

國立中央圖書館



0039052

查目

中學叢書

新三角學講義精解

幾何

著



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國家圖書館典藏  
由國家圖書館數位化

5717  
7541

317.02  
8355

33

### 習題

(6—8頁)

1, 2, 3. 從略.

4.  $v = 10$  周/秒  $= 20\pi$  本位弧/秒,  $t = 2/v = 2/20\pi = 7/220$  秒.

5. 高(弧長)  $= R \times \theta = 1760 \times \frac{\pi}{180} = 30.73$  碼.

(5, 6, 8 題均假定目的物為圓弧, 觀測點為圓心.)

6.  $R = 6 \sqrt{\frac{\pi}{180}} = \frac{1080}{\pi} = 344$  呎.

7. 弧長  $= R\theta = 3950 \times \frac{\pi}{180 \times 60} = 1.149$  哩.

8. 月之直徑(弧長)  $= 238793 \times 1868 \times \frac{\pi}{180 \times 60 \times 60}$   
 $= 2162$  哩.

9.  $\therefore$  10 秒間轉過之弧長為  $20 \left( \frac{10}{3600} \right) = \frac{1}{18}$  哩.

$\therefore \theta = \left( \frac{1}{18} / \frac{1}{2} \right) \text{rad} = \frac{1}{9} \left( \frac{180^\circ}{\pi} \right) = 6.37^\circ$

10. 今分針移過 30 格時, 時針在第  $15 + \frac{30}{12} = 17.5$  格.

$\therefore$  兩針相差為 12.5 格.

又因每格為  $6^\circ$ , 故兩針相差  $12.5 \times 6 = 75^\circ = \frac{5}{12} \pi \text{ rad}$ .

11. 設五段弧長依次為

$x - 2y, x - y, x, x + y, x + 2y$

$$\begin{cases} x - 2y + x - y + x + x + y + x + 2y = 2\pi & (1) \\ 6(x - 2y) = x + 2y & (2) \end{cases}$$



南京 039052

由(1)得  $5x = 2\pi \quad \therefore x = \frac{2}{5}\pi$

代入(2)  $y = \frac{5}{14}x = \frac{1}{7}\pi$

故最小弧所對中心角為

$$x - 2y = \left(\frac{2}{5} - \frac{2}{7}\right)\pi = \frac{4}{35}\pi \text{ rad.}$$

12. 從幾何學知  $n$  邊正多邊形之一內角為

$$\frac{(n-2) \cdot 180^\circ}{n} = \frac{(n-2)\pi}{n} \text{ rad}$$

故五邊形之內角為  $\frac{3}{5}\pi \text{ rad}$ , .....

13.  $\therefore x$  本位弧 =  $\frac{180x}{\pi}$  度,  $x$  百分度 =  $\frac{180x}{200}$  度

$$\therefore x + \frac{180x}{\pi} + \frac{180x}{200} = 180^\circ$$

$$\therefore x = 180 / \left(1 + \frac{180}{\pi} + \frac{9}{10}\right)$$

故最小角為  $\frac{9}{10}x = \frac{9}{10} \cdot \frac{180}{1 + \frac{180}{\pi} + \frac{9}{10}} = 2^\circ 44' 12''$

14. 灣曲之長度為  $(2100 \times \frac{37^\circ}{2} + 2800 \times 21^\circ) \frac{\pi}{180^\circ}$

$$= 9765 \times \frac{\pi}{18} = 1705 \text{ 尺.}$$

## 習題二 (12頁)

1. 今  $a = \frac{2}{3}c$ , 又  $a^2 + b^2 = c^2$ ,  $\therefore b = \sqrt{c^2 - a^2} = \frac{\sqrt{5}}{3}c$

從此可求矣。

2. 今  $a = \sqrt{c^2 - b^2} = \sqrt{p^2 + q^2 - q^2} = p$  下從略。

3.  $c = \sqrt{a^2 + b^2} = \sqrt{\frac{4x^2y^2}{(x-y)^2} + (x+y)^2} = \sqrt{\frac{4x^2y^2 + (x^2 - y^2)^2}{(x-y)^2}}$   
 $= \frac{x^2 + y^2}{x-y}$  下從略。

4.  $a - b = \frac{1}{4}c \dots\dots\dots(1)$        $a^2 + b^2 = c^2 \dots\dots\dots(2)$

(1)<sup>2</sup> - (2),       $-2ab = -\frac{15}{16}c^2 \dots\dots\dots(3)$

$\sqrt{(2) - (3)}$ ,       $a + b = \frac{\sqrt{31}}{4}c \dots\dots\dots(4)$

從(1)(4)得  $a = \frac{1}{8}(\sqrt{31} + 1)c$ ,  $b = \frac{1}{8}(\sqrt{31} - 1)c$

故  $\sin A = \frac{a}{c} = \frac{1}{8}(\sqrt{31} + 1)$

$\cos A = \frac{b}{c} = \frac{1}{8}(\sqrt{31} - 1)$

$\tan A = \frac{\sqrt{31} + 1}{\sqrt{31} - 1} = \frac{16 + \sqrt{31}}{15}$

$\cot A = \frac{\sqrt{31} - 1}{\sqrt{31} + 1} = \frac{16 - \sqrt{31}}{15}$

$\sec A = \frac{4}{15}(\sqrt{31} + 1)$        $\csc A = \frac{4}{15}(\sqrt{31} - 1)$

5.  $\therefore b = \sqrt{c^2 - a^2} = \sqrt{16 - 8 + 2\sqrt{12}} = \sqrt{8 + 2\sqrt{12}}$   
 $= \sqrt{(\sqrt{6} + \sqrt{2})^2} = \sqrt{6} + \sqrt{2}$

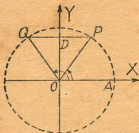
$$\therefore \cos A = \frac{b}{c} = \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

$$\begin{aligned} \tan A &= \frac{a}{b} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{(\sqrt{6} - \sqrt{2})^2}{6 - 2} \\ &= \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3} \end{aligned}$$

### 習題三 (17—18 頁)

1.

a.



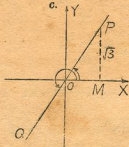
$$OA = 1, OD = \frac{4}{5}$$

b.



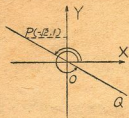
$$OA = 1, OD = \frac{1}{2}\sqrt{2}$$

c.



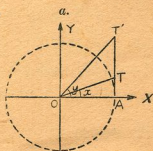
兩角  $\angle XOP, \angle XOQ$

d.

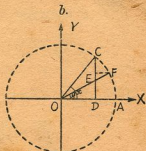


$\angle XOP$  及  $\angle XOQ$

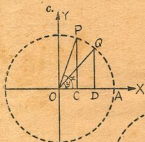
2.



$AT' = 3AT$ ,  $y$  爲  $\angle AOT'$



$DE = \frac{1}{m}DC$ ,  $y$  爲  $\angle AOF$

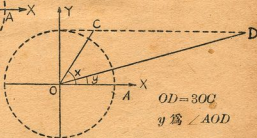


$OD = 2OC$

(設  $45^\circ < x < 90^\circ$ )

$y$  爲  $\angle AOQ$

d.



$OD = 3OC$

$y$  爲  $\angle AOD$

3. 今  $\sin \theta = x + \frac{1}{x}$ , 即  $x^2 - x \sin \theta + 1 = 0$

若  $x$  爲實數, 則  $\Delta \geq 0$ ,

即  $\sin^2 \theta - 4 \geq 0$ , 即  $|\sin \theta| \geq 2$

但此不合理, 故  $x$  必不能爲實數.

或從證  $x + \frac{1}{x} = \frac{x^2 + 1}{x} > 1$ , 因而得  $\sin \theta > 1$  爲不可能.

$$4. \quad \therefore x^2 + y^2 \geq 2xy \quad \therefore (x+y)^2 \geq 4xy$$

$$\therefore \frac{4xy}{(x+y)^2} \leq 1, \quad \text{即} \quad \sec^2 \theta \leq 1$$

但  $\sec \theta$  必大於 1, 即  $\sec^2 \theta$  不能小於 1, 至少等於 1.

此時  $4xy = (x+y)^2$ , 即  $(x-y)^2 = 0$ ,  $\therefore x=y$ .

### 習 題 四 (22—23 頁)

$$1. \quad (a) \cos x = \frac{1}{\sec x} = -\frac{3}{5}, \quad \sin x = \pm \sqrt{1 - \left(-\frac{3}{5}\right)^2} = \pm \frac{4}{5},$$

$$\tan x = \left(\pm \frac{4}{5}\right) / \left(-\frac{3}{5}\right) = \mp \frac{4}{3}, \quad \cot x = \frac{1}{\tan x} = \mp \frac{3}{4},$$

$$\csc x = \frac{1}{\sin x} = \pm \frac{5}{4}. \quad (\text{注意本題之符號})$$

$$(c) \quad \sin x = \frac{1}{\csc x} = \frac{1}{-1} = -1,$$

$$\cos x = \pm \sqrt{1 - \sin^2 x} = 0, \quad \tan x = \frac{-1}{0} = \infty,$$

$$\cot x = \frac{1}{\infty} = 0, \quad \sec x = \frac{1}{0} = \infty.$$

(b) (d) 從略.

(e) 在第三象限中, 除正切, 餘切外均為負值解從略.

$$2. \quad \therefore \text{vers } x = 1 - \cos x = \frac{\sqrt{2} - 1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$

$$\therefore \cos x = \frac{1}{\sqrt{2}}$$

故  $x$  在第一或第四象限內, 若  $x$  在第一象限內, 則可求其餘各函數之值, 代入右端後即得所求之答案; 若  $x$  在第四象限內, 則所得之答案為  $-2$ .



$$\begin{aligned}
 3. \text{ 左邊} &= (\sin \theta + \cos^2 \theta / \sin \theta) + (\cos \theta + \sin^2 \theta / \cos \theta) - \sec \theta \\
 &= \frac{1}{\sin \theta} + \frac{1}{\cos \theta} - \sec \theta = \csc \theta = \sqrt{1 + \cot^2 \theta} \\
 &= \sqrt{1 + \frac{a^2 - b^2}{b^2}} = \sqrt{\frac{a^2}{b^2}} = \frac{a}{b}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \sec^2 \phi &= 1 + \left( \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \right)^2 = 1 + \frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{b^{\frac{2}{3}}}, \\
 \therefore \sec \phi &= \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}.
 \end{aligned}$$

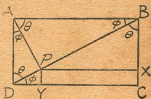
同理 
$$\csc \phi = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}}.$$

故 
$$\begin{aligned}
 a \csc \phi + b \sec \phi &= a^{\frac{2}{3}} (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}} + b^{\frac{2}{3}} (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}} \\
 &= (a^{\frac{2}{3}} + b^{\frac{2}{3}}) (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}} \\
 &= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.
 \end{aligned}$$

5. 如圖在  $\triangle PBX$ ,  $\triangle PAB$ ,

$\triangle ADB$  中,  $\sin \theta = \frac{PX}{PB},$

$\sin \theta = \frac{PB}{BA}, \quad \sin \theta = \frac{BA}{BD},$



故  $\sin^3 \theta = \frac{PX}{BD}, \quad \therefore \sin \theta = \left( \frac{PX}{BD} \right)^{\frac{1}{3}}.$

又在  $\triangle PDY$ ,  $\triangle PAD$ ,  $\triangle ADB$  中,

$\sin \phi = \frac{PY}{PD}, \quad \sin \phi = \frac{PD}{AD}, \quad \sin \phi = \frac{AD}{BD}.$

$$\text{故 } \sin^3 \phi = \frac{PY}{BD}, \quad \therefore \sin \phi = \left(\frac{PY}{BD}\right)^{\frac{1}{3}}$$

$$\text{但在 } \triangle ADB \text{ 中, } \sin \theta = \frac{AB}{BD}, \quad \sin \phi = \frac{AD}{BD}$$

$$\sin^2 \theta + \sin^2 \phi = \frac{AB^2}{BD^2} + \frac{AD^2}{BD^2} = \frac{AB^2 + AD^2}{BD^2} = \frac{BD^2}{BD^2} = 1$$

$$\text{故 } \left(\frac{PX}{BD}\right)^{\frac{2}{3}} + \left(\frac{PY}{BD}\right)^{\frac{2}{3}} = 1$$

$$\text{即 } \frac{PX^{\frac{2}{3}}}{BD^{\frac{2}{3}}} + \frac{PY^{\frac{2}{3}}}{BD^{\frac{2}{3}}} = \frac{BD^{\frac{2}{3}}}{BD^{\frac{2}{3}}} = \frac{AC^{\frac{2}{3}}}{BD^{\frac{2}{3}}}. \quad (\because AC = BD)$$

### 習題五 (34—36 頁)

$$1. \text{ 左邊} = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$2. \text{ 右邊} = \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = 1 + \cot^2 x$$

$$3. \text{ 左邊} = (\cos^2 3A + \sin^2 3A)(\cos^4 3A - \cos^2 3A \sin^2 3A + \sin^4 3A)$$

$$= (\cos^2 3A + \sin^2 3A)^2 - 3 \cos^2 3A \sin^2 3A$$

$$= 1 - 3 \sin^2 3A \cos^2 3A = \text{右邊}$$

$$4. \because \sin^2(x+y) + \cos^2(x+y) = \sin^2(x-y) + \cos^2(x-y) (=1)$$

$$\text{移項得 } \cos^2(x+y) - \sin^2(x-y)$$

$$= \cos^2(x-y) - \sin^2(x+y)$$

$$5. \text{左邊} = \frac{1 - \tan A}{1 + \tan A} \cdot \frac{\cot A}{\cot A} = \frac{\cot A - 1}{\cot A + 1}$$

$$6. \text{左邊} = \frac{2 \csc^2 A}{\csc^2 A - 1} = \frac{2 \csc^2 A}{\cot^2 A} = \frac{2}{\cos^2 A} = 2 \sec^2 A$$

$$7. \text{左邊} = \frac{(\sec A + \tan A)(\sec A - \tan A)}{\sec A - \tan A}$$

$$= \frac{\sec^2 A - \tan^2 A}{\sec A - \tan A} = \frac{1}{\sec A - \tan A}$$

或從  $\sec^2 A - \tan^2 A = (\sec A - \tan A)(\sec A + \tan A) = 1$

兩邊以  $\sec A - \tan A$  除之即得。

$$8. \text{左邊} = \sin^2 x + 2 + \csc^2 x + \cos^2 x + 2 + \sec^2 x \\ = 5 + (1 + \cot^2 x) + (1 + \tan^2 x) = \text{右邊}$$

$$9. \text{左邊} = [\sec \beta (\sec x + \tan x) + \tan \beta (\sec x + \tan x)] \\ \times [\sec \beta (\sec x - \tan x) + \tan \beta (\tan x - \sec x)] \\ = (\sec \beta + \tan \beta) (\sec x + \tan x) (\sec \beta - \tan \beta) \\ \times (\sec x - \tan x) \\ = (\sec^2 x - \tan^2 x) (\sec^2 \beta - \tan^2 \beta) = 1 \cdot 1 = 1$$

$$10. \text{左邊} = \frac{(\csc x + \cot x)(\sec x - \tan x)}{\sec^2 x - \tan^2 x} \\ = \frac{(\csc^2 x - \cot^2 x)(\sec x - \tan x)}{\csc x - \cot x} = \frac{\sec x - \tan x}{\csc x - \cot x}$$

或從  $\csc^2 x - \cot^2 x = \sec^2 x - \tan^2 x (=1)$  做。

(用例一，證五法)

$$11. \text{左邊} = \frac{(\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)} \quad (\text{以 } \sin^2 x + \cos^2 x \text{ 代 } 1)$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - \tan x}{1 + \tan x} \quad (\text{以 } \cos x \text{ 除分子, 分母})$$

$$\begin{aligned} 12. \text{ 左邊} &= \frac{(\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)} = \frac{\cos x - \sin x}{\cos x + \sin x} \\ &= \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} = \frac{\cos^2 x - \sin^2 x}{1 + 2 \sin x \cos x} \end{aligned}$$

$$\begin{aligned} 13. \text{ 左邊} &= (2 + \sin A)(\sec A - 2 \tan A) \\ &= 2 \sec A + \tan A - 4 \tan A - \frac{2 \sin^2 A}{\cos A} \\ &= 2(1 - \sin^2 A) \sec A - 3 \tan A \\ &= 2 \cos A - 3 \tan A \end{aligned}$$

$$\begin{aligned} 14. \text{ 今 } \sec \theta &= \frac{\sqrt{2}}{\cos x + \sin x} \quad \therefore \cos \theta = \frac{\sqrt{2}(\sin x + \cos x)}{2} \\ \therefore \sin \theta &= \tan \theta \cos \theta = \frac{\sqrt{2}(\sin x - \cos x)}{2} \\ \therefore \sqrt{2} \sin \theta &= \sin x - \cos x \end{aligned}$$

$$\begin{aligned} 15. \text{ 從已知式移項得 } & (\sqrt{2} + 1) \sin \theta = \cos \theta \\ \text{兩邊以 } \sqrt{2} - 1 \text{ 乘之 } & \sin \theta = (\sqrt{2} - 1) \cos \theta \\ \text{移項得 } & \sin \theta + \cos \theta = \sqrt{2} \cos \theta \end{aligned}$$

$$\begin{aligned} 16. \text{ 今 } \quad 2 \tan x &= m + n, \quad 2 \sin x = m - n \\ \text{前式除後式得 } & \cos x = (m - n) / (m + n) \end{aligned}$$

$$\begin{aligned} 17. \quad \therefore \frac{\cos^3 \theta}{\cos \alpha} + \frac{\sin^3 \theta}{\sin \alpha} &= \sin^2 \alpha + \cos^2 \alpha \\ \text{移項 } \frac{\cos^3 \theta - \cos^3 \alpha}{\cos \alpha} &= \frac{\sin^3 \alpha - \sin^3 \theta}{\sin \alpha} \quad (1) \\ \text{又因 } \frac{\cos^3 \theta}{\cos \alpha} + \frac{\sin^3 \theta}{\sin \alpha} &= \sin^2 \theta + \cos^2 \theta \end{aligned}$$

$$\therefore \frac{\cos^2 \theta (\cos \theta - \cos \alpha)}{\cos \alpha} = \frac{\sin^2 \theta (\sin \alpha - \sin \theta)}{\sin \alpha} \quad (2)$$

$$(1) \quad \frac{\cos^3 \theta - \cos^3 \alpha}{\cos^2 \theta (\cos \theta - \cos \alpha)} = \frac{\sin^3 \alpha - \sin^3 \theta}{\sin^2 \theta (\sin \alpha - \sin \theta)}$$

$$\begin{aligned} \text{即} \quad & \frac{\cos^2 \theta + \cos \theta \cos \alpha + \cos^2 \alpha}{\cos^2 \theta} \\ & = \frac{\sin^2 \alpha + \sin \alpha \sin \theta + \sin^2 \theta}{\sin^2 \theta} \end{aligned}$$

$$\text{即} \quad \frac{\cos^2 \alpha}{\cos^2 \theta} - \frac{\sin^2 \alpha}{\sin^2 \theta} + \frac{\cos \alpha}{\cos \theta} - \frac{\sin \alpha}{\sin \theta} = 0$$

$$\text{即} \quad \left( \frac{\cos \alpha}{\cos \theta} - \frac{\sin \alpha}{\sin \theta} \right) \left( \frac{\cos \alpha}{\cos \theta} + \frac{\sin \alpha}{\sin \theta} + 1 \right) = 0$$

$$\begin{aligned} 18. \text{ 左邊} &= \begin{vmatrix} 1 & \cos^4 \theta & 0 \\ 1 & (1 + \sin^2 \theta)^2 & 0 \\ 1 & \cos^4 \theta & (1 + \cos^2 \theta)^2 - \sin^4 \theta \end{vmatrix} \\ &= [(1 + \cos^2 \theta)^2 - \sin^4 \theta][(1 + \sin^2 \theta)^2 - \cos^4 \theta] \\ &= (1 + \cos^2 \theta + \sin^2 \theta)(1 + \cos^2 \theta - \sin^2 \theta) \\ &\quad \times (1 + \sin^2 \theta + \cos^2 \theta)(1 + \sin^2 \theta - \cos^2 \theta) \\ &= 2 \cdot 2 \cos^2 \theta \cdot 2 \cdot 2 \sin^2 \theta = 16 \sin^2 \theta \cos^2 \theta \end{aligned}$$

## 習題六 (37—38 頁)

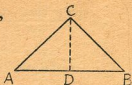
1. 2. 3. 4. 從略。

## 習題七 (42—43 頁)

1. a, b, c 從略。 (a 之答數  $c=3.1194$ ,  $b=2.2223$ )

2. 今  $\overline{AC} = 30\sqrt{2}$  尺,  $AB = 60$  尺,  
 $AD = 30$  尺.

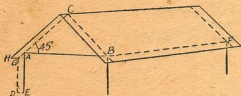
$$\cos A = \frac{30}{30\sqrt{2}} = \frac{\sqrt{2}}{2}$$



$\therefore A = 45^\circ$  (屋頂之傾斜)

$\therefore CD = 30 \tan 45^\circ = 30$  尺 (屋脊距屋簷之高)

3.  $\therefore AB = 40$  尺,  $BF = 80$  尺,  $\angle A = 45^\circ$   
 $DE = 1$  尺,  $\angle H = \angle A = 45^\circ$   
 $AC = 20\sqrt{2}$  尺,  $GA = \sqrt{2}$  尺.



$\therefore$  椽子  $CG$  長  $20\sqrt{2} + \sqrt{2} = 21\sqrt{2} = 29.694$  尺.

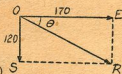
全屋面之面積為  $21\sqrt{2} \times 82 \times 2 = 3444\sqrt{2}$

$= 4870.5$  平方尺.

4. 合力  $OR = \sqrt{170^2 + 120^2} = 208$  磅

$$\therefore \tan \theta = \frac{120}{170} = \frac{12}{17} = .7059$$

$\therefore \theta = 35^\circ 13'$  (東  $35^\circ 13'$  南)



5.  $\therefore \angle AOC = 32^\circ 3'$ ,  $\therefore \angle AOB = 57^\circ 57'$

$$\begin{aligned}\therefore \overline{AB} &= \overline{AO} \sin 57^{\circ} 57' \\ &= 4000 \times .8476 \\ &= 3390.4 \text{ 英里}\end{aligned}$$

\(\therefore\) 緯度圈為以 \(BA\) 為半徑之一圓周長為

$$2\pi \times 3390.4 = 21310 \text{ 英里}$$

6. 桿高 =  $80 \tan 50^{\circ} = 80 \times 1.1918$   
= 95.34 尺.



7. 設  $\overline{AB} = c$ ,  $\overline{BE} = b$ ,  $\overline{BD} = \frac{c}{2}$

則  $\angle BOE = \angle AOE = \frac{180^{\circ}}{n}$

今  $\angle AOE$  以  $\widehat{AE}$  度之

即  $\angle EBA$  以  $\frac{1}{2}\widehat{AE}$  度之

故  $\angle EBA = \frac{1}{2} \cdot \frac{180^{\circ}}{n} = \frac{90^{\circ}}{n}$

$$\therefore \frac{c/2}{b} = \cos \frac{90^{\circ}}{n}$$

$$\therefore b = \frac{c}{2} \cos \frac{1}{n} 90^{\circ}$$

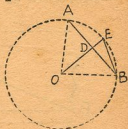
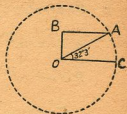
8. 設  $p$  為周界，則

$$\frac{p}{720} \div 1 = \sin 30'$$

(\(\because\) 中心角為一度)

$$\therefore p = 720 \times 0.00873 = 6.283$$

9.  $\therefore \overline{AB} = \overline{BO} \tan \frac{180^{\circ}}{n} = r \tan \frac{180^{\circ}}{n}$



$$\therefore S = nr \cdot r \tan \frac{180^\circ}{n} = nr^2 \tan \frac{180^\circ}{n}$$

$$\therefore A'B' = r \sin \frac{180^\circ}{n}$$

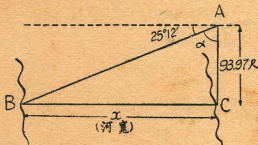
$$B'O' = r \cos \frac{180^\circ}{n}$$

$$\therefore S' = nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}$$



### 習題八 (53—57 頁)

1.  $x = BC = \text{河寬} = AC \tan \alpha = 93.97 \tan(90^\circ - 25^\circ 12')$   
 $= 93.97 \tan 64^\circ 48' = 93.97 \times 2.1251 = 200 \text{ 尺}$



2. 今  $\angle A$  爲直角

$$\angle L = 45^\circ + 15^\circ = 60^\circ$$

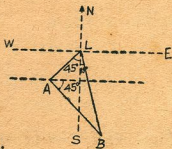
又  $LA = 4$  哩

故二舟距離爲

$$AB = LA \tan 60^\circ = 4\sqrt{3}$$

$$= 6.928$$

故  $A, B$  之距離爲 6.928 哩。





3. 今  $AC$  (N.N.W.) 與  $AB$  (W.S.W.) 成直角

$$\angle N'BD = 37^\circ 30'$$

$$\angle N'BC = 7^\circ 30'$$

故知,

$$\angle CBD = 30^\circ$$

$$\begin{aligned} \angle DBA &= 90^\circ - 37^\circ \frac{1}{2} - 22^\circ \frac{1}{2} \\ &= 30^\circ \end{aligned}$$

$$\therefore \angle BDA = 90^\circ - 30^\circ = 60^\circ$$

$$\therefore \angle DCB = \angle CBD,$$

$$\therefore CD = DB = 1.5 \text{ 哩},$$

$$\therefore AB = DB \cdot \cos DBA$$

$$= \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

$$\therefore \text{火車之速率爲 } \frac{3\sqrt{3}}{8} \text{ 哩/分} = 39 \text{ 哩/時}$$

4. 今  $\angle AOB = 180^\circ - 28^\circ - 62^\circ = 90^\circ$

$$\text{又 } OA = 2 \times 10 = 20 \text{ 哩},$$

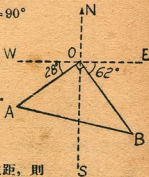
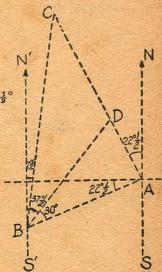
$$OB = 2 \times 10 \frac{1}{2} = 21 \text{ 哩}.$$

$$\therefore AB = \sqrt{20^2 + 21^2} = 29 \text{ 哩}.$$

5.  $\angle ACB = 54^\circ 24' - 27^\circ 12'$

$$= 27^\circ 12'$$

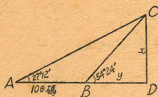
設  $x$  爲山高;  $y$  爲近點至山之距, 則



$$x = \frac{300 \sin 27^\circ 12' \sin 54^\circ 24'}{\sin 27^\circ 12'} \quad \text{〔從公式甲二 (48 頁)]}$$

$$= 300 \sin 54^\circ 24'$$

$$y = x \cot 54^\circ 24' = 300 \cos 54^\circ 24' = 174.6 \text{ 呎}$$



6. 從公式甲三 (48 頁)

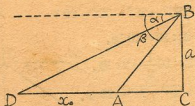
$$\text{河闊} = \frac{30 \tan 45^\circ}{\tan 60^\circ - \tan 45^\circ} = \frac{30}{1.732 - 1} = \frac{30}{.732} = 41 \text{ 尺}$$

7. 今  $A, D$  二點間之距離為  $x$  尺

$$DC = a \tan(90^\circ - \alpha) = a \cot \alpha$$

$$AC = a \tan(90^\circ - \beta) = a \cot \beta$$

$$\therefore x = DC - AC = a(\cot \alpha - \cot \beta)$$



〔照例二之解二法可求得答數之另一形式〕

8. 距離 =  $150(\cot 30^\circ - \cot 45^\circ) = 150(\sqrt{3} - 1) = 109.8 \text{ 尺}$

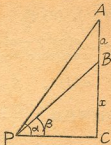
9. 設  $AB = a, \quad BC = x, \quad PC = y,$

$$\frac{x+a}{y} = \tan \alpha, \quad \frac{x}{y} = \tan \beta$$

則  $\frac{x+a}{x} = \frac{\tan \alpha}{\tan \beta}$

$$\therefore x = \frac{a \tan \beta}{\tan \alpha - \tan \beta}$$

10. 塔高 =  $\frac{25}{\sqrt{3}-1} = \frac{25}{2}(\sqrt{3}+1)$   
 $= \frac{25}{2} \times 2.732 = 34.15$  尺。



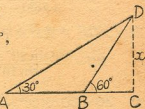
11. 今  $AB=1$  哩,  $\angle DAC=30^\circ$ ,  
 $\angle DBC=60^\circ$ .

設此屋距路遠  $x$  哩, 則

$$(1+x \cot 60^\circ) = x \cot 30^\circ$$

$$x = \frac{1}{\cot 30^\circ - \cot 60^\circ} = \frac{1}{\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

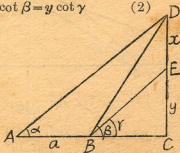
$$= 0.866 \text{ 哩} = 1524 \text{ 碼}.$$



12. 設旗桿高  $x$  尺, 屋高  $y$  尺, 則

$$(x+y) \cot \alpha - y \cot \gamma = a \quad (1)$$

$$(x+y) \cot \beta = y \cot \gamma \quad (2)$$



解 (1), (2) 得

$$x = \frac{a(\cot \gamma - \cot \beta)}{\cot \gamma(\cot \alpha - \cot \beta)} = \frac{a \tan \alpha(\tan \beta - \tan \gamma)}{\tan \beta - \tan \alpha} \text{ 尺.}$$

13. 今  $\alpha = 30^\circ$ ,  $\beta = 60^\circ$ ,  $\gamma = 50^\circ$ ,  $a = 100$  尺.

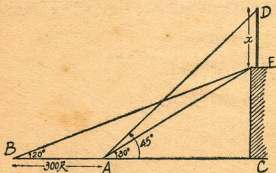
故旗桿長為  $\frac{100(\tan 60^\circ - \tan 50^\circ)\tan 30^\circ}{\tan 60^\circ - \tan 30^\circ}$

$$= \frac{100(\sqrt{3} - 1.1918)0.5774}{1.7321 - 0.5774} = 27 \text{ 尺.}$$

14. 在  $\triangle BCE$  中從公式甲二知

$$EC = \frac{300 \sin 20^\circ \sin 30^\circ}{\sin 10^\circ}$$

則  $AC = EC \cot 30^\circ = \frac{300 \sin 20^\circ \cos 30^\circ}{\sin 10^\circ}$



又

$$DC = AC \tan 45^\circ = AC$$

$$\therefore x = DC - EC = \frac{300 \sin 20^\circ (\cos 30^\circ - \sin 30^\circ)}{\sin 10^\circ}$$

$$= \frac{300 \times 3420 \times (.866 - .5)}{.1736} = 216 \text{ 尺.}$$

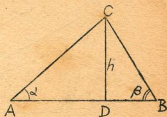
(此題如在 VI 章之後則應化為對數式做)

15. 令此氣球高度為  $h$  尺

$$\text{則 } AD = h \cot \alpha, \quad BD = h \cot \beta$$

$$\begin{aligned} \therefore h(\cot \alpha + \cot \beta) \\ = AD + BD = a \end{aligned}$$

$$\begin{aligned} \therefore h &= \frac{a}{\cot \alpha + \cot \beta} \\ &= \frac{a \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} \end{aligned}$$



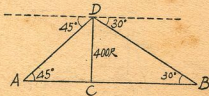
16. 今  $\alpha = 44^\circ 21'$ ,  $\beta = 62^\circ 30'$ ,  $AB = 1800$  尺.

$$\text{代入得 } h = 1166 \text{ 尺.}$$

(如在 VI 章之後則應化  $h = \frac{a \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$  用對數做)

17. 今  $DC =$  山高為 400 尺,  $A, B$  為兩人立處.

$$AC = 400 \cot 45^\circ, \quad BC = 400 \cot 30^\circ$$



$$\therefore AB = AC + BC = 400(1 + \sqrt{3}) = 1098$$

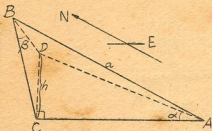
即兩人之直線距為 1098 尺.

18. 設山高為  $h$  尺, 則  $CB$  向北,  $CA$  向東.

則  $AC = h \cot \alpha, \quad BC = h \cot \beta$

$$\therefore h^2(\cot^2 \alpha + \cot^2 \beta) = \overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2 = a^2$$

$$\therefore h = \frac{a}{\sqrt{\cot^2 \alpha + \cot^2 \beta}} = \frac{a \tan \alpha \tan \beta}{\sqrt{\tan^2 \alpha + \tan^2 \beta}}$$

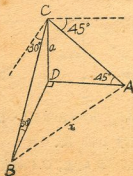


19. 山高 =  $\frac{a}{\sqrt{\cot^2 \alpha + \cot^2 \beta}} = \frac{1000}{\sqrt{3+1}} = 500$  尺。

20. 設  $A, B$  為船之兩位置，其距離為  $x$  尺

$$AD = a \cot 45^\circ = a, \quad BD = a \cot 30^\circ = a\sqrt{3}$$

$$\therefore x^2 = \overline{AD}^2 + \overline{BD}^2 = a^2 + 3a^2 = 4a^2 \quad (AD \perp BD)$$



$\therefore x = 2a$  即船之速度為每時  $2a$  尺。

21. 設  $D, E$  為氣球之兩位置

高  $BD = CE$

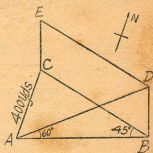
今  $AC = 400$  碼向北,

$AB$  向東,  $BC$  向西北,

故  $\angle A = 90^\circ$

$\angle CBA = 45^\circ$

$\angle ACB = 45^\circ$



$\therefore AB = AC = 400$  碼。

今  $BD = AB \tan 60^\circ = 400\sqrt{3} = 693$  碼。

22. 今  $\alpha = 30^\circ$ ,  $\beta = 60^\circ$ ,  $a = 50$  尺, 代入公式丙三, 則

$$\text{屋高} = \frac{50 \tan 60^\circ}{\tan 60^\circ - \tan 30^\circ} = \frac{50\sqrt{3}}{\sqrt{3} - \frac{1}{\sqrt{3}}} = 3 \times 25 = 75 \text{ 尺。}$$

23. 設  $ED$  為屋,  $BA$  為塔 (高為  $h$  尺)。

則  $h = BC + CA = DC(\tan \beta + \tan \alpha)$

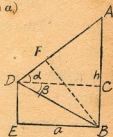
$$= a(\tan \alpha + \tan \beta)$$

如作  $BF \perp AD$

則  $BD = DC / \cos \beta = a / \cos \beta$

$$BF = BD \sin(\alpha + \beta)$$

$$AB = BF / \cos \alpha$$



$$\therefore h = AB = \frac{a \sin(\alpha + \beta)}{\cos \alpha \cos \beta}.$$

24. 如上圖中  $\alpha=60^\circ$ ,  $\beta=30^\circ$ ,  $a=30$  尺.

$$\text{則屋高 } AB=30(\tan 60^\circ + \tan 30^\circ)$$

$$=40\sqrt{3}=69.28 \text{ 尺.}$$

25. 設街闊為  $DB$ ,  $A, C$  為兩窗口.

$$\therefore DO=\sqrt{40^2-21^2}$$

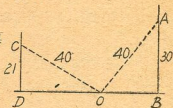
$$=\sqrt{1159}=34.04$$

$$OB=\sqrt{40^2-30^2}$$

$$=\sqrt{700}=26.45$$

$$\therefore \text{街闊}=DO+OB$$

$$=26.45+34.04=60 \text{ 尺.}$$



26. 設  $CD$  為桿高,  $DA$  為土堆高.

$$\text{則 } AC=AB \tan 60^\circ = \sqrt{3} AB$$

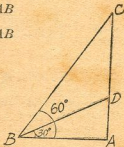
$$AD=AB \tan 30^\circ = \frac{1}{\sqrt{3}} AB$$

$$= \frac{\sqrt{3}}{3} AB$$

$$\therefore AC=3AD$$

$$\text{但 } AD+CD=3AD$$

$$\text{故 } CD=2AD.$$

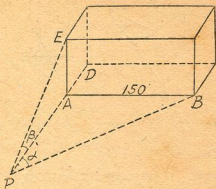


27. 如圖測點  $P$  在  $DA$  之延長線上,  $AB=150$  尺,  $AE$  為屋高.

$$\text{今 } \cos \alpha = \sqrt{\frac{1}{5}} \quad \therefore \sin \alpha = \frac{2}{\sqrt{5}}$$

$$\tan \alpha = 2 = \frac{AB}{AP} = \frac{150}{AP} \quad \therefore AP = 75 \text{ 尺.}$$



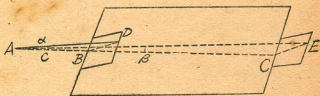


$$\text{再 } \sin \beta = \frac{3}{\sqrt{34}}, \quad \cos \beta = \sqrt{1 - \frac{9}{34}} = \frac{5}{\sqrt{34}}$$

$$\therefore \tan \beta = \frac{3}{5} = \frac{AE}{AP}, \quad \therefore AE = 75 \times \frac{3}{5} = 45 \text{ 尺.}$$

$$28. \quad DB = c \tan \alpha, \quad DB = CE = (c + \overline{BC}) \tan \beta$$

$$\therefore c \tan \alpha = (c + \overline{BC}) \tan \beta$$

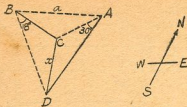


$$\therefore BC = \text{球場長度} = \frac{c \tan \alpha - c \tan \beta}{\tan \beta}$$

$$= c(\tan \alpha \cot \beta - 1) \text{ 尺.}$$

$$29. \quad \text{今 } AD \text{ 南向, } AB \text{ 西向. 故 } \angle BAD = 90^\circ$$

$$\text{又設塔高 } CD \text{ 爲 } x \text{ 尺, 則 } \overline{BD}^2 - \overline{AD}^2 = \overline{AB}^2 = x^2$$



$$\text{但 } BD = x \cot 18^\circ = x \sqrt{\frac{5}{5-2\sqrt{5}}} = x \sqrt{5+2\sqrt{5}}$$

$$AD = x \cot 30^\circ = x\sqrt{3}$$

$$\text{代入上式得 } x^2(5+2\sqrt{5}-3) = a^2,$$

$$\therefore x = \frac{a}{\sqrt{2\sqrt{5}+2}}$$

30. 今  $\angle CAD = 45^\circ$ ,  $CD \perp AD$

$$\therefore AD = CD \cot 45^\circ = x$$

在  $\text{rt } \triangle ABD$  中,

$$\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = a^2 + x^2$$

$$\text{故 } \frac{CD}{BD} = \frac{x}{\sqrt{a^2+x^2}} = \tan 15^\circ$$

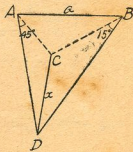
$$= 2 - \sqrt{3}$$

$$\therefore x^2 = (7-4\sqrt{3})(a^2+x^2)$$

$$\therefore x^2 = \frac{a^2(7-4\sqrt{3})}{4\sqrt{3}-6} = \frac{a^2(7-4\sqrt{3})(4+2\sqrt{3})}{\sqrt{3}(4-2\sqrt{3})(4+2\sqrt{3})}$$

$$= \frac{a^2(4-2\sqrt{3})}{4\sqrt{3}} = \frac{a^2(\sqrt{3}-1)^2}{4\sqrt{3}}$$

$$\therefore x = \frac{a}{2} \left( \frac{\sqrt{3}-1}{\sqrt[4]{3}} \right) = \frac{a}{2} (3^{\frac{1}{4}} - 3^{-\frac{1}{4}}).$$



31. 今  $\angle ABD = 60^\circ$ ,  $\angle CBD = 30^\circ$ ,  
 $\angle ABC = 30^\circ$ ,  $\angle BCA = 135^\circ$ ,  
 $\angle CAB = 15^\circ$ ,  $BC = 5280$  呎.

