II. *On the Empirical Laws of the Tides in the Port of London; with some Reflexions on the Theory. By the Rev.* William Whewell, *A.M., F.R.S., Fellow and Tutor of Trinity College, Cambridge.*

Received November 13, 1833,—Read January 9, 1834.

THE present state of our knowledge of the tides is remarkably at variance with the complete and scientific character which Physical Astronomy is, in common opinion, supposed to have attained. We may, perhaps, most easily figure to ourselves the real condition of this subject, by imagining what the condition of other branches of astronomy would be, if some great natural or moral convulsion should sweep away our existing science, and replunge us in the ignorance of the dark ages, leaving extant only a few general notions concerning the theories which are at present established. In such a state of things, we may suppose that some tradition of the doctrine of universal gravitation would survive the change, and that learned men would still go on asserting that the various astronomical phenomena of the universe were owing to that cause; but the resources of mathematical art being, for the time, lost, they would be unable to prove the truth of such assertions: and, both the collected stores of observation, and the habit and apparatus of observing, being, in such a case, supposed to be annihilated, it would be long before there would arise persons able and willing to supply such deficiency; the more so as those who might make such collections would have still to seek for the mode of turning them to any use. If, in this state of things, a few persons should, by their own sagacity and labour, or by the aid of some traditionary secret, attain to the power of predicting phenomena with tolerable correctness, we may imagine that they would use their peculiar skill for purposes of gain, and that they would not readily admit the world at large to the knowledge of the secret which gave them a superiority over the rest of their countrymen.

Our knowledge of the tides, at the present time, exactly realizes this imaginary condition which we have supposed for astronomy in general. Our philosophers assert, without hesitation, that this phenomenon is the result of the law of the universal gravitation of matter; yet no one has hitherto deduced, from this law, the laws by which the phenomena are actually regulated with regard to time and place. Analysis has been largely used; but it has been employed only to deduce the consequences of certain assumed suppositions, which suppositions are acknowledged to be utterly different from the real state of the case: and where is the immediate advantage, for the purposes of sound philosophy, of analysis which does not solve the problem proposed, over no analysis at all? Some observations of the tides have no doubt been made, and more are now making; but it is not too much to say, that these are only a commencement of the collections which the subject will require, to place it on a par with the other provinces of physical astronomy. The laws which connect the course of the observed tides with the motions and distances of the sun and moon are not known for any single port; and the tables, which in every other province of physics are the result of the knowledge which our men of science have accumulated for us, are, in this department, published by persons possessing and professing no theoretical views on the subject; and the methods by which they are calculated are not only not a portion of our published knowledge, but are guarded as secrets, and handed down as private property from one generation to another*.

Of course it cannot be intended here to speak with any disrespect of the persons who have calculated tide tables under these circumstances. Their labours are useful to the community in proportion as their tables are exact, which some of them are to a very remarkable degree. And, as no one thinks of condemning other persons who make a profit of any peculiar and secret knowledge which they may possess connected with any of the useful arts, there would be no justice in blaming those who do the same with respect to secrets which concern one of the most important arts, namely, navigation. But the circumstance most worthy of remark is, that there *should be* secrets in such a matter; that on such a subject our men of science should be ignorant of, and unable to discover, that which persons of much less elevated pretensions know and apply; that the laws which are to be collected either by the observation of facts, or by the deductions of theory, should not be known to our philosophers by either method, and yet should be in the possession of other persons, to a considerable extent. This circumstance makes our knowledge of the tides assume the character rather of a mere practical art, than of a portion of that complete and perfect science of which the other consequences of the law of universal gravitation supply examples.

Some persons may conceive that, in what has been said, I am disparaging too much the labours of the great mathematicians, NEWTON, BERNOULLI, LAPLACE and others, who have employed their skill on this subject. But this opinion cannot, I conceive, be maintained with justice. It is well known that all the mathematical solutions of the problem have confessedly gone upon suppositions very remote from the real facts: NEWTON and BERNOULLI, for instance, have assumed the form of the fluid spheroid, under the influence of the sun and moon, to be the form of equilibrium: LAPLACE has supposed the whole globe to be covered with water of an uniform depth. It is in no degree clear, that investigations conducted on such assumptions will give us even an approximation to the true result; and the only way in which the assumptions could be justified, would be by our finding, from observation, that the laws of the facts are such, or nearly such, as these hypothetical calculations give. If this agreement were

* What is here asserted was strictly true till the publication of Mr. LUBBOCK's Memoir on the Tides of the Port of London, and his Tide Tables, founded on his discussion of these. At present his Tide Tables are calculated by published methods; but the laws which these methods imply have not yet been compared with theory.

established, it would then, no doubt, become highly probable that the simplifications hypothetically introduced into the natural state of things were not such as materially to alter the general course of the phenomena.

But this has not been done by any of the theoretical writers above referred to. Undoubtedly most of them have undertaken to show that *some* of the known laws of the facts are accounted for by the theory, and that the measures of *some* of the phenomena agree with those which theoretical calculations give. But this has been executed only with respect to a few of the circumstances of the case. It has not been shown, by any writer, that the *general course* of the effects produced upon the tides, by the changes of position and distance of the heavenly bodies, is such as, according to the mathematical reasoning, it ought to be. In short, the mathematicians who have treated this subject have not completed their task by giving rules for the calculation of tide tables, and showing that the tables so produced agree with the general course of the observations in all essential circumstances.

The task just mentioned would consist of two parts; the theoretical deduction of the effects produced in the tides by changes of distance and position of the sun and moon; and the examination of the laws which such changes appear to follow in the observations; with a comparison of the two sets of results. The latter part of the task had not been executed, so far as I am aware, by any one, previously to Mr. LUBBOCK's discussion of the Tides of the Port of London, inserted in the Philosophical Transactions for 1831; and that memoir is hitherto the only published record of such an examination. The establishment, on theoretical grounds, of rules for the calculation of tide tables, has been attempted by BERNOULLI and by LAPLACE. The methods recommended by the former are probably the foundation of those at present used by the calculators of such tables. The method of LAPLACE is complicated, and would be very laborious in practice. He has unfortunately, as appears to me, not put his process in such a form as to give a principal term, with smaller corrections for declination, parallax, and other circumstances if necessary, to be combined with the principal term. When the results of such an investigation are not made to assume this shape, the comparison of the formula with observation becomes a work of very repulsive labour and trouble.

It has already been stated, that some of the published tide tables are found to be not very incorrect when compared with observation. If any tide tables were so good that they might be considered as representing the general laws of the actual phenomena, we might discuss such tables, and compare them with theory, in the same manner as if they were the records of observation; and with this additional advantage, that they would be free from the effect of the accidental causes, as wind and other circumstances, which produce irregularities in the actual tide. Nor would it be difficult, by such a discussion, to discover the rules which are followed in the construction of such tables.

It may, however, be doubted whether there are any tables which are worth this trouble. Original tide tables are very few: I know of none except those which are published for Liverpool, and those for London. The former are remarkably exact; they are calculated according to rules obtained by Mr. HOLDEN, some years ago, from the examination of five years of observations made at the Liverpool Docks by Mr. HUTCHINSON, at that time harbour-master. The calculations are at present conducted by the Rev. GEORGE HOLDEN, of Maghull, a descendant of the person who first invented the rules. Other Liverpool tide tables are also calculated by Mr. WOFFINDEN. Of London tides several, apparently independent, tables are annually published; and though the differences of these are considerable, I do not know that any one set is considered as possessing a decided superiority in the general result. I am not aware that any tide tables are published for Brest, though so large a collection of observations has been made at that port, and though so much labour has been employed in the discussion of these, for the purpose of comparing certain points of LAPLACE's theory with them: nor have, I believe, tide tables for any place been calculated according to the method recommended in the Mécanique Celeste.

The method generally practised in England for the construction of tide tables for other places has been, to take the time which is stated in the London or the Liverpool tables, and, if necessary, to add or subtract some constant quantity, according to the place. The Liverpool tide tables are in this manner used, generally without correction, for the whole of the north-western coast of England: and tables are published professing to give the hours at most of the principal ports of England, in parallel columns; the hours at different places having constant differences. Thus the hour of high water at Plymouth is stated as always $1^h\ 55^m$ later than the hour in the same half-day at London. This assumption of a constant difference in the hours of high water at different places is, however, inexact; as we should expect it to be from considering the mode in which the tide is transmitted from one place to another, and as it appears to be from observation.

It appears, therefore, that the most promising mode of advancing our knowledge of the tides, is to examine the laws which can be collected from observation, taking so great a number of observations, that the effects of all accidental causes may disappear in the average results. The collection of observations discussed by Mr. DESSIOU, under the direction of Mr. LUBBOCK, affords us an admirable opportunity for this examination; the collection including 13073 observations, and a period of nineteen years, from January 1st, 1808, to December 31st, 1826. Our object in this examination being to ascertain the manner in which the positions and distances of the heavenly bodies affect the time and height of high water, the mode of proceeding must be to examine how these two quantities depend upon the right ascension, declination and parallax of the sun and moon, and upon other astronomical elements, if such are found to be needed. The mean time of high water will be found to be affected by inequalities, depending on the elements just mentioned; and the law and amount of these inequalities may be collected from observations, without any reference to theory, (provided the observations are sufficiently numerous and their circumstances sufficiently varied,) in the same manner in which the greater inequalities of the moon, the variation, evection and annual equation, were detected by observation, long before the motions of the heavenly bodies were referred to their true causes. Indeed, I believe the instances are comparatively few in the history of philosophy, in which the general laws of the phenomena have been pointed out by the theory before they had been gathered by observation. The laws of the tides, thus empirically obtained, may be used either as tests of the extant theories, or as suggestions for the improvement of those portions of mathematical hydraulics on which the true theory must depend. And this is the way in which we are most likely to discover how the theory must be applied. The problems regarding the motion of fluids, which we are unable to solve directly, are far too numerous to allow us to be surprised that we should be obliged to desert the *à priori* road in this case. The phenomena of waves, the motions of water in tubes, in canals, in rivers, the motion of winds, the resistance of fluids to bodies in motion, are all cases in which we are yet far from having drawn our analytical mechanics into a coincidence with experiment, or even a tolerable proximity to it. The theoretical analysis of the tides is, at present, in an equally imperfect state. It is not at all improbable that, as in many other cases, this problem in the mechanism of the solar system (for such it is) may be found in the end less complex and difficult than similar problems concerning the motions of smaller masses; but the problem remains still to be solved, or at least it remains still to be shown that the solution has been approximated to. I shall therefore here proceed to examine the empirical laws of the tides of the port of London, as they appear from the records of the nineteen years of observations above mentioned.

Chap. I. *On the Empirical Laws of the Time of High Water.*

The point which I have first to determine is, the manner in which the time of high water is affected by the right ascensions, declinations, and parallaxes of the sun and moon. For this purpose I shall have to consider *the establishment, the semimenstrual inequality, the corrections for lunar parallax, lunar declination,* and *solar parallax and declination.*

1. *The Establishment.*—The vulgar establishment of any port is the interval of time by which the time of high water follows the moon's transit *on the day of the new and full moon.* But it is the *mean* value of this interval of time which we must here employ, in order to simplify our discussion. This is what Laplace calls the *fundamental hour* of the port: I have termed it, in a former paper on this subject, the *corrected establishment,* since it is the lunar hour of high water, freed from the semimenstrual inequality. Its value at the London Docks is $1^{h}\ 26^{m}$, by the mean of all the observations.

