THE PRINCIPLES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS

SIMPLIFIED

AND APPLIED TO

THE SOLUTION OF VARIOUS USEFUL PROBLEMS IN PRACTICAL MATHEMATICS AND MECHANICS

BY

THOMAS TATE

MATHEMATICAL MASTER OF THE NATIONAL SOCIETY'S TRAINING COLLEGE, BATTERSEA;
AUTHOR OF "EXERCISES ON MECHANICS," ETC.

NEW EDITION

LONDON
LONGMANS, GREEN, AND CO.
1867

LONDON
PRINTED BY SPOTTISWOODE AND CO.
NEW STREET SQUARE

TO

THE REV. H. MOSELEY, F.R.S.

&c. &c.

ONE OF HER MAJESTY'S INSPECTORS OF SCHOOLS.

REVEREND AND DEAR SIR,

This work I inscribe to you, as a tribute of esteem, on account of the discoveries with which you have enriched physical science; and as an acknowledgment of the value which I attach to your services in promoting elementary education.

That you may long continue to be a distinguished instrument, under the blessing of Divine Providence, in elevating the intellectual and moral condition of the people of this country, is the earnest wish of

Your humble and obliged servant,

THOMAS TATE.

PREFACE.

The object of this work is to explain, illustrate, and apply the fundamental principles of the Calculus, in such a manner as to bring them within the comprehension of a student, having merely a knowledge of ordinary Algebra and Trigonometry, and to enable him to undertake the perusal of such valuable practical works as "Moseley's Principles of Engineering," "Navier's L'Application de la Mécanique," "Whewell's Mechanics," "Hann's Treatise on the Steam Engine," &c.

To a person merely acquainted with ordinary algebra, the Calculus must, at first, appear mysterious and metaphysical; for he has to view abstract quantities, not only in an isolated form, but as admitting of continuous changes, and of taking certain finite ratios as they approach zero or infinity. The principle involved in a limiting ratio must, however, be eventually understood by every student who wishes to make a satisfactory progress in this branch of analysis. I have adopted the method of limits almost exclusively in this work, because it appears to be the most natural and consistent foundation of the Calculus; and with the view to simplify this method as much as possible, I have fully explained and applied it in the preliminary portion of this treatise, apart from the conventional and abstract notation by which the condition of a limit is usually expressed.

It is highly desirable that Teachers and Practical Men should possess some knowledge of this most important branch of pure mathematics, in order to enable them to understand our best works on mechanical and experimental philosophy. The great physical laws, by which it has pleased the Almighty to govern the universe, must always form a lofty subject of contemplation to his intelligent creatures; but these laws can only be duly interpreted by the aid of the symbolic language of the higher analysis.

As a complete knowledge of a great subject, like this, cannot be obtained from the perusal of one book, those who aspire to a further acquaintance with the higher parts of the Calculus must study the works of De Morgan *, Moigno, O'Brien, Hymers, Gregory, Price †, Hall, Moseley ‡, and Young.

T. TATE.

Battersea, Feb. 1849.

* "De Morgan's Calculus" is the largest work on the subject in our language.

† On the method of Infinitesimals.

‡ On Definite Integrals, published in the "Encyclopedia Metropolitana."

TABLE OF CONTENTS.

THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

INTRODUCTORY PRINCIPLES, &c.

1. THE quantities used in the following work are of two kinds, *constant* and *variable.*

A *constant* quantity always preserves the same value throughout an investigation, whereas a *variable* quantity in general admits of taking any value we may please to give it. Constants are expressed by the first letters of the alphabet, and variables by the last letters: thus, in the expression $a+bx^2$, the letters a and b are constant quantities, and x is the variable.

2. When a quantity changes gradually, or passes from one value to another, by going through all the intermediate values, we say that it varies *continuously;* and, on the contrary, if the quantity changes abruptly in going from one value to another, we say that it varies *discontinuously.* If a body move in a curve, its distance from a fixed point will vary continuously; but if, on the contrary, the body should, as it were, leap from one point of the curve to another, its distance from the fixed point will vary discontinuously. All quantities are supposed to vary continuously in the differential calculus.

3. When a variable and constants are in any way combined in an expression, that expression is said to be a *function* of the variable; thus $\sqrt{a+bx+cx^2}$, $(a+bx)^n$, a^x, sin. $(a+x)$, are all functions of x. Functions are expressed by placing the symbols f, F, &c. before the variable. Thus, if $y=a+bx-cx^3$, we say that y is a function of x, and we should express this relation generally by writing $y=f(x)$. The letters f, F, used as the symbols of a function, denote the way in which the variable is combined with the constants which enter the expression. It is important to observe that $f(x)$ denotes an expression different from $F(x)$; and, moreover, in the functions $f(x)$ and $f(y+h)$, we are to understand that, whatever may be the *form* of the first expression with respect to x, the second will have the same form with respect to $y+h$.

THE BINOMIAL THEOREM.

4. This theorem enables us to raise a binomial to any power, without going through the process of multiplication. The following is the simplest form of this theorem: —

$$(1+x)^n=1+\frac{n}{1}x+\frac{n\,(n-1)}{1\,.\,2}x^2+\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}x^3+\&c.$$

It will be afterwards shown that all other cases may be reduced to this form.

For the sake of conciseness, putting A_1, A_2, &c., for the coefficients of x, x^2, &c., we have

$$(1+x)^n=1+A_1x+A_2x^2+\ .\ .\ .+A_rx^r,\ .\ .\ .$$

where the coefficient of the general term is

$$A_r=\frac{n\,(n-1)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots r}.$$

5. *To prove the binomial theorem when the index is a positive integer.*

By multiplication we know, that

$$(1+x)^2=1+2x+x^2,$$
$$(1+x)^3=1+3x+3x^2+x^3,$$

where it will be seen that the coefficients are formed according to the law in the preceding development.

We shall now show that if the *law* for the coefficients be *assumed* to be true for any one power of $1+x$, it will also be true for the next higher power, that is, if it be true for $(1+x)^{n-1}$, it will also be true for $(1+x)^n$.

Let us suppose that $(1+x)^{n-1}=1+a_1x+a_2x^2+\&c. \ldots (1)$, where a_1, a_2, &c. have the same form in reference to $n-1$, that A_1, A_2, &c. have to n, that is,

$$a_1=n-1,\ a_2=\frac{(n-1)(n-2)}{1 \cdot 2}, \&c.$$

Multiplying both sides of eq. (1) by $1+x$, we find

$$(1+x)^n=1+(1+a_1)x+(a_1+a_2)x^2+(a_2+a_3)x^3+\&c.,$$

where the coefficient of x^r will obviously be $a_{r-1}+a_r$.

But $1+a_1=1+n-1=n=A_1$,

$$a_1+a_2=n-1+\frac{(n-1)(n-2)}{1 \cdot 2}=(n-1)\left\{1+\frac{n-2}{2}\right\}$$
$$=\frac{n(n-1)}{1 \cdot 2}=A_2,$$

and so on. Generally the coefficient of x^r, or $a_{r-1}+a_r$

$$=\frac{(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots (r-1)}+\frac{(n-1)(n-2)\ldots(n-r)}{1 \cdot 2 \cdot 3 \ldots r}$$
$$=\frac{(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots (r-1)}\left\{1+\frac{n-r}{r}\right\}$$
$$=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r}=A_r;$$

$$\therefore\ (1+x)^n=1+A_1x+A_2x^2+\ldots+A_rx^r+\&c.$$

Hence it appears that if this formula be true for any one power, it is also true for the next higher power; but we know that it is true for the 2nd and 3rd powers, therefore it is true for the 4th power, and therefore for the 5th power, and so on by continued inductions we conclude that it is true for the nth power.

6. *To prove the binomial theorem, when the index is fractional or negative.*

Multiplying together the developments of $(1+x)^n$ and $(1+x)^m$, established in the preceding article, we have,

$$(1+x)^n \times (1+x)^m = (1+x)^{n+m} = \left\{1+\frac{n}{1}x+\frac{n(n-1)}{1\,.\,2}x^2+\&\text{c.}\right\}$$
$$\times\left\{1+\frac{m}{1}x+\frac{m(m-1)}{1\,.\,2}x^2+\&\text{c.}\right\}.$$

$$\text{But } (1+x)^{n+m} = 1+\frac{(n+m)}{1}x+\frac{(n+m)(n+m-1)}{1\,.\,2}x^2+\&\text{c.}$$

$$\therefore \{1+\frac{n}{1}x+\frac{n(n-1)}{1\,.\,2}x^2+\&\text{c.}\} \times \{1+\frac{m}{1}x+\frac{m(m-1)}{1\,.\,2}x^2+\&\text{c.}\}$$
$$=1+\frac{n+m}{1}x+\frac{(n+m)(n+m-1)}{1\,.\,2}x^2+\&\text{c.} \;\ldots\; (1).$$

Now although n and m were, in the development of $(1+x)^n$ &c., restricted to integer values, yet it is evident that the expression for the above product, if true at all, must be true for any values that may be given to n and m. This result might be verified by *actual* multiplication, but the operation is tedious.

$$\text{Let} \quad f(n)=1+\frac{n}{1}x+\frac{n(n-1)}{1\,.\,2}x^2+\&\text{c.} \;\ldots\; (2),$$

whatever may be the value of n; then

$$f(m)=1+\frac{m}{1}x+\frac{m(m-1)}{1\,.\,2}x^2+\&\text{c.},$$

$$\text{and } f(n+m)=1+\frac{n+m}{1}x+\frac{(n+m)(n+m-1)}{1\,.\,2}x^2+,\ \&\text{c.}$$

Hence by eq. (1) we have,

$$f(n)\times f(m)=f(n+m),$$

whatever may be the values of n and m. In like manner,

$$f(n)\times f(m)\times f(p)=f(n+m)\times f(p)=f(n+m+p),$$

and so on to any number of factors; thus it appears that the product of any number of series, such as that expressed by $f(n)$, will produce another series having the same form. Hence we have,

$$\left\{f\left(\frac{n}{m}\right)\right\}^m=f\left(\frac{n}{m}\right)\times f\left(\frac{n}{m}\right)\times \text{\&c. to } m \text{ factors.}$$

$$=f\left(\frac{n}{m}+\frac{n}{m}+\text{\&c. to } m \text{ terms}\right)$$

$$=f(n);$$

but when n is a plus integer $f(n)=(1+x)^n$;

$$\therefore \left\{f\left(\frac{n}{m}\right)\right\}^m=(1+x)^n; \quad \therefore f\left(\frac{n}{m}\right)=(1+x)^{\frac{n}{m}},$$

taking the mth root of both sides of the equality.

Hence the series symbolised by $f(n)$ is the development of $(1+x)^n$, when n is a positive integer or fraction.

Let us now consider the case $f(-n)$, where n may be integral or fractional.

$$f(n)\times f(-n)=f(n-n)=f(0)=1,$$

(because the series (2) becomes unity when $n=0$),

$$\therefore f(-n)=\frac{1}{f(n)}=\frac{1}{(1+x)^n}=(1+x)^{-n}.$$

Hence the series symbolised by $f(n)$ is the development of $(1+x)^n$, when n is *negative* and any integral or fractional number.

Generally, therefore, *whatever* be the value of the index n, we always have,

$$(1+x)^n=f(n)=1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\text{\&c.}$$

Lastly, because

$$a \pm x = a\left(1 \pm \frac{x}{a}\right),$$

$$\therefore\ (a \pm x)^n = a^n\left(1 \pm \frac{x}{a}\right)^n = a^n\left\{1 \pm \frac{n}{1}\,\frac{x}{a} + \frac{n(n-1)}{1\,.\,2}\,\frac{x^2}{a^2} \pm \text{\&c.}\right\}$$

$$= a^n \pm \frac{n}{1} a^{n-1}x + \frac{n(n-1)}{1\,.\,2} a^{n-2}x^2 + \text{\&c.}$$

which is the most general form of the binomial theorem where

$$\text{the } (r+1)\text{th term} = \frac{n(n-1)\,.\,.\,.\,(n-r+1)}{1\,.\,2\,.\,3\,.\,.\,.\,r} a^{n-r}x^r.$$

INDETERMINATE COEFFICIENTS.

7. If the equality $a + bx = A + Bx$ be true for *every valu* of x; then $a = A$, and $b = B$.

For as the equation holds true for any value that may b given to x, let $x = n$,

$$\therefore\ a + bn = A + Bn;$$

subtracting this from the proposed equality,

$$b(x-n) = B(x-n);$$

$$\therefore\ b = B, \text{ and } \therefore\ a = A.$$

The equation $a + bx = A + Bx$, differs essentially from ord nary equations where x admits of being determined in term of the constants; whereas in the equation here considered admitting of all values, is an indeterminate quantity; thu we find $x = \frac{A-a}{b-B}$, but since $a = A$ and $b = B$, we have $x = \frac{0}{0}$ a result which, in the present case, is the symbol of an inde terminate quantity.

In general, if $a + bx + cx^2 + \text{\&c.} = A + Bx + Cx^2 + \text{\&c.}$, b true for *every value* of x; then $a = A$, $b = B$, $c = C$, &c., tha is, the coefficients of the like powers of x are equal.

As *any* value may be put for x, let $x=0$; then $a=A$; taking a and A from the original eq., we have,

$$bx+cx^2+\text{ \&c.}=Bx+Cx^2+\text{ \&c.};$$

here x may take *any* value we please, hence we may now consider it as some definite number; therefore dividing each side of the equality by x, we have,

$$b+cx+\text{ \&c.}=B+Cx+\text{ \&c.};$$

here again as *any* value may be put for x, let $x=0$; then $b=B$. Proceeding in this way, we find, $c=C$, $d=D$, and so on.

Ex. 1. Required three terms of the quotient of $\frac{1-2x}{1+4x}$.

$$\text{Let } \frac{1-2x}{1+4x}=A+Bx+Cx^2+\text{ \&c.};$$

where the coefficients A, B, C, &c., remain to be determined. Multiplying each side of the equality by $1+4x$, we have,

$$\begin{aligned}1-2x&=A+Bx+Cx^2+\text{ \&c.}\\ &\qquad\;\; +4Ax+4Bx^2+\text{ \&c.}\\ &=A+(4A+B)x+(4B+C)x^2+\text{ \&c.}\end{aligned}$$

therefore equating the coefficients of the like powers of x,

$$A=1;\ 4A+B=-2,\ \therefore\ B=-2-4A=-6;$$

$$4B+C=0,\ \therefore\ C=-4B=24;\ \text{and so on.}$$

$$\therefore\ \frac{1-2x}{1+4x}=1-6x+24x^2-\text{ \&c.}$$

Ex. 2. Resolve $\frac{1}{(x+1)(x+2)(x+3)}$ into its partial fractions.

$$\text{Let } \frac{1}{(x+1)(x+2)(x+3)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3};$$

where the coefficients A, B, and C remain to be determined; multiplying both sides of this eq. by $(x+1)(x+2)(x+3)$,

$$1=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2).$$

Since *any* value may be put for x, let $x=-1$,

$$\text{then } 1=A(-1+2)(-1+3), \therefore A=\tfrac{1}{2};$$

$$\text{let } x=-2, \text{ then } 1=B(-2+1)(-2+3), \therefore B=-1;$$

$$\text{let } x=-3, \text{ then } 1=C(-3+1)(-3+2), \therefore C=\tfrac{1}{2}.$$

$$\therefore \frac{1}{(x+1)(x+2)(x+3)}=\frac{1}{2(x+1)}-\frac{1}{x+2}+\frac{1}{2(x+3)}.$$

Ex. 3. Resolve $\dfrac{A+Bx}{(1+ax)(1+bx)}$ into its partial fractions.

$$\text{Let } \frac{A+Bx}{(1+ax)(1+bx)}=\frac{P}{1+ax}+\frac{Q}{1+bx};$$

multiplying both sides of this eq. by $(1+ax)(1+bx)$,

$$A+Bx=P(1+bx)+Q(1+ax).$$

Since *any* value may be put for x, let $x=-\dfrac{1}{a}$, so as to make

$$1+ax=0, \text{ then } A-\frac{B}{a}=P\left(1-\frac{b}{a}\right),$$

$$\therefore P=\frac{Aa-B}{a-b};$$

let $x=-\dfrac{1}{b}$, so as to make $1+bx=0$, then $A-\dfrac{B}{b}=Q\left(1-\dfrac{a}{b}\right)$,

$$\therefore Q=\frac{Ab-B}{b-a}=-\frac{Ab-B}{a-b}$$

$$\therefore \frac{A+Bx}{(1+ax)(1+bx)}=\frac{Aa-B}{(a-b)(1+ax)}-\frac{Ab-B}{(a-b)(1+bx)}.$$

8. To expand a^x, or to prove the truth of the exponential theorem.

$a^x = \{1+(a-1)\}^x$ (expanding by the binomial)

$$=1+x(a-1)+\frac{x(x-1)}{1.2}(a-1)^2+\frac{x(x-1)(x-2)}{1.2.3}(a-1)^3+\&c.$$

$$=1+x(a-1)+\frac{x^2-x}{1.2}(a-1)^2+\frac{x^3-3x^2+2x}{1.2.3}(a-1)^3+\&c.$$

$$=1+\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.\}\,x+A_2x^2+\&c.$$

$$=1+Ax+A_2x^2+A_3x^3+\&c.;$$

where $A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.$,

and A_2, A_3, &c. also depend upon powers of $a-1$.

We are therefore at liberty to assume that

$$a^x=1+Ax+A_2x^2+A_3x^3+\&c. \quad . \quad . \quad . \quad (1)$$

and putting $2x$ for x, we have,

$$a^{2x}=1+2Ax+4A_2x^2+8A_3x^3+\&c. \quad . \quad . \quad . \quad (2)$$

But by actually squaring both sides of eq. (1), we have,

$$a^{2x}=1+2Ax+(A^2+2A_2)x^2+(2A_3+2AA_2)x^3+\&c. \quad . \quad . \quad . \quad (3),$$

Hence equating the coefficients of the like powers of x in eq. (2) and (3), we have,

$$4A_2=A^2+2A_2, \ \therefore\ A_2=\frac{A^2}{2};$$

$$8A_3=2A_3+2AA_2, \ \therefore\ 6A_3=2AA_2=A^3, \ \therefore\ A_3=\frac{A^3}{2.3};$$

and so on to the other coefficients.

$$\therefore\ a^x=1+\frac{Ax}{1}+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\frac{A^4x^4}{1.2.3.4}+\&c.$$

Cor. 1 If e be put for that value of a which makes $A=1$, then

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\&c.$$

In order to find the numerical value of e, let $x=1$, then

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.=2{\cdot}71828, \&c.$$

It is important to observe that e or 2·71828, &c. is taken for the base of what is called the Napierian or hyperbolic logarithms.

Throughout this work the logarithms of any number n to the bases a and e, are thus expressed, $\log_a n$, and $\log_e n$. Hence $\log_a a=1$, because the exponent of the base a to produce the number a must be unity, or $a=a^1$.

9. In the general equation for a^x let $x=\frac{1}{\text{A}}$, or $\text{A}x=1$, then

$$a^{\frac{1}{\text{A}}}=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.=2{\cdot}71828, \&c.=e.$$

$$\therefore\ e=a^{\frac{1}{\text{A}}},\ \therefore\ a=e^{\text{A}},$$

taking the logarithms of each side of these two equations, viz., of the first when a is the base, and of the second when e is the base,

$$\log_a e=\frac{1}{\text{A}},\ \text{and}\ \log_e a=\text{A}\ .\ .\ .\ (1)$$

by multiplication,

$$\log_a e\times\log_e a=1,\ \therefore\ \log_a e=\frac{1}{\log_e a}\ .\ .\quad(2)$$

From eq. (1) we have,

$$\log_e a=\text{A}=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.\ .\ .\ .\ (3).$$

10. To find the sum of the series $1^2+2^2+3^2+\ .\ .\ .\ +n^2$.

Let S_2 be put for the sum of the proposed series, and S_1 for the sum of the arithmetical series $1+2+\ .\ .\ \ +n$, then in the expansion,

$$(x-1)^3=x^3-3x^2+3x-1,$$

take x successively equal to 1, 2, 3, . . . n, add the resulting equations, and cancel the common terms, and we shall have,

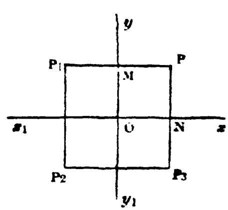
$$\begin{aligned}
0 &= 1^3-3 \cdot 1^2+3 \cdot 1-1 \\
1^3 &= 2^3-3 \cdot 2^2+3 \cdot 2-1 \\
2^3 &= 3^3-3 \cdot 3^2+3 \cdot 3-1 \\
\vdots & \qquad\qquad \vdots \\
(n-1)^3 &= n^3-3 \cdot n^2+3 \cdot n-1 \\
\hline
0 &= n^3-3\,s_2 + 3 \cdot s_1-n
\end{aligned}$$

$$\therefore\ s_2=\frac{n^3}{3}+s_1-\frac{n}{3}=\frac{n^3}{3}+(n+1)\frac{n}{2}-\frac{n}{3}$$

$$=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}.$$

EQUATIONS TO CURVES.

11. We now proceed to treat of the equations to the straight line, the circle, and those other curves which are of the greatest practical importance.



The position of a point is determined when its distance from two given straight lines is known. Thus, let Ox, and Oy be two straight lines, perpendicular to each other, then the position of a point P will be known, when the perpendicular distances PN and PM, from the given lines Ox, and Oy are known. The lines Ox and Oy are called *axes*, and ON (=MP) and PN, the *ordinates* of the point P. The perpendicular PN is called the *ordinate*, and its length is expressed by y; ON is called the *abscissa*, and its length is expressed by x; PN and ON taken together, are called the coordinates of P; and the point O, the origin of coordinates.

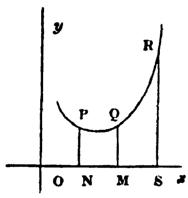
Ex. If ON=4, and PN=3, then to find the point P; from a scale of equal parts take ON$=4$; draw NP perpendicular to Ox, and from the same scale of equal parts, take NP$=3$; and P will be the point required.

12. If ON$=a$, and NP$=b$, then $x=a$, and $y=b$, are the equations to the point P.

$x=-a$, and $y=b$, are equations of a point P_1, situated in the angle yOx_1.

$x=-a$, and $y=-b$, are equations of a point P_2 situated in the angle y_1Ox_1.

$x=+a$, and $y=-b$, are equations of a point P_3 situated in the angle y_1Ox. *See the Principles of Geometry and Mensuration*, p. 94.

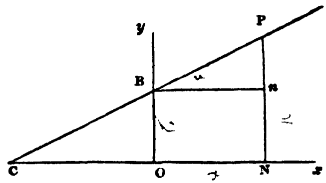


13. If in the curve PQR, the relation of the ordinates to their corresponding abscissa be expressed by an equation, the curve itself may be drawn by finding different points in it. Thus, let $y=x^2-3x+3$ be the equation of the curve, or the equation expressing the relation between any ordinate and its corresponding abscissa: then taking x or ON$=1$, we have y or NP$=1^2-3\times1+3=1$, hence the point P is determined. Taking x or OM$=2$, we have y or MQ$=2^2-3\times2+3=1$, hence the point Q is determined. Taking x or OS$=3$, we have y or SR$=3^2-3\times3+3=3$, hence the point R is determined. And so on to any number of points.

In general the ordinate of a curve is always some function of its abscissa, hence the equation of a curve is generally expressed by the equation $y=f(x)$. We propose to determine the peculiar form of this equation for various curves having some given property or mode of generation.

The Equation to the straight Line.

14. Let CP be the straight line, and Ox and Oy the axes. Put ON$=x$, NP$=y$, OB$=b$, and tan $\angle$ C$=a$; then drawing Bn parallel to Ox, we have



$$\frac{\text{P}n}{\text{B}n}=\tan\angle\text{PB}n=\tan\angle\text{C}=a.$$

But Pn=PN$-$BO$=y-b$, and Bn=ON$=x$,

$$\therefore\ \frac{y-b}{x}=a$$

$$\therefore\ y=ax+b.$$

Since P is *any* point in the straight line CP, the relation of x and y, expressed in this equation, also contains the relation of the co-ordinates of *every* point in CP; hence the above expression is called the equation of the straight line.

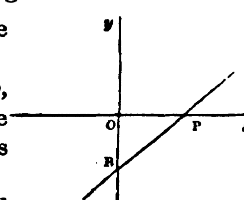
Cor. 1. If the line be drawn through the origin O, then OB or $b=0$, and $\therefore y=ax$;

this is the eq. of a line drawn through the origin.

15. Given, the equation to a straight line to construct it.

For example, let $y=ax-b$, be the equation.

When $x=0$, then $y=a\times 0-b=-b$, that is, when the abscissa is 0, the ordinate OB is $-b$. Or the line cuts the axis of y below the origin.



When $y=0$, then $ax-b=0$, and $x=\frac{b}{a}=$OP.

16. *The equation to a straight line drawn through a give. point.*

Let $y=ax+b$ be the equation to the line, and $y_{,}$ and $x_{,}$ the co-ordinates to the given point; then, as $y_{,}$ and $x_{,}$ are co-ordinates of a point in the straight line, we must have,

$$y_{,}=ax_{,}+b$$
$$\text{and } y=ax+b$$
$$\therefore\ y-y_{,}=a(x-x_{,}).$$

17. *To find the equation to a straight line which passes through two given points.*

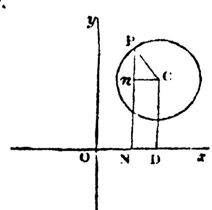
Let $y=ax+b$ be the equation to the line, where a and b are to be eliminated by means of the given co-ordinates $x_{,}$ and $y_{,}$ of the one point, and $x_{,,}$ and $y_{,,}$ of the other.

$$\therefore\ y_{,}=ax_{,}+b\ \ldots (1)$$
$$\text{and } y_{,,}=ax_{,,}+b\ \ldots (2)$$
$$\therefore\ y_{,}-y_{,,}=a(x_{,}-x_{,,})\,;$$
$$\therefore\ a=\frac{y_{,}-y_{,,}}{x_{,}-x_{,,}}.$$

$$\text{Since } y=ax+b$$
$$\text{and } y_{,}=ax_{,}+b\,;$$
$$\therefore\ y-y_{,}=a(x-x_{,})=\frac{y_{,}-y_{,,}}{x_{,}-x_{,,}}(x-x_{,}),$$

by substituting the value of a.

Equation to the Circle.



18. Let $\text{OD}=\alpha$, $\text{DC}=\beta$, be the co-ordinates of the centre C; and $\text{ON}=x$, and $\text{NP}=y$, the co-ordinates of the point P in the circumference;

then $\text{P}n^2+\text{C}n^2=\text{CP}^2$ or r^2;
but $\text{P}n=\text{NP}-\text{DC}=y-\beta$,
and $\text{C}n=\text{OD}-\text{ON}=\alpha-x$;
$\therefore\ (y-\beta)^2+(\alpha-x)^2=r^2.$

Cor. 1. If the centre C be on the axis Ox, and the circumference be on the origin O, then DC or $\beta=0$, and OD or $\alpha=r$;

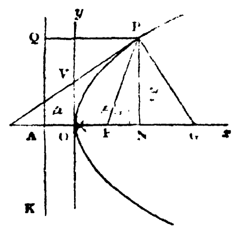
$$\therefore\ y^2+(r-x)^2=r^2,$$
$$\text{and}\ \therefore\ y^2=2rx-x^2.$$

Cor. 2. If the centre C be in the origin O, then $\beta=0$, and $\alpha=0$;

$$\therefore\ y^2+x^2=r^2.$$

Equation to the Parabola.

19. Let QK be a given fixed line called the directrix, F a fixed point called the focus, and P *any* point in the curve, then the characteristic property of the parabola is, that the distance PQ of the point P from QK is equal to the distance PF of this point P from the focus.



Through F draw AFx perpendicular to QK; bisect FA in O, then O will be a point in the curve.

Let OA$=$OF$=a$, ON$=x$, and NP$=y$.

Now FP$^2=$FN$^2+$NP^2;

but FP$=$QP$=$AN$=$OA$+$ON$=a+x$, FN$=$ON$-$OF$=x-a$, and NP$=y$; therefore, by substitution,

$$(a+x)^2=(x-a)^2+y^2;$$
$$\therefore\ y^2=4ax.$$

Now, as this equation is true for *any* point P in the curve, it will express the relation of the co-ordinates of *every* point in the curve; hence this relation is called the equation of the curve.

Cor. Let FP$=r$, and $\angle$PFN$=\theta$,

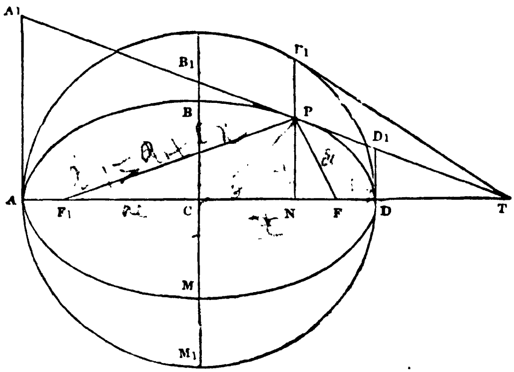
then $r=$FP$=$AN$=$AF$+$FN$=2a+r\cos$PFN$=2a+r\cos\theta$;

$$\therefore\ r=\frac{2a}{1-\cos\theta}.$$

This is called the polar equation of the parabola.

Equation to the Ellipse.

20. If two lines FP and F_1P revolve about two fixed points F and F_1 called the foci, in such a way that we always have



$FP+F_1P=2a$, a constant; the curve traced by the point of intersection P is called an ellipse.

Bisect FF_1 in C, and take $CA=CD=a$, then the curve will pass through A and D; through C draw BCM perpendicular to AD; with F as a centre, and radius equal to a, describe an arc cutting this line in the points B and M, then the curve will pass through B and M, because $FB=F_1B=a$, and $\therefore$ $FB+F_1B=2a$.

Let $CN=x$, $NP=y$, $CB=b$, $FP=d$, $F_1P=d_{\prime}$; and as CF must be a certain part of CD, let CF or $CF_1=ae$, which is called the eccentricity. From the right-angled triangles, FNP and F_1NP we have,

$$FP^2 \text{ or } d^2=NF^2+NP^2=(ae-x)^2+y^2,$$
$$F_1P \text{ or } d_{\prime}^2=NF_1^2+NP^2=(ae+x)^2+y^2.$$

Hence we have, by addition and subtraction,

$$d_{\prime}^2+d^2=2(a^2e^2+x^2+y^2) \ldots\ldots\ldots (1)$$
$$d_{\prime}^2-d^2 \text{ or } (d_{\prime}+d)(d_{\prime}-d)=4aex \ldots (2).$$

$$\text{But } d_{,}+d=2a \;\ldots\; (3)$$

by substituting in (2), $2a(d_{,}-d)=4aex$,

$$\therefore\; d_{,}-d=2ex \;\ldots\; (4);$$

adding and subtracting (3) and (4),

$$d_{,}=a+ex,\; d=a-ex,$$

substituting these values in eq. (1),

$$2(a^2e^2+x^2+y^2)=(a+ex)^2+(a-ex)^2$$

$$\therefore\; y^2=(1-e^2)a^2-(1-e^2)x^2$$

$$=(1-e^2)\,(a^2-x^2).$$

In order to eliminate e from this equation, we have, $CF^2+CB^2=FB^2$, that is $a^2e^2+b^2=a^2$, and

$$\therefore\; 1-e^2=\frac{b^2}{a^2}.$$

$$\therefore\; y^2=\frac{b^2}{a^2}(a^2-x^2) \;\ldots\; (1)$$

$$\text{and } \frac{y^2}{b^2}+\frac{x^2}{a^2}=1 \;\ldots\; (2)$$

Cor. 1. Let A be the origin, and put $AN=x_1$; then $x_1=AC+CN=a+x$, $\therefore\; x=x_1-a$. Substituting this value of x in eq. (1), we find, after a little reduction,

$$y^2=\frac{b^2}{a^2}(2ax_1-x_1{}^2).$$

This is the equation to the ellipse for the co-ordinates AN and NP, or when the origin is taken at A.

Cor. 2. Let a circle be described upon the major diameter AD of an ellipse, ABDM, then putting $x_1=AN$, and $NP_1=y_1$, and $AD=2r=2a$, we have by *Cor.* 1. to the equation of the circle, Art. 18.

$$y_1{}^2=2rx_1-x_1{}^2,$$

but by the last *Cor.* the equation to the ellipse is

$$y^2=\frac{b^2}{a^2}(2rx_1-x_1^2),$$

hence by division, $\frac{y}{y_1}=\frac{b}{a}$, or NP : NP$_1$ :: BC : AC.

Cor. 3. Let C be the pole; join C and P; put CP$=r$, and $\angle$PCD$=\theta$. Then CN or $x=r\cos\theta$, and NP or $y=r\sin\theta$; substituting these values in eq. (2) of the ellipse, we have,

$$\frac{r^2\sin^2\theta}{b^2}+\frac{r^2\cos^2\theta}{a^2}=1;$$

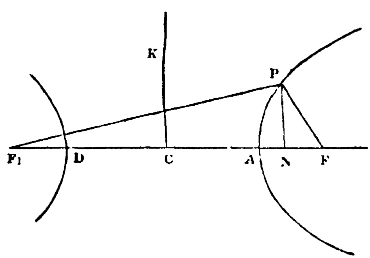
$$\therefore\ r=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{b}{\sqrt{1-e^2\cos^2\theta}},$$

by substituting $1-e^2$ for $\frac{b^2}{a^2}$ and reducing.

This is called the polar equation to the ellipse.

Equation to the Hyperbola.

21. In the hyperbola the difference between FP and F$_1$P is a constant quantity, F and F$_1$ being the fixed foci.



Let F$_1$P$-$FP$=2a$. Bisect F$_1$F in C, and take CA$=$CD$=a$, then one branch of the curve will pass through A, and the other through D. Put CN$=x$, NP$=y$, CF or CF$_1=e$ times CA$=ea$, FP$=d$, and F$_1$P$=d_{\prime}$.

Then from the right-angled triangles NPF and NPF$_1$, we have,

$$d^2=\text{NF}^2+\text{NP}^2=(ea-x)^2+y^2$$

$$d_{\prime}^2=\text{NF}_1^2+\text{NP}^2=(ea+x)^2+y^2.$$

Proceeding in the same manner as in deriving the equation to the ellipse, we have,

$$d_{\prime}^2+d^2=2(a^2e^2+x^2+y^2) \ldots (1)$$

$$d_{\prime}^2-d^2 \text{ or } (d_{\prime}+d)(d_{\prime}-d)=4aex.$$

$$\text{But } d_{\prime}-d=2a,$$

$$\therefore\ d_{\prime}+d=2ex;$$

hence by adding and subtracting,

$$d_{\prime}=a+ex, \text{ and } d=ex-a;$$

substituting these values in (1) and reducing, we find,

$$y^2=(e^2-1)(x^2-a^2) \ldots (2)$$

Making $(e^2-1)=\frac{b^2}{a^2}$, or $b^2=a^2(e^2-1)$, in order to sustain the analogy between the equations of the hyperbola and ellipse, we find,

$$y^2=\frac{b^2}{a^2}(x^2-a^2) \ldots (3)$$

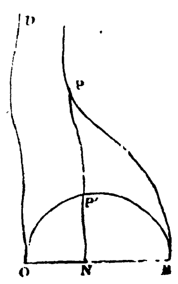
$$\therefore\ \frac{y^2}{b^2}-\frac{x^2}{a^2}=-1 \ldots (4)$$

Cor. Let A be the origin, and put $\text{AN}=x_1$, then $x_1=\text{CN}-\text{AC}=x-a$, and $\therefore\ x=x_1+a$. Substituting this value of x in eq. (3),

$$y^2=\frac{b^2}{a^2}(2ax_1+x_1^2).$$

Equation to the Witch.

22. If OP′B be a semicircle, and NP be taken a *fourth proportional to* ON, OB, and NP′, *then the locus of* P *is the* ***witch.***



Put $\text{ON}=x$, $\text{NP}=y$, $\text{OB}=2r$, then by the equation to the circle $\text{NP}'=\sqrt{2rx-x^2}$;

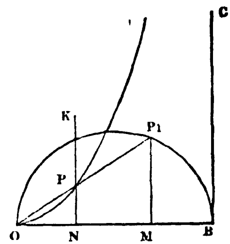
$$\text{but ON : OB :: NP}' \text{ : NP},$$

$$\text{that is, } x : 2r :: \sqrt{2rx-x^2} : y,$$

$$\therefore\ y=\frac{2r\sqrt{2rx-x^2}}{x}=2r\sqrt{\frac{2r-x}{x}}.$$

Equation to the Cissoid.

23. If OP_1B be a semicircle, of which P_1M is any ordinate, take $ON=BM$, and draw NPK perpendicular to OB, and join OP_1 cutting NK in P; then the locus of P is the *cissoid.*



Put $ON=MB=x$, $NP=y$, and $OB=2r$, then from the equation of the circle $MP_1=\sqrt{2rx-x^2}$; but from the similar triangles ONP and OMP_1, we have,

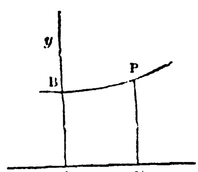
$$ON : NP :: OM : MP_1,$$

$$\text{that is, } x : y :: 2r-x : \sqrt{2rx-x^2};$$

$$\therefore\ y^2=\frac{x^3}{2r-x}.$$

Equation to the Logarithmic Curve.

24. In this curve any abscissa ON is always the logarithm of its corresponding ordinate NP; thus, let $ON=x$, $NP=y$, and a be the base of the system, then



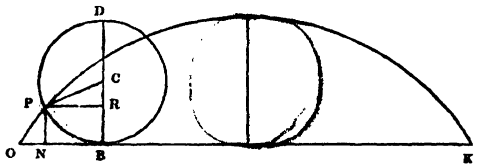
$$x=\log_a y,\ \therefore\ y=a^x.$$

If $x=ON=0$, then y becomes OB, hence $OB=a^0=1$; that is, the ordinate at the origin is always unity.

Cor. If the abscissa increases arithmetically, the ordinate increases geometrically; thus, if $x=1$, $y=a^1$; if $x=2$, $y=a^2$; if $x=3$, $y=a^3$; and so on.

Equation to the Cycloid.

25. *If the circle PBD roll along the straight line OK, a* point P in the circumference of the circle will describe a curve OPK, called the *cycloid.*



Let C be the centre of the circle, BCD a diameter perpendicular to OK; join PC, and draw PR perpendicular to BD, and PN to OK. As the circle is supposed to have rolled from O to B, the arc PB must be equal to OB. Put $\text{ON}=x$, $\text{NP}=y$, $\text{BD}=2r$, $\angle \text{PCB}=\theta$, or what is the same thing, arc of $\angle \text{PCB}$ radius being unity $=\theta$; then we have

$$x=\text{OB}-\text{NB}=\text{arc PB}-\text{PR},$$

$$\text{but arc PB}=\text{rad.}\times\theta=r\theta,$$

$$\text{and PR}=\text{rad. sin }\theta=r\,.\sin\theta.$$

Hence by substitution we have,

$$x=r\theta-r\sin\theta=r(\theta-\sin\theta)\ldots(1)$$

$$y=\text{NP}=\text{BR}=\text{CB}-\text{CR},$$

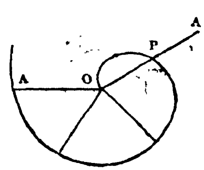
$$\text{but CB}=r,\text{ and CR}=r\cos\theta;$$

$$\therefore\ y=r-r\cos\theta=r(1-\cos\theta)\ldots(2).$$

Equations (1) and (2) express the equation of the cycloid. The arc θ cannot be eliminated between these equations.

Equation to the Spiral of Archimedes.

26. If the line OA revolve uniformly round O as a centre, while a point P moves uniformly from O along OA, then the point P will describe the spiral of Archimedes.



Let $\text{OP}=r$, the radius vector, $\angle \text{POA}=\theta$, the angle described by the radius vector; and let a be the value of r, when the radius vector has made one revolution; then

$$\theta : 2\pi :: r : a; \therefore r = \frac{a}{2\pi} \cdot \theta.$$

ON THE LIMITING VALUES OF QUANTITIES.

27. Any definite quantity multiplied by 0 is equal to 0. For example, let us take the product $10 \times x$, then as we decrease x the product will also be decreased; thus we find, $10 \times \cdot 1 = 1$, $10 \times \cdot 01 = \cdot 1$, $10 \times \cdot 001 = \cdot 01$, $10 \times \cdot 0001 = \cdot 001$, and so on; so that when x is taken 0, the product $10 \times 0 = 0$. And generally $a \times 0 = 0$.

Any definite quantity divided by 0 is infinite or ∞. For example, let us take the fraction $\frac{10}{x}$, then as we decrease x the fraction will be increased; thus we have, $\frac{10}{\cdot 1} = 100$, $\frac{10}{\cdot 01} = 1000$, $\frac{10}{\cdot 001} = 10000$, $\frac{10}{\cdot 0001} = 100000$, and so on; so that when x is taken 0, the fraction $\frac{10}{0} = \infty$ or infinite. And generally $\frac{a}{0} = \infty$.

Any definite quantity multiplied by a quantity infinitely great, is infinite. Let us take the product $10 \times x$, then as we increase x the product will also be increased; thus we have $10 \times 1 = 10$, $10 \times 10 = 100$, $10 \times 100 = 1000$, and so on; so that when x is taken ∞, the product $10 \times \infty = \infty$. And generally $a \times \infty = \infty$.

Any definite quantity divided by a quantity infinitely great is equal to 0. Let us take the fraction $\frac{1}{x}$, then as we increase x the fraction will be decreased; thus we have $\frac{1}{10} = \cdot 1$,

$\frac{1}{100}=\cdot01$, $\frac{1}{1000}=\cdot001$, and so on; so that when x is taken ∞ the fraction $\frac{1}{\infty}=0$. And generally $\frac{a}{\infty}=0$.

The magnitude of any algebraic expression depends upon the value which we assign to the variable contained in it. Thus the magnitude of the expression $3x+4$, will increase or decrease with the value given to variable x. If x be taken infinite, the expression will become infinite; and if x be taken 0, the expression will be equal to 4.

28. *Definition.* The quantity towards which an expression continually approaches or converges, by making the variable continually approach a certain value, is called ***the limiting value*** of the expression.

Illustrations and Applications.

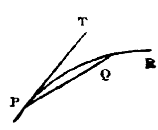
1. Thus the limiting value of $ax+b$ is b, when x approaches 0; because as x decreases the value of ax approaches nearer and nearer to 0.

Again the limiting value of $ax+b$ is ∞, when x approaches ∞; because as x increases the value of ax also increases, and when x becomes very great, or approaches infinity, ax becomes very great or approaches infinity.

2. The limit of $\sin x$ is 0, when x approaches 0; for as x is decreased the smaller and smaller its sine becomes. And similarly the limit of $\tan x$ is 0, when x approaches 0.

3. The limit of $\cos x$ is 1, when x approaches 0; because as the arc x is decreased the nearer and nearer $\cos x$ approaches radius or unity.

4. In the curve PQR, the secant PQ approaches nearer and nearer to the tangent PT as the point Q approaches P; hence the tangent PT is defined to be the limiting position of the secant PQ, when Q approaches P, or what is the same thing, as the $\angle$QPT approaches 0.



5. If a regular polygon be inscribed in a circle, and if another be inscribed having twice the number of sides, it is evident that the surface of the second will approach more nearly to the surface of the circle than that of the first. If the number of sides be continually doubled, the polygon will approach nearer and nearer to the circle, until at length their difference must become less than any quantity that can be assigned; hence the circle is the limit of the inscribed polygon when the number of sides approaches ∞.

6. The limiting value of $\frac{a}{1+x}$ is a, when $x=0$; because for *every* value of x greater than 0 it is less than a, yet it is also to be observed that the nearer x approaches to 0, the nearer does $\frac{a}{1+x}$ approach to a.

Again the limiting value of $\frac{a}{1+x}$ is 0, when $x=\infty$; because as x increases the fraction becomes less and less, and finally when x is greater than any finite quantity, the fraction becomes less than any finite quantity.

It will readily be seen that a is the *actual* value of $\frac{a}{1+x}$ when $x=0$, as well as the *limiting* value. In like manner 0 is the *actual* value of $\frac{a}{1+x}$ when $x=\infty$, as well as the limiting value. In general the *limiting value* of any function, when the variable approaches 0 or ∞, is nothing else, in fact, than the *actual value* of the function when the variable is taken 0 or ∞. This being the case, it may be asked,—why make any distinction between the limiting value of an expression, and its actual value? why say that an expression *approaches* a certain value, when we might more simply say that it *takes* that value? In answer to this, it may be stated, that there are many expressions which assume indeterminate, if not illogical, forms, when the variable *actually* takes a particular value; whereas, in such cases, the limit ing values of the expressions would be perfectly intelligible. Thus, in the expression $\frac{2ax+x^2}{x}$, if we make $x=0$, we find that the expression becomes $\frac{0}{0}$, which has no meaning whatever; however we have by division, $\frac{2ax+x^2}{x}=2a+x$, for *all* values of x;

$$\therefore \text{ the limiting value of } \frac{2ax+x^2}{x}=2a, \text{ when } x=0;$$

thus by resorting to the idea of a limiting value we are enabled to avoid the indeterminate form $\frac{0}{0}=2a$, which we must otherwise have come to. However, for the sake of conciseness, when no ambiguity can arise, we shall hereafter sometimes speak of a limiting value as if it were an actual value.

7. To find the limits of $\frac{a+cx}{x}$.

$$\text{Here } \frac{a+cx}{x}=\frac{a}{x}+c.$$

$$\text{If } x=0, \text{ then } \frac{a}{x}=\frac{a}{0}=\infty;$$

$$\therefore \text{ the limit of } \frac{a+cx}{x}=\infty, \text{ when } x=0.$$

$$\text{If } x=\infty, \text{ then } \frac{a}{x}=\frac{a}{\infty}=0,$$

$$\therefore \text{ the limit of } \frac{a+cx}{x}=c, \text{ when } x=\infty.$$

8. The limit of $\frac{x}{x+a}$ is 1, when $x=\infty$.

Here $\frac{x}{x+a}=\frac{1}{1+\frac{a}{x}}$; and when $x=\infty$, $\frac{a}{x}=0$; hence, &c.

9. The limiting value of $\frac{x^2-a^2}{x-a}$ is $2a$, when $x=a$; because $\frac{x^2-a^2}{x-a}=x+a$, for *all* values of x, and $\therefore$ as x approaches a, the nearer and nearer does $x+a$ approach $2a$.

This example shows that, though the two terms of a ratio may respectively approach 0, yet the limiting value of that ratio may be finite. This is quite in keeping with our most ordinary notions of ratios; for the value of a ratio does not depend upon the *absolute* magnitude of the terms composing it, but upon their *relative* magnitude. Thus one line A may be three times the length of another line B, without regard to their absolute lengths. If the length of A is 3 feet, the length of B will be 1 foot; if the length of A is 3 inches, the length of B will be 1 inch; in fact, if the length of A were inconceivably small, the length of B would be still one-third of that inconceivably small length.

10. Required the limiting value of $1+\frac{1}{x}+\frac{1}{x^2}+\ldots$ to n terms, when $n=\infty$.

Here (Alg., page 92.) the sum of this geometrical series is $\frac{1-\frac{1}{x^n}}{1-\frac{1}{x}}$; but when $n=\infty$, $\frac{1}{x^n}=\frac{1}{\infty}=0$, and hence the limiting value becomes $\frac{1}{1-\frac{1}{x}}=\frac{x}{x-1}$, which is the sum of the proposed series continued *in infinitum*.

11. Required the limiting value of $\frac{1}{n^2}(1+2+3+\ldots+n)$ when $n=\infty$.

Here (Alg., page 89.) $1+2+3+\ldots+n=(n+1)\frac{n}{2}$;

$\therefore$ the proposed expression $=\frac{1}{n^2}\times(n+1)\frac{n}{2}=\frac{1}{2}+\frac{1}{2n}$, for all values of n.

But when $n=\infty$, $\frac{1}{2}+\frac{1}{2n}=\frac{1}{2}+\frac{1}{\infty}=\frac{1}{2}$, which is the limiting value required.

12. Required the limiting value of $\frac{1}{n^3}(1^2+2^2+\ldots+n^2)$ when $n=\infty$.

By Art. **10.**, $1^2+2^2+\ldots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$;

$\therefore$ the proposed expression

$$=\frac{1}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)=\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}.$$

But when $n=\infty$, this becomes $=\frac{1}{3}$.

13. Required the limiting value of $\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$ when h approaches 0.

By the binomial theorem,

$$\left(1+\frac{h}{x}\right)^{\frac{x}{h}}=1+\frac{x}{h}\cdot\frac{h}{x}+\frac{\frac{x}{h}\left(\frac{x}{h}-1\right)}{1.2}\cdot\frac{h^2}{x^2}+\&c.$$

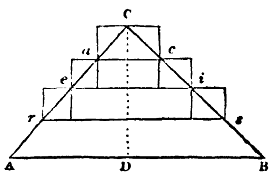
$$=1+1+\frac{\left(1-\frac{h}{x}\right)}{1.2}+\frac{\left(1-\frac{h}{x}\right)\left(1-\frac{2h}{x}\right)}{1.2.3}+\&c.$$

Now, when $h=0$, $\frac{h}{x}=\frac{0}{x}=0$, and hence the limiting value of this expression,

$$=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{ \&c.}$$

$$=2{\cdot}7182, \text{ \&c.}=e, \text{ by Art. 8. } Cor. \text{ 1.}$$

14. To find the area of the triangle ABC, by the method of limits.



Put a=the base AB, and p =the perpendicular CD. Let CD be divided into n equal parts, and upon each of these parts let circumscribed and inscribed rectangles be drawn; then the area of the triangle will be less than the sum of the circumscribed rectangles, and greater than the sum of the inscribed ones.

The height of each of these rectangles will be $\frac{p}{n}$, the base ac of the first will be $\frac{a}{n}$, the base ei of the second $\frac{2a}{n}$, the base rs of the third $\frac{3a}{n}$, and so on; the base of the nth being $\frac{na}{n}$.

$\therefore$ the sum of the circumscribed rectangles

$$=\frac{a}{n}\cdot\frac{p}{n}+\frac{2a}{n}\cdot\frac{p}{n}+\frac{3a}{n}\cdot\frac{p}{n}+\ldots+\frac{na}{n}\cdot\frac{p}{n}$$

$$=\frac{ap}{n^2}(1+2+3+\ldots+n)=\frac{ap}{2}\left(1+\frac{1}{n}\right).$$

In like manner, the sum of the inscribed rectangles,

$$=0+\frac{a}{n}\cdot\frac{p}{n}+\frac{2a}{n}\cdot\frac{p}{n}+\frac{3a}{n}\cdot\frac{p}{n}+\ldots+\frac{(n-1)a}{n}\cdot\frac{p}{n}$$

$$=\frac{ap}{n^2}\{1+2+3+\ldots+(n-1)\}=\frac{ap}{2}\left(1-\frac{1}{n}\right).$$

Now the area of the triangle lies between these two areas, whatever may be the value assigned to n. But when the number of parts into which CD is divided is increased without limit, or, what is the same thing, when $n = \infty$ in the above expressions, the areas of the circumscribed and inscribed rectangles become equal to one another; that is, $= \frac{ap}{2}$. Therefore the area of the triangle $= \frac{ap}{2} = \frac{\text{AB} \cdot \text{CD}}{2}$.

15. To find the space s described by a falling body in a given time t.

Gravity being a constantly acting force, it adds equal increments of velocity to a descending body in equal intervals of time. Thus at the end of one second it is found that the velocity acquired is $32\frac{1}{6}$ ft.; at the end of 2 seconds, 2 times $32\frac{1}{6}$ ft.; at the end of 3 seconds, 3 times $32\frac{1}{6}$ ft.; and so on, the velocity increasing with the time. Hence if v be put for the velocity acquired at the end of t seconds, and g be put for $32\frac{1}{6}$, we have the general relation,

$$v = tg \ldots (1).$$

If the body be projected vertically downwards with any given velocity, it is evident that we must add this velocity to that which is due to gravity, to obtain the total velocity of the body at any instant of its descent.

Let V be the velocity with which a body is projected vertically downwards, and let t be divided into n equal intervals, each being equal to $\frac{t}{n}$; then as v is the velocity acquired in t seconds, the velocity communicated to the body by gravity in each interval will be the nth part of $v = \frac{v}{n}$. Hence the velocity at the commencement of the motion will be V; at the end of the 1st interval $\text{V} + \frac{v}{n}$; at the end of the 2d interval $\text{V} + \frac{2v}{n}$; and so on. Now let us suppose that the motion of the body is uniform between each interval. then taking the velocities at the *beginning* of each interval, the space described in the time t

$$=\mathrm{v}\times\frac{t}{n}+\left(\mathrm{v}+\frac{v}{n}\right)\frac{t}{n}+\left(\mathrm{v}+\frac{2v}{n}\right)\frac{t}{n}+\ldots \text{ to } n \text{ terms}$$

$$=t\mathrm{v}+\frac{tv}{n^2}\{1+2+3+\ldots \text{ to } (n-1) \text{ terms}\}=t\mathrm{v}+\frac{tv}{2}\left(1-\frac{1}{n}\right)$$

This result is evidently *less* than the actual space s which the body will describe.

Now taking the velocities at the *end* of each interval, the space described in the time t

$$=\left(\mathrm{v}+\frac{v}{n}\right)\frac{t}{n}+\left(\mathrm{v}+\frac{2v}{n}\right)\frac{t}{n}+\left(\mathrm{v}+\frac{3v}{n}\right)\frac{t}{n}+\ldots \text{ to } n \text{ terms}$$

$$=t\mathrm{v}+\frac{tv}{2}\left(1+\frac{1}{n}\right).$$

This result is evidently *greater* than the actual space s which the body will describe.

Now the space s lies between these two spaces, whatever may be the value assigned to n. But when the intervals are increased without limit, or, what is the same thing, when $n=\infty$, these two expressions for the spaces become equal to one another, that is $=t\mathrm{v}+\frac{tv}{2}$.

$$\therefore\ \mathrm{s}=t\mathrm{v}+\frac{tv}{2}\ \ldots\ (2).$$

Substituting the value of v given in eq. (1),

$$\mathrm{s}=t\mathrm{v}+t^2\times\frac{g}{2}\ \ldots\ (3).$$

If $\mathrm{v}=0$, or if the body fall from a state of rest, then we have,

$$\mathrm{s}=t^2\times\frac{g}{2}\ \ldots\ (4).$$

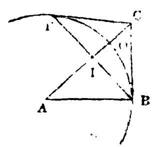
16. To find the limiting relation of tan h to sin h, when the arc h approaches 0.

Here we always have, $\frac{\tan h}{\sin h}=\frac{1}{\cos h}$.

Now when h is indefinitely decreased, the value of cos h $=1$;

∴ the limiting value of $\frac{\tan h}{\sin h}=1$, when h approaches 0; that is, sin h and tan h tend to an equality as h approaches 0.

17. To show that the arc of a circle is greater than its sine and less than its tangent.



Let BF be any arc of a circle whose radius is AB. Draw BC and FC tangents to the points B and F; join AC cutting the circle in the point O; and join BF cutting AC in I; then BC=FC, BI=FI, and arc BO=arc FO.

Now Art. 73. Geo.,

$$\text{arc FOB} > \text{FIB} < \text{FC}+\text{CB}$$

$$\therefore\ 2 \text{ arc BO} > 2\ \text{BI} < 2\ \text{BC}$$

$$\therefore\ \text{arc BO} > \ \text{BI} < \text{BC}.$$

But BI is the sine of the arc BO, and BC is its tangent: therefore the arc is greater than its sine, and less than its tangent; or putting h for arc BO,

$$\therefore\ h > \sin h < \tan h.$$

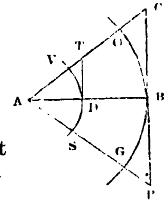
Let us now enquire what this relation becomes when the arc approaches 0.

