+..+b_k+a_1+..+a_p)^*.$$

Let $h(a_1,\ldots,a_p) = E\cap(a_1+\ldots+a_p)^*$, the regular subset of words of E in which no b_j appears. For words of the

form wb_jv in E such that $w \varepsilon (a_1+\ldots+a_p)^*$, there exist L_i, R_i such that $wb_j \varepsilon L_i$, $v \varepsilon R_i$, and we let

$f'_{i_j}(a_1,\ldots,a_p).b_j.g_{i_j}(b_1,\ldots,b_k,a_1,\ldots,a_p) =$

$L_i \cap (a_1+\ldots+a_p)^*b_j. \quad R_i \cap (b_1+\ldots+b_k+a_1+\ldots+a_p)^*.$

<u>Theorem 3.15</u>: For a class of regulators Π in a standard algebra S such that $\mathcal{R}(\Pi)$ is a regular algebra of regulators, then

$$\mathcal{R}\ (\Pi \cup \mathcal{R}_T[S]) \subseteq \mathcal{R}[S].$$

<u>Proof</u>: For a regular function of p+k variables, $\Omega_1,\ldots,\Omega_p$, operators in Π and total regulators $\Psi_1,\ldots,\Psi_k$, the lemma implies that there exist regular functions h, f'_{i_j}, g_{i_j}, such that

$$f(\Psi_1,\ldots,\Psi_k,\Omega_1,\ldots,\Omega_p) = h(\Omega_1,\ldots,\Omega_p) +$$

$$+ \sum_{j=1}^{k} \sum_{i_j=1}^{q_j} f'_{i_j}(\Omega_1,\ldots,\Omega_p).\Psi_j.g_{i_j}(\Psi_1,\ldots,\Psi_k,\Omega_1,\ldots,\Omega_p)\ .$$

By hypothesis, $h(\Omega_1,\ldots,\Omega_p)$ is a regulator, and for each $j = 1,\ldots,k$, $i_j = 1,\ldots,q_j$, and each regular event E,

$f'_{i_j}(\Omega_1,\ldots,\Omega_p).\Psi_j.g_{i_j}(\Psi_1,\ldots,\Psi_k,\Omega_1,\ldots,\Omega_p)[E] =$

$f'_{i_j}(\Omega_1,\ldots,\Omega_p).\Psi_j[g_{i_j}(\Psi_1,\ldots,\Psi_k,\Omega_1,\ldots,\Omega_p)[E]\ ] = F$, say,

and, as Ψ_j is a total regulator, F is a regular event.

(Note that the proof also shows that for a function f as above such that $f(b_1,..,b_k,a_1,..,a_p) \cap (a_1+..+a_p)^* = 0$, then $f(\Psi_1,...,\Psi_k,\Omega_1,...,\Omega_p)$ is a total regulator as h is then the empty function.)

In other words, we can enlarge any regular algebra of regulators in such a way as to include the total regulators. We now present a few examples of total regulators.

Definition: Let E be an event in a standard algebra S. We define the operators δ^{E^ℓ} and δ^{E^r} by

$$\delta^{E^\ell}[F] = \delta_{F^\ell}[E] \quad \text{and} \quad \delta^{E^r}[F] = \delta_{F^r}[E]$$

for an event F in S. For a class of events $\mathcal{X}$, let $\delta^{\mathcal{X}^\ell}$ denote the operator class $\{\delta^{E^\ell} | E \varepsilon \mathcal{X}\}$, and $\delta^{\mathcal{X}^r}$ the operator class $\{\delta^{E^r} | E \varepsilon \mathcal{X}\}$.

Lemma 3.16: In a standard algebra S, the operator classes $\delta^{\mathcal{R}^\ell}$ and $\delta^{\mathcal{R}^r}$ consist of finite valued total regulators, that is, total regulators which assume only a finite number of values over S.

Proof: Let E and F be events in S such that E is regular. Then $\delta^{E^\ell}[F] = \delta_{F^\ell}[E]$ and $\delta^{E^r}[F] = \delta_{F^r}[E]$ are regular events, and the regularity of E also implies that the operators map events in S to the finitely many event derivates of E. Note also that the operators are linear as $\delta^{E^\ell}[F+G] = \delta_{(F+G)^\ell}[E] = \delta_{F^\ell}[E] + \delta_{G^\ell}[E]$ for events F and G. (Similarly for δ^{E^r}.)

Corollary 3.16.1: For a class of regulators Π in a standard algebra S such that $\mathcal{R}(\Pi)$ is a regular algebra of regulators, then

$$\mathcal{R}((\Pi \cup \delta^{\mathcal{R}^\ell} \cup \delta^{\mathcal{R}^r}) \cap \mathcal{S}[S]) \subseteq \mathcal{R}[S].$$

Another example, generalizing a result of Haines [15], of a large class of total regulators may be obtained by introducing the concept of "divisibility" for words.

Definition: Between words over a finite alphabet V, a divisibility relation $|$ is a relation on V* such that

(i) $1|1$,

(ii) $w|v \Rightarrow w|av$ for $a \varepsilon V$,

(iii) $w|v \Rightarrow aw|av$ for $a \varepsilon V$,

(iv) $w|v,\ v|u \Rightarrow w|u$.

We prove the following lemma about divisibility relations.

Lemma 3.17: There is no infinite 'division free' sequence $w_0, w_1, w_2, \ldots,$ of words over a finite alphabet V such that $i < j$ implies that $w_i \nmid w_j$.

Proof: Lexicographically order the words of V* first by length and then by a trivial order on V. Among all such infinite division free sequences, if they exist, there will be one with a minimal first word, $\overline{w}_0$ say. Among all such sequences beginning with $\overline{w}_0$, there will be one with a minimal second word, $\overline{w}_1$. Continuing in this fashion, we obtain a minimal sequence $\overline{t} = \overline{w}_0, \overline{w}_1, \overline{w}_2, \ldots,$. Now as $\overline{t}$ is an infinite sequence and V is a finite alphabet, there exists $a \varepsilon V$ such that infinitely many words in $\overline{t}$ begin with the letter a. Let $\{av_j | j \varepsilon N\}$ be the subsequence of $\overline{t}$ with this property, where $av_0 = \overline{w}_k$ is the first word in $\overline{t}$ beginning with a. Then it is easy to see that $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{k-1}, \overline{v}_0, \overline{v}_1, \ldots,$ is a division free sequence

contradicting the minimality of $\bar{t}$.

The lemma is equivalent to a result of G. Higman [16] which states:

Theorem: If E is any set of words formed from a finite alphabet, it is possible to find a finite subset E_0 of E such that, given a word $w \varepsilon E$, it is possible to find $w_0 \varepsilon E_0$ such that $w_0 | w$.

However, the lemma as given enables us to prove a result on "divisibility" operators. For a divisibility relation $|$ on V* and an event E in V*, let DIV[E] be the event $\{w| \quad w|v \quad \text{for some } v \varepsilon E\}$, the set of divisors of E.

Theorem 3.18: DIV[E] is a regular event and hence DIV is a total regulator for V* events.

Proof: We prove that DIV[E] is regular by showing that it has only finitely many word derivates. If $w|v$ for $v \varepsilon \delta_{a^\ell}[DIV[E]]$, then (ii) implies that $aw|av$, so that aw is in DIV[E]. Hence w is in $\delta_{a^\ell}[DIV[E]]$, and the word derivates of DIV[E] are divisor closed.

Now if there are infinitely many word derivates of DIV[E], there is an infinite sequence of letters, a,b,c,..., in V such that

$$\mathrm{DIV}[E],\quad \delta_{a^{\ell}}[\mathrm{DIV}[E]],\quad \delta_{(ab)^{\ell}}[\mathrm{DIV}[E]],\ldots,$$

is an infinite sequence of distinct events (and a decreasing sequence as shown in the first part of the proof). But if we select

$$w_0 \varepsilon \mathrm{DIV}[E] \setminus \delta_{a^{\ell}}[\mathrm{DIV}[E]],\ w_1 \varepsilon \delta_{a^{\ell}}[\mathrm{DIV}[E]] \setminus \delta_{(ab)^{\ell}}[\mathrm{DIV}[E]],\ldots,$$

then $w_0, w_1, \ldots,$ is an infinite division free sequence, a contradiction.

<u>Corollary 3.18.1</u>: For a standard algebra S over a finite alphabet V and for a linear operator Ω in $\mathcal{O}[S]$, let $\Psi = \begin{pmatrix} V \\ 1 \end{pmatrix}^{\dagger} \times \Omega$. Then Ψ^* is a total regulator. In particular, $\begin{pmatrix} V \\ 1 \end{pmatrix}^{*}$ is a total regulator.

<u>Proof</u>: For $\begin{bmatrix} v \\ w \end{bmatrix} \varepsilon \Psi^*$, $\begin{bmatrix} av \\ w \end{bmatrix} + \begin{bmatrix} av \\ aw \end{bmatrix} \varepsilon \Psi^*$ for $a \varepsilon V$, and starring the operator supplies the transitivity and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon \Psi^*$ required for Ψ^* to be a divisibility relation.

Before concluding our discussion of total regulators, we point out that, although there are only countably many regular events in a standard algebra S, our results for the total regulators are not dependent on the fact that the operators take only countably many values over the events in S.

Let $\mathcal{R}_C[S]$ be the class of <u>countably-valued regulators</u> where operators in $\mathcal{R}_C[S]$ take only countable many values over events in S, and in addition, are linear regulators. Clearly, $\mathcal{R}_T[S] \subseteq \mathcal{R}_C[S]$, but an attempted generalization of 3.15 fails for this class as shown by the following example.

Let S be a standard algebra over a single letter a and let $a^P = \{a^{p_m} | p_m \text{ the } m^{\underline{th}} \text{ prime}\}$. Then we define the operator Ω by $\Omega[a^n] = a^*$, for $a^n \not\subseteq a^P$, and $\Omega[a^{p_m}] = \{a^{p_s} | 1 \leq s \leq m+1\}$, that is, Ω takes the $m^{\underline{th}}$ prime exponent to the first m+1 prime exponents. Then Ω is trivially a linear regulator as it maps all regular sets to a* or to a finite set, and similarly, Ω is countably valued as a^P is the only additional set in its range. However, $\Omega^*[a^2] = a^P$ is a non-regular set so that Ω cannot

be inserted into any regular algebra of regulators.

Another class of regulators, the $\dotplus_{\mathcal{R}}$ operators, although not total regulators, can also be inserted into any regular algebra of regulators.

