DIFFERENTIAL EQUATIONS

MAURUS

AN ELEMENTARY COURSE IN DIFFERENTIAL EQUATIONS

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PREFACE

The aim of the author, in preparing this work, has been to afford his classes an easy, condensed course in *ordinary differential equations*, and to serve as a review of Integral Calculus.

With few exceptions, the numerous problems are new, though fashioned after the old models.

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DIFFERENTIAL EQUATIONS

CHAPTER I

DEFINITIONS. DERIVATION OF A DIFFERENTIAL EQUATION

Any equation containing differentials or derivatives is called a *differential equation.* If only one independent variable is involved, the equation is an *ordinary differential equation.* The following are examples of *ordinary differential equations:*

$$x\frac{dy}{dx} + y = xy\cot x. \tag{1}$$

$$e^x dy + ye^x dx = dy. \tag{2}$$

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + a. \tag{3}$$

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0. \tag{4}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2. \tag{5}$$

The *order* of a differential equation is the order of the highest derivative in the equation, — for example, (1), (2), and (3) are of the first order; (4) and (5) are of the second order.

The *degree* of a differential equation is the degree of the derivative of highest order in the equation, — for example, (1), (2), and (4) are of the first degree; (3) and (5) are of the second degree.

A *solution* of a differential equation is an equation of relationship between the variables, free from differentials and derivatives, which will satisfy the differential equation. In the process of solving, one or more integrations must be performed, each involving an arbitrary constant. The *general*, or *complete*, solution thus contains an arbitrary constant for each unit in the order of the equation; that is, the complete solution of a differential equation of the second order will contain two arbitrary constants, etc. The differential equation may be considered as derived from its complete solution by the elimination of the arbitrary constants.

Example 1. Form a differential equation by eliminating the constant c from the equation $xy = c \sin x$. Since two equations are required to eliminate one arbitrary constant, a second equation is needed. This may be formed by differentiating the given equation: $x\,dy + y\,dx = c \cos x\,dx$. Eliminating the c, we have the differential equation $x\dfrac{dy}{dx} + y = xy \cot x$.

Example 2. Form a differential equation by eliminating the constants c_1 and c_2 from the equation $y = c_1e^x + c_2e^{2x}$. Since there are two constants to eliminate, three equations are needed. The two extra equations are formed by differentiating the given equation twice, giving

$$\frac{dy}{dx} = c_1e^x + 2\,c_2e^{2x} \quad \text{and} \quad \frac{d^2y}{dx^2} = c_1e^x + 4\,c_2e^{2x}.$$

Eliminating, by determinants or otherwise, we have

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\,y = 0.$$

EXERCISE 1

Eliminate the constants from the following equations:

1. $x^2 + y^2 = cx$.
2. $my = m^2x + 4$.
3. $y^2 = 3c^2x + c^3$.
4. $y = c_1e^{2x} + c_2e^{5x}$.
5. $y = c_1 \log x + c_2$.
6. $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$.
7. $e^y = c_1e^x + c_2e^{-x}$.
8. $y = c_1 \sin 3x + c_2 \cos 3x$.
9. $y = c_1 \sec x + c_2 \tan x$.
10. $y = c_1x\sqrt{1-x^2} + c_2(1-2x^2)$.
11. $(x-\alpha)^2 + (y-\beta)^2 = 25$.

CHAPTER II

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

CASE I. When the variables can be separated, the solution becomes a problem in direct integration.

Example. Solve the equation $x\,dy - y\,dx = dx - dy$.

Transposing and uniting, $(x+1)\,dy = (y+1)\,dx$.

Dividing by $(x+1)(y+1)$, $\dfrac{dy}{y+1} = \dfrac{dx}{x+1}$.

The variables are now separated.

Integrating, $\log(y+1) = \log(x+1) + c$.

The c is an arbitrary constant of integration and may be put in various forms suitable to the problem. The simplicity of the solution generally depends upon the selection of a proper form for the c.

In the example given, $\log c$ is as much an arbitrary constant as c. Replacing the c by $\log c$, the answer assumes the form

$$\log(y+1) = \log(x+1) + \log c \quad \text{or} \quad y+1 = c(x+1).$$

EXERCISE 2

Solve:

1. $2\,dx + x\,dy = 0$.
2. $\dfrac{dy}{dx} + 2\,xy = 0$.
3. $\cos^2 y\,dx + \cos^2 x\,dy = 0$.
4. $\tan x \cos^2 y\,dx + \tan y \cos^2 x\,dy = 0$.
5. $x\,dy + y\,dx = 0$.

6. $xy\,dx + dy = x^2 dy.$

7. $\tan y \cos^2 y\,dx + \tan x \cos^2 x\,dy = 0.$

8. $e^x dy + ye^x dx = dy.$

9. $\sin x \cos y\,dx + \cos x \sin y\,dy = 0.$

10. $\cos^2 x\,dy + \cos^2 y\,dx = \cos^2 x \cos^2 y\,dx.$

11. $x^2 dy + y^2 dx + dx + dy = 0.$

12. $x\,dy = \sqrt{y^2+1}\,dx.$

13. $x\dfrac{dy}{dx} + y = xy \cot x.$

14. $y\,dx - x\,dy = (x^2+1)\,dy - 2\,xy\,dx.$

15. $x^2 y\,dy + xy^2 dx + x\,dx + y\,dy = 0.$

16. $(x^2 - 7\,x + 12)\dfrac{dy}{dx} + xy - 5\,y - 2\,x + 10 = 0.$

17. $(1 - y \cot y)\dfrac{dy}{dx} + \dfrac{y}{x} = 0.$

Case II. If the equation is *homogeneous* in x and y, substitute $y = vx$. Then separate the variables and solve by Case I.

Example. $2\,x^3 dy = 3\,x^2 y\,dx - y^3 dy.$

Put $y = vx$; then $dy = v\,dx + x\,dv.$

Substituting in the given equation,

$$2\,vx^3 dx + 2\,x^4 dv = 3\,vx^3 dx - v^4 x^3 dx - v^3 x^4 dv.$$

Dividing by x^3, $2\,vdx + 2\,xdv = 3\,vdx - v^4 dx - v^3 x dv.$

Separating the variables,

$$\frac{dx}{x} + \frac{v^3+2}{v\,(v^3-1)}\,dv = 0,$$

or

$$\frac{dx}{x} + \frac{dv}{v-1} - \frac{2\,dv}{v} + \frac{(2\,v+1)\,dv}{v^2+v+1} = 0.$$

Integrating, $\log x + \log(v-1) - 2\log v + \log(v^2+v+1) = \log c.$

Uniting and dropping the logarithms,

$$\frac{x(v^3-1)}{v^2} = c.$$

Substituting $\dfrac{y}{x} = v, \qquad x^3 - y^3 = cy^2.$

EXERCISE 3

Solve:

1. $(2x+3y)\,dx + (4x+5y)\,dy = 0.$
2. $(4x+5y)\,dx + (6x+7y)\,dy = 0.$
3. $(2x-3y)\,dx + (4x-5y)\,dy = 0.$
4. $(x^2+y^2)\,dx + xy\,dy = 0.$
5. $(x^2-y^2)\,dx + xy\,dy = 0.$
6. $(x^2+2xy+3y^2)\,dx + (x^2+6xy+5y^2)\,dy = 0.$
7. $x\,dx + x\,dy + 2y\,dx = 0.$
8. $x\,dx + y\,dy = (x^2+y^2)^{\frac{1}{2}}dx.$
9. $2xy\,dx - (2x^2+y^2)\,dy = 0.$
10. $(2x^2+6xy+3y^2)\,dx + (3x^2+6xy+7y^2)\,dy = 0.$
11. $x^2 - y^2 + 2xy\dfrac{dy}{dx} = 0.$
12. $(x^3-2y^3)\,dx + 3xy^2dy = 0.$
13. $\dfrac{dy}{dx} = \dfrac{3x^2y}{3x^3+y^3}.$
14. $\dfrac{dy}{dx} = \dfrac{y(2x^3+y^3)}{x(x^3+2y^3)}.$

CASE III. If the equation is of the form

$$(a_1x+b_1y+c_1)\,dx + (a_2x+b_2y+c_2)\,dy = 0,$$

it is called *nonhomogeneous of the first degree.* Substitute $x = x' + x_0$ and $y = y' + y_0$. Then $dx = dx'$ and $dy = dy'$, and the equation becomes

$$(a_1x' + b_1y' + a_1x_0 + b_1y_0 + c_1)\,dx'$$
$$+ (a_2x' + b_2y' + a_2x_0 + b_2y_0 + c_2)\,dy' = 0^{(a)}.$$

Place $a_1x_0 + b_1y_0 + c_1 = 0$ and $a_2x_0 + b_2y_0 + c_2 = 0$. If these equations are not identical or contradictory, they can be solved for x_0 and y_0. These values, being substituted in equation (a), reduce it to

$$(a_1x' + b_1y')\,dx' + (a_2x' + b_2y')\,dy' = 0,$$

which is homogeneous, and solvable by Case II.

If the auxiliary equations are either identical or contradictory, place $a_1x + b_1y = v$, eliminating x or y, and solve the resulting equation.

EXERCISE 4

1. $(2x + 3y + 4)\,dx + (4x + 5y + 6)\,dy = 0.$
2. $(x + 3y - 2)\,dx + (5x + 7y - 10)\,dy = 0.$
3. $(x - 2y + 3)\,dx + (3x - 6y + 10)\,dy = 0.$
4. $(6x - 8y + 1)\,dx + (9x - 12y + 10)\,dy = 0.$
5. $(5x + 6y + 8)\,dx + (7x + 8y + 10)\,dy = 0.$
6. $(x + 2y + 3)\,dx + (4x + 5y + 6)\,dy = 0.$
7. $(3x - 4y - 5)\,dx + (5x - 6y - 7)\,dy = 0.$
8. $(4x - 5y + 6)\,dx - (5x + 3y - 11)\,dy = 0.$
9. $\dfrac{dy}{dx} = \dfrac{7x - 9y + 7}{9x + 2y + 9}.$
10. $\dfrac{dy}{dx} = \dfrac{3x - 5y + 8}{21x - 35y + 72}.$

CASE IV. A differential equation formed from its primitive by differentiation without reduction is called an *exact* equation. If it is of the first order, it can be put in the form $Mdx + Ndy = 0$.

Example 1. $xe^y + 2x^3y + 3y\cos x + y^3 = k.$

Differentiating, we obtain

$$(e^y + 6x^2y - 3y\sin x)\,dx + (xe^y + 2x^3 + 3\cos x + 3y^2)\,dy = 0.$$

Call the original expression u. Then $M = e^y + 6x^2y - 3y\sin x$, which is $\frac{\partial u}{\partial x}$, and $N = xe^y + 2x^3 + 3\cos x + 3y^2$, which is $\frac{\partial u}{\partial y}$. Since $\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$, the test for an exact differential equation of the first order is as follows: Obtain $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If these results are identical, the equation is exact.

To solve: Integrate Mdx with respect to x. This will give all the terms of the solution which contain x. If the solution has any terms not containing x, they will not appear in Mdx, but will appear in Ndy, from which they may be obtained.