Now it has been shown in the last problem, that sin h and tan h tend to an equality as h approaches 0; but as h *always* lies between sin h and tan h, we have for the limiting values when h approaches 0,

$$h=\sin h=\tan h;$$

or the sine and tangent of an arc, in their limiting state, are in a ratio of equality with the arc itself.

18. To find the circumference of a circle whose radius is r.



Let A be the centre of the circle GBO, &c., and PC the side of a regular circumscribed polygon of n sides, touching the circle in the point B. Let SDV be another circle whose radius AD$=1$, and DT a tangent to the point D, cutting AC in T; then, if we put 2π for the circumference of this circle, the arc VDS$=\frac{2\pi}{n}$, $\therefore$ arc DV$=\frac{1}{2}$ of $\frac{2\pi}{n}=\frac{\pi}{n}$, and DT$=\tan\frac{\pi}{n}$. From the similar triangles ADT and ABC, we have,

$$\text{AD} : \text{DT} :: \text{AB} : \text{BC}, \text{ or}$$

$$1 : \tan\frac{\pi}{n} :: r : \text{BC}=r\tan\frac{\pi}{n};$$

$$\therefore \text{PC}=2\text{BC}=2r\tan\frac{\pi}{n};$$

$$\therefore \text{the perimeter of the polygon}=2r\tan\frac{\pi}{n}\times n.$$

Now if n be continually increased, the arc $\frac{\pi}{n}$ will be decreased, and the sides of the polygon will approach nearer and nearer to the circumference of the circle.

$$\therefore \text{Circum. circle}=\text{limiting value of } 2r\tan\frac{\pi}{n}\times n,$$

when $n=\infty$, or $\frac{\pi}{n}=0$; but by the preceding problem $\tan\frac{\pi}{n}$ approaches $\frac{\pi}{n}$ as $\frac{\pi}{n}$ approaches 0;

$$\therefore \text{Circum. circle}=\text{limiting value of } 2r\frac{\pi}{n}\times n=2r\pi,$$

where 2π is put for the circumference of a circle whose radius is 1. Hence *it follows that the circumferences of circles are to one another as their radii.*

19. To find the area of a circle.

Adopting the notation and fig. of the preceding problem, we have

$$\text{area triangle PAC} = \text{AB} \cdot \text{BC} = r^2 \tan \frac{\pi}{n}.$$

But there are n of these triangles making up the polygon,

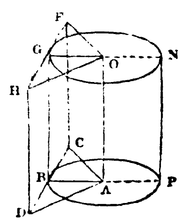
$$\therefore \text{ area polygon} = nr^2 \tan \frac{\pi}{n}.$$

Now when $n = \infty$, $\tan \frac{\pi}{n} = \frac{\pi}{n}$, and then the polygon coincides with the circle, or, more strictly speaking, the limit of the polygon is the circle, when $n = \infty$; $\therefore$ taking the limits of both sides of the equality, we have,

$$\text{area circle} = nr^2 \frac{\pi}{n} = r^2 \pi = \left(\frac{d}{2}\right)^2 \pi = d^2 \frac{\pi}{4}.$$

Hence the areas of circles are to one another as the squares of their radii or diameters.

20. To find the solidity and surface of a right cylinder, PBGN.



Let DCFH be the side of a prism of n equal sides, circumscribed about the cylinder; put r = the radius AB, and h = the perpendicular height AO; then by problem 19, area triangle $\text{ADC} = r^2 \tan \frac{\pi}{n}$;

$$\therefore \text{ solidity prism ADCFHO} = \text{area base ADC} \times \text{perpen. AO}$$
$$= r^2 \tan \frac{\pi}{n} \times h.$$

But the solidity of the whole circumscribed prism will be n times this result;

$$\therefore \text{ solidity circum. prism} = nr^2 \tan \frac{\pi}{n} \times h.$$

Now when $n=\infty$, tan $\frac{\pi}{n}=\frac{\pi}{n}$, and the limit of the circumscribed prism is the cylinder;

$$\therefore \text{ solidity cylinder} = \text{limiting value of } nr^2 \tan\frac{\pi}{n}\times h$$

$$= nr^2 \frac{\pi}{n}\times h = r^2\pi h.$$

That is, the solidity of the cylinder is equal to the area of the base multiplied by the perpendicular height.

Again, the area of the face DCFH $=$ DC $\times$ DH $=2$ BC $\times$ AO $=2r\tan\frac{\pi}{n}\times h$.

But there are n side faces in the whole circumscribed prism.

$\therefore$ Surface in the side faces of the prism $=2nr\tan\frac{\pi}{n}\times h$.

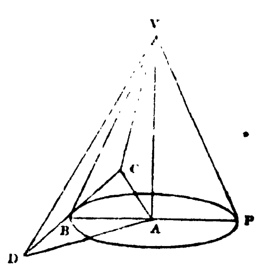
Hence we have, by taking the limits as before,

$$\text{convex surface cylinder} = 2nr\frac{\pi}{n}h = 2r\pi h.$$

Now, by problem 18., $2r\pi$ is the circumference of the base; therefore the convex surface of a right cylinder is equal to the circumference of the base multiplied by the perpendicular height.

21. To find the solidity and surface of a right cone PBV.

Let DCV be one of the faces of a pyramid of n equal sides, circumscribed about the cone; put $r=$ AB, the radius of the base, and $h=$ AV, the perpendicular height; then, proceeding as in the last problem, we have,



$$\text{area triangle DAC} = r^2\tan\frac{\pi}{n};$$

$\therefore$ solidity pyramid ADCV $= \frac{1}{3} \times$ area base $\times$ perpend. height $= \frac{1}{3} r^2 \tan \frac{\pi}{n} \times h$.

But the solidity of the whole circumscribed pyramid will be n times this result;

$$\therefore \text{ solidity circum. pyramid} = \tfrac{1}{3} n r^2 \tan \frac{\pi}{n} \times h.$$

Now when $n = \infty$, $\tan \frac{\pi}{n} = \frac{\pi}{n}$, and the limit of the circumscribed pyramid is the cone;

$$\therefore \text{ solidity cone} = \text{limiting value of } \tfrac{1}{3} n r^2 \tan \frac{\pi}{n} \times h$$

$$= \tfrac{1}{3} n r^2 \frac{\pi}{n} \times h = \tfrac{1}{3} r^2 \pi h.$$

Hence the solidity of a cone is equal to one third the area of the base multiplied by the perpendicular height.

Again, to find the convex surface of the cone, put $s =$ BV, the slant height, then we have,

$$\text{area face DCV} = \tfrac{1}{2} \text{DC} \times \text{BV} = \text{BC} \times \text{BV} = r \tan \frac{\pi}{n} \times s;$$

$\therefore$ whole surface in the side faces of the circumscribed pyramid $= n r \tan \frac{\pi}{n} \times s$.

Therefore, taking the limit as before, we have,

$$\text{convex surface cone} = n r \frac{\pi}{n} s = r \pi s.$$

But $r\pi$ is equal to one half the circumference of the base; therefore the convex surface of the cone is equal to one half the circumference of the base multiplied by the slant height.

INCREMENTS AND THEIR LIMITING RATIO.

29. When the variable in an expression undergoes an *increase*, or takes an *increment*, the expression itself necessarily undergoes a change, and the quantity by which it is thus increased is called its increment.

Thus if x, in the expression ax, takes the increment h, then ax will become $a(x+h)$, and

$$\therefore \text{ the increment of } ax = a(x+h) - ax = ah,$$

that is, if x be increased by h, the function ax will be increased by a times h.

Let $f(x) = \frac{x^2}{x+1}$, then if x takes the increment h, the function $f(x)$ will become $\frac{(x+h)^2}{x+h+1}$, and

$$\therefore \text{ the increment of } f(x) = \frac{(x+h)^2}{x+h+1} - \frac{x^2}{x+1}.$$

In general, if x, in the function $f(x)$, takes the increment h, then $f(x)$ will become $f(x+h)$, and

$$\therefore \text{ the increment of } f(x) = f(x+h) - f(x).$$

In the following exercises, &c. we shall invariably suppose h to be the increment of the independent variable; and, for the sake of conciseness, we shall write "Incr. $f(x)$" for "The increment of $f(x)$," and "Incr. y" for "The increment of y."*

Ex. 1. If x receives the increment h, what will be the increment or increase of $y = ax^2$?

* In the calculus of finite differences, the symbol Δy is used for expressing Incr. y, or, as it is called, the difference of y.

Here when x becomes $x+h$, the function ax^2 becomes $a(x+h)^2$;

$$\therefore \text{ incr. } y = a(x+h)^2 - ax^2 = 2axh + ah^2.$$

2. What will be the limiting ratio of incr. y to incr. x, in the last example, when h or incr. x approaches 0?

Here, dividing each side of the last equality by h,

$$\frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x} = 2ax + ah.$$

Now this equation must hold true whatever may be the value given to h. If h be taken very small, ah will also be very small; in fact, if we suppose h to become smaller and smaller the nearer and nearer will $2ax+ah$ approach $2ax$;

$$\therefore \text{ *the limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = 2ax.$$

3. If x receives the increment h, required the increment of $y = x^2 - 3x + 2$.

Here when x becomes $x+h$, the function y becomes $(x+h)^2 - 3(x+h) + 2$;

$$\therefore \text{ incr. } y = (x+h)^2 - 3(x+h) + 2 - \{x^2 - 3x + 2\}$$
$$= (2x-3)h + h^2.$$

4. Required the limiting ratio of incr. y to incr. x, in the last example.

Dividing each side of the last equality by h, we have,

$$\frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x} = 2x - 3 + h.$$

When h approaches 0, we have,

* It must be observed that a limiting value of a ratio is not a mere approximation; for while we speak of h approaching 0, or approaching it as nearly as we please, we do this merely to aid our conception of the terms of the ratio, yet we actually take $h=0$ in finding the value of the limit of the ratio.

$$\text{limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = 2x - 3.$$

5. Required the increment of $y = \frac{3x}{x+2}$, when x becomes $x+h$.

$$\text{Incr. } y = \frac{3(x+h)}{x+h+2} - \frac{3x}{x+2} = \frac{6h}{(x+h+2)(x+2)}.$$

6. Required the limiting ratio of the increments in the last example.

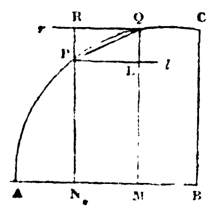
Dividing each side of the last equality by h,

$$\frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x} = \frac{6}{(x+h+2)(x+2)}.$$

Therefore when h approaches 0, we have,

$$\text{limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = \frac{6}{(x+0+2)(x+2)} = \frac{6}{(x+2)^2}.$$

7. In the curve APQ, let AN($=x$), and NP($=y$) be the co-ordinates of the point P; and let $y=4x^2$ be the equation to the curve. Required the limiting ratio of the increment of NP to the increment of AN.



Let NP move parallel to itself until it comes to the position MQ. Draw PL parallel to AN, then if NM($=h$) be the increment of AN or x, LQ will be the increment of NP or y. By the equation to the curve, we have,

$$\text{NP} = 4\text{AN}^2, \text{ and } \text{MQ} = 4\text{AM}^2$$
$$= 4x^2 \qquad\qquad = 4(x+h)^2;$$

$$\therefore \text{LQ} = \text{MQ} - \text{NP} = 4(x+h)^2 - 4x^2 = 8xh + 4h^2.$$

Dividing each side of this equality by h,

$$\frac{\text{LQ}}{h} \text{ or } \frac{\text{incr. NP}}{\text{incr. AN}} = 8x + 4h.$$

Now these increments will have a definite ratio, however small h may be taken, or, what is the same thing, however near MQ may be taken to NP. When h approaches 0, we have the limiting value of $\frac{\text{incr. NP}}{\text{incr. AN}}=8x$.

8. Let $y=x^2-\frac{1}{x}$. Required the limiting ratio of incr. y to incr. x, or the limit of $\frac{\text{incr. } y}{\text{incr. } x}$.

Here, when x becomes $x+h$, then y becomes $(x+h)^2-\frac{1}{x+h}$,

$$\therefore \text{ incr. } y=(x+h)^2-\frac{1}{x+h}-\left\{x^2-\frac{1}{x}\right\}=2xh+h^2+\frac{h}{(x+h)x};$$

$$\therefore \quad \frac{\text{incr. } y}{h} \text{ or } \frac{\text{incr. } y}{\text{incr. } x}=2x+h+\frac{1}{(x+h)x}.$$

Therefore when h approaches 0, we have,

$$\text{limiting value of } \frac{\text{incr. } y}{\text{incr. } x}=2x+0+\frac{1}{(x+0)x}=2x+\frac{1}{x^2}.$$

Exercises for the Student.

1. If x receives the increment of h, required the increments of the following functions, $5x^2+2$, $7x^2-6x$, ax^2-bx+c, ax^3-c.

Answers. $10xh+5h^2$, $(14x-6)h+7h^2$, $(2ax-b)h+ah^2$, $3ax^2h+3axh^2+ah^3$.

2. If the side of a square be x, and it be increased by h, by what quantity will the surface of the square be increased? *Ans.* $2xh+h^2$.

3. If the radius of a circle be x, and it be increased by h, what will be the increment of the circle? *Ans.* $(2xh+h^2)\pi$.

4. In *Ex.* 2., what will be the increment given to the diagonal? *Ans.* $h\sqrt{2}$.

5. Required the increments of the following functions, $\frac{1-a}{x}$, $\frac{2x}{a+x}$, $(1+x)^2$.

Answers. $\frac{(a-1)h}{x(x+h)}$, $\frac{2ah}{(a+x+h)\,(a+x)}$, $2(1+x)h+h^2$.

6. Required the limiting ratio of the increment of the function to the increment of the variable, in the following functions, $2x^2$, x^3-ax, $5x^2+a$, $3x^4+c$, x^3+x^2-ax+c, $\frac{x+a}{x}$, $x-\frac{3}{x}$.

Answers. $4x$, $3x^2-a$, $10x$, $12x^3$, $3x^2+2x-a$, $\frac{-a}{x^2}$, $1+\frac{3}{x^2}$.

7. Required the limiting ratio of the increments in examples 2 and 3. *Ans.* $2x$, $2x\pi$.

8. Let $y=x^2-3x+c$, be the equation of a curve, and x receive the increment h, what will be the increment of the ordinate y? *Ans.* $(2x-3)h+h^2$.

9. Required the limiting ratio of the increment of y to the increment of x, in the last example. *Ans.* $2x-3$.

DIFFERENTIAL CALCULUS.

NOTATION OF THE DIFFERENTIAL CALCULUS.

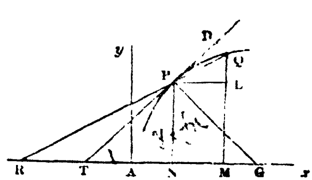
30. It now becomes desirable that we should have some notation for expressing the limiting ratio of two simultaneous increments. It matters not what this notation is, provided it expresses the thing signified without ambiguity, at the same time that it admits of being readily extended to all cases which can arise, without any restriction as to its interpretation.

31. In *Exs.* 3. and 4., p. 37., it is shown that the limiting value of $\frac{\text{incr. } y}{\text{incr. } x}=2x-3$, where $y=x^2-3x+2$. Now if dx

be put for the incr. x or h in the limiting value; then, in order to maintain the uniformity of notation, dy must be put for incr. y. Thus $\frac{dy}{dx}=2x-3$, or, putting in the value of y, $\frac{d(x^2-3x+2)}{dx}=2x-3$. By $\frac{dy}{dx}$, therefore, we simply mean the limiting value of $\frac{\text{incr. } y}{\text{incr. } x}$, or the limiting value of the ratio of the increment of y to the increment of x, when the increment of x approaches 0. As y is here understood to be a function of x, we also have $\frac{df(x)}{dx}=$ limiting value of $\frac{\text{incr. } f(x)}{\text{incr. } x.}=\frac{f(x+h)-f(x)}{h}$, when $h=0$. Now the limit of $\frac{f(x+h)-f(x)}{h}$ is in general a finite quantity, if the division by h is performed before h is made 0.

Geometrical representation of $\frac{df(x)}{dx}$, or the limiting value of $\frac{\text{incr. } f(x)}{\text{incr. } x}$.

32. Let $AN=x$, and $NP=y$, be the co-ordinates of the point P in the curve PQ. Take a point Q in the curve near to P, its co-ordinates being AM and MQ; from P draw PL parallel to Ax; draw the chord QP, and produce it to meet the axis in R.



Let $y=f(x)$ be the equation to the curve, then when AN or x takes the increment $h=NM$, NP or y will take the increment LQ.

$$\therefore \ NP=f(x), \text{ and } MQ=f(x+h);$$

$$\therefore \text{LQ}=\text{MQ}-\text{NP}=f(x+h)-f(x);$$

$$\therefore \frac{\text{LQ}}{\text{PL}}, \text{ or } \frac{\text{LQ}}{h}=\frac{f(x+h)-f(x)}{h};$$

$$\text{but } \frac{\text{LQ}}{\text{PL}}=\tan \text{LPQ}=\tan \text{NRP};$$

$$\therefore \tan \text{NRP}=\frac{f(x+h)-f(x)}{h} \text{ or } \frac{\text{incr. } f(x)}{\text{incr. } x}.$$

Now this equality is true for all values of h: when the point Q approaches P, the value of h or incr. x approaches 0, the secant QPR approaches nearer and nearer to the tangent PT, and therefore the ∠ QPD approaches 0, while the ∠ NRP approaches nearer and nearer to an equality with the ∠ NTP; hence we have, taking the limits of the above equation,

$$\tan \text{NTP}=\text{limiting value of } \frac{\text{incr. } f(x)}{\text{incr. } x}$$

$$=\frac{df(x)}{dx} \text{ or } \frac{dy}{dx} \quad . \; . \; . \quad (1),$$

which equation in general gives us the geometrical interpretation of the *differential coefficient* $\frac{dy}{dx}$.

33. This investigation leads us to the following *definition of a tangent to a curve.* The line PT is said to be the tangent at P, when PT is the limiting position of the secant QPR, on the supposition that Q approaches P, or, what is the same thing, when the ∠ QPD approaches 0.

34. It is necessary that we should have names given to these symbols, in order to speak of them with precision; thus dx is called the *differential* of x, and dy the *differential** of y. The symbol $\frac{dy}{dx}$ is called the *differential coeffi-*

* In the calculus of differences the increment of y is called the first *difference* of y; hence, in the differential calculus, this difference taken indefinitely small, has been called the *differential* of y.

cient, and the *process* by which $\frac{dy}{dx}$ is obtained is called *differentiation.* The letter *d*, therefore, placed before any expression or function, indicates that the function is to be *differentiated*, so that *d* is a symbol of an operation, and not of a quantity.

In the examples given in Art. **29.**, the limiting ratios there found are the values of the differential coefficients of the respective functions; thus, in *Ex.* 6. $\frac{dy}{dx}=\frac{6}{(x+2)^2}$, and so on.

35. The first object of the differential calculus is to determine rules for finding $\frac{df(x)}{dx}$, or the limiting value of $\frac{f(x+h)-f(x)}{h}$, when h approaches 0; and then to show the use of these limiting values in the solution of various problems in pure and mixed mathematics.

RULES FOR THE DIFFERENTIATION OF FUNCTIONS.

36. *Rule* 1. A constant quantity, connected with a function of x by the process of multiplication or division, remains as a multiplier or divisor after differentiation.

$$\text{Let } y=ax^3, \text{ then } \frac{dy}{dx}=3ax^2.$$

$$\text{Incr. } y=a(x+h)^3-ax^3=3ax^2h+3axh^2+ah^3;$$

$$\therefore \frac{\text{incr. } y}{h}=3ax^2+3axh+ah^2;$$

$$\therefore \text{ limiting value of } \frac{\text{incr. } y}{\text{incr. } x}=3ax^2,$$

$$\text{that is, } \frac{dy}{dx}=3ax^2.$$

$$\text{Let } y=\frac{1}{c}x^2, \text{ then } \frac{dy}{dx}=\frac{2x}{c}.$$

$$\text{Incr. } y=\frac{1}{c}(x+h)^2-\frac{1}{c}x^2=\frac{2xh}{c}+\frac{h^2}{c};$$

$$\therefore \frac{\text{incr. } y}{h}=\frac{2x}{c}+\frac{h}{c};$$

$$\therefore \text{ limiting value of } \frac{\text{incr. } y}{\text{incr. } x}=\frac{2x}{c},$$

$$\text{that is, } \frac{dy}{dx}=\frac{2x}{c}.$$

Generally, let $y=af(x)$, then $\dfrac{dy}{dx}=a\dfrac{df(x)}{dx}$.

$$\text{Incr. } y=af(x+h)-af(x)=a\{f(x+h)-f(x)\}$$
$$=a \text{ incr. } f(x).$$

Dividing by h or incr. x,

$$\frac{\text{incr. } y}{\text{incr. } x}=a\frac{\text{incr. } f(x)}{\text{incr. } x}.$$

This equation being true for all values of h or incr. x, it will also be true when h approaches 0; therefore, taking the limiting values, by putting the ratios of the differentials for the ratios of the increments, we have,

$$\frac{dy}{dx}=a\frac{df(x)}{dx}.$$

37. *Rule* 2. A constant quantity, connected with a function of x by the sign of addition or subtraction, disappears after differentiation.

Let $y=x^2+c$, then $\dfrac{dy}{dx}=2x$.

$$\text{Incr. } y=(x+h)^2+c-(x^2+c)=2xh+h^2;$$

$$\therefore \frac{\text{incr. } y}{h}=2x+h;$$

$$\therefore \text{ limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = 2x,$$

$$\text{that is, } \frac{dy}{dx} = 2x.$$

$$\text{Let } y = ax - c, \text{ then } \frac{dy}{dx} = a.$$

$$\text{Incr. } y = a(x+h) - c - (ax - c) = ah\,;$$

$$\therefore \frac{\text{incr. } y}{h} = a, \therefore \frac{dy}{dx} = a.$$

$$\text{Generally, let } y = f(x) + c, \text{ then } \frac{dy}{dx} = \frac{df(x)}{dx}.$$

$$\text{Incr. } y = f(x+h) + c - \{f(x) + c\} = f(x+h) - f(x)\,;$$

$$\therefore \frac{\text{incr. } y}{h} = \frac{f(x+h) - f(x)}{h}.$$

Taking the limiting values when h approaches 0,

$$\frac{dy}{dx} = \frac{df(x)}{dx}.$$

38. *Rule* 3. To obtain the differential coefficient of any constant power of x, multiply together the exponent, and x with its exponent diminished by unity.

Thus if $y = x^4$, then $\frac{dy}{dx} = 4x^3$.

$$\text{Let } y = x^n, \text{ then } \frac{dy}{dx} = nx^{n-1}.$$

$$\text{Incr. } y = (x+h)^n - x^n, \text{ by the binomial theorem,}$$

$$= nx^{n-1}h + \frac{n(n-1)}{1\,.\,2}x^{n-2}h^2 + \&c.$$

$$\therefore \frac{\text{incr. } y}{h} = nx^{n-1} + \frac{n(n-1)}{1\,.\,2}x^{n-2}h + \&c.$$

$$\therefore \text{ limiting value of } \frac{\text{incr. } y}{\text{incr. } x} = nx^{n-1},$$

$$\text{that is, } \frac{dy}{dx} = nx^{n-1}.$$

Multiplying by dx, we also have, $dy = nx^{n-1}dx$, which is an expression for the differential of x^n. This form is convenient for algebraic calculation; strictly speaking, however, the dx should not be separated from the dy.

This result is true, whether n be a whole number, a fraction, or a minus quantity.

EXAMPLES.

1. Let $y=3x^7$, then $\dfrac{dy}{dx}=3\times 7\times x^{7-1}=21x^6$. (See *Rule* 1.)

2. If $y=\frac{2}{3}ax^5+c$, required the differential coefficient, or the value of $\dfrac{dy}{dx}$. (See *Rule* 2.) *Ans.* $\frac{10}{3}ax^4$.

3. What is the differential coefficient of $\dfrac{3}{a}(2x^3-1)$?

Ans. $\dfrac{18x^2}{a}$.

4. Let $y=2x^{\frac{3}{2}}$, then $\dfrac{dy}{dx}=2\times\frac{3}{2}\times x^{\frac{3}{2}-1}=3x^{\frac{1}{2}}$, because *Rule* 3. is true for fractional indices.

5. Required the differential coefficient of $4x^{\frac{1}{2}}$.

$$\text{Here, } \frac{d(4x^{\frac{1}{2}})}{dx}=4\times\tfrac{1}{2}\times x^{\frac{1}{2}-1}=2x^{-\frac{1}{2}}=\frac{2}{x^{\frac{1}{2}}}.$$

6. Required the value of $\dfrac{d(6x^{\frac{2}{3}}+5)}{dx}$. *Ans.* $\dfrac{4}{x^{\frac{1}{3}}}$.

7. If $y=\dfrac{a}{x^2}=ax^{-2}$, then $\dfrac{dy}{dx}=-2ax^{-2-1}=-\dfrac{2a}{x^3}$; because *Rule* 3. is true for minus indices.

8. Required the differential coefficients of the following functions, $ax^{\frac{5}{2}}$, $6x^{\frac{1}{3}}+c$, $\dfrac{3a}{x^3}$, and $3a\left(b-\dfrac{1}{x^{\frac{1}{3}}}\right)$.

Answers. $\dfrac{5}{2}ax^{\frac{3}{2}}$, $\dfrac{2}{x^{\frac{2}{3}}}$, $-\dfrac{9a}{x^4}$, and $\dfrac{a}{x^{\frac{4}{3}}}$.

9. What is the differential of ax^{n-2}?

Let $y=ax^{n-2}$, then $\frac{dy}{dx}=a(n-2)x^{n-3}$; multiplying each side of the equality by dx, we have, $dy=a(n-2)x^{n-3}dx$.

10. Required the differentials of the following functions, $\frac{2}{3}x^6$, $\frac{5}{6}x^{\frac{6}{5}}$, $\frac{3}{4}x^{\frac{5}{4}}$, $a-\frac{1}{x^5}$, $9\sqrt[3]{x}$.

Answers. $4x^5dx$, $x^{\frac{1}{5}}dx$, $\frac{15}{16}x^{\frac{1}{4}}dx$, $\frac{5dx}{x^6}$, $\frac{3dx}{x^{\frac{2}{3}}}$.

39. *Rule* 4. The differential coefficient of the sum of any functions is equal to the sum of the several differential coefficients of the functions; and the differential of the sum of the functions is equal to the sum of the differentials of the functions.

Let $y=ax+bx^2$, then $\frac{dy}{dx}=a+2bx$, and $dy=adx+2bxdx$.

$$\text{Incr. } y=a(x+h)+b(x+h)^2-(ax+bx^2)$$
$$=ah+2bxh+bh^2,$$
$$\therefore \frac{\text{incr. } y}{h}=a+2bx+bh$$
$$\therefore \text{ limiting value of } \frac{\text{incr. } y}{\text{incr. } x}=a+2bx,$$
$$\text{that is, } \frac{dy}{dx}=a+2bx;$$
$$\therefore dy=adx+2bxdx.$$

Where adx is the differential of ax, and $2bxdx$ is the differential of bx^2; hence the rule as applied to this case.

$$\text{Generally, let } y=f(x)\pm F(x)\pm \&c.$$
$$\text{Incr. } y=f(x+h)\pm F(x+h)\pm \&c.-\{f(x)\pm F(x)\pm \&c.\}$$
$$=f(x+h)-f(x)\pm \{F(x+h)-F(x)\}\pm \&c.$$
$$=\text{incr. } f(x)\pm \text{incr. } F(x)\pm \&c.;$$

$$\therefore \quad \frac{\text{incr. } y}{h} = \frac{\text{incr. } f(x)}{h} \pm \frac{\text{incr. F}(x)}{h} \pm \text{\&c.}$$

This equation being true for all values of h, it will also be true when h approaches 0; therefore taking the limiting values, by putting the ratios of the differentials for the ratios of the increments, we have,

$$\frac{dy}{dx} = \frac{df(x)}{dx} \pm \frac{d\text{F}(x)}{dx} \pm \text{\&c.},$$

$$\text{or } dy = df(x) \pm d\text{F}(x) \pm \text{\&c.}$$

Examples.

1. Let $y = x^n + ax^m + c$, then $\dfrac{dy}{dx} = nx^{n-1} + max^{m-1}$, and $dy = nx^{n-1}dx + max^{m-1}dx$.

2. What is the differential coefficient of $ax^2 - x^{\frac{2}{3}}$?

Let $y = ax^2 - x^{\frac{2}{3}}$, then $\dfrac{dy}{dx} = 2ax^{2-1} - \frac{2}{3}x^{\frac{2}{3}-1} = 2ax - \frac{2}{3} \cdot \dfrac{1}{x^{\frac{1}{3}}}$.

3. What is the differential coefficient of $x^{\frac{1}{2}}\left(x^{\frac{3}{2}} - a\right)$?

Let $y = x^{\frac{1}{2}}\left(x^{\frac{3}{2}} - a\right) = x^2 - ax^{\frac{1}{2}}$, then

$$\frac{dy}{dx} = 2x^{2-1} - \tfrac{1}{2}ax^{\frac{1}{2}-1} = 2x - \frac{a}{2x^{\frac{1}{2}}}.$$

4. What is the differential coefficient of $ax^2 - x + \dfrac{1}{x^2}$?

Let $y = ax^2 - x + \dfrac{1}{x^2} = ax^2 - x + x^{-2}$;

$$\therefore \quad \frac{dy}{dx} = 2ax - 1 - 2x^{-2-1} = 2ax - 1 - \frac{2}{x^3}.$$

5. What are the differential coefficients of the following functions, $\frac{1}{5}(2x^5-6x^3)$, x^3+x^2-x+1, $\frac{1}{2}x^4-\frac{1}{3}ax^3$, $(2x-1)\times(2x+1)$, $(a-x)^2+ax$, $\frac{1}{2}ax^4-\frac{1}{x}$, $x(1-x)^2$?

Answers. $2x^4-\frac{18}{5}x^2$, $3x^2+2x-1$, $2x^3-ax^2$, $8x$, $2x-a$,

$$2ax^3+\frac{1}{x^2},\ 3x^2-4x+1.$$

6. What is the differential of $(a+x)^3-a(3x^2+a^2)$?

Let $y=(a+x)^3-a(3x^2+a^2)=x^3+3a^2x$;

$$\therefore\ \frac{dy}{dx}=3x^2+3a^2,\ \therefore\ dy=3x^2dx+3a^2dx.$$

7. What are the differentials of the following functions.

$$\frac{1}{x}(x^3-2x^{\frac{3}{2}}),\ (x^{\frac{3}{4}}+1)(x^{\frac{3}{4}}-1),\ (a-x)^3+3a^2x\,?$$

Answers. $2xdx-\frac{dx}{x^{\frac{1}{2}}}$, $\frac{3}{2}x^{\frac{1}{2}}dx$, $6axdx-3x^2dx$.

40. *Rule* 5. To find the differential of the product of two functions, multiply each function by the differential of the other, and add the products.

Required the differential of $y=(a+x)x$.

This expression may be readily differentiated by multiplying the factors, and then applying *Rule* 4.; but we propose to go through the operation so as to illustrate the method by which the rule here given is established.

$$\text{Incr. } y=(a+x+h)(x+h)-(a+x)x.$$

Subtracting and adding $(a+x)(x+h)$ and reducing,

$$\text{incr. } y=\{(a+x+h)-(a+x)\}\,(x+h)+\{(x+h)-x\}\,(a+x)$$
$$=h(x+h)+h(a+x)$$

$$\therefore\ \frac{\text{incr. } y}{h}=(x+h)+(a+x).$$

When h approaches 0, we have,

$$\frac{dy}{dx}=x+(a+x)=a+2x,$$

$$\therefore\ dy=dx\times x+(a+x)dx=(a+2x)dx.$$

Let us now apply this method of demonstration to the general formula, $y=f(x)\times \text{F}(x)$.

If x takes the increment h, we have,

$$\text{incr. } y=f(x+h)\times \text{F}(x+h)-f(x)\times \text{F}(x).$$

Subtracting and adding $f(x)\times \text{F}(x+h)$, and reducing,

$$\begin{aligned}\text{incr. } y &= \{f(x+h)-f(x)\}\,\text{F}(x+h)+\{\text{F}(x+h)-\text{F}(x)\}f(x)\\ &=\text{incr. } f(x)\times \text{F}(x+h)+\text{incr. } \text{F}(x)\times f(x),\end{aligned}$$

$$\therefore\ \frac{\text{incr. } y}{h}=\frac{\text{incr. } f(x)}{h}\times \text{F}(x+h)+\frac{\text{incr. } \text{F}(x)}{h}\times f(x).$$

As this equation holds true for all values of h, it will also be true when h approaches 0, and then the limiting value of $\frac{\text{incr. } y}{h}=\frac{dy}{dx}$, the limiting value of $\frac{\text{incr. } f(x)}{h}=\frac{df(x)}{dx}$, and so on.

$$\therefore\ \frac{dy}{dx}=\frac{df(x)}{dx}\times \text{F}(x)+\frac{d\text{F}(x)}{dx}\times f(x),$$

and multiplying each side of this equality by dx,

$$dy=df(x)\times \text{F}(x)+d\text{F}(x)\times f(x).$$

41. It will now be easy to differentiate the product of three or more functions of the same variable.

Let $y=f(x)\times \text{F}(x)\times \phi(x)$,

then if $\Phi(x)$ be put for $f(x)\times \text{F}(x)$,

$$y=\Phi(x)\times\phi(x).$$

Differentiating by the rule just proved,

$$dy=d\Phi(x)\times\phi(x)+d\phi(x)\times\Phi(x),$$

$$\text{but } \Phi(x)=f(x)\times F(x)$$

$$\therefore\ d\Phi(x)=df(x)\times F(x)+dF(x)\times f(x),$$

therefore by substitution and reduction,

$$dy=df(x)\times F(x)\times\phi(x)+dF(x)\times f(x)\times\phi(x)$$
$$+d\phi(x)\times f(x)\times F(x).$$

Where we multiply the differential of each factor by all the other factors, and add the results. The rule will obviously apply to any number of factors.

EXAMPLES.

1. Let $y=(ax+x^2)(a+x^3)$, then by the rule,
$$dy=d(ax+x^2)\times(a+x^3)+d(a+x^3)\times(ax+x^2)$$
$$=(adx+2xdx)\times(a+x^3)+3x^2dx\times(ax+x^2)$$
$$=(5x^4+4ax^3+2ax+a^2)dx, \text{ by reduction.}$$

Dividing each side of this equation by dx will give us the value of the differential coefficient.

2. Let $y=(1+x^2)(1+x^3)$, then $dy=(5x^4+3x^2+2x)dx$.

3. Let $y=(a+bx^3)(bx^3-a)$, then $dy=6b^2x^5dx$.

4. Let $y=(1+x+x^3)(1-x^3)+x^6-1$, then
$$dy=(dx+3x^2dx)(1-x^3)-3x^2dx\times(1+x+x^3)+6x^5dx$$
$$=(1-4x^3)dx, \text{ and } \therefore\ \frac{dy}{dx}=1-4x^3.$$

5. Let $y=(a+2x^2)(a-3x^3)+6x^5$, then $\dfrac{dy}{dx}=4ax-9ax^2$.

6. Let $y=x^2(1-ax^2)(1+ax^2)$, then $\dfrac{dy}{dx}=2x-6a^2x^5$.

42. *Rule* 6. To find the differential of a fraction, multiply the differential of the numerator into the denominator, from this product subtract the differential of the denominator multiplied by the numerator, and divide the remainder by the square of the denominator.

Let $y=\frac{x}{a+x}$, then when x takes the increment h,

$$\text{incr.}\,y=\frac{x+h}{a+x+h}-\frac{x}{a+x}=\frac{(x+h)(a+x)-x(a+x+h)}{(a+x+h)(a+x)}.$$

Subtracting and adding $x(a+x)$ to the numerator,

$$\text{incr.}\,y=\frac{\{(x+h)-x\}(a+x)-x\{(a+x+h)-(a+x)\}}{(a+x+h)(a+x)}$$

$$=\frac{h(a+x)-xh}{(a+x+h)(a+x)},$$

$$\therefore\ \frac{\text{incr.}\,y}{h}=\frac{a}{(a+x+h)(a+x)},\quad \therefore\ \frac{dy}{dx}=\frac{a}{(a+x)^2}.$$

Let us now apply this method of demonstration to the general formula, $y=\frac{f(x)}{F(x)}$.

Let x take the increment h, then

$$\text{incr.}\,y=\frac{f(x+h)}{F(x+h)}-\frac{f(x)}{F(x)}$$

$$=\frac{f(x+h)F(x)-f(x)F(x+h)}{F(x+h)F(x)}.$$

Subtracting and adding $f(x)F(x)$, and reducing,

$$\text{incr.}\,y=\frac{\{f(x+h)-f(x)\}F(x)-\{F(x+h)-F(x)\}f(x)}{F(x+h)F(x)}$$

$$=\frac{\text{incr.}\,f(x)\times F(x)-\text{incr.}\,F(x)\times f(x)}{F(x+h)F(x)}.$$

Dividing both sides of this equation by h,

$$\frac{\text{incr.}\,y}{h}=\frac{\frac{\text{incr.}\,f(x)}{h}\times F(x)-\frac{\text{incr.}\,F(x)}{h}\times f(x)}{F(x+h)F(x)}$$

As this equation is true for all values of h, it will also be true when h approaches 0, and then the limiting value of

$\frac{\text{incr. } y}{h}=\frac{dy}{dx}$; the limiting value of $\frac{\text{incr. } f(x)}{h}=\frac{df(x)}{dx}$; and so on.

$$\therefore \frac{dy}{dx}=\frac{\frac{df(x)}{dx}\times \text{F}(x)-\frac{d\text{F}(x)}{dx}\times f(x)}{\{\text{F}(x)\}^2}.$$

Multiplying both sides of this equation by dx,

$$dy=\frac{df(x)\times \text{F}(x)-d\text{F}(x)\times f(x)}{\{\text{F}(x)\}^2}.$$

Examples.

1. Let $y=\frac{3x^2-1}{x^2+1}$; then by the rule,

$$dy=\frac{d(3x^2-1)\times(x^2+1)-d(x^2+1)\times(3x^2-1)}{(x^2+1)^2}$$

$$=\frac{6xdx\times(x^2+1)-2xdx\times(3x^2-1)}{(x^2+1)^2}=\frac{8xdx}{(x^2+1)^2},$$

$$\text{and } \frac{dy}{dx}=\frac{8x}{(x^2+1)^2}.$$

2. Let $y=\frac{x^2}{a^2+x^2}$, then $\frac{dy}{dx}=\frac{2a^2x}{(a^2+x^2)^2}$.

3. Let $y=\frac{x(x+1)}{x^2+x+1}$, then $\frac{dy}{dx}=\frac{2x+1}{(x^2+x+1)^2}$.

4. Let $y=\frac{x^3}{x^2-1}-\frac{x^2}{x-1}$, then $\frac{dy}{dx}=\frac{2x}{(x^2-1)^2}$.

5. Let $y=ax^3-\frac{x^3}{x^3-x+c}$, then $\frac{dy}{dx}=3ax^2-\frac{x^2(3c-2x)}{(x^3-x+c)^2}$.

6. Let $y=\frac{x}{1+x}$, then $dy=\frac{dx}{(1+x)^2}$.

7. Let $y=\frac{1+x^2}{1-x^2}$, then $dy=\frac{4xdx}{(1-x^2)^2}$.

43. *Rule* 7. To find the differential of any power of a function, multiply together the index of the power, the function itself with its index diminished by unity, and the differential of the function or root.

$$\text{Let } y=z^3, \text{ where } z=a+x^2.$$

When x takes the increment h, let the increment of z be k, then we have,

$$\text{incr. } y=(z+k)^3-z^3=(3z^2+3zk+k^2)k$$
$$=(3z^2+3zk+k^2)(2xh+h^2),$$

because k or incr. $z=a+(x+h)^2-(a+x^2)=2xh+h^2$.

$$\therefore \frac{\text{incr. } y}{h}=(3z^2+3zk+k^2)(2x+h).$$

Now when h approaches 0, k or incr. z also approaches 0, since the magnitude of k depends upon the magnitude of h; hence, by making h approach 0, we have,

$$\frac{dy}{dx}=3z^2\times 2x=3(a+x^2)^2\times 2x,$$

$$\therefore dy \text{ or } d(a+x^2)^3=3(a+x^2)^2\times 2xdx,$$

$$\text{where } 2xdx=d(a+x^2).$$

Let us now consider the general case, $y=z^n$, where $z=f(x)$.

When x takes the increment h, let the increment of z be k, then we have,

$$\text{incr. } y=(z+k)^n-z^n=\left\{nz^{n-1}+\frac{n(n-1)}{1\,.\,2}z^{n-2}k+\&c.\right\}k.$$

$$\therefore \frac{\text{incr. } y}{h}=\left\{nz^{n-1}+\frac{n(n-1)}{1\,.\,2}z^{n-2}k+\&c.\right\}\frac{k}{h}.$$

Now when h approaches 0, k or incr. z also approaches 0, since the magnitude of incr. z depends upon the magnitude of h; then the limiting value of $\frac{\text{incr. } y}{h}=\frac{dy}{dx}$; and the limiting value of $\frac{k}{h}$ or $\frac{\text{incr. } z}{h}=\frac{dz}{dx}$;

$$\therefore \frac{dy}{dx}=nz^{n-1}\frac{dz}{dx}, \text{ or } dy=nz^{n-1}dz.$$

EXAMPLES.

1. Let $y=(a+bx+cx^2)^n$, then $z=a+bx+cx^2$,

$$\therefore dy=n(a+bx+cx^2)^{n-1}\,d(a+bx+cx^2)$$
$$=n(a+bx+cx^2)^{n-1}(b+2cx)dx.$$

2. Let $y=\sqrt{a^2+x^2}=(a^2+x^2)^{\frac{1}{2}}$,

$$\therefore dy=\tfrac{1}{2}(a^2+x^2)^{\frac{1}{2}-1}\ d(a^2+x^2)=\frac{xdx}{\sqrt{a^2+x^2}},$$

$$\therefore \frac{dy}{dx}=\frac{x}{\sqrt{a^2+x^2}}.$$

3. Let $y=(1+x+x^2)^3$, then $\frac{dy}{dx}=3(1+x+x^2)^2(1+2x)$.

4. Let $y=\sqrt[3]{1+x^3}$, then $\frac{dy}{dx}=\frac{x^2}{(1+x^3)^{\frac{2}{3}}}$.

5. Let $y=(2+3x^2)\,(1-x^2)^3$, then by *Rule 5*,

$$dy=d(2+3x^2)\times(1-x^2)^3+d(1-x^2)^3\times(2+3x^2)$$
$$=-6x(1-x^2)^2(4x^2+1)dx.$$

6. Let $y=x^3(a+x)^2$, then $dy=(3a+5x)\,(a+x)x^2dx$.

7. Let $y=(1+x^2)^3(1+x)^4$, then

$$dy=(4+6x+10x^2)\,(1+x^2)^2\,(1+x)^3dx.$$

8. Let $y=\frac{(1+x)^3}{1+x^2}$, then by *Rule* 6,

$$dy=\frac{3(1+x)^2dx\times(1+x^2)-2xdx\times(1+x)^3}{(1+x^2)^2}$$

$$=\frac{(1+x)^2(3-2x+x^2)dx}{(1+x^2)^2}.$$

9. Let $y=\frac{\sqrt{a^2-x^2}}{x}$, then $dy=\frac{-a^2dx}{x^2\sqrt{a^2-x^2}}$.

10. Let $y=\frac{x^2}{\sqrt{1+x^4}}$, then $dy=\frac{2xdx}{(1+x^4)^{\frac{3}{2}}}$.

11. Let $y=\frac{x^n}{(1+x)^n}=x^n(1+x)^{-n}$, then by *Rule* 5,

$$dy=nx^{n-1}dx\times(1+x)^{-n}-n(1+x)^{-n-1}dx\times x^n$$

$$=\frac{nx^{n-1}dx}{(1+x)^{n+1}}.$$

12. Let $y=\frac{1}{(1-x)^3}$, then $dy=\frac{3dx}{(1-x)^4}$.

13. Let $y=\frac{1}{(1-x)^{\frac{3}{2}}}$, then $dy=\frac{3dx}{2(1-x)^{\frac{5}{2}}}$.

14. Let $y=(1+x)(1-x)^{\frac{1}{2}}$, then $dy=\frac{(1-3x)dx}{2(1-x)^{\frac{1}{2}}}$.

15. Let $y=\frac{x^n-1}{x^n+1}$, then $\frac{dy}{dx}=\frac{2nx^{n-1}}{(x^n+1)^2}$.

16. Let $y=\frac{x}{\sqrt{1-x}}+\frac{\sqrt{1-x}}{x}$, then $\frac{dy}{dx}=\frac{(x^2+x-1)(2-x)}{2x^2(1-x)^{\frac{3}{2}}}$.

MAXIMA AND MINIMA.

44. The maximum state of a function, is that particular value which is *greater* than any of the values which immediately precede or follow it. On the contrary, a minimum state of a function, is that particular value which is *less* than any of the values which immediately precede or follow it.

When a railway train starts, its motion is very slow; the speed goes on increasing until it attains a certain limit, which we call the maximum speed; and when the steam is being turned off, the motion becomes gradually less and less until it attains a minimum when the steam is being turned on again.

In the circle, the sine increases with the arc until it arrives at 90°, when the sine = radius, and afterwards the sine decreases as the arc increases until it arrives at 180°. In this case the sine is a maximum when the arc = 90°.

In fig. 1. Art. **49.**, DI is the maximum ordinate; and in fig. 2., DI is the minimum ordinate.

These illustrations show, that just before a quantity attains its maximum it is increasing, but just after it has passed the maximum it is decreasing; and the contrary takes place with respect to the minimum.

45. The following example shows that while x increases continually, the value of the proposed function of x increases only up to a certain value of x, and afterwards decreases.

Let $f(x)=6x-x^2$, then we have, by actual calculation,

values of x . . . . 0, 1, 2, 3, 4, &c.,

corresponding values of $f(x)$, 0, 5, 8, 9, 8, &c.

Here $6x-x^2$ is a maximum when $x=3$.

46. The following example shows that while x increases continually, the value of the proposed function of x decreases to a certain value of x, and afterwards increases.

Let $f(x)=x^2+(4-x)^2$, then we have, by actual calculation,

values of x . . . 0, 1, 2, 3, &c.,
corresponding values of $f(x)$, 16, 10, 8, 10, &c.

Here $x^2+(4-x)^2$ is a minimum when $x=2$.

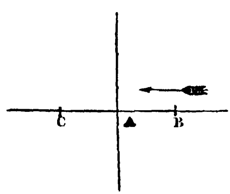
The differential calculus supplies us with the means of determining the maximum or minimum value of any function.

47. *If a quantity changes its sign it must have passed through* 0 *or* ∞.

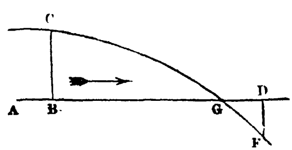
If a tradesman's profit continually decreases from day to day until it becomes minus, that is, until he loses by his trade, then it is evident that his profit must have been zero before it could change its sign from plus to minus.

The expression $(x-2a)^3$ is minus for all values of x less than $2a$, and plus for all values of x greater than $2a$. Now if we suppose x at first very small and to increase continually, this change of sign can only take place by x passing through the value $x=2a$, and then $(x-2a)^3=0$.*

Let a point B move along the line BC; then so long as the point is on the right of A, its distance from A is positive, when it arrives at A its distance is 0, and when it has moved on to the left of A its distance from A becomes minus, that is, the distance in passing from the plus to the minus state, has gone through 0.



Let the ordinate BC of the curve CGF move from the point A; then at B this ordinate has a plus value, when it arrives at G it becomes 0, and at D the ordinate DF has a minus value, that is, in passing from plus to minus the ordinate has passed through 0.

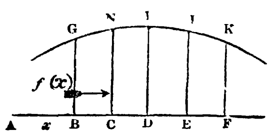


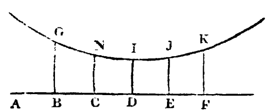
* In like manner $\frac{1}{(x-2a)^3}$ must pass through $x=2a$, in order to change its sign, and then $\frac{1}{(x-2a)^3}=\infty$.

48. *If x increase continually, then $\frac{df(x)}{dx}$ will be positive or negative, according as $f(x)$ is increasing or decreasing.*

In the function $f(x)$, let x take the increment h, then $\frac{f(x+h)-f(x)}{h}$ will obviously be positive or negative according as $f(x)$ is increasing or decreasing, and this will be the case however small h may be taken, that is, the limiting value of $\frac{f(x+h)-f(x)}{h}$ or $\frac{df(x)}{dx}$ will be positive or negative according as $f(x)$ is increasing or decreasing.

49. Let the ordinate BG, of the curve GNIJK, move uniformly from A towards F, and let the ordinate become a maximum when it arrives at the position DI in fig. 1., and a minimum in fig. 2. Now as the ordinate of a curve is always some function of its abscissa, let $x=$ AB the variable abscissa, and $f(x)=$ BG the corresponding ordinate.





In fig. 1., the ordinate is increasing before it becomes a maximum, that is, $f(x)$ is increasing, and therefore, by Art. **48.**, $\frac{df(x)}{dx}$ will be positive before the ordinate arrives at the maximum position. On the contrary, after the ordinate has passed the maximum position, it is decreasing, that is, $f(x)$ is decreasing, and therefore, by Art. **48.**, $\frac{df(x)}{dx}$ will be negative after the ordinate has passed the maximum position. Thus it appears that $\frac{df(x)}{dx}$ changes its sign from $+$ to $-$ in passing through the maximum position of the ordinate; therefore by Art. **47.**, $\frac{df(x)}{dx}=0$, when $f(x)$ is a maximum.

Similarly in fig. 2., $\frac{df(x)}{dx}$ changes its sign from $-$ to $+$ in passing through the minimum position of the ordinate, therefore by Art. 47., $\frac{df(x)}{dx}=0$, when $f(x)$ is a minimum.

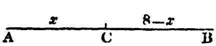
Hence we have the following rule for finding the maximum or minimum value of a function.

50. *Rule.* Find the differential coefficient of the function $f(x)$, and put the result equal to 0; then the value of x, determined from the solution of this equation, will be the value of x, which will render the proposed function a maximum or minimum, should it admit of becoming so.* If, as x continually increases, $\frac{df(x)}{dx}$ changes its sign from $+$ to $-$, there is a corresponding maximum value; and, on the contrary, there is a corresponding minimum value if $\frac{df(x)}{dx}$ changes its sign from $-$ to $+$: but if there is no change of sign, the function does not admit of a maximum or minimum.

51. The following considerations will frequently simplify the operation in finding the maximum or minimum value of a function. When a quantity is a maximum or minimum, it is obvious that any power, root, multiple, or part of the quantity, will also be a maximum or minimum.

EXAMPLES.

1. Divide a line, whose length $AB=8$, into two parts, AC and CB, so that their product may be a maximum.



Let $x=AC$, then $8-x=CB$. Hence we have to make

* Independently of the criterion here given, the peculiar nature of certain geometrical as well as other kinds of problems will indicate whether the proposed quantity admits of a maximum or minimum state.

$$f(x)=x(8-x)=8x-x^2, \text{ a max.}$$

Differentiating and putting the result equal to 0,

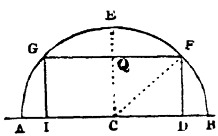
$$\frac{df(x)}{dx}=8-2x=0,$$

solving the equation, $8-2x=0$,

we find $x=\frac{8}{2}=4$.

Hence it appears that the two parts must be equal to one another. This result may be verified by arithmetic; thus $3\times5=15$, $6\times2=12$, &c.; whereas $4\times4=16$.

2. To inscribe the greatest rectangle IGFD in the semi-circle ABFG, whose radius CF$=r$.



Let DF$=x$, then CD$=\sqrt{r^2-x^2}$, and ID$=2$CD$=2\sqrt{r^2-x^2}$;

$\therefore$ area rect. IGFD$=$DF$\times$ID$=2x\sqrt{r^2-x^2}=$a max.;

therefore, by Art. **51.**, omitting the constant factor 2, and squaring,

$$x^2(r^2-x^2)=r^2x^2-x^4=\text{a max.}$$

Differentiating and making the result equal to 0,

$$2r^2x-4x^3=0;$$

$$\therefore\ x^2=\frac{r^2}{2} \text{ and } x=\frac{r}{\sqrt{2}}.$$

3. Given the hypothenuse ($=c$) of a right-angled triangle; to find the other sides, when the area is a maximum.

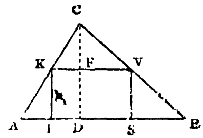
Let $x=$one of the sides, then the other side$=\sqrt{c^2-x^2}$;

$$\therefore \text{ area triangle}=\frac{x}{2}\sqrt{c^2-x^2}=\text{a max.},$$

$$x^2(c^2-x^2)=c^2x^2-x^4=\text{a max.}$$

Hence we find, as in the last ex., $x=\frac{c}{\sqrt{2}}$, and the other side $=\sqrt{c^2-x^2}=\frac{c}{\sqrt{2}}$; therefore the required sides are equal.

4. In the given triangle ABC, to inscribe the greatest rectangle IKVS.



Let AB $=c$, the perpend. CD $=b$, and IK $=$ DF $=x$, then, by the similar triangles ABC and KVC, we have,

$$\text{CD} : \text{AB} :: \text{CF} : \text{KV}$$

$$b : c :: b-x : \text{KV}=\frac{c}{b}(b-x);$$

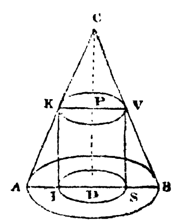
$$\therefore \text{ area rect.}=\text{KV}\times\text{IK}=\frac{c}{b}(b-x)x=\text{a max.}$$

Neglecting the constant multiplier, we have,

$$y=(b-x)x=bx-x^2=\text{a max.};$$

$$\therefore \frac{dy}{dx}=b-2x=0; \therefore x=\frac{b}{2}.$$

5. Required the length of the greatest roller ISVK which can be cut out of the given right cone ABC.



Let AB $=a$, CD $=b$, and CP $=x$; then we have, by similar triangles,

$$\text{CD} : \text{AB} :: \text{CP} : \text{KV}$$

$$b : a :: x : \text{KV}=\frac{ax}{b}.$$

Solidity cy. ISVK $=$ area base $\times$ length

$$=\cdot 7854\ \text{KV}^2\times\text{IK}$$

$$=\frac{\cdot 7854a^2}{b^2}\times x^2(b-x)=\text{a max.}$$

Neglecting the constant factor, we have

$$y=x^2(b-x)=bx^2-x^3=\text{a max.}$$

$$\therefore \frac{dy}{dx}=2bx-3x^2=0, \therefore x=\frac{2}{3}b,$$

$$\text{and } \text{IK}=\text{CD}-\text{CP}=b-\frac{2}{3}b=\frac{1}{3}b.$$

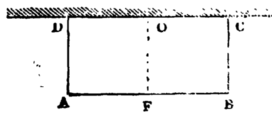
6. The perimeter, or sum of the sides, of a rectangle is p, required the sides when the area is a max.

Let x= the base, then the perpendicular$=\frac{1}{2}p-x$,

$$\therefore \text{Area}=x(\tfrac{1}{2}p-x)=\tfrac{1}{2}px-x^2=\text{a max.}$$

Differentiating, &c., we find $x=\frac{1}{4}p$; hence it follows that the greatest rectangle is a square.

7. A rectangular sheep-fold ABCD is to be built against an old wall DC, so as to enclose a given area, viz., a square feet. Required its dimensions, so that it may be built with the least expense.



Here the expense will be a minimum when the length of the walling CBAD is a minimum.

Let x=AB, then AB×AD$=a$, ∴ AD$=\frac{a}{x}$,

$$\therefore \text{the length of the walling}=\text{AB}+2\,\text{AD}=x+\frac{2a}{x}=\text{a min.}$$

$$\text{Let } y=x+\frac{2a}{x}=\text{a min.}$$

$$\frac{dy}{dx}=1-\frac{2a}{x^2}=0, \text{ and } x=\sqrt{2a},$$

$$\text{but AD}=\frac{a}{x}=\frac{a}{\sqrt{2a}}=\tfrac{1}{2}\sqrt{2a}.$$

Hence it follows that the breadth must be half the length.