2. *The Semimenstrual Inequality.*—The interval of tide and moon's transit is affected by a considerable inequality, which goes through its period twice in the space of one month: it may be considered as depending upon the moon's distance from the sun in right ascension; or, which is the same thing, on the solar time of the moon's transit. It has been examined by Mr. Lubbock, and shown to agree, with remarkable exactness, with the formula,

$$\tan 2(\theta' - \lambda') = -\frac{h \sin 2(\phi - \alpha)}{h' + h \cos 2(\phi - \alpha)},$$

in which λ' is the mean interval of the tide and transit, and θ' the correct interval; ϕ the solar time of the moon's transit, and α a constant quantity. The ratio of the quantity h' to h is 2·9884 : 1; the quantity α is 2 hours.

According either to the method of BERNOULLI or to that of LAPLACE, there would result from the theory an expression of the above form, for the interval of tide and moon's transit. By assuming suitably the values of $\frac{h}{h'}$ and α, the results of observation at other places may also be made to agree very closely with the above formula. The curves which represent by their ordinates the successive values of the above formula, when constructed for different places, exhibit a remarkable general similarity, as may be seen in the Philosophical Transactions, 1831, where Mr. LUBBOCK has given these curves for Portsmouth, Plymouth, Sheerness, London and Brest. The curve is symmetrical with respect to the axis, intersecting it when $\phi = \alpha$, and when $\phi = \alpha +$ a fourth of a circumference. Its ordinate has a negative minimum and a positive maximum, which are equal in magnitude; but these values are not midway between the values 0, consequently the ordinate increases more rapidly after the minimum and before the maximum, than it diminishes before the minimum and after the maximum. This property appears very clearly in the curves constructed for all the above ports.

But in other respects the result of the observations, thus compared, does not agree with the theory. According to the theory, the quantities h and h' express the amount of the separate solar and lunar tides respectively, and as the ratio of these effects must be the same for all places, the maximum value of the semimenstrual inequality ought to be the same in all the above cases; namely, the time corresponding to half the angle whose tangent is $\frac{h}{\sqrt{h'^2 - h^2}}$. If, as LAPLACE finds from the Brest observations, $\frac{h'}{h} = 2·6157$, the angle corresponding to the above tangent is 22° 28′; the maximum value of the inequality is 45^{m}, and the double of this, or $1^{h}\ 30^{m}$, is the difference of the greatest and least interval of the tide and moon's transit.

According to observation, the difference of the greatest and least intervals is as follows*:

London	$1^{h}\ 28^{m}$
Sheerness	1 29
Portsmouth	1 21
Plymouth	1 36
Brest	1 19

It appears unlikely that the difference in these values for Plymouth and Brest, or even Plymouth and Portsmouth, can depend upon accidental causes, or too limited a number of observations. It would appear, therefore, that the *coefficient of the semimenstrual inequality*, $\left(\frac{h}{h'}\right)$, is different at different places; a circumstance which no extant theory would have led us to expect. This subject, however, deserves further

* We suppose here the effects of parallax and declination to be eliminated by the averages of the observations.

examination; and it would be important for this, as well as for other purposes, to discuss some large collection of observations for other places than London, in the mode which Mr. LUBBOCK has applied to the London observations. Such collections are known to exist for Brest and for Liverpool.

The quantity α in the formula is undoubtedly different at different places. It is what Mr. LUBBOCK, following LAPLACE, calls the *retard,* and depends upon what I have termed the *age of the tide.* It cannot be determined with certainty or exactness without the use of a large body of observations. Its value at London is 2^{h}, at Brest $1^{h}\ 12^{m}$; at Portsmouth it is intermediate between the value at Brest and at London, as we should expect, being about $1^{h}\ 30^{m}$; but at Plymouth it is greater than it is at London, which, as Mr. LUBBOCK observes, is at present a very inexplicable circumstance; probably to be explained only by the determination of the value of this quantity for several other places.

3. *The Correction for Lunar Parallax.*—Mr. LUBBOCK has classified the tide observations which he has discussed according to the value of the moon's horizontal parallax which existed at the time when the tide occurred, and also according to the hour of the moon's transit, so as to form a table of double entry of the differences from the mean interval: this is Table XVII. in his Memoir of 1831, which I here insert.

TABLE showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) for every Minute of the Moon's Horizontal Parallax.

Moon's Transit.	H. P. 54′.	H. P. 55′.	H. P. 56′.	H. P. 57′.	H. P. 58′.	H. P. 59′.	H. P. 60′.	H. P. 61′.
h m	m	m	m	m	m	m	m	m
0 0	+12	+ 9	+ 4	....	− 3	− 4	−13	−14
0 30	+12	+ 9	+ 2	+ 2	− 3	− 5	− 9	−11
1 0	+10	+ 8	+ 3	+ 5	− 1	− 4	− 9	−11
1 30	+ 8	+ 5	+ 3	+ 5	− 1	− 3	−10	−11
2 0	+ 8	+ 6	+ 2	+ 3	+ 1	− 1	− 8	− 9
2 30	+ 7	+ 5	+ 1	+ 1	+ 2	− 2	− 6	− 8
3 0	+ 6	+ 4	+ 2	....	+ 2	− 2	− 6	
3 30	+ 6	+ 4	+ 3	+ 1	+ 3	− 2	− 5	
4 0	+ 4	+ 3	+ 2	− 1	+ 2	− 2	− 6	
4 30	....	+ 1	+ 3	− 1	− 1	− 2	− 8	
5 0	+ 1	+ 1	+ 3	+ 1	0	− 1		
5 30	+ 1	0	− 1	+ 2	− 1	− 1		
6 0	+ 1	+ 1	− 3	+ 1	− 2	− 2		
6 30	+ 2	+ 4	− 3	− 1	− 3	− 3		
7 0	+ 4	+ 2	− 3	− 2	− 5	− 4		
7 30	+ 9	− 2	− 2	− 4	− 7	− 7	− 7	
8 0	+16	0	0	− 3	− 8	− 6	−11	
8 30	+21	+ 8	+ 3	+ 1	− 7	− 6	−12	
9 0	+19	+ 9	+ 4	+ 1	− 9	−10	−16	
9 30	+17	+11	+ 7	+ 1	− 8	−11	−18	
10 0	+16	+12	+ 8	0	− 5	−12	−17	
10 30	+15	+13	+ 8	− 1	− 2	−10	−14	−17
11 0	+13	+12	+ 7	− 2	− 2	− 8	−14	−18
11 30	+13	+10	+ 6	− 1	− 2	− 4	−16	−16

On examining this Table, it appears that in the column corresponding to H. P. 57′, the differences from the mean, or corrections for parallax, are very small for all hours of the moon's transit, (ranging from $+ 5^m$ to $- 4^m$,) and that the positive nearly balance the negative values. We may suppose, therefore, that for H. P. 57′ nearly, the correction is 0. It appears also that the correction is generally negative when the H. P. is greater than 57′, and positive when it is less, the exceptions being of small amount compared with the general mass of observations; and if we take the sums in each vertical column of Table XVII. we shall find that they are nearly as the difference of the parallax from the mean value 57′. It appears, therefore, that this correction must involve a factor $(P - p)$, when P is the mean horizontal parallax of the moon (or 57′), and p any other value of her horizontal parallax.

If we take any vertical column of this Table, and thus follow the correction through the various hours of the moon's transit, we find that for all values of the parallax the correction is very small, when the moon passes at $5^h\ 30^m$ or 6^h, and that the positive and negative values in that case nearly balance each other. In each column, when the hour of transit is either greater or less than this, the correction increases with the difference of hour, and proceeds to a maximum, which appears to occur about 9^h or 10^h transit. As a simple way of satisfying these conditions, we may suppose the correction to involve the factor $\sin^2(\phi - \beta)$ when β is a constant quantity: and combining this factor with the one already found, we shall have $B\,(P - p)\sin^2(\phi - \beta)$ for this correction in minutes of time.

It appears that in order to give the maximum value of this correction when it occurs at about 10^h, β must not be much different from 4^h. In order to determine B, take the formula $B\,(P - p)\sin^2(\phi - \beta)$ for every half-hour: its value is

$$2\,B\,(P - p)\,\{\sin^2 7\tfrac{1}{2}^\circ + \sin^2 15^\circ + \sin^2 22\tfrac{1}{2}^\circ + \sin^2 30^\circ + \sin^2 37\tfrac{1}{2}^\circ + \sin^2 45^\circ$$
$$+ \sin^2 82\tfrac{1}{2}^\circ + \sin^2 75^\circ + \sin^2 67\tfrac{1}{2}^\circ + \sin^2 60^\circ + \sin^2 52\tfrac{1}{2}^\circ\}$$
$$= 11\,B\,(P - p).$$

Comparing this with the sums for H. P. 54′, 55′, 56′, 57′, 58′, 59′, (the other columns being incomplete,) we have,

Horizontal parallax..	54′	55′	56′	57′	58′	59′
Formula	33 B	22 B	11 B	0	−11 B	− 22 B
Observed sums	221	135	59	8	−60	−112

Hence, taking the sums, $99\,B = 595$, whence $B = 6$, and the expression is $6\,(P - p)\sin^2(\theta - 4^h)$.

This may be put in the form $3\,(P - p)\left(1 - \cos 2\,(\theta - 4^h)\right)$,

or $$3\,(P - p)\left(1 + \sin 2\,(\theta - 1^h)\right).$$

The agreement with the sums observed is as follows:

Horizontal parallax..	54′	55′	56′	57′	58′	59′
Formula	198	132	66	0	−66	−132
Observed sums......	221	135	59	8	−60	−112

And on applying this correction, the residual quantities are, with one or two exceptions, within the limits $+ 4^{m}$ and $- 4^{m}$.

4. *The Correction for Lunar Declination.*—In Table IX. of his Memoir, Mr. LUBBOCK has arranged the intervals of the time of tide and moon's transit according to the declination of the moon, taken for every three degrees; and in Table XIX. he has given the difference of these intervals from the mean, arranged according to declination and time of transit. On inspecting this Table, it appears that this difference from the mean, or correction for lunar declination, is, for all values of the time of transit, 0 when the declination has its mean value of about 16°, positive when the declination is less, and negative when it is greater than this. The correction for a given declination, as shown in the vertical columns, is not constant, but it appears difficult to determine whether the variations are accidental or are the consequences of the form of the correction. Till we have better data, I will neglect these variations.

Taking, then, the sums of the vertical columns in Table XIX., we find as follows (the sums being expressed in minutes):

TABLE showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) for every Three Degrees of the Moon's Declination.