<u>Definition</u>: For events E and F, let

$$\dotplus_E[F] = E + F \quad \text{if } F \neq 0, \text{ and}$$
$$= 0 \quad \text{if } F = 0,$$

and for a class of events $\mathcal{X}$, let $\dotplus_{\mathcal{X}}$ denote the operator class $\{\dotplus_E | E \varepsilon \mathcal{X}\}$.

The condition that $\dotplus_E[0] = 0$ is necessary for the linearity of the $\dotplus$ operators. Observe that if we define $\hat{\dotplus}_E[F] = E{+}F$ for events E and F, then $\dotplus_E = \partial_{\partial \hat{\dotplus}_E}$, and the $\dotplus$ operators are the linearized $\hat{\dotplus}$ operators.

<u>Lemma 3.19</u>: For a standard algebra S, an operator Ω in $\mathcal{L}[S]$, and events E and F in S,

(1) $\Omega . \dotplus_E \quad = \quad \dotplus_{\Omega[E]} . \Omega$,

(2) $\dotplus_E . \dotplus_F \quad = \quad \dotplus_{E+F}$,

(3) $(\dagger_E . \Omega)^* = \Omega^* . \dagger_E = \dagger_{\Omega^*[E]} . \Omega^*$,

(4) $(\dagger_E)^* = \dagger_E$.

Proof: Immediate.

Theorem 3.20: For a class of regulators Π in a standard algebra S such that $\mathcal{R}(\Pi)$ is a regular algebra of regulators, then

$$\mathcal{R}(\Pi \cup (\dagger_{\mathcal{R}} \cap \mathcal{O}[S])) \subseteq \mathcal{R}[S].$$

Proof: As in the proof of 3.13, we show that every operator in the class is an operator of the form $\dagger_E . \Omega$ for E a regular event and $\Omega \varepsilon \mathcal{R}(\Pi)$.

For regular events E,F, operators Ω, Ψ in $\mathcal{R}(\Pi)$, and an event X,

$$(\dagger_E . \Omega)+(\dagger_F . \Psi)[X] = \Omega[X]+E+\Psi[X]+F = \dagger_{E+F} . (\Omega+\Psi)[X].$$

The similar result for $(\dagger_E . \Omega).(\dagger_F . \Psi)$ and $(\dagger_E . \Omega)^*$ follows from the lemma and the fact that $\mathcal{R}(\Pi)$ operators are regulators. The proof of the theorem is now immediate.

Consideration of the intersection operators proves more complex. For a regular event E, $\cap_E$ is a linear

operator, and we may also show the analogous results of 3.19 (2) and (4), that is,

$$\cap_E \cdot \cap_F = \cap_{E \cap F},$$

$$(\cap_E)^* = \Delta + \cap_E = \Delta ,$$

$$\cap_E + \cap_F = \cap_{E+F} .$$

However, the methods of 3.20 cannot be applied to this operator class, for given a linear operator Ω,

$$\Omega . \cap_E [F] = \Omega [F \cap E]$$

which need not be the same event as $\Omega[F] \cap \Omega[E]$. By similar reasoning, $(\cap_E . \Omega)^*$ need not be equal to $\Omega^* . \cap_E$, and the question remains open as to whether this operator class can be inserted into any regular algebra of regulators.

Some Remarks on Context-Free Languages:

We conclude the chapter with an examination of the effect of regulators on context-free events, and the context-free preserving operators, $\mathcal{C}$ [S] for a standard algebra S.

I. $\mathcal{L}[S] \cap \mathcal{C}[S]$ is not closed under the biregular operations of +, ×, and †: Let V be the alphabet {a,b,c} and L the context-free event $\{wcw^T | w\varepsilon(a+b)^*\}$. The operator $\Omega = \sum_{v\varepsilon(a+b)^*} \begin{bmatrix} v \\ v^T \end{bmatrix}$ is in $\mathcal{C}[S]$ as well as $\mathcal{R}[S]$, although it is not biregular. Let

$\Psi = \Delta_{\{a,b\}} \times \begin{bmatrix} c \\ c \end{bmatrix} \times \Omega$, and then $\Psi[L] = \{wcw | w\varepsilon(a+b)^*\}$,

a non-context-free event as noted in the proof of 1.13. Observe that this example also shows that $\mathcal{R}[S] \not\subseteq \mathcal{C}[S]$.

II. A slight modification of the argument preceding 1.11 shows that $\partial(CVX_{\Delta[S]} \cap \partial\mathcal{R}[S])$ operators do not preserve $\mathcal{C}$-class events: Let L_1 and L_2 be the context-free languages defined in the example of Ginsburg and Spanier.

Then $\Omega = \Delta_{\{a,b,c,d\}} \times \begin{bmatrix} 1 \\ L_2+1 \end{bmatrix} \times \Delta_{\{a,b,c,d\}}$ is an open convex operator whose dual is a regulator but $\partial_\Omega[L_1]$ is not context-free as $\cap_{a^*} \cdot \partial_\Omega[L_1]$ yields the same non-context-free event $\{a^n | n=4.6^i, i \geq 0\}$. Similarly $\Omega^\dagger$ is an increasing substitution whose dual does not preserve $\mathcal{C}$-class events.

III. Although $\mathcal{R}^{\ell}$, $\mathcal{R}^{r}$, $\delta_{\mathcal{R}^{\ell}}$, $\delta_{\mathcal{R}^{r}}$, are operator classes preserving $\mathcal{C}$-class events as they are $\begin{bmatrix}\mathcal{R}\\ \mathcal{R}\end{bmatrix}$ operators, $\mathcal{R}(\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{r}})$ and $\mathcal{R}(\mathcal{R}^{r} \cup \delta_{\mathcal{R}^{\ell}})$ operators do not preserve $\mathcal{C}$-class events as the proof of 3.12 showed that these operator classes generate all (coded) recursively enumerable sets.

The example of Ginsburg and Spanier may be used again to show that

$$\mathcal{R}((\delta_{\mathcal{R}^{\ell}} \cup \delta_{\mathcal{R}^{r}}) \cap \mathcal{O}[S]) \not\subseteq \mathcal{C}[S] :$$

Let L be the context-free language generated by the productions

$S \rightarrow aSb^2$ $\quad$ $S \rightarrow bSa^3$ $\quad$ $S \rightarrow cSabc$ $\quad$ $S \rightarrow d$

and let $\Omega = ((\delta_{a^{\ell}} \cdot \delta_{a^{r}}) + (\delta_{b^{\ell}} \cdot \delta_{b^{r}}) + (\delta_{c^{\ell}} \cdot \delta_{c^{r}}))^*$, an operator in $\mathcal{R}(\delta_{\mathcal{R}^{\ell}} \cup \delta_{\mathcal{R}^{r}})$. Then $\Omega[L]$ is not context-free as $\cap_{a^*} \cdot (\begin{bmatrix}a\\a\end{bmatrix} + \begin{bmatrix}b\\b\end{bmatrix} + \begin{bmatrix}c\\c\end{bmatrix} + \begin{bmatrix}d\\1\end{bmatrix})^{\dagger} \cdot \Omega[L] = \{a^n | n=4.6^i, i \geq 0\}$.

$\mathcal{R}((\mathcal{R}^{\ell} \cup \mathcal{R}^{r}) \cap \mathcal{O}[S]) \subseteq \mathcal{R}[S]$ since this class of operators corresponds to Gruska's algebra of word pairs as noted in

the proof of 3.9. As $\mathcal{C}$ is closed under full substitution, Gruska's result also implies that $\mathcal{R}((\mathcal{C}^{\ell} \cup \mathcal{C}^{r}) \cap \mathcal{O}[S]) \subseteq \mathcal{C}[S]$.

The normal form of 3.11.1 for $\mathcal{R}(\mathcal{R}^{\ell} \cup \delta_{\mathcal{R}^{\ell}})$ operators (similarly for $\mathcal{R}(\mathcal{R}^{r} \cup \delta_{\mathcal{R}^{r}})$ operators) shows that this class of operators preserves $\mathcal{C}$-class events. As $\delta_{\mathcal{C}^{\ell}}[\mathcal{C}] \nsubseteq \mathcal{C}$, this cannot be extended to $\mathcal{R}(\mathcal{R}^{\ell} \cup \delta_{\mathcal{C}^{\ell}})$ operators. The case for $\mathcal{R}(\mathcal{C}^{\ell} \cup \delta_{\mathcal{R}^{\ell}})$ operators is still open.

IV. The proofs of 3.15 and 3.19 show that $\mathcal{R}_T[S]$ and $+_{\mathcal{R}} \cap \mathcal{O}[S]$ can be included in any regular algebra of operators preserving context-free languages. Indeed, the same holds for $\mathcal{C}_T[S]$ and $+_{\mathcal{C}} \cap \mathcal{O}[S]$ operators. The example following 3.18.1 shows that the similar result doesn't hold for $\mathcal{R}_C[S]$ and $\mathcal{C}_C[S]$.

V. It can be shown that open convex members of $\begin{bmatrix}\mathcal{R}\\ \mathcal{C}\end{bmatrix}$ are closed under the regular operations.

Chapter 4

The Algebra of Commutative Events

Before considering the theory of operators for commutative events, we examine the algebra of regular commutative events and the corresponding regular expressions. Let S^+ be a standard algebra with a finite alphabet V in which the letters are allowed to commute. For formal expressions E and F, let $\langle E \rangle$, $\langle F \rangle$ denote the events represented by E, F respectively. As usual, an event in S is <u>regular</u> if it can be obtained from a finite number of applications of the regular operations +, ., and *, over 0, 1, and a finite number of non-trivial words in V*. Let $\mathcal{S}^+$ denote the class of <u>commutative</u> standard events and $\mathcal{R}^+$ the class of <u>commutative</u> regular events.

In [17], Redko lists an axiom scheme for S^+, essentially the one below, but we have added C0 and C1 for completeness (and then C12 and C13 are redundant).