8. Required the same as in the last example, when the enclosed space is divided into two compartments by a wall FQ.

Adopting the same notation as in the last example, we have

$$\text{the length of the walling} = AB + 3AD = x + \frac{3a}{x}.$$

Hence we find $AD = \frac{1}{3}AB$.

9. A cistern, open at the top, having a square base, is to be covered with a sup. ft. of lead; required the dimensions of the cistern when its content is a maximum.

Let $x=$ the side of the base, and $z=$ the perpend. height; then the sup. ft. in the cistern $= x^2 + 4xz = a$;

$$\therefore\ z = \frac{a - x^2}{4x};$$

and solidity of the cistern = area base × perpend. height

$$= x^2 \times z = \tfrac{1}{4}(ax - x^3) = \text{a max., then we have,}$$

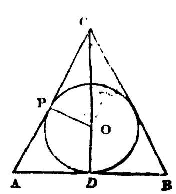
$$y = ax - x^3 = \text{a max.};$$

$$\therefore\ \frac{dy}{dx} = a - 3x^2 = 0,\ \therefore\ x = \sqrt{\frac{a}{3}},$$

$$\text{and } z = \frac{a - x^2}{4x} = \frac{a - \frac{a}{3}}{4\sqrt{\frac{a}{3}}} = \frac{1}{2}\sqrt{\frac{a}{3}}.$$

Hence the height must be half the side of the base.

10. To describe the least isosceles triangle ABC, about a given circle, whose radius $OP = OD = r$.



Let $x = CO$, then $CP = \sqrt{x^2 - r^2}$, and from the similar triangles CPO and CDA, we have

$$CP : OP :: DC : AD$$

$$\sqrt{x^2 - r^2} : r :: x + r : AD = \frac{r(x + r)}{\sqrt{x^2 - r^2}};$$

$$\therefore \text{ Area } \triangle \text{ ABC}=\text{AD}\times\text{DC}=\frac{r(x+r)^2}{\sqrt{x^2-r^2}}=\text{ a min.}$$

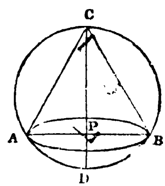
$$\text{and } y=\frac{(x+r)^4}{x^2-r^2}=\frac{(x+r)^3}{x-r}=\text{ a min.}$$

$$\therefore \frac{dy}{dx}=\frac{3(x+r)^2\,(x-r)-(x+r)^3}{(x-r)^2}=0,$$

$$\therefore\ 3(x+r)^2(x-r)-(x+r)^3=0,$$

$$\therefore\ x=2r, \text{ and CD}=x+r=3r.$$

11. Required the altitude of the greatest cone ABC, which can be cut out of the given sphere ADBC, whose diameter CD$=2r$.

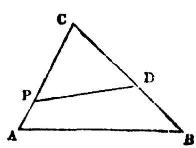


Let $x=$CP, then PD$=2r-x$,
and AP$^2=$CP$\times$PD$=x(2r-x)$,
$\therefore$ area base cone $=\pi\times$AP$^2=\pi x(2r-x)$,
$\therefore$ solidity cone$=\frac{1}{3}$ area base$\times$perpend.

$$=\frac{\pi}{3}x^2(2r-x)=\text{ a max.}$$

Differentiating, &c., $4rx-3x^2=0$, $\therefore\ x=\frac{4}{3}r$.

12. To bisect the triangle ABC by the shortest line PD.



Let BC$=a$, AC$=b$, CP$=x$, and CD$=y$,
then by the problem,
area $\triangle$ ABC$=2$ area $\triangle$ PDC,
$\therefore\ \frac{1}{2}ab\sin C=2\times\frac{1}{2}xy\sin C$;
$\therefore\ 2xy=ab$, and $y=\frac{ab}{2x}$.

Now, by Trigonometry, page 125., we have

$$\text{PD}^2=\text{CP}^2+\text{CD}^2-2\text{CP}\,.\,\text{CD}\,.\cos C$$

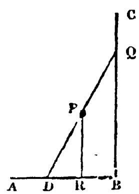
$$=x^2+\frac{a^2b^2}{4x^2}-ab\cos C=\text{ a min.}$$

Differentiating, &c., we have $2x-\frac{a^2b^2}{2x^3}=0$,

$$\therefore\ x=\sqrt{\frac{ab}{2}},\ \text{and}\ y=\frac{ab}{2x}=\sqrt{\frac{ab}{2}}.$$

Hence it follows that CP=CD.

13. Through a given point P within a given right angle ABC, to draw the line DQ which shall cut off the least triangle DBQ.



From P draw PR parallel to BC.

Let BR$=a$, RP$=b$, BD$=x$; then DR$=x-a$, and by the similar triangles DBQ and DRP, we have

DR : RP :: BD : BQ

$$x-a:\ b::\ x:\text{BQ}=\frac{bx}{x-a}.$$

$$\text{Area}\ \triangle\ \text{DBQ}=\tfrac{1}{2}\,\text{BD}\,.\,\text{BQ}=\frac{bx^2}{2(x-a)}=\text{a min.}$$

$$\text{and}\ y=\frac{x^2}{x-a}=\text{a min.}$$

$$\therefore\ \frac{dy}{dx}=\frac{2x(x-a)-x^2}{(x-a)^2}=0\ ;\ \therefore\ x=2a.$$

Hence it follows that the line DQ is bisected in the point P.

14. The whole surface of a right cone is c sup. ft.; required its dimensions when the content is a maximum.

Let $x=$ the radius of the base, and $z=$ the slant height; then we have,

$$\text{circum. base}=2\pi x\ ;\ \text{area base}=\pi x^2\ ;$$

$$\text{convex surface}=\tfrac{1}{2}\ \text{circum. base}\times\text{slant height}=\pi x\times z\ ;$$

$\therefore$ Total surface cone $= \pi x^2 + \pi x \times z = c$;

$$\therefore\ z = \frac{c}{\pi x} - x;$$

$$\therefore \text{ perpend. height cone} = \sqrt{z^2 - x^2} = \sqrt{\frac{c^2}{\pi^2 x^2} - \frac{2c}{\pi}}.$$

Solidity cone $= \frac{1}{3}$ area base $\times$ perpend. height

$$= \frac{1}{3}\pi x^2 \sqrt{\frac{c^2}{\pi^2 x^2} - \frac{2c}{\pi}} = \text{a max.}$$

Squaring and neglecting the constant factor, we have,

$$y = x^4\left(\frac{c}{\pi x^2} - 2\right) = \frac{c}{\pi}x^2 - 2x^4 = \text{a max.};$$

$$\therefore\ \frac{dy}{dx} = \frac{2c}{\pi}x - 8x^3 = 0;\ \therefore\ x = \tfrac{1}{2}\sqrt{\frac{c}{\pi}},$$

$$\text{and } z = \frac{c}{\pi x} - x = \frac{3}{2}\sqrt{\frac{c}{\pi}}.$$

Hence it appears that the slant height is 3 times the radius of the base.

15. Let r be the radius of a circular sheet of tin; it is required to find the dimensions of a sector cut out of it, which will form a conical vessel of the greatest capacity.

Let x = the length of the arc of the sector.

Now when the sector is coiled up so as to form the cone, x will be the circumference of the base, and r will be the slant height;

$$\therefore \text{ diam. of the base} = \frac{x}{\pi}, \text{ and rad. base} = \frac{x}{2\pi};$$

$$\therefore \text{ perpend. height of the cone} = \sqrt{\left(r^2 - \frac{x^2}{4\pi^2}\right)},$$

$$\text{and area base} = \frac{\pi}{4} \times \frac{x^2}{\pi^2} = \frac{x^2}{4\pi};$$

$\therefore$ content $= \frac{1}{3}$ area base $\times$ perpend. $= \frac{1}{3} \cdot \frac{x^2}{4\pi}\sqrt{r^2 - \frac{x^2}{4\pi^2}}$ $=$ a max. Squaring and neglecting the constant factors,

$$y = x^4\left(r^2 - \frac{x^2}{4\pi^2}\right) = \text{a max.};$$

$$\therefore \frac{dy}{dx} = 4r^2x^3 - \frac{3}{2\pi^2}x^5 = 0; \quad \therefore \quad x = 2r\pi\sqrt{\frac{2}{3}}.$$

16. Of all triangles upon the same base a, and having the same perimeter $2p$, the isosceles has the greatest area.

Let $x =$ one of the sides, then the other side $= 2p - a - x$. By Mensuration, prob. 3., we have,

$$\text{area } \Delta = \sqrt{p(p-a)(p-x)(p - \overline{2p-a-x})} = \text{a max.};$$

$$\therefore \quad y = (p-x)(a+x-p) = \text{a max.}$$

Hence we find, by the usual process, $x = p - \frac{a}{2}$, and the other side $= 2p - a - x = p - \frac{a}{2}$.

17. To inscribe the greatest parallelogram in a given parabola.

Let ABFG be the given parabola, and IGFD the required parallelogram. (See *fig.* to *Ex.* 2.)

Put the height $\text{CE} = b$, and $x = \text{EQ}$, then by the property of the parabola, Art. 19.,

$$\text{GQ}^2 = 4ax, \text{ and } \therefore \text{ GF} = 2\sqrt{4ax}.$$

$$\text{Again, IG} = \text{CQ} = \text{CE} - \text{EQ} = b - x.$$

$$\text{area IGFD} = \text{GF} \times \text{IG} = 2\sqrt{4ax}(b-x) = \text{a max.}$$

$$\therefore \quad y = x(b-x)^2 = \text{a max.}$$

$$\therefore \frac{dy}{dx} = (b-x)^2 - 2x(b-x) = 0,$$

$$\therefore\ x=\frac{b}{3},\ \text{and } \mathrm{CQ}=b-x=\tfrac{2}{3}b.$$

That is, the height of the rectangle must be two-thirds of the height of the parabola.

18. To determine whether $y=x^2-6x$ admits of a max. or min. value.

In the preceding examples the peculiar nature of the figure has invariably indicated, with sufficient certainty, whether the proposed quantity became a max. or min.; now in the following examples we shall find it necessary to employ the test of a max. or min. given in the Rule.

$$y=x^2-6x;\ \therefore\ \frac{dy}{dx}=2x-6=0;$$

$$\therefore\ x=3.$$

In order to ascertain whether this value of x gives a max. or min. value to y, we have to observe that $2x-6$ will be *negative* for all values of x less than 3, and *positive* for all values of x greater than 3; that is, $\frac{dy}{dx}$ will be *increasing* as x is continually increased; hence we conclude that y admits of a minimum. Or we may substitute 3 for x in the proposed expression x^2-6x, and ascertain, by an easy trial whether this value of x renders the expression a max. or min.

19. To determine whether $y=ax-x^2$ admits of a max. or min. value.

$$\text{Here } \frac{dy}{dx}=a-2x=0,\ \therefore\ x=\frac{a}{2}.$$

Now $a-2x$ will be *positive* for all values of x less than $\frac{a}{2}$, and *negative* for all values of x greater than $\frac{a}{2}$; that is, $\frac{dy}{dx}$

will be *decreasing* as x is continually increased, therefore $x=\frac{a}{2}$ makes y a max.

20. To determine the maxima and minima values of the function $y=3x^3-x+c$.

$$\frac{dy}{dx}=9x^2-1=0\ ;\ \therefore\ x=\pm\tfrac{1}{3}.$$

Where $x=\frac{1}{3}$ makes the proposed function a min., and $x=-\frac{1}{3}$ a max.

$$\text{Let } y=\frac{x}{x^2+1};\ \therefore\ \frac{dy}{dx}=\frac{1-x^2}{(x^2+1)^2},$$

$$\therefore\ 1-x^2=0,\text{ and } x=\pm 1.$$

In addition to the criterion of a max. or min., given in the Rule, the following one may be advantageously used.

Now if the value of $\frac{dy}{dx}$ be *decreasing*, then it follows by Art. **48.**, that the differential coefficient of this quantity will be *negative*, which will therefore indicate that the function admits of a max.; and in like manner it may be shown that if the result of the second differentiation is a *positive* quantity, then the function admits of a min.

In the above example, put $y'=\frac{1-x^2}{(x^2+1)^2}$, then we find by differentiating,

$$\frac{dy'}{dx}=\frac{2x^3-6x}{(x^2+1)^3}.$$

$$\text{If } x=+1,\ \frac{dy'}{dx}=-\tfrac{1}{2},\ \therefore\ y=\frac{x}{x^2+1}=\tfrac{1}{2},\text{ a max.}$$

$$x=-1,\ \frac{dy'}{dx}=+\tfrac{1}{2},\ \therefore\ y=-\tfrac{1}{2},\text{ a min.}$$

This process is equivalent to finding the second differential coefficient of y, or the value of $\frac{d^2y}{dx^2}$. (*See* Art. **59.**)

21. To divide a given number a into two parts, such that the third power of the one multiplied by the second power of the other shall be a maximum.

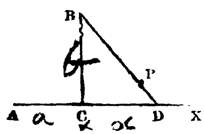
Let x= one part, then $a-x$=the other;

$$\therefore\ y=x^3(a-x)^2=\text{a max.}$$

$$\therefore\ \frac{dy}{dx}=3x^2(a-x)^2-2x^3(a-x)=0.$$

$$\therefore\ 3a-5x=0, \text{ and } x=\frac{3a}{5}.$$

22. Let a ship sail from a given place A, in the direction AX, at the same time that a boat sets out from another place B to approach the ship; it is required to find the direction in which the boat must sail in order to come as near the ship as possible, the velocity of the ship being to that of the boat as m to n.



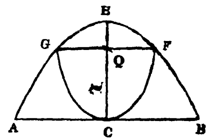
Let D and P be the position of the two vessels when nearest to each other, then DPB must obviously be a straight line. Draw BC perpendicular to AX, and put AC$=a$, BC$=b$, and CD$=x$, then BD$=\sqrt{\text{BC}^2+\text{CD}^2}=\sqrt{b^2+x^2}$; moreover,

$$\text{AD} : \text{BP} :: m : n,\ \therefore\ \text{BP}=\frac{n(a+x)}{m};$$

$$\therefore\ \text{PD}=\text{BD}-\text{BP}=\sqrt{b^2+x^2}-\frac{n(a+x)}{m}=\text{a min.}$$

Hence we have by differentiation, &c.,

$$\frac{x}{\sqrt{b^2+x^2}}-\frac{n}{m}=0,\ \therefore\ x=\frac{nb}{\sqrt{m^2-n^2}}.$$



23. Within a given parabola AEB, to inscribe the greatest parabola GFC having its vertex C in the middle of the base AB.

Let CE $=b$, and EQ $=x$, then, Art. **19.**, $GQ^2=4ax$, and $\therefore$ GF $=2\sqrt{4ax}$.

$$\therefore \text{ area parabola GCF} = \tfrac{2}{3}\, \text{GF . QC}$$

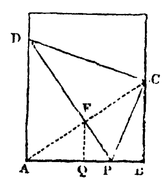
$$= \tfrac{2}{3}\times 2\sqrt{4ax}\times(b-x) = \text{a max.}$$

$$\therefore\ y=x(b-x)^2=\text{a max.}$$

$$\therefore\ \frac{dy}{dx}=(b-x)^2-2x(b-x)=0,$$

$$\therefore\ x \text{ or EQ} = \frac{b}{3}, \text{ and QC} = \text{CE} - \text{EQ} = \tfrac{2}{3}\, b.$$

24. The corner A of a leaf is turned over, so as just to reach the edge of the page at C; it is required to find when the length of the crease PD is a minimum.



Let AB $=a$, and AP $=x$. Join AC, cutting PD in F; draw FQ parallel to BC; then, since AD $=$ CD, and AP $=$ PC, therefore AF $=$ FC, AQ $=$ QB $=\frac{a}{2}$, and $\angle$ AFP $=$ a right angle. Now from the similar triangles APF and PQF, we have,

$$\text{AP} : \text{PF} :: \text{PF} : \text{PQ},$$

$$\therefore\ \text{PF} = \sqrt{\text{AP . PQ}} = \sqrt{x\left(x-\frac{a}{2}\right)}.$$

Again, from the similar triangles APD and QPF,

$$\text{AP} : \text{PD} :: \text{PQ} : \text{PF};$$

$$\therefore\ \text{PD} = \frac{\text{AP . PF}}{\text{PQ}} = \frac{x\sqrt{x\left(x-\frac{a}{2}\right)}}{x-\frac{a}{2}} = \text{a min.}$$

$$\therefore\ y=\frac{x^3}{x-\frac{a}{2}}=\frac{2x^3}{2x-a}=\text{a min.}$$

$$\therefore\ \frac{dy}{dx}=\frac{6x^2(2x-a)-4x^3}{(2x-a)^2}=0;$$

$$\therefore\ 6x^2(2x-a)-4x^3=0,\text{ and } x=\tfrac{3}{4}a.$$

25. Find when the area of the part turned down is a minimum.

$$\text{AF}=\sqrt{\text{AP}^2-\text{PF}^2}=\sqrt{\tfrac{1}{2}ax},$$

$$\therefore\ \text{area APD}=\tfrac{1}{2}\text{PD}\,.\,\text{AF}=\frac{\sqrt{\frac{1}{2}a}\times x^2}{2\sqrt{x-\frac{a}{2}}}=\text{a min.}$$

$$\therefore\ y=\frac{x^4}{2x-a}=\text{a min.},\ \therefore\ x=\frac{2a}{3}.$$

26. If $z=b-ax$, represent the relation of the speed and traction of a horse, where z is the traction in lbs., and x the rate in miles per hour; required the rate x so that the horse may perform the greatest amount of *work*.

Work per hour $=5280xz=5280x(b-ax)=$ a max.

$$\therefore\ y=x(b-ax)=bx-ax^2=\text{ a max.}$$

$$\therefore\frac{dy}{dx}=b-2ax=0,\ \therefore\ x=\frac{b}{2a}.$$

If $b=250$, and $a=41\frac{2}{3}$, then $x=\dfrac{250}{2\times 41\frac{2}{3}}=3$. (*See* Tate's Mechanics, Art. 6.)

EXERCISES.

1. Divide 15 into two parts, such that the product of the less by the square of the greater, shall be a maximum. *Ans.* 10 *and* 5.

2. The greatest rectangle inscribed in a quadrant of a circle is a square. Prove also that the same is true for the whole circle.

3. Required the same as in *Ex.* 9., when the cistern is *closed* at the top.

Ans. The vessel must have the form of a cube.

4. Required the same as in *Ex.* 9., when the cistern has the form of a right cylinder.

Ans. The height must be half the diameter of the base.

5. Supposing the vessel in *Ex.* 9., to be made of tin, and that it is divided into two compartments, what will then be its dimensions?

Ans. The height must be $\frac{2}{5}$ *of the side of the base.*

6. Required the altitude of the greatest cylinder which can be cut out of a sphere whose diameter is D. *Ans.* $D\sqrt{\frac{1}{3}}$.

7. Given the same as in *Ex.* 13., to draw DQ so that BD+BQ shall be a minimum. *Ans.* $BD=\sqrt{ab}$.

Next show that BD=BQ when the area of the triangle DBQ is a minimum.

8. To find a point in a semicircle, such that the sum of the lines drawn from it to the extremities of the diameter shall be a maximum. *Ans. The point will bisect the arc.*

9. Of all the cones whose convex surface is given ($=c$) to find that whose solidity is a maximum.

Ans. The radius of the base $=\sqrt{\dfrac{c}{\pi\sqrt{3}}}$.

10. At what point in the line ($=D$) joining the centres of two spheres, whose radii are r and r_1, can the greatest amount of both surfaces be seen? *Ans.* $x=\dfrac{r^{\frac{3}{2}}D}{r^{\frac{3}{2}}+r_1^{\frac{3}{2}}}$.

11. The altitude of the least cone circumscribed about a given sphere is equal to twice the diameter of the sphere.

12. If two bodies, A and C, move at the same time from two given points, A and C, in the directions AC and CB, and with the velocities m and n; it is required to find the dis-

tance moved over by C, when they are at the least distance from each other.

Ans. $x=\dfrac{an(m+n\cos C)}{m^2+n^2+2mn\cos C}$, where $a=AC$.

13. The altitude of the greatest parabola that can be formed by cutting a right cone is $\frac{3}{4}$ of the slant height of the cone.

14. Required the base of the greatest rectangle which can be inscribed in a semiellipse, whose major axis is $2a$, and minor axis $2b$. *Ans.* $a\sqrt{2}$.

15. Let $y=x^2+3x+2$; to find when y is a max. or min. *Ans.* $x=-\frac{3}{2}$ makes y a min.

16. $y=3x^2-4x$ is a min. when $x=\frac{2}{3}$.

17. $y=\dfrac{x^2-x+1}{x^2+x-1}$ is a min. when $x=2$, and a max. when $x=0$.

18. $y=1+3x-x^3$ is a max. when $x=1$, and a min. when $x=-1$.

RULES FOR THE DIFFERENTIATION OF FUNCTIONS.

[*Continued from page 56.*]

52. *Rule* 8. To differentiate a compound function, or the function of a function. If $y=F(z)$, where $z=f(x)$, then $\dfrac{dy}{dx}=\dfrac{dy}{dz}\times\dfrac{dz}{dx}$; that is, the differential coefficient of y is found by taking the differential coefficient of y with respect to z, and then multiplying this result by the differential coefficient of z with respect to x.

First, taking a particular case in order to illustrate the process of reasoning, let $y=\dfrac{x^2}{(1+x)^2}$, or putting z for $\dfrac{x}{1+x}$, $y=z^2$.

Now, when x takes the increment h, let z be increased by k, then

$$\text{incr. } y = \text{incr. } z^2 = \frac{\text{incr. } z^2}{k} \times k;$$

$$\therefore \frac{\text{incr. } y}{h} = \frac{\text{incr. } z^2}{k} \times \frac{k}{h} \ldots (1);$$

but incr. $z^2 = (z+k)^2 - z^2 = 2zk + k^2$,

$$\therefore \frac{\text{incr. } z^2}{k} = 2z + k;$$

$$\text{and } z = \frac{x}{1+x},$$

$$\therefore \text{incr. } z \text{ or } k = \frac{x+h}{1+x+h} - \frac{x}{1+x} = \frac{h}{(1+x+h)(1+x)},$$

$$\therefore \frac{k}{h} = \frac{1}{(1+x+h)(1+x)};$$

substituting these values in eq. (1),

$$\frac{\text{incr. } y}{h} = (2z+k)\frac{1}{(1+x+h)(1+x)},$$

which is true for all values of h. Now when h approaches 0, k also approaches 0, for the magnitude of k depends upon the magnitude of h; hence, taking the limiting value of $\frac{\text{incr. } y}{h}$, we have,

$$\frac{dy}{dx} = 2z \times \frac{1}{(1+x)^2}, \text{ that is, } \frac{dz^2}{dz} \times \frac{dz}{dx};$$

$$\therefore \frac{dy}{dx} = \frac{2x}{1+x} \times \frac{1}{(1+x)^2} = \frac{2x}{(1+x)^3}.$$

Generally, let $y = \text{F}(z)$, where $z = f(x)$.

Supposing, as usual, x to become $x+h$,

$$\text{incr. } y = \text{incr. F}(z) = \frac{\text{incr. F}(z)}{\text{incr. } z} \times \text{incr. } z,$$

multiplying and dividing by incr. z;

$$\therefore \frac{\text{incr. } y}{h} = \frac{\text{incr. F}(z)}{\text{incr. } z} \times \frac{\text{incr. } z}{h}.$$

As this equality is true for all values of h, it will therefore be true when h approaches 0, that is, it will be true when the ratios are taken at their limiting values. But when h approaches 0, incr. z also approaches 0, since the magnitude of incr. z depends upon the magnitude of h; and then the limiting value of $\frac{\text{incr. F}(z)}{\text{incr. } z} = \frac{d\text{F}(z)}{dz}$, the limiting value of $\frac{\text{incr. } z}{h}$ or $\frac{\text{incr. } f(x)}{h} = \frac{df(x)}{dx}$, and so on.

$$\therefore \frac{dy}{dx} = \frac{d\text{F}(z)}{dz} \times \frac{df(x)}{dx} \text{ or } \frac{dy}{dz} \times \frac{dz}{dx}.$$

Ex. Let $y = a + x^n - \sqrt{a + x^n}$.*

Here, putting z for $a + x^n$, we have,

$$y = z - z^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = 1 - \frac{1}{2z} = 1 - \frac{1}{2\sqrt{a + x^n}},$$

$$\text{and } \frac{dz}{dx} = \frac{d(a + x^n)}{dx} = nx^{n-1};$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \left\{1 - \frac{1}{2\sqrt{a + x^n}}\right\} nx^{n-1}.$$

In practice the operation may be conducted after the following manner: —

$$dy = dz - \frac{dz}{2z^{\frac{1}{2}}} = \left\{1 - \frac{1}{2z^{\frac{1}{2}}}\right\} dz;$$

* This may also be differentiated by rule 7.

$$\text{but } z=a+x^n; \ \therefore \ dz=nx^{n-1}dx;$$

$$\therefore \ dy=\left\{1-\frac{1}{2\sqrt{a+x^n}}\right\}nx^{n-1}dx;$$

$$\therefore \ \frac{dy}{dx}=\left\{1-\frac{1}{2\sqrt{a+x^n}}\right\}nx^{n-1}.$$

As exercises on this rule the student may work out any of the examples under Rule 7., which is merely a particular form of the one here given.

53. *Rule* 9. If $y=f(x)$, then $\dfrac{dy}{dx}=\dfrac{1}{\frac{dx}{dy}}$.

First, as an example, let $y=\dfrac{1}{x-1}$, then solving this equation for x, we find $x=\dfrac{1+y}{y}$, thereby showing that if y be a function of x, then x must be a function of y. When x takes the increment h, let y take the increment k, then

$$\text{incr. } x \text{ or } h=\frac{1+y+k}{y+k}-\frac{1+y}{y}=-\frac{k}{(y+k)y};$$

$$\text{but } \frac{\text{incr. } y}{h}=\frac{k}{h}=k\div\frac{-k}{(y+k)y}=-(y+k)y.$$

Now when h approaches 0, k also approaches 0, hence we have by taking the limiting values,

$$\frac{dy}{dx}=-y^2 \text{ or } \frac{1}{\frac{dx}{dy}}.$$

Let us now take the general function $y=f(x)$. It will be readily understood, since y is a function of x, that x may be found in terms of y, or what is the same thing, x must be a function of y. By simple algebra we have

$$\frac{\text{incr. } y}{\text{incr. } x}=\frac{1}{\frac{\text{incr. } x}{\text{incr. } y}}.$$

Now as y is a function of x, the limiting value of $\frac{\text{incr. } y}{\text{incr. } x}$, or $\frac{dy}{dx}$, may be found when incr. x or h approaches 0; and in like manner, as x is a function of y the limiting value of $\frac{\text{incr. } x}{\text{incr. } y}$, or $\frac{dx}{dy}$, may be found when incr. y or k approaches 0, which it does when h approaches 0; therefore taking the limiting values of both sides of the equality, we have,

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}, \text{ and } \frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Examples.

1. Let $y=x^2+3x+a$, required $\frac{dx}{dy}$.

Here we might find the value of x in terms of y, and then proceed to determine the differential coefficient by the preceding rules; but the process, in general, will be much more simple by the present rule.

$$\frac{dy}{dx}=2x+3, \quad \therefore \frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}=\frac{1}{2x+3}.$$

In practice the operation may be conducted in the following manner; differenting the proposed function,

$$dy=2x\,dx+3dx=(2x+3)dx;$$

dividing each side by dy,

$$\frac{dx}{dy}(2x+3)=1, \quad \therefore \frac{dx}{dy}=\frac{1}{2x+3}.$$

2. Let $y=ax+c$, then $\frac{dx}{dy}=\frac{1}{a}$.

3. Let $y=\frac{x}{1-x}$, then $\frac{dx}{dy}=(1-x)^2$.

4. Let $y=x^n-x$, then $\frac{dx}{dy}=\frac{1}{nx^{n-1}-1}$.

5. Let $x=\frac{y^2}{1+y}$, then $\frac{dy}{dx}=\frac{(1+y)^2}{y^2+2y}$.

54. *Rule* 10. To find the differential of an exponential function, multiply together the hyp. log. of the base, the exponential itself, and the differential of the exponent.

If $y=a^x$, then $dy=\log_e a \,.\, a^x dx$.

Let x take the increment h, then we have,

$$\text{incr. } y=a^{x+h}-a^x=a^x(a^h-1)\ldots(1).$$

Developing by the binomial theorem, $a^h=\{1+(a-1)\}^h$

$$=1+h(a-1)+\frac{h(h-1)}{1\,.\,2}(a-1)^2+\&c.;$$

$$\therefore\ a^h-1=h(a-1)+\frac{h(h-1)}{1\,.\,2}(a-1)^2+\&c.$$

Substituting this in eq. (1), and dividing by h,

$$\frac{\text{incr. } y}{h}=a^x\left\{(a-1)+\frac{h-1}{1\,.\,2}(a-1)^2+\frac{(h-1)(h-2)}{1\,.\,2\,.\,3}(a-1)^3+\&c.\right\}$$

Now when h approaches 0, the limit of $\frac{\text{incr. } y}{h}=\frac{dy}{dx}$;

$$\therefore\ \frac{dy}{dx}=a^x\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.\}.$$

But by Art. 9., $(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.=\log_e a$;

$$\therefore\ \frac{dy}{dx}=\log_e a\,.\,a^x, \text{ and } dy=da^x=\log_e a\,.\,a^x dx.$$

If $y=a^z$, where $z=f(x)$, then by *Rule* 8.,

$$\frac{dy}{dx}=\frac{da^z}{dz}\times\frac{dz}{dx}=\log_e a \,.\, a^z\frac{dz}{dx};$$

$$\therefore\ dy=da^z=\log_e a \,.\, a^z dz.$$

Cor. If e be put for a, then, since $\log_e e=1$, we have,

$$de^z=e^z dz,$$

that is, *the differential of e^z is the product of e^z and the differential of the exponent.*

EXAMPLES.

1. Let $y=e^{x^2}$, then $dy=e^{x^2}d(x^2)=2xe^{x^2}dx$.

2. Let $y=e^{nx}$, then $dy=ne^{nx}dx$.

3. Let $y=a^{cx^3+x}$, then $dy=\log_e a \,.\, a^{cx^3+x}d(cx^3+x)$
$=\log_e a \,.\, a^{cx^3+x}(3cx^2+1)dx$.

4. Let $y=x^n e^x$, then by *Rule* 5,
$dy=nx^{n-1}dx\times e^x+e^x dx\times x^n=x^{n-1}e^x(n+x)dx$.

5. Let $y=e^x(x-1)$, then $dy=e^x x dx$.

6. Let $y=e^x(x^2-2x+2)$, then $dy=e^x x^2 dx$.

7. Let $y=\frac{e^x}{1+x}$, then $\frac{dy}{dx}=\frac{e^x x}{(1+x)^2}$.

8. Let $y=(1+e^x)^n$, then we have, by *Rule* 7.,
$dy=n(1+e^x)^{n-1}\times d(1+e^x)=n(1+e^x)^{n-1}e^x dx$.

9. Let $y=(x+e^x)^2$, then $dy=2(x+e^x)(1+e^x)dx$.

55. *Rule* 11. To find the differential of the logarithm of a quantity, divide the differential of the quantity by the hyp. log. of the base $\times$ the quantity itself.

$$\text{If } y=\log_a x, \text{ then } dy=\frac{dx}{\log_e a \,.\, x}.$$

Let x take the increment h, then we have,

$$\text{incr. } y = \log_a (x+h) - \log_a x = \log_a \frac{x+h}{x} = \log_a \left(1+\frac{h}{x}\right);$$

$$\therefore \frac{\text{incr. } y}{h} = \frac{1}{h} \log_a \left(1+\frac{h}{x}\right) = \frac{1}{x} \cdot \frac{x}{h} \log_a \left(1+\frac{h}{x}\right)$$

$$= \frac{1}{x} \log_a \left(1+\frac{h}{x}\right)^{\frac{x}{h}}.$$

Now when h approaches 0, the limit of $\frac{\text{incr. } y}{h} = \frac{dy}{dx}$, and by

Art. **28.** *Ex.* 13., $\log_a \left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \log_a e = \frac{1}{\log_e a}$, (see Art. **9.**),

$$\therefore \frac{dy}{dx} = \frac{d \log_a x}{dx} = \frac{1}{\log_e a \cdot x}, \text{ and } dy = d \log_a x = \frac{dx}{\log_e a \cdot x}.$$

Or thus:—From the nature of logarithms, $x = a^y$, therefore by *Rule* 10, $dx = \log_e a \cdot a^y \, dy$; $\therefore dy = \frac{dx}{\log_e a \cdot x}$.

If $y = \log_a z$, where $z = f(x)$, then by *Rule* 8,

$$\frac{dy}{dx} = \frac{d \log_a z}{dz} \times \frac{dz}{dx} = \frac{1}{\log_e a \cdot z} \times \frac{dz}{dx},$$

$$\text{and } dy = d \log_a z = \frac{dz}{\log_e a \cdot z}.$$

Cor. If the base be e, then $\log_e e = 1$, and

$$d \log_e z = \frac{dz}{z};$$

that is, *the differential of the hyp. log. of a quantity is equal to the differential of the quantity divided by the quantity itself.*

Examples.

1. Let $y = \log_a cx^3$, then $dy = \frac{d(cx^3)}{\log_e a \cdot cx^3} = \frac{3dx}{\log_e a \cdot x}$.

2. Let $y = \log_e (a+x)$, then $dy = \frac{d\,(a+x)}{a+x} = \frac{dx}{a+x}$.

3. Let $y = \log_e (1+x^2)$, then $dy = \frac{d(1+x^2)}{1+x^2} = \frac{2xdx}{1+x^2}$,

$$\text{and } \frac{dy}{dx} = \frac{2x}{1+x^2}.$$

4. Let $y = \log_e ax^n$, then $\frac{dy}{dx} = \frac{n}{x}$.

5. Let $y = \log_e \{x + (x^2-1)^{\frac{1}{2}}\}$, then

$$dy = \frac{d\{x+(x^2-1)^{\frac{1}{2}}\}}{x+(x^2-1)^{\frac{1}{2}}} = \frac{dx + \frac{1}{2}(x^2-1)^{\frac{1}{2}-1}2xdx}{x+(x^2-1)^{\frac{1}{2}}}$$

$$= \frac{\{(x^2-1)^{\frac{1}{2}} + x\}dx}{(x^2-1)^{\frac{1}{2}}\{x+(x^2-1)^{\frac{1}{2}}\}} = \frac{dx}{(x^2-1)^{\frac{1}{2}}}.$$

6. Let $y = \log_e \frac{x}{\sqrt{1+x^2}} = \log_e x - \frac{1}{2} \log_e (1+x^2)$, then

$$dy = \frac{dx}{x} - \frac{xdx}{1+x^2} = \frac{dx}{x(1+x^2)}.$$

7. Let $y = \log_e \frac{1+x}{1-x}$, then $dy = \frac{2dx}{1-x^2}$.

8. Let $y = \log_e \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$, then $dy = \frac{2dx}{x\sqrt{x^2+1}}$

9. Let $y=x^n\left(\log_e x-\frac{1}{n}\right)$, then

$$dy=nx^{n-1}dx\times\left(\log_e x-\frac{1}{n}\right)+\frac{dx}{x}\times x^n$$
$$=nx^{n-1}\log_e x\,dx.$$

10. Let $y=x(\log_e x)^n$, then $dy=(\log_e x)^{n-1}\{\log_e x+n\}\,dx.$

11. Let $y=e^x\log_e x$, then $dy=e^x dx\times\log_e x+\frac{dx}{x}\times e^x$

$$=e^x\left\{\log_e x+\frac{1}{x}\right\}dx.$$

12. Let $y=e^{\log_e x}$, then $dy=e^{\log_e x}\frac{dx}{x}.$

13. Let $y=\log_e\frac{e^x-1}{e^x+1}=\log_e(e^x-1)-\log(e^x+1)$,

$$\text{then } dy=\frac{e^x dx}{e^x-1}-\frac{e^x dx}{e^x+1}=\frac{2e^x dx}{e^{2x}-1}.$$

14. Let $y=e^x\sqrt{\frac{1+x}{1-x}}.$

A complicated product may often be conveniently differentiated by first taking its logarithm.

$$\log_e y=x+\tfrac{1}{2}\log_e(1+x)-\tfrac{1}{2}\log_e(1-x);$$

$$\therefore\ \frac{dy}{y}=dx+\tfrac{1}{2}\cdot\frac{dx}{1+x}+\tfrac{1}{2}\cdot\frac{dx}{1-x}=\frac{2-x^2}{1-x^2}dx;$$

$$\therefore\ \frac{dy}{dx}=y\times\frac{2-x^2}{1-x^2}=e^x\sqrt{\frac{1+x}{1-x}}\cdot\frac{2-x^2}{1-x^2}.$$

15. Let $y=x^x$, then $\log_e y=x\log_e x$,

$$\therefore\ \frac{dy}{y}=dx\,.\,\log_e x+x\,.\,\frac{dx}{x},\ \therefore\ \frac{dy}{dx}=x^x(\log_e x+1).$$

56. *Rule* 12. To find the differentials of the trigonometrical functions, sin x, cos x, tan x, &c.

$$d \sin x = \cos x\, dx,\ d \cos x = -\sin x\, dx,$$

$$d \tan x = \sec^2 x\, dx, \text{ \&c.}$$

(1.) Let $y=\sin x$, then if x takes the increment h,

$$\text{incr. } y = \sin(x+h) - \sin x = 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2},$$

by Trigo. Art. **32.** page 121.

$$\therefore\ \frac{\text{Incr. } y}{h} = \cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

Now when h approaches 0 we have at the limits (Art. **28.** *Ex.* 17.),

$$\frac{\sin\frac{h}{2}}{\frac{h}{2}} = 1, \text{ and } \frac{\text{incr. } y}{h} = \frac{dy}{dx},$$

$$\therefore\ \frac{dy}{dx} = \frac{d \sin x}{dx} = \cos x,$$

$$\text{and } dy = d \sin x = \cos x\, dx.$$

If $y=\sin z$, where $z=f(x)$, then, by Rule 8,

$$\frac{dy}{dx} = \frac{d \sin z}{dz} \times \frac{dz}{dx} = \cos z \times \frac{dz}{dx},$$

$$\text{and } dy = d \sin z = \cos z dz.$$

(2.) Since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$,

$$\therefore\ d \cos x = d \sin\left(\frac{\pi}{2}-x\right) = \cos\left(\frac{\pi}{2}-x\right) d\left(\frac{\pi}{2}-x\right)$$

$$= \sin x \times -dx = -\sin x dx.$$

And generally, $d \cos z = -\sin z dz$, where $z = f(x)$.

(3.) Since $\tan x = \frac{\sin x}{\cos x}$, then, by *Rule* 6,

$$d \tan x = \frac{d \sin x \times \cos x - d \cos x \times \sin x}{\cos^2 x}$$

$$= \frac{(\cos^2 x + \sin^2 x)\, dx}{\cos^2 x} = \frac{dx}{\cos^2 x} = \sec^2 x dx.$$

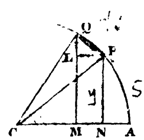
And generally, $d \tan z = \sec^2 z dz$, where $z = f(x)$.

(4.) Similarly $d \cot z = -\frac{dz}{\sin^2 z}$.

From these cases the differential of any other trigonometrical functions may be readily found.*

---

* The method of infinitesimals invented by Leibnitz enables us to arrive at these results, as well as those in Art. 58., with great simplicity. Let C be the centre of the circular arc AP, of which PN is the sine, and CN the cosine, the radius CA being unity. Then, according to this method, we may suppose Q to be taken so near to P that PQ may be regarded as a straight line, perpendicular to CP.



Let $AP = s$, $PN = y$, and $CN = x$; then $PQ = ds$, it being the indefinitely small increment of s; similarly $LQ = dy$, and MN or $LP = -dx$. By the similar triangles PLQ and PNC we have

PQ : LQ :: CP : CN,

or $ds : dy :: 1 : x$,

$\therefore dy = x \times ds$, that is, $d \sin. s = \cos. s \times ds$.

And PQ : LP :: CP : PN,

or $ds : -dx :: 1 : y$,

$\therefore dx = -y \times ds$, that is, $d \cos. s = -\sin. s \times ds$.

Again, from the equation $dy = x \times ds$ we have

$$ds = \frac{dy}{x} = \frac{dy}{\sqrt{1-y^2}}, \text{ that is, } d \sin^{-1} y = \frac{dy}{\sqrt{1-y^2}}.$$

And from the equation $dx = -y \times ds$ we have

$$ds = -\frac{dx}{y} = -\frac{dx}{\sqrt{1-x^2}}, \text{ that is, } d \cos^{-1} x = -\frac{dx}{\sqrt{1-x^2}}.$$

EXAMPLES.

1. Let $y=\sin nx$, then $dy=\cos nx\times d(nx)=n\cos nxdx$.

2. Let $y=\sin^3 x$, then by *Rule* 7.,

$$dy=3\sin^2 x\, d\sin x=3\sin^2\cos x\, dx.$$

3. Let $y=(\cos x)^n$, then $dy=n(\cos x)^{n-1}\times d\cos x$

$$=-n(\cos x)^{n-1}\sin x\, dx.$$

4. Let $y=\sin 2x\cos x$, then, by *Rule* 5,

$$\frac{dy}{dx}=2\cos 2x\cos x-\sin x\sin 2x$$

$$=\cos 2x\cos x+\cos 2x\cos x-\sin 2x\sin x$$

$$=\cos 2x\cos x+\cos 3x.$$

5. Let $y=x-\sin x\cos x$, then $dy=2\sin^2 x\, dx$.

6. Let $y=e^{\sin x}$, then $dy=e^{\sin x}\, d\sin x=e^{\sin x}\cos x\, dx$.

7. Let $y=\log_e\sin 2x$, then, by *Rule* 11.,

$$dy=\frac{d\sin 2x}{\sin 2x}=\frac{2\cos 2x\, dx}{\sin 2x}=2\cot 2x\, dx.$$

8. Let $y=\log_e\sqrt{\frac{1+\sin x}{1-\sin x}}$

$$=\tfrac{1}{2}\log_e(1+\sin x)-\tfrac{1}{2}\log_e(1-\sin x),$$

$$\therefore\ dy=\tfrac{1}{2}\frac{d(1+\sin x)}{1+\sin x}-\tfrac{1}{2}\frac{d(1-\sin x)}{1-\sin x}=\frac{dx}{\cos x}.$$

In a similar manner the other differentials may be obtained. It must, however, be observed that the correctness of the results, obtained by this method, arises from the principle of the compensation of errors. The first error that we adopt is, that PQ is a straight line perpendicular to CP. Now as this will be more and more nearly true as the arc PQ approaches 0, we compensate for this error by taking the magnitudes depending upon PQ as if PQ were really 0. This method invariably leads to correct results; and, with due care, it forms one of the most powerful instruments in the application of the differential calculus.

9. Let $y=\log_e\sqrt{\dfrac{1-\cos x}{1+\cos x}}$, then $\dfrac{dy}{dx}=\dfrac{1}{\sin x}$.

10. Let $y=\tan^n x$, then $\dfrac{dy}{dx}=\dfrac{n\tan^{n-1}x}{\cos^2 x}$.

11. Let $y=\tan x-x$, then $\dfrac{dy}{dx}=\tan^2 x$.

12. Let $y=\sqrt{\tan 2x}$, then $\dfrac{dy}{dy}=\dfrac{\sec^2 2x}{\sqrt{\tan 2x}}$.

13. Let $y=\cos^2 x-\sin^2 x$, then $dy=-2\sin 2x\,dx$.

14. Let $y=\sin x\cos x\tan x$, then

$\log_e y=\log_e\sin x+\log_e\cos x+\log_e\tan x$;

$$\therefore\ \frac{dy}{y}=\frac{\cos x\,dx}{\sin x}-\frac{\sin x\,dx}{\cos x}+\frac{dx}{\cos^2 x\tan x};$$

$$\therefore\ \frac{dy}{dx}=\sin x\cos x\tan x\left\{\cot x-\tan x+\frac{2}{\sin 2x}\right\}.$$

15. Let $y=x^{\sin. x}$; $\therefore\ \log_e y=\sin x\log_e x$;

$$\therefore\ \frac{dy}{y}=\cos.\ xdx\times\log_e x+\frac{dx}{x}\sin x;$$

$$\therefore\ \frac{dy}{dx}=x^{\sin. x}\left\{\cos x\ .\ \log_e x+\frac{\sin x}{x}\right\}.$$

16. $y=e^{ax}\sin rx$, $\dfrac{dy}{dx}=e^{ax}(a\sin rx+r\cos rx)$.

17. $y=e^x\sin^m x$, $\dfrac{dy}{dx}=e^x\sin^{m-1}x(\sin x+m\cos x)$.

18. Let $y=\sec x$; required $\dfrac{dy}{dx}$.

$y=\sec x=\dfrac{1}{\cos x}$, then, by *Rule* 6.,

$$\frac{dy}{dx}=\frac{\sin x}{\cos^2 x}=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}=\tan x\,.\,\sec x.$$

After the same manner prove the following formulæ: —

$$\frac{d(\text{versin}\, x)}{dx}=\sin x,$$

$$\frac{d\,\text{cosec}\, x}{dx}=-\text{cosec}\, x\,.\,\cot x.$$

57. *Notation of Inverse Functions.* If $y=F(x)$ be called he direct function, then $x=F^{-1}(y)$ is the notation expressing he inverse function. Hence, if we have given the inverse unction $x=f^{-1}(y)$, we immediately return to the direct unction $y=f(x)$; thus if $y=\sin^{-1}x$, we have $\sin y=x$; herefore *the expression* $\sin^{-1}$x *indicates an arc whose sine s* x, and so on to other inverse functions.

58. *Rule* 13. To find the differentials of inverse trigonometrical functions.

(1.) If $y=\sin^{-1}\frac{x}{a}$, then $dy=\frac{dx}{\sqrt{a^2-x^2}}$.

$$\text{For if } y=\sin^{-1}\frac{x}{a};\ \therefore\ \sin y=\frac{x}{a};$$

$$\therefore\ d\sin y=\frac{dx}{a};\ \therefore\ \cos y\, dy=\frac{dx}{a};$$

$$\text{and } dy=\frac{dx}{a\cos y}=\frac{dx}{a\sqrt{1-\sin^2 y}}=\frac{dx}{a\sqrt{1-\frac{x^2}{a^2}}}=\frac{ax}{\sqrt{a^2-x^2}}.$$

(2.) If $y=\cos^{-1}\frac{x}{a}$, then $dy=\frac{-dx}{\sqrt{a^2-x^2}}$.

$$\text{For } \cos y=\frac{x}{a};\ \therefore\ -\sin y\, dy=\frac{dx}{a};$$

$$\therefore\ dy=\frac{-dx}{a\sin y}=\frac{-dx}{a\sqrt{1-\cos^2 y}}=\frac{-dx}{\sqrt{a^2-x^2}}.$$

(3.) If $y=\tan^{-1}\frac{x}{a}$, then $dy=\frac{adx}{a^2+x^2}$.

$$\text{For } \tan y=\frac{x}{a}; \ \therefore \ \sec^2 y \, dy=\frac{dx}{a};$$

$$\therefore \ dy=\frac{dx}{a\sec^2 y}=\frac{dx}{a(1+\tan^2 y)}=\frac{adx}{a^2+x^2}.$$

Similarly we have $d \text{ versin}^{-1}\frac{x}{a}=\frac{dx}{\sqrt{2ax-x^2}}$,

$$\text{and } d\sec^{-1}\frac{x}{a}=\frac{adx}{x\sqrt{x^2-a^2}}.$$

In these formulæ the radius of the circle is a in reference to the arc x; but they are at once reduced to radius unity by making $a=1$; hence we have,

$$dy=d\sin^{-1}x=\frac{dx}{\sqrt{1-x^2}} \ \dots . \ (1.)$$

$$dy=d\cos^{-1}x=\frac{-dx}{\sqrt{1-x^2}} \ \dots . \ (2.)$$

$$dy=d\tan^{-1}x=\frac{dx}{1+x^2} \ \dots . \ (3.)$$

Examples.

1. Let $y=\sin^{-1}\frac{x}{\sqrt{1+x^2}}$, to find dy;

$\therefore \sin y=\frac{x}{\sqrt{1+x^2}}$; differentiating we find,

$$\cos y \, dy=d\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{dx}{(1+x^2)^{\frac{3}{2}}};$$

$$\therefore \ dy=\frac{1}{\cos y}\times\frac{dx}{(1+x^2)^{\frac{3}{2}}}.$$

$$\text{But } \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{x^2}{1+x^2}} = \frac{1}{(1+x^2)^{\frac{1}{2}}};$$

hence we have by substitution

$$dy = (1+x^2)^{\frac{1}{2}} \times \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \frac{dx}{1+x^2}.$$

Or thus, by the immediate application of formula (1), where we must put $\frac{x}{\sqrt{1+x^2}}$ for x, and $\therefore$ $\frac{x^2}{1+x^2}$ for x^2, thus we have

$$dy = d\sin^{-1}\frac{x}{\sqrt{1+x^2}} = d\left(\frac{x}{\sqrt{1+x^2}}\right) \div \sqrt{1-\frac{x^2}{1+x^2}}$$
$$= \frac{dx}{1+x^2}.$$

2. Let $y = \sin^{-1}\frac{1-x^2}{1+x^2}$, then $dy = -\frac{2dx}{1+x^2}$.

The following examples admit of concise forms of solution; at the same time it should be observed that they may all be solved by the methods given in *Ex.* 1.

3. Let $y = \cos^{-1}(4x^3-3x)$, then $\cos y = 4x^3-3x$;

= +3 cos x

hence by Trigo. Art. **31.**, p.121., $x = \cos\frac{y}{3}$, and $y = 3\cos^{-1}x$,

$$\therefore \text{ by formula (2), } dy = \frac{-3dx}{\sqrt{1-x^2}}.$$

4. Let $y = \sin^{-1}(3x-4x^3)$, then $dy = \frac{3dx}{\sqrt{1-x^2}}$.

= 3 sin⁻¹ x

5. Let $y = \tan^{-1}\frac{2x}{1-x^2}$, then $dy = \frac{2dx}{1+x^2}$.

6. Let $y = \sin^{-1}(2x-1)$, then $dy = \frac{dx}{\sqrt{x-x^2}}$.

SUCCESSIVE DIFFERENTIATION.

59. In the preceding articles we have framed a system of rules whereby the differential coefficient of the ordinary functions of x may be calculated; this *differential coefficient* we have defined to be the limiting ratio of the increment of the function to the increment of the variable x, and have designated it by the symbol $\frac{df(x)}{dx}$, or $\frac{dy}{dx}$ where y is put for $f(x)$. This symbol represents an *operation*, which is given for each particular form of the function in the rules already established; thus if $y=x^n$, we find $\frac{dy}{dx}$ by decreasing the exponent by unity and multiplying by n, that is, $\frac{dy}{dx}=nx^{n-1}$, and so on to other cases. But this operation may be repeated until the expression operated upon becomes zero. If $\frac{dy}{dx}$ represents one operation, then by an extension of the meaning of the symbol, $\frac{ddy}{dx\times dx}$ or more concisely $\frac{d^2y}{dx^2}$ will represent two operations; and generally $\frac{d^ny}{dx^n}$ will symbolise n operations; hence this symbol is called the nth differential coefficient. For example, if $y=x^n$ we have

$$\frac{dy}{dx}=nx^{n-1},\ \frac{d^2y}{dx}=n(n-1)x^{n-2},$$

$$\frac{d^3y}{dx^3}=n(n-1)(n-2)x^{n-3}, \text{ and so on.}$$

$\frac{d^ny}{dx^n} = n \cdot n-1 \cdot n-(n-1)\ x^0 = \lfloor n$

EXAMPLES.

1. If $y=ax^3+x^2$, then $\frac{dy}{dx}=3ax^2+2x$,

$$\frac{d^2y}{dx^2}=6ax+2,\ \frac{d^3y}{dx^3}=6a,\ \text{and}\ \frac{d^4y}{dx^4}=0.$$

2. Let $y=a-bx^2$, then $\dfrac{d^2y}{dx^2}=-2b$.

3. Let $y=x^4+x^3+x^2+x+1$, then $\dfrac{d^4y}{dx^4}=1.2.3.4$.

4. Let $y=\dfrac{1}{x^2}=x^{-2}$, then

$$\frac{dy}{dx}=-2x^{-3},\ \frac{d^2y}{dx^2}=2.3x^{-4},\ \frac{d^3y}{dx^3}=-2.3.4x^{-5},$$

$\dfrac{d^4y}{dx^4}=2.3.4.5x^{-6}=\dfrac{2.3.4.5}{x^6}$, hence generally we have

$$\frac{d^ny}{dx^n}=\frac{(-1)^n 2.3.4...(n+1)}{x^{n+2}}.$$

5. Let $y=\dfrac{1}{1+x^2}$, then $\dfrac{d^3y}{dx^3}=\dfrac{24x(1-x^2)}{(1+x^2)^4}$.

6. Let $y=c+b(x-a)^n$, then $\dfrac{d^ny}{dx^n}=n(n-1)...2.1.b$.

7. Let $y=a^x$, then $\dfrac{dy}{dx}=\log_e a.a^x$,

$$\frac{d^2y}{dx^2}=\log_e a.\log_e a.a^x=(\log_e a)^2a^x,\ \text{and so on,}$$

$$\frac{d^ny}{dx^n}=(\log_e a)^n a^x.$$

8. Let $y=e^{mx}$, then $\dfrac{d^3y}{dx^3}=m^3e^{mx}$.

9. Let $y=x^ne^x$, then

$$\frac{dy}{dx}=nx^{n-1}e^x+x^ne^x=(x^n+nx^{n-1})e^x,$$

$$\frac{d^2y}{dx^2}=\{nx^{n-1}+n(n-1)x^{n-2}\}\,e^x+(x^n+nx^{n-1})e^x$$

$$=\{x^n+2nx^{n-1}+n(n-1)x^{n-2}\}\,e^x.$$

10. Let $y=xe^x$, then $\frac{d^3y}{dx^3}=(3+x)e^x$, and generally

$$\frac{d^ny}{dx^n}=(n+x)e^x.$$

In Lagrange's method of derived functions, the symbol $f'(x)$ is used in the place of $\frac{df(x)}{dx}$, and is called the first derived function; $f''(x)$ is used in the place of $\frac{d^2f(x)}{dx^2}$, and is called the second derived function, and so on, $f^n(x)$ being the nth derived function, and equivalent to $\frac{d^nf(x)}{dx^n}$.

Maclaurin's Theorem.

60. If $y=f(x)$ admits of being expanded in the ascending positive powers of x, let

$$y=\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+\&\text{c}.$$

where A, B, C, &c. are called constants, the values of which we proceed to determine.

By successive differentiation we have

$$\frac{dy}{dx}=\text{B}+2\text{C}x+3\text{D}x^2+4\text{E}x^3+\&\text{c}.$$

$$\frac{d^2y}{dx^2}=2\text{C}+2\,.\,3\,.\,\text{D}x+3\,.\,4\,.\,\text{E}x^2+\&\text{c}.$$

$$\frac{d^3y}{dx^3}=2\,.\,3\,.\,\text{D}+2\,.\,3\,.\,4\text{E}x+\&\text{c}.$$

$$\&\text{c}.=\&\text{c}.$$

Now since A, B, C, &c. do not involve x, they must remain the same whatever value may be given to x. Make $x=0$ in these several equations, and let $(y)_0$, $\left(\frac{dy}{dx}\right)_0$, $\left(\frac{d^2y}{dx^2}\right)_0$, &c. represent the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. when x is taken 0, then we have

$$(y)_0=\text{A}\,;\ \left(\frac{dy}{dx}\right)_0=\text{B};\ \left(\frac{d^2y}{dx^2}\right)=2\text{C},$$

$$\therefore\ \text{C}=\left(\frac{d^2y}{dx^2}\right)_0\cdot\frac{1}{1\,.\,2};\ \left(\frac{d^3y}{dx^3}\right)_0=2\,.\,3\,.\,\text{D},$$

$$\therefore\ \text{D}=\left(\frac{d^3y}{dx^3}\right)_0\cdot\frac{1}{1\,.\,2\,.\,3};\ \text{and so on.}$$

Hence, by substituting these values of the constants in the assumed equation, we have

$$y=(y)_0+\left(\frac{dy}{dx}\right)_0\cdot\frac{x}{1}+\left(\frac{d^2y}{dx^2}\right)_0\cdot\frac{x^2}{1\,.\,2}+\left(\frac{d^3y}{dx^3}\right)_0\cdot\frac{x^3}{1\,.\,2\,.\,3}+\&\text{c.}$$

This development is commonly known by the name of Maclaurin's Theorem.*

Application of Maclaurin's Theorem to the development of functions.