Moon's Transit.	0	3° Decl.	6° Decl.	9° Decl.	12° Decl.	15° Decl.	18° Decl.	21° Decl.	24° Decl.	27° Decl.
h m	m	m	m	m	m	m	m	m	m	m
0 0	+ 8	+ 5	+ 7	+ 5	+ 2	+ 2	− 3	− 6	−11	−10
0 30	+ 9	+ 6	+ 9	+ 7	− 2	....	− 2	− 4	− 7	−11
1 0	+ 8	+ 8	+10	+ 5	+ 2	+ 2	− 3	− 5	− 6	−11
1 30	+ 5	+ 8	+ 7	+ 2	+ 4	+ 3	− 4	− 7	− 6	−12
2 0	+ 6	+ 8	+ 6	+ 4	+ 3	+ 3	− 2	− 5	− 5	−10
2 30	+ 6	+ 8	+ 5	+ 7	+ 2	+ 2	− 1	− 4	− 5	− 9
3 0	+ 9	+ 9	+ 6	+ 7	+ 4	+ 2	− 2	− 7	− 7	−11
3 30	+11	+11	+ 9	+ 7	+ 5	+ 2	− 2	− 9	−10	−13
4 0	+ 9	+10	+10	+ 7	+ 7	+ 1	0	− 7	−10	−16
4 30	+ 8	+ 8	+ 8	+ 8	+ 8	− 1	0	− 6	−11	−18
5 0	+13	+12	+11	+12	+ 9	0	0	− 4	− 9	−16
5 30	+17	+14	+12	+13	+ 6	0	− 3	− 5	−10	−14
6 0	+20	+16	+13	+13	+ 6	+ 1	− 5	− 7	−13	−17
6 30	+21	+19	+12	+13	+ 8	....	− 7	− 9	−18	−21
7 0	+21	+19	+12	+17	+10	+ 4	− 6	− 8	−19	−27
7 30	+16	+16	+14	+18	+10	+ 5	− 8	− 6	−20	−34
8 0	+14	+18	+16	+16	+ 8	+ 4	− 6	− 6	−23	−30
8 30	+16	+15	+15	+12	+ 8	+ 2	− 1	− 5	−24	−22
9 0	+13	+ 9	+12	+ 7	+ 7	− 2	− 5	− 8	−18	−17
9 30	+13	+ 7	+11	+ 6	+ 7	− 2	− 6	− 7	− 7	−12
10 0	+11	+ 6	+ 6	+ 6	+ 4	− 2	− 8	− 7	− 5	−13
10 30	+11	+ 8	+ 3	+ 6	+ 3	0	− 7	− 5	− 5	−13
11 0	+ 9	+ 7	+ 3	+ 4	+ 3	+ 1	− 6	− 6	− 9	−13
11 30	+ 8	+ 3	+ 4	+ 3	+ 4	+ 3	− 3	− 8	−14	−12
Sums	+282	+250	+221	+205	+128	+30	−90	−151	−272	−382

It is tolerably manifest that these sums decrease faster for the large declinations than for the small ones; and we shall probably see the law more clearly by referring the correction to declination 0° than to the mean declination. For this purpose subtract from each sum the correction for declination 0°, and we have

Declination..	0	3°	6°	9°	12°	15°	18°	21°	24°	27°
Corrected sum	0	−32	−61	−77	−154	−252	−372	−433	−554	−664

It appears that the numbers here are not very remote from the ratio of the squares of the sines of the declinations. In fact, if we take the formula $-3168 \sin^2 \delta$, we have

For declination	0	3°	6°	9°	12°	15°	18°	21°	24°	27°
Formula	0	−9	−34	−77	−138	−229	−303	−408	−525	−654

which may pass for a first approximation to the observed result, when it is considered how much the errors are increased by addition. Dividing by 24, this gives $-132^m \sin^2 \delta$ for the correction to be applied to each result calculated for declination 0. When δ is about 16°, the mean value, this correction is 11^m. Hence $11^m - 132^m \sin^2 \delta$ is the correction to be applied to the mean value. This gives us the following Table, which is a first approximation to Table XIX. of Mr. LUBBOCK.

For declination ..	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°
Correction	+ 11	+ 10	+ 9	+ 8	+ 5	+ 2	− 2	− 6	− 11	− 16
The sums........	+264	+240	+216	+192	+120	+48	−48	−144	−264	−384

Which agree nearly with

+282	+250	+221	+205	+128	+30	−90	−151	−272	−382

the observed results, with sufficient accuracy.

We may observe that the expression $11 - 132 \sin^2 \delta$ is 0 when $\sin^2 \delta = \frac{1}{12}$ or $\delta =$ 16° 45′. This is the mean value of Δ, because the correction is applied to the mean. Therefore the expression is $132 (\sin^2 \Delta - \sin^2 \delta)$.

In each vertical column of Table XIX., the value appears to be greatest and least when the time of the moon's transit is about 7^h and 1^h. Hence we shall take the correction given by the above formula, and try whether the *residual phenomenon*, after this correction has been applied, is governed by any fixed rule.

For this purpose apply the above correction with an opposite sign to Mr. LUBBOCK'S Table XIX. The numbers in the columns are minutes.

TABLE XIX. freed from the Term $11 - 132 \sin^2 \delta = 132 (\sin^2 \Delta - \sin^2 \delta)$.

Decl.	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°	Sums.	
Corr.	−11	−10	− 9	− 8	− 5	− 2	+ 2	+ 6	+11	+16		
☽'s Transit.											0° to 15°.	18° to 27°.
h m												
0 0	− 3	− 5	− 2	− 3	− 3	0	− 1	0	0	+ 6	−16	+ 5
0 30	− 2	− 4	0	− 1	− 7	....	0	+ 2	+ 4	+ 5	−14	+11
1 0	− 3	− 2	+ 1	− 3	− 3	0	− 1	+ 1	+ 5	+ 5	−10	+10
1 30	− 6	− 2	− 2	− 6	− 1	+ 1	− 2	− 1	+ 5	+ 4	−16	+ 6
2 0	− 5	− 2	− 3	− 4	− 2	+ 1	0	+ 1	+ 6	+ 6	−12	+13
2 30	− 5	− 2	− 4	− 1	− 3	0	+ 1	+ 2	+ 6	+ 7	−15	+16
3 0	− 2	− 1	− 3	− 1	− 1	0	0	− 1	+ 4	+ 5	− 8	+ 8
3 30	0	+ 1	0	− 1	0	0	0	− 3	+ 1	+ 3	+ 0	+ 1
4 0	− 2	0	+ 1	− 1	+ 2	− 1	+ 2	− 1	+ 1	0	− 1	+ 2
4 30	− 3	− 2	− 1	0	+ 3	− 3	+ 2	0	0	− 2	− 6	0
5 0	+ 2	+ 2	+ 2	+ 4	+ 4	− 2	+ 2	+ 2	+ 2	0	+12	+ 6
5 30	+ 6	+ 4	+ 3	+ 5	+ 1	− 2	− 1	+ 1	+ 1	+ 2	+17	+ 3
6 0	+ 9	+ 6	+ 4	+ 4	+ 1	− 1	− 3	− 1	− 2	− 1	+23	− 7
6 30	+10	+ 9	+ 3	+ 4	+ 2	....	− 5	− 3	− 7	− 5	+29	−20
7 0	+10	+ 9	+ 3	+ 6	+ 5	+ 2	− 4	− 2	− 8	−11	+35	−25
7 30	+ 5	+ 6	+ 5	+ 7	+ 5	+ 3	− 6	− 0	− 9	−18	+31	−33
8 0	+ 3	+ 8	+ 7	+ 5	+ 3	+ 2	− 4	− 0	−12	−14	+26	−30
8 30	+ 5	+ 5	+ 6	+ 1	+ 3	0	+ 1	+ 1	−13	− 6	+20	−17
9 0	+ 2	− 1	+ 3	− 1	+ 2	− 4	− 3	− 2	− 7	− 1	+ 1	−13
9 30	+ 2	− 3	+ 2	− 2	+ 2	− 4	− 4	− 1	+ 4	+ 4	− 3	− 5
10 0	0	− 4	− 3	− 2	− 1	− 4	− 6	− 1	+ 6	+ 3	−14	+ 2
10 30	0	− 2	− 6	− 2	− 2	− 2	− 5	+ 1	+ 6	+ 3	−14	+ 5
11 0	− 2	− 3	− 6	− 4	− 2	− 1	− 4	0	+ 2	+ 3	−20	− 1
11 30	− 3	− 7	− 5	− 5	− 1	+ 1	− 1	− 2	− 3	+ 4	−20	− 2

From the changes of magnitude and sign, it appears that each vertical column of differences may be represented nearly by a term $A \sin 2 (\phi - \gamma)$, ϕ being the hour-angle of the first column, and γ a certain other angle. Also it appears that in each horizontal line, A passes from positive to negative, and *vice versâ,* when the declination passes through its mean value. Hence there is a factor $\delta - \Delta$, Δ being the mean value of the declination.

For declinations less than the mean, the maximum values of the correction are about the hours of transit $0^h\ 0^m$ and $7^h\ 0^m$. This would give for γ the value $3^h\ 30^m$.

For declinations greater than the mean, the maximum values of the correction would occur nearly when the hour of transit is $1^h\ 30^m$ or $7^h\ 30^m$. This would give for γ, $4^h\ 30^m$; the mean of this and the other value is 4^h.

Hence the formula for the above residual quantities will be

$$D (\Delta - \delta) \sin 2 (\phi - \gamma);$$

where, however, instead of $\Delta - \delta$, we may have other functions, as $\sin \Delta - \sin \delta$, $\sin^2 \Delta - \sin^2 \delta$. Hence the whole correction for lunar declination appears to be

$$132 (\sin^2 \delta - \sin^2 \Delta) + D (\delta - \Delta) \sin 2 (\phi - \gamma),$$

which will be simplified if we put $\sin^2 \delta - \sin^2 \Delta$ for $\delta - \Delta$; the expression then becomes,

$$(\sin^2\delta - \sin^2\Delta)\left(132 + \mathrm{D}\sin 2(\varphi - \gamma)\right).$$

We have to find D. For that purpose take the formula

$$(\sin^2\delta - \sin^2\Delta)\,\mathrm{D}\sin 2(\varphi - \gamma),$$

which expresses the residual phenomenon just given from Table XIX. Take the case where $\delta = 0$, and we have for the first vertical column, the expression

$$-\mathrm{D}\sin^2\Delta\sin 2(\varphi - \gamma).$$

The vertical column contains all the values of this for every half-hour of the value of $\varphi - \gamma$, that is, for values of $\sin 2(\varphi - \gamma)$ taken at intervals of 15° round the circumference. Taking the sum of these values for one semicircle, it is, by known formulæ,

$$\frac{\sin 90^\circ \times \sin 82\tfrac{1}{2}^\circ}{\sin 7\tfrac{1}{2}^\circ} = \tan 82\tfrac{1}{2}^\circ = 7{\cdot}5957.$$

Now this sum in the Table is 45 if we take the mean of the positive and negative values; observing, however, that this value compared with the succeeding columns appears to be smaller than the general course of the numbers would give it. Hence,

$$\mathrm{D}\sin^2\Delta \times 7{\cdot}5957 = 45\,;\ \mathrm{D} = 72 \text{ nearly}.$$

Hence $\mathrm{D}\sin^2\Delta = 6$. But it will agree better with the general numbers to make $\mathrm{D} = 7$, and the expression for the residual phenomenon is

$$84(\sin^2\delta - \sin^2\Delta)\sin 2(\varphi - \gamma).$$

The values of $84(\sin^2\Delta - \sin^2\delta)$ for the successive values of δ are hence found; and hence the corrections.

