Myers

A STUDY

OF

THE LIGHT CURVE OF THE VARIABLE STAR U PEGASI

BASED ON

THE OBSERVATIONS OF HARVARD COLLEGE OBSERVATORY CIRCULAR NO. 23.

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G. W. MYERS, DIRECTOR.

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A STUDY OF THE LIGHT CURVE

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THE VARIABLE STAR U PEGASI.

Professor Pickering has shown in Harvard College Observatory Circular No. 23, that U Pegasi no longer deserves the distinction of being considered the variable of shortest known period. Contrary to the usual form of contestant, in the present instance, the disputant for pre-eminence in this particular is not a newly discovered variable of shorter period than any hitherto known, but is the variable ω Centauri 19, discovered by Baily some time since and found to have the period 7^h 11^m. Manifestly, therefore, U Pegasi, whose period has until recently been regarded as lying between $3^h.0$ and $5^h.6$, has been turned down the list, not because of the excessive shortness of the period of some other star. The reason for the change lies in the fact that the inequality of brightness of the alternate minima of U Pegasi escaped detection, until Professor Pickering's discussion revealed it last winter. His observations, published in the form of a light curve and reproduced in substance in Plate I. accompanying this paper, showed the most probable period based upon all preceding observations to be about $4^h.5$; but that, in view of the failure of former observers to recognize the difference of brightness of the minima, this period should be doubled. Applying a slight correction to the double value, shown to be justified by more recent observations, he states, as the best value for the period-length of this star 8^h 59^m 41^s. The mean value of the brightness at the two approximately equal maxima is $9^m.30$; at the secondary minimum, the brightness is $9^m.75$, and at the primary it is $9^m.90$. The plate referred to gives the observations on such a scale that one division in the ordinates corresponds to 0.1 magnitude and, in the abscissas, to half an hour. The above mentioned circular states that the total number of settings here represented is 2784 and that the time of observation, including rests, is 30 hours. Each dot in the plate represents 80 settings, the dots being formed by the method of overlapping means.

The least difference of stellar brightness of whose existence the eye can be certain, being about 0.1 of a magnitude, and the difference of brightness between the primary and secondary minima, as stated in the Circular, lying so near this limit, i. e. = 0.15 of a magnitude, there would seem to be just cause for suspicion that this apparent difference has arisen from the rather large accidental errors always attaching to photometric observations. In view of the almost uniformly high degree of excellence attained in the past by Professor Pickering's forms of photometer, it cannot be denied that the results of photometric measures are on the whole to be ascribed a far higher measure of accuracy than belongs to photometric estimates. A recent personal study of β Lyrae's light variation made with one of Professor Pickering's polarization photometers removes from the writer's mind the last vestige of doubt as to the certainty of the existence of this difference of brightness at the minima. But whatever doubt may have existed for a time as to its reality, it would seem that the following statements of Professor Pickering in the Ap. J. for March of this year, ought to dispel it quite effectually. "Twelve observations, each consisting of sixteen settings, were made when the star was within twenty minutes of its primary minimum.

Deriving from each of these, by means of the light curve, the magnitude of this minimum, we obtain on Oct. 18, 1897, 9.89, 9.94, and 9.96; on Dec. 30, 9.90, 9.95, and 9.93; on Jan. 1, 1898, 9.93, 9.86, and 9.85; on Jan. 5, 9.85, and on Jan. 7, 9.86 and 9.88. Mean of all = 9.90; greatest value = 9.96; least value = 9.85 and average deviation = ± 0.035. Similarly, fourteen observations were taken within twenty minutes of the secondary minimum with the results on Oct. 18, 1897, 9.75 and 9.71; on Oct. 29, 9.74, 9.69, 9.70 and 9.70; on Dec. 28, 9.78, 9.77, 9.76 and 9.80; on Jan. 3, 1898, 9.77, 9.77, 9.74 and 9.78. Mean of all = 9.75; greatest value = 9.80; least value = 9.69, and average deviation = ± 0.029." The probable errors would, of course, be smaller than the "average deviations." Obviously, average deviations, probable errors, and the like, mean nothing at all here, or they mean that an error in the greatest value of the primary minimum large enough to make it equal to even the least value at the secondary cannot be entertained as a probability, since it would mean the commission of a systematic error nearly twice as great as the average deviation and more than twice as great as the probable error. The chances against this would be a little worse than 1 to 5.2. The internal evidence of the observations is, it would seem, quite conclusive in favor of the reality of the discrepancy. The statements just quoted show, moreover, that especial attention was directed to the point in question, and it seems therefore scarcely reasonable to suspect that, under such circumstances, an error of 0.15 of a magnitude could elude certain detection and confirmation.

Assuming the reality of this difference, the light curve appears to be susceptible of treatment by essentially the same method as that adapted and used by the writer in his recent discussion of Beta Lyrae's light curve entitled: UNTERSUCHUNGEN UEBER DEN LICHTWECHSEL DES STERNES β LYRAE, Muenchen, 1896. It is the purpose of this Bulletin to present the results and an outline of the method used in a recent study of U Pegasi, based essentially upon the observations of Pickering's Circular No. 23, and by the method more fully developed in the foregoing dissertation. The fundamental hypothesis underlying the whole discussion is that the light curve of U Pegasi is capable of being explained on the satellite theory.

ECCENTRICITY.

The uncertainty in the instants of maximum brightness as indicated by the light curve of Plate I., obviously precludes the possibility of deriving an approximate value of the orbital eccentricity of the component from the chief epochs of light variation, as was done with β Lyrae. One may readily convince himself by considerations adduced below, however, that this eccentricity must be quite small.

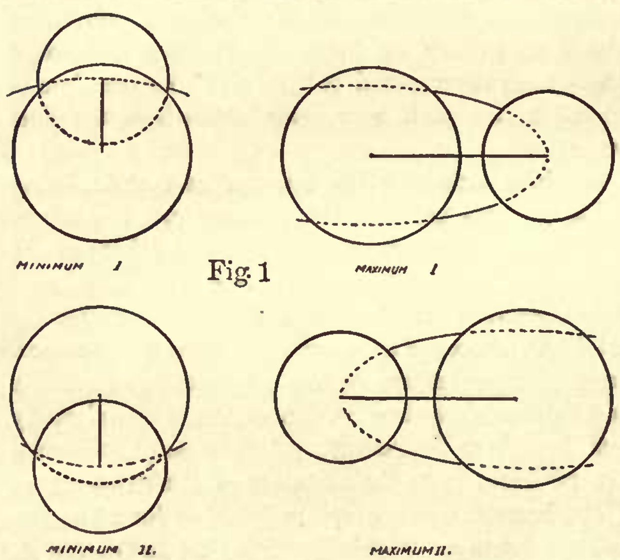


Fig. 1

Assuming the light fluctuations to be due to the mutual eclipses of two unequally bright bodies, we should have the chief epochs occurring when the relative positions of the components are as indicated in the subjoined figure. That the bodies are unequally bright, follows at once from the consideration that at Min. I. the brightness of the star is reduced by 41 per cent of its maximum brightness, and at Min. II. by only 31 per cent; unless the orbital eccentricity is assumed quite large. It will now be shown that the latter cannot be the case.

Assuming also provisionally, that both bodies are spheres, a lower limit for the eclipse-duration at Min. I. can be easily obtained from the observational curve given in Fig. 1. A little reflection will make it clear that the shorter the eclipse-duration be taken, the larger will be the corresponding distance between centres of the components. If, then, we assume that the eclipse has not begun until the light curve has fallen quite appreciably and that it has ended shortly before the curve ceases to rise, we shall obtain a value for the duration of the eclipse, at all events short enough, — perhaps too short, — and the corresponding value of the distance of centres must be at all events great enough — perhaps too great. Proceeding thus, I obtain $3^h.3$ for the interval shorter than which the eclipse-duration at Min. I. cannot be. The corresponding value of the distance between centres may then be regarded as fixing a superior limit for this orbital element.