61. Let $y=(a+x)^n$;

then making $x=0$, $(y)_0=a^n$;

* Adopting the notation of Lagrange (see page 94.), this theorem may be written,

$$f(x)=f(0)+f'(0)\,.\,\frac{x}{1}+f''(0)\,.\,\frac{x^2}{1\,\,2}+f'''(0)\,.\,\frac{x^3}{1\,.\,2\,.\,3}+\&\text{c.}$$

$$\frac{dy}{dx}=n(a+x)^{n-1}, \therefore \left(\frac{dy}{dx}\right)_0=na^{n-1};$$

$$\frac{d^2y}{dx^2}=n(n-1)(a+x)^{n-2}, \therefore \left(\frac{d^2y}{dx^2}\right)_0=n(n-1)a^{n-2};$$

$$\&c.=\&c. \qquad , \therefore \quad \&c.=\&c.$$

Hence, by substituting these values in Maclaurin's Theorem, we have

$$y \text{ or } (a+x)^n=a^n+\frac{n}{1}a^{n-1}x+\frac{n(n-1)}{1.2}a^{n\ 2}x^2+\&c.,$$

which is the binomial theorem.*

62. Let $y=\log_e(1+x)$;

then $(y)_0=\log_e 1=0$;

differentiating by *Rule* 11,

$$\frac{dy}{dx}=\frac{1}{1+x}, \therefore \left(\frac{dy}{dx}\right)_0=1;$$

$$\frac{d^2y}{dx^2}=-\frac{1}{(1+x)^2}, \therefore \left(\frac{d^2y}{dx^2}\right)_0=-1;$$

$$\frac{d^3y}{dx^3}=\frac{2}{(1+x)^3}, \therefore \left(\frac{d^3y}{dx^2}\right)_0=2;$$

and so on. Therefore, by Maclaurin's theorem,

$$y \text{ or } \log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c.$$

Cor. Putting $x-1$ for x we have

$$\log_e x=(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3-\tfrac{1}{4}(x-1)^4+\&c.$$

* Although this theorem has been used in establishing the rules of differentiation, yet it will be instructive to see how the differential calculus may be applied in proving the binomial theorem.

63. Let $y=a^x$;

$$\text{then } (y)_0=a^0=1;$$

differentiating by rule 10,

$$\frac{dy}{dx}=\log_e a\,.\,a^x, \quad \therefore \left(\frac{dy}{dx}\right)_0=\log_e a;$$

$$\frac{d^2y}{dx^2}=(\log_e a)^2a^x, \quad \therefore \left(\frac{d^2y}{dx}\right)_0=(\log_e a)^2;$$

$$\vdots \qquad \vdots$$

$$\frac{d^ny}{dx^n}=(\log_e a)^n a^x, \quad \therefore \left(\frac{d^ny}{dx^n}\right)_0=(\log_e a)^n.$$

$$\therefore\ a^x=1+\frac{\log_e a\,.\,x}{1}+\frac{(\log_e a)^2\,.\,x^2}{1\,.\,2}+\frac{(\log_e a)^3\,.\,x^3}{1\,.\,2\,.\,3}+\&c.$$

64. To expand sin x and cos x in terms of the arc x.

Let $y=\sin x$, $\therefore\ (y)_0=\sin 0=0$;

differentiating by rule 12,

$$\frac{dy}{dx}=\cos x, \quad \therefore \left(\frac{dy}{dx}\right)_0=\cos 0=1;$$

$$\frac{d^2y}{dx^2}=-\sin x, \quad \therefore \left(\frac{d^2y}{dx^2}\right)_0=-\sin 0=0;$$

$$\frac{d^3y}{dx^3}=-\cos x, \quad \therefore \left(\frac{d^3y}{dx^3}\right)_0=-\cos 0=-1;$$

nd so on; where it is obvious that all the even orders of ifferentiation will become 0, and the odd ones alternately lus and minus,

$$\therefore\ \sin x=x-\frac{x^3}{1\,.\,2\,.\,3}+\frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}-\&c.$$

Differentiating both sides of this equality,

$$\cos x=1-\frac{x^2}{1\,.\,2}+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}-\&c.$$

65. Let $y=\sin^{-1}x$;

then by Art. **58.**, eq. (1), and expanding by the binomial,

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$$

$$=1+\tfrac{1}{2}x^2+\frac{1\,.\,3}{2\,.\,4}x^4+\,\ldots\,+\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n}x^{2n}+\text{\&c.}\ldots(1)$$

Differentiating this equation for $2n$ times, and then making $x=0$, we shall obviously have,

$$\left(\frac{d^{2n+1}y}{dx^{2n+1}}\right)_0=\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n}\,.\,2n(2n-1)\ldots 2\,.\,1.$$

Hence the general term of Maclaurin's theorem is

$$\left(\frac{d^{2n+1}y}{dx^{2n+1}}\right)_0\cdot\frac{x^{2n+1}}{1\,.\,2\ldots(2n+1)}=\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1}.$$

Therefore taking n successively $=1, 2, 3$, &c., we obtain the 3rd, 4th, &c. terms in the development. Moreover we have $(y)_0=\sin^{-1}0=0$, and from eq. (1), $\left(\frac{dy}{dx}\right)_0=1$,

$$\therefore\ \sin^{-1}x=x+\frac{1}{1\,.\,2}\cdot\frac{x^3}{3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7}+\text{\&c.}$$

Developments of this kind may often be obtained more simply in the following manner.

Assume $y=A_0+A_1x+A_2x^2+$ &c.

$$\therefore\ \frac{dy}{dx}=A_1+2A_2x+3A_3x^2+\text{\&c.}$$

Equating the coefficients in this expansion, with those in eq. (1), we have,

$$A_1=1;\ A_2=0;\ 3A_3=\tfrac{1}{2},\ \therefore\ A_3=\tfrac{1}{2}\,.\,\tfrac{1}{3};\ A_4=0;$$

$$5A_5=\frac{1\,.\,3}{2\,.\,4},\ \therefore\ A_5=\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5};\ \text{and so on.}$$

Moreover when $x=0$, $A_0=(y)_0=\sin^{-1}0=0$.

Substituting these values in the assumed equation, we find the same expression for $\sin^{-1}x$ as that above given.

Cor. The length of a circular arc may be readily found by means of this series; thus, let the arc contain 30°, then $\sin^{-1}x=\text{arc } 30°$, and

$$\therefore x \text{ or } \sin 30=\tfrac{1}{2};$$

$$\therefore \text{arc } \underset{\sim}{30}=\frac{1}{2}+\frac{1}{1.2}\cdot\frac{1}{3.2^3}+\frac{1.3}{2.4}\cdot\frac{1}{5.2^5}+\&c.=\cdot523598 \&c.$$

But the arc 30° is $\frac{1}{12}$ of the whole circumference,

$$\therefore \text{circumference to rad. } 1=\cdot523598\times 12$$

$$\therefore \text{circumference to diam. } 1=3\cdot14159 \&c.$$

66. Let $y=\tan^{-1}x$;

then by Art. **58.**, eq. (3), and dividing,

$$\frac{dy}{dx}=\frac{1}{1+x^2}=1-x^2+x^4-x^6+\&c.$$

Hence we have by successive differentiation,

$$\frac{d^2y}{dx^2}=-2x+4x^3-6x^5+\&c.$$

$$\frac{d^3y}{dx^3}=-2+3.4x^2-5.6x^4+\&c.$$

$$\frac{d^4y}{dx^4}=\ 2.3.4x-4.5.6x^3+\&c.$$

$$\frac{d^5y}{dx^5}=2.3.4-3.4.5.6x^2+\&c.$$

$$\&c.=\&c.$$

Now make $x=0$, then $(y)_0=\tan^{-1}0=0$;

$$\left(\frac{dy}{dx}\right)_0=1;\ \left(\frac{d^2y}{dx^2}\right)_0=0;\ \left(\frac{d^3y}{dx^3}\right)_0=-2;\ \left(\frac{d^4y}{dx^4}\right)_0=0;$$

$\frac{d^5y}{dx^5}=2.3.4$; and so on. Substituting these values in Maclaurin's theorem,

$$y=x-\frac{2x^3}{2.3}+\frac{2.3.4x^5}{2.3.4.5}-\&c.,$$

$$\text{or } \tan^{-1}x=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\&c.,$$

which is an expression for the length of an arc in terms of its tangent.

This development may also be obtained, by differentiating both sides of Maclaurin's theorem, in order to derive the value of $\frac{dy}{dx}$, and then equating the coefficients of this series with those in the series for $\frac{dy}{dx}$ just given.

67. By Art. **8.** *Cor.* 1., we have,

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\&c.$$

In this expansion put $x\sqrt{-1}$, and $-x\sqrt{-1}$ successively for x, then

$$e^{x\sqrt{-1}}=1+x\sqrt{-1}-\frac{x^2}{2}-\frac{x^3\sqrt{-1}}{2.3}+\frac{x^4}{2.3.4}+\&c.$$

$$e^{-x\sqrt{-1}}=1-x\sqrt{-1}-\frac{x^2}{2}+\frac{x^3\sqrt{-1}}{2.3}+\frac{x^4}{2.3.4}-\&c.$$

first adding these equations, and then subtracting, &c.,

$$e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}=2\left\{1-\frac{x^2}{2}+\frac{x^4}{2.3.4}-\&c.\right\}$$

$$=2\cos x \ldots (1), \text{ by Art. } \mathbf{64}.$$

$$e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}=2\sqrt{-1}\left\{x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-\&c.\right\}$$

$$=2\sqrt{-1}\sin x \ldots (2), \text{ by Art. } \mathbf{64}.$$

Adding (1) and (2), and dividing by 2,

$$e^{x\sqrt{-1}}=\cos x+\sqrt{-1}\,\sin x\ldots(3)$$

Subtracting (2) from (1), and dividing by 2,

$$e^{-x\sqrt{-1}}=\cos x-\sqrt{-1}\,\sin x\ldots(4).$$

Cor. 1. Hence from (1) and (2), we have,

$$\cos x=\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2},$$

$$\text{and } \sin x=\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}.$$

Cor. 2. Dividing the latter eq. by the former,

$$\tan x=\frac{1}{\sqrt{-1}}\,\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}=\frac{1}{\sqrt{-1}}\,\frac{e^{2x\sqrt{-1}}-1}{e^{2x\sqrt{-1}}+1}.$$

These remarkable formulæ were discovered by Euler.

Cor. 3. In eq. (3) put nx for x,

$$\therefore\ \cos nx+\sqrt{-1}\,\sin nx=e^{nx\sqrt{-1}}=(e^{x\sqrt{-1}})^n$$

$$=(\cos x+\sqrt{-1}\,\sin x)^n, \text{ from (3).}$$

This is called Demoivre's Formula.

TAYLOR'S THEOREM.

68. Let $f(x)=ax^n+bx^m+cx^p+\&c.,$

where n, m, p, &c. may represent *any* constant quantities, whether integral, fractional, positive, or negative.

Let x become $x+h$, then we have,

$$f(x+h)=a(x+h)^n+b(x+h)^m+c(x+h)^p+\&c.,$$

expanding by the binomial theorem, and arranging the terms according to the ascending powers of h,

$$f(x+h)=\left.\begin{array}{l}ax^n\\bx^m\\cx^p\\\&c.\end{array}\right. + \left.\begin{array}{l}nax^{n-1}\\mbx^{m-1}\\pcx^{p-1}\\\&c.\end{array}\right\}\frac{h}{1} + \left.\begin{array}{l}n(n-1)ax^{n-2}\\m(m-1)bx^{m-2}\\p(p-1)cx^{p-2}\\\&c.\end{array}\right\}\frac{h^2}{1.2}+\&c.$$

Here it will be observed that the first column in the expansion is the proposed function $f(x)$; the second column is derived from the first by differentiation; the third column is derived from the second by differentiation; and so on, any column in the series being derived from its preceding column by the process of differentiation,

$$\therefore f(x+h)=f(x)+\frac{df(x)}{dx}\frac{h}{1}+\frac{d^2f(x)}{dx^2}\frac{h^2}{1.2}+\frac{d^3f(x)}{dx^3}\frac{h^3}{1.2.3}+\&c$$

This important development was first given by Dr. Taylor, and hence it is called *Taylor's Theorem.*

The following proof is usually given by writers on the differential calculus:

69. Let $y=f(x+h)$, where x and h are independent of each other, then $\frac{dy}{dx}=\frac{dy}{dh}$; the former being the differential coefficient of y on the supposition that x is the variable, and h constant; and the latter that h is the variable, and x constant.*

---

* Put $s=x+h$, $\therefore y=f(s)$, then by *Rule* 8.

$$\frac{dy}{dx}=\frac{df(s)}{ds}\cdot\frac{ds}{dx}=\frac{df(s)}{ds},$$

$$\text{because } \frac{ds}{dx}=\frac{d(x+h)}{dx}=\frac{dx}{dx}=1.$$

$$\text{Again, } \frac{dy}{dh}=\frac{df(s)}{ds}\cdot\frac{ds}{dh}=\frac{df(s)}{ds},$$

$$\text{because } \frac{ds}{dh}=\frac{d(x+h)}{dh}=\frac{dh}{dh}=1.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dh}, \text{ each being equal to } \frac{df(s)}{ds}.$$

This principle depends upon the circumstance that x and h are involved in precisely the same manner. The following illustrations will render the truth of the principle sufficiently apparent.

1. Let $y=a(x+h)^n+b(x+h)^m+$ &c.;

$$\therefore\ \frac{dy}{dx}=na(x+h)^{n-1}+mb(x+h)^{m-1}+\text{\&c.}$$

$$\text{and } \frac{dy}{dh}=na(x+h)^{n-1}+mb(x+h)^{m-1}+\text{\&c.}$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dh}.$$

2. Let $y=\log_e(x+h)\cdot\sin(x+h)$.

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dh}=\frac{1}{x+h}\sin(x+h)+\cos(x+h)\cdot\log_e(x+h).$$

Let us now assume,

$$f(x+h)=f(x)+\text{N}h^a+\text{P}h^b+\text{Q}h^c+\text{\&c.},$$

Where the quantities N, P, Q, &c. are functions of x not involving h, and a, b, c, &c. are constant indices which remain to be found. None of these indices can be negative, for if any one term were of the form $\text{R}h^{-e}=\frac{\text{R}}{h^e}$, that term would become infinite when $h=0$, while the left hand member of the equation is reduced to $f(x)$. All the exponents therefore being positive, they may be supposed to be arranged in an ascending order, that is, $a>b$, $b>c$, and so on. The first term, as in the assumption, must be $f(x)$, for when $h=0$ we must have the equality $f(x)=f(x)$.

Differentiating first with respect to x, and then with respect to h,

$$\frac{df(x+h)}{dx}=\frac{df(x)}{dx}+\frac{d\text{N}}{dx}h^a+\frac{d\text{P}}{dx}h^b+\frac{d\text{Q}}{dx}h^c+\text{\&c.}$$

$$\frac{df(x+h)}{dh}=a\text{N}h^{a-1}+b\text{P}h^{b-1}+c\text{Q}h^{c-1}+\text{\&c.}$$

Now, by Art. 69. these two series are identical, whatever may be the value of h, and therefore the exponents of the several powers of h, as well as the corresponding coefficients, must be the same in both series ; hence we have from the identity of the exponents,

$$a-1=0,\ b-1=a,\ c-1=b,\ \&c.;$$

$$\therefore\ a=1,\ b=a+1=2,\ c=b+1=3,\ \&c$$

and from the coefficients, we have,

$$a\text{N}=\frac{df(x)}{dx},\ b\text{P}=\frac{d\text{N}}{dx},\ c\text{Q}=\frac{d\text{P}}{dx},\ \&c.;$$

$$\therefore\ \text{N}=\frac{df(x)}{dx},\ \text{P}=\tfrac{1}{2}\frac{d\text{N}}{dx}=\frac{1}{1\,.\,2}\ \frac{d^2f(x)}{dx^2},$$

$$\text{Q}=\tfrac{1}{3}\ \frac{d\text{P}}{dx}=\frac{1}{1\,.\,2\,.\,3}\ \frac{d^3f(x)}{dx^3},\ \text{and so on.}$$

$$\therefore f(x+h)=f(x)+\frac{df(x)}{dx}\frac{h}{1}+\frac{d^2f(x)}{dx^2}\ \frac{h^2}{1\,.\,2}+\frac{d^3f(x)}{dx^3}\ \frac{h^3}{1\,.\,2\,.\,3}+\&c.$$

Or adopting the notation of Lagrange (see page 94.), this theorem may be written,

$$f(x+h)=f(x)+f'(x)\ \frac{h}{1}+f''(x)\ \frac{h^2}{1\,.\,2}+f'''(x)\frac{h^3}{1.\,2\,.\,3}+\&c.$$

Putting $y=f(x)$, and $y_1=f(x+h)$, this theorem may also be written,

$$y_1=y+\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\ \frac{h^2}{1\,.\,2}+\frac{d^3y}{dx^3}\ \frac{h^3}{1\,.\,2\,.\,3}+\&c.$$

Taylor's theorem will give the expansion of $f(x+h)$ in all cases, so long as x retains its general value ; but particular values may be given to x, in certain functions, which will render some of the differential coefficients infinite ; in such cases the theorem is said to *fail* in giving the development according to the ascending integral powers of h.

Maclaurin's theorem may be readily obtained from Taylor's, by making $x=0$, and then putting x for h.

Application of Taylor's Theorem.

70. To expand $(x+h)^3$ by this theorem.

$$\text{Let } y=x^3, \text{ then } \frac{dy}{dx}=3x^2,\ \frac{d^2y}{dx^2}=6x,\ \frac{d^3y}{dx^3}=6.$$

Substituting these values in Taylor's theorem,

$$y_1=y+\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1\,.\,2}+\frac{d^3y}{dx^3}\frac{h^3}{1\,.\,2\,.\,3}+\&c.$$

$$y_1=(x+h)^3=x^3+3x^2\frac{h}{1}+6x\frac{h^2}{1\,.\,2}+6\frac{h^3}{1\,.\,2\,.\,3}$$

$$=x^3+3x^2h+3xh^2+h^3.$$

71. To expand $\sin(x+h)$.

Let $y=\sin x$, then $\frac{dy}{dx}=\cos x$, $\frac{d^2y}{dx^2}=-\sin x$, $\frac{d^3y}{dx^3}=-\cos x$, $\frac{d^4y}{dx^4}=\sin x$, after this the values recur. Substituting these values in the theorem,

$$y_1=y+\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1\,.\,2}+\frac{d^3y}{dx^3}\frac{h^3}{1\,.\,2\,.\,3}+\&c.$$

$$y_1=\sin(x+h)=\sin x+\cos x\frac{h}{1}-\sin x\frac{h^2}{1\,.\,2}-\cos x\frac{h^3}{1\,.\,2\,.\,3}$$

$$+\sin x\frac{h^4}{1\,.\,2\,.\,3\,.\,4}+\cos x\frac{h^5}{1\,.\,2\ldots 5}-\&c.$$

$$=\sin x\left\{1-\frac{h^2}{1\,.\,2}+\frac{h^4}{1\,.\,2\,.\,3\,.\,4}-\&c.\right\}$$

$$+\cos x\left\{h-\frac{h^3}{1\,.\,2\,.\,3}+\frac{h^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}-\&c.\right\}.$$

When $x=0$ this expression becomes the same as that given in Art. **64**.

Similarly $\cos(x+h)$ may be expanded.

72. To expand $\log_e(x+h)$.

Let $y=\log_e x$, then $\frac{dy}{dx}=\frac{1}{x}$, $\frac{d^2y}{dx^2}=-\frac{1}{x^2}$, $\frac{d^3y}{dx^3}=\frac{2}{x^3}$,

$$\frac{d^4y}{dx^4}=-\frac{2.3}{x^4}, \text{ \&c.}$$

$$\therefore \log_e(x+h)=\log_e x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\text{\&c.}$$

73. To determine a series for the calculation of logarithms.

In the preceding series, let $x=1$, then $\log_e x=0$, and

$$\therefore \log_e(1+h)=h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4}+\text{\&c.} \dots (1)$$

This series is of no practical use in the calculation of logarithms, since it obviously becomes divergent when h exceeds unity.

Changing h into $-h$ in this series,

$$\log_e(1-h)=-h-\frac{h^2}{2}-\frac{h^3}{3}-\frac{h^4}{4}+\text{\&c.} \dots (2)$$

Subtracting (2) from (1), we have,

$$\log_e\left(\frac{1+h}{1-h}\right)=2\left\{h+\frac{h^3}{3}+\frac{h^5}{5}+\text{\&c.}\right\} \dots (3)$$

In order to render this equation convergent, put $\frac{1}{2x+1}$ for h, then $\frac{1+h}{1-h}=\frac{x+1}{x}$;

$$\therefore \log_e\left(\frac{x+1}{x}\right)=\log_e(x+1)-\log_e x=2\left\{\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\text{\&c.}\right\}$$

$$\therefore \log_e(x+1)=\log_e x+2\left\{\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\text{\&c.}\right\}.$$

A series of considerable convergency, which enables us to calculate the logarithms of numbers by means of those which immediately precede them. For example, if $x=10$, then the first four terms of the series will give the value of $\log_e 11$ correct to ten decimal places. Other series have been determined which are still more convergent.

Vanishing Fractions.

74. The substitution of a particular value for x, in a fraction, sometimes makes both the numerator and denominator vanish; such fractions are called vanishing fractions. Thus $\frac{x-1}{x^2-1}$ becomes $\frac{0}{0}$ when $x=1$; however, we have by division $\frac{x-1}{x^2-1}=\frac{1}{x+1}=\frac{1}{2}$ when $x=1$, therefore $\frac{1}{2}$ is the true value of the fraction when $x=1$. Here both numerator and denominator vanish when $x=1$, because they both contain the factor $x-1$, which becomes 0 when $x=1$. In like manner

$$\frac{(a^2-x^2)^{\frac{1}{2}}+(a-x)}{(a-x)^{\frac{1}{2}}+(a^2-x^2)^{\frac{1}{2}}}=\frac{0}{0}, \text{ when } x=a,$$

but dividing numerator and denominator by the common factor $(a-x)^{\frac{1}{2}}$, we have

$$\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{1+(a+x)^{\frac{1}{2}}}=\frac{\sqrt{2a}}{1+\sqrt{2a}}, \text{ when } x=a.$$

Thus by an easy algebraic process we may frequently find the value of a vanishing fraction; the method, however, derived from the differential calculus is more general, and in many difficult cases much more simple in practice.

Let $u=\frac{f(x)}{\text{F}(x)}$ be a vanishing fraction which becomes $\frac{0}{0}$ when $x=a$.

$\therefore\ u\,\mathrm{F}(x)=f(x)$; differentiating by rule 5,

$$\mathrm{F}(x)\frac{du}{dx}+u\,\frac{d\mathrm{F}(x)}{dx}=\frac{df(x)}{dx};$$

but $\mathrm{F}(x)=0$ when $x=a$,

$$\therefore\ u\,\frac{d\mathrm{F}(x)}{dx}=\frac{df(x)}{dx},\quad \therefore\ u=\frac{\frac{df(x)}{dx}}{\frac{d\mathrm{F}(x)}{dx}}.$$

Hence we have the following rule. To find the value of a vanishing fraction, divide the differential coefficient of the numerator by the differential coefficient of the denominator, and then substitute the given value for the variable. Should it be found, after this process, that the fraction still vanishes the process may obviously be repeated until the fraction ceases to have the vanishing form.

Ex. 1. Find the value of $u=\frac{x^3-1}{x^3+2x^2-x-2}$, when $x=1$.

Here the differential coefficient of the numerator is $3x^2$; and that of the denominator $3x^2+4x-1$,

$$\therefore\ u=\frac{3x^2}{3x^2+4x-1}=\frac{1}{2},\ \text{when } x=1.$$

2. $u=\frac{x^3+3x^2-4x-12}{x^2-2x}=5$, when $x=2$.

3. $u=\frac{1-x^n}{1-x}=n$ when $x=1$.

4. Find the value of $u=\frac{x^3-x^2-x+1}{x^4-2x^3+x^2}$, when $x=1$.

Here it will be necessary to differentiate twice.

The result of the first differentiations is $\frac{3x^2-2x-1}{4x^3-6x^2+2x}$.

and that of the second $\frac{6x-2}{12x^2-12x+2}=2=u$, when $x=1$.

5. The sum of the series $x+2x^2+3x^2+...+nx^n$, is $u=\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$; required its value when $x=1$.

By two differentiations we find $u=\frac{0}{0}=\frac{n(n+1)}{2}$.

6. $u=\frac{x-x^{2n+1}}{1-x^2}=n$, when $x=1$.

7. $u=\frac{3x^2-6x+3}{2x^2-4x+2}=\frac{3}{2}$, when $x=1$.

8. Find the value of $\frac{a^x-b^x}{x}$, when $x=0$.

Here the differential coefficient of the numerator is $a^x \log_e a-b^x \log_e b$; and that of the denominator is 1;

$$\therefore\ u=a^x \log_e a-b^x \log_e b=\log_e \frac{a}{b},\ \text{when } x=0.$$

9. $u=\frac{e^x-e^a}{x-a}=e^a$, when $x=a$.

10. $u=\frac{\log_e x}{(1-x)^{\frac{1}{2}}}=0$, when $x=1$.

11. $u=\frac{a^{\frac{3}{2}}x^{\frac{1}{2}}-x^2}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}}=3a$, when $x=a$.

12. Find the value of $u=\frac{\cos ax-\cos a}{1-x^2}$, when $x=1$.

Diff. coeff. num^r. $=-a\sin ax$.

" " deno^r. $=-2x$.

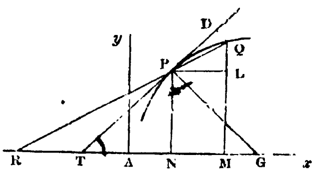
$$\therefore\ u=\frac{-a\sin ax}{-2x}=\frac{a\sin a}{2},\ \text{when } x=1.$$

13. $u=\frac{1-\cos x}{x\log_e(1+x)}$, when $x=0$.

After two differentiations we find $u=\frac{1}{2}$.

TANGENTS TO CURVES.

75. To draw a tangent to a given point P of the plane curve APQ referred to the rectangular axes Ax and Ay.



Let PT be the tangent, cutting the axis of x in the point T, then by Art. **32.** we have

$$\tan \text{NTP} = \frac{dy}{dx} \dots (1)$$

In order to draw the tangent PT it is only necessary that we should find the point T, or the distance NT which is called the *subtangent*; for this purpose we have

$$\tan \text{NTP} \times \text{NT} = \text{NP}, \text{ or } \frac{dy}{dx} \times \text{NT} = y,$$

$$\therefore \text{NT or subtangent} = y \div \frac{dy}{dx} \dots (2)$$

The length of the tangent is found from the equation

$$\text{PT} = \sqrt{\text{NP}^2 + \text{NT}^2} \dots (3)$$

If PG be drawn perpendicular to the tangent at P, and cutting the axis in G, then PG is called the *normal*, and NG the *subnormal*.

$$\text{Since } \angle \text{NPG} = \angle \text{NTP},$$

$$\therefore \text{NG or subnormal} = \text{NP} \times \tan \text{NPG} = y \times \tan \text{NTP}$$

$$= y \cdot \frac{dy}{dx} \dots (4)$$

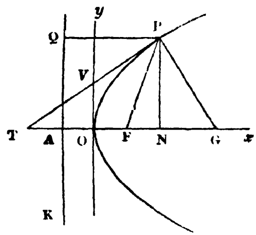
The length of the normal is found from the equation.

$$\text{PG or normal} = \sqrt{\text{NP}^2+\text{NG}^2} = \sqrt{y^2+y^2\left(\frac{dy}{dx}\right)^2}$$

$$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \ldots . (5)$$

EXAMPLES.

1. To draw a tangent PT to any point P in the parabola.



From the equation of the curve Art. **19.**,

$$y^2=4ax;$$

hence by differentiation we have

$$2y\,dy=4a\,dx, \therefore \frac{dy}{dx}=\frac{2a}{y};$$

substituting in eq. (2.) Art. **75.**,

$$\text{NT}=y\div\frac{2a}{y}=\frac{y^2}{2a}=\frac{4ax}{2a}=2x=2\,\text{ON}.$$

Thus it appears that the subtangent NT is equal to twice the abscissa ON; and ∴ OT=ON.

Again from eq. (4.) Art. **75.**, we have

$$\text{NG}=y\,.\,\frac{dy}{dx}=y\,.\,\frac{2a}{y}=2a=2\,\text{OF}.$$

From this, we learn, that the subnormal is a constant quantity, being always equal to twice the distance of the focus from the vertex of the curve.

2. To draw a tangent P_1T to any point P_1 in the circle AP_1DM_1. (See *fig.* p. 16.)

Let A be the origin of co-ordinates, AN$=x$, and NP$_1=y$, then by Art. **18.** *Cor.* 1.

$$y^2=2rx-x^2, \therefore \frac{dy}{dx}=\frac{r-x}{y};$$

substituting this in eq. (2.) Art. **75.**,

$$\text{NT}=y\div\frac{r-x}{y}=\frac{y^2}{r-x}=\frac{2rx-x^2}{r-x}.$$

3. To draw a tangent PT to any point P in the ellipse ABDM. (See *fig.* p. 16.)

Taking A as the origin, AN$=x$, and NP$=y$, we have by *Cor.* 1. Art. **20.**

$$y^2=\frac{b^2}{a^2}(2ax-x^2),\ \therefore\ \frac{dy}{dx}=\frac{b^2}{a^2}\cdot\frac{a-x}{y};$$

substituting in eq. (2.), Art. **75.**

$$\text{NT}=y\div\frac{b^2}{a^2}\cdot\frac{a-x}{y}=\frac{a^2y^2}{b^2(a-x)}=\frac{2ax-x^2}{a-x}.$$

Here it will be observed that as the value of NT is independent of b, it will remain the same whatever may be the magnitude of b or the minor axis; hence it follows that if a circle AP_1DM_1, or any ellipse, be described upon the major diameter AD, and the ordinates NPP_1 be drawn cutting the curves in the points P and P_1, then the tangents PT and P_1T will intersect the axis in the same point T. This property gives us an easy geometrical method for drawing a tangen to any point in a given ellipse.

4. To find the subtangent and subnormal in the cissoid.

Here by Art. **23.** the equation to the curve is

$$y^2=\frac{x^3}{2r-x};\ \text{hence by differentiation,}$$

$$2y\,dy=\frac{x^2(6r-2x)dx}{(2r-x)^2};\ \therefore\ \frac{dy}{dx}=\frac{x^2(3r-x)}{y(2r-x)^2};$$

substituting in eq. (2.) Art. **75.**, we have

$$\text{subtangent}=y\div\frac{dy}{dx}=y\div\frac{x^2(3r-x)}{y(2r-x)^2}=\frac{y^2(2r-x)^2}{x^2(3r-x)}$$

$$=\frac{x(2r-x)}{3r-x},\ \text{by substituting for } y^2.$$

$$\text{Subnormal} = y \cdot \frac{dy}{dx} = \frac{x^2(3r-x)}{(2r-x)^2}.$$

5. To draw a tangent to the point P in the cycloid. (See *fig.* p. 21.)

Let ON $=x$, NP $=y$, and BD $=2r$, then by Art. **25.**,

$$x = \text{arc PB} - \text{PR},$$

but arc PB $= r$ versin$^{-1}\frac{y}{r}$, and PR $= \sqrt{\text{BR}\cdot\text{RD}} = \sqrt{2ry-y^2}$,

$$\therefore\ x = r\ \text{versin}^{-1}\frac{y}{r} - \sqrt{2ry-y^2};$$

differentiating by *Rule* 13, &c.,

$$dx = \frac{rdy}{\sqrt{2ry-y^2}} - \frac{(r-y)dy}{\sqrt{2ry-y^2}} = \frac{ydy}{\sqrt{2ry-y^2}},$$

$$\therefore\ \frac{dx}{dy} = \frac{y}{\sqrt{2ry-y^2}},\ \text{or}\ \frac{dy}{dx} = \frac{\sqrt{2ry-y^2}}{y},\ \text{see Art. } \mathbf{53.};$$

which is the differential equation to the cycloid. Substituting this in eq. (4.), we have

$$\text{subnormal} = y \cdot \frac{dy}{dx} = \sqrt{2ry-y^2} = \text{PR} = \text{NB}.$$

Hence the chord PB is perpendicular to the curve; and as the angle formed by the chords BP and PD is a right-angle, PD is the direction of the tangent to the point P of the cycloid.

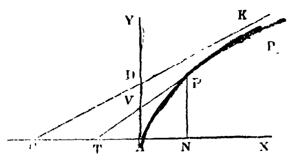
6. The equation to the cubical parabola is, $y^3 = px$.

Show that the subtangent $=3x$, and subnormal $=\frac{p^2}{3y}$.

7. In the witch (see Art. **22.**) the subtangent $= -\frac{2rx-x^2}{r}$, and subnormal $= -\frac{4r^3}{x^2}$.

8. In the curve whose equation is $y=\frac{x}{1+x^2}$, the sub-normal$=\frac{x(1-x^2)}{(1+x^2)^3}$.

Asymptotes to Curves.



76. Asymptotes are tangents to the curve at a point which is at an infinite distance from the origin. Thus if CK be a tangent to the curve APP_1, at a point infinitely distant from the origin A, then CK is an asymptote; which may be drawn when the values of AC or AD are finite when x and y, or x or y are infinite.

Ex. 1. To draw an asymptote to the hyperbola.

Here Art. **21.**, $y=\frac{b}{a}\sqrt{2ax+x^2}$, $\therefore\ \frac{dy}{dx}=\frac{b}{a}\cdot\frac{a+x}{\sqrt{2ax+x^2}}$;

Hence by eq. (2.), Art. **75.**, we have

$$\text{NT}=y\div\frac{b}{a}\,\frac{a+x}{\sqrt{2ax+x^2}}=\frac{2ax+x^2}{a+x},$$

$$\therefore\ \text{AT}=\text{NT}-\text{AN}=\frac{2ax+x^2}{a+x}-x=\frac{a}{1+\frac{a}{x}}.$$

From the similar triangles TAV and TNP, we have

$$\text{AV}=\frac{\text{NP}.\text{AT}}{\text{NT}}=y\cdot\frac{ax}{a+x}\div\frac{2ax+x^2}{a+x}=\frac{b}{\sqrt{\frac{2a}{x}+1}}.$$

Now when x is infinite, AT$=a$, and AV$=b$. Hence take AC$=a$, AD$=b$, and join CD, then CD produced is the asymptote to the curve. In this case the asymptote passes through the centre C of the curve.

2. Draw the asymptote to the curve whose equation is $y=\frac{x}{1+x^2}$.

Here we find $\text{NT}=\frac{x(1+x^2)}{1-x^2}$;

$$\therefore \text{AT}=\frac{x(1+x^2)}{1-x^2}-x=\frac{2x^3}{1-x^2}=\frac{2}{\frac{1}{x^3}-\frac{1}{x}};$$

$$\text{and AV}=\frac{\text{NP}\,.\,\text{AT}}{\text{NT}}=\frac{2}{x\left(\frac{1}{x^2}+1\right)^2}.$$

Therefore, when $x=\infty$, $\text{AT}=\infty$, and $\text{AV}=0$. The latter result shows that the asymptote must pass through the origin A; while the former result shows that the asymptote does not *cut* the axis of x; that is, it must coincide with this axis.

The method, given in these examples, is sometimes difficult of application; the following may be frequently used with advantage.

Let (if possible) the equation of the curve be put into the form $y=ax+b+\frac{c}{x}+\frac{d}{x^2}+$ &c., then as x increases the terms after b decrease, and when $x=\infty$ they vanish, leaving the equation $y=ax+b$ for the infinite branch of the curve. But this is an equation to a straight line CK (see last fig.), cutting the axis of y at a point $y=b$, and forming an $\angle$ C, such that $\tan \text{C}=a$. (See Art. **14.**) Thus it appears that the infinite branch of the curve coincides with the straight line CK represented by the equation $y=ax+b$.

3. To draw the asymptote to the curve, $y^3=x^3+x^2$.

Taking the root, and expanding by the binomial,

$$y=x\left(1+\frac{1}{x}\right)^{\frac{1}{3}}=x\left\{1+\frac{1}{3x}-\frac{1}{9x^2}+\text{\&c.}\right\}$$

$$=x+\frac{1}{3}-\frac{1}{9x}+\text{\&c.}$$

When $x=\infty$, we have $y=x+\frac{1}{3}$ for the equation to the asymptote. Taking $x=0$, in this equation, $y=\frac{1}{3}$; and taking $y=0$, $x=-\frac{1}{3}$; hence it follows that the asymptote cuts the axis at an angle of 45° at the distances AD=AC=$\frac{1}{3}$ from the origin.

4. Show that the extreme ordinate BC (see *fig.* page 20.) is an asymptote to the cissoid.

5. Show that the curve, whose equation is $y=\frac{x^2+x}{x-1}$, has an asymptote which cuts the axis at an angle of 45°.

Equation to the Tangent.

77. Let PT be the tangent (see *fig.* page 110.); NP$=y$, AN$=x$, the co-ordinates of the given point P of the curve; and $x_,$ and $y_,$, the co-ordinates of *any* point in the straight line PT; then we have, by Art. **16.**, for the equation of this line

$$y_,-y=a(x_,-x),$$

where $a=\tan$ NTP; but by Art. **32.**, $\frac{dy}{dx}=\tan$ NTP,

$$\therefore\ a=\frac{dy}{dx};\ \text{hence by substitution,}$$

$$y_,-y=\frac{dy}{dx}(x_,-x)\ \ldots\ (1),$$

which is the equation to the tangent PT.

Since the normal PG is drawn at right angles to the tangent PT,

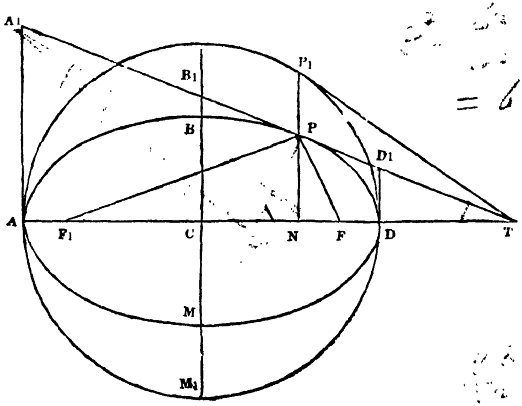
$$\therefore\ \tan \text{PG}x=-\cot \text{NTP}=-\frac{1}{a}=-\frac{dx}{dy},$$

hence the equation to the normal PG is

$$y_,-y=-\frac{dx}{dy}\ (x_,-x)\ \ldots\ (2)$$

EXAMPLES.

1. To find the equation to the tangent PT in the ellipse, the centre C being the origin.



Art. 20., $\frac{y^2}{b^2}+\frac{x^2}{a^2}=1$, $\therefore \frac{dy}{dx}=-\frac{b^2}{a^2}\cdot\frac{x}{y}$,

substituting this in eq. (1.) Art. 77. for the tangent,

$$y_{\prime}-y=-\frac{b^2}{a^2}\cdot\frac{x}{y}(x_{\prime}-x),$$

$$\therefore\ yy_{\prime}-y^2=\frac{b^2}{a^2}x^2-\frac{b^2}{a^2}xx_{\prime}=b^2-y^2-\frac{b^2}{a^2}xx_{\prime}.$$

$$\therefore\ a^2yy_{\prime}+b^2xx_{\prime}=a^2b^2\ \ldots\ (1)$$

which is the equation to the tangent PT, where the variable co-ordinates are $y_{\prime}$ and $x_{\prime}$.

Cor. 1. Make $y_{\prime}=0$ in this equation, then we find

$$x_{\prime} \text{ or } \mathrm{CT}=\frac{a^2}{x}=\frac{\mathrm{CD}^2}{\mathrm{CN}},\ \therefore\ \mathrm{CT}\,.\,\mathrm{CN}=\mathrm{CD}^2.$$

Cor. 2. Make $x_{,}=a$, then we find

$$y_{,} \text{ or DD}_1=\frac{a^2b^2-b^2xa}{a^2y}=\frac{b^2(a-x)}{ay};$$

again, make $x_{,}=-a$, then in like manner,

$$y_{,} \text{ or AA}_1=\frac{b^2(a+x)}{ay};$$

$$\therefore \text{ AA}_1 \cdot \text{DD}_1=\frac{b^4(a^2-x^2)}{a^2y^2}=\frac{b^4(a^2-x^2)}{b^2(a^2-x^2)}=b^2=\text{CB}^2;$$

$$\text{and } \frac{\text{AA}_1}{\text{DD}_1}=\frac{a+x}{a-x}=\frac{\text{AN}}{\text{DN}},$$

therefore, if A_1N and D_1N be joined, the triangles AA_1N and DD_1N will be similar, and therefore the lines A_1N and D_1N will form equal angles with the axis AD;

$$\text{also } \frac{\text{AA}_1}{\text{DD}_1}=\frac{\text{AT}}{\text{DT}}; \therefore \frac{\text{AN}}{\text{DN}}=\frac{\text{AT}}{\text{DT}},$$

$$\therefore \text{AN} \cdot \text{DT}=\text{DN} \cdot \text{AT}.$$

Cor. 3. The equation (1.) to the tangent may be written

$$y_{,}=-\frac{b^2x}{a^2y}x_{,}+\frac{b^2}{y}.$$

$$\text{Let } m=-\frac{b^2x}{a^2y};\ m^2a^2=\frac{b^4x^2}{a^2y^2};$$

$$\therefore m^2a^2+b^2=\frac{b^4x^2}{a^2y^2}+b^2=\frac{b^2}{a^2y^2}\{b^2x^2+a^2y^2\}=\frac{b^4}{y^2};$$

$$\therefore \sqrt{m^2a^2+b^2}=\frac{b^2}{y}; \text{ substituting these values,}$$

$$y_{,}=mx_{,}+\sqrt{m^2a^2+b^2} \ldots (2);$$

where m must be the tangent of the angle which the line makes with the axis of x. This form of the equation to the tangent is often convenient in the solution of problems. As an illustration of its application, let us take the following problem.

Prob. Pairs of tangents to an ellipse intersect each other at right angles; required the equation to the curve passing through the points of intersection.

Here we have for the equation of one of the tangents

$$y=mx+\sqrt{m^2a^2+b^2},$$

$$\therefore\ y^2-2mxy+m^2x^2=m^2a^2+b^2 \ldots (1)$$

Since the other tangent is at right angles to this, we shall obtain the equation to the former from the equation to the latter by substituting $-\frac{1}{m}$ for m,

$$\therefore\ y=-\frac{1}{m}x+\sqrt{\frac{a^2}{m^2}+b^2},$$

$$\therefore\ m^2y^2+2mxy+x^2=a^2+m^2b^2 \ldots (2)$$

ling (1) and (2) and reducing, we have

$$x^2+y^2=a^2+b^2,$$

which is an equation to a circle, whose radius $=\sqrt{a^2+b^2}$, and centre the same as that of the ellipse.

2. To find the equation to the tangent PT of a parabola. (See *fig.* page 111.)

$$\text{Here } y^2=4ax,\ \therefore\ \frac{dy}{dx}=\frac{2a}{y};$$

substituting this in the general eq. (1.) of the tangent, we have

$$y_{\prime}-y=\frac{2a}{y}(x_{\prime}-x),$$

which is the equation to the tangent PT; or eliminating y^2, we also have

$$yy_{\prime}=2a(x_{\prime}+x).$$

Also by the general eq. (2.), the equation to the normal is

$$y_{\prime}-y=-\frac{y}{2a}(x_{\prime}-x).$$

3. Show that the equation to the tangent in a hyperbola is, $a^2yy_{\prime}-b^2xx_{\prime}=-a^2b^2$, taking the origin at the centre C.

4. Show that the equation to the tangent in a cissoid is

$$y_{\prime}=\frac{x^{\frac{1}{2}}}{(2r-x)^{\frac{3}{2}}}\{(3r-x)\,x_{\prime}-rx\}.$$

Inclination of Curves and Tangents.

78. The angle which a curve makes with its axis is obviously the same as that which the tangent makes.

Therefore $\frac{dy}{dx}$ is also equal to the tangent of the angle which the curve makes with the axis of x.

Ex. 1. Required the angle at which the parabola cuts the axis of x at the vertex.

$$\text{Here } \frac{dy}{dx}=\frac{2a}{y}=\tan \text{ inclination to axis } x;$$

but when $y=0$, $\frac{2a}{y}=\infty=\tan 90°$, therefore at the vertex the angle is 90°, that is, the curve at the vertex is perpendicular to the axis.

2. At what point in the ellipse is the curve parallel to the axis of x.

Here by *Ex.* 1., Art. **77.**,

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\cdot\frac{x}{y}=\tan 0, \text{ when } x=0\;;$$

that is, the curve at the extremity of the minor axis is parallel to the major axis.

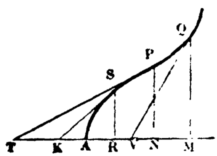
3. Let $y=\frac{x}{1+x^2}$ be the equation to the curve. At what angle does the curve cut the axis of x?

$$\frac{dy}{dx}=\frac{1-x^2}{(1+x^2)^2}=1=\tan 45°, \text{ when } x=0, \text{ and } \therefore\ y=0\;;$$

therefore the curve cuts the origin at an angle of 45° to the axis.

Points of Contrary Flexure.

79. *Definition.* A point of inflexion or *contrary flexure* is a point where a curve passes from concavity to convexity, or from convexity to concavity.



Let ASPQ be a curve, concave from A to P, and convex from P to Q, then P will be a point of contrary flexure. Draw the tangents SK, PT, and QV to the points S, P, and Q; then if the ordinate SR move from A along AM, it is evident that the angle SKR will be decreasing until SR comes to the position PN, but after this the angle will be increasing; thus the angle PTN is less than either of the angles SKR or QVM. Hence it follows, that at a point P of inflexion the angle PTN, and consequently its tangent, or $\frac{dy}{dx}$, must be a maximum or a minimum, that is, we shall have $\frac{d^2y}{dx^2}=0$ or ∞, at the same time observing the criterion given in Art. **50.**

The curve is concave from A to P, and here we observe that the angle SKR is decreasing, and consequently its tangent, or $\frac{dy}{dx}$, is decreasing; that is, $\frac{d^2y}{dx^2}$ is negative; and this takes place throughout the concave curve AP. (See Art. **48.**) In like manner, it may be shown that $\frac{d^2y}{dx^2}$ is positive from P to Q, where the curve is convex to the axis of x. This change of sign from $+$ to $-$, or $-$ to $+$, is the most direct indication of a point of inflexion.

Ex. 1. Required the point P of contrary flexure in the curve APQ, whose equation is $y=2(x-a)^3$.

$$\frac{dy}{dx}=6(x-a)^2, \text{ and } \frac{d^2y}{dx^2}=12(x-a);$$

$$\therefore\ 12(x-a)=0, \text{ and } \therefore\ x=a\,;$$

$\therefore$ take AN $=a$, and the ordinate NP will cut the curve in the point P, which may be that of contrary flexure. To assure ourselves of this, we observe that $\frac{dy}{dx}$, or $6(x-a)^2$, is decreasing as we augment x up to a, and on the contrary $\frac{dy}{dx}$ is increasing as we augment x above a; therefore the curve is concave from A to P, and convex from P to Q, and hence P is the point of contrary flexure. Or we may also observe that for all values of x less than a, the value of $\frac{d^2y}{dx^2}$, or $12(x-a)$, is minus, and, on the contrary, for all values of x greater than a it is plus; hence, &c.

$$\text{Also } \tan \text{PTN}=\frac{dy}{dx}=6(x-a)^2=0, \text{ when } x=a\,,$$

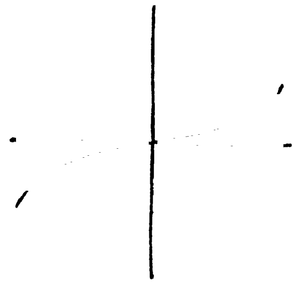
therefore the tangent at the point of contrary flexure is parallel to the axis AM.

2. The equation to the cubical parabola is $a^2y=x^3$.

$$\therefore\ \frac{dy}{dx}=\frac{3x^2}{a^2}, \text{ and } \frac{d^2y}{dx^2}=\frac{6x}{a^2}\,;$$

$$\therefore\ \frac{6x}{a^2}=0, \text{ gives } x=0,$$

hence the contrary flexure, if any, must take place at the origin.



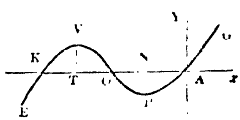
In the equation $a^2y=x^3$, when x is plus, y is also plus; and when x is minus, y is also minus; therefore the curve consists of two identical branches, as shown in the annexed cut. Now $\frac{dy}{dx}=\frac{3x^2}{a^2}$ is increasing as x is being increased, therefore the right branch must be convex to the axis of x; and from the identity of the two branches, the left branch must also be convex to the same axis; therefore the curve has a point of contrary flexure at the origin.

Or we may arrive at the same result, by observing that the value of $\frac{d^2y}{dx^2}$ is minus when x is minus, and plus when x is plus; that is, $\frac{d^2y}{dx^2}$ changes its sign as x passes through 0.

80. TRACING OF CURVES.

Ex. 1. Required the form of the curve, whose equation is

$$y=x(x+1)(x+2)=x^3+3x^2+2x.$$



(1.) To find where the curve meets the axis Ax.

When $y=0$, we have

$$x(x+1)(x+2)=0$$

$$\therefore x=0, -1, \text{ or } -2;$$

therefore the curve must pass through the origin A, also through O, and K; where AO$=1$, and AK$=2$.

(2.) To find y for particular values of x.

For all positive values of x, the values of y will be positive (which will be shown by substitution in the proposed equation); hence the curve extends to infinity in the right branch AG.

For all minus values of x less than AO or 1, the values of y will be minus; hence the curve from A to O lies *below* the axis of x.

For all minus values of x greater than AO, and less than AK or 2, the values of y will be positive; hence the curve from O to K lies above the axis of x.

For all minus values of x greater than AK or 2, the values of y will be negative; hence the remainder of the curve KE extends indefinitely below the axis of x.

(3.) To find the inclination of the curve at the points A, O, and K.

$$\tan \angle \text{ inclination} = \frac{dy}{dx} = 3x^2 + 6x + 2;$$

$$\text{when } x = \ \ 0, \tan \angle \text{ at A} = \frac{dy}{dx} = \ \ 2, \therefore \angle \text{ at A} = 63° \ 26';$$

$$\text{when } x = -1, \tan \angle \text{ at O} = \frac{dy}{dx} = -1, \therefore \angle \text{ at O} = 135°;$$

$$\text{when } x = -2, \tan \angle \text{ at K} = \frac{dy}{dx} = \ \ 2, \therefore \angle \text{ at K} = 63° \ 26'.$$

(4.) To find the points in the curve which run parallel to the axis of x.

In this case, the angle which the direction of the curve makes with the axis of x must be 0, therefore $\frac{dy}{dx}$ must be 0,

$$\therefore 3x^2 + 6x + 2 = 0, \text{ whence } x = -1 \pm \sqrt{\tfrac{1}{3}};$$

$\therefore$ take AN $= 1 - \sqrt{\frac{1}{3}}$, and AT $= 1 + \sqrt{\frac{1}{3}}$; and the ordinates NP and TV will cut the curve in the points required. At these points the ordinates obviously attain their maximum values.

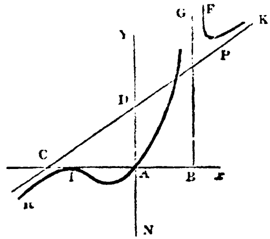
(5.) To find the points of contrary flexure.

$$\text{Here } \frac{d^2y}{dx^2} = 6x + 6 = 0, \therefore x = -1,$$

therefore there is a point of contrary flexure at O.

2. Required the form of the curve whose equation is

$$y = \frac{x(x+1)^2}{(x-1)^2} \quad . \ . \ . \ (1)$$



(1.) To find where the curve meets the axis of x.

This is readily done by finding the values of x corresponding to $y=0$; thus we have

$x(x+1)^2=0$, $\therefore$ $x=0$, or -1;

hence the origin is a point in the curve; take $\text{AI}=1$, then I is another point.

(2.) To find y for particular values of x.

For all minus values of x, the values of y are minus; therefore the left branch lies entirely below the axis of x.

For all positive values of x, the values of y are also positive; therefore the right branch lies entirely above the axis of x. Moreover when $x=1$, $y=\infty$; hence take $\text{AB}=1$, and draw BG perpendicular to Ax, then the curve tends continually towards BG. But when x is taken greater than 1 or AB, the curve reappears in the form FK; and when $x=\infty$, we also have $y=\infty$, that is, the curve here branches off to infinity.

(3.) To find the asymptote to the curve.

$$y=\frac{x(x+1)^2}{(x-1)^2}=x+4+\frac{8}{x}+\&c., \text{ by division;}$$

$\therefore$ $y=x+4$, is the equation to the asymptote.

To construct this line. Take $x=0$, then $y=4$; take $y=0$, then $x=-4$; $\therefore$ take $\text{AD}=4$, and $\text{AC}=4$; join DC, then this line produced is the asymptote to the two branches IR and PK.

(4.) To find the inclination of the curve at the points A and I.

Differentiating the proposed equation to the curve,

$$\frac{dy}{dx}=\frac{x^3-3x^2-5x-1}{(x-1)^3}=\tan\angle \text{ inclination}$$

$$\text{when } x=-1, \tan \angle \text{ at I}=\frac{dy}{dx}=0, \therefore \angle \text{ at I}=0,$$

that is, the curve touches the axis of x at I;

$$\text{when } x=0, \tan \angle \text{ at A}=\frac{dy}{dx}=1, \therefore \angle \text{ at A}=45^\circ.$$

(5.) To find the points of contrary flexure.

Differentiating the value of $\frac{dy}{dx}$ above found,

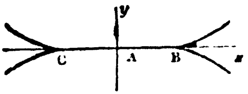
$$\frac{d^2y}{dx^2}=\frac{8(2x+1)}{(x-1)^4}, \text{ which must be taken 0 or } \infty;$$

$$\text{then } 8(2x+1)=0, \therefore x=-\tfrac{1}{2};$$

therefore a point of contrary flexure takes place between A and I at the distance $\frac{1}{2}$ from A. To assure ourselves of this, we observe that minus values of x less than $\frac{1}{2}$ render $\frac{d^2y}{dx^2}$ plus, whereas minus values of x greater than $\frac{1}{2}$ render it minus.

3. Let $y=(x^2-1)^{\frac{3}{2}}$ be the equation to the curve.

Then when $y=0$, $x=\pm 1$; hence take AB$=1$, and AC$=1$, then B and C are points in the curve. When x is less than ± 1, the value of y is impossible; therefore the curve does not approach the origin A nearer than B or C. But for *all* + or − values of x greater than 1, the values of y are possible, and are either plus or minus; hence the curve extends indefinitely above as well as below the axis of x. When the curve runs parallel to the axis of x, we have



$$\frac{dy}{dx}=3(x^2-1)^{\frac{1}{2}}=0, \therefore x=\pm 1;$$

therefore the right hand branches touch Ax at B, and the left hand ones at C, that is, they meet in a common tangent, BC, without intersecting. Singular points of this kind are called *cusps*.