Assuming $\gamma = 4^{\mathrm{h}}$, the following Table represents the table of the residual phenomenon.

TABLE of the Expression $84(\sin^2\Delta - \sin^2\delta)\sin 2(\varphi - \gamma)$.

	$\delta =$	0°	3°	6°	9°	12°	15°	18°	21°	23°	27°
	$84(\sin^2\Delta - \sin^2\delta) =$	7	7	6	5	3	1	0	− 4	− 7	−10
φ.	$\sin 2(\varphi - \gamma)$.										
1^{h}	−1·000	− 7	− 7	− 6	− 5	− 3	− 1	0	+ 4	+ 7	+10
2	−0·866	− 6	− 6	− 5	− 4	− 3	− 1	0	+ 3	+ 6	+ 9
3	−0·100	− 4	− 4	− 4	− 3	− 2	− 1	0	+ 2	+ 4	+ 5
4	0·000	0	0	0	0	0	0	0	0	0	0
5	0·500	+ 4	+ 4	+ 4	+ 3	+ 2	+ 1	0	− 2	− 4	− 5
6	0·866	+ 6	+ 6	+ 5	+ 4	+ 3	+ 1	0	− 3	− 6	− 9
7	1·000	+ 7	+ 7	+ 6	+ 5	+ 3	+ 1	0	− 4	− 7	−10
8	0·866	+ 6	+ 6	+ 5	+ 4	+ 3	+ 1	0	− 3	− 6	− 9
9	0·500	+ 4	+ 4	+ 4	+ 3	+ 2	+ 1	0	− 2	− 4	− 5
10	·000	0	0	0	0	0	0	0	0	0	0
11	−0·500	− 4	− 4	− 4	− 3	− 2	− 1	0	+ 2	+ 4	+ 5
12	−0·866	− 6	− 6	− 5	− 4	− 3	− 1	0	+ 3	+ 6	+ 9

This agrees as to its changes of magnitude and sign, and as to the mean of the numbers, with the table of the residual quantities, p. 25. The formula,

$$(\sin^2\delta - \sin^2\Delta)\{132 + 84\sin 2(\varphi - 4^{\mathrm{h}})\}$$

will therefore express, with considerable accuracy, the general course and average values of the numbers in Mr. Lubbock's Table XIX.

But if the correction, instead of being applied to the *mean* value of the interval (of tide and transit), had been applied to the interval calculated for declination 0, it is clear the correction would have been

$$\{132 + 84 \sin 2 (\varphi - 4^h)\} \sin^2 \delta.$$

Mr. Lubbock has given other tables also, from which the mean correction for lunar declination may be collected. His Table XV., which contains the differences of the intervals of the time of moon's transit and high water from the mean interval, arranged according to the calendar months and to times of the moon's transit, is in fact principally a table of the correction for lunar declination. For by examination of that table, it will be seen that the correction in each month goes through its cycle of 0, +, 0, —, in one semirevolution of the moon; that is, while the declination passes from its maximum north, to its maximum south, value: and since these results are the mean of nineteen years, the moon will have been nearly as much on the north as on the south of the ecliptic, and the result will be nearly the same as if she had moved in the ecliptic. It may be observed, however, that it appears by what has been shown above, that the corrections increase faster than the declinations; and therefore the corrections due to the high declinations will not be quite balanced by those due to the declinations which correspond to an equal opposite celestial latitude.

It is to be noticed, also, that this Table XV., being arranged for calendar months, contains the effect of solar declination and parallax as well as of lunar declination. It also contains the effect of the equation of time; the times of the moon's transit being given in *mean* solar time, whereas we suppose the tide to depend on the hour-angle of the moon from the sun, that is, on the transit in true solar time. These effects may be eliminated, and the effect of the changes of lunar declination upon the tide-hour may be determined from this table in an approximate manner; but the accuracy of such a determination is necessarily less than that of the one already obtained, and I shall therefore not insert it here.

5. *The Solar Correction.*—The sums of the positive and of the negative numbers in each vertical column of Mr. Lubbock's Table XV. would be equal, if the inequality depended on the moon alone, since each column contains the corrections which occur in a half-revolution of the moon. Therefore the difference of these sums is due to a solar inequality, and the mean excess or defect must be subtracted or added in order to obtain the corrections due to the moon. These means are as follow:

	Jan.	Feb.	March.	April.	May.	June.	July.	August.	Sept.	October.	Nov.	Dec.	
	+105 — 61	+ 48 —147	+ 63 —144	+64 —96	+90 —79	+120 — 71	+128 — 62	+110 — 68	+90 —90	+ 99 —111	+ 98 —120	+103 — 74	
	+ 44	— 99	— 81	—32	+11	+ 49	+ 66	+ 42	0	— 12	— 22	+ 29	
Means	+ 2	— 4	— 3	— 1	0	+ 2	+ 3	+ 2	0	— 1	— 1	+ 1	Solar inequality.

It appears that this correction changes from positive to negative four times in the course of the year, and hence may be approximately represented by $m \sin 2 (\theta - \mu)$ when θ is the sun's right ascension. But the maximum and minimum values in different parts of the year are of unequal magnitude and at unequal intervals. This may be reconciled with an expression of the form $m \sin 2 (\theta - \mu) + n \sin (\theta - \nu)$; and we might determine m, n, μ, ν, so as to make the expression agree nearly with the result of observation. In fact, however, this would not be worth while, except we had the empirical law confirmed by the results of observations at other places; for the greatest values of this correction are $- 4^{m}$ and $+ 3^{m}$. We here exclude the effects of the equation of time.

It is not difficult to see why the solar correction assumes such a form as this. It includes the corrections due both to the sun's declination and his parallax. The former effect is twice a minimum and twice a maximum in the course of a year; the latter once only. The two effects are not immediately separated in the tables, because the sun's perigee being nearly stationary, the cycle of changes due to solar parallax and the double cycle of changes due to solar declination coincide.

When the form and amount of the solar correction are more exactly determined, it may be more exactly compared with the theory.

CHAP. II. *On the Empirical Laws of the Height of High Water.*

The same kind of discussion of the observations which has enabled us to obtain approximately the laws of the *times* of high water, will also give similar information with respect to the *heights*, since these have been observed at the docks, and the results tabulated by Mr. LUBBOCK, in the same way as the others. The heights will be affected in the same way as the tides, by a *semimenstrual inequality*, by corrections for *lunar parallax and declination*, and by a *solar correction*.

1. *Of the Mean Level of the Water.*—The quantities which are wanted for the comparison of observed heights with the theory, are the total height of the tide, that is, the difference of high and low water. The heights of low water are not given in the London observations, and we have, therefore, only the differences of the high waters to reduce to their laws.

A comparison of these with the theory, supposes the mean level of the water to be constant, that is, the mean of the heights of high and low water to be the same, whatever be the height of the tide. I do not know whether this permanency of the mean level has been verified at the London Docks. It has been ascertained to be true in several other cases, and is probably universal, or at least liable to few and peculiar exceptions.

This mean level may be determined by the mean of many observations, and is a more fixed and distinct level than any level depending on a smaller number of observations. It is, moreover, free from the irregularities to which levels selected in any other way are exposed. Thus the level of high water, or of low water, at spring tides, or at neap tides, is different according to the different effects of lunar and solar parallax and declination.

I proceed to discuss the variations of the heights of the London tides.

2. *The Semimenstrual Inequality.*—The law of the heights is contained in the last column of Mr. LUBBOCK's Table V., which gives the heights for every half-hour, mean time, of the moon's transit, taking a mean of the months of the year.

In order to obtain a formula expressing this series of quantities, we observe that the maximum and minimum are nearly when the times of transit are 2^h and 8^h respectively; that the mean of the extreme heights is 21·1 feet, and the difference of the extremes 3·4 feet. Hence the height may be expressed approximately by the formula $21{\cdot}1 + 1{\cdot}7 \cos 2 (\phi - 30°)$. It appears, however, that the maximum and minimum occur a little earlier than 2^h and 8^h. I shall therefore assume for the height the formula $21{\cdot}1 + 1{\cdot}7 \cos (2 \phi - 51°)$; we shall then have the following comparison with observations.

ϕ	$\cos (2 \phi - 51)$	Height observed.	Height calculated.	Diff.	Diff. — ·23
h m		Feet.	Feet.		
2 0	·9877	22·78	22·80	·02	— ·21
3 0	·7771	22·42	22·59	·17	— ·06
4 0	·3584	21·71	22·10	·39	+ ·16
5 0	— ·1564	20·83	21·28	·45	+ ·22
6 0	— ·6293	20·03	20·37	·34	+ ·21
7 0	— ·9335	19·51	19·56	·05	— ·18
8 0	— ·9877	19·42	19·43	·01	— ·22
9 0	— ·7771	19·78	20·10	·32	+ ·09
10 0	— ·3584	20·49	20·92	·43	+ ·20
11 0	·1564	21·37	21·85	·48	+ ·25
0 0	·6293	22·17	22·46	·29	+ ·06
1 0	·9335	22·69	22·72	·03	— ·20

It is evident that this is a first approximation. Also the difference, which is always positive, follows the law of a sine. To show this, subtract from this difference ·23, as is done in the last column.

The difference in the last column will be 0 when ϕ is $0^h\ 30^m$ and $6^h\ 30^m$ nearly, and will, in the course of 6 hours of ϕ, go through all the values of $\sin \phi$. Hence it may be represented by $- c \sin (4 \phi - 30°)$, and it is evident that c is nearly ·23.

Comparison of the residual Phenomenon of the Semimenstrual Series of Heights, freed of the Terms $21{\cdot}1 + 1{\cdot}7 \cos (2 \phi - 51)$, with the Formula $\cdot 23 - \cdot 23 \sin (4 \phi - 30)$.

ϕ	$\sin (4 \phi - 30)$	Form.	Obs.	Excess of Obs.
h	Feet.	Feet.	Feet.	Feet.
0	— ·5	·35	·29	—·06
1	+ ·5	·11	·03	—·08
2	+ 1·0	0	·02	+·02
3	+ ·5	·11	·17	+·06
4	— ·5	·35	·39	+·04
5	— 1·0	·46	·45	—·01
6	— ·5	·35	·34	—·01
7	+ ·5	·11	·05	—·06
8	+ 1·0	0	·01	+·01
9	+ ·5	·11	·32	+·21
10	— ·5	·35	·43	+·08
11	— 1·0	·46	·48	+·02

It appears from the nature of the still residual differences, that we might bring our formula still nearer to observation; but as the differences do not exceed $\frac{1}{10}$th of a foot, except in one instance, this exactness would be, in the present state of our knowledge, superfluous.

Hence the mean height in feet of the tide at London Dock is represented by

$$21{\cdot}1 \;+ 1{\cdot}7 \cos (2\,\phi - 51^\circ) + {\cdot}23 - {\cdot}23 \sin (4\,\phi - 30^\circ),$$
$$\text{or } 21{\cdot}33 + 1{\cdot}7 \cos (2\,\phi - 51^\circ) - {\cdot}23 \sin (4\,\phi - 30^\circ)$$

where ϕ is the hour-angle of the moon's transit, mean time.

3. *Correction of the Heights for Lunar Parallax.*—Table XVIII. of Mr. LUBBOCK contains the effect of variations of the moon's distance.