Calling the radius of the larger component unity and of the smaller κ, the radius vector of the true orbit, r, one-half the distance between the nearest points of the positions of the companions at the beginning and end of the eclipse, x, and for this roughly approximate purpose, assuming e to be zero, we have from the figure:

$$CPC' \geqq 3.3\,\mu = 132° \qquad (\mu = 2\,\pi/P = 360°/9 = 40°)$$

Hence,

$$CPD \geqq 66° \text{ and} \qquad r \leqq (x + \kappa) \csc 66°$$
$$\leqq 1.0946\,(x + \kappa)$$

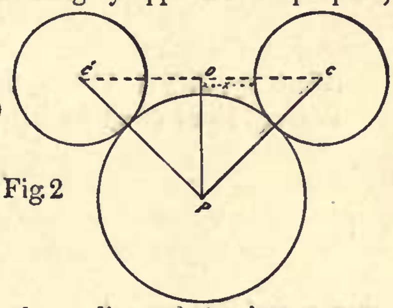


Fig 2

But since $x \leqq 1$ and $\kappa \leqq 1$, we shall have $r \leqq 2.189$ times the radius of the larger companion. So small a distance of centres relative to the dimensions of the primary, coupled with a large orbital eccentricity, would be highly improbable theoretically in any case, and assuming distinct duplicity, would be a physical impossibility on any other hypothesis than that the extent of the secondary is quite inconsiderable compared with that of the primary. The approximately equal fall of brightness at the minima, together with the similarity of form of the light curve in the neighborhood of these two chief epochs, argues strongly for the view that the form and dimensions of the companions cannot be widely different, and this latter view is still further supported by the fact that the relative brightness of the components is found later, independently of any hypothesis regarding the ratio of the radii, to be about 0.8.

It may therefore be assumed as a first approximation that $e = 0$, and we shall now proceed to determine the value of the ratio of the brightness of the companions and to fix the limits within which the ratio of the radii must be comprised. We shall then undertake to find the most probable value of this latter ratio by direct reference to the light curve of the star.

Circular Orbital Elements and Light Ratio of the Components of U Pegasi.

The chief epochs of the light curve shall be designated in order from left to right in Figure 1 as Min. I., Max. I., Min. II. and Max. II. From the curve Max. I. is seen to have a brightness of 9.32 magnitude and Max. II. of 9.34 magnitude, so that the mean value $9^m.33$ has been used throughout the discussion for the brightness at both the maxima. For the brightness at Min. I., the value 9.90 magnitude has been used and for Min. II., 9.75 magnitude. Reducing these differences in stellar magnitudes at the chief epochs of variability to their equivalent light ratios, by the aid of Pogson's scale, we obtain:

$$\frac{\text{Brightness at Min. II.}}{\text{Brightness at Min. I.}} = c = 1.1480$$

$$\frac{\text{Brightness at Mean Max.}}{\text{Brightness at Min. I.}} = m = 1.6904$$

Retaining the nomenclature of the foregoing paragraph, calling the light ratio of the components λ and the portion of the discs common to both bodies at the middle of the eclipses a, the preceding equations give the following:

$$(1)\qquad \frac{1+\kappa^2\lambda-a\kappa^2\lambda}{1-a\kappa^2+\kappa^2\lambda}=c$$

$$(2)\qquad \frac{1+\kappa^2\lambda}{1-a\kappa^2+\kappa^2\lambda}=m.$$

If it be thought desirable to include the possibility of a flattening of the discs, we may assume, as a means of making a first approximation to the general effect of such deformation, that the bodies are similar ellipsoids of revolution and designate by q, the common ratio of the semi-major to the semi-minor axis, whereupon equation (2) must be replaced by

$$(2a)\qquad q\frac{1+\kappa^2\lambda}{1-a\kappa^2+\kappa^2\lambda}=m$$

(Conf. Ap. J. Vol. VII., p. 13, where $a\kappa^2$ should be stricken from the numerator of (e).)

From (1) and (2a) we find readily

$$(3)\qquad \frac{a\kappa^2\lambda}{1+\kappa^2\lambda}=(m-cq)/m,$$

and

$$(4)\qquad \frac{a\kappa^2}{1+\kappa^2\lambda}=(m-q)/m,$$

whence, dividing, we get

$$(5)\qquad \lambda=(m-cq)/(m-q).$$

Neglecting the flattening provisionally, i.e., putting $q=1$, (5) gives, when the foregoing values of c and m are substituted,

$$\lambda=0.7865.$$

From (3) and (1), we obtain

$$(6)\qquad \frac{1}{\kappa^2}=\frac{m}{m-q}a-\frac{m-cq}{m-q},$$

and (4) gives

$$m-q=m\frac{a\kappa^2}{1+\kappa^2\lambda}.$$

Since now, $a\kappa^2$ and $1+\kappa^2\lambda$ are essentially positive, being quantities of light, this latter relation shows that m must be greater than q. Consequently,

$$\frac{d\left(\frac{1}{\kappa^2}\right)}{da}=\frac{m}{m-q},$$

is also a positive magnitude. $(1/\kappa^2)$ and a therefore, increase and decrease together, so that the maximum value of a corresponds to the maximum value of $(1/\kappa^2)$.

If now,

$$\kappa^2\leqq 1,\ \text{then}\ a\leqq 1\ \text{(from geometrical considerations)},$$

and it follows from (6) that,

$$\frac{1}{\kappa^2}\leqq\frac{cq}{m-q},\ \text{or}\ \kappa^2\geqq\frac{m-q}{cq}.$$

But if,

$$\kappa^2\geqq 1,\ \text{we may put}\ a\leqq\frac{1}{\kappa^2}\ \text{(also for geometrical reasons)},$$

We have from (6),

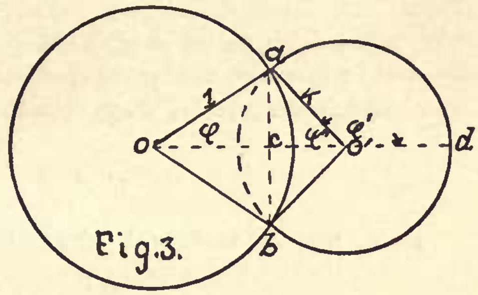
$$\frac{1}{\kappa^2} \leqq \frac{m}{m-q} \cdot \frac{1}{\kappa^2} - \frac{m-c\,q}{m-q}$$

and hence,

$$\kappa^2 \leqq \frac{q}{m-c\,q}.$$

Summing up both contingencies into a single condition, there results:

$$\text{(7)} \qquad \frac{m-q}{c\,q} \leqq \kappa^2 \leqq \frac{q}{m-c\,q}.$$



From this, we have also,

$$\frac{m-q}{c\,q} \leqq \frac{q}{m-c\,q}.$$

and finally,

$$q \geqq \frac{m}{c+1}.$$

Substituting now the former values of m and c, we obtain

$$q \geqq 0.787.$$

It does not therefore appear to be necessary to assume the existence of a flattening for U Pegasi, such as was shown to be necessary in my Dissertation on Beta Lyrae, p. 30, for the latter star.

Taking again the value of q as unity, and substituting in (7) we find:

$$0.6014 \leqq \kappa^2 \leqq 1.845, \quad \text{or} \quad 0.7755 \leqq \kappa \leqq 1.358.$$

The following test values distributed linearly over this interval were, therefore, selected for criteria to an approximation to κ:

0.80, 0.85, 1.00, 1.15 and 1.35,

and for each of these values a light curve was computed by the method and with the results given below.

Using the portion of the light curve lying within 1.5 hours before and after Min. I., and the notation (v, Fig. 3) and equations developed in my dissertation and published in the Ap. J. for Jan., 1898, I have to compute the values of M and H from the data furnished by the light curve and then for $\kappa < 1$, to solve the transcendental equations:

$$\text{(8)} \qquad \begin{cases} M = \phi + \kappa^2 \phi'' - \kappa \sin(\phi'' + \phi) \\ H = \kappa^2 \phi' - \phi + \kappa \sin(\phi' - \phi) \end{cases}$$

and for $\kappa > 1$,

$$\text{(9)} \qquad \begin{cases} M = \phi + \kappa^2 \phi'' - \kappa \sin(\phi' + \phi) \\ H = \phi_1 - \kappa^2 \phi'' + \kappa \sin(\phi_1 - \phi'') \end{cases}$$

for ϕ and ϕ'' and when $\kappa = 1$,

$$\text{(9}a\text{)} \qquad M = 2\,\phi - \kappa \sin 2\,\phi = 2\,\phi - \sin 2\,\phi \quad (H \text{ being here zero}).$$

These solutions may be made most conveniently by means of tables giving the values of M and H for suitably chosen values of ϕ and ϕ', from which approximate values of ϕ and ϕ' may be interpolated, which may then be corrected by the following differential formulæ:

$$\text{(10)} \qquad \delta\,\phi = \frac{\delta\,M}{2\,\kappa\,\mathrm{tg}\,\phi''\,\sin(\phi'' + \phi)} \quad \text{and } \delta\,\phi = \frac{\delta\,H}{2\,\kappa\,\mathrm{tg}\,\phi'\,(\sin\phi' - \phi)}, \text{ for } \kappa < 1.$$

When $\kappa > 1$, we shall have to use instead of the latter,

$$(11) \qquad \delta \phi = \frac{\delta H}{2 \kappa \operatorname{tg} \phi' \sin (\phi - \phi')}.$$

If it be desired to assume a value of q a little greater than unity, it will then be necessary to compute λ from equation (5) above. Differentiating (5) with respect to q, I obtain

$$\frac{d \lambda}{d q} = - \frac{m (1 + c)}{(m - q)^2},$$

an essentially negative magnitude.