4. Show that the witch has two branches symmetrical with respect to the axis of x, that the axis of y forms an asymptote to the curve, and that there are two points of contrary flexure at the distance $x=\frac{3r}{2}$ from the origin.

THE INTEGRAL CALCULUS.

INTEGRATION.

81. *Integration* is the converse of Differentiation; thus as the *differential* of ax^3 is $3ax^2dx$, so the *integral* of $3ax^2dx$ is ax^3. The primary object of the Integral Calculus, therefore, is from a given differential expression to find the function from which it has been derived; this process is called Integration, and the symbol ($\int$) by which it is represented, is consequently the converse of the symbol (d) which represents differentiation; thus $\int(dy)=y$, and generally if $df(x)$ is the differential of $f(x)$, then $\int\{df(x)\}=f(x)$ is the integral of $df(x)$.

Since $4ax^3dx$ is the differential of either ax^4 or $ax^4+\text{C}$, where C is a constant, it follows that the integral of $4ax^3dx$ is generally expressed by

$$\int 4ax^3dx=ax^4+\text{C},$$

where C is called an *arbitrary constant*, the value of which remains to be determined from the peculiar nature of the problem.

Since the integral of any given differential expression is the function from which the given expression is obtained by differentiation, it follows that we can integrate those functions only to which we are led by differentiation. The method to be pursued in integration may be resolved into two divisions:—I. To derive certain elementary rules or forms of integration from a simple inspection of the results of differentiation. II. By various artifices to bring the forms of other functions to be integrated within those so determined.

Elementary Rules of Integration.

82. *Rule* 1. A constant multiplier is not changed by integration; and hence, in a differential expression, it may be written without the sign of integration: thus $\int ax^2dx = a\int x^2dx = \frac{ax^3}{3}$.

$$\text{Since } d\{af(x)\} = adf(x);$$

$$\therefore \int adf(x) = af(x).$$

83. *Rule* 2. To integrate $ax^n dx$, where the index n may be any number except -1; add unity to the index, divide by the index so increased, and the differential of the variable.

$$\text{Since } d\left\{\frac{ax^{n+1}}{n+1} + c\right\} = ax^n dx$$

$$\therefore \int ax^n dx = \frac{ax^{n+1}}{n+1} + c.$$

EXAMPLES.

1. $\int 8x^3dx = \frac{8x^{3+1}}{3+1} + c = 2x^4 + c.$

2. $\int 6ax^2dx = \frac{6ax^{2+1}}{2+1} + c = 2ax^3 + c.$

3. $\int x^5 dx = \frac{x^6}{6} + c.$

4. $\int \sqrt{x} \times dx = \int x^{\frac{1}{2}}\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2x^{\frac{3}{2}}}{3} + c.$

5. $\int \frac{dx}{x^{\frac{1}{3}}} = \int x^{-\frac{1}{3}}\, dx = \frac{3x^{\frac{2}{3}}}{2} + c.$

6. $\int \frac{dx}{x^4} = \int x^{-4} dx = \frac{x^{-4+1}}{-4+1} = -\frac{x^{-3}}{3} = -\frac{1}{3x^3}.$*

7. Required the integrals of the following expressions,

$$5x^4 dx,\ 2axdx,\ \frac{adx}{\sqrt{x}},\ 2x^{\frac{3}{4}}\, dx,\ \frac{2dx}{x^3}.$$

$$\textit{Answers},\ x^5,\ ax^2,\ 2ax^{\frac{1}{2}},\ \frac{8x^{\frac{7}{4}}}{7},\ -\frac{1}{x^2}.$$

84. *Rule* 3. The integral of the sum of any number of functions is equal to the sum of the integrals of the several functions.

$$\text{Since } df(x) + dF(x) + \&c. = d\{f(x) + F(x) + \&c.\}$$

$$\therefore \int\{df(x) + dF(x) + \&c.\} = f(x) + F(x) + \&c.$$

Examples.

1. $\int (6ax^2 + x^3)dx = \int 6ax^2 dx + \int x^3 dx = 2ax^3 + \frac{x^4}{4} + c.$

2. $\int \left(x^n - \frac{2}{x^3}\right) dx = \int x^n dx - \int 2x^{-3} dx = \frac{x^{n+1}}{n+1} + \frac{1}{x^2} + c.$

---

* In all the examples hereafter given, it must be *understood* that the arbitrary constant c is always to be added, although it may not in all cases be printed.

3. Required the integrals of the following expressions:—

$(1+8x^3)dx$, $(a+bx+cx^3)dx$, $(1+b\sqrt{x})dx$, $(x^3-x^{\frac{1}{3}})dx$,

$(x^{-2}+2x)dx$, $\frac{4dx}{x^3}-\frac{6adx}{x^2}$. *Answers*, $x+2x^4$,

$ax+\frac{bx^2}{2}+\frac{cx^4}{4}$, $x+\frac{2bx^{\frac{3}{2}}}{3}$, $\frac{x^4-3x^{\frac{4}{3}}}{4}$, $x^2-\frac{1}{x}$, $\frac{6ax-2}{x^2}$.

4. $\int(a+x^2)^3x^2dx=\int(a^3+3a^2x^2+3ax^4+x^6)x^2dx$
$=\frac{1}{3}a^3x^3+\frac{3}{5}a^2x^5+\frac{3}{7}ax^7+\frac{1}{9}x^9.$

5. $\int(1+x)^2x^3dx=\frac{1}{4}x^4+\frac{2}{5}x^5+\frac{1}{6}x^6.$

85. *Rule* 4. To integrate $a\{f(x)\}^n df(x)$, where $df(x)$ is the differential of the root. Add unity to the index, divide by the index so increased and the differential of the root.

Since, by Art. **43.**, $d\,\frac{a\{f(x)\}^{n+1}}{n+1}=a\{f(x)\}^n df(x)$;

$$\therefore \int a\{f(x)\}^n df(x)=\frac{a\{f(x)\}^{n+1}}{n+1}.$$

Examples.

1. $\int(x+ax^2)^n(1+2ax)dx=\frac{(x+ax^2)^{n+1}}{n+1}$; here it will be observed that $(1+2ax)dx=d(x+ax^2)$, or the differential of the root.

2. $\int(1+x^m)^n x^{m-1}dx=\frac{(1+x^m)^{n+1}x^{m-1}dx}{(n+1)\times mx^{m-1}dx}=\frac{(1+x^m)^{n+1}}{m(n+1)}.$

3. $\int(a+bx+cx^2)^n(b+2cx)dx=\frac{1}{n+1}(a+bx+cx^2)^{n+1}.$

4. $\int\frac{xdx}{\sqrt{1+x^2}}=\int(1+x^2)^{-\frac{1}{2}}xdx=\frac{(1+x^2)^{\frac{1}{2}}xdx}{\frac{1}{2}\times 2xdx}=(1+x^2)^{\frac{1}{2}}.$

5. $\int(1-x^4)^{\frac{5}{3}}x^3dx=\frac{(1-x^4)^{\frac{8}{3}}x^3dx}{\frac{8}{3}\times(-4x^3dx)}=-\frac{3}{32}(1-x^4)^{\frac{8}{3}}.$

6. $\int\frac{dx}{2(1+x)^{\frac{3}{2}}}=\frac{1}{2}\int(1+x)^{-\frac{3}{2}}dx=\frac{-1}{\sqrt{1+x}}.$

7. $\int\frac{x^3dx}{(a^4-x^4)^2}=\frac{1}{4(a^4-x^4)}.$

8. Integrate the following expressions, $(1+x)^{\frac{1}{2}}dx$,

$(1+3x^2)^n xdx$, $(1+2x+3x^2)^3(2+6x)dx$, $(1-2x)^{-n}dx$.

Answers, $\frac{2}{3}(1+x)^{\frac{3}{2}}$, $\frac{(1+3x^2)^{n+1}}{6(n+1)}$, $\frac{1}{4}(1+2x+3x^2)^4$,

$$\frac{1}{2(n-1)(1-2x)^{n-1}}.$$

Expressions which do not appear in the form for the direct application of this rule, may sometimes be brought to that form by an easy reduction.

9. $\int\frac{dx}{x^3(1+x^3)^{\frac{1}{3}}}=\int\frac{dx}{x^4(x^{-3}+1)^{\frac{1}{3}}}=\int(x^{-3}+1)^{-\frac{1}{3}}x^{-4}dx$

$$=\frac{(x^{-3}+1)^{\frac{2}{3}}x^{-4}dx}{\frac{2}{3}(-3x^{-4})dx}=\frac{(x^{-3}+1)^{\frac{2}{3}}}{-2}=-\frac{(1+x^3)^{\frac{2}{3}}}{2x^2}.$$

10. $\int\frac{dx}{(1\pm x^2)^{\frac{3}{2}}}=\frac{x}{\sqrt{1\pm x^2}}.$

11. $\int\frac{dx}{x^n(1+x^n)^{\frac{1}{n}}}=\frac{(1+x^n)^{\frac{n-1}{n}}}{(1-n)x^{n-1}}.$

12. $\int\frac{dx}{x\sqrt{2ax-x^2}}=-\frac{\sqrt{2ax-x^2}}{ax}$; the proper form for integration, in this case, is $(2ax^{-1}-1)^{-\frac{1}{2}}x^{-2}dx$.

13. $\int \frac{dx}{\sqrt{x^4 - x^3}} = 2\sqrt{1 - \frac{1}{x}}$.

14. $\int \frac{dx}{x^2(1-x^2)^{\frac{1}{2}}} = \int \frac{x^{-3}dx}{(x^{-2}-1)^{\frac{1}{2}}} = -\frac{(1-x^2)^{\frac{1}{2}}}{x}$.

86. *Rule* 5. When the numerator of a fraction is the differential of the denominator, the integral of the fraction is the logarithm of the denominator to the base e.

$$\text{Since } d \log_e z = \frac{dz}{z};$$

$$\therefore \int \frac{dz}{z} = \log_e z + \text{C}.$$

EXAMPLES.

1. $\int \frac{2axdx}{1+ax^2} = \log_e(1 + ax^2) + \text{C}$;

here $2axdx$ is the differential of $1 + ax^2$.

2. $\int \frac{x^{n-1}dx}{1+x^n} = \frac{1}{n}\int \frac{nx^{n-1}dx}{1+x^n} = \frac{1}{n} \log_e(1 + x^n)$.

3. $\int \frac{adx}{a+bx} = \frac{a}{b}\int \frac{bdx}{a+bx} = \frac{a}{b}\int \frac{d(a+bx)}{a+bx}$

$$= \frac{a}{b} \log_e(a + bx).$$

4. $\int \left\{\frac{1}{a+x} - \frac{1}{b+x}\right\} dx = \int \frac{dx}{a+x} - \int \frac{dx}{b+x}$

$$= \log_e(a+x) - \log_e(b+x) = \log_e \frac{a+x}{b+x}.$$

5. $\int \left\{\frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x+3}\right\} dx = \log_e \frac{x^2-1}{x+3}$.

6. $\int \frac{(1+x^{-\frac{1}{2}})dx}{x+2x^{\frac{1}{2}}} = \int \frac{d(x+2x^{\frac{1}{2}})}{x+2x^{\frac{1}{2}}} = \log_e (x+2x^{\frac{1}{2}})$.

7. $\int \frac{(6x^2-2)dx}{x^3-x+1} = 2 \log_e (x^3-x+1) = \log_e (x^3-x+1)^2$.

8. $\int \frac{xdx}{x^2-a^2} = \log_e \sqrt{x^2-a^2}$.

9. $\int \frac{x^2dx}{1+x^3} = \log_e (1+x^3)^{\frac{1}{3}}$.

10. Required the integral of $\frac{x^3\,dx}{1+x}$.

Here the index of x in the numerator being greater than that in the denominator, we first divide as follows :

$$\frac{x^3}{1+x} = x^2 - x + 1 - \frac{1}{1+x},$$

$$\therefore \int \frac{x^3\,dx}{1+x} = \int \left(x^2 - x + 1 - \frac{1}{1+x}\right)dx = \frac{x^3}{3} - \frac{x^2}{2} + x - \log_e(1+x).$$

11. $\int \frac{x^2dx}{x-1} = \frac{x^2}{2} + x + \log_e (x-1)$.

12. $\int (1+\log_e x)^n \frac{dx}{x} = \frac{(1+\log_e x)^{n+1}}{n+1}$. Here the differential of the root, or $d(1+\log_e x) = \frac{dx}{x}$; hence *Rule* 4 applies.

The following examples are important *logarithmic forms.*

13. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \int \frac{dx}{\sqrt{x^2 \pm a^2}} \cdot \frac{x+\sqrt{x^2 \pm a^2}}{x+\sqrt{x^2 \pm a^2}} = \int \frac{(x^2 \pm a^2)^{-\frac{1}{2}} x\,dx + dx}{x+\sqrt{x^2 \pm a^2}}$

$= \log_e (x+\sqrt{x^2 \pm a^2})$, the num^r. being the differential of the denom^r.

14. $\int \frac{dx}{x(a^2 \pm x^2)^{\frac{1}{2}}} = \frac{1}{a} \int \frac{ax^{-2}dx}{(a^2x^{-2} \pm 1)^{\frac{1}{2}}} = -\frac{1}{a} \int \frac{d(ax^{-1})}{\{(ax^{-1})^2 \pm 1\}^{\frac{1}{2}}}$

$$=-\frac{1}{a}\log_e\{ax^{-1}+(a^2x^{-2}\pm 1)^{\frac{1}{2}}\},\text{ by } Ex.\ 13.,$$

$$=\frac{1}{a}\log_e\frac{x}{a+(a^2\pm x^2)^{\frac{1}{2}}}.$$

15. $$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log_e\frac{x-a}{x+a}.$$

For $$\frac{1}{x^2-a^2}=\frac{1}{2a}\left\{\frac{1}{x-a}-\frac{1}{x+a}\right\},$$

$$\therefore\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\int\frac{dx}{x-a}-\frac{1}{2a}\int\frac{dx}{x+a}=\frac{1}{2a}\log_e\frac{x-a}{x+a}.$$

16. $$\int\frac{dx}{a^2-x^2}=-\int\frac{dx}{x^2-a^2}=-\frac{1}{2a}\log_e\frac{x-a}{x+a},$$

$$=\frac{1}{2a}\log_e\frac{x+a}{x-a}.$$

17. $$\int\frac{dx}{(1+x+x^2)^{\frac{1}{2}}}=\int\frac{2dx}{(4+4x+4x^2)^{\frac{1}{2}}}=\int\frac{d(2x+1)}{\{(2x+1)^2+(3^{\frac{1}{2}})^2\}^{\frac{1}{2}}}$$

$$=\log_e\{2x+1+2(1+x+x^2)^{\frac{1}{2}}\},\text{ by } Ex.\ 13.$$

For a *general* formula of integration, see *Ex.* 4. Art. **91.**

18. $$\int\frac{dx}{(x^2-x-1)^{\frac{1}{2}}}=\log_e\{2x-1+2(x^2-x-1)^{\frac{1}{2}}\}.$$

19. $$\int\frac{dx}{1+3x+2x^2}=\log_e\frac{2x+1}{2(x+1)}.\ *$$

87. ***To integrate elementary exponential expressions.***

By Art. **54.** $da^z=\log_e a\,.\,a^z dz$,

$$\therefore\int a^z dz=\frac{1}{\log_e a}\,.\,a^z+\text{C}.$$

* It must be always understood that the symbol log indicates the hyp log., or the log. to the base e.

$$\text{If } a=e, \int e^z dz = e^z + \text{C},$$

where z is any function of x. The rule expressed in this formula admits of immediate application when the differential factor in the proposed integral is the differential of the variable index, or bears some constant ratio to it.

EXAMPLES.

1. $\int e^{ax^2} x dx = \frac{1}{2a} \int e^{ax^2} \times d(ax^2) = \frac{e^{ax^2}}{2a}$.

2. $\int b e^{2x} dx = \frac{be^{2x}}{2}$.

3. $\int a^{nx} dx = \frac{1}{n} \int a^{nx} d(nx) = \frac{a^{nx}}{n \log a}$.

Differenting by *Rules* 5. and 10., we have,

$$d(e^x z) = e^x \left\{ \frac{dz}{dx} + z \right\} dx,$$

$$\therefore \int e^x \left\{ \frac{dz}{dx} + z \right\} dx = e^x z,$$

where the factor of $e^x dx$ is composed of two parts, one of which is the differential coefficient of the other.

4. $\int e^x(3x^2 + x^3 - 1)dx = e^x(x^3 - 1)$, where $3x^2$ is the differential coefficient of $x^3 - 1$.

5. $\int e^x(2x + x^2)dx = e^x x^2$.

6. $\int (1 + e^x)^{\frac{1}{2}} e^x dx = \frac{2}{3}(1 + e^x)^{\frac{3}{2}}$; by *Rule* 4, p. 130.

88. ***To integrate elementary trigonometrical expressions.***

(1.) Since $d \sin z = \cos z dx$,

$$\therefore \int \cos z dz = \sin z + \text{C}.$$

(2.) Since $d \cos z = -\sin z dz$,

$$\therefore \int \sin z dz = -\cos z + \text{C}.$$

(3.) Since $d \tan z = \sec^2 z dz$ or $\frac{dz}{\cos^2 z}$,

$$\therefore \int \sec^2 z dz \text{ or } \int \frac{dz}{\cos^2 z} = \tan z + C.$$

(4.) Since $d \cot z = -\frac{dz}{\sin^2 z}$,

$$\therefore \int \frac{dz}{\sin^2 z} = -\cot z.$$

EXAMPLES.

1. $\int \cos nx \times dx = \frac{1}{n} \int \cos nx \times d(nx) = \frac{1}{n} \sin nx.$

$$\int \sin (m+n)x dx = -\frac{\cos (m+n)x}{m+n}.$$

2. $\int \tan x dx = \int \frac{\sin x dx}{\cos x} = \int \frac{-d \cos x}{\cos x} = -\log \cos x.$

3. $\int \cot x dx = \log \sin x.$

4. $\int \frac{dx}{1+\cos x} = \int \frac{d(\frac{1}{2} x)}{(\cos \frac{1}{2} x)^2} = \tan \frac{1}{2} x$, by (3).

5. $\int \frac{\sin x dx}{1+\cos x} = \int \frac{-d(1+\cos x)}{1+\cos x} = -\log (1+\cos x).$

6. $\int \frac{dx}{\sin x \cos x} = \int \frac{\sec^2 x dx}{\tan x} = \int \frac{d \tan x}{\tan x} = \log \tan x.$

7. As $\frac{1}{\sin x} = \frac{1}{2 \sin \frac{1}{2} x \cos \frac{1}{2} x} = \frac{(\sec \frac{1}{2} x)^2}{2 \tan \frac{1}{2} x}$,

$$\therefore \int \frac{dx}{\sin x} = \int \frac{(\sec \frac{1}{2} x)^2 d(\frac{1}{2} x)}{\tan \frac{1}{2} x} = \int \frac{d \tan \frac{1}{2} x}{\tan \frac{1}{2} x} = \log \tan \frac{1}{2} x.$$

8. $\int \frac{dx}{\cos x} = \int \frac{dx}{\sin (\frac{1}{2} \pi + x)} = \log \tan \left(\frac{1}{4} \pi + \frac{x}{2}\right).$

9. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x.$

10. $\displaystyle\int \frac{dx}{\cos^2 x \sin^2 x} = \int \left\{ \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right\} dx = \tan x - \cot x.$

11. $\int \sin mx \cos nx\, dx = \int \frac{1}{2} \{\sin (m+n)x + \sin (m-n)x\} dx$

$$= -\tfrac{1}{2} \left\{ \frac{\cos (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n} \right\}.$$

By expressing the product of sines and cosines of angles in terms of the sums and differences of sines and cosines of angles, as we have here done, various other formulæ may readily be found.

89. *To integrate elementary circular functions.*

(1.) Since $d \sin^{-1} \dfrac{x}{a} = \dfrac{dx}{\sqrt{a^2 - x^2}}$, Art. **58.**

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + \text{C}.$$

(2.) Since $d \cos^{-1} \dfrac{x}{a} = \dfrac{-dx}{\sqrt{a^2 - x^2}}$,

$$\therefore \int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1} \frac{x}{a} + \text{C}.$$

(3.) Since $d \tan^{-1} \dfrac{x}{a} = \dfrac{adx}{a^2 + x^2}$,

$$\therefore \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + \text{C}.$$

(4.) Since $d \text{ versin}^{-1} \dfrac{x}{a} = \dfrac{dx}{\sqrt{2ax - x^2}}$,

$$\therefore \int \frac{dx}{\sqrt{2ax - x^2}} = \text{versin}^{-1} \frac{x}{a} + \text{C}.$$

(5.) Since $d \sec^{-1}\frac{x}{a}=\frac{adx}{x\sqrt{x^2-a^2}}$,

$$\therefore \int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}+\text{C}.$$

EXAMPLES.

1. $\displaystyle\int\frac{adx}{(a-bx^2)^{\frac{1}{2}}}=\frac{a}{b^{\frac{1}{2}}}\int\frac{dx}{\left(\frac{a}{b}-x^2\right)^{\frac{1}{2}}}$

$$=\frac{a}{b^{\frac{1}{2}}}\sin^{-1}\frac{x}{\sqrt{\frac{a}{b}}}=\frac{a}{b^{\frac{1}{2}}}\sin^{-1}x\sqrt{\frac{b}{a}},$$

by form (1), where we put $\frac{a}{b}$ for a^2.

2. $\displaystyle\int\frac{dx}{a+bx^2}=\frac{1}{b}\int\frac{dx}{\frac{a}{b}+x^2}$, by form (3),

$$=\frac{1}{b}\cdot\frac{1}{\sqrt{\frac{a}{b}}}\tan^{-1}\frac{x}{\sqrt{\frac{a}{b}}}=\frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{b}{a}}.$$

3. $\displaystyle\int\frac{xdx}{(a^4-x^4)^{\frac{1}{2}}}=\frac{1}{2}\int\frac{d(x^2)}{\{a^4-(x^2)^2\}^{\frac{1}{2}}}=\frac{1}{2}\sin^{-1}\frac{x^2}{a^2}.$

4. $\displaystyle\int\frac{xdx}{a^4+x^4}=\frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}.$

5. $\displaystyle\int\frac{dx}{1-2x+2x^2}=\int\frac{2dx}{2-4x+4x^2}=\int\frac{d(2x-1)}{(2x-1)^2+1}$

$$=\tan^{-1}(2x-1)\text{; by form (3).}$$

For a general method of integrating expressions of this form, see *Ex.* 3. Art. 91.

6. $\int \frac{dx}{1+x+x^2} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{1+2x}{\sqrt{3}}.$

7. $\int \frac{dx}{(1+2x-x^2)^{\frac{1}{2}}} = \int \frac{d(x-1)}{\{2-(x-1)^2\}^{\frac{1}{2}}} = \sin^{-1} \frac{x-1}{\sqrt{2}}$; by (1).

8. $\int \frac{dx}{(1-x-x^2)^{\frac{1}{2}}} = \sin^{-1} \frac{2x+1}{\sqrt{5}}.$

Fundamental Formulæ.

90. Collecting the results of the preceding articles, we have

(*a*.) $\int x^n dx = \frac{x^{n+1}}{n+1}$, except when $n=-1$, and then

(*b*.) $\int \frac{dx}{x} = \log x.$

(*c*.) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}.$

(*d*.) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}.$

(*e*.) $\int \frac{dx}{(a^2-x^2)^{\frac{1}{2}}} = \sin^{-1} \frac{x}{a}$, and $\int \frac{-dx}{(a^2-x^2)^{\frac{1}{2}}} = \cos^{-1} \frac{x}{a}.$

(*f*.) $\int \frac{dx}{(x^2 \pm a^2)^{\frac{1}{2}}} = \log \{x + (x^2 \pm a^2)^{\frac{1}{2}}\}.$

(*g*.) $\int \frac{dx}{x(x^2-a^2)^{\frac{1}{2}}} = \frac{1}{a} \sec^{-1} \frac{x}{a}.$

(*h*.) $\int \frac{dx}{x(a^2 \pm x^2)^{\frac{1}{2}}} = \frac{1}{a} \log \frac{x}{a+(a^2 \pm x^2)^{\frac{1}{2}}}.$

(*i*.) $\int a^x dx = \frac{a^x}{\log a}$, and $\int e^{ax} dx = \frac{e^{ax}}{a}.$

(*j.*) $\int \sin mx dx = -\frac{1}{m} \cos mx.$

(*k.*) $\int \cos mx dx = \frac{1}{m} \sin mx.$

(*l.*) $\int \sec^2 mx dx = \frac{1}{m} \tan mx.$

All other integrals are reduced to some one of these forms.

91. By certain easy algebraic processes, many integrals may be brought to some one of these elementary forms.

I. By various modes of transformation.

1. $$\int \frac{dx}{(x^2+2ax)^{\frac{1}{2}}} = \int \frac{d(x+a)}{\{(x+a)^2-a^2\}^{\frac{1}{2}}}$$
$$= \log\{(x^2+2ax)^{\frac{1}{2}}+x+a\}, \text{ by form } (f).$$

2. $$\int \frac{xdx}{\{(x^2-a^2)(b^2-x^2)\}^{\frac{1}{2}}} = \int \frac{d(x^2-a^2)^{\frac{1}{2}}}{\{b^2-a^2-(x^2-a^2)\}^{\frac{1}{2}}}$$
$$= \sin^{-1}\left(\frac{x^2-a^2}{b^2-a^2}\right)^{\frac{1}{2}}, \text{ by form } (e).$$

3. $$\int \frac{dx}{a+bx+cx^2} = 2\int \frac{2cdx}{4ac+4bcx+4c^2x^2}$$
$$= 2\int \frac{d(2cx+b)}{(2cx+b)^2+4ac-b^2},$$

which is integrated by (c) or by (d), according as $4ac-b^2$ is positive or negative. See examples, Art. **89.** and **86.**, for this and the two succeeding formulæ.

4. $$\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}} = \frac{1}{c^{\frac{1}{2}}}\int \frac{2cdx}{(4ac+4bcx+4c^2x^2)^{\frac{1}{2}}}$$

$$=\frac{1}{c^{\frac{1}{2}}}\int\frac{d(2cx+b)}{\{(2cx+b)^2+4ac-b^2\}^{\frac{1}{2}}}$$

$$=\frac{1}{c^{\frac{1}{2}}}\log\{2cx+b+(4c\times\overline{a+bx+cx^2})^{\frac{1}{2}}\},$$
by (f).

5. Similarly $\displaystyle\int\frac{dx}{(a+bx-cx^2)^{\frac{1}{2}}}=\frac{1}{c^{\frac{1}{2}}}\int\frac{d(2cx-b)}{\{4ac+b^2-(2cx-b)^2\}^{\frac{1}{2}}}$

$$=\frac{1}{c^{\frac{1}{2}}}\sin^{-1}\frac{2cx-b}{(4ac+b^2)^{\frac{1}{2}}},\text{ by }(e).$$

6. $\displaystyle\int\frac{dx}{x(a+bx+cx^2)^{\frac{1}{2}}}=\int\frac{x^{-2}dx}{(ax^{-2}+bx^{-1}+c)^{\frac{1}{2}}},$

$$=-\int\frac{d(x^{-1})}{(ax^{-2}+bx^{-1}+c)^{\frac{1}{2}}},$$

which has the same form as Example 4. Thus we have.

$$\int\frac{dx}{x(1+x+x^2)^{\frac{1}{2}}}=-\int\frac{d(x^{-1})}{(x^{-2}+x^{-1}+1)^{\frac{1}{2}}}$$

$$=-\log\{2x^{-1}+1+(4\times\overline{1+x^{-1}+x^{-2}})^{\frac{1}{2}}\}$$

$$=\log\frac{x}{2+x+2(1+x+x^2)^{\frac{1}{2}}}.$$

II. By splitting an expression.

1. $\displaystyle\int\frac{(c+bx)dx}{a^2+x^2}=c\int\frac{dx}{a^2+x^2}+b\int\frac{xdx}{a^2+x^2}$

$$=\frac{c}{a}\tan^{-1}\frac{x}{a}+\frac{b}{2}\log(a^2+x^2)$$

2. $\displaystyle\int\frac{xdx}{(x^2+a)(x^2+b)}=\int\frac{xdx}{b-a}\left\{\frac{1}{x^2+a}-\frac{1}{x^2+b}\right\}$

$$=\frac{1}{2(b-a)}\log\frac{x^2+a}{x^2+b}.$$

3. $\displaystyle\int\frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{1}{b^2-a^2}\left\{\frac{1}{a}\tan^{-1}\frac{x}{a}-\frac{1}{b}\tan^{-1}\frac{x}{b}\right\}.$

4. $\displaystyle\int\frac{xdx}{(x+1)^2}=\int\left\{\frac{1}{x+1}-\frac{1}{(x+1)^2}\right\}dx=\log(x+1)+\frac{1}{x+1}$

5. $\displaystyle\int\frac{dx}{x(a+bx^3)}=\frac{1}{3a}\log\frac{x^3}{a+bx^3}.$

Here $\displaystyle\frac{1}{x(a+bx^3)}=\frac{1}{a}\left\{\frac{1}{x}-\frac{bx^2}{a+bx^3}\right\}.$

6. $\displaystyle\int\frac{e^x xdx}{(1+x)^2}=\int\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}e^x dx=\frac{e^x}{1+x},$

since the second term within the brackets is the differential coefficient of the first. See *Ex.* 4., Art. **87**.

III. By adding and subtracting the same quantity.

1. $\displaystyle\int\frac{xdx}{(2ax-x^2)^{\frac{1}{2}}}=\int\left\{\frac{adx}{(2ax-x^2)^{\frac{1}{2}}}-\frac{(a-x)dx}{(2ax-x^2)^{\frac{1}{2}}}\right\}$

$\displaystyle=a\text{ versin}^{-1}\frac{x}{a}-(2ax-x^2)^{\frac{1}{2}}.$

2. $\displaystyle\int(a+bx)^{\frac{1}{2}}xdx=\frac{1}{b}\int(\overline{a+bx}-a)(a+bx)^{\frac{1}{2}}dx$

$\displaystyle=\frac{1}{b}\int(a+bx)^{\frac{3}{2}}dx-\frac{a}{b}\int(a+bx)^{\frac{1}{2}}dx=\frac{2}{5b^2}(a+bx)^{\frac{5}{2}}-\frac{2a}{3b^2}(a+bx)^{\frac{3}{2}}.$

3. $\displaystyle\int(1+x)^n x^2dx=\int(1+x)^n(\overline{1+x}-1)^2dx$

$\displaystyle=\int(1+x)^{n+2}dx-2\int(1+x)^{n+1}dx+\int(1+x)^n dx$

$\displaystyle=\frac{(1+x)^{n+3}}{n+3}-\frac{2(1+x)^{n+2}}{n+2}+\frac{(1+x)^{n+1}}{n+1}.$

4. $\displaystyle\int\frac{x^2dx}{(1+x)^{\frac{3}{2}}}=\{x^2-4x-8\}\frac{2}{3(1+x)^{\frac{1}{2}}}.$

6. $$\int\frac{(x+1)^{\frac{1}{2}}dx}{x(x-1)^{\frac{1}{2}}}=\int\frac{(x+1)dx}{x(x^2-1)^{\frac{1}{2}}}=\int\frac{dx}{(x^2-1)^{\frac{1}{2}}}+\int\frac{dx}{x(x^2-1)^{\frac{1}{2}}}$$
$$=\log\{x+(x^2-1)^{\frac{1}{2}}\}+\sec^{-1}x.$$

7. $$\int\frac{(1+x)^{\frac{1}{2}}xdx}{(1-x)^{\frac{1}{2}}}=\tfrac{1}{2}\sin^{-1}x-\frac{x+2}{2}(1-x^2)^{\frac{1}{2}}.$$ See Art. **93.**

8. $$\int\frac{(2ax-x^2)^{\frac{1}{2}}dx}{x}=\int\frac{(2a-x)dx}{(2ax-x^2)^{\frac{1}{2}}}$$
$$=\int\frac{(a-x)dx}{(2ax-x^2)^{\frac{1}{2}}}+a\int\frac{dx}{(2ax-x^2)^{\frac{1}{2}}}=(2ax-x^2)^{\frac{1}{2}}+a\,\text{versin}^{-1}\frac{x}{a}.$$

9. $$\int\left(\frac{x+a}{x}\right)^{\frac{1}{2}}dx=\int\frac{(x+a)dx}{(x^2+ax)^{\frac{1}{2}}}$$
$$=\frac{1}{2}\int\frac{(2x+a)dx}{(x^2+ax)^{\frac{1}{2}}}+\frac{a}{2}\int\frac{dx}{(x^2+ax)^{\frac{1}{2}}}$$
$$=(x^2+ax)^{\frac{1}{2}}+\frac{a}{2}\log\{(x^2+ax)^{\frac{1}{2}}+x+\tfrac{1}{2}a\}.$$

(See *Ex.* 1. p. 140.)

10. $$\int\frac{(1-x^2)^{\frac{1}{2}}dx}{x}=\log\frac{x}{1+(1-x^2)^{\frac{1}{2}}}+(1-x^2)^{\frac{1}{2}}.$$

V. By substitution.

1. Let the integral be $u=\int\frac{1-x^{\frac{1}{2}}}{1-x^{\frac{1}{3}}}dx.$

Let $x=z^6$, where the exponent of z is equal to the product of the denominators of the exponents of x, $\therefore\ dx=6z^5dz$; hence we have, by substitution and division,

$$u=\int\frac{1-z^3}{1-z^2}\cdot 6z^5dz=6\int\left(z^6+z^4-z^3+z^2-z+1-\frac{1}{1+z}\right)dz$$
$$=6\left\{\frac{x^{\frac{7}{6}}}{7}+\frac{x^{\frac{5}{6}}}{5}-\frac{x^{\frac{2}{3}}}{4}+\frac{x^{\frac{1}{2}}}{3}-\frac{x^{\frac{1}{3}}}{2}-x^{\frac{1}{6}}-\log(1+x^{\frac{1}{6}})\right\}.$$

2. Let the integral be $u=\int\frac{dx}{(2+x)(1+x)^{\frac{1}{2}}}$.

Assume $(1+x)^{\frac{1}{2}}=z$, $\therefore$ $1+x=z^2$, $dx=2zdz$, and $2+x=z^2+1$; therefore, by substitution, $u=\int\frac{2dz}{z^2+1}=2\tan^{-1}z=2\tan^{-1}(1+x)^{\frac{1}{2}}$.

After the same method $\int\frac{dx}{(c+ex)(a+bx)^{\frac{1}{2}}}$ is found.

3. $u=\int\frac{dx}{(c+ex)(a+bx^2)^{\frac{1}{2}}}$; let $c+ex=z$; taking the log of each side and differentiating, $\frac{edx}{c+ex}=\frac{dz}{z}$;

and $a+bx^2=a+\frac{b}{e^2}(z-c)^2=\frac{1}{e^2}(ae^2+bc^2-2bcz+bz^2)$,

$$\therefore \text{ by subst., } u=\int\frac{dz}{z(ae^2+bc^2-2bcz+bz^2)^{\frac{1}{2}}},$$

which is integrated as in *Ex.* 6, Art. **91.**

4. $$u=\int\frac{dx}{(c+ex^2)(a+bx^2)^{\frac{1}{2}}}=\int\frac{(ax^{-2}+b)^{-\frac{1}{2}}x^{-3}dx}{cx^{-2}+e}$$

$$=-\int\frac{d(ax^{-2}+b)^{\frac{1}{2}}}{a(cx^{-2}+e)}=-\int\frac{dz}{cz^2+ae-cb},$$ by making $(ax^{-2}+b)^{\frac{1}{2}}=z$. This will be integrated by form (c) or (d) according as $ae-cb$ is $+$ or $-$.

If $ae=cb$, then $u=-\int\frac{dz}{cz^2}=\frac{1}{cz}=\frac{x}{c(a+bx^2)^{\frac{1}{2}}}$.

For example $\int\frac{dx}{(1+2x^2)(2+4x^2)^{\frac{1}{2}}}=\frac{x}{(2+4x^2)^{\frac{1}{2}}}$.

5. $$\int \frac{dx}{(1-x^2)(1+x^2)^{\frac{1}{2}}} = \frac{1}{2^{\frac{1}{2}}} \log_e \frac{(1+x^2)^{\frac{1}{2}} + 2^{\frac{1}{2}}x}{(1-x^2)^{\frac{1}{2}}},$$

here $c=1$, $e=-1$, $a=1$, $b=1$; then by (d) and reducing.

6. Similarly $$\int \frac{xdx}{(c+ex^2)(a+bx^2)^{\frac{1}{2}}} = \int \frac{d(a+bx^2)^{\frac{1}{2}}}{b(c+ex^2)}$$

$$= \int \frac{dz}{ez^2+bc-ae}, \text{ by making } (a+bx^2)^{\frac{1}{2}} = z.$$

7. $$u = \int \frac{\{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{m}{n}} dx}{(1+x^2)^{\frac{1}{2}}}; \text{ let } x+(1+x^2)^{\frac{1}{2}} = z^n;$$

$\therefore \dfrac{dx}{(1+x^2)^{\frac{1}{2}}} = \dfrac{ndz}{z}$ by taking the log of each side, and then differentiating. Hence by subst., &c.

$$u = \int nz^{m-1}dz = \frac{n}{m} z^m = \frac{n}{m}\{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{m}{n}}.$$

8. $\displaystyle\int \frac{dx}{(a+bx)(cx^2+ex+f)^{\frac{1}{2}}}$; this integral may be brought to the form $\displaystyle\int \frac{dz}{a(z^2-2\beta z+\gamma)^{\frac{1}{2}}}$ by making $z = \dfrac{1}{a+bx}$, and proceeding as in the foregoing examples.

RATIONAL FRACTIONS.

92. In a rational fraction the indices of x are all positive integers. Expressions of this kind may be integrated by resolving them into a series of simpler fractions, called partial fractions. This can always be done by the method of indeterminate coefficients explained in (Art. **7.**), *Exs.* 2., 3., 4.

EXAMPLES.

1st. *When the factors in the denominator are all different.*

1. $\int \frac{dx}{x^2-2x} = \int \frac{dx}{x(x-2)}$.

Here by decomposition, $\frac{1}{x(x-2)} = \frac{1}{2}\left\{\frac{1}{x-2} - \frac{1}{x}\right\}$

$$\therefore \int \frac{dx}{x(x-2)} = \frac{1}{2}\int \left\{\frac{dx}{x-2} - \frac{dx}{x}\right\}$$

$$= \frac{1}{2}\left\{\log(x-2) - \log x\right\} = \frac{1}{2}\log\frac{x-2}{x}.$$

2. $\int \frac{a\,dx}{x^2-ax} = \log\frac{x-a}{x}$.

3. $\int \frac{x\,dx}{x^2+6x+8} = \int \frac{x\,dx}{(x+4)(x+2)}$.

Let $\frac{x}{(x+4)(x+2)} = \frac{A}{x+4} + \frac{B}{x+2}$; clearing of fractions,

$$x = A(x+2) + B(x+4);$$

to find A, let $x=-4$; then $-4 = A(-4+2)$, $\therefore A=2$;
to find B, let $x=-2$, then $-2=2B$, $\therefore B=-1$;

substituting the values of A and B, we have

$$\frac{x}{(x+4)(x+2)} = \frac{2}{x+4} - \frac{1}{x+2},$$

$$\therefore \int \frac{x\,dx}{(x+4)(x+2)} = \int \frac{2dx}{x+4} - \int \frac{dx}{x+2}$$

$$= 2\log(x+4) - \log(x+2) = \log\frac{(x+4)^2}{x+2}.$$

4. $\int \frac{dx}{x^2+6x+8} = \frac{1}{2}\log\frac{x+2}{x+4}$.

5. $\int \frac{a\,dx}{x^2-5x+6} = a \log \frac{x-3}{x-2}.$

6. $\int \frac{(x^2+2)dx}{x^3+7x^2+14x+8} = \int \frac{(x^2+2)dx}{(x+1)(x+2)(x+4)}.$

Let $\frac{x^2+2}{(x+1)(x+2)(x+4)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+4};$

clearing of fractions, by multiplying by $(x+1)(x+2)(x+4)$,

$x^2+2 = A(x+2)(x+4) + B(x+1)(x+4) + C(x+1)(x+2);$

to find A, let $x=-1$, then $3 = 3A$, $\therefore A = 1$;

to find B, let $x=-2$, then $6 = -2B$, $\therefore B = -3$;

to find C, let $x=-4$, then $18 = 6C$, $\therefore C = 3$;

substituting these values of A, B, and C,

$$\frac{x^2+2}{(x+1)(x+2)(x+4)} = \frac{1}{x+1} - \frac{3}{x+2} + \frac{3}{x+4};$$

$$\therefore \int \frac{(x^2+2)dx}{(x+1)(x+2)(x+4)} = \int \frac{dx}{x+1} - \int \frac{3dx}{x+2} + \int \frac{3dx}{x+4}$$

$$= \log(x+1) - 3\log(x+2) + 3\log(x+4)$$

$$= \log \frac{(x+1)(x+4)^3}{(x+2)^3}.$$

7. $\int \frac{x^2dx}{x^3+6x^2+11x+6} = \log \frac{\sqrt{(x+1)(x+3)^9}}{(x+2)^4}.$

Here the factors of the denominator are $(x+1), (x+2), (x+3)$.

8. $\int \frac{(1+3x^2)dx}{x-x^3} = \log \frac{x}{(1-x^2)^2}.$

Obs. In like manner, if the denominator contains four factors, we should resolve the expression into four partial fractions; and so on to other cases.

2d. *When some of the factors are equal.*

When the denominator has the form of $(x-a)^n(x-b)(x-c)\ldots$&c., we may readily decompose the fraction into its partial fractions by assuming it equal to

$$\frac{A}{(x-a)^n}+\frac{B}{(x-a)^{n-1}}+\text{\&c.}+\frac{N}{x-a}+\frac{P}{x-b}+\frac{Q}{x-c}+\text{\&c.}$$

1. Required the integral of $\int\frac{(x^2+x)dx}{(x-2)^2(x-1)}$.

$$\text{Let } \frac{x^2+x}{(x-2)^2(x-1)}=\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x-1},$$

$$\therefore\ x^2+x=A(x-1)+B(x-2)(x-1)+C(x-2)^2\ldots(1);$$

to find A, let $x=2$; then $2^2+2=A\ldots(2)$.

Subtracting eq. (2) from (1) we have,

$$x^2-2^2+x-2=A(x-2)+B(x-2)(x-1)+C(x-2)^2,$$

and dividing by $x-2$,

$$x+2+1=A+B(x-1)+C(x-2);$$

substituting the value of $A(=6)$ derived from (2), and reducing,

$$x-3=B(x-1)+C(x-2);$$

to find B, let $x=2$, then $B=-1$;

to find C, let $x=1$, then $C=2$;

$$\therefore\ \frac{x^2+x}{(x-2)^2(x-1)}=\frac{6}{(x-2)^2}-\frac{1}{x-2}+\frac{2}{x-1}$$

$$\therefore\int\frac{(x^2+x)dx}{(x-2)^2(x-1)}=\int\frac{6dx}{(x-2)^2}-\int\frac{dx}{x-2}+\int\frac{2dx}{x-1}$$

$$=-\frac{6}{x-2}-\log(x-2)+2\log(x-1)$$

$$=-\frac{6}{x-2}+\log\frac{(x-1)^2}{x-2}.$$

The method employed in this investigation will apply to all similar cases.

2. $\int \frac{x\,dx}{(x+2)^2(x+1)} = -\frac{2}{x+2} + \log\frac{x+2}{x+1}$.

3. $\int \frac{dx}{(x+2)^2(x-1)} = \frac{1}{3(x+2)} + \frac{1}{9}\log\frac{x-1}{x+2}$.

4. To integrate $\frac{(x^2-2)dx}{x^4-x^3}$ or $\frac{(x^2-2)dx}{x^3(x-1)}$.

$$\text{Let } \frac{x^2-2}{x^3(x-1)} = \frac{\text{A}}{x^3} + \frac{\text{B}}{x^2} + \frac{\text{C}}{x} + \frac{\text{D}}{x-1},$$

$$\therefore\ x^2-2 = \text{A}(x-1) + \text{B}x(x-1) + \text{C}x^2(x-1) + \text{D}x^3 \ldots (1);$$

to find A, let $x=0$, then $\text{A}=2$;

substituting this value of A in (1), transposing and dividing by x,

$$x = 2 + \text{B}(x-1) + \text{C}x(x-1) + \text{D}x^2 \ldots (2);$$

to find B, let $x=0$, then $\text{B}=2$;

substituting this value of B in (2), reducing and dividing by x,

$$1 = 2 + \text{C}(x-1) - \text{D}x \ldots (3);$$

to find C, let $x=0$, then $\text{C}=1$;

to find D, let $x=1$, then $\text{D}=-1$;

$$\therefore\ \frac{x^2-2}{x^3(x-1)} = \frac{2}{x^3} + \frac{2}{x^2} + \frac{1}{x} - \frac{1}{x-1};$$

$$\therefore \int \frac{(x^2-2)dx}{x^3(x-1)} = \int \frac{2\,dx}{x^3} + \int \frac{2\,dx}{x^2} + \int \frac{dx}{x} - \int \frac{dx}{x-1}$$

$$= -\frac{1}{x^2} - \frac{2}{x} + \log x - \log(x-1) = -\frac{1+2x}{x^2} + \log\frac{x}{x-1}.$$

5. $\int \frac{(x+10)dx}{x^3-3x^2} = \frac{10}{3x} + \log\left(\frac{x-3}{x}\right)^{\frac{13}{9}}$.

3rd. *When there is a quadratic factor having impossible roots.*

1. $\int \frac{x^2\,dx}{x^3+x^2+x+1} = \int \frac{x^2\,dx}{(x+1)(x^2+1)}$; where x^2+1 is a quadratic factor whose roots are impossible.

$$\text{Let } \frac{x^2}{(x+1)(x^2+1)} = \frac{\text{A}}{x+1} + \frac{\text{M}x+\text{N}}{x^2+1};$$

clearing of fractions,

$$x^2 = \text{A}(x^2+1) + (x+1)(\text{M}x+\text{N}) \ldots (1),$$

to find A, let $x=-1$, then $\text{A}=\frac{1}{2}$;

substituting this value of A in (1), and transposing,

$$(x+1)(\text{M}x+\text{N}) = x^2 - \frac{x^2+1}{2} = \frac{x^2-1}{2};$$

dividing by $x+1$,

$$\text{M}x+\text{N} = \frac{x-1}{2},$$

$$\therefore \frac{x^2}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{x-1}{2(x^2+1)},$$

$$\therefore \int \frac{x^2\,dx}{x^3+x^2+x+1} = \int \frac{dx}{2(x+1)} + \int \frac{(x-1)dx}{2(x^2+1)}$$

$$= \tfrac{1}{2}\log(x+1) + \tfrac{1}{2}\int \frac{x\,dx}{x^2+1} - \tfrac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \tfrac{1}{2}\log(x+1) + \tfrac{1}{4}\log(x^2+1) - \tfrac{1}{2}\tan^{-1}x.$$

2. $\int \frac{dx}{x^3-x^2+x-1} = -\frac{1}{2}\log\frac{\sqrt{x^2+1}}{x-1} - \frac{1}{2}\tan^{-1}x.$

3. $\int \frac{(x^2-x+1)dx}{x^3+x^2+x+1} = \log\frac{(x+1)^{\frac{3}{2}}}{(x^2+1)^{\frac{1}{4}}} - \frac{1}{2}\tan^{-1}x.$

4. To integrate $\frac{x^2\,dx}{(x^2+1)(x^2+2)}$, where there are two quadratic factors having impossible roots.

Here we must assume

$$\frac{x^2}{(x^2+1)(x^2+2)}=\frac{\text{A}x+\text{B}}{x^2+1}+\frac{\text{C}x+\text{D}}{x^2+2};$$

clearing of fractions and arranging the terms, in the left hand member, according to the powers of x,

$$x^2=(\text{A}+\text{C})x^3+(\text{B}+\text{D})x^2+(2\text{A}+\text{C})x+2\text{B}+\text{D};$$

hence we have, by equating the coefficients of the like powers of x,

$$\text{A}+\text{C}=0,\text{ and }2\text{A}+\text{C}=0,\ \therefore\ \text{A}=0,\text{ and }\text{C}=0;$$

$$\text{B}+\text{D}=1,\text{ and }2\text{B}+\text{D}=0,\ \therefore\ \text{B}=-1,\text{ and }\text{D}=2;$$

$$\therefore\ \frac{x^2}{(x^2+1)(x^2+2)}=-\frac{1}{x^2+1}+\frac{2}{x^2+2};$$

$$\therefore\int\frac{x^2\,dx}{(x^2+1)(x^2+2)}=-\int\frac{dx}{x^2+1}+2\int\frac{dx}{x^2+2}$$

$$=-\tan^{-1}x+\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}},\text{ by form }(c).$$

5. $\displaystyle\int\frac{dx}{(x^2+1)(x^2+4)}=\tfrac{1}{3}\tan^{-1}x-\tfrac{1}{6}\tan^{-1}\frac{x}{2}.$

6. $\displaystyle\int\frac{x^2\,dx}{(x+1)(x-1)(x^2+2)}=\tfrac{1}{6}\log\frac{1-x}{1+x}+\frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}.$

Here we must assume

$$\frac{x^2}{(x+1)(x-1)(x^2+2)}=\frac{\text{A}}{x+1}+\frac{\text{B}}{x-1}+\frac{\text{C}x+\text{D}}{x^2+2},$$

and then determine the coefficients by the method followed in *Example* 4, or in *Example* 1.

7. To integrate $\dfrac{xdx}{(x+1)^2(x^2+4)}$.

Assume $\dfrac{x}{(x+1)^2(x^2+4)}=\dfrac{A}{(x+1)^2}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2+4}$,

$\therefore\ x=A(x^2+4)+B(x+1)(x^2+4)+(Cx+D)(x+1)^2$;

to find A, let $x=-1$, then $-1=5A$, $\therefore\ A=-\frac{1}{5}$;

ubstituting this value of A, and adding 1 to each side of the quality, in order to render the equation divisible by $x+1$,

$$x+1=\tfrac{1}{5}(1-x^2)+B(x+1)(x^2+4)+(Cx+D)(x+1)^2,$$

$$\therefore\ 1=\tfrac{1}{5}(1-x)+B(x^2+4)+(Cx+D)(x+1);$$

to find B, let $x=-1$, then $1=\frac{2}{5}+5B$, $\therefore\ B=\frac{3}{25}$;

ubstituting the value of B in the last equation and reducing,

$$Cx+D=-\tfrac{1}{25}\cdot\frac{3x^2-5x-8}{x+1}=-\tfrac{1}{25}(3x-8);$$

$$\therefore\int\frac{x\,dx}{(x+1)^2(x^2+4)}=-\tfrac{1}{5}\int\frac{dx}{(x+1)^2}+\tfrac{3}{25}\int\frac{dx}{x+1}-\tfrac{1}{25}\int\frac{(3x-8)dx}{x^2+4}$$

$$=\tfrac{1}{25}\left\{\frac{5}{x+1}+\tfrac{3}{2}\log\frac{(x+1)^2}{x^2+4}+4\tan^{-1}\frac{x}{2}\right\}.$$ See *Ex.* 1.

8. $\displaystyle\int\frac{x^2\,dx}{(x-1)^2(x^2+1)}=\tfrac{1}{4}\log\frac{(x-1)^2}{x^2+1}-\frac{1}{2(x-1)}$.

4th. *When the highest index of x in the numerator exceeds that in the denominator.*

Fractions of this class (as well as some others) may be brought to a form admitting of integral by actual division.

1. To integrate $\dfrac{x^2dx}{a+bx}$

Here, by actual division, we have

$$\frac{x^2}{bx+a}=\frac{x}{b}-\frac{a}{b^2}+\frac{a^2}{b^2}\cdot\frac{1}{a+bx},$$

$$\therefore \int\frac{x^2dx}{a+bx}=\int\frac{xdx}{b}-\int\frac{adx}{b^2}+\int\frac{a^2}{b^2}\frac{dx}{a+bx}$$

$$=\frac{x^2}{2b}-\frac{ax}{b^2}+\frac{a^2}{b^3}\log(a+bx).$$

2. $\int\frac{x^3dx}{a+bx}=\frac{x^3}{3b}-\frac{ax^2}{2b^2}+\frac{a^2x}{b^3}-\frac{a^3}{b^4}\log(a+bx).$

3. $\int\frac{xdx}{a+bx}=\frac{x}{b}-\frac{a}{b^2}\log(a+bx).$

4. To integrate $\frac{x^2dx}{x^2-1}$.

$$\text{Here } \frac{x^2}{x^2-1}=1+\frac{1}{x^2-1},$$

$$\therefore \int\frac{x^2dx}{x^2-1}=x+\int\frac{dx}{x^2-1}=x+\tfrac{1}{2}\log\frac{x-1}{x+1}.$$

5. $\int\frac{(x^3-2x)dx}{x^2-1}=\frac{x^2}{2}-\tfrac{1}{2}\log(x^2-1).$

INTEGRATION BY PARTS.

93. Since $d(zv)=zdv+vdz$ (Art. **40.**), where z and v are functions of the variable x,

$$\therefore\ zv=\int zdv+\int vdz,$$

$$\therefore\ \int zdv=zv-\int vdz.$$

This is called the formula of integration by parts; by it we are enabled to integrate any function zdv, provided the function vdz admits of integration.

EXAMPLES.

1. $\int x^3(1+x^2)^4dx = \int x^2(1+x^2)^4 xdx$.

Here we must put $(1+x^2)^4xdx = dv$, since it is obviously the differential of a known function.

$$\text{Let } x^2 = z, \text{ and } (1+x^2)^4xdx = dv,$$

$$\therefore\ 2xdx = dz, \text{ and } \tfrac{1}{10}(1+x^2)^5 = v;$$

$$\begin{aligned}\therefore\ \int x^3(1+x^2)^4dx &= \int zdv \\ &= zv - \int vdz \\ &= x^2 \times \tfrac{1}{10}(1+x^2)^5 - \int \tfrac{1}{10}(1+x^2)^5 \times 2xdx \\ &= \tfrac{1}{60}(1+x^2)^5(5x^2-1).\end{aligned}$$

2. $\int(x^2+a^2)^{\frac{1}{2}}dx = \frac{x}{2}(x^2+a^2)^{\frac{1}{2}} + \frac{a^2}{2}\log\{x+(x^2+a^2)^{\frac{1}{2}}\}$.

$$\text{Let } (x^2+a^2)^{\frac{1}{2}} = z, \text{ and } dx = dv,$$

$$\therefore\ \frac{xdx}{(x^2+a^2)^{\frac{1}{2}}} = dz, \text{ and } x = v,$$

$$\begin{aligned}\therefore\ \int(x^2+a^2)^{\frac{1}{2}}dx &= \int zdv \\ &= zv - \int vdz \\ &= (x^2+a^2)^{\frac{1}{2}}\,.\,x - \int x\,.\,\frac{x\,dx}{(x^2+a^2)^{\frac{1}{2}}} \\ &= x(x^2+a^2)^{\frac{1}{2}} - \int\frac{(x^2+a^2)dx - a^2dx}{(x^2+a^2)^{\frac{1}{2}}} \\ &= x(x^2+a^2)^{\frac{1}{2}} - \int(x^2+a^2)^{\frac{1}{2}}dx + a^2\int\frac{dx}{(x^2+a^2)^{\frac{1}{2}}};\end{aligned}$$

by transposition and form (f), Art. 90., we have

$$2\int(x^2+a^2)^{\frac{1}{2}}dx = x(x^2+a^2)^{\frac{1}{2}} + a^2\log\{x+(x^2+a^2)^{\frac{1}{2}}\},$$

by dividing by 2, the proposed integral is found.

3. $\int(a^2-x^2)^{\frac{1}{2}}dx=\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

Proceeding exactly as in the last example, we find,

$$2\int(a^2-x^2)^{\frac{1}{2}}dx=x(a^2-x^2)^{\frac{1}{2}}+a^2\int\frac{dx}{(a^2-x^2)^{\frac{1}{2}}},$$

whence by form (*e*) and dividing by 2, the integral is found.

