Cosmology

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- verbal averaging: can we do better?

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\+ observations of expansion $\Rightarrow$ hot big bang

$\Rightarrow$ black body radiation, nucleosynthesis

- but: $\exists$ galaxies $\Rightarrow$ inhomogeneous, anisotropic spatial slices
- standard model: density perturbations (anisotropy)
- scalar (GR) averaging: statistically homogeneous spatial slices

verbal averaging

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     - g = a (pseudo-Riemannian) metric

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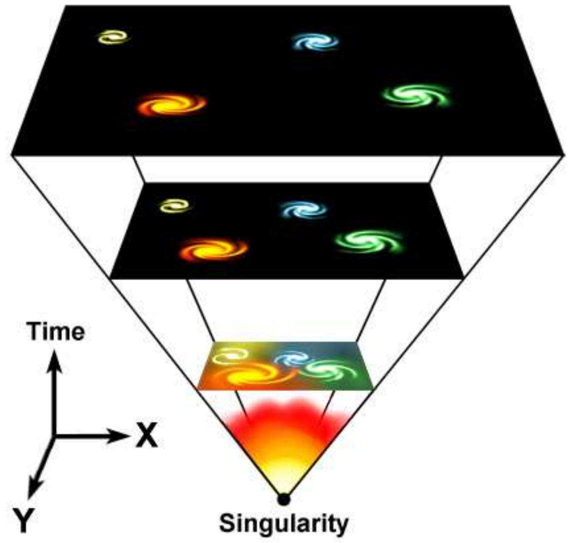
3. assume that $(M, \mathbf{g})$ remains unchanged if we add density perturbations to an early time slice

verbal averaging

- w:Comoving coordinates

verbal averaging



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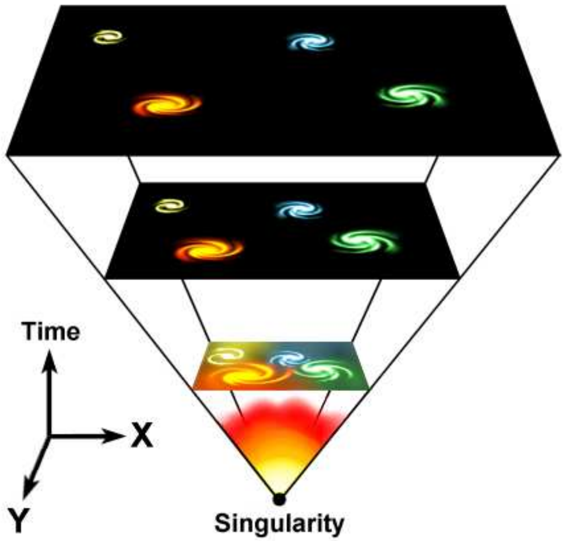

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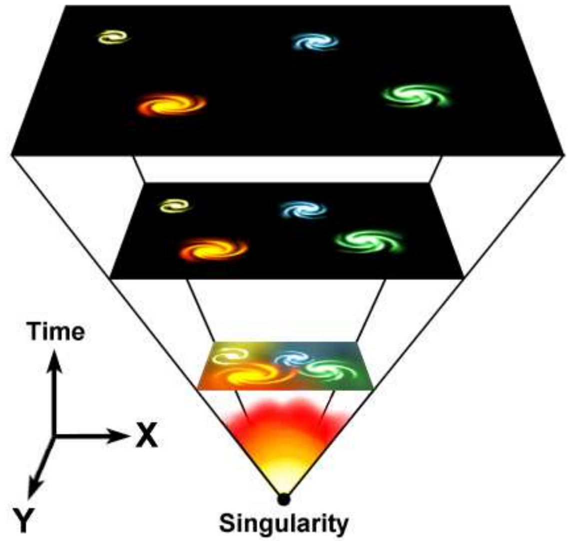
-

$$\Delta x(t) = a(t)\Delta r$$

verbal averaging



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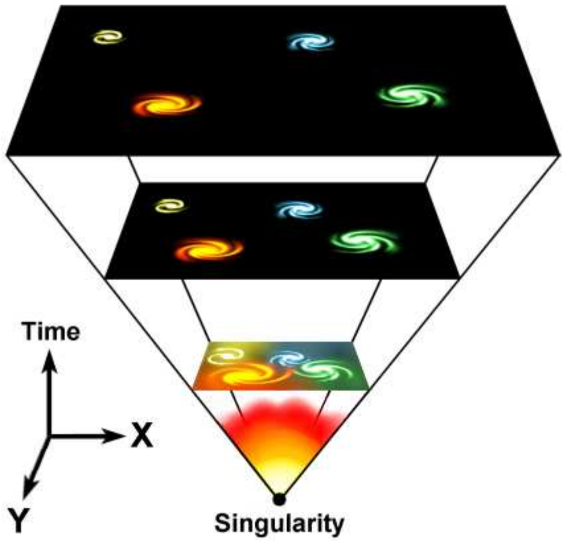
- spherical coordinates for spatial slice

verbal averaging



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- $$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} \mathrm{d}s = a(t)\Delta r = a(t)|r_2 - r_1|$$

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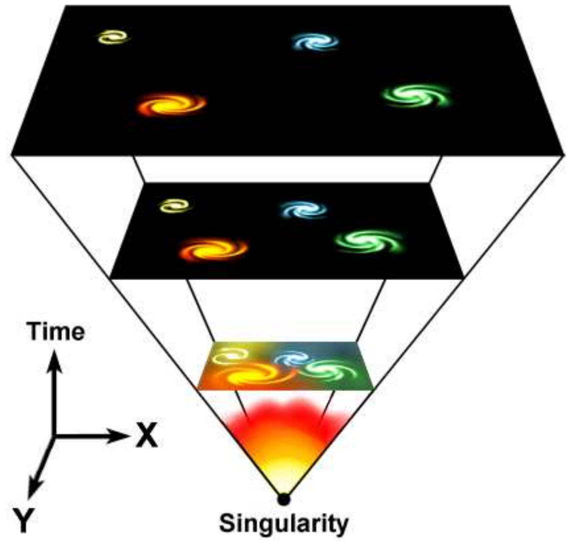

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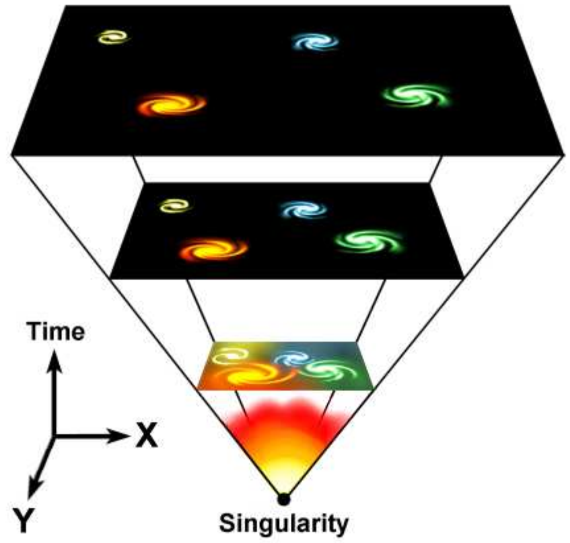

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- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

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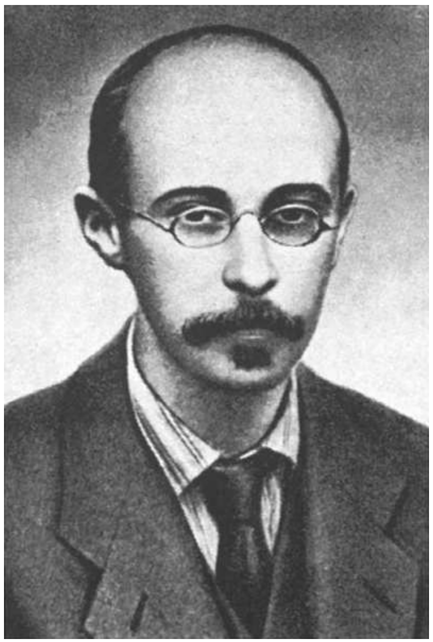
- w: w:
  w:Howard Percy Robertson
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FLRW metric

