NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

STABILITY ANALYSIS OF SHIP STEERING IN CANALS

by

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March, 1997

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REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE March 1997	3. REPORT TYPE AND DATES COVERED Master's Thesis	
4. TITLE AND SUBTITLE STABILITY ANALYSIS OF SHIP STEERING IN CANALS		5. FUNDING NUMBERS	
6. AUTHOR(S) Panos E. Kapasakis			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey CA 93943-5000		8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.			
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.		12b. DISTRIBUTION CODE	

13. ABSTRACT *(maximum 200 words)*

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14. SUBJECT TERMS SHIP STEERING BIFURCATIONS LIMIT CYCLES			15. NUMBER OF PAGES 94
			16. PRICE CODE
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18 298-102

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STABILITY ANALYSIS OF SHIP STEERING IN CANALS

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Submitted in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
March 1997

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The problem of ship steering in canals and confined waters is analyzed
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and nonlinear techniques in order to assess its stability under finite distur-
bances. The results indicate that for certain regions of parameters, limit cycle
oscillations may develop which could compromise system stability and safety
of operations.

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I. INTRODUCTION

A. PROBLEM STATEMENT

The problem of motion stability of ships and other marine vehicles has been the subject of extensive studies in the past (Comstock, 1977). Most of these studies are with regards the ship's directional stability in open waters under open loop conditions. It is well known that in restricted waters such as canals or rivers, although it is possible to have positional stability in principle, in reality it is never the case. This is due to the destabilizing effects of the bank suction forces and moments. These act always in a destabilizing fashion; i.e., after a small initial disturbance they produce a force or a moment which tends to increase the ship's path deviation from nominal (Comstock, 1977). Open loop conditions, important as they are, rarely represent real life applications. Ships traveling in canals are under closed loop control, typically provided through the helmsman or an autopilot. It becomes then important to assess the stability of the system under finite disturbances and incorporating some modeling of closed loop operations.

B. OBJECTIVES AND OUTLINE

This thesis utilizes both linear and nonlinear techniques in order to analyze the problem of closed loop dynamic stability of ships in canals. Ship modeling

is discussed in Chapter II and it follows standard ship maneuvering equations (Clayton and Bishop, 1982) augmented by a model for bank suction effects (Beck, 1976). A Mariner class ship is selected as a model because of the availability of data for its hydrodynamic coefficients (Comstock, 1977), (Bernitsas and Kekridis, 1984). A heading control law based on Nomoto's first–order model, coupled with a line of sight guidance law is chosen in order to model the fundamental behavior of closed loop conditions. Since, in practice control actions are hardly ever instantaneous, appropriate time lags are introduced in the feedback. These are modeled using the techniques outlined in (Venne, 1992). Linear eigenvalue analysis is performed in Chapter III in order to reveal parametric stability boundaries. It is established that loss of stability occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis which results in periodic solutions or limit cycles (Guckenheimer and Holmes, 1983). A nonlinear analysis based on Hopf bifurcation methods (Chow and Mallet–Paret, 1977) and (Hassard and Wan, 1978) is performed in Chapter IV. Conclusions derived from this study and some recommendations for further extensions are discussed in Chapter V. The basic programs that were used to produce the results are included in the Appendix.

II. PROBLEM FORMULATION

A. SHIP DYNAMICS

Restricting our attention to the horizontal plane, the mathematical model consists of the nonlinear sway (translational motion parallel to the vehicle's longitudinal axis) and yaw (rotational motion about the vertical axis) equations of motion. If we consider a moving Cartesian coordinate frame located at the geometric center of the vehicle, Newton's equations of motion are:

$$m(\dot{v} + ur + x_G\dot{r}) \;=\; Y \;, \tag{1}$$

$$I_Z\dot{r} + mx_G(\dot{v} + ur) \;=\; N \;, \tag{2}$$

where v and r are the relative sway and yaw velocities of the moving vehicle with respect to the water, m is the vehicle's mass, I_Z is its moment of inertia with respect to the vertical axis, and u is the constant forward speed. Since all quantities will be considered as dimensionless in this study, in the following we consider u to be equal to one. x_G is the longitudinal position of the ship's center of gravity with respect to its centroid, and Y,N represent the total excitation sway force and yaw moment respectively. Following standard ship maneuvering assumptions, these forces can be expressed as the sum of quadratic drag terms and first–order memoryless polynomials in v and r, which represent added mass and damping due to water. In this way they can

be expressed by:

$$
\begin{aligned}
Y &= Y_{\dot{v}}\dot{v} + Y_{\dot{r}}\dot{r} + Y_{v}v + \frac{1}{6}Y_{vvv}v^3 + \frac{1}{2}Y_{vrr}vr^2 + \\
&\quad \frac{1}{2}Y_{rvv}rv^2 + Y_{r}r + Y(\psi, y) + Y_{\delta}\delta \,,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
N &= N_{\dot{v}}\dot{v} + N_{\dot{r}}\dot{r} + N_{v}v + \frac{1}{6}N_{vvv}v^3 + \frac{1}{2}N_{vrr}vr^2 + \\
&\quad \frac{1}{2}N_{rvv}rv^2 + N_{r}r + N(\psi, y) + N_{\delta}\delta \,,
\end{aligned} \tag{4}
$$

where Y_a, N_a represent the partial derivatives with respect to the indicated variable a and δ is the effective rudder angle. $Y(\psi, y)$, $N(\psi, y)$ represent the interaction/proximity forces and moments that arise due to the presence of the canal, and shallow water effects.

The resulting nonlinear differential equations can be non-dimensionalized with respect to the constant forward speed of the ship, u and its length, l. The dimensionless time variable is then equal to tu/l. A standard inertial system (x, y) is introduced here where the x–axis points in the assumed nominal straight line path and the y-axis is the distance from this nominal path, see Figure 1. We assume that the nominal straight line path is along the centerline of the canal. In cases where the concept of a geometric centerline is not applicable, we assume that the nominal path is along the zero bank suction location. This is consistent with recommended navigation practices that are currently in use.

The complete model is presented by the following equations:

$$
(m - Y_{\dot{v}})\dot{v} - (Y_{\dot{r}} - mx_G)\dot{r} = Y_{v}v + \frac{1}{6}Y_{vvv}v^3 + \frac{1}{2}Y_{vrr}vr^2 +
$$
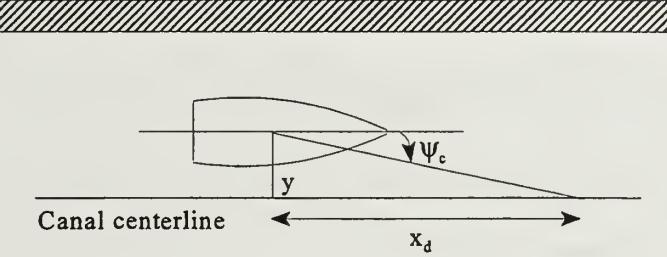
Figure 1: Vehicle geometry and definitions of symbols

$$\frac{1}{2}Y_{rvv}rv^2 + (Y_r - m)r + Y(\psi, y) + Y_\delta\delta , \tag{5}$$

$$-(N_{\dot{v}} - mx_G)\dot{v} + (I_z - N_{\dot{r}})\dot{r} = N_v v + \frac{1}{6}N_{vvv}v^3 + \frac{1}{2}N_{vrr}vr^2 + , $$

$$\frac{1}{2}N_{rvv}rv^2 + (N_r - mx_G)r + N(\psi, y) + N_\delta\delta , \tag{6}$$

$$\dot{\psi} = r , \tag{7}$$

$$\dot{y} = \sin\psi + v\cos\psi , \tag{8}$$

where the expressions for the ship's yaw rate as well as its inertial position rate have been added, and ψ is the local heading angle. The interaction/proximity forces and moments are expanded to include up to third order terms,

$$Y(\psi, y) = Y_\psi\psi + Y_y y + \frac{1}{6}Y_{\psi\psi\psi}\psi^3 + \frac{1}{6}Y_{yyy}y^3 , \tag{9}$$

$$N(\psi, y) = N_\psi\psi + N_y y + \frac{1}{6}N_{\psi\psi\psi}\psi^3 + \frac{1}{6}N_{yyy}y^3 , \tag{10}$$

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as explained in more detail in the folowing section.

B. HYDRODYNAMIC COEFFICIENTS

We chose a Mariner class ship as a representative model. Its hydrodynamic coefficients and geometric properties were taken from (Comstock, 1977) and (Bernitsas and Kekridis, 1984). Results from (Beck, 1976) were used in order to model the bank suction forces and moments. Typical results are shown in Figure 2. These show force and moment coefficients versus ship deviation (η), and for canal width $w = 0.4L$. Force and moment coefficients are nondimensionalized with respect to the water density and the ship's speed and length, as is customary in ship maneuvering. Since the suction force and moment must change their sign as either y or ψ changes its sign, they must be modeled by odd power polynomials, as was done in the previous section. Numerical values for the coefficients Y_y, Y_{yyy}, N_y, and N_{yyy} can be found by curve fitting of the results of Figure 2. Using a depth to draft ratio of 1.9 we were able to estimate,

$$Y_y = 0.02 \,,$$

$$Y_{yyy} = 0.468 \,,$$

$$N_y = -0.0025 \,,$$

$$N_{yyy} = 0 \,.$$

The value of N_ψ was estimated by "discretizing" the ship in two segments,

FIGURE 13
VARIATION OF SIDE FORCE AND MOMENT WITH
WALL POSITION RATIO FOR THE MARINER
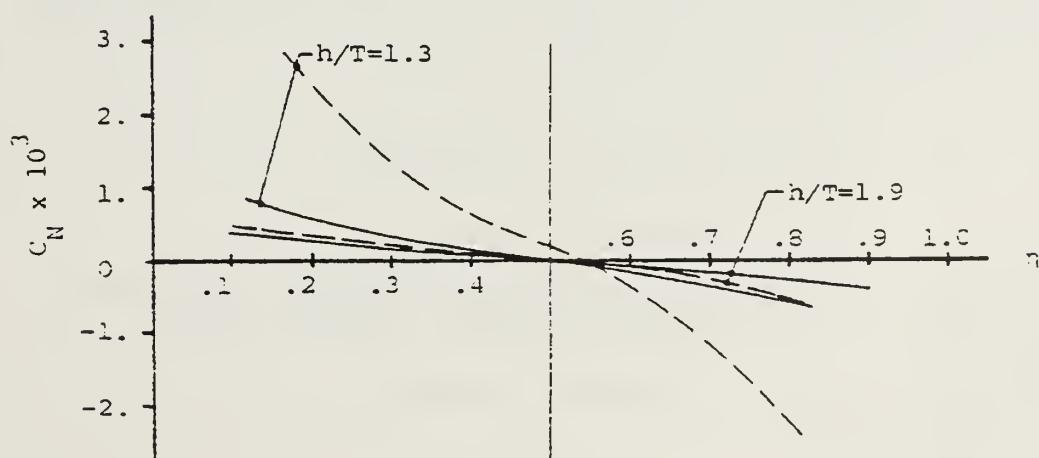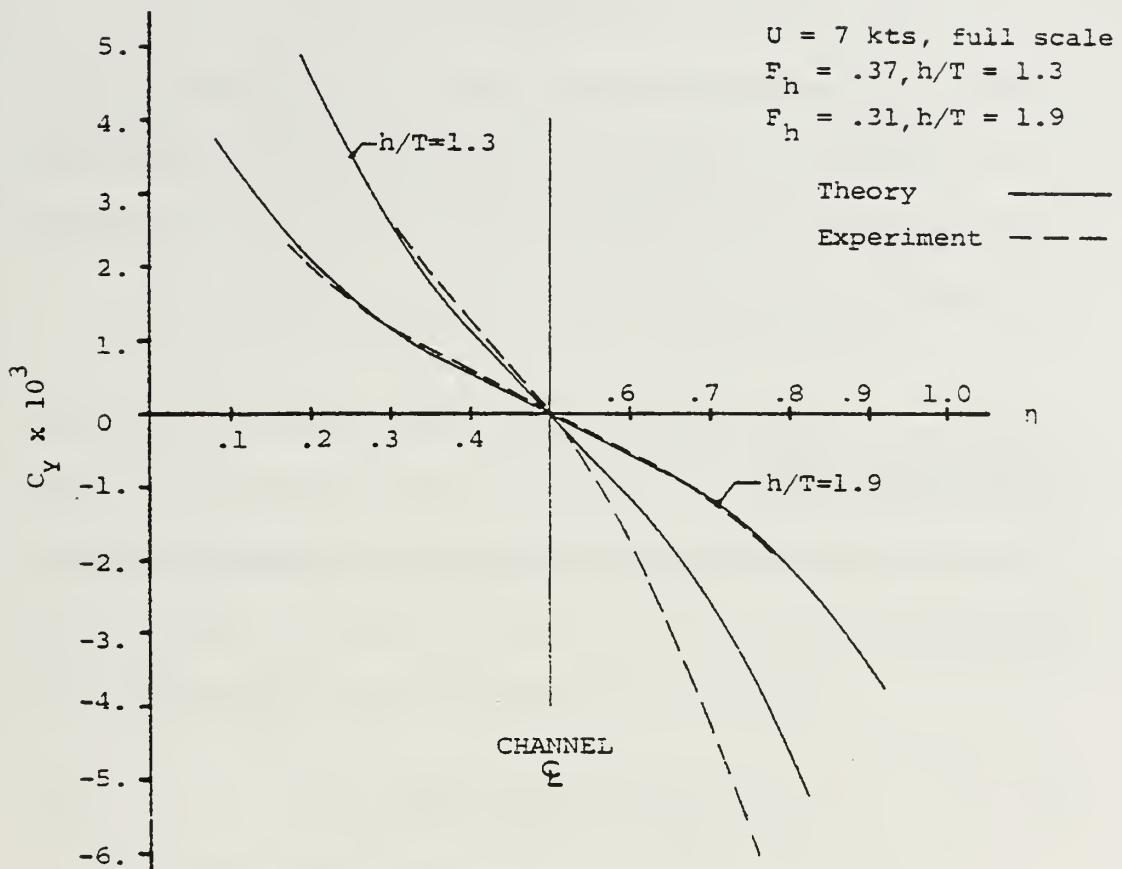
Figure 2: Forces and moments due to canal [Beck (1976)]

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at the bow and the stern, and calculating the suction forces on each part individually. Their resultant moment produced,

$$N_\psi = 0.01 \ .$$

The value of $N_{\psi\psi\psi}$ was taken to be zero, due to lack of reliable data. The value of Y_ψ was estimated to be equal to $-Y_v = 0.014$ which is true for motions along the centerline of the canal (Comstock, 1977). Again, due to lack of reliable data we took $Y_{\psi\psi\psi} = 0$.

C. CONTROL LAW

The linearized set of equations (5) through (7) can be expressed in the following form:

$$\dot{\psi} \ = \ r \ ,$$

$$\dot{v} \ = \ a_{11}\psi + a_{12}v + a_{13}r + a_{14}y + b_1\delta \ , \tag{11}$$

$$\dot{r} \ = \ a_{21}\psi + a_{22}v + a_{23}r + a_{24}y + b_2\delta \ ,$$

where the coefficients a_{ij}, b_i are functions of the vehicle hydrodynamic coefficients and geometric properties and are given below:

$$a_{11} \ = \ \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_\psi + (Y_{\dot{r}} - mx_G)N_\psi] \ ,$$

$$a_{12} \ = \ \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_v + (Y_{\dot{r}} - mx_G)N_v] \ ,$$

$$a_{13} \ = \ \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})(Y_r - m) + (Y_{\dot{r}} - mx_G)(N_r - mx_G)] \ ,$$

$$a_{14} \ = \ \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_y + (Y_{\dot{r}} - mx_G)N_y] \ ,$$

$$a_{21} = \frac{1}{D_{vr}}[(N_{\dot{v}} - mx_G)Y_\psi + (m - Y_{\dot{v}})N_\psi] ,$$

$$a_{22} = \frac{1}{D_{vr}}[(N_{\dot{v}} - mx_G)Y_v + (m - Y_{\dot{v}})N_v] ,$$

$$a_{23} = \frac{1}{D_{vr}}[(N_{\dot{v}} - mx_G)(Y_r - m) + (m - Y_{\dot{v}})(N_r - mx_G)] ,$$

$$a_{24} = \frac{1}{D_{vr}}[(N_{\dot{v}} - mx_G)Y_y + (m - Y_{\dot{v}})N_y] ,$$

$$b_1 = \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_\delta + (Y_{\dot{r}} - mx_G)N_\delta] ,$$

$$b_2 = \frac{1}{D_{vr}}[(N_{\dot{v}} - mx_G)Y_\delta + (m - Y_{\dot{v}})N_\delta] ,$$

$$D_{vr} = (m - Y_{\dot{v}})(I_Z - N_{\dot{r}}) - (N_{\dot{v}} - mx_G)(Y_{\dot{r}} - mx_G) .$$

A control law that could model a human operator should not be based on feedback of the side slip velocity v. Instead, it is more likely that human operators will respond to changes in the ship's heading angle ψ and rate of change of heading, r. Therefore, we choose to base our control law on Nomoto's model, which follows by assuming negligible sway velocity v,

$$\dot{r} = ar + c\psi + b\delta , \tag{12}$$

where the coefficients are

$$a = a_{23} ,$$

$$b = b_2 ,$$

$$c = a_{21} .$$

A linear control law can be introduced in the form,

$$\delta_0 = k_1(\psi - \psi_c) + k_2 r , \tag{13}$$

where ψ_c is the commanded heading angle and the gains k_1, k_2 are computed such that the closed loop system (7), (12), and (13) has the desired dynamics. The existence of the difference $(\psi - \psi_c)$ in the control law (13) has the effect of trying to point the ship's longitudinal axis towards the desired heading. The characteristic equation of the system is obtained from (7), (12), and (13) as

$$s^2 - (a + bk_2)s - (c + bk_1) = 0 .$$

If the desired characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 ,$$

the control gains can be computed from

$$
\begin{aligned}
k_1 &= -\frac{\omega_n^2}{b} - \frac{c}{b} , \\
k_2 &= -\frac{a + 2\zeta\omega_n}{b} .
\end{aligned}
$$

The natural frequency ω_n and damping ratio ζ are selected based on general properties of second order systems. Relatively high values of ω_n and low values of ζ will correspond to a responsive operator, while the opposite is true for combinations of low ω_n and high ζ.