Example 2. $(2ax + 2hy)\,dx + (2hx + 2by)\,dy = 0.$

Here $\qquad M = 2ax + 2hy \quad$ and $\quad N = 2hx + 2by.$

$$\frac{\partial M}{\partial y} = 2h \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 2h.$$

Hence the equation is exact.

Integrating the first part with respect to x, we have $ax^2 + 2hxy$. This gives all the terms involving x. The second part contains one term, $2by\,dy$, without an x. Integrated, this gives by^2. Hence the complete solution is $ax^2 + 2hxy + by^2 = c$.

EXERCISE 5

1. $(4x - 3y)\,dx - (3x - 8y)\,dy = 0.$
2. $(7x + 8y)\,dx + (8x + 9y)\,dy = 0.$
3. $(2x + 3y)\,dx + (3x + 14y)\,dy = 0.$
4. $(x^2 + 2xy)\,dx + x^2dy = 0.$
5. $(2x^2 + 6xy + 3y^2)\,dx + (3x^2 + 6xy + 7y^2)\,dy = 0.$

6. $xy^2dx + x^2ydy + xdx + ydy = 0.$

7. $(4x^3y - 3x^2y^2)dx + (x^4 - 2x^3y)dy = 0.$

8. $\sin x \cos y dx + \cos x \sin y dy = 0.$

9. $(3x^2y - 3e^y - 2xe^{3y} + y^3)dx$
$+ (x^3 - 3xe^y - 3x^2e^{3y} + 3xy^2)dy = 0.$

10. $\tan y \cdot dx + x\sec^2 y dy = 0.$

11. $\sin^{-1} y dx + \dfrac{xdy}{\sqrt{1-y^2}} = 0.$

12. $\dfrac{dy}{(x+1)^n} - \dfrac{nydx}{(x+1)^{n+1}} = 0.$

13. $e^x(2e^x + e^y)dx + e^y(e^x + 2e^y)dy = 0.$

14. $2x^2dy + 4xydx = e^xdy - e^ydx + ye^xdx - xe^ydy.$

15. $\dfrac{xdx + ydy}{\sqrt{x^2+y^2}} = dx.$

16. $\dfrac{2xdx - 2ydy}{x^2 - y^2} = \dfrac{xdy + ydx}{xy}.$

CASE V. If the differential equation has been reduced after differentiation, it may no longer be exact.

Example. $$2x^3y^4 - 3x^5y^2 = c. \quad (1)$$

Differentiating,

$$(6x^2y^4 - 15x^4y^2)dx + (8x^3y^3 - 6x^5y)dy = 0. \quad (2)$$

This is exact. Dividing by x^2y,

$$(6y^3 - 15x^2y)dx + (8xy^2 - 6x^3)dy = 0. \quad (3)$$

Here $M = 6y^3 - 15x^2y$ and $N = 8xy^2 - 6x^3$,

and $\dfrac{\partial M}{\partial y} = 18y^2 - 15x^2$ and $\dfrac{\partial N}{\partial x} = 8y^2 - 18x^2.$

Hence equation (3) is *not* exact. The factor x^2y, which would reduce equation (3) to (2) and make it exact, is called an *integrating factor*. Sometimes the integrating factor for a given equation can be readily found by inspection. Various rules might be given for other cases. The following will prove useful in many problems:

Multiply the given equation by $x^m y^n$. In the new equation get $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If the method is possible, similar terms will be found. Equate the corresponding coefficients and solve for m and n.

EXERCISE 6

1. $(9x + 14y)dx + 7xdy = 0.$
2. $(x + 2y)dx + xdy = 0.$
3. $(12xy - 15y^2)dx + (10x^2 - 18xy)dy = 0.$
4. $(16xy - 15y^2)dx + (8x^2 - 21xy)dy = 0.$
5. $\frac{dy}{dx} = \frac{3y^2 - 4xy}{x^2 - 2xy}.$
6. $(5xy - 2y^2)dx + (5x^2 - 4xy)dy = 0.$
7. $(35xy + 20y^2)dx + (18x^2 + 33xy)dy = 0.$
8. $(x^2y^2 - y)dx + (2x^3y + x)dy = 0.$
9. $(2xy^4 - y)dx + (x^2y^3 + 2x)dy = 0.$
10. $\frac{xdy}{ydx} + \frac{15y^3 - 35x^2}{21y^3 - 20x^2} = 0.$

CASE VI. An equation in the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x or constants, is called *linear*. $e^{\int P dx}$ is an integrating factor.

Example. $\frac{dy}{dx} - \frac{y}{x} = xe^x$. Here $P = -\frac{1}{x}$; $\int P dx = -\log x = \log x^{-1}$; $e^{\int P dx} = x^{-1}$. x^{-1} is the integrating factor. Multiply by this and also by dx, and the equation becomes $x^{-1}dy - x^{-2}y dx = e^x dx$. This is exact. Integrating, $x^{-1}y = e^x + c$, or $y = x(e^x + c)$.

EXERCISE 7

1. $\frac{dy}{dx} + \frac{y}{x} = 3x.$

2. $\frac{dy}{dx} - \frac{y}{x} = 3x^3.$

3. $\frac{dy}{dx} + \frac{3y}{x} = x^{-3}.$

4. $\frac{dy}{dx} + y \tan x = \sec x.$

5. $\cos x \frac{dy}{dx} + y = 1 - \sin x.$

6. $\cos x \left(\frac{dy}{dx} - \cos x\right) = 1 - y \sin x.$

7. $x dy + (xy + y) dx = dx.$

8. $x^2 dy + xy(x+2) dx = e^x dx.$

9. $x dy + y dx + dy = e^x (x+1) dx.$

10. $\frac{dy}{dx} + \frac{y}{\sqrt{x^2+1}} = \sqrt{x^2+1} - x.$

11. $x^2 dy - y dx + dy = e^{\tan^{-1} x}(x^2+1) dx.$

12. $x(x^2+1) dy + y(2x^2+1) dx = x(x^2+1)^{\frac{3}{2}} dx.$

CASE VII. If the equation is of the form

$$\frac{dy}{dx} + Py = Qy^n,$$

where P and Q are functions of x or constants, it can be reduced to the linear form as follows:

Multiply by

$$(-n+1)y^{-n}: (-n+1)y^{-n}\frac{dy}{dx} + P(-n+1)y^{-n+1} = Q(-n+1).$$

Substitute $y^{-n+1} = u$, $(-n+1)y^{-n}\frac{dy}{dx} = \frac{du}{dx}\cdot$ The equation becomes $\frac{du}{dx} + P(-n+1)u = Q(-n+1)$, which is linear and can be solved by the preceding case.

EXERCISE 8

1. $\frac{dy}{dx} + \frac{3y}{x} = 6xy^{\frac{4}{3}}.$

2. $3x\frac{dy}{dx} + y + x^2y^4 = 0.$

3. $3xy^2\frac{dy}{dx} + 2y^3 = 5x^3.$

4. $3x\frac{dy}{dx} - y = xy^4(2e^x + 3).$

5. $y\,dx - x\,dy = y^2e^x dx.$

6. $3xy^2\frac{dy}{dx} + x^3 - 2y^3 = 0.$

7. $2xy\,dx - 2x^2dy = y^3e^x dx.$

8. $y^3 - 2x^3 - 3xy^2\frac{dy}{dx} = 0.$

9. $4y^{\frac{1}{3}}\frac{dy}{dx} - 3y^{\frac{4}{3}}\tan x = 3\sec x.$

10. $(2x+3)^3dy = 4y(2x+3)^2dx + 2y^{\frac{3}{2}}dx.$

11. $x^2e^ydy = x\,dy - y\,dx.$

EXERCISE 9

MISCELLANEOUS PROBLEMS

1. $x(1 - \log xy)\frac{dy}{dx} + y = 0.$

2. $y(1 - \log xy) + x\frac{dy}{dx} = 0.$

3. $(6x - 22y + 22)\,dx + (17x - 6y - 7)\,dy = 0.$

4. $\frac{dy}{dx} + \frac{y}{x^2-1} = \sqrt{x+1}.$

5. $\frac{dy}{dx} = \frac{y}{x} + xe^y\frac{dy}{dx}.$

6. $2y^2dx = x(2y - x^2e^y)\,dy.$

7. $x\,dy = 2y\,dx + x^3e^xdx.$

8. $(e^{x+y} + e^y)\,dy = (e^{x+y} + e^x)\,dx.$

9. $\frac{dy}{dx} = e^{x-y}.$

10. $y\frac{dy}{dx} = 2xy - x^2\frac{dy}{dx}.$

11. $(4x - 3y)dx = (3x - 10y)dy.$

12. $2x\frac{dy}{dx} = y + 2\sqrt{x}.$

13. $\sin 2x \cdot dy = 2\tan y \cdot dx.$

14. $3x(3y\,dx + 4x\,dy) = 10y^2(y\,dx + 3x\,dy).$

15. $(x - 2y + 4)dx + (7x - 14y + 1)dy = 0.$

16. $2x\frac{dy}{dx} = y\left(2 - \sqrt{9y^2 - 4x^2}\right).$

17. $\left(\frac{2x}{y^2} - \frac{1}{y}\right)dx - \left(\frac{2x^2}{y^3} - \frac{x}{y^2}\right)dy = 0.$

18. $(ax + b)dy = (by + a)dx.$

19. $2x\tan\frac{y}{x} + y = \frac{x\,dy}{dx}.$

20. $y(xy + 1)dx + x(xy - 1)dy = 0.$

21. $3x^2y^2dy + xy^3(x + 2)dx = 2e^x dx.$

22. $2\tan y\,dx + \tan x\,dy = 0.$

23. $\frac{dy}{dx} = \frac{\sin y + 2xy + y\sin x}{\cos x - x^2 - x\cos y}.$

24. $(y^2 + 2y + 4)dx + (x^2 + 2x + 4)dy = 0.$

25. $x(1 - \log xy)dy + y(1 + \log xy)dx = 0.$

26. $(3 - 5x^2y^2)dx = 2x^3y\,dy.$

27. $2\sin^{-1}\frac{y}{x}dx = \frac{x\,dy - y\,dx}{\sqrt{x^2 - y^2}}.$

28. $y^2dx = 2xy\,dy + 2xy^4dx.$

29. $6xy\,dy = (1 - 3y^2 + 5x^4)dx.$

30. $xy(x\,dy + y\,dx) + (x\,dy - y\,dx)\sqrt{1 - x^2y^2} = 0.$

CHAPTER III

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER BUT NOT OF THE FIRST DEGREE

When the equation is of the first order but not of the first degree, $\frac{dy}{dx}$ will generally be represented by p. Three cases will be considered: (1) the equation solvable for p; (2) the equation solvable for y; (3) the equation solvable for x.

Case I. Solvable for p. If the equation can be resolved into factors of the form

$$[p - f_1(x, y)][p - f_2(x, y)] \text{ etc.,}$$

each factor may be put equal to 0 and these equations solved by the preceding chapter.

Example. $p^2 - 2p - 35 = 0$. This can be factored into

$$(p + 5)(p - 7) = 0.$$

Placing each factor in turn equal to 0, and solving, we have $y + 5x + c = 0$ and $y - 7x + c = 0$.

Since the differential equation is of the first order, the solution can contain but one arbitrary constant. Hence the c must be the same in each separate part of the complete solution. This solution in the given example may also be written $(y + 5x + c)(y - 7x + c) = 0$.