作  $CF \perp AB$

則  $BF = 5280 \cos 30^\circ$

$$CF = 5280 \sin 30^\circ = 2640,$$

$$AF = CF \cot 15^\circ = 2640 \cot 15^\circ.$$

故山高  $AD = AB \sin 60^\circ$

$$= (BF + AF) \sin 60^\circ$$

$$= 2640(2 \cos 30^\circ + \cot 15^\circ) \sin 60^\circ$$

$$= 2640 \times 5.4641 \times .866 = 12492.6 \text{ 呎.}$$



(參考 290 頁例十一)

32.  $E$  為  $CD$  之中點,  $O$  為底之中心, 故所求之角為  $AEO$ .

$$\text{今 } AE = \sqrt{AC^2 - CE^2}$$

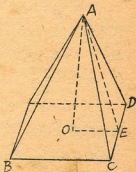
$$= \sqrt{150^2 - 100^2}$$

$$= \sqrt{12500}$$

$$\cos AEO = \frac{OE}{AE} = \frac{100}{50\sqrt{5}}$$

$$= \frac{2}{5}\sqrt{5} = .8944$$

$\therefore \angle AEO = 26^\circ 34'$  (銳角).

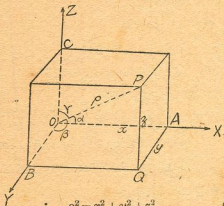


33. 如圖  $OA = x$ ,  $AQ = y$ ,  $QP = z$ , 又  $OP = \rho$  為長方體

$ABOP$  之對角線，今  $\angle OAP = \text{直角}$

$$\overline{OP}^2 = \overline{OA}^2 + \overline{AP}^2 = x^2 + \overline{AP}^2$$

但  $\overline{AP}^2 = \overline{AQ}^2 + \overline{QP}^2 = y^2 + z^2$  ( $\angle PQA = \text{直角}$ )



$$\therefore \rho^2 = x^2 + y^2 + z^2 \quad (1)$$

$$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma$$

代入(1), 則  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

設  $\alpha = \beta = \gamma$ , 則  $3 \cos^2 \alpha = 1$

$$\therefore \cos \alpha = \frac{1}{\sqrt{3}} \sqrt{3} = .5773$$

$$\therefore \alpha = 54^\circ 41' 20''.$$

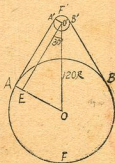
34. 如圖皮帶之長爲

$$AA' + BB' + \widehat{AFB} + \widehat{A'F'B'}$$

今  $OA = 70$ ,  $O'A' = 10$ ,

作  $O'E \parallel AA'$ , 則  $AE = 10$ ,

$OE = 60$ ,  $OO' = 120$



故  $\angle EO'O = 30^\circ$ , 則  $\angle EOO' = 60^\circ$

$$(\because \angle OEO' = 90^\circ)$$

故  $\angle AOB = 120^\circ$ ,  $\angle A'O'B' = 120^\circ$

$$(\because OA \parallel O'A')$$

則  $\widehat{AFB} = 240^\circ = \frac{2}{3}(\text{大圓周}) = \frac{280\pi}{3}$

$$\widehat{A'F'B'} = 120^\circ = \frac{1}{3}(\text{小圓周}) = \frac{20\pi}{3}$$

又  $AA' = BB' = OO' \cos 30^\circ = 60\sqrt{3}$

故皮帶長為  $120\sqrt{3} + \frac{20\pi}{3} + \frac{280\pi}{3}$

$$= 120\sqrt{3} + 100\pi = 522 \text{ 呎.}$$

35. 如例八可得皮帶長為

$$\frac{4\pi}{3}(1+2) + 2(3\sqrt{3}) = 4\pi + 6\sqrt{3} = 22.96 \text{ 尺.}$$

## 習題九 (66—68 頁)

1, 2, 3a, b, ……g 從略.

$$\begin{aligned} 3). \text{Covers}(-300^\circ) &= 1 - \sin(-300^\circ) = 1 + \sin 300^\circ \\ &= 1 + \sin(360^\circ - 60^\circ) = 1 - \sin 60^\circ \\ &= 1 - \frac{\sqrt{3}}{2} = \frac{1}{2}(2 - \sqrt{3}) \end{aligned}$$

4. 化簡下式

$$\begin{aligned} \text{a. 原式} &= a \cos(90^\circ - x) - b \sin x - a \sin x + b \sin x \\ &= a \sin x - a \sin x = 0 \end{aligned}$$

$$\begin{aligned} \text{b. 原式} &= (-\cos x)(-\sin x) + (-\tan y)\cos y - \tan x \\ &= \sin x \cos x - \sin y - \tan x \end{aligned}$$

$$\begin{aligned} \text{c. 原式} &= \frac{\cos x \sin x}{-\cos x} + \frac{-\sin x \sin x}{-\sin x} \\ &= -\sin x + \sin x = 0 \end{aligned}$$

$$\begin{aligned} \text{d. 原式} &= \frac{0}{-1} + \sin(720^\circ + 180^\circ) - \cot(360^\circ + 90^\circ) \\ &\quad + \cos(5 \cdot 360^\circ) \\ &= \sin 180^\circ - \cot 90^\circ + \cos 0^\circ = 0 - 0 + 1 = 1 \end{aligned}$$

5. 略.

$$6. \quad \therefore \frac{11\pi}{4} = 2\pi + \frac{3\pi}{4} \text{ 代入式中得}$$

$$\begin{aligned} \sin^2 \frac{3\pi}{4} - \cos^2 \frac{3\pi}{4} + 2 \tan \frac{3\pi}{4} - \sec^2 \frac{3\pi}{4} \\ = \frac{1}{2} - \frac{1}{2} - 2 - 2 = -4 \end{aligned}$$

$$\begin{aligned} 7. \quad \therefore \cos 134^\circ 42' &= \cos(450^\circ - 315^\circ 18') = \sin 315^\circ 18' \\ &= -70 \end{aligned}$$

$$\begin{aligned} \therefore \cot 315^\circ 18' &= -\sqrt{\csc^2 315^\circ 18' - 1} \quad (\text{因在第四象限}) \\ &= -\sqrt{1.021} = -1.01 \end{aligned}$$

$$\begin{aligned} 8. \text{ a. } \therefore A + B + C &= 180^\circ \quad \therefore A = 180 - \overline{B + C} \\ \therefore \cos A &= \cos(180^\circ - \overline{B + C}) = -\cos(B + C). \end{aligned}$$

$$\begin{aligned} \text{b. } \therefore 2A + B + C &= 180^\circ + A \\ \therefore \sin(2A + B + C) &= \sin(180^\circ + A) = -\sin A. \end{aligned}$$

$$\text{c. } \therefore A + B + C = 180^\circ \quad \therefore \frac{3A}{2} = 270^\circ - \frac{3}{2}(B + C)$$

$$\begin{aligned}\therefore \sin \frac{3A}{2} &= \sin \left( 270^\circ - \frac{3}{2}(B+C) \right) \\ &= -\cos \frac{3}{2}(B+C).\end{aligned}$$

$$d. \quad \therefore \frac{A+B}{4} = \frac{\pi-C}{4}$$

$$\begin{aligned}\therefore \tan \frac{A+B}{4} &= \tan \frac{\pi-C}{4} = \tan \left( \frac{\pi}{2} - \frac{\pi+C}{4} \right) \\ &= \cot \frac{\pi+C}{4} = -\tan \left( \frac{\pi}{2} + \frac{\pi+C}{4} \right) \\ &= -\tan \frac{3\pi+C}{4}.\end{aligned}$$

$$e. \quad \therefore \cos \left( \frac{\pi}{4} - \frac{A}{2} \right) = \sin \left( \frac{\pi}{2} - \frac{\pi}{4} - \frac{A}{2} \right) \\ = \sin \left( \frac{\pi}{4} + \frac{A}{2} \right)$$

$$\text{又因} \quad \frac{A}{2} = \frac{\pi}{2} - \frac{1}{2}(B+C)$$

$$\frac{\pi}{4} - \frac{A}{2} = -\frac{\pi}{4} + \frac{B+C}{2} = -\left( \frac{\pi}{4} - \frac{B+C}{2} \right)$$

$$\begin{aligned}\therefore \cos \left( \frac{\pi}{4} - \frac{A}{2} \right) &= \cos \left[ -\left( \frac{\pi}{4} - \frac{B+C}{2} \right) \right] \\ &= \cos \left( \frac{\pi}{4} - \frac{B+C}{2} \right)\end{aligned}$$

$$\text{故} \quad \cos \left( \frac{\pi}{4} - \frac{A}{2} \right) = \sin \left( \frac{\pi}{4} + \frac{A}{2} \right) = \cos \left( \frac{\pi}{4} - \frac{B+C}{2} \right).$$

## 習題十 (74—77 頁)

1. 在圖中設  $OA=1$ ,  $\angle AOB=x$ ,  $\angle COB=y$ .

$$\text{今 } \sin(x-y) = CF = PE - GC$$

$$\text{但 } \frac{PE}{OP} = \sin x$$

$$\therefore PE = \sin x \cdot OP = \sin x \cos y$$

$$\frac{GC}{CP} = \cos x$$

$$\therefore GC = \cos x \cdot CP = \cos x \sin y$$

$$\text{故 } \sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\text{又 } \cos(x-y) = OF = OE + PG$$

$$\text{但 } \frac{OE}{OP} = \cos x, \quad \therefore OE = \cos x \cdot OP = \cos x \cos y$$

$$\frac{PG}{CP} = \sin x, \quad \therefore PG = \sin x \cdot CP = \sin x \sin y$$

$$\text{故 } \cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$2. \quad \sin x = \sin(x-y) \cos y + \cos(x-y) \sin y \quad (1)$$

$$\cos x = \cos(x-y) \cos y - \sin(x-y) \sin y \quad (2)$$

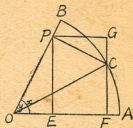
$\cos y(1) - \sin y(2)$  得

$$\sin x \cos y - \cos x \sin y = \sin(x-y) (\sin^2 y + \cos^2 y)$$

$$\therefore \sin(x-y) = \sin x \cos y - \cos x \sin y$$

$\sin y(1) + \cos y(2)$  得

$$\sin x \sin y + \cos x \cos y = \cos(x-y) (\sin^2 y + \cos^2 y)$$





$$\therefore \cos(x-y) = \sin x \sin y + \cos x \cos y$$

$$3. \quad \sin \alpha = \frac{1}{\sqrt{5}} \quad \therefore \cos \alpha = \frac{2}{\sqrt{5}}$$

( $\because$   $\alpha$  爲銳角,  $\cos \alpha$  取正號)

$$\cos \beta = \frac{1}{\sqrt{10}} \quad \therefore \sin \beta = \frac{3}{\sqrt{10}}$$

( $\because$   $\beta$  爲銳角,  $\sin \beta$  取正號)

$$\sin(\beta - \alpha) = \sin \beta \cos \alpha - \cos \beta \sin \alpha$$

$$= \frac{3}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\therefore \beta - \alpha = \frac{\pi}{4}$$

$$4. \quad \text{今} \quad \sin A = \frac{1}{2}, \quad \cos A = \frac{\sqrt{3}}{2},$$

$$\sin B = \frac{1}{4}, \quad \cos B = \frac{1}{4}\sqrt{15}.$$

$$\text{又因} \quad \cos C = \sqrt{\frac{2}{3}} \quad \therefore \sin C = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \cos D = \frac{\sqrt{2} + \sqrt{3}}{2\sqrt{3}}$$

$$\therefore \sin D = \sqrt{1 - \frac{5 + 2\sqrt{6}}{12}} = \frac{\sqrt{7 - 2\sqrt{6}}}{2\sqrt{3}} = \frac{\sqrt{6} - 1}{2\sqrt{3}}$$

$$\text{故} \quad \frac{\sin(A+B)}{\sin(C+D)} = \frac{\sin A \cos B + \cos A \sin B}{\sin C \cos D + \cos C \sin D}$$

$$= \frac{\sqrt{15} + \sqrt{3}}{8} \cdot \frac{6}{\sqrt{2} + \sqrt{3} + \sqrt{12} - \sqrt{2}}$$

$$= \frac{\sqrt{3}(\sqrt{5} + 1)}{4} \cdot \frac{3}{3\sqrt{3}} = \frac{\sqrt{5} + 1}{4}.$$

$$5. \tan(A-B) = \frac{\frac{\sqrt{3}}{4-\sqrt{3}} - \frac{\sqrt{3}}{4+\sqrt{3}}}{1 + \frac{\sqrt{3}}{4-\sqrt{3}} \cdot \frac{\sqrt{3}}{4+\sqrt{3}}} = \frac{\frac{6}{13}}{1 + \frac{3}{13}} = .375$$

$$6. \sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \sin 30^\circ \cos 45^\circ \\ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

$$\tan 615^\circ = \tan(720^\circ - 105^\circ) = -\tan 105^\circ \\ = -\tan(45^\circ + 60^\circ) \\ = -\frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ} = -\frac{1 + \sqrt{3}}{1 - \sqrt{3}} = 2 + \sqrt{3}$$

$$7. \sin(2\pi - x) = \sin 2\pi \cos x - \cos 2\pi \sin x = -\sin x \\ \cos(270^\circ - x) = \cos 270^\circ \cos x + \sin 270^\circ \sin x = -\sin x \\ \cot(\pi - x) = \frac{\cot \pi \cot x + 1}{\cot x - \cot \pi} = \frac{\cot x + \frac{1}{\cot \pi}}{\frac{\cot x}{\cot \pi} - 1} = \frac{\cot x + \frac{1}{\infty}}{\frac{\cot x}{\infty} - 1} \\ = \frac{\cot x}{-1} = -\cot x$$

$$8. \text{ a. 左邊} = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y} \\ = \frac{\sin(x+y)}{\cos x \cos y}$$

$$\text{ b. 左邊} = (\sin x \cos y + \sin y \cos x) \\ \times (\sin x \cos y - \sin y \cos x) \\ = \sin^2 x \cos^2 y - \sin^2 y \cos^2 x$$

$$= \sin^2 x (1 - \sin^2 y) - \sin^2 y \cos^2 x$$

$$= \sin^2 x - \sin^2 y (\cos^2 x + \sin^2 x)$$

$$= \sin^2 x - \sin^2 y$$

c. 左邊  $= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

$$= \cos^2 x (1 - \sin^2 y) - \sin^2 x \sin^2 y$$

$$= \cos^2 x - \sin^2 y (\cos^2 x + \sin^2 x)$$

$$= \cos^2 x - \sin^2 y$$

d. 
$$\frac{a}{\cot \alpha - \cot \beta} = \frac{a}{\frac{1}{\tan \alpha} - \frac{1}{\tan \beta}} = \frac{a}{\frac{\tan \beta - \tan \alpha}{\tan \alpha \tan \beta}}$$

$$= \frac{a \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$$

又 
$$\frac{a}{\cot \alpha - \cot \beta} = \frac{a}{\frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta}}$$

$$= \frac{a \sin \alpha \sin \beta}{\sin \beta \cos \alpha - \sin \alpha \cos \beta} = \frac{a \sin \alpha \sin \beta}{\sin(\beta - \alpha)}$$

e. 左邊  $= \sin 45^\circ \cos x + \cos 45^\circ \sin x + \cos 45^\circ \cos x$

$$- \sin 45^\circ \sin x$$

$$= \frac{\sqrt{2}}{2} (2 \cos x) = \sqrt{2} \cos x$$

f.  $\sin x - \cos x = -(\cos x - \sin x)$

$$= -\sqrt{2} \left( \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x \right)$$

$$= -\sqrt{2} (\cos x \cos 45^\circ - \sin x \sin 45^\circ)$$

$$= -\sqrt{2} \cos \left( x + \frac{\pi}{4} \right)$$

$$g. \text{ 左邊} = \frac{\tan \frac{\pi}{4} \pm \tan x}{1 \mp \tan \frac{\pi}{4} \tan x} = \frac{1 \pm \tan x}{1 \mp \tan x} \quad (\because \tan \frac{\pi}{4} = 1)$$

$$h. \text{ 左邊} = \frac{\tan \frac{9\pi}{4} + \tan \theta}{1 - \tan \frac{9\pi}{4} \tan \theta} \cdot \frac{\tan \frac{3\pi}{4} + \tan \theta}{1 - \tan \frac{3\pi}{4} \tan \theta}$$

$$\therefore \tan \frac{9\pi}{4} = \tan \left( 2\pi + \frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1$$

$$\tan \frac{3\pi}{4} = \tan \left( \pi - \frac{\pi}{4} \right) = -\tan \frac{\pi}{4} = -1$$

$$\therefore \text{上式即爲} \frac{1 + \tan \theta}{1 - \tan \theta} \cdot \frac{-1 + \tan \theta}{1 + \tan \theta} = -1$$

$$\text{或從} \tan \left[ \left( \frac{9\pi}{4} + \theta \right) - \left( \frac{3\pi}{4} + \theta \right) \right] = \tan \frac{5\pi}{4} = \infty,$$

其展開式之分母應爲 0，亦可證明。

$$\begin{aligned} i. \text{ 左邊} &= \cos A \cos B + \sin A \sin B - \sin A \cos B \\ &\quad - \cos A \sin B \\ &= \cos A (\cos B - \sin B) - \sin A (\cos B - \sin B) \\ &= (\cos A - \sin A) (\cos B - \sin B) \end{aligned}$$

$$\begin{aligned} j. \text{ 左邊} &= \frac{\sin \beta \cos \gamma - \sin \gamma \cos \beta}{\cos \beta \cos \gamma} \\ &\quad + \frac{\sin \gamma \cos \alpha - \sin \alpha \cos \gamma}{\cos \alpha \cos \gamma} \\ &\quad + \frac{\sin \alpha \cos \beta - \sin \beta \cos \alpha}{\cos \alpha \cos \beta} = \frac{\sin \beta}{\cos \beta} - \frac{\sin \gamma}{\cos \gamma} \\ &\quad + \frac{\sin \gamma}{\cos \gamma} - \frac{\sin \alpha}{\cos \alpha} + \frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta} = 0 \end{aligned}$$

$$k. \quad \therefore \tan(5A - 3A) = \frac{\tan 5A - \tan 3A}{1 + \tan 5A \tan 3A}$$

$$\therefore \text{左邊} = \tan(5A - 3A) = \tan 2A$$

$$l. \quad \therefore \sin[(x+y) - y] = \sin(x+y)\cos y \\ - \cos(x+y)\sin y$$

$$\therefore \text{左邊} = \sin[(x+y) - y] = \sin x$$

$$m. \quad \text{今} \quad \cos \alpha = \cos \left[ \left( \frac{\alpha}{2} + nx \right) + \left( \frac{\alpha}{2} - nx \right) \right] \\ = \cos \left( \frac{\alpha}{2} + nx \right) \cos \left( \frac{\alpha}{2} - nx \right) \\ - \sin \left( \frac{\alpha}{2} + nx \right) \sin \left( \frac{\alpha}{2} - nx \right)$$

$$n. \quad \text{今} \quad \cos(\alpha + \beta + \gamma) = \cos(\alpha + \beta)\cos \gamma \\ - \sin(\alpha + \beta)\sin \gamma \\ = (\cos \alpha \cos \beta - \sin \alpha \sin \beta)\cos \gamma \\ - (\sin \alpha \cos \beta + \sin \beta \cos \alpha)\sin \gamma \\ = \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \beta \cos \gamma$$

$$o. \quad \text{今} \quad \frac{\sin(\alpha - \beta + \gamma)}{\cos \alpha \cos \beta \cos \gamma} \\ = \frac{\sin(\alpha - \beta)\cos \gamma + \cos(\alpha - \beta)\sin \gamma}{\cos \alpha \cos \beta \cos \gamma} \\ = \frac{(\sin \alpha \cos \beta - \cos \alpha \sin \beta)\cos \gamma}{\cos \alpha \cos \beta \cos \gamma} \\ + \frac{(\cos \alpha \cos \beta + \sin \alpha \sin \beta)\sin \gamma}{\cos \alpha \cos \beta \cos \gamma} \\ = \tan \alpha - \tan \beta + \tan \gamma + \tan \alpha \tan \beta \tan \gamma$$

$$\begin{aligned}
 9. \quad a. \quad \cot(a+\beta+\gamma) &= \cot(\overline{a+\beta+\gamma}) \\
 &= \frac{\cot(a+\beta)\cot\gamma-1}{\cot(a+\beta)+\cot\gamma} = \frac{\frac{\cot a \cot \beta - 1}{\cot a + \cot \beta} \cdot \cot \gamma - 1}{\frac{\cot a \cot \beta - 1}{\cot a + \cot \beta} + \cot \gamma} \\
 &= \frac{\cot a \cot \beta \cot \gamma - \cot a - \cot \beta - \cot \gamma}{\cot a \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot a - 1}
 \end{aligned}$$

b. 從 (a), 以  $a+\beta+\gamma=90^\circ$  代入

$$\text{則得} \quad \cot 90^\circ = 0 = \frac{\Pi \cot a - \Sigma \cot a}{\Sigma \cot a \cot \beta - 1}$$

即  $\Pi \cot a - \Sigma \cot a = 0$ , 故  $\Sigma \cot a = \Pi \cot a$

$$10. \quad \text{今} \quad \tan \frac{A+B+C}{2} = \tan \frac{\pi}{2} = \infty$$

$$\text{但} \quad \tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) = \frac{\Pi \tan \frac{A}{2} - \Sigma \tan \frac{A}{2}}{\Sigma \tan \frac{A}{2} \tan \frac{B}{2} - 1} \quad (\text{公式 16})$$

故其分母必為 0, 即  $\Sigma \tan \frac{A}{2} \tan \frac{B}{2} = 1$

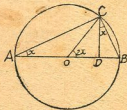
## 習題十一 (84—87 頁)

1. 今半徑為 1,  $\therefore AB=2$ ,

$$AC=2 \cos x, \quad BC=2 \sin x$$

$$\begin{aligned}
 \text{故} \quad \sin 2x &= CD = AC \sin x \\
 &= 2 \cos x \sin x
 \end{aligned}$$

$$\begin{aligned}
 \cos 2x &= OD = OB - BD \\
 &= 1 - BC \sin x
 \end{aligned}$$



$$= 1 - 2 \sin^2 x = \cos^2 x + \sin^2 x - 2 \sin^2 x \\ = \cos^2 x - \sin^2 x$$

$$2. \text{ 今 } \cos 2010^\circ = -\sqrt{1 - \frac{1}{4}} = -\sqrt{\frac{3}{4}} = -\frac{1}{2}\sqrt{3}$$

(在第三象限取負號)

$$\therefore \tan 1005^\circ = \tan \frac{1}{2}(2010^\circ) = \frac{1 - \cos 2010^\circ}{\sin 2010^\circ} \\ = \frac{1 + \frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = -2 - \sqrt{3}$$

$$3. \text{ 今 } \cos x = \frac{2}{\sqrt{5}},$$

$$\therefore \tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{1 - \frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}} = \sqrt{5} - 2$$

$$\text{又 } \cos 2x = 1 - 2 \sin^2 x = 1 - \frac{2}{5} = \frac{3}{5}$$

$$\therefore \cos 3x = 4 \cos^3 x - 3 \cos x = \frac{32}{5\sqrt{5}} - \frac{6}{\sqrt{5}} = \frac{2\sqrt{5}}{25}$$

$$\text{故 } \sin \frac{3x}{2} = \sqrt{\frac{1 - \cos 3x}{2}} = \sqrt{\frac{25 - 2\sqrt{5}}{50}} \\ = \frac{1}{10}\sqrt{50 - 4\sqrt{5}}$$

(此題亦可從  $\sin(x + \frac{x}{2})$  着手)

$$4. \text{ 今 } \tan^2(\alpha - \beta) = \frac{1 - \cos 2(\alpha - \beta)}{1 + \cos 2(\alpha - \beta)} \\ = \frac{1 - \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta}{1 + \cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta}$$

$$= \frac{\sec 2\alpha \sec 2\beta - (1 + \tan 2\alpha \tan 2\beta)}{\sec 2\alpha \sec 2\beta + (1 + \tan 2\alpha \tan 2\beta)}$$

$$\begin{aligned} \text{但 } \sec 2\alpha &= \sqrt{1 + \tan^2 2\alpha} = \sqrt{1 + \frac{4(ab+cd)^2}{(a^2-b^2+c^2-d^2)^2}} \\ &= \frac{\sqrt{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]}^*}{(a^2-d^2) - (b^2-c^2)} \end{aligned}$$

$$\sec 2\beta = \frac{\sqrt{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]}}{(a^2-d^2) + (b^2-c^2)}$$

$$\therefore \sec 2\alpha \sec 2\beta = \frac{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]}{(a^2-d^2)^2 - (b^2-c^2)^2}$$

$$\begin{aligned} 1 + \tan 2\alpha \tan 2\beta &= 1 + \frac{4(ab+cd)(ac+bd)}{(a^2-d^2)^2 - (b^2-c^2)^2} \\ &= \frac{[(a+d)^2 - (b-c)^2][(a-d)^2 + (b+c)^2]}{(a^2-d^2)^2 - (b^2-c^2)^2} \end{aligned}$$

$$\begin{aligned} \therefore \tan^2(\alpha - \beta) &= \frac{2(b-c)^2[(a-d)^2 + (b+c)^2]}{2(a+d)^2[(a-d)^2 + (b+c)^2]} \\ &= \frac{(b-c)^2}{(a+d)^2} \end{aligned}$$

$$\therefore \tan(\alpha - \beta) = \frac{b-c}{a+d} \quad (\text{此題相當之繁})$$

$$\begin{aligned} * \therefore & (a^2 - b^2 + c^2 - d^2)^2 + 4(ab+cd)^2 \\ &= (a^2 + b^2 + c^2 + d^2 - 2(b^2 + d^2))^2 + 4(ab+cd)^2 \\ &= (a^2 + b^2 + c^2 + d^2)^2 + 4(b^2 + d^2)(b^2 + d^2 - a^2 - b^2 - c^2 - d^2) \\ &\quad + 4(ab+cd)^2 \\ &= (a^2 + b^2 + c^2 + d^2)^2 + 4(ab+cd)^2 - 4(b^2 + d^2)(a^2 + c^2) \\ &= (a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2 \\ &= (a^2 + b^2 + c^2 + d^2 - 2ad + 2bc)(a^2 + b^2 + c^2 + d^2 + 2ad - 2bc) \\ &= [(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2] \end{aligned}$$



$$5. \quad \therefore \tan x = \frac{5}{12}, \quad \therefore \cos x = \frac{12}{13}, \quad \sin x = \frac{5}{13}.$$

$$\text{又} \quad \cos^2 y = \frac{1}{2}(1 + \cos 2y) = \frac{1}{2} \cdot \frac{1152}{625} = \frac{576}{625}$$

$$\therefore \cos y = \frac{24}{25}, \quad \sin y = \frac{7}{25}$$

$$\therefore \cos(x+y) = \frac{12 \cdot 24}{13 \cdot 25} - \frac{5 \cdot 7}{13 \cdot 25} = \frac{253}{325}$$

$$\begin{aligned} \text{則} \quad \sin \frac{x+y}{2} &= \sqrt{\frac{1 - \cos(x+y)}{2}} \\ &= \sqrt{\frac{1 - (253/325)}{2}} = \frac{6}{5\sqrt{13}} \end{aligned}$$

$$\text{故} \quad \csc \frac{x+y}{2} = \frac{5}{6} \sqrt{13}$$

$$\begin{aligned} 6. \quad \tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{\sin^2 \theta}{\cos \theta \sin \theta} = \frac{2(1 - \cos^2 \theta)}{2 \sin \theta \cos \theta} \\ &= \frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{b}{a} \end{aligned}$$

$$\therefore a(1 - \cos 2\theta) = b \sin 2\theta$$

$$\text{即} \quad a \cos 2\theta + b \sin 2\theta = a$$

(或從求  $\sin x$ ,  $\cos x$  着手做亦可)

$$\begin{aligned} 7. \quad \text{a. 左邊} &= \cos(2x+x) = \cos 2x \cos x - \sin 2x \sin x \\ &= (2 \cos^2 x - 1) \cos x - 2 \sin^2 x \cos x \\ &= 2 \cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x \\ &= 4 \cos^3 x - 3 \cos x \end{aligned}$$

$$b. \text{ 左邊} = \sin 2(2x) = 2 \sin 2x \cos 2x$$

$$= 4 \sin x \cos x (1 - 2 \sin^2 x)$$

$$= 4 \sin x \cos x - 8 \sin^3 x \cos x$$

$$c. \text{ 左邊} = \tan(2x+x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$= \frac{\frac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2 \tan x}{1 - \tan^2 x} \cdot \tan x} = \frac{\frac{3 \tan x - \tan^3 x}{1 - \tan^2 x}}{\frac{1 - 3 \tan^2 x}{1 - \tan^2 x}}$$

$$= \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$d. \cos 2x = \frac{\cos^2 x - \sin^2 x}{1} = \frac{(\cos^2 x - \sin^2 x) \div \cos^2 x}{(\cos^2 x + \sin^2 x) \div \cos^2 x}$$

$$= \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$e. \cos 4x = 1 - 2 \sin^2 2x = 1 - 8 \sin^2 x \cos^2 x$$

$$= 1 - 8 \cos^2 x + 8 \cos^4 x \quad (\because \sin^2 x = 1 - \cos^2 x)$$

同理可證

$$\cos 4x = 1 - 8 \sin^2 x + 8 \sin^4 x.$$

$$f. \text{ 今 } \cot \frac{x}{2} = \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} = \frac{\pm \sqrt{\frac{1 + \cos x}{2}}}{\pm \sqrt{\frac{1 - \cos x}{2}}} = \pm \sqrt{\frac{1 + \cos x}{1 - \cos x}}$$

$$= \pm \frac{\sqrt{1 - \cos^2 x}}{1 - \cos x} = \frac{\sin x}{1 - \cos x}$$

$$= \pm \frac{1 + \cos x}{\sqrt{1 - \cos^2 x}} = \frac{1 + \cos x}{\sin x}$$

$$\begin{aligned}
 \text{g. 左邊} &= \frac{2 \sin \frac{1}{4} x \cos \frac{1}{4} x}{1} = \frac{2 \sin \frac{1}{4} x \cos \frac{1}{4} x}{\cos^2 \frac{1}{4} x + \sin^2 \frac{1}{4} x} \\
 &= \frac{2 \sin \frac{1}{4} x}{\cos \frac{1}{4} x} = \frac{2 \tan \frac{1}{4} x}{1 + \tan^2 \frac{1}{4} x}
 \end{aligned}$$

$$\begin{aligned}
 \text{h. } \sin \frac{1}{2} x \pm \cos \frac{1}{2} x &= \sqrt{(\sin \frac{1}{2} x \pm \cos \frac{1}{2} x)^2} \\
 &= \sqrt{\sin^2 \frac{1}{2} x + \cos^2 \frac{1}{2} x \pm 2 \sin \frac{1}{2} x \cos \frac{1}{2} x} \\
 &= \sqrt{1 \pm \sin x}
 \end{aligned}$$

$$\begin{aligned}
 \text{i. 左邊} &= \sin \frac{1}{2} x + \cos \frac{1}{2} x \\
 &= 2 \sin \frac{1}{4} x \cos \frac{1}{4} x + 1 - 2 \sin^2 \frac{1}{4} x \\
 &= 1 + 2 \sin \frac{1}{4} x (\cos \frac{1}{4} x - \sin \frac{1}{4} x) \\
 &= 1 + 2 \sin \frac{1}{4} x \sqrt{(\cos \frac{1}{4} x - \sin \frac{1}{4} x)^2} \\
 &= 1 + 2 \sin \frac{1}{4} x \sqrt{\cos^2 \frac{1}{4} x + \sin^2 \frac{1}{4} x - 2 \sin \frac{1}{4} x \cos \frac{1}{4} x} \\
 &= 1 + 2 \sin \frac{1}{4} x \sqrt{1 - \sin \frac{1}{2} x}
 \end{aligned}$$

∴ 左邊 = 右邊

$$\begin{aligned}
 \text{j. 左邊} &= 3 \sin x - (3 \sin x - 4 \sin^3 x) \\
 &= 4 \sin^3 x = 2 \sin x (2 \sin^2 x) \\
 &= 2 \sin x (1 - \cos 2x)
 \end{aligned}$$

$$\begin{aligned}
 \text{k. 左邊} &= \frac{1}{\sin 2x} + \frac{\cos 4x}{\sin 4x} = \frac{2 \cos 2x + 2 \cos^2 2x - 1}{2 \sin 2x \cos 2x} \\
 &= \frac{2 \cos 2x (1 + \cos 2x)}{2 \sin 2x \cos 2x} - \frac{1}{\sin 4x} \\
 &= \frac{1 + \cos 2x}{\sin 2x} - \csc 4x = \frac{2 \cos^2 x}{2 \sin x \cos x} - \csc 4x \\
 &= \cot x - \csc 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{l. 右邊} &= 2 \cdot \frac{2 + (1 + \cos 4x)}{2 \sin^2 2x} = 2 \cdot \frac{2 + 2 \cos^2 2x}{8 \sin^2 x \cos^2 x} \\
 &= \frac{1 + \cos^2 2x}{2 \sin^2 x \cos^2 x} \\
 &= \frac{(\cos^2 x + \sin^2 x)^2 + (\cos^2 x - \sin^2 x)^2}{2 \sin^2 x \cos^2 x} \\
 &= \frac{2(\cos^4 x + \sin^4 x)}{2 \sin^2 x \cos^2 x} = \frac{\cos^4 x + \sin^4 x}{\sin^2 x \cos^2 x} \\
 &= \tan^2 x + \cot^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{m. 左邊} &= \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\
 &= \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{n. } \cos x + \sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} + 2 \cos^2 \frac{x}{2} - 1 \\
 &= 2 \cos \frac{x}{2} \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right) - 1 \\
 &\quad \left( \text{同理化 } \sin \frac{x}{2} + \cos \frac{x}{2} \right) \\
 &= 2 \cos \frac{x}{2} \left[ 2 \cos \frac{x}{4} \left( \sin \frac{x}{4} + \cos \frac{x}{4} \right) - 1 \right] - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{8. a. 今左邊} &= 8 \sin \frac{\theta}{8} \cos \frac{\theta}{8} \cos \frac{\theta}{4} \cos \frac{\theta}{2} \\
 &= 4 \sin \frac{\theta}{4} \cos \frac{\theta}{4} \cos \frac{\theta}{2} = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \theta
 \end{aligned}$$

$$b. \because 2 \cos^2 x = 1 + \cos 2x \quad \therefore 4 \cos^2 x = 2 + 2 \cos 2x$$

$$\text{則 } 2 \cos x = \sqrt{2 + 2 \cos 2x}$$

$$\text{同理推之 } 2 \cos 2x = \sqrt{2 + 2 \cos 4x}$$

$$\text{代入則 } 2 \cos x = \sqrt{2 + \sqrt{2 + 2 \cos 4x}}$$

$$c. x = 11\frac{1}{4}^\circ \quad \text{則 } 4x = 45^\circ \quad \text{代入上式即得}$$

$$d. \because 2 \cos \theta - 1 = \frac{4 \cos^2 \theta - 1}{2 \cos \theta + 1} = \frac{2 \cos 2\theta + 1}{2 \cos \theta + 1}$$

$$\text{同理 } 2 \cos 2\theta - 1 = \frac{2 \cos 4\theta + 1}{2 \cos 2\theta + 1},$$

$$2 \cos 4\theta - 1 = \frac{2 \cos 8\theta + 1}{2 \cos 4\theta + 1}$$

$$\text{故 } (2 \cos \theta - 1)(2 \cos 2\theta - 1)(2 \cos 4\theta - 1)$$

$$= \frac{2 \cos 2\theta + 1}{2 \cos \theta + 1} \cdot \frac{2 \cos 4\theta + 1}{2 \cos 2\theta + 1} \cdot \frac{2 \cos 8\theta + 1}{2 \cos 4\theta + 1}$$

$$= \frac{2 \cos 8\theta + 1}{2 \cos \theta + 1}$$

$$e. 1 + \sec \theta = \frac{1 + \cos \theta}{\cos \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{\cos \theta} \cdot \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= \frac{\sin \theta \cdot \cos \frac{\theta}{2}}{\cos \theta \cdot \sin \frac{\theta}{2}} = \tan \theta \cot \frac{\theta}{2}$$

$$\text{同理 } 1 + \sec \frac{\theta}{2} = \tan \frac{\theta}{2} \cot \frac{\theta}{4},$$

$$1 + \sec \frac{\theta}{4} = \tan \frac{\theta}{4} \cot \frac{\theta}{8}.$$

相乘即得 左邊 =  $\tan \theta \cot \frac{\theta}{8}$  = 右邊.

$$\begin{aligned} \text{f. } \therefore \tan x - \cot x &= \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = -\frac{\cos^2 x - \sin^2 x}{\sin x \cos x} \\ &= -2 \cot 2x \end{aligned}$$

$$2 \tan 2x - 2 \cot 2x = -4 \cot 4x$$

$$4 \tan 4x - 4 \cot 4x = -8 \cot 8x$$

相加得

$$\tan x + 2 \tan 2x + 4 \tan 4x = \cot x - 8 \cot 8x$$

$$\text{g. 設 } x = 11^\circ 15', \text{ 則 } 2x = 22^\circ 30', \quad 4x = 45^\circ.$$

$$\therefore \cot 8x = \cot 90^\circ = 0$$

故從上題即可證得 左邊 = 右邊

$$\begin{aligned} \text{9. a. 原式} &= [\cos x \cos \alpha - \cos(\alpha + x)]^2 + \cos^2 x (1 - \cos^2 \alpha) \\ &= [\cos x \cos \alpha - (\cos x \cos \alpha - \sin x \sin \alpha)]^2 \\ &\quad + \cos^2 x \sin^2 \alpha = \sin^2 \alpha (\cos^2 x + \sin^2 x) = \sin^2 \alpha \end{aligned}$$

$$\begin{aligned} \text{b. 原式} &= \frac{1 + \cos 2x}{2} + \frac{1 + \cos(240^\circ + 2x)}{2} \\ &\quad + \frac{1 + \cos(240^\circ - 2x)}{2} \\ &= \frac{3 + \cos 2x + 2 \cos 240^\circ \cos 2x}{2} \end{aligned}$$