Finally, in order to take into account the effect of rudder saturation, the commanded rudder angle is given by

$$\delta = \delta_{\text{sat}} \tanh\left(\frac{\delta_0}{\delta_{\text{sat}}}\right) , \tag{14}$$

where δ_0 is the slope of δ at the origin given by (13), and δ_{sat} is the saturation limit on δ, typically around 0.4 radians. The hyperbolic tangent function is used instead of a hard saturation function, because of its differentiability.

D. GUIDANCE SCHEME

Since the previous control law stabilizes the ship to any commanded heading angle, it must be coupled with an appropriate orientation guidance scheme to provide path keeping along the desired track. The simplest guidance scheme which models some fundamental aspects of helmsman behavior is a pure pursuit guidance where the commanded heading angle equals the line of sight angle,

$$\psi_c = -\tan^{-1}\left(\frac{y}{x_d}\right),$$ (15)

as shown in Figure 1. The ship is located at (x, y) and attempts to point its longitudinal axis towards a target point which is located ahead of the ship on the reference path at a constant preview distance x_d. Pursuit guidance is achieved by commanding a heading angle ψ_c equal to the line of sight angle (15).

The positional error information y in equation (15), is assumed to lag the actual error y by an amount of T seconds, in other words,

$$y = y(t - T).$$ (16)

In this equation, the time lag T models the necessary time that it takes for the helmsman to process his path deviation and initiate appropriate corrective actions.

III. LINEAR ANALYSIS

In this Chapter, we present a linearized analysis of the equations of motion. The purpose of this analysis is to assess stability or instability of the equations in response to small deviations from the straight line reference track. No attempt will be made here to characterize the transient response of the system. This can be accomplished through numerical simulations.

A. COMBINED SYSTEM

If we incorporate the interaction/proximity forces (9) and (10) in the ship dynamics model, equations (5) through (8), we end up with the combined system,

$$\dot{\psi} = r ,$$
$$(m - Y_{\dot{v}})\dot{v} - (Y_{\dot{r}} - mx_G)\dot{r} = Y_v v + \frac{1}{6}Y_{vvv}v^3 + \frac{1}{2}Y_{vrr}vr^2 + \frac{1}{2}Y_{rvv}rv^2 +$$
$$(Y_r - m)r + Y_\psi\psi + Y_y y + \frac{1}{6}Y_{\psi\psi\psi}\psi^3 + \frac{1}{6}Y_{yyy}y^3 + Y_\delta\delta ,$$
$$-(N_{\dot{v}} - mx_G)\dot{v} + (I_z - N_{\dot{r}})\dot{r} = N_v v + \frac{1}{6}N_{vvv}v^3 + \frac{1}{2}N_{vrr}vr^2 +$$
$$\frac{1}{2}N_{rvv}rv^2 + (N_r - mx_G)r + N_\psi\psi + N_y y + \frac{1}{6}N_{\psi\psi\psi}\psi^3 +$$
$$\frac{1}{6}N_{yyy}y^3 + N_\delta\delta ,$$
$$\dot{y} = \sin\psi + v\cos\psi .$$

$$(17)$$

Study of the asymptotic properties of this system is the subject of this and the following chapters.

B. LINEARIZATION

The linearized form of equations (17) is the following,

$$\dot{\psi} = r \, ,$$
$$(m - Y_{\dot{v}})\dot{v} - (Y_{\dot{r}} - mx_G)\dot{r} = Y_v v + (Y_r - m)r +$$
$$Y_\psi \psi + Y_y y + Y_\delta \delta \, ,$$
$$-(N_{\dot{v}} - mx_G)\dot{v} + (I_z - N_{\dot{r}})\dot{r} = N_v v + (N_r - mx_G)r + \qquad (18)$$
$$N_\psi \psi + N_y y + N_\delta \delta \, ,$$
$$\dot{y} = \psi + v \, .$$

The rudder angle δ has one of the forms that are developed below. The time delay operator can be expressed in terms of its Taylor series expansion,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} - \frac{1}{6}T^3 y^{(iii)} + \cdots . \qquad (19)$$

Practical computations can be performed by truncating equation (19) to first, second, or third order.

1. First order approximation in y

We have,

$$y(t - T) = y - T\dot{y} \, ,$$

or

$$y(t - T) = y - T(\psi + v) \, . \qquad (20)$$

In its linear form,

$$\tanh\left(\frac{\delta_0}{\delta_{\text{sat}}}\right) = \frac{\delta_0}{\delta_{\text{sat}}} \, ,$$

and, therefore,

$$\delta = \delta_0 \, .$$

Furthermore, in its linear form equation (15) can be written as

$$\psi_c = -\frac{y(t - T)}{x_d} \, .$$

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If we incorporate (20) into (13), we derive the linearized first order approximation in δ,

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - \frac{k_1T}{x_d}(\psi + v) . \tag{21}$$

2. Second order approximation in y

Keeping the second order terms in $y(t - T)$ we get,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} . \tag{22}$$

If we incorporate (22) into (13) along with the linear equations $\delta = \delta_0$ and $\psi_c = -y(t - T)/x_d$, we get

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - \frac{k_1T}{x_d}(\psi + v) + \frac{k_1T^2}{2x_d}(r + \dot{v}) . \tag{23}$$

3. Third order approximation in y

In this case,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} - \frac{1}{6}T^3y^{(iii)} , \tag{24}$$

and

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - \frac{k_1T}{x_d}(\psi + v) + \frac{k_1T^2}{2x_d}(r + \dot{v}) - \frac{k_1T^3}{6x_d}(\dot{r} + \ddot{v}) . \tag{25}$$

4. First order approximation in δ

If a time lag, T, exists on δ instead of simply y, then

$$\delta = \delta_{\text{sat}} \tanh\left(\frac{\delta_1}{\delta_{\text{sat}}}\right) ,$$

where,

$$\delta_1 = \delta_0(t - T) = \delta_0 - T\dot{\delta}_0 .$$

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This equation models a time lag associated with the entire application of the necessary corrective control action and not just its positional error. Using the above equations along with the linear equations $\delta = \delta_1$ and $\psi_c = -y/x_d$, we get

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - k_1Tr - \frac{k_1T}{x_d}(\psi + v) - k_2T\dot{r} . \tag{26}$$

5. First order approximation in both y and δ

In a more general setting, we can assume that a time lag, T_1, exists in the control law δ, and a different time lag, T_2, is present in the processing of the positional error y. Assuming a first order approximation for both, we have

$$\delta_1 = \delta_0(t - T_1) = \delta_0 - T_1\dot{\delta}_0 ,$$

and

$$y(t - T_2) = y - T_2\dot{y} .$$

Therefore, in this case using the above equations along with the linear equations $\delta = \delta_1$ and $\psi_c = -y(t - T_2)/x_d$,we obtain

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - k_1T_1r - \frac{k_1T_1}{x_d}(\psi + v) -$$
$$\frac{k_1T_2}{x_d}(\psi + v) + \frac{k_1T_1T_2}{x_d}(r + \dot{v}) - k_2T_1\dot{r} . \tag{27}$$

Using one of the above expressions, the linearized equations of motion can be written as

$$\begin{aligned}
\dot{\psi} &= r , \\
\dot{v} &= a_{11}\psi + a_{12}v + a_{13}r + a_{14}y + b_1\delta , \\
\dot{r} &= a_{21}\psi + a_{22}v + a_{23}r + a_{24}y + b_2\delta , \\
\dot{y} &= \psi + v ,
\end{aligned} \tag{28}$$

where all coefficients have been previously defined.

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C. SERIES EXPANSIONS OF TIME LAG

1. First order approximation in y

In this case we have,

$$y(t - T) = y - T\dot{y} \ .$$

Substitution of (21) into (28) yields the following matrix system,

$$
\begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,1} & A_{22,1} & A_{23,1} & A_{24,1} \\ A_{31,1} & A_{32,1} & A_{33,1} & A_{34,1} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix} ,
\tag{29}
$$

where,

$$A_{21,1} = a_{11} + b_1 k_1 - \frac{b_1 k_1 T}{x_d}$$

$$A_{22,1} = a_{12} - \frac{b_1 k_1 T}{x_d}$$

$$A_{23,1} = a_{13} + b_1 k_2$$

$$A_{24,1} = a_{14} + \frac{b_1 k_1}{x_d}$$

$$A_{31,1} = a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d}$$

$$A_{32,1} = a_{22} - \frac{b_2 k_1 T}{x_d}$$

$$A_{33,1} = a_{23} + b_2 k_2$$

$$A_{34,1} = a_{24} + \frac{b_2 k_1}{x_d}$$

2. Second order approximation in y

In this case we have,

$$y(t - T) = y - T\dot{y} + \frac{1}{2} T^2 \ddot{y} \ .$$

Substitution of (23) into (28) yields the following matrix system,

$$
\begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,2} & A_{22,2} & A_{23,2} & A_{24,2} \\ A_{31,2} & A_{32,2} & A_{33,2} & A_{34,2} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix} ,
\tag{30}
$$

where,

$$
A_{21,2} = \frac{a_{11}x_d + b_1 k_1 x_d - b_1 k_1 T}{x_d - 0.5 b_1 k_1 T^2}
$$

$$
A_{22,2} = \frac{a_{12}x_d - b_1 k_1 T}{x_d - 0.5 b_1 k_1 T^2}
$$

$$
A_{23,2} = \frac{a_{13}x_d + b_1 k_2 x_d + 0.5 b_1 k_1 T^2}{x_d - 0.5 b_1 k_1 T^2}
$$

$$
A_{24,2} = \frac{a_{14}x_d + b_1 k_1}{x_d - 0.5 b_1 k_1 T^2}
$$

$$
A_{31,2} = a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d} + \frac{b_2 k_1 T^2}{2 x_d} A_{21,2}
$$

$$
A_{32,2} = a_{22} - \frac{b_2 k_1 T}{x_d} + \frac{b_2 k_1 T^2}{2 x_d} A_{22,2}
$$

$$
A_{33,2} = a_{23} + b_2 k_2 + \frac{b_2 k_1 T^2}{2 x_d} + \frac{b_2 k_1 T^2}{2 x_d} A_{23,2}
$$

$$
A_{34,2} = a_{24} + \frac{b_2 k_1}{x_d} + \frac{b_2 k_1 T^2}{2 x_d} A_{24,2}
$$

3. Third order approximation in y

A third order approximation in y is based on,

$$
y(t - T) = y - T\dot{y} + \frac{1}{2}T^2 \ddot{y} - \frac{1}{6}T^3 y^{(iii)} .
$$

Similar algebraic steps as in the previous two approximations result in the following eigenvalue problem,

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & B_{33,3} & 0 & B_{35,3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & B_{53,3} & 0 & B_{55,3} \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \\ \dot{v}_2 \end{bmatrix} =
$$

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$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
A_{31,3} & A_{32,3} & A_{33,3} & A_{34,3} & A_{35,3} \\
1 & 1 & 0 & 0 & 0 \\
A_{51,3} & A_{52,3} & A_{53,3} & A_{54,3} & A_{55,3}
\end{bmatrix}
\begin{bmatrix}
\psi \\
v_1 \\
r \\
y \\
v_2
\end{bmatrix}
, \tag{31}
$$

where, $v_1 = v$, v_2 is the rate of change of v, and the entries of the generalized eigenvalue problem (31) are given bellow. Higher order approximations in T can not produce any usable stability results, since the B matrix in (31) becomes singular.

$$
A_{31,3} = a_{11} + b_1 k_1 - \frac{b_1 k_1 T}{x_d}
$$

$$
A_{32,3} = a_{12} - \frac{b_1 k_1 T}{x_d}
$$

$$
A_{33,3} = a_{13} + b_1 k_2 + \frac{b_1 k_1 T^2}{2 x_d}
$$

$$
A_{34,3} = a_{14} + \frac{b_1 k_1}{x_d}
$$

$$
A_{35,3} = \frac{b_1 k_1 T^2}{2 x_d} - 1
$$

$$
A_{51,3} = a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d}
$$

$$
A_{52,3} = a_{22} - \frac{b_2 k_1 T}{x_d}
$$

$$
A_{53,3} = a_{23} + b_2 k_2 + \frac{b_2 k_1 T^2}{2 x_d}
$$

$$
A_{54,3} = a_{24} + \frac{b_2 k_1}{x_d}
$$

$$
A_{55,3} = \frac{b_2 k_1 T^2}{2 x_d}
$$

$$
B_{33,3} = \frac{b_2 k_1 T^3}{6 x_d}
$$

$$
B_{35,3} = B_{33,3}
$$

$$
B_{53,3} = \frac{b_2 k_1 T^3}{6 x_d} + 1
$$

$$B_{55,3} = \frac{b_2 k_1 T^3}{6 x_d}$$

4. First order approximation in δ

Substitution of (26) into (28) yields the following matrix system,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & B_{23,4} & 0 \\ 0 & 0 & B_{33,4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,4} & A_{22,4} & A_{23,4} & A_{24,4} \\ A_{31,4} & A_{32,4} & A_{33,4} & A_{34,4} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix}, \qquad (32)$$

where,

$$A_{21,4} = a_{11} + b_1 k_1 - \frac{b_1 k_1 T}{x_d}$$

$$A_{22,4} = a_{12} - \frac{b_1 k_1 T}{x_d}$$

$$A_{23,4} = a_{13} + b_1 k_2 - b_1 k_1 T$$

$$A_{24,4} = a_{14} + \frac{b_1 k_1}{x_d}$$

$$A_{31,4} = a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d}$$

$$A_{32,4} = a_{22} - \frac{b_2 k_1 T}{x_d}$$

$$A_{33,4} = a_{23} + b_2 k_2 - b_2 k_1 T$$

$$A_{34,4} = a_{24} + \frac{b_2 k_1}{x_d}$$

$$B_{23,4} = b_1 k_2 T$$

$$B_{33,4} = 1 + b_2 k_2 T$$

5. First order approximation in both y and δ

Substitution of (27) into (28) yields the following matrix system,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & B_{22,5} & B_{23,5} & 0 \\ 0 & B_{32,5} & B_{33,5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,5} & A_{22,5} & A_{23,5} & A_{24,5} \\ A_{31,5} & A_{32,5} & A_{33,5} & A_{34,5} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix}, \qquad (33)$$

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where,

$$A_{21,5} = a_{11} + b_1 k_1 - \frac{b_1 k_1 T_1}{x_d} - \frac{b_1 k_1 T_2}{x_d}$$

$$A_{22,5} = a_{12} - \frac{b_1 k_1 T_1}{x_d} - \frac{b_1 k_1 T_2}{x_d}$$

$$A_{23,5} = a_{13} + b_1 k_2 - b_1 k_1 T_1 + \frac{b_1 k_1 T_1 T_2}{x_d}$$

$$A_{24,5} = a_{14} + \frac{b_1 k_1}{x_d}$$

$$A_{31,5} = a_{21} + b_2 k_1 - \frac{b_2 k_1 T_1}{x_d} - \frac{b_2 k_1 T_2}{x_d}$$

$$A_{32,5} = a_{22} - \frac{b_2 k_1 T_1}{x_d} - \frac{b_2 k_1 T_2}{x_d}$$

$$A_{33,5} = a_{23} + b_2 k_2 - b_2 k_1 T_1 + \frac{b_2 k_1 T_1 T_2}{x_d}$$

$$A_{34,5} = a_{24} + \frac{b_2 k_1}{x_d}$$

$$B_{22,5} = 1 - \frac{b_1 k_1 T_1 T_2}{x_d}$$

$$B_{23,5} = b_1 k_2 T_1$$

$$B_{32,5} = -\frac{b_2 k_1 T_1 T_2}{x_d}$$

$$B_{33,5} = 1 + b_2 k_2 T_1$$

D. EXACT COMPUTATION OF TIME LAG

The previous analysis using Taylor series expansions of $y(t - T)$ breaks down for approximations beyond third order, as the corresponding generalized eigenvalue problem becomes singular. Thus, in order to obtain an exact computation of the stability curves and check the validity of the calculations, a different technique will be performed in this section. This technique is based

on frequency response methods and it utilizes Nyquist's criterion for stability [Friedland (1986)]. We write the system of equations (11) along with equations $\dot{y} = \psi + v$ and $\delta = k_1\psi + k_2 r + \frac{k_1}{x_d}y(t-T)$, in the Laplace domain, where

$$y(t-T) \longrightarrow ye^{-Ts} , \tag{34}$$

is the time delay operator. The characteristic equation of the system is,

$$s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 + (\beta_2 s^2 + \beta_1 s + \beta_0)\frac{1}{x_d}e^{-Ts} = 0 , \tag{35}$$

where

$$\alpha_1 = -a_{12} - a_{23} - b_2 k_2 ,$$

$$\alpha_2 = -a_{14} + a_{12}a_{23} - a_{22}b_1 k_2 - a_{22}a_{13} + a_{12}b_2 k_2 - b_2 k_1 - a_{21} ,$$

$$\alpha_3 = -a_{24} - a_{13}a_{24} - a_{24}b_1 k_2 + a_{23}a_{14} + a_{14}b_2 k_2 - b_1 k_1 a_{22} -$$

$$a_{11}a_{22} + b_2 k_1 a_{12} + a_{21}a_{12} ,$$

$$\alpha_4 = a_{21}a_{14} + a_{12}a_{24} - a_{22}a_{14} - a_{11}a_{24} + b_2 k_1 a_{14} - b_1 k_1 a_{24} ,$$

$$\beta_2 = -b_1 k_1 ,$$

$$\beta_1 = -a_{13}b_2 k_1 + b_1 k_1 a_{23} - b_2 k_1 ,$$

$$\beta_0 = a_{12}b_2 k_1 - b_1 k_1 a_{22} - b_2 k_1 a_{11} + b_1 k_1 a_{21} ,$$

The characteristic equation is written as,

$$1 + \frac{1}{x_d}G(s) = 0 ,$$

where

$$G(s) = \frac{(\beta_2 s^2 + \beta_1 s + \beta_0)e^{-Ts}}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \tag{36}$$

is the open loop transfer function , and $\frac{1}{x_d}$ denotes the effective position gain. For stability, we can utilize the Nyquist criterion which states that the critical value of x_d can be computed from,

$$\frac{1}{x_d}|(G(j\omega)| = 1 \qquad \text{for} \qquad \arg G(j\omega) = -\pi \qquad (37)$$

where j denotes the imaginary unit. The argument, $\arg G(j\omega)$, of(36) is,

$$\arg G(j\omega) = -\omega T + tan^{-1}\frac{\beta_1\omega}{\beta_0 - \beta_2\omega^2} - tan^{-1}\frac{\alpha_3\omega - \alpha_1\omega^3}{\omega^4 - \alpha_2\omega^2 + \alpha_4} . \qquad (38)$$

The set of equations (37) result in the critical visibility distance,

$$x_d = \sqrt{\frac{\beta_1^2\omega^2 + (\beta_0 - \beta_2\omega^2)^2}{(\alpha_3\omega - \alpha_1\omega^3)^2 + (\omega^4 - \alpha_2\omega^2 + \alpha_4)^2}} . \qquad (39)$$

The value of ω in(38) such that $arg G(j\omega) = -\pi$ is the value of the phase crossover frequency. After solving for the phase crossover frequency, the critical value of x_d is obtained by direct substitution of this value of ω into equation (39).