EXERCISE 10

1. $p^2 - 6p + 8 = 0.$
2. $p^2 - p - 12 = 0.$
3. $p^3 - p^2 - 12p = 0.$
4. $2p^3 - 3p^2 - 3p + 2 = 0.$
5. $p^3 - 9p^2 + 26p - 24 = 0.$
6. $p^2 + px + py + xy = 0.$
7. $p^3 - 6p^2x + 11px^2 - 6x^3 = 0.$
8. $p^2(x+1) + y = p(x+y) + 1.$
9. $x^2p^2 - 1 = x^2p - p.$
10. $x^2p^2 + x^2 = p^2.$
11. $p^2x^2 - y^2 = 0.$
12. $xyp^2 + p(x^2 + y^2) + xy = 0.$

CASE II. If the equation can be solved for y, do so, and then differentiate with respect to x, remembering that $\frac{dy}{dx} = p$. If possible, solve this equation by the second chapter. By eliminating the p between this result and the original we obtain the required solution.

Example. $p^2x = 2py + x.$

Solving for y, $$y = \frac{p^2x - x}{2p}.$$

Differentiating with respect to x,

$$p = \frac{p\left(2px\frac{dp}{dx} + p^2 - 1\right) - (p^2x - x)\frac{dp}{dx}}{2p^2}.$$

Clearing of fractions and collecting terms,

$$x(p^2+1)\frac{dp}{dx} = p(p^2+1), \text{ or } x\frac{dp}{dx} = p.$$

The solution for this equation is $p = \frac{x}{c}$.

Eliminating the p between this and the original equation,

$$x^2 = 2cy + c^2.$$

The equation $y = px + f(p)$ is known as *Clairaut's equation.* It comes under this case. Differentiating with respect to x, we have $p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$ or $\frac{dp}{dx}[x + f'(p)] = 0$. Placing the first factor equal to 0, we have $\frac{dp}{dx} = 0$, or $p = c$. Hence the solution of the differential equation is $y = cx + f(c)$.

If the second factor is placed equal to 0, and the p eliminated between this and the differential equation, the result will be a solution satisfying the differential equation. It will contain no constant of integration, and hence is not the complete solution. Neither can it be obtained from the complete solution by giving a particular value to the constant. It is called the *singular solution*, and is, generally, the envelope of the family of lines represented by the complete solution.

EXERCISE 11

1. $y = 2px - p^2$.
2. $y = p + p^2e^{-2x}$.
3. $y = px \log x + p^2x^2$.
4. $4x^2y = 2px^3 + p^2$.
5. $6px^2 + 4p^2 = 9xy$.
6. $p^{\frac{3}{2}}x = 3p^{\frac{1}{2}}y + x$.

The following are types of *Clairaut's equation.* Solve and obtain the singular solution if possible.

7. $y = px + p^2$.
8. $y = px + \tan^{-1}p$.
9. $y = px + 6\sqrt{p^2+1}$.
10. $py = p^2x + a$.
11. $y = px + p\sqrt{p^2+1}$.
12. $y^2 - 1 + p^2(x^2-1) = 2pxy$.

CASE III. If the equation is solvable for x, differentiate with respect to y, remembering that $\frac{dx}{dy} = \frac{1}{p}$.

Solve this equation, if possible, by the preceding chapter, and eliminate the p between the resultant and the original equation.

Example. $y = 3\,px + p^{\frac{3}{2}}y.$

Solving for x, $$x = \frac{y - p^{\frac{3}{2}}y}{3\,p}.$$

Differentiating with respect to y,

$$\frac{1}{p} = \frac{p\left[1 - p^{\frac{3}{2}} - \frac{3}{2}p^{\frac{1}{2}}y\frac{dp}{dy}\right] - (y - p^{\frac{3}{2}}y)\frac{dp}{dy}}{3\,p^2}.$$

Clearing of fractions and collecting terms,

$$y\frac{dp}{dy}(p^{\frac{3}{2}} + 2) + 2\,p\,(p^{\frac{3}{2}} + 2) = 0,$$

or $$\frac{dp}{p} = -2\frac{dy}{y}.$$

Solving, $$p = \frac{c^2}{y^2}.$$

Eliminating the p by substituting in the original equation,

$$y^3 = 3\,c^2x + c^3.$$

EXERCISE 12

1. $y = 2\,px + 4\,p^2y.$
2. $x - 2\,p + 3\,p^2 = 0.$
3. $y = 2\,px + y^{\frac{1}{3}}(2\,p)^{\frac{4}{3}}.$
4. $2\,p^3x - 3\,p^2y = 2\,y.$
5. $p^2e^{2y} + 2\,px = 1.$
6. $x - y = p^3.$*
7. $p^2x = p + e^{-2y}.$
8. $9\,xyp^2 - 6\,y^2p + 4 = 0.$
9. $p^2e^{2y} = pe^{2y} + e^x.$
10. $4\,p^2xy^2 = 2\,py^3 + 1.$

* When the elimination of p is impossible, the solution is indicated by obtaining x and y in terms of p.

EXERCISE 13

MISCELLANEOUS PROBLEMS

1. $px^2 = 2xy + x^{\frac{2}{3}}p^{\frac{1}{3}}$.

2. $e^y = pe^{px}$.

3. $p^2x = py \log y + y^2$.

4. $p^2x^2 + y^2 = 2p(xy + 2a^2)$.

5. $xy(p^2+1) = p(x^2 + y^2)$.

6. $p^2(x^2 - a^2) + y^2 = b^2 + 2pxy$.

7. $e^{2x} = pe^{2x} + p^2e^y$.

8. $2pxy = x^2p^2 + y^2 - a^2(p^2+1)$.

9. $2x^3y = 3x^4p + y^4p^3$. (Substitute $y^3 = v$ and solve for v.)

10. $y^2 + y^4 + x^2p^2 = 2xyp + 2xy^3p$.

11. $x(p^2+2) = p(2x^2+1)$.

12. $y(1-4xp^2) = 2p(x - y^2 + 1)$. (Substitute $y^2 = v$ and solve for v.)

13. $4x^2 + 13xyp + 9y^2p^2 = 0$.

14. $y^2(p^2+1) = 4(x+yp)^2$. (Substitute $4x^2 + 3y^2 = v^2$.)

15. $4x + 9yp^2 = 6p(x+y)$.

16. $y + \left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$.

17. $xp^2(x+y) = y^2(1-p)$.

18. $2x^2p^2 + 5xyp + 2y^2 = 0$.

19. $x^2y + yp^2 = x^3p$.

20. $y^2 = (e^{-2x} - 1)(p^2 + 2py)$.

21. $p(x^2 - y^2) + xy(p^2 - 1) = a^2p$. $\left(\text{Substitute } \begin{matrix} x^2 = u \\ y^2 = v \end{matrix}\right)$

CHAPTER IV

DIFFERENTIAL EQUATIONS OF ORDERS HIGHER THAN THE FIRST

If the dependent variable and its derivatives are all of the first degree and are not multiplied together, the equation is called *linear.* Equations of this kind can all be put into the general form

$$A_1 \frac{d^n y}{dx^n} + A_2 \frac{d^{n-1} y}{dx^{n-1}} + A_3 \frac{d^{n-2} y}{dx^{n-2}} \cdots A_{n+1} y = f(x),$$

where A_1, A_2, A_3, etc. are functions of x or constants. We shall consider first the equation all of whose coefficients are constants and whose second member is 0.

To solve, substitute $y = e^{kx}$. This will lead to an auxiliary equation in k. Obtain the roots of this equation, k_1, k_2, etc. The differential equation is satisfied by any of these values in $y = e^{kx}$ or by the combined form $y = c_1 e^{k_1 x} + c_2 e^{k_2 x} +$ etc. The latter is the complete solution.

Example. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0.$

Substitute $y = e^{kx}$; then $\frac{dy}{dx} = ke^{kx}$ and $\frac{d^2y}{dx^2} = k^2 e^{kx}$. These, in the differential equation, give $e^{kx}(k^2 - 2k - 8) = 0$, or $k^2 - 2k - 8 = 0$. The roots of this equation are 4 and -2, and the complete solution is $y = c_1 e^{4x} + c_2 e^{-2x}$.

Three cases are to be considered: (1) when the roots of the auxiliary equation are all real and different; (2) when some or all of the roots are complex; (3) when some of the roots are repeated.

CASE I. This case is solved like the problem given.

EXERCISE 14

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 8y = 0.$

2. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0.$

3. $\frac{d^2y}{dx^2} - 25y = 0.$

4. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 20y = 0.$

5. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0.$

6. $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0.$

7. $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0.$

8. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 15y = 0.$

9. $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 24y = 0.$

10. $\frac{d^3y}{dx^3} + 48y = 28\frac{dy}{dx}.$

CASE II. Some of the roots complex. The complex roots may be written in the form $a \pm b\sqrt{-1}$, and these occur as conjugate pairs. The terms corresponding would be

$$c_1e^{(a+b\sqrt{-1})x} + c_2e^{(a-b\sqrt{-1})x} \text{ or } e^{ax}(c_1e^{b\sqrt{-1}x} + c_2e^{-b\sqrt{-1}x}).$$

The quantity in the parentheses may also be written

$$c_1 \cos bx + c_1\sqrt{-1} \sin bx + c_2 \cos bx - c_2\sqrt{-1} \sin bx,$$

or $$(c_1 + c_2) \cos bx + (c_1\sqrt{-1} - c_2\sqrt{-1}) \sin bx,$$

or, again, $$k_1 \cos bx + k_2 \sin bx.$$

The terms then become $e^{ax}(k_1 \cos bx + k_2 \sin bx)$.

Example. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$. The auxiliary equation is $k^2 + 2k + 4 = 0$, of which the roots are $-1 + \sqrt{-3}$ and $-1 - \sqrt{-3}$. Here $a = -1$ and $b = \sqrt{3}$, and the complete solution is $y = e^{-x}(c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x)$.

EXERCISE 15

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0.$

2. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 20y = 0.$

3. $\frac{d^2y}{dx^2} + 4y = 0.$

4. $\frac{d^2y}{dx^2} + a^2y = 0.$

5. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4y = 0.$

6. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

7. $\frac{d^3y}{dx^3} = 8y.$

8. $\frac{d^4y}{dx^4} - 16y = 0.$

9. $3\frac{d^4y}{dx^4} - 25\frac{d^3y}{dx^3} + 50\frac{d^2y}{dx^2} - 50\frac{dy}{dx} + 12y = 0.$

10. $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + y = 0.$

If the operator $\frac{d}{dx}$ be replaced by D, $\frac{d^2}{dx^2}$ by D^2, etc., the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$$

may be written

$$D^2y - 2Dy - 8y = 0, \quad \text{or} \quad (D^2 - 2D - 8)y = 0.$$

The coefficient of y is the same function of D as the auxiliary equation is of k. If this coefficient is factored as an ordinary quantic, the equation becomes

$$(D-4)(D+2)y=0.$$

If the y is operated on by the near factor $(D+2)$, the result is $\frac{dy}{dx}+2y$; operating on this result with the other factor $(D-4)$, we obtain

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-4\frac{dy}{dx}-8y, \quad \text{or} \quad \frac{d^2y}{dx^2}-2\frac{dy}{dx}-8y,$$

the original differential expression. It can easily be shown that the order of the factors may be changed without altering the result.