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$$u := \int \frac{\mathrm{d}t}{a(t)}$$

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- but $\int \mathrm{d}u \neq$ proper time; *more:* arXiv:astro-ph/0707.2106

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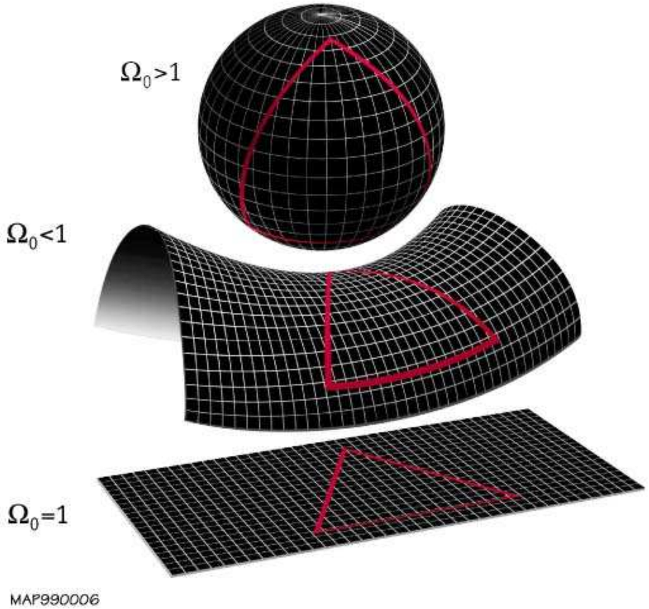
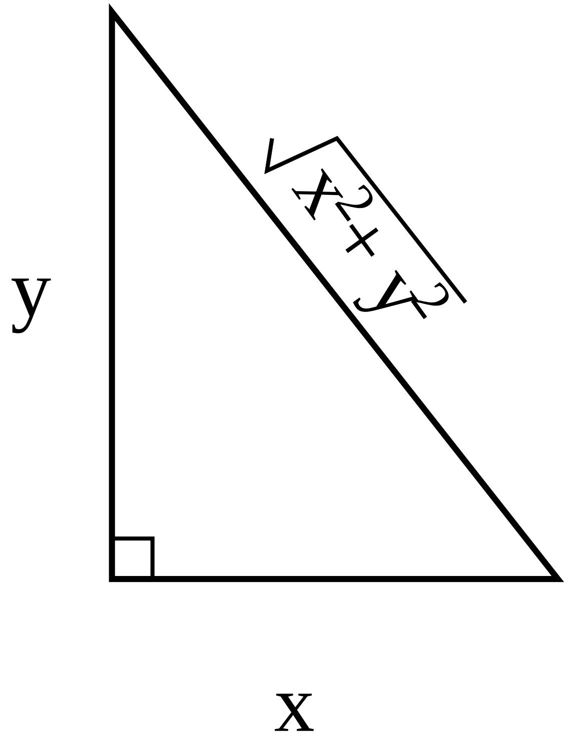


where $$r_\perp := \begin{cases} R_\mathrm{C}\sinh\frac{r}{R_\mathrm{C}} & k<0 \\ r & k=0 \\ R_\mathrm{C}\sin\frac{r}{R_\mathrm{C}} & k>0 \end{cases}$$

for a comoving radius of curvature R_C and curvature of sign k

curvature

- on a spatial slice (fixed value of t):



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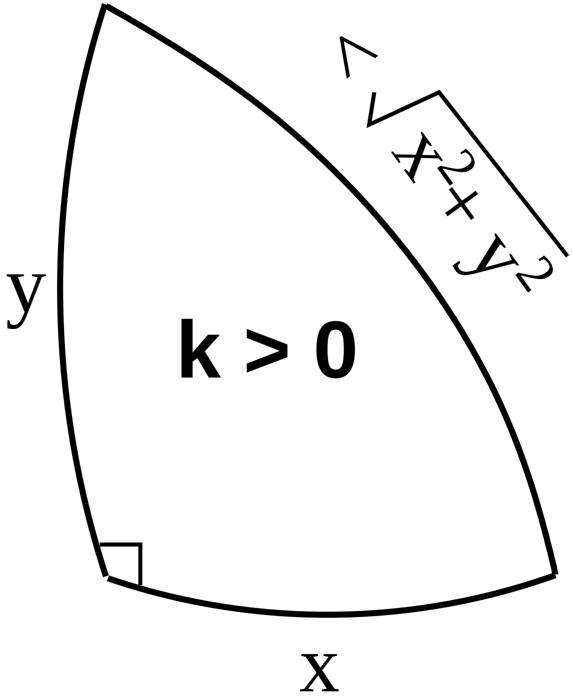

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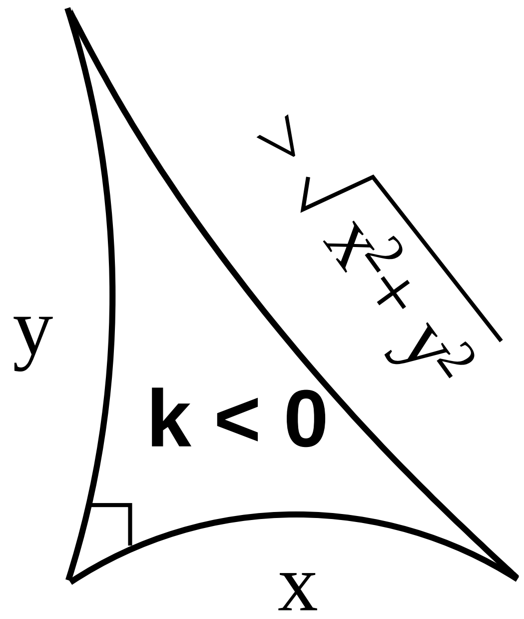

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2D curvature intuition: $k > 0$

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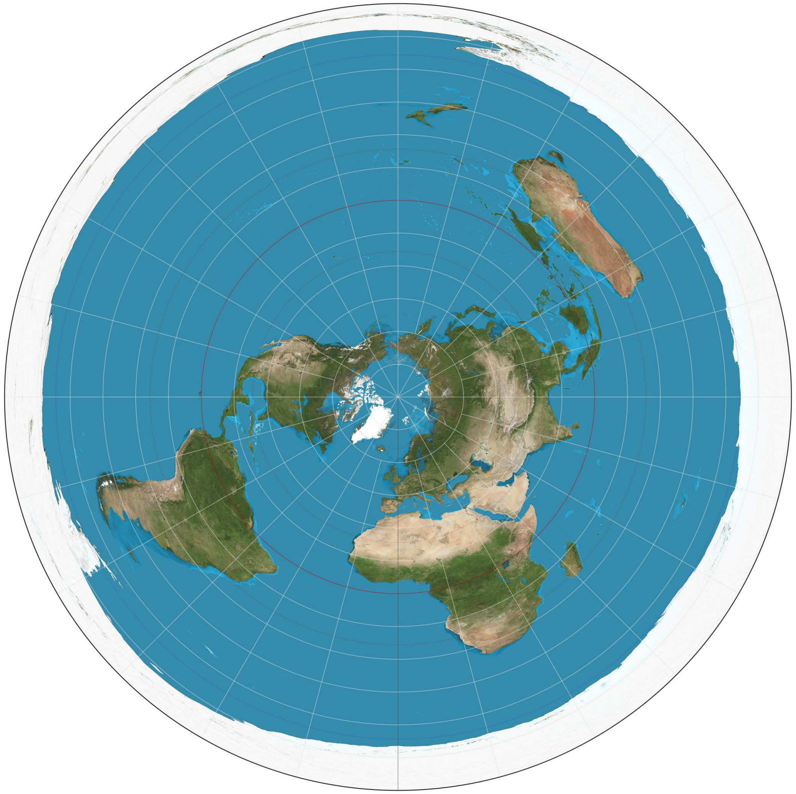

w: ______________________ (al-Biruni, c. 1000 CE)