E. RESULTS AND DISCUSSION

Typical results are presented in Figures 3 through 6. All results shown are nondimensional unless otherwise stated. The critical value of x_d versus ω_n for zero time lag, and parametrized for different values of the damping ratio ζ is shown in Figure 3. Stability is ensured for values of x_d greater than its critical value. It can be seen that lower values of ω_n require higher values of x_d for stability. This means that a less responsive helmsman will need a
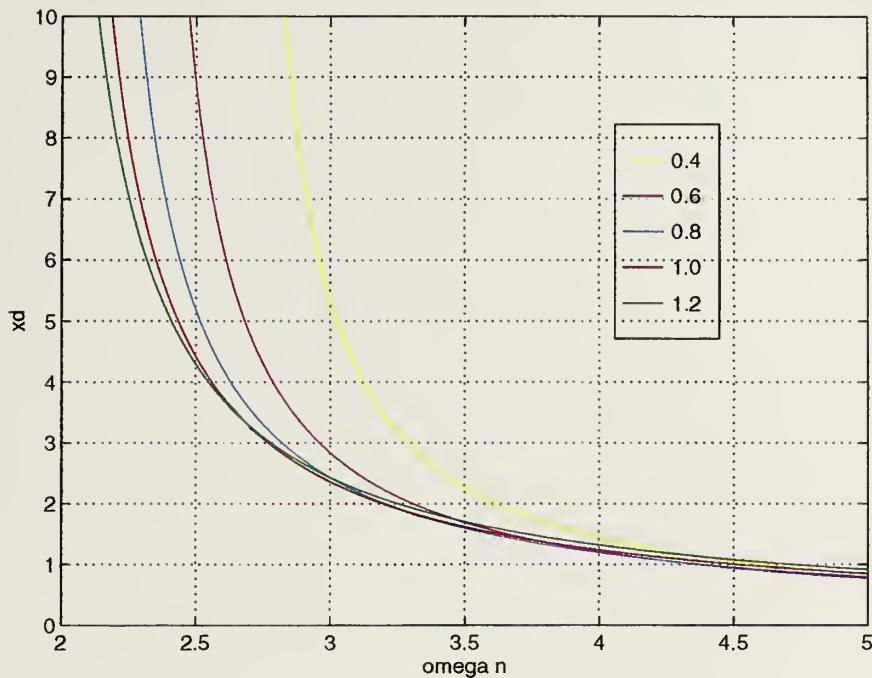
Figure 3: Critical x_d versus ω_n for $T_2 = 0$ and various values of ζ
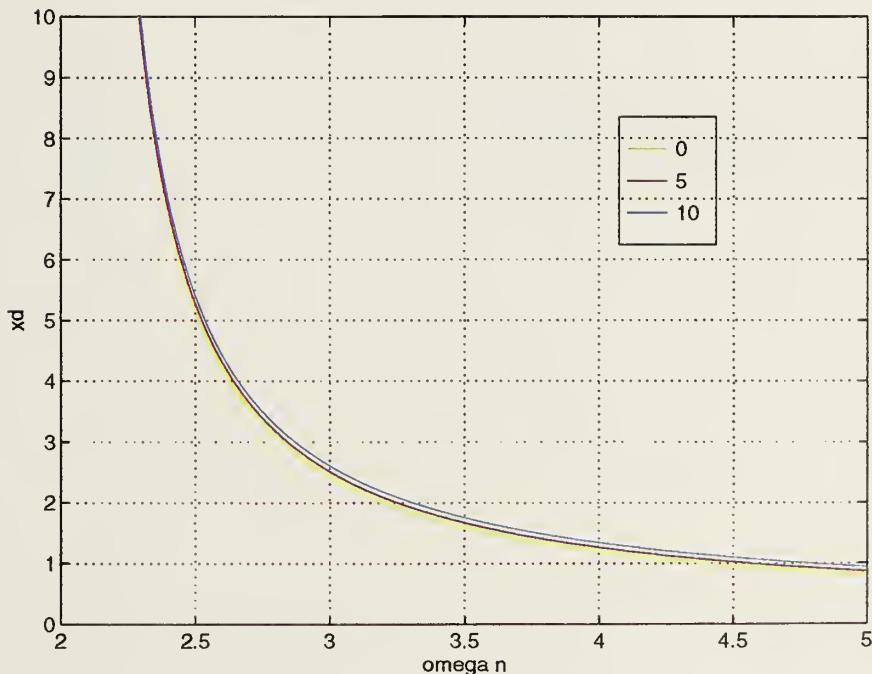
Figure 4: Critical x_d versus ω_n for $\zeta = 0.8$ and various values of T_2
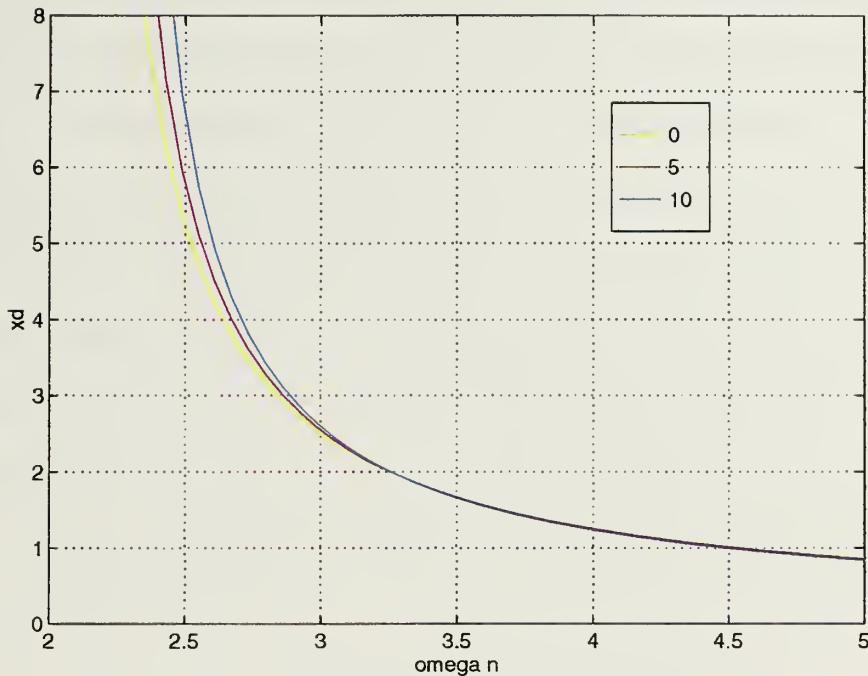
Figure 5: Critical x_d versus ω_n for $\zeta = 0.8$, $T_2 = 5$sec and various values of T_1
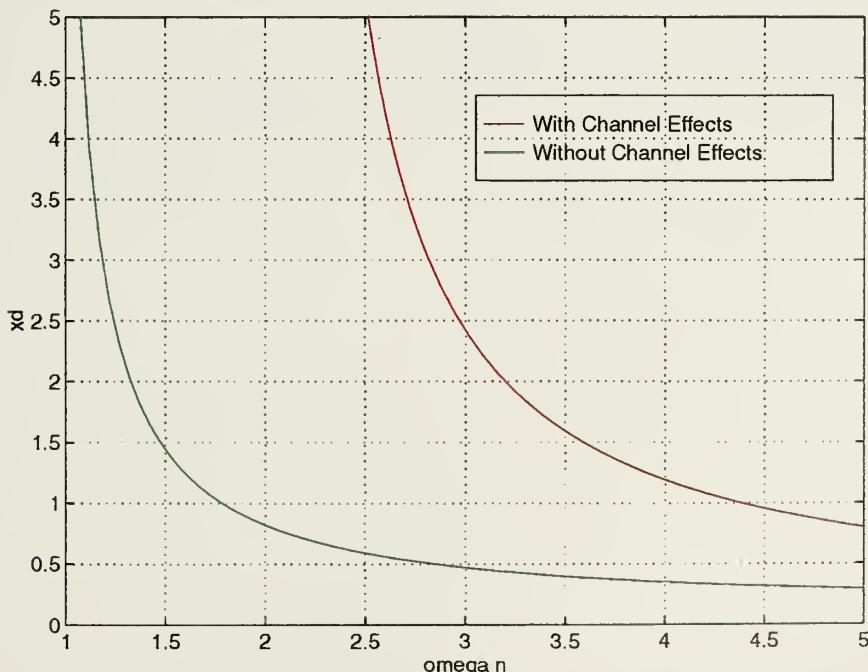
Figure 6: Critical x_d versus ω_n for $\zeta = 0.8$ and zero time lag: Channel effects

longer lookahead distance for a stable operation. Similar conclusions hold for variations in the damping ratio ζ. In this case, higher values of ζ correspond to better helmsman response which requires less lookahead distance.

The effect of time lag T_2 is shown in Figure 4. Time lags are in seconds. It can be seen that reasonable amounts of time lag do not have a serious effect on stability, at least in a linear sense. Of course, an amount of time lag may have a serious effect on the transient response of the system as well as its ability to reject an external disturbance. This can only be established by a systematic series of numerical simulations. Similar conclusions hold for non–zero time lags T_1, as shown in Figure 5.

Finally, the severe destabilizing effect of the canal is demonstrated by the results of Figure 6 where a comparison between canal and open water results is presented. It can be seen that an order of magnitude increase in the lookahead distance may be required if the same control parameters are to be used in both open water and a canal.

IV. NONLINEAR ANALYSIS

A. INTRODUCTION

It can be numerically confirmed that in all cases of stability loss of the previous chapter, one pair of complex conjugate eigenvalues of the corresponding eigenvalue problem crosses transversally the imaginary axis. A situation like this in which a certain parameter is varied such that the real part of one pair of complex conjugate eigenvalues of the linearized system matrix crosses zero, results in the system leaving its steady state in an oscillatory manner. This loss of stability is called Hopf bifurcation and generically occurs in one of two ways, supercritical or subcritical. In the supercritical case, stable limit cycles are generated after the nominal straight line motion loses its stability. The amplitudes of these limit cycles are continuously increasing as the parameter distance from its critical value is increased. For small values of this criticality distance the resulting limit cycle is of small amplitude and differs little from the initial nominal state. In the subcritical case, however, stable limit cycles are generated before the nominal state loses its stability. Therefore, depending on the initial conditions it is possible to diverge away from the nominal straight line path and converge towards a limit cycle even before the nominal motion loses its stability. This means that in the subcritical Hopf bifurcation case the domain of attraction of the nominal state is decreasing and in fact it shrinks to zero as the critical point is approached. Random external dis-

turbances of sufficient magnitude can throw the vehicle off to an oscillatory steady state even though the nominal state may still remain stable. After the nominal state becomes unstable, a discontinuous increase in the magnitude of motions is observed as there exist no simple stable nearby attractors for the vehicle to converge to. Distinction between these two qualitatively different types of bifurcation is, therefore, essential in design. The computational procedure requires higher order approximations in the equations of motion and is the subject of the next section.

B. DETAILED CALCULATIONS

The nonlinear equations of motion are written as,

$$\dot{\psi} = r , \tag{40}$$

$$\dot{v} = a_{11}\psi + a_{12}v + a_{13}r + a_{14}y + b_1\delta' , \tag{41}$$

$$\dot{r} = a_{21}\psi + a_{22}v + a_{23}r + a_{24}y + b_2\delta' , \tag{42}$$

$$\dot{y} = \sin\psi + v\cos\psi , \tag{43}$$

where the control law is,

$$\delta = k_1\psi + k_2r + k_1\tan^{-1}\frac{y(t-T)}{x_d} , \tag{44}$$

and, including saturation,

$$\delta' = \delta_{\text{sat}}\tanh\left(\frac{\delta}{\delta_{\text{sat}}}\right) . \tag{45}$$

We perform a Taylor series expansion of the equations, keeping the first non-vanishing nonlinear terms. Due to port/starboard symmetry in the problem

this means expansion to third order terms,

$$\sin \psi = \psi - \tfrac{1}{6}\psi^3 , \quad \cos \psi = 1 - \tfrac{1}{2}\psi^2 , \tag{46}$$

$$\delta' = \delta - \frac{1}{3\delta_{\text{sat}}^2}\delta^3 , \tag{47}$$

$$\delta = k_1\psi + k_2 r + \frac{k_1}{x_d}y(t - T) - \frac{k_1}{3x_d^3}y^3(t - T) , \tag{48}$$

where for consistency, the term $y(t - T)$ is approximated by its first order expansion in T,

$$y(t - T) = y - T\dot{y} = y - T\psi + \tfrac{1}{6}T\psi^3 - Tv + \tfrac{1}{2}Tv\psi^2 . \tag{49}$$

Substitution of equations (46) to (49) into equations (40) to (45), produces the system,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}) , \tag{50}$$

where the state variables vector is,

$$\mathbf{x} = [\psi, v, r, y]^T ,$$

$\mathbf{A}$ is the linearized matrix at equilibrium, and $\mathbf{g}(\mathbf{x})$ contains all third order terms.

If $\mathbf{T}$ is the matrix of eigenvectors of $\mathbf{A}$ evaluated at the critical point x_d, the linear change of coordinates,

$$\mathbf{x} = \mathbf{T}\mathbf{z} , \quad \mathbf{z} = \mathbf{T}^{-1}\mathbf{x} , \tag{51}$$

transforms system (50) into its normal coordinate form,

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} + \mathbf{T}^{-1}\mathbf{g}(\mathbf{T}\mathbf{z}) . \tag{52}$$

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At the Hopf bifurcation point, matrix $\mathbf{T}^{-1}\mathbf{AT}$ takes the form,

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & -\omega_0 & 0 & 0 \\ \omega_0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & q \end{bmatrix},$$

where ω_0 is the imaginary part of the critical pair of eigenvalues, and the remaining two eigenvalues p and q are negative(or complex conjugate with negative real parts) . Therefore in the new system of coordinates (51) the dynamics of (50) are governed by a reduced two-dimensional system z_1 and z_2 since the coordinates z_3 and z_4 correspond to the eigenvalues p and q and are asymptotically stable. For values of x_d close to the bifurcation point x_d, matrix $\mathbf{T}^{-1}\mathbf{AT}$ is,

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} \alpha'\varepsilon & -(\omega_0 + \omega'\varepsilon) & 0 & 0 \\ \omega_0 + \omega'\varepsilon & \alpha'\varepsilon & 0 & 0 \\ 0 & 0 & p + p'\varepsilon & 0 \\ 0 & 0 & 0 & q + q'\varepsilon \end{bmatrix},$$

where, ε denotes the criticality difference,

$$\varepsilon = x_d - x_{d_{\text{critical}}}, \tag{53}$$

ω' = derivative of the real part of the critical eigenvalues with respect to ε ,

α' = derivative of the imaginary part of the critical eigenvalues

with respect to ε ,

p' = derivative of p with respect to ε ,

q' = derivative of q with respect to ε .

Due to continuity, the eigenvalues $p + p'\varepsilon$ and $q + q'\varepsilon$ remain negative for small nonzero values of ε. Therefore, the coordinates z_3, z_4 correspond to negative

eigenvalues and are asymptotically stable. Center manifold theory predicts that the relationship between the critical coordinates z_1, z_2 and the stable coordinates z_3, z_4 is at least of quadratic order. In fact, due to the symmetry in our problem the relationship is cubic,

$$z_1 = \mathcal{O}(z_3^3, z_4^3) , \quad z_2 = \mathcal{O}(z_3^3, z_4^3) .$$

It follows that because of this order, z_3, z_4 do not influence the third order Taylor series expansions, and can be dropped from the equations. Therefore, we can write the reduced system that describes the essential dynamics of (52) in the form,

$$\dot{z}_1 = \alpha'\varepsilon z_1 - (\omega_0 + \omega'\varepsilon)z_2 + F_1(z_1, z_2) , \tag{54}$$

$$\dot{z}_2 = (\omega_0 + \omega'\varepsilon)z_1 + \alpha'\varepsilon z_2 + F_2(z_1, z_2) , \tag{55}$$

where F_1, F_2 contain the third order terms,

$$F_1(z_1, z_2) = r_{11}z_1^3 + r_{12}z_1^2 z_2 + r_{13}z_1 z_2^2 + r_{14}z_2^3 , \tag{56}$$

$$F_2(z_1, z_2) = r_{21}z_1^3 + r_{22}z_1^2 z_2 + r_{23}z_1 z_2^2 + r_{24}z_2^3 . \tag{57}$$

The coefficients r_{ij} are computable from the previous Taylor expansions, and they are given below.