For formal expressions E, F, G over V (events in S),

C0 : $E + 0 = E$

C1 : $E.0 = 0$

C2 : $E + F = F + E$

C3 : $(E+F)+G = E+(F+G)$

C4 : $E.F = F.E$

C5 : $(E.F).G = E.(F.G)$

C6 : $(E+F).G = E.G + F.G$

C7 : $E.1 = E$

C8 : $1^* = 1$

C9 : $(E.F^*)^* = 1 + E.E^*.F^*$

C10: $(E+F)^* = (E.F)^*(E^*+F^*)$

C11: $E^* = (E^k)^*(1+E+\ldots+E^{k-1})$, $k=1,2,\ldots$

C12: $E + E = E$

C13: $(E+F)^* = E^*.F^*$

By relying on the theory of real vector cones and a questionable induction step (see Theorem 5 [17]), Redko has deduced that, given two regular expressions E, F such that $\langle E \rangle = \langle F \rangle$, the equality of the expressions is provable from the axioms (written $E \underset{pr}{=} F$). Ginsburg [18], by interpreting the words in V^* as vectors over N, has shown that $\mathcal{R}^+$ is closed under intersection and boolean difference

(see the remark following the proof of 4.1 for the equivalence of regular events and Ginsburg's "semi-linear" events), and that the result of these operations was effectively calculable for arbitrary regular events. Below, we show that the sufficiency of the axiom scheme may be shown by a lengthy reduction process, and that this process also proves the closure of the regular events under intersection and difference.

Regular Expressions and the Normal Form:

In this section we examine the algebra of regular expressions and (i) show that there is a normal form for these expressions in which the starred words appearing in any single term form a "linearly independent" set, and (ii) prove that the axiom scheme suffices for relationships between single terms in this normal form.

The following are easily deducible from C0-C13:

C14: $0^* = 1$

C15: $E^* = 1 + E.E^*$

C16: $E^* = 1 + E + \ldots + E^{k-1} + E^k E^*$, $k=1,2,\ldots$, and we write $1 + E + \ldots + E^{k-1}$ as $E^{<k}$, where $E^{<0} = 0$,

C17: $(E^*)^* = E^*$

C18: $E^*.E^* = E^*$

C19: $E^*.F^*.G^* = (E^{\alpha}.F^{\beta}.G^{\gamma})^*.(E^{<\alpha}.F^*.G^*+E^*.F^{<\beta}.G^*+E^*.F^*.G^{<\gamma})$

where α, β, $\gamma \varepsilon N$ such that $\alpha+\beta+\gamma > 0$.

Definition: $E \leq F \iff E + F = F$.

C20: $E \leq F \Rightarrow E.G \leq F.G$

C21: $E \leq F^*.E$

C22: $E \leq F,\ F \leq E \iff E = F$.

Definition: For words w_i in V*, a regular expression of the form $w_1{}^*w_2{}^*\ldots w_n{}^*w_0$, or a finite sum of expressions in this form, denoted $\sum_f w_1{}^*w_2{}^*\ldots w_n{}^*w_0$, is said to be in almost-normal form.

Lemma 4.1: Any regular expression, E, can be expressed in almost-normal form.

Proof: We show that the set of expressions in almost-normal form is closed under the regular operations. A finite sum of expressions in almost-normal form is already in almost-normal form, and a finite product can be rearranged by the commutivity and distributivity of S.

We can then transform the star of an expression in almost-normal form into almost-normal form by C13 and C9. As all finite expressions, that is, finite sums of words, are in almost-normal form, the lemma then follows by induction on the complexity of the expression E.

It is clear that the commutative regular events (expressed in almost-normal form) correspond exactly to the semi-linear events where the word $a_1^{\alpha} \ldots a_p^{\beta}$ corresponds to the vector $(\alpha, \ldots, \beta)$, α, $\beta \in N$.

<u>Definition</u>: A set of words $\{w_1, \ldots, w_n\}$ over the alphabet V is said to be <u>independent</u> if

$$w_1^{\alpha_1} \ldots w_n^{\alpha_n} = w_1^{\beta_1} \ldots w_n^{\beta_n}$$

for $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in N$, then $\alpha_i = \beta_i$, $i = 1, \ldots, n$.

An expression $\sum_f w_1^* \ldots w_n^* w_0$ is said to be in <u>normal form</u> if the starred words in each term of the sum form an independent set.

<u>Lemma 4.2</u>: Any regular expression, E, can be expressed in normal form.

Proof: In view of 4.1 it suffices to consider a single expression of the form $w_1*\ldots w_n*w_0$, $n \geq 2$. If the starred words do not form an independent set, then there exist $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n \in N$ such that

$$w_1^{\alpha_1}\ldots w_n^{\alpha_n} = w_1^{\beta_1}\ldots w_n^{\beta_n},$$

and at least two pairs, (α_i,β_i), of exponents differ. Without loss of generality, we assume that

$$v = w_1^{\alpha_1}\ldots w_s^{\alpha_s} = w_{s+1}^{\beta_{s+1}}\ldots w_n^{\beta_n}$$

such that

(i) $1 \leq s \leq n-1$,

(ii) $\alpha_1+\ldots+\alpha_s > 0$, $\beta_{s+1}+\ldots+\beta_n > 0$.

A generalized form of C19 allows us to write

$$w_1*..w_n* = (w_1^{\alpha_1}..w_s^{\alpha_s})*(w_1^{<\alpha_1}w_2*..w_s*+..+w_1*..w_{s-1}*w_s^{<\alpha_s}) \times$$

$$(w_{s+1}^{\beta_{s+1}}..w_n^{\beta_n})*(w_{s+1}^{<\beta_{s+1}}w_{s+2}*..w_n*+..+w_{s+1}*..w_{n-1}*w_n^{<\beta_n})$$

$$=v*(w_1^{<\alpha_1}w_2*..w_s*+..+w_1*..w_{s-1}*w_s^{<\alpha_s}) \times$$

$$(w_{s+1}^{<\beta_{s+1}}w^*_{s+2}..w_n*+..+w^*_{s+1}..w^*_{n-1}w_n^{<\beta_n})$$

which is a finite sum of normal terms with one less starred word in each. If any of these expressions is not in normal form, we repeat the above process for that term, and observe that in a finite number of steps, the process will terminate, leaving a finite number of normal terms.

By convention, we call an expression which consists of a finite sum of words <u>sub-normal</u>, and regular expressions which consist solely of a sum of starred disjunctions of words <u>super-normal</u>.

<u>Theorem 4.3</u>: For an arbitrary regular expression F and a single normal term E such that $\langle F \rangle \subseteq \langle E \rangle$, then $F \underset{pr}{\leq} E$.

We first prove the following lemmas.

<u>Lemma 4.4</u>: For a regular expression E such that $w \varepsilon \langle E \rangle$, then $w \underset{pr}{\leq} E$.

<u>Proof</u>: It is sufficient to consider E a single normal term, $v_1{}^*\ldots v_m{}^* v_0$ say, and if $w \varepsilon \langle E \rangle$, then $w = v_1{}^{\alpha}\ldots v_m{}^{\beta} v_0$ for some $\alpha,\ldots,\beta \varepsilon N$. We can replace E by

$$(v_1^{<\alpha} + v_1^{\alpha} v_1^*) \ldots (v_n^{<\beta} + v_n^{\beta} v_n^*) v_0,$$

a finite sum of terms, one of which is $v_1^* .. v_n^* w$. C16 and C12 then imply that $w \underset{\rho r}{\leq} v_1^* \ldots v_n^* w$.

Definition: Let $|$ be the relation defined on V^* by

$$a_1^{\alpha_1} \ldots a_p^{\alpha_p} \;\Big|\; a_1^{\beta_1} \ldots a_p^{\beta_p}$$

if and only if $\alpha_i \leq \beta_i$ for each i.

Lemma 4.5: Every set of pairwise incomparable words in V^* (with respect to $|$) is finite, and hence for any event in V^*, the set of minimal words is finite.

Proof: $|$ is a divisibility relation on V^* and the result follows from 3.17.

Lemma 4.6: If $\langle w_1^* \ldots w_n^* w \rangle \cap \langle v_1^* \ldots v_m^* v \rangle = \langle X \rangle$ is an infinite event, then

$$\langle w_i^* \ldots w_n^* \rangle \cap \langle v_1^* \ldots v_m^* \rangle \neq 1.$$

Proof: As $\langle X \rangle$ is infinite, there exist an infinite number of identities of the form

$$u = w_1^{\alpha_1} \ldots w_n^{\alpha_n} w = v_1^{\beta_1} \ldots v_m^{\beta_m} v$$

with $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \varepsilon N$. As the set of minimal words with respect to $|$ is finite, let u' be a minimal word in $\langle X \rangle$ of the form

$$u' = w_1^{\alpha'_1} \ldots w_n^{\alpha'_n} w = v_1^{\beta'_1} \ldots v_m^{\beta'_m} v ,$$

such that u' divides an infinite number of words in $\langle X \rangle$. Then for $u \varepsilon \langle X \rangle$ such that $u' | u, u' \neq u$,

$$u'' = w_1^{\alpha_1 - \alpha'_1} \ldots w_n^{\alpha_n - \alpha'_n} = v_1^{\beta_1 - \beta'_1} \ldots v_m^{\beta_m - \beta'_m}$$

is a non-trivial word in $\langle w_1^* \ldots w_n^* \rangle \cap \langle v_1^* \ldots v_m^* \rangle$.

Lemma 4.7: Such a u'' can be effectively found.

Proof: By consideration of the system of the homogeneous linear equations associated with $w_1^* \ldots w_n^*$ and $v_1^* \ldots v_m^*$.

Lemma 4.8: For arbitrary expressions t*X, t*Y,

$$t^*X \leq t^*Y \iff X \leq t^*Y .$$

Proof: $\Rightarrow$: C21 and C12. $\Leftarrow$: C20 and C18.

Definition: For a regular expression F in normal form, the dimension of F, denoted dim(F), is the maximum number of non-trivial words in the super-normal part of any of its normal terms.

Proof of 4.3: It is sufficient to consider F a single normal term and the proof is by induction on dim(F). If F is sub-normal, the result follows from 4.4. Consider $E = v_1{*}\ldots v_m{*}v$, $F = w_1{*}\ldots w_{n+1}{*}w$, where we assume the result for n. As $\langle E\rangle \supseteq \langle F\rangle$, 4.6 and 4.7 imply that there exists a common word, t say, in the super-normal parts of the expressions. By C16, we may replace F by an expression of the form $\sum_f t{*}\ u_1{*}\ldots u_n{*}u$ and as $t{*}E = E$, the result follows from 4.8.

Sufficiency of the Axiom Scheme:

We now show that we can effectively replace (by a lengthy process of reduction) a regular expression E by regular expressions corresponding to the intersection and boolean difference of the event $\langle E\rangle$ with the regular

event $\langle X \rangle$, for a single normal term X. This enables us to prove the sufficiency of the axiom scheme.