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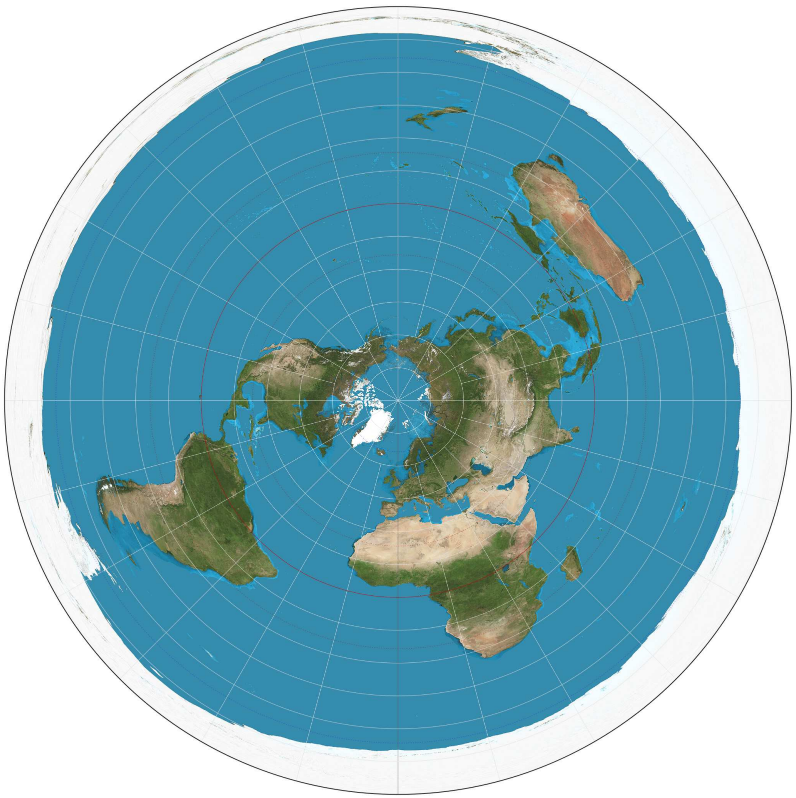

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- intuition switch: S^2 easier vs S^3 more physical

2D topology intuition ($k = 0$)

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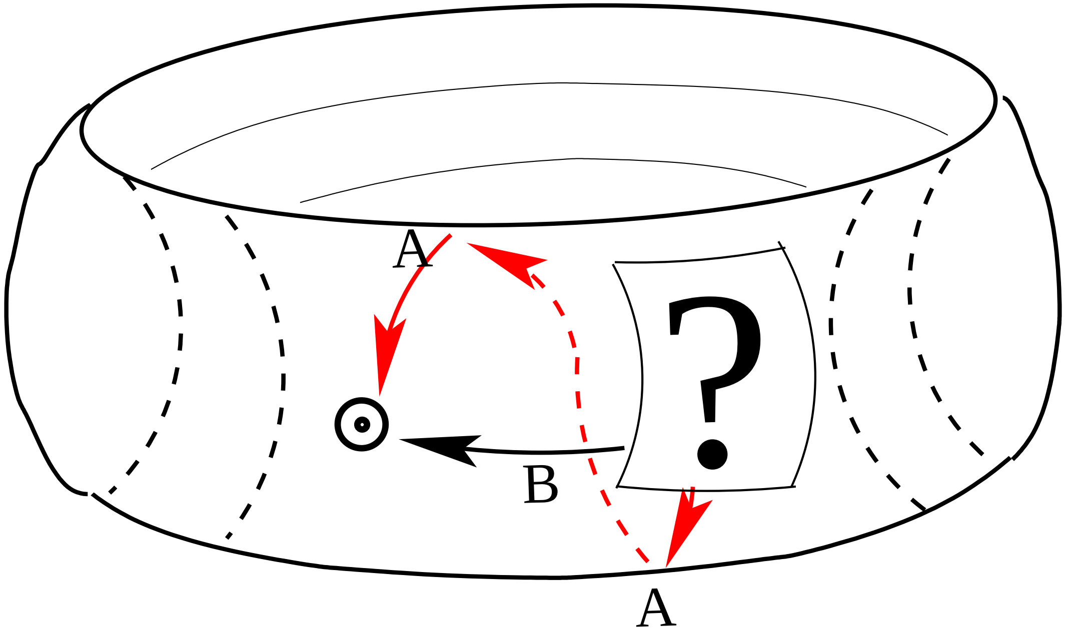

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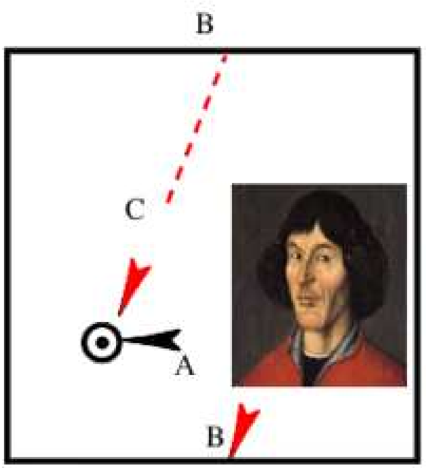

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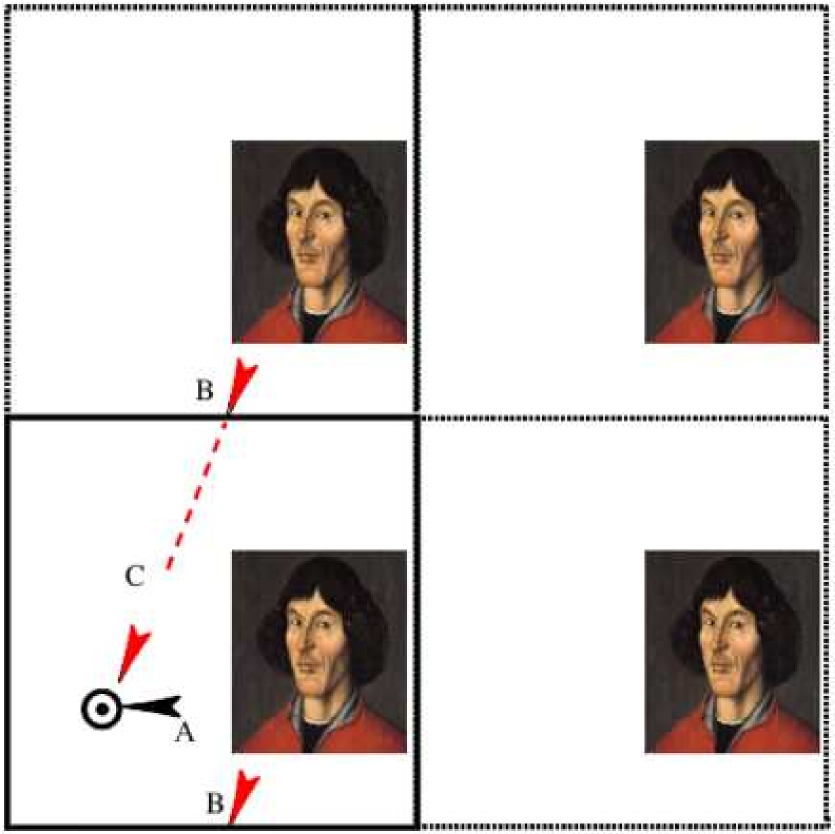