TABLE showing the Difference in the Height of High Water, and the Mean Height for every Minute of the Moon's Horizontal Parallax.

Moon's Transit.	H. P. 54′.	H. P. 55′.	H. P. 56′.	H. P. 57′.	H. P. 58′.	H. P. 59′.	H. P. 60′.	H. P. 61′.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	−0·52	−·33	−·30	+·10	+·06	+·23	+ ·33	+·53
0 30	− ·50	−·28	−·20	+·08	+·17	+·41	+ ·36	+·53
1 0	− ·58	−·44	−·09	−·07	+·26	+·24	+ ·31	+·61
1 30	− ·55	−·46	+·03	−·07	+·23	+·23	+ ·41	+·77
2 0	− ·57	−·52	−·15	−·05	+·12	+·24	+ ·50	+·75
2 30	− ·54	−·45	−·28	+·10	+·08	+·34	+ ·69	+·79
3 0	− ·68	−·37	−·25	+·06	+·05	+·42	+ ·83	
3 30	− ·63	−·23	−·13	+·08	+·13	+·56	+1·00	
4 0	− ·57	−·28	−·12	+·14	+·20	+·66	+ ·98	
4 30	− ·49	−·29	−·13	+·23	+·67	+·76	+ ·83	
5 0	− ·50	−·40	−·30	+·25	+·53	+·80		
5 30	− ·47	−·40	−·37	+·37	+·50	+·86		
6 0	− ·45	−·28	−·19	+·27	+·50	+·77		
6 30	− ·54	−·21	−·09	+·05	+·40	+·62	+ ·43	
7 0	− ·64	−·29	−·11	+·05	+·31	+·54	+ ·53	
7 30	− ·75	−·48	−·28	+·01	+·15	+·33	+ ·58	
8 0	− ·68	−·30	−·25	+·05	+·02	−·08	+ ·54	
8 30	− ·54	−·27	−·26	+·01	−·20	+·42	+ ·44	
9 0	− ·26	−·23	−·21	−·11	−·23	−·76	+ ·39	
9 30	+ ·03	−·24	−·15	−·26	−·09	+·30	+ ·45	+·87
10 0	− ·09	−·30	−·11	−·14	+·14	+·36	+ ·49	+·69
10 30	− ·20	−·37	−·07	+·03	+·32	+·36	+ ·48	+·55
11 0	− ·31	−·31	−·14	+·21	+·26	+·26	+ ·51	+·61
11 30	− ·43	−·22	−·25	+·27	+·13	+·14	+ ·46	+·68

If we take the means of the vertical columns, they are, in hundredths of feet,

Horizontal parallax	54′	55′	56′	57′	58′	59′	60′	61′
Means	−47	−33	−18	+7	+20	+37	+47	+67

These are very nearly as the differences of the parallax. We shall find that the formula $1{\cdot}7\,(p - \mathrm{P})$ when p is the parallax, and P is $57'$, will very nearly give this result. It gives, in fact,

−51 −34 −17 0 +17 +34 +51 +68

Hence, applying these corrections with opposite signs to Table XVIII., we obtain the residual phenomenon as follows, in hundredths of feet.

TABLE XVIII. freed from the Sum 1·7 $(p - P)$.

Hor. Par...	54′	55′	56′	57′	58′	59′	60′	61′
Corr.	+51	+34	+17	0	−17	−34	−51	−68
h m								
0 0	− 1	+ 1	−13	+10	−11	+ 11	−18	−15
0 30	+ 1	+ 6	− 3	+ 8	0	7	−15	−15
1 0	− 7	−10	+ 8	− 7	+ 9	− 10	−20	− 7
1 30	− 4	−12	+20	− 7	+ 6	− 11	−10	+ 9
2 0	− 6	−18	+ 2	− 5	− 5	− 10	− 1	+ 7
2 30	− 3	−11	−11	+10	− 9	− 11	+18	+11
3 0	−17	− 3	− 8	+ 6	−12	+ 8	+32	
3 30	−12	+11	+ 4	+ 8	− 4	+ 32	+49	
4 0	− 3	+ 6	+ 5	+14	+ 3	+ 32	+47	
4 30	+ 2	+ 5	+ 4	+23	+50	+ 42		
5 0	+ 1	− 6	−13	+25	+36	+ 46		
5 30	+ 4	− 6	−20	+37	+33	+ 52		
6 0	+ 6	+ 6	− 2	+27	+33	+ 43		
6 30	− 3	+13	+ 8	+ 5	+23	+ 28	− 8	
7 0	−13	+ 5	+ 6	+ 5	+14	+ 20	+ 2	
7 30	−24	−14	−11	+ 1	− 2	− 1	+ 7	
8 0	−14	+ 4	− 8	+ 5	−15	− 42	+ 3	
8 30	− 3	+ 7	− 9	+ 1	−37	+ 8	+ 7	
9 0	+26	+11	− 4	−11	−40	−110	−12	
9 30	+54	+10	+ 2	−26	−26	− 4	− 6	+19
10 0	+42	+ 4	+ 6	−14	− 3	+ 2	− 2	+ 1
10 30	+31	− 3	+10	+ 3	+15	+ 2	− 3	− 6
11 0	+20	+ 3	+ 3	+21	+ 9	− 6	0	− 7
11 30	+ 8	+12	− 8	+27	− 4	− 20	− 5	0

There is no very manifest rule in this Table. There appear to be many large positive terms between the hours $2^h\,30^m$ and 7^h, and for parallaxes greater than 57′, while the terms for other hours are more generally negative.

But it appears to be better to wait for the examination of other observations than to attempt to found a formula on these circumstances.

4. *Correction of the Heights for Lunar Declination.*—Table XX. of Mr. LUBBOCK will supply the means of determining the law of the effect of the lunar declination upon the height, being a Table of the heights arranged according to intervals of 3° of declination, and according to the time of the moon's transit. We shall place here the Table, expressed in hundredths of feet.

TABLE showing the Difference in the Height of High Water, and the Mean Height for every Three Degrees of the Moon's Declination.

Moon's Transit.	0	3° Decl.	6° Decl.	9° Decl.	12° Decl.	15° Decl.	18° Decl.	21° Decl.	24° Decl.	27° Decl.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	− 10	+ 07	+ 06	+ 02	+ 04	− 06	− 10	− 08	− 34	− 42
0 30	+ 09	− 01	+ 06	− 05	+ 14	+ 10	− 03	− 15	− 10	− 20
1 0	+ 23	− 10	+ 11	− 01	− 02	+ 07	+ 02	+ 03	− 07	− 35
1 30	+ 34	− 13	+ 16	+ 08	− 04	+ 07	+ 12	+ 13	− 06	− 45
2 0	+ 40	− 25	+ 10	+ 15	+ 03	− 01	+ 15	− 03	− 15	− 25
2 30	+ 39	− 25	+ 08	+ 25	+ 22	+ 01	+ 24	− 18	− 18	+ 01
3 0	+ 27	− 10	+ 03	+ 27	+ 21	+ 04	+ 21	− 09	− 28	− 09
3 30	+ 22	+ 13	....	+ 31	+ 16	+ 10	+ 17	+ 01	− 34	− 20
4 0	+ 14	− 01	+ 54	+ 20	....	+ 08	+ 12	+ 03	− 32	− 18
4 30	+ 17	− 05	+ 16	+ 10	− 10	+ 06	+ 14	− 08	− 13	− 08
5 0	+ 21	− 08	+ 37	+ 07	− 01	+ 09	+ 12	− 28	− 12	− 08
5 30	+ 22	− 09	+ 53	+ 04	+ 04	+ 08	+ 04	− 05	− 08	− 21
6 0	+ 13	+ 15	+ 51	+ 18	+ 16	+ 13	+ 03	+ 11	− 19	− 37
6 30	− 09	+ 37	+ 19	+ 27	+ 22	+ 11	− 03	− 06	− 45	− 49
7 0	+ 19	− 02	+ 31	+ 56	+ 26	+ 04	0	− 46	− 24	− 53
7 30	+ 52	+ 38	+ 32	+ 64	+ 15	− 13	− 08	− 35	− 19	− 70
8 0	+ 48	+ 60	+ 44	+ 57	+ 47	+ 17	− 07	− 28	− 20	− 61
8 30	+ 22	+ 71	+ 43	+ 25	+ 74	+ 36	− 10	− 28	− 37	− 64
9 0	+ 43	+ 65	+ 32	+ 15	+ 53	+ 25	− 10	− 28	− 50	− 41
9 30	+ 66	+ 54	+ 17	+ 28	+ 37	+ 06	− 12	− 31	− 56	− 43
10 0	+ 58	+ 58	+ 37	+ 44	+ 36	+ 06	− 06	− 26	− 37	− 44
10 30	+ 32	+ 54	+ 44	+ 42	+ 21	+ 02	− 15	− 29	− 19	− 64
11 0	+ 15	+ 41	+ 31	+ 31	+ 13	+ 05	− 15	− 14	− 30	− 61
11 30	+ 12	+ 34	+ 26	+ 28	+ 12	− 08	+ 02	+ 18	− 38	− 44
Sums {	+648 − 19	+547 −109	+627 0	+584 − 6	+461 − 17	+195 − 28	+138 − 99	+ 49 −365	0 −611	+ 1 −882
	+629	+438	+627	+578	+444	+167	+ 39	−316	−611	−811

It is clear that these sums decrease faster for the large declinations than for the small ones, and the series is tolerably regular with the expression of the number corresponding to declination 3°, which appears to be affected by some anomaly. If we reject this term, and subtract 629 from each of the terms, we find for

Declination ..	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°
Correction ..	0	(−191)	−2	−51	−185	−462	−590	−945	−1240	−1440

It will appear that the law may be expressed nearly by $- 7300 \sin^2 \delta$, which gives

−0 −20 −79 −178 −315 −490 −697 −937 −1207 −1504.

This agrees pretty well, except for the smaller numbers, which are obviously irregular. Hence, if Δ be the mean declination, we shall have the correction to be applied to the mean sums $= 7300 (\sin^2 \Delta - \sin^2 \delta)$; and the correction to the single terms will be $304 (\sin^2 \Delta - \sin^2 \delta)$.

If we suppose the mean declination to be 16° 45′, as appeared in the correction for the times, $7300 \sin^2 \Delta = 608$, and the corrections are,

Declination ..	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°
Correction ..	608	588	529	430	293	118	− 89	−329	−599	−896

Hence

25	24	22	17	11	4	−4	−14	−25	−38

are the corrections for the single terms of the Table.

We shall now apply this correction with a negative sign, in order to consider the law of the residual phenomenon.

TABLE XX. freed from the Term 304 ($\sin^2 \Delta - \sin^2 \delta$).