Case III. When the auxiliary equation has multiple roots.

If two of the roots, k_1, k_2, of the auxiliary equation are equal, we cannot use $c_1e^{k_1x}+c_2e^{k_1x}$, as this could be written $(c_1+c_2)e^{k_1x}$, and the c_1+c_2 is equivalent to *one* constant. The way to handle this case can best be illustrated by an example,

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0.$$

The roots of the auxiliary equation are 3, 3. Using the D symbol, the differential equation may be written

$$(D^2-6D+9)y=0, \quad \text{or} \quad (D-3)(D-3)y=0.$$

Placing $$(D-3)y=v, \qquad (1)$$

the equation becomes

$$(D-3)v=0, \quad \text{or} \quad \frac{dv}{dx}-3v=0.$$

This equation is linear with e^{-3x} as the integrating factor. Its solution is $v = c_1 e^{3x}$. Substituting this in (1), we have

$$(D-3)y = c_1 e^{3x}, \quad \text{or} \quad \frac{dy}{dx} - 3y = c_1 e^{3x}.$$

This equation is again linear with e^{-3x} as the integrating factor. The solution is $y = (c_1 x + c_2) e^{3x}$, which is the complete solution of the original equation. In the same way, if three of the roots of the auxiliary equation were equal, $y = e^{k_1 x}(c_1 x^2 + c_2 x + c_3)$ would be the corresponding part of the complete solution, etc.

EXERCISE 16

1. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0.$

2. $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 25y = 0.$

3. $\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} - 4y = 0.$

4. $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 4\dfrac{d^2y}{dx^2} = 0.$

5. $\dfrac{d^5y}{dx^5} - 5\dfrac{d^4y}{dx^4} = 0.$

6. $\dfrac{d^4y}{dx^4} - 13\dfrac{d^3y}{dx^3} + 63\dfrac{d^2y}{dx^2} - 135\dfrac{dy}{dx} + 108y = 0.$

7. $(4D^2 - 4D + 1)y = 0.$

8. $(D^3 + 5D^2 + 3D - 9)y = 0.$

9. $(D^3 + D^2 - 5D + 3)y = 0.$

10. $(D^3 - 6D^2 + 12D - 8)y = 0.$

CASE IV. When the second member is not 0, the complete solution will consist of two parts: the *general* part, which contains the constants of integration; and the *particular* part, which accounts for the second member. The *complete* solution is the sum of the two parts. The *general* part is obtained by one of the three preceding cases; the *particular* part depends upon the second member, and various methods might be given for obtaining it. We shall confine ourselves to a second member comprising terms of the forms x^n, e^{nx}, $\sin nx$, $\cos nx$, and their products, and shall use an *inspection* method to arrive at a corresponding solution.

Example 1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6\,y = 3\,x^2 - 5\,x + 6$. Here the auxiliary equation is $k^2 - k - 6 = 0$, its roots are 3 and -2, and the *general* part of the solution is $y = c_1e^{3x} + c_2e^{-2x}$. The second member of the given equation must have come from a power series of the form $c_3x^2 + c_4x + c_5$. Substituting $y = c_3x^2 + c_4x + c_5$ in the given equation, we have

$$2\,c_3 - 2\,c_3x - c_4 - 6\,c_3x^2 - 6\,c_4x - 6\,c_5 = 3\,x^2 - 5\,x + 6.$$

Equating the coefficients of like powers of x,

$$-6\,c_3 = 3; \quad -2\,c_3 - 6\,c_4 = -5; \quad 2\,c_3 - c_4 - 6\,c_5 = 6.$$

Solving these equations,

$$c_3 = -\tfrac{1}{2}; \quad c_4 = 1; \quad c_5 = -\tfrac{4}{3}.$$

Hence the *particular* part of the solution is $-\frac{x^2}{2} + x - \frac{4}{3}$, and the *complete* solution is $y = c_1e^{3x} + c_2e^{-2x} - \frac{x^2}{2} + x - \frac{4}{3}$.

Example 2. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4\,y = 2\,e^{3x}$. Here the auxiliary equation is $k^2 - 5\,k + 4 = 0$ and the *general* part of the solution is $y = c_1e^{4x} + c_2e^{x}$. The second member of the given equation, $2\,e^{3x}$, must have come from a term of the form ce^{3x}. Substituting $y = ce^{3x}$ in the given equation,

$$9\,ce^{3x} - 15\,ce^{3x} + 4\,ce^{3x} = 2\,e^{3x}, \quad \text{or} \quad -2\,c = 2.$$

$$\therefore c = -1.$$

The *particular* part of the solution is, then, $-e^{3x}$, and the *complete* solution is $y = c_1e^{4x} + c_2e^x - e^{3x}$.

Example 3. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6\,y = \sin x$. Here the *general* part of the solution is $y = c_1e^{2x} + c_2e^{3x}$. The second member, $\sin x$, must have come from an expression of the form $c_3 \sin x + c_4 \cos x$. Substituting $y = c_3 \sin x + c_4 \cos x$ in the given equation,

$$(5\,c_3 + 5\,c_4)\sin x + (5\,c_4 - 5\,c_3)\cos x = \sin x.$$

Equating corresponding coefficients,

$$5\,c_3 + 5\,c_4 = 1; \quad 5\,c_4 - 5\,c_3 = 0.$$

Solving these equations, $c_3 = c_4 = \frac{1}{10}$.

Hence the *complete* solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{\sin x + \cos x}{10}.$$

Example 4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = e^{-2x}$. Here the *general* part of the solution is $y = c_1 + c_2e^{-2x}$. It might appear that the second member must have come from a term of the form ce^{-2x}; but this is already found in the *general* part of the solution. In this case assume $y = cxe^{-2x}$. Substituting in the given equation,

$$4\,cxe^{-2x} - 4\,ce^{-2x} + 2\,ce^{-2x} - 4\,cxe^{-2x} = e^{-2x}.$$

$$\therefore -2\,c_3 = 1, \quad \text{or} \quad c_3 = -\tfrac{1}{2}.$$

Hence the *complete* solution is

$$y = c_1 + c_2e^{-2x} - \tfrac{1}{2}\,xe^{-2x}.$$

If the second member consists of several terms, the corresponding *particular* integral may be found for the parts taken separately.

EXERCISE 17

1. $(D^3 - 3D^2 + 3D - 1)y = e^{2x}$.

2. $(D^3 - 3D^2 + 3D - 9)y = 7e^{2x}$.

3. $(D^3 - 6D^2 + 11D - 6)y = e^{4x}$.

4. $\dfrac{d^4y}{dx^4} - 16y = \sin x$.

5. $(D^2 - 12D + 27)y = 81x - 171$.

6. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 4y = e^x$.

7. $(D^2 - 1)y = 3x - 2x^2$.

8. $(D^2 + 4)y = \cos 2x$.

9. $(D^2 - D - 12)y = e^{4x} + \sin 3x + e^{2x}\sin 3x$.

10. $(6D^2 - 7D + 2)y = e^x$.

Case V. When the equation is of the form

$$A_1x^n\frac{d^ny}{dx^n} + A_2x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots = 0,$$

A_1, A_2, etc. being constants, substitute $x = e^t$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\dfrac{dy}{dt}}{e^t},$$

or
$$x\frac{dy}{dx} = \frac{dy}{dt}.$$

Similarly,
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt},$$

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt},\text{ etc.}$$

Using the symbol D for $\frac{d}{dt}$, we have

$$x\frac{dy}{dx}=Dy;$$

$$x^2\frac{d^2y}{dx^2}=(D^2-D)\,y=D(D-1)\,y;$$

$$x^3\frac{d^3y}{dx^3}=(D^3-3\,D^2+2\,D)\,y$$
$$=D(D-1)(D-2)\,y, \text{ etc.}$$

This substitution will reduce the equation to one of the preceding forms. In the same way, if the equation is of the form $A_1(ax+b)^n\frac{d^ny}{dx^n}+A_2(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots=0$, substitute $ax+b=e^t$.

EXERCISE 18

1. $x^2\frac{d^2y}{dx^2}+3\,x\frac{dy}{dx}+y=0.$

2. $\frac{d^2y}{dx^2}-\frac{4}{x}\frac{dy}{dx}+\frac{4\,y}{x^2}=1.$

3. $x^3\frac{d^3y}{dx^3}-3\,x^2\frac{d^2y}{dx^2}+6\,x\frac{dy}{dx}-6\,y=0.$

4. $x^3\frac{d^3y}{dx^3}+3\,x^2\frac{d^2y}{dx^2}-27\,x\frac{dy}{dx}+48\,y=0.$

5. $x^3\frac{d^3y}{dx^3}+3\,x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8\,y=\log x.$

6. $x^4\frac{d^4y}{dx^4}+16\,x^3\frac{d^3y}{dx^3}+72\,x^2\frac{d^2y}{dx^2}+96\,x\frac{dy}{dx}+24\,y=0.$

7. $\frac{d^2s}{dt^2}=\frac{k}{t^2}.$

8. $(ax+b)^2\frac{d^2y}{dx^2}-12a^2y=0.$

9. $(2x-3)^2\frac{d^2y}{dx^2}+3(2x-3)\frac{dy}{dx}-6y=0.$

10. $(x-1)^3\frac{d^3y}{dx^3}+(x-1)^2\frac{d^2y}{dx^2}+3(x-1)\frac{dy}{dx}-8y=0.$

CASE VI. If the equation does not contain y directly, represent the derivative of lowest order by p. Then the next derivative will be $\frac{dp}{dx}$, etc. Solve this new equation for p in terms of x, and ultimately for y.

Example. $x\frac{d^2y}{dx^2}=\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3.$

Putting $\frac{dy}{dx}=p$, $\qquad x\frac{dp}{dx}=p+p^3.$

Separating the variables,

$$\frac{dp}{p^3+p}=\frac{dx}{x},$$

or

$$\frac{dp}{p}-\frac{p\,dp}{p^2+1}=\frac{dx}{x}.$$

Solving for $p=\frac{dy}{dx}$, $\qquad \frac{dy}{dx}=\frac{\pm x}{\sqrt{c_1^2-x^2}};$

whence $\qquad y=\pm\sqrt{c_1^2-x^2}+c_2.$

Clearing of radicals,

$$x^2+(y-c_2)^2=c_1^2.$$

EXERCISE 19

1. $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0.$

2. $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0.$

3. $\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-1=0.$

4. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0.$

5. $\frac{d^2y}{dx^2} = 2 \tan x \frac{dy}{dx}.$

6. $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}.$

7. $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0.$

8. $\frac{d^2y}{dx^2} = (2 \tan x + \cot x)\frac{dy}{dx}.$

9. $\log \frac{d^2y}{dx^2} = 1 - \frac{dy}{dx}.$

10. $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = a.$

Case VII. If the equation does not contain x directly, put $\frac{dy}{dx} = p$; then $\frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p\frac{dp}{dy}.$

Example. $(1-y^2)\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 = 0.$

Putting $\frac{dy}{dx} = p,\qquad (1-y^2)p\frac{dp}{dy} + p^2y = 0.$

Separating the variables,

$$\frac{dp}{p} = -\frac{y\,dy}{(1-y^2)}.$$

Solving, $\qquad p = c_1\sqrt{1-y^2} = \frac{dy}{dx};$

whence $\qquad \sin^{-1}y = c_1x + c_2.$

EXERCISE 20

1. $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0.$

2. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0.$

3. $y\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2.$

4. $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=4y^2\log y.$

5. $3y\frac{d^2y}{dx^2}+6\left(\frac{dy}{dx}\right)^2-4y^2=0.$

6. $y\left(\frac{d^2y}{dx^2}-y\right)=2\left(\frac{dy}{dx}\right)^2.$

7. $3y\frac{d^2y}{dx^2}+36y^2=2\left(\frac{dy}{dx}\right)^2.$

8. $\frac{d^2y}{dx^2}-\tan y\left(\frac{dy}{dx}\right)^2=\tan y.$

9. $\frac{d^2y}{dx^2}+2\tan y\left(\frac{dy}{dx}\right)^2=\frac{1}{2}\sin 2y.$

10. $(1-y^2)^{\frac{3}{2}}\frac{d^2y}{dx^2}+ay\left(\frac{dy}{dx}\right)^3=0.$

CASE VIII. When the equation is of the form $\frac{d^2y}{dx^2}=f(y)$, multiply by $2\frac{dy}{dx}dx$ and integrate. This gives $\left(\frac{dy}{dx}\right)^2=2\int f(y)\,dy+c.$ Extract the square root, separate the variables, and solve.