4. $\int(x^2+2ax)^{\frac{1}{2}}dx=\int\{(x+a)^2-a^2\}^{\frac{1}{2}}d(x+a)$

$$=\frac{x+a}{2}(x^2+2ax)^{\frac{1}{2}}-\frac{a^2}{2}\log\{x+a+(x^2+2ax)^{\frac{1}{2}}\},$$

by making a^2 minus in *Ex.* 2. and then substituting $x+a$ for x.

5. $\int(2ax-x^2)^{\frac{1}{2}}dx=\int\{a^2-(x-a)^2\}^{\frac{1}{2}}d(x-a)$

$$=\frac{x-a}{2}(2ax-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x-a}{a},$$

by substituting $x-a$ for x in *Ex.* 3.

6. $\int\frac{x^3dx}{(1+x^2)^{\frac{3}{2}}}=\int x^2\cdot(1+x^2)^{-\frac{3}{2}}xdx.$

Let $x^2=z$, and $(1+x^2)^{-\frac{3}{2}}xdx=dv$,

$$\therefore\ 2xdx=dz,\text{ and }\frac{-1}{(1+x^2)^{\frac{1}{2}}}=v;$$

$$\therefore\int\frac{x^3dx}{(1+x^2)^{\frac{3}{2}}}=\int zdv$$

$$=zv-\int vdz$$

$$=-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}+\int\frac{2xdx}{(1+x^2)^{\frac{1}{2}}}$$

$$=-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}+2(1+x^2)^{\frac{1}{2}}=\frac{x^2+2}{(1+x^2)^{\frac{1}{2}}}.$$

7. $$\int \frac{x^3 dx}{(1+x^2)^{\frac{5}{2}}} = -\frac{3x^2+2}{3(1+x^2)^{\frac{3}{2}}}.$$

8. $$\int \frac{x^{\frac{3}{2}} dx}{(2a-x)^{\frac{1}{2}}} = -2x^{\frac{3}{2}}(2a-x)^{\frac{1}{2}} + 3\int (2ax-x^2)^{\frac{1}{2}} dx.$$
(See *Ex.* 5.)

9. $$\int x \log x dx = \frac{x^2}{2}(\log x - \tfrac{1}{2}).$$

Let $\log x = z$, and $x dx = dv$; $\therefore \frac{dx}{x} = dz$, and $\frac{x^2}{2} = v$;

$$\therefore \int x \log x dx = \int z dv$$
$$= zv - \int v dz$$
$$= \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x} \text{ *}$$
$$= \log x \cdot \frac{x^2}{2} - \frac{x^2}{4} = \text{\&c.}$$

10. $$\int x^n \log x dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$$

11. $$\int x e^{ax} dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$$

Let $x = z$, and $e^{ax} dx = dv$; $\therefore dx = dz$, and $\frac{e^{ax}}{a} = v$;

$$\therefore \int x e^{ax} dx = \int z dv$$
$$= zv - \int v dz$$
$$= x \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot dx$$
$$= \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} = \text{\&c.}$$

* The student should endeavour to acquire the power of writing down this equality without going over the intermediate steps given in these examples; thus

$$\int x \log x \, dx = \int \log x \times x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}.$$

12. $\int \sin^{-1} x dx = x \sin^{-1} x + (1-x^2)^{\frac{1}{2}}$.

Let $\sin^{-1} x = z$, and $dx = dv$;

$$\therefore \frac{dx}{(1-x^2)^{\frac{1}{2}}} = dz, \text{ and } x = v;$$

$$\therefore \int \sin^{-1} x dx = \int z dv$$

$$= zv - \int v dz$$

$$= \sin^{-1} x \cdot x - \int \frac{x dx}{(1-x^2)^{\frac{1}{2}}} = \&c.$$

13. $\int \frac{\sin^{-1} x \cdot x dx}{(1-x^2)^{\frac{1}{2}}} = x - (1-x^2)^{\frac{1}{2}} \sin^{-1} x$.

14. $\int \tan^{-1} x dx = x \tan^{-1} x - \log (1+x^2)^{\frac{1}{2}}$.

Rationalization.

94. Functions of the form $\int x^{m-1}(a+bx^n)^{\frac{p}{q}} dx$ may be rationalized when $\frac{m}{n}$ or $\frac{m}{n}+\frac{p}{q}$ is an integer.

(I.) Assume $a+bx^n = z^q$;

$$\therefore x^m = \frac{(z^q - a)^{\frac{m}{n}}}{b^{\frac{m}{n}}}; \text{ then by differentiation,}$$

$$x^{m-1} dx = \frac{q}{nb^{\frac{m}{n}}} z^{q-1}(z^q - a)^{\frac{m}{n}-1} dz;$$

multiplying by $(a+bx^n)^{\frac{p}{q}}$ or z^p, and integrating,

$$\int x^{m-1}(a+bx^n)^{\frac{p}{q}} dx = \frac{q}{nb^{\frac{m}{n}}} \int z^{p+q-1}(z^q - a)^{\frac{m}{n}-1} dz.$$

Now, when $\frac{m}{n}$ is an integer, the binomial $(z^q - a)^{\frac{m}{n}-1}$ can be expressed in a finite series of powers of z, and hence the integral can be obtained in finite terms. The condition $\frac{m}{n}$ = a positive integer, is called the *first criterion.*

(II.) Again, $x^{m-1}(a+bx^n)^{\frac{p}{q}}dx = x^{m+\frac{np}{q}-1}(ax^{-n}+b)^{\frac{p}{q}}$; now, from what has just been shown, this latter expression is integrable if $\left\{m+\frac{np}{q}\right\} \div (-n)$ = a positive integer; that is, if $\frac{m}{n}+\frac{p}{q}$ = a negative integer. This condition is called the *second criterion.*

In this case, therefore, we must first put the expression under the form above given, and then assume $ax^{-n}+b=z^q$.

EXAMPLES.

1. Let $\int x^3(1+x^2)^{\frac{3}{2}}dx$ be required.

Here $n=2$, $m-1=3$; $\therefore$ $m=4$, and $\frac{m}{n}=\frac{4}{2}$ = a positive integer; hence the first criterion is satisfied. Since $p=3$ and $q=2$,

$$\therefore \text{ assume } 1+x^2=z^2; \quad \therefore x^4=(z^2-1)^2;$$

$$(1+x^2)^{\frac{3}{2}}=z^3, \text{ and } x^3dx=(z^2-1)zdz;$$

$$\therefore \int x^3(1+x^2)^{\frac{3}{2}}dx=\int(z^2-1)z^4dz=\frac{z^7}{7}-\frac{z^5}{5}$$

$$=\frac{5z^2-7}{35}\cdot z^5=\frac{5x^2-2}{35}(1+x^2)^{\frac{5}{2}}.$$

2. $\int \frac{dx}{x^4(1+x^2)^{\frac{1}{2}}}=\int x^{-4}(1+x^2)^{-\frac{1}{2}}dx.$

Here $n=2$, $m-1=-4$; $\therefore$ $m=-3$, and $\frac{m}{n}=-\frac{3}{2}$; therefore the first criterion does not apply; but $q=2$, $\frac{p}{q}=-\frac{1}{2}$, and $\frac{m}{n}+\frac{p}{q}=-\frac{3}{2}-\frac{1}{2}=-2$; which shows that the second criterion applies.

Then $\int\frac{dx}{x^4(1+x^2)^{\frac{1}{2}}}=\int x^{-5}(x^{-2}+1)^{-\frac{1}{2}}dx$;

$\therefore$ assume $x^{-2}+1=z^2$; $\therefore$ $x^{-4}=(z^2-1)^2, x^{-5}dx=-z(z^2-1)dz$

$$\therefore \int\frac{x^{-5}dx}{(x^{-2}+1)^{\frac{1}{2}}}=-\int\frac{z(z^2-1)dz}{z}=-\frac{z^3}{3}+z$$

$$=\frac{(2x^2-1)(1+x^2)^{\frac{1}{2}}}{3x^3}.$$

3. $\int x^2(1+x)^{\frac{1}{2}}dx.$

Here $\frac{2+1}{1}=3$, therefore the first criterion applies;

$\therefore$ let $1+x=z$, $\therefore$ $x^3=(z-1)^3$, $\therefore$ $x^2dx=(z-1)^2dz$;

$$\therefore \int x^2(1+x)^{\frac{1}{2}}dx=\int z^{\frac{1}{2}}(z-1)^2dz,$$

$$=\tfrac{2}{35}z^{\frac{3}{2}}(5z^2-14z+\tfrac{35}{3})=\tfrac{2}{35}(1+x)^{\frac{3}{2}}(5x^2-4x+\tfrac{8}{3}).$$

4. $\int\frac{x^{\frac{1}{3}}dx}{(1+x^{\frac{2}{3}})^{\frac{1}{2}}}=4(1+x^{\frac{1}{3}})^{\frac{1}{2}}\{\tfrac{1}{5}(1+x^{\frac{1}{3}})^2-\tfrac{2}{3}(1+x^{\frac{1}{3}})+1\}$;

this comes under the first criterion.

5. $\int\frac{(1-x^2)\,dx}{x^6}=-\frac{(1-x^2)^{\frac{3}{2}}(3+2x^2)}{15x^5}$, by 2nd criterion.

6. $\int x(a+x)^{\frac{3}{2}}dx=\tfrac{2}{35}(a+x)^{\frac{5}{2}}(5x-2a).$

7. $\int\frac{x^2dx}{(1+x)^{\frac{5}{2}}}=\tfrac{2}{3}\frac{3(1+x)^2+6(1+x)-1}{(1+x)^{\frac{3}{2}}}.$

8. $\int\frac{x^{3n-1}dx}{(a+bx^n)^{\frac{1}{2}}}=\frac{2(a+bx^n)^{\frac{1}{2}}}{15nb^3}(3b^2x^{2n}-4abx^n+8a^2.)$

Here $\frac{(3n-1)+1}{n}=3$; hence we must assume

$$a+bx^n=z^2, \therefore x^{3n-1}dx=\frac{2}{nb^3}\cdot z(z^2-a)^2dz;$$

$$\therefore \int \frac{x^{3n-1}dx}{(a+bx^n)^{\frac{1}{2}}}=\frac{2}{nb^3}\int (z^2-a)^2dz=\frac{2}{nb^3}\left\{\frac{z^5}{5}-\frac{2az^3}{3}+a^2z\right\}$$

$$=\frac{2z}{15nb^3}(3z^4-10az^2+15a^2)=\&c.$$

9. $\int \frac{x^m dx}{(a+bx)^n}=\frac{1}{b^{m+1}}\int \frac{(z-a)^m dz}{z^n}$, assuming $a+bx=z$.

When m is positive, this can be expanded and integrated.

10. $\int \frac{dx}{x^m(a+bx)^n}=\int x^{-(m+n)}(ax^{-1}+b)^{-n}dx$

$$=-\frac{1}{a^{m+n-1}}\int \frac{(z-b)^{m+n-2}dz}{z^n}, \text{ by making } ax^{-1}+b=z.$$

Which can be expanded and integrated when $m+n-2$ is a positive integer.

Method of Reduction.

95. This method consists in making the proposed integral depend upon another of the same form, in which the indices are diminished, so that by repeating the process, we at length arrive at an integral which can be determined by forms already established.

(I.) To diminish m in the formula $u_m=\int x^m(a+bx^n)^r dx$, where r may be either positive or negative.

Integrating by parts, Art. **93.**, we have

$$u_m=\int x^{m-n+1}\cdot(a+bx^n)^r x^{n-1}dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{r+1}}{nb(r+1)}-\frac{m-n+1}{nb(r+1)}\int x^{m-n}(a+bx^n)^{r+1}dx \ldots (\text{A})$$

$$\text{Now} \int x^{m-n}(a+bx^n)^{r+1}dx = \int x^{m-n}(a+bx^n)(a+bx^n)^r dx$$
$$= a\int x^{m-n}(a+bx^n)^r dx + b\int x^m(a+bx^n)^r dx$$
$$= au_{m-n} + bu_m.$$

Therefore, by substituting this in eq. (A), and solving the resulting equation for u_m, we obtain

$$u_m = \frac{x^{m-n+1}(a+bx^n)^{r+1}}{b(nr+m+1)} - \frac{a(m-n+1)}{b(nr+m+1)} u_{m-n},$$

where u_m is made to depend on u_{m-n}, that is $\int x^m(a+bx^n)^r dx$ on $\int x^{m-n}(a+bx^n)^r dx$.

By repeating this process, u_{m-n} may be made to depend on u_{m-2n}, and so on.

In the formula (A) the integral is made to depend on another of the same form, in which m is diminished by n, and r is increased by unity.

(II.) To diminish the index r in the formula

$$u_r = \int x^m(a+bx^n)^r dx,$$

where m may be either positive or negative.

$$u_r = a\int x^m(a+bx^n)^{r-1}dx + b\int x^{m+n}(a+bx^n)^{r-1}dx;$$

but by integrating by parts, we have

$$\int x^{m+n}(a+bx^n)^{r-1}dx = \int x^{m+1}.(a+bx^n)^{r-1}x^{n-1}dx$$
$$= \frac{x^{m+1}(a+bx^n)^r}{nbr} - \frac{m+1}{nbr}\int x^m(a+bx^n)^r dx;$$

substituting this in the preceding equation,

$$u_r = au_{r-1} + \frac{x^{m+1}(a+bx^n)^r}{nr} - \frac{m+1}{nr}u_r,$$

$$\therefore u_r = \frac{x^{m+1}(a+bx^n)^r}{nr+m+1} + \frac{anr}{nr+m+1}u_{r-1};$$

where u_r is made to depend upon u_{r-1}, that is

$$\int x^m(a+bx^n)^r dx \text{ on } \int x^m(a+bx^n)^{r-1}dx.$$

By repeating the process, u_{r-1} may be made to depend on u_{r-2}, and so on.

(III.) To diminish m in the formula $u_m=\int\frac{dx}{x^m(a+bx^n)^r}$, where r may be either positive or negative.

Multiplying num^r^. and deno^r^., by $a+bx^n$, and splitting,

$$\frac{dx}{x^m(a+bx^n)^{r-1}}=\frac{adx}{x^m(a+bx^n)^r}+\frac{bdx}{x^{m-n}(a+bx^n)^r};$$

by integration and transposition, we have

$$u_m=\frac{1}{a}\int\frac{dx}{x^m(a+bx^n)^{r-1}}-\frac{b}{a}u_{m-n};$$

now integrating by parts, we have

$$\int\frac{dx}{x^m(a+bx^n)^{r-1}}=\int\frac{1}{(a+bx^n)^{r-1}}\cdot\frac{dx}{x^m}$$

$$=-\frac{1}{(m-1)x^{m-1}(a+bx^n)^{r-1}}-\frac{nb(r-1)}{m-1}\int\frac{dx}{x^{m-n}(a+bx^n)^r};$$

substituting this in the expression for u_m and reducing,

$$u_m=-\frac{1}{a(m-1)x^{m-1}(a+bx^n)^{r-1}}+\frac{n(1-r)-m+1}{a(m-1)}bu_{m-n};$$

where u_m is made to depend upon u_{m-n}, and hence, by repeating the process, m may be reduced by any multiple of n.

(IV.) To diminish the index r in the formula

$$u_r=\int\frac{x^m dx}{(a+bx^n)^r},$$

where m may be either positive or negative.

Multiplying numerator and denominator by $a+bx^n$ and splitting,

$$\frac{x^m dx}{(a+bx^n)^{r-1}}=\frac{ax^m dx}{(a+bx^n)^r}+\frac{bx^{m+n}dx}{(a+bx^n)^r};$$

by integration and transposition, we have,

$$u_r=-\frac{b}{a}\int\frac{x^{m+n}dx}{(a+bx^n)^r}+\frac{1}{a}u_{r-1};$$

now integrating by parts, we have,

$$\int\frac{x^{m+n}dx}{(a+bx^n)^r}=\int x^{m+1}\cdot\frac{x^{n-1}dx}{(a+bx^n)^r}$$

$$=-\frac{x^{m+1}}{nb(r-1)(a+bx^n)^{r-1}}+\frac{m+1}{nb(r-1)}\int\frac{x^m dx}{(a+bx^n)^{r-1}};$$

substituting this in the expression for u_r, and reducing,

$$u_r=\frac{x^{m+1}}{na(r-1)(a+bx^n)^{r-1}}-\frac{1}{a}\left\{\frac{m+1}{n(r-1)}-1\right\}u_{r-1};$$

where u_r is made to depend upon u_{r-1}, and by repeating the process, r may be reduced by any number of units.

96. Let the function be $u_m=\int\frac{x^m dx}{(1-x^2)^{\frac{1}{2}}}$.

Here method (I.) applies; $a=1$, $b=-1$, $r=-\frac{1}{2}$, and $n=2$. Therefore the formula of reduction becomes

$$u_m=-\frac{x^{m-1}(1-x^2)^{\frac{1}{2}}}{m}+\frac{m-1}{m}\int\frac{x^{m-2}dx}{(1-x^2)^{\frac{1}{2}}}.$$

It will be highly instructive to the student to apply the *general method* to each particular case.

Here $u_m=\int\frac{x^m dx}{(1-x^2)^{\frac{1}{2}}}=\int x^{m-1}\cdot\frac{x dx}{(1-x^2)^{\frac{1}{2}}};$

hence we have by the formula for integration by parts (Art. **93.**),

$$u_m=-x^{m-1}(1-x^2)^{\frac{1}{2}}+(m-1)\int x^{m-2}(1-x^2)^{\frac{1}{2}}dx.$$

$$\text{Now } \int x^{m-2}(1-x^2)^{\frac{1}{2}}dx = \int\frac{x^{m-2}dx}{(1-x^2)^{\frac{1}{2}}} - \int\frac{x^m dx}{(1-x^2)^{\frac{1}{2}}},$$

by multiplying and dividing by $(1-x^2)^{\frac{1}{2}}$ and splitting

$$\therefore\ u_m = -x^{m-1}(1-x^2)^{\frac{1}{2}} + (m-1)\int\frac{x^{m-2}dx}{(1-x^2)^{\frac{1}{2}}} - (m-1)u_m;$$

$$\therefore\ u_m = -\frac{x^{m-1}(1-x^2)^{\frac{1}{2}}}{m} + \frac{m-1}{m}\int\frac{x^{m-2}dx}{(1-x^2)^{\frac{1}{2}}}.$$

By putting $m-2$, $m-4$, &c., for m, this integral is finally reduced to

$$\int\frac{dx}{(1-x^2)^{\frac{1}{2}}} = \sin^{-1}x \text{ when } m \text{ is even,}$$

$$\text{and to } \int\frac{xdx}{(1-x^2)^{\frac{1}{2}}} = -(1-x^2)^{\frac{1}{2}} \text{ when } m \text{ is odd.}$$

Ex. 1. Let $\int\frac{x^3dx}{(1-x^2)^{\frac{1}{2}}}$ be required. Here $m=3$.

$$\therefore \int\frac{x^3dx}{(1-x^2)^{\frac{1}{2}}} = -\frac{x^2(1-x^2)^{\frac{1}{2}}}{3} + \tfrac{2}{3}\int\frac{xdx}{(1-x^2)^{\frac{1}{2}}}.$$

$$= -\frac{x^2(1-x^2)^{\frac{1}{2}}}{3} - \tfrac{2}{3}(1-x^2)^{\frac{1}{2}} = -\tfrac{1}{3}(1-x^2)^{\frac{1}{2}}(x^2+2).$$

2. Let $\int\frac{x^4dx}{(1-x^2)^{\frac{1}{2}}}$ be required. Here $m=4$.

$$\therefore \int\frac{x^4dx}{(1-x^2)^{\frac{1}{2}}} = -\frac{x^3(1-x^2)^{\frac{1}{2}}}{4} + \tfrac{3}{4}\int\frac{x^2dx}{(1-x^2)^{\frac{1}{2}}};$$

then substituting $m=2$ in the formula,

$$\int\frac{x^2dx}{(1-x^2)^{\frac{1}{2}}} = -\frac{x(1-x^2)^{\frac{1}{2}}}{2} + \tfrac{1}{2}\int\frac{dx}{(1-x^2)^{\frac{1}{2}}}$$

$$= -\tfrac{1}{2}x(1-x^2)^{\frac{1}{2}} + \tfrac{1}{2}\sin^{-1}x,$$

substituting and reducing, we have

$$\int\frac{x^4dx}{(1-x^2)^{\frac{1}{2}}}=-(1-x^2)^{\frac{1}{2}}\left\{\frac{x^3}{4}+\frac{3x}{8}\right\}+\tfrac{3}{8}\sin^{-1}x.$$

97. Let $u_r=\int\frac{dx}{(1+x^2)^r}$ be required.

Here method (IV.) applies; $a=1$, $b=1$, $n=2$, and $m=0$ Therefore the formula of reduction becomes,

$$\int\frac{dx}{(1+x^2)^r}=\frac{x}{2(r-1)(1+x^2)^{r-1}}+\frac{2r-3}{2(r-1)}\int\frac{dx}{(1+x^2)^{r-1}}.$$

By this formula the integral is finally reduced to

$$\int\frac{dx}{1+x^2}=\tan^{-1}x.$$

Ex. 1. Let $\int\frac{dx}{(1+x^2)^3}$ be required. Here $r=3$,

$$\therefore\int\frac{dx}{(1+x^2)^3}=\frac{x}{4(1+x^2)^2}+\tfrac{3}{4}\int\frac{dx}{(1+x^2)^2};$$

then making $r=2$ in the formula,

$$\int\frac{dx}{(1+x^2)^2}=\frac{x}{2(1+x^2)}+\tfrac{1}{2}\int\frac{dx}{1+x^2}$$

$$=\frac{x}{2(1+x^2)}+\tfrac{1}{2}\tan^{-1}x,$$

therefore substituting this value, and reducing,

$$\int\frac{dx}{(1+x^2)^3}=\frac{x}{4(1+x^2)^2}+\frac{3}{8}\cdot\frac{x}{1+x^2}+\frac{3}{8}\tan^{-1}x.$$

98. Let $\int\frac{dx}{x^m(x^2-1)^{\frac{1}{2}}}$ be required.

Here method (III.) applies; $a=-1$, $b=1$, $n=2$, and $r=\frac{1}{2}$. Hence the formula of reduction becomes,

$$\int\frac{dx}{x^m(x^2-1)^{\frac{1}{2}}}=\frac{1}{m-1}\frac{(x^2-1)^{\frac{1}{2}}}{x^{m-1}}+\frac{m-2}{m-1}\int\frac{dx}{x^{m-2}(x^2-1)^{\frac{1}{2}}}.$$

By this formula the integral is finally reduced to

$$\int\frac{dx}{x(x^2-1)^{\frac{1}{2}}}=\sec^{-1}x \text{ when } m \text{ is odd.}$$

$$\text{and to} \int\frac{dx}{x^2(x^2-1)^{\frac{1}{2}}}=\frac{(x^2-1)}{x} \text{ when } m \text{ is even.}$$

Ex. 1. Let $m=3$; then

$$\int\frac{dx}{x^3(x^2-1)^{\frac{1}{2}}}=\frac{(x^2-1)^{\frac{1}{2}}}{2x^2}+\tfrac{1}{2}\sec^{-1}x.$$

2. Let $m=5$; then

$$\int\frac{dx}{x^5(x^2-1)^{\frac{1}{2}}}=\frac{(x^2-1)^{\frac{1}{2}}}{4x^4}+\tfrac{3}{4}\int\frac{dx}{x^3(x^2-1)^{\frac{1}{2}}},$$

$$\text{but} \int\frac{dx}{x^3(x^2-1)^{\frac{1}{2}}}=\frac{(x^2-1)^{\frac{1}{2}}}{2x^2}+\tfrac{1}{2}\sec^{-1}x,$$

$$\therefore \int\frac{dx}{x^5(x^2-1)^{\frac{1}{2}}}=\frac{(x^2-1)^{\frac{1}{2}}}{4x^4}+\tfrac{3}{8}\,\frac{(x^2-1)^{\frac{1}{2}}}{x^2}+\tfrac{3}{8}\sec^{-1}x.$$

99. Let $\int(a^2-x^2)^{\frac{n}{2}}dx$, n being odd.

Here method (II.) applies; $a=a^2$, $b=-1$, $n=2$, $r=\frac{n}{2}$, and $m=0$. Hence the formula of reduction becomes,

$$\int(a^2-x^2)^{\frac{n}{2}}dx=\frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1}+\frac{na^2}{n+1}\int(a^2-x^2)^{\frac{n-2}{2}}dx.$$

Ex. 1. Let $n=1$, and $a=1$, then

$$\int(1-x^2)^{\frac{1}{2}}dx=\frac{x(1-x^2)^{\frac{1}{2}}}{2}+\tfrac{1}{2}\int\frac{dx}{(1-x^2)^{\frac{1}{2}}}$$

$$=\frac{x(1-x^2)^{\frac{1}{2}}}{2}+\tfrac{1}{2}\sin^{-1}x.$$

100. Let $\int\frac{x^m dx}{(1+x^2)^r}$ be required.

Here formula (A) of method (I.) applies; $a=1$, $b=1$, $r=-r$, $n=2$. Hence the formula of reduction becomes

$$\int\frac{x^m dx}{(1+x^2)^r}=-\frac{1}{2r-2}\,\frac{x^{m-1}}{(1+x^2)^{r-1}}+\frac{m-1}{2r-2}\int\frac{x^{m-2}dx}{(1+x^2)^{r-1}}.$$

Ex. 1. Let $m=4$, $r=2$;

$$\int\frac{x^4 dx}{(1+x^2)^2}=-\frac{x^3}{2(1+x^2)}+\frac{3}{2}\int\frac{x^2 dx}{1+x^2}$$

$$=-\frac{x^3}{2(1+x^2)}+\frac{3x}{2}-\frac{3}{2}\tan^{-1}x.$$

101. Sometimes it is requisite to employ a combination of the methods of reduction. Thus let $\int\frac{x^5 dx}{(1+x^2)^{\frac{3}{2}}}$ be required.

Here we shall first reduce the exponent of the denominator by the general formula (Art. **100.**); where $m=5$, and $r=\frac{3}{2}$;

$$\therefore \int\frac{x^5 dx}{(1+x^2)^{\frac{3}{2}}}=-\frac{x^4}{(1+x^2)^{\frac{1}{2}}}+4\int\frac{x^3 dx}{(1+x^2)^{\frac{1}{2}}}.$$

In order to obtain a formula of reduction for this last expression, we have, by the general formula of method (I.), making $a=1$, $b=1$, $n=2$, and $r=-\frac{1}{2}$;

$$\therefore \int\frac{x^m dx}{(1+x^2)^{\frac{1}{2}}}=\frac{1}{m}x^{m-1}(1+x^2)^{\frac{1}{2}}-\frac{m-1}{m}\int\frac{x^{m-2}dx}{(1+x^2)^{\frac{1}{2}}},$$

let $m=3$, then,

$$\int\frac{x^3 dx}{(1+x^2)^{\frac{1}{2}}}=\tfrac{1}{3}x^2(1+x^2)^{\frac{1}{2}}-\tfrac{2}{3}\int\frac{x\,dx}{(1+x^2)^{\frac{1}{2}}}$$

$$=\tfrac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2);$$

hence we have by substitution,

$$\int\frac{x^5 dx}{(1+x^2)^{\frac{3}{2}}}=-\frac{x^4}{(1+x^2)^{\frac{1}{2}}}+\tfrac{4}{3}(1+x^2)^{\frac{1}{2}}(x^2-2).$$

102. Let $u_m = \int \frac{dx}{x^m(a+bx)^{\frac{1}{2}}}$ be required.

Here method (III.) applies; $n=1$, and $r=\frac{1}{2}$; hence the formula of reduction becomes

$$u_m = -\frac{1}{a(m-1)}\,\frac{(a+bx)^{\frac{1}{2}}}{x^{m-1}} - \frac{b(2m-3)}{2a(m-1)}\int \frac{dx}{x^{m-1}(a+bx)^{\frac{1}{2}}}.$$

By means of this expression the integral is reduced to

$$\int \frac{dx}{x(a+bx)^{\frac{1}{2}}}.$$ See *Ex.* 2. p. 145.

Ex. 1. Let $m=2$, $a=1$, and $b=1$, then

$$\int \frac{dx}{x^2(1+x)^{\frac{1}{2}}} = -\frac{(1+x)^{\frac{1}{2}}}{x} - \tfrac{1}{2}\int \frac{dx}{x(1+x)^{\frac{1}{2}}}$$

$$= -\frac{(1+x)^{\frac{1}{2}}}{x} - \tfrac{1}{2}\log \frac{(1+x)^{\frac{1}{2}}-1}{(1+x)^{\frac{1}{2}}+1}.$$

103. Let $u_m = \int \frac{x^m dx}{(2ax-x^2)^{\frac{1}{2}}}$ be required.

This expression is the same as $\int \frac{x^{m-\frac{1}{2}}dx}{(2a-x)^{\frac{1}{2}}}$, to which method (I.) applies; $a=2a$, $b=-1$, $n=1$, $r=-\frac{1}{2}$, and $m=m-\frac{1}{2}$; hence the formula of reduction becomes,

$$u_m = -\frac{x_{m-\frac{1}{2}}\,(2a-x)^{\frac{1}{2}}}{m} + \frac{a(2m-1)}{m}\int \frac{x^{m-\frac{3}{2}}dx}{(2a-x)^{\frac{1}{2}}},$$

$$= -\frac{x^{m-1}(2ax-x^2)^{\frac{1}{2}}}{m} + \frac{a(2m-1)}{m}\int \frac{x^{m-1}dx}{(2ax-x^2)^{\frac{1}{2}}}.$$

By means of this expression the integral is finally reduced to

$$\int \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = \text{vers}^{-1}\frac{x}{a}.$$

Ex. 1. Let $m=1$, then

$$\int\frac{xdx}{(2ax-x^2)^{\frac{1}{2}}}=-(2ax-x^2)^{\frac{1}{2}}+a\int\frac{dx}{(2ax-x^2)^{\frac{1}{2}}}$$

$$=-(2ax-x^2)^{\frac{1}{2}}+a\text{ vers}^{-1}\frac{x}{a}.$$

2. Let $m=2$, then

$$\int\frac{x^2dx}{(2ax-x^2)^{\frac{1}{2}}}=-\frac{x}{2}(2ax-x^2)^{\frac{1}{2}}+\frac{3a}{2}\int\frac{xdx}{(2ax-x^2)^{\frac{1}{2}}}$$

$$=-(2ax-x^2)^{\frac{1}{2}}\left(\frac{x}{2}+\frac{3a}{2}\right)+\frac{3a^2}{2}\text{ vers}^{-1}\frac{x}{a},$$

by substitution and reduction.

104. *To integrate exponential and logarithmic functions.*

Integrating by parts, we have,

$$\int e^{ax}x^ndx=\frac{e^{ax}x^n}{a}-\frac{n}{a}\int e^{ax}x^{n-1}dx.$$

Ex. 1. If $n=1$;

$$\int e^{ax}xdx=\frac{e^{ax}x}{a}-\frac{e^{ax}}{a^2}.$$

2. If $n=2$;

$$\int e^{ax}x^2dx=\frac{e^{ax}x^2}{a}-\frac{2}{a}\int e^{ax}xdx$$

$$=\frac{e^{ax}x^2}{a}-\frac{2}{a}\left\{\frac{e^{ax}x}{a}-\frac{e^{ax}}{a^2}\right\}$$

$$=e^{ax}\left\{\frac{x^2}{a}-\frac{2x}{a^2}+\frac{1\cdot 2}{a^3}\right\}.$$

3. $\int e^{ax}x^4dx=e^{ax}\left\{\frac{x^4}{a}-\frac{4x^3}{a^2}+\frac{4\cdot 3x^2}{a^3}-\frac{4\cdot 3\cdot 2x}{a^4}+\frac{4\cdot 3\cdot 2\cdot 1}{a^5}\right\}.$

In like manner, we have,

$$\int \frac{e^{ax}dx}{x^n} = \int e^{ax} \cdot \frac{dx}{x^n}$$

$$= -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}dx}{x^{n-1}},$$

which finally reduces the integral to $\int \frac{e^{ax}dx}{x}$.

4. If $n=2$, and $a=1$;

$$\int \frac{e^x dx}{x^2} = -\frac{e^x}{x} + \int \frac{e^x dx}{x}.$$

5. If $n=3$, and $a=1$;

$$\int \frac{e^x dx}{x^3} = -\frac{e^x}{2x^2} + \tfrac{1}{2}\int \frac{e^x dx}{x^2}, \text{ by subst.,}$$

$$= -\frac{e^x}{2x^2} + \tfrac{1}{2}\left\{ -\frac{e^x}{x} + \int \frac{e^x dx}{x} \right\}$$

$$= -\frac{e^x}{2x^2}(1+x) + \tfrac{1}{2}\int \frac{e^x dx}{x}.$$

To integrate $\int \frac{e^x dx}{x}$ we have, by multiplying the development of e by $\frac{dx}{x}$,

$$\frac{e^x dx}{x} = \frac{dx}{x} + \frac{dx}{1} + \frac{x dx}{1.2} + \frac{x^2 dx}{1.2.3} + \text{\&c.}$$

$$\therefore \int \frac{e^x dx}{x} = \log x + x + \frac{x^2}{1.2^2} + \frac{x^3}{1.2.3^2} + \text{\&c.}$$

105. Integrating by parts,

$$\int x^m (\log x)^n dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx.$$

Ex. 1. If $m=4$, and $n=1$;

$$\int x^4 \log x dx = \frac{x^5 \log x}{5} - \tfrac{1}{5}\int x^4 dx = \frac{x^5}{5}\{\log x - \tfrac{1}{5}\}.$$

2. $\int x^2 \log x dx = \frac{x^3}{3}\{\log x - \frac{1}{3}\}$.

3. $\int x^2 (\log x)^2 dx = \frac{x^3 (\log x)^2}{3} - \frac{2}{3}\int x^2 (\log x) dx$

$$= \frac{x^3 (\log x)^2}{3} - \tfrac{2}{3}\left\{\frac{x^3 \log x}{3} - \tfrac{1}{3}\cdot\frac{x^3}{3}\right\}$$

$$= \frac{x^3}{3}\{(\log x)^2 - \tfrac{2}{3}\log x + \tfrac{2}{9}\}.$$

4. $\int x^3 (\log x)^2 dx = \frac{x^4}{4}\{(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}\}$.

Integration of Circular Functions.

106. Functions of the form $\sin^m x \cos^n x dx$ may be integrated by methods similar to those applied in the preceding sections.

(I.) When one of the indices is an odd positive integer, as $m = 2r + 1$.

Since $\sin^m x = (\sin^2 x)^r \sin x = (1 - \cos^2 x)^r \sin x$,

$$\therefore \int \sin^m x \cos^n x dx = \int \cos^n x (1 - \cos^2 x)^r \sin x dx$$

$$= \int \cos^n x (1 - \cos^2 x)^r d \cos x.$$

Since r is an integer, by expanding the binomial, the expression will consist of a finite number of terms, each of which may be integrated by rule 3. This will of course apply to $\int \sin^{2r+1} x dx$.

Ex. 1. $\int \sin^5 x \cos^4 x dx = \int \cos^4 x (1 - \cos^2 x)^2 \sin x dx$

$$= -\int \cos^4 x (1 - \cos^2 x)^2 d \cos x = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9}.$$

2. $\int \sin^3 x \cos^2 x dx = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}$.

3. $\int \sin^5 x dx = \int (1-\cos^2 x)^2 \sin x dx$

$$= -\int (1 - 2\cos^2 x + \cos^4 x) d\cos x = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}.$$

4. $\int \sin^3 x dx = \frac{\cos^3 x}{3} - \cos x$.

5. $\int \sin^3 x \cos^n x dx = -\frac{\cos^{n+1} x}{n+1} + \frac{\cos^{n+3} x}{n+3}$.

Similarly, when $n = 2r+1$, we have,

$$\int \sin^m x \cos^n x dx = \int \sin^m x (1-\sin^2 x)^r \cos x dx$$
$$= \int \sin^m x (1-\sin^2 x)^r d \sin x,$$

which is integrated as in the last case.

Ex. 1. $\int \sin^2 x \cos^5 x dx = \int \sin^2 x (1-\sin^2 x)^2 \cos x dx$

$$= \int \{\sin^2 x - 2\sin^4 x + \sin^6 x\} d \sin x$$
$$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7}.$$

2. $\int \sin^4 x \cos^3 x dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7}$.

3. $\int \sin^n x \cos^3 x dx = \frac{\sin^{n+1} x}{n+1} - \frac{\sin^{n+3} x}{n+3}$.

(II.) When $m+n = -2r$, an even negative integer,

$$\sin^m x \cos^n x = \tan^m x \cos^{m+n} x = \tan^m x \sec^{2r} x$$
$$= \tan^m x (1+\tan^2 x)^{r-1} \sec^2 x;$$

$$\therefore \int \sin^m x \cos^n x dx = \int \tan^m x (1+\tan^2 x)^{r-1} \sec^2 x dx$$
$$= \int \tan^m x (1+\tan^2 x)^{r-1} d \tan x,$$

which is integrable as before.

$$Ex. \int \frac{dx}{\sin^6 x \cos^4 x} = \int \frac{\sec^{10} x dx}{\tan^6 x} = \int \frac{(1+\tan^2 x)^4 d \tan x}{\tan^6 x}$$

$$= \int \frac{(1+z^2)^4 dz}{z^6}, \text{ putting } z \text{ for } \tan x,$$

$$= -\frac{1}{5z^5} - \frac{4}{3z^3} - \frac{6}{z} + 4z + \frac{z^3}{3}.$$

(III.) The expressions $\int \sin^n x dx$ and $\int \cos^n x dx$ can always be integrated when n is an integer, by developing the power of $\sin x$ or $\cos x$, as the case may be, in a series according to the multiples of the arc x.

$$Ex. 1. \int \cos^5 x dx = \int \left\{ \frac{\cos 5x}{16} + \frac{5 \cos 3x}{16} + \frac{5 \cos x}{8} \right\} dx$$

$$= \frac{\sin 5x}{80} + \frac{5 \sin 3x}{48} + \frac{5 \sin x}{8}.$$

2. $\int \sin^3 x dx = \frac{\cos 3x}{12} - \frac{3 \cos x}{4}.$

3. $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}.$

4. $\int \sin^4 x dx = \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + \frac{3x}{8}.$

107. When neither of the conditions of Art. **106.** is satisfied, we must proceed by the method of reduction.

(I.) To reduce m in $u_m = \int \sin^m x \cos^n x dx$, where n is either positive or negative.

Integrating by parts, we have,

$$u_m = -\int \sin^{m-1} x \cos^n x d \cos x$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x dx \ldots (A).$$

But $\int \sin^{m-2} x \cos^{n+2} x dx = \int \sin^{m-2} x \cos^n x (1 - \sin^2 x) dx$

$$= u_{m-2} - u_m.$$

Therefore, substituting this in eq. (A), and solving for u_m,

$$u_m = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} u_{m-2}.$$

After the same method the index n is reduced, supposing it to be positive, and m either positive or negative.

Ex. 1. Let $m=3$, and $n=-8$; then

$$\int \frac{\sin^3 x dx}{\cos^8 x} = \frac{\sin^2 x}{5 \cos^7 x} - \frac{2}{5} \int \frac{\sin x dx}{\cos^8 x}$$

$$= \frac{\sin^2 x}{5 \cos^7 x} - \frac{2}{5 \,.\, 7 \cos^7 x}.$$

2. $\displaystyle\int \frac{\sin^3 x dx}{\cos^4 x} = \frac{1}{\cos^3 x}(\sin^2 x - \tfrac{2}{3}).$

3. Let $n=0$, then

$$u_m = \int \sin^m x dx = -\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m} u_{m-2};$$

a formula by which $\int \sin^m x dx$ is reduced to $-\cos x$ or x, according as m is odd or even.

4. $\displaystyle\int \frac{\sin^3 x dx}{\cos^2 x} = \cos x + \sec x.$

(II.) To reduce n in $u_n = \displaystyle\int \frac{\sin^m x dx}{\cos^n x}$ where m is either positive or negative.

Multiplying by $\cos^2 x + \sin^2 x = 1$, and splitting, we have,

$$u_n = \int \frac{\sin^m x (\cos^2 x + \sin^2 x) dx}{\cos^n x} = u_{n-2} + \int \frac{\sin^{m+2} x dx}{\cos^n x}.$$

Now, integrating by parts, we have,

$$\int\frac{\sin^{m+2}x dx}{\cos^n x}=-\int\sin^{m+1}x\,.\,\frac{d\cos x}{\cos^n x}$$

$$=\frac{\sin^{m+1}x}{(n-1)\cos^{n-1}x}-\frac{m+1}{n-1}u_{n-2}.$$

Substituting this in the equation for u_n, and reducing,

$$u_n=\frac{\sin^{m+1}x}{(n-1)\cos^{n-1}x}+\frac{n-m-2}{n-1}u_{n-2}.$$

After the same method m is reduced in the formul $\int\frac{\cos^n x dx}{\sin^m x}$.

Ex. 1. Let $m=0$, then

$$\int\frac{dx}{\cos^n x}=u_n=\frac{\sin x}{(n-1)\cos^{n-1}x}+\frac{n-2}{n-1}u_{n-2};$$

$$\therefore\int\frac{dx}{\cos^6 x}=u_6=\frac{\sin x}{5\cos^5 x}+\frac{4}{5}u_4,$$

$$u_4=\frac{\sin x}{3\cos^3 x}+\frac{2}{3}u_2,$$

$$u_2=\int\frac{dx}{\cos^2 x}=\tan x;$$

$$\therefore\int\frac{dx}{\cos^6 x}=\frac{\sin x}{5\cos^5 x}+\frac{4\sin x}{3\,.\,5\cos^3 x}+\frac{4\,.\,2}{5\,.\,3}\tan x.$$

2. $$\int\frac{dx}{\cos^3 x}=\frac{\sin x}{2\cos^2 x}+\frac{1}{2}\log\tan\left\{\frac{\pi}{4}+\frac{x}{2}\right\}.$$

(III.) Since $d\tan x=(1+\tan^2 x)dx$;

$$\therefore\int\tan^m x dx=\int\tan^{m-2}x(\overline{1+\tan^2 x}-1)dx$$

$$=\int\tan^{m-2}x d\tan x-\int\tan^{m-2}x dx$$

$$=\frac{\tan^{m-1}x}{m-1}-\int\tan^{m-2}x dx.$$

Ex. 1. $\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \int \tan^2 x dx$

$$= \tfrac{1}{3}\tan^3 x - \tan x + x. \text{ (Ex. 9., p. 137.)}$$

2. $\int \tan^3 x dx = \frac{1}{2}\tan^2 x + \log\cos x.$

In like manner, we have,

$$\int \frac{dx}{\tan^m x} = \int \frac{(\overline{1+\tan^2 x} - \tan^2 x)dx}{\tan^m x}$$

$$= \frac{-1}{(m-1)\tan^{m-1} x} - \int \frac{dx}{\tan^{m-2} x}.$$

3. $\displaystyle\int \frac{dx}{\tan^5 x} = \frac{-1}{4\tan^4 x} + \frac{1}{2\tan^2 x} + \log\sin x.$

108. Functions of the form $x^n \cos x$ and $x^n \sin x$ may be integrated, by repeating the operation of integration by parts.

Ex. 1. $\int x \, . \sin x dx = -x\cos x + \int \cos x dx$

$$= -x\cos x + \sin x.$$

2. $\int x^2 . \cos x dx = x^2 \sin x - 2\int x \sin x dx$

$$= x^2 \sin x + 2(x\cos x - \sin x).$$

3. $\int x \cos x dx = x \sin x + \cos x.$

4. $\int x^3 \cos x dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x.$

109. To integrate $e^{ax} \sin nx dx$, &c.

By a double integration by parts, we have,

$$\int e^{ax} \sin nx dx = \frac{1}{a} e^{ax} \sin nx - \frac{n}{a}\int e^{ax} \cos nx dx$$

$$= \frac{1}{a} e^{ax} \sin nx - \frac{n}{a}\left\{\frac{1}{a} e^{ax} \cos nx + \frac{n}{a}\int e^{ax} \sin nx dx\right\}.$$

Solving this eq. for the value of $\int e^{ax} \sin nx dx$, we obtain

$$\int e^{ax} \sin nx dx = e^{ax} \frac{a\sin nx - n\cos nx}{n^2 + a^2}.$$

Similarly, $\int e^{ax} \cos nx dx = e^{ax} \dfrac{a \cos nx + n \sin nx}{n^2 + a^2}$.

110. By substituting for $\cos^n x$ and $\sin^n x$, their developments in sines and cosines of multiple arcs, the integrals of $e^{ax} \sin^n x dx$ and $e^{ax} \cos^n x dx$ may be obtained.

Ex. $\int e^x \sin^2 x = \int e^x(\frac{1}{2} - \frac{1}{2} \cos 2x)dx$

$$= \tfrac{1}{2}\int e^x dx - \tfrac{1}{2}\int e^x \cos 2x dx$$

$$= \frac{e^x}{2} - \frac{e^x}{2} \cdot \frac{\cos 2x + 2 \sin 2x}{5}.$$

DEFINITE INTEGRALS.—INTEGRATION BY SERIES.

111. In order to determine the value of the arbitrary constant C in an integral expression, we must first ascertain, *from the problem* proposed, what particular value of the variable x makes the integral 0; by this means we shall have obtained two equations, each containing C, from which C may therefore be eliminated. Thus we have $\int x^3 dx = \frac{x^4}{4} + C$ for the *general* value of the integral: now suppose the problem indicates that the integral becomes 0 when $x=a$, then $0 = \frac{a^4}{4} + C$; therefore, by subtraction, we find the corrected integral to be $\int_a^x x^3 dx = \frac{x^4}{4} - \frac{a^4}{4}$. Here the symbol $\int_a^x$ is now prefixed to indicate that the integration is taken from the limit $x=a$, that is, the value of the integral commences when $x=a$. This form is called the corrected integral, which, as we have seen, assumes the commencement of the integral, but does not assign any particular value to x, so as to fix the final limit of the integral; now if we suppose x to take some particular value, say b, we have $\int_a^b x^3 dx = \frac{b^4}{4} - \frac{a^4}{4}$, which is called the *definite integral* taken between the limits $x=a$ and $x=b$; where a is called the inferior limit, and b the superior limit. Generally, let $\int f(x)dx = \text{F}(x) + \text{C}$; now if the integral is 0 when $x=a$, then $0 = \text{F}(a) + \text{C}$; therefore, by subtraction,

$$\int_a^x f(x)dx = \text{F}(x) - \text{F}(a), \text{ which is the corrected integral}$$

If x now takes the particular value b, then

$$\int_a^b f(x)dx = \text{F}(b) - \text{F}(a), \text{ which is the definite integral.}$$

Hence it follows, that the definite integral of an expression is equal to the *difference* of the values assumed by the general integral, when b and a (the limits) are successively substituted for the variable x. Moreover as every function of x may be supposed to represent the ordinate of a curve, the abscissa being x, the problem of finding the value of $\int_a^b f(x)dx$, is equivalent to finding the area of a curve included between the ordinates corresponding to $x=a$, $x=b$. (See examples on the area of curves.)

EXAMPLES.

1. To find the value of the definite integral $\int_0^1 (1+x)^n dx$.

Here the general integral is

$$\int (1+x)^n dx = \frac{1}{n+1}(1+x)^{n+1} + \text{C},$$

therefore making successively $x=1$, $x=0$, and subtracting the results, we find

$$\int_0^1 (1+x)^n dx = \frac{1}{n+1} \cdot 2^{n+1} - \frac{1}{n+1} = \frac{1}{n+1}(2^{n+1} - 1).$$

2. To find the value of $\int_0^{\frac{1}{2}\pi} \cos x dx$.

The general integral is $\int \cos x dx = \sin x + \text{C}$; therefore, making successively $x = \frac{1}{2}\pi$, $x=0$, and subtracting the results, we find

$$\int_0^{\frac{1}{2}\pi} \cos x dx = \sin \tfrac{1}{2}\pi - \sin 0 = 1.$$

3. To find the value of $\int_0^1 (1-x^2)^{\frac{1}{2}} dx$.

Here, by *Ex.* 1., Art. **99.**, the general value of the integral is $\int (1-x^2)^{\frac{1}{2}} dx = \frac{x(1-x^2)^{\frac{1}{2}}}{2} + \frac{1}{2} \sin^{-1} x + \text{C}$,

$$\therefore \int_0^1 (1-x^2)^{\frac{1}{2}} = \tfrac{1}{2} \sin^{-1} 1 - \tfrac{1}{2} \sin^{-1} 0 = \tfrac{1}{2} \cdot \frac{\pi}{2}.$$

4. To find the value of the definite integral $\int_0^1 \frac{x^{2n} dx}{(1-x^2)^{\frac{1}{2}}}$.

In the formula of reduction, Art. **96.**, put $2n$ for m, then we have

$$u_{2n} = \int \frac{x^{2n} dx}{(1-x^2)^{\frac{1}{2}}} = -\frac{x^{2n-1}(1-x^2)^{\frac{1}{2}}}{2n} + \frac{2n-1}{2n} \int \frac{x^{2n-2} dx}{(1-x^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{2n} q_{2n-1} + \frac{2n-1}{2n} u_{2n-2},$$

by putting, for the sake of conciseness, q_{2n-1} for $x^{2n-1}(1-x^2)^{\frac{1}{2}}$.

Let $n-1$, $n-2$, ..., 1, 0 be put successively for n, in the above equation, then we have

$$u_{2n} = -\frac{1}{2n} q_{2n-1} + \frac{2n-1}{2n} u_{2n-2}$$

$$u_{2n-2} = -\frac{1}{2n-2} q_{2n-3} + \frac{2n-3}{2n-2} u_{2n-4}$$

$$u_{2n-4} = -\frac{1}{2n-4} q_{2n-5} + \frac{2n-5}{2n-4} u_{2n-6}$$

$$\vdots \qquad \vdots$$

$$u_2 = -\tfrac{1}{2} q_1 + \tfrac{1}{2} u_0$$

$$u_0 = \sin^{-1} x.$$

In order to eliminate all the u's, excepting the first, multiply the second equation by the coefficient of u_{2n-2} in the first, the third by the resulting coefficient of u_{2n-4} in the second, and so on; then add the equations thus obtained, and strike out the terms common to both sides of the resulting equation; hence we find

$$u_{2n}=-\left\{\frac{q_{2n-1}}{2n}+\frac{(2n-1)q_{2n-3}}{2n(2n-2)}+\&c.\right\}+\frac{(2n-1)\ldots3.1}{2n(2n-2)\ldots4.2}\sin^{-1}x+c,$$

which is the general value of the integral.

Now if the integral becomes 0, when $x=0$; then $c=0$, for $q_{2n-1}=x^{2n-1}\sqrt{(1-x^2)}=0$, when $x=0$, and so on to all the other q's.

When $x=1$, all the q's become 0, and $\sin^{-1}x=\frac{\pi}{2}$;

$$\therefore\int_0^1\frac{x^{2n}dx}{\sqrt{(1-x^2)}}=\frac{(2n-1)(2n-3)\ldots3.1}{2n(2n-2)\ldots4.2}\cdot\frac{\pi}{2}.$$

This result may be more readily obtained by the method employed in the following examples.

5. To find the value of $q_m=\int_0^{\frac{1}{2}\pi}\sin^m x dx.$

Here, by *Ex.* 3. p. 175., the general formula of reduction is,

$$u_m=-\frac{\sin^{m-1}x\cos x}{m}+\frac{m-1}{m}u_{m-2},$$

in which, if we make successively $x=\frac{1}{2}\pi$, $x=0$, and subtract the results, (or, what is the same thing, take the integral of both sides between the same limits,) we shall find that the integrated part vanishes by both substitutions, and then we have

$$q_m=\frac{m-1}{m}q_{m-2},$$

where q_m is the definite integral, u_m being the general one.

Now, making m successively 2, 4, 6 . . . m, we have

$$q_2=\tfrac{1}{2}q_0=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}dx=\tfrac{1}{2}\cdot\frac{\pi}{2},$$

$$q_4=\tfrac{3}{4}q_2,$$

$$q_6=\tfrac{5}{6}q_4,$$

$$\vdots \qquad \vdots$$

$$q_{m-2}=\frac{m-3}{m-2}\,q_{m-4},$$

$$q_m=\frac{m-1}{m}\,q_{m-2};$$

multiplying all these equations together, and striking out the factors common to both sides, we have, when m is even,

$$q_m \text{ or } \int_0^{\frac{1}{2}\pi}\sin^m x dx=\frac{1\,.\,3\,.\,5\ldots(m-3)\,(m-1)}{2\,.\,4\,.\,6\ldots(m-2)\,m}\cdot\frac{\pi}{2}.$$

When m is odd, the first integral in the above series is

$$q_3=\tfrac{2}{3}q_1=\tfrac{2}{3}\int_0^{\frac{1}{2}\pi}\sin x dx=\tfrac{2}{3}(-\cos\tfrac{1}{2}\pi+\cos 0)=\tfrac{2}{3},$$

$$\therefore \int_0^{\frac{1}{2}\pi}\sin^m x dx=\frac{2\,.\,4\,.\,6\ldots(n-1)}{3\,.\,5\,.\,7\ldots n}.$$

6. To find the value of $q_n=\int_0^a (a^2-x^2)^{\frac{n}{2}}$, n being odd.

Here the general formula of reduction is given in Art. **99.** If we make successively $x=a$, $x=0$, in this formula, we shall find that the integrated part vanishes by both substitutions; hence, by taking the integration on both sides between the same limits, we find

$$\int_0^a (a^2-x^2)^{\frac{n}{2}}dx=\frac{na^2}{n+1}\int_0^a (a^2-x^2)^{\frac{n-2}{2}}dx;$$

making n successively 1, 3, 5, &c., we find

$$\int_0^a (a^2-x^2)^{\frac{1}{2}}dx=\frac{a^2}{2}\int_0^a (a^2-x^2)^{-\frac{1}{2}}dx=\tfrac{1}{2}\cdot\frac{\pi a^2}{2},$$

$$\int_0^a (a^2-x^2)^{\frac{3}{2}}dx=\frac{3a^2}{4}\int_0^a (a^2-x^2)^{\frac{1}{2}}dx,$$

$$\int_0^a (a^2-x^2)^{\frac{5}{2}}dx=\frac{5a^2}{6}\int_0^a (a^2-x^2)^{\frac{3}{2}}dx,$$

$$\vdots \qquad\qquad \vdots$$

$$\int_0^a (a^2-x^2)^{\frac{n}{2}}dx=\frac{na^2}{n+1}\int_0^a (a^2-x^2)^{\frac{n-2}{2}}dx;$$

multiplying these equations together, and then striking out the factors common to both sides, we find

$$\int_0^a (a^2-x^2)^{\frac{n}{2}}dx=\frac{1.3.5\ldots n}{2.4.6\ldots(n+1)}\cdot\frac{\pi a^{n+1}}{2}.$$

112. When a proposed differential expression cannot be integrated by any of the ordinary methods, it must be expanded in an infinite series, and then each term can be separately integrated. There are also many important expansions which may be obtained from the integration of a series.

EXAMPLES.