Lemma 4.9: For single normal terms $X = x_1^* \ldots x_m^* x$ and $E = v_1^* \ldots v_n^* v$ such that $v_1^* \ldots v_n^* \underset{pr}{\leq} x_1^* \ldots x_m^*$, there exist regular expressions Y and Z such that

(1) $E \underset{pr}{=} Y + Z$,

(2) $\langle Y \rangle \subseteq \langle X \rangle$ and $\langle Z \rangle \subseteq \langle E \rangle \setminus \langle X \rangle$.

Proof: By induction on dim(E). For n = 0, the result is trivial. Now if $\langle E \rangle \cap \langle X \rangle \neq 0$, let $y = v_1^{\alpha_1} \ldots v_n^{\alpha_n} v$ be a word in the intersection and by C16, we may replace E by $(v_1^{<\alpha_1} + v_1^{\alpha_n} v_1^*) \ldots (v_n^{<\alpha_n} + v_n^{\alpha_n} v_n^*) v =$ $v_1^* \ldots v_n^* y + H$, where H is a finite sum of normal terms of dimension $\leq n-1$ (thus satisfying our induction hypothesis), and $v_1^* \ldots v_n^* y \underset{pr}{\leq} X$.

Lemma 4.10: For a single normal term $X = x_1^* \ldots x_m^* x$, and a regular expression E, there are regular expressions Y and Z such that

(1) $E \underset{pr}{=} Y + Z$,

(2) $\langle Y \rangle \subseteq \langle X \rangle$ and $\langle Z \rangle \subseteq \langle E \rangle \setminus \langle X \rangle$.

Proof: It suffices to consider E a single normal term of the form $u_1 * \ldots u_n * u$, and the proof is by induction on dim(E). The case for n = 0 is trivial and we proceed by induction. The idea involved is to express the words in <E> in terms of the words $x_1, \ldots, x_m$ with rational (possibly negative) exponents and to decompose the expression E with this dependence in mind. If the words of <E> are not expressible in this form, we "complete" the set $\{x_1, \ldots, x_m\}$ with words $x_{m+1}, \ldots, x_p$ such that $x_1 * \ldots x_m * x_{m+1} * \ldots x_p *$ is normal (that is, the set $\{x_1, \ldots, x_p\}$ is independent), and such that, for a word $v \varepsilon$<E>, there exist rational $r_1, \ldots, r_p$ such that $v = x_1^{r_1} \ldots x_p^{r_p}$ (that is, $\{x_1, \ldots, x_p\}$ is a "basis" for the words in <E>).

We define the x_i-index of a word v in <E>, $\mathrm{ind}_{x_i}(v)$, as the exponent of x_i in the expression for v in terms of $x_1, \ldots, x_p$. Now for the word x_i, if there are u_j, u_k in $\{u_1, \ldots, u_n\}$ such that $\mathrm{ind}_{x_i}(u_j) < 0$, $\mathrm{ind}_{x_i}(u_k) > 0$,

then there exist positive integers p and q such that $\mathrm{ind}_{x_i}(u_j^p\, u_k^q) = 0$, and we may replace E by C19 in terms of the word $u_j^p\, u_k^q$. It is clear that after a

finite number of steps in this reduction process, we will have replaced E by a finite number of normal terms, each of which satisfies exactly one of the following:

(1) each of the starred words has zero x_i-index,

(2) the starred words have non-negative x_i-index, and at least one starred word has strictly positive x_i-index,

(3) the starred words have non-positive x_i-index, and at least one starred word has strictly negative x_i-index.

Now repeating this process, in turn, for $x_1, x_2, .., x_p$, we observe that the reduction process for the word x_j does not affect the decomposition (1), (2), (3) above for $i < j$. Thus, in a finite number of steps, we have replaced E by a finite sum of normal terms such that, if (non-trivial) starred words appear in an expression, then either they all have non-negative x_i-index, $i=1,\ldots,p$, or they all have non-positive x_i-index. By induction, we need only consider those terms of dimension n. As a starred word, v_j say, in such a term is a word over $x_1,\ldots,x_p$ with rational coefficients, by C11 we may

replace $(v_j)^*$ by $(v_j^{q_j})^* \, v_j^{<q_j}$, where q_j is the l.c.m. of the (absolute value of) the divisors of the rational exponents in the representation of v_j over $x_1,\ldots,x_p$. Hence if $v_1^*\ldots v_n^*v$ is such a term obtained from the reduction process, we may assume that the v_j are <u>integral</u> combinations of the $x_1,\ldots,x_p$.

Now for the case where the v_j have non-positive index, it is clear that we can find <u>positive</u> integers $s_1,\ldots,s_n$ sufficiently large such that, if we replace $v_1^*\ldots v_n^*v$ by $(v_1^{<s_1}+v_1^{s_1}v_1^*)\ldots(v_n^{<s_n}+v_n^{s_n}\,v_n^*)v$, then no word in $\langle v_1^*\ldots v_n^*v_1^{s_1}\ldots v_n^{s_n}\,v\rangle$ is in $\langle X\rangle$. As this is the only term of dimension n in the expanded expression, the result follows for the negative case by induction.

Similarly, for the positive case, if some v_j has a strictly positive x_i-index for $i = m+1,\ldots,p$, we may replace $v_1^*..v_n^*v$ by $v_1^*..v_{j-1}^*(v_j^{<s_j}+v_j^{s_j}\,v_j^*)\ldots v_n^*v$ for a positive integer s_j sufficiently large so that $\langle v_1^*\ldots v_j^*\ldots v_n^*v_j^{s_j}\,v\rangle$ has no word in $\langle X\rangle$ and again, as this is the only term in the expansion of $v_1^*\ldots v_n^*v$ of dimension n, the result follows by induction for this case.

Thus, to complete the proof, we must only consider terms of the form $v_1^* \ldots v_n^* v$ where each v_j has x_i-index ≥ 0 for $i=1,\ldots,p$, and x_i-index $= 0$, $i=m+1,\ldots,p$. The result for this case follows from 4.9.

<u>Theorem 4.11</u>: For a regular expression E and F such that $\langle F\rangle \supseteq \langle E\rangle$, then $F \geq_{pr} E$.

<u>Proof</u>: By induction on the number of terms in the expression F. For a single term, the result follows from 4.3. We assume the result for expressions F' with $<q$ terms. Then for an expression F of the form $F' + x_1^* \ldots x_m^* x$, 4.10 implies that $E =_{pr} Y + Z$ such that $\langle Y\rangle \subseteq \langle x_1^* \ldots x_m^* x\rangle$ and $\langle Z\rangle \subseteq \langle F'\rangle$ and the induction step follows from another use of 4.3.

<u>The Boolean Operations for Regular Expressions</u>:

Lemma 4.10 also implies that commutative regular events are closed under intersection and boolean difference. However, the reduction process in actually computing the intersection and difference for two regular events is lengthy and we provide here a shorter algorithm for this.

<u>Lemma 4.12</u>: For regular expressions t*G and t*H,

$$t^*G \cap t^*H = t^*(t^*G \cap H + G \cap t^*H).$$

<u>Proof</u>:

$$t^*G \cap t^*H = \sum_{i,j}^{\infty} (t^iG \cap t^jH)$$

$$= \sum_{i,k}^{\infty} (t^iG \cap t^{i+k}H) + \sum_{j,k}^{\infty} (t^{j+k}G \cap t^jH)$$

$$= \sum_{i,k}^{\infty} t^i(G \cap t^kH) + \sum_{j,k}^{\infty} t^j(t^kG \cap H)$$

$$= t^*(G \cap t^*H) + t^*(t^*G \cap H)$$

$$= t^*((t^*G \cap H) + (G \cap t^*H))$$

as was to be shown.

<u>Lemma 4.13</u>: For single supernormal terms t*A, t*B such that $t^*A \geq t^*B$ and $A \geq B$, then

$$t^*A \setminus t^*B = t^*(A \setminus B).$$

Proof:

$$t^*A \setminus t^*B = \sum_i^{\infty} t^iA \setminus t^*B = \sum_i^{\infty} t^iA \setminus (t^{<i} + t^it^*)B$$

$$= \sum_i^{\infty} \Big(\bigcap_{0 \le k < i} (t^iA \setminus t^kB)\Big) \cap (t^iA \setminus t^it^*B)$$

$$= \sum_i^{\infty} \Big(\bigcap_{0 \le k < i} t^k(t^{i-k}A \setminus B)\Big) \cap t^i(A \setminus (t^*B \cap A)).$$

As $A \geq B$, then $t^*B \cap A = B$, and as B is a super-normal expression,

$$t^{i-k}A \setminus B = t^{i-k}A \setminus (t^{i-k}A \cap B) = t^{i-k}A \setminus 0 = t^{i-k}A$$

for $i > k$. Then we have that

$$t^*A \setminus t^*B = \sum_i^{\infty} \Big(\bigcap_{0 \le k < i} t^iA\Big) \cap t^i(A \setminus B)$$

$$= \sum_i^{\infty} t^i(A \cap A \setminus B) = t^*(A \setminus B).$$

Theorem 4.14: For each pair of regular events E and F, $E \cap F$ and $E \setminus F$ are regular events.

Proof: It suffices to consider E and F as single normal terms. The result for intersection follows immediately from 4.6, C19, and 4.12, by induction on the dimension of E and F.