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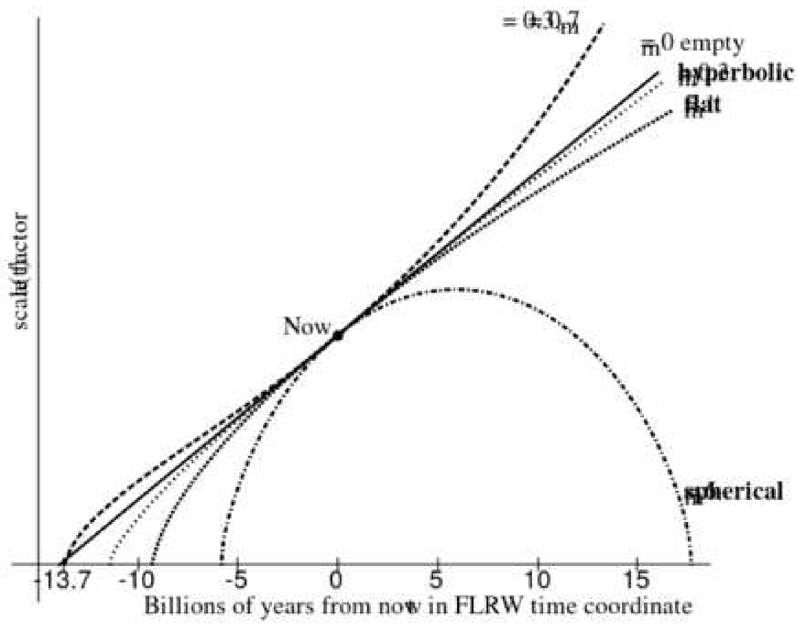

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$\Leftrightarrow$

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  (Defn: $a_0 := 1$)

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  (Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_{\mathrm{r}} \propto a^{-4} = (1+z)^4$

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- $\Rightarrow$ black body + primordial nucleosynthesis

CMB discovery: McKellar 1941

- $T \approx 2.3$ K — Andrew McKellar (1941; ADS:1941PDAO....7..251M) from observations by Walter S. Adams (1941; ADS:1941ApJ....93...11A)
- Penzias & Wilson rediscovery (1965 + Nobel prize)

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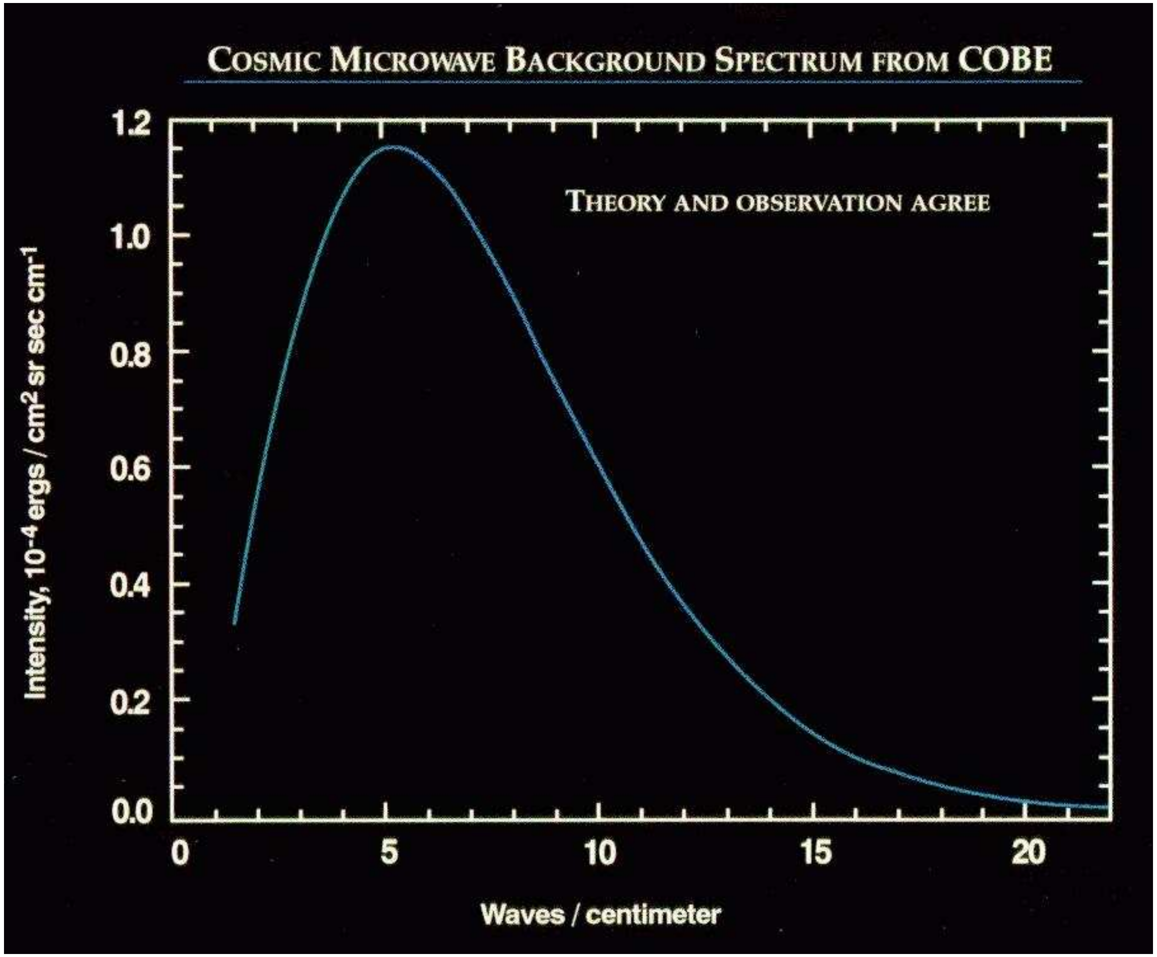

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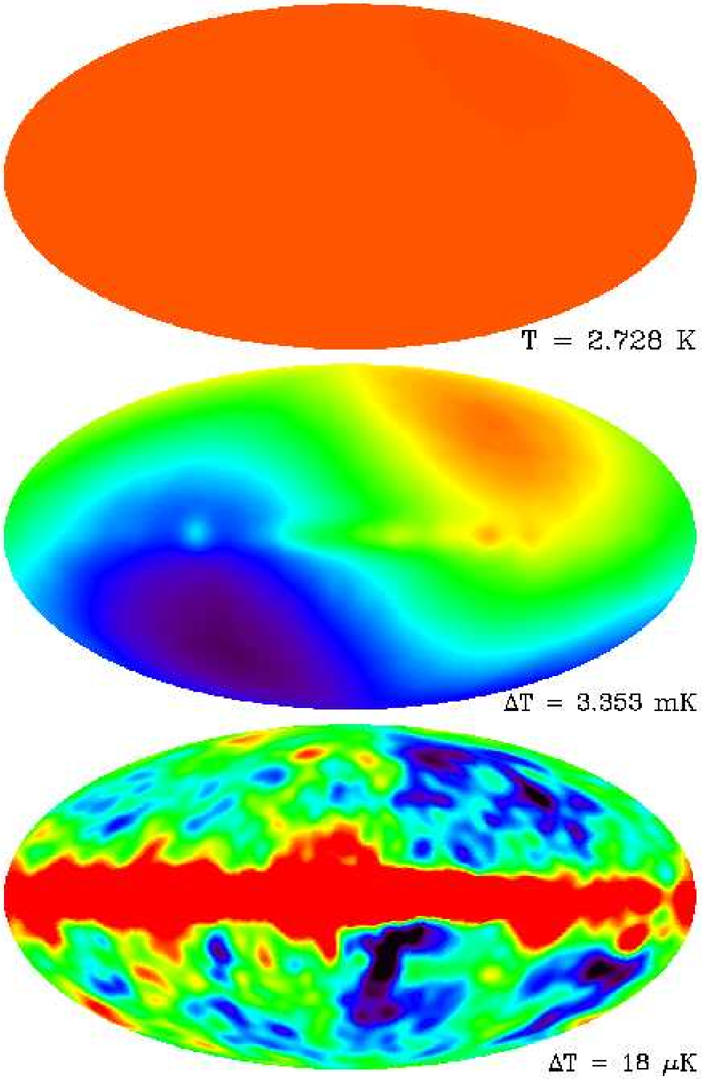
- COBE /DMR (Differential Microwave Radiometer)