1. Let $\tan^{-1} x=\int \frac{dx}{1+x^2}+C$ be required in a series.

By division, we have

$$\frac{dx}{1+x^2}=(1-x^2+x^4-x^6+\&c.)dx,$$

$$\therefore\ \tan^{-1} x=\int \frac{dx}{1+x^2}=x-\frac{x^3}{3}+\frac{x^5}{5}-\&c.+C;$$

but when $x=0$, $\tan^{-1} x=\tan^{-1} 0=0$, $\therefore\ C=0$;

$$\therefore\ \tan^{-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\&c.$$

2. Let $\sin^{-1} x=\int \frac{dx}{(1-x^2)^{\frac{1}{2}}}+C$ be required in a series.

By the binomial theorem,

$$(1-x^2)^{-\frac{1}{2}}dx=\left(1+\frac{x^2}{2}+\frac{1\,.\,3x^4}{2\,.\,4}+\text{ \&c.}\right)dx,$$

$$\therefore\ \sin^{-1}x=\int(1-x^2)^{-\frac{1}{2}}dx$$

$$=x+\frac{x^3}{2\,.\,3}+\frac{1\,.\,3x^5}{2\,.\,4\,.\,5}+\frac{1\,.\,3\,.\,5x^7}{2\,.\,4\,.\,6\,.\,7}+\text{ \&c.};\ (c=0).$$

3. Let $\log_e(1+x)=\int\frac{dx}{1+x}+c$ be required in a series.

By division, $\frac{dx}{1+x}=(1-x+x^2-x^3+\text{ \&c.})dx,$

$$\therefore\ \log_e(1+x)=\int\frac{dx}{1+x}=x-\frac{x^2}{2}+\frac{x^3}{3}-\text{ \&c. }+c;$$

but when $x=0$, $\log_e(1+x)=\log_e 1=0$, $\therefore\ c=0$;

$$\therefore\ \log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\text{ \&c.}$$

4. To find $\int\frac{a^x\,dx}{1-x}$.

By division, $\frac{1}{1-x}=1+x+x^2+\text{ \&c.}\ \ldots\ (1),$

also by Art. **63**.

$$a^x=1+\frac{\log a\,.\,x}{1}+\frac{(\log a)^2\,.\,x^2}{1\,.\,2}+\text{ \&c.}\ \ldots\ (2);$$

multiplying (1) and (2)

$$\frac{a^x}{1-x}=1+(1+\log a)x+\left(1+\frac{\log a}{1}+\frac{(\log a)^2}{1\,.\,2}\right)x^2+\text{ \&c.};$$

multiplying both sides by dx, and integrating,

$$\int \frac{a^x dx}{1-x} = x + (1+\log a)\frac{x^2}{2} + \left(1 + \frac{\log a}{1} + \frac{(\log a)^2}{2}\right)\frac{x^3}{3} + \&c. + C.$$

5. To find $\int \frac{dx}{(c-x)^{\frac{1}{2}}(2ax-x^2)^{\frac{1}{2}}}$.

By the binomial theorem,

$$\frac{1}{(c-x)^{\frac{1}{2}}} = \frac{1}{c^{\frac{1}{2}}}\left(1-\frac{x}{c}\right)^{-\frac{1}{2}} = \frac{1}{c^{\frac{1}{2}}}\left(1+\frac{1}{2}\cdot\frac{x}{c}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{c^2}+\&c.\right),$$

$$\therefore \int \frac{dx}{(c-x)^{\frac{1}{2}}(2ax-x^2)^{\frac{1}{2}}} =$$

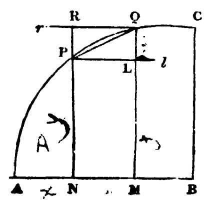
$$\frac{1}{c^{\frac{1}{2}}}\int \frac{dx}{(2ax-x^2)^{\frac{1}{2}}}\left\{1+\frac{1}{2}\cdot\frac{x}{c}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{c^2}+\&c.\right\},$$

where the integrations evidently depend upon the general formula of reduction given in Art. **103.**

APPLICATION OF THE INTEGRAL CALCULUS.

TO FIND THE AREAS OF PLANE SURFACES.

113. *Differential of areas.* Let AN and NP be the coordinates of the point P in the plane curve APQC; AM and MQ those of the point Q. Draw Qr and Pl parallel to AB, and produce NP to meet Qr in R. Put AN$=x$, NP$=y$, NM$=h$, and area ANP$=$A. Now, conceiving the ordinate NP to move from N to M, we shall have Incr. $x=h$, Incr. $y=$QL, Incr. A$=$area NPQM; and since the magnitude of A depends upon x (for as x changes A also changes), it follows that A must be some function of x.



$$\frac{\text{area NRQM}}{\text{area NPLM}} = \frac{\text{NM . MQ}}{\text{NM . NP}} = \frac{\text{MQ}}{\text{NP}} = \frac{y+\text{incr. } y}{y} = 1 + \frac{\text{incr. } y}{y}.$$

Now as h approaches 0, Incr. y approaches 0; hence, by taking the limiting values of both sides of this equality, we have,

$$\text{limit } \frac{\text{area NRQM}}{\text{area NPLM}} = 1;$$

but Incr. A, or area NPQM, is always greater than area NPLM and less than area NRQM, $\therefore$ *à fortiori*,

$$\text{limit } \frac{\text{area NPQM}}{\text{area NPLM}} = 1, \ldots (1).$$

Now area NPQM=Incr. A, and area NPLM$=y \cdot h$,

$$\therefore \text{ limit } \frac{1}{y} \cdot \frac{\text{incr. A}}{h} = 1, \;\therefore\; \frac{1}{y} \cdot \frac{dA}{dx} = 1,$$

$$\therefore \frac{dA}{dx} = y, \text{ or } dA = ydx \ldots (2),$$

which is the differential expression of the area of any plane curve. By taking the integral, we have,

$$A = \int ydx \ldots (3).$$

This integral, after being corrected, by means of the limits in the proposed problem, gives the expression for the area of a plane curve related to rectangular co-ordinates. When the equation to the curve is given, the value of y may, in general, be found in terms of x; and then ydx, the differential of the area, may be integrated by means of the rules given in the preceding articles.

In order to show the connection between an area and its differential, let us take a simple *illustration.* If the base of a right-angled triangle be x, and its perpendicular $y=2x$, then

$$\text{the area } A = \tfrac{1}{2}xy = x^2;$$

differenting this, we find the differential of the area to be expressed by

$$x, \text{ or } ydx;$$

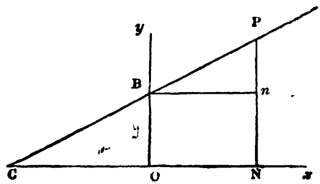
hence, by integration,

$$A = \int 2x\,dx = x^2,$$

which we already know to be the case.

EXAMPLES.

1. To find the area of the right-angled triangle CNP.



Let $CN = b$, $NP = l$, $CO = x$, and $OB = y$; then, by similar triangles,

$$x : y :: b : l, \therefore y = \frac{lx}{b},$$

$$\therefore A = \int y\,dx = \int \frac{lx\,dx}{b} = \frac{lx^2}{2b} + C.$$

In order to find C, let $x = 0$, then area $= 0$, and this equation becomes $0 = 0 + C$, $\therefore C = 0$.

Hence $A = \frac{lx^2}{2b}$, which is the expression for the area of the triangle COB. When $x = b$, that is when CO becomes equal to CN, we have

$$\text{area CNP} = \int_0^b y\,dx = \frac{lb^2}{2b} = \frac{lb}{2}.*$$

2. To find the area of the parabola. (See *fig.* p. 15.)

Let $ON = x$, and $NP = y$, then

$$y^2 = 4ax, \therefore y = 2a^{\frac{1}{2}}x^{\frac{1}{2}},$$

---

* The area included between the ordinates OB and NP of any curve, is found from the general integral $\int y\,dx$ by making successively $x = CN = b$, $x = CO = a$, and subtracting the latter result from the former; but this is equivalent to taking the integral between the limits $x = a$, $x = b$;

$$\therefore \int_a^b y\,dx = \text{area OBNP}.$$

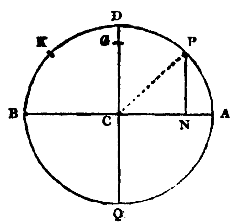
$\therefore$ area ONP $=\int ydx=\int 2a^{\frac{1}{2}}x^{\frac{1}{2}}dx=\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}+C$.

Here, as in the last example, $C=0$, since area $=0$, when $x=0$,

$$\therefore \text{ area ONP}=\int_0^x ydx=\tfrac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}=\tfrac{2}{3}xy,$$

substituting the value of y. Hence the area of a parabola is equal to $\frac{2}{3}$ of the circumscribed rectangle.

3. To find the area of the circle.



Let CN $=x$, NP $=y$, CP or radius $=a$; then $y=(a^2-x^2)^{\frac{1}{2}}$,

$\therefore$ area CNPD $=\int ydx=\int(a^2-x^2)^{\frac{1}{2}}dx$

$=\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}+\dfrac{x}{2}(a^2-x^2)^{\frac{1}{2}}+C$,

See *Ex.* 3. p. 156.

Here $C=0$, since area $=0$, when $x=0$.

The value of $\sin^{-1}\dfrac{x}{a}$ can only be calculated by approximation from an infinite series. See Art. **65**.

If $x=a$, the area CNPD becomes the quadrant ACD,

$$\therefore \text{ quadrant ACD}=\int_0^a(a^2-x^2)^{\frac{1}{2}}=\frac{a^2}{2}\sin^{-1}1=\frac{\pi a^2}{4},$$

$$\therefore \text{ area whole circle ADBQ}=\pi a^2.$$

Cor. 1. If AN $=x$, NP $=y=(2ax-x^2)^{\frac{1}{2}}$,

$$\therefore \text{ area ANP}=\int_0^x(2ax-x^2)^{\frac{1}{2}}dx. \quad \text{(See } Ex. \text{ 5. p. 156.)}$$

Obs. It should be remembered that $\int(a^2-x^2)^{\frac{1}{2}}dx$ expresses the circular area CNPD, where x is the cosine to radius a, and $\int(2ax-x^2)^{\frac{1}{2}}dx$ the circular area ANP, where x is the versine to radius a.

4. To find the area of the ellipse. (See *fig.* p. 16.)

Let CN$=x$, NP$=y$, then $y=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$,

$$\therefore \text{ area CNPB}=\int y\,dx=\frac{b}{a}\int(a^2-x^2)^{\frac{1}{2}}dx$$

$$=\frac{b}{a}\text{ circ. area CNP}_1\text{B}_1,$$

where AB_1DM_1 is a circle described upon the major axis.

$$\therefore \text{ elliptic quadrant DCB}=\frac{b}{a}\text{ circ. quadrant DCB}_1$$

$$=\frac{b}{a}\frac{\pi a^2}{4}=\frac{\pi ab}{4},$$

$$\therefore \text{ area whole ellipse}=\pi ab.$$

5. To find the area of the hyperbola. (See *fig.* p. 114.)

Taking the centre C for the origin, CN$=x$, NP$=y$,

$$\therefore\ y=\frac{b}{a}(x^2-a^2)^{\frac{1}{2}},$$

$\therefore$ area ANP$=\int y\,dx=\frac{b}{a}\int(x^2-a^2)^{\frac{1}{2}}dx$. (See *Ex.* 2. p. 155.)

$$=\frac{b}{a}\left\{\frac{x}{2}(x^2-a^2)^{\frac{1}{2}}-\frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)\right\}+\text{C}.$$

Now, when $x=$ CA $=a$, area ANP$=0$, $\therefore 0=-\frac{ab}{2}\log a+\text{C}$,

$\therefore$ C$=\frac{ab}{2}\log a$; substituting this value of C, and reducing, we have

$$\text{area ANP}=\frac{xy}{2}-\frac{ab}{2}\log\frac{x+(x^2-a^2)^{\frac{1}{2}}}{a}.$$

Cor. Let the points C and P be joined, then

$$\text{area sector CAP}=\triangle\text{CNP}-\text{area ANP}$$

$$=\frac{ab}{2}\log\frac{x+(x^2-a^2)^{\frac{1}{2}}}{a}$$

$$=\frac{ab}{2}\log\left\{\frac{x}{a}+\frac{y}{b}\right\}.$$

6. To find the area of the witch. (See *fig.* p. 19.)

Let ON$=x$, NP$=y$, OB$=2r$, then

$$y=\frac{2r(2rx-x^2)^{\frac{1}{2}}}{x}. \quad \text{(See Art. 22.)}$$

$$\therefore \text{ area }=\int y\,dx=2r\int\frac{(2rx-x^2)^{\frac{1}{2}}dx}{x}. \quad \text{(See } Ex. \text{ 8. p. 144.)}$$

$$=2r\left\{(2rx-x^2)^{\frac{1}{2}}+r\text{ versin}^{-1}\frac{x}{r}\right\}+\text{C},$$

and C$=0$, since area $=0$, when $x=0$.

Let $x=2r$, then the whole area OBPD$=2r\times r\pi$

$=2\pi r^2=2$(area semicircle OP^{1}B).

7. To find the area of the cissoid. (See *fig.* p. 20.)

Here $y=\dfrac{x^{\frac{3}{2}}}{(2r-x)^{\frac{1}{2}}}$. (See Art. **23.**)

$$\therefore \text{ area ONP}=\int y\,dx=\int\frac{x^{\frac{3}{2}}dx}{(2r-x)^{\frac{1}{2}}}. \quad \text{(See } Ex. \text{ 8. p. 157.)}$$

$$=-2x^{\frac{3}{2}}(2r-x)^{\frac{1}{2}}+3\int(2rx-x^2)^{\frac{1}{2}}dx$$

$$=-2x(2rx-x^2)^{\frac{1}{2}}+3\text{ (circ. area ONK)}.$$

(See *Obs.* to *Ex.* 3. p. 188.) Let $x=2r$ then the whole area OBCP contained between the curve and its asymptote $=$ **3** (area semicircle OKB).

8. To find the area of the logarithmic curve.

Here $y=a^x$. See Art. **24**.

$$\therefore \text{ area ONPB}=\int y dx=\int a^x dx$$

$$=\frac{a^x}{\log a}+\text{C},$$

$$\text{now when } x=0, \text{ area}=0, \;\therefore\; 0=\frac{a^0}{\log a}+\text{C},$$

$$\therefore\; \text{C}=-\frac{a^0}{\log a},$$

$$\therefore \text{ area ONPD}=\frac{1}{\log a}\{a^x-a^0\}=\frac{1}{\log a}(\text{NP}-\text{OB})$$

$$=\text{subtan. to P}\times(\text{NP}-\text{OB}).$$

9. To find the area of the catenary, the equation being

$$y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}});$$

$$\therefore \text{ area}=\int y dx=\int \frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})dx$$

$$=\frac{a^2}{2}(e^{\frac{x}{a}}-e^{\;\frac{x}{a}})+\text{C}(=0).$$

10. To find the areas of the spaces OVKO and APOA in *Ex.* 1., page 123.

Here $y=x^3+3x^2+2x$; hence we have the following general expression for the area,

$$\text{A}=\int y dx=\int (x^3+3x^2+2x)dx$$

$$=\tfrac{1}{4}x^4+x^3+x^2+\text{C}\;.\;.\;.\;(1)$$

In order to find the area OVKO, let AT and TV be the co-ordinates limiting the area OTV; then when $x=\text{AO}=-1$, the area$=0$; $\therefore\; 0=\frac{1}{4}-1+1+\text{C}$, $\therefore\; \text{C}=-\frac{1}{4}$;

$$\therefore \text{ area OTV}=\int_{-1}^{-x} y dx=\tfrac{1}{4}x^4-x^3+x^2-\tfrac{1}{4};$$

and when $x = \text{AK} = 2$, we have,

$$\text{area OVKO} = \int_{-1}^{-2} y dx = \tfrac{1}{4} \cdot 16 - 8 + 4 - \tfrac{1}{4} = -\tfrac{1}{4}.$$

This result is *minus*, because x and y in the curve OVK have *different* signs. Irrespective of position the area is $\frac{1}{4}$.

To find the area APOA; let AN and PN be the co-ordinates limiting the area APN; then in eq. (1) $C=0$, since $x=0$, when area $=0$;

$$\therefore \text{ area APN} = \int_0^{-x} y dx = \tfrac{1}{4}x^4 - x^3 + x^2;$$

and when $x = \text{AO} = 1$, we have,

$$\text{area APOA} = \int_0^{-1} y dx = \tfrac{1}{4} - 1 + 1 = \tfrac{1}{4}.$$

This result is plus, because x and y throughout the curve APO have the same sign.

TO FIND THE LENGTHS OF CURVES.

114. *Lemma.* If s' be the length of the arc PQ, and c the length of the chord PQ, then $\dfrac{s'}{c} = 1$, when s' and c approach 0. (See *fig.*, page 41.)

Let TPD be the tangent to the point P; draw QD perpendicular to PD, and put $\angle \text{QPD} = \theta$; then, Geo. Art. **73.**, we have,

$$\text{arc PQ} > \text{chord PQ} < \text{PD} + \text{QD};$$

but from the right-angled triangle PDQ, we have,

$$\text{PD} = c \cos\theta, \text{ and QD} = c \sin\theta;$$

$$\therefore s' > c < c\cos\theta + c\sin\theta;$$

$$\therefore \frac{s'}{c} > 1 < \cos\theta + \sin\theta.$$

Now, by the definition of a tangent, Art. **33.**, when s' approaches 0, the limiting value of θ is 0, and therefore the limiting value of $\cos\theta+\sin\theta$ is 1;

$$\therefore \text{ limit } \frac{s'}{c} > 1 < 1, \text{ that is, limit } \frac{s'}{c} = 1.$$

115. To find the differential of the arc of the curve. Let AN$=x$, NP$=y$, NM$=$PL$=h$ or incr. x, arc AP$=s$; then LQ$=$incr. y, and arc PQ$=$incr. s.

Now, since the magnitude of s depends upon x, it follows that s must be some function of x.

From the right-angled triangle PLQ, we have,

$$(\text{chord PQ})^2 = \text{PL}^2 + \text{LQ}^2, \text{ or}$$

$$c^2 = h^2 + (\text{incr. } y)^2; \therefore \frac{c^2}{h^2} = 1 + \left(\frac{\text{incr. } y}{h}\right)^2;$$

multiplying by $\left(\frac{\text{incr. } s}{c}\right)^2$, we have,

$$\left(\frac{\text{incr. } s}{h}\right)^2 = \left(\frac{\text{incr. } s}{c}\right)\left\{1 + \left(\frac{\text{incr. } y}{h}\right)^2\right\}.$$

Now when h approaches 0, the limiting value of $\frac{\text{incr. } s}{h}$ $=\frac{ds}{dx}$, that of $\frac{\text{incr. } s}{c}=1$, by Art. **114.**, and that of $\frac{\text{incr. } y}{h}$ $=\frac{dy}{dx}$; hence we have,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

$$\therefore ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx, \text{ or } ds^2 = dx^2 + dy^2.*$$

---

* If, according to the method of infinitesimals, we may be allowed to consider an infinitely small arc PQ as a straight line coinciding with its chord, and to put PL$=dx$, LQ$=dy$, and arc PQ$=ds$, which conditions really obtain at the limits, then we readily find

$$\text{PQ}^2 = \text{PL}^2 + \text{LQ}^2, \text{ that is, } ds^2 = dx^2 + dy^2.$$

Hence we have by integration,

$$s=\int\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx.$$

EXAMPLES.

1. To find the length of an arc of a parabola.

Here the equation of the curve is $y^2=4ax$;

$$\therefore\ \frac{dy}{dx}=\frac{2a}{y};\ \therefore\left(\frac{dy}{dx}\right)^2=\frac{4a^2}{y^2}=\frac{a}{x};$$

$$s=\int\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\int\left(1+\frac{a}{x}\right)^{\frac{1}{2}}dx$$

$$=(x^2+ax)^{\frac{1}{2}}+\frac{a}{2}\log\ \{(x^2+ax)^{\frac{1}{2}}+x+\tfrac{1}{2}a\}+\text{C}.$$ *Ex.* 9., p. 144.

And $s=0$, when $x=0$; $\therefore\ 0=\frac{a}{2}\log\frac{a}{2}+\text{C}$; hence, eliminating C between these two equations, and reducing,

$$s=(x^2+ax)^{\frac{1}{2}}+\frac{a}{2}\log\frac{a+2x+2(x^2+ax)^{\frac{1}{2}}}{a}.$$

2. To find the length of the arc of the semicubical parabola, whose equation is $y^2=a^2x^3$.

Here $y=ax^{\frac{3}{2}}$, $\therefore\ \frac{dy}{dx}=\frac{3ax^{\frac{1}{2}}}{2}$;

$$\therefore\ s=\int\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\int\left\{1+\frac{9a^2x}{4}\right\}^{\frac{1}{2}}dx$$

$$=\frac{(4+9a^2x)^{\frac{3}{2}}}{27a^2}+\text{C}.$$

If $s=0$, $x=0$; $\therefore\ 0=\frac{8}{27a^2}+\text{C}$; $\therefore\ \text{C}=-\frac{8}{27a^2}$;

$$\therefore\ s=\frac{(4+9a^2x)^{\frac{3}{2}}-8}{27a^2}.$$

This was the first curve which was rectified. The honour of the discovery is due to W. Neil.

3. To find the length of a circular AP. See fig. p. 188.

Let AN$=x$, PN$=y$, AC the radius $=a$, and arc AP$=s$;

$$\therefore\ y=(2ax-x^2)^{\frac{1}{2}};\ \therefore\ \frac{dy}{dx}=\frac{a-x}{(2ax-x^2)^{\frac{1}{2}}};$$

$$\therefore\ ds=\left\{1+\frac{(a-x)^2}{2ax-x^2}\right\}^{\frac{1}{2}}dx=\frac{adx}{(2ax-x^2)^{\frac{1}{2}}};$$

$$\therefore\ s=a\int\frac{dx}{(2ax-x^2)^{\frac{1}{2}}}=a\text{ versin}^{-1}\frac{x}{a}+\text{C}; \text{ (See Art. 89.)}$$

and $\text{C}=0$, since $s=0$, when $x=0$.

When $x=a$, the length of quadrant AD$=\frac{\pi a}{2}$, and the whole circumference $=2\pi a$.

Obs. The various expressions for the differential of a circular arc whose radius is a, should be carefully remembered; as for example,

$$ds=\frac{adx}{(2ax-x^2)^{\frac{1}{2}}}=\frac{adx}{y}.$$

4. To find the length of the arc of an ellipse.

Here $y=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}\ \therefore\ \frac{dy}{dx}=-\frac{b}{a}\frac{x}{(a^2-x^2)^{\frac{1}{2}}}$;

$$\therefore\ 1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^2x^2}{a^2(a^2-x^2)}=\frac{a^2-e^2x^2}{a^2-x^2},$$

by putting e^2 for $\frac{a^2-b^2}{a^2}$;

$$\therefore\ s=\int\left\{\frac{a^2-e^2x^2}{a^2-x^2}\right\}^{\frac{1}{2}}dx=a\int\left\{\frac{1-e^2z^2}{1-z^2}\right\}dz \text{ (putting } az \text{ for } x\text{)}$$

$$=a\int\frac{dz}{(1-z^2)^{\frac{1}{2}}}\left\{1-\frac{e^2z^2}{2}-\frac{e^4z^4}{2.4}-\frac{1.3e^6z^6}{2.4.6}-\&c.\right\},$$

expanding $(1-e^2z^2)^{\frac{1}{2}}$ by the binomial theorem.

Here the integration depends on $\int\frac{z^{2n}dx}{(1-z^2)^{\frac{1}{2}}}$. Let the length of the quadrant be required; then we must integrate from $x=0$ to $x=a$, that is, since $az=x$, from $z=0$ to $z=1$; hence we have, by *Ex.* 4., p. 180.,

$$\int_0^1\frac{z^{2n}dz}{(1-z^2)^{\frac{1}{2}}}=\frac{1\,.\,3\,...\,(2n-1)}{2\,.\,4\,...\,2n}\cdot\frac{\pi}{2};$$

hence making successively $n=1, 2, 3$, &c.,

$$\int_0^1\frac{z^2dz}{(1-z^2)^{\frac{1}{2}}}=\frac{1}{2}\cdot\frac{\pi}{2},\int_0^1\frac{z^4dz}{(1-z^2)^{\frac{1}{2}}}=\frac{1\,.\,3}{2\,.\,4}\cdot\frac{\pi}{2}, \&c.;$$

therefore the length of the quadrant of the ellipse

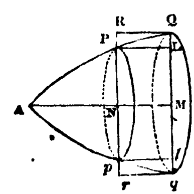
$$=\frac{\pi a}{2}\left\{1-\frac{1}{2^2}e^2-\frac{1\,.\,3}{2^2\,.\,4^2}e^4-\frac{1\,.\,3^2\,.\,5}{2^2\,.\,4^2\,.\,6^2}e^6-\&c.\right\},$$

a series which converges rapidly when e is a small fraction.

TO FIND THE VOLUMES AND SURFACES OF SOLIDS.

Differential of the Volume of a Solid of Revolution.

116. Let V = the volume of the solid generated by the revolution of APN round the axis AM; AN $=x$; NP $=y$; NM $=h=$ Incr. x; then QL or RP = Incr. y, and Incr. V = the solid generated by the revolution of NPQM. Now the solids generated by the revolution of the rectangles PM and RM, are the cylinders PLlp and RQqr, whose solidities may be found by *Ex.* 20. p. 33.



$$\therefore \frac{\text{solidity cyl. RQ}qr}{\text{solidity cyl. PL}lp} = \frac{\pi \cdot \text{RN}^2 \cdot \text{NM}}{\pi \cdot \text{PN}^2 \cdot \text{NM}} = \frac{\text{RN}^2}{\text{PN}^2}$$

$$= \frac{(y + \text{Incr. } y)^2}{y^2} = \left(1 + \frac{\text{Incr. } y}{y}\right)^2.$$

Now as h approaches 0, Incr. y also approaches 0,

$$\therefore \text{ limiting value of } \frac{\text{solidity cyl. RQ}qr}{\text{solidity cyl. PL}lp} = 1;$$

But the solid PQqp is always intermediate between the two cylinders, $\therefore$ *à fortiori*,

$$\text{limit } \frac{\text{solidity PQ}qp}{\text{solidity cyl. PL}lp} = 1;$$

now solidity PQqp=Incr. V, and solidity cyl. PL$lp = \pi y^2 h$,

$$\therefore \text{ limit } \frac{1}{\pi y^2} \cdot \frac{\text{Incr. V}}{h} = 1, \quad \therefore \frac{1}{\pi y^2} \cdot \frac{d\text{V}}{dx} = 1,$$

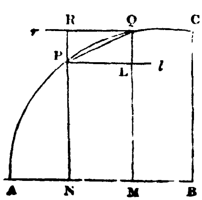
$$\therefore \frac{d\text{V}}{dx} = \pi y^2, \text{ or } d\text{V} = \pi y^2 dx \quad . \quad . \quad . \quad (1)$$

which is the differential expression of the volume of any solid of revolution. By taking the integral we have

$$\text{V} = \pi \int y^2 dx \quad . \quad . \quad . \quad (2)$$

Differential of the Surface of a Solid of Revolution.

117. Let S=the surface generated by the revolution of the arc AP($=s$) round the axis AB; and Incr. S=the surface generated by the arc PQ or Incr. s, due to the increment NM given to x.



On PL and QR produced take Pl and Qr each equal to the length of the arc PQ; then Pl and Qr will generate cylindrical surfaces: therefore by *Ex.* 20. p. 33., we have

$$\frac{\text{surf. gen. by Q}r}{\text{surf. gen. by P}l}=\frac{2\pi\,.\,\text{QM}\,.\,\text{Q}r}{2\pi\,.\,\text{PN}\,.\,\text{P}l}=\frac{\text{QM}}{\text{PN}}$$

$$=\frac{y+\text{Incr. }y}{y}=1+\frac{\text{Incr. }y}{y};$$

$$\therefore \text{ limiting value of } \frac{\text{surf. gen. by Q}r}{\text{surf. gen. by P}l}=1.$$

But the surface generated by the arc PQ is obviously greater than the surface generated by Pl, and less than that generated by Qr; therefore *à fortiori*

$$\text{limiting value of } \frac{\text{surf. gen. by PQ}}{\text{surf. gen. by P}l}=1, \text{ that is,}$$

$$\text{limiting value of } \frac{1}{2\pi y}\,.\,\frac{\text{Incr. }S}{\text{Incr. }s}=1.$$

Now since S is some function of s, at the same time s is some function of x, therefore when h or NM approaches 0, Incr. s as well as Incr. S approaches 0; hence the limiting value of $\frac{\text{Incr. }S}{\text{Incr. }s}$ is $\frac{dS}{ds}$,

$$\therefore \frac{1}{2\pi y}\,.\,\frac{dS}{ds}=1, \text{ or } \frac{dS}{ds}=2\pi y,$$

$$\therefore dS=2\pi y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx \quad . \; . \; . \; (1)$$

by substituting the value of ds.

Hence by integration, we have

$$S=2\pi\int y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx \quad . \; . \; . \; (2)$$

118. The differential expressions contained in the two preceding articles admit of taking the following forms.

Thus in eq. (1) Art., **116.** since πy^2=area section Pp,

$\therefore$ dV or element of the solid=area section P$p\times dx$.

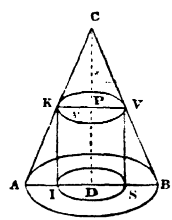
And in eq. (1) Art. **117.**, since $2\pi y$=perimeter section Pp,

$\therefore$ dS or element of the surface=perimeter section P$p \times ds$.

Now it is important to observe, that the reasoning, employed in establishing these results, holds true whatever may be the form of the section Pp of the solid, provided that all the sections parallel to Pp are similar figures, or otherwise that they may be expressed by the same general equation.

EXAMPLES.

1. To find the volume and surface of an upright cone.



Let AD$=r$, DC$=a$, CP$=x$, and KP$=y$; then

AD : DC :: KP : CP, that is

$$r : a :: y : x, \therefore y=\frac{rx}{a};$$

$$\therefore \text{V}=\pi\int y^2dx=\pi\int\frac{r^2x^2dx}{a^2}=\frac{\pi r^2x^3}{3a^2}+\text{C},$$

and C$=0$, since V$=0$, when $x=0$;

Let $x=a$; $\therefore$ whole cone CAB$=\dfrac{\pi r^2a}{3}=\frac{1}{3}$ cylinder of the same base and altitude.

To find the convex surface, we have $y=\dfrac{rx}{a}$,

$$\therefore \left(\frac{dy}{dx}\right)^2=\frac{r^2}{a^2},$$

$$\therefore S=2\pi\int y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\frac{2\pi r(a^2+r^2)^{\frac{1}{2}}}{a^2}\int xdx$$

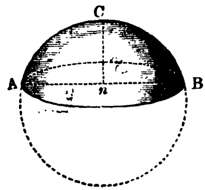
$$=\frac{\pi r(a^2+r^2)^{\frac{1}{2}}x^2}{a^2}+\text{C},$$

and C$=0$, since $S=0$, when $x=0$.

Let $x=a$; $\therefore$ convex surface cone CAB $=\pi r(a^2+r^2)^{\frac{1}{2}}$

$=\frac{1}{2}$ circum. base $\times$ slant height AC.

2. To find the volume and surface of a sphere.



Let $Cn=x$, $An=y$, and the radius of the sphere $=r$, then

$$y^2=2rx-x^2 \ldots (1)$$

$$\therefore \text{ V}=\pi\int y^2dx=\pi\int(2rx-x^2)dx$$

$$=\pi(rx^2-\tfrac{1}{3}x^3)+\text{C, and C}=0,$$

$\therefore$ solidity segment ACB $=\pi x^2(r-\frac{1}{3}x)$.

Let $x=2r$; then whole sphere $=4\pi r^2(r-\frac{2}{3}r)=\frac{4}{3}\pi r^3$.

But the solidity of the circumscribing cylinder $=2\pi r^3$;

$\therefore$ solidity sphere $=\frac{2}{3}$ the circum. cylinder.

To find the surface, we have by differentiating (1)

$$\frac{dy}{dx}=\frac{r-x}{(2rx-x^2)^{\frac{1}{2}}}, \quad \therefore 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(r-x)^2}{2rx-x^2}=\frac{r^2}{y^2},$$

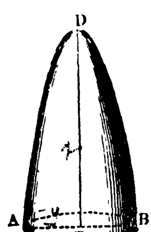
$$\therefore S=2\pi\int y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=2\pi\int y\cdot\frac{r}{y}dx=2\pi rx+\text{C},$$

and C $=0$; $\therefore$ surface segment ACB $=2\pi rx$.

Let $x=2r$; then surface whole sphere $=4\pi r^2$.

Hence the surface of the sphere is equal to the convex surface of the circumscribing cylinder. (See *Ex.* 20. p. 33.)

3. To find the volume and surface of a paraboloid ABD, generated by the revolution of the parabola ADB about its axis mD.



Let D$m=x$, and $am=y$;

$$\text{then } y^2=4ax \ldots (1),$$

$$\therefore \text{ volume ABD}=\pi\int y^2dx=\pi\int 4axdx$$

$$=2\pi ax^2+\text{C, and C}=0,$$

$\therefore$ volume ABD $=2\pi ax^2=\frac{1}{2}\pi y^2x=\frac{1}{2}$ area base $\times$ perpendicular height.

To find the convex surface, we have by differentiating (1),

$$\frac{dy}{dx}=\frac{2a}{y}, \ \therefore \ 1+\left(\frac{dy}{dx}\right)^2=1+\frac{4a^2}{y^2}=\frac{x+a}{x};$$

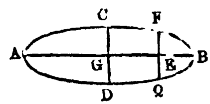
$$\therefore \ S=2\pi\int y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=4\pi a^{\frac{1}{2}}\int(x+a)^{\frac{1}{2}}dx.$$

$$=\tfrac{8}{3}\pi a^{\frac{1}{2}}(x+a)^{\frac{3}{2}}+\text{C},$$

and $S=0$, when $x=0$; $\therefore$ $\text{C}=-\frac{8}{3}\pi a^{\frac{1}{2}}a^{\frac{3}{2}}$;

$\therefore$ surface $\text{ABD}=\frac{8}{3}\pi a^{\frac{1}{2}}\{(x+a)^{\frac{3}{2}}-a^{\frac{3}{2}}\}$.

4. To find the volume of a prolate spheroid, formed by the revolution of an ellipse ACBD about its major axis AB.



Let $\text{GE}=x$, and $\text{EF}=y$;

then $y^2=\frac{b^2}{a^2}(a^2-x^2)$;

$$\therefore \ \text{V}=\pi\int y^2dx=\pi\int\frac{b^2}{a^2}(a^2-x^2)dx$$

$$=\frac{\pi b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right)+\text{C}, \text{ and } \text{C}=0,$$

$$\therefore \text{ volume gen. by CFQD}=\frac{\pi b^2x}{a^2}\left(a^2-\frac{x^2}{3}\right);$$

when $x=\text{GB}=a$, then volume $\text{CDB}=\frac{2}{3}\pi b^2a$, and whole solid $=\frac{4}{3}\pi b^2a$.

Cor. 1. Comparing this with the expression in *Ex.* 2,

sphere on major axis : prolate spheroid :: a^2 : b^2.

5. To find the volume of an oblate spheroid, formed by the revolution of an ellipse ACBD about its minor axis CD. (See last *fig.*)

In this case, let GE $=y$, and EF $=x$; then

$$x^2=\frac{b^2}{a^2}(a^2-y^2), \therefore\ y^2=\frac{a^2}{b^2}(b^2-x^2);$$

$$\therefore\ \text{V}=\pi\int\frac{a^2}{b^2}(b^2-x^2)dx=\frac{\pi a^2}{b^2}\left(b^2x-\frac{x^3}{3}\right),$$

taking $x=b$, and doubling the result, we have the whole solid $=\frac{4}{3}\pi a^2 b$.

Cor. 2. Comparing this with the expression of the last example, we find

prolate spheroid : oblate spheroid :: b : a.

6. To find the surface of a prolate spheroid.

Adopting the notation and figure of *Ex.* 4., we have

$$1+\left(\frac{dy}{dx}\right)^2=\frac{a^2-e^2x^2}{a^2-x^2}, \text{ see } Ex.\ 4.,\ \text{p. } 195.;$$

$$\therefore\ S=2\pi\int y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\frac{2\pi be}{a}\int\left(\frac{a^2}{e^2}-x^2\right)^{\frac{1}{2}}dx$$

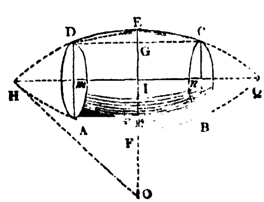
$$=\frac{\pi be}{a}\left\{x\left(\frac{a^2}{e^2}-x^2\right)^{\frac{1}{2}}+\frac{a^2}{e^2}\sin^{-1}\frac{ex}{a}\right\},$$

by *Ex.* 3., page 156. This is the expression for the surface included by the ordinates CD and FQ.

Making $x=a$, and doubling the result, we obtain

$$\text{surface whole spheroid}=\frac{2\pi ab}{e}\{e(1-e^2)^{\frac{1}{2}}+\sin^{-1}e\}.$$

7. To find the volume of a circular spindle, formed by the revolution of the arc HLQ about its chord HQ.



From O, the centre of the circle, draw OIE perpendicular to the chord HQ; let OE $=r$, OI $=c$, Im

$=$ GD $=x$, D$m=$ GI $=y$; then $\text{OD}^2=\text{GD}^2+\text{OG}^2$, that is,

$$r^2=x^2+(y+c)^2;$$

$$\therefore\ y^2=r^2-c^2-x^2-2cy;$$

$$\therefore\ \text{V}=\pi\int y^2dx=\pi\int(r^2-c^2-x^2-2cy)dx$$

$$=\pi\left\{(r^2-c^2)x-\frac{x^3}{3}-2c\int ydx\right\}$$

$$=\pi\left\{(r^2-c^2)x-\frac{x^3}{3}-2c\,(\text{gen. area I}m\text{DE})\right\}+\text{C},$$

and $\text{C}=0$; which is the volume of the frustum generated by the revolution of ImDE, and this being doubled will give the expression for whole frustum ABCD. When $x=$ IH, the volume of the semispindle $\text{HEF}=2\pi\{\frac{1}{3}\text{IH}^3-\text{OI}\times\text{area IHE}\}$.

8. To find the volume of a parabolic spindle, formed by the revolution of a parabola HEQ about its ordinate HQ, taken perpendicular to the axis FO of the curve. (See last fig.)

Let I$m=$ GD $=x$, D$m=$ GI $=y$, IE $=b$, and IH $=l$; then $\text{GD}^2=4a\times\text{EG}$, that is, $x^2=4a(b-y)$;

$$\therefore\ y^2=\frac{1}{16a^2}(4ab-x^2)^2=\frac{1}{16a^2}(l^2-x^2)^2;$$

$$\therefore\ \text{V}=\pi\int y^2dx=\frac{\pi}{16a^2}\int(l^2-x^2)^2dx$$

$$=\frac{\pi}{16a^2}\left\{l^4x-\frac{2l^2x^3}{3}+\frac{x^5}{5}\right\}+\text{C, and C}=0,$$

which is an expression for the volume of the frustum generated by the revolution of ImDE. When $x=l$, this expression gives the volume of the half spindle HEF; hence volume half spindle $\text{HEF}=\frac{\pi l^5}{16a^2}(1-\frac{2}{3}+\frac{1}{5})=\frac{8}{15}\pi lb^2$, eliminating a^2 by means of the equation to the curve.

9. To find the volume of a conical solid, the base being any given curve. (See *fig.* p. 199.)

Let CP$=x$, CD$=a$, A$=$area base AB whatever may be its form, and $A_1=$area section KV; then, from the similar figures, we have

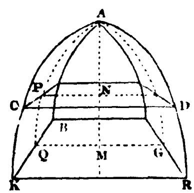
$$\frac{A}{A_1}=\frac{AB^2}{KV^2}=\frac{CD^2}{CP^2}=\frac{a^2}{x^2}, \therefore A_1=\frac{Ax^2}{a^2};$$

$$\therefore \text{ Art. 118., } dV \text{ or element solid}=A_1dx=\frac{Ax^2dx}{a^2},$$

$$\therefore V=\frac{A}{a^2}\int x^2dx=\frac{Ax^3}{3a^2};$$

when $x=a$, volume ABC$=\frac{1}{3}Aa=$area base $\times\frac{1}{3}$ altitude.

10. To find the volume and surface of a groin, formed by the intersection of two semicircular arches with each other.



Here all horizontal sections, such as CDNP, will be squares; the vertical section, APQMG, parallel to the line KR, will be a semicircle; and if AM be a vertical line, APQM will be a quadrant.

Let AN$=x$, NP$=y$, AM$=$MQ$=a$; then the generating area CPDN$=$CD$^2=4y^2$; therefore, Art. 118., dV or element solid $=$generating area $\times\, dx=4y^2dx=4(2ax-x^2)dx$;

$$\therefore V=4\int(2ax-x^2)dx=4\left(ax^2-\frac{x^3}{3}\right),$$

which expresses the volume of the part ACD. When $x=a$, the whole volume $=\frac{8}{3}a^3$.

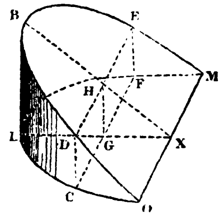
To find the surface. Perimeter section CDN$=8$NP$=8y$.

$\therefore dS$ or element surface$=$perimeter section CDN $\times\, ds$

$$=8y\times\frac{adx}{y}=8adx, \text{ see } Ex. \text{ 3. p. 195.}$$

$$\therefore \text{S} = \int 8a\,dx = 8ax + (\text{const.} = 0),$$

when $x = a$, the whole curved surface $= 8a^2$.



11. To find the volume and surface of the solid OMBL, cut off from a right cylinder, by a plane OMB passing through the centre X of the base, and inclined at an angle α to the plane of the base OML.

From X draw XL perpendicular to OM, and let CDEF be a section perpendicular to the base, and parallel to OM; then CDEF will be a rectangle, and we may regard the solid as being generated by the motion of this rectangle parallel to itself.

Let $\text{XG} = x$, $\text{XO} = \text{XL} = \text{XM} = r$; then $\text{CF} = 2\text{CG} = 2(r^2 - x^2)^{\frac{1}{2}}$, and GH or $\text{CD} = x \tan \alpha$;

$$\therefore \text{area gen}^{\text{g}}\text{. rect. CDEF} = \text{CF} \cdot \text{CD} = 2(r^2 - x^2)^{\frac{1}{2}} x \tan \alpha\,;$$

$$\therefore d\text{V or element solid} = 2 \tan \alpha (r^2 - x^2)^{\frac{1}{2}} x\,dx,$$

$$\therefore \text{V} = 2 \tan \alpha \int (r^2 - x^2)^{\frac{1}{2}} x\,dx = -\frac{2 \tan \alpha}{3} (r^2 - x^2)^{\frac{3}{2}} + \text{C}\,;$$

$$\text{and V} = 0, \text{ when } x = 0, \therefore 0 = -\frac{2 \tan \alpha\, r^3}{3} + \text{C},$$

$$\therefore \text{C} = \frac{2 \tan \alpha\, r^3}{3};$$

$$\therefore \text{V or volume OCDEM} = \frac{2 \tan \alpha}{3} \{r^3 - (r^2 - x^2)^{\frac{3}{2}}\}.$$

When $x = r$, the whole solid $\text{OLBM} = \dfrac{2 \tan \alpha \cdot r^3}{3}$.

To find the convex surface. Here the generating line is CD: putting s, therefore, for the arc OC,

$$d\text{S or element surface} = \text{CD} \times ds = x \tan \alpha \times \frac{r\,dx}{(r^2 - x^2)^{\frac{1}{2}}};$$

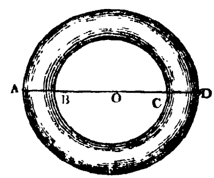
$$\therefore\ s = r \tan \alpha \int \frac{x dx}{(r^2 - x^2)^{\frac{1}{2}}} = -r \tan \alpha (r^2 - x^2)^{\frac{1}{2}} + c,$$

and $s=0$, when $x=0$, $\therefore$ $c = r^2 \tan \alpha$;

$$\therefore\ s \text{ or surface OCD} = r \tan \alpha \{r - (r^2 - x^2)^{\frac{1}{2}}\};$$

taking $x=r$, and doubling the result, we find the whole convex surface OLBM$=2r^2 \tan \alpha$.

12. To find the volume of a cylindrical ring, formed by the revolution of a circle, whose diameter is AB, round O as an axis.



Put $2r=$AB, the diameter of the revolving circle, and $b=$the distance of the centre of this circle from O.

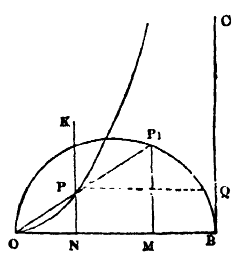
Conceive a horizontal section to be made, passing through the centre of revolution O, and dividing the ring into two equal parts. Parallel to this plane, and at the distance x from it, let another section be made; then this section will form a plane ring, whose half-breadth we shall represent by y, and therefore its area$=\pi(b+y)^2 - \pi(b-y)^2 = 4\pi by$;

$$\therefore\ dv = \text{gen}^{\text{g}}.\ \text{area} \times dx = 4\pi by dx,$$

$$\therefore\ v = 4\pi b \int_0^r y dx = 4\pi b \times \frac{\pi r^2}{4} = \pi^2 r^2 b,$$

which is the volume of the half ring; therefore the volume of the whole ring$=2\pi^2 r^2 b$.

13. To find the volume of the solid formed by the revolution of the cissoid about its asymptote BC.



Taking B as the origin, let BQ$=$NP$=x$, QP$=$BN$=y$, OB$=2r$. The equation of the curve, given in Art. **23.**, may be thus expressed:

$$\text{NP}^2 = \frac{\text{ON}^3}{\text{BN}} = \frac{(\text{OB} - \text{BN})^3}{\text{BN}};$$

$$\text{that is, } x^2 = \frac{(2r-y)^3}{y} \ldots (1).$$

By the formula of parts,

$$\text{V} = \pi \int y^2 dx = \pi y^2 x - 2\pi \int xy dy;$$

$$\text{but from eq. (1), } xy = (2r-y)(2ry-y^2)^{\frac{1}{2}},$$

$$\therefore \int xy dy = \int (2r-y)(2ry-y^2)^{\frac{1}{2}} dy$$

$$= \int (r-y)(2ry-y^2)^{\frac{1}{2}} dy + r\int (2ry-y^2)^{\frac{1}{2}} dy$$

$$= \tfrac{1}{3}(2ry-y^2)^{\frac{3}{2}} + r(\text{cir. area BNK});$$

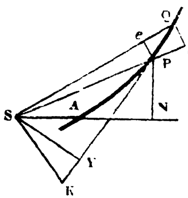
$$\therefore \text{V} = \pi \{y^2 x - \tfrac{2}{3}(2ry-y^2)^{\frac{3}{2}} - 2r(\text{cir. area BNK})\} + \text{C},$$

and $\text{V}=0$, when $y=2r$, $\therefore 0 = -2\pi r(\text{cir. area BKO}) + \text{C}$;

$$\therefore \text{V} = \pi \{\tfrac{1}{3}(2ry-y^2)^{\frac{3}{2}} + 2r(\text{cir. area NKO})\};$$

when $y=0$, the whole solid $=2\pi r$(area semicircle BKO) $= 2\pi r \times \dfrac{\pi r^2}{2} = \pi^2 r^3$.

POLAR CO-ORDINATES.



119. Let APQ be a curve referred to polar co-ordinates, the origin being at S; on S as a centre describe the circular arcs Pe and Qq

Let SP the radius vector $=r$, $\angle$ ASP $=\theta$, area ASP$=$A, and AP$=s$, then Q$e=$ incr. r, $\angle$ PSQ $=$ incr. θ, area SPQ $=$ incr. A, and PQ $=$ incr. s.

$$\frac{\text{area S}q\text{Q}}{\text{area SP}e} = \frac{\text{S}q^2}{\text{SP}^2} = \left\{\frac{r+\text{incr. } r}{r}\right\}^2 = \left\{1 + \frac{\text{incr. } r}{r}\right\}^2,$$

$$\therefore \text{limiting value of } \frac{\text{area S}q\text{Q}}{\text{area SP}e} = 1;$$

now area SPQ is greater than area SPe, and less than area SqQ; therefore, *à fortiori*,

$$\text{limit } \frac{\text{area SPQ}}{\text{area SP}e}=1, \text{ or limit } \frac{\text{incr. A}}{\frac{1}{2}r^2 \text{ incr. } \theta}=1,$$

$$\therefore \frac{1}{\frac{1}{2}r^2} \cdot \frac{d\text{A}}{d\theta}=1, \text{ or } d\text{A}=\frac{r^2 d\theta}{2} \ldots (1),$$

which is the differential of the area ASP.

By Art. **115.**, $ds=\sqrt{dx^2+dy^2}$;

$$\text{but SN}=\text{SP} . \cos\theta, \text{ and NP}=\text{SP} . \sin\theta;$$

$$\text{that is, } x=r\cos\theta, \text{ and } y=r\sin\theta;$$

differentiating these two equations, we have

$$dx=dr\cos\theta-r\sin\theta\, d\theta,$$

$$dy=dr\sin\theta+r\cos\theta\, d\theta,$$

$$\text{squaring and adding } dx^2+dy^2=dr^2+r^2d\theta^2,$$

$$\therefore ds=\sqrt{dr^2+r^2d\theta^2} \ldots (2).$$

This result may also be proved after the method of limits.

120. These, as well as other important formulæ, may be readily derived by the method of infinitesimals.

Let, as in Art. **56.**, p. 86., PQ$=ds$, Q$e=dr$, P$e=rd\theta$; then

$$d\text{A}=\text{area SPQ}=\tfrac{1}{2}\,\text{SP} . \text{P}e=\tfrac{1}{2}\,r . rd\theta=\tfrac{1}{2}r^2d\theta,$$

$$ds=\text{PQ}=\sqrt{\text{Q}e^2+\text{P}e^2}=\sqrt{dr^2+r^2d\theta^2}.$$

Let fall SY perpendicular to QP produced, and draw SK perpendicular to SP; then, when PQ is infinitely small, PK becomes the tangent to the point P, and SK is called the polar subtangent. Put SY$=p$, and $\angle$ SQP or $\angle$ SPK$=\varphi$; then

$$\sin\varphi=\frac{\text{P}e}{\text{PQ}}=\frac{rd\theta}{ds} \ldots (1),$$

$$\tan\varphi = \frac{\text{P}e}{\text{Q}e} = \frac{r\,d\theta}{dr} \ldots (2),$$

$$p = \text{SY} = r\sin\varphi = \frac{r^2 d\theta}{ds} \ldots (3),$$

$$\text{polar subtangent} = \text{SK} = r\tan\varphi = \frac{r^2 d\theta}{dr} \ldots (4).$$

Examples.

Polar Areas, &c.

1. To find the area of OFP in the common parabola, the focus F being the origin. See *fig.* p. 15.

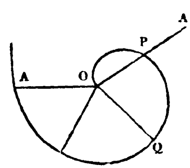
By Cor. Art. **19.**, $r = \dfrac{2a}{1+\cos\theta} = \dfrac{a}{\cos^2\frac{\theta}{2}}$; then by (1) Art. **119.**

$$\text{A} = \tfrac{1}{2}\int r^2 d\theta = a^2\int \frac{\frac{1}{2}d\theta}{\cos^4\frac{\theta}{2}}$$

$$= a^2\int\left(1+\tan^2\frac{\theta}{2}\right)\frac{\frac{1}{2}d\theta}{\cos^2\frac{\theta}{2}} = a^2\int\left(1+\tan^2\frac{\theta}{2}\right)d\tan\frac{\theta}{2}$$

$$= a^2\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right).$$

2. To find the area of the spiral of Archimedes.



By Art. **26.**, $r = \dfrac{a}{2\pi}\cdot\theta$,

$$\therefore\ \text{A} = \tfrac{1}{2}\int r^2 d\theta$$

$$= \frac{a^2}{8\pi^2}\int\theta^2 d\theta = \frac{a^2\theta^3}{24\pi^2} = \frac{\pi r^3}{3a};$$

when the radius vector OP has made one revolution, $r = \text{OA} = a$; $\therefore$ the area $\text{OPQA} = \dfrac{\pi a^2}{\text{''}}$.

3. To find the area of the lemniscata of Bernoulli.

Here the equation to the curve is $r^2=a^2\cos 2\theta$;

$$\therefore\quad A=\tfrac{1}{2}\int r^2 d\theta=\frac{a^2}{2}\int\cos 2\theta\, d\theta=\frac{a^2}{4}\sin 2\theta+(\text{const.}=0).$$

If $\theta=45°$, then $\sin 2\theta=1$, $\therefore$ $\frac{1}{4}$ lemniscata$=\frac{a^2}{4}$, and whole area $=a^2$.

4. To find the length of the spiral of Archimedes, the equation being $r=a\theta$.

By eq. (2), Art. 119.

$$s=\int\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}d\theta=\frac{1}{a}\int(r^2+a^2)^{\frac{1}{2}}dr$$

$$=\frac{r}{2a}(r^2+a^2)^{\frac{1}{2}}+\frac{a}{2}\log\{r+(r^2+a^2)^{\frac{1}{2}}\}+C,$$

by *Ex.* 2. page 155. When $r=0$, $s=0$, $\therefore$ $0=\frac{a}{2}\log a+C$,

$\therefore$ the length of the arc from the origin is

$$\frac{r(r^2+a^2)^{\frac{1}{2}}}{2a}+\frac{a}{2}\log\frac{r+(r^2+a^2)^{\frac{1}{2}}}{a}.$$

5. The equation to the hyberbolic spiral is $r\theta=a$.

Show that the area swept out by the radius vector from 0 to r is $\frac{1}{2}\,ar$.

6. In the logarithmic spiral $r=ae^{m\theta}$; show that

$$s=(1+m^2)^{\frac{1}{2}}\frac{r}{m}.$$

7. The curve represented by the equation $r=a\sin 3\theta$, has six identical loops formed about the centre of a circle whose radius is a; show that the area of one of these loops is $\frac{1}{12}\pi a^2$.

Polar Tangents, &c.

8. To draw a tangent to the spiral of Archimedes.

Here $r=\frac{a}{2\pi}\cdot\theta$, and by (4) Art. **120.**

$$\text{polar subtangent}=r^2\cdot\frac{d\theta}{dr}=r^2\cdot\frac{2\pi}{a}=\frac{2\pi r^2}{a}.$$

When $r=\text{OA}=a$ (see *fig.* to *Ex.* 2), the subtangent at $\text{A}=2\pi a=$ the circumference of a circle described with the radius OA.

9. In the logarithmic spiral $r=ae^{m\theta}$;

$$\therefore\ dr=mae^{m\theta}d\theta,\ \text{and}\ \frac{d\theta}{dr}=\frac{1}{mr};$$

$$\text{polar subtangent}=r^2\cdot\frac{d\theta}{dr}=\frac{r}{m}.$$

By (2) Art. **120.**, $\tan\varphi=r\cdot\frac{d\theta}{dr}=\frac{1}{m}$;

hence in this curve the tangent always makes the same angle with the radius vector.

By (3) Art. **120.**,

$$p=\frac{r^2d\theta}{ds}=\frac{r^2d\theta}{\sqrt{dr^2+r^2d\theta^2}}=\frac{r^2}{\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}}$$

$$=\frac{r^2}{\sqrt{m^2r^2+r^2}}=\frac{r}{\sqrt{m^2+1}}.$$

10. In the hyperbolic spiral $r\theta=a$;

$$\therefore\ dr\theta+rd\theta=0,\ \text{and}\ \frac{d\theta}{dr}=-\frac{\theta}{r};$$

$$\therefore\ \text{polar subtangent}=r^2\cdot\frac{d\theta}{dr}=-r\theta=-a$$

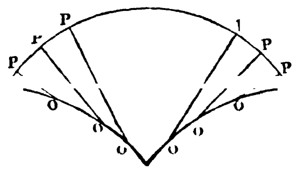
a constant quantity; hence the locus of the extremity of the subtangent is a circle whose radius is a.

11. Show that the subtangent to the lituus is $\frac{2a}{r}$; the equation of the curve being $r^2\theta=a$.

12. In the lemniscata of Bernoulli, the perpendicular p upon the tangent is $\frac{r^3}{a^2}$.

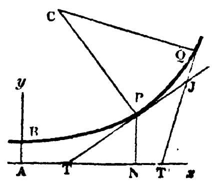
RADIUS OF CURVATURE.

121. Conceive a fine cord, fixed at one extremity, to be gradually unwound from the convex curve BOO, then the extremity B will describe a curve BPV, in which every successive portion will be swept by continually increasing radii OP, OP, &c. Here OP is called the *radius of curvature* to the point P, and O the centre of curvature. The curve BOO, which forms the locus of the centres of curvature, is called the evolute; and it is obvious that the radii of curvature are all perpendicular to the curve, and at the same time form tangents to the evolute; hence the direction of the curve at P is always perpendicular to the direction of the evolute at O. Hence, also, if any two points be taken, on the curve BV, infinitely near to each other, then the lines drawn perpendicular to the curve at these points will intersect in the centre of curvature.