The nonlinear expansion coefficients r_{ij} that are utilized in the definition of the cubic stability coefficient $\mathcal{K}$ are given by the following:

$$r_{11} = n_{12}\ell_{21} + n_{13}\ell_{31} + n_{14}\ell_{41} + n_{12}d_{11} + n_{13}d_{21} ,$$

$$r_{12} = n_{12}\ell_{22} + n_{13}\ell_{32} + n_{14}\ell_{42} + n_{12}d_{12} + n_{13}d_{22} ,$$

$$r_{13} = n_{12}\ell_{23} + n_{13}\ell_{33} + n_{14}\ell_{43} + n_{12}d_{13} + n_{13}d_{23} ,$$

$$r_{14} = n_{12}\ell_{24} + n_{13}\ell_{34} + n_{14}\ell_{44} + n_{12}d_{14} + n_{13}d_{24} ,$$

$$r_{21} = n_{22}\ell_{21} + n_{23}\ell_{31} + n_{24}\ell_{41} + n_{22}d_{11} + n_{23}d_{21} ,$$

$$r_{22} = n_{22}\ell_{22} + n_{23}\ell_{32} + n_{24}\ell_{42} + n_{22}d_{12} + n_{23}d_{22} ,$$

$$r_{23} = n_{22}\ell_{23} + n_{23}\ell_{33} + n_{24}\ell_{43} + n_{22}d_{13} + n_{23}d_{23} ,$$

$$r_{24} = n_{22}\ell_{24} + n_{23}\ell_{34} + n_{24}\ell_{44} + n_{22}d_{14} + n_{23}d_{24} ,$$

where we denote,

$$\mathbf{T} = [m_{ij}] , \ \mathbf{T}^{-1} = [n_{ij}] , \ i,j = 1,\ldots,4 .$$

For numerical purposes the critical eigenvector of $\mathbf{T}$ must be normalized so that its first nonvanishing coefficient is identically equal to 1. The coefficients ℓ_{ij} are given by,

$$
\begin{aligned}
\frac{\ell_{21}}{b_1} = \ & \delta_{\psi\psi v}m_{11}^2 m_{21} + \delta_{\psi vv}m_{11}m_{21}^2 + \delta_{\psi\psi r}m_{11}^2 m_{31} + \delta_{\psi rr}m_{11}m_{31}^2 \\
& + \delta_{\psi\psi y}m_{11}^2 m_{41} + \delta_{\psi yy}m_{11}m_{41}^2 + \delta_{vvr}m_{21}^2 m_{31} + \delta_{vrr}m_{21}m_{31}^2 \\
& + \delta_{vvy}m_{21}^2 m_{41} + \delta_{vyy}m_{21}m_{41}^2 + \delta_{rry}m_{31}^2 m_{41} + \delta_{ryy}m_{31}m_{41}^2 \\
& + \delta_{\psi vr}m_{11}m_{21}m_{31} + \delta_{\psi vy}m_{11}m_{21}m_{41} + \delta_{\psi ry}m_{11}m_{41}m_{31} \\
& + \delta_{vry}m_{21}m_{41}m_{31} + \delta_{\psi\psi\psi}m_{11}^3 + \delta_{vvv}m_{21}^3 + \delta_{rrr}m_{21}^3 + \delta_{yyy}m_{41}^3 , \\
\frac{\ell_{22}}{b_1} = \ & \delta_{\psi\psi v}(m_{11}^2 m_{22} + 2m_{11}m_{12}m_{21}) + \delta_{\psi vv}(m_{12}m_{21}^2 + 2m_{11}m_{12}m_{22}) \\
& + \delta_{\psi\psi r}(m_{11}^2 m_{32} + 2m_{11}m_{12}m_{31}) + \delta_{\psi rr}(m_{12}m_{31}^2 + 2m_{11}m_{31}m_{32}) \\
& + \delta_{\psi\psi y}(m_{11}^2 m_{42} + 2m_{11}m_{12}m_{41}) + \delta_{\psi yy}(m_{41}^2 m_{12} + 2m_{11}m_{41}m_{42}) \\
& + \delta_{vvr}(m_{21}^2 m_{32} + 2m_{31}m_{21}m_{22}) + \delta_{vrr}(m_{22}m_{31}^2 + 2m_{31}m_{32}m_{21})
\end{aligned}
$$

$$+\delta_{vvy}(m_{21}^2 m_{42} + 2m_{41}m_{21}m_{22}) + \delta_{vyy}(m_{22}m_{41}^2 + 2m_{41}m_{42}m_{21})$$

$$+\delta_{rry}(m_{31}^2 m_{42} + 2m_{41}m_{31}m_{32}) + \delta_{ryy}(m_{32}m_{41}^2 + 2m_{41}m_{42}m_{31})$$

$$+\delta_{\psi vr}[m_{11}m_{21}m_{32} + m_{31}(m_{11}m_{22} + m_{12}m_{21})]$$

$$+\delta_{\psi vy}[m_{11}m_{21}m_{42} + m_{41}(m_{11}m_{22} + m_{12}m_{21})]$$

$$+\delta_{\psi ry}[m_{11}m_{41}m_{32} + m_{31}(m_{11}m_{42} + m_{12}m_{41})]$$

$$+\delta_{vry}[m_{21}m_{41}m_{32} + m_{31}(m_{21}m_{42} + m_{22}m_{41})]$$

$$+\delta_{\psi\psi\psi}(3m_{11}^2 m_{12}) + \delta_{vvv}(3m_{21}^2 m_{22}) + \delta_{rrr}(3m_{31}^2 m_{32}) + \delta_{yyy}(3m_{41}^2 m_{42}) \ ,$$

$$\frac{\ell_{23}}{b_1} = \delta_{\psi\psi v}(m_{12}^2 m_{21} + 2m_{11}m_{12}m_{22}) + \delta_{\psi vv}(m_{11}m_{22}^2 + 2m_{12}m_{21}m_{22})$$

$$+\delta_{\psi\psi r}(m_{12}^2 m_{31} + 2m_{11}m_{12}m_{32}) + \delta_{\psi rr}(m_{11}m_{32}^2 + 2m_{12}m_{31}m_{32})$$

$$+\delta_{\psi\psi y}(m_{12}^2 m_{41} + 2m_{11}m_{12}m_{42}) + \delta_{\psi yy}(m_{42}^2 m_{11} + 2m_{12}m_{41}m_{42})$$

$$+\delta_{vvr}(m_{22}^2 m_{31} + 2m_{21}m_{22}m_{32}) + \delta_{vrr}(m_{21}m_{32}^2 + 2m_{31}m_{22}m_{32})$$

$$+\delta_{vvy}(m_{22}^2 m_{41} + 2m_{21}m_{22}m_{42}) + \delta_{vyy}(m_{21}m_{42}^2 + 2m_{41}m_{42}m_{22})$$

$$+\delta_{rry}(m_{32}^2 m_{41} + 2m_{42}m_{31}m_{32}) + \delta_{ryy}(m_{31}m_{42}^2 + 2m_{41}m_{42}m_{32})$$

$$+\delta_{\psi vr}[m_{12}m_{22}m_{31} + m_{32}(m_{11}m_{22} + m_{12}m_{21})]$$

$$+\delta_{\psi vy}[m_{12}m_{22}m_{41} + m_{42}(m_{11}m_{22} + m_{12}m_{21})]$$

$$+\delta_{\psi ry}[m_{12}m_{42}m_{31} + m_{32}(m_{11}m_{42} + m_{12}m_{41})]$$

$$+\delta_{vry}[m_{22}m_{42}m_{31} + m_{32}(m_{21}m_{42} + m_{22}m_{41})]$$

$$+\delta_{\psi\psi\psi}(3m_{11}m_{12}^2) + \delta_{vvv}(3m_{21}m_{22}^2) + \delta_{rrr}(3m_{31}m_{32}^2) + \delta_{yyy}(3m_{41}m_{42}^2) \ ,$$

$$\frac{\ell_{24}}{b_1} = \delta_{\psi\psi v}m_{12}^2 m_{22} + \delta_{\psi vv}m_{12}m_{22}^2 + \delta_{\psi\psi r}m_{12}^2 m_{32} + \delta_{\psi rr}m_{12}m_{32}^2$$

$$+\delta_{\psi\psi y}m_{12}^2 m_{42} + \delta_{\psi yy}m_{12}m_{42}^2 + \delta_{vvr}m_{22}^2 m_{32} + \delta_{vrr}m_{22}m_{32}^2$$

$$+\delta_{vvy}m_{22}^2 m_{42} + \delta_{vyy}m_{22}m_{42}^2 + \delta_{rry}m_{32}^2 m_{42} + \delta_{ryy}m_{32}m_{42}^2$$

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$$+\delta_{\psi vr}m_{12}m_{22}m_{32} + \delta_{\psi vy}m_{12}m_{22}m_{42} + \delta_{\psi ry}m_{12}m_{42}m_{32} + \delta_{vry}m_{22}m_{42}m_{32}$$

$$+\delta_{\psi\psi\psi}m_{12}^3 + \delta_{vvv}m_{22}^3 + \delta_{rrr}m_{32}^3 + \delta_{yyy}m_{42}^3 \ ,$$

$$\frac{\ell_{31}}{b_2} = \frac{\ell_{21}}{b_1} \ ,$$

$$\frac{\ell_{32}}{b_2} = \frac{\ell_{22}}{b_1} \ ,$$

$$\frac{\ell_{33}}{b_2} = \frac{\ell_{23}}{b_1} \ ,$$

$$\frac{\ell_{34}}{b_2} = \frac{\ell_{24}}{b_1} \ ,$$

$$\ell_{41} = -\frac{1}{2}m_{11}^2\left(m_{21} + \frac{1}{3}m_{11}\right) \ ,$$

$$\ell_{42} = -m_{11}\left(m_{12}m_{21} + \frac{1}{2}m_{11}\dot{m}_{22} + \frac{1}{2}m_{12}m_{11}\right) \ ,$$

$$\ell_{43} = -m_{12}\left(m_{11}m_{22} + \frac{1}{2}m_{12}m_{21} + \frac{1}{2}m_{12}m_{11}\right) \ ,$$

$$\ell_{44} = -\frac{1}{2}m_{12}^2\left(m_{22} + \frac{1}{3}m_{12}\right) \ .$$

The coefficients d_{ii} are given by,

$$d_{11} = \frac{1}{D_{vr}}[c_{11}(I_z - N_{\dot{r}}) + c_{21}(Y_{\dot{r}} - mx_G)] \ ,$$

$$d_{12} = \frac{1}{D_{vr}}[c_{12}(I_z - N_{\dot{r}}) + c_{22}(Y_{\dot{r}} - mx_G)] \ ,$$

$$d_{13} = \frac{1}{D_{vr}}[c_{13}(I_z - N_{\dot{r}}) + c_{23}(Y_{\dot{r}} - mx_G)] \ ,$$

$$d_{14} = \frac{1}{D_{vr}}[c_{14}(I_z - N_{\dot{r}}) + c_{24}(Y_{\dot{r}} - mx_G)] \ ,$$

$$d_{21} = \frac{1}{D_{vr}}[c_{11}(N_{\dot{v}} - mx_G) + c_{21}(m - Y_{\dot{v}})] \ ,$$

$$d_{22} = \frac{1}{D_{vr}}[c_{12}(N_{\dot{v}} - mx_G) + c_{22}(m - Y_{\dot{v}})] \ ,$$

$$d_{23} = \frac{1}{D_{vr}}[c_{13}(N_{\dot{v}} - mx_G) + c_{23}(m - Y_{\dot{v}})] \ ,$$

$$d_{24} = \frac{1}{D_{vr}}[c_{14}(N_{\dot{v}} - mx_G) + c_{24}(m - Y_{\dot{v}})] \ ,$$

where

$$c_{11} = \frac{1}{6}Y_{vvv}m_{21}^3 + \frac{1}{2}Y_{vrr}m_{31}^2m_{21} + \frac{1}{2}Y_{rvv}m_{21}^2m_{31}$$
$$+ \frac{1}{6}Y_{\psi\psi\psi}m_{11}^3 + \frac{1}{6}Y_{yyy}m_{41}^3 \ ,$$

$$c_{12} = \frac{1}{6}Y_{vvv}3m_{21}^2m_{22} + \frac{1}{2}Y_{vrr}(m_{31}^2m_{22} + 2m_{31}m_{32}m_{21})$$
$$+ \frac{1}{2}Y_{rvv}(m_{21}^2m_{32} + 2m_{31}m_{21}m_{22}) + \frac{1}{6}Y_{\psi\psi\psi}3m_{11}^2m_{12}$$
$$+ \frac{1}{6}Y_{yyy}3m_{41}^2m_{42} \ ,$$

$$c_{13} = \frac{1}{6}Y_{vvv}3m_{22}^2m_{21} + \frac{1}{2}Y_{vrr}(m_{32}^2m_{21} + 2m_{31}m_{32}m_{22})$$
$$+ \frac{1}{2}Y_{rvv}(m_{22}^2m_{31} + 2m_{32}m_{21}m_{22}) + \frac{1}{6}Y_{\psi\psi\psi}3m_{12}^2m_{11}$$
$$+ \frac{1}{6}Y_{yyy}3m_{42}^2m_{41} \ ,$$

$$c_{14} = \frac{1}{6}Y_{vvv}m_{22}^3 + \frac{1}{2}Y_{vrr}m_{32}^2m_{22} + \frac{1}{2}Y_{rvv}m_{22}^2m_{32}$$
$$+ \frac{1}{6}Y_{\psi\psi\psi}m_{12}^3 + \frac{1}{6}Y_{yyy}m_{42}^3 \ ,$$

$$c_{21} = \frac{1}{6}N_{vvv}m_{21}^3 + \frac{1}{2}N_{vrr}m_{31}^2m_{21} + \frac{1}{2}N_{rvv}m_{21}^2m_{31}$$
$$+ \frac{1}{6}N_{\psi\psi\psi}m_{11}^3 + \frac{1}{6}N_{yyy}m_{41}^3 \ ,$$

$$c_{22} = \frac{1}{6}N_{vvv}3m_{21}^2m_{22} + \frac{1}{2}N_{vrr}(m_{31}^2m_{22} + 2m_{31}m_{32}m_{21})$$
$$+ \frac{1}{2}N_{rvv}(m_{21}^2m_{32} + 2m_{31}m_{21}m_{22}) + \frac{1}{6}N_{\psi\psi\psi}3m_{11}^2m_{12}$$
$$+ \frac{1}{6}N_{yyy}3m_{41}^2m_{42} \ ,$$

$$c_{23} = \frac{1}{6}N_{vvv}3m_{22}^2m_{21} + \frac{1}{2}N_{vrr}(m_{32}^2m_{21} + 2m_{31}m_{32}m_{22})$$
$$+ \frac{1}{2}N_{rvv}(m_{22}^2m_{31} + 2m_{32}m_{21}m_{22}) + \frac{1}{6}N_{\psi\psi\psi}3m_{12}^2m_{11}$$
$$+ \frac{1}{6}N_{yyy}3m_{42}^2m_{41} \ ,$$

$$c_{24} = \frac{1}{6}N_{vvv}m_{22}^3 + \frac{1}{2}N_{vrr}m_{32}^2m_{22} + \frac{1}{2}N_{rvv}m_{22}^2m_{32}$$
$$+ \frac{1}{6}N_{\psi\psi\psi}m_{12}^3 + \frac{1}{6}N_{yyy}m_{42}^3 \ .$$

The coefficients in a third order expansion of the control law are defined by

$$\delta_{\psi\psi v} = -\frac{1}{\delta_{\text{sat}}^2}(k_1')^2 k_2' + 0.5 k_1 \frac{T}{x_d} + \frac{k_1 T^3}{x_d^3} ,$$

$$\delta_{\psi vv} = -\frac{1}{\delta_{\text{sat}}^2} k_1'(k_2')^2 + \frac{T^3 k_1}{x_d^3} ,$$

$$\delta_{\psi\psi r} = -\frac{1}{\delta_{\text{sat}}^2}(k_1')^2 k_2 ,$$

$$\delta_{\psi rr} = -\frac{1}{\delta_{\text{sat}}^2} k_1' k_2^2 ,$$

$$\delta_{\psi\psi y} = -\frac{1}{\delta_{\text{sat}}^2}(k_1')^2 \frac{k_1}{x_d} - \frac{T^2 k_1}{x_d^3} ,$$

$$\delta_{\psi yy} = -\frac{1}{\delta_{\text{sat}}^2} k_1' \frac{k_1^2}{x_d^2} + \frac{T k_1}{x_d^3} ,$$

$$\delta_{vvr} = -\frac{1}{\delta_{\text{sat}}^2}(k_2')^2 k_2 ,$$

$$\delta_{vrr} = -\frac{1}{\delta_{\text{sat}}^2} k_2' k_2^2 ,$$

$$\delta_{vvy} = -\frac{1}{\delta_{\text{sat}}^2}(k_2')^2 \frac{k_1}{x_d} - \frac{k_1 T^2}{x_d^3} ,$$

$$\delta_{vyy} = -\frac{1}{\delta_{\text{sat}}^2} k_2' \frac{k_1^2}{x_d^2} + \frac{T k_1}{x_d^3} ,$$

$$\delta_{rry} = -\frac{1}{\delta_{\text{sat}}^2} k_2^2 \frac{k_1}{x_d} ,$$

$$\delta_{ryy} = -\frac{1}{\delta_{\text{sat}}^2} k_2 \frac{k_1^2}{x_d^2} ,$$

$$\delta_{\psi vr} = -\frac{1}{\delta_{\text{sat}}^2} 2 k_1' k_2' k_2 ,$$

$$\delta_{\psi vy} = -\frac{1}{\delta_{\text{sat}}^2} 2 k_1' k_2' \frac{k_1}{x_d} - 2\frac{T^2 k_1}{x_d^3} ,$$

$$\delta_{\psi ry} = -\frac{1}{\delta_{\text{sat}}^2} 2 k_1' k_2 \frac{k_1}{x_d} ,$$

$$\delta_{vry} = -\frac{1}{\delta_{\text{sat}}^2} 2 k_2' k_2 \frac{k_1}{x_d} ,$$

$$\delta_{\psi\psi\psi} = -\frac{1}{\delta_{\text{sat}}^2}\frac{1}{3}(k_1')^3 + \frac{k_1 T}{6x_d} + \frac{k_1 T^3}{3x_d^3} ,$$

$$\delta_{vvv} = -\frac{1}{\delta_{\text{sat}}^2}\frac{1}{3}(k_2')^3 + \frac{k_1 T^3}{3x_d^3} ,$$

$$\delta_{rrr} = -\frac{1}{\delta_{\text{sat}}^2}\frac{1}{3}k_2^3 ,$$

$$\delta_{yyy} = -\frac{1}{\delta_{\text{sat}}^2}\frac{1}{3}\frac{k_1^3}{x_d^3} - \frac{2k_1}{3x_d^3} ,$$

and

$$k_1' = k_1 - k_1\frac{T}{x_d} ,$$

$$k_2' = -k_1\frac{T}{x_d} .$$

C. NONLINEAR COEFFICIENT $\mathcal{K}$

If we introduce polar coordinates in the form,

$$z_1 = R\cos\theta , \quad z_2 = R\sin\theta , \tag{58}$$

we can use (54) and (55) to produce an equation describing the rate of change of the radial coordinate R,

$$\dot{R} = \alpha'\varepsilon R + \mathcal{P}(\theta)R^3 , \tag{59}$$

and a similar equation in the rate of change of the angular coordinate θ,

$$\dot{\theta} = \omega_0 + \omega'\varepsilon + \mathcal{Q}(\theta)R^2 . \tag{60}$$