Example. $\frac{d^2y}{dx^2}=y.$

Multiplying by $2\frac{dy}{dx}$, $2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=2y\frac{dy}{dx}.$

Integrating, $$\left(\frac{dy}{dx}\right)^2=y^2+c_1,$$

or $$\frac{dy}{dx}=\pm\sqrt{y^2+c_1};$$

$$\pm\frac{dy}{\sqrt{y^2+c_1}}=dx.$$

Using the positive sign,

$$\log\left(y+\sqrt{y^2+c_1}\right)=x+\log c_2,$$

$$y+\sqrt{y^2+c_1}=c_2e^x.$$

Clearing of radicals, $\qquad y^2+c_1=y^2-2\,c_2ye^x+c_2^2e^{2x}.$

Solving for y, $\qquad y=\frac{c_2e^x}{2}-\frac{c_1}{2\,c_2}e^{-x},$

or $\qquad y=k_1e^x+k_2e^{-x}.$

EXERCISE 21

1. $\frac{d^2y}{dx^2}=a^2y.$

2. $\frac{d^2y}{dx^2}=-a^2y.$

3. $y^3\frac{d^2y}{dx^2}=1.$

4. $\frac{d^2y}{dx^2}=2\,e^y.$

5. $\frac{d^2y}{dx^2}=\frac{1}{\sqrt{y}}.$

6. $y^2\frac{d^2y}{dx^2}=1.$

7. $\frac{d^2y}{dx^2}=e^{-ay}.$

8. $y^{\frac{1}{3}}\frac{d^2y}{dx^2}=1.$

9. $y^{\frac{2}{3}}\frac{d^2y}{dx^2}=1.$

10. $\cos^3 y\frac{d^2y}{dx^2}=\sin y.$

CASE IX. An *exact* differential equation has been defined as one obtained from its primitive by differentiation without further reduction. If the equation is of an order higher than the first, and is of the form $A_1\frac{d^ny}{dx^n}+A_2\frac{d^{n-1}y}{dx^n}+\cdots A_{n+1}y=f(x)$, where A_1, A_2, etc. are functions of x, the test for exactness is as follows: Differentiate A_1 with respect to x and subtract the result from A_2; differentiate this remainder with respect to x and subtract from A_3, etc. If the last remainder is 0, the equation is exact. If the test is satisfied, the first integral is obtained as follows: The first term will be $A_1 \dfrac{d^{n-1}y}{dx^{n-1}}$; the coefficient of the second term will be the first remainder obtained in the test; the coefficient of the next term the next remainder, and so on. The second member will be $\int f(x)\,dx + c$. If the resulting equation is again exact, repeat the process. Otherwise, solve, if possible, by one of the preceding cases.

Example. $(3x^2 - 5x)\dfrac{d^2y}{dx^2} + (12x - 10)\dfrac{dy}{dx} + 6y = 0$

Following the test,

$$\begin{array}{cc} 6x - 5 & 6 \\ \hline 6x - 5 & 0 \end{array}$$

Since the last remainder is 0, the test is satisfied. The first integral is

$$(3x^2 - 5x)\frac{dy}{dx} + (6x - 5)y = c_1$$

$$\begin{array}{c} 6x - 5 \\ \hline 0 \end{array}$$

The test is again satisfied, and the complete solution is

$$(3x^2 - 5x)y = c_1x + c_2.$$

EXERCISE 22

1. $(x^2 - 3x)\dfrac{d^3y}{dx^3} + (6x - 9)\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = e^x + 6.$

2. $(2x + 5)^2\dfrac{d^2y}{dx^2} + 8(2x + 5)\dfrac{dy}{dx} + 8y = \sin x.$

3. $\sin x\dfrac{d^2y}{dx^2} + 2\cos x\dfrac{dy}{dx} - y\sin x = e^{\frac{x}{2}}.$

4. $(1 + x^2)\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = 9x^2 + 3.$

5. $(3x-4)^2\frac{d^2y}{dx^2}+6(3x-4)\frac{dy}{dx}=0.$

6. $(3x^2-5x+7)\frac{d^2y}{dx^2}+(12x-10)\frac{dy}{dx}+6y=e^x.$

7. $x^7\frac{d^3y}{dx^3}+21x^6\frac{d^2y}{dx^2}+126x^5\frac{dy}{dx}+210x^4y=3e^x+\sin x.$

8. $\frac{d^3y}{dx^3}=\sin x+8e^{2x}.$

9. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=e^{-x}.$

10. $(a^2-x^2)^{\frac{3}{2}}\frac{d^2y}{dx^2}=ax.$

CASE X. In Case V a change of the *independent* variable reduced the equation to a solvable form. In many other problems a proper change of either the dependent or independent variable will effect a reduction of the given equation. This is especially true of equations of the second order. We shall consider two cases, and shall illustrate the methods by examples.

Example. $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}-y=0.$

Putting $y=uv, \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}, \frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\cdot\frac{dv}{dx}+v\frac{d^2u}{dx^2}$ in the given equation and collecting terms,

$$u\frac{d^2v}{dx^2}-\left(2\frac{du}{dx}-2\tan x\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}-2\tan u\frac{du}{dx}u\right)v=0. \quad (1)$$

Placing the coefficient of $\frac{dv}{dx}$ equal to 0, and solving without constants of integration,

$$u=\sec x.$$

Then $$\frac{du}{dx} = \sec x \tan x,$$

and $$\frac{d^2u}{dx^2} = \sec^3 x + \sec x \tan^2 x.$$

Substituting in equation (1) and simplifying,

$$\frac{d^2v}{dx^2} = 0;$$

whence $$v = c_1 x + c_2.$$

Therefore $$y = uv = (c_1 x + c_2) \sec x.$$

EXERCISE 23

1. $\dfrac{d^2y}{dx^2} - 2 \cot x \dfrac{dy}{dx} + y(2 \cot^2 x + 5) = 0.$

2. $\dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + y(x^2 + 8) = 0.$

3. $\dfrac{d^2y}{dx^2} - (1 + 2 \cot x) \dfrac{dy}{dx} + y(1 + \cot x + 2 \cot^2 x) = 0.$

4. $x^2 \dfrac{d^2y}{dx^2} - 2x(x + 3) \dfrac{dy}{dx} + y(x^2 + 6x + 12) = 0.$

5. $4x^2 \dfrac{d^2y}{dx^2} - 20x \dfrac{dy}{dx} + y(35 + 36x^2) = 0.$

6. $\dfrac{d^2y}{dx^2} + 2a \cot ax \dfrac{dy}{dx} - 2a^2 y = 0.$

7. $4 \dfrac{d^2y}{dx^2} - 4x^{-\frac{1}{2}} \dfrac{dy}{dx} + y(x^{-\frac{3}{2}} + x^{-1}) = 0.$

8. $x^2 \dfrac{d^2y}{dx^2} - 2(x^2 + x) \dfrac{dy}{dx} + y(5x^2 + 2x + 2) = 0.$

9. $(x \log x)^2 \dfrac{d^2y}{dx^2} - 2x \log x \dfrac{dy}{dx} - y[(x \log x)^2 - \log x - 2] = 0.$

10. $x^2 \log x \left(\dfrac{d^2y}{dx^2} + a^2 y\right) = y - 2x \dfrac{dy}{dx}.$

CASE XI. In this case the equation is reduced by changing the *independent* variable.

Example. $9x\frac{d^2y}{dx^2}+6\frac{dy}{dx}+4x^{-\frac{1}{3}}y=0.$

Putting $$x=f(t),$$

then $$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx},$$

and $$\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2+\frac{dy}{dt}\cdot\frac{d^2t}{dx^2}.$$

Substituting in the given equation and collecting terms,

$$9x\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2+\frac{dy}{dt}\left(9x\frac{d^2t}{dx^2}+6\frac{dt}{dx}\right)+4x^{-\frac{1}{3}}y=0. \qquad (1)$$

Placing the coefficient of $\frac{dy}{dt}$ equal to 0 and solving,

$$t=3x^{\frac{1}{3}}.$$

Substituting in (1) and reducing,

$$9\frac{d^2y}{dt^2}+4y=0.$$

The solution of this equation is

$$y=c_1\cos\tfrac{2}{3}t+c_2\sin\tfrac{2}{3}t.$$

Hence $$y=c_1\cos 2x^{\frac{1}{3}}+c_2\sin 2x^{\frac{1}{3}}.$$

This result may also be written

$$y=c_1\sin(2x^{\frac{1}{3}}+\alpha).$$

Placing the coefficient of $\frac{d^2y}{dt^2}$ equal to the coefficient of y in (1) will often effect an easy reduction.

EXERCISE 24

1. $\frac{d^2y}{dx^2} - \tan x \frac{dy}{dx} + 4y \sec^2 x = 0.$

2. $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0.$

3. $\frac{d^2y}{dx^2} + \frac{2x}{x^2+1}\frac{dy}{dx} + \frac{y}{(x^2+1)^2} = 0.$

4. $\frac{d^2y}{dx^2} - 2\csc 2x\frac{dy}{dx} - 4y\tan^2 x = 0.$

5. $x^2(x^2-1)\frac{d^2y}{dx^2} + (2x^3 - x)\frac{dy}{dx} + y = 0.$

6. $(a^2 + x^2)^2\frac{d^2y}{dx^2} + 2x(a^2+x^2)\frac{dy}{dx} + a^2y = 0.$

7. $\frac{d^2y}{dx^2} + 2\csc 2x\frac{dy}{dx} - y\cot^2 x = 0.$

8. $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - a^2x^3y = 0.$

9. $\frac{d^2y}{dx^2} - \tan x\frac{dy}{dx} - y\sec^2 x = 0.$

10. $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 0.$

EXERCISE 25

MISCELLANEOUS PROBLEMS

1. $\frac{d^2y}{dx^2} - a^2y = 0.$

2. $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = 1.$

3. $(D^4 - 10D^3 + 25D^2)y = 0.$

4. $3\left(\frac{d^2y}{dx^2}\right)^2 - 2\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} = 0.$

5. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$

6. $\frac{d^2y}{dx^2} - (\cot x + 4\tan x)\frac{dy}{dx} - 4y\tan^2 x = 0.$
(Substitute $z = \log\sec x$.)