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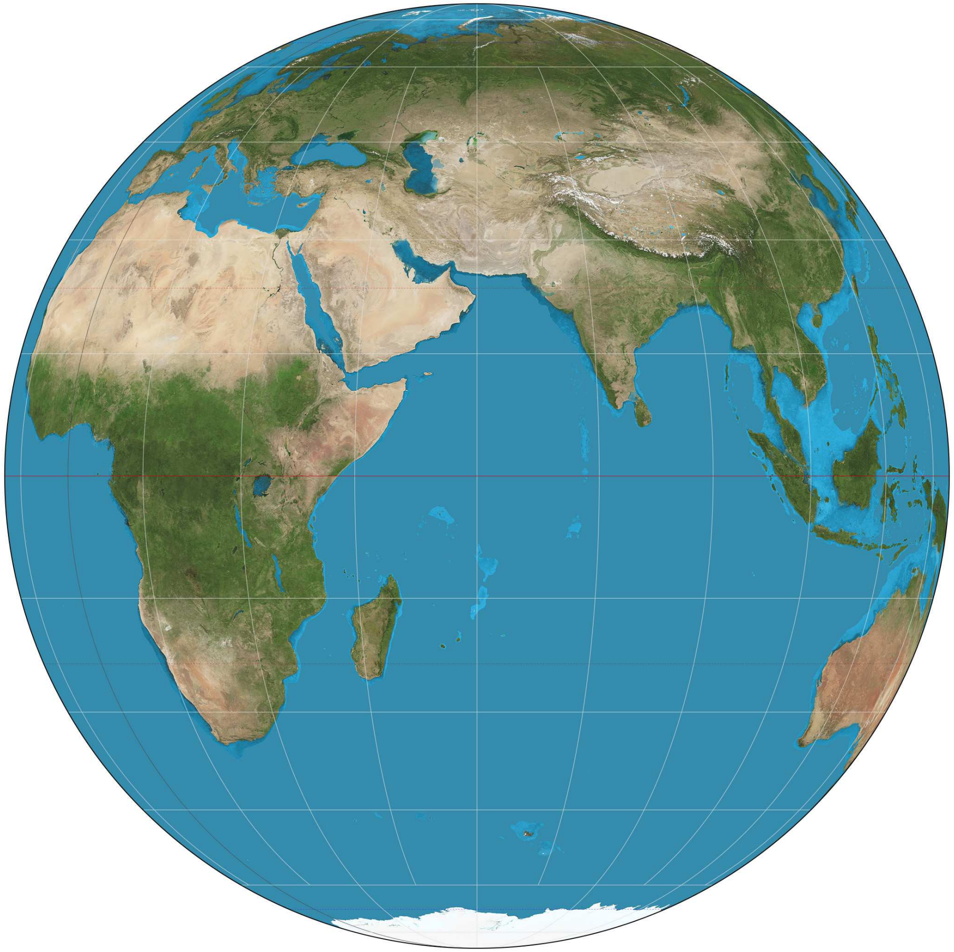

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- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: https://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- convenient conversion: 1 km/s ≈ 1.04 kpc/Gyr ≈ 1 kpc/Gyr

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1} $\Rightarrow t_0(\mathrm{EdS}) \approx 13.0$ or 6.5 Gyr, resp.

FLRW: $\rho_{\rm crit}$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

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FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \Omega_{\mathrm{m}} H^2 - \frac{c^2\, k}{a^2}$

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FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 + \Omega_k H^2$

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FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$\boxed{1 = \Omega_{\rm m} + \Omega_k}$$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m0}} = 1 \Leftrightarrow k = 0$ flat

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$

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  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

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Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
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- $\Omega_{\rm tot} :=$

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- $\Omega_{\rm tot} := \Omega_{\rm m} +$

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Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m0}} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m0}} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m0}} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{m}} + \Omega_{\mathrm{r}} +$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

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  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic
- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

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- $\Omega_{\rm tot} := \Omega_{\rm b} + \Omega_{\rm nbDM} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - k{r_\perp}^2 = 0$ coord singularity at equator

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  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - k{r_\perp}^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1$

FLRW curvature constant

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- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$

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- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\frac{\Omega_{k\,0} H_0^2}{c^2}$

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $R_{\mathrm{C}}^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{1-\Omega_{\mathrm{tot}\,0}}$

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical*

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat*

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - k{r_\perp}^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot0}} - 1}}}$
- $\Omega_{\mathrm{tot0}} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot0}} = 1$ *flat* R_{C} undefined

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic*

FLRW curvature constant

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  - orthographic coords: k
- orthographic: $1 - k{r_\perp}^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_\mathrm{C} = \frac{c}{H_0}\frac{1}{\sqrt{\Omega_\mathrm{tot0} - 1}}}$
- $\Omega_\mathrm{tot0} > 1$ *spherical* R_C real
- $\Omega_\mathrm{tot0} = 1$ *flat* R_C undefined
- $\Omega_\mathrm{tot0} < 1$ *hyperbolic* R_C imaginary (or use $|R_\mathrm{C}|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Einstein's free parameter: Λ

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  https://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  https://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- *hint:* mixed index form of $\mathbf{g}$ is easy

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Friedmann Eqn ($\Lambda \neq 0$):
$$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

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acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3} + \frac{c^2\,\Lambda}{3}$$

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Defn: "dust solution": $p(t) = 0$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$):
$$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\, \pi\, G\, \rho}{3} + \frac{c^2\, \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\, \Lambda}{3\, H^2}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\frac{\rho}{\rho_{\rm crit}} + \Omega_\Lambda\,H^2$$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2\,\Omega_{\rm m}}{2} + \Omega_\Lambda\,H^2$$

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$):
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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$):
$$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\rm m}}{2} + \Omega_\Lambda$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\, \Lambda}{3\, H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} \frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda$$

Defn: $q := -\frac{\ddot{a} a}{\dot{a}^2}$

Einstein's free parameter: Λ

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acceleration Eqn ($\Lambda \neq 0$): $$q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
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Low z limit Hubble law

- use r = comoving radial distance

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$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $r \approx \dfrac{z\,c}{H_0}$

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EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

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$$= \boxed{\frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right)}$$

$$\approx 6h^{-1}\mathrm{Gpc} \left(1 - \frac{1}{\sqrt{1+z}}\right)$$

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- high-level frontends (e.g. python) should be easy to write