λ and q therefore change against each other, so that an increase in q will necessitate a decrease in λ. Again designating the maximum value of κ^2 by K^2 and the minimum by k^2 we have

$$k^2 = \frac{m - q}{c q} \quad \text{and} \quad K^2 = \frac{q}{m - c q}.$$

Differentiating these with respect to q, we find,

$$\frac{dk^2}{d q} = - \frac{m}{c q^2} \quad \text{and} \quad \frac{d K^2}{d q} = + \frac{m}{(m - c q)^2}.$$

The former of these differential coefficients is essentially negative, and the latter is essentially positive. An augmentation of q will therefore depress the minor and elevate the major limit of κ^2; and to be able to include a value of q somewhat larger than unity, values of M and H were also computed for $\kappa = 0.70$. The table of computed M's and H's is given here.

AUXILIARY TABLES FOR INTERPOLATING APPROXIMATE VALUES OF ϕ AND ϕ''.

ϕ for $\kappa < 1$ or ϕ'' for $\kappa > 1$	$\kappa = 0.70$		$\kappa = 0.80$		$\kappa = 0.85$		$\kappa = 1.15$		$\kappa = 1.35$	
	M	H	M	H	M	H	M	H	M	H
° ′										
0 00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2 00	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000
4 00	0.0005	0.0001	0.0006	0.0001	0.0005	0.0003	0.0007	0.0001	0.0009	0.0001
6 00	0.0018	0.0007	0.0017	0.0005	0.0017	0.0003	0.0022	0.0003	0.0031	0.0010
8 00	0.0043	0.0009	0.0040	0.0006	0.0038	0.0005	0.0051	0.0007	0.0077	0.0014
10 00	0.0086	0.0013	0.0080	0.0009	0.0077	0.0008	0.0100	0.0008	0.0152	0.0021
12 00	0.0148	0.0029	0.0137	0.0017	0.0133	0.0014	0.0171	0.0013	0.0261	0.0041
14 00	0.0319	0.0044	0.0216	0.0025	0.0210	0.0018	0.0274	0.0019	0.0414	0.0068
16 00	0.0353	0.0065	0.0325	0.0037	0.0314	0.0026	0.0405	0.0026	0.0618	0.0090
18 00	0.0503	0.0093	0.0461	0.0054	0.0444	0.0038	0.0577	0.0038	0.0880	0.0148
20 00	0.0690	0.0136	0.0631	0.0076	0.0610	0.0056	0.0792	0.0062	0.1207	0.0201
22 00	0.0933	0.0202	0.0838	0.0104	0.0805	0.0072	0.1050	0.0082	0.1607	0.0274
24 00	0.1195	0.0247	0.1087	0.0142	0.1043	0.0096	0.1366	0.0107	0.2086	0.0360
26 00	0.1522	0.0328	0.1378	0.0184	0.1320	0.0126	0.1735	0.0129	0.2648	0.0473
28 00	0.1908	0.0422	0.1715	0.0231	0.1644	0.0161	0.2137	0.0175	0.3112	0.0613
30 00	0.2344	0.0542	0.2104	0.0293	0.2014	0.0280	0.2620	0.0228	0.4079	0.0779
32 00	0.2876	0.0693	0.2549	0.0367	0.2433	0.0251	0.3163	0.0281	0.4959	0.0983
34 00	0.3307	0.0710	0.3053	0.0458	0.2906	0.0309	0.4022	0.0343	0.5969	0.1240
36 00	0.4176	0.1120	0.3621	0.0555	0.3436	0.0381	0.4457	0.0425	0.7116	0.1548
38 00	0.4998	0.1438	0.4055	0.0696	0.4024	0.0465	0.5222	0.0518	0.8447	0.1972
40 00	0.5977	0.1863	0.4968	0.0853	0.4679	0.0569	0.6059	0.0622	0.9942	0.2446

ϕ for $\kappa<1$ or ϕ'' for $\kappa>1$	$\kappa=0.70$		$\kappa=0.80$		$\kappa=0.85$		$\kappa=1.15$		$\kappa=1.35$	
	M	H	M	H	M	H	M	H	M	H
° ′										
42 00	0.7218	0.2503	0.5772	0.1047	0.5397	0.0681	0.6984	0.0752	1.1765	0.3103
43 00	0.8028	0.2994	—	—	—	—	—	—	1.2694	0.3591
43 30	0.8460	0.3328	—	—	—	—	—	—	—	—
44 00	0.9281	0.3818	0.6658	0.1296	0.6162	0.0858	0.7996	0.0908	1.3790	0.4010
44 20	0.9797	0.4388	—	—	—	—	—	—	—	—
44 26	1.0297	0.4783	—	—	—	—	—	—	—	—
45 00	—	—	—	—	—	—	—	—	1.5036	0.4638
46 00	—	—	0.7666	0.1600	0.7066	0.1002	0.9103	0.1087	1.6499	0.5445
47 00	—	—	0.8229	0.1731	—	—	—	—	1.8417	0.6700
47 30	—	—	—	—	—	—	—	—	1.9983	0.7763
47 48	—	—	—	—	—	—	—	—	2.1843	0.9573
48 00	—	—	0.8810	0.1999	0.8028	0.1217	1.0308	0.1303	—	—
49 00	—	—	0.9488	0.2288	—	—	—	—	—	—
50 00	—	—	1.0215	0.2609	0.9094	0.1490	1.1641	0.1578	—	—
51 00	—	—	1.1055	0.3032	—	—	—	—	—	—
51 30	—	—	1.1534	0.3301	—	—	—	—	—	—
52 00	—	—	1.2082	0.3632	1.0290	0.1842	1.3129	0.1929	—	—
52 30	—	—	1.2726	0.4083	—	—	—	—	—	—
53 00	—	—	1.3736	0.4847	—	—	—	—	—	—
53 8.7	—	—	1.4531	0.5575	—	—	—	—	—	—
54 00	—	—	—	—	1.1675	0.2480	1.4749	0.2376	—	—
56 00	—	—	—	—	1.3320	0.3044	1.6574	0.2991	—	—
57 00	—	—	—	—	1.4389	0.3728	—	—	—	—
57 30	—	—	—	—	1.5066	0.4060	—	—	—	—
58 00	—	—	—	—	1.5992	0.4734	1.8764	0.3874	—	—
58 12.7	—	—	—	—	1.7040	0.5666	—	—	—	—
58 30	—	—	—	—	—	—	1.9414	0.4192	—	—
59 00	—	—	—	—	—	—	2.0187	0.4567	—	—
59 30	—	—	—	—	—	—	2.0970	0.5050	—	—
60 00	—	—	—	—	—	—	2.2032	0.5786	—	—
60 25	—	—	—	—	—	—	2.3976	0.7439	—	—

The table gives the values of M and H, of course, only up to $\phi = \sin^{-1}\kappa$, or to $\phi' = \sin^{-1}\frac{1}{\kappa}$, according as $\kappa \lesseqgtr 1$.

The expression for M when $\kappa = 1$ is so simple as to render the use of an auxiliary table unnecessary, and this case has therefore not been included in the foregoing lists.

M and H are connected with the observations by means of the relations:

$$(12) \qquad M = \pi\,(1 + \kappa^2\lambda)\,(1 - J) \quad \text{and} \quad H = \pi\kappa^2 - \pi\,(1 + \kappa^2\lambda)\,(1 - J),$$

where J is obtained from the light curve by subtracting the ordinate of the curve for any given instant from the mean ordinate for the maxima, calling this difference ΔG and substituting in the equation:

$$\log J = 0.04\,\Delta G \qquad (\Delta G \text{ being in tenths of a magnitude}).$$

The values of M and H, on the various hypotheses for κ and for the times preceding and following Min. I. given in the first column, are tabulated here.