The proof for boolean difference is again by induction on the dimension of E and F, and for $\dim(E) + \dim(F) = 0$, the result follows trivially. If F is a finite sum of words, the result follows by repeated use of C16, so that it suffices to consider E and F single normal terms of the form,

$$E = t^*w_1^* \ldots w_n^*w, \qquad F = t^*v_1^* \ldots v_m^*v,$$

such that $E \geq F$. In this case,

$$v = t^{\alpha}w_1^{\alpha_1} \ldots w_n^{\alpha_n} w,$$

for some $\alpha, \alpha_1, \ldots, \alpha_n \varepsilon N$, and then,

$$(E \setminus F) = w(t^*w_1^* \ldots w_n^* \setminus t^*v_1^* \ldots v_m^*t^{\alpha} w_1^{\alpha_1} \ldots w_n^{\alpha_n}).$$

If $z = t^{\alpha} w_1^{\alpha_1} \ldots w_n^{\alpha_n} \neq 1$, then $\alpha+\alpha_1+\ldots+\alpha_n > 0$, and by C16, we may replace $t^*w_1^* \ldots w_n^*$ by,

$$(t^{<\alpha} + t^{\alpha}t^*)(w_1^{<\alpha_1} + w_1^{\alpha_1}w_1^*) \ldots (w_n^{<\alpha_n} + w_n^{\alpha_n} w_n^*).$$

As this is a sum of normal terms, all of which have dimension $<n$, with the exception of $t^*w_1^* \ldots w_n^*z$, by induction the problem is reduced to consideration of

$t^*w_1^*\ldots w_n^* \setminus t^*v_1^*\ldots v_m^*$. Now as $t^*w_1^*..w_n^* \geq t^*v_1^*..v_m^*$, there exist $\alpha^{(1)}, \alpha_1^{(1)}, .., \alpha_n^{(1)} .., \alpha^{(m)}, \alpha_1^{(m)}, .., \alpha_n^{(m)} \in N$ such that

$$v_i = t^{\alpha^{(i)}} w_1^{\alpha_1^{(i)}} \ldots w_n^{\alpha_n^{(i)}}, \quad i=1,\ldots,m,$$

and we let $u_i = \quad w_1^{\alpha_1^{(i)}} \ldots w_n^{\alpha_n^{(i)}}$. Then, $t^*w_1^*..w_n^* \geq t^*u_1^*\ldots u_m^* \geq t^*v_1^*\ldots v_m^*$, and

$$t^*w_1^*..w_n^* \setminus t^*v_1^*..v_m^* = t^*w_1^*..w_n^* \setminus t^*u_1..u_m^* + t^*u_1^*..u_m^* \setminus t^*v_1^*..v_m^*, = A + B \text{ say.}$$

$t^*w_1^*..w_n^* \setminus t^*u_1^*..u_m^*$ satisfies the condition of 4.13 and then the induction hypothesis provides the result for event A. Now if $m < n$, another use of the induction hypothesis implies that B is regular, and if $m = n$, we may replace $t^*u_1^*\ldots u_m^*$ by repeated use of C12 to obtain a finite sum of normal terms, all but one having an intersection with $t^*v_1^*\ldots v_m^*$ of strictly smaller dimension (and hence satisfying the induction hypothesis) and the exceptional term being $t^*v_1^*\ldots v_m^*$ itself, and so B is also a regular event in this case.

We add a note on the questionable proofs of both Redko and Salomaa [19]. These authors rely, as we do, on the normal form for regular expressions and the associated linear independence. The result for expressions involving only single letters in the super-normal parts of each term is proved by induction as in 4.11. They then conclude that the general case reduces to this special case.

However, we show that this reduction is not straightforward, even if the result might be proved in this fashion.

It is clear that each regular tautology in S^+ can be proved if we can prove relations of the form

$$F = \sum_{j=1}^{q} F_j \underset{\rho r}{\geq} E ,$$

for E a single normal term of the form $w_1^* \ldots w_n^* w$, and as

$$Xw \underset{\rho r}{\geq} Yw \iff X \underset{\rho r}{\geq} Y$$

for a single word w, we can even assume that E is a super-normal expression. Then by 4.9, this is equivalent to showing that

$$F \cap E \underset{\rho r}{\geq} E ,$$

and as $F \cap E$ is regular, there is a regular function f such that $F \cap E = f(w_1,\ldots,w_n)$ and the expression $f(w_1,\ldots,w_n)$ has the same number of normal terms as the expression $F \cap E$. Hence the problem is reduced to showing that

$$f(a_1,\ldots,a_n) \underset{\rho r}{\geq} a_1^*\ldots a_n^* ,$$

where the a_i, $i=1,\ldots,n$, are single letters (and we call $a_1^*\ldots a_n^*$ a universal event). As $a_1^*\ldots a_n^* \underset{\rho r}{\geq} g(a_1,\ldots,a_n)$ for any regular function g, it is clear that we are required to prove tautologies in which only one of the expressions has starred words consisting of single letters, and in fact,

Proposition: 'Every regular tautology in S may be reduced to a 'universal' tautology of the form

$$f(a_1,\ldots,a_n) \underset{\rho r}{=} a_1^*\ldots a_n^* .'$$

With this in mind, consider proving, for example,

$$(ab)^*a^*+(ab)^*b^*+(a^2b^3)^*(a^3b^2)^*a^{12}b^{16}+(a^4b^2)^*(a^2b^8)^*a^{14}b^2=a^*b^*.$$

The expressions are obviously equivalent as the first two terms in the left hand side are equivalent to a^*b^*, but

any attempt to decompose a*b* by repeated use of C10, as suggested by Redko and Salomaa, to cater for the remaining two terms, makes the problem considerably more complex, and seemingly impossible.

Chapter 5

Operators and Regular Commutative Events

In [20], Parikh has shown that the commutative image of a context-free language is a semi-linear event--that is, a regular event if considered as an event over a commutative alphabet. This result has been described as "among the most fundamental and subtly difficult to prove in the theory (of context-free languages)" ([20]-editorial footnote), and Parikh's theorem relies on induction over generation trees. However, because of the 'context-freeness', the commutative image of a context-free language generated by a grammar Γ is the same as the event generated by the 'commutative' context-free grammar Γ', obtained from Γ by allowing all letters in $V_N \cup V_T$ to commute. In this chapter, we first show that Parikh's result follows easily from a more general theorem concerning regular solutions of regular equations and then show that the latter result naturally leads to a theorem concerning the closure of a class of regular substitutions. We conclude the chapter with some conjectures about the closure of a large class of regulators for the commutative events.

<u>Commutative Regular Equations and Parikh's Theorem</u>:

It is clear that the language generated by a grammar is the minimal solution of a corresponding system of equations. This is simplest seen in terms of an example--the language generated by the grammar

$$A \to A^3Ba \qquad A \to AB^2a \qquad A \to b$$
$$B \to AB^2 \qquad B \to a$$

with $V_N = \{A,B\}$, $V_T = \{a,b\}$, and A the initial letter, is the smallest event X for which there is an event Y satisfying:

$$X = X^3Ya + XY^2a + b$$
$$Y = XY^2 + a .$$

Parikh's result then is a theorem about solutions of finite equations over a commutative alphabet. It is therefore a special consequence of the following theorem on more general regular equations in variables representing commutative events.

<u>Theorem 5.1</u>: A system of <u>regular equations</u>

$$X_0 = f_0(X_0,\ldots,X_r,X_{r+1},\ldots,X_m)$$
$$\ldots\ldots \qquad (1)$$
$$X_r = f_r(X_0,\ldots,X_r,X_{r+1},\ldots,X_m)$$

in which the f_i are regular functions of their arguments, has a <u>regular minimal solution</u> (for any given events $X_{r+1},\ldots,X_m$),

$$X_0 = g_0(X_{r+1},\ldots,X_m)$$
$$\ldots\ldots \qquad (2)$$
$$X_r = g_r(X_{r+1},\ldots,X_m)$$

(in which the g_i are regular functions) in the sense that the events $X_0,\ldots,X_r$ defined by (2) satisfy (1) and any sequence of events $Y_0,\ldots,Y_r$ satisfying (1) has $Y_i \supseteq g_i(X_{r+1},\ldots,X_m)$.

<u>Proof</u>: We first consider a single equation $X_0 = f_0(X_0,X_1,\ldots,X_m)$ Using C0-C13 and the techniques of Chapter 4, we can put this equation into a form, $X_0 = E(X_1,\ldots,X_m)+F(X_0,X_1,\ldots,X_m).X_0$ (3), where E and F are regular functions--we abbreviate this to

$$X_0 = E + F(X_0).X_0 \quad .$$

Now this implies

$$X_0 \supseteq [F(E)]^*.E = G^*E, \text{ say,}$$

where $G = F(E)$. We now show that G^*E is in fact a solution of (3), and so is the required minimal solution. If $\phi(X_0, X_1, \ldots, X_m)$ is any word in $X_0, X_1, \ldots, X_m$ which involves X_0, then using the relation $Y^*Y^* = Y^*$ and the commutative law we have

$$\phi(Y^*Z, X_1, \ldots, X_m) = Y^*.\phi(Z, X_1, \ldots, X_m).$$

Using this we derive

$$E + F(G^*E).G^*E = E + G^*.F(E).E = E + G^*.G.E = G^*E,$$

proving our assertion.

To solve systems containing more than one equation we use this as the induction step and the fact that regular functions of regular events are regular to eliminate the variables one by one.

As an example of the technique, we consider the system of two equations which we discussed earlier where $(X_0, X_1, X_2, X_3) = (Y, X, a, b)$:

$$X = X^3Ya + XY^2a + b$$
$$Y = XY^2 + a.$$

The Y-equation in form (3) is $Y = [a] + [XY]Y$ and so has the solution $Y = (Xa)^*a$. Substituting this in the X-equation, we get $X = X^3(Xa)^*a^2 + X(Xa)^*a^3 + b$, or in the form of (3),

$$X = [b] + [(Xa)^*X^2a^2 + (Xa)^*a^3]X$$

whose solution is

$$X = \{(ba)^*b^2a^2 + (ba)^*a^3\}^*b,$$

whence from $Y = (Xa)^*a$ we get

$$Y = [\{(ba)^*b^2a^2 + (ba)^*a^3\}^*ba]^*a.$$

Expressed in normal form, these solutions may be written:

$$X = b + (ba)^*(a^3)^*a^2b^3 + (ba)^*(a^3)^*a^3b$$

$$Y = a + (ba)^*(a^3)^*a^2b \ .$$

We now formalize the translation from grammars to equations.

Corollary 5.1.1: Let Γ be a context-free grammar in which V_N and V_T are considered as generating a commutative semi-group. Then L_Γ is a regular event.

Proof: To each A in V_N, we associate the formal expression $\phi_1+\ldots+\phi_s$ where $A \to \phi_1,\ldots,A \to \phi_s$ are the productions in P with A as the left hand side. We then consider the system of equations:

$$X_0 = f_0(X_0,X_1,\ldots,X_p,a_1,\ldots,a_q)$$

$$\vdots$$

$$X_p = f_p(X_0,X_1,\ldots,X_p,a_1,\ldots,a_q) ,$$

where (i) $X_0,X_1,\ldots,X_p$ are events corresponding to $A_0,A_1,\ldots,A_p$, respectively, and

(ii) $f_i(X_0,X_1,\ldots,X_p,a_1,\ldots,a_q)$ is the (trivially regular) function obtained by replacing each A_j by X_j in the formal expression associated with A_i, $i,j=0,1,\ldots,p$.

It is clear that $X_i = Im_\Gamma(A_i)$, $i=0,1,\ldots,p$, is the minimal solution of this system of equations, and from Theorem 5.1 the minimal solution consists of regular events. In particular, for the X_i corresponding to the initial letter in V_N, $X_i = L_\Gamma$ is regular as was to be shown.