The system of equations (59) and (60) contains two variables, one of which, R, is slowly varying in time, whereas the other one, θ is a fast variable. Then,

equation (59) can be averaged over one cycle in θ to produce an equation with constant coefficients and similar stability properties,

$$\dot{R} = \alpha' \varepsilon R + \mathcal{K} R^3 \, , \tag{61}$$

where,

$$\mathcal{K} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(\theta) \, d\theta = \tfrac{1}{8} \left(3r_{11} + r_{13} + r_{22} + 3r_{24} \right) \, . \tag{62}$$

Nontrivial equilibrium solutions of (61) correspond to limit cycles in the original system. The nontrivial equilibrium of (61), R_0, is given by,

$$R_0 = \sqrt{-\frac{\alpha' \varepsilon}{\mathcal{K}}} \, . \tag{63}$$

In our problem, since the trivial equilibrium gains its stability for $x_d > x_{d_{\text{critical}}}$, the coefficient α' is always negative. Therefore, it can be seen from (63) that a limit cycle will exist provided $\mathcal{K}$ and ε have the same sign. The stability properties of this limit cycle can be determined by linearization of (61) around R_0. The linearized system is,

$$\dot{R} = -2\alpha' \varepsilon (R - R_0) \, , \tag{64}$$

and its eigenvalue is,

$$\beta = -2\alpha' \varepsilon \, . \tag{65}$$

We can see that the Floquet exponent (65) is negative if ε is negative, which means that $\mathcal{K}$ must be negative. Therefore, location and stability of limit cycles depends entirely on the cubic coefficient $\mathcal{K}$ which is computable from (62). We can summarize our findings as,
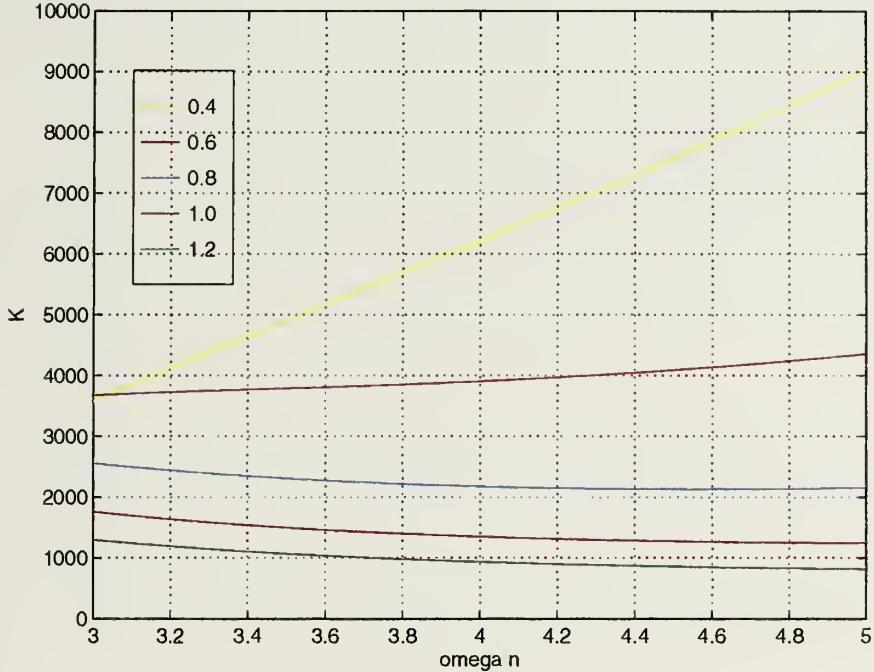
Figure 7: Nonlinear coefficient $\mathcal{K}$ versus ω_n for $\delta_{\text{sat}} = 0.4$ and various values of ζ

- If $\mathcal{K} < 0$ then limit cycles exist for $\varepsilon < 0$ $\left(x_d < x_{d_{\text{critical}}}\right)$ and they are stable.

- If $\mathcal{K} > 0$ then limit cycles exist for $\varepsilon > 0$ $\left(x_d > x_{d_{\text{critical}}}\right)$ and they are unstable.

D. RESULTS AND DISCUSSION

Typical results in terms of the nonlinear stability coefficient $\mathcal{K}$ are shown in Figures 7 through 10. The general conclusion from this series of graphs is that the bifurcations are all strongly subcritical. This means that any linearized stability results should be viewed with extreme caution. Limit cycles exist
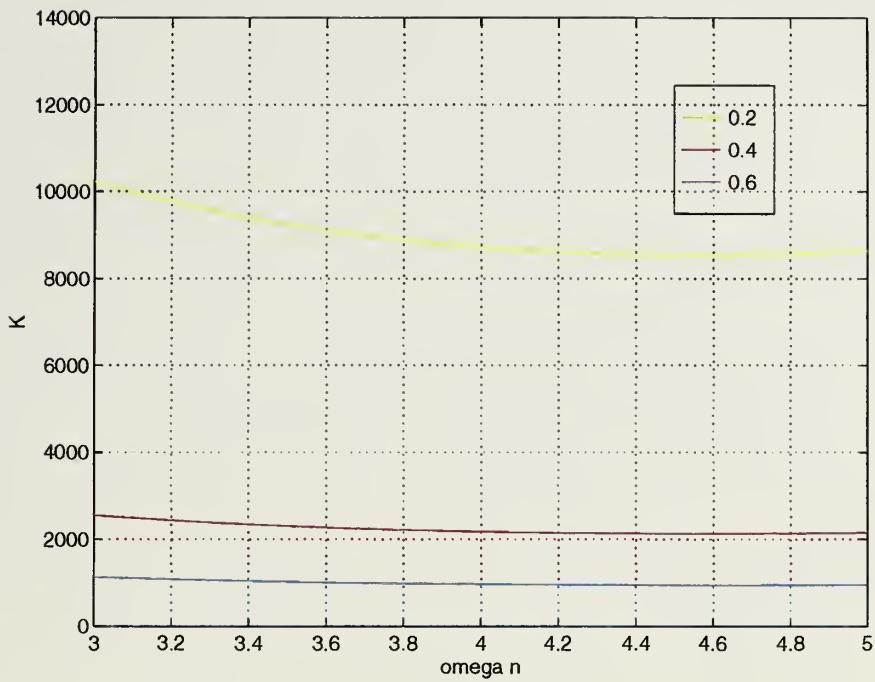
Figure 8: Nonlinear coefficient $\mathcal{K}$ versus ω_n for $\zeta = 0.8$ and various values of δ_{sat}
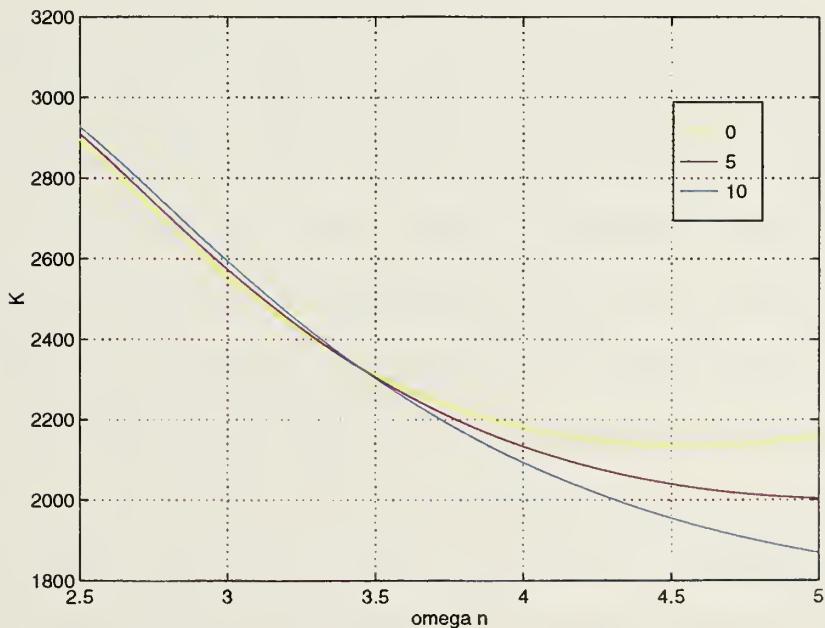
Figure 9: Nonlinear coefficient $\mathcal{K}$ versus ω_n for $\zeta = 0.8$, $\delta_{\text{sat}} = 0.4$, $T_1 = 0$, and various values of T_2 (sec)
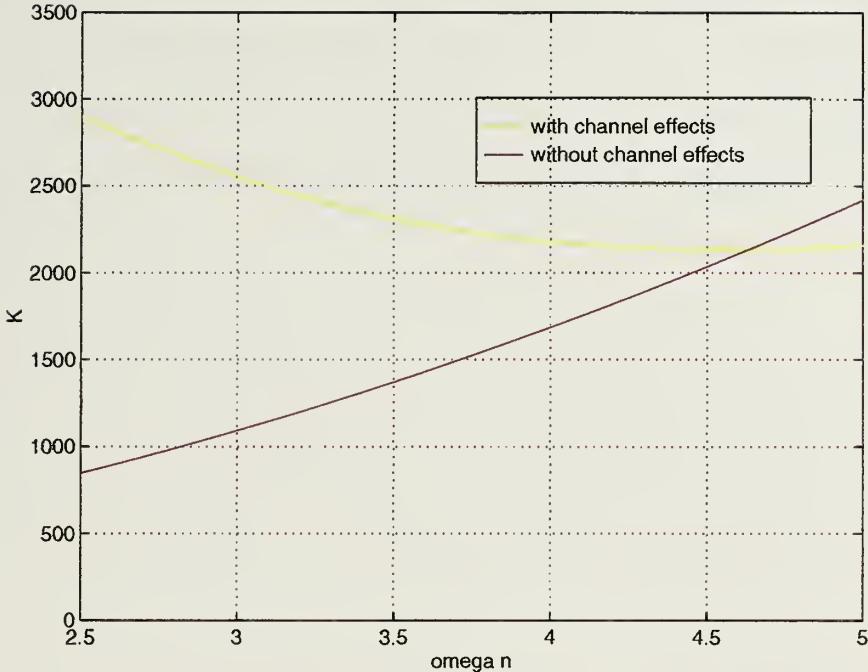
Figure 10: Nonlinear coefficient $\mathcal{K}$ versus ω_n with and without channel effects

before stability in the linear sense is lost and a self sustained oscillation in the system may develop as a result of an external disturbance even if the nominal equilibrium state is still stable. The bifurcations become more subcritical; i.e., $\mathcal{K}$ is more positive, for smaller values of ζ, as Figure 7 shows. Figure 8 demonstrates the effect that the saturation limit δ_{sat} has on the value of $\mathcal{K}$. Higher values of δ_{sat}, although are not related to the critical value of x_d result in significant changes in the value of $\mathcal{K}$. The general trend is that higher δ_{sat} results in less subcritical bifurcations as evidenced by the overall decrease of $\mathcal{K}$. Figure 9 shows the effect of non–zero time lag on the nonlinear stability coefficient. It can be seen that, like the linear stability results, the effect is minimal. Finally, Figure 10 shows the canal effect on $\mathcal{K}$. This figure was

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produced for zero time lag, $\zeta = 0.8$, and $\delta_{sat} = 0.4$. It can be seen that the existence of a canal causes the bifurcations to be much more subcritical than the open water case. This demonstrates the severe destabilizing effect that the canal introduces in both the linearized and the nonlinear analysis.

V. CONCLUSIONS AND RECOMMENDATIONS

This thesis presented a comprehensive study of linear and nonlinear stability properties of straightline motion of surface ships in confined waters. The classical maneuvering equations of motion incorporating canal suction effects, were coupled with appropriate navigation, gauidance, and control laws in order to mimic the helmsman's behavior. The main conclusions from this study can be summarized as follows:

1. There exists a critical preview distance for straightline positional stability. For values less than the critical distance, the system is unstable.

2. Including canal effects, the critical preview distance may be an order of magnitude higher than in open water. If the same preview distance is to be used in both cases, the corresponding control law for canal maneuvering must be considerably more responsive than in open waters.

3. The critical preview distance is monotonically decreasing for increasing control law responsiveness. This means that in order to accomodate smaller values of the preview distance, more responsive control laws are required .

4. Physically realizable time lags do not seem to have a significant effect on the values of the preview distance.

5. As the preview distance becomes less than its critical value, one pair of

complex conjugate eigenvalues of the linearized system matrix crosses the imaginary axis. This corresponds to a bifurcation to periodic solutions (Hopf bifurcation) and the system exhibits oscillatory behavior, also known as limit cycles.

6. Higher order approximations in the equations of motion were utilized in order to assess the stability of the resulting limit cycles. It was found that in all cases, the limit cycles were unstable. This has the following implications:

   (a) It is possible for the system to lose its stability even *before* the critical preview distance is crossed. This means that all linearized stability analysis results should be viewed with extreme caution.

   (b) As the critical preview distance is crossed, it is expected that the system will develop limit cycles of large amplitude. This clearly presents a dangerous situation which should be avoided in practice, by appropriate changes in the design parameters.

Recommendations for further research include the following:

1. Simulation studies in order to verify the limit cycle behavior.

2. Study of different ship characteristics and canal geometry.

APPENDIX

The following is a list and description of the computer programs used in this thesis. The programs are written in FORTRAN. Complete printouts of the programs follow. Standard eigenvalue/eigenvector numerical analysis subroutines are required for all programs.

- **THESIS1.FOR**

  Calculation of critical x_d. First order approximation for time lag T_2.

- **THESIS2.FOR**

  Calculation of critical x_d. Second order approximation for time lag T_2.

- **THESIS3.FOR**

  Calculation of critical x_d. Third order approximation for time lag T_2.

- **THESIS4.FOR**

  Calculation of critical x_d. First order approximation for time lag T_1.

- **THESIS5.FOR**

  Calculation of critical x_d. First order approximation for time lags T_1 and T_2.