(將餘弦複角函數展開合併)

$$= \frac{3 + \cos 2x - 2 \cos 60^\circ \cos 2x}{2}$$

$$= \frac{3 + \cos 2x - \cos 2x}{2} = \frac{3}{2}$$

$$\begin{aligned}
 10. \quad a. \quad \tan \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \\
 &= \pm \sqrt{\frac{1 - [(\cos u - e)/(1 - e \cos u)]}{1 + [(\cos u - e)/(1 - e \cos u)]}} \\
 &= \pm \sqrt{\frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)}} = \pm \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \\
 &\quad \left( \because \tan \frac{u}{2} = \pm \sqrt{\frac{1-\cos u}{1+\cos u}} \right)
 \end{aligned}$$

b. 在上題中以  $\cos \alpha$  換  $\cos u$ , 又以  $\cos \beta$  換  $e$ , 即可推得本題之結果。

## 習題十二 (91—94 頁)

1. a 至 k 各題均可用公式 (28—35) 一次或二次直接推得, 解法從略。

$$\begin{aligned}
 2. \quad a. \quad \sin 3x &= (\sin 3x - \sin x) + \sin x \\
 &= 2 \cos 2x \sin x + \sin x = 2 \sin x (\cos 2x + \frac{1}{2}) \\
 &= 2 \sin x (\cos 2x - \cos 120^\circ) \\
 &= 2 \sin x [2 \sin(x + 60^\circ) \sin(60^\circ - x)] \\
 &= 4 \sin x \sin(60^\circ + x) \sin(60^\circ - x)
 \end{aligned}$$

$$\begin{aligned}
 b. \quad \text{右邊} &= 2 \cos x (\cos 120^\circ + \cos 2x) \\
 &= -\cos x + 2 \cos x \cos 2x \\
 &= -\cos x + (\cos 3x + \cos x) = \cos 3x
 \end{aligned}$$

$$\begin{aligned}
 c. \quad \text{左邊} &= -2[\cos(\theta - 2\alpha + m\alpha) - \cos(\theta - m\alpha)] \\
 &\quad \times \cos(\theta - m\alpha)
 \end{aligned}$$

$$\begin{aligned}
 &= -2[\cos(\theta - 2a + ma)\cos(\theta - ma) \\
 &\quad - \cos^2(\theta - ma)] \\
 &= -\cos(2\theta - 2a) - \cos(2ma - 2a) \\
 &\quad + 2\cos^2(\theta - ma) \\
 &= 1 + \cos(2\theta - 2ma) - \cos(2\theta - 2a) \\
 &\quad - \cos(2ma - 2a)
 \end{aligned}$$

d. 左邊 =  $\frac{\cos A + \cos 11A}{\cos A - \cos 11A}$  (分子, 分母各用公式)  
 (30, 31 化出合併)

$$= \frac{\cos 6A \cos 5A}{\sin 6A \sin 5A} = \cot 6A \cot 5A$$

e. 左邊 =  $\frac{2\cos(2\theta - A)\cos A + 2\cos(2\theta - B - C)\cos(B - C)}{2\sin(2\theta - A)\cos A + 2\cos(2\theta - B - C)\sin(B - C)}$

$$= \frac{\cos(B + C) + \cos(B - C)}{\sin(B + C) + \sin(B - C)}$$

$$= \frac{2\cos B \cos C}{2\sin B \cos C} = \cot B$$

f. 左邊 =  $\frac{2\sin nA \cos(A - n) \pm \sin nA}{2\cos nA \cos(A - n) \pm \sin nA}$

$$= \frac{\sin nA [2\cos(A - n) \pm 1]}{\cos nA [2\cos(A - n) \pm 1]} = \tan nA$$

g. 左邊 =  $\cos x + 2\cos 120^\circ \cos x$

$$= \cos x + 2\left(-\frac{1}{2}\right)\cos x = 0$$

h. 左邊 =  $2\sin 30^\circ \cos 20^\circ + 2\sin 30^\circ \cos 10^\circ$

$$= 2\sin 30^\circ (\cos 20^\circ + \cos 10^\circ)$$

$$= \cos 20^\circ + \cos 10^\circ$$



$$= \cos(90^\circ - 70^\circ) + \cos(90^\circ - 80^\circ)$$

$$= \sin 70^\circ + \sin 80^\circ$$

$$i. \text{ 左邊} = \frac{2 \sin 2A \sin A + 2 \sin 4A \sin A + 2 \sin 6A \sin A}{2 \sin A}$$

$$= \frac{\cos A - \cos 3A + \cos 3A - \cos 5A + \cos 5A - \cos 7A}{2 \sin A}$$

$$= \frac{\cos A - \cos 7A}{2 \sin A}$$

$$j. \text{ 左邊} = \frac{\cos 2na + \cos 2a}{2} + \sin^2 a$$

$$= \frac{(2 \cos^2 na - 1) + (1 - 2 \sin^2 a)}{2} + \sin^2 a$$

$$= \cos^2 na$$

$$k. \text{ 左邊} = (\sin 5x + \sin x) - \sin x$$

$$= 2 \sin 3x \cos 2x - \sin x$$

$$l. \quad \therefore \cos 5x - \cos x = -2 \sin 3x \sin 2x$$

$$\therefore \cos 5x = \cos x - 2 \sin 3x \sin 2x$$

$$m. \text{ 左邊} = \cos(x-y) - \cos\left[\frac{\pi}{2} + (x+y)\right]$$

$$= 2 \sin\left(\frac{\pi}{4} + y\right) \sin\left(\frac{\pi}{4} + x\right)$$

$$n. \text{ 左邊} = \cos\left[\frac{\pi}{2} - (x+y)\right] - \cos(x-y)$$

$$= 2 \sin\left(\frac{\pi}{2} - y\right) \sin\left(x - \frac{\pi}{4}\right)$$

$$= -2 \sin\left(x - \frac{\pi}{4}\right) \sin\left(y - \frac{\pi}{4}\right)$$

$$\begin{aligned}
 \text{o. 左邊} &= \cos\left(\frac{\pi}{2} - \alpha\right) - \cos \beta \\
 &= 2 \sin\left[\frac{\pi}{4} - \frac{1}{2}(\alpha - \beta)\right] \sin\left[\frac{1}{2}(\alpha + \beta) - \frac{\pi}{4}\right] \\
 \text{又} &= -\left[\cos\left(\frac{\pi}{2} + \alpha\right) + \cos \beta\right] \\
 &= -2 \cos\left[\frac{\pi}{4} + \frac{1}{2}(\alpha + \beta)\right] \cos\left[\frac{\pi}{4} + \frac{1}{2}(\alpha - \beta)\right]
 \end{aligned}$$

$$\begin{aligned}
 \text{p. 左邊} &= \frac{1}{2}(\cos 2\beta - \cos 2\alpha) \\
 &= \frac{1}{2}(1 - 2 \sin^2 \beta - (1 - 2 \sin^2 \alpha)) = \sin^2 \alpha - \sin^2 \beta
 \end{aligned}$$

$$\begin{aligned}
 \text{q. 左邊} &= \frac{1}{2}[\cos(2n+2)A + \cos 2nA] \\
 &= \frac{1}{2}(2 \cos^2(n+1)A - 1 + 1 - 2 \sin^2 nA) \\
 &= \cos^2(n+1)A - \sin^2 nA
 \end{aligned}$$

$$\begin{aligned}
 \text{r. 左邊} &= \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)] \\
 &= \frac{1}{2}(2 \cos^2 \frac{1}{2}(\alpha + \beta) + 2 \cos^2 \frac{1}{2}(\alpha - \beta) - 2) \\
 &= \text{右邊}
 \end{aligned}$$

$$\begin{aligned}
 \text{s. 設} \quad 2\alpha - \beta &= A, \quad 2\beta - \alpha = B, \\
 \text{則} \quad A + B &= \alpha + \beta, \quad A - B = \alpha - 3\beta, \quad \text{其餘如 } 2(p).
 \end{aligned}$$

$$\begin{aligned}
 \text{t. 左邊} &= 2 \cos(x+y) \cos z + 2 \cos z \cos(x-y) \\
 &= 2 \cos z [\cos(x+y) + \cos(x-y)] = 4 \Pi \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{u. } \therefore \cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma) \\
 = 4 \Pi \cos \frac{1}{2}(\alpha + \beta), \quad (\text{見上題})
 \end{aligned}$$

$$\text{故 } \Sigma \cos \alpha = 4 \Pi \cos \frac{1}{2}(\alpha + \beta) - \cos(\alpha + \beta + \gamma)$$

$$\text{3. a. 原式} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \sin \beta} + \dots$$

$$= \cot \beta - \cot \alpha + \cot \gamma - \cot \beta + \cot \alpha - \cot \gamma = 0$$

$$\begin{aligned} \text{b. 左邊} &= \frac{1}{2}[\sin(\alpha + \beta - \gamma - \delta) - \sin(\alpha + \gamma - \beta - \delta) \\ &\quad + \dots] = \frac{1}{2} \cdot 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{c. 原式} &= \frac{\sin \beta \sin(\alpha + \beta)}{\cos \beta [2 \sin^2 \frac{1}{2}(\alpha + \beta)]} + \frac{\sin \frac{1}{2}(\alpha - \beta)}{\cos \beta \sin \frac{1}{2}(\alpha + \beta)} \\ &= \frac{\sin \beta \cos \frac{1}{2}(\alpha + \beta)}{\cos \beta \sin \frac{1}{2}(\alpha + \beta)} + \frac{\sin[\frac{1}{2}(\alpha + \beta) - \beta]}{\cos \beta \sin \frac{1}{2}(\alpha + \beta)} \\ &= \frac{\cos \beta \sin \frac{1}{2}(\alpha + \beta)}{\cos \beta \sin \frac{1}{2}(\alpha + \beta)} = 1 \end{aligned}$$

### 習題十三 (127—138 頁)

$$\begin{aligned} \text{1. 左邊} &= 2 \sin^2 \alpha \cdot 2 \sin \alpha \cos 3\alpha + 2 \cos^2 \alpha \cdot 2 \cos \alpha \sin 3\alpha \\ &= 2 \sin^2 \alpha (\sin 4\alpha - \sin 2\alpha) + 2 \cos^2 \alpha (\sin 4\alpha + \sin 2\alpha) \\ &= 2 \sin 4\alpha (\sin^2 \alpha + \cos^2 \alpha) + 2 \sin 2\alpha (\cos^2 \alpha - \sin^2 \alpha) \\ &= 2 \sin 4\alpha + 2 \sin 2\alpha \cos 2\alpha = 3 \sin 4\alpha \end{aligned}$$

$$\begin{aligned} \text{2. 左邊} &= \frac{1}{\tan 3\theta + \tan \theta} - \frac{\tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} = \frac{1 - \tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} \\ &= \frac{1}{\tan(3\theta + \theta)} = \frac{1}{\tan 4\theta} = \cot 4\theta \end{aligned}$$

$$\begin{aligned} \text{3. 左邊} &= \frac{\sin \theta (\cos \theta - \sin \phi)}{\cos^2 \theta - \sin^2 \phi} + \frac{\sin \phi (\cos \phi + \sin \theta)}{\cos^2 \phi - \sin^2 \theta} \\ \cos^2 \theta - \sin^2 \phi &= 1 - \sin^2 \theta - (1 - \cos^2 \phi) = \cos^2 \phi - \sin^2 \theta \end{aligned}$$

因上式分母同，故可改變分子之排列。

$$\therefore \text{左邊} = \frac{\sin \theta (\cos \theta + \sin \phi)}{\cos^2 \theta - \sin^2 \phi} + \frac{\sin \phi (\cos \phi - \sin \theta)}{\cos^2 \phi - \sin^2 \theta}$$

$$= \frac{\sin \theta}{\cos \theta - \sin \phi} + \frac{\sin \phi}{\cos \phi + \sin \theta}$$

$$4. \text{ 左邊} = \left( \frac{1 - \tan^2 \frac{\theta}{2}}{\tan \frac{\theta}{2}} \right)^2 = 4 \left( \frac{1 - \tan^2 \frac{\theta}{2}}{2 \tan \frac{\theta}{2}} \right)^2$$

$$= 4 \left( \frac{1}{\tan \theta} \right)^2 = \frac{4}{\tan^2 \theta}$$

$$\text{右邊} = 4 / \left( 1 - 2 \tan \theta \times \frac{1 - \tan^2 \theta}{2 \tan \theta} \right) = \frac{4}{\tan^2 \theta}$$

$$5. \text{ 左邊} = \tan(2x+x)\tan(2x-x)$$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \cdot \frac{\tan 2x - \tan x}{1 + \tan 2x \tan x}$$

$$= \frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x}$$

$$6. \cos 2x = \frac{\cos x \cos 2x}{\cos(2x-x)} = \frac{\cos x \cos 2x}{\cos x \cos 2x + \sin x \sin 2x}$$

$$= \frac{1}{1 + \tan x \tan 2x} \quad (\text{分子, 分母同以 } \cos x \cos 2x \text{ 除之})$$

$$7. \text{ 左邊} = \frac{\sin 3x \sin x}{\sin x} = \frac{\cos 2x - \cos 4x}{2 \sin x}$$

$$= \frac{1 - 2 \sin^2 x - (1 - 2 \sin^2 2x)}{2 \sin x} = \frac{\sin^2 2x - \sin^2 x}{\sin x}$$

$$8. \text{ 左邊} = \left( \frac{1}{\cos x} + \frac{\cos x + 1}{\sin x \cos x} \right) \cdot \frac{\cos x}{\cos^2 \frac{x}{2}} \cdot \frac{\cos \frac{x}{2}}{\cos^2 \frac{x}{4}}$$

$$= \frac{\sin x + (\cos x + 1)}{\sin x \cos \frac{x}{2}} \cdot \frac{1}{\cos^2 \frac{x}{4}}$$

$$\begin{aligned}
 &= \frac{2 \cos \frac{x}{2} \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)}{2 \sin \frac{x}{2} \cos^2 \frac{x}{2}} \cdot \frac{1}{\cos^2 \frac{x}{4}} \\
 &= \frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}} \cdot \sec^2 \frac{x}{4} = \left( \sec \frac{x}{2} + \csc \frac{x}{2} \right) \sec^2 \frac{x}{4}
 \end{aligned}$$

$$\begin{aligned}
 9. \text{ 左邊} &= \left( \frac{\sin^2 3A}{\cos^2 3A \sin^2 A} - \frac{1}{\cos^2 3A} \right) \cdot \frac{\cos 4A}{\sin 4A} \\
 &= \frac{(\sin 3A - \sin A)(\sin 3A + \sin A)}{\cos^2 3A \sin^2 A} \cdot \frac{\cos 4A}{2 \sin 2A \cos 2A} \\
 &= \frac{2 \sin A \cos A \cos 4A}{\cos^2 3A \sin^2 A} = \frac{\cos 2A \cos 4A}{\cos^2 3A \sin^2 A} \cdot \frac{\sin 2A}{\cos 2A} \\
 &= \frac{\cos 6A + \cos 2A}{2 \cos^2 3A \sin^2 A} \tan 2A = \frac{\cos^2 3A - \sin^2 A}{\cos^2 3A \sin^2 A} \tan 2A \\
 &= (\csc^2 A - \sec^2 3A) \tan 2A \\
 &= \left( \csc^2 \frac{x}{6} - \sec^2 \frac{x}{2} \right) \tan \frac{x}{3}
 \end{aligned}$$

$$\begin{aligned}
 10. \text{ 左邊} &= \sqrt{(1 - \cos \alpha)(1 - \cos \beta)} = 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \\
 &= \cos \frac{1}{2}(\alpha - \beta) - \cos \frac{1}{2}(\alpha + \beta) \\
 &= [1 - \cos \frac{1}{2}(\alpha + \beta)] - [1 - \cos \frac{1}{2}(\alpha - \beta)] \\
 &= \text{vers} \frac{1}{2}(\alpha + \beta) - \text{vers} \frac{1}{2}(\alpha - \beta)
 \end{aligned}$$

$$\begin{aligned}
 11. \therefore \cos 5\alpha + \cos \alpha &= 2 \cos 3\alpha \cos 2\alpha \\
 &= 2(4 \cos^3 \alpha - 3 \cos \alpha)(2 \cos^2 \alpha - 1)
 \end{aligned}$$

$$\therefore \cos 5\alpha = 16 \cos^5 \alpha - 20 \cos^3 \alpha + 5 \cos \alpha$$

$$12. \therefore 4 \cos^3 x = \cos 3x + 3 \cos x, \quad 2 \cos^2 x = \cos 2x + 1$$

$$\therefore 16 \cos^5 x = 2 \cos 3x \cos 2x + 6 \cos 2x \cos x$$

$$+ 2 \cos 3x + 6 \cos x$$

$$= \cos 5x + 5 \cos 3x + 10 \cos x$$

$$13. \text{左邊} = 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ = \cos 20^\circ - \cos 20^\circ = 0$$

$$14. \text{左邊} = \frac{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)} + \frac{\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$= \frac{\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right) + \cos^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$$= \frac{2}{\sin\left(\frac{\pi}{2} + x\right)} = \frac{2}{\cos x}$$

$$15. \text{右邊} = \frac{2(\sin 2x + \cos 2x)}{2(\cos 2x \sin x + \sin 2x \sin x)}$$

$$= \frac{\sin 2x + \cos 2x}{\sin x(\sin 2x + \cos 2x)} = \frac{1}{\sin x} = \csc x$$

$$16. \text{左邊} = \frac{1}{\cos x} (\sin 8x + \sin 6x + 8 \sin 2x \cos^2 x)$$

$$= \frac{\sin 6x + 4 \sin 2x \cos 2x (2 \cos^2 2x - 1) + 8 \sin 2x \cos^2 x}{\cos x}$$

$$= \frac{1}{\cos x} [\sin 6x + 8 \sin 2x \cos^3 2x - 4 \sin 2x (2 \cos^2 x - 1 - 2 \cos^2 x)]$$

$$= \frac{\sin 6x + 4 \sin 2x (1 + 2 \cos^3 2x)}{\cos x}$$

17. 左邊 =  $\csc^2 x \sec^2 x (\sin nx \cos x - \cos nx \sin x)$

$$= \frac{4}{(2 \sin x \cos x)^2} \cdot \sin (nx - x) = \frac{4}{\sin^2 2x} \cdot \sin (n-1)x$$

$$= 4 \sin (n-1)x \cdot \csc^2 2x$$

18. 左邊 =  $\frac{\sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)} = \frac{\cos \beta - \cos \alpha}{\cos \alpha + \cos \beta}$

$$= \frac{2 \sin \alpha \sin \beta (\cos \beta - \cos \alpha)}{2 \sin \alpha \sin \beta (\cos \alpha + \cos \beta)}$$

$$= \frac{\sin 2\beta \sin \alpha - \sin 2\alpha \sin \beta}{\sin 2\beta \sin \alpha + \sin 2\alpha \sin \beta}$$

$$= \frac{\csc 2\alpha \csc \beta - \csc 2\beta \csc \alpha}{\csc 2\alpha \csc \beta + \csc 2\beta \csc \alpha}$$

(分子分母同以  $\sin \alpha \sin \beta \sin 2\alpha \sin 2\beta$  除之)

19. 左邊 = ..... (乘出)

$$= x^2 [\cos 2\alpha \cos 2\beta - \cos^2(\alpha + \beta)]$$

$$+ y^2 [\sin 2\alpha \sin 2\beta - \sin^2(\alpha + \beta)] + \sin^2(\alpha - \beta)$$

$$= -x^2 \sin^2(\alpha - \beta) - y^2 \sin^2(\alpha - \beta) + \sin^2(\alpha - \beta)$$

(參考習題十, 8. b, c.)

$$= (1 - x^2 - y^2) \sin^2(\alpha - \beta)$$

或從  $x^2$  之係數  $\frac{1}{2} \cos 2(\alpha + \beta) + \frac{1}{2} \cos 2(\alpha - \beta)$

$$- \cos^2(\alpha + \beta) = \dots = -\sin^2(\alpha - \beta) \text{ 做}$$

20. 設  $2^{2n-2}\alpha = A$  則  $2^{2n-1}\alpha = 2A$ ,  $2^{2n}\alpha = 4A$ .

故本題可變爲  $\frac{\sec 4A - 1}{\sec 2A - 1} = \frac{\tan 4A}{\tan A}$  便易做矣

註：例八(99頁)亦以用上法令  $2^{2n-1} = B$  代入做爲簡便。

$$21. \text{ 今 } (\cos x + i \sin x)^7 = \cos^7 x + 7i \cos^6 x \sin x \\ - 21 \cos^5 x \sin^2 x \dots\dots$$

$$\text{但 } (\cos x + i \sin x)^7 = \cos 7x + i \sin 7x$$

比較含  $i$  之項，則得

$$\begin{aligned} \sin 7x &= 7 \cos^6 x \sin x - 35 \cos^4 x \sin^3 x \\ &\quad + 21 \cos^2 x \sin^5 x - \sin^7 x \\ &= 7(1 - \sin^2 x)^3 \sin x - 35(1 - \sin^2 x)^2 \sin^3 x \\ &\quad + 21(1 - \sin^2 x) \sin^5 x - \sin^7 x \\ &= 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x \end{aligned}$$

$$22. \text{ 左邊} = 2 \sin 3x \cos 3x$$

$$\begin{aligned} &= 2(3 \sin x - 4 \sin^3 x)(4 \cos^3 x - 3 \cos x) \\ &= 2 \sin x (3 - 4(1 - \cos^2 x))(4 \cos^3 x - 3 \cos x) \\ &= 2 \sin x (16 \cos^5 x - 16 \cos^3 x + 3 \cos x) \end{aligned}$$

$$23. \text{ 設 } y = \cos x + i \sin x, \quad \text{則 } y^{-1} = \cos x - i \sin x$$

$$\therefore y - y^{-1} = 2i \sin x, \quad \text{又 } y^n + y^{-n} = 2 \cos nx$$

$$\text{今 } (y - y^{-1})^6 = -64 \sin^6 x = (y^6 + y^{-6}) - 6(y^4 + y^{-4}) \\ + 15(y^2 + y^{-2}) - 20$$

$$\therefore -32 \sin^6 x = \cos 6x - 6 \cos 4x + 15 \cos 2x - 10.$$

$$24. \text{ 如上題用棣美弗定理做，從 } (4 \cos^3 x)^2 \cos^2 x \text{ 做亦可。}$$

$$25. \text{ 左邊} = \sin(4x + x) + \cos(4x + x)$$



$$\begin{aligned}
 &= \cos 4x(\sin x + \cos x) + \sin 4x(\cos x - \sin x) \\
 &= (\sin x + \cos x)(\cos 4x + 2 \sin 2x(\cos x - \sin x)^2) \\
 &= (\sin x + \cos x)(\cos 4x + 2 \sin 2x(1 - \sin 2x)) \\
 &= (\sin x + \cos x)(\cos 4x + 2 \sin 2x - 2 \sin^2 2x) \\
 &= (\sin x + \cos x)(\cos 4x + 2 \sin 2x - (1 - \cos 4x)) \\
 &= \text{左邊}
 \end{aligned}$$

$$\begin{aligned}
 26. \text{ 左邊} &= \{ \sin \beta (\sin(\alpha + 2\beta) \sin(\alpha + 3\beta) + \sin \alpha \sin(\alpha + 3\beta) \\
 &\quad + \sin \alpha \sin(\alpha + \beta)) \} \div [ \sin(\alpha + \beta) \sin(\alpha + 2\beta) \\
 &\quad \cdot \sin(\alpha + 3\beta) ]
 \end{aligned}$$

$$\begin{aligned}
 \text{被除數} &= \frac{1}{2} \sin \beta (\cos \beta - \cos(2\alpha + 5\beta) + \cos 3\beta \\
 &\quad - \cos(2\alpha + 3\beta) + \cos \beta - \cos(2\alpha + \beta)) \\
 &= \frac{1}{2} \sin \beta \sin(\alpha + \beta) [2 \sin(\alpha + 2\beta) \\
 &\quad + 2 \sin(\alpha + 4\beta) + 2 \sin \alpha] \\
 &= \frac{1}{2} \sin(\alpha + \beta) [\cos(\alpha - \beta) - \cos(\alpha + 5\beta)] \\
 &= \sin(\alpha + \beta) \sin(\alpha + 2\beta) \sin 3\beta
 \end{aligned}$$

$$\text{故原式左邊} = \frac{\sin 3\beta}{\sin(\alpha + 3\beta)} = \sin 3\beta \csc(\alpha + 3\beta)$$

$$\begin{aligned}
 27. \text{ 右邊} &= \frac{\sec^2 \frac{1}{2} \theta}{a \sec^2 \frac{1}{2} \theta + b(1 - \tan^2 \frac{1}{2} \theta)} \\
 &= \frac{\sec^2 \frac{1}{2} \theta}{a \sec^2 \frac{1}{2} \theta + b(\cos^2 \frac{1}{2} \theta - \sin^2 \frac{1}{2} \theta) \sec^2 \frac{1}{2} \theta} = \frac{1}{a + b \cos \theta} \\
 \therefore \text{ 左邊} &= \text{右邊}
 \end{aligned}$$

$$\begin{aligned}
 28. \text{ 左邊} &= \frac{1}{4} \cos 20^\circ \cos 40^\circ \cos 80^\circ \\
 &= \frac{1}{4} (\cos 60^\circ + \cos 20^\circ) \cos 80^\circ \\
 &= \frac{1}{4} (\cos 60^\circ \cos 80^\circ + \cos 20^\circ \cos 80^\circ)
 \end{aligned}$$



$$= \frac{1}{8} (\cos 80^\circ + \cos 100^\circ + \cos 60^\circ)$$

$$= \frac{1}{8} (\cos 80^\circ - \cos 80^\circ + \cos 60^\circ) = 1/16$$

29.  $\therefore \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = 3/16$  (例十三)

又  $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1/16$  (28 題)

$\therefore \tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ = 3$

30. 左邊  $= (\tan 9^\circ + \tan 81^\circ) - (\tan 27^\circ + \tan 63^\circ)$

$$= \frac{\sin 90^\circ}{\cos 9^\circ \cos 81^\circ} - \frac{\sin 90^\circ}{\cos 27^\circ \cos 63^\circ}$$

$$= \frac{2}{\cos 72^\circ} - \frac{2}{\cos 36^\circ} = 2 \left( \frac{\cos 36^\circ - \cos 72^\circ}{\cos 72^\circ \cos 36^\circ} \right)$$

$$= 2 \left( \frac{2 \sin 18^\circ \sin 54^\circ}{\sin 18^\circ \sin 54^\circ} \right) = 4$$

31. 左邊  $= \frac{\cos 10^\circ - \sqrt{3} \sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$$= \frac{4(\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ)}{\sin 20^\circ}$$

$$= \frac{4 \sin 20^\circ}{\sin 20^\circ} = 4$$

32. 左邊  $= \frac{1}{2} [\cos 120^\circ + \cos 40^\circ + \cos 240^\circ + \cos 80^\circ$

$$+ \cos 200^\circ + \cos 120^\circ]$$

$$= \cos 120^\circ + \frac{1}{2} \left( -\frac{1}{2} \right) + \frac{1}{2} \cos 200^\circ + \cos 60^\circ \cos 20^\circ$$

$$= -\frac{1}{2} - \frac{1}{4} - \frac{1}{2} \cos 20^\circ + \frac{1}{2} \cos 20^\circ = -\frac{3}{4}$$

33.  $\therefore \sqrt{3} = \tan 60^\circ = \frac{\tan 20^\circ + \tan 40^\circ}{1 - \tan 20^\circ \tan 40^\circ}$  去分母移項即得

$$\begin{aligned} 34. \quad \cos 9^\circ + \sin 9^\circ &= \sqrt{1 + \sin 18^\circ} = \sqrt{1 + \frac{1}{4}(\sqrt{5} - 1)} \\ &= \frac{1}{2}\sqrt{3 + \sqrt{5}} \end{aligned}$$

$$\cos 9^\circ - \sin 9^\circ = \sqrt{1 - \sin 18^\circ} = \frac{1}{2}\sqrt{5 - \sqrt{5}}$$

$$\therefore \cos 9^\circ = \frac{1}{4}(\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}})$$

$$35. \quad \text{左邊} = \tan\left(\frac{\pi}{4} - \frac{\pi}{10}\right) = \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{10}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{10}} = \frac{1 - \tan \frac{\pi}{10}}{1 + \tan \frac{\pi}{10}}$$

$$= \frac{\left(1 - \tan \frac{\pi}{10}\right)^2}{1 - \tan^2 \frac{\pi}{10}} = \frac{\left(1 - \frac{1}{5}\sqrt{25 - 10\sqrt{5}}\right)^2}{1 - \frac{1}{25}(25 - 10\sqrt{5})}$$

$$= \frac{2 - \frac{2\sqrt{5}}{5}\sqrt{5 - 2\sqrt{5}} - \frac{2}{5}\sqrt{5}}{\frac{2}{5}\sqrt{5}}$$

$$= \sqrt{5} - 1 - \sqrt{5 - 2\sqrt{5}} = \text{右邊}$$

$$\begin{aligned} 36. \quad \text{左邊} &= \frac{1}{2}[3 + \cos 2A + \cos(120^\circ + 2A) + \cos(120^\circ - 2A)] \\ &= \frac{3}{2} + \frac{1}{2}(\cos 2A + 2 \cos 120^\circ \cos 2A) = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} 37. \quad \text{左邊} &= \cos^8 \frac{\pi}{8} + \cos^8\left(\pi - \frac{3\pi}{8}\right) + \cos^8 \frac{3\pi}{8} + \cos^8\left(\pi - \frac{\pi}{8}\right) \\ &= 2\left(\cos^8 \frac{\pi}{8} + \cos^8 \frac{3\pi}{8}\right) \\ &= \frac{1}{8}\left[\left(1 + \cos \frac{\pi}{4}\right)^4 + \left(1 + \cos \frac{3\pi}{4}\right)^4\right] \end{aligned}$$

$$= \frac{1}{8} \left[ \left( \frac{2 + \sqrt{2}}{2} \right)^4 + \left( \frac{2 - \sqrt{2}}{2} \right)^4 \right] = \frac{1}{8} \left( \frac{136}{16} \right) = \frac{17}{16}$$

38. 設  $\frac{2}{7}\pi = \alpha$ , 又設原式為  $x$ , 兩邊乘  $2 \sin \alpha$

$$\begin{aligned} \text{則 } 2 \sin \alpha \cdot x &= \sin 2\alpha + (\sin 3\alpha - \sin \alpha) + (\sin 4\alpha - \sin 2\alpha) \\ &= \sin 3\alpha + \sin 4\alpha - \sin \alpha \end{aligned}$$

$$\text{但 } \sin 4\alpha = \sin \frac{8}{7}\pi = -\sin \frac{1}{7}\pi = -\sin \frac{6}{7}\pi = -\sin 3\alpha$$

$$\therefore 2 \sin \alpha \cdot x = -\sin \alpha \quad \therefore x = -\frac{1}{2}$$

$$\text{或 } x = 2 \cos 2\alpha \cos \alpha + (2 \cos^2 \alpha - 1)$$

$$= 2 \cos \alpha (\cos 2\alpha + \cos \alpha) - 1$$

$$= 2 \cos \alpha (\cos 2\alpha + \cos 6\alpha) - 1$$

$$= 4 \cos \alpha \cos 2\alpha \cos 4\alpha - 1$$

$$= \frac{\sin 8\alpha}{2 \sin \alpha} - 1 = \frac{\sin \alpha}{2 \sin \alpha} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

註：例十六亦以用上兩法較便，尤以第一法為簡。

$$39. \text{ 左邊} = \frac{1}{2} \left( 2 \cos \frac{2\pi}{7} - 2 \cos \frac{\pi}{7} - 2 \cos \frac{3\pi}{7} \right)$$

$$= - \left( \cos \frac{3\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{\pi}{7} \right) \quad \text{以下如 38 題}$$

$$40. \text{ 左邊} = 2 \cos \frac{\pi}{3} \cos \frac{\pi}{36} - \cos \frac{\pi}{36} = \cos \frac{\pi}{36} - \cos \frac{\pi}{36} = 0$$

$$41. \text{ 左邊} = \frac{1}{2} \left( \cos \frac{2\pi}{3} + \cos \frac{\pi}{18} \right) \left( -\cos \frac{\pi}{36} \right)$$

$$= \frac{1}{4} \cos \frac{\pi}{36} - \frac{1}{4} \left( \cos \frac{\pi}{12} + \cos \frac{\pi}{36} \right) = -\frac{1}{4} \cos \frac{\pi}{12}$$

$$= -\frac{1}{4} \cdot \frac{\sqrt{6} + \sqrt{2}}{4} = -\frac{1 + \sqrt{3}}{8\sqrt{2}} \quad (102 \text{ 頁表 } 13)$$

42. 從  $8 \cos^4 \theta = 2(\cos 2\theta + 1)^2$

求到  $\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)$

再以  $\alpha, \frac{\pi}{3} + \alpha, \frac{\pi}{3} - \alpha$  次第代  $\theta$ , 再合併之, 則原式

$$\begin{aligned} \text{左邊} &= \frac{1}{8}(\Sigma \cos 4\alpha + 4\Sigma \cos 2\alpha + 9) \\ &= \frac{1}{8}[9 + \cos 4\alpha + 2 \cos \frac{4\pi}{3} \cos 4\alpha + 4(\cos 2\alpha \\ &\quad + 2 \cos \frac{2\pi}{3} \cos 2\alpha)] \\ &= \frac{1}{8}[9 + \cos 4\alpha - \cos 4\alpha + 4(\cos 2\alpha - \cos 2\alpha)] = \frac{9}{8} \end{aligned}$$

43. 設  $\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13} = x,$

$$\cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{11\pi}{13} = y.$$

則  $x + y = \cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{9\pi}{13} + \cos \frac{11\pi}{13}$

$$= 2 \cos \frac{\pi}{13} \left( \cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{10\pi}{13} \right)$$

$$= \frac{\sin \frac{2\pi}{13} \left( \cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{10\pi}{13} \right)}{\sin \frac{\pi}{13}}$$

$$= \frac{\frac{1}{2} \left( \sin \frac{4\pi}{13} + \sin \frac{8\pi}{13} - \sin \frac{4\pi}{13} + \sin \frac{12\pi}{13} - \sin \frac{8\pi}{13} \right)}{\sin \frac{\pi}{13}}$$

$$= \frac{\frac{1}{2} \sin \frac{12\pi}{13}}{\sin \frac{\pi}{13}} = \frac{\frac{1}{2} \sin \frac{\pi}{13}}{\sin \frac{\pi}{13}} = \frac{1}{2}$$

$$\begin{aligned} xy &= \left( \cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13} \right) \left( \cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{11\pi}{13} \right) \\ &= - \left( \cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13} \right) \left( \cos \frac{8\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{2\pi}{13} \right) \end{aligned}$$

乘出再用公式(30)化積爲和式,又化其間之

$$\cos \frac{15\pi}{13} = \cos \frac{11\pi}{13}, \quad \cos \frac{17\pi}{13} = \cos \frac{9\pi}{13}$$

合併之得

$$\begin{aligned} xy &= -\frac{3}{2} \left( \cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{9\pi}{13} + \cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{11\pi}{13} \right) \\ &= -\frac{3}{2} (x+y) = -\frac{3}{2} \left( \frac{1}{2} \right) = -\frac{3}{4} \end{aligned}$$

因此解  $x, y$  得

$$x - y = \frac{1}{2} \sqrt{13} \quad (\text{取} + \text{號})$$

$$\therefore x = \frac{1}{4} (1 + \sqrt{13}),$$

$$y = \frac{1}{4} (1 - \sqrt{13}).$$

$$\begin{aligned} 44. \text{ 左邊} &= 2 \cos \frac{1}{2} (x-z) \cos \frac{1}{2} (x+z-2y) + 2 \cos^2 \frac{1}{2} (z-x) - 1 \\ &= -1 + 2 \cos \frac{1}{2} (x-z) \left[ \cos \frac{1}{2} (x+z-2y) \right. \\ &\quad \left. + \cos \frac{1}{2} (x-z) \right] \end{aligned}$$

$$= -1 + 4 \cos \frac{1}{2}(x-z) \cos \frac{1}{2}(x-y) \cos \frac{1}{2}(z-y)$$

$$\begin{aligned} 45. \text{ 右邊} &= 2 \cos(\alpha + \beta) \cos \gamma + 2 \cos \gamma \cos(\alpha - \beta) \\ &= 2 \cos \gamma [\cos(\alpha + \beta) + \cos(\alpha - \beta)] = 4 \Pi \cos \alpha \end{aligned}$$

$$\begin{aligned} 46. \text{ 左邊} &= 2 \cos(\alpha + \beta) \cos(\alpha - \beta) + 2 \cos(\alpha + \beta + 2\gamma) \\ &\quad \times \cos(\alpha + \beta) \\ &= 2 \cos(\alpha + \beta) [\cos(\alpha - \beta) + \cos(\alpha + \beta + 2\gamma)] \\ &= 4 \Pi \cos(\alpha + \beta) \end{aligned}$$

$$\begin{aligned} 47. \text{ 左邊} &= \frac{1}{2 \sin(\beta - \gamma)} [\sin(2\beta - \alpha) + \sin(2\beta - 2\gamma - \alpha) \\ &\quad - \sin(2\gamma - \alpha) + \sin(2\beta - 2\gamma + \alpha)] \\ &= \frac{1}{\sin(\beta - \gamma)} [\sin(2\beta - \alpha - \gamma) \cos \gamma \\ &\quad + \sin(\alpha + \beta - 2\gamma) \cos \beta] \\ &= \frac{1}{\sin(\beta - \gamma)} [\sin(\alpha + \beta - 2\gamma) \cos \beta \\ &\quad - \sin(\alpha + \gamma - 2\beta) \cos \gamma] \end{aligned}$$

$$\begin{aligned} 48. \text{ 左邊} &= \frac{\Sigma \sin \alpha \sin(\beta - \gamma)}{\Pi \sin(\alpha - \beta)} \\ &= \frac{\Sigma [\cos(\alpha + \gamma - \beta) - \cos(\alpha + \beta - \gamma)]}{2 \Pi \sin(\alpha - \beta)} = 0 \end{aligned}$$

$$\begin{aligned} 49. \text{ 原式} &= - \left[ \frac{\tan \alpha \tan(\beta - \gamma) + \tan \beta \tan(\gamma - \alpha)}{\Pi \tan(\alpha - \beta)} \right. \\ &\quad \left. + \frac{\tan \gamma \tan(\alpha - \beta)}{\Pi \tan(\alpha - \beta)} \right] \end{aligned}$$

設分子爲  $N$ , 又令  $a = \tan \alpha$ ,  $b = \tan \beta$ ,  $c = \tan \gamma$ , 則

$$\begin{aligned}
 N &= \frac{a(b-c)}{1+bc} + \frac{b(c-a)}{1+ca} + \frac{c(a-b)}{1+ab} \\
 &= \frac{a(b-c)(1+ca)(1+ab) + b(c-a)(1+bc)(1+ab)}{\Pi(1+bc)} \\
 &\quad + \frac{c(a-b)(1+bc)(1+ca)}{\Pi(1+bc)} \\
 &= \frac{[\Sigma a(b-c) + \Sigma a^2(b^2-c^2) + \Sigma a^3bc(b-c)]}{\Pi(1+bc)} \\
 &= \frac{[0+0+abc \Sigma a^2(b-c)]}{\Pi(1+bc)}
 \end{aligned}$$