7. $\cos y \dfrac{d^2y}{dx^2} = \sin y \left(\dfrac{dy}{dx}\right)^2.$

8. $3\left(\dfrac{d^2y}{dx^2}\right)^2 = \dfrac{dy}{dx} \cdot \dfrac{d^3y}{dx^3}.$

9. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = \dfrac{2x^2(n-1)}{x^2+a^2}\dfrac{dy}{dx}.$

10. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 8y = 0.$

11. $(2x-3)^2\dfrac{d^2y}{dx^2} + 2(2x-3)\dfrac{dy}{dx} - 4a^2y = 0.$

12. $\dfrac{d^2y}{dx^2} + (\tan x - 2)\dfrac{dy}{dx} + y(1-\tan x) = 0.$

(Substitute $y = ve^x$.)

13. $(D^3 - 2D^2 - 15D)y = 0.$

14. $\dfrac{d^2y}{dx^2} = 2\tan x\dfrac{dy}{dx}.$

15. $\dfrac{d^2y}{dx^2} + (\tan x - 2\cot x)\dfrac{dy}{dx} + 2y\cot^2 x = 0.$

(Substitute $z = \log \sin x$.)

16. $\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3.$

17. $(x\log x)^2\dfrac{d^2y}{dx^2} + x\log x(\log x - 4)\dfrac{dy}{dx} + 6y = 0.$

(Substitute $z = \log\log x$.)

18. $(D^4 + 4)y = 0.$

19. $(D^2 - 11D + 30)y = e^{6x} - e^{5x}.$

20. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1.$

21. $(y^2+1)\dfrac{d^2y}{dx^2} - 2y\left(\dfrac{dy}{dx}\right)^2 = (y^2+1)^2\tan^{-1}y.$

22. $\dfrac{d^2s}{dt^2} = -a^2s.$

23. $(D^4 - 10D^3 + 50D^2 - 130D + 169)y = 0.$

24. $y\dfrac{d^2y}{dx^2} + \dfrac{1-\log y}{\log y}\left(\dfrac{dy}{dx}\right)^2 = 2a^2y^2\log y.$

25. $x^5 \frac{d^3y}{dx^3} + 15x^4 \frac{d^2y}{dx^2} + 60x^3 \frac{dy}{dx} + 60x^2y = 0.$

26. $(D^3 - 4D^2 + 4D)y = 0.$

27. $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + 2a^2y = 0.$

28. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \tan x \cdot \frac{dy}{dx} = 0.$

29. $(a^2 + x^2)^{\frac{3}{2}} \frac{d^2y}{dx^2} = a^2.$

30. $(D^5 + 11D^4 + 32D^3 + 4D^2 - 57D + 9)y = 0.$

EXERCISE 26

APPLICATIONS

1. Find the equation of the curve whose subnormal is constant.

2. Find the equation of the curve whose subnormal is proportional to the square of the abscissa of the point of contact.

3. Find the equation of the curve whose subnormal is equal to the abscissa of the point of contact.

4. Find the equation of the curve in which the slope at any point varies directly as the abscissa of the point.

5. Find the equation of the curve in which the slope at any point varies inversely as the ordinate of the point.

6. Find the equation of the curve in which the slope at any point varies directly as the ordinate of the point.

7. Find the equation of the curve whose slope at any point is $-\frac{4x}{9y}$.

8. A point moves in a path always perpendicular to the line joining its position to the origin. Find the equation of its path.

9. The tangent to a curve cuts intercepts from the coördinate axes whose sum is constant. Find the equation of the curve. (Singular solution.)

10. The tangent to a curve cuts intercepts from the coördinate axes whose product is constant. Find the equation of the curve. (Singular solution.)

11. Find the equation of the family of curves all of which cut the hyperbola $x^2 - y^2 = a^2$ at right angles.

12. Find the equation of the family of lines all of which cut the circle $x^2 + y^2 = a^2$ at right angles.

13. A point moves so that its acceleration varies inversely as the cube of its distance from the initial point. Find the equation of its motion.

14. A point moves so that its acceleration varies inversely as the square of its distance from the point of starting. Find the equation of its motion.

15. A point moves so that its acceleration varies directly as the distance from the initial point and is negative. Find the equation of its motion.

16. Find the equation of the curve whose radius of curvature is constant.

17. Find the equation of the curve whose radius of curvature is twice the length of the normal.

18. Find the equation of the curve in which the radius of curvature at any point varies directly as the slope at the point.

19. If a horizontal beam is supported at both ends and a load is uniformly distributed along its length, the upper fibers are in compression and the lower in tension. Between the two is a neutral surface, and its intersection with a vertical plane, parallel to the axis of the beam, is called the neutral line. Its differential equation is

$$EI\frac{d^2y}{dx^2}=\frac{w}{2}\left(\frac{l^2}{4}-x^2\right),$$

the origin being at the middle of the beam, x horizontal, and y vertical, l the length of the beam, w the weight per unit of length, E the modulus of elasticity of the beam, I the moment of inertia. Find the equation of the neutral line, if $\frac{dy}{dx}=0$ when $x=0$, and $y=0$ when $x=\frac{l}{2}$.

20. If a load P is concentrated at the center of the beam and the weight of the beam neglected, the differential equation of the neutral line is

$$EI\frac{d^2y}{dx^2}=\frac{P}{2}\left(\frac{l}{2}-x\right).$$

Solve the equation under the same conditions as above.

21. A cantilever beam has one end unsupported. When uniformly loaded, the differential equation of the neutral line is

$$EI\frac{d^2y}{dx^2}=\frac{w}{2}(l-x)^2,$$

the origin being at the fixed end. Solve the equation under the conditions, when $x=0$, $y=0$, $\frac{dy}{dx}=0$.

22. If the weight P is concentrated at the free end and the weight of the beam neglected, the differential equation of the neutral line is

$$EI\frac{d^2y}{dx^2} = P(l - x).$$

Solve under the same conditions as above.

23. If R is the resistance of an electrical current, C is the current, V the electromotive force, L the coefficient of self-induction, $L\frac{dC}{dt} + RC = V$. Find C in terms of t.

24. The equation for the current C in a circuit consisting of a charged condenser and a resistance R is

$$\frac{dC}{dt} + \frac{C}{RK} = \frac{1}{R}f'(t),$$

where K is the capacity of the condenser. Find C in terms of t.

25. The equation for the quantity of electricity discharged by a condenser in a circuit consisting of a condenser of capacity K and a resistance R is $\frac{dq}{dt} + \frac{q}{RK} = \frac{1}{R}f(t)$. Find q in terms of t.

ANSWERS

EXERCISE 1

1. $x^2 + 2xy\dfrac{dy}{dx} = y^2.$
2. $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + 4.$
3. $8y\left(\dfrac{dy}{dx}\right)^3 = 27\left(y - 2x\dfrac{dy}{dx}\right)^2.$
4. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = 0.$
5. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$
6. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0.$
7. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1.$
8. $\dfrac{d^2y}{dx^2} + 64y = 0.$
9. $\dfrac{d^2y}{dx^2} - \tan x\dfrac{dy}{dx} - y\sec^2 x = 0.$
10. $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0.$
11. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3 = 25\left(\dfrac{d^2y}{dx^2}\right)^2.$

EXERCISE 2

1. $x^2 = ce^{-y}.$
2. $y = ce^{-x^2}.$
3. $\tan x + \tan y = c.$
4. $\tan^2 x + \tan^2 y = c.$
5. $xy = c.$
6. $y = c\sqrt{x^2 - 1}.$
7. $\tan x \cdot \tan y = c.$
8. $y(e^x - 1) = c.$
9. $\sec x \cdot \sec y = c.$
10. $\tan x + \tan y = x + c.$
11. $x + y = c(1 - xy).$
12. $\sqrt{y^2 + 1} = cx - y.$
13. $xy = c \sin x.$
14. $y = c(x^2 + x + 1).$
15. $(x^2 + 1)(y^2 + 1) = c.$
16. $(y - 2)(x - 3)^2 = c(x - 4).$
17. $xy = c \sin y.$

EXERCISE 3

1. $(5y + 2x)^2(y + x) = c.$
2. $(7y + 4x)^2(y + x) = c.$
3. $\log c(5y^2 - xy - 2x^2)$
$$= \frac{7}{\sqrt{41}}\log\frac{10y - x - x\sqrt{41}}{10y - x + x\sqrt{41}}.$$
4. $x^2(x^2 + 2y^2) = c.$
5. $x = ce^{-\frac{y^2}{2x^2}}.$
6. $x^3 + 3x^2y + 9xy^2 + 5y^3 = c.$
7. $x^2(x + 3y) = c.$

8. $\sqrt{x^2+y^2} = x + c.$

9. $y = ce^{\frac{x^3}{y^3}}.$

10. $2x^3 + 9x^2y + 9xy^2 + 7y^3 = c.$

11. $x^2 + y^2 = cx.$

12. $x^3 + y^3 = cx^2.$

13. $y = ce^{\frac{x^3}{y^3}}.$

14. $x^3 - y^3 = cxy.$

EXERCISE 4

1. $(2x + 5y + 8)^2(x + y + 1) = c.$
2. $(x + 7y - 2)^2(x + y - 2) = c.$
3. $5x + 15y + c = \log(5x - 10y + 16).$
4. $2x + 3y + c = \log(3x - 4y + 2).$
5. $(x + y + 1)(5x + 8y + 14)^2 = c.$
6. $(x + y + 1)(x + 5y + 9)^3 = c.$
7. $\log c(6y^2 - xy - 3x^2 + 23y - 8x + 19)$
$$= \frac{9}{\sqrt{73}} \log \frac{12y - x + 23 - (x+1)\sqrt{73}}{12y - x + 23 + (x+1)\sqrt{73}}.$$
8. $4x^2 - 10xy - 3y^2 + 12x + 22y = c.$
9. $7x^2 - 18xy - 2y^2 + 14x - 18y = c.$
10. $x - 7y + c = \log(3x - 5y + 11).$

EXERCISE 5

1. $2x^2 - 3xy + 4y^2 = c.$
2. $7x^2 + 16xy + 9y^2 = c.$
3. $x^2 + 3xy + 7y^2 = c.$
4. $x^3 + 3x^2y = c.$
5. $2x^3 + 9x^2y + 9xy^2 + 7y^3 = c.$
6. $x^2y^2 + x^2 + y^2 = c.$
7. $x^4y - x^3y^2 = c.$
8. $\cos x \cdot \cos y = c.$
9. $x^3y - 3xe^y + x^2e^{3y} + xy^3 = c.$
10. $x \tan y = c.$
11. $x \sin^{-1} y = c.$
12. $y = c(x+1)^n.$
13. $e^{2x} + e^{x+y} + e^{2y} = c.$
14. $2x^2y = ye^x - xe^y + c.$
15. $\sqrt{x^2+y^2} = x + c.$
16. $x^2 - y^2 = cxy.$

EXERCISE 6

1. $3x^3 + 7x^2y = c.$
2. $x^3 + 3x^2y = c.$
3. $2x^6y^5 - 3x^5y^6 = c.$
4. $x^{\frac{5}{3}}y^{\frac{4}{3}}(2x - 3y) = c.$
5. $x^3y(x - y) = c.$
6. $5x^3y^3 - 3x^2y^4 = c.$
7. $x^{\frac{7}{3}}y^{\frac{6}{5}} + x^{\frac{4}{3}}y^{\frac{11}{5}} = c.$
8. $x^2y^2 + y = cx.$
9. $x^2y^3 - x = cy^2.$
10. $3x^5y^7 - 5x^7y^4 = c.$

EXERCISE 7

1. $xy = x^3 + c$.
2. $y = x(x^3 + c)$.
3. $x^3y = x + c$.
4. $y = \sin x + c \cdot \cos x$.
5. $y(\sec x + \tan x) = x + c$.
6. $y = \sin x + (x + c)\cos x$.
7. $e^x(xy - 1) = c$.
8. $2x^2y = e^x + ce^{-x}$.
9. $y(x + 1) = xe^x + c$.
10. $y(x + \sqrt{x^2 + 1}) = x + c$.
11. $y = (x + c)e^{\tan^{-1}x}$.
12. $4y\sqrt{x^2 + 1} = x^3 - 2x + cx^{-1}$.