As a further geometrical illustration of the principle of curvature; let different circles be swept so as to touch the ellipse ABA′B′ in the point B (see *fig.* p. 216.); then so long as the circles fall within the curve of the ellipse on each side of B, the radius of curvature at B must be greater than the radius of any of these circles; and, on the contrary, when these circles fall without the ellipse, the radius of curvature at B must be less than the radius of any of these circles. Hence the circle of curvature is that circle which is intermediate between those circles which fall within the curve and those which fall without it.

122. Let PC and QC be two normals to a curve at the points P and Q; PT and QT′ tangents to these points. Let AN$=x$, NP$=y$, $s=$ the length of the arc BP, $c=$ the chord joining P and Q, $\angle$PT$x=\psi$, $\angle$QT′$x=\psi'$, and R$=$ the radius of curvature at P, which, from what has been explained, is the limiting value of PC as PQ approaches O.



$$\text{(1) R} = \text{ the limiting value of } \frac{c}{\angle C}.$$

For by simple algebra and trigonometry, we have

$$\frac{c}{C} = \frac{c}{\sin C} \cdot \frac{\sin C}{C}$$

$$\frac{PC}{\sin PQC} \quad \frac{\sin C}{C}$$

Now as the point Q approaches P, the $\angle$PQC will approach nearer and nearer to 90° or $\frac{\pi}{2}$ as its limit; therefore the limiting value of sin PQC is 1: and moreover, by (17) Art. **28.**, the limiting value of $\frac{\sin C}{C}=1$; therefore the limiting value of $\frac{c}{C}=$ the limiting value of PC$=$R.

(2) Conceiving the point P to move to Q, then

$$PQ = \text{incr.}\, s, \text{ and } \psi' - \psi = \text{incr.}\, \psi.$$

Since $\angle$ P and $\angle$ Q are right angles,

$$\angle \text{C} = \angle \text{TJT}', \text{ but } \psi' - \psi = \angle \text{TJT}',$$

$$\therefore \angle \text{C} = \psi' - \psi = \text{incr.}\,\psi;$$

hence, by simple algebra, we have

$$\frac{\text{incr.}\,\psi}{\text{incr.}\,s} = \frac{\angle \text{C}}{\text{incr.}\,s} = \frac{c}{\text{incr.}\,s} \div \frac{c}{\angle \text{C}}.$$

Now as PQ or incr. s is diminished, the limiting value of $\frac{\text{incr.}\,\psi}{\text{incr.}\,s}$ is $\frac{d\psi}{ds}$; from what has just been proved, that of $\frac{c}{\angle \text{C}}$ is R, and that of $\frac{c}{\text{incr.}\,s}$ is 1, by Art. **144.**; hence we have

$$\frac{d\psi}{ds} = \frac{1}{\text{R}} \ldots (1).$$

Let us now proceed to find R in terms of x and y. By Art. **32.**, we have

$$\tan\psi = \frac{dy}{dx}, \therefore \psi = \tan^{-1}\frac{dy}{dx};$$

therefore, by form (3) Art. **58.**,

$$d\psi = \frac{d\,\frac{dy}{dx}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{dx^2}{dx^2 + dy^2} \cdot d\left(\frac{dy}{dx}\right)$$

$$= \frac{dx^2}{ds^2} \cdot d\left(\frac{dy}{dx}\right);$$

therefore, substituting in (1) we have

$$\frac{1}{\text{R}} = \frac{d\psi}{ds}$$

$$= \frac{dx^2}{ds^3} \cdot d\left(\frac{dy}{dx}\right) \ldots (2).$$

If x be the independant variable in this expression, then

$$\frac{1}{R}=\frac{dx^2}{ds^3}\cdot\frac{d^2y}{dx}=\frac{dx\,d^2y}{ds^3}\ldots(3);$$

$$\therefore R^2=\frac{ds^6}{dx^2(d^2y)^2}=\frac{(dx^2+dy^2)^3}{dx^2(d^2y)^2}$$

$$=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3}{\left(\frac{d^2y}{dx^2}\right)^2}\ldots(4).$$

$\rho=\frac{\{1+(y')\}^2}{y''}$

When the equation of the curve is given, the radius of curvature, to any point in the curve, may be found from any of the last three formulæ. In the following examples, formula (4) is only employed.

EXAMPLES.

1. To find the radius of curvature of the parabola.

Here $y^2=4ax$, and $y=2a^{\frac{1}{2}}x^{\frac{1}{2}}$;

$$\therefore \frac{dy}{dx}=a^{\frac{1}{2}}x^{-\frac{1}{2}}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}},$$

$$\text{and } \frac{d^2y}{dx^2}=-\tfrac{1}{2}a^{\frac{1}{2}}x^{-\frac{3}{2}}=-\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}};$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2=\frac{a+x}{x}, \text{ and } \left(\frac{d^2y}{dx^2}\right)^2=\frac{a}{4x^3};$$

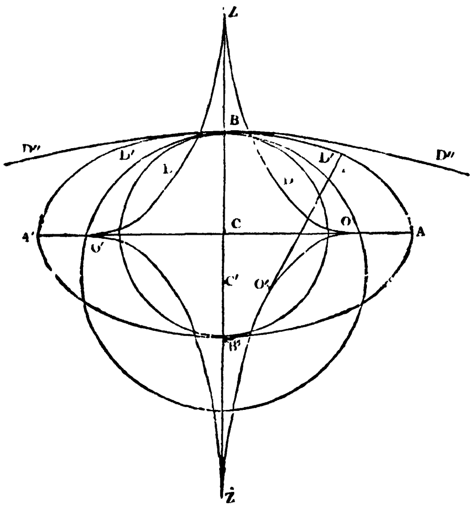
substituting these values in formula (4), we have

$$R^2=\frac{(a+x)^3}{x^3}\div\frac{a}{4x^3}=\frac{4(a+x)^3}{a},$$

$$\therefore R=\frac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Cor. When $x=0$, $R=2a$; that is, the radius of curvature at the vertex is equal to twice the distance of the focus.

2. To find the radius of curvature of the ellipse ABA′B. Taking the centre C as the origin,



$$y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}};$$

$$\therefore \frac{dy}{dx} = -\frac{b}{a}\frac{x}{(a^2 - x^2)^{\frac{1}{2}}},$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{ab}{(a^2 - x^2)^{\frac{3}{2}}},$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^2x^2}{a^2(a^2 - x^2)} = \frac{a^2 - e^2x^2}{a^2 - x^2}, \text{ where } e^2 = \frac{a^2 - b^2}{a^2};$$

and $\left(\frac{d^2y}{dx^2}\right)^2 = \frac{a^2b^2}{(a^2 - x^2)^3}$; substituting these values in the general formula (4), we have

$$R^2 = \frac{(a^2 - e^2x^2)^3}{(a^2 - x^2)^3} \div \frac{a^2b^2}{(a^2 - x^2)^3} = \frac{(a^2 - e^2x^2)^3}{a^2b^2};$$

$$\therefore\ R=\frac{(a^2-e^2x^2)^{\frac{3}{2}}}{ab}.$$

Let x and y be the co-ordinates of the point P in the ellipse; draw PO′ perpendicular to the curve, or, what is the same thing, perpendicular to the tangent to the point P; take off PO′=to the value of R above found; then O′ will be the centre of the circle of curvature to the point P, and will consequently be a point in the evolute OO′Z. In this way we may obtain any number of points in the evolute; however, it should be observed, that its equation may be generally expressed.

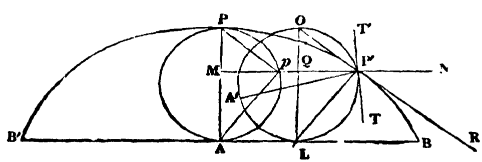
Cor. 1. When $x=0$, $R=\frac{a^3}{ab}=\frac{a^2}{b}$; that is, the radius of curvature at $B=ZB=\frac{a^2}{b}$.

Cor. 2. When $x=a$, $R=\frac{(a^2-e^2a^2)^{\frac{3}{2}}}{ab}=\frac{b^2}{a}$; that is, the radius of curvature at $A=OA=\frac{b^2}{a}$.

Cor. 3. Since ZB is equal to the length of the evolute ZO′O added to OA, therefore the length of the evolute ZO′O $=ZB-OA=\frac{a^2}{b}-\frac{b^2}{a}=\frac{a^3-b^3}{ab}$.

The Cycloid.

3. As this curve is not only interesting in itself, but important in its application to mechanical science, we shall here notice some of its most remarkable properties.



Let BP′PB be the cycloid; LP′O the position of the generating circle, LO being the diameter perpendicular to B′B; ApP the position of the generating circle when it has rolled over one half its circumference; then AP is the diameter of the generating circle and perpendicular to BB′; hence B′B and AP are the axes of the curve; from P′ let fall P′M perpendicular to AP cutting LO in Q, and the circle ApP in p.

(1.) Since PM=OQ, &c., the triangles PMp and OQP′ are identical; therefore OP′ is equal and parallel to Pp; and similarly P′L is equal and parallel to pA.

(2.) It has been shown (*Ex.* 5., page 113.) that P′L is the normal to the point P, and therefore OP′R forms a tangent to the curve at P′. Hence we have the following easy rule for drawing a tangent to any point P′ in the cycloid: from P′ draw PM perpendicular to the axis AP, and from the point p, where that line cuts the generating circle on the axis, draw the chord pP, and through P′ draw OP′R parallel to pP, and it will be the tangent required.

(3.) Let TT′ touch the generating circle OP′LA′ in the point P′, then, from an obvious property of the circle, the tangent RP′O will bisect the ∠ MP′T′.

(4.) To find the equation to the cycloid, taking the vertex P as the origin.

Since circum. semicircle ApP=AB, and arc Ap=arc LP′ =LB, therefore, by subtraction, arc Pp=AL=P′p.

Let PM=x, MP′=y, AP=$2r$; then

$$y=\text{M}p+\text{P}'p=\text{M}p+\text{arc P}p;$$

but arc Pp=r versin$^{-1}\frac{x}{r}$, and Mp= $\sqrt{\text{AM . MP}}=(2rx-x^2)^{\frac{1}{2}}$

$$\therefore\ y=(2rx-x^2)^{\frac{1}{2}}+r\ \text{versin}^{-1}\frac{x}{r};$$

$$\therefore\ dy=\frac{(r-x)dx}{(2rx-x^2)^{\frac{1}{2}}}+\frac{rdx}{(2rx-x^2)^{\frac{1}{2}}}=\frac{(2rx-x^2)^{\frac{1}{2}}dx}{x};$$

$$\therefore\ \frac{dy}{dx}=\frac{(2rx-x^2)^{\frac{1}{2}}}{x}\quad ..(1),$$

which is the differential equation of the cycloid, taking the vertex P as the origin.

(5.) To find the length of the cycloid.

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{2rx-x^2}{x^2}=\frac{2r}{x};$$

$$\therefore\ s=\text{PP}'=\int\left(\frac{2r}{x}\right)^{\frac{1}{2}}dx=2\sqrt{2rx}+\text{C};$$

and $\text{C}=0$, since $s=0$ when $x=0$;

$$\therefore\ \text{arc PP}'=2\sqrt{2rx}=2\text{P}p;$$

that is, the arc of a cycloid is equal to twice the chord of the corresponding arc of the generating circle.

When $x=2r$, the arc of the semicycloid $=4r=$twice the diameter of the generating circle.

The rectification of this curve was discovered by Wren.

(6.) To find the area of the cycloid.

Here, by integration of parts,

$$\text{area PP}'\text{M}=\int ydx=yx-\int xdy$$

$$=yx-\int(2rx-x^2)^{\frac{1}{2}}dx, \text{ by eq. (1)},$$

$$=yx-\text{cir. area P}p\text{M}.$$

When $x=2r$, $y=\text{AB}=\pi r$, and cir. area PpM becomes area semicircle $\text{P}p\text{A}=\frac{1}{2}\pi r^2$; therefore area of the semicycloid $\text{PBA}=2\pi r^2-\frac{1}{2}\pi r^2=\frac{3}{2}\pi r^2$, and area of the whole cycloid$=3\pi r^2=$three times the area of the generating circle.

(7.) To find the radius of curvature of the cycloid.

By *Ex.* 5., p. 113., we have,

$$\frac{dy}{dx}=\frac{(2ry-y^2)^{\frac{1}{2}}}{y},\ \therefore\ 1+\left(\frac{dy}{dx}\right)^2=\frac{2r}{y};$$

differentiating this latter result, we have,

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=\frac{-\frac{dy}{dx}\cdot 2r}{y^2}=-\frac{2r}{y^2}\cdot\frac{dy}{dx};$$

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{r}{y^2};$$

substituting these values in the formula for R^2, we have,

$$R^2=\left(\frac{2r}{y}\right)^3\div\left(-\frac{r}{y^2}\right)^2=8ry;$$

$$\therefore\ R=2\sqrt{2ry}=2\sqrt{BD\cdot BR}=2PB \text{ (see } \textit{fig.} \text{ p. 21.).}$$

Hence it appears that the radius of curvature to any point in the cycloid is double the normal to the same point. Thus, in *fig.* p. 212., if BVB′ be the cycloid, and the lines PO the radii of curvature, then the axis BB′ will bisect all these radii. From this property it may readily be proved by common geometry, that the evolute BOO is a cycloid precisely the same as the semicycloid BPV.

4. In the cubical parabola, $3a^2y=x^3$, and the radius of curvature $R=\frac{(a^4+x^4)^{\frac{3}{2}}}{2a^4x}$.

5. In the rectangular hyperbola referred to its asymptotes,

$xy=m^2$, and $R=\frac{(x^2+y^2)^{\frac{3}{2}}}{2m^2}$.

6. *In the catenary,* $y=\frac{c}{2}(e^{\frac{x}{c}}+e^{-\frac{x}{c}})$,

$$\frac{dy}{dx}=\tfrac{1}{2}(e^{\frac{x}{c}}-e^{-\frac{x}{c}}),\ \frac{d^2y}{dx^2}=-\frac{a}{y^2},\text{ and }\therefore\ R=\frac{y^2}{2c}.$$

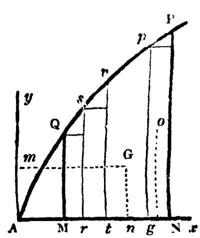
To express the limiting Value of the Sum of a Series in the Form of a definite Integral.

123. The sum of a series may be expressed by placing the symbol Σ before its general term. Thus, if Δx be put for the increment of the variable x, and $f(x)\,\Delta x$ for the general term of a series, we have

$$\Sigma_a^{a+(n-1)\Delta x}\{f(x)\,\Delta x\}=f(a)\Delta x+f(a+\Delta x)\Delta x+f(a+2\Delta x)\Delta x$$
$$+\ldots+f(a+\overline{n-1}\Delta x)\,\Delta x\ldots(1),$$

where Σ symbolises the word *sum*, and the general term $f(x)\Delta x$ is the *type* of a series of terms, connected by the sign of addition, taken from $x=a$ to $x=a+(n-1)\Delta x$, the increment of x in passing from one term to the next succeeding one being Δx; hence we say that the whole symbol indicates the sum of a series taken between the limits $x=a$ and $x=a+(n-1)\Delta x$. With the view of leaving the limits indefinite, in order that they may be assigned to suit the peculiar conditions of a problem, the sum of a series is sometimes simply expressed by $\Sigma f(x)\Delta x$.

In the curve AQP let $\text{AM}=a$, $\text{AN}=x$, $\text{NP}=y=f(x)$. Let MN be divided into n equal parts, viz., $\text{M}r=rt=$ &c. $=\Delta x$; and on these bases let rectangles be constructed as in the annexed figure. Then $\text{MN}=x-a$;



$\text{M}r$ or $\Delta x=\dfrac{x-a}{n}$, $\therefore\ n\Delta x=x-a$.

Also from the equation to the curve, $y=f(x)$, $\text{MQ}=f(a)$, $rs=f(a+\Delta x)$, $tv=f(a+2\Delta x)$, and so on, the $(n-1)$th ordinate $gp=f(a+\overline{n-1}\,\Delta x)$; hence area rectangle $\text{Q}r=\text{MQ}.\text{M}r=f(a)\Delta x$, area rectangle $st=rs\,.\,rt=f(a+\Delta x)\Delta x$, and so on; the area rectangle $p\text{N}=gp\,.\,g\text{N}=f(a+\overline{n-1}\Delta x)\Delta x$. Hence it appears that series (1) expresses the sum of all the rectangles inscribed in the curved space MQPN. But since $a+(n-1)\Delta x = \text{AN} - g\text{N} = x - \Delta x$, this series may also be expressed by $\sum_a^{x-\Delta x}\{f(x)\Delta x\}$.

Now, Art. **113.** eq. (1.), the sum of these rectangles approaches nearer and nearer to the area of the curve MQPN as Δx is diminished, or, what is the same thing, as the number of parts n is increased. But, by Art. **113.**, area MQPN$=\int_a^x f(x)dx$;

$$\therefore \int_a^x f(x)dx = \text{limit of } \sum_a^{x-\Delta x}\{f(x)\Delta x\} \ldots (2).$$

This theorem is very important as it regards the application of the calculus to mechanics, and indeed to almost every branch of general physics.

The indefinitely small rectangles (or other portions into which we suppose the integral quantity to be divided) are called *elements*; thus the limit of $f(x)\Delta x$ or $f(x)dx$ is the element.

124. As a geometrical illustration of this theorem, let AQP be a straight line (see the last *fig.*), and $y=x$ its equation; then we have for the sum of all the rectangles inscribed in MQPN,

$$\sum_a^{x-\Delta x}\{x\Delta x\} = a\Delta x + (a+\Delta x)\Delta x + \ldots + (x-\Delta x)\Delta x$$

$$= (a+x-\Delta x)\frac{n\Delta x}{2}, \text{ by summing the series,}$$

$$= \frac{x^2-a^2}{2} - \frac{(x-a)^2}{2n}, \text{ since } \Delta x = \frac{x-a}{n},$$

$$= \frac{x^2-a^2}{2}, \text{ when } n=\infty;$$

which is the area of the trapezoid MQPN.

Now we have also by integration

$$\text{area MQPN} = \int_a^x x dx = \frac{x^2-a^2}{2};$$

which verifies the theorem in this particular case.

125. The method of dividing a quantity into *elements,* and then taking their sum, may be readily employed for finding expressions for areas, length of arcs, &c. For example, let it be required to find an expression for the volume of the solid generated by the revolution of the curve APQM about the axis AM. (See *fig.* page 196.) Here, taking as our element the cylinder generated by the revolution of the rectangle NMLP; then assuming that the solid AQq is the limit of the sum of all the infinitely small cylinders into which we suppose it divided, and of which $\pi y^2\Delta x$ is the type or general form, we have

$$\text{limit } \sum_0^{x-\Delta x}\{\pi y^2\Delta x\} = \pi\int_0^x y^2 dx = \text{V},$$

or the volume of the solid AQq.

In all formulæ connected with the application of the calculus, it is important to observe, that the second, as well as all higher powers of Δx, may be neglected, since, when the limiting value of such formulæ is taken, any error arising from this source must vanish.

APPLICATION OF THE CALCULUS TO MECHANICS, &c.

Centre of Gravity of Plane Surfaces.

126. The property of the centre of gravity of any plane surface MQPN($=m$) is as follows (see *fig.* page 221.): Let the whole area be divided into any number of parts; then, supposing the surface to turn about Ay as an axis, the sum of the moments of these parts is equal to the moment of the whole area considered as acting in its centre of gravity G. From this property the distance, Gm ($=$X), of the centre of gravity G from the axis Ay is determined; and in like manner the distance, G$n=$Y, from the axis Ax is determined.

Let the surface MQPN be divided into elements or indefinitely small rectangles, as in Art. **123.**; then the moment of any one of these rectangles will be its area multiplied by its distance from the axis of motion Ay: hence, adopting the notation there given, we have

moment element Np = area Np × AN* $= xy\Delta x$,

∴ the sum of the moments of all the rectangles Qr, st, . . ., N$p = \sum_a^{x-\Delta x}\{xy\Delta x\}$, where xy is some function of x; but moment area MQPN = area MQPN × G$m = m$.X,

$$\therefore\ m.\text{X} = \text{limit of } \sum_a^{x-\Delta x}\{xy\Delta x\} = \int_a^x xy\,dx,$$

by theorem (2) Art. **123.**;

$$\therefore\ \text{X} = \frac{\int_a^x xy\,dx}{m} \dots (1).$$

A similar expression may be found for the value of Y; but one more convenient for calculation is determined as follows:—

Conceive the surface ANP to turn about Ax as an axis; then the moment of any one of the rectangles Qr, st, . . . , Np, will be its area multiplied by the distance of its centre of gravity from Ax: hence, if O be the centre of gravity of the rectangle Np, we have

moment Np = area Np × O$g = y\Delta x \times \frac{1}{2}y = \frac{1}{2}y^2\Delta x$,

$$\therefore\ m\,.\,\text{Y} = \text{limit of } \sum_a^{x-\Delta x}\{\tfrac{1}{2}y^2\Delta x\} = \tfrac{1}{2}\int_a^x y^2dx,$$

$$\therefore\ \text{Y} = \tfrac{1}{2}.\ \frac{\int_a^x y^2dx}{m} \ \dots (2).$$

When AM $= a = 0$, then we have for the centre of gravity of APN,

$$\text{G}m \text{ or X} = \frac{\int_0^x xy\,dx}{m} \ \dots (3),$$

* Here AN may be taken as the distance of the centre of gravity of the element Np from the axis Ay, since when the limits are taken any error from this assumption vanishes.

$$\text{and G}n\text{ or Y} = \frac{1}{2}\frac{\int_0^x y^2 dx}{m} \quad \ldots (4).$$

Ex. 1. Let AQP be a parabola; required the centre of gravity of ANP.

Here $y^2 = 4ax$, and by *Ex.* 2., Art. **113.**,

$$m = \text{area APN} = \tfrac{2}{3}xy = \tfrac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}},$$

therefore by formula (3),

$$\text{G}m\text{ or X} = \frac{\int_0^x xydx}{m} = \frac{\int_0^x 2a^{\frac{1}{2}}x^{\frac{3}{2}}dx}{\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}}$$

$$= \frac{\frac{4}{5}a^{\frac{1}{2}}x^{\frac{5}{2}}}{\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}} = \tfrac{3}{5}x = \tfrac{3}{5}\text{AN}.$$

Also from formula (4),

$$\text{G}n\text{ or Y} = \tfrac{1}{2}\cdot\frac{\int_0^x y^2dx}{m} = \tfrac{1}{2}\cdot\frac{\int_0^x 4axdx}{\frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}}$$

$$= \tfrac{3}{4}a^{\frac{1}{2}}x^{\frac{1}{2}} = \tfrac{3}{8}y = \tfrac{3}{8}\text{NP}.$$

2. Let ANP be a portion of a circle, AN being a line passing through the centre.

Here $y = (2ax - x^2)^{\frac{1}{2}}$ is the equation to the circle,

$$\therefore \int_0^x xydx = \int_0^x x(2ax - x^2)^{\frac{1}{2}}dx$$

$$= -\int_0^x (a - x)(2ax - x^2)^{\frac{1}{2}}dx + \int_0^x a(2ax - x^2)^{\frac{1}{2}}dx$$

$$= -\tfrac{1}{3}(2ax - x^2)^{\frac{3}{2}} + a.\ \text{circ. area ANP};$$

$$\therefore \text{X} = \frac{\int_0^x xydx}{m} = \frac{-\frac{1}{3}(2ax - x^2)^{\frac{3}{2}} + a.\ \text{cir. area ANP}}{\text{cir. area ANP}}$$

$$=\frac{-(2ax-x^2)^{\frac{3}{2}}}{3\text{ circ. area ANP}}+a.$$

$$\text{Also } Y=\tfrac{1}{2}\cdot\frac{\int_0^x y^2dx}{m}=\tfrac{1}{2}\cdot\frac{\int_0^x (2ax-x^2)dx}{m}$$

$$=\frac{ax^2-\frac{1}{3}x^3}{2\text{ cir. area ANP}}.$$

When $x=a$, ANP becomes a quadrant; then $X=a-\frac{4a}{3\pi}$,

and $Y=\frac{4a}{3\pi}$.

Centre of Gravity of a Solid of Revolution.

127. Conceive APN to revolve about the axis Ax, then the solid that will thus be formed will obviously have its centre of gravity somewhere in the axis Ax; let n be the centre of gravity, put An=X, and volume solid $=m$. Now regarding the solid to be made up of a series of cylindrical laminæ, formed by the revolution of the infinitely small rectangles Qr, st, ..., NP; the moment of the whole solid, supposed to turn upon Ay as a fulcrum, will be equal to the sum of the moments of these cylindrical laminæ. But the moment of the cylindrical lamina formed by the revolution of NP = solidity lamina × AN$=\pi y^2\Delta x\times x=\pi y^2 x\Delta x$;

$\therefore$ sum of all the moments of these laminæ

$=\sum_a^{x-\Delta x}\{\pi y^2x\Delta x\}$, where y^2x is some function of x;

$$\therefore\ m\cdot X=\sum_a^{x-\Delta x}\{\pi y^2x\Delta x\}=\pi\int_a^x y^2xdx,$$

$$\therefore\ X=\frac{\pi\int_a^x y^2xdx}{m}\quad\ldots(1).$$

Ex. 1. Let the body be a segment of a sphere:

$$y^2=2ax-x^2;$$

then, integrating between $x=0$, and $x=x$, we have

$$\int_0^x y^2 x dx=\int_0^x (2ax-x^2)x dx=\frac{2ax^3}{3}-\frac{x^4}{4},$$

and *Ex.* 2, p. 200., $m=\pi(ax^2-\frac{1}{3}x^3)$;

$$\therefore \text{x}=\text{A}n=\frac{\pi\int_0^x y^2 x dx}{m}=\frac{\pi(\frac{2}{3}ax^3-\frac{1}{4}x^4)}{\pi(ax^2-\frac{1}{3}x^3)}=\frac{x(8a-3x)}{4(3a-x)}.$$

When the segment becomes a hemisphere, then $x=a$, and $\text{x}=\frac{5a}{8}$.

2. Let the body be a paraboloid:

$$y^2=4ax,$$

$$\therefore \int_0^x y^2 x dx=\int_0^x 4ax^2 dx=\tfrac{4}{3}ax^3,$$

and *Ex.* 3. p. 200., $m=\frac{1}{2}\pi y^2 x=2\pi a x^2$,

$$\therefore \text{x}=\frac{\pi\int_0^x y^2 x dx}{m}=\frac{\frac{4}{3}\pi a x^3}{2\pi a x^2}=\tfrac{2}{3}x.$$

Centre of Gravity of a Curved Line.

128. Let arc QP$=s$, arc pP$=\Delta x$ (see *fig.* p. 221.), the other notation being the same as in Art. **123.** Supposing the curve to turn upon Ax as an axis or fulcrum, then the moment of the whole arc QP will be equal to the sum of the moments of the arcs Qs, sv,..., pP. Now the moment of the arc Pp=PN . P$p=y\Delta s$;

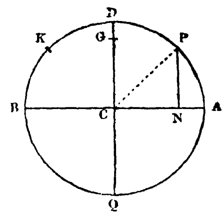
$\therefore$ sum of all the moments of Qs, sv, ..., pP$=\Sigma y\Delta s$;

$$\therefore\ s\,.\,\text{Y}=\text{limit of } \Sigma y\Delta s=\int yds,$$

$$\therefore\ \text{Y}=\frac{\int yds}{s}\ \ldots\ (1).$$

$$\text{Similarly, X}=\frac{\int xds}{s}\ \ldots\ (2).$$

Ex. 1. To find the centre of gravity of the circular arc DP.



Let CN$=x$, NP$=y$, CP or CA$=a$, and DP$=s$; then

$$y^2=a^2-x^2, \text{ and } ds=\frac{adx}{(a^2-x^2)^{\frac{1}{2}}}=\frac{adx}{y};$$

$$\therefore \int_0^x yds=\int_0^x adx=ax, \therefore \text{ by eq. (1)}$$

$$\text{Y}=\frac{\int_0^x yds}{s}=\frac{ax}{s}.$$

If DK be taken equal to DP, then the centre of gravity G of the arc PDK must obviously lie in the line CD,

$$\therefore\ \text{Y}=\text{CG}=\frac{ax}{s}=\frac{a\times 2x}{2s}=\frac{\text{radius . chord PK}}{\text{arc PK}}.$$

Acceleration of Motion by given moving Forces.

129. When a body descends freely by the force of gravity, the *moving force* is measured by the *weight* of the body, or, what is the same thing, by the *pressure* which it would exert upon any obstacle. If the resistance of the air be taken into account, then the moving force, at any instant, is measured by the weight or pressure of the body minus the opposing pressure or resistance of the air. Thus, therefore, moving forces are measured by the *unbalanced pressure* exerted on the moving body. In the case of gravity, the moving force, or pressure producing motion, being constant, the descending body acquires equal increments of motion in equal times.

Let P and $P_{,}$ be the moving pressures exerted on two equal bodies, and f and $f_{,}$ the increments of velocity which these moving pressures respectively produce in the same interval of time (P and $P_{,}$ being supposed constant during this interval), then it is determined by observation and experiment, that $P : P_{,} :: f : f_{,}$, that is, the moving forces or pressures are measured by the increments of motion they communicate in the same time.

Let the pressure $P_{,}$ be the weight W of the body, then $f_{,}$ will be equal to g $(=32\frac{1}{6})$, the increment of velocity communicated by gravity in one second,

$$\therefore P : W :: f : g; \quad \therefore P = \frac{W}{g} . f \ldots (1).$$

For example, if $f = \frac{1}{2} g$, then $P = \frac{1}{2} W$, that is, the moving force is one-half that of gravity. Now regarding the increment of motion, communicated to a body by a moving pressure in one second, as the measure of that moving pressure; let v be the velocity of the body acquired in t seconds, the moving pressure at the end of that time being measured by f; and let Δv and Δf be the corresponding increments of v and f in the increment of time Δt; then if the force f were acting uniformly for Δt seconds, $f \times \Delta t$ would be the increment of velocity, and if the force $f + \Delta f$ were acting uniformly for Δt seconds $(f + \Delta f)\, \Delta t$ would be the increment of velocity; and it is obvious that Δv is intermediate between these. Hence as Δt approaches 0, we have

$$\text{limit of } \frac{(f+\Delta f)\Delta t}{f \times \Delta t} = \text{limit of } \left(1 + \frac{\Delta f}{f}\right) = 1,$$

$$\therefore \textit{à fortiori}, \text{ limit of } \frac{\Delta v}{f \times \Delta t} = 1, \text{ that is,}$$

$$\frac{1}{f} \cdot \frac{dv}{dt} = 1, \quad \therefore f = \frac{dv}{dt} \ldots (2).$$

Substituting this value of f in eq. (1)

$$P=\frac{W}{g}.f=\frac{W}{g}\cdot\frac{dv}{dt}\quad\ldots\quad(3).$$

Let s be the space described in t seconds, and v the velocity acquired in that time; and also let Δs and Δv be the increments of space and velocity in the increment of time Δt. The space which would be described in Δt seconds with the uniform velocity v is $v\Delta t$, and that which would be described in the same time with the uniform velocity $v+\Delta v$ is $(v+\Delta v)\Delta t$. It is obvious that Δs is intermediate between these.

Hence, as Δt approaches 0, we have,

$$\text{limit of }\frac{(v+\Delta v)\Delta t}{v\Delta t}=\text{limit of }\left(1+\frac{\Delta v}{v}\right)=1;$$

$$\therefore\ \textit{à fortiori}, \text{limit of }\frac{\Delta s}{v\,.\,\Delta t}=1, \text{ that is,}$$

$$\frac{1}{v}\cdot\frac{ds}{dt}=1;\quad\therefore\ v=\frac{ds}{dt}\ \ldots\ (4).$$

Differentiating this equation with respect to t, we have, $\frac{dv}{dt}=\frac{d^2s}{dt^2}$; therefore by eq. (2),

$$f=\frac{dv}{dt}=\frac{d^2s}{dt^2}\ \ldots\ (5).$$

Multiplying (2) and (4),

$$f\,ds=v\,dv,\ \text{or}\ f=\frac{vdv}{ds}\ \ldots\ (6).$$

Substituting this value of f in (1),

$$P=\frac{W}{g}f=\frac{W}{g}\cdot\frac{vdv}{ds}\ \ldots\ (7).$$

Equations (6) and (7) are generally most easily applied to the solution of problems.

EXAMPLES.

1. Let the accelerating force be constant, as in the case of gravity.

Here, putting g for f, we have, by eq. (2),

$dv=gdt$; therefore by integration,

$$v=gt+\text{C};$$

if $t=0$ when $v=0$, that is, if the motion commences with the time, then $\text{C}=0$, and

$$v=gt.$$

By eq. (4), $ds=vdt=gtdt$, by substituting the value of v; hence by integration,

$$s=\frac{gt^2}{2}=\frac{vt}{2}.$$

These results are the same as those derived in (15) page 29.

2. A body falls towards the centre of the earth; it is required to find the motion of the body, assuming the force of attraction to vary inversely as the square of the distance from the centre.

Let r=the radius of the earth, g=the force of attraction at its surface, a=the distance of the body from the centre at the commencement of its motion, and x=its distance at the end of t seconds; then

$$f:g::\frac{1}{x^2}:\frac{1}{r^2};\ \therefore f=\frac{r^2g}{x^2};$$

substituting this value of f in eq. (6), observing that in this case $s=a-x$, and $\therefore\ ds=-dx$, we find

$$-\frac{r^2gdx}{x^2}=vdv;$$

therefore by integration and reducing,

$$v^2=\frac{2r^2g}{x}+\text{C}.$$

The constant C depends upon the initial velocity of the body, that is, upon the velocity which it has at the commencement of the motion. Let $v=0$ when $x=a$, then $0=\frac{2r^2g}{a}+C$;

$$\therefore\ v^2=\frac{2r^2g}{x}-\frac{2r^2g}{a}=\frac{2r^2g(a-x)}{ax}.$$

When the body arrives at the surface of the earth, then $x=r$, and

$$v=\sqrt{\frac{2rg(a-r)}{a}}.$$

If a be infinite, then $\frac{a-r}{a}=1$, and

$$\therefore\ v=\sqrt{2rg}.$$

Supposing, therefore, that there is no resisting medium, if a body be projected vertically upwards with this velocity, it would never return to the earth. Taking the radius of the earth to be 4000 miles, this velocity will be about 7 miles per second.

3. To find the vertical motion of a body near the surface of the earth, supposing the resistance of the air to vary as the square of the velocity.

When the velocity of the body is unity, let the resisting force of the air be m, the accelerating force of gravity being g; then the resisting force of the air, when the velocity is v, will be represented by mv^2; therefore the force accelerating the body when its velocity is v, will be

$$f=g-mv^2;$$

therefore by eq. (2) we have,

$$g-mv^2=\frac{dv}{dt};$$

$$\therefore\ dt=\frac{dv}{g-mv^2}=\frac{1}{2g^{\frac{1}{2}}}\left(\frac{1}{g^{\frac{1}{2}}+m^{\frac{1}{2}}v}+\frac{1}{g^{\frac{1}{2}}-m^{\frac{1}{2}}v}\right)dv;$$

$$\therefore\ t=\frac{1}{2\sqrt{mg}}\log\frac{g^{\frac{1}{2}}+m^{\frac{1}{2}}v}{g^{\frac{1}{2}}-m^{\frac{1}{2}}v}+\mathrm{C},$$

where the constant is 0, since $v=0$ when $t=0$.

This expression determines the time in terms of the velocity acquired; and by an easy algebraic artifice, we can find, from this equation, the velocity in terms of the time.

To find the space s, we have by eq. (6),

$$vdv=f\,ds=(g-mv^2)ds;$$

$$\therefore\ ds=\frac{vdv}{g-mv^2}$$

$$\therefore\ s=\int\frac{vdv}{g-mv^2}=-\frac{1}{2m}\log\,(g-mv^2)+\mathrm{C},$$

and $v=0$, when $s=0$, $\therefore\ 0=-\frac{1}{2m}\log g+\mathrm{C}$,

$$\therefore\ s=\frac{1}{2m}\left\{\log g-\log\,(g-mv^2)\right\}=\frac{1}{2m}\log\frac{g}{g-mv^2}.$$

Formulæ relative to Work done by a Variable Pressure.

130. Let P lbs. be the variable pressure applied to the body when it has moved over x feet, and U the work done over that space; then $d\mathrm{U}=\mathrm{P}dx$.

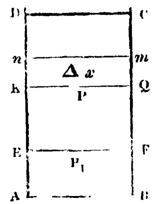
Let Δx be the increment of space, corresponding to $\Delta\mathrm{U}$ and $\Delta\mathrm{P}$ the increments of work and pressure respectively; then $\Delta\mathrm{U}$ will obviously be greater than $\mathrm{P}\times\Delta x$, and less than $(\mathrm{P}+\Delta\mathrm{P})\Delta x$; but we have

$$\text{limit of }\frac{(\mathrm{P}+\Delta\mathrm{P})\Delta x}{\mathrm{P}\,.\,\Delta x}=\text{limit of }\left(1+\frac{\Delta\mathrm{P}}{\mathrm{P}}\right)=1,$$

$$\therefore\ \textit{à fortiori},\ \text{limit of }\frac{\Delta\mathrm{U}}{\mathrm{P}\,.\,\Delta x}=1,\ \text{that is,}$$

$$\frac{1}{\mathrm{P}}\cdot\frac{d\mathrm{U}}{dx}=1,\ \therefore\ d\mathrm{U}=\mathrm{P}dx\ \ldots\ (1).$$

Or thus: suppose the piston of a steam engine to be moved from EF to DC under *any variable pressure.*



Let l=AD the total length of the stroke; a=AE the space through which the steam acts with a uniform pressure; P_1 = the pressure at EF, where the steam is cut off from the boiler; x=AK the height of the piston at KQ when P is the pressure of the steam; Δx=Kn where nm is the position of the piston indefinitely near to KQ; and U= the work done from EF to DC.

Now suppose the pressure P to act uniformly through the small space Δx, then $P\times\Delta x$ will represent the work done through this space, and $\Sigma P\Delta x$ will represent the sum of all the elements of work done from EF to DC; moreover since P is always some function of x, we have by theorem Art. **123.**

$$U = \text{limit of } \Sigma P\Delta x = \int_a^l P\,dx \ldots (2),$$

which is the same as the preceding expression.

The work done from AB to EF will obviously be represented by aP_1, hence for the total work we have

$$U_1 = aP_1 + \int_a^l P\,dx \ldots (3).$$

Cor. 1. Let W be the weight of the mass moved,

then $U = \frac{v^2 W}{2g}$. (See "Exercises on Mechanics," p. 89)

Differentiating this equation, we find

$$dU = \frac{W}{g}.\,vdv \ldots (4),$$

therefore by equating with (1)

$$P\,dx = \frac{W}{g}vdv \quad .. (5),$$

which is the same relation as (7) Art. **129.**

Substituting in this equation the value of *ds* or *dx* obtained from (4) Art. **129.**, we find

$$P=\frac{W}{g}\cdot\frac{dv}{dt}\ldots(6),$$

which is the same relation as (3) Art. **129.**

Examples.

1. To find a general expression for the work done upon 1 inch of the piston of a steam engine, when the steam acts expansively; assuming that the law of Mariotte applies to the expansion of steam.

Here, using the notation and figure of the preceding article, we have

By Mariotte's law, $P=\frac{aP_1}{x}$,

$$\therefore \int P\,dx=\int\frac{aP_1\,dx}{x}=aP_1\log x+C,$$

hence by formula (3), we have

$$U_1=aP_1+\int_a^l P\,dx=aP_1\left\{1+\log\frac{l}{a}\right\}.$$

2. Let W lbs. be the weight of a railway train, v' its velocity in feet per second at the moment the steam is turned off, q the coefficient of the friction of the rail, p lbs. the resistance of the atmosphere to the whole train when the speed is 1 foot per second. Required the space which the train will move over before it stops, &c.

In this case we shall employ formulæ (5) and (6). Let v

be the velocity of the train when it has passed over x feet, then the retarding force or pressure at this point is

$$P = -qW - pv^2,$$

therefore by formula (5)

$$-(qW + pv^2)dx = \frac{W}{g} \cdot vdv,$$

$$\therefore \ dx = -\frac{W}{g} \cdot \frac{vdv}{qW + pv^2},$$

hence by integration, we find

$$x = -\frac{W}{2pg} \log (qW + pv^2) + C,$$

and since $x = 0$, when $v = v_{\prime}$,

$$\therefore \ 0 = -\frac{W}{2pg} \log (qW + pv_{\prime}^2) + C,$$

$$\therefore \ x = \frac{W}{2pg} \log \frac{qW + pv_{\prime}^2}{qW + pv^2},$$

which is an expression for the space moved over when the velocity is v. When $v = 0$, that is, when the train comes to a state of rest, we find the whole space moved over to be $\frac{W}{2pg} \log \left\{ 1 + \frac{pv_{\prime}^2}{qW} \right\}$.

To find the time t, we have by eq. (6)

$$-qW - pv^2 = \frac{W}{g} \cdot \frac{dv}{dt},$$

$$\therefore \ dt = -\frac{W}{g} \cdot \frac{dv}{qW + pv^2} = -\frac{W}{pg} \cdot \frac{dv}{\frac{qW}{p} + v^2},$$

$$\therefore \ t = -\frac{W^{\frac{1}{2}}}{p^{\frac{1}{2}} g q^{\frac{1}{2}}} \tan^{-1} \sqrt{\frac{p}{qW}} v + C,$$

by forms (c) page 139.

When $t=0$, $v=v_{\prime}$, and

$$\therefore \; 0=-\frac{W^{\frac{1}{2}}}{p^{\frac{1}{2}}gq^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{p}{qW}}\,v_{\prime}+C,$$

$$\therefore \; t=\frac{W^{\frac{1}{2}}}{p^{\frac{1}{2}}gq^{\frac{1}{2}}}\left\{\tan^{-1}\sqrt{\frac{p}{qW}}\,v_{\prime}-\tan^{-1}\sqrt{\frac{p}{qW}}\,v\right\},$$

which expresses the time corresponding to the velocity v. When $v=0$, that is, when the train comes to a state of rest, we find the whole time to be $\frac{W^{\frac{1}{2}}}{p^{\frac{1}{2}}gq^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{p}{qW}}\,v_{\prime}$.

Centre of Gyration.

131. It is shown in the Author's "Exercises on Mechanics," &c., p. 92., that the work accumulated in a rotating body is not altered when the whole mass is *collected* in its centre of gyration; and moreover that the work in any rotating particle is equal to its weight in lbs. multiplied by the square of its distance from the axis of rotation divided by $2g$, the velocity of a particle at 1 foot from the axis having a velocity of 1 foot per second. Let m be put for the volume of the body, w the weight of each unit, k the distance of the centre of gyration from the axis, Δm the volume of any small particle at the distance r from the axis; then the weight of the whole body$=wm$, the weight of the particle $\Delta m=w\Delta m$, the accumulated work in this particle $=\frac{w\Delta mr^2}{2g}$, and the accumulated work in the whole body $=\frac{wmk^2}{2g}$; therefore the work accumulated in all the particles composing the body may be represented by $\frac{w}{2g}\Sigma r^2\Delta m$,

$$\therefore \frac{wmk^2}{2g} = \text{limit of } \frac{w}{2g} \Sigma r^2 \Delta m ;$$

now since r is always some function of m, therefore by theorem, Art. 123., and striking out the common factors, we find

$$mk^2 = \text{limit of } \Sigma r^2 \Delta m = \int r^2 dm \ldots (1),$$

$$\therefore k^2 = \frac{\int r^2 dm}{m} \ldots (2)$$

where the limits of the integration are left to be assigned by the nature of the problem which may be proposed.

The moment of inertia of a body is equal to its volume multiplied by the square of its radius of gyration. Thus if I be put for the moment of inertia, then $I = mk^2$; and eq. (1) shows that the inertia of a body is equal to the sum of all the moments of inertia of the particles composing it.

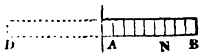
Hence if v be put for the velocity of a point 1 foot from the axis, the work accumulated in the rotating body $= U = \frac{\text{weight} \times \text{velocity}^2}{2g} = \frac{wm \times (vk)^2}{2g} = \frac{wv^2}{2g} \cdot I.$

EXAMPLES.

1. To find the radius of gyration of a uniform rod AB revolving about its extremity A.

Let us first solve the question without the aid of the calculus.

Put a = the cross section of the rod, l = AB; and let AB be divided into n small equal portions, then $\frac{al}{n}$ will be the



mass of each, and their distances from A will be $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}, \ldots, \frac{nl}{n}$; therefore the sum of their moments of inertia will be represented by

$$\left(\frac{l}{n}\right)^2 \cdot \frac{al}{n} + \left(\frac{2l}{n}\right)^2 \cdot \frac{al}{n} + \left(\frac{3l}{n}\right)^2 \cdot \frac{al}{n} + \ldots + \left(\frac{nl}{n}\right)^2 \cdot \frac{al}{n}$$

$$= al^3 \cdot \frac{1}{n^3}\left\{1^2+2^2+\ldots+n^2\right\} = \frac{al^3}{3}, \text{ when } n=\infty,$$

by (12) page 27.

Now this is equal to the moment of inertia of the whole rod, supposing its matter collected in its centre of inertia, hence we have

$$\text{I} = mk^2 = alk^2 \ldots (1)$$

$$\therefore\ alk^2 = \frac{al^3}{3}, \therefore\ k = \frac{l}{\sqrt{3}} = \frac{\text{AB}}{\sqrt{3}},$$

which is the radius of gyration.

Let the rod extend to the left of A, making AD=AB; then if $k_{\prime}$ be put for the radius of gyration of the rod DB, rotating upon its middle point A as an axis, we have the moment of inertia of the two parts AB and AD$=2alk^2$, by eq. (1);

and moment of inertia of the whole rod DB$=2alk_{\prime}^2$;

$$\therefore\ 2alk_{\prime}^2 = 2alk^2,$$

$$\therefore\ k_{\prime} = k = \frac{l}{\sqrt{3}} = \frac{\text{DB}}{2\sqrt{3}}.$$

Again, let AN$=x$, and the length of an indefinitely small portion at N$=\Delta x$; then the volume of this small portion $=a\Delta x$, and the volume of the rod AN$=ax$, hence by formula (1),

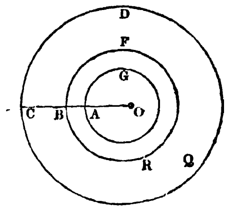
$$ax \times k^2 = \text{limit of } \Sigma(x^2 \times a\Delta x)$$

$$= a\int_0^x x^2 dx = \frac{ax^3}{3},$$

$$\therefore\ k^2 = \frac{x^2}{3}, \text{ and } k = \frac{\quad}{\sqrt{3}}$$

$$\text{when } x=l, \text{ then } k = \frac{l}{\sqrt{3}} = \frac{\text{AB}}{\sqrt{3}}.$$

2. To determine the radius of gyration, &c. of a circle CDQ revolving about its centre O as an axis.



In order to illustrate the process of reasoning pursued in this subject, we shall solve the problem independently of the general formula (2).

Here let us suppose that the circle is made up of a series of concentric rings as ABFGR; put $OA=x$, $AB=\Delta x$, and $OC=a$; then the volume of the ring $ABFG=\pi(x+\Delta x)^2-\pi x^2=2\pi x\Delta x$, neglecting $(\Delta x)^2$ according to Art. 125.; hence the moment of inertia of this ring $=x^2\times$ (vol. ring) $2\pi x^3\Delta x$; and as this may be regarded as the general type of the moments of every ring in the series making up the circle, the sum of the moments of all the concentric rings may be represented by $2\pi\Sigma x^3\Delta x$,

$$\therefore\ I=\text{limit of } 2\pi\Sigma x^3\Delta x$$

$$=2\pi\int x^3dx=\frac{\pi x^4}{2}+C,$$

taking this between the limits of $x=x$, $x=a$, that is, making successively $x=a$, $x=x$, and subtracting, we find

$$I=\frac{\pi a^4}{2}-\frac{\pi x^4}{2}=\frac{\pi}{2}(a^4\quad x^4)\ldots(1),$$

which is an expression for the moment of inertia of the circular ring BCDFQ. Let k be the radius of gyration of this ring, then

$$I=\text{area ring}\times k^2=\pi(a^2-x^2)k^2;$$

$$\therefore\ \pi(a^2-x^2)k^2=\frac{\pi}{2}(a^4-x^4),$$

$$\therefore\ k=\{\tfrac{1}{2}(a^2+x^2)\}^{\frac{1}{2}}\ldots(2),$$

which is an expression for the radius of gyration of the circular ring BCDGF.

Making $x=0$, we obtain

$$k=\frac{a}{\sqrt{2}}\ldots(3)$$

which is an expression for the radius of gyration of the whole circle CDQ.

Making $x=a$ in (2), we obtain

$$k=a\ldots(4).$$

It is obvious that these formulæ will not at all be altered by supposing the circle to have any given thickness; hence formula (2) is an expression for the radius of gyration of the rim of a fly wheel, (3) that of a circular wheel of uniform thickness, (4) that of a circular hoop revolving on an axis passing through its centre and perpendicular to the plane of the hoop.

If h be put for the thickness of a hollow cylinder, whose external radius is a, and interior radius b; then by eq. (1) we have the moment of inertia

$$I=\frac{\pi h}{2}(a^4-b^4)\ldots(5).$$

If $b=0$, then we have for the moment of inertia of a cylinder,

$$I=\frac{\pi a^4 h}{2}\ldots(6)$$

3. To determine the radius of gyration, &c. of a sphere revolving about its diameter.

Let Δx be the thickness of a thin lamina, formed by planes perpendicular to the diameter, x being put for its distance from the centre of the sphere; also let $a=$ the radius of the sphere, and $y=$ the radius of the thin lamina. Now, since this lamina may be considered as a thin cylinder, its moment of inertia $=\frac{1}{2}\pi y^4\Delta x$, by eq. (6), *Ex.* 2. But the *moment of* inertia of any zone of the sphere is equal to the sum of the moments of all such thin laminæ making up that zone,

$$\therefore \ \text{I}=\text{limit of } \Sigma(\tfrac{1}{2}\pi y^4 \Delta x)=\tfrac{1}{2}\pi\int_0^x y^4 dx.$$

Now by the equation of the circle $y^2=a^2-x^2$,

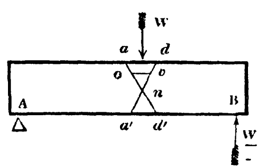
$$\therefore \ \text{I}=\tfrac{1}{2}\pi\int_0^x (a^2-x^2)^2 dx=\tfrac{1}{2}\pi(a^4x-\tfrac{2}{3}a^2x^3+\tfrac{1}{5}x^5),$$

which is an expression for the moment of inertia of a zone, whose breadth, measured from the centre of the sphere, is x.

When $x=a$, we have the moment of the semi-sphere $=\frac{4}{15}\pi a^5$; and therefore the moment of the whole sphere $=\frac{8}{15}\pi a^5$.

Strength of Material.

132. When a beam A B undergoes a transverse strain, by the pressure of a weight W placed upon it, the material on the upper side $a\,d$ is compressed, while that on the under side $a'\,d'$ is extended. That imaginary point n within the *section of rupture* $a\,d$, which neither undergoes compression nor extension, is called the *neutral axis* of rupture.



When the lower fibres of a beam are upon the point of yielding to the force of extension, at the same time that the upper fibres are upon that of yielding to the force of compression, then (supposing the beam to remain nearly horizontal) the sum of the forces extending the fibres on the under side of the neutral axis are equal to the sum of the forces compressing the fibres on the upper side.

Let d = the depth of the beam;

b = its breadth;

a, $a_{\prime}$ = the respective distances of the neutral axis n from the top and bottom of the beam;

$f, f_{,}$ = the compressive and tensile forces respectively exerted by a sq. in. of the material at the distances a and $a_{,}$ from the neutral axis;

l = AB, the distance between the supports;

$x = nv$, a variable distance from n.

To find the Position of the Neutral Axis in Rectangular Beams.

Assuming that the force with which a fibre resists compression or extension, as the case may be, is in proportion to the extent of compression or extension of that fibre, hence we have

$$\text{compressive force per sq. in. at } v = \frac{fx}{a};$$

$$\text{area of the element of surface at } v = b\Delta x;$$

$$\therefore \text{ compressive force of the element of surface} = \frac{fx}{a} \times b\Delta x$$

$$= \frac{fb}{a} x\Delta x,$$

$$\therefore \text{ sum of all the compressive forces} = \frac{fb}{a} \Sigma_0^a x\Delta x$$

$$= \frac{fb}{a}\int_0^a xdx = \frac{fba}{2}.$$

Similarly we have

$$\text{sum of all the tensile forces} = \frac{f_{,}b}{a_{,}}\int_0^{a_{,}} xdx = \frac{f_{,}ba_{,}}{2}.$$

But these are the only horizontal forces acting upon the fibres;

$$\therefore \frac{fba}{2} = \frac{f_{,}ba_{,}}{2},$$

$$\therefore\ fa = f_{,}a_{,} \ldots (1),$$

which expresses the relation of the distances of the neutra axis from the upper and under sides of the beam.

Conditions of Rupture.

Now when rupture is about to take place, the beam turns upon the neutral axis n, as a fulcrum; hence the tendency of any fibre to resist the moment of the force $\frac{W}{2}$, tending tc rupture the beam, is the moment of the force of that fibre referred to n as the centre of motion.

$\therefore$ Moment of the element undergoing compression = force of the element undergoing compression $\times$ its distance from the neutral axis $= \frac{fb}{a} x\Delta x \times x = \frac{fb}{a} x^2 \Delta x$;

$\therefore$ sum of all the moments of the forces of compression

$$= \frac{fb}{a} \Sigma_0^a x^2 \Delta x = \frac{fb}{a} \int_0^a x^2 dx = \frac{fba^2}{3} \ldots (2).$$

Similarly we have

sum of all the moments of the forces of extension

$$= \frac{f_{,}b}{a_{,}} \int_0^{a_{,}} x^2 dx = \frac{f_{,}ba_{,}^2}{3} \ldots (3).$$

But the moment of the pressure tending to rupture the beam, is expressed by $\frac{W}{2} \times \frac{l}{2}$. Now this moment must be equal to the sum of the two moments expressed by eq. (2) and (3),

$$\therefore\ \frac{Wl}{4} = \frac{fba^2}{3} + \frac{f_{,}ba_{,}^2}{3} = \frac{b}{3}\{fa^2 + f_{,}a_{,}^2\},$$

but by eq. (1), $fa = f_{,}a_{,}$

$$\therefore \frac{wl}{4}=\frac{b}{3}\{fa^2+f_,a_,^2\}=\frac{fab}{3}(a+a_,)=\frac{fabd}{3},$$

$$\therefore w=\frac{4fabd}{3l} \text{ or } \frac{4f_,a_,bd}{3l}\ldots(4).$$

Now $f_,bd$ is the direct tensile strength of the beam, therefore the transverse strength of a beam, loaded in the middle and supported at the extremities, varies as the direct tensile strength, multiplied by the depth of the neutral axis divided by the distance between the points of support.

Cor. 1. When the force f with which the fibres resist compression is equal to the force $f_,$ with which they resist extension, we have, by eq. (1),

$$fa=f_,a_,, \quad \therefore a=a_,=\tfrac{1}{2}d,$$

therefore eq. (4) becomes

$$w=\frac{2fbd^2}{3l}\ldots(5).$$

Cor. 2. If the beam is absolutely incompressible, or $f=\infty$, then $a=0$, and $a_,=d$. In this case eq. (4) becomes

$$w=\frac{4f_,bd^2}{3l}\ldots(6).$$

On this important subject, the student may consult Moseley's "Mechanical Principles of Engineering," and Hodgkinson's edition of "Tredgold on the Strength of Materials."

THE END.

N

LONDON
PRINTED BY SPOTTISWOODE AND CO.
NEW-STREET SQUARE