但從代數知  $\Sigma a^2(b-c) = -\Pi(b-c)$

$$\begin{aligned}
 \text{故 } N &= \frac{-abc \Pi(b-c)}{\Pi(1+bc)} = -abc \cdot \frac{b-c}{1+bc} \cdot \frac{c-a}{1+ca} \cdot \frac{a-b}{1+ab} \\
 &= -\tan \alpha \tan \beta \tan \gamma \tan(\beta-\gamma) \tan(\gamma-\alpha) \\
 &\quad \times \tan(\alpha-\beta)
 \end{aligned}$$

$$\text{故原式} = -\frac{-\Pi \tan \alpha \cdot \Pi \tan(\alpha-\beta)}{\Pi \tan(\alpha-\beta)} = \Pi \tan \alpha$$

$$\begin{aligned}
 50. \text{ 左邊} &= \Sigma \frac{\sin(\alpha-\beta)[\cos(\alpha-\beta) - \cos(2\theta - \alpha - \beta)]}{-2 \sin(\alpha-\beta) \sin(\beta-\gamma) \sin(\gamma-\alpha)} \\
 &= -\Sigma \frac{\sin 2(\alpha-\beta) - \sin 2(\theta-\beta) + \sin 2(\theta-\alpha)}{4\Pi \sin(\alpha-\beta)} \\
 &= -\frac{1}{4\Pi \sin(\alpha-\beta)} [\sin 2(\alpha-\beta) + \sin 2(\beta-\gamma) \\
 &\quad + \sin 2(\gamma-\alpha)] \\
 &= -\frac{1}{4\Pi \sin(\alpha-\beta)} [-4 \sin(\alpha-\beta) \sin(\beta-\gamma) \\
 &\quad \times \sin(\gamma-\alpha)] = 1 \quad (\text{見 108 頁例十七})
 \end{aligned}$$



$$\begin{aligned}
 51. \text{ 左邊} &= \sum \frac{\tan^2 \alpha - \tan^2 \beta}{\tan \alpha \tan \beta} = \sum \frac{\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta}{\sin \alpha \sin \beta \cos \alpha \cos \beta} \\
 &= \sum \frac{4(\cos^2 \beta - \cos^2 \alpha)}{\sin 2\alpha \sin 2\beta} = \frac{2\sum \sin 2\gamma (\cos 2\beta - \cos 2\alpha)}{\Pi \sin 2\alpha} \\
 &= \frac{\sum [\sin 2(\gamma + \beta) + \sin 2(\gamma - \beta) + \sin 2(\alpha - \gamma) - \sin 2(\alpha + \gamma)]}{\Pi \sin 2\alpha} \\
 &= \frac{2\sum \sin 2(\beta - \alpha)}{\Pi \sin 2\alpha}
 \end{aligned}$$

$$\begin{aligned}
 52. \text{ 左邊} &= \sum \frac{\sin(\theta - \beta - \alpha) \sin(\theta - \gamma - \alpha)}{\sin(\beta - \alpha) \sin(\gamma - \alpha)} \\
 &= -\sum \frac{\cos(\beta - \gamma) - \cos(2\theta - 2\alpha - \beta - \gamma)}{2 \sin(\alpha - \beta) \sin(\gamma - \alpha)} \\
 &= -\frac{\sum [\cos(\beta - \gamma) - \cos(2\theta - 2\alpha - \beta - \gamma)] \sin(\beta - \gamma)}{2\Pi \sin(\beta - \gamma)}
 \end{aligned}$$

$$\begin{aligned}
 \text{分子} &= -\frac{1}{2} [\sin 2(\beta - \gamma) - \sin 2(\theta - \alpha - \gamma) \\
 &\quad + \sin 2(\theta - \alpha - \beta) + \dots] \\
 &= -\frac{1}{2} \sum \sin 2(\beta - \gamma) = 2\Pi \sin(\beta - \gamma) \quad (\text{見例十七})
 \end{aligned}$$

故原式爲 1

$$\begin{aligned}
 53. \text{ 原式} &= \sin(\beta - \gamma) [\sin^2 \alpha \sin(\gamma - \delta) \sin(\delta - \beta) \\
 &\quad - \sin^2 \delta \sin(\alpha - \beta) \sin(\gamma - \alpha)] \\
 &\quad + \sin(\delta - \alpha) [\dots] \\
 &= \frac{1}{4} \sin(\beta - \gamma) \{ [\cos(\alpha - \gamma + \delta) - \cos(\alpha + \gamma - \delta)] \\
 &\quad \times [\cos(\alpha + \beta - \delta) - \cos(\alpha - \beta + \delta)] - \dots \} + \dots \\
 &= \frac{1}{4} \sin(\beta - \gamma) [\cos(\alpha - \gamma + \delta) \cos(\alpha + \beta - \delta) \\
 &\quad - \cos(\alpha + \gamma - \delta) \cos(\alpha + \beta - \delta) + \dots] + \dots
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \sin(\beta - \gamma) [\cos(2\alpha + \beta - \gamma) + 2 \cos(\beta + \gamma - 2\delta) \\
&\quad - \cos(2\alpha + \beta + \gamma - 2\delta) + \cos(2\alpha - \beta + \gamma) + \dots] \\
&\quad + \dots \\
&= \frac{1}{16} [\sin 2(\alpha + \beta - \gamma) - \sin 2(\alpha + \beta - \delta) \\
&\quad + \sin 2(\alpha + \gamma - \delta) - \sin 2(\alpha - \beta + \gamma) + \dots] \\
&\quad + \frac{1}{8} [\sin 2(\beta - \delta) - \sin 2(\gamma - \delta) - \sin 2(\alpha - \gamma) \\
&\quad + \sin 2(\alpha - \beta)] + \dots = 0
\end{aligned}$$

54. 因  $\Sigma A^3 - 3\Pi A = A^3 + B^3 + C^3 - 3ABC$   
 $= \Sigma A \cdot (\Sigma A^2 - \Sigma AB)$

故如  $\Sigma A = 0$ , 則  $\Sigma A^3 = 3\Pi A$

今  $\Sigma \cos(\beta + \gamma) \sin(\beta - \gamma) = \frac{1}{2} \Sigma (\sin 2\beta - \sin 2\gamma) = 0$

故  $\Sigma \cos^3(\beta + \gamma) \sin^3(\beta - \gamma) = 3\Pi \cos(\beta + \gamma) \sin(\beta - \gamma)$

55. 原式 = 
$$\begin{vmatrix}
1 & 0 & 0 \\
\sin \alpha & \cos^2 \alpha & \sin \gamma - \sin \alpha \sin \beta \\
\sin \beta & \sin \gamma - \sin \alpha \sin \beta & \cos^2 \beta
\end{vmatrix}$$

$$\begin{aligned}
&= \cos^2 \alpha \cos^2 \beta - (\sin \gamma - \sin \alpha \sin \beta)^2 \\
&= (\cos \alpha \cos \beta - \sin \gamma + \sin \alpha \sin \beta) \\
&\quad \times (\cos \alpha \cos \beta + \sin \gamma - \sin \alpha \sin \beta) \\
&= [\cos(\alpha - \beta) - \sin \gamma] [\cos(\alpha + \beta) + \sin \gamma] \\
&= \left[ \cos(\alpha - \beta) + \cos\left(\frac{\pi}{2} + \gamma\right) \right] \\
&\quad \times \left[ \cos(\alpha + \beta) + \cos\left(\frac{\pi}{2} - \gamma\right) \right]
\end{aligned}$$

$$= 4 \cos \left[ \frac{1}{2}(\alpha + \beta + \gamma) - \frac{1}{4}\pi \right] \\ \times \Pi \left[ \cos \frac{1}{2}(\alpha + \beta - \gamma) + \frac{1}{4}\pi \right]$$

56. 第一列乘  $\cos A$  減第三列, 又第二列乘  $\sin A$  減第三列得

$$\begin{aligned} \text{左邊} &= \frac{2}{\sin A \cos A} \begin{vmatrix} 0 & 0 & \sin A \cos A \\ \sin B(\cos A - \cos B) & \cos B(\sin A - \sin B) & \sin B \cos B \\ \sin C(\cos A - \cos C) & \cos C(\sin A - \sin C) & \sin C \cos C \end{vmatrix} \\ &= 8 \begin{vmatrix} \sin B \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(B-A) & \cos B \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B) \\ \sin C \sin \frac{1}{2}(A+C) \sin \frac{1}{2}(C-A) & \cos C \cos \frac{1}{2}(A+C) \sin \frac{1}{2}(A-C) \end{vmatrix} \\ &= 2 \sin \frac{1}{2}(A-B) \sin \frac{1}{2}(C-A) \left\{ \left[ \sin(B+C) + \sin(B-C) \right] \right. \\ &\quad \times \left[ \sin \frac{2A+B+C}{2} + \sin \frac{B-C}{2} \right] - \left[ \sin(B+C) - \sin(B-C) \right] \\ &\quad \left. \times \left[ \sin \frac{2A+B+C}{2} - \sin \frac{B-C}{2} \right] \right\} \\ &= 2 \sin \frac{A-B}{2} \sin \frac{C-A}{2} \left[ 2 \sin(B-C) \sin \frac{2A+B+C}{2} \right. \\ &\quad \left. + 2 \sin(B+C) \sin \frac{B-C}{2} \right] \\ &= 4 \sin \frac{A-B}{2} \sin \frac{B-C}{2} \sin \frac{C-A}{2} \left[ 2 \sin \frac{2A+B+C}{2} \cos \frac{B-C}{2} \right. \\ &\quad \left. + \sin(B+C) \right] \\ &= 4 \sin \frac{A-B}{2} \sin \frac{B-C}{2} \sin \frac{C-A}{2} \left[ \sin(A+B) + \sin(B+C) \right. \\ &\quad \left. + \sin(C+A) \right] \end{aligned}$$

57. 照左邊行列式之第一行展開之得

$$\begin{aligned} \text{左邊} &= \cos \frac{1}{2}(B-C) [\cos \frac{1}{2}(C+A) \sin \frac{1}{2}(A+B) \\ &\quad - \sin \frac{1}{2}(C+A) \cos \frac{1}{2}(A+B)] - \dots\dots \\ &= \cos \frac{1}{2}(B-C) \sin \frac{1}{2}(B-C) - \cos \frac{1}{2}(C-A) \sin \frac{1}{2}(A-C) \\ &\quad + \cos \frac{1}{2}(A-B) \sin \frac{1}{2}(A-B) \\ &= \frac{1}{2} [\sin(B-C) - \sin(A-C) + \sin(A-B)] \end{aligned}$$

$$\begin{aligned} \text{右邊} &= \frac{1}{2} \left\{ \begin{vmatrix} \sin B & \cos B \\ \sin C & \cos C \end{vmatrix} - \begin{vmatrix} \sin A & \cos A \\ \sin C & \cos C \end{vmatrix} + \begin{vmatrix} \sin A & \cos A \\ \sin B & \cos B \end{vmatrix} \right\} \\ &= \frac{1}{2} [\sin(B-C) - \sin(A-C) + \sin(A-B)] \end{aligned}$$

故兩邊相等。

58. 照分子之第一列展開之得

$$\begin{aligned} \text{左邊分子} &= \cos(\beta-\gamma) \sin(\gamma-\beta) - \cos(\gamma-\alpha) \sin(\gamma-\alpha) \\ &\quad + \cos(\alpha-\beta) \sin(\beta-\alpha) \\ &= -\frac{1}{2} [\sin 2(\beta-\gamma) + \sin 2(\gamma-\alpha) \\ &\quad + \sin 2(\alpha-\beta)] \\ &= 2 \sin(\beta-\gamma) \sin(\gamma-\alpha) \sin(\alpha-\beta) \quad (\text{見例十七}) \\ &= 16\Pi \sin \frac{1}{2}(\alpha-\beta) \cdot \Pi \cos \frac{1}{2}(\alpha-\beta) \end{aligned}$$

$$\text{左邊分母} = -[\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)]$$

(見上題之右邊化法)

$$= 4\Pi \sin \frac{1}{2}(\alpha-\beta) \quad (\text{見例十七})$$

$$\text{故原式} = 4\Pi \cos \frac{1}{2}(\alpha-\beta)$$

59. 今  $\sin 2^n \theta = \sin 2(2^{n-1} \theta) = 2 \sin 2^{n-1} \theta \cos 2^{n-1} \theta$

$$\sin 2^{n-1}\theta = 2 \sin 2^{n-2}\theta \cos 2^{n-2}\theta$$

.....

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

相乘得  $\sin 2^n\theta = 2^n \cos 2^{n-1}\theta \cos 2^{n-2}\theta \cdots \cdots \cos \theta \sin \theta$

$$60. \text{ 從例二十七得 } \sin \theta = 2^n \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cdots \cdots \cos \frac{\theta}{2^n} \sin \frac{\theta}{2^n}$$

$$\text{即 } \frac{\sin \theta}{\theta} = \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cdots \cdots \cos \frac{\theta}{2^n} \left( \sin \frac{\theta}{2^n} / \frac{\theta}{2^n} \right)$$

$$\text{當 } n = \infty \quad \lim_{n \rightarrow \infty} \left( \sin \frac{\theta}{2^n} / \frac{\theta}{2^n} \right) = 1$$

$$\text{故 } \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cdots \cdots \cos \frac{\theta}{2^n} = \frac{\sin \theta}{\theta}$$

$$61. \therefore 2 \cos^2 \frac{45^\circ}{2^n} = 2 \cos^2 \frac{1}{2} \left( \frac{45^\circ}{2^{n-1}} \right) = 1 + \cos \frac{45^\circ}{2^{n-1}}$$

$$\therefore 2 \cos \frac{45^\circ}{2^n} = \sqrt{2 + 2 \cos \frac{45^\circ}{2^{n-1}}}$$

$$\text{同理 } 2 \cos \frac{45^\circ}{2^{n-1}} = \sqrt{2 + 2 \cos \frac{45^\circ}{2^{n-2}}}$$

.....

$$\text{故 } 2 \cos \frac{45^\circ}{2^n} = \sqrt{2 + 2 \cos \frac{45^\circ}{2^{n-1}}}$$

$$= \sqrt{2 + \sqrt{2 + 2 \cos \frac{45^\circ}{2^{n-2}}}}$$

$$= \sqrt{2 + \sqrt{2 + \cdots \sqrt{2 + 2 \cos 90^\circ}}}$$

$$= \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}$$

$$62. \therefore 2 \sin^2 A = 1 - \cos 2A \quad \therefore 2 \sin A = \sqrt{2 - 2 \cos 2A}$$

$$\begin{aligned}
 \text{今 } 2 \cos\left(60^\circ + \frac{30^\circ}{2^n}\right) &= 2 \sin\left(30^\circ - \frac{30^\circ}{2^n}\right) \\
 &= \sqrt{2 - 2 \cos\left(60^\circ - \frac{30^\circ}{2^{n-1}}\right)} \\
 &= \sqrt{2 - 2 \sin\left(30^\circ + \frac{30^\circ}{2^{n-1}}\right)} \\
 &= \sqrt{2 - \sqrt{2 - 2 \cos\left(60^\circ + \frac{30^\circ}{2^{n-2}}\right)}}
 \end{aligned}$$

設  $n$  爲偶數則至第  $n$  次得

$$\begin{aligned}
 2 \sin\left(30^\circ + \frac{30^\circ}{2^{n-(n-1)}}\right) &= \sqrt{2 - 2 \cos\left(60^\circ + \frac{30^\circ}{2^{n-n}}\right)} \\
 &= \sqrt{2 - 2 \cos 90^\circ} = \sqrt{2}
 \end{aligned}$$

$$\text{故 } 2 \cos\left(60^\circ + \frac{30^\circ}{2^n}\right) = \sqrt{2 - \sqrt{2 - \dots - \sqrt{2}}}$$

63. 如例二十四，以  $d=x$  代入即行。

64. 第  $n$  項爲  $\sin \pi$ ，今加入計算，如例二十四，設  $x=d=\frac{\pi}{n}$ ，

即得總和爲

$$\begin{aligned}
 &\sin \frac{1}{2} n \left(\frac{\pi}{n}\right) \sin \left[\frac{\pi}{n} + \frac{1}{2} (n-1) \frac{\pi}{n}\right] / \sin \frac{1}{2} \left(\frac{\pi}{n}\right) \\
 &= \sin \frac{1}{2} \pi \sin \left(\frac{\pi}{2} + \frac{\pi}{2n}\right) / \sin \frac{\pi}{2n} \\
 &= \cos \frac{\pi}{2n} / \sin \frac{\pi}{2n} = \cot \frac{\pi}{2n}
 \end{aligned}$$

65. 設和爲  $S$

$$\text{今 } 2 \cos x \sin \frac{1}{2} d = \sin(x + \frac{1}{2} d) - \sin(x - \frac{1}{2} d)$$

$$2 \cos(x+d) \sin \frac{1}{2} d = \sin(x + \frac{3}{2} d) - \sin(x + \frac{1}{2} d)$$

$$\dots\dots\dots$$

$$2 \cos(x + \overline{n-1}d) \sin \frac{1}{2} d = \sin(x + \overline{n-\frac{1}{2}}d) - (\sin x + \overline{n-\frac{3}{2}}d)$$

相加得  $2S \sin \frac{1}{2} d = \sin(x + \overline{n-\frac{1}{2}}d) - \sin(x - \frac{1}{2}d)$

$$\therefore S = \sin \frac{1}{2} nd \cos[x + \frac{1}{2}(n-1)d] / \sin \frac{1}{2} nd.$$

66.  $\therefore -\cos(A+B) = \cos(\pi + A+B),$

$$\cos(A+2B) = \cos(2\pi + A+2B), \dots\dots$$

故  $S = \cos A + \cos[A + (\pi + B)]$

$$+ \cos[A + 2(\pi + B)] + \dots\dots\dots$$

在上題中，以  $A$  代  $x$ ， $\pi + B$  代  $d$ ，則得本題之證矣。

67.  $\csc A = \frac{1}{\sin A} = \frac{\sin \frac{1}{2} A}{\sin A \sin \frac{1}{2} A} = \frac{\sin(A - \frac{1}{2} A)}{\sin A \sin \frac{1}{2} A}$

$$= \frac{\sin A \cos \frac{1}{2} A - \cos A \sin \frac{1}{2} A}{\sin A \sin \frac{1}{2} A} = \cot \frac{1}{2} A - \cot A$$

$$\csc 2A = \cot A - \cot 2A, \quad \csc 2^2 A = \cot 2A - \cot 2^2 A,$$

$\dots\dots\dots$

相加得  $S = \cot \frac{1}{2} A - \cot 2^{n-1} A$

68.  $\therefore \tan \frac{1}{2} A \sec A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A \cos A} = \frac{\sin(A - \frac{1}{2} A)}{\cos \frac{1}{2} A \cos A}$

$$= \frac{\sin A \cos \frac{1}{2} A - \sin \frac{1}{2} A \cos A}{\cos \frac{1}{2} A \cos A}$$

$$= \tan A - \tan \frac{1}{2} A$$

同理  $\tan \frac{1}{2^2} A \sec \frac{1}{2} A = \tan \frac{1}{2} A - \tan \frac{1}{2^2} A$

$$\tan \frac{1}{2^n} A \sec \frac{1}{2^{n-1}} A = \tan \frac{A}{2^{n-1}} - \tan \frac{1}{2^n} A$$

相加得  $S = \tan A - \tan \frac{1}{2^n} A$

69. 因  $2 \cos \theta - 1 = \frac{4 \cos^2 \theta - 1}{2 \cos \theta + 1} = \frac{2 \cos 2\theta + 1}{2 \cos \theta + 1}$

同理得  $2 \cos \frac{\theta}{2} - 1 = \frac{2 \cos \theta + 1}{2 \cos \frac{1}{2} \theta + 1}$

$$2 \cos \frac{\theta}{2^2} - 1 = \frac{2 \cos \frac{1}{2} \theta + 1}{2 \cos \frac{1}{2^2} \theta + 1}$$

$$2 \cos \frac{\theta}{2^{n-1}} - 1 = \frac{2 \cos \frac{1}{2^{n-2}} \theta + 1}{2 \cos \frac{1}{2^{n-1}} \theta + 1}$$

相乘得  $(2 \cos \theta - 1)(2 \cos \frac{1}{2} \theta - 1) \cdots (2 \cos \frac{1}{2^{n-1}} \theta - 1)$

$$= \frac{2 \cos 2\theta + 1}{2 \cos \frac{1}{2^{n-1}} \theta + 1}$$

70. 因  $1 + \sec 2\theta = \frac{\cos 2\theta + 1}{\cos 2\theta} = \frac{2 \cos^2 \theta}{\cos 2\theta} = \frac{\sin 2\theta \cos \theta}{\cos 2\theta \sin \theta}$

即  $1 + \sec 2\theta = \tan 2\theta \cot \theta$ ,  $1 + \sec 4\theta = \tan 4\theta \cot 2\theta$ ,

.....  $1 + \sec 2^n \theta = \tan 2^n \theta \cot 2^{n-1} \theta$

即  $(1 + \sec 2\theta)(1 + \sec 4\theta) \cdots (1 + \sec 2^n \theta)$

$$= \tan 2^n \theta \cot \theta$$



71.  $\therefore 8 \cos^4 A = \cos 4A + 4 \cos 2A + 3$  (從 20 節化法)

$$\begin{aligned} \text{故 } 8 \times \text{左邊} &= \left[ \cos 4A + \cos 4\left(A + \frac{2\pi}{n}\right) \right. \\ &\quad \left. + \cos 4\left(A + \frac{4\pi}{n}\right) + \dots \right] \\ &\quad + 4 \left[ \cos 2A + \cos 2\left(A + \frac{2\pi}{n}\right) \right. \\ &\quad \left. + \cos 2\left(A + \frac{4\pi}{n}\right) + \dots \right] + 3n \\ &= \frac{\sin 4\pi \cos 4\left(A + \frac{n-1}{n}\pi\right)}{\sin \frac{4\pi}{n}} \\ &\quad + \frac{4 \sin 2\pi \cos 2\left(A + \frac{n-1}{n}\pi\right)}{\sin \frac{2\pi}{n}} + 3n \quad (\text{見 65 題}) \end{aligned}$$

$$\therefore \text{左邊} = \frac{3n}{8} \quad (\because \sin 4\pi = 0, \quad \sin 2\pi = 0)$$

$$72. \quad \therefore \cos \frac{A}{2} + \cos \frac{B}{2} = \frac{\cos A - \cos B}{2\left(\cos \frac{A}{2} - \cos \frac{B}{2}\right)}$$

$$\text{同理 } \cos \frac{1}{2^2} A + \cos \frac{1}{2^2} B = \frac{\cos \frac{1}{2} A - \cos \frac{1}{2} B}{2\left(\cos \frac{1}{2^2} A - \cos \frac{1}{2^2} B\right)}$$

$$\dots\dots\dots$$

$$\cos \frac{1}{2^n} A + \cos \frac{1}{2^n} B = \frac{\cos \frac{1}{2^{n-1}} A - \cos \frac{1}{2^{n-1}} B}{2\left(\cos \frac{1}{2^n} A - \cos \frac{1}{2^n} B\right)}$$

$$\text{相乘得 左邊} = \frac{\cos A - \cos B}{2^n \left( \cos \frac{1}{2^n} A - \cos \frac{1}{2^n} B \right)}$$

$$73. \text{ 令 } x_r = \cos \frac{y}{2^r}, \quad \text{則 } \cos \frac{y}{2^{r+1}} = \sqrt{\frac{1+x_r}{2}} = x_{r+1}$$

$$\text{即 } \cos y = x_0, \quad \cos \frac{y}{2} = x_1, \quad \cos \frac{y}{2^2} = x_2 \cdots \cdots$$

$$\text{相乘得 } \cos \frac{y}{2} \cos \frac{y}{2^2} \cdots \cdots \cos \frac{y}{2^r} = x_1 \cdot x_2 \cdot x_3 \cdots \cdots x_r$$

$$\text{即 } 2^r \cos \frac{y}{2} \cos \frac{y}{2^2} \cdots \cdots \cos \frac{y}{2^r} \sin \frac{y}{2^r}$$

$$= 2^r x_1 \cdot x_2 \cdot x_3 \cdots \cdots x_r \sin \frac{y}{2^r}$$

$$= y x_1 \cdot x_2 \cdot x_3 \cdots \cdots x_r \sin \frac{y}{2^r} / \frac{y}{2^r}$$

$$\text{但 } \sin y = 2^r \cos \frac{y}{2} \cos \frac{y}{2^2} \cdots \cdots \cos \frac{y}{2^r} \sin \frac{y}{2^r} \quad (\text{見例二十七})$$

$$\text{又 } \lim_{r \rightarrow \infty} \left( \sin \frac{y}{2^r} / \frac{y}{2^r} \right) = 1$$

$$\text{故 } \sin y = y x_1 x_2 x_3 \cdots \cdots \text{至 } \infty$$

$$\therefore \cos y = x_0$$

$$\therefore \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x_0^2}, \quad y = \cos^{-1} x_0$$

$$\text{即 } \sqrt{1 - x_0^2} = \cos^{-1} x_0 (x_1 x_2 \cdots \cdots \infty)$$

$$\therefore \cos^{-1} x_0 = \frac{\sqrt{1 - x_0^2}}{x_1 x_2 \cdots \cdots \text{至 } \infty}$$

74. 在 61 題設  $n = \infty$ , 則  $\cos \frac{45^\circ}{2^n} = \cos 0^\circ = 1$ , 即得此證矣

或令此式爲  $x$

$$\text{則 } x = \sqrt{2+x}, \quad \text{即 } x^2 - x - 2 = 0$$

$$\text{即 } (x-2)(x+1) = 0, \quad \therefore x = 2 \quad (-1 \text{ 根不合})$$

$$\begin{aligned} 75. \text{ 從 } \quad & \sin(x+y-z) - \sin(z+x-y) \\ & = \sin(z+x-y) - \sin(y+z-x) \\ \therefore \quad & 2 \cos x \sin(y-z) = 2 \cos z \sin(x-y) \\ \text{即 } \quad & \cos x \sin y \cos z - \cos x \cos y \sin z \\ & = \sin x \cos y \cos z - \sin y \cos x \cos z \end{aligned}$$

兩邊各除以  $\cos x \cos y \cos z$

$$\text{得 } \quad \tan y - \tan z = \tan x - \tan y$$

$$\text{即 } \quad \tan z - \tan y = \tan y - \tan x \quad \text{故如題云}$$

$$76. \text{ 從 } \quad \cot \gamma - \cot \beta = \cot \beta - \cot \alpha \quad (1)$$

$$\text{亦即 } \quad \cot \gamma = 2 \cot \beta - \cot \alpha \quad (2)$$

$$\text{今 } \quad \cot(\beta - \gamma) + \cot(\beta - \alpha) = \cot(\beta - \gamma) - \cot(\alpha - \beta)$$

$$= \frac{1 + \cot \beta \cot \gamma}{\cot \gamma - \cot \beta} - \frac{1 + \cot \alpha \cot \beta}{\cot \beta - \cot \alpha}$$

$$= \frac{\cot \beta (\cot \gamma - \cot \alpha)}{\cot \beta - \cot \alpha} \quad \text{〔從(1)式〕}$$

$$= \frac{2 \cot \beta (\cot \beta - \cot \alpha)}{\cot \beta - \cot \alpha} \quad \text{〔從(2)式〕}$$

$$= 2 \cot \beta$$

$$\text{故 } \quad \cot(\beta - \gamma) - \cot \beta = \cot \beta - \cot(\beta - \alpha)$$

$$77. \therefore \quad z - y = y - x \quad \therefore \quad y = \frac{1}{2}(x+z)$$

$$\text{即 } \quad y - z = \frac{1}{2}(x+z) - z = \frac{1}{2}(x-z)$$

$$\text{故 } \frac{\tan y}{\tan(y-z)} = \frac{\tan \frac{1}{2}(x+z)}{\tan \frac{1}{2}(x-z)}$$

$$\text{亦等於 } \frac{\sin \frac{1}{2}(x+z) \cos \frac{1}{2}(x-z)}{\cos \frac{1}{2}(x+z) \sin \frac{1}{2}(x-z)} = \frac{\sin x + \sin z}{\sin x - \sin z} \quad \text{故如題云}$$

$$\begin{aligned} 78. \text{ 因 } \tan y &= \frac{\tan z}{2} = \frac{\tan \frac{z}{2}}{1 - \tan^2 \frac{z}{2}} = \frac{\tan \frac{z}{2} \left(1 + \tan^2 \frac{z}{2}\right)}{1 - \tan^4 \frac{z}{2}} \\ &= \frac{\tan \frac{z}{2} + \tan \frac{x}{2}}{1 - \tan \frac{z}{2} \tan \frac{x}{2}} = \tan \frac{z+x}{2}, \quad \text{故 } y = \frac{z+x}{2} \end{aligned}$$

$$79. \text{ 今 } \frac{1}{\cos \phi} - \frac{1}{\cos(\phi-\alpha)} = \frac{1}{\cos(\phi+\alpha)} - \frac{1}{\cos \phi}$$

$$\text{即 } \cos \phi = \frac{2 \cos(\phi-\alpha) \cos(\phi+\alpha)}{\cos(\phi-\alpha) + \cos(\phi+\alpha)} = \frac{\cos 2\phi + \cos 2\alpha}{2 \cos \phi \cos \alpha}$$

$$\text{故 } 2 \cos \alpha \cos^2 \phi = \cos 2\phi + \cos 2\alpha$$

$$= (2 \cos^2 \phi - 1) + (2 \cos^2 \alpha - 1)$$

$$\text{即 } \cos^2 \phi (1 - \cos \alpha) = 1 - \cos^2 \alpha$$

$$\therefore \cos^2 \phi = 1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2}$$

$$\therefore \cos \phi = \sqrt{2} \cos \frac{\alpha}{2}$$

80. 以  $a, b$  作為未知數，從十字法(見附錄二)得

$$\frac{a}{\begin{vmatrix} \sin \beta & c \sin \gamma \\ \cos \beta & c \cos \gamma \end{vmatrix}} = \frac{b}{\begin{vmatrix} c \sin \gamma & \sin \alpha \\ c \cos \gamma & \cos \alpha \end{vmatrix}} = \frac{1}{\begin{vmatrix} \sin \alpha & \sin \beta \\ \cos \alpha & \cos \beta \end{vmatrix}}$$

$$\text{即 } \frac{a}{\sin(\beta-\gamma)} = \frac{b}{\sin(\gamma-\alpha)} = \frac{c}{\sin(\alpha-\beta)} \quad (\text{化出且以 } c \text{ 乘之})$$

81. 平方第一式兩邊且以公式 26 代入得

$$\frac{1 - \cos x}{1 + \cos x} = \frac{(1 + C)(1 - \cos y)}{(1 - C)(1 + \cos y)} = \frac{1 + C - \cos y - C \cos y}{1 - C + \cos y - C \cos y}$$

$$\text{即 } \frac{2}{2 \cos x} = \frac{2(1 - C \cos y)}{2(\cos y - C)} \quad (\text{附錄二比例中合分之理})$$

$$\text{即 } \cos x = \frac{\cos y - C}{1 - C \cos y}$$

82. 從第一式即  $\frac{\tan A - \tan B}{\tan A(1 + \tan A \tan B)} + \sin^2 C \csc^2 A = 1$ ,

$$\begin{aligned} \text{即 } \frac{\sin^2 C(1 + \tan^2 A)}{\tan^2 A} &= 1 - \frac{\tan A - \tan B}{\tan A(1 + \tan A \tan B)} \\ &= \frac{\tan B(1 + \tan^2 A)}{\tan A(1 + \tan A \tan B)}, \end{aligned}$$

約去  $(1 + \tan^2 A)/\tan A$  且去分母移項得

$$\sin^2 C = \tan A \tan B(1 - \sin^2 C) = \tan A \tan B \cos^2 C$$

$$\therefore \tan^2 C = \tan A \tan B$$

$$83. \text{ 即 } \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \cdot \frac{1 - \cos 2\phi}{1 + \cos 2\phi} = \frac{a - b}{a + b}$$

$$\begin{aligned} \therefore (1 - \cos 2\theta - \cos 2\phi + \cos 2\theta \cos 2\phi)(a + b) \\ = (1 + \cos 2\theta + \cos 2\phi + \cos 2\theta \cos 2\phi)(a - b) \end{aligned}$$

$$\text{即 } 2a(\cos 2\theta + \cos 2\phi) = 2b(1 + \cos 2\theta \cos 2\phi)$$

$$\text{即 } a^2 - ab(\cos 2\theta + \cos 2\phi) + b^2 \cos 2\theta \cos 2\phi = -b^2 + a^2$$

$$\therefore (a - b \cos 2\theta)(a - b \cos 2\phi) = a^2 - b^2$$

84.  $\therefore \cos A = 1 - x, \quad \cos B = 1 - mx, \quad \cos C = m$

$$\therefore \sin A = \sqrt{2x - x^2}, \quad \sin C = \sqrt{1 - m^2}$$

從  $B=C-A$ , 則  $\cos B = \cos C \cos A + \sin C \sin A$

$$\text{即 } 1 - mx = m(1-x) + \sqrt{1-m^2} \cdot \sqrt{2x-x^2}$$

$$\text{即 } 1 - m = \sqrt{(1-m^2)(2x-x^2)}$$

$$\text{即 } 1 - m = (1+m)(2x-x^2)$$

$$\therefore x^2 - 2x + \frac{1-m}{1+m} = 0, \quad \text{故 } x = 1 \pm \sqrt{\frac{2m}{1+m}}$$

$$85. \therefore \sin \theta = \frac{a}{b} \sin \phi, \quad \cos \theta = \frac{c}{d} \cos \phi$$

$$\text{則 } \frac{a^2}{b^2} \sin^2 \phi + \frac{c^2}{d^2} \cos^2 \phi = 1$$

$$\therefore \sin^2 \phi = \frac{1 - \frac{c^2}{d^2}}{\frac{a^2}{b^2} - \frac{c^2}{d^2}} = \frac{b^2(d^2 - c^2)}{a^2 d^2 - b^2 c^2}$$

$$\text{又 } \frac{a^2}{b^2} - \frac{a^2}{b^2} \cos^2 \phi + \frac{c^2}{d^2} \cos^2 \phi = 1$$

$$\therefore \cos^2 \phi = \frac{\frac{a^2}{b^2} - 1}{\frac{a^2}{b^2} - \frac{c^2}{d^2}} = \frac{d^2(a^2 - b^2)}{a^2 d^2 - b^2 c^2}$$

$$\text{今 } \cos(\theta \mp \phi) = \cos \theta \cos \phi \pm \sin \phi \sin \theta$$

$$= \frac{c}{d} \cos^2 \phi \pm \frac{a}{b} \sin^2 \phi$$

$$\begin{aligned} \therefore \cos(\theta \mp \phi) &= \frac{c}{d} \cdot \frac{d^2(a^2 - b^2)}{a^2 d^2 - b^2 c^2} \pm \frac{a}{b} \cdot \frac{b^2(d^2 - c^2)}{a^2 d^2 - b^2 c^2} \\ &= \frac{cd(a^2 - b^2) \pm ab(d^2 - c^2)}{a^2 d^2 - b^2 c^2} \end{aligned}$$

$$= \frac{(ad \mp bc)(ac \pm bd)}{(ad - bc)(ad + bc)} = \frac{ac \pm bd}{ad \pm bc}$$

(約去時同取上號或下號)

$$86. \text{ 今 } \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\frac{\sin A}{\cos A} - \frac{n \sin A \cos A}{1 - n \sin^2 A}}{1 + \frac{\sin A}{\cos A} \cdot \frac{n \sin A \cos A}{1 - n \sin^2 A}}$$

$$= \frac{\sin A(1 - n \sin^2 A - n \cos^2 A)}{\cos A(1 - n \sin^2 A + n \sin^2 A)}$$

$$= (1 - n) \tan A$$

$$87. \text{ 今 } \cot(\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta} = \frac{x(x + x^{-1} + 1) - 1}{(x + x^{-1} + 1)^{\frac{1}{2}}(x + 1)}$$

$$= \frac{x(x + 1)}{(x + 1)(x + x^{-1} + 1)^{\frac{1}{2}}} = x(x + x^{-1} + 1)^{-\frac{1}{2}}$$

$$\text{又 } \cot \gamma = 1 / \tan \gamma = (x^{-3} + x^{-2} + x^{-1})^{-\frac{1}{2}}$$

$$= [x^{-2}(x + x^{-1} + 1)]^{-\frac{1}{2}}$$

$$= x(x + x^{-1} + 1)^{-\frac{1}{2}}$$

$$\text{故 } \cot(\alpha + \beta) = \cot \gamma \quad \text{即} \quad \alpha + \beta = \gamma$$

88. 化已知兩式爲

$$x = \frac{\tan \theta}{\tan \theta \cos \phi + \sin \phi}, \quad y = \frac{\tan \phi}{\tan \phi \cos \theta + \sin \theta}$$

$$\text{故 } \frac{x}{y} = \frac{\tan \theta (\tan \phi \cos \theta + \sin \theta)}{\tan \phi (\tan \theta \cos \phi + \sin \phi)}$$

$$= \frac{\sin \theta (\tan \phi + \tan \theta)}{\sin \phi (\tan \phi + \tan \theta)} = \frac{\sin \theta}{\sin \phi}$$

89.  $\therefore \frac{u_{n+2} - u_{n+1}}{u_n} = \frac{\sin^{n+2}\theta + \cos^{n+2}\theta - \sin^{n+4}\theta - \cos^{n+4}\theta}{\sin^n\theta + \cos^n\theta}$

$$= \frac{\sin^{n+2}\theta \cos^2\theta + \cos^{n+2}\theta \sin^2\theta}{\sin^n\theta + \cos^n\theta}$$

$$= \frac{\sin^2\theta \cos^2\theta (\sin^n\theta + \cos^n\theta)}{\sin^n\theta + \cos^n\theta}$$

$$= \sin^2\theta \cos^2\theta = \text{定數}$$

今以  $n=1$  時爲左邊， $n=3$  時爲右邊故如題云。

90. 左邊  $= (\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \cdots \cdots$

$$\cdots \cdots (\cos 2n\theta + i \sin 2n\theta)$$

$$= \cos(\theta + 2\theta + \cdots + 2n\theta)$$

$$+ i \sin(\theta + 2\theta + \cdots + 2n\theta) \quad (\text{從棣美弗定理})$$

但  $\theta + 2\theta + \cdots + 2n\theta = \frac{2n}{2}(1+2n)\theta = n(2n+1)\theta$

(A.P.  $2n$  項之和)