EXERCISE 8

1. $y^{-\frac{1}{3}} = x(c - 2x)$.
2. $y^{-3} = x(x + c)$.
3. $x^2y^3 = x^5 + c$.
4. $2y^{-3} = x^{-1}(4e^x + c) - (4e^x + 3x)$.
5. $x = y(e^x + c)$.
6. $x^3 + y^3 = cx^2$.
7. $x^2 = y^2(e^x + c)$.
8. $x^3 + y^3 = cx$.
9. $y^{\frac{4}{3}} = (x + c)\sec x$.
10. $2y^{-\frac{1}{2}}(2x + 3)^2 = c(2x + 3) - 1$.
11. $y = x(e^y + c)$.

EXERCISE 9

1. $y = c\log xy$.
2. $x = c\log xy$.
3. $(3x + 2y - 5)^2 = c(2x - 3y + 2)$.
4. $3y\sqrt{x - 1} = 2(x - 1)\sqrt{x^2 - 1} + c\sqrt{x + 1}$.
5. $y = x(e^y + c)$.
6. $y^2 = x^2(e^y + c)$.
7. $y = x^2(e^x + c)$.
8. $e^y + 1 = c(e^x + 1)$.
9. $e^y = e^x + c$.
10. $y = ce^{\frac{x^2}{y}}$.
11. $2x^2 - 3xy + 5y^2 = c$.
12. $x = ce^{\frac{y}{\sqrt{x}}}$.
13. $\sin y = c\tan x$.
14. $3x^3y^4 - 5x^2y^6 = c$.
15. $\log(x - 2y + 1)^3 + x + 7y + c = 0$.
16. $\sin^{-1}\dfrac{2x}{3y} = x + c$.
17. $x^2 - xy = cy^2$.
18. $(by + a)^a = c(ax + b)^b$.
19. $\sin\dfrac{y}{x} = cx^2$.
20. $y = cxe^{xy}$.
21. $x^2y^3 = e^x + ce^{-x}$.
22. $\sin^2 x \cdot \sin y = c$.
23. $x\sin y + x^2y - y\cos x = c$.
24. $x + y + 2 = c(xy + x + y - 2)$.
25. $x\log xy = cy$.
26. $x^3 - x^5y^2 = c$.
27. $\sin^{-1}\dfrac{y}{x} = cx^2$.
28. $x = y^2(x^2 + c)$.
29. $x - 3xy^2 + x^5 = c$.
30. $\sin^{-1}xy + \log\dfrac{y}{x} = c$.

EXERCISE 10

1. $(y-2x+c)(y-4x+c)=0.$
2. $(y-4x+c)(y+3x+c)=0.$
3. $(y-c)(y-4x+c)(y+3x+c)=0.$
4. $(y-2x+c)(y+x+c)(2y-x+c)=0.$
5. $(y-2x+c)(y-3x+c)(y-4x+c)=0.$
6. $(y-ce^{-x})(x^2+2y+c)=0.$
7. $(x^2-2y+c)(x^2-y+c)(3x^2-2y+c)=0.$
8. $y=x+c;\ y-1=c(x+1).$
9. $(y-x+c)(xy-cx-1)=0.$
10. $x^2+(y-c)^2=1.$
11. $(xy-c)(y-cx)=0.$
12. $(xy-c)(x^2+y^2-c)=0.$

EXERCISE 11

1. $x=\frac{2p}{3}+\frac{c}{p^2};$
   $y=\frac{p^2}{3}+\frac{2c}{p}.$
2. $y=ce^x+c^2.$
3. $y=c\log x+c^2.$
4. $y=cx^2+c^2.$
5. $cx^3=(y-c)^2.$
6. $c^3x^3=3cy+1.$
7. $y=cx+c^2$; singular solution, $x^2+4y=0.$
8. $y=cx+\tan^{-1}c$; no singular solution.
9. $y=cx+6\sqrt{c^2+1}$;
   singular solution, $x^2+y^2=36.$
10. $y=cx+\frac{a}{c}$; singular solution, $y^2=4ax.$
11. $y=cx+c\sqrt{c^2+1}.$
12. $y=cx+\sqrt{c^2+1}$; singular solution, $x^2+y^2=1.$

EXERCISE 12

1. $y^2=cx+c^2.$
2. $y=p^2-2p^3+c;$
   $x=2p-3p^2.$
3. $y^2=cx+c^{\frac{4}{3}}.$
4. $27cy^2=8(x-c)^3.$
5. $e^{2y}=2cx+c^2.$
6. $y=c-p^3-\frac{3p^2}{2}-3p-3\log(p-1);$
   $x=c-\frac{3p^2}{2}-3p-3\log(p-1).$
7. $x=ce^y+c^2.$
8. $cy^3=(x+c)^2.$
9. $e^x=ce^y+c^2.$
10. $x=cy^2+c^2.$

EXERCISE 13

1. $x^2 = 2cy + c^{\frac{2}{3}}$.
2. $y = cx + \log c$.
3. $x = c \log y + c^2$.
4. $y = cx \pm 2a\sqrt{c}$.
5. $(x^2 - y^2 + c)(y - cx) = 0$.
6. $y = cx \pm \sqrt{a^2c^2 + b^2}$.
7. $e^y = ce^x + c^2$.
8. $y = cx \pm a\sqrt{c^2 + 1}$.
9. $2y^3 = 3cx^2 + c^3$.
10. $y^2 = 2cxy^2 + c^2x^2$.
11. $(y - x^2 + c)(y - \log x + c) = 0$.
12. $y^2 = cx + \dfrac{c}{c+1}$.
13. $(4x^2 + 9y^2 - c)(x^2 + y^2 - c) = 0$.
14. $(x + c)^2 + y^2 = 4c^2$.
15. $(2x - 3y + c)(2x^2 - 3y^2 + c) = 0$.
16. $y = cx - c^2 + c^3$.
17. $(xy - c)(y - ce^{\frac{x}{y}}) = 0$.
18. $(xy^2 - c)(x^2y - c) = 0$.
19. $y^2 + c^2 = cx^2$.
20. $y^2e^{2x} + 2cy + c^2 = 0$.
21. $y^2 = cx^2 - \dfrac{a^2c}{1+c}$.

EXERCISE 14

1. $y = c_1e^{3x} + c_2e^{-x}$.
2. $y = c_1e^{4x} + c_2e^{2x}$.
3. $y = c_1e^{5x} + c_2e^{-5x}$.
4. $y = c_1e^{5x} + c_2e^{-4x}$.
5. $y = c_1e^{-4x} + c_2$.
6. $y = c_1e^{2x} + c_2e^{\frac{x}{2}}$.
7. $y = c_1e^{2x} + c_2e^{3x} + c_3e^{4x}$.
8. $y = c_1e^{5x} + c_2e^{x} + c_3e^{-3x}$.
9. $y = c_1e^{6x} + c_2e^{4x} + c_3e^{-x}$.
10. $y = c_1e^{4x} + c_2e^{2x} + c_3e^{-6x}$.

EXERCISE 15

1. $y = e^{3x}(c_1 \cos x + c_2 \sin x)$.
2. $y = e^{4x}(c_1 \cos 2x + c_2 \sin 2x)$.
3. $y = c_1 \cos 2x + c_2 \sin 2x$.
4. $y = c_1 \cos ax + c_2 \sin ax$.
5. $y = c_1e^{2x} + e^{-\frac{x}{2}}\left(c_2 \cos \dfrac{\sqrt{7}}{2}x + c_3 \sin \dfrac{\sqrt{7}}{2}x\right)$.
6. $y = e^{-\frac{x}{2}}\left(c_1 \cos \dfrac{\sqrt{3}}{2}x + c_2 \sin \dfrac{\sqrt{3}}{2}x\right)$.
7. $y = c_1e^{2x} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$.
8. $y = c_1e^{2x} + c_2e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$.
9. $y = c_1e^{6x} + c_2e^{\frac{x}{3}} + e^{x}(c_3 \cos x + c_4 \sin x)$.
10. $y = e^{\frac{x}{2}}\left(c_1 \cos \dfrac{\sqrt{3}}{2}x + c_2 \sin \dfrac{\sqrt{3}}{2}x\right) + e^{-\frac{x}{2}}\left(c_3 \cos \dfrac{\sqrt{3}}{2}x + c_4 \sin \dfrac{\sqrt{3}}{2}x\right)$.

EXERCISE 16

1. $y = e^{-2x}(c_1x + c_2)$.
2. $y = e^{5x}(c_1x + c_2)$.
3. $y = e^{2x}(c_1x + c_2) + c_3e^{x}$.
4. $y = e^{2x}(c_1x + c_2) + c_3x + c_4$.
5. $y = c_1e^{5x} + c_2x^3 + c_3x^2 + c_4x + c_5$.
6. $y = e^{3x}(c_1x^2 + c_2x + c_3) + c_4e^{4x}$.
7. $y = e^{\frac{x}{2}}(c_1x + c_2)$.
8. $y = e^{-3x}(c_1x + c_2) + c_3e^{x}$.
9. $y = e^{x}(c_1x + c_2) + c_3e^{-3x}$.
10. $y = e^{2x}(c_1x^2 + c_2x + c_3)$.

EXERCISE 17

1. $y = e^{x}(c_1x^2 + c_2x + c_3) + e^{2x}$.
2. $y = c_1e^{3x} + c_2\cos\sqrt{3}\,x + c_3\sin\sqrt{3}\,x - e^{2x}$.
3. $y = c_1e^{3x} + c_2e^{2x} + c_3e^{x} + \frac{1}{6}e^{4x}$.
4. $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x - \frac{\sin x}{15}$.
5. $y = c_1e^{3x} + c_2e^{9x} + 3x - 5$.
6. $y = c_1e^{4x} + e^{x}(c_2 - \frac{1}{3}x)$.
7. $y = c_1e^{x} + c_2e^{-x} + 2x^2 - 3x - 4$.
8. $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x\cos 2x$.
9. $y = c_1e^{-3x} + e^{4x}\left(c_2 + \frac{x}{7}\right) + \frac{\cos 3x - 7\sin 3x}{150} - \frac{e^{2x}}{442}(19\sin 3x + 9\cos 3x)$.
10. $y = c_1e^{\frac{2x}{3}} + c_2e^{\frac{x}{2}} + e^{x}$.