Hence, for the grammar considered earlier, we have that $L_\Gamma = b + (ab)^*(a^3)^*a^2b^3 + (ab)^*(a^3)^*a^3b$, and thus the following corollary.

<u>Corollary 5.1.2</u>: (Parikh's Theorem). Let Γ be a context-free grammar generating a language, L_Γ. Then the commutative image of L_Γ is a regular (commutative) event whose normal form can be found effectively from Γ.

<u>Biregular Operators over $\mathcal{R}^+$</u>:

A context-free grammar can be interpreted as an operator of the form,

$$\Omega = \bigcap_{V_T^*} \cdot (\Delta_V \times \Sigma\begin{bmatrix}A\\w\end{bmatrix} \times \Delta_V)^*,$$

(where $V = V_N \cup V_T$ and the sum is taken over all productions $A \to w$ in P) operating on the initial symbol A_0 of the grammar. The theorem above then implies that Ω is a regulator when $V_N \cup V_T$ is considered as a commutative alphabet, and as Ω is the star of a biregular operator (followed by an intersection operator), it is natural to investigate the properties of the $\begin{bmatrix}\mathcal{R}^+\\\mathcal{R}^+\end{bmatrix}$ operators for commutative events. In this section we examine the analogous results of Chapter 2 for the biregular

operators over commutative populations, and in particular, show that the open regular substitutions are closed under star. As the starred operator above is an operator of this type, we extend the results of the previous section and motivate the conjectures of the final section of the chapter.

$\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$ operators are defined as in the non-commutative case for biregular operators; we observe immediately that these operators over an alphabet of n letters correspond to commutative regular events over an alphabet of 2n letters.

<u>Lemma 5.2</u>: For a finite alphabet, V say,

(1) There exists a normal form for $\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$ operators. In other words, every event in $\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$ is of the form,

$$\sum_{f} \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}^{\dagger} \times \cdots\cdots \times \begin{bmatrix} w_n \\ v_n \end{bmatrix}^{\dagger} \times \begin{bmatrix} w \\ v \end{bmatrix},$$

where $w_1, \ldots, w_n, w, v_1, \ldots, v_n, v$ are words in V^*.

(2) All decision problems for $\mathcal{R}^+$ correspond to decision problems for $\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$ operators, and in particular, $\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$ has soluble equivalence, emptiness, and membership decision problems.

(3) $\begin{bmatrix} \mathcal{Q}^+ \\ \mathcal{Q}^+ \end{bmatrix}$ is closed under intersection and complement, and hence forms a boolean algebra of operators.

Proof: As in the similar results for $\mathcal{Q}^+$ of Chapter 4.

Lemma 5.3: Let S^+ be a commutative standard algebra. Then for operators Ω and Ψ in $\mathcal{L}[S^+]$, and words u,v,w in S^+,

$$\begin{bmatrix} w \\ u \end{bmatrix}^{\dagger} \times \Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\dagger} \times \Psi = \begin{bmatrix} v \\ u \end{bmatrix}^{\dagger} \times \left(\begin{bmatrix} w \\ u \end{bmatrix}^{\dagger} \times \Omega . \Psi + \Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\dagger} \times \Psi \right).$$

Proof: $\begin{bmatrix} w \\ u \end{bmatrix}^{\dagger} \times \Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\dagger} \times \Psi = \sum_{i,j}^{\infty} \left(\begin{bmatrix} w \\ u \end{bmatrix}^{\times i} \times \Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\times j} \times \Psi \right)$

$$= \sum_{i,k}^{\infty} \begin{bmatrix} w \\ u \end{bmatrix}^{\times i} \times \Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\times i+k} \times \Psi + \sum_{j,k}^{\infty} \begin{bmatrix} w \\ u \end{bmatrix}^{\times j+k} \times \Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\times j} \times \Psi$$

$$= \sum_{i,k}^{\infty} \begin{bmatrix} v \\ u \end{bmatrix}^{\times i} \times \left(\Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\times k} \times \Psi \right) + \sum_{j,k}^{\infty} \begin{bmatrix} v \\ u \end{bmatrix}^{\times j} \times \left(\begin{bmatrix} w \\ u \end{bmatrix}^{\times k} \times \Omega . \Psi \right)$$

$$= \begin{bmatrix} v \\ u \end{bmatrix}^{\dagger} \times \left(\Omega . \begin{bmatrix} v \\ w \end{bmatrix}^{\dagger} \times \Psi + \begin{bmatrix} w \\ u \end{bmatrix}^{\dagger} \times \Omega . \Psi \right).$$

($\begin{bmatrix} v \\ w \end{bmatrix}^{\times 2}$ means $\begin{bmatrix} v \\ w \end{bmatrix} \times \begin{bmatrix} v \\ w \end{bmatrix}$ as distinct from $\begin{bmatrix} v \\ w \end{bmatrix} . \begin{bmatrix} v \\ w \end{bmatrix}$).

Theorem 5.4: $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix} \cdot \begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix}$.

Proof: As in the proof of 4.14 the result follows from 5.3 and induction on the 'dimension' of the operators.

Corollary 5.4.1: $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix} [\mathcal{R}^+] \subseteq \mathcal{R}^+$ and $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix} \cap \mathcal{O}[S^+] \subseteq \dot{\mathcal{R}}^+[S^+]$, that is, the $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix}$ operators are regulators.

Proof: As in the similar proof for $\begin{bmatrix} \mathcal{R} \\ \mathcal{R} \end{bmatrix}$ operators.

Corollary 5.4.2: The commutative regular events are closed under regular event differentiation, inverse regular substitution, and regular substitution.

We are now in a position to consider the star of the open regular substitutions, and we first prove some necessary lemmas, the first of which is of further interest in the final section of the chapter.

Lemma 5.5: For a commutative standard algebra S^+ and operators Ω and Ψ in $\mathcal{L}[S^+]$,

(1) $\Delta\times\Omega.\Delta\times\Psi \geq \Delta\times\Omega\times\Psi$,

(2) $(\Delta\times\Omega)^* = (\Delta\times\Omega^\dagger)^*$.

<u>Proof</u>: (1) Let $\begin{bmatrix} e' \\ f' \end{bmatrix}$ and $\begin{bmatrix} e \\ f \end{bmatrix}$ be arbitrary word-pairs in Ω and Ψ respectively. Then $\Delta\times\Omega\times\Psi$ consists of word pairs of the form $\begin{bmatrix} we'e \\ wf'f \end{bmatrix}$ and as $\begin{bmatrix} w \\ w \end{bmatrix} \times \begin{bmatrix} e' \\ f' \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix} =$ $\begin{bmatrix} wf \\ wf \end{bmatrix} \times \begin{bmatrix} e' \\ f' \end{bmatrix} . \begin{bmatrix} we' \\ we' \end{bmatrix} \times \begin{bmatrix} e \\ f \end{bmatrix}$ is in the operator product $\Delta\times\Omega.\Delta\times\Psi$, (1) follows.

(2) Trivially $\Delta\times\Omega^\dagger \geq \Delta\times\Omega$, so that $(\Delta\times\Omega^\dagger)^* \geq (\Delta\times\Omega)^*$. The first part of the lemma implies that $(\Delta\times\Omega)^* \geq \Delta\times\Omega^\dagger$.

<u>Lemma 5.6</u>: Let Ω_0 be the operator $\binom{a}{A}$, that is to say $\Delta \times \begin{bmatrix} a \\ A \end{bmatrix}$, where A is a (commutative) regular event and $a\varepsilon V$. Then $\Omega_0^*[a]$ is a regular event.

<u>Proof</u>: As in the proof of 5.1, A may be considered as an event of the form

$$f(w_1,\ldots,w_k,a).a + E$$

where (i) $w_1,..,w_k$ are words in $(V\setminus a)^*$,

(ii) E is a regular event in $(V\setminus a)^*$, and

(iii) f is a regular function, which is a non-empty function if $a\varepsilon A$.

The minimal solution for equations of the form $X = f(w_1,..,w_k,X).X+E$ is $(f(w_1,..,w_k,E))^*.E$. Then as

$$\Omega_O^*[a] \geq f(w_1,..,w_k,\Omega_O^*[a]).\Omega_O^*[a] + (a+E),$$

we have that

$$\Omega_O^*[a] = (f(w_1,..,w_k,(a+E)))^*.(a+E), = \alpha \text{ say},$$

is a regular event.

Corollary 5.6.1: Ω_O^* is an open regular substitution.

Proof: 5.5(2) implies that $(\Delta \times \begin{bmatrix} a \\ A \end{bmatrix})^* = (\Delta \times \begin{bmatrix} a \\ A \end{bmatrix}^\dagger)^*$ and as $\Delta \times \begin{bmatrix} a \\ A \end{bmatrix}^\dagger$ is an open substitution, 3.5.1 implies that its star is also. Hence $\Omega_O^* = \Delta \times \begin{bmatrix} a \\ \Omega_O^*[a] \end{bmatrix}^\dagger = (\begin{smallmatrix} a \\ \alpha \end{smallmatrix})^\dagger$ is an open regular substitution.

Note that 5.6 and its corollary also hold for the empty word in place of the letter a as $(\frac{1}{E})^*[1] = E^*$ is a regular event if E is regular.

<u>Theorem 5.7</u>: Ψ^* is an open regular substitution if Ψ is.

<u>Proof</u>: Let $\phi_1 = \binom{b}{B} + \ldots + \binom{d}{D} + \binom{1}{E}$, and

$$\phi_2 = \begin{pmatrix} b \\ \binom{a}{A} * [B] \end{pmatrix} + \ldots + \begin{pmatrix} d \\ \binom{a}{A} * [D] \end{pmatrix} + \begin{pmatrix} 1 \\ \binom{a}{A} * [E] \end{pmatrix}$$

for events A,B,...,D,E in V* and letters a,b,...,d in V. Then for $\Omega_1 = \Delta + \Omega_0 = \Delta + \binom{a}{A}$, we have

$$\Omega_1 \cdot \binom{b}{B} = \binom{b}{B} + \binom{a}{A} \times \binom{b}{B} + \begin{pmatrix} b \\ \Omega_1 [B] \end{pmatrix} \leq \begin{pmatrix} b \\ \Omega_1 [B] \end{pmatrix} \cdot \Omega_1 ,$$

and hence,

$$\Omega_1^* \cdot \binom{b}{B} = \sum_n \Omega_1^n \cdot \binom{b}{B} \leq \sum_n \begin{pmatrix} b \\ \Omega_1^n [B] \end{pmatrix} \cdot \Omega_1^n \leq \begin{pmatrix} b \\ \Omega_1^* [B] \end{pmatrix} \cdot \Omega_1^* .$$

It then follows that $\Omega_1^* \cdot \phi_1 \cdot \Omega_1^* = \phi_2 \cdot \Omega_1^*$ by noting

(i) $\Omega_1^* = (\Delta + \binom{a}{A})^* = \binom{a}{A}^*$,

(ii) $\Omega_1^* \cdot \Omega_1^* = \Omega_1^*$ for the left hand inclusion ($\leq$), and

(iii) $\Omega_1^* \cdot \binom{b}{B} \geq \begin{pmatrix} b \\ \Omega_1^* [B] \end{pmatrix}$ for the converse ($\geq$).