故原式左邊  $= \cos n(2n+1)\theta + i \sin n(2n+1)\theta$

$$= U_{n(2n+1)}$$

91.  $\tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$

$$= \frac{\frac{x \sin \alpha}{y - x \cos \alpha} + \frac{y \sin \alpha}{x - y \cos \alpha}}{1 - \frac{x \sin \alpha}{y - x \cos \alpha} \cdot \frac{y \sin \alpha}{x - y \cos \alpha}}$$

$$= \frac{\sin \alpha [(x^2 + y^2) - 2xy \cos \alpha]}{xy - (x^2 + y^2) \cos \alpha + xy(\cos^2 \alpha - \sin^2 \alpha)}$$



$$\begin{aligned}
 &= \frac{[(x^2 + y^2) - 2xy \cos \alpha] \sin \alpha}{-[(x^2 + y^2) - 2xy \cos \alpha] \cos \alpha} \\
 &= -\tan \alpha, \quad \text{故如題云。}
 \end{aligned}$$

$$\begin{aligned}
 92. \quad \text{今} \quad &\sin^2 \theta \cos(\theta - \alpha) \cos(\theta - \beta) \\
 &= \cos^2 \theta \sin(\theta - \alpha) \sin(\theta - \beta) \\
 \therefore &\sin^2 \theta [\cos(2\theta - \alpha - \beta) + \cos(\alpha - \beta)] \\
 &= \cos^2 \theta [\cos(\alpha - \beta) - \cos(2\theta - \alpha - \beta)]
 \end{aligned}$$

$$\text{故} \quad \cos(2\theta - \alpha - \beta) = \cos(\alpha - \beta) \cos 2\theta$$

$$\begin{aligned}
 \text{即} \quad &\cos 2\theta \cos(\alpha + \beta) + \sin 2\theta \sin(\alpha + \beta) \\
 &= \cos 2\theta \cos(\alpha - \beta)
 \end{aligned}$$

$$\therefore \tan 2\theta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{\sin(\alpha + \beta)} = \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$93. \quad \tan z = \frac{2 \tan \frac{z}{2}}{1 - \tan^2 \frac{z}{2}} = \frac{2 \tan \frac{x}{2} \tan \frac{y}{2}}{1 - \tan^2 \frac{x}{2} \tan^2 \frac{y}{2}}$$

$$\begin{aligned}
 &\frac{2 \sin \frac{x}{2} \sin \frac{y}{2}}{\cos \frac{x}{2} \cos \frac{y}{2}} \\
 &= \frac{\cos^2 \frac{x}{2} \cos^2 \frac{y}{2} - \sin^2 \frac{x}{2} \sin^2 \frac{y}{2}}{\cos^2 \frac{x}{2} \cos^2 \frac{y}{2}}
 \end{aligned}$$

$$= \frac{4 \sin \frac{x}{2} \sin \frac{y}{2} \cos \frac{x}{2} \cos \frac{y}{2}}{2 \cos \left( \frac{x}{2} + \frac{y}{2} \right) \cos \left( \frac{x}{2} - \frac{y}{2} \right)} = \frac{\sin x \sin y}{\cos x + \cos y}$$

$$94. \quad \text{今 } \sin^2 \theta = \frac{\cos \beta - \cos \alpha}{\cos \beta}, \quad \sin^2 \phi = \frac{\cos \beta - \cos \gamma}{\cos \beta}$$

$$\text{則 } \tan^2 \theta = \frac{\cos \beta - \cos \alpha}{\cos \alpha}, \quad \tan^2 \phi = \frac{\cos \beta - \cos \gamma}{\cos \gamma}$$

$$\therefore \frac{\tan^2 \theta}{\tan^2 \phi} = \frac{\tan^2 \alpha}{\tan^2 \gamma} = \frac{\cos \gamma (\cos \beta - \cos \alpha)}{\cos \alpha (\cos \beta - \cos \gamma)}$$

$$\therefore \sin^2 \alpha \cos \gamma (\cos \beta - \cos \gamma) \\ = \sin^2 \gamma \cos \alpha (\cos \beta - \cos \alpha)$$

$$\therefore \cos \beta = \frac{\sin^2 \alpha \cos^2 \gamma - \sin^2 \gamma \cos^2 \alpha}{\sin^2 \alpha \cos \gamma - \sin^2 \gamma \cos \alpha} \\ = \frac{\cos^2 \gamma - \cos^2 \alpha}{(\cos \gamma - \cos \alpha)(1 + \cos \alpha \cos \gamma)} \\ = \frac{\cos \gamma + \cos \alpha}{1 + \cos \alpha \cos \gamma}$$

$$\text{則 } \frac{1 - \cos \beta}{1 + \cos \beta} = \frac{(1 - \cos \alpha)(1 - \cos \gamma)}{(1 + \cos \alpha)(1 + \cos \gamma)}$$

$$\text{即 } \tan \frac{\beta}{2} = \tan \frac{\alpha}{2} \tan \frac{\gamma}{2}$$

$$95. \quad \therefore \frac{\tan(\theta + \alpha)}{x} = \frac{\tan(\theta + \beta)}{y} = \frac{\tan(\theta + \alpha) + \tan(\theta + \beta)}{x + y} \\ = \frac{\tan(\theta + \alpha) - \tan(\theta + \beta)}{x - y}$$

$$\therefore \frac{x + y}{x - y} = \frac{\tan(\theta + \alpha) + \tan(\theta + \beta)}{\tan(\theta + \alpha) - \tan(\theta + \beta)} = \frac{\sin(2\theta + \alpha + \beta)}{\sin(\alpha - \beta)}$$

$$\therefore \frac{x + y}{x - y} \sin^2(\alpha - \beta) = \sin(\alpha - \beta) \sin(2\theta + \alpha + \beta) \\ = \frac{1}{2} [\cos 2(\theta + \beta) - \cos 2(\theta + \alpha)]$$

$$\text{同理 } \frac{y+z}{y-z} \sin^2(\beta-\gamma) = \frac{1}{2} [\cos 2(\theta+\gamma) - \cos 2(\theta+\beta)]$$

$$\frac{z+x}{z-x} \sin^2(\gamma-\alpha) = \frac{1}{2} [\cos 2(\theta+\alpha) - \cos 2(\theta+\gamma)]$$

$$\therefore \Sigma \frac{x+y}{x-y} \sin^2(\alpha-\beta) = 0$$

$$96. \text{ 今 } \frac{\sin \alpha}{\cos \alpha} = \frac{n}{m} \quad \text{故 } \frac{m-n}{m+n} = \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha}$$

$$\begin{aligned} \text{故 } \sqrt{\frac{m-n}{m+n}} + \sqrt{\frac{m+n}{m-n}} &= \sqrt{\frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha}} + \sqrt{\frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha}} \\ &= \frac{\cos \alpha - \sin \alpha}{\sqrt{\cos^2 \alpha - \sin^2 \alpha}} + \frac{\cos \alpha + \sin \alpha}{\sqrt{\cos^2 \alpha - \sin^2 \alpha}} = \frac{2 \cos \alpha}{\sqrt{\cos 2\alpha}} \end{aligned}$$

$$\begin{aligned} 97. \therefore \cot^2 x &= (\sqrt{2} \cot \theta \sin \phi - \cos \phi)^2 \\ &= \frac{(\sqrt{2} \cos \theta \sin \phi - \cos \phi \sin \theta)^2}{\sin^2 \theta} \\ \therefore \csc^2 x &= \frac{2 \cos^2 \theta \sin^2 \phi - \frac{1}{2} \sqrt{2} \sin 2\phi \sin 2\theta}{\sin^2 \theta} \\ &\quad + \frac{\cos^2 \phi \sin^2 \theta + \sin^2 \theta}{\sin^2 \theta} \end{aligned}$$

$$\begin{aligned} \therefore \cot^2 y &= (\sqrt{2} \cot \phi \sin \theta - \cos \theta)^2 \\ &= \frac{(\sqrt{2} \cos \phi \sin \theta - \cos \theta \sin \phi)^2}{\sin^2 \phi} \end{aligned}$$

$$\begin{aligned} \therefore \csc^2 y &= \frac{2 \cos^2 \phi \sin^2 \theta - \frac{1}{2} \sqrt{2} \sin 2\phi \sin 2\theta}{\sin^2 \phi} \\ &\quad + \frac{\cos^2 \theta \sin^2 \phi + \sin^2 \phi}{\sin^2 \phi} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(1 - \sin^2 \phi)(1 - \cos^2 \theta) - \frac{1}{2} \sqrt{2} \sin 2\phi \sin 2\theta}{\sin^2 \phi} \\
 &\quad + \frac{\cos^2 \phi \sin^2 \theta + \cos^2 \theta \sin^2 \phi + \sin^2 \phi}{\sin^2 \phi} \\
 &= \frac{2 \cos^2 \theta \sin^2 \phi - \frac{1}{2} \sqrt{2} \sin 2\phi \sin 2\theta + \cos^2 \phi \sin^2 \theta + \sin^2 \theta}{\sin^2 \phi}
 \end{aligned}$$

$$\therefore \frac{\csc^2 y}{\csc^2 x} = \frac{\sin^2 x}{\sin^2 y} = \frac{\sin^2 \theta}{\sin^2 \phi}$$

98. 從  $\tan(\overline{\alpha + \beta} - \beta) = m \tan(\overline{\alpha + \beta} - \alpha)$   
 即  $\tan(\gamma - \beta) = m \tan(\gamma - \alpha)$   
 即  $\sin(\gamma - \beta) \cos(\gamma - \alpha) = m \sin(\gamma - \alpha) \cos(\gamma - \beta)$   
 即  $\sin(2\gamma - \alpha - \beta) + \sin(\alpha - \beta)$   
 $= m[\sin(2\gamma - \alpha - \beta) - \sin(\alpha - \beta)]$   
 即  $\sin \gamma + \sin(\alpha - \beta) = m[\sin \gamma - \sin(\alpha - \beta)]$   
 $(\because \alpha + \beta = \gamma)$

$$\therefore \sin \gamma = \frac{m+1}{m-1} \sin(\alpha - \beta)$$

99. 今  $(1 + \sin x)^2 (1 + \sin y)^2 (1 + \sin z)^2$   
 $= (1 - \sin^2 x)(1 - \sin^2 y)(1 - \sin^2 z)$

故  $\Pi(1 + \sin x) = \Pi(1 - \sin x)$

乘出得  $\Sigma \sin x = -\Pi \sin x$

令  $\sin x = u, \quad \sin y = v, \quad \sin z = w,$

$\cos x = k, \quad \cos y = m, \quad \cos z = n.$

則  $u + v + w = -uvw$

原

書

缺

頁

$$= 2 \cos C [\sin(A+B) - \cos(A-B)]$$

$$(\because \sin C = \sin \overline{A+B})$$

$$= 2 \cos C \left[ \sin\left(\frac{\pi}{4} + A + B - \frac{\pi}{4}\right) \right.$$

$$\left. - \sin\left(\frac{\pi}{4} + A - B + \frac{\pi}{4}\right) \right]$$

$$= 4 \cos C \cos\left(\frac{\pi}{4} + A\right) \sin\left(B - \frac{\pi}{4}\right)$$

$$7. \because \cos(A+B+C) = \cos 180^\circ = -1$$

$$\therefore \cos(A+B+C) = \cos A \cos B \cos C$$

$$- \Sigma \sin A \sin B \cos C = -1 \quad (\text{公式 15})$$

$$\text{或} \quad \Sigma \sin A \sin B \cos C = \Pi \cos A + 1$$

$$\text{即} \quad \frac{\Sigma \sin A \sin B \cos C}{\sin A \sin B \sin C} = \frac{\Pi \cos A + 1}{\sin A \sin B \sin C}$$

$$\text{故} \quad \Sigma \cot A = \Pi \cot A + \Pi \csc A$$

此係特別做法，普通可從化  $\Sigma \frac{\cos A}{\sin A}$  着手。

$$8. \text{ 從} \quad \sin \frac{1}{2}(A+B+C) = \sin \frac{1}{2}\pi = 1$$

$$\text{故} \quad \Sigma \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} - \Pi \sin \frac{A}{2} = 1$$

(公式 14, 見 73 頁)

移  $\Pi \sin \frac{A}{2}$  至右邊, 再兩邊以  $\Pi \cos \frac{1}{2}A$  除之

$$\text{即得} \quad \Sigma \tan \frac{1}{2}A = \Pi \tan \frac{1}{2}A + \Pi \sec \frac{1}{2}A$$

$$\begin{aligned}
 9. \quad \Sigma \cos^2 \alpha &= \Sigma \frac{1}{2} (1 + \cos 2\alpha) \\
 &= \frac{3}{2} + \frac{1}{2} \Sigma \cos 2\alpha \quad (\text{參考 143 頁例七}) \\
 &= \frac{3}{2} + \frac{1}{2} (-1 - 4\Pi \cos \alpha) \quad (\text{見 142 頁例六}) \\
 &= 1 - 2\Pi \cos \alpha
 \end{aligned}$$

$$10. \quad \text{今 } \sin A = \sin(B+C)$$

$$\begin{aligned}
 \text{故右邊} &= \sin A [\sin(B+C) + \sin(B-C)] \\
 &= \sin^2 A + \sin(B+C)\sin(B-C) \\
 &= \sin^2 A - \frac{1}{2}(\cos 2B - \cos 2C) \\
 &= \sin^2 A - \frac{1}{2}(1 - 2\sin^2 B - 1 + 2\sin^2 C) \\
 &= \sin^2 A + \sin^2 B - \sin^2 C \quad (\text{此題亦可如 9 題法做})
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \Sigma \cos^2 \frac{1}{2} A &= \frac{1}{2} \Sigma (1 + \cos A) = \frac{3}{2} + \frac{1}{2} \Sigma \cos A \\
 &= \frac{3}{2} + \frac{1}{2} \left( 1 + 4\Pi \sin \frac{1}{2} A \right) \quad (\text{見 138 頁例一}) \\
 &= 2 + 2\Pi \sin \frac{1}{2} A
 \end{aligned}$$

$$12. \quad \because 3A + 3B + 3C = 3\pi \quad \therefore \frac{3}{2}(B+C) = \frac{3}{2}\pi - \frac{3}{2}A$$

$$\text{故} \quad \sin \frac{3}{2}(B+C) = -\cos \frac{3}{2}A,$$

$$\cos \frac{3}{2}A = -\sin \frac{3}{2}(B+C)$$

$$\text{今左邊} = 2 \sin \frac{3}{2}A \cos \frac{3}{2}A + 2 \sin \frac{3}{2}(B+C) \cos \frac{3}{2}(B-C)$$

$$\begin{aligned}
 &= -2 \cos \frac{3}{2} (B+C) \cos \frac{3}{2} A - 2 \cos \frac{3}{2} A \cos \frac{3}{2} (B-C) \\
 &= -2 \cos \frac{3}{2} A [\cos \frac{3}{2} (B+C) + \cos \frac{3}{2} (B-C)] \\
 &= -4 \Pi \cos \frac{3}{2} A
 \end{aligned}$$

13.  $\therefore \cos(\beta + \gamma) = \cos(2\pi - \alpha) = \cos \alpha$

又  $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$

$$\begin{aligned}
 \text{今 } \Sigma \sin^2 \alpha &= \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) \\
 &= \frac{3}{2} - \frac{1}{2}[2 \cos^2 \alpha - 1 + 2 \cos(\beta + \gamma) \cos(\beta - \gamma)] \\
 &= 2 - [\cos \alpha \cos(\beta + \gamma) + \cos \alpha \cos(\beta - \gamma)] \\
 &= 2 - \cos \alpha [\cos(\beta + \gamma) + \cos(\beta - \gamma)] \\
 &= 2 - 2 \Pi \cos \alpha
 \end{aligned}$$

14.  $\therefore \cot(\alpha + \beta) = \cot(\frac{1}{2}\pi - \gamma) = \tan \gamma = 1/\cot \gamma$

即  $\frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha} = \frac{1}{\cot \gamma}$

去分母移項即得  $\Sigma \cot \alpha = \Pi \cot \alpha$

$$\begin{aligned}
 15. \quad \therefore \frac{1}{\cot \frac{C}{2}} &= \tan \frac{C}{2} = \tan\left(\frac{\pi}{2} - \frac{A+B}{2}\right) \\
 &= \cot \frac{A+B}{2} = \frac{\cot \frac{A}{2} \cot \frac{B}{2} - 1}{\cot \frac{B}{2} + \cot \frac{A}{2}}
 \end{aligned}$$

$$\therefore \Sigma \cot \frac{A}{2} = \Pi \cot \frac{A}{2}$$



但  $\Sigma \tan \frac{A}{2} = \Pi \tan \frac{A}{2} + \Pi \sec \frac{A}{2}$  (由題 8)

故  $(\Sigma \tan \frac{A}{2})(\Sigma \cot \frac{A}{2}) = 1 + \Pi \csc \frac{A}{2}$

16. 今  $\frac{\Sigma \tan \alpha}{(\Sigma \sin \alpha)^2} = \frac{\Pi \tan \alpha}{(4\Pi \cos \frac{\alpha}{2})^2}$  (見 140 頁例三及附錄一公式 50 A)

$$= \frac{\Pi(\sin \alpha / \cos \alpha)}{16(\Pi \cos \frac{\alpha}{2})^2}$$

$$= \frac{8 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2}}{16 \cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} (\Pi \cos \alpha)}$$

$$= \frac{1}{2} \left( \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \right) \cdot \frac{1}{\Pi \cos \alpha}$$

$$= \frac{\Pi \tan \frac{\alpha}{2}}{2\Pi \cos \alpha}$$

17. 左邊 =  $\Sigma \frac{\cos A}{\sin B \sin C} = \frac{\Sigma \sin A \cos A}{\Pi \sin A} = \frac{\Sigma \sin 2A}{2\Pi \sin A}$   
 $= \frac{4\Pi \sin A}{2\Pi \sin A} = 2$  (見題 3) 故左邊 = 右邊

18. 左邊去括號, 且以

$$\sin A \cos B + \cos A \sin B = \sin(A+B), \dots\dots\dots$$

$$\text{則左邊} = \sin(A+B) + \sin(B+C) + \sin(C+A)$$

$$= \sin(\pi - C) + \sin(\pi - B) + \sin(\pi - A)$$

$$= \sin C + \sin B + \sin A = \Sigma \sin A$$

如以左邊 =  $\frac{1}{2}(\Sigma(\sin \overline{A+B} + \sin \overline{A-B})$   
 $+ \Sigma(\sin \overline{A+C} + \sin \overline{A-C}))$  亦可做。

$$\begin{aligned} 19. \text{ 左邊} &= 2 \sin 2\alpha \cos 2\alpha + 2 \sin 2(\beta + \gamma) \cos 2(\beta - \gamma) \\ &= 2 \sin 2\alpha \cos(2\pi - 2\beta - 2\gamma) \\ &\quad + 2 \sin(2\pi - 2\alpha) \cos 2(\beta - \gamma) \\ &= 2 \sin 2\alpha [\cos 2(\beta + \gamma) - \cos 2(\beta - \gamma)] \\ &= -4\pi \sin 2\alpha \end{aligned}$$

$$\begin{aligned} 20. \because 2\alpha + 2\beta + 2\gamma = 2\pi \quad \therefore \cos 2(\alpha + \beta) &= \cos 2\gamma \\ \Sigma \sin^2 2\alpha &= 1 - \frac{1}{2}(\cos 4\alpha + \cos 4\beta) + (1 - \cos^2 2\gamma) \\ &= 2 - \cos 2(\alpha + \beta) \cos 2(\alpha - \beta) - \cos^2 2\gamma \\ &= 2 - \cos 2\gamma [\cos 2(\alpha - \beta) + \cos 2(\alpha + \beta)] \\ &= 2 - 2\pi \cos 2\alpha \end{aligned}$$

$$\begin{aligned} 21. \text{ 設} \quad x + y + z &= \frac{1}{2}\pi \\ \text{則} \quad \cos z &= \sin(x + y) \\ \text{又} \quad \frac{\pi}{2} - x &= y + z, \\ \frac{\pi}{2} - y &= z + x \end{aligned}$$

故得

$$\begin{aligned} \Sigma \cos x &= 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) \\ &\quad + 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x + y) \\ &= 2 \cos \frac{1}{2}(x + y) \left[ \cos \frac{1}{2}(x - y) + \sin \frac{1}{2}(x + y) \right] \end{aligned}$$

$$\begin{aligned}
 &= 2 \cos \frac{1}{2}(x+y) \left[ \cos \frac{1}{2}(x-y) + \cos \left( \frac{\pi}{2} - \frac{x+y}{2} \right) \right] \\
 &= 2 \cos \frac{x+y}{2} \left[ 2 \cos \frac{1}{2} \left( \frac{\pi}{2} - x \right) \cos \frac{1}{2} \left( \frac{\pi}{2} - y \right) \right] \\
 &= 4 \Pi \cos \frac{x+y}{2}
 \end{aligned}$$

今因

$$\left( \frac{3A}{2} + B - 2C \right) + \left( \frac{3B}{2} + C - 2A \right) + \left( \frac{3C}{2} + A - 2B \right) = \frac{\pi}{2}$$

故以  $\frac{3A}{2} + B - 2C$  代  $x$ , 又第二, 第三括號中之數各代  $y$  及  $z$ , 即得本題之證也。

22. 從  $\cos^4 \alpha = \frac{1}{8}(\cos 4\alpha + 4 \cos 2\alpha + 3)$  見 20 節(100 頁)

$$\begin{aligned}
 \text{故 } \Sigma \cos^4 \alpha &= \frac{1}{8} \Sigma \cos 4\alpha + \frac{1}{2} \Sigma \cos 2\alpha + \frac{9}{8} \\
 &= \frac{1}{8}(4\Pi \cos 2\alpha - 1) + \frac{1}{2}(-4\Pi \cos \alpha - 1) + \frac{9}{8} \\
 &\quad (\text{令題 25 中之 } m \text{ 爲 } 2 \text{ 及 } 1 \text{ 而得}) \\
 &= \frac{1}{2}(1 - 4\Pi \cos \alpha + \Pi \cos 2\alpha)
 \end{aligned}$$

23. 左邊  $= \frac{1}{2} \Sigma [\sin(3\alpha + \beta - \gamma) - \sin(3\alpha - \beta + \gamma)]$

$$= \frac{1}{2} \Sigma [\sin 2(\gamma - \alpha) + \sin 2(\alpha - \beta)]$$

(以  $\alpha + \beta + \gamma = \pi$  之關係代入)

$$= \sin 2(\gamma - \alpha) + \sin 2(\beta - \gamma) + \sin 2(\alpha - \beta)$$

$$= -4\Pi \sin(\alpha - \beta) \quad (\text{見 } 108 \text{ 頁例十七})$$

$$24. \therefore \frac{\tan \gamma + \tan \alpha}{\tan \beta + \tan \beta} = \frac{\sin(\gamma + \alpha)}{\tan \beta \cos \gamma \cos \alpha} = \frac{\sin(\pi - \beta)}{\tan \beta \cos \gamma \cos \alpha}$$

$$= \frac{\sin \beta}{\tan \beta \cos \gamma \cos \alpha} = \frac{\cos \beta}{\cos \gamma \cos \alpha}$$

$$\therefore \text{原式左邊} = \frac{\sum \cos^2 \alpha}{\Pi \cos \alpha} = \frac{1 - 2\Pi \cos \alpha}{\Pi \cos \alpha} \quad (\text{見題 9})$$

$$= \Pi \sec \alpha - 2$$

$$25. \therefore \cos m\alpha = \cos(m\pi - m\beta + \gamma) = (-1)^m \cos m(\beta + \gamma)$$

$$\therefore \cos m(\beta + \gamma) = (-1)^m \cos m\alpha$$

$$\text{故 } \sum \cos 2m\alpha = 2 \cos^2 m\alpha - 1 + 2 \cos m(\beta + \gamma) \cos m(\beta - \gamma)$$

$$= 2 \cos m\alpha [(-1)^m \cos m(\beta + \gamma)]$$

$$+ 2 [(-1)^m \cos m\alpha \cos m(\beta - \gamma)] - 1$$

$$= (-1)^m \cdot 2 \cos m\alpha [\cos m(\beta + \gamma)$$

$$+ \cos m(\beta - \gamma)] - 1$$

$$= (-1)^m \cdot 4\Pi \cos m\alpha - 1$$

$$26. \therefore \sin \frac{n(\alpha + \beta)}{2} = \sin \frac{n(\pi - \gamma)}{2} = \sin \left( \frac{4m \pm 1}{2} \pi - \frac{n\gamma}{2} \right)$$

$$= \sin \left[ 2m\pi \pm \left( \frac{\pi}{2} \mp \frac{n\gamma}{2} \right) \right]$$

$$= \pm \sin \left( \frac{\pi}{2} \mp \frac{n\gamma}{2} \right) = \pm \cos \frac{n\gamma}{2}$$

$$\text{亦即 } \cos \frac{n\gamma}{2} = \pm \sin \frac{n(\alpha + \beta)}{2}$$

$$\text{故 } \sum \sin n\alpha = 2 \sin \frac{n(\alpha + \beta)}{2} \cos \frac{n(\alpha - \beta)}{2}$$

$$+ 2 \sin \frac{n\gamma}{2} \cos \frac{n\gamma}{2}$$

$$\begin{aligned}
 &= \pm 2 \cos \frac{n\gamma}{2} \left[ \cos \frac{n(\alpha - \beta)}{2} + \cos \frac{n(\alpha + \beta)}{2} \right] \\
 &= \pm 4 \Pi \cos \frac{n\alpha}{2} = 4 \sin \frac{n\pi}{2} \cdot \Pi \cos \frac{n\alpha}{2}
 \end{aligned}$$

因  $\sin \frac{n}{2} \pi = \sin \left( \frac{4m \pm 1}{2} \right) \pi = \sin \left( 2m\pi \pm \frac{\pi}{2} \right)$

$$= \sin \left( \pm \frac{\pi}{2} \right) = \pm 1 \text{ 也.}$$

27. 因  $\sin B + \sin C - \sin A = 4 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2}$  (見題 2)

又  $\Sigma \sin A = 4 \Pi \cos \frac{A}{2}$

(公式 50A 或在 26 題中設  $n=1$ )

故  $(\Sigma \sin A)(\sin B + \sin C - \sin A)$

$$= 4 \sin B \sin C \cos^2 \frac{A}{2}, \text{ 故如題云.}$$

28. 因  $\Sigma \sin^3 \alpha = \frac{3}{4} \Sigma \sin \alpha - \frac{1}{4} \Sigma \sin 3\alpha$  (見 143 頁例八)

又因  $\alpha + \beta + \gamma = 2\pi$ , 故於 145 頁例十一中, 以  $\alpha$  代  $2A$ , ……

並令  $m=1$  及 3, 則得

$$\Sigma \sin \alpha = 4 \Pi \sin \frac{\alpha}{2}, \quad \Sigma \sin 3\alpha = 4 \Pi \sin \frac{3\alpha}{2}, \quad \text{代入之即得.}$$

29.  $\therefore \Sigma \sin A(1 + 2 \cos B) = \Sigma [\sin A + \sin(A+B)$

$$+ \sin(A-B)]$$

$$= \sin A - \sin C + \sin(A-B) + \dots$$

$$[\because \sin \overline{A+B} = \sin(2\pi - C) = -\sin C, \dots]$$

$$= \sin(A-B) + \sin(B-C) + \sin(C-A)$$

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36. 將左邊乘出,再用公式 14,15 (附錄一)得

$$2\Pi \sin \alpha + 2\Pi \cos \alpha + \sin(\alpha + \beta + \gamma) - \cos(\alpha + \beta + \gamma)$$

$$\text{因 } \alpha + \beta + \gamma = \begin{cases} 2n\pi \\ (2n - \frac{1}{2})\pi \end{cases}$$

$$\text{則 } \sin(\alpha + \beta + \gamma) - \cos(\alpha + \beta + \gamma) = \begin{cases} 0 - 1 \\ -1 + 0 \end{cases} = -1$$

故如題云。

37. 今  $\delta = 2\pi - (\alpha + \beta + \gamma) \quad \therefore \sin \delta = -\sin(\alpha + \beta + \gamma)$

$$\therefore \sin \alpha + \sin \beta + \sin \gamma + \sin \delta$$

$$= \sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma)$$

$$= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha + \beta + 2\gamma}{2}$$

$$= 2 \sin \frac{\alpha + \beta}{2} \left[ \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta + 2\gamma}{2} \right]$$

$$= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

38. 今  $\frac{\alpha + \beta + \gamma + \delta}{2} = \pi \quad \text{即} \quad \frac{\alpha + \beta}{2} = \pi - \frac{\gamma + \delta}{2}$

$$\therefore \cos \frac{\alpha + \beta}{2} = \cos \left( \pi - \frac{\gamma + \delta}{2} \right) = -\cos \frac{\gamma + \delta}{2}$$

$$\text{即} \quad \cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \sin \frac{\beta}{2}$$

$$= -\cos \frac{\gamma}{2} \cos \frac{\delta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2} \quad \text{移項即得}$$

39. 左邊 = 
$$\begin{vmatrix} 1 & \cos \gamma & \cos \beta \\ 0 & \sin^2 \gamma & \cos \alpha - \cos \beta \cos \gamma \\ 0 & \cos \alpha - \cos \beta \cos \gamma & \sin^2 \beta \end{vmatrix}$$

$$\begin{aligned}
&= \sin^2 \gamma \sin^2 \beta - (\cos \alpha - \cos \beta \cos \gamma)^2 \\
&= (\sin \gamma \sin \beta - \cos \beta \cos \gamma + \cos \alpha) \\
&\quad \times (\sin \gamma \sin \beta + \cos \beta \cos \gamma - \cos \alpha) \\
&= [\cos \alpha - \cos(\beta + \gamma)][\cos(\beta - \gamma) - \cos \alpha] \\
&= 4 \sin \frac{\beta + \gamma + \alpha}{2} \sin \frac{\beta + \gamma - \alpha}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{\alpha + \gamma - \beta}{2} \\
&= 4 \sin S \cdot \Pi \sin(S - \alpha) \\
&\quad (\because \beta + \gamma - \alpha = 2(S - \alpha), \dots\dots\dots)
\end{aligned}$$

40. 左邊 =  $\frac{\sin(2S - \alpha - \beta)}{\cos(S - \alpha)\cos(S - \beta)} + \frac{\sin(S - \gamma - S)}{\cos(S - \gamma)\cos S}$

$$\begin{aligned}
&= \frac{2 \sin \gamma}{\cos \gamma + \cos(\alpha - \beta)} - \frac{2 \sin \gamma}{\cos(\alpha + \beta) + \cos \gamma} \\
&= \frac{2 \sin \gamma [\cos(\alpha + \beta) - \cos(\alpha - \beta)]}{\cos^2 \gamma + \cos \gamma [\cos(\alpha - \beta) + \cos(\alpha + \beta)] + \cos(\alpha - \beta)\cos(\alpha + \beta)} \\
&= \frac{4\Pi \sin \alpha}{1 - \Sigma \cos^2 \alpha - 2\Pi \cos \alpha}
\end{aligned}$$

41. 化已知式得

$$2 \cos(\gamma + \delta) \cos(\alpha - \beta) = 2 \cos(\alpha + \beta) \cos(\gamma - \delta)$$

即  $\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{\cos(\gamma - \delta)}{\cos(\gamma + \delta)}$

故  $\frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{\cos(\alpha - \beta) + \cos(\alpha + \beta)} = \frac{\cos(\gamma - \delta) - \cos(\gamma + \delta)}{\cos(\gamma - \delta) + \cos(\gamma + \delta)}$

(比例合分之理)

即  $\frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} = \frac{\sin \gamma \sin \delta}{\cos \gamma \cos \delta}$

即  $\tan \alpha \tan \beta = \tan \gamma \tan \delta$



$$42. \text{ 今 } \sin \alpha + \sin \gamma = 2 \sin \beta = 2 \sin(\alpha + \gamma)$$

$$\text{即 } \cos \frac{\alpha - \gamma}{2} = 2 \cos \frac{\alpha + \gamma}{2}$$

$$\begin{aligned} \text{即 } \cos \frac{\alpha}{2} \cos \frac{\gamma}{2} + \sin \frac{\alpha}{2} \sin \frac{\gamma}{2} \\ = 2 \left( \cos \frac{\alpha}{2} \cos \frac{\gamma}{2} - \sin \frac{\alpha}{2} \sin \frac{\gamma}{2} \right) \end{aligned}$$

$$\text{即 } 3 \sin \frac{\alpha}{2} \sin \frac{\gamma}{2} = \cos \frac{\alpha}{2} \cos \frac{\gamma}{2}$$

$$\therefore \tan \frac{\alpha}{2} \tan \frac{\gamma}{2} = \frac{1}{3}$$

$$43. \text{ 今 } \tan \alpha + \tan \gamma = 2 \tan \beta = -2 \tan(\alpha + \gamma)$$

$$= \frac{-2(\tan \alpha + \tan \gamma)}{1 - \tan \alpha \tan \gamma}$$

$$\therefore 1 - \tan \alpha \tan \gamma = -2 \quad \text{即} \quad \tan \alpha \tan \gamma = 3$$

$$\therefore \tan \alpha = 3 \cot \gamma \quad \text{或} \quad \tan^2 \alpha = 9 \cot^2 \gamma$$

$$\text{故 } \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha} = \frac{9(1 + \cos 2\gamma)}{1 - \cos 2\gamma}$$

$$\text{即 } \frac{1}{\cos 2\alpha} = \frac{5 + 4 \cos 2\gamma}{-(4 + 5 \cos 2\gamma)} \quad (\text{合分之理})$$

$$\text{但 } \cos 2\alpha = -\cos(\pi - 2\alpha) = -\cos(\beta + \gamma - \alpha)$$

$$\therefore \cos(\beta + \gamma - \alpha) = \frac{4 + 5 \cos 2\gamma}{5 + 4 \cos 2\gamma}$$

$$44. \quad (1) \text{ 左邊} = \begin{vmatrix} \sin^2 A - \sin^2 B & \cot A - \cot B & 0 \\ \sin^2 B - \sin^2 C & \cot B - \cot C & 0 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \sin(A+B)\sin(A-B) & \frac{\sin(B-A)}{\sin A \sin B} & 0 \\ \sin(B+C)\sin(B-C) & \frac{\sin(C-B)}{\sin C \sin B} & 0 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$= \frac{\sin(A-B)\sin(B-C)}{\sin A \sin B \sin C} \begin{vmatrix} \sin C & -\sin C \\ \sin A & -\sin A \end{vmatrix} = 0$$

(2) 展開行列式得  $2 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}$

即  $2 \left( \Pi \sin \frac{1}{2} A \right)^2$

但從 138 頁例一得  $\Sigma \cos A = 1 + 4 \Pi \sin \frac{1}{2} A$

(因  $A+B+C=\pi$ )

故  $\left( \Pi \sin \frac{1}{2} A \right)^2 = \frac{1}{16} (\Sigma \cos A - 1)^2$  代入上式即得,

45. 令  $x = \tan \alpha$ ,  $y = \tan \beta$ ,  $z = \tan \gamma$

如 147 頁例十五推到  $\alpha + \beta + \gamma = n\pi$

則  $3\alpha + 3\beta + 3\gamma = 3n\pi = m\pi$

再如 145 頁例十二推到  $\Sigma \tan 3\alpha = \Pi \tan 3\alpha$

但  $\tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha} = \frac{3x - x^3}{1 - x^2}, \dots \dots$  (公式 23)

故  $\Sigma \frac{3x - x^3}{1 - x^2} = \Pi \frac{3x - x^3}{1 - x^2}$

46. 設  $x = \tan \alpha$ ,  $y = \tan \beta$ ,  $z = \tan \gamma$

則  $\Sigma \tan \alpha \tan \beta = 1$

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b. 設  $\operatorname{vers} x = y$  則  $x = \operatorname{vers}^{-1} y = \operatorname{vers}^{-1}(\operatorname{vers} x)$

又設  $\operatorname{vers}^{-1} x = \beta$  則  $x = \operatorname{vers} \beta = \operatorname{vers}(\operatorname{vers}^{-1} x)$

c. 設  $\sin^{-1} x = \alpha$  則  $\sin \alpha = x$

故  $\cos \alpha = \sqrt{1-x^2}$

又設  $\cos^{-1} x = \beta$  則  $\cos \beta = x$

故  $\sin \beta = \sqrt{1-x^2}$

$\therefore \cos \alpha = \sin \beta$ , 即  $\cos \sin^{-1} x = \sin \cos^{-1} x$

d. 設  $\sin^{-1} x = \alpha$  則  $x = \sin \alpha$

$\therefore -x = -\sin(\sin^{-1} x)$

又設  $y = -\sin x = \sin(-x)$

故  $-x = \sin^{-1} y = \sin^{-1}(-\sin x)$

故 左邊 = 右邊

e. 設  $\tan^{-1} x = \alpha$ ,  $\cot^{-1} x = \beta$

則  $\tan \alpha = x = \cot \beta$ , 即  $\tan \alpha = \tan\left(\frac{\pi}{2} - \beta\right)$

$\therefore \alpha = \frac{\pi}{2} - \beta$ , 即  $\alpha + \beta = \frac{\pi}{2}$

故如題云。(參考 170 頁例五)

f. 設  $\sec^{-1} \frac{y}{x} = \alpha$  則  $\sec \alpha = \frac{y}{x}$   $\therefore \cos \alpha = \frac{x}{y}$

又設  $\sin^{-1} \frac{x}{y} = \beta$  則  $\sin \beta = \frac{x}{y}$  以後做法同例五。

g. 設  $\sin^{-1} x = \alpha$  則  $\sin \alpha = x$

今  $\cos 2\alpha = 1 - 2\sin^2 \alpha = 1 - 2x^2$

$\therefore 2\alpha = \cos^{-1}(1 - 2x^2)$  故如題云。

h. 設  $\cos^{-1}x = \alpha$  則  $\cos \alpha = x$

$$\text{今 } \cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha = 4x^3 - 3x$$

$$\therefore 3\alpha = \cos^{-1}(4x^3 - 3x)$$

$$\text{即 } 3 \cos^{-1}x = \cos^{-1}(4x^3 - 3x)$$

i. 設  $\tan^{-1}x = \alpha$  則  $\tan \alpha = x$

$$\text{今 } \tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha} = \frac{3x - x^3}{1 - 3x^2}$$

$\therefore 3\alpha = \tan^{-1}[(3x - x^3):(1 - 3x^2)]$  故如題云。

j. 設第一及第二角各為  $\alpha, \beta$ , 則  $\tan \alpha = \frac{1}{2}$ ,  $\tan \beta = \frac{1}{3}$

$$\begin{aligned} \therefore \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - (\frac{1}{2})(\frac{1}{3})} \\ &= \frac{5/6}{5/6} = 1 \end{aligned}$$

$$\therefore \alpha + \beta = \frac{\pi}{4} \quad \text{故 } \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$$

k. 設  $\tan^{-1} \frac{3}{5} = \alpha$  則  $\tan \alpha = \frac{3}{5}$   $\therefore \cot \alpha = \frac{5}{3}$

$$\text{又設 } \cot^{-1} \frac{7}{3} = \beta \quad \text{則 } \cot \beta = \frac{7}{3}$$

代入  $\cot(\alpha - \beta)$  公式得

$$\cot(\alpha - \beta) = \frac{\frac{5}{3} \cdot \frac{7}{3} + 1}{\frac{7}{3} - \frac{5}{3}} = \frac{44}{6} = \frac{22}{3}$$

$$\therefore \alpha - \beta = \cot^{-1} \frac{22}{3} \quad \text{故如題云。}$$

l. 設第一, 第二角各為  $\alpha, \beta$

$$\text{則 } \tan \alpha = 2 + \sqrt{3}, \quad \tan \beta = 2 - \sqrt{3}$$

$$\begin{aligned}\therefore \tan(\alpha - \beta) &= \dots = \frac{(2 + \sqrt{3}) - (2 - \sqrt{3})}{1 + (2 + \sqrt{3})(2 - \sqrt{3})} \\ &= \frac{2\sqrt{3}}{2} = \sqrt{3}\end{aligned}$$

$$\text{故 } \sec(\alpha - \beta) = \sqrt{1+3} = 2 \quad \therefore \alpha - \beta = \sec^{-1}2$$