EXERCISE 18

1. $xy = c_1\log x + c_2$.
2. $y = c_1x + c_2x^4 - \frac{x^2}{2}$.
3. $y = c_1x + c_2x^2 + c_3x^3$.
4. $y = c_1x^2 + c_2x^4 + \frac{c_3}{x^6}$.
5. $y = \frac{c_1}{x^2} + x\left(c_2\cos\sqrt{3}\log x + c_3\sin\sqrt{3}\log x\right) + \frac{1}{8}\log x$.
6. $x^4y = c_1x^3 + c_2x^2 + c_3x + c_4$.
7. $s = c_1t + c_2 - k\log t$.
8. $y = c_1(ax + b)^4 + c_2(ax + b)^{-3}$.
9. $y = c_1(2x - 3) + c_2(2x - 3)^{-\frac{3}{2}}$.
10. $y = c_1(x - 1)^2 + c_2\cos\log(x - 1)^2 + c_3\sin\log(x - 1)^2$.

EXERCISE 19

1. $y = c_1\log x + c_2$.
2. $y = c_1\sin^{-1}x + c_2$.
3. $y = \log\sec(x + c_1) + c_2$.
4. $y = \log(x + c_1) + c_2$.
5. $y = c_1\tan x + c_2$.
6. $y = x + c_2 - \cos(x + c_1)$.
7. $y = c_1\sec^{-1}x + c_2$.
8. $y = c_1\sec x + c_2$.
9. $y = (x + c_1)\log(x + c_1) + c_2$.
10. $(x - c_1)^2 + (y - c_2)^2 = a^2$.

EXERCISE 20

1. $(x + c_1)^2 + y^2 = c_2^2$.
2. $e^y = c_1 \sin x + c_2 \cos x$.
3. $y = c_1 e^{c_2 x}$.
4. $\log y = c_1 e^{2x} + c_2 e^{-2x}$.
5. $y^8 = c_1 e^{2x} + c_2 e^{-2x}$.
6. $y = c_1 \sec(x + c_2)$.
7. $y^{\frac{1}{3}} = c_1 \sin(2x + c_2)$.
8. $\sin y = c_1 e^x + c_2 e^{-x}$.
9. $\tan y = c_1 e^x + c_2 e^{-x}$.
10. $a \sin^{-1} y = x + c_1 y + c_2$.

EXERCISE 21

1. $y = c_1 e^{ax} + c_2 e^{-ax}$.
2. $y = c_1 \sin(ax + c_2)$.
3. $(x + c_2)^2 - c_1^2 y^2 = c_1^4$.
4. $x + c_2 = \frac{1}{2c_1} \log \frac{\sqrt{e^y + c_1^2} - c_1}{\sqrt{e^y + c_1^2} + c_1}$.
5. $2(y^{\frac{1}{2}} - 2c_1^2)\sqrt{y^{\frac{1}{2}} + c_1^2} = 3x + c_2$.
6. $c_1 y^{\frac{1}{2}} \sqrt{y - 2c_1^2} + 2c_1^3 \log\left(y^{\frac{1}{2}} + \sqrt{y - 2c_1^2}\right) = x + c_2$.
7. $4c_1 c_2 e^{\frac{ay}{2}} = 2c_1^2 e^{c_2 x} + ae^{-c_2 x}$.
8. $y^{\frac{1}{3}} \sqrt{3y^{\frac{2}{3}} + 3c_1^2} - c_1^2 \sqrt{3} \log\left(y^{\frac{1}{3}} + \sqrt{y^{\frac{2}{3}} + c_1^2}\right) = 2x + c_2$.
9. $2(3y^{\frac{2}{3}} - 4c_1 y^{\frac{1}{3}} + 8c_1^2)\sqrt{y^{\frac{1}{3}} + c_1} = 5\sqrt{6}x + c_2$.
10. $\sin y = \sqrt{1 + c_1^2} \sin \frac{x + c_2}{c_1}$.

EXERCISE 22

1. $(x^2 - 3x)y = e^x + x^3 + c_1 x^2 + c_2 x + c_3$.
2. $(2x + 5)^2 y = -\sin x + c_1 x + c_2$.
3. $y \sin x = 4e^{\frac{x}{2}} + c_1 x + c_2$.
4. $y\sqrt{1 + x^2} = (1 + x^2)^{\frac{3}{2}} + c_1 \log\left(x + \sqrt{1 + x^2}\right) + c_2$.
5. $y = \frac{c_1}{3x - 4} + c_2$.
6. $(3x^2 - 5x + 7)y = e^x + c_1 x + c_2$.
7. $x^7 y = c_1 x^2 + c_2 x + c_3 + 3e^x + \cos x$.
8. $y = \cos x + e^{2x} + c_1 x^2 + c_2 x + c_8$.
9. $e^x(y + c_1 x + c_2) = x + c_3$.
10. $y = a \sin^{-1} \frac{x}{a} + c_1 x + c_2$.

EXERCISE 23

1. $y = c_1 \sin x \cdot \sin(2x + \alpha)$.
2. $y = e^{\frac{x^2}{2}}(c_1 \cos 3x + c_2 \sin 3x)$.
3. $y = \sin x(c_1 e^x + c_2)$.
4. $y = e^x(c_1 x^4 + c_2 x^3)$.
5. $y = x^{\frac{5}{2}}(c_1 \cos 3x + c_2 \sin 3x)$.
6. $y \sin ax = c_1 e^{ax} + c_2 e^{-ax}$.
7. $y = e^{\sqrt{x}}(c_1 x + c_2)$.
8. $y = xe^x(c_1 \cos 2x + c_2 \sin 2x)$.
9. $y = \log x \cdot (c_1 e^x + c_2 e^{-x})$.
10. $y \log x = c_1 \cos ax + c_2 \sin ax$.

EXERCISE 24

1. $y = c_1 \sin[2 \log(\sec x + \tan x) + \alpha]$.
2. $y = c_1 x \sqrt{1 - x^2} + c_2(1 - 2x^2)$.
3. $y\sqrt{x^2 + 1} = c_1 x + c_2$.
4. $y = c_1 \sec^2 x + c_2 \cos^2 x$.
5. $xy = c_1 + c_2\sqrt{x^2 - 1}$.
6. $y\sqrt{x^2 + a^2} = c_1 x + c_2$.
7. $y = c_1 \sin x + c_2 \csc x$.
8. $y = c_1 e^{\frac{ax^2}{2}} + c_2 e^{-\frac{ax^2}{2}}$.
9. $y = c_1 \sec x + c_2 \tan x$.
10. $y = c_1 e^{x^2} + c_2 e^{-x^2}$.

EXERCISE 25

1. $y = c_1 e^{ax} + c_2 e^{-ax}$.
2. $y = \pm \cos(x + c_1) + c_2$.
3. $y = c_1 x + c_2 + e^{5x}(c_3 x + c_4)$.
4. $xy + c_1 x + c_2 y + c_3 = 0$.
5. $y = c_1 x + c_2\sqrt{1 - x^2}$.
6. $y = c_1 \sec^4 x + c_2 \cos x$.
7. $\sin y = c_1 x + c_2$.
8. $y^2 + c_1 x + c_2 y + c_3 = 0$.
9. $y = c_1(x^2 + a^2)^n + c_2$.
10. $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x)$.
11. $y = c_1(2x - 3)^a + c_2(2x - 3)^{-a}$.
12. $y = e^x(c_1 \sin x + c_2)$.
13. $y = c_1 + c_2 e^{5x} + c_3 e^{-3x}$.
14. $y = c_1 \tan x + c_2$.
15. $y = c_1 \sin^2 x + c_2 \sin x$.
16. $e^x = c_1 e^y + c_2 e^{-y}$.
17. $y = c_1(\log x)^2 + c_2(\log x)^3$.
18. $y = e^x(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$.
19. $y = e^{6x}(c_1 + x) + e^{5x}(c_2 + x)$.
20. $e^y = c_1 e^x + c_2 e^{-x}$.
21. $\tan^{-1} y = c_1 e^x + c_2 e^{-x}$.
22. $s = c_1 \sin(at + \alpha)$.
23. $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + e^{3x}(c_3 \cos 2x + c_4 \sin 2x)$.
24. $(\log y)^2 = c_1 e^{2ax} + c_2 e^{-2ax}$.
25. $x^5 y = c_1 + c_2 x + c_3 x^2$.
26. $y = c_1 + e^{2x}(c_2 x + c_3)$.
27. $y = e^{ax}(c_1 \cos ax + c_2 \sin ax)$.
28. $e^y = c_1 \sin x + c_2$.
29. $y = \sqrt{x^2 + a^2} + c_1 x + c_2$.
30. $y = c_1 e^x + e^{-8x}(c_2 x + c_3) + e^{-3x}\left(c_4 e^{\sqrt{10}x} + c_5 e^{-\sqrt{10}x}\right)$.

EXERCISE 26

1. $y^2 = 4kx + c.$
2. $y^2 = 2kx^3 + c.$
3. $y^2 = x^2 + c^2.$
4. $y = kx^2 + c.$
5. $y^2 = 2kx + c.$
6. $y = ce^{kx}.$
7. $4x^2 + 9y^2 = c^2.$
8. $x^2 + y^2 = c^2.$
9. $(x - y)^2 - 2k(x + y) + k^2 = 0.$
10. $xy = k^2.$
11. $xy = c^2.$
12. $y = cx.$
13. $c_1 s^2 = k + (c_1 t + c_2)^2.$
14. $c_1\sqrt{c_1 s^2 - 2ks} + 2k\sqrt{c_1}\log\left(\sqrt{c_1}s + \sqrt{c_1 s - 2k}\right) = c_1^2 t + c_2.$
15. $s = c\sin(kt + \alpha).$
16. $(x - c_1)^2 + (y - c_2)^2 = a^2.$
17. $(x + c_1)^2 = c_2(2y - c_2).$
18. $y + c_2 = u + k\log\sqrt{\dfrac{u-k}{u+k}}\cdot\ [u^2 = k^2 - (x + c_1)^2.]$
19. $EIy = \dfrac{w}{2}\left(\dfrac{l^2x^2}{8} - \dfrac{x^4}{12}\right) - \dfrac{5wl^4}{384}.$
20. $EIy = \dfrac{P}{2}\left(\dfrac{lx^2}{4} - \dfrac{x^3}{6}\right) - \dfrac{Pl^3}{32}.$
21. $EIy = \dfrac{w}{24}(l - x)^4 + \dfrac{wl^3x}{6} - \dfrac{wl^4}{24}.$
22. $EIy = \dfrac{P}{6}(l - x)^3 + \dfrac{Pl^2x}{6} - \dfrac{Pl^3}{6}.$
23. $C = \dfrac{V}{R} + ce^{-\frac{R}{L}t}.$
24. $C = ce^{-\frac{t}{RK}} + \dfrac{e^{-\frac{t}{RK}}}{R}\int e^{\frac{t}{RK}}f'(t)\,dt.$
25. $q = ce^{-\frac{t}{RK}} + \dfrac{e^{-\frac{t}{RK}}}{R}\int e^{\frac{t}{RK}}f(t)\,dt.$

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