Now let $\phi = \binom{a}{A} + \phi_1$; then we have that

$$\phi^* = \left(\binom{a}{A}^* \cdot \phi_1 \right)^* \cdot \binom{a}{A}^* = \sum_n \left(\binom{a}{A}^* \cdot \phi_1 \right)^n \cdot \binom{a}{A}^*$$

$$= \sum_n \left(\phi_2 \cdot \binom{a}{A}^* \right)^n \cdot \binom{a}{A}^* .$$

By the first part of the proof, $\binom{a}{A}^* \cdot \phi_2 \cdot \binom{a}{A}^* = \phi_2 \cdot \binom{a}{A}^*$, and so $\phi^* = \phi_2^* \cdot \binom{a}{A}^*$. Then by induction on the number of letters in V, we may replace ϕ^* by the iterated composition

$$\binom{1}{E'}^* \cdot \binom{d}{D'}^* \cdot \ldots \cdot \binom{b}{B'}^* \cdot \binom{a}{A}^* \quad ,$$

where $B' = \binom{a}{A}^*[B], \ldots, E' = \binom{d}{D'}^* \cdot \ldots \cdot \binom{b}{B'}^* \cdot \binom{a}{A}^*[E]$.

Also $\phi^* = (\phi^{\dagger})^*$ and as $\phi^{\dagger}$ is an open substitution, ϕ^* is also. Hence, we may assume that ϕ^* is of the form

$$(*) \quad \left(\begin{bmatrix} a \\ \phi^*[a] \end{bmatrix} + \begin{bmatrix} b \\ \phi^*[b] \end{bmatrix} + \ldots + \begin{bmatrix} d \\ \phi^*[d] \end{bmatrix} + \begin{bmatrix} 1 \\ \phi^*[1] \end{bmatrix} \right)^{\dagger} .$$

The theorem now follows, for if Ψ is an increasing <u>regular</u> substitution, then Ψ is of the form $\phi^{\dagger}$, where the

events A,B,...,D,E are regular, and 5.6.1 implies that B',...,D',E' are all regular events. Hence $\Psi^*[a],..,\Psi^*[1]$ are all regular events and as Ψ^* is of the form (*) above, it is a regular substitution.

Corollary 5.7.1: Parikh's Theorem (again !).

Proof: The operator--mentioned in our initial remarks--that corresponded to the grammar of a context-free language was an open regular substitution.

Some Conjectures:

As in the non-commutative case for biregular operators, $\begin{bmatrix} a \\ a^2 \end{bmatrix}^\dagger$ is an example of an operator in $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix}$ whose star is not. However, we consider the stars of operators in a special sub-class of $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix}$.

Definition: Let $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ denote the class of operators of the form

$$\Delta_V \times \Omega$$

where Ω is an $\begin{bmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{bmatrix}$ operator over a finite alphabet V.

Analogous to the non-commutative case, we call $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operators <u>open</u>.

<u>Lemma 5.8</u>: $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ is closed under the operation . , as well as +, ×, †.

<u>Proof</u>: The proof of closure of $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ under the biregular operations is a straightforward result of the axioms for $\mathcal{R}^+$. For $\Delta\times\Psi$, $\Delta\times\Psi$ in $\binom{\mathcal{R}^+}{\mathcal{R}^+}$, 5.4 implies that $\Delta\times\Omega.\Delta\times\Psi$, $= \phi$ say, is in $\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$, and observe that $\phi \leq \Delta\times\phi$. Then for $\begin{bmatrix}e\\ f\end{bmatrix} \varepsilon\, \phi$, there is a word v in V* such that $\begin{bmatrix}v\\ f\end{bmatrix}$ is in $\Delta\times\Omega$ and $\begin{bmatrix}e\\ v\end{bmatrix}$ is in $\Delta\times\Psi$. Then for any word w,
$\begin{bmatrix}we\\ wf\end{bmatrix} \doteq \begin{bmatrix}w\\ w\end{bmatrix} \times \begin{bmatrix}v\\ f\end{bmatrix} . \begin{bmatrix}w\\ w\end{bmatrix} \times \begin{bmatrix}e\\ v\end{bmatrix}$ is in the operator product $\Delta\times\Omega.\Delta\times\Psi$. Hence $\Delta\times\phi \leq \phi$ and so $\phi = \Delta\times\phi$ is in $\binom{\mathcal{R}^+}{\mathcal{R}^+}$.

It is clear that the grammars of (commutative) context-free languages may be considered as $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operators, and in fact, a grammar for <u>any</u> finite rewriting system, that is, a finite list of rewriting rules of the form $w \to v$, is an $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operator. We now make the following conjectures.

<u>Conjecture A</u>: For $\Psi \in \binom{\mathcal{R}^+}{\mathcal{R}^+}$, Ψ^* is a regulator.

In view of 5.3.1, A is implied by:

<u>Conjecture B</u>: For $\Psi \in \binom{\mathcal{R}^+}{\mathcal{R}^+}$, $\Psi^* \in \binom{\mathcal{R}^+}{\mathcal{R}^+}$ (the <u>open mapping conjecture</u>).

It is obvious from the proof of 5.8 that $\Psi^* = \Delta \times \Psi^*$ and hence, if Ψ^* is in $\begin{bmatrix}\mathcal{R}^+\\ \mathcal{R}^+\end{bmatrix}$, then Ψ^* is an $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operator. 5.8 also implies that B is equivalent to:

<u>Conjecture B'</u>: $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ is closed under +, ., and *.

The following example shows that, in settling these conjectures, we cannot hope to use all of the structure theory for operators over a non-commutative population.

<u>Lemma 5.9</u>: For a standard algebra S,

(1) $\mathcal{R}^+[S^+] \cap \mathcal{L}[S^+]$ is not closed under the biregular operations,

(2) $\begin{bmatrix}\mathcal{S}^+\\ \mathcal{R}^+\end{bmatrix} \cap \mathcal{O}[S^+] \not\subseteq \mathcal{R}^+[S^+]$.

<u>Proof</u>: Let Ω be the $\begin{bmatrix}\mathcal{S}^+\\ \mathcal{R}^+\end{bmatrix}$ operator, $\{\begin{bmatrix}a^p\\ 1\end{bmatrix} \mid p \text{ a prime}\}$, and then for an event E, $\Omega[E] = \Omega[a^* \cap E]$. As regular events over a single letter are both $\mathcal{R}$- and $\mathcal{R}^+$-events,

the fact that $\begin{bmatrix} \mathcal{S} \\ \mathcal{R} \end{bmatrix}$ is a class of regulators implies that Ω is in $\mathcal{R}^+[S^+]$. However,

$$\Omega \times \begin{bmatrix} b \\ b \end{bmatrix}^{\dagger} [(ab)^*] = \{b^p \mid p \text{ a prime}\}$$

is a non-regular event, whence (1) and (2).

We now list our evidence in support of the conjectures.

I. Open regular substitutions are $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ operators and the regular closure of this class is a subclass of $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$.

II. The single variable case:

First, we note that for words w and v in V* (V a finite alphabet), $\Delta \times \begin{bmatrix} w \\ v \end{bmatrix}$ corresponds to the operator $v\delta_w$ of Chapter 3, but in this case we are considering commutative differentiation. When V is a single letter, a say,

$$(\Delta \times \sum_f \begin{bmatrix} a^n \\ a^m \end{bmatrix})^*$$

is an $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ operator, as it is in the regular closure of the class of (right or left) differential operators over a single letter, and the normal form for operators of this type (3.11.1) provides the result.

When we consider the general single variable problem, that is, operators of the form,

$$\Psi = \Delta \times \sum_{f} \begin{bmatrix} a^{n_1} \\ m_1 \\ a \end{bmatrix}^{\dagger} \times \ldots \times \begin{bmatrix} a^{n_p} \\ m_p \\ a \end{bmatrix}^{\dagger} \times \begin{bmatrix} a^{n} \\ \\ a^{m} \end{bmatrix},$$

we may no longer assert that this operator is a regular function over the operators a and δ_a. However

$$\Delta \times \begin{bmatrix} a^{n_1} \\ m_1 \\ a \end{bmatrix}^{\dagger} \times \ldots \times \begin{bmatrix} a^{n_p} \\ m_p \\ a \end{bmatrix}^{\dagger} \times \begin{bmatrix} a^{n} \\ \\ a^{m} \end{bmatrix}$$ does correspond to a

context-free function over the operators a and δ_a of the following form:

$$L = \{a^m(a^{m_1 q_1}(\ldots(a^{m_p q_p}\,\delta_{a^{n_p q_p}})\ldots)\delta_{a^{n_1 q_1}})\delta_{a^n} \mid q_1,\ldots,q_p \varepsilon N\}.$$

Hence Ψ^* above corresponds to a context-free function $f(a,\delta_a) = (\sum_f L)^*$ of a and δ_a. Now for the <u>operators</u> a and δ_a, we have the relation $\delta_a . a = \Delta$. It follows that Ψ^* corresponds to

$$(*)\ \bigcap_{a^*\delta_{a^*}} \cdot \begin{pmatrix} \delta_a . a \\ 1 \end{pmatrix}^* [f(a,\delta_a)]$$

in computing which we consider a and δ_a as <u>letters</u>.

In Chapter 3 we noted that operators of the form $\begin{pmatrix} \delta_a \cdot a \\ 1 \end{pmatrix}^*$ were $\left[\begin{matrix}\mathcal{S}\\ \mathcal{R}\end{matrix}\right]$ operators and hence regulators, but we could say little of their properties for preserving context-freeness. We suspect that operators of this type do preserve context-freeness but are unable to prove this at present. A positive answer to this question would settle another open problem in the theory of context-free languages which has arisen from an investigation of the Dyck languages [M. Nivat, private communication].