- **HOPF.FOR**

  Calculation of the nonlinear cubic stability coefficient $\mathcal{K}$. Requires the output of one of the previous programs as its input.

```fortran
C     PROGRAM THESIS1.FOR (Time Delay-1st Order Approx. T2)
C     BIFURCATION ANALYSIS
C
      PROGRAM THESIS1
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2,L,NR,NV,NDELTA,NPSI,NY,IZ,MASS,
     &               NRDOT,NVDOT
      DIMENSION A(4,4),FV1(4),IV1(4),ZZZ(4,4),WR(4),WI(4)
C
      OPEN (11,FILE='BIF1.RES')
      OPEN (12,FILE='BIF2.RES')
      OPEN (13,FILE='BIF3.RES')
      open (15,file='eig.res1')
C
C     Vehicle Parameters:
      IZ     =0.0
      L      =528
      RHO    =1.94
      XG     =0.0
      MASS   =0.0088
      U      =1.0
C
      YRDOT = 0.00000
      YVDOT =-0.00912
      YR     =+0.00456
      YV     =-0.01434
      YPSI  = 0.01400
      YY    = 0.02000
      YDELTA= 0.00278
      NRDOT =-0.00115
      NVDOT = 0.00000
      NR     =-0.00296
      NV     =-0.00460
      NPSI  = 0.01000
      NY     =-0.00250
      NDELTA=-0.00139
      WRITE (*,1001)
      READ  (*,*)    WNMIN,WNMAX,IWN
      WRITE (*,1002)
      READ  (*,*)    XDMIN,XDMAX,IXD
      WRITE (*,1003)
      READ  (*,*)    ZETA
      WRITE(*,1100)
      READ (*,*) TL
      TL=TL*U/L
C
      DVR =(IZ-NRDOT)*(MASS-YVDOT)-
     &     (MASS*XG-YRDOT)*(MASS*XG-NVDOT)
```

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```
      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
     &     (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
     &     (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR
      BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2 =((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR
C
      ANOM=AA23
      BNOM=BB2
      CNOM=AA21
      EPS =1.D-5
      ILMAX=1500
C
      DO 1 I=1,IWN
        WRITE (*,2001) I,IWN
        WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
        K1=-(WN**2.D0/BNOM)-(CNOM/BNOM)
        K2=-(ANOM+2.D0*ZETA*WN)/BNOM
        DO 2 J=1,IXD
          XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
        CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
     &  AA24,BB1,BB2,A,TL)
        CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
        CALL DSTABL(DEOS,WR,WI,FREQ)
        write(15,*)DEOS,XD,WN
C
        IF (J.GT.1) GO TO 10
        DEOSOO=DEOS
        XDOO  =XD
        LL=0
        GO TO 2
   10   DEOSNN=DEOS
        XDNN  =XD
        PR=DEOSNN*DEOSOO
        IF (PR.GT.0.D0) GO TO 3
        LL=LL+1
        IF (LL.GT.3) STOP 1000
        IL=0
        XDO=XDOO
        XDN=XDNN
        DEOSO=DEOSOO
```

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```fortran
      DEOSN=DEOSNN
6     XDL=XDO
      XDR=XDN
      DEOSL=DEOSO
      DEOSR=DEOSN
      XD=(XDL+XDR)/2.DO
C

      CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
     &  AA22,AA23,AA24,BB1,BB2,A,TL)
      CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
      CALL DSTABL(DEOS,WR,WI,FREQ)
      DEOSM=DEOS
      XDM=XD
      PRL=DEOSL*DEOSM
      PRR=DEOSR*DEOSM
      IF (PRL.GT.0.DO) GO TO 5
      XDO=XDL
      XDN=XDM
      DEOSO=DEOSL
      DEOSN=DEOSM
      IL=IL+1
      IF (IL.GT.ILMAX) STOP 3100
      DIF=DABS(XDL-XDM)
      IF (DIF.GT.EPS) GO TO 6
      XD=XDM
      GO TO 4
5     IF (PRR.GT.0.DO) STOP 3200
      XDO=XDM
      XDN=XDR
      DEOSO=DEOSM
      DEOSN=DEOSR
      IL=IL+1
      IF (IL.GT.ILMAX) STOP 3100
      DIF=DABS(XDM-XDR)
      IF (DIF.GT.EPS) GO TO 6
      XD=XDM
4     LLL=10+LL
      WRITE (LLL,*) XD,WN
3     XDOO=XDNN
      DEOSOO=DEOSNN
2     CONTINUE
1 CONTINUE
C
 1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
 1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
 1003 FORMAT (' ENTER DAMPING RATIO')
 1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
 2001 FORMAT (2I5)
```

```fortran
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
C
C     EVALUATES THE EIGENVALUE WITH THE MAXIMUM REAL PART
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
    1 CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
     & AA22,AA23,AA24,BB1,BB2,A,TL)
C
C     FORMS THE LINEARIZED MATRIX A (time delay 1st order approximation
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
      A(2,1)=AA11+BB1*K1-BB1*K1*TL/XD
      A(2,2)=AA12-BB1*K1*TL/XD
      A(2,3)=AA13+BB1*K2
      A(2,4)=AA14+BB1*K1/XD
      A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD
      A(3,2)=AA22-BB2*K1*TL/XD
      A(3,3)=AA23+BB2*K2
      A(3,4)=AA24+BB2*K1/XD
      A(4,1)=1.0D0
      A(4,2)=1.0D0
      A(4,3)=0.0D0
      A(4,4)=0.0D0
      RETURN
      END
```

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```
C       PROGRAM THESIS2.FOR (Time Delay-2nd Order Approx T2)
C       BIFURCATION ANALYSIS
C
        PROGRAM THESIS2
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION K1,K2,L,NR,NV,NDELTA,NPSI,NY,IZ,MASS,
     &                   NRDOT,NVDOT
        DIMENSION A(4,4),FV1(4),IV1(4),ZZZ(4,4),WR(4),WI(4)
C


        OPEN (11,FILE='BIF1.RES')
        OPEN (12,FILE='BIF2.RES')
        OPEN (13,FILE='BIF3.RES')

        open (15,file='eig.res1')
C
C     Vehicle Parameters:
        IZ    =0.0
        L     =528
        RHO   =1.94
        XG    =0.0
        MASS  =0.0088
        U     =1.0
C
        YRDOT = 0.00000
        YVDOT =-0.00912
        YR    =+0.00456
        YV    =-0.01434

        YPSI  = 0.01400
        YY    = 0.02000
        YDELTA= 0.00278

        NRDOT =-0.00115
        NVDOT = 0.00000
        NR    =-0.00296
        NV    =-0.00460

        NPSI  = 0.01000
        NY    =-0.00250

        NDELTA=-0.00139
        WRITE (*,1001)

        READ  (*,*)    WNMIN,WNMAX,IWN
        WRITE (*,1002)
        READ  (*,*)    XDMIN,XDMAX,IXD
        WRITE (*,1003)
```

```fortran
      READ  (*,*)     ZETA
      WRITE(*,1100)
      READ (*,*) TL
      TL=TL*U/L
C
      DVR =(IZ-NRDOT)*(MASS-YVDOT)-
     &     (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
     &     (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
     &     (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

      BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2 =((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR
C
      ANOM=AA23
      BNOM=BB2
      CNOM=AA21

      EPS  =1.D-5
      ILMAX=1500
C
      DO 1 I=1,IWN
        WRITE (*,2001) I,IWN
        WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
        K1=-(WN**2.D0/BNOM)-(CNOM/BNOM)

        K2=-(ANOM+2.D0*ZETA*WN)/BNOM

        DO 2 J=1,IXD
          XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
        CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
     &    AA24,BB1,BB2,A,TL)
        CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
        CALL DSTABL(DEOS,WR,WI,FREQ)
        write(15,*)DEOS,XD,WN
C
        IF (J.GT.1) GO TO 10
        DEOSOO=DEOS
        XDOO  =XD
```

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```
            LL=0
            GO TO 2
10          DEOSNN=DEOS
            XDNN =XD
            PR=DEOSNN*DEOSOO
            IF (PR.GT.0.D0) GO TO 3
            LL=LL+1
            IF (LL.GT.3) STOP 1000
            IL=0
            XDO=XDOO
            XDN=XDNN
            DEOSO=DEOSOO
            DEOSN=DEOSNN
6           XDL=XDO
            XDR=XDN
            DEOSL=DEOSO
            DEOSR=DEOSN
            XD=(XDL+XDR)/2.D0
C
            CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
     &      AA22,AA23,AA24,BB1,BB2,A,TL)
            CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
            CALL DSTABL(DEOS,WR,WI,FREQ)
            DEOSM=DEOS
            XDM=XD
            PRL=DEOSL*DEOSM
            PRR=DEOSR*DEOSM
            IF (PRL.GT.0.D0) GO TO 5
            XDO=XDL
            XDN=XDM
            DEOSO=DEOSL
            DEOSN=DEOSM
            IL=IL+1
            IF (IL.GT.ILMAX) STOP 3100
            DIF=DABS(XDL-XDM)
            IF (DIF.GT.EPS) GO TO 6
            XD=XDM
            GO TO 4
5           IF (PRR.GT.0.D0) STOP 3200
            XDO=XDM
            XDN=XDR
            DEOSO=DEOSM
            DEOSN=DEOSR
            IL=IL+1
            IF (IL.GT.ILMAX) STOP 3100
            DIF=DABS(XDM-XDR)
            IF (DIF.GT.EPS) GO TO 6
            XD=XDM
```

```fortran
    4     LLL=10+LL
          WRITE (LLL,*) XD,WN
    3     XDOO=XDNN
          DEOSOO=DEOSNN
    2   CONTINUE
    1 CONTINUE
C
 1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
 1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
 1003 FORMAT (' ENTER DAMPING RATIO')
 1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
 2001 FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
C
C     EVALUATES THE EIGENVALUE WITH THE MAXIMUM REAL PART
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
         IF (WR(I).LT.DEOS) GO TO 1
         DEOS=WR(I)
         IJ=I
    1 CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
     &        AA22,AA23,AA24,BB1,BB2,A,TL)
C
C     FORMS THE LINEARIZED MATRIX A (time delay 1st order approximation)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
      A(2,1)=(AA11*XD+BB1*K1*XD-BB1*K1*TL)/(XD-0.50D0*BB1*K1*TL*TL)
      A(2,2)=(AA12*XD-BB1*K1*TL)/(XD-0.50D0*BB1*K1*TL*TL)
      A(2,3)=(AA13*XD+BB1*K2*XD+0.50D0*BB1*K1*TL*TL)/
     &        (XD-0.50D0*BB1*K1*TL*TL)
      A(2,4)=(AA14*XD+BB1*K1)/(XD-0.50D0*BB1*K1*TL*TL)
```

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```fortran
      A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD+(BB2*K1*TL*TL/(2.0D0*XD))*A(2,1)
      A(3,2)=AA22-BB2*K1*TL/XD+(BB2*K1*TL*TL/(2.0D0*XD))*A(2,2)
      A(3,3)=AA23+BB2*K2+(BB2*K1*TL*TL/(2.0D0*XD))+
     &          (BB2*K1*TL*TL/(2.0*XD))* A(2,3)
      A(3,4)=AA24+BB2*K1/XD+(BB2*K1*TL*TL/(2.0D0*XD))*A(2,4)
      A(4,1)=1.0D0
      A(4,2)=1.0D0
      A(4,3)=0.0D0
      A(4,4)=0.0D0
      RETURN
      END
```

```
C       PROGRAM THESIS3.FOR (Time Delay-3rd Order Approx T2)
C       BIFURCATION ANALYSIS
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION K1,K2,L,NR,NV,IZ,MASS,NDELTA,NPSI,NY,
       &                 NRDOT,NVDOT
        DIMENSION A(5,5),B(5,5),FV1(5),IV1(5),ZZZ(5,5),ALFR(5),
       &                 ALFI(5),BETA(5),WR(5),WI(5)
C
        OPEN (11,FILE='BIF1.RES')
        OPEN (12,FILE='BIF2.RES')
        OPEN (13,FILE='BIF3.RES')
C
C    Vehicle Parameters:
        IZ      =0.0
        L       =528
        RHO     =1.94
        XG      =0.0
        MASS    =0.0088
        U       =1.0
C
        YRDOT = 0.00000
        YVDOT =-0.00912
        YR      =+0.00456
        YV      =-0.01434
        YPSI    = 0.01400
        YY      = 0.02000
        YDELTA= 0.00278
        NRDOT =-0.00115
        NVDOT = 0.00000
        NR      =-0.00296
        NV      =-0.00460
        NPSI    = 0.01000
        NY      =-0.00250
        NDELTA=-0.00139
        WRITE (*,1001)
        READ  (*,*)     WNMIN,WNMAX,IWN
        WRITE (*,1002)
        READ  (*,*)     XDMIN,XDMAX,IXD
        WRITE (*,1003)
        READ  (*,*)     ZETA
        WRITE(*,1100)
        READ (*,*) TL
        TL=TL*U/L
C
        DVR =(IZ-NRDOT)*(MASS-YVDOT)-
       &     (MASS*XG-YRDOT)*(MASS*XG-NVDOT)
```

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```fortran
      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
     &      (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
     &      (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

      BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2 =((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

      ANOM=AA23
      BNOM=BB2
      CNOM=AA21

      EPS =1.D-5
      ILMAX=1500
C
      DO 1 I=1,IWN
        WRITE (*,2001) I,IWN
        WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
        K1=-(WN**2.D0/BNOM)-(CNOM/BNOM)
        K2=-(ANOM+2.D0*ZETA*WN)/BNOM
        DO 2 J=1,IXD
          XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
          CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,
     &                AA21,AA22,AA23,AA24,BB1,BB2,A,B,TL)
          CALL RGG(5,5,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
          DO 11 IJE=1,5
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
   11     CONTINUE
          CALL DSTABL(DEOS,WR,WI,FREQ)
C
          IF (J.GT.1) GO TO 10
          DEOSOO=DEOS
          XDOO  =XD
          LL=0
          GO TO 2
   10     DEOSNN=DEOS
          XDNN  =XD
          PR=DEOSNN*DEOSOO
          IF (PR.GT.0.D0) GO TO 3
          LL=LL+1
```

```
              IF (LL.GT.3) STOP 1000
              IL=0
              XDO=XDOO
              XDN=XDNN
              DEOSO=DEOSOO
              DEOSN=DEOSNN
    6         XDL=XDO
              XDR=XDN
              DEOSL=DEOSO
              DEOSR=DEOSN
              XD=(XDL+XDR)/2.DO
C
              CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,
    &             AA23,AA24,BB1,BB2,A,B,TL)
              CALL RGG(5,5,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
              DO 12 IJE=1,5
                WR(IJE)=ALFR(IJE)/BETA(IJE)
                WI(IJE)=ALFI(IJE)/BETA(IJE)
    12        CONTINUE
              CALL DSTABL(DEOS,WR,WI,FREQ)
C
              DEOSM=DEOS
              XDM=XD
              PRL=DEOSL*DEOSM
              PRR=DEOSR*DEOSM
              IF (PRL.GT.0.DO) GO TO 5
              XDO=XDL
              XDN=XDM
              DEOSO=DEOSL
              DEOSN=DEOSM
              IL=IL+1
              IF (IL.GT.ILMAX) STOP 3100
              DIF=DABS(XDL-XDM)
              IF (DIF.GT.EPS) GO TO 6
              XD=XDM
              GO TO 4
    5         IF (PRR.GT.0.DO) STOP 3200
              XDO=XDM
              XDN=XDR
              DEOSO=DEOSM
              DEOSN=DEOSR
              IL=IL+1
              IF (IL.GT.ILMAX) STOP 3100
              DIF=DABS(XDM-XDR)
              IF (DIF.GT.EPS) GO TO 6
              XD=XDM
    4         LLL=10+LL
              WRITE (LLL,*) XD,WN
```

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```
      3     XDOO=XDNN
            DEOSOO=DEOSNN
      2   CONTINUE
      1 CONTINUE
C
 1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
 1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
 1003 FORMAT (' ENTER DAMPING RATIO')
 1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
 2001 FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(5),WI(5)
      DEOS=-1.0D+20
      DO 1 I=1,5
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
      1 CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,AA24
     &                 ,BB1,BB2,A,B,TL)

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(5,5),B(5,5)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
      A(1,5)=0.0D0
      A(2,1)=0.0D0
      A(2,2)=0.0D0
      A(2,3)=0.0D0
      A(2,4)=0.0D0
      A(2,5)=1.0D0
      A(3,1)=AA11-(BB1*K1*TL/XD)+BB1*K1
      A(3,2)=AA12-BB1*K1*TL/XD
      A(3,3)=AA13+BB1*K2+(BB1*K1*TL*TL/(2.D0*XD))
      A(3,4)=AA14+BB1*K1/XD
      A(3,5)=-1.D0+(BB1*K1*TL*TL/(2.D0*XD))
      A(4,1)=1.0D0
```

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```fortran
      A(4,2)=1.0D0
      A(4,3)=0.0D0
      A(4,4)=0.0D0
      A(4,5)=0.0D0
      A(5,1)=AA21+BB2*K1-BB2*K1*TL/XD
      A(5,2)=AA22-BB2*K1*TL/XD
      A(5,3)=AA23+BB2*K2+(BB2*K1*TL*TL/(2.D0*XD))
      A(5,4)=AA24+BB2*K1/XD
      A(5,5)=(BB2*K1*TL*TL/(2.D0*XD))
C
      B(1,1)=1.0D0
      B(1,2)=0.0D0
      B(1,3)=0.0D0
      B(1,4)=0.0D0
      B(1,5)=0.0D0
      B(2,1)=0.0D0
      B(2,2)=1.0D0
      B(2,3)=0.0D0
      B(2,4)=0.0D0
      B(2,5)=0.0D0
      B(3,1)=0.0D0
      B(3,2)=0.0D0
      B(3,3)=(BB1*K1*TL*TL*TL/(6.D0*XD))
      B(3,4)=0.0D0
      B(3,5)=(BB1*K1*TL*TL*TL/(6.D0*XD))
      B(4,1)=0.0D0
      B(4,2)=0.0D0
      B(4,3)=0.0D0
      B(4,4)=1.0D0
      B(4,5)=0.0D0
      B(5,1)=0.0D0
      B(5,2)=0.0D0
      B(5,3)=1.0D0+(BB2*K1*TL*TL*TL/(6.D0*XD))
      B(5,4)=0.0D0
      B(5,5)=(BB2*K1*TL*TL*TL/(6.D0*XD))
      RETURN
      END
```

```
C       PROGRAM THESIS4.FOR (Time Delay-1st Order Approx in delta ,T1)
C       BIFURCATION ANALYSIS
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION K1,K2,L,NR,NV,IZ,MASS,NDELTA,NPSI,NY,
       &                 NRDOT,NVDOT
        DIMENSION A(4,4),B(4,4),FV1(4),IV1(4),ZZZ(4,4),ALFR(4),
       &                 ALFI(4),BETA(4),WR(4),WI(4)
C
        OPEN (11,FILE='BIF1.RES')
        OPEN (12,FILE='BIF2.RES')
        OPEN (13,FILE='BIF3.RES')
C
C    Vehicle Parameters:
        IZ    =0.0
        L     =528
        RHO   =1.94
C       G     =32.2
        XG    =0.0
        MASS  =0.0088
        U     =3.0


C

        YRDOT = 0.00000
        YVDOT =-0.00912
        YR    =+0.00456
        YV    =-0.01434

        YPSI  = 0.01400
        YY    = 0.02000

        YDELTA= 0.00278
        NRDOT =-0.00115
        NVDOT = 0.00000
        NR    =-0.00296
        NV    =-0.00460

        NPSI  =  0.01000
        NY    = -0.00250

        NDELTA=-0.00139

        WRITE (*,1001)
        READ  (*,*)    WNMIN,WNMAX,IWN
        WRITE (*,1002)
        READ  (*,*)    XDMIN,XDMAX,IXD
        WRITE (*,1003)
```

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```
      READ  (*,*)    ZETA
      WRITE(*,1100)
      READ  (*,*) TL
      TL=TL*U/L
C

      DVR =(IZ-NRDOT)*(MASS-YVDOT)-
     &    (MASS*XG-YRDOT)*(MASS*XG-NVDOT)