VALUES OF M AND H COMPUTED FROM THE LIGHT CURVE FOR THE EPOCHS t.

t	$\kappa = 0.80$		$\kappa = 0.85$		$\kappa = 1.00$		$\kappa = 1.15$		$\kappa = 1.35$	
	M	H	M	H	M	H	M	H	M	H
h.										
−1.50	0.1624	1.8482	0.1694	2.1004	0.1930	2.9486	0.2203	3.9345	0.2628	5.4628
−1.25	0.3910	1.6196	0.4077	1.8621	0.4625	2.6771	0.5303	3.6245	0.6325	5.0931
−1.00	0.6903	1.3203	0.7200	1.5498	0.8201	2.3215	0.9364	3.2184	1.1168	4.6088
−0.75	1.0671	0.9435	1.1129	0.1569	1.2677	1.8739	1.4476	2.7072	1.7264	3.9992
−0.50	1.4496	0.5610	1.5118	0.7580	1.7221	1.4195	1.9664	2.1884	2.3452	3.3804
−0.25	1.8458	0.1648	1.9250	0.3448	2.1927	0.9489	2.5038	2.6510	2.9861	2.7395
−0.12½	1.9067	0.1039	1.9885	0.2813	2.2650	0.8766	2.5864	1.5684	3.0847	2.6409
0.00	1.9326	0.0780	2.0156	0.2542	2.2959	0.8457	2.6216	1.5332	3.1267	2.5989
+0.12½	1.8986	0.1120	1.9801	0.2897	2.2555	0.8861	2.5755	1.5793	3.0727	2.6529
+0.25	1.7192	0.2914	1.7930	0.4768	2.0423	1.0993	2.3321	1.8227	2.7814	2.9442
+0.50	1.2380	0.7726	1.2912	0.9786	1.4708	1.6708	1.6794	2.4754	2.0030	3.7226
+0.75	0.8358	1.1758	0.8716	1.3982	0.9928	2.1488	1.1337	3.0211	1.3521	4.3735
+1.00	0.5383	1.4623	0.5614	1.7084	0.6395	2.5021	0.7302	3.4246	0.8708	4.8548
+1.25	0.2748	1.7258	0.2866	1.9832	0.3265	2.8151	0.3728	3.7820	0.4446	5.2810
+1.50	−0.0642	1.9464	0.0670	2.2028	0.0763	3.0653	0.0871	4.0677	0.1039	5.6217

The distance of centres, r, is seen from the accompanying figure to be given by

(13) $$\rho = \frac{\kappa \sin(\phi'' \pm \phi)}{\sin \phi}, \text{ where } \sin\phi = \kappa \sin\phi'' \text{ and } \phi' + \phi'' = 180^\circ.$$

The figure relates only to the case in which $\kappa < 1$ and $\phi'' < 90^\circ$, but the modifications necessary to adapt it to the cases where $\kappa > 1$ and $\phi'' \gtrless 90^\circ$, are so obvious, that they may be left to the reader.

Assuming now a circular orbit, and denoting by α and $\beta + \frac{\pi}{2}$, the longitude in the apparent orbit and the true anomaly in the real orbit respectively, both counted from the node, and calling r and ρ the radii vectores in the true and apparent orbits, we may write,

$$\rho \cos\alpha = r \sin\beta \text{ and } \operatorname{tg}\alpha = \cos i \cot\beta,$$

whence,

(14) $$\rho^2 = r^2 \sin^2\beta + r^2 \cos^2 i \cos^2\beta.$$

Calling, for brevity,

(14 a) $$x = r^2 \text{ and } y = r^2 \cos^2 i = x \cos^2 i,$$

and we then have the following simple relation between the various magnitudes;

(15) $$\rho^2 = x \sin^2\beta + y \cos^2\beta. \quad (\beta = \mu\, t = 40^\circ\, t, \; t \text{ in hours from Min. I.}),$$

which holds for all cases except when the smaller disc is projected wholly upon the larger at the epoch of Min. I.

The solution of equations (8), (9) and (9a) for the five hypothetical values of κ gave the results here tabulated.

TABULATED VALUES OF ϕ AND ρ.

t	$\kappa = 0.80$		$\kappa = 0.85$		$\kappa = 1.00$		$\kappa = 1.15$		$\kappa = 1.35$	
	ϕ	ρ	ϕ	ρ	ϕ	ρ	ϕ''	ρ	ϕ''	ρ
	° ′		° ′		° ′		° ′		° ′	
−1.50	27 30.3	1.5403	28 18.5	1.5440	30 40.0	1.7202	28 17.0	1.8510	25 56.0	2.0210
−1.25	36 56.0	1.3276	38 10.4	1.3697	41 46.5	1.4916	38 12.5	1.6066	34 39.0	1.7515
−1.00	44 30.5	1.0987	46 17.5	1.1381	51 25.5	1.2472	46 26.6	1.3449	41 15.0	1.4710
−0.75	50 33.0	0.8440	53 16.0	0.8813	60 45.0	0.9774	53 43.0	1.0557	46 26.2	1.1379
−0.50	53 8.0	0.6012	57 32.1	0.6397	68 43.0	0.7260	58 40.3	0.7848	36 18.0	0.5558
−0.25	46 12.6	0.3482	56 44.0	0.3951	76 8.5	0.4790	45 51.0	0.2360	47 35.0	0.8292
−0.12	41 52.0	0.3038	55 28.6	0.3577	77 14.0	0.4420	51 47.3	0.2833	47 46.2	0.8805
0.00	39 8.0	0.2841	54 44.0	0.3411	77 42.0	0.4260	53 29.0	0.3024	47 47.2	0.8897
+0.12	42 40.0	0.3101	55 40.0	0.3626	77 5.5	0.4474	51 10.3	0.2771	47 45.4	0.8726
+0.25	50 45.2	0.4321	58 1.4	0.4745	73 50.0	0.5568	60 21.8	0.6006	46 28.8	0.7254
+0.50	52 13.0	0.7342	55 33.3	0.7718	64 26.3	0.8630	56 14.0	0.9326	47 33.3	0.9982
+0.75	47 13.6	0.9969	49 19.0	1.0356	55 16.1	1.1394	49 33.3	1.2298	43 45.5	1.3331
+1.00	41 4.3	1.1781	42 34.0	1.2512	46 54.4	1.3664	42 39.0	1.4729	38 23.0	1.6037
+1.25	32 49.0	1.4288	30 50.2	1.4729	36 51.0	1.6004	33 50.0	1.7226	30 52.0	1.8803
+1.50	. . . .	. . . .	20 40.0	1.7093	22 18.1	1.8518	20 40.0	1.9909	19 1.0	2.2309

A comparison of the values of ρ on the last two hypotheses for κ, shows at once that these values of κ need not be further considered, since the values of ρ in both cases fall for a time, reach a minimum before Min. I., rise to a maximum value about the time $t = 0$, fall to a second minimum value, and then rise continuously; and since ρ denotes the radius vector of the apparent orbit, which latter must be an ellipse, obviously such a variation of it must be impossible. The value of κ, i. e., the radius of the darker body cannot, therefore, have either of these latter values.

Substituting the values of ρ for the first three assumptions for κ in equation (15) above, we shall have the following 15 observation equations:

OBSERVATION EQUATIONS.

			$\kappa = 0.80$	$\kappa = 0.85$	$\kappa = 1.00$	$r = \sqrt{x}$ for $\kappa = 0.80$,	$\kappa = 0.85$,	$\kappa = 1.00$
(1)	$0.7500\,x$	$+0.2500\,y$	=2.3725;	=2.3839;	=2.9588	1.4915	1.7764	2.0207
(2)	0.5868	+0.4132	=1.7625;	=1.8761;	=2.2249	1.7561	1.8052	1.9543
(3)	0.4132	+0.5868	=1.2071;	=1.2953;	=1.5555	1.7281	1.7776	1.9270
(4)	0.2500	+0.7500	=0.7123;	=0.7767;	=0.9553	1.6411	1.6878	1.8358
(5)	0.1170	+0.8830	=0.3614;	=0.4092;	=0.5271	1.6746	1.7274	1.8767
(6)	0.0302	+0.9698	=0.1213;	=0.1561;	=0.2294	1.1693	1.1693	1.2300
(7)	0.0076	+0.9924	=0.0923;	=0.1279;	=0.1954	1.2670	1.2825	1.4142
(8)	0.0000	+1.0000	=0.0807;	=0.1163;	=0.1815	1.4545	1.4554	1.6264
(9)	0.0076	+0.9924	=0.0962;	=0.1315;	=0.2002	2.0176	2.0610	2.2410
(10)	0.0302	+0.9698	=0.1867;	=0.2252;	=0.3100	2.0303	1.0893	2.2417
(11)	0.1170	+0.8830	=0.5390;	=0.5957;	=0.7448	1.8277	1.9393	2.1018
(12)	0.2500	+0.7500	=0.9938;	=1.0727;	=1.2982	1.6723	1.8275	1.9780
(13)	0.4132	+0.5868	=1.3879;	=1.5655;	=1.8670	1.8966	1.8992	2.0524
(14)	0.5868	+0.4132	=2.0415;	=2.1694;	=2.5613	—	2.0184	2.1772
(15)	0.7500	+0.2500	= —	=2.9217;	=3.4292			