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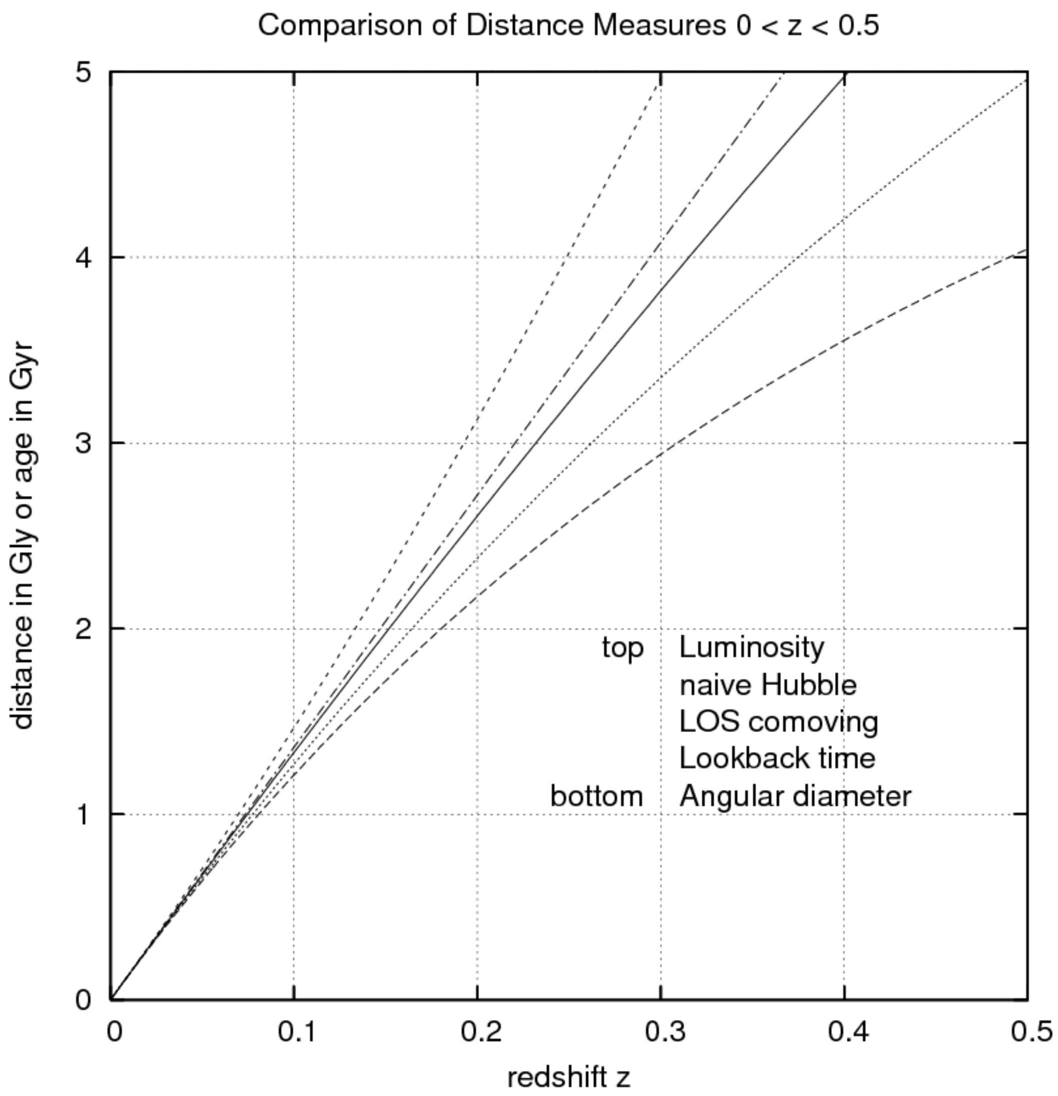
- w:Distance measures (cosmology)

FLRW distances e.g. ΛCDM



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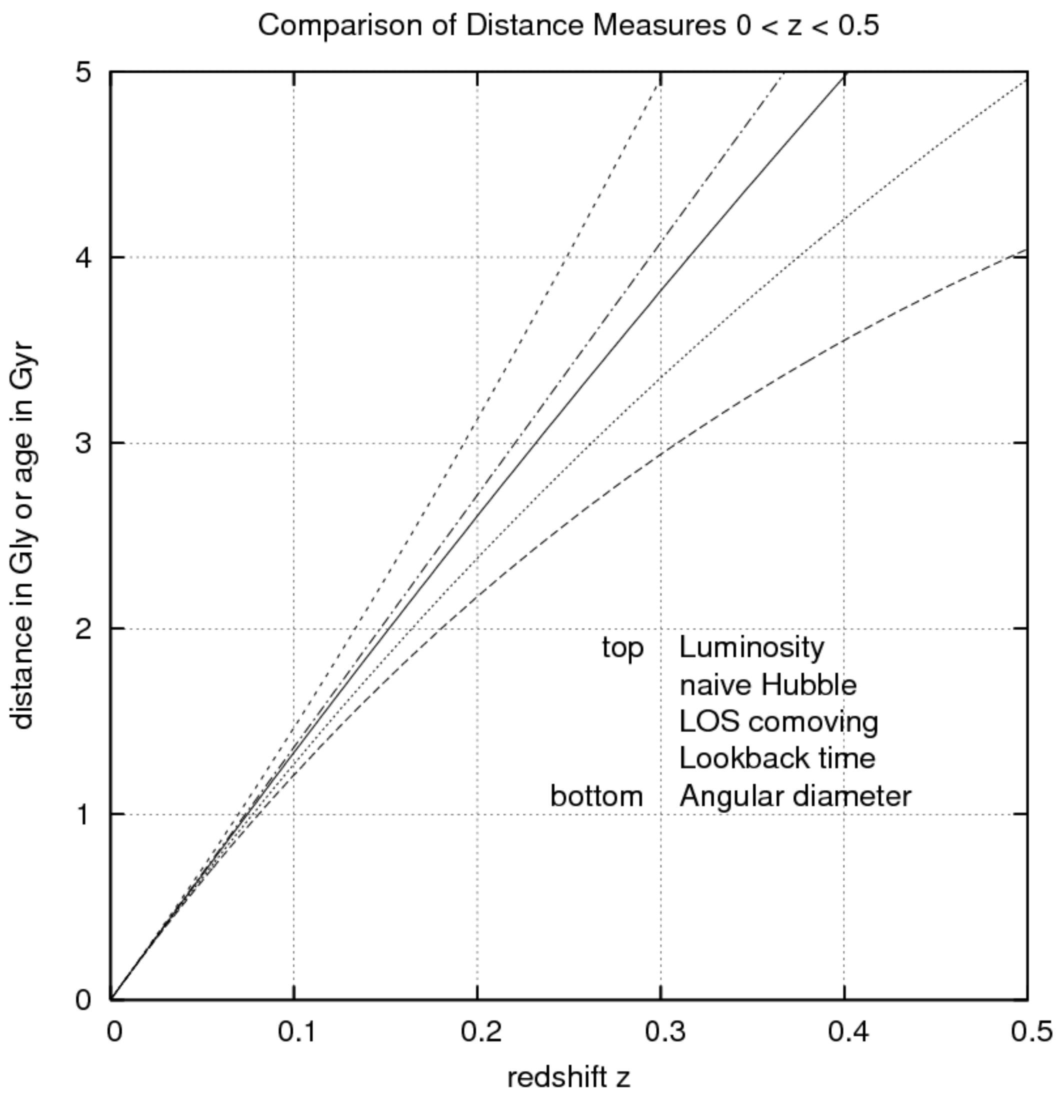

Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on h)