故如題云。(此題若從求  $\sec(\alpha - \beta)$  入手, 則必先化正切爲正餘弦, 便無此簡)

m. 設三角依次爲  $\alpha, \beta, \gamma$

$$\text{則 } \tan \alpha = \frac{1}{2}, \quad \tan \beta = \frac{1}{5}, \quad \tan \gamma = \frac{1}{8}$$

$$\text{則 } \tan(\alpha + \beta) = \frac{7}{9}$$

$$\text{又 } \tan(\overline{\alpha + \beta} + \gamma) = 1 \quad (\text{法同 2. j 題})$$

$$\therefore \alpha + \beta + \gamma = \frac{\pi}{4} \quad \text{故如題云。 (參考 172 頁例八)}$$

$$\text{n. 設 } \tan^{-1} \frac{1}{408} = \alpha$$

$$\text{則 } \tan \alpha = \frac{1}{408}, \quad \tan 2\alpha = \frac{816}{166463}$$

$$\text{又 } \tan^{-1} \frac{1}{1393} = \beta, \quad \text{則 } \tan \beta = \frac{1}{1393}$$

$$\begin{aligned}\text{今 } \tan(2\alpha - \beta) &= \frac{\frac{816}{166463} - \frac{1}{1393}}{1 - \frac{816}{166463} \cdot \frac{1}{1393}} \\ &= \frac{970225}{231883775} = \frac{1}{239}\end{aligned}$$

$$\therefore 2\alpha - \beta = \tan^{-1} \frac{1}{239}$$

故如題云。(此題數字甚大容易算錯)

o. 設兩角爲  $\alpha, \beta$

$$\text{得 } \tan(\alpha + \beta) = \frac{1}{2}, \quad \tan 2(\alpha + \beta) = \frac{4}{3}$$

$$\therefore \sec 2(\alpha + \beta) = \frac{5}{3} \quad \text{即 } \cos 2(\alpha + \beta) = \frac{3}{5}$$

$$\therefore \alpha + \beta = \frac{1}{2} \cos^{-1} \frac{3}{5} \quad \text{如題所云。}$$

p. 設四角依次爲  $\alpha, \beta, \gamma, \delta$

$$\text{則 } \tan(\alpha + \beta) = \frac{7}{11}, \quad \tan(\gamma + \delta) = \frac{2}{9}$$

再求  $\tan(\overline{\alpha + \beta + \gamma + \delta})$  即得。(此題類似 172 頁例八)

q. 設第一, 第二角各爲  $\alpha, \beta$

$$\text{則 } \cot \alpha = (a^3 + a^2 + a)^{\frac{1}{2}} = \sqrt{a(a^2 + a + 1)}$$

$$\cot \beta = (a + a^{-1} + 1)^{\frac{1}{2}} = \sqrt{\frac{a^2 + a + 1}{a}}$$

$$\text{故 } \cot(\alpha + \beta) = \frac{(a^2 + a + 1) - 1}{\left(\sqrt{a} + \frac{1}{\sqrt{a}}\right)\sqrt{a^2 + a + 1}}$$

$$= \frac{a(a+1)\sqrt{a}}{(a+1)\sqrt{a^2+a+1}}$$

$$\therefore \tan(\alpha + \beta) = \frac{\sqrt{a^2 + a + 1}}{a\sqrt{a}} = \sqrt{\frac{a^2 + a + 1}{a^3}}$$

$$= (a^{-1} + a^{-2} + a^{-3})^{\frac{1}{2}}$$

即  $\alpha + \beta = \tan^{-1}(a^{-3} + a^{-2} + a^{-1})^{\frac{1}{2}}$  如題所云。

r. 設兩角爲  $A, B$

$$\text{則 } \tan A = (\sqrt{2} + 1)\tan \alpha, \quad \tan B = (\sqrt{2} - 1)\tan \alpha$$

$$\begin{aligned} \therefore \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{2 \tan \alpha}{1 + \tan^2 \alpha} \\ &= \frac{2 \sin \alpha / \cos \alpha}{\sec^2 \alpha} = 2 \sin \alpha \cos \alpha \\ &= \sin 2\alpha \end{aligned}$$

$\therefore A - B = \tan^{-1}(\sin 2\alpha)$  故如題云。

s. 設左邊之角爲  $\alpha$

$$\text{則 } \sin \alpha = \sqrt{\frac{x-y}{x-z}}, \quad \cos \alpha = \sqrt{\frac{y-z}{x-z}}$$

$$\text{故 } \tan \alpha = \sqrt{\frac{x-y}{y-z}}$$

$\therefore \alpha = \tan^{-1} \sqrt{(x-y):(y-z)}$   $\therefore$  左邊 = 右邊

t. 設兩角依次爲  $\alpha, \beta$ , 則  $\sin \alpha = \frac{1}{\sqrt{82}}$ ,

$$\cos \alpha = \frac{9}{\sqrt{82}}, \quad \sin \beta = \frac{4}{\sqrt{41}}, \quad \cos \beta = \frac{5}{\sqrt{41}}$$

$$\therefore \tan \alpha = \frac{1}{9}, \quad \tan \beta = \frac{4}{5}$$

今  $\tan(\alpha + \beta) = \dots = 1 \quad \therefore \alpha + \beta = \frac{\pi}{4}$

故如題云。

此題之右邊爲  $\frac{\pi}{4}$ , 且  $\tan \alpha$  等亦比  $\sin \alpha$  等爲簡,

故用  $\tan(\alpha + \beta)$  做。否則仍以從  $\sin(\alpha + \beta)$  或  $\cos(\alpha + \beta)$  入手較爲適宜也。



u. 設第一, 第二兩角爲  $\alpha, \beta$

$$\text{則 } \sec \alpha = \frac{5}{3}, \quad \cos \alpha = \frac{3}{5}, \quad \sin \alpha = \frac{4}{5};$$

$$\sec \beta = \frac{13}{12}, \quad \cos \beta = \frac{12}{13}, \quad \sin \beta = \frac{5}{13}$$

$$\therefore \sin(\alpha + \beta) = \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65} = .9692$$

$$\text{故 } \alpha + \beta = 75^\circ 45'$$

v. 設第一, 第二兩角各爲  $\alpha, \beta$

$$\text{則 } \text{vers } \alpha = 1 - \cos \alpha = a$$

$$\text{即 } \cos \alpha = 1 - a, \quad \sin \alpha = \sqrt{2a - a^2}$$

$$\text{同理 } \cos \beta = 1 - b, \quad \sin \beta = \sqrt{2b - b^2}$$

$$\text{今 } \text{vers}(\alpha + \beta) = 1 - \cos(\alpha + \beta)$$

$$= 1 - [(1-a)(1-b) - \sqrt{(2a-a^2)(2b-b^2)}]$$

$$= a + b - ab + \sqrt{(2a-a^2)(2b-b^2)}$$

$$\therefore \alpha + \beta = \text{vers}^{-1} \left\{ a + b - ab + \sqrt{(2a-a^2)(2b-b^2)} \right\}$$

w. 令第一, 第二角各爲  $\alpha, \beta$ , 再求  $\cos(\alpha - \beta)$  即得, 法同上兩題。

x. 設  $\tan^{-1} x = \alpha$  則  $\tan \alpha = x, \quad \cos \alpha = \frac{1}{\sqrt{x^2 + 1}}$

$$\text{設 } \cot^{-1} \cos \alpha = \beta \quad \text{則 } \cot \beta = \frac{1}{\sqrt{x^2 + 1}}$$

$$\text{即 } \csc \beta = \frac{\sqrt{x^2 + 2}}{\sqrt{x^2 + 1}} \quad \therefore \sin \beta = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 2}}$$

$$\text{即 } \sin \cot^{-1} \cos \tan^{-1} x = [(x^2 + 1)/(x^2 + 2)]^{\frac{1}{2}}$$

y. 設第一, 第二角各爲  $\alpha, \beta$ , 求  $\sin(\alpha - \beta)$  即得 (法與  $u, v, w$  等題同).

z. 設第一, 第二, 第三角各爲  $\alpha, \beta, \gamma$

$$\text{則 } \tan \alpha = \frac{1}{3}, \quad \tan \gamma = \frac{1}{26}, \quad \tan \beta = \frac{1}{7}$$

$$\text{則 } \tan 3\beta = \frac{\frac{3}{7} - (\frac{1}{7})^3}{1 - 3(\frac{1}{7})^2} = \frac{147 - 1}{343 - 21} = \frac{73}{161}$$

(見附錄公式 23)

$$\tan(\alpha + \gamma) = \dots = \frac{29}{77}, \quad \tan\left(3\beta - \frac{\pi}{4}\right) = \dots = -\frac{44}{117}$$

$$\therefore \tan\left(\alpha + \gamma + 3\beta - \frac{\pi}{4}\right) = \frac{1}{2057}$$

$$\text{即 } \alpha + \gamma + 3\beta - \frac{\pi}{4} = \tan^{-1} \frac{1}{2057}$$

$$\text{故 } \tan^{-1} \frac{1}{3} + 3 \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{26} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2057}$$

$$\begin{aligned} 3. \text{ a. 設第一角爲 } \alpha, \text{ 則 } \tan \alpha &= \sqrt{\frac{a(a+b+c)}{bc}} \\ &= \frac{1}{bc} \sqrt{abc(a+b+c)} \end{aligned}$$

$$\begin{aligned} \text{第二角爲 } \beta, \text{ 則 } \tan \beta &= \sqrt{\frac{b(a+b+c)}{ca}} \\ &= \frac{1}{ca} \sqrt{abc(a+b+c)} \end{aligned}$$

$$\text{今 } \tan(\alpha + \beta) = \frac{\frac{a+b}{abc} \sqrt{abc(a+b+c)}}{1 - \frac{1}{abc^2} [abc(a+b+c)]}$$

$$\begin{aligned}
 &= \frac{\frac{a+b}{abc} \sqrt{abc(a+b+c)}}{-\frac{a+b}{c}} = -\frac{1}{ab} \sqrt{abc(a+b+c)} \\
 &= -\sqrt{\frac{c(a+b+c)}{ab}}
 \end{aligned}$$

即  $\tan(\pi - \alpha + \beta) = \sqrt{\frac{c(a+b+c)}{ab}}$

$\therefore \pi - \alpha + \beta = \tan^{-1} \sqrt{\frac{c(a+b+c)}{ab}}$  故如題云。

b. 設第一, 第二, 第三角各為  $\alpha, \beta, \gamma$

則  $\tan \alpha = \frac{1}{5}, \quad \tan 2\alpha = \frac{2/5}{1-1/25} = \frac{5}{12},$

$\tan 4\alpha = \frac{10/12}{1-25/144} = \frac{120}{119}, \quad \tan(4\alpha - \beta) = \frac{49}{50}$

又  $\tan\left(\frac{\pi}{4} - \gamma\right) = \frac{1-1/99}{1+1/99} = \frac{49}{50}$

故  $4\alpha - \beta = \frac{\pi}{4} - \gamma$  故如題云。

c. 設角依次為  $\alpha, \beta, \gamma, \delta$ , 求得

$\tan(\alpha + \beta) = \dots = x^2/2, \quad \tan(\gamma + \delta) = \dots = 2/x^2$

故  $\tan(\alpha + \beta + \gamma + \delta) = \frac{x^2/2 + 2/x^2}{1 - x^2/2 \cdot 2/x^2} = \infty$

$\therefore \alpha + \beta + \gamma + \delta = n\pi + \frac{\pi}{2}$

d. 設第一, 第二角各為  $\alpha, \beta$ , 則  $\sin \alpha = 4/5,$   
 $\cos \alpha = 3/5, \quad \cos \beta = 12/13, \quad \sin \beta = 5/13$

$$\text{故 } \sin(\alpha + \beta) = \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65}$$

$$\text{即 } \cos\left(\frac{\pi}{2} - \alpha + \beta\right) = \frac{63}{65}$$

$$\therefore \frac{\pi}{2} - \alpha - \beta = \cos^{-1} \frac{63}{65} \quad \text{故如題云。}$$

e. 設角依次爲  $\alpha, \beta, \gamma$  則  $\cos \alpha = \frac{4}{5}$ ,

$$\sin \beta = \frac{\sqrt{10}}{10}, \quad \sin \alpha = \frac{3}{5}, \quad \cos \beta = \frac{3\sqrt{10}}{10}$$

$$\text{故 } \tan \alpha = \frac{3}{4}, \quad \tan \beta = \frac{1}{3} \quad (\text{以下做法同 } 2.m)$$

f. 設第一, 二, 三角各爲  $\alpha, \beta, \gamma$ , 又  $\tan^{-1} x = \delta$

$$\text{則 } \tan \beta = x - 1, \quad \tan \gamma = x + 1,$$

$$\tan \alpha = \frac{2x}{2 + x^2 + x^4}$$

$$\text{故 } \tan(\beta + \gamma) = \frac{2x}{1 - (x^2 - 1)} = \frac{2x}{2 - x^2}$$

$$\begin{aligned} \tan(\alpha + \beta + \gamma) &= \frac{\frac{2x}{2 + x^2 + x^4} + \frac{2x}{2 - x^2}}{1 - \frac{2x}{2 + x^2 + x^4} \cdot \frac{2x}{2 - x^2}} \\ &= \frac{2x(4 + x^4)}{4 - 4x^2 + x^4 - x^6} = \frac{2x(4 + x^4)}{(1 - x^2)(4 + x^4)} \\ &= \frac{2x}{1 - x^2} \quad \text{又 } \tan 2\delta = \frac{2x}{1 - x^2} \end{aligned}$$

$$\therefore \alpha + \beta + \gamma = 2\delta \quad \text{故如題云。}$$

g. 設第一,二,三角各爲  $A, B, C$

$$\text{則 } \tan A = \frac{1}{2} \tan 2\alpha = \frac{\tan \alpha}{1 - \tan^2 \alpha}, \dots\dots$$

$$\begin{aligned} \text{今 } \tan(B+C) &= \frac{\cot \alpha + \cot^3 \alpha}{1 - \cot^4 \alpha} = \frac{\cot \alpha}{1 - \cot^2 \alpha} \\ &= \frac{\tan \alpha}{\tan^2 \alpha - 1} = -\tan A = \tan(-A) \end{aligned}$$

$\therefore B+C = -A$  即  $A+B+C=0$  故如題云。

h. 設  $\frac{1}{2} \cos^{-1} \frac{a}{b} = \alpha$  則  $\cos 2\alpha = \frac{a}{b}$

$$\text{又因 } \frac{\pi}{2} - \left(\frac{\pi}{4} - \alpha\right) = \frac{\pi}{4} + \alpha$$

$$\begin{aligned} \text{故 } \tan\left(\frac{\pi}{4} + \alpha\right) + \tan\left(\frac{\pi}{4} - \alpha\right) \\ &= \tan\left(\frac{\pi}{4} + \alpha\right) + \cot\left(\frac{\pi}{4} + \alpha\right) \\ &= \frac{1}{\sin\left(\frac{\pi}{4} + \alpha\right) \cos\left(\frac{\pi}{4} + \alpha\right)} = \frac{2}{\sin\left(\frac{\pi}{2} + 2\alpha\right)} \\ &= \frac{2}{\cos 2\alpha} = \frac{2b}{a} \end{aligned}$$

i. 設第一,二角爲  $\alpha, \beta$

$$\text{則 } \sin \alpha = \frac{2ab}{a^2 + b^2}, \quad \cos \alpha = \frac{a^2 - b^2}{a^2 + b^2}, \dots\dots$$

$$\begin{aligned} \therefore \sin(\alpha + \beta) &= \frac{2ab(c^2 - d^2)}{(a^2 + b^2)(c^2 + d^2)} + \frac{2cd(a^2 - b^2)}{(a^2 + b^2)(c^2 + d^2)} \\ &= \frac{2(ac - bd)(bc + ad)}{(ac - bd)^2 + (bc + ad)^2} \end{aligned}$$

$$\therefore \alpha + \beta = \sin^{-1} \frac{2xy}{x^2 + y^2} \quad \text{故如題云。}$$

j. 設第一, 第二角爲  $\alpha, \beta$

$$\text{則 } \tan \alpha = t, \quad \tan \beta = 2t/(1-t^2)$$

$$\begin{aligned} \text{今 } \tan(\alpha + \beta) &= \frac{t + \frac{2t}{1-t^2}}{1 - \frac{2t^2}{1-t^2}} = \frac{3t-t^3}{1-3t^2} \\ &= \frac{t(t+\sqrt{3})(t-\sqrt{3})}{(\sqrt{3}t+1)(\sqrt{3}t-1)} \quad (=x) \end{aligned}$$

$$\text{今 } t > 0, \text{ 則 } t + \sqrt{3} > 0, \quad \sqrt{3}t + 1 > 0$$

(1) 設  $t > \sqrt{3}$  或  $t < 1/\sqrt{3}$ , 則  $t - \sqrt{3}$  與  $\sqrt{3}t - 1$  同號, 故  $x$  爲正。

$$\therefore \tan(\alpha + \beta) = \frac{3t-t^3}{1-3t^2} \quad \therefore \alpha + \beta = \tan^{-1} \frac{3t-t^3}{1-3t^2}$$

(2) 設  $\sqrt{3} > t > 1/\sqrt{3}$ , 則  $t - \sqrt{3}$  與  $\sqrt{3}t - 1$  異號, 故  $x$  爲負, 但  $\alpha + \beta$  通常假定爲銳角。

$$\therefore \tan(\alpha + \beta) = -\frac{3t-t^3}{1-3t^2}$$

$$\text{即 } \tan(\pi - \alpha + \beta) = \frac{3t-t^3}{1-3t^2}$$

$$\text{即 } \pi - \alpha - \beta = \tan^{-1} \frac{3t-t^3}{1-3t^2}$$

$$\text{故 } \alpha + \beta = \pi - \tan^{-1} \frac{3t-t^3}{1-3t^2}$$

k. 設  $\tan^{-1}(\tan^3 \alpha) = \alpha \quad \therefore \tan \alpha = \tan^3 \alpha$

$$\text{則 } \tan(a+\alpha) = \frac{\tan a + \tan^3 a}{1 - \tan^4 a} = \frac{\tan a}{1 - \tan^2 a}$$

$$\text{又 } 2 \tan(a+\alpha) = \frac{2 \tan a}{1 - \tan^2 a} = \tan 2a$$

$$\therefore \tan^{-1}\{2 \tan(a+\alpha)\} = 2a \quad \text{故如題云.}$$

$$1. \text{ 設 } \cot^{-1}x = \alpha \quad \therefore \cot \alpha = x, \quad \tan \alpha = \frac{1}{x},$$

$$\tan 3\alpha = \frac{3/x - 1/x^3}{1 - 3/x^2} = \frac{3x^2 - 1}{x^3 - 3x}$$

$$\text{設 } \cos^{-1} \tan 3\alpha = \beta \quad \therefore \cos \beta = \frac{3x^2 - 1}{x^3 - 3x}$$

$$\text{則 } \sin \beta = \frac{\sqrt{x^6 - 15x^4 + 15x^2 - 1}}{x^3 - 3x}$$

$$\therefore \sin 2\beta = \frac{2(3x^2 - 1)\sqrt{x^6 - 15x^4 + 15x^2 - 1}}{x^3(x^3 - 3)^2}$$

故如題云.

$$m. \text{ 設 } \tan^{-1}x = \alpha \quad \text{則 } \tan \alpha = x = \cot\left(\frac{\pi}{2} - \alpha\right)$$

$$\text{故 } \cot^{-1}x = \frac{\pi}{2} - \alpha, \quad \csc \alpha = \frac{\sqrt{x^2 + 1}}{x},$$

$$\tan\left(\frac{\pi}{2} - \alpha\right) = \cot \alpha = \frac{1}{x}$$

$$\therefore \csc \tan^{-1}x - \tan \cot^{-1}x = (\sqrt{x^2 + 1} - 1)/x$$

$$\text{再設 } \tan^{-1} \frac{\sqrt{x^2 + 1} - 1}{x} = \beta$$

$$\therefore \tan \beta = \frac{\sqrt{x^2 + 1} - 1}{x}$$

$$\begin{aligned} \text{即 } \tan 2\beta &= \frac{2\left(\frac{\sqrt{x^2+1}-1}{x}\right)}{1-\frac{x^2+2-2\sqrt{x^2+1}}{x^2}} \\ &= \frac{2x(\sqrt{x^2+1}-1)}{2(\sqrt{x^2+1}-1)} = x \end{aligned}$$

$$\text{即 } 2\beta = \tan^{-1}x$$

$$\text{故 } \tan^{-1}x = 2 \tan^{-1}(\csc \tan^{-1}x - \tan \cot^{-1}x)$$

$$\text{n. 設 } \tan^{-1}\frac{a}{b} = 2\alpha, \quad \tan^{-1}\frac{b}{a} = 2\beta$$

$$\text{則 } \tan 2\alpha = \frac{a}{b}, \quad \tan 2\beta = \frac{b}{a}, \quad \cos 2\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore \cos^2\alpha = \frac{1+\cos 2\alpha}{2} = \frac{\sqrt{a^2+b^2}+b}{2\sqrt{a^2+b^2}}$$

$$\therefore \sec^2\alpha = \frac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}+b} = \frac{2\sqrt{a^2+b^2}}{a^2}(\sqrt{a^2+b^2}-b)$$

$$\text{同理 } \csc^2\beta = \frac{2\sqrt{a^2+b^2}}{b^2}(\sqrt{a^2+b^2}+a)$$

$$\begin{aligned} \text{今原式左邊} &= \frac{a^3}{2}\sec^2\alpha + \frac{b^3}{2}\csc^2\beta = [a(\sqrt{a^2+b^2}-b) \\ &\quad + b(\sqrt{a^2+b^2}+a)]\sqrt{a^2+b^2} \\ &= (a+b)(a^2+b^2) \end{aligned}$$

$$\text{o. 設 } \frac{\sin^{-1}(3 \sin x) + x}{4} = A$$

$$\text{則 } \frac{\sin^{-1}(3 \sin x) - 3x}{4} = A - x$$



$$\text{今} \quad \sin^{-1}(3 \sin x) + x = 4A$$

$$\therefore 3 \sin x = \sin(4A - x)$$

$$\text{即} \quad 2 \sin x = \sin(4A - x) - \sin x$$

$$= 2 \cos 2A \sin(2A - x)$$

$$\text{即} \quad \sin x (\sin^2 A + \cos^2 A)$$

$$= (\cos^2 A - \sin^2 A) \sin(2A - x)$$

$$\therefore \sin^2 A [\sin x + \sin(2A - x)]$$

$$= \cos^2 A [\sin(2A - x) - \sin x]$$

$$\text{即} \quad \sin^2 A [2 \sin A \cos(A - x)]$$

$$= \cos^2 A [2 \cos A \sin(A - x)]$$

$$\text{故} \quad \tan^2 x = \tan(A - x) \quad \text{故如題云。}$$

$$\text{p. 設} \quad \frac{2\pi}{3} + \cos^{-1} \frac{a}{b} = \alpha, \quad \frac{2\pi}{3} - \cos^{-1} \frac{a}{b} = \beta$$

$$\text{則原式左邊} = \sin \alpha \sin \beta - \cos \alpha \cos \beta$$

$$= -\cos(\alpha + \beta) = -\cos \frac{4\pi}{3}$$

$$= -\left[-\cos \frac{\pi}{3}\right] = \cos \frac{\pi}{3} = \frac{1}{2} \quad \text{故如題云。}$$

$$\text{q. 設} \quad \tan^{-1} x = \alpha \quad \text{則} \quad \tan \alpha = x$$

$$\therefore \tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha} = x^2 \quad \text{故} \quad \cos 2\alpha = \frac{1 - x^2}{1 + x^2}$$

$$\text{今} \quad \cos 6\alpha = 4 \cos^3 2\alpha - 3 \cos 2\alpha \quad \text{代入即得}$$

$$\text{r. 設左邊之角爲} A, \quad \text{則} \quad \cos A = \frac{\tan \alpha + \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\begin{aligned}\tan \frac{A}{2} &= \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{1 + \tan \alpha \tan \beta - \tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta + \tan \alpha + \tan \beta}} \\ &= \sqrt{\frac{(1 - \tan \alpha)(1 - \tan \beta)}{(1 + \tan \alpha)(1 + \tan \beta)}} \\ &= \sqrt{\tan\left(\frac{\pi}{4} - \alpha\right)\tan\left(\frac{\pi}{4} - \beta\right)} \quad (76 \text{ 頁 } g)\end{aligned}$$

$\therefore A = 2 \tan^{-1} \sqrt{\tan\left(\frac{\pi}{4} - \alpha\right)\tan\left(\frac{\pi}{4} - \beta\right)}$  故如題云。

s. 設右邊之角爲  $x$ , 則  $\cos x = \frac{\sin 2\alpha + \cos \beta}{1 + \sin 2\alpha \cos \beta}$

$$\begin{aligned}\tan^2 \frac{x}{2} &= \frac{1 - \cos x}{1 + \cos x} = \frac{(1 - \sin 2\alpha)(1 - \cos \beta)}{(1 + \sin 2\alpha)(1 + \cos \beta)} \\ &= \frac{(\cos \alpha - \sin \alpha)^2 \tan^2 \frac{\beta}{2}}{(\cos \alpha + \sin \alpha)^2} \\ &= \left(\frac{1 - \tan \alpha}{1 + \tan \alpha}\right)^2 \tan^2 \frac{\beta}{2}\end{aligned}$$

$$\therefore \tan \frac{x}{2} = \tan\left(\frac{\pi}{4} - \alpha\right)\tan \frac{\beta}{2}$$

$\therefore x = 2 \tan^{-1} \left[ \tan\left(\frac{\pi}{4} - \alpha\right)\tan \frac{\beta}{2} \right]$  故如題云。

t. 本題做法與上題一樣, 亦是從右邊推起。

u. 設  $\tan^{-1}(\cot x) = \alpha$  則  $\tan \alpha = \cot x$

設  $\tan^{-1}(\tan x) = \beta$  則  $\tan \beta = \tan x$

$$\text{今 } \tan(\alpha - \beta) = \frac{\cot x - \tan x}{1 + 1} = \frac{1 - \tan^2 x}{2 \tan x} = \cot 2x$$

$$\text{又 } \tan(2x + \alpha - \beta) = \frac{\tan 2x + \cot 2x}{1 - 1} = \infty$$

$$\therefore 2x + \alpha - \beta = n\pi + \frac{\pi}{2} = \frac{\pi}{2}(2n+1)$$

$$\alpha + 2x = \beta + \frac{1}{2}(2n+1)\pi \quad \text{故如題云。}$$

$$\text{v. 今 } \tan^{-1} \frac{c_1 x - y}{c_1 y + x} = \tan^{-1} \frac{\frac{x}{y} - \frac{1}{c_1}}{1 + \frac{1}{c_1} \cdot \frac{x}{y}}$$

$$= \tan^{-1} \frac{x}{y} - \tan^{-1} \frac{1}{c_1}$$

$$\text{又 } \tan^{-1} \frac{c_2 - c_1}{c_2 c_1 + 1} = \tan^{-1} \frac{\frac{1}{c_1} - \frac{1}{c_2}}{1 + \frac{1}{c_2} \cdot \frac{1}{c_1}}$$

$$= \tan^{-1} \frac{1}{c_1} - \tan^{-1} \frac{1}{c_2}$$

$$\text{同理 } \tan^{-1} \frac{c_3 - c_2}{c_3 c_2 + 1} = \tan^{-1} \frac{1}{c_2} - \tan^{-1} \frac{1}{c_3}$$

.....

$$\text{又 } \tan^{-1} \frac{c_n - c_{n-1}}{c_n c_{n-1} + 1} = \tan^{-1} \frac{1}{c_{n-1}} - \tan^{-1} \frac{1}{c_n}$$

$$\therefore \text{原式左邊} = \left( \tan^{-1} \frac{x}{y} - \tan^{-1} \frac{1}{c_1} \right)$$

$$+ \left( \tan^{-1} \frac{1}{c_1} - \tan^{-1} \frac{1}{c_2} \right)$$

$$+ \left( \tan^{-1} \frac{1}{c_2} - \tan^{-1} \frac{1}{c_3} \right) + \dots$$

$$+ \left( \tan^{-1} \frac{1}{c_{n-1}} - \tan^{-1} \frac{1}{c_n} \right) + \tan^{-1} \frac{1}{c_n}$$

$$= \tan^{-1} \frac{x}{y}$$

$$\begin{aligned} \text{w. } \therefore \tan^{-1} \frac{1}{2n^2} &= \tan^{-1} \frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} \\ &= \tan^{-1}(2n+1) - \tan^{-1}(2n-1) \end{aligned}$$

如  $n=1, 2, \dots, n$ , 則

$$\tan^{-1} \frac{1}{2} = \tan^{-1} 3 - \tan^{-1} 1, \quad \tan^{-1} \frac{1}{8} = \tan^{-1} 5 - \tan^{-1} 3$$

.....

$$\tan^{-1} \frac{1}{2n^2} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$$\begin{aligned} \text{相加, 得 } \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{8} + \dots + \tan^{-1} \frac{1}{2n^2} \\ &= \tan^{-1}(2n+1) - \tan^{-1} 1 \\ &= \tan^{-1}(2n+1) - \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{x. } \therefore \tan^{-1} \frac{x}{1 + (n-1)nx^2} &= \tan^{-1} \frac{nx - (n-1)x}{1 + n(n-1)x^2} \\ &= \tan^{-1} nx - \tan^{-1}(n-1)x \end{aligned}$$

如  $n=1, 2, \dots, n$

則  $\tan^{-1} x = \tan^{-1} x$

$$\tan^{-1} \frac{x}{1 + 1 \cdot 2x^2} = \tan^{-1} 2x - \tan^{-1} x, \quad \dots$$

$$\tan^{-1} \frac{x}{1 + (n-1)nx^2} = \tan^{-1} nx - \tan^{-1}(n-1)x$$

相加, 得原式左邊  $= \tan^{-1} nx$

4. 設  $\cot^{-1} \sqrt{\cos \alpha} = A$  則  $\cot A = \sqrt{\cos \alpha}$

$$\therefore \sin A = \frac{1}{\sqrt{1 + \cos \alpha}}; \quad \cos A = \frac{\sqrt{\cos \alpha}}{\sqrt{1 + \cos \alpha}}$$

又設  $\tan^{-1}\sqrt{\cos \alpha} = B$  則  $\tan \beta = \sqrt{\cos \alpha}$

$$\therefore \sin B = \frac{\sqrt{\cos \alpha}}{\sqrt{1 + \cos \alpha}}; \quad \cos B = \frac{1}{\sqrt{1 + \cos \alpha}}$$

$\therefore \sin u = \sin A \cos B - \cos A \sin B$

$$= \frac{1}{1 + \cos \alpha} - \frac{\cos \alpha}{1 + \cos \alpha} = \left( \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \right)^2 = \tan^2 \frac{\alpha}{2}$$

$$\therefore \sin u = \tan^2 \frac{\alpha}{2}$$

5. 設三角爲  $\alpha, \beta, \gamma$ , 則  $\tan \alpha = a$ ,  $\tan \beta = b$ ,  $\tan \gamma = c$

因  $\alpha + \beta + \gamma = \pi$

則  $\Sigma \tan \alpha = \Pi \tan \alpha$  (見 140 頁例三)

$\therefore a + b + c = abc$

6. 設三角依次爲  $\alpha, \beta, \gamma$

則  $\sin \alpha = \frac{x}{a}$ ,  $\sin \beta = \frac{y}{b}$ ,  $\sin \gamma = \frac{c^2}{ab}$ ,

$$\cos \gamma = \frac{\sqrt{a^2 b^2 - c^4}}{ab}$$

今  $\alpha + \beta = \gamma$   $\therefore \cos \alpha \cos \beta = \cos \gamma + \sin \alpha \sin \beta$

即  $(1 - \sin^2 \alpha)(1 - \sin^2 \beta) = \cos^2 \gamma + \sin^2 \alpha \sin^2 \beta$

$$+ 2 \cos \gamma \sin \alpha \sin \beta$$

即  $-\sin^2 \alpha - \sin^2 \beta + \sin^2 \gamma = 2 \sin \alpha \sin \beta \cos \gamma$

$$\therefore -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{c^4}{a^2 b^2} = \frac{2xy}{ab} \cdot \frac{\sqrt{a^2 b^2 - c^4}}{ab}$$

故  $b^2 x^2 + a^2 y^2 + 2xy(a^2 b^2 - c^4)^{\frac{1}{2}} = c^4$

7. 設兩角爲  $A, B$ , 則  $\cos A = \frac{x}{a}$ ,  $\cos B = \frac{y}{b}$

今  $A+B=\alpha$   $\therefore \cos A \cos B - \sin A \sin B = \cos \alpha$

即  $\cos A \cos B - \cos \alpha = \sin A \sin B$

即  $\cos^2 A \cos^2 B + \cos^2 \alpha - 2 \cos A \cos B \cos \alpha$   
 $= (1 - \cos^2 A)(1 - \cos^2 B)$

$\therefore \cos^2 A + \cos^2 B - 2 \cos A \cos B \cos \alpha$   
 $= 1 - \cos^2 \alpha = \sin^2 \alpha$

即  $\frac{x^2}{a^2} - \frac{2xy}{ab} \cos \alpha + \frac{y^2}{b^2} = \sin^2 \alpha$

8. 設  $\cos^{-1} \frac{x}{a} = \alpha, \quad \sin^{-1} \frac{y}{a} = \beta$

則  $\cos \alpha = \frac{x}{a}, \quad \sin \beta = \frac{y}{a}$

今  $\alpha = 2\beta$  故  $\cos \alpha = \cos 2\beta = 1 - 2 \sin^2 \beta$

即  $x/a = 1 - 2y^2/a^2$  故  $a^2 = ax + 2y^2$

9. 今  $\frac{\tan(\theta - \alpha)}{\tan \theta} = \frac{\tan \theta}{\tan(\theta - \beta)}$

即  $\frac{\tan(\theta - \alpha) + \tan \theta}{\tan \theta - \tan(\theta - \alpha)} = \frac{\tan \theta + \tan(\theta - \beta)}{\tan(\theta - \beta) - \tan \theta}$

即  $\frac{\sin(2\theta - \alpha)}{\sin \alpha} = \frac{\sin(2\theta - \beta)}{\sin(-\beta)} = -\frac{\sin(2\theta - \beta)}{\sin \beta}$

故  $-\sin \beta(\sin 2\theta \cos \alpha - \cos 2\theta \sin \alpha)$   
 $= \sin \alpha(\sin 2\theta \cos \beta - \cos 2\theta \sin \beta)$

$\therefore \sin 2\theta(\sin \alpha \cos \beta + \sin \beta \cos \alpha)$   
 $= 2 \cos 2\theta \sin \alpha \sin \beta$

$\therefore \tan 2\theta = \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$

$$\text{即 } \theta = \frac{1}{2} \tan^{-1} \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

10. 設第一, 二, 三角依次爲  $\alpha, \beta, \gamma$

$$\text{則 } \text{vers } \alpha = 1 - \cos \alpha = \frac{x}{a} \quad \therefore \cos \alpha = 1 - \frac{x}{a}$$

$$\text{同理 } \cos \beta = 1 - \frac{bx}{a}, \quad \cos \gamma = b$$

$$\text{今 } \alpha - \beta = \gamma \quad \text{即 } \gamma - \alpha = -\beta$$

$$\text{故 } \cos \gamma \cos \alpha + \sin \gamma \sin \alpha = \cos \beta$$

$$\text{即 } \left(1 - \frac{x}{a}\right)b + \sqrt{\frac{2x}{a} - \frac{x^2}{a^2}} \sqrt{1 - b^2} = 1 - \frac{bx}{a}$$

$$\text{即 } \sqrt{\frac{2x}{a} - \frac{x^2}{a^2}} = \frac{\sqrt{1-b}}{\sqrt{1+b}}$$

$$\text{即 } \frac{x^2}{a^2} - 2\left(\frac{x}{a}\right) + \frac{1-b}{1+b} = 0$$

$$\text{依 } \frac{x}{a} \text{ 之二次方程式解得 } \frac{x}{a} = 1 \pm \sqrt{\frac{2b}{1+b}}$$

11. 設第一, 二角爲  $A, B$

$$\text{則 } \sin A = \sin \theta + \sin \phi, \quad \sin B = \sin \theta - \sin \phi$$

$$\text{故 } \sin^2 A + \sin^2 B = 2(\sin^2 \theta + \sin^2 \phi) = 2\left(\frac{1}{2}\right) = 1$$

$$\therefore \sin^2 A = 1 - \sin^2 B = \cos^2 B$$

$$\text{即 } \sin A = \cos B \quad (\text{取正號})$$

$$\text{即 } \sin A = \sin\left(\frac{\pi}{2} - B\right)$$

$$\text{即 } A + B = \frac{\pi}{2} \quad (\text{取主值}) \quad \text{故如題云.}$$

12. 今  $\tan \theta = x\sqrt{3}/(2c-x), \quad \tan \phi = (2x-c)/c\sqrt{3}$

$$\begin{aligned} \text{故 } \tan(\theta - \phi) &= \frac{x\sqrt{3}/(2c-x) - (2x-c)/c\sqrt{3}}{1+x\sqrt{3}(2x-c)/c\sqrt{3}(2c-x)} \\ &= \frac{2(c^2 - cx + x^2)}{2\sqrt{3}(c^2 - cx + x^2)} = \frac{1}{\sqrt{3}} \end{aligned}$$

$$\text{即 } \theta - \phi = \frac{\pi}{6} \quad (\text{取主值})$$

## 習題十七 (210—219 頁)

1. 化  $\cos 3x$  爲  $\cos x$  函數，則得  $12 \cos^3 x - 3 \cos x = 0$

$$\text{即 } 3 \cos x(2 \cos x - 1)(2 \cos x + 1) = 0$$

$$\therefore \cos x = 0, \frac{1}{2}, -\frac{1}{2}$$

$$\therefore x = n \cdot 360^\circ \pm 90^\circ, n \cdot 360^\circ \pm 60^\circ, n \cdot 360^\circ \pm 120^\circ$$

故比  $360^\circ$  小之正角爲  $60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ, 300^\circ$ 。

2. 因  $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$  (見 80 頁例 4)

$$\text{故 } \tan^2 x = \frac{2 \tan x}{1 + \tan^2 x}$$

$$\text{即 } \tan x(\tan x - 1)(\tan^2 x + \tan x + 2) = 0$$

$$1. \tan x = 0 \quad \therefore x = n \cdot 180^\circ$$

$$2. \tan x = 1 \quad \therefore x = n \cdot 180^\circ + 45^\circ$$

$$3. \tan^2 x + \tan x + 2 = 0. \quad \Delta = 1 - 8 < 0, \text{ 故爲虛數。}$$

故比  $360^\circ$  小之正角爲  $0^\circ, 45^\circ, 180^\circ, 225^\circ$ 。

3. 即  $\cos 4x + \cos 2x + 1 = 0$  即  $2 \cos^2 2x + \cos 2x = 0$

$$\text{即 } \cos 2x(2 \cos 2x + 1) = 0 \quad \therefore \cos 2x = 0, -\frac{1}{2}$$



$$\text{即 } \begin{cases} 2x = 2n \cdot 180^\circ \pm 90^\circ \\ 2x = 2n \cdot 180^\circ \pm 120^\circ \end{cases} \quad \text{即 } \begin{cases} x = n \cdot 180^\circ \pm 45^\circ \\ x = n \cdot 180^\circ \pm 60^\circ \end{cases}$$

小於  $360^\circ$  正角有  $45^\circ, 60^\circ, 120^\circ, 135^\circ, 225^\circ, 240^\circ, 300^\circ, 315^\circ$ .

$$4. \quad \text{即 } \frac{1}{\cos x} - \frac{1}{\sin x} = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}$$

設  $\sin x \cos x \neq 0$

$$\text{則 } (\cos x - \sin x)(\sin x + \cos x + 1) = 0$$

$$1. \quad \cos x - \sin x = 0 \quad \therefore \tan x = 1$$

$$\therefore x = n \cdot 180^\circ + 45^\circ$$

$$2. \quad \sin x + \cos x + 1 = 0 \quad \therefore \sin x + \cos x = -1$$

平方化簡得  $\sin 2x = 0$ , 即  $\sin x \cos x = 0$  不合。

故比  $360^\circ$  小之正角有  $45^\circ, 225^\circ$ .

$$5. \quad \text{今 } 2 \cos^2 x - 1 = a(1 - \cos x)$$

$$\text{即 } 2 \cos^2 x + a \cos x - (1 + a) = 0$$

$$\therefore \cos x = \frac{1}{4} [-a \pm \sqrt{a^2 + 8(1+a)}]$$

$$\text{即 } x = \cos^{-1} \frac{1}{4} (-a \pm \sqrt{a^2 + 8a + 8})$$

$$6. \quad \therefore \tan\left(\frac{1}{4}\pi + x\right) = \dots\dots\dots = \frac{1 + \tan x}{1 - \tan x}$$

$$\tan\left(\frac{1}{4}\pi - x\right) = \frac{1 - \tan x}{1 + \tan x}$$

代入去分母得

$$(1 + \tan x)^2 + (1 - \tan x)^2 = 4(1 - \tan x)(1 + \tan x)$$

$$\text{即 } \tan^2 x = 1/3 \quad \text{即 } \tan x = \pm \sqrt{3}/3$$

$$\therefore x = n \cdot 180^\circ \pm 30^\circ$$

故比  $360^\circ$  小之正角爲  $30^\circ, 150^\circ, 210^\circ, 330^\circ$ . (或可從公式 27 着手)

7. 即 
$$\frac{\cos x - \sin x}{\cos x + \sin x} = \cos^2 x - \sin^2 x$$

即 
$$(\cos x - \sin x)[(\cos x + \sin x)^2 - 1] = 0$$

1.  $\cos x - \sin x = 0 \quad \therefore \tan x = 1$

$$\therefore x = n \cdot 180^\circ + 45^\circ$$

2.  $(\cos x + \sin x)^2 = 1 \quad \text{即} \quad \sin 2x = 0$

即  $\sin x \cos x = 0 \quad \therefore x = n \cdot 180^\circ, n \cdot 360^\circ \pm 90^\circ$

故小於  $360^\circ$  之正角爲  $0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, 270^\circ$ .