      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
     &    (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
     &    (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR


      BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2 =((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

      ANOM=AA23
      BNOM=BB2


      EPS  =1.D-5
      ILMAX=1500
C
      DO 1 I=1,IWN
        WRITE (*,2001) I,IWN
        WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
        K1=-WN**2.D0/BNOM
        K2=-(ANOM+2.D0*ZETA*WN)/BNOM
        DO 2 J=1,IXD
          XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
          CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,
     &               AA21,AA22,AA23,AA24,BB1,BB2,A,B,TL)
          CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
          DO 11 IJE=1,4
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
  11      CONTINUE
          CALL DSTABL(DEOS,WR,WI,FREQ)
C
          IF (J.GT.1) GO TO 10
          DEOSOO=DEOS
          XDOO  =XD
```

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```fortran
         LL=0
         GO TO 2
10       DEOSNN=DEOS
         XDNN =XD
         PR=DEOSNN*DEOSOO
         IF (PR.GT.0.D0) GO TO 3
         LL=LL+1
         IF (LL.GT.3) STOP 1000
         IL=0
         XDO=XDOO
         XDN=XDNN
         DEOSO=DEOSOO
         DEOSN=DEOSNN
6        XDL=XDO
         XDR=XDN
         DEOSL=DEOSO
         DEOSR=DEOSN
         XD=(XDL+XDR)/2.D0
C
         CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,
     &       AA23,AA24,BB1,BB2,A,B,TL)
         CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
         DO 12 IJE=1,4
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
12       CONTINUE
         CALL DSTABL(DEOS,WR,WI,FREQ)
C
         DEOSM=DEOS
         XDM=XD
         PRL=DEOSL*DEOSM
         PRR=DEOSR*DEOSM
         IF (PRL.GT.0.D0) GO TO 5
         XDO=XDL
         XDN=XDM
         DEOSO=DEOSL
         DEOSN=DEOSM
         IL=IL+1
         IF (IL.GT.ILMAX) STOP 3100
         DIF=DABS(XDL-XDM)
         IF (DIF.GT.EPS) GO TO 6
         XD=XDM
         GO TO 4
5        IF (PRR.GT.0.D0) STOP 3200
         XDO=XDM
         XDN=XDR
         DEOSO=DEOSM
         DEOSN=DEOSR
```

```
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDM-XDR)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
   4    LLL=10+LL
        WRITE (LLL,*) XD,WN
   3    XDOO=XDNN
        DEOSOO=DEOSNN
   2  CONTINUE
   1 CONTINUE
C
 1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
 1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
 1003 FORMAT (' ENTER DAMPING RATIO')
 1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
 2001 FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
   1  CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,AA24
     &                  ,BB1,BB2,A,B,TL)

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4),B(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0

      A(2,1)=AA11+BB1*K1-BB1*K1*TL/XD
      A(2,2)=AA12-BB1*K1*TL/XD
      A(2,3)=AA13+BB1*K2-BB1*K1*TL
      A(2,4)=AA14+BB1*K1/XD
```

```fortran
      A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD
      A(3,2)=AA22-BB2*K1*TL/XD
      A(3,3)=AA23+BB2*K2-BB2*K1*TL
      A(3,4)=AA24+BB2*K1/XD

      A(4,1)=1.0D0
      A(4,2)=1.0D0
      A(4,3)=0.0D0
      A(4,4)=0.0D0
C
      B(1,1)=1.0D0
      B(1,2)=0.0D0
      B(1,3)=0.0D0
      B(1,4)=0.0D0

      B(2,1)=0.0D0
      B(2,2)=1.0D0
      B(2,3)=BB1*K2*TL
      B(2,4)=0.0D0

      B(3,1)=0.0D0
      B(3,2)=0.0D0
      B(3,3)=1.0D0+(BB2*K2*TL)
      B(3,4)=0.0D0

      B(4,1)=0.0D0
      B(4,2)=0.0D0
      B(4,3)=0.0D0
      B(4,4)=1.0D0

      RETURN
      END
```

```
C      PROGRAM THESIS5.FOR (Time Delay-1st Order Approx in delta & y ie. in T1 & T2))
C      BIFURCATION ANALYSIS
C
       IMPLICIT DOUBLE PRECISION (A-H,O-Z)
       DOUBLE PRECISION K1,K2,L,NR,NV,IZ,MASS,NDELTA,NPSI,NY,
      &                 NRDOT,NVDOT
       DIMENSION A(4,4),B(4,4),FV1(4),IV1(4),ZZZ(4,4),ALFR(4),
      &                 ALFI(4),BETA(4),WR(4),WI(4)
C
       OPEN (11,FILE='BIF1.RES')
       OPEN (12,FILE='BIF2.RES')
       OPEN (13,FILE='BIF3.RES')
C
C      Vehicle Parameters:
       IZ    =0.0
       L     =528
       RHO   =1.94
C      G     =32.2
       XG    =0.0
       MASS  =0.0088
       U     =3.0


C

       YRDOT = 0.00000
       YVDOT =-0.00912
       YR    =+0.00456
       YV    =-0.01434

       YPSI  = 0.01400
       YY    = 0.02000

       YDELTA= 0.00278
       NRDOT =-0.00115
       NVDOT = 0.00000
       NR    =-0.00296
       NV    =-0.00460

       NPSI  =  0.01000
       NY    = -0.00250
       NDELTA= -0.00139

       WRITE (*,1001)
       READ  (*,*)     WNMIN,WNMAX,IWN
       WRITE (*,1002)
       READ  (*,*)     XDMIN,XDMAX,IXD
       WRITE (*,1003)
       READ  (*,*)     ZETA
```

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```
        WRITE(*,1100)
        READ (*,*)      TL1
        TL1=TL1*U/L
        WRITE(*,1101)
        READ (*,*)      TL2
        TL2=TL2*U/L
C
        DVR =(IZ-NRDOT)*(MASS-YVDOT)-
     &      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

        AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
        AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
        AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
        AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
        AA13=((IZ-NRDOT)*(YR-MASS)+
     &      (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
        AA23=((NVDOT-MASS*XG)*(YR-MASS)+
     &      (MASS-YVDOT)*(NR-MASS*XG))/DVR
        AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
        AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

        BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
        BB2 =((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

        ANOM=AA23
        BNOM=BB2
        CNOM=AA21

        EPS  =1.D-5
        ILMAX=1500
C
        DO 1 I=1,IWN
          WRITE (*,2001) I,IWN
          WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
          K1=-(WN**2.D0/BNOM)-(CNOM/BNOM)
          K2=-(ANOM+2.D0*ZETA*WN)/BNOM
          DO 2 J=1,IXD
            XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
          CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,
     &                AA21,AA22,AA23,AA24,BB1,BB2,A,B,TL1,TL2)
          CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
          DO 11 IJE=1,4
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
   11     CONTINUE
          CALL DSTABL(DEOS,WR,WI,FREQ)
C
```

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```fortran
            IF (J.GT.1) GO TO 10
            DEOSOO=DEOS
            XDOO  =XD
            LL=0
            GO TO 2
 10         DEOSNN=DEOS
            XDNN  =XD
            PR=DEOSNN*DEOSOO
            IF (PR.GT.0.D0) GO TO 3
            LL=LL+1
            IF (LL.GT.3) STOP 1000
            IL=0
            XDO=XDOO
            XDN=XDNN
            DEOSO=DEOSOO
            DEOSN=DEOSNN
 6          XDL=XDO
            XDR=XDN
            DEOSL=DEOSO
            DEOSR=DEOSN
            XD=(XDL+XDR)/2.D0
C
            CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,
     &          AA23,AA24,BB1,BB2,A,B,TL1,TL2)
            CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
            DO 12 IJE=1,4
              WR(IJE)=ALFR(IJE)/BETA(IJE)
              WI(IJE)=ALFI(IJE)/BETA(IJE)
 12         CONTINUE
            CALL DSTABL(DEOS,WR,WI,FREQ)
C
            DEOSM=DEOS
            XDM=XD
            PRL=DEOSL*DEOSM
            PRR=DEOSR*DEOSM
            IF (PRL.GT.0.D0) GO TO 5
            XDO=XDL
            XDN=XDM
            DEOSO=DEOSL
            DEOSN=DEOSM
            IL=IL+1
            IF (IL.GT.ILMAX) STOP 3100
            DIF=DABS(XDL-XDM)
            IF (DIF.GT.EPS) GO TO 6
            XD=XDM
            GO TO 4
 5          IF (PRR.GT.0.D0) STOP 3200
            XDO=XDM
```

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```fortran
            XDN=XDR
            DEOSO=DEOSM
            DEOSN=DEOSR
            IL=IL+1
            IF (IL.GT.ILMAX) STOP 3100
            DIF=DABS(XDM-XDR)
            IF (DIF.GT.EPS) GO TO 6
            XD=XDM
    4       LLL=10+LL
            WRITE (LLL,*) XD,WN
    3       XDOO=XDNN
            DEOSOO=DEOSNN
    2    CONTINUE
    1 CONTINUE
C
 1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
 1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
 1003 FORMAT (' ENTER DAMPING RATIO')
 1100 FORMAT (' ENTER TIME LAG TL1 (dimensional)')
 1101 FORMAT (' ENTER TIME LAG TL2 (dimensional)')
 2001 FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
    1 CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,AA24
     &                 ,BB1,BB2,A,B,TL1,TL2)

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4),B(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
```

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```fortran
      A(2,1)=AA11+BB1*K1-BB1*K1*TL2/XD-BB1*K1*TL1/XD
      A(2,2)=AA12-BB1*K1*TL2/XD-BB1*K1*TL1/XD
      A(2,3)=AA13+BB1*K2-BB1*K1*TL1+BB1*K1*TL1*TL2/XD
      A(2,4)=AA14+BB1*K1/XD

      A(3,1)=AA21+BB2*K1-BB2*K1*TL2/XD-BB2*K1*TL1/XD
      A(3,2)=AA22-BB2*K1*TL2/XD-BB2*K1*TL1/XD
      A(3,3)=AA23+BB2*K2-BB2*K1*TL1+BB2*K1*TL1*TL2/XD
      A(3,4)=AA24+BB2*K1/XD

      A(4,1)=1.0D0
      A(4,2)=1.0D0
      A(4,3)=0.0D0
      A(4,4)=0.0D0
C
      B(1,1)=1.0D0
      B(1,2)=0.0D0
      B(1,3)=0.0D0
      B(1,4)=0.0D0

      B(2,1)=0.0D0
      B(2,2)=1.0D0-(BB1*K1*TL1*TL2/XD)
      B(2,3)=BB1*K2*TL1
      B(2,4)=0.0D0

      B(3,1)=0.0D0
      B(3,2)=-BB2*K1*TL1*TL2/XD
      B(3,3)=1.0D0+(BB2*K2*TL1)
      B(3,4)=0.0D0

      B(4,1)=0.0D0
      B(4,2)=0.0D0
      B(4,3)=0.0D0
      B(4,4)=1.0D0

      RETURN
      END
```

```
C       PROGRAM HOPF.FOR
C
C       Hopf Bifurcation Analysis
C
C       Third Order Expansions:  First Order Approximation
C
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION K1,K2,K3,L,NR,NV,NDELTA,NPSI,NY,IZ,MASS,
       &                 NRDOT,NVDOT,K1P,K2P,NVVV,NVRR,NRVV,NYYY,NPPP
        DOUBLE PRECISION M11,M12,M13,M14,M21,M22,M23,M24,
       1                 M31,M32,M33,M34,M41,M42,M43,M44,
       2                 N11,N12,N13,N14,N21,N22,N23,N24,
       3                 N31,N32,N33,N34,N41,N42,N43,N44,
       4                 L21,L22,L23,L24,L31,L32,L33,L34,
       5                 L41,L42,L43,L44
C
        DIMENSION A(4,4),T(4,4),TINV(4,4),FV1(4),IV1(4),ZZZ(4,4)
        DIMENSION WR(4),WI(4),TSAVE(4,4),TLUD(4,4),IVLUD(4),SVLUD(4)
        DIMENSION ASAVE(4,4)


        OPEN (11,FILE='BIF1.RES')
        OPEN (15,FILE='HOPF.RES')
C
C       Vehicle Parameters:

        IZ    =0.0
        L     =528
        RHO   =1.94
        G     =32.2
        XG    =0.0
        MASS  =0.0088
        U     =1.0
C
        YRDOT = 0.00000
        YVDOT =-0.00912
        YR    =+0.00456
        YV    =-0.01434
        YVVV  =-0.15391
        YVRR  =-0.05476
        YRVV  = 0.04608
        YYYY  = 0.46800
        YPPP  = 0.00000
        YPSI  = 0.01400
        YY    = 0.02000
        YDELTA=+0.00278
```

```fortran
      NRDOT =-0.00115
      NVDOT = 0.00000
      NR    =-0.00296
      NV    =-0.00460
      NVVV  =-0.00336
      NVRR  = 0.00759
      NRVV  =-0.05160
      NYYY  = 0.00000
      NPPP  = 0.00000
      NPSI  = 0.01000
      NY    =-0.00250
      NDELTA=-0.00139


      WRITE (*,1003)
      READ  (*,*)    ZETA
      WRITE(*,1100)
      READ  (*,*)    TL
      TL=TL*U/L
      WRITE (*,1006)
      READ  (*,*)    DO

C

      DVR =(IZ-NRDOT)*(MASS-YVDOT)-
     &     (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
     &     (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
     &     (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

      BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2 =((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR
C
      ANOM=AA23
      BNOM=BB2
      CNOM=AA21

      EPS  =1.D-5
      ILMAX=1500
C
      IWN=1000
```

```fortran
      DO 1 II=1,IWN
        WRITE (*,2001) II
        READ(11,*) XD,WN
        K1=-(WN**2.D0/BNOM)-(CNOM/BNOM)
        K2=-(ANOM+2.D0*ZETA*WN)/BNOM
        K3=K2

C       Start Hopf Bifurcation Analysis
C
C       Evaluate Nonlinear Rudder Expansion Coefficients
C
        K2=0.0D0
        K1P=K1-K1*TL*U/XD
        K2P=K2-K1*TL/XD
C
        DPPV=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1P*K2P
     &       + 0.5D0*K1*TL/XD + 3.D0*K1*(TL*U)**3/(3.D0*XD**3)
        DPVV=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K2P*K2P
     &       + 3.D0*U*TL*TL*TL*K1/(3.D0*XD**3)
        DPPR=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1P*K3
        DPRR=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K3*K3
        DPPY=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1P*K1/XD
     &       - 3.D0*TL*TL*U*U*K1/(3.D0*XD**3)
        DPYY=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1*K1/(XD**2)
     &       + 3.D0*TL*U*K1/(3.D0*XD**3)
        DVVR=- (1.D0/(3.D0*D0**2))*3.D0*K2P*K2P*K3
        DVRR=- (1.D0/(3.D0*D0**2))*3.D0*K2P*K3*K3
        DVVY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K2P*K2P/XD
     &       - 3.D0*K1*TL*TL/(3.D0*XD**3)
        DVYY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K1*K2P/(XD**2)
     &       + 3.D0*TL*K1/(3.D0*XD**3)
        DRRY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K3*K3/XD
        DRYY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K1*K3/(XD**2)
        DPVR=- (1.D0/(3.D0*D0**2))*6.D0*K1P*K2P*K3
        DPVY=- (1.D0/(3.D0*D0**2))*6.D0*K1P*K1*K2P/XD
     &       - 6.D0*TL*TL*U*K1/(3.D0*XD**3)
        DPRY=- (1.D0/(3.D0*D0**2))*6.D0*K1P*K1*K3/XD
        DVRY=- (1.D0/(3.D0*D0**2))*6.D0*K1*K2P*K3/XD
        DPPP=- (1.D0/(3.D0*D0**2))*1.D0*K1P*K1P*K1P
     &       + K1*TL*U/(6.D0*XD) + (K1*(TL*U)**3)/(3.D0*XD**3)
        DVVV=- (1.D0/(3.D0*D0**2))*1.D0*K2P*K2P*K2P
     &       + K1*(TL**3)/(3.D0*XD**3)
        DRRR=- (1.D0/(3.D0*D0**2))*1.D0*K3*K3*K3
        DYYY=- (1.D0/(3.D0*D0**2))*(K1/XD)**3-K1/(3.D0*XD**3)
     &       - K1/(3.D0*XD**3)
C
C       Evaluate Transformation Matrix of Eigenvectors
C
```

```fortran
      CALL LINEAR(K1,K3,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
     &    AA24,BB1,BB2,A,TL)

      DO 11 I=1,4
        DO 12 J=1,4
          ASAVE(I,J)=A(I,J)
12      CONTINUE
11    CONTINUE
      CALL RG(4,4,A,WR,WI,1,ZZZ,IV1,FV1,IERR)
      CALL DSOMEG(IEV,WR,WI,OMEGA,CHECK)
      OMEGA0=OMEGA
      DO 50 I=1,4
        T(I,1)= ZZZ(I,IEV)
        T(I,2)=-ZZZ(I,IEV+1)
50    CONTINUE
      IF (IEV.EQ.1) GO TO 13
      IF (IEV.EQ.2) GO TO 14
      IF (IEV.EQ.3) GO TO 15
      STOP 3004
14    DO 60 I=1,4
        T(I,3)=ZZZ(I,1)
        T(I,4)=ZZZ(I,4)
60    CONTINUE
      GO TO 17
15    DO 70 I=1,4
        T(I,3)=ZZZ(I,1)
        T(I,4)=ZZZ(I,2)
70    CONTINUE
      GO TO 17
13    DO 16 I=1,4
        T(I,3)=ZZZ(I,3)
        T(I,4)=ZZZ(I,4)
16    CONTINUE
17    CONTINUE
C
C     Normalization of the Critical Eigenvector
C
      CALL NORMAL(T)
C
C     Definition of Mij
C
      M11=T(1,1)
      M21=T(2,1)
      M31=T(3,1)
      M41=T(4,1)
      M12=T(1,2)
      M22=T(2,2)
```