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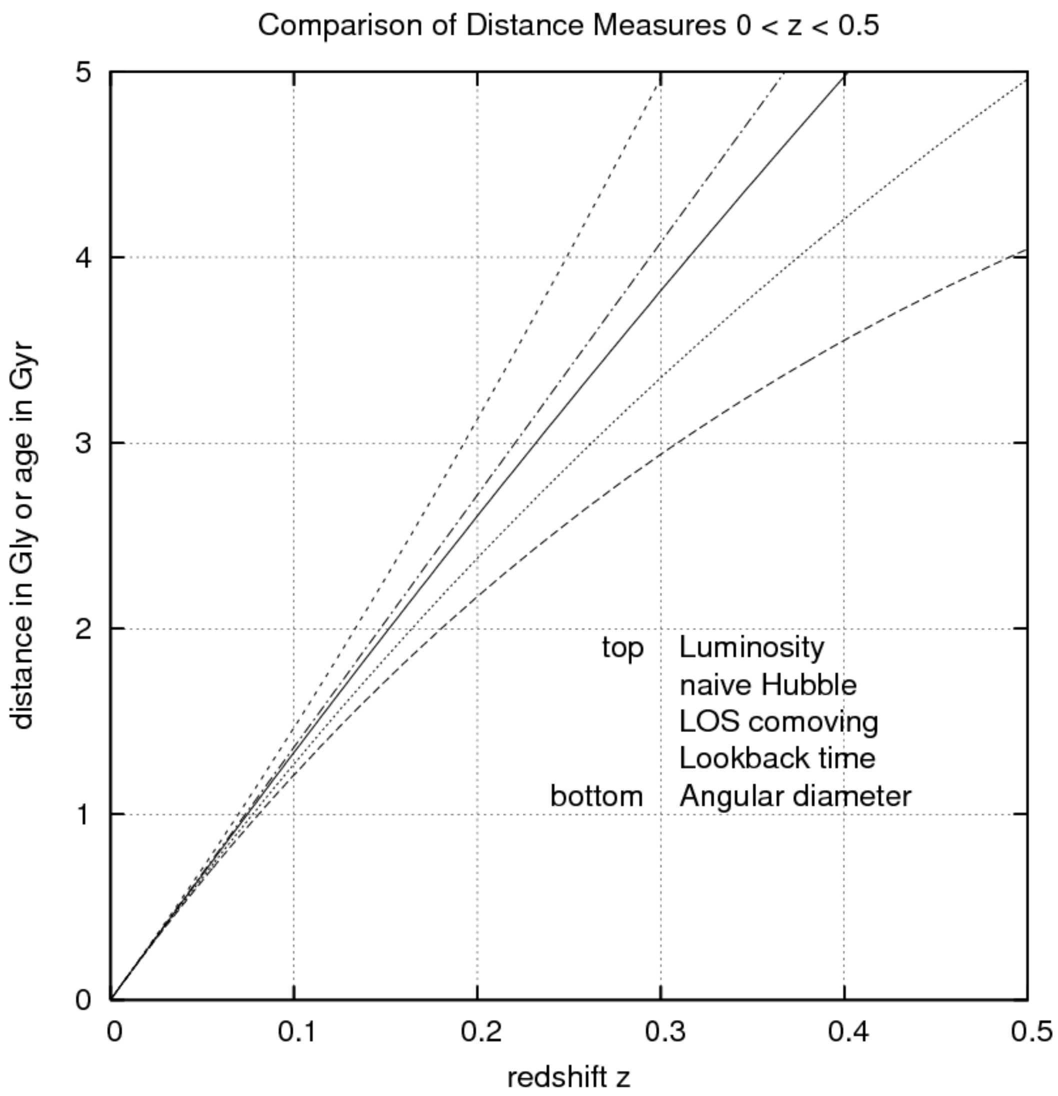

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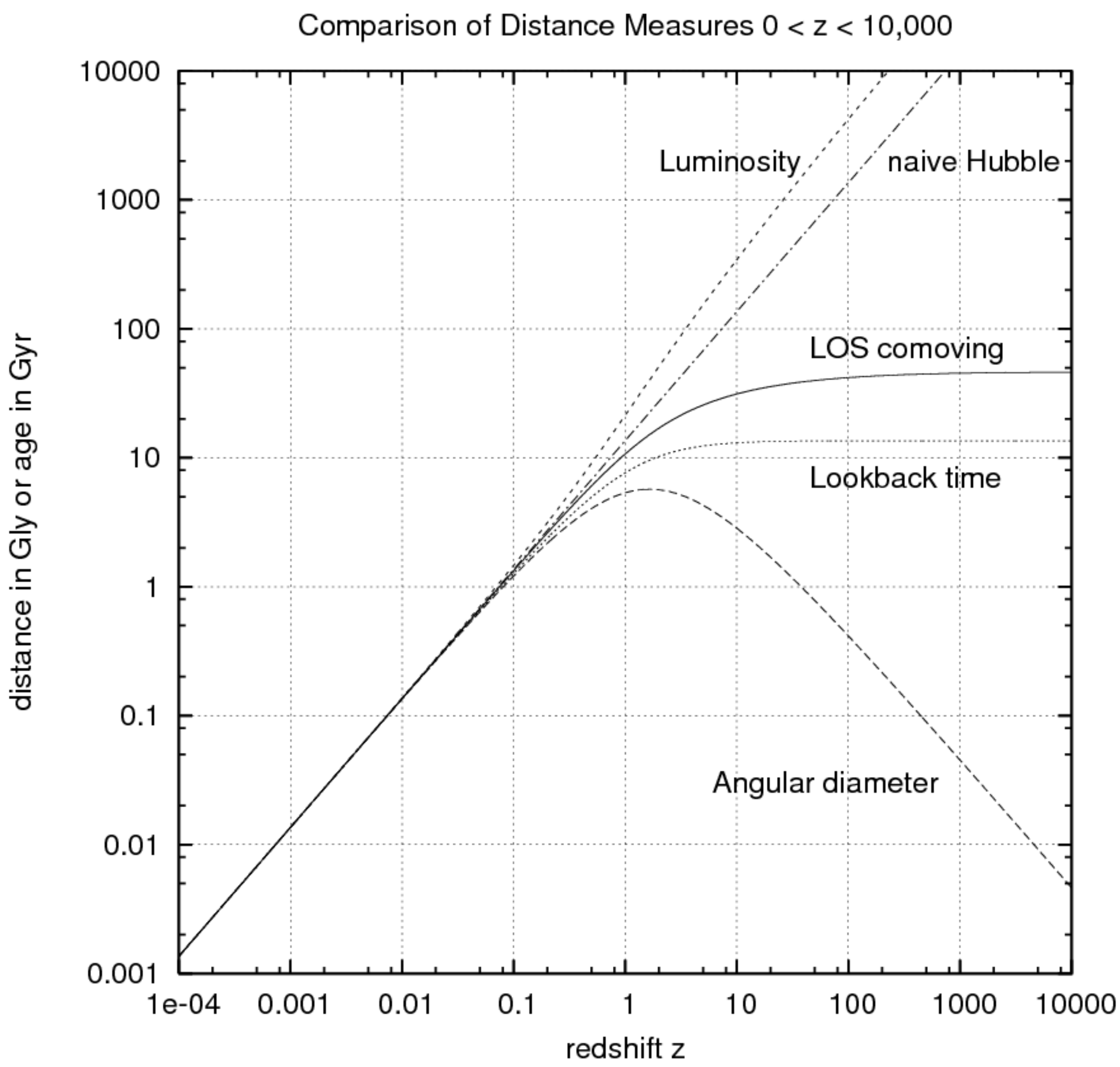

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  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = rH_0$ or
  4. $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq c/H_0 \approx 3\mathrm{h}^{-1}$ Gpc
- $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq ct_0 \approx 4.5\mathrm{h}^{-1}$ Gpc
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"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
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$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature $(+, 0, -)$?

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

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Warning: r is a geodesic distance **on** S^2; it is **not** a radial distance from the S^2 "centre".