Each pair of these equations furnishes a value for both x and y, and from the results of their solution the values of r and $\cos^2 i$ may be obtained with the help of (14a). The assumption of a circular form of the orbit requires that the different values of r and of $\cos^2 i$, on the correct hypothesis for κ, shall all be approximately equal. The values for r obtained by solving (1) and (2), (2) and (3), (3) and (4), etc., in succession for the various values of κ are tabulated in the last three columns of the foregoing table. The mean values and probable errors for each of the assumptions for κ are: for $\kappa = 0.80$, $r = 1.6636 \pm 0.0485$; for $\kappa = 0.85$, $r = 1.7512 \pm 0.0494$, and for $\kappa = 1.00$, $r = 1.9341 \pm 0.0535$. The individual determinations of $\cos^2 i$ are not given here, but the corresponding means and probable errors are, for the respective cases:

$$\cos^2 i = +0.0275 \pm 0.0069; \ = +0.0482 \pm 0.0072; \ = +0.0547 \pm 0.0074.$$

The difference of the probable errors is not great in any case, but both r and $\cos^2 i$ agree in their testimony favoring the smallest value of κ as being the most probable. Assuming this value of κ however, a physical peculiarity, though not an impossibility, is met in the circumstance that the most probable distance of centres (1.6634) is considerably less than the sum of the radii (=1.8), i. e., the masses must interpenetrate, and consequently form a single body (Poincaré's apiod).

The probable errors not differing by enough to enable them to pronounce with sufficient emphasis for any one of the hypotheses, it seemed desirable to approach the problem also indirectly to see whether the conclusions will be the same as those given by this direct solution. That the foregoing discussion, however, indicates conclusively that the correct value of κ is smaller than 0.85, there can be no doubt.

Indirect Solution.

The mode of procedure here is to read from the light curve for suitably chosen epochs, the instantaneous brightnesses in stellar magnitudes, to form the differences between these brightnesses and the maximum brightness, to convert these differences, by means of the Pogson scale, into their equivalent light ratios, to compare these ratios with the corresponding ratios, computed from certain assumed elements, and finally, after finding sufficiently close approximations to the correct values of the elements, to adjust these differences in the sense computation minus observation, by the method of Least Squares.

Letting J' and J'' denote the instantaneous brightnesses in the neighborhood of Min. I. and Min. II. respectively, and M', H', M'', and H'', the corresponding values of the M and H defined by equations (12), it will be seen by referring to my article on Beta Lyrae, in the January Astrophysical Journal, that

$$(16) \qquad 1-J' = \frac{M'}{\pi\,(1+\lambda\,\kappa^2)}; \qquad 1-J'' = \frac{\lambda\,M''}{\pi\,(1+\lambda\,\kappa^2)};$$

and hence, there is an obvious advantage in adjusting $1-J'$ and $1-J''$ instead of J' and J''. The former quantities were therefore used throughout the reductions.

The equations for computing M', or M'' are:

$$(17) \quad \begin{cases} (a)\ \beta = 40^\circ t. \\ (b)\ \rho = r\sqrt{\sin^2\beta + \cos^2 i \cos^2\beta}. \ \text{ If } i = \frac{\pi}{2} - i' \text{ is near } \frac{\pi}{2},\ i' \text{ is small and} \\ (c)\ \rho = r\sqrt{\sin^2\beta + i'^2\cos^2\beta}. \ \text{ If } i' = 0,\ \rho = r\sin\beta. \\ (d)\ \cos\phi = \dfrac{1+\rho^2-\kappa^2}{2\rho}. \\ (e)\ \sin\phi' = \dfrac{1}{\kappa}\sin\phi. \\ (f)\ M', \text{ or } M'' = \phi + \kappa^2\phi'' - \kappa\sin(\phi+\phi'') = \phi+\kappa^2\phi'' - \rho\sin\phi. \end{cases}$$

These, together with (16), determine $1 - J'$ and $1 - J''$ from the light curve. The value of $\cos^2 i$ as found above, was small, and as a first approximation i was taken $\frac{\pi}{2}$, or $i' = 90 - i = 0$.

To neglect the effect of orbital eccentricity requires Min. II. to fall at the middle point of the period. Disregarding provisionally the slight displacement of this chief epoch from the middle point, taking ordinates equidistant from Min. I. and Min. II. before and after these epochs, forming the means for each epoch separately and computing the corresponding values of $1 - J'$ and $1 - J''$, the results here tabulated were obtained.

t	Minimum I.			Minimum II.			$1-J'$.	$1-J''$.
	Before.	After.	Mean.	Before.	After.	Mean.		
1.50	9.32	9.34	9.33	9.36	9.35	9.35	0.0228	0.0000
1.25	9.37	9.37	9.37	9.41	9.40	9.40	0.0742	0.0362
1.00	9.42	9.44	9.43	9.50	9.47	9.48	0.1330	0.0896
0.75	9.51	9.53	9.52	9.61	9.55	9.58	0.2072	0.1606
0.50	9.62	9.64	9.63	9.73	9.67	9.70	0.2901	0.2380
0.25	9.72	9.72	9.72	9.84	9.84	9.84	0.3749	0.3018
0.00	9.75	9.75	9.75	9.90	9.90	9.90	0.4084	0.3208

The results of this table are shown graphically on Plate II.

The values of M' and M'' for various assumptions for κ both greater and less than the minimum value given above (0.7755) were computed from formula (16). For values of κ less than 0.7755, it was of course necessary to assume a flattening of the discs and to compute λ from the formula $\lambda = \dfrac{m - c\,q}{m - q}$, ($q$ depending on the flattening). After having computed a number of light curves for various assumptions for r and κ, it became evident that the discs must lie extremely close together. The attempt was then made to ascertain what value of κ would best satisfy the observations on the hypothesis of contact of discs. The residuals $M_o - M_c = \Delta M$ were formed for the values of t in the above table and compared, that hypothesis furnishing the smallest mean residual, ΔM, being assumed to lie nearest the truth. M_o was computed from the observed values of J' and J'' by the aid of the formulas,

$$M' = (1 + \kappa^2 \lambda)\,\pi\,(1 - J')\,; \qquad M'' = (1 + \kappa^2 \lambda)\,\frac{\pi}{\lambda}\,(1 - J'')\,; \qquad M_o = 1/2\,(M' + M'')$$

and M_c by the aid of

$$M_c = \phi + \kappa^2 \phi'' - \rho' \sin \phi, \qquad \rho' = \frac{1}{f}\rho \text{ and } \frac{1}{f} = \sqrt{\sin^2 \beta + q^2 \cos^2 \beta}.$$

The value $\kappa = 0.8$, involving a somewhat simpler hypothesis than $\kappa = 0.75$, viz: $q = 1$, was taken as a basis for further experimentation, notwithstanding the fact that the latter value of κ gave a somewhat smaller mean residual. If, moreover, as seems now probable, a part of the light change be ascribed to a flattening, the larger value of κ will probably be nearer the truth. The distance of centres, r, was then assumed to be $1 + 1.1\,\kappa$ and $1 + 0.9\,\kappa$ in turn, and the mean residuals computed on these hypotheses. The results of these six hypotheses are here tabulated. Unless otherwise explicitly stated, it is to be understood that $r = 1 + \kappa$, that $q = 1$, and that the components are similar ellipsoids of revolution. For the case where $\kappa = 0.75$, the smallest possible value of q ($= 1.0271$) was used in obtaining M_o.

VALUES OF M_o, M_c AND ΔM ON VARIOUS HYPOTHESES FOR κ.

M_o	M_c	ΔM	M_o	M_c	ΔM
$\kappa = 0.75$, $q = 1.0271$			$\kappa = 0.8$		
0.0514	0.1372	−0.0858	0.0538	0.1456	−0.0918
0.2728	0.2926	−0.0248	0.2839	0.3303	−0.0464
0.5612	0.5720	−0.0108	0.5732	0.6097	−0.0265
0.9358	0.9189	+0.0169	0.9746	0.9810	−0.0064
1.3486	1.3300	+0.0186	1.3998	1.4274	−0.0276
1.7259	1.7299	−0.0040	1.7915	1.8949	−0.1034
Mean residual		0.0302	Mean residual		0.0503
$\kappa = 0.9$			$\kappa = 0.95$		
0.0586	0.1634	−0.1048	0.0612	0.1690	−0.1078
0.3092	0.3703	−0.0611	0.3229	0.3911	−0.0682
0.6349	0.6850	−0.0501	0.6632	0.7228	−0.0596
1.0579	1.1051	−0.0472	1.1049	1.0957	−0.0092
1.5242	1.6394	−0.1152	1.5919	1.7098	−0.1179
1.9509	2.1904	−0.2395	2.0375	2.3206	−0.2831
Mean residual		0.1030	Mean residual		0.1076
$\kappa = 0.8$, $r = 1 + 0.9\,\kappa$			$\kappa = 0.8$, $r = 1 + 1.1\,\kappa$		
0.0538	0.2116	−0.1578	0.0538	0.0882	−0.0344
0.2839	0.4029	−0.1190	0.2839	0.2672	+0.0167
0.5732	0.6806	−0.1074	0.5732	0.5179	+0.0553
0.9746	1.0313	−0.0567	0.9746	0.8897	+0.0849
1.3998	1.6714	−0.2716	1.3998	1.3835	+0.0163
1.7915	1.9134	−0.1219	1.7915	1.8760	−0.0845
Mean residual		0.1391	Mean residual		0.0487

The low value of the mean residual of the first hypothesis, viz.: $\kappa = 0.75$ and $q = 1.0271$, renders a further investigation into the general effect of a flattening desirable. The light curves for $q = 1.01$, 1.02, 1.03, 1.04 and 1.05 were now computed and the residuals, ΔM_{o-c}, formed with the proper values of λ, from $\lambda = (m - c\,q)/(m - q)$, with $\kappa = 0.7785$ and $r = 1.7814$.