8. 即 
$$(\cos x - \sin x)(\cos^2 x - \sin^2 x) = a(\cos x + \sin x)$$

即 
$$(\cos x + \sin x)[(\cos x - \sin x)^2 - a] = 0$$

1.  $\tan x = -1 \quad \therefore x = n \cdot 180^\circ - 45^\circ$  爲  $135^\circ$  及  $315^\circ$

2.  $\sin 2x = 1 - a \quad \therefore x = \frac{1}{2}[n\pi + (-1)^n \sin^{-1}(1 - a)]$

9. 今左邊 =  $(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)$

$$= 1 - 3 \sin^2 x \cos^2 x = 1 - \frac{3}{4} \sin^2 2x$$

故 
$$1 - \frac{3}{4} \sin^2 2x = \frac{7}{12} \sin^2 2x$$

即 
$$\sin^2 2x = 3/4 \quad \therefore \sin 2x = \pm \sqrt{3}/2$$

$\therefore 2x = n \cdot 180^\circ \pm 60^\circ \quad \therefore x = n \cdot 90^\circ \pm 30^\circ$

所求之角爲  $30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ$ .

10. 今  $\tan(a-x) = -1/\tan x = -\cot x = \tan(\frac{1}{2}\pi + x)$

$$\therefore a - x = k\pi + \left(\frac{1}{2}\pi + x\right)$$

$$\text{即 } 2x = a - \frac{1}{2}(2k+1)\pi \quad \text{故 } x = \frac{1}{4}[(2n+1)\pi + 2a]$$

(因  $(2k+1), 2n+1$  均表示奇數也.)

$$11. \text{ 今左邊} = \frac{\sin x \sin 3x}{\cos x \cos 3x} = \frac{-\cos 4x - \cos 2x}{\cos 4x + \cos 2x} = -\frac{2}{5}$$

$$\text{即 } 3 \cos 4x - 7 \cos 2x = 0$$

$$\text{即 } 6 \cos^2 2x - 7 \cos 2x - 3 = 0$$

$$\text{即 } (2 \cos 2x - 3)(3 \cos 2x + 1) = 0$$

$$\therefore \cos 2x = -\frac{1}{3}, \frac{3}{2} \quad (\text{但 } \frac{3}{2} \text{ 不合})$$

當  $\cos 2x = -\frac{1}{3}$ , 即  $\cos(180^\circ - 2x) = \frac{1}{3} = 0.3333$  時,

主角爲  $70^\circ 32'$ .

故  $2x$  之主角爲  $180^\circ - 70^\circ 32' = 109^\circ 28'$

$$\text{故 } 2x = n \cdot 360^\circ \pm 109^\circ 28'$$

$$\therefore x = n \cdot 180^\circ \pm 54^\circ 44'$$

所求之角爲  $54^\circ 44', 125^\circ 16', 234^\circ 44', 305^\circ 16'$ .

$$12. \text{ 平方兩邊得 } 2 - 2\sqrt{\cos^2 x} = 4 \cos^2 x$$

$$\text{即 } 2 \cos^2 x + \cos x - 1 = 0$$

$$\text{即 } (2 \cos x - 1)(\cos x + 1) = 0$$

$$\therefore \cos x = \frac{1}{2}, -1$$

$$\therefore x = n \cdot 360^\circ \pm 60^\circ, n \cdot 360^\circ \pm 180^\circ$$

比  $360^\circ$  小之正角適合原式者只有  $60^\circ$ .

$$13. \text{ 即 } 2 \sin\left(\frac{1}{2}\pi + x\right) \cos \frac{1}{6}\pi = \frac{8}{9} \quad \text{即 } \cos x = \frac{1}{2}\sqrt{3}$$

$$\therefore x = n \cdot 360^\circ \pm 30^\circ \quad \text{故所求角爲 } 30^\circ, 330^\circ.$$

14. 即  $\sin 2x + \sin 24^\circ = \sin 90^\circ + \sin 24^\circ \quad \therefore \sin 2x = 1$

$\therefore x = n \cdot 90^\circ + (-1)^n \cdot 45^\circ$ , 故所求角為  $45^\circ, 225^\circ$ .

15. 左邊 =  $\frac{\cos(x+120^\circ) + \cos(x-120^\circ)}{\cos(x+120^\circ)\cos(x-120^\circ)} = \frac{4 \cos x \cos 120^\circ}{\cos 2x + \cos 240^\circ}$

$$= \frac{-2 \cos x}{2 \cos^2 x - 1 - \frac{1}{2}} = \frac{-4 \cos x}{4 \cos^2 x - 3}$$

$$\therefore \frac{-4 \cos x}{4 \cos^2 x - 3} = 2 \cos x$$

即  $2 \cos x(4 \cos^2 x - 1) = 0 \quad \therefore \cos x = 0, \pm \frac{1}{2}$

$\therefore x = n \cdot 360^\circ \pm 90^\circ, n \cdot 360^\circ \pm 60^\circ, n \cdot 360^\circ \pm 120^\circ$

比  $360^\circ$  小之正角有  $60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ, 300^\circ$ .

16. 即  $\frac{\sin^2 x \cos^2 2x}{\cos^2 x \sin^2 2x} = 1$  即  $\tan^2 2x = \tan^2 x$  兩平方

即  $\tan 2x = \pm \tan x = \tan(\pm x)$

$\therefore 2x = n\pi \pm x \quad \therefore x = n\pi, \frac{1}{3}n\pi$

故所求角為  $60^\circ, 120^\circ, 240^\circ, 300^\circ$  ( $0^\circ$  及  $180^\circ$  不合).

17. 即  $2 \cos 3x \cos 2x + \cos 2x = 0$

即  $\cos 2x(2 \cos 3x + 1) = 0$

1.  $\cos 2x = 0 \quad \therefore 2x = n \cdot 360^\circ \pm 90^\circ$

故  $x = n \cdot 180^\circ \pm 45^\circ$

2.  $\cos 3x = -\frac{1}{2} \quad \therefore 3x = n \cdot 360^\circ \pm 120^\circ$  以  $120^\circ$

故  $x = n \cdot 120^\circ \pm 40^\circ$   ~~$x = \frac{n \cdot 360^\circ \pm 120^\circ}{3} + \frac{(6n\pi + 2\pi)}{3}$~~

比  $360^\circ$  小之正角為  $40^\circ, 45^\circ, 80^\circ, 135^\circ, 160^\circ, 200^\circ,$

$225^\circ, 280^\circ, 315^\circ, 320^\circ$ .

$$18. \text{ 即 } 2(\cos 4x \cos 3x + \cos 4x \cos x) = 0$$

$$\text{即 } (\cos 3x + \cos x)\cos 4x = 0$$

$$\text{即 } \cos x \cos 2x \cos 4x = 0$$

$$\therefore x = n \cdot 360^\circ \pm 90^\circ, n \cdot 180^\circ \pm 45^\circ, n \cdot 90^\circ \pm 22^\circ 30'$$

故比  $360^\circ$  小之正角爲  $22^\circ 30', 45^\circ, 67^\circ 30', 90^\circ, 112^\circ 30',$

$135^\circ, 157^\circ 30', 202^\circ 30', 225^\circ, 247^\circ 30', 270^\circ, 292^\circ 30',$

$315^\circ, 337^\circ 30'$  十四個角。

$$19. \text{ 即 } 2 \sin 3\theta \cos \alpha - 2 \sin \theta \cos \alpha = \cos \alpha$$

因  $\cos \alpha$  爲定數，設不爲 0， 則  $2(\sin 3\theta - \sin \theta) = 1$

$$\text{即 } 8 \sin^3 \theta - 4 \sin \theta + 1 = 0$$

$$\text{即 } (2 \sin \theta - 1)(4 \sin^2 \theta + 2 \sin \theta - 1) = 0$$

$$1. \sin \theta = \frac{1}{2} \quad \therefore \theta = n \cdot 180^\circ + (-1)^n \cdot 30^\circ$$

$$2. \sin \theta = \frac{-1 \pm \sqrt{5}}{4} \quad \therefore \theta = n \cdot 180^\circ + (-1)^n \cdot 18^\circ$$

$$\text{或 } \theta = n \cdot 180^\circ - (-1)^n \cdot 54^\circ$$

故小於  $360^\circ$  之正角爲  $18^\circ, 30^\circ, 150^\circ, 162^\circ, 234^\circ, 306^\circ$ 。

$$20. \text{ 即 } \cos 6\theta + \cos 2\alpha + \cos 10\theta + \cos 2\alpha = 2 \cos 2\alpha$$

$$\text{即 } 2 \cos 8\theta \cos 2\theta = 0$$

$$\therefore \theta = \frac{1}{8} \left( 2n\pi \pm \frac{\pi}{2} \right) \text{ 或 } \frac{1}{2} \left( 2n\pi \pm \frac{\pi}{2} \right)$$

$$\text{上式亦可寫爲 } \frac{1}{16} (2k+1)\pi, \frac{1}{4} (2k+1)\pi$$

故比  $2\pi$  小之正角爲  $\frac{\pi}{16}, \frac{3\pi}{16}$  等 20 角。

$$21. \text{ 即 } 4 \cos^2 \theta - 3 = 2 \sin \theta$$

$$\text{即 } 4 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\therefore \sin \theta = \frac{-1 \pm \sqrt{5}}{4}$$

$$\therefore \theta = n \cdot 180^\circ + (-1)^n \cdot 18^\circ \text{ 或 } n \cdot 180^\circ - (-1)^n \cdot 54^\circ$$

故比  $360^\circ$  小之正角爲  $18^\circ, 162^\circ, 234^\circ, 306^\circ$ .

$$22. \text{ 即 } -2 \sin(n-1)\theta \sin \theta = \sin \theta$$

$$\text{即 } \sin \theta [2 \sin(n-1)\theta + 1] = 0$$

$$1. \sin \theta = 0 \quad \therefore \theta = m\pi$$

$$2. \sin(n-1)\theta = -\frac{1}{2}$$

$$\therefore \theta = \frac{1}{n-1} \left[ m\pi + (-1)^m \cdot \frac{7\pi}{6} \right]$$

$$23. \text{ 即 } [\sin n\theta + \sin(n-1)\theta][\sin n\theta - \sin(n-1)\theta] = \sin^2 \theta$$

$$\text{即 } 2 \sin \frac{1}{2}(2n-1)\theta \cos \frac{1}{2}\theta \cdot 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}(2n-1)\theta = \sin^2 \theta$$

$$\text{即 } \sin \theta [\sin(2n-1)\theta - \sin \theta] = 0$$

$$\text{即 } 2 \sin \theta \sin(n-1)\theta \cos n\theta = 0$$

$$\text{故 } \theta = m\pi, \frac{1}{n-1} m\pi \text{ 或 } \frac{1}{n} \left[ 2m\pi \pm \frac{\pi}{2} \right]$$

$$24. \text{ 即 } \sin \theta + 4 \sin \theta \cos \theta + 3(3 \sin \theta - 4 \sin^3 \theta) = 0$$

$$\text{即 } \sin \theta (10 + 4 \cos \theta - 12 \sin^2 \theta) = 0$$

$$\text{即 } \sin \theta (6 \cos^2 \theta + 2 \cos \theta - 1) = 0$$

$$1. \sin \theta = 0 \quad \therefore \theta = n\pi$$

$$2. \cos \theta = \frac{-1 \pm \sqrt{7}}{6} = 0.2743, -0.6076$$

$$\therefore \theta = \begin{cases} n \cdot 360^\circ \pm 74^\circ 5' \\ n \cdot 360^\circ \pm 127^\circ 25' \end{cases}$$

故六角爲  $0^\circ, 74^\circ 5', 127^\circ 25', 180^\circ, 232^\circ 35', 288^\circ 55'$ .

25. 因右邊  $= \sin \theta \left( 1 + \frac{\sin \theta}{\cos \theta} \cdot \frac{1 - \cos \theta}{\sin \theta} \right) = \frac{\sin \theta}{\cos \theta} = \tan \theta$

故原式爲  $\tan \theta + 2 \cot 2\theta - \tan \theta$  即  $\cot 2\theta = 0$

$$\therefore 2\theta = k\pi \pm \frac{\pi}{2} = \frac{1}{2}(2k \pm 1)\pi = \frac{1}{2}(2n + 1)\pi$$

$$\therefore \theta = \frac{1}{4}(2n + 1)\pi$$

26. 即  $\tan \theta = 1 \quad \therefore \theta = n \cdot 180^\circ + 45^\circ$

比  $540^\circ$  小之正角爲  $45^\circ, 225^\circ, 405^\circ$ .

27. 即  $\sin(\alpha - x)\cos(\alpha - x) = \sin x \cos x$

即  $\sin(2\alpha - 2x) = \sin 2x$

$$\therefore 2x = k\pi + (-1)^k(2\alpha - 2x)$$

1.  $k = 2n + 1$  則  $0 = k\pi - 2\alpha$  不合理

2.  $k = 2n$  則  $4x = 2n\pi + 2\alpha \quad \therefore x = \frac{1}{2}(n\pi + \alpha)$

28. 即  $\cos^2 x - \sin^2 x = \sin x \cos x (\sin x + \cos x)$

即  $(\cos x + \sin x)(\cos x - \sin x - \sin x \cos x) = 0$

1.  $\cos x + \sin x = 0$  即  $\tan x = -1$

$$\therefore x = n \cdot 180^\circ + 45^\circ$$

2.  $\cos x - \sin x = \sin x \cos x$

即  $1 - \sin 2x = \frac{1}{4} \sin^2 2x$  即  $\sin^2 2x + 4 \sin 2x - 4 = 0$

$\therefore \sin 2x = -2 + 2\sqrt{2}$  (負號不合因小於  $-1$  也)

即  $\sin 2x = 0.8284 \quad \therefore 2x = n \cdot 180^\circ + (-1)^n \cdot 55^\circ 56'$

$$\therefore x = n \cdot 90^\circ + (-1)^n \cdot 27^\circ 58'$$

故小於  $360^\circ$  之正角爲  $27^\circ 58'$ ,  $135^\circ$ ,  $242^\circ 2'$ ,  $315^\circ$ . (有不適合之二角已棄去)

$$29. \text{ 即 } \frac{1 + \tan \theta}{1 - \tan \theta} = 8 \tan \theta \quad \text{即 } 8 \tan^2 \theta - 7 \tan \theta + 1 = 0$$

$$\therefore \tan \theta = \frac{1}{16}(7 \pm \sqrt{17}) = 0.1798, 0.6951$$

$$\therefore \theta = n \cdot 180^\circ + 10^\circ 12' \text{ 或 } n \cdot 180^\circ + 34^\circ 48'$$

故比  $360^\circ$  小之正角有  $10^\circ 12'$ ,  $34^\circ 48'$ ,  $190^\circ 12'$ ,  $214^\circ 48'$ .

$$30. \text{ 即 } \frac{\tan \theta + \tan^2 \theta}{1 - \tan \theta} = 2 \quad \text{即 } \tan^2 \theta + 3 \tan \theta - 2 = 0$$

$$\therefore \tan \theta = \frac{1}{2}(-3 \pm \sqrt{17}) = 0.5616, -3.5616$$

$$\therefore \theta = n \cdot 180^\circ + 29^\circ 19', \theta = n \cdot 180^\circ + 105^\circ 41'$$

故比  $360^\circ$  小之正角有  $29^\circ 19'$ ,  $105^\circ 41'$ ,  $209^\circ 19'$ ,  $285^\circ 41'$ .

$$31. \text{ 因 } \tan 3\theta = \frac{\tan \theta + \tan 2\theta}{1 - \tan \theta \tan 2\theta} = \frac{\tan 3\theta}{1 - \tan \theta \tan 2\theta}$$

$$\text{即 } \tan \theta \tan 2\theta \tan 3\theta = 0 \quad \therefore \theta = n\pi, \frac{1}{2}n\pi \text{ 或 } \frac{1}{3}n\pi$$

即爲  $n \cdot 90^\circ$  及  $n \cdot 60^\circ$  ( $n \cdot 180^\circ$  包含於  $n \cdot 90^\circ$  中)

故角爲  $0^\circ, 60^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, 270^\circ, 300^\circ$ .

$$32. \text{ 即 } 2 \sin^2 \theta = \cos^2 \frac{3\theta}{2} \quad \text{即 } 4(1 - \cos^2 \theta) = 1 + \cos 3\theta$$

$$\text{即 } 4 \cos^3 \theta + 4 \cos^2 \theta - 3 \cos \theta - 3 = 0$$

$$\text{即 } (\cos \theta + 1)(4 \cos^2 \theta - 3) = 0$$

$$\therefore \cos \theta = -1, \pm \frac{1}{2} \sqrt{3}$$

$$\therefore \theta = 2k\pi \pm \pi, 2k\pi \pm \frac{1}{6}\pi \text{ 或 } 2k\pi \pm \frac{5}{6}\pi$$



故比  $360^\circ$  小之正角爲  $30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ$ .

$$33. \text{ 即 } (2 \sin \theta - 1)(2 \sin \theta - \sqrt{3}) = 0$$

$$\therefore \sin \theta = 1/2, \quad \sqrt{3}/2$$

$$\therefore \theta = n \cdot 180^\circ + (-1)^n \cdot 30^\circ \text{ 或 } n \cdot 180^\circ + (-1)^n \cdot 60^\circ$$

故比  $360^\circ$  小之正角爲  $30^\circ, 60^\circ, 120^\circ, 150^\circ$ .

$$34. (4 - \sqrt{3})(\sin \theta + \cos \theta) = 4(\sin^3 \theta + \cos^3 \theta)$$

$$= 4(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)$$

$$= (\sin \theta + \cos \theta)(4 - 2 \sin 2\theta)$$

$$\text{即 } (\sin \theta + \cos \theta)(2 \sin 2\theta - \sqrt{3}) = 0$$

$$1. \tan \theta = -1 \quad \therefore \theta = n \cdot 180^\circ - 45^\circ$$

$$2. \sin 2\theta = \sqrt{3}/2 \quad \therefore \theta = n \cdot 90^\circ + (-1)^n \cdot 30^\circ$$

故小於  $360^\circ$  之正角爲  $30^\circ, 60^\circ, 135^\circ, 210^\circ, 240^\circ, 315^\circ$ .

$$35. \sin(\theta + \alpha) = \cos(\theta - \alpha) = \sin\left(\frac{\pi}{2} - \theta + \alpha\right)$$

$$\therefore \theta + \alpha = k\pi + (-1)^k \left(\frac{1}{2}\pi - \theta + \alpha\right) \text{ (參考 195 頁例十六)}$$

$k$  爲奇數時不合, 故  $k$  必爲偶數, 設爲  $2n$ , 則

$$\theta + \alpha = 2n\pi + \frac{1}{2}\pi - \theta + \alpha \quad \therefore \theta = \frac{1}{4}(4n+1)\pi$$

$$36. \text{ 今 } \cos\left(\frac{3x}{2} - \frac{\pi}{4}\right) = -\sin\left(\frac{2x}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{2x}{3} + \frac{\pi}{4}\right)$$

$$\therefore \frac{3x}{2} - \frac{\pi}{4} = 2n\pi \pm \left(\frac{2x}{3} + \frac{\pi}{4}\right)$$

$$1. \frac{5x}{6} = 2n\pi + \frac{\pi}{2} \quad \therefore x = \frac{3}{5}(4n+1)\pi$$

$$2. \frac{13x}{6} = 2n\pi \quad \therefore x = \frac{12}{13}n\pi$$

$$37. \text{ 即 } \tan m\theta = \tan n\theta \quad \therefore m\theta = k\pi + n\theta$$

$$\text{即 } (m-n)\theta = k\pi \quad \therefore \theta = \frac{k\pi}{m-n}$$

$$38. \text{ 今 } \pi\theta = n\pi + \frac{\pi}{\theta} \quad \therefore \theta^2 - n\theta - 1 = 0$$

$$\therefore \theta = \frac{1}{2}(n \pm \sqrt{n^2 + 4})$$

$$39. \text{ 即 } \cos\left(x + \frac{\pi}{4}\right) = \pm \sin(\pi - 2x)$$

今於正負兩號分別討論之

$$1. \cos\left(x + \frac{\pi}{4}\right) = \sin(\pi - 2x) = \cos\left(2x - \frac{\pi}{2}\right)$$

$$\therefore x + \frac{\pi}{4} = 2k\pi \pm \left(2x - \frac{\pi}{2}\right)$$

$$\text{故 } x = \frac{3}{4}\pi - 2k\pi, \frac{2k\pi}{3} + \frac{\pi}{12}$$

$$2. \cos\left(x + \frac{\pi}{4}\right) = -\sin(\pi - 2x) = \sin(2x - \pi)$$

$$= \cos\left(\frac{3\pi}{2} - 2x\right)$$

$$\therefore x + \frac{\pi}{4} = 2k\pi \pm \left(\frac{3\pi}{2} - 2x\right)$$

$$\text{故 } x = \frac{2k\pi}{3} + \frac{5}{12}\pi, \frac{7\pi}{4} - 2k\pi$$

故比  $2\pi$  小之角爲  $\frac{\pi}{12}$ ,  $\frac{5\pi}{12}$ ,  $\frac{3\pi}{4}$ ,  $\frac{13\pi}{12}$ ,  $\frac{17\pi}{12}$ ,  $\frac{7\pi}{4}$ .

$$40. \text{ 今 } \sin 3\theta + \sin 2\theta = 2 \sin 2\theta \cos \theta = \sin 3\theta + \sin \theta$$

$$\text{即 } \sin 2\theta - \sin \theta = 0 \quad \text{即 } 2 \sin \frac{\theta}{2} \cos \frac{3\theta}{2} = 0$$

$$\therefore \theta = 2k\pi \text{ 或 } \frac{2}{3}\left(2k\pi \pm \frac{\pi}{2}\right) = \frac{1}{3}(4k \pm 1)\pi = \frac{1}{3}(2n+1)\pi$$

比  $2\pi$  小之正角爲  $0, \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$ . (如從  $\sin 2\theta = \sin \theta$

由例十六亦可做)

$$41. \therefore 2 \csc^2 \theta = \frac{2}{\sin^2 \theta} = 1 + \frac{2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}{2}$$

$$\text{故去原式分母得 } \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = \frac{\sqrt{3}}{2}$$

$$\text{即 } \cos \theta = \frac{\sqrt{3}}{2} \quad \therefore \theta = n \cdot 360^\circ \pm 30^\circ$$

故所求角爲  $30^\circ, 330^\circ$ .

$$42. \text{ 即 } \frac{1}{\sqrt{2}}(\cos 3\theta + \sin 3\theta) = 1$$

$$\text{即 } \cos 3\theta \cos \frac{\pi}{4} + \sin 3\theta \sin \frac{\pi}{4} = 1$$

$$\text{即 } \cos\left(3\theta - \frac{\pi}{4}\right) = 1 \quad \text{即 } 3\theta - \frac{\pi}{4} = 2n\pi$$

$$\therefore \theta = \frac{1}{3}\left(2n + \frac{1}{4}\right)\pi$$

比  $2\pi$  小之正角爲  $\frac{1}{12}\pi, \frac{3}{4}\pi, \frac{17}{12}\pi$ . (或從原式兩邊平

方亦可做)

$$43. \text{ 即 } \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} = 3 \tan \theta$$

$$\text{即 } 8 \tan^3 \theta = 0 \quad \text{即 } \tan \theta = 0$$

$$\therefore \theta = n\pi \quad \text{故所求角爲 } 0, \pi.$$

$$44. \frac{8 \cos \theta}{\sin \theta} = \frac{1}{\sin^2 \frac{1}{2} \theta} + \frac{1}{\cos^2 \frac{1}{2} \theta} = \frac{1}{\sin^2 \frac{1}{2} \theta \cos^2 \frac{1}{2} \theta} = \frac{4}{\sin^2 \theta}$$

$$\text{即 } 2 \sin \theta \cos \theta = 1 \quad \text{即 } \sin 2\theta = 1$$

$$\therefore \theta = \frac{1}{2} \left[ n\pi + (-1)^n \cdot \frac{\pi}{2} \right] = \frac{\pi}{4} [2n + (-1)^n]$$

$$= \frac{\pi}{4} (4k+1) = k\pi + \frac{\pi}{4}$$

( $\because$  在  $2n + (-1)^n$  中, 若  $n = 2k$ , 則為  $4k+1$ ,  
若  $n = 2k+1$ , 則亦為  $4k+1$  也)

故比  $2\pi$  小之正角為  $\frac{1}{4}\pi, \frac{5}{4}\pi$ .

$$45. \text{ 今 } 16 \cos^6 \theta = 16 \cos^6 \theta - 20 \cos^4 \theta + 5 \cos^2 \theta$$

(見 128 頁題 11)

$$\text{即 } \cos \theta (2 \cos \theta - 1)(2 \cos \theta + 1) = 0$$

$$\therefore \theta = n \cdot 360^\circ \pm 90^\circ, \quad n \cdot 360^\circ \pm 60^\circ, \quad n \cdot 360^\circ \pm 120^\circ$$

比  $360^\circ$  小之正角為  $60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ, 300^\circ$ .

$$46. \text{ 即 } 2 \sin 4\theta \cos \theta + 2 \sin(\theta + 45^\circ) \cos \theta = 0$$

(參考 188 頁例二解三)

$$\text{即 } 2 \cos \theta [\sin 4\theta + \sin(\theta + 45^\circ)] = 0$$

$$\text{即 } 4 \cos \theta \sin \frac{1}{2}(5\theta + 45^\circ) \cos \frac{1}{2}(3\theta - 45^\circ) = 0$$

$$1. \theta = n \cdot 360^\circ \pm 90^\circ$$

$$2. 5\theta + 45^\circ = 2n \cdot 180^\circ \quad \therefore \theta = n \cdot 72^\circ - 9^\circ$$

$$3. 3\theta - 45^\circ = 2n \cdot 360^\circ \pm 180^\circ$$

$$\therefore \theta = n \cdot 240^\circ \pm 60^\circ + 15^\circ$$



比  $360^\circ$  小之正角爲  $63^\circ, 75^\circ, 90^\circ, 135^\circ, 195^\circ, 207^\circ,$   
 $270^\circ, 279^\circ, 315^\circ, 351^\circ.$

$$47. \text{ 即 } 2 \sin\left(\theta - \frac{1}{3}\pi\right)(\cos 3\theta + 3 \cos \theta) + 2 \cos\left(\theta - \frac{1}{3}\pi\right)$$

$$\times (3 \sin \theta - \sin 3\theta) - 6 \sin\left(2\theta - \frac{1}{3}\pi\right) = \sqrt{3}$$

$$[\text{用 } \cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3 \cos \theta), \dots]$$

$$\text{即 } -2 \sin\left(2\theta + \frac{1}{3}\pi\right) + 6 \sin\left(2\theta - \frac{1}{3}\pi\right)$$

$$- 6 \sin\left(2\theta - \frac{1}{3}\pi\right) = \sqrt{3}$$

$$\text{即 } \sin\left(2\theta + \frac{1}{3}\pi\right) = -\frac{1}{2}\sqrt{3}$$

$$\therefore 2\theta + \frac{1}{3}\pi = n\pi + (-1)^n \frac{4\pi}{3}$$

$$\therefore \theta = \frac{1}{6}(3n-1)\pi + \frac{2}{3}(-1)^n \pi$$

$$48. \therefore \cos(\alpha - \beta) = \cos(\alpha - \beta) + \cos \alpha - \cos \alpha$$

$$= 2 \cos\left(\alpha - \frac{\beta}{2}\right) \cos \frac{\beta}{2} - \cos \alpha$$

代入原式得

$$x^2 \cos \alpha \cos\left(\alpha - \frac{\beta}{2}\right) + x \left[ 2 \cos\left(\alpha - \frac{\beta}{2}\right) \cos \frac{\beta}{2} - \cos \alpha \right]$$

$$- 2 \cos \frac{\beta}{2} = 0$$

$$\text{即 } x \cos\left(\alpha - \frac{\beta}{2}\right) \left( x \cos \alpha + 2 \cos \frac{\beta}{2} \right)$$

$$- \left( x \cos \alpha + 2 \cos \frac{\beta}{2} \right) = 0$$

$$\text{即 } \left[ x \cos \left( \alpha - \frac{\beta}{2} \right) - 1 \right] \left[ x \cos \alpha + 2 \cos \frac{\beta}{2} \right] = 0$$

$$\therefore x = \sec \left( \alpha - \frac{\beta}{2} \right) \quad \text{或} \quad 2 \sec \alpha \cos \frac{\beta}{2}$$

[或即解二次方程式再化簡]

$$49. \text{ 即 } 2(\cos^2 \theta - \cos^2 \alpha) = (\cos 3\theta + 3 \cos \theta)(\cos \theta - \cos \alpha) \\ + (\sin 3\theta - 3 \sin \theta)(\sin \theta - \sin \alpha)$$

$$\text{即 } \cos 2\theta - \cos 2\alpha = (\cos 3\theta \cos \theta + \sin 3\theta \sin \theta) \\ + 3(\cos^2 \theta - \sin^2 \theta) \\ - (\cos 3\theta \cos \alpha + \sin 3\theta \sin \alpha) \\ - 3(\cos \theta \cos \alpha - \sin \theta \sin \alpha) \\ = \cos 2\theta + 3 \cos 2\theta - \cos(3\theta - \alpha) \\ - 3 \cos(\theta + \alpha)$$

$$\text{即 } \cos(3\theta - \alpha) - \cos 2\alpha - 3[\cos 2\theta - \cos(\theta + \alpha)] = 0$$

$$\text{即 } 2 \sin \frac{1}{2}(3\theta + \alpha) [\sin \frac{1}{2}(3\alpha - 3\theta) - 3 \sin \frac{1}{2}(\alpha - \theta)] = 0$$

$$1. \quad \sin \frac{1}{2}(3\theta + \alpha) = 0 \quad \therefore 3\theta + \alpha = 2n\pi$$

$$\therefore \theta = \frac{1}{3}(2n\pi - \alpha)$$

$$2. \quad 3 \sin \frac{1}{2}(\alpha - \theta) - 4 \sin^2 \frac{1}{2}(\alpha - \theta) = 3 \sin \frac{1}{2}(\alpha - \theta)$$

$$\text{即 } \sin \frac{1}{2}(\alpha - \theta) = 0 \quad \therefore \theta - \alpha = 2n\pi$$

$$\therefore \theta = 2n\pi + \alpha$$

$$50. \text{ 從提示 } 4 \sin^3 \frac{1}{6} x - 3 \sin \frac{1}{6} x = 4 \sin^2 \frac{1}{6} x$$

$$\text{令 } \frac{1}{6} x = A, \quad \text{則 } 4 \sin^3 A - 4 \sin^2 A - 3 \sin A = 0$$

$$\text{即 } \sin A(2 \sin A - 3)(2 \sin A + 1) = 0$$

$$\sin A = \frac{3}{2} \text{ 不合} \quad \therefore A = k\pi \text{ 或 } k\pi - (-1)^k \cdot \frac{\pi}{6}$$

$$\therefore x = 6k\pi \text{ 或 } 6k\pi - (-1)^k \pi$$

$$51. \text{ 即 } \tan \theta + \frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \tan \theta} + \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} = 3$$

$$\text{即 } \tan^3 \theta - 3 \tan^2 \theta - 3 \tan \theta + 1 = 0$$

$$\text{即 } (\tan \theta + 1)(\tan^2 \theta - 4 \tan \theta + 1) = 0$$

$$\therefore \tan \theta = -1, 2 \pm \sqrt{3}$$

$$\therefore \theta = n\pi + \frac{3\pi}{4}, n\pi + \tan^{-1}(2 \pm \sqrt{3})$$

$$52. \therefore \tan(\overline{x + \alpha + x + \beta + x + \gamma})$$

$$= \frac{\Sigma \tan(x + \alpha) - \Pi \tan(x + \alpha)}{1 - \Sigma \tan(x + \alpha) \tan(x + \beta)}$$

今分母爲 0, 故  $\tan(x + \alpha + x + \beta + x + \gamma) = \infty$

$$\therefore 3x + \alpha + \beta + \gamma = n\pi + \frac{\pi}{2}$$

$$\therefore x = \frac{1}{3} \left( n\pi + \frac{\pi}{2} - \alpha - \beta - \gamma \right)$$

$$53. \text{ 即 } \frac{\sin 5\theta}{\cos \theta \cos 4\theta} + \frac{\sin 5\theta}{\cos 2\theta \cos 3\theta} = 0$$

$$\text{即 } \sin 5\theta (\cos \theta \cos 4\theta + \cos 2\theta \cos 3\theta) = 0$$

$$\text{即 } \sin 5\theta \cos \theta [(2 \cos^2 2\theta - 1) + \cos 2\theta (4 \cos^2 \theta - 3)] = 0$$

$$\text{即 } \sin 5\theta \cos \theta (4 \cos^2 2\theta - \cos 2\theta - 1) = 0$$

$$\therefore \theta = \frac{n\pi}{5}, 2k\pi \pm \frac{\pi}{2} \left( \text{即 } \frac{1}{2} \overline{2n+1\pi} \right)$$

$$\text{或 } \frac{1}{2} \cos^{-1} \frac{1 \pm \sqrt{17}}{8}$$

54. 令  $\theta/5 = \alpha$  則  $\tan 5\alpha = 5 \tan \alpha$

即  $\sin 5\alpha \cos \alpha = 5 \sin \alpha \cos 5\alpha$

即  $3 \sin 4\alpha = 2 \sin 6\alpha$

即  $3 \sin 2\alpha \cos 2\alpha = 3 \sin 2\alpha - 4 \sin^3 2\alpha$

即  $\sin 2\alpha (\cos 2\alpha - 1)(4 \cos 2\alpha + 1) = 0$

$\therefore \alpha = \frac{1}{2} n\pi, n\pi$  (此包含在  $\frac{1}{2} n\pi$  內)

或  $\frac{1}{2} \left[ 2n\pi + \cos^{-1} \left( -\frac{1}{4} \right) \right]$

即  $\theta = \frac{5}{2} n\pi, 5 \left[ n\pi + \frac{1}{2} \cos^{-1} \left( -\frac{1}{4} \right) \right]$

55. 即  $\frac{\sin 2^x \alpha}{\sin 2^x \alpha \sin 2^{x+1} \alpha} = \frac{1}{\sin 2^3 \alpha}$  (參考 197 頁例十九)

即  $\sin 2^{x+1} \alpha = \sin 2^3 \alpha$

故  $2^{x+1} \alpha = n\pi + (-1)^n \cdot (2^3 \alpha)$

當  $n=0$  時,  $2^{x+1} \alpha = 2^3 \alpha$

$\therefore x+1=3$ , 即  $x=2$

56. 即  $7346 \cdot 7^{\sec x} + 7 \cdot 7^{\sec x} - 7010 \cdot 7^{2 \sec x} - 343 \cdot 7^{2 \sec x}$   
 $+ 147 \cdot 7^{3 \sec x} = 147$

即  $49 \cdot 7^{3 \sec x} - 2451 \cdot 7^{2 \sec x} + 2451 \cdot 7^{\sec x} - 49 = 0$

即  $(7^{\sec x} - 1)(7^{\sec x} - 49)(49 \cdot 7^{\sec x} - 1) = 0$

$\therefore 7^{\sec x} = 7^0, 7^2, 7^{-2}$

即  $\sec x = 0, 2, -2$  即  $\cos x = \infty, \frac{1}{2}, -\frac{1}{2}$

今  $\cos x = \infty$  不合, 因  $|\cos x| \leq 1$  也



$$\therefore x = 2n\pi \pm \frac{2}{3}\pi \text{ 或 } 2n\pi \pm \frac{1}{3}\pi$$

57. 即  $5 \cdot 5^{\tan x} + 5/5^{\tan x} = 23$   
 即  $5 \cdot 5^{2 \tan x} - 26 \cdot 5^{\tan x} + 5 = 0$   
 即  $(5^{\tan x} - 5)(5 \cdot 5^{\tan x} - 1) = 0$   
 即  $5^{\tan x} = 5^1, 5^{-1}$   
 $\therefore \tan x = \pm 1 \qquad \therefore x = n \cdot 180^\circ \pm 45^\circ$

58.  $\therefore 4 - 4 \cos 2\theta = 8 \sin^2 \theta$  又  $6 - 4 \cos^2 \theta = 2 + 4 \sin^2 \theta$   
 即  $2^{8 \sin^2 \theta} - 3 \cdot 2^{2+4 \sin^2 \theta} + 32 = 0$   
 即  $(2^{4 \sin^2 \theta})^2 - 12 \cdot 2^{4 \sin^2 \theta} + 32 = 0$  (參考 197 頁例二十)  
 即  $(2^{4 \sin^2 \theta} - 4)(2^{4 \sin^2 \theta} - 8) = 0$   
 $\therefore 2^{4 \sin^2 \theta} = 2^2, 2^3 \qquad \therefore \sin^2 \theta = \frac{1}{2}, \frac{3}{4}$   
 $\therefore \sin \theta = \pm \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$   
 $\therefore \theta = (n \pm 1/4)\pi \text{ 或 } (n \pm 1/3)\pi$

59. 如圖  $\lambda = \sqrt{29}$ ,  $\tan \gamma = \frac{2}{5} = 4 \qquad \therefore \gamma = 21^\circ 48'$

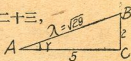
參考 198 頁例二十一及 199 頁例二十三,

$$\therefore \sin(\theta + \gamma) = \frac{5}{\sqrt{29}} = \sin 68^\circ 12'$$

$$\therefore \theta = -21^\circ 48' + n \cdot 180^\circ + (-1)^n (68^\circ 12')$$

故角為  $46^\circ 24', 90^\circ$ .