Decl.	0°	3°	6°	9°	12°	15°	18°	21°	24°	27°
Corr.	25	24	22	17	11	4	−4	−14	−25	−38
h m										
0 0	−35	−17	−16	−15	− 7	−10	− 6	+ 6	− 9	− 4
0 0	−15	−25	−16	−22	+ 3	+ 6	+ 1	− 1	+15	+18
1 0	− 2	−35	−11	−18	−13	+ 3	+ 6	+17	+18	+ 3
1 30	+ 9	−37	− 6	− 9	−15	+ 3	+16	+27	+19	− 7
2 0	+15	−49	−12	− 2	− 8	− 5	+19	+11	+10	+13
2 30	+14	−49	−14	+ 8	+11	+ 3	+28	− 4	+ 7	+39
3 0	+ 2	−34	−18	+10	+10	0	+25	+ 5	− 3	+29
3 30	− 3	−11	....	+04	+ 5	+ 6	+21	+15	− 9	+18
4 0	−11	−25	+32	+ 3	....	+ 4	+16	+17	− 8	+20
4 30	− 8	−29	−06	− 7	−21	+ 2	+18	+ 6	+12	+30
5 0	− 4	−32	+15	−10	−12	+ 5	+16	−14	+13	+30
5 30	− 3	−33	+31	−13	− 7	+ 4	+ 8	+ 9	+17	+17
16 0	−12	−19	+29	− 1	+ 5	+ 9	+ 7	+25	+ 6	+ 1
6 30	−34	+13	− 3	+10	+11	+ 7	+ 1	+ 8	−20	−11
7 0	− 5	−26	+ 9	+39	+15	0	+ 4	−32	+ 1	−15
7 30	+27	+14	+10	+47	+ 4	−11	− 4	−21	+ 6	−32
8 0	+23	+36	+22	+40	+36	+13	− 3	−14	+ 5	−23
8 30	− 3	+47	+21	+ 8	+63	+32	− 7	−14	−12	−26
9 0	+18	+41	+10	− 2	+47	+21	− 7	−14	−25	− 3
9 30	+41	+30	− 5	+11	+36	+ 2	− 8	−17	−31	− 5
10 0	+33	+34	+15	+27	+25	+ 2	− 2	−12	−12	− 6
10 30	+ 7	+30	+22	+25	+10	− 2	−11	−15	+ 6	−26
11 0	−10	+17	+ 9	+ 4	+ 2	+ 1	−11	0	− 5	−23
11 30	−13	+10	+ 4	+11	+ 1	−12	+ 6	+32	−13	− 6

Though this Table exhibits great anomalies, it appears clear that all the heavy minus terms, and only small positive terms, are in the upper left-hand and lower right-hand quarter; and that all the heavy positive terms, and only small negative terms, occur in the upper right-hand and lower left-hand quarters. Also the terms are on the whole larger in the outer than in the inner columns. It appears probable, therefore, that the law from which this proceeds involves a term $d\ (\sin^2 \delta - \sin^2 \Delta) \sin 2\varphi$, which would give such a result; but the coefficient of this term cannot be determined satisfactorily; and hence the effect of declination in the moon is probably of the form

$$(\sin^2 \Delta - \sin^2 \delta)\ (7300 - d \sin 2\varphi).$$

Hence the correction to be applied to the height calculated for declination 0, is

$$- \sin^2 \delta\ (7300 - d \sin 2\varphi),$$

or

$$- \sin^2 \delta \left(7300 + d \cos 2\ (\varphi + 45^\circ)\right).$$

We may make a remark with respect to Mr. LUBBOCK'S Tables for the heights, similar to one we made with respect to those for the times. Table XVI., which gives the differences of height for each hour of the moon's transit in the different calendar months, is in reality composed mainly of the effects of the moon's declination. In order to obtain these effects from the Table, we should have to eliminate the effects of the sun (including the effects of the equation of time). By this means we should obtain a result agreeing in part with that which we have obtained from Table XX.; but the accuracy of this result would necessarily be less than of that already obtained, and I shall omit it.

5. *The Solar Correction of the Heights.*—If we take the means of each month in Table XVI., we have the sun's effect on the heights in that month. These are as follow:

Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
−7	−23	−8	0	+17	+8	+8	−9	−1	+9	+8	+4

This, like the solar correction for the time, passes from positive to negative, and from negative to positive, four times in the course of the year, but has its maxima and minima of unequal magnitude, and at unequal intervals. Hence we may, as in the case of the times, express it by $m \sin 2(\theta - \mu) + n \sin(\theta - \nu)$, where θ is the sun's longitude; and we may account for this form by considering that the former term is the effect of declination, and the latter term the effect of parallax. To this is to be added the effect of the equation of time, in order to obtain the whole of the solar correction.

Recapitulation.—Hence it appears that the result of the London Dock observations, which we have now examined, may be expressed in the following manner.

If λ' be the corrected establishment, S′ the semimenstrual inequality of the time of high water, P′ the correction for lunar parallax, Q′ the correction for lunar declination, Q the solar correction, and if φ be the mean time of the moon's transit, we have for the time of high water

$$\varphi + \lambda' + S' + P' + Q' + Q.$$

In this expression it has appeared that

$$\tan 2\,S' = \frac{h \sin 2(\varphi - \alpha)}{h' + h \cos 2(\varphi - \alpha)}; \quad \frac{h'}{h} = 2{\cdot}9887; \quad \alpha = 2 \text{ hours.}$$

$$P' = (P - p)\{B + B \sin 2(\varphi - \beta)\}; \quad B = 3^{m}; \quad \beta = 1 \text{ hour.}$$

$$Q' = (\sin^2 \Delta - \sin^2 \delta)\{C + D \sin 2(\varphi - \gamma)\};$$

$$\Delta = 16^\circ\, 45', \ C = 132^{m}, \ D = 84^{m}, \ \gamma = 4 \text{ hours.}$$

$$Q = m \sin 2(\varphi - \mu) + n \sin(\varphi - \nu),$$

m, *n* being small, and their determination here omitted.

In like manner, if l be the height of the mean high water, s' the semimenstrual change, p' the correction due to lunar parallax, q' the correction due to lunar declination, q the solar correction, the height of high water is

$$l + s' + p' + q' + q.$$

In this expression, the numbers being feet, l at the London Docks is 21·33 above the origin of the measures used in the Tables.

$$s' = 1{\cdot}7 \cos (2\,\varphi - 51^{\circ}) - {\cdot}23 \sin (4\,\varphi - 30^{\circ})$$
$$p' = (p - \mathrm{P}) \{{\cdot}17 + b \cos 2\,\varphi\}$$
$$q' = (\sin^2 \Delta - \sin^2 \delta) \{73 - d \cos 2\,(\varphi - 45)\}$$
$$q = m \sin 2\,(\varphi - \mu) + n \sin (\varphi - \nu),$$

b, d being not clearly shown by the London observations to be constant terms; and m, n, as before, being small, and their determination for the present omitted.

CHAP. III. *Comparison of the preceding Results with the Theory.*

I shall now compare the preceding results with the theory of DANIEL BERNOULLI, according to which the waters of the ocean assume nearly the form in which they would be in equilibrium under the action of the sun and moon; and a supposition being made that the pole of the fluid spheroid follows the pole of the spheroid of equilibrium at a certain distance (namely, at an hour-angle λ'), and that the equilibrium corresponds to the configuration of the sun and moon, not at the moment of the tide, but at a previous moment, at which the right ascension of the moon was less by a quantity α.

I take this theory rather than that of LAPLACE, not only because of the difficulty and labour of the comparison in the latter case, but also because the hypothesis on which LAPLACE's solution proceeds appears to me likely to affect the results, so as to make them differ altogether from those of the real case; and because the assumption, by means of which his solution is obtained, appears to me to be very insecure.

According to the theory of BERNOULLI, we have

$$\tan 2\,(\theta - \lambda') = \frac{h \sin 2\,(\varphi - \alpha)}{h' + h \cos 2\,(\varphi - \alpha)}\,; \qquad \ldots\ldots\ldots\ldots \quad (1.)$$

where θ is the hour-angle corresponding to the place of high water measured from the moon, φ the hour-angle of the moon from the sun, h, h' the heights of the solar and lunar tides, λ' the hour-angle by which the tide follows the pole of equilibrium, α the *retardation*, or difference of right ascension of the moon due to the age of the tide.

Neglecting the effects of parallax and declination, this expression gives the law of the semimenstrual inequality; and this, as we have already said, agrees very clearly with observation, assuming proper values of $\frac{h}{h'}$, and of α.

But we find here some circumstances in which the theory and observation are difficult to reconcile. The value of $\frac{h'}{h}$, the ratio of the lunar and solar tide, ought to be the same at all places. We find, however, that the Brest observations give 2·6167, while the London observations require it to be 2·9887; and other places give values still more different.

Also the differences of the value of α for different places might be supposed to depend necessarily upon the time of the transmission of the tide from one place to another, and therefore to increase as we follow the tide. But it appears that though at Portsmouth this retardation is intermediate between that at Brest and at London, as it might be expected to be from the course of the tide-hours, yet that at Plymouth, where the tide is five or six hours earlier than it is at Portsmouth, the retardation implies a tide as late as London.

Leaving, however, these anomalies to be removed or confirmed by the accumulation and discussion of observations, I proceed to the effects of parallax and declination.

1. If from any cause h' receive a small increment $\delta h'$, we can easily find the corresponding change, $\delta \tan 2(\theta' - \lambda')$, in the first side of equation (1.). We have

$$\delta \tan 2(\theta' - \lambda') = \frac{h \sin 2(\phi - \alpha) \, . \, \delta h'}{\left(h' + h \cos 2(\phi - \alpha)\right)^2}.$$

Let h' represent the mean value, and $\delta h'$ any deviation from the mean; and let S′ represent the semimenstrual inequality, that is, the value of $\theta' - \lambda'$ freed from effects of declination and parallax. Then

$$\tan 2 \mathrm{S}' = \frac{h \sin 2(\phi - \alpha)}{h' + h \cos 2(\phi - \alpha)}; \text{ whence}$$

$$\delta \tan 2(\theta' - \lambda') = -\frac{\tan^2 2 \mathrm{S}'}{\sin 2(\phi - \alpha)}, \frac{\delta h'}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Now, *cæteris paribus*, h' is as the cube of the parallax. If, therefore, P be the mean parallax, and p any other,

$$h' + \delta h' = h' \frac{p^3}{\mathrm{P}^3} = h'\left(1 + \frac{p - \mathrm{P}}{\mathrm{P}}\right)^3 = h' + 3 h' \frac{p - \mathrm{P}}{\mathrm{P}},$$

omitting other terms, because $p - \mathrm{P}$ is small.

Hence $\delta h' = 3 h' \frac{(p - \mathrm{P})}{\mathrm{P}}$; and when we make this substitution, equation (2.) gives the change in the first side due to the effect of lunar parallax.

Since the arc $\theta' - \lambda'$ is small, we may put it for its tangent; hence, making the above substitution and calling the effect of lunar parallax P′,

$$2 \mathrm{P}' = -\frac{h'}{h} \frac{\tan^2 2 \mathrm{S}'}{\sin 2(\phi - \alpha)} \cdot 3 \frac{(p - \mathrm{P})}{\mathrm{P}}.$$

As a first approximation to the general form of the result, we may put $\frac{h}{h'} \sin 2(\phi - \gamma)$ for $\tan 2 \mathrm{S}'$, since $\frac{h}{h'}$ is a fraction (about one third), and since the general course of the two functions, $\sin 2(\phi - \alpha)$ and $\tan 2 \mathrm{S}'$, agrees.