VALUES OF ΔM_{o-c} FOR VARIOUS VALUES OF q.

$q = 1.01$	$q = 1.02$	$q = 1.03$	$q = 1.04$	$q = 1.05$
−0.0835	−0.0607	−0.0774	−0.0737	−0.0705
−0.0182	−0.0274	−0.0086	−0.0017	+0.0046
−0.0031	+0.0018	+0.0295	+0.0377	+0.0483
+0.0229	+0.0244	+0.0557	+0.0676	+0.0788
+0.0149	+0.0113	+0.0389	+0.0493	+0.0609
−0.0409	−0.0533	−0.0391	−0.0330	−0.0262
Mean Resid. 0.0306	0.0298	0.0415	0.0438	0.0482

These latter values of κ and r resulted from a Least Square solution of a set of observation equations connecting $d\kappa$, dr, and di' $(=-di)$, which gave $di' = \sqrt{-0.0096}$. This value of di' being imaginary but small, it was put $= 0$. The residuals for the five hypotheses are collected in the table last preceding.

For the case in which $q = 1.00$, the third column for $\kappa = 0.8$ above may be examined.

Both the run of the individual values of ΔM_{o-c} and the magnitude of the mean residual indicate $q = 1.02$ to be most approximate.

Differential equations were now derived in such form as to connect dk, dr, dq, and di'^2, with $dM = \Delta M_{o-c}$. The derivation of these relations was made as follows:

Differentiating $M = \phi + \kappa^2\ \phi'' - \rho' \sin\phi$, where $\rho' = (1/f)\ \rho$ and $(1/f) = \sqrt{\sin^2\beta + q^2\cos^2\beta}$, and reducing by means of $\cos\phi'' = \dfrac{\rho'^2+\kappa^2-1}{2\rho'\kappa}$, $\cos\phi = \dfrac{1+\rho'^2-\kappa^2}{2\rho'}$ and $\cos(\phi+\phi'') = \dfrac{\rho'^2-1-\kappa^2}{2\kappa}$, we find,

$$dM = (1 - \rho'\cos\phi)\ d\phi + \kappa^2\ d\phi'' + 2\ \kappa\phi''\ d\kappa - \sin\phi\ .\ d\rho',$$

$$d\phi = \frac{\kappa\, d\kappa}{\rho'\sin\phi} - \frac{\kappa\cos\phi''}{\rho'\sin\phi}\, d\rho' \quad \text{and} \quad d\phi'' = \left(\frac{\cos\phi''}{\rho'\tan\phi} - \frac{\sin\phi''}{\rho'}\right) d\kappa - \frac{\cos\phi}{\rho'\sin\phi}\, d\rho'$$

which give, after some simplifications,

$$dM = 2\ \phi''\ \kappa\, d\kappa - 2\sin\phi\, d\rho'.$$

But $d\rho' = \dfrac{1}{f}\, d\rho + \rho\, d\left(\dfrac{1}{f}\right)$, where $\rho = r\sqrt{\sin^2\beta + i'^2\cos^2\beta}$ for small values of $i' = \dfrac{\pi}{2} - i$.

Differentiating and substituting we obtain finally,

$$\text{(A)} \qquad dM = 2\kappa\phi''\ d\kappa - R\,dr - Q\,dq - I\,di'^2$$

in which

$$R = 2\,\frac{\rho\sin\phi}{rf}; \qquad Q = 2\,q\,\rho\,f\cos^2\beta\sin\phi; \qquad \text{and } I = \frac{r}{f}\sin\phi\cos^2\beta\csc\beta.$$

Computing the coefficients $2\ \kappa\ \phi''$, R, Q, and I with the values $q = 1.02$, $\lambda = 0.7748$, $\kappa = 0.7785$, $r = 1.7816$, and $i' = o$, for the epochs used in the foregoing tables, the following six observation equations were obtained: —

$0.9154\,dk$	$-0.7517\,dr$	$-0.3403\,dq$	$-0.2232\,di'^2$	-0.0607	$= 0$
1.2387	−0.8591	−0.6383	−0.5389	−0.0274	= 0
1.5827	−0.8616	−0.9019	−1.0897	+0.0018	= 0
1.9702	−0.7537	−1.0031	−2.0139	+0.0244	= 0
2.4611	−0.5420	−0.8119	−3.6442	+0.0113	= 0
3.4477	−0.2205	−0.3762	−6.3170	−0.0533	= 0

These were rendered homogeneous by putting

$$dk = 0.2901\,x;\ dr = 1.1606\,y;\ dq = 0.9969\,z;\ di'^2 = 0.1583\,w, \text{ and } \nu = 0.0607.$$

The equations then were: —

$0.2655\,x$	$-0.8724\,y$	$-0.3392\,z$	$-0.0353\,w$	$-1.0000\,\nu$	$= 0$
0.3593	−0.9971	−0.6363	−0.0855	−0.4514	= 0
0.4591	−1.0000	−0.8991	−0.1725	+0.0297	= 0
0.5715	−0.8747	−1.0000	−0.3188	+0.4020	= 0
0.7138	−0.6290	−0.8094	−0.5769	+0.1862	= 0
1.0000	−0.2529	−0.3750	−1.0000	−0.8781	= 0

These resulted in the following Normal Equations:

$+2.2467\,x$	$-2.2508\,y$	$-2.2557\,z$	$-1.7133\,w$	$-0.9296\,\nu$	$=0$
-2.2508	$+3.9800$	$+3.3081$	$+1.1833$	$+1.0462$	$=0$
-2.2557	$+3.3081$	$+3.1241$	$+1.3822$	$+0.3763$	$=0$
-1.7133	$+1.1833$	$+1.3822$	$+1.4727$	$+0.7113$	$=0$

The solution of these normals gave

$$x = -0.302;\ y = +0.056;\ z = -0.183;\ \text{and } w = -0.194$$

and hence,

$$d\kappa = -0.087;\ dr = +0.064;\ dq = -0.183;\ \text{and } d i'^2 = i'^2 = -0.031.$$

Inasmuch as an imaginary value of i' $(= d\,i')$ can have no physical significance, the $d\,i'^2$ can only be put equal to zero, and the equations solved on this hypothesis gave,

$$d\kappa = -0.088;\ dr = -0.125;\ \text{and } dq = -0.170.$$

The assumed value of q was 1.02, and consequently the maximum allowable negative value for $d\,q = -0.02$, a magnitude considerably smaller than that resulting from the Least Square solution.

Another important contravention of the physical conditions involved in the problem is that $d\kappa$ and $d\,q$ should be of unlike signs. This can be readily shown. We have seen above that $\kappa^2 \geqq \frac{m-q}{c\,q}$. Taking the least value of κ^2 consistent with physical conditions, viz., $\kappa^2 = \frac{m-q}{c\,q}$, and differentiating it, we obtain $\frac{d\kappa^2}{d\,q} = -\frac{m}{2\,c\,\kappa\,q^2}$, an essentially negative magnitude. The latter relationship would be emphasized more strongly by using κ^2 greater than this least value.

While, therefore, a Least Square adjustment can add nothing to the accuracy of the values obtained experimentally, the magnitudes and signs of the values of $d\kappa$ and $d\,r$ furnished by the adjustment indicate that κ and r should be corrected toward the values of these quantities derived in the direct solution of the first part of this paper. Inasmuch as the foregoing results are the best that could be obtained after having computed twenty-five or thirty different light curves in which the elements were shifted in almost every conceivable way, it may be asserted with confidence that the following results may be regarded as the best attainable in the present state of the observational material:

1. The light curve of U Pegasi given in Harvard College Observatory Circular, No. 23, is satisfactorily represented by the satellite theory.