If $\begin{pmatrix} \delta_a \cdot a \\ 1 \end{pmatrix}^*$ is an operator with this property, then (*) would be a context-free event in $a^*\delta_{a^*}$, and by Parikh's theorem, the commutative image of this event would be a regular event of the form

$\sum_{f'} (a^{i_1} \delta_{a^{j_1}})^* \ldots (a^{i_k} \delta_{a^{j_k}})^* a^i \delta_{a^j}$ say, over the <u>letters</u> a and δ_a. It follows that

$$\Psi^* = \Delta \times \sum_{f'} \begin{bmatrix} a^{j_1} \\ a^{i_1} \end{bmatrix}^{\dagger} \times \ldots \times \begin{bmatrix} a^{j_k} \\ a^{i_k} \end{bmatrix}^{\dagger} \times \begin{bmatrix} a^{j} \\ a^{i} \end{bmatrix}$$

is an operator in $\binom{\mathcal{R}^+}{\mathcal{R}^+}$. Note that we do <u>not</u> assert that

$$\Psi^* = \sum_{f'} (a^{i_1} \delta_{a^{j_1}})^* \ldots (a^{i_k} \delta_{a^{j_k}})^* a^i \delta_{a^j}$$

holds as an operator equality.

III. The general case:

When we consider operators of the form,

$$(\Delta \times \sum_{f} \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}^{\dagger} \times \ldots \times \begin{bmatrix} w_n \\ v_n \end{bmatrix}^{\dagger} \times \begin{bmatrix} w \\ v \end{bmatrix})^* ,$$

as in II, we can interpret the operator as the star of a finite sum of linear operator events, $(\sum_f L)^*$ say, over the operator alphabet $\{a, \delta_a | \ a \varepsilon V\}$. However, if V has more than a single letter, V = {a,b} say, then a typical operator word might be δ_a.b.a, and here we do not have the commutivity of the single letter case. Thus, if we were to parallel the above argument, operating on $(\sum_f L)^*$ with

$$\begin{pmatrix} \delta_a.a + \delta_b.b \\ 1 \end{pmatrix}^*$$ would not suffice as this operator would have no effect on $\delta_a.b.a$. A satisfactory form of an operator would be

$$\Omega = \left(\Delta_V \times \begin{bmatrix} \delta_a \\ 1 \end{bmatrix} \times \Delta_V \times \begin{bmatrix} a \\ 1 \end{bmatrix} \times \Delta_V + \Delta_V \times \begin{bmatrix} \delta_b \\ 1 \end{bmatrix} \times \Delta_V \times \begin{bmatrix} b \\ 1 \end{bmatrix} \times \Delta_V \right)^*$$

but unfortunately, this operator does not preserve context-freeness as the example below illustrates.

Let $V' = \{a,b,c,\delta_a\}$ and consider the context-free event

$$\theta = (b.(\delta_a)^2)^* . \sum_{n \geq 0} (ac)^n . (ab)^n .$$

Then $\cap_{b^*c^*b^*} . \Omega[\theta] = \sum_{m \geq 0} b^m c^m b^m$, a non-context free event.

However, the damage is not irreparable as we note that in the intended interpretation θ corresponds to an $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ operator as does $\Omega[\theta]$. Hence Ω might be a suitable form of operator, if we could prove a theorem asserting that for a context-free event M and an operator of the form Ω, the commutative image of $\cap_E.\Omega[M]$ is a regular

event for regular E (although $\Omega[M]$ may not be context-free). This appears as difficult as the basic conjecture.

As an alternative to the above approach, we conclude the chapter with a possible 'zero$^{\underline{th}}$' step in an induction proof, that is, we examine $(\Delta\times\Omega)^*$ when Ω is of the form $\sum_f \begin{bmatrix} w \\ v \end{bmatrix}$.

<u>Definition</u>: For a word w in V*, let V(w) be the finite subset (of the alphabet V) $\{a \mid a \text{ is a letter in the word } w\}$.

<u>Lemma 5.10</u>: $(\Delta \times \begin{bmatrix} w \\ v \end{bmatrix})^* \in \begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$.

<u>Proof</u>: As $\Delta \times \begin{bmatrix} w \\ v \end{bmatrix} = \Delta \times \begin{bmatrix} 1 \\ v \end{bmatrix} . \Delta \times \begin{bmatrix} w \\ 1 \end{bmatrix}$,

$$(\Delta\times\begin{bmatrix} w \\ v \end{bmatrix})^* = (\Delta\times\begin{bmatrix} 1 \\ v \end{bmatrix}.\Delta\times\begin{bmatrix} w \\ 1 \end{bmatrix})^* = \Delta+\Delta\times\begin{bmatrix} 1 \\ v \end{bmatrix}.(\Delta\times\begin{bmatrix} w \\ 1 \end{bmatrix}.\Delta\times\begin{bmatrix} 1 \\ v \end{bmatrix})^*.\Delta\times\begin{bmatrix} w \\ 1 \end{bmatrix}.$$

By 5.8, the problem is then reduced to consideration of $(\Delta \times \begin{bmatrix} w \\ 1 \end{bmatrix} . \Delta \times \begin{bmatrix} 1 \\ v \end{bmatrix})^*$. Now there exist subwords w', v' of w and v respectively such that

(i) $V(w') \cap V(v') = 0$, and

(ii) $w = w'x$ and $v = v'x$ for some word x in V*.

Then for $z, u \in V^*$,

$$\begin{bmatrix} zw'x \\ z \end{bmatrix} \cdot \begin{bmatrix} u \\ uv'x \end{bmatrix} \neq 0 \iff zw'x = uv'x \iff z = yv' \text{ and}$$

$$u = yw' \text{ for some word } y \text{ in } V^*.$$

Hence $(\Delta \times \begin{bmatrix} w'x \\ 1 \end{bmatrix} \cdot \Delta \times \begin{bmatrix} 1 \\ v'x \end{bmatrix})^* = (\Delta \times \begin{bmatrix} w' \\ v' \end{bmatrix})^*$, and as

$V(w') \cap V(v') = 0$,

$$\Delta \times \begin{bmatrix} w' \\ v' \end{bmatrix} \cdot \Delta \times \begin{bmatrix} w' \\ v' \end{bmatrix} = \Delta \times \begin{bmatrix} w' \\ v' \end{bmatrix} \times \begin{bmatrix} w' \\ v' \end{bmatrix},$$

which implies that $(\Delta \times \begin{bmatrix} w' \\ v' \end{bmatrix})^* = \Delta \times \begin{bmatrix} w' \\ v' \end{bmatrix}^\dagger$, an operator in $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ as was to be shown.

<u>Theorem 5.11</u>: Let Ω be an arbitrary (operator) event over the operators $a, b, \ldots, \delta_a, \delta_b, \ldots$, and $\Psi = (\Omega + zu\delta_u)$ for words z, u. Then $\Psi^* = \Omega^* \cdot (zu\delta_u)^* \cdot \Omega^*$, and if Ω^* is an $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ operator, then 5.8 and 5.10 imply that Ψ^* is an $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ operator.

<u>Proof</u>: Let s and t be words in V*, and then for n, m, $p \in N$ such that $p \neq 0$, it is clear from elementary vector space theory that

$$t \varepsilon (zu\delta_u)^n . \Omega^m . (zu\delta_u)^p [s] \implies t \varepsilon \Omega^m . (zu\delta_u)^{n+p} [s].$$

It follows that any product of the terms Ω, $zu\delta_u$ is

equivalent to one of the form $\Omega^n.(zu\delta_u)^m.\Omega^p$ for some n,m,p N and the theorem is immediate.

Corollary 5.11.1: For words w, v, z_i, u_i, x_j, y_j, i=1,...,n, j=1,...,m, in V*.

$$(v\delta_w + z_1u_1\,\delta_{u_1} + \ldots + z_nu_n\,\delta_{u_n} + y_1\,\delta_{x_1y_1} + \ldots + y_m\,\delta_{x_my_m})^*$$

is an $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operator.

Proof: The dual result of 5.11 is that

$\partial_{\Psi^*} = \partial_{\Omega^*}.(u\delta_{zu})^*.\partial_{\Omega^*}$. The corollary then follows from 5.8, 5.10 and induction on n+m.

We conclude the chapter with the remark that this approach also doesn't appear very promising, since the following type of $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operator shows that we cannot hope for a 'formula' or 'standard expression' in decomposing the $\binom{\mathcal{R}^+}{\mathcal{R}^+}$ operators.

Let $\Omega_{n,m,p} = (a^{n-p}\,b^{m+1}\,\delta_{a^nb^m} + a^{n+1}\,b^m\,\delta_{a^{n-p}b^{m+1}})$ where n,m,pϵN such that $n \geq p$.

$\Omega^*_{n,m,p}$ is an $\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^+ \end{pmatrix}$ operator (easily proved by appealing to a geometric argument) but it is not as 'tame' as the operators of the type in 5.11.1. It can be shown that

$$a^n\, b^{m+1} = a^{n-p}\, b^{m+1}\delta_{a^n b^m} \cdot (a^{n+1}b^m \delta_{a^{n-p}b^{m+1}} \cdot a^{n-p}b^{m+1}\delta_{a^n b^m})^p [a^n b^m]$$

and that no other choice in the application of the suboperators in $\Omega_{n,m,p}$ would suffice. In other words, when looking for a 'standard' form for these operators as in 5.11, we might be forced to apply the sub-operators in a certain pattern for an arbitrary number of times, q say, before there would be a choice of which operator to apply at the $(q+1)^{\underline{th}}$ step in a derivation.

In conclusion, we summarize our evidence for the conjecture:

(1) open regular substitutions obey the conjecture;

(2) the conjecture holds for operators of the form $\sum_f \begin{pmatrix} a^n \\ a^m \end{pmatrix}$;

(3) if $\begin{pmatrix} ba \\ 1 \end{pmatrix}^*$ is an operator which preserves context-free events, then the conjecture holds for all one variable operators;

(4) operators of the form

$$(v\delta_{w} + z_1u_1\delta_{u_1} + \ldots + z_nu_n\delta_{u_n} + y_1\delta_{x_1y_1} + \ldots + y_m\delta_{x_my_m})$$

satisfy the conjecture.

However,

(5) the class of linear regulators is not closed under the biregular operations;

(6) the operator arising naturally in a generalization of (3) does <u>not</u> preserve context-freeness.

(5) and (6) show that some natural methods of proof cannot succeed.

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