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```
                M32=T(3,2)
                M42=T(4,2)
                M13=T(1,3)
                M23=T(2,3)
                M33=T(3,3)
                M43=T(4,3)
                M14=T(1,4)
                M24=T(2,4)
                M34=T(3,4)
                M44=T(4,4)
C
C       Definition of Lij
C
        L21= DPPV*M11*M11*M21 + DPVV*M11*M21*M21 + DPPR*M11*M11*M31
     &      + DPRR*M11*M31*M31 + DPPY*M11*M11*M41 + DPYY*M11*M41*M41
     &      + DVVR*M21*M21*M31 + DVRR*M21*M31*M31 + DVVY*M21*M21*M41
     &      + DVYY*M21*M41*M41 + DRRY*M31*M31*M41 + DRYY*M31*M41*M41
     &      + DPVR*M11*M21*M31 + DPVY*M11*M21*M41 + DPRY*M11*M41*M31
     &      + DVRY*M21*M41*M31 + DPPP*M11*M11*M11 + DVVV*M21*M21*M21
     &      + DRRR*M31*M31*M31 + DYYY*M41*M41*M41
C
        PPV = M11*M11*M22 + 2.0*M11*M12*M21
        PVV = M12*M21*M21 + 2.0*M11*M21*M22
        PPR = M11*M11*M32 + 2.0*M11*M12*M31
        PRR = M12*M31*M31 + 2.0*M11*M31*M32
        PPY = M11*M11*M42 + 2.0*M11*M12*M41
        PYY = M41*M41*M12 + 2.0*M11*M41*M42
        VVR = M21*M21*M32 + 2.0*M31*M21*M22
        VRR = M22*M31*M31 + 2.0*M31*M32*M21
        VVY = M21*M21*M42 + 2.0*M41*M21*M22
        VYY = M22*M41*M41 + 2.0*M41*M42*M21
        RRY = M31*M31*M42 + 2.0*M41*M31*M32
        RYY = M32*M41*M41 + 2.0*M41*M42*M31
        PVR = M11*M21*M32+M31*(M11*M22+M12*M21)
        PVY = M11*M21*M42+M41*(M11*M22+M12*M21)
        PRY = M11*M41*M32+M31*(M11*M42+M12*M41)
        VRY = M21*M41*M32+M31*(M21*M42+M22*M41)
        PPP = 3.0*M11*M11*M12
        VVV = 3.0*M21*M21*M22
        RRR = 3.0*M31*M31*M32
        YYY = 3.0*M41*M41*M42
C
        L22=DPPV*PPV+DPVV*PVV+DPPR*PPR+DPRR*PRR+DPPY*PPY+DPYY*PYY
     &      +DVVR*VVR+DVRR*VRR+DVVY*VVY+DVYY*VYY+DRRY*RRY+DRYY*RYY
     &      +DPVR*PVR+DPVY*PVY+DPRY*PRY+DVRY*VRY+DPPP*PPP+DVVV*VVV
     &      +DRRR*RRR+DYYY*YYY
C
        PPV = M12*M12*M21 + 2.0*M11*M12*M22
```

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```
              PVV = M11*M22*M22 + 2.0*M12*M21*M22
              PPR = M12*M12*M31 + 2.0*M11*M12*M32
              PRR = M11*M32*M32 + 2.0*M12*M31*M32
              PPY = M12*M12*M41 + 2.0*M11*M12*M42
              PYY = M11*M42*M42 + 2.0*M12*M41*M42
              VVR = M22*M22*M31 + 2.0*M21*M22*M32
              VRR = M21*M32*M32 + 2.0*M22*M31*M32
              VVY = M22*M22*M41 + 2.0*M21*M22*M42
              VYY = M21*M42*M42 + 2.0*M22*M41*M42
              RRY = M32*M32*M41 + 2.0*M31*M32*M42
              RYY = M31*M42*M42 + 2.0*M32*M41*M42
              PVR = M12*M22*M31 + M32*(M11*M22+M12*M21)
              PVY = M12*M22*M41 + M42*(M11*M22+M12*M21)
              PRY = M12*M42*M31 + M32*(M11*M42+M12*M41)
              VRY = M22*M42*M31 + M32*(M21*M42+M22*M41)
              PPP = 3.0*M11*M12*M12
              VVV = 3.0*M21*M22*M22
              RRR = 3.0*M31*M32*M32
              YYY = 3.0*M41*M42*M42
C
              L23=DPPV*PPV+DPVV*PVV+DPPR*PPR+DPRR*PRR+DPPY*PPY+DPYY*PYY
     &          +DVVR*VVR+DVRR*VRR+DVVY*VVY+DVYY*VYY+DRRY*RRY+DRYY*RYY
     &          +DPVR*PVR+DPVY*PVY+DPRY*PRY+DVRY*VRY+DPPP*PPP+DVVV*VVV
     &          +DRRR*RRR+DYYY*YYY
C
              L24= DPPV*M12*M12*M22 + DPVV*M12*M22*M22 + DPPR*M12*M12*M32
     &          + DPRR*M12*M32*M32 + DPPY*M12*M12*M42 + DPYY*M12*M42*M42
     &          + DVVR*M22*M22*M32 + DVRR*M22*M32*M32 + DVVY*M22*M22*M42
     &          + DVYY*M22*M42*M42 + DRRY*M32*M32*M42 + DRYY*M32*M42*M42
     &          + DPVR*M12*M22*M32 + DPVY*M12*M22*M42 + DPRY*M12*M42*M32
     &          + DVRY*M22*M42*M32 + DPPP*M12*M12*M12 + DVVV*M22*M22*M22
     &          + DRRR*M32*M32*M32 + DYYY*M42*M42*M42
C
              L31=L21
              L32=L22
              L33=L23
              L34=L24
C
              L21=L21*BB1*U*U
              L22=L22*BB1*U*U
              L23=L23*BB1*U*U
              L24=L24*BB1*U*U
              L31=L31*BB2*U*U
              L32=L32*BB2*U*U
              L33=L33*BB2*U*U
              L34=L34*BB2*U*U
              L41=-0.5*M11*M11*(M21+U*M11/3.0)
              L42=-M11*(M12*M21+0.5*M11*M22+0.5*U*M12*M11)
```

```fortran
      L43=-M12*(M11*M22+0.5*M12*M21+0.5*U*M11*M12)
      L44=-0.5*M12*M12*(M22+U*M12/3.0)
C

      C11=(1/6.0)*YVVV*(M21*M21*M21)+0.5*YVRR*(M31*M31*M21)+
     & 0.5*YRVV*M21*M21*M31+(1/6.0)*YPPP*M11*M11*M11+(1/6.0)*YYYY*M41*M41*M41

      C12=(1/6.0)*YVVV*(3*M21*M21*M22)+0.5*YVRR*(M31*M31*
     & M22+2*M31*M32*M21)+0.5*YRVV*(M21*M21*M32+2*M31*M21*M22)+(1/6.0)*YPPP*3*M11*M11*M12
     & +(1/6.0)*YYYY*3*M41*M41*M42


      C13=(1/6.0)*YVVV*(3*M21*M22*M22)+0.5*YVRR*(M21*M32*
     & M32+2*M31*M32*M22)+0.5*YRVV*(M22*M22*M31+2*M32*M21*M22)+(1/6.0)*YPPP*3*M11*M12*M12
     & +(1/6.0)*YYYY*3*M41*M42*M42

      C14=(1/6.0)*YVVV*(M22*M22*M22)+0.5*YVRR*(M32*M32*M22)+
     & 0.5*YRVV*M22*M22*M32+(1/6.0)*YPPP*M12*M12*M12+(1/6.0)*YYYY*M42*M42*M42

      C21=(1/6.0)*NVVV*(M21*M21*M21)+0.5*(NVRR*M31*M31*M21)+
     & 0.5*NRVV*M21*M21*M31+(1/6.0)*NPPP*M11*M11*M11+(1/6.0)*NYYY*M41*M41*M41

      C22=(1/6.0)*NVVV*(3*M21*M21*M22)+0.5*NVRR*(M31*M31*
     & M22+2*M31*M32*M21)+0.5*NRVV*(M21*M21*M32+2*M31*M21*M22)+(1/6.0)*NPPP*3*M11*M11*M12
     & +(1/6.0)*NYYY*3*M41*M41*M42

      C23=(1/6.0)*NVVV*(3*M21*M22*M22)+0.5*NVRR*(M21*M32*
     & M32+2*M31*M32*M22)+0.5*NRVV*(M22*M22*M31+2*M32*M21*M22)+(1/6.0)*NPPP*3*M11*M12*M12
     & +(1/6.0)*NYYY*3*M41*M42*M42

      C24=(1/6.0)*NVVV*(M22*M22*M22)+0.5*NVRR*M32*M32*M22+
     & 0.5*NRVV*M22*M22*M32+(1/6.0)*NPPP*M12*M12*M12+(1/6.0)*NYYY*M42*M42*M42



      D11=(C11*(IZ-NRDOT)+C21*(YRDOT-M*XG))/DVR
      D12=(C12*(IZ-NRDOT)+C22*(YRDOT-M*XG))/DVR
      D13=(C13*(IZ-NRDOT)+C23*(YRDOT-M*XG))/DVR
      D14=(C14*(IZ-NRDOT)+C24*(YRDOT-M*XG))/DVR

      D21=(C11*(NVDOT-M*XG)+C21*(M-YVDOT))/DVR
      D22=(C12*(NVDOT-M*XG)+C22*(M-YVDOT))/DVR
      D23=(C13*(NVDOT-M*XG)+C23*(M-YVDOT))/DVR
      D24=(C14*(NVDOT-M*XG)+C24*(M-YVDOT))/DVR
C
C     Invert Transformation Matrix
C
      DO 20 I=1,4
        DO 30 J=1,4
```

```fortran
             TINV(I,J)=0.0
             TSAVE(I,J)=T(I,J)
   30      CONTINUE
   20    CONTINUE
         CALL DLUD(4,4,TSAVE,4,TLUD,IVLUD)
         DO 40 I=1,4
           IF (IVLUD(I).EQ.0) STOP 3003
   40    CONTINUE
         CALL DILU(4,4,TLUD,IVLUD,SVLUD)
         DO 80 I=1,4
           DO 90 J=1,4
             TINV(I,J)=TLUD(I,J)
   90      CONTINUE
   80    CONTINUE
C
C        Check Inv(T)*A*T
C
         IMULT=2
         IF (IMULT.EQ.1) CALL MULT(TINV,ASAVE,T)
         IF (IMULT.EQ.0) STOP
C
C        Definition of Nij
C
         N11=TINV(1,1)
         N21=TINV(2,1)
         N31=TINV(3,1)
         N41=TINV(4,1)
         N12=TINV(1,2)
         N22=TINV(2,2)
         N32=TINV(3,2)
         N42=TINV(4,2)
         N13=TINV(1,3)
         N23=TINV(2,3)
         N33=TINV(3,3)
         N43=TINV(4,3)
         N14=TINV(1,4)
         N24=TINV(2,4)
         N34=TINV(3,4)
         N44=TINV(4,4)
C
         R11=N12*L21+N13*L31+N14*L41+N12*D11+N13*D21
         R12=N12*L22+N13*L32+N14*L42+N12*D12+N13*D22
         R13=N12*L23+N13*L33+N14*L43+N12*D13+N13*D23
         R14=N12*L24+N13*L34+N14*L44+N12*D14+N13*D24
         R21=N22*L21+N23*L31+N24*L41+N22*D11+N23*D21
         R22=N22*L22+N23*L32+N24*L42+N22*D12+N23*D22
         R23=N22*L23+N23*L33+N24*L43+N22*D13+N23*D23
         R24=N22*L24+N23*L34+N24*L44+N22*D14+N23*D24
```

```fortran
C
C       Evaluate Alpha' and Omega'
C
        DINC=0.001
        XDR =XD+DINC
        XDL =XD-DINC
        XD  =XDR

        CALL LINEAR(K1,K3,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
     &     AA24,BB1,BB2,A,TL)

        CALL RG(4,4,A,WR,WI,O,ZZZ,IV1,FV1,IERR)
        CALL DSTABL(DEOS,WR,WI,FREQ)
        ALPHR=DEOS
        OMEGR=FREQ

        XD=XDL
C
        CALL LINEAR(K1,K3,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
     &     AA24,BB1,BB2,A,TL)

        CALL RG(4,4,A,WR,WI,O,ZZZ,IV1,FV1,IERR)
        CALL DSTABL(DEOS,WR,WI,FREQ)
        ALPHL=DEOS
        OMEGL=FREQ
C
        DALPHA=(ALPHR-ALPHL)/(XDR-XDL)
        DOMEGA=(OMEGR-OMEGL)/(XDR-XDL)
C
C       Evaluation of Hopf Bifurcation Coefficients
C
        COEF1=3.0*R11+R13+R22+3.0*R24
        COEF2=3.0*R21+R23-R12-3.0*R14
        AMU2 =-COEF1/(8.0*DALPHA)
        BETA2=0.25*COEF1
        TAU2 =-(COEF2-DOMEGA*COEF1/DALPHA)/(8.0*OMEGA0)
        PER  =2.0*3.1415927/OMEGA0
        PER  =PER*U/L
C
        WRITE (15,2002) XD,WN,COEF1,DALPHA,PER


    1 CONTINUE
C

 1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
 1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
 1003 FORMAT (' ENTER DAMPING RATIO')
```

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```
1006 FORMAT (' ENTER DSAT ')
1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
2001 FORMAT (2I5)
2002 FORMAT (5E15.5)


     END
C====================================================================

     SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
C
C    EVALUATES THE EIGENVALUE WITH THE MAXIMUM REAL PART
C
     IMPLICIT DOUBLE PRECISION (A-H,O-Z)
     DIMENSION WR(4),WI(4)
     DEOS=-1.0D+20
     DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
   1 CONTINUE
     OMEGA=WI(IJ)
     OMEGA=DABS(OMEGA)
     RETURN
     END
C
     SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
    & AA22,AA23,AA24,BB1,BB2,A,TL)
C
C    FORMS THE LINEARIZED MATRIX A (time delay 1st order approximation
C
     IMPLICIT DOUBLE PRECISION (A-H,O-Z)
     DOUBLE PRECISION K1,K2
     DIMENSION A(4,4)
     A(1,1)=0.0D0
     A(1,2)=0.0D0
     A(1,3)=1.0D0
     A(1,4)=0.0D0
     A(2,1)=AA11+BB1*K1-BB1*K1*TL/XD
     A(2,2)=AA12-BB1*K1*TL/XD
     A(2,3)=AA13+BB1*K2
     A(2,4)=AA14+BB1*K1/XD
     A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD
     A(3,2)=AA22-BB2*K1*TL/XD
     A(3,3)=AA23+BB2*K2
     A(3,4)=AA24+BB2*K1/XD
     A(4,1)=1.0D0
     A(4,2)=1.0D0
```

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```fortran
      A(4,3)=0.0D0
      A(4,4)=0.0D0
      RETURN
      END




C------------------------------------------------------------------



      SUBROUTINE DSOMEG(IJK,WR,WI,OMEGA,CHECK)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      CHECK=-1.0D+25
      DO 1 I=1,4
        IF (WR(I).LT.CHECK) GO TO 1
        CHECK=WR(I)
        IJ=I
    1 CONTINUE
      OMEGA=DABS(WI(IJ))
      IF (WI(IJ).GT.0.D0) IJK=IJ
      IF (WI(IJ).LT.0.D0) IJK=IJ-1
      RETURN
      END
C========================================================================


C========================================================================
      SUBROUTINE NORMAL(T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION T(4,4),TNOR(4,4)
      CFAC=T(1,1)**2+T(1,2)**2
      IF (DABS(CFAC).LE.(1.D-10)) STOP 4001
      TNOR(1,1)=1.D0
      TNOR(2,1)=(T(1,1)*T(2,1)+T(2,2)*T(1,2))/CFAC
      TNOR(3,1)=(T(1,1)*T(3,1)+T(3,2)*T(1,2))/CFAC
      TNOR(4,1)=(T(1,1)*T(4,1)+T(4,2)*T(1,2))/CFAC
      TNOR(1,2)=0.D0
      TNOR(2,2)=(T(2,2)*T(1,1)-T(2,1)*T(1,2))/CFAC
      TNOR(3,2)=(T(3,2)*T(1,1)-T(3,1)*T(1,2))/CFAC
      TNOR(4,2)=(T(4,2)*T(1,1)-T(4,1)*T(1,2))/CFAC
      DO 1 I=1,4
        DO 2 J=1,2
          T(I,J)=TNOR(I,J)
    2   CONTINUE
    1 CONTINUE
      RETURN
```

```
      END
C=====================================================================
      SUBROUTINE MULT(TINV,A,T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION TINV(4,4),A(4,4),T(4,4),A1(4,4),A2(4,4)
      DO 1 I=1,4
        DO 2 J=1,4
          A1(I,J)=0.D0
          A2(I,J)=0.D0
    2   CONTINUE
    1 CONTINUE
      DO 3 I=1,4
        DO 4 J=1,4
          DO 5 K=1,4
            A1(I,J)=A(I,K)*T(K,J)+A1(I,J)
    5     CONTINUE
    4   CONTINUE
    3 CONTINUE
      DO 6 I=1,4
        DO 7 J=1,4
          DO 8 K=1,4
            A2(I,J)=TINV(I,K)*A1(K,J)+A2(I,J)
    8     CONTINUE
    7   CONTINUE
    6 CONTINUE
      DO 11 I=1,4
C         WRITE (*,101) (A(I,J),J=1,4)
   11 CONTINUE
      DO 12 I=1,4
C         WRITE (*,101) (T(I,J),J=1,4)
   12 CONTINUE
      DO 10 I=1,4
        WRITE (*,101) (A2(I,J),J=1,4)
   10 CONTINUE
C     WRITE (*,101) A2(1,1)
      RETURN
  101 FORMAT (4E15.5)
      END
```

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