2. The distance of centres does not materially differ from the sum of the radii of the components, suggesting the probable concrete existence of the "apiodal" form of Poincaré.

3. The smaller companion is about 0.77 as bright as the larger, and the ratio of radii is approximately 1:0.78.

4. The inclination of the orbit is very nearly 90°, and the disc of one or both bodies, if separate, is slightly flattened.

5. The accuracy of present observations does not suffice to determine the elements of the "system" completely, since the foregoing discussion shows the residuals to be incapable of adjustment by Least Squares.

6. The manner of rise and fall of the observed curve after and before the minima, which portions of the curve were determined with especial care, fails to confirm one's first impression on examining the curve, viz.: that the components are separated enough to remain apart for an appreciable time at the maxima. The difference between the durations of uniform brightness at the maxima, as shown by the curve, would seem to indicate a considerable orbital eccentricity, whereas the small distance of centres nullifies the possibility of its existence. It, therefore, seems desirable to direct attention to the importance of a careful photometric study

of U Pegasi's light curve near the maxima, with a view to ascertaining whether or not the form of this curve near these epochs is real.

The appended plates will assist in forming a quick judgment of the degree of approximation of the theoretical to the observed curve.

Plate I. gives the points of Pickering's curve, the continuous curve drawn through these points and used as the basis of the present discussion, together with the computed points (marked with circles) of the theoretical curve. The position of the points of the derived curve would conform much more closely to the curve of observations, by shifting the entire observed curve before and after Min. II. forward by about 1.20 minutes, which, in view of the short period of time over which the observations on which the constants of the equation of the light changes depend, would be allowable. Plate II. shows the effect of this slight shift. This, of course, amounts to assuming that Min. II. lies midway of the period, and yet, since especial attention was directed to the study of the light change in this vicinity, it does not seem that the difficulty would be likely to lie here. From private conversation with Professor Pickering, I learn that the scarcity of observations at command and the shortness of the interval over which his available observations were distributed made a definite determination of the first constant of the equation of the light variation of this star impossible, and that only the third decimal of a day can be relied on. This suggests the removal of the difficulty by shifting the entire computed curve forward, or, what amounts to the same thing, the entire theoretical curve backward by the above mentioned amount, and this gives a wholly satisfactory accord of theory and observation for the entire curve save at the maxima.

Plate III. accordingly represents the observed curve in full line, the derived curve in dotted line, and the latter, after the shift referred to, in a long dash followed by two shorter ones. The computed points, enclosed in circles, are also given in their true (unshifted) positions.

Barring the vicinity of the maxima, for which further observations must be awaited, the representation may, the writer thinks, be regarded as provisionally satisfactory, and that U Pegasi is to be regarded as varying by reason of the mutual occultations of revolving components.

The following table contains for corresponding epochs the grade values of the ordinates of both the computed and observed curves, together with the residuals in the sense observation — computation for the final unshifted curve. Aside from the fact that the errors are systematic, i. e., all of same sign (which are almost wholly gotten rid of by the aforesaid shift), but little more could be desired, and the exceedingly small values of the average deviations deprives the residuals of almost all significance. Applying the mean residual as a correction to the individual residuals, which is the same as adopting the shifted curve as final, the representation becomes entirely satisfactory.

COMPARISON OF COMPUTED WITH OBSERVED CURVE.

t	J_o	J_c	ΔJ_{o-c}	J_o	J_c	ΔJ_{o-c}
	m.	m.	m.	m.	m.	m.
1.50	9.35	9.36	−0.01	9.34	9.36	−0.02
1.25	9.41	9.41	0.00	9.37	9.39	−0.02
1.00	9.48	9.48	0.00	9.43	9.45	−0.02
.75	9.58	9.58	0.00	9.52	9.52	0.00
.50	9.70	9.72	0.02	9.62	9.62	0.00
.25	9.84	9.87	0.03	9.72	9.73	−0.01
.00	9.90	9.90	0.00	9.75	9.75	0.00
Average Deviations = 0.009						= 0.01

Figure 4 illustrates the geometrical relations prevailing in the system, on the hypothesis of separation of discs. The resemblance to β Lyrae is quite apparent, though there is an essential difference in that, with the latter star, the smaller component is the brighter, while with U Pegasi the reverse is the case.

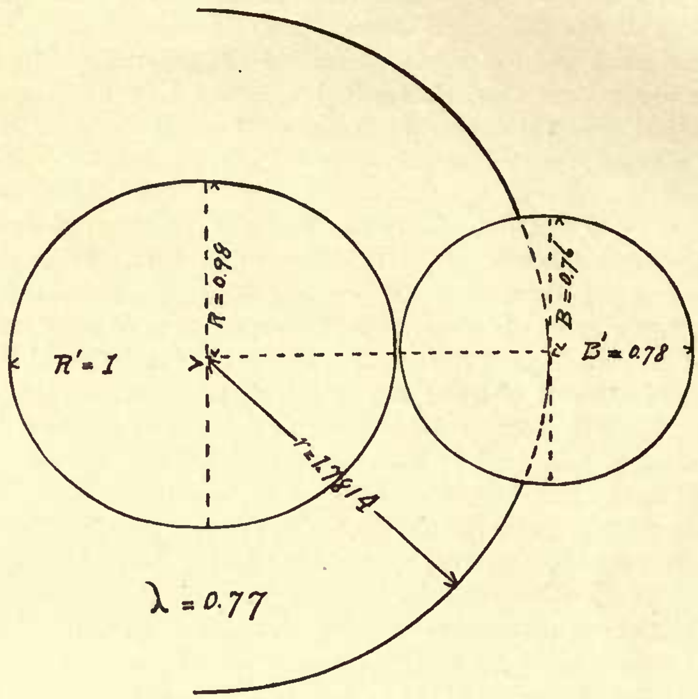


FIG. 4. — THE SYSTEM OF U PEGASI.

In conclusion, the writer would thank Dean Ricker of this University and Professor Pickering of Harvard College Observatory for valuable assistance rendered during the prosecution of this inquiry: the former, by the loan of a computing machine, without which the laborious computations involved in this paper could hardly have been made during the progress of regular University work; and the latter, by granting the writer every possible means of acquainting himself personally with the working methods and of forming an idea of the attainable accuracy of the polarizing photometer.