Non-radial spatial geodesics

- positive curvature

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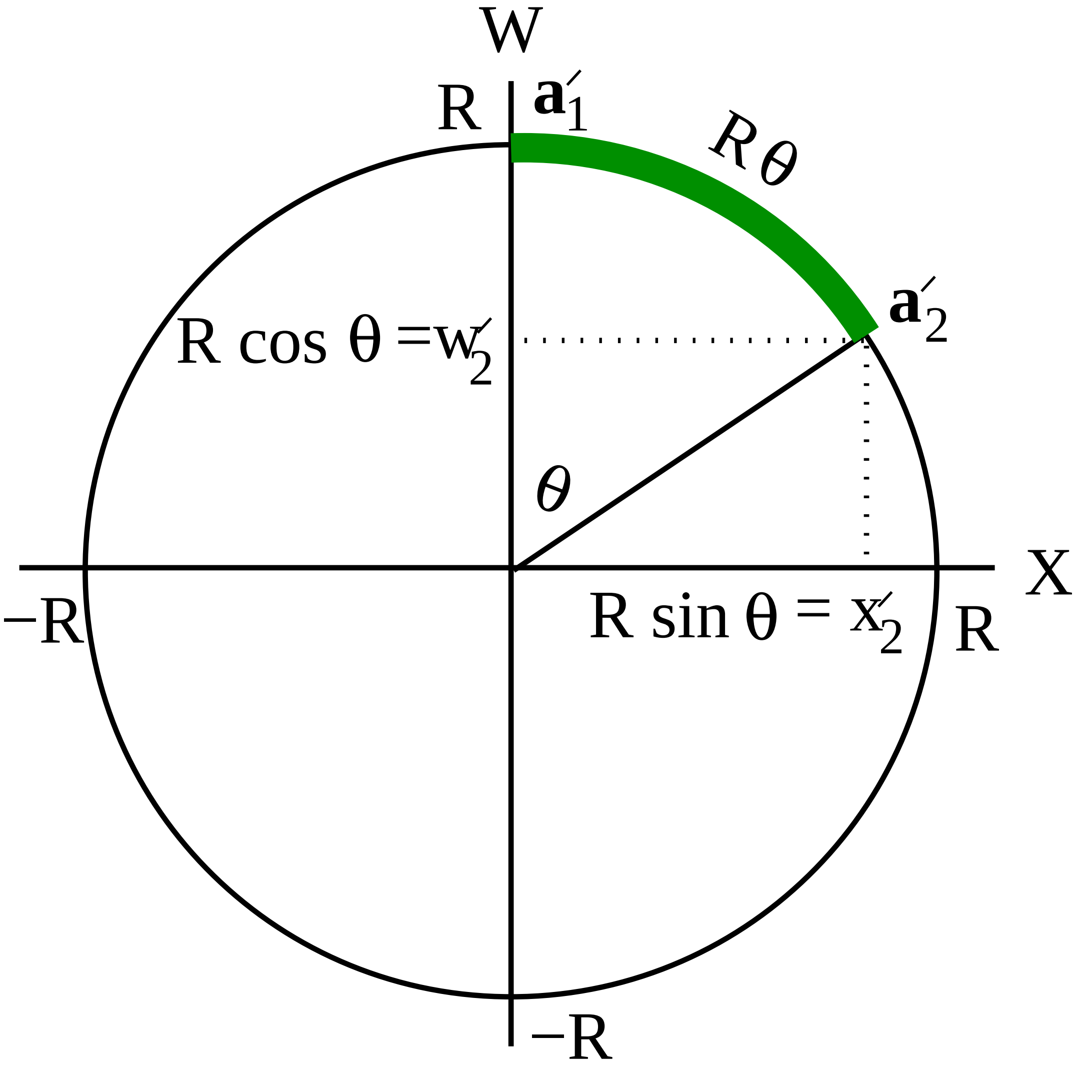

Non-radial spatial geodesics

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Non-radial spatial geodesics

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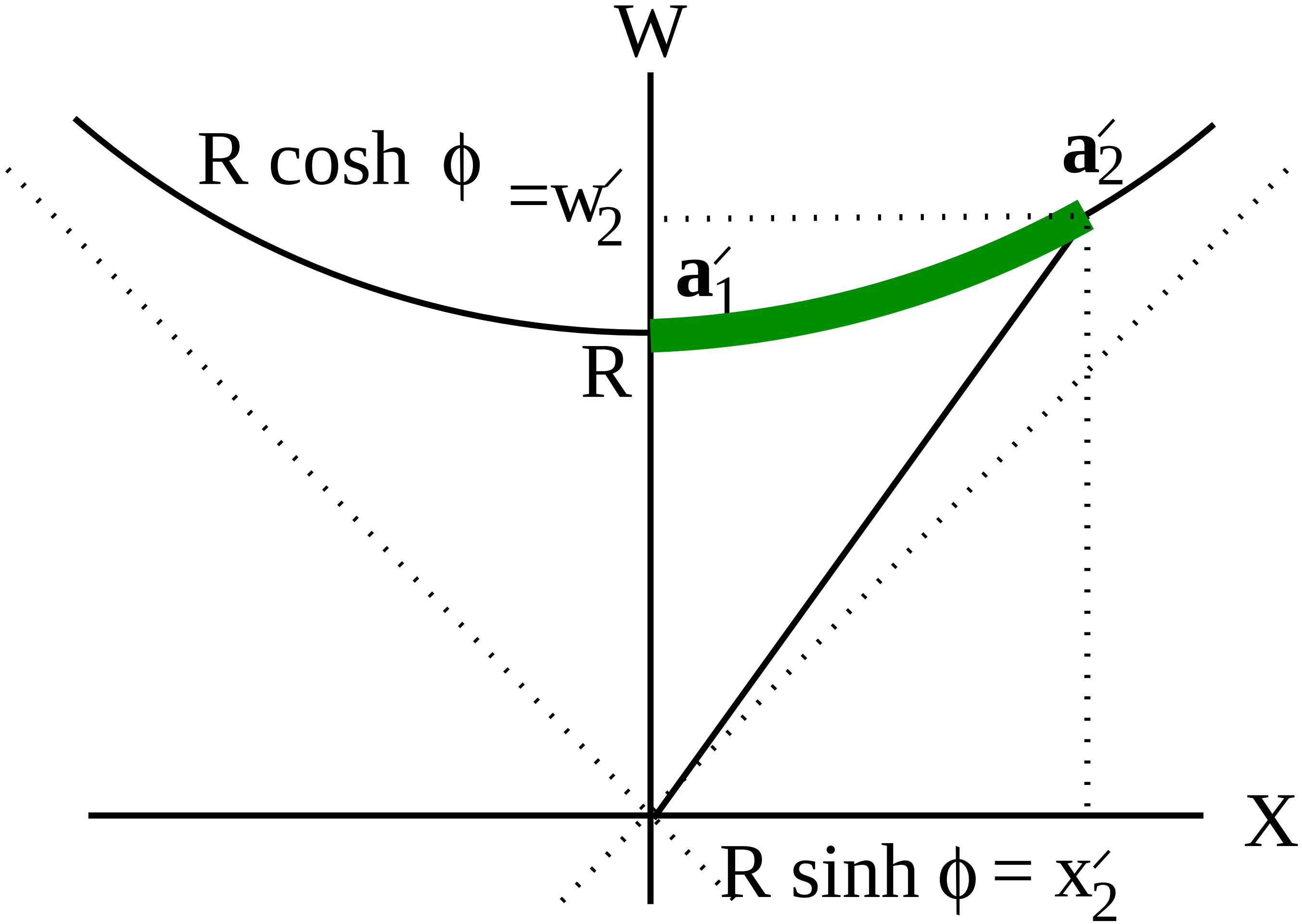

-

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

$$\Sigma(r_i) := \begin{cases} R_\mathrm{C}\ \sinh(r_i/R_\mathrm{C}) & k < 0 \\ r_i & k = 0 \\ R_\mathrm{C}\ \sin(r_i/R_\mathrm{C}) & k > 0 \end{cases}$$

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- metric on S^3 (or $\mathbb{R}^3$ or H^3):
$$\mathrm{d}s^2 = \begin{cases} (k/|k|)\ (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \mathrm{d}w^2 & k \neq 0 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & k = 0 \end{cases}$$

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