Hence we should have

$$\mathrm{P}' = -\frac{3h}{h'} \sin 2(\phi - \alpha) \, . \, \frac{p - \mathrm{P}}{\mathrm{P}}$$

$$\mathrm{P}' = (\mathrm{P} - p) \, . \, \mathrm{B} \sin 2(\phi - \alpha);$$

B being a constant quantity.

The expression which we obtained from observation was

$$P' = (P - p)\left(B + B \sin 2 (\varphi - \beta)\right),$$

of which the second term agrees in form with the one given by the theory, except that the angle β is different from α; but the first term should not occur according to the theory as hitherto stated.

2. Again, for the effects of lunar declination on the time of high water.

If k' be the value of h' when the place of observation and the pole of the lunar tide spheroid are both in the equator, and if ε be the difference of declination of the place and the pole in any other situation, we shall have $h' = k' \cos^2 \varepsilon$ nearly.

In the course of a tide-day there are two tides, corresponding to two positions of the tidal spheroid; and if l be the latitude of the place, δ the declination of the moon, the two corresponding differences of declination will be $l - \delta$ and $l + \delta$, the pole of the spheroid being supposed to have the same declination as the moon has at the moment of the origin of the tide (that is, when the moon's right ascension was less by α than it is at the moment of the tide).

Then, in the first case,

$$\begin{aligned} h' &= k' \cos^2 \varepsilon = k' \cos^2 (l - \delta) = k' (\cos l \cos \delta - \sin l \sin \delta)^2 \\ &= k' \{\cos^2 l - 2 \sin l \cos l \sin \delta \cos \delta - (\cos^2 l - \sin^2 l) \sin^2 \delta\} \\ &= k' \cos^2 l - \tfrac{1}{2} k' \sin 2 l \sin 2 \delta - k' \cos 2 l \sin^2 \delta. \end{aligned}$$

In the second case, similarly,

$$h' = k' \cos^2 l + \tfrac{1}{2} k' \sin 2 l \sin 2 \delta - k' \cos 2 l \sin^2 \delta.$$

In order to find the effect of the declination upon each tide, we should put for $\delta h'$ the quantities $-\frac{1}{2} k' \sin 2 l \sin 2 \delta - k' \cos 2 l \sin^2 \delta$, and $+\frac{1}{2} k' \sin 2 l \sin 2 \delta - k' \cos 2 l \sin^2 \delta$ respectively.

Thus, according to the theory, the effect of declination on the two tides of the same day should be different. This difference is very much modified by the circumstances in which the actual state of the ocean differs from the theoretical state: the difference of the diurnal tides may, however, be detected in the observations at most places of the earth's surface, perhaps at almost all. But there are peculiar circumstances in the port of London which affect this difference, and obliterate it: the tide at London is composed of two tides, which differ by half a day from each other, and hence the difference of the two semidiurnal tides disappears altogether. Therefore, instead of the effects of declination on the two semidiurnal tides, we must take the mean of these effects, which is $- k' \cos 2 l \sin^2 \delta$.

Hence if Q' represent the effect of lunar declination on the time of high water, we have by equation (2.) (substituting $- k' \cos 2 \theta \sin^2 \delta$ for δh, and putting the arc for $\tan 2 (\theta' - \lambda')$),

$$2\,Q' = \frac{\tan^2 2\,S'}{\sin 2 (\varphi - \alpha)} \frac{k'}{h} \cos 2 l \sin^2 \delta.$$

In this expression we have $k' \cos^2 l$ for h' in the value of $\tan S'$; but it is clear that in considering the effect of solar declination, we should in like manner have $k \cos^2 l$ for h, whence the value of $\tan 2\,\mathrm{S}'$ in equation (1.) would remain unaltered.

Putting, as before, $\frac{h}{p'} \sin 2 (\phi - \alpha)$ for $\tan 2\,\mathrm{S}'$, the equation becomes

$$2\,\mathrm{Q}' = \frac{k'}{h'} \cos l \sin 2 (\phi - \alpha) \,.\, \sin^2 \delta; \text{ or}$$

$$\mathrm{Q}' = \mathrm{D} \sin 2 (\phi - \alpha) \,.\, \sin^2 \delta,$$

where D is a constant quantity.

In this expression the correction for declination is supposed to be applied to the time of high water, calculated for the moon and the place, both in the equator. But in our tables this correction is applied to the mean place. Let Δ be the value of the declination at this mean place; then the correction for that case is $\mathrm{D} \sin 2 (\phi - \gamma) \sin^2 \Delta$, and therefore,

$$\mathrm{Q}' = (\sin^2 \delta - \sin^2 \Delta)\, \mathrm{D} \sin 2 (\phi - \alpha)$$

is the correction to be applied to the mean.

The correction according to observation was

$$\mathrm{Q}' = (\sin^2 \delta - \sin^2 \Delta) \left(\mathrm{C} + \mathrm{D} \sin 2 (\phi - \gamma)\right).$$

The second term agrees with the theory, except that the arc γ is different from α: the first term, $\mathrm{C} (\sin^2 \delta - \sin^2 \Delta)$, has nothing corresponding to it in the theory.

3. We now proceed to the theoretical laws which regulate the height of high water.

If θ, θ' be the distance of any place in the equator from the places to which the sun and moon are vertical (these luminaries being supposed to be in the equinoctial), the height of the water at the place will be $\frac{1}{2} \left(h \cos 2 (\theta - \lambda) + h' \cos 2 (\theta' - \lambda')\right)$ above the mean level; and if θ' be taken the distance of the highest water from the moon, then

$$h \cos (2\, \theta - \lambda) + h' \cos 2 (\theta' - \lambda')$$

will be the whole tide, which call y.

Now we have

$$\tan 2 (\theta' - \lambda') = \frac{h \sin 2 (\phi - \alpha)}{h' + h \cos 2 (\phi - \alpha)},$$

where $\theta + \theta' = \phi$.

Hence we find

$$\cos 2 (\theta' - \lambda') = \frac{h' + h \cos 2 (\phi - \alpha)}{\surd \{h'^2 + 2\, h\, h' \cos 2 (\phi - \alpha) + h^2\}}$$

$$\cos 2 (\theta - \lambda) = \frac{h + h' \cos 2 (\phi - \alpha)}{\surd \{h'^2 + 2\, h\, h' \cos 2 (\phi - \alpha) + h^2\}}$$

$$y = \surd \{h'^2 + 2\, h\, h' \cos 2 (\phi - \alpha) + h^2\} \quad \ldots\ldots \quad (3.)$$

If, as before, δy represent the variation of y in virtue of any variation of h',

$$\delta y = \frac{h' + h \cos 2 (\phi - \alpha)}{y} \,.\, \delta h'. \quad \ldots\ldots \quad (4.)$$

The semimenstrual inequality of the heights is given by equation (3.).

Expanding, we have

$$y = \sqrt{(h'^2 + h^2)}\left\{1 + \frac{h\,h'}{(h'^2 + h^2)}\cos 2(\phi - \alpha) - \frac{h^2\,h'^2}{8\,(h'^2 + h^2)^2}\cos^2 2(\phi - \alpha) +, \&c.\right\}$$

$$= \sqrt{(h'^2 + h^2)} - \frac{h^2\,h'^2}{16\,(h'^2 + h^2)^{\frac{3}{2}}} + \frac{h\,h'}{\sqrt{(h'^2 + h^2)}}\cos 2(\phi - 2\alpha) - \frac{h^2\,h'^2}{16\,(h'^2 + h^2)^{\frac{3}{2}}}\cos(4\phi - 4\alpha);$$

omitting ulterior terms, since the coefficients diminish according to powers of $\dfrac{\frac{h}{h'}}{1 + \left(\frac{h}{h'}\right)^2}$.

Hence the variable part of this expression is of the form

$$\text{K} \cos(2\phi - 2\alpha) - \text{L} \cos(4\phi - 4\alpha).$$

The expression of the semimenstrual inequality of heights, found from observation, was (in feet)

$$\text{S}' = 1{\cdot}7 \cos(2\phi - 51^\circ) - {\cdot}23 \cos(4\phi - 30^\circ),$$

which agrees with the theoretical expression, except as to the values of the arcs which take the place of 2α and 4α.

4. To find the effect of lunar parallax on the heights, substitute as before for $\delta h'$, in equation (3.), and let p' be this effect; then

$$p' = \frac{h' + h\cos 2(\phi - \alpha)}{y} \,.\, 3\,h' \frac{p - \text{P}}{\text{P}}.$$

Here y is the mean height.

Therefore p' is of the form $(p - \text{P})\big(a + b\cos 2(\phi - \alpha)\big)$.

The form as given by observation is $(p - \text{P})(a + b\cos 2\phi)$, where, however, the existence and constancy of b are doubtful.

5. To find the effect of lunar declination on the heights, substitute for δh, as before, $- k' \cos 2\,l \sin^2\delta$. We thus find from equation (3.), q' being the effect,

$$q' = -\frac{h' + h\cos 2(\phi - \alpha)}{y} k' \cos 2\,l \sin^2\delta;$$

and referring the correction to the mean declination Δ, it becomes of the form

$$q' = (\sin^2\Delta - \sin^2\delta)\big(c + d\cos 2(\phi - \alpha)\big).$$

The form given by observation was

$$q' = (\sin^2\Delta - \sin^2\delta)\big(c + d\cos 2(\phi + 45^\circ)\big),$$

where, however, d was not determined as to quantity, the observations being too anomalous.

It appears, therefore, that the results of observation and theory for the variations of height agree as to form, with the exception of the epochs α, β, γ.

CHAP. IV. *Reflexions on the Theory.*

It would be unsafe to attempt to deduce any general views concerning the laws of the tides from the preceding investigations. It is very unlikely that the discussion of observations at any one place, and those the very first set which have been systematically discussed, should exhibit clearly the true principles of the theory: and besides this, it so happens, that the phenomena of the tides at London are in some measure masked by a curious combination of circumstances, namely, by the mouth of its river being on the side of an island, turned away from the side on which the tide comes, and so situated that the path of the tide round one end of the island is just twelve hours longer than round the other. It will require the accumulation and discussion of many large masses of observations, at various places, to put us in firm possession of the laws of the phenomena as given by experience; and this road, whether or not it be the *only* practicable way of arriving at the true theory, is at least that to which, founding our expectations on the past history of science, we may look with most hope. When we consider the enormous accumulation of observed phenomena and empirical laws which preceded the discovery of the true principles of the heavenly motions, we may easily suppose that we are only at the outset of what we have to do, in order to obtain the same success with regard to the tides: and we may, from the same consideration, find additional motives to desire that such observations may be made, and such existing observations may be discussed, as may most speedily lead us to a complete and scientific knowledge of the subject.

But though we cannot make our inferences from the preceding investigation with confidence, there are some reflexions concerning the mode in which the forces of the sun and moon manifest themselves in the tides, which are suggested by the comparison made in the foregoing pages, and which I will venture to state. The confirmation or refutation of these views must depend on future investigations of the same nature as that contained in this memoir: in the mean time, the views seem fitted to give some additional impulse to the curiosity with which all men of science must now look upon the progress of this subject.