CAMBRIDGE, MASS., *August, 1898.*

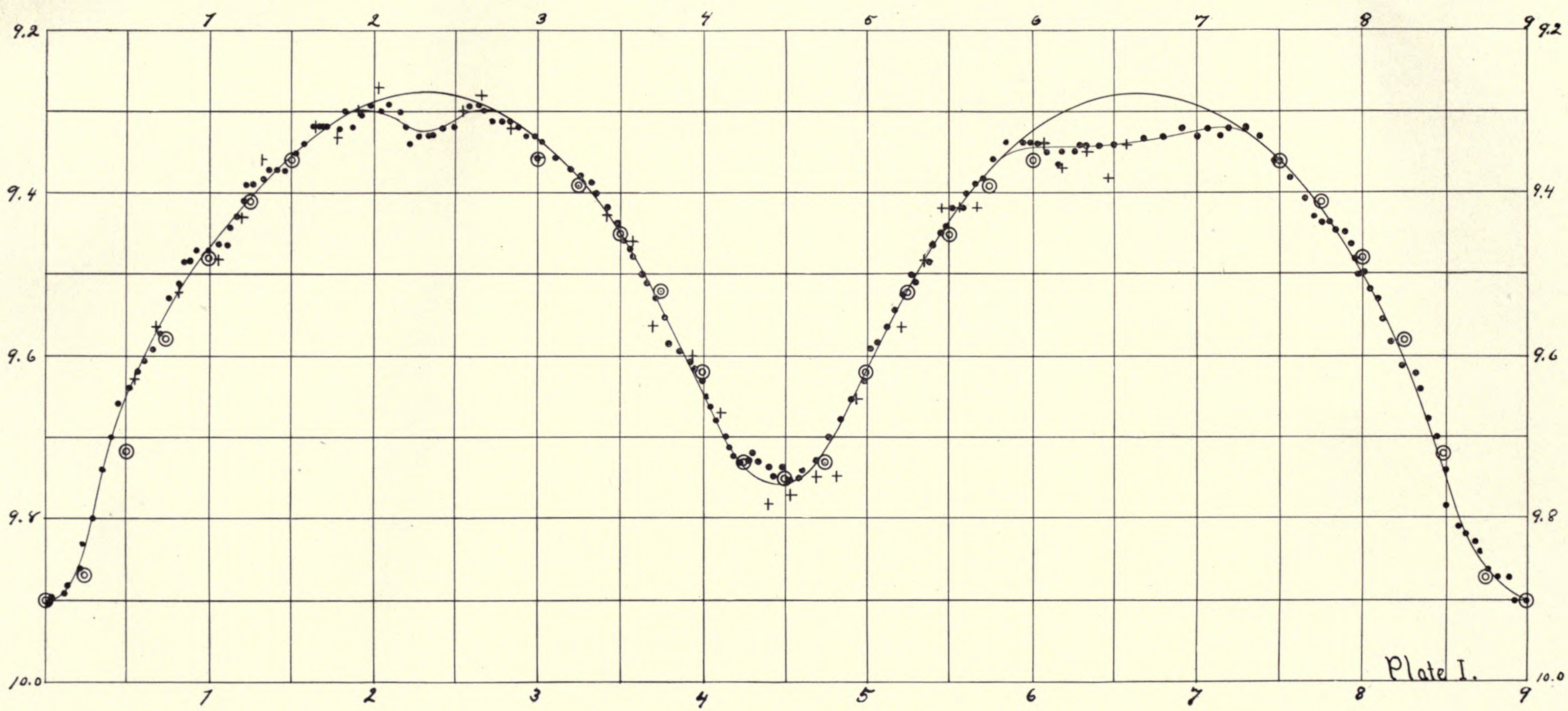
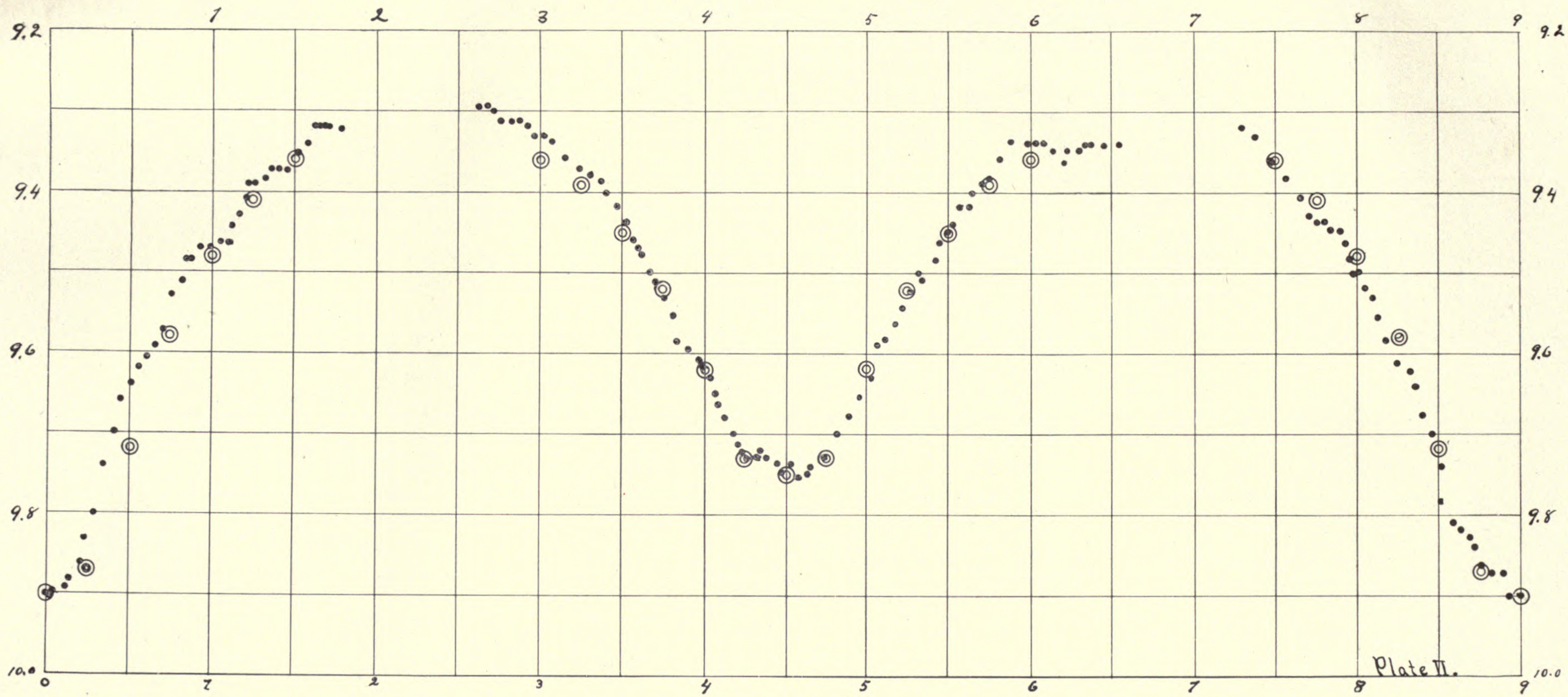
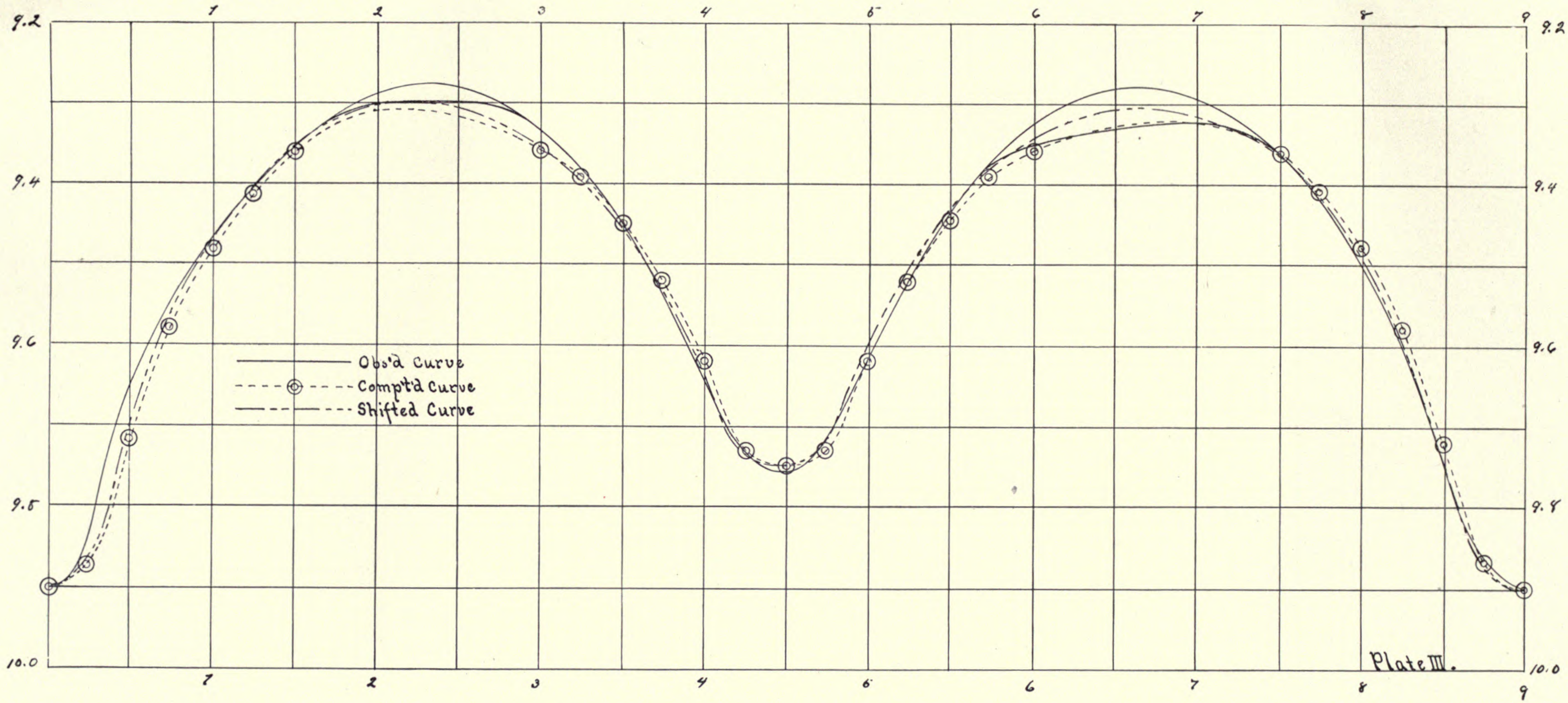


Plate I.



0
1
2
3
4
5
6
7
8
9
9.2
9.4
9.6
9.8
10.0
Plate II.



Obs'd Curve
Comptd Curve
Shifted Curve
9.2
9.4
9.6
9.8
10.0
1
2
3
4
5
6
7
8
9

Plate III.