Among the inequalities considered in this memoir, those in which the empirical laws are the clearest and the anomalies the smallest (after the semimenstrual inequalities,) are the inequalities of the time of high water, depending on the moon's parallax and declination. In these the comparison of the law, from theory and from observation, may be stated as follows:

Observation.	Theory.
$P' = (P - p)\left(B + B \sin 2(\phi - \beta)\right);$	$(P - p) B \sin 2(\phi - \alpha).$
$Q' = (\sin^2\delta - \sin^2\Delta)\left(C + D \sin 2(\phi - \gamma)\right);$	$(\sin^2\delta - \sin^2\Delta) D \sin 2(\phi - \alpha).$

It will be observed, that in each of these cases observation gives, in P′ and Q′, a term depending on the parallax and on the declination, (namely, the terms $(P - p)$ B and $(\sin^2\delta - \sin^2\Delta)$ C,) which term is not given by the theory; besides giving another term which coincides with the one given by the theory. The latter term depends on the hour of the moon's transit, and vanishes twice in the course of a semilunation; the former term in each case is independent of the time of the moon's transit, and depends only on the parallax and on the declination.

Now P' and Q' are the corrections to which $\theta' - \lambda'$ is subject, where θ' is the hour-angle of the tide from the moon. In the theory, λ' is supposed to be constant, so that the variation of $\theta' - \lambda'$ alone affects θ'.

But since $\theta' = \lambda' + (\theta' - \lambda')$, if λ' were affected by an inequality arising from parallax equal to $(P - p)$ A, we should have, taking the theoretical value of the variation of $\theta' - \lambda'$ due to this cause, and adding it to the value resulting from the common theory, the whole variation of $\theta' = (P - p)\left(A + B \sin 2(\varphi - \gamma)\right)$.

In like manner, if λ' were affected by an inequality equal to $(\sin^2\delta - \sin^2\Delta)$ C, and $\theta' - \lambda'$ by the inequality resulting from the theory, we should have for the whole inequality in θ' arising from declination, $(\sin^2\delta - \sin^2\Delta)\{C + D \sin 2(\varphi - \gamma)\}$.

Now these expressions agree with those which we have obtained from observation, excepting that we have other arcs in the place of the arc α. It appears, therefore, that the empirical laws will be verified by supposing λ' to be affected by inequalities depending upon the parallax and declination of the moon, but having an epoch different from that of the semimenstrual inequality.

The quantity λ' is the hour-angle by which the lunar tide follows the high water of the lunar spheroid of equilibrium. It appears, therefore, that the physical statement of the result just obtained is this, that the distance at which the actual elevation of the waters follows the position of equilibrium, varies as the parallax and declination of the disturbing luminary vary.

This distance was, in the theory, assumed to be constant; but there is no obvious physical reason why it may not change with changes of the force by which the fluid spheroid of equilibrium is determined. This distance, or *lagging*, of the pole of the watery spheroid behind the place which it would occupy if the earth and luminary were at rest, is owing to the resistance of the shores and of the parts of the water amongst each other; and its amount is determined by the amount of these resistances. But we are very far from being able to trace the mode in which these causes operate, so as to be entitled to affirm that changes, and even small changes, in the force or velocity of the disturbing body, may not produce corresponding changes in the extent of this lagging.

In fact, there seems to be good reason to suppose, from other circumstances, that the force and velocity of the disturbing body do affect the distance by which the actual elevation lags behind the elevation of equilibrium. For λ and λ', the lagging in the case of the solar and of the lunar tide, are quite different; the former (for the London Docks) being $3^h\ 25^m$, the latter $1^h\ 25^m$ *. It is true that this difference of 2^h is, in the

* See Mr. LUBBOCK's Memoir, Philosophical Transactions, 1831, p. 387.

theory, got rid of by supposing the actual tide to be referred to a configuration of the sun and moon anterior by $2\frac{1}{2}$ days to the configuration at the time of the tide. But since we refer the effects of parallax and declination to the parallax and declination contemporaneous with the tide, we must look for an analogy only when we do the same in the other case.

In the semimenstrual inequality, as determined from observation, there is no such discrepancy with theory as compels us to suppose a change in λ'. But this forms no objection to our view; for if, in the course of a semirevolution of the moon, there were a periodical change in λ', this must have the same cycle as the change in $\theta' - \lambda'$, and would therefore be confounded with that change, and would not result in a separate *form* from our discussion.

But, moreover, the difference of the *quantity* of the semimenstrual inequality at different places, which we have already shown to exist, supplies a confirmation of the opinion here put forwards. For this difference implies that the tide travels from one given place to another in different times at different periods of a semilunation; that is, it implies that the velocity of the tide-wave is different in different configurations of the sun and moon, that is, under different circumstances of the tide-producing forces. And this agrees with our doctrine, that the amount of lagging is different under different circumstances of those forces; for if the amount of lagging of the tide elevation go through a cycle of changes in a certain period, the velocity with which this elevation travels will also go through a cycle of changes in the same period. And this difference of the semimenstrual inequality at different places, does appear to betray a semimenstrual inequality affecting λ', the amount of the inequality varying with the place; and this variation, added to the theoretical semimenstrual inequality which affects $\theta' - \lambda'$, and which is the same for all places, makes up the empirical semimenstrual inequality of θ', given by our mode of investigation, which thus appears to be different at different places.

Taking all these reflexions into consideration, there appears to be good reason to believe that the amount of the lagging of the tide behind the equilibrium-tide is really affected by changes in the distances and velocities of the disturbing luminaries.

There is another circumstance in which the empirical differ from the theoretical laws: the epochs β, γ of the changes due to parallax and declination are different from the epoch α of the semimenstrual inequality.

The physical statement of this result is, that the time required to transmit to any port the general effect of the tide-producing forces, and the time required to transmit to the same port the effects of particular changes in these forces, are different. And of this result we may say, in the same manner as of the former, that we see far too obscurely the causes which determine the amount of this interval in one case, to assert that it must necessarily be the same under different circumstances. But we may illustrate this subject somewhat further. We may suppose an imaginary mean moon, moving uniformly in the equator, at a constant distance from the earth, to produce the mean tide; another auxiliary moon, by moving directly to or from the earth, in the line of the mean moon, to produce the inequality which arises from parallax; another auxiliary moon, by moving north and south in the meridian of the moon to produce the inequality which arises from declination. Now the tides produced by all these moons will require some time for the operation of the forces to take effect; that is, they will correspond to positions of the moon at a time anterior to the actual time. But there seems not to be the smallest reason to conclude that these anterior times will all be anterior by the same interval: the contrary, rather, is obvious. It is clear, for instance, that a tide oscillating in a north and south direction in the Atlantic and Pacific Oceans, will take a different portion of time to obey the forces which produce it, from the general tide which travels from east to west round the earth in virtue of the diurnal motion, and impinges against the broad sides of the great continents. We may therefore expect to find the epochs of all these partial tides different; and as every separate term in the expression of an inequality may be considered as representing a different tide, there will be nothing inconsistent with the best physical notions we can yet form on the subject in finding the epochs of the arguments of every separate term of our formulæ different from one another.

It appears, then, that though the equilibrium theory, taken in combination with the preceding considerations, may very probably give us the general form of the terms, and the variable part of the arcs on which they depend, the constant epoch which occurs in each of these arcs, and which determines when the inequality vanishes and reaches its maximum, will probably have to be determined in all cases by observation.

I will observe further, that not only the epochs, but the coefficients of each of these terms will probably have to be determined for the most part from observation. For the tides, though in the theory to which we refer considered as representing positions of equilibrium of a fluid, are in fact the results of its motion; and it is not at all clear that the elevation which results from the motion will be equal to the elevation which would be requisite for equilibrium. It is true that there must be always a tendency to this equilibrium-elevation so long as the actual elevation is greater or less than that; but this tendency may never fully appear in the circumstance of the tide; since the tide-producing forces have to supply also a residue of force which must be employed in producing the motion of the fluid.

Moreover, the motion of the fluid is of the nature of an oscillation, so that series of increasing and diminishing oscillations at intervals of a half-day, a day, and other intervals, pass through any given part of the ocean. Now it is physically, not only possible but certain, that each oscillation in each series is affected by those which precede it in the same series, and affects those which succeed it, so that their relative magnitude is different from what it would otherwise be. And the effects thus produced will depend upon the depth of the ocean, the form of its shores, and other causes, of which it is impossible to estimate the result *à priori*.

Even in the case of the semimenstrual inequality, which in its form agrees so closely with the theory, and which in its amount appears to depend only on the ratio of the forces of the sun and moon, we find that in fact its amount is different at different places, as we have already stated. We cannot expect, therefore, that the amount of the corrections for parallax and declination will agree very exactly with those from theory; and till the empirical corrections are more certainly and generally determined, I have not thought it worth while to make the comparison.

But though there is at present this uncertainty respecting the amount of the inequalities of the tide, I do not conceive that there can be any doubt that the forms of these corrections are such as I have stated them. In the case of the times of high water especially, the general course of the variations of the quantities is as regular as can be expected, and as is requisite for the establishment of our formulæ. The heights are much more anomalous; probably they are more affected by winds, &c., than the times are: and when we reflect that the tide at London may be affected by the operation of causes in a remote part of the ocean, propagating their effect by the progression of the tide-wave, we shall not be surprised at considerable deviations from rule. The trade winds and other winds of the tropical regions may be felt in our tides, and may even affect the means of long series of observations; for it is to be recollected that the averages which we obtain are not the averages of the effects of the sun and moon alone, but the averages of their effect, together with that of meteorological causes; and it is very conceivable that the latter average may not vanish in the long run. It is moreover to be observed, that the peculiar circumstances of London, in having a tide compounded of two tides, arriving by different roads after journeys of different lengths, may easily be supposed to give rise to additional chances of irregularity.

It may not be superfluous to remark, that, independently of such a combination of circumstances, there is nothing in the situation of the port of London to diminish the value of tide observations there. The length and windings of the river by which the tide reaches the port present no objection to the comparison of the observations with theory. These circumstances may modify the tide, but they modify it alike every day, or at least alike at like periods of the tidal cycles, and therefore they introduce no irregularity. Indeed, there are some reasons for believing that the tides in rivers and deep sounds are more regular than those on the open coast; and at any rate, as they are generally larger in such situations, their variations are more observable.

Concluding Observations.

It appears from the preceding investigations and considerations, that the following are now the most important steps from which any great improvement of our knowledge on this subject may be hoped.

Large collections of observations at other places must be discussed in a manner resembling that employed for the London observations by Mr. DESSIOU. The Brest and the Liverpool observations would be excellent materials for such operations. We must thus ascertain whether the empirical forms of the corrections for parallax and declination, deduced from these, agree with those obtained for London in the preceding pages. If they do, the coefficients must be compared with each other and with the theory, in order to determine the most promising mode of pursuing the latter.

The empirical formulæ obtained in the preceding pages represent the observations with tolerable exactness; probably they agree with them almost as well as any formulæ would do, and as well as the observations agree with each other. These formulæ might be used in calculating tide tables as readily as any other empirical rules; and the tables so calculated might be compared with observations made at London. Such a comparison, continued long enough, would disclose any additional corrections which may be requisite in this mode of calculating tide tables.

Tide observations are now made at the Katharine Docks, with good apparatus and a judicious system; and, so far as I can judge, with proper care. These will hereafter form materials for a better discussion of the London tides than the London Dock observations, made in a ruder manner, could allow.