60. 即  $1 + \frac{1}{2}(1 - \cos 2\theta) = \frac{3}{2} \sin 2\theta$



$$\text{即} \quad 3 \sin 2\theta + \cos 2\theta = 3$$

以下做法如上題可求得

$$\cos(2\theta - \alpha) = \frac{3}{\sqrt{10}} = \cos 18^\circ 26'$$

$$\therefore 2\theta - \alpha = n \cdot 360^\circ \pm 18^\circ 26'$$

$$\therefore \theta = 35^\circ 47' + n \cdot 180^\circ \pm 9^\circ 13'$$



故比  $360^\circ$  小之正角爲  $26^\circ 34'$ ,  $45^\circ$ ,  $206^\circ 34'$ ,  $225^\circ$ .

$$61. \quad \therefore \quad 2 + \sqrt{3} = \tan 75^\circ$$

$$\text{即} \quad \sin \theta \cos 75^\circ + \cos \theta \sin 75^\circ = \cos 75^\circ$$

$$\text{即} \quad \sin(\theta + 75^\circ) = \sin 15^\circ$$

$$\therefore \quad \theta + 75^\circ = n \cdot 180^\circ + (-1)^n \cdot 15^\circ$$

故比  $360^\circ$  小之正角爲  $90^\circ$ ,  $300^\circ$ . (此題或可照上兩題法做)

$$62. \quad \text{今} \quad \tan \theta + \tan 2\theta = \sqrt{3} (1 - \tan \theta \tan 2\theta)$$

$$\therefore \quad \frac{\tan \theta + \tan 2\theta}{1 - \tan \theta \tan 2\theta} = \sqrt{3} \quad \text{即} \quad \tan 3\theta = \sqrt{3}$$

$$\therefore \quad 3\theta = n\pi + \frac{\pi}{3} \quad \text{即} \quad \theta = \frac{1}{3} \left( n + \frac{1}{3} \right) \pi$$

故比  $2\pi$  小之正角有  $\frac{\pi}{9}$ ,  $\dots, \frac{16\pi}{9}$  六角.

$$63. \quad \text{即} \quad m \sin 2(\alpha - \theta) = n \sin 2\theta$$

$$\text{即} \quad m(\sin 2\alpha \cos 2\theta - \cos 2\alpha \sin 2\theta) = n \sin 2\theta$$

$$\text{即} \quad \sin 2\theta(m \sin 2\alpha \cot 2\theta - m \cos 2\alpha - n) = 0$$

$$1. \quad \sin 2\theta = 0 \quad \text{即} \quad 2 \sin \theta \cos \theta = 0$$

但  $\cos \theta$  不可爲 0  $\therefore \sin \theta = 0 \quad \therefore \theta = n\pi$

$$2. \quad \cot 2\theta = \frac{m \cos 2\alpha + n}{m \sin 2\alpha}$$

$$\therefore \theta = \frac{1}{2} \tan^{-1} \frac{m \sin 2\alpha}{m \cos 2\alpha + n}$$

$$64. \quad \text{即} \quad \frac{\tan(\theta - \alpha)}{\tan \theta} = \frac{\tan \theta}{\tan(\theta - \beta)}$$

$$\text{即} \quad \frac{\tan(\theta - \alpha) + \tan \theta}{\tan \theta + \tan(\theta - \beta)} = \frac{\tan(\theta - \alpha) - \tan \theta}{\tan \theta - \tan(\theta - \beta)}$$

$$\text{即} \quad \frac{\sin(2\theta - \alpha)}{\sin(-\alpha)} = \frac{\sin(2\theta - \beta)}{\sin \beta}$$

$$\begin{aligned} \text{即} \quad \sin \beta (\sin 2\theta \cos \alpha - \cos 2\theta \sin \alpha) \\ = -\sin \alpha (\sin 2\theta \cos \beta - \cos 2\theta \sin \beta) \end{aligned}$$

$$\text{即} \quad \sin(\alpha + \beta) \sin 2\theta = 2 \sin \alpha \sin \beta \cos 2\theta$$

$$\therefore \quad \tan 2\theta = \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$\therefore \quad \theta = \frac{1}{2} n\pi + \frac{1}{2} \tan^{-1} \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$65. \quad \text{今} \quad 4 \cos \theta \sin(\theta - \alpha) \sin^3 \alpha = \cos^2 \alpha (\sin^2 \alpha - \cos^2 \theta) \\ = \cos^2 \alpha (\sin^2 \alpha \sin^2 \theta - \cos^2 \theta \cos^2 \alpha)$$

$$\begin{aligned} \text{即} \quad 4 \cos \theta \sin^3 \alpha (\sin \theta \cos \alpha - \cos \theta \sin \alpha) \\ = \cos^2 \alpha (\sin^2 \alpha \sin^2 \theta - \cos^2 \theta \cos^2 \alpha) \end{aligned}$$

全式除以  $\cos^4 \alpha \cos^2 \theta$ , 得

$$4 \tan^3 \alpha (\tan \theta - \tan \alpha) = \tan^2 \alpha \tan^2 \theta - 1$$

$$\text{即} \quad \tan^2 \theta \tan^2 \alpha - 4 \tan^3 \alpha \tan \theta + 4 \tan^4 \alpha - 1 = 0$$

$$\therefore \quad \tan \theta = 2 \tan \alpha \pm \cot \alpha$$

$$\therefore \theta = n\pi + \tan^{-1}(2 \tan \alpha \pm \cot \alpha)$$

66. 即  $35(3 \sin \theta - 4 \sin^3 \theta) - 40 \sin 2\theta \sin \theta + 39 \sin \theta = 0$

即  $4 \sin \theta (36 - 10 \sin 2\theta - 35 \sin^2 \theta) = 0$

1.  $\sin \theta = 0 \quad \therefore \theta = n\pi$

2.  $36(\sin^2 \theta + \cos^2 \theta) - 20 \sin \theta \cos \theta - 35 \sin^2 \theta = 0$

即  $\tan^2 \theta - 20 \tan \theta + 36 = 0$

即  $(\tan \theta - 2)(\tan \theta - 18) = 0$

$\therefore \tan \theta = 2, \tan \theta = 18$

故  $\theta = n\pi + \tan^{-1} 2, n\pi + \tan^{-1} 18$

67. 即  $\left[ 1 + \frac{\cos(\theta + \alpha) + \cos(\theta - \alpha)}{2} \right] \cdot \frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)}$   
 $+ \frac{\sin 2\alpha}{2} + \frac{\sin(\theta + \alpha) - \sin(\theta - \alpha)}{2} = 0$

即  $\sin(\theta + \alpha) + \frac{1}{4} \sin(2\theta + 2\alpha) + \frac{1}{2} \sin(\theta + \alpha) \cos(\theta - \alpha)$   
 $+ \frac{1}{2} \cos(\theta + \alpha) \sin 2\alpha + \frac{1}{4} \sin(2\theta + 2\alpha)$   
 $- \frac{1}{2} \cos(\theta + \alpha) \sin(\theta - \alpha) = 0$

即  $2 \sin(\theta + \alpha) + 2 \sin(\theta + \alpha) \cos(\theta + \alpha)$   
 $+ \sin 2\alpha + \cos(\theta + \alpha) \sin 2\alpha = 0$

即  $2 \sin(\theta + \alpha) [1 + \cos(\theta + \alpha)]$   
 $+ \sin 2\alpha [1 + \cos(\theta + \alpha)] = 0$

即  $[2 \sin(\theta + \alpha) + \sin 2\alpha] [1 + \cos(\theta + \alpha)] = 0$

$\therefore \theta + \alpha = n\pi + (-1)^n \sin^{-1}(-\frac{1}{2} \sin 2\alpha)$  或  $2n\pi \pm \pi$

68. 左邊 =  $[\tan(\alpha + \theta) - \tan(\alpha - \theta)][\tan(\alpha + \theta) + \tan(\alpha - \theta)]$

$$= \frac{\sin 2\alpha \sin 2\theta}{\cos^2(\alpha + \theta) \cos^2(\alpha - \theta)} = \frac{4 \sin 2\alpha \sin 2\theta}{(\cos 2\alpha + \cos 2\theta)^2}$$

故  $\sin \alpha \sin \theta [16 \cos^2 \alpha \cos^2 \theta - (\cos 2\alpha + \cos 2\theta)^2] = 0$

1.  $\sin \theta = 0 \quad \therefore \theta = n\pi \quad (\text{設 } \sin \alpha \neq 0)$

2.  $4 \cos \alpha \cos \theta = \pm 2(\cos^2 \alpha + \cos^2 \theta - 1)$

$\therefore (\cos \theta \mp \cos \alpha)^2 = 1, \quad \text{即 } \cos \theta \mp \cos \alpha = 1$

$\therefore \theta = \cos^{-1}(1 \pm \cos \alpha)$

69. 即  $\sec^2 \theta - 1 = 2 \tan \alpha \tan \beta \sec \theta + \tan^2 \alpha + \tan^2 \beta$

即  $(\sec \theta - \tan \alpha \tan \beta)^2 = (1 + \tan^2 \alpha)(1 + \tan^2 \beta)$

即  $\sec \theta - \tan \alpha \tan \beta = \pm \sec \alpha \sec \beta$

$\therefore \theta = \sec^{-1}(\tan \alpha \tan \beta \pm \sec \alpha \sec \beta)$

70. 設  $\theta = 6\alpha$  則  $\cos 6\alpha = -1$

今  $\cos 3\alpha = \pm \sqrt{\frac{1 + \cos 6\alpha}{2}} = 0$

即  $4 \cos^3 \alpha - 3 \cos \alpha = 0$  即  $\cos \alpha (4 \cos^2 \alpha - 3) = 0$

$\therefore \cos \alpha = 0, \pm \frac{\sqrt{3}}{2}$  故  $\cos \frac{\theta}{6} = 0, \pm \frac{\sqrt{3}}{2}$

本題亦可從  $\theta = (2n+1)\pi$  着手, 但若化  $\cos \theta$  爲  $\cos(\theta/6)$  函數做則繁矣。

71. 令五角依次爲  $x - 2y, x - y, x, x + y, x + 2y$  (公差爲  $y$ )

則  $\cos \omega = \cos(x - 2y) + \cos(x - y) + \cos(x + y)$   
 $+ \cos(x + 2y)$

$\therefore \cos x = 2 \cos x \cos 2y + 2 \cos x \cos y$

即  $\cos x [1 - 2 \cos y - 2(2 \cos^2 y - 1)] = 0$   $\cos x$  不能爲 0

$$\therefore 4 \cos^2 y + 2 \cos y - 3 = 0 \quad \therefore \cos y = \frac{\pm \sqrt{13} - 1}{4}$$

$$\text{但 } \frac{-\sqrt{13} - 1}{4} < -1, \quad \text{故公差爲 } y = \cos^{-1} \frac{\sqrt{13} - 1}{4}.$$

72. 從(1),(2); (2),(3)

$$x = \frac{\cos(\beta - \gamma) - \cos(\alpha - \beta)}{\cos(\alpha + \beta) - \cos(\beta + \gamma)} = \frac{\cos(\gamma - \alpha) - \cos(\beta - \gamma)}{\cos(\beta + \gamma) - \cos(\gamma + \alpha)}$$

$$\text{即 } \frac{\sin\left(\beta - \frac{\alpha + \gamma}{2}\right) \sin\left(\gamma - \frac{\alpha + \beta}{2}\right)}{\sin\left(\beta + \frac{\alpha + \gamma}{2}\right) \sin\left(\gamma + \frac{\alpha + \beta}{2}\right)}$$

$$\text{即 } \frac{\sin \beta \cos \frac{\alpha + \gamma}{2} - \cos \beta \sin \frac{\alpha + \gamma}{2}}{\sin \beta \cos \frac{\alpha + \gamma}{2} + \cos \beta \sin \frac{\alpha + \gamma}{2}}$$

$$= \frac{\sin \gamma \cos \frac{\alpha + \beta}{2} - \cos \gamma \sin \frac{\alpha + \beta}{2}}{\sin \gamma \cos \frac{\alpha + \beta}{2} + \cos \gamma \sin \frac{\alpha + \beta}{2}}$$

$$\text{即 } \frac{\tan \beta - \tan \frac{1}{2}(\alpha + \gamma)}{\tan \beta + \tan \frac{1}{2}(\alpha + \gamma)} = \frac{\tan \gamma - \tan \frac{1}{2}(\alpha + \beta)}{\tan \gamma + \tan \frac{1}{2}(\alpha + \beta)}$$

從比例合分之理得

$$\frac{\tan \beta}{\tan \frac{1}{2}(\gamma + \alpha)} = \frac{\tan \gamma}{\tan \frac{1}{2}(\alpha + \beta)}$$

再從(1),(2); (1),(3) 即得本題之證。

$$73. \text{ 以 } \beta, \gamma \text{ 之值代 } x \text{ 則 } \begin{cases} bc \cos \alpha \cos \beta + ca \sin \alpha \sin \beta = ab \\ bc \cos \alpha \cos \gamma + ca \sin \alpha \sin \gamma = ab \end{cases}$$

就  $\cos \alpha, \sin \alpha$  解之 (見附錄二代數 2)

$$\frac{\cos \alpha}{a^2bc(\sin \beta - \sin \gamma)} = \frac{\sin \alpha}{ab^2c(\cos \gamma - \cos \beta)}$$

$$= \frac{-1}{abc^2\sin(\gamma - \beta)}$$

即  $\frac{\cos \alpha}{a \cos \frac{1}{2}(\beta + \gamma)} = \frac{\sin \alpha}{b \sin \frac{1}{2}(\beta + \gamma)} = \frac{1}{c \cos \frac{1}{2}(\beta - \gamma)}$

即  $\frac{\cos^2 \alpha + \sin^2 \alpha}{a^2 \cos^2 \frac{1}{2}(\beta + \gamma) + b^2 \sin^2 \frac{1}{2}(\beta + \gamma)} = \frac{1}{c^2 \cos^2 \frac{1}{2}(\beta - \gamma)}$

即  $a^2[1 + \cos(\beta + \gamma)] + b^2[1 - \cos(\beta + \gamma)]$   
 $= c^2[1 + \cos(\beta - \gamma)]$

展開複角函數整理之即得本題之證。

74. 今  $\sin x \cos \alpha + \cos x \sin \alpha = m \sin \alpha$

$$\therefore \tan \alpha = \frac{\sin x}{m - \cos x}$$

以  $\beta, \gamma$  代入則  $\frac{\sin \beta}{m - \cos \beta} = \frac{\sin \gamma}{m - \cos \gamma}$

即  $\sin(\beta - \gamma) = m(\sin \beta - \sin \gamma)$

故  $\cos \frac{1}{2}(\beta - \gamma) = m \cos \frac{1}{2}(\beta + \gamma)$

[設  $\sin \frac{1}{2}(\beta - \gamma) \neq 0$ ]

75. 從根與係數之關係得  $\Sigma \tan \alpha = 0$  (見附錄二代數 4)

$$\therefore \tan \gamma = -(\tan \alpha + \tan \beta) = -h$$

今  $\tan \gamma$  爲  $\tan \theta$  之一值，亦即  $-h$  代  $\tan \theta$  必適合。

故  $ah^3 + (2a - x)h = y$

76. 今  $\frac{1 + \tan \theta}{1 - \tan \theta} = 3 \cdot \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$

$$\text{即 } 3 \tan^4 \theta - 6 \tan^2 \theta + 8 \tan \theta - 1 = 0$$

故由根與係數之關係得  $\Sigma \tan \alpha = 0$

(因三次項之係數為 0)

$$77. \text{ 今 } \sin \theta \cos \lambda + \cos \theta \sin \lambda = 2 \sin \theta \cos \theta + b$$

$$\text{令 } \tan \frac{1}{2} \theta = x$$

$$\text{則 } \sin \theta = \frac{2x}{1+x^2}, \quad \cos \theta = \frac{1-x^2}{1+x^2}$$

$$\text{則 } (b + \sin \lambda)x^4 - 2(2 + \cos \lambda)x^3 + 2b\lambda^2 + 2(2 - \cos \lambda)x + (b - \sin \lambda) = 0 \quad (\text{參考 202 頁例二十七})$$

$$\text{則 } S_1 = \Sigma \tan \frac{\alpha}{2} = \frac{2(2 + \cos \lambda)}{b + \sin \lambda}, \quad S_2 = \frac{2b}{b + \sin \lambda}$$

$$S_3 = \frac{2(\cos \lambda - 2)}{b + \sin \lambda}, \quad S_4 = \frac{b - \sin \lambda}{b + \sin \lambda}$$

$$\text{今 } \tan \frac{\alpha + \beta + \gamma + \delta}{2} = \frac{S_1 - S_3}{1 - S_2 + S_4} = \dots = \frac{8}{0} = \infty$$

$$\therefore \Sigma \frac{\alpha}{2} = n\pi + \frac{\pi}{2} = (2n + 1) \frac{\pi}{2}$$

$$\therefore \Sigma \alpha = (2n + 1)\pi$$

$$78. \therefore \cot(\theta + \alpha) - \csc(\theta + \alpha) = \frac{\cos(\theta + \alpha) - 1}{\sin(\theta + \alpha)}$$

$$= -\tan \frac{\theta + \alpha}{2}$$

$$\text{故原式爲 } \Sigma \tan \frac{\theta + \alpha}{2} = 0 \quad \text{即 } \Sigma \frac{\tan \frac{\theta}{2} + \tan \frac{\alpha}{2}}{\tan \frac{\theta}{2} \tan \frac{\alpha}{2} - 1} = 0$$

$$\text{設 } \tan \frac{\theta}{2} = x$$



$$\text{故 } \Sigma \left( x + \tan \frac{\alpha}{2} \right) \left( x \tan \frac{\beta}{2} - 1 \right) \left( x \tan \frac{\gamma}{2} - 1 \right) = 0$$

$$\text{即 } \Sigma \left[ x^3 \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \left( \Pi \tan \frac{\alpha}{2} - \tan \frac{\beta}{2} - \tan \frac{\gamma}{2} \right) x^2 \right. \\ \left. + \left( 1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} - \tan \frac{\alpha}{2} \tan \frac{\gamma}{2} \right) x + \tan \frac{\alpha}{2} \right] = 0$$

$$\therefore \Sigma \tan \frac{\theta_1}{2} = \frac{2\Sigma \tan \frac{\alpha}{2} - 3\Pi \tan \frac{\alpha}{2}}{\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}$$

$$\Sigma \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{3 - 2\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}$$

$$\Pi \tan \frac{\theta_1}{2} = - \frac{\Sigma \tan \frac{\alpha}{2}}{\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}$$

$$\tan \frac{\theta_1 + \theta_2 + \theta_3}{2} = \frac{\Sigma \tan \frac{\theta_1}{2} - \Pi \tan \frac{\theta_1}{2}}{1 - \Sigma \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2}}$$

$$= \frac{2\Sigma \tan \frac{\alpha}{2} - 3\Pi \tan \frac{\alpha}{2} + \Sigma \tan \frac{\alpha}{2}}{\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2} - 3 + 2\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}$$

$$= \frac{3 \left( \Sigma \tan \frac{\alpha}{2} - \Pi \tan \frac{\alpha}{2} \right)}{-3 \left( 1 - \Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right)}$$

$$= -\tan \frac{\alpha + \beta + \gamma}{2} = \tan \left( -\frac{\alpha + \beta + \gamma}{2} \right)$$

$$\therefore \frac{\theta_1 + \theta_2 + \theta_3}{2} = k\pi - \frac{\alpha + \beta + \gamma}{2}$$

$$\therefore \theta_1 + \theta_2 + \theta_3 + \alpha + \beta + \gamma = 2k\pi$$

79. 令  $\tan \frac{1}{2}x = t$

則  $\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$

即  $A \left( \frac{2t}{1+t^2} \right)^3 + B \left( \frac{1-t^2}{1+t^2} \right)^3 + C = 0$

即  $F(t) = (C-B)t^6 + 3(B+C)t^4 + 8At^3 + 3(C-B)t^2 + (C+B) = 0$

$$F'(t) = 6(C-B)t^5 + 12(B+C)t^3 + 24At^2 + 6(C-B)t$$

今  $F(t), F'(t)$  無公因式，故本題有六個不同之根。

設六根為  $t_1, t_2, \dots, t_6$ ，從根與係數關係得：

$$S_1 = 0, \quad S_2 = 3(B+C)/(C-B), \quad S_3 = -8A/(C-B),$$

$$S_4 = 3, \quad S_5 = 0, \quad S_6 = (C+B)/(C-B)$$

故  $\tan \frac{\sum t_i}{2} = \frac{S_1 - S_3 + S_5}{1 - S_2 + S_4 - S_6} = -\frac{A}{B}$

80. 今以  $\theta$  代  $x$  可適合，且  $\theta$  與  $\alpha, \beta, \gamma$  相差均不為  $2\pi$ ，故

知  $\theta$  為此方程式之一根。今以  $\tan(x/2) = A$ ，則

$$\sin x = 2A/(1+A^2), \quad \cos x = (1-A^2)/(1+A^2)$$

代入原式則  $\sin 2\theta \left[ \frac{2aA}{1+A^2} + \frac{b(1-A^2)}{1+A^2} \right] = \frac{2A(1-A^2)}{(1+A^2)^2}$

$$\times (a \sin \theta + b \cos \theta)$$

$$\begin{aligned} \text{即 } \sin 2\theta(1+A^2)(b+2aA-bA^2) \\ = 2A(1-A^2)(a \sin \theta + b \cos \theta) \end{aligned}$$

$$\text{即 } -b \sin 2\theta \cdot A^4 + \dots + b \sin 2\theta = 0$$

$$\therefore \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\theta}{2} = \frac{b \sin 2\theta}{-b \sin 2\theta} = -1$$

故如題云。

$$81. \text{ 設 } \tan \frac{\theta}{2} = x \quad \text{則 } \sin \theta = \frac{2x}{1+x^2}, \quad \cos \theta = \frac{1-x^2}{1+x^2}$$

$$\text{則 } a \left[ \frac{(1-x^2)^2 - 4x^2}{(1+x^2)^2} \right] + b \frac{4x(1-x^2)}{(1+x^2)^2} + c \frac{1-x^2}{1+x^2} + d = 0$$

$$\begin{aligned} \text{即 } (a-c+d)x^4 - 4bx^3 + (-6a+2d)x^2 \\ + 4bx + (a+c+d) = 0 \end{aligned}$$

$$\therefore \sum \tan \frac{\alpha}{2} = \frac{4b}{a-c+d}$$

$$\text{及 } \sum \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} = -\frac{4b}{a-c+d}$$

$$\therefore \sum \tan \frac{\alpha}{2} + \sum \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} = 0$$

$$\begin{aligned} \text{即 } \left( \tan \frac{\alpha}{2} + \tan \frac{\delta}{2} \right) \left( 1 + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \right) \\ + \left( \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \right) \left( 1 + \tan \frac{\alpha}{2} \tan \frac{\delta}{2} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{即 } \left( \sin \frac{\alpha}{2} \cos \frac{\delta}{2} + \cos \frac{\alpha}{2} \sin \frac{\delta}{2} \right) \\ \times \left( \cos \frac{\beta}{2} \cos \frac{\gamma}{2} + \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \right) \end{aligned}$$

$$\begin{aligned}
 & + \left( \sin \frac{\beta}{2} \cos \frac{\gamma}{2} + \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \right) \\
 & \times \left( \cos \frac{\alpha}{2} \cos \frac{\delta}{2} + \sin \frac{\alpha}{2} \sin \frac{\delta}{2} \right) = 0
 \end{aligned}$$

$$\text{即 } \sin \frac{\alpha+\delta}{2} \cos \frac{\beta-\gamma}{2} + \sin \frac{\beta+\gamma}{2} \cos \frac{\alpha-\delta}{2} = 0$$

$$\begin{aligned}
 \text{即 } \sin \frac{\alpha+\delta+\beta-\gamma}{2} + \sin \frac{\alpha+\delta-\beta+\gamma}{2} + \sin \frac{\beta+\gamma+\alpha-\delta}{2} \\
 + \sin \frac{\beta+\gamma-\alpha+\delta}{2} = 0
 \end{aligned}$$

$$82. (1) \text{ 今 } \sin x(2 \cos x - n) = m \cos x - k$$

$$\begin{aligned}
 \text{即 } (1 - \cos^2 x)(4 \cos^2 x - 4n \cos x + n^2) \\
 = m^2 \cos^2 x - 2km \cos x + k^2
 \end{aligned}$$

$$\text{即 } 4 \cos^4 x - 4n \cos^3 x + \dots = 0$$

$$\therefore \Sigma \cos \alpha = 4n/4 = n$$

$$\begin{aligned}
 (2) \text{ 今 } (\sin 2x + k)^2 &= (m \cos x + n \sin x)^2 \\
 &= \frac{1}{2} m^2 (1 + \cos 2x) \\
 &\quad + \frac{1}{2} n^2 (1 - \cos 2x) + mn \sin 2x
 \end{aligned}$$

$$\text{設 } \sin 2x = y$$

$$\text{則 } 2(y+k)^2 - 2mny = m^2 + n^2 + (m^2 - n^2) \cos 2x$$

$$\text{即 } 2(y+k)^2 - 2mny - m^2 - n^2 = (m^2 - n^2) \cos 2x$$

$$\begin{aligned}
 \text{即 } [2y^2 + (4k - 2mn)y + 2k^2 - m^2 - n^2]^2 \\
 = (m^2 - n^2)^2 (1 - y^2)
 \end{aligned}$$

$$\text{即 } 4y^4 + 8(2k - mn)y^3 + \dots = 0$$

$$\therefore \Sigma \sin 2\alpha = -2(2k - mn) = 2mn - 4k$$

83. 今  $\sin \theta + \cos \theta = c \sin \theta \cos \theta$

即  $1 + 2 \sin \theta \cos \theta = c^2 \sin^2 \theta \cos^2 \theta$

$$c^2 \sin^2 2\theta - 4 \sin 2\theta - 4 = 0$$

故  $\sin 2\theta = \frac{2 \pm 2\sqrt{1+c^2}}{c^2} \quad \because 1+c^2 > 0$

$\therefore \sin 2\theta$  可有兩解，惟  $|\sin 2\theta| < 1$ ，故必須

$$-c^2 < 2 + 2\sqrt{1+c^2} < c^2 \quad (1)$$

或  $-c^2 < 2 - 2\sqrt{1+c^2} < c^2 \quad (2)$

(1) 就(1)言，則  $2 + 2\sqrt{1+c^2} > -c^2$  必可成立

(因左邊必為正)

故只須  $c^2 > 2 + 2\sqrt{1+c^2}$

即  $c^2 - 2 > 2\sqrt{1+c^2}$

即  $(c^2 - 2)^2 > 4(1+c^2)$  即  $c^2(c^2 - 8) > 0$

$\therefore$  只須  $c^2 > 8$

即在  $c^2 > 8$  時，必可以成立，而  $\sin 2\theta$  可有一解。

(2) 就(2)言，則  $c^2 > 2 - 2\sqrt{1+c^2}$  必可成立

(因右邊必為負)

故只須  $2 - 2\sqrt{1+c^2} > -c^2$

即  $c^2 + 2 > 2\sqrt{1+c^2}$

即  $(c^2 + 2)^2 > 4(1+c^2)$  即  $c^4 > 0$

不論  $c^2 \geq 8$  均可成立。

綜上所述則  $c^2 > 8$  時， $\sin 2\theta$  有兩解，即在  $0, 2\pi$  間

$\theta$  有四根。

又如  $e^2 < 8$ , 則  $\theta$  在  $0$  與  $2\pi$  之間祇有兩根 (根指主角而言).

$$84. \text{ 即 } 32 \cos^3 x - 24 \cos x + 3\sqrt{6} = 0$$

$$\text{即 } (4 \cos x - \sqrt{6})(8 \cos^2 x + 2\sqrt{6} \cos x - 3) = 0$$

$$(1) \quad \cos x = \frac{\sqrt{6}}{4} = \frac{\sqrt{6}}{2} \sin \frac{\pi}{6}$$

$$(2) \quad 8 \cos^2 x + 2\sqrt{6} \cos x - 3 = 0$$

$$\therefore \cos x = \frac{-\sqrt{6} \pm \sqrt{30}}{8} = \frac{\sqrt{6}}{2} \left[ \frac{-1 \pm \sqrt{5}}{4} \right]$$

$$= \frac{\sqrt{6}}{2} \sin \frac{\pi}{10}, \quad -\frac{\sqrt{6}}{2} \sin \frac{3\pi}{10} \quad \text{故如題云.}$$

85. 如例二十九, 令  $x = \rho \cos \theta$  比較下兩式之相當係數

$$4 \cos^3 \theta - 3 \cos \theta - \cos 3\theta = 0$$

$$\rho^3 \cos^3 \theta - 3\rho \cos \theta + 1 = 0$$

$$\text{得 } \frac{\rho^3}{4} = \frac{3\rho}{3} = \frac{1}{-\cos 3\theta} \quad \therefore \rho = 2, \quad \cos 3\theta = -\frac{1}{2}$$

$$\therefore \theta = n \cdot 120^\circ \pm 40^\circ$$

故當  $\theta$  為  $80^\circ, 160^\circ, 320^\circ$  時,  $\cos \theta$  有不同之值

故  $x = 2 \cos 80^\circ, 2 \cos 160^\circ$  或  $2 \cos 320^\circ$

即為  $2 \cos 80^\circ, -2 \cos 20^\circ$  或  $2 \cos 40^\circ$ .

(此係三角解法)

$$86. \text{ 如上題得 } \rho = 2\sqrt{2}, \quad \cos 3\theta = \frac{1}{2} \sqrt{2}$$

$$\therefore \theta = n \cdot 120^\circ \pm 15^\circ$$

故當  $\theta$  為  $15^\circ, 105^\circ, 135^\circ$  時,  $\cos \theta$  有不同之值

故  $x = 2\sqrt{2} \cos 15^\circ, 2\sqrt{2} \cos 105^\circ, 2\sqrt{2} \cos 135^\circ$

即  $x = 1 + \sqrt{3}, 1 - \sqrt{3},$  及  $-2.$

[因  $\cos 15^\circ = \frac{1}{4}(\sqrt{6} + \sqrt{2})$ ]

87. 如上兩題求到  $\rho = 2a, \quad \cos 3\theta = \cos 3A$

$\therefore \theta = n \cdot 120^\circ \pm A$

$\theta = A, 120^\circ \pm A$  ( $\cos \theta$  不同之值)

故  $x = 2a \cos A, \quad 2a \cos(120^\circ \pm A)$

88. 今  $(\tan x - \sqrt{3})(\tan x - 1) < 0$

即  $\sqrt{3} > \tan x > 1$

可知  $\tan x$  恆爲正, 故  $x$  必在第一, 三象限內. 且正切之值隨角而增.

(1) 設在第一象限:

因  $\tan \frac{\pi}{3} = \sqrt{3}, \quad \tan \frac{\pi}{4} = 1$  故  $\frac{\pi}{3} > x > \frac{\pi}{4}$

(2) 設在第三象限:

因  $\tan \frac{4\pi}{3} = \sqrt{3}, \quad \tan \frac{5\pi}{4} = 1$  故  $\frac{4\pi}{3} > x > \frac{5\pi}{4}$

89. 即  $\frac{(\sin x + 1)(2 \sin x - 1)}{(\sqrt{2} \sin x + 1)(\sqrt{2} \sin x - 1)} < 0$

因  $|\sin x| \leq 1$  故  $\sin x + 1 > 0$

今以  $(2 \sin^2 x - 1)^2$  乘兩邊, 則

$(2 \sin x - 1)(\sqrt{2} \sin x - 1)(\sqrt{2} \sin x + 1) < 0$

$\therefore \frac{1}{2} < \sin x < \frac{1}{\sqrt{2}}$  或  $-1 < \sin x < -\frac{1}{\sqrt{2}}$

(因  $\sin x$  必大於  $-1$ )

本題假定  $x < 180^\circ$ ，即  $\sin x$  必為正，故祇有

$$\frac{1}{2} < \sin x < \frac{1}{\sqrt{2}}$$

(1) 在第一象限： $\pi/6 < x < \pi/4$

(第一象限內角增正弦亦增加)

(2) 在第二象限： $3\pi/4 < x < 5\pi/6$

(第二象限內角增正弦反減小)

90. 今 
$$\frac{-4 \sin^2 x + 2(\sqrt{3} - \sqrt{2}) \sin x - (\sqrt{6} - 2)}{4 \sin^2 x - 1} < 0$$

今分子之二次項之係數為負且其判別式

$$\Delta = 4(\sqrt{3} - \sqrt{2})^2 - 16(\sqrt{6} - 2) = 52 - 24\sqrt{6} < 0$$

故其分子必為負 故  $\frac{1}{4 \sin^2 x - 1} > 0$

兩邊乘以  $(4 \sin^2 x - 1)^2$

則  $(2 \sin x - 1)(2 \sin x + 1) > 0$

$$\therefore \sin x > \frac{1}{2} \text{ 或 } \sin x < -\frac{1}{2}$$

故  $150^\circ > x > 30^\circ$  或  $330^\circ > x > 210^\circ$  (今  $x < 360^\circ$ )

91. 設  $x$  有不等之實根，則此式之判別式大於 0

即  $24^2 \sin^2 \alpha - 64(-3 \sin \alpha + 2) > 0$

即  $(3 \sin \alpha + 2)(3 \sin \alpha - 1) > 0$

$$\therefore \sin \alpha > \frac{1}{3} \text{ 或 } \sin \alpha < -\frac{2}{3}$$

但  $\alpha$  限於  $180^\circ$  之內，即  $\sin \alpha$  恆為正，故只有  $\sin \alpha > \frac{1}{3}$

$\therefore 90^\circ > \alpha > \sin^{-1} \frac{1}{3}$  及  $\pi - \sin^{-1} \frac{1}{3} > \alpha > 90^\circ$

即  $\alpha$  在  $\pi - \sin^{-1} \frac{1}{3}$  與  $\sin^{-1} \frac{1}{3}$  之間。



92. 今條件有二：

$$(1) \Delta = 4b^2 - 4c > 0 \quad \text{即} \quad b^2 > c$$

$$(2) \because |\sin x| \leq 1 \quad \therefore \left| \frac{-2b \pm 2\sqrt{b^2 - c}}{2} \right| \leq 1$$

$$\text{即} \quad | -b \pm \sqrt{b^2 - c} | \leq 1$$

今  $b$  爲正，故  $-b \pm \sqrt{b^2 - c}$  爲負

$$\text{故即} \quad | b \mp \sqrt{b^2 - c} | \leq 1$$

$$\text{惟} \quad b + \sqrt{b^2 - c} \geq b - \sqrt{b^2 - c}$$

$$\text{故第二條件爲} \quad | b + \sqrt{b^2 - c} | \leq 1$$

$$\text{而} \quad b + \sqrt{b^2 - c} \text{ 必爲正}$$

$$\text{故即} \quad b + \sqrt{b^2 - c} \leq 1$$

是故原式中設  $\sin x$  有兩不等之實根時其條件爲：

$$b^2 \geq c \text{ 及 } b + \sqrt{b^2 - c} \leq 1$$

$$93. \text{ 今} \quad \tan \theta = \tan \beta \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}$$

$$\text{即} \quad \tan \alpha \tan^2 \theta + \tan \theta (\tan \beta - 1) + \tan \alpha \tan \beta = 0$$

今  $\theta$  爲實數，即  $\tan \theta$  爲實數，故其判別式爲正

$$\text{即} \quad (\tan \beta - 1)^2 - 4 \tan^2 \alpha \tan \beta > 0$$

$$\tan^2 \beta - 2(1 + 2 \tan^2 \alpha) \tan \beta + (\sec^2 \alpha - \tan^2 \alpha)^2 > 0$$

$$\text{故} \quad [\tan \beta - (\sec \alpha - \tan \alpha)^2]$$

$$\times [\tan \beta - (\sec \alpha + \tan \alpha)^2] > 0$$

即  $\tan \beta$  不能介於  $(\sec \alpha - \tan \alpha)^2$  及  $(\sec \alpha + \tan \alpha)^2$  之間。

94. 令 
$$x = \frac{\tan(\theta + \alpha)}{\tan(\theta - \alpha)}$$

則 
$$\frac{x+1}{x-1} = \frac{\tan(\theta + \alpha) + \tan(\theta - \alpha)}{\tan(\theta + \alpha) - \tan(\theta - \alpha)}$$

即 
$$\frac{x+1}{x-1} = \frac{\sin 2\theta}{\sin 2\alpha}$$

因  $\theta$  爲實數，則  $\left| \frac{(x+1)\sin 2\alpha}{x-1} \right| < 1$

即 
$$\frac{(x+1)^2 \sin^2 2\alpha}{(x-1)^2} < 1$$

即 
$$(x+1)^2 \sin^2 2\alpha < (x-1)^2$$

即 
$$(1 - \sin^2 2\alpha)x^2 - 2(1 + \sin^2 2\alpha)x + (1 - \sin^2 2\alpha) > 0$$

即 
$$[(1 - \sin 2\alpha)x - (1 + \sin 2\alpha)]$$

$$\times [(1 + \sin 2\alpha)x - (1 - \sin 2\alpha)] > 0$$

即  $x$  之值不能介於  $\frac{1 + \sin 2\alpha}{1 - \sin 2\alpha}$  及  $\frac{1 - \sin 2\alpha}{1 + \sin 2\alpha}$  之間。

95. 設原式爲  $x$ ，則  $(a-x)\tan^2\theta + b\tan\theta + (c-x) = 0$

今  $\theta$  爲實值，則  $\Delta = b^2 - 4(a-x)(c-x) > 0$

即 
$$4x^2 - 4(a+c)x + (4ac - b^2) < 0$$

即 
$$[2x - (a+c) + \sqrt{b^2 + (a-c)^2}]$$

$$\times [2x - (a+c) - \sqrt{b^2 + (a-c)^2}] < 0$$

故  $x$  之值在  $\frac{1}{2}[a+c - \sqrt{b^2 + (a-c)^2}]$  及  $\frac{1}{2}[a+c + \sqrt{b^2 + (a-c)^2}]$  之間。

96. a. 設 
$$x = p \cot \theta + q \tan \theta$$

則 
$$q \tan^2 \theta - x \tan \theta + p = 0$$

$$\therefore \tan \theta = \frac{x \pm \sqrt{x^2 - 4pq}}{2q}$$

$$\therefore \Delta \geq 0 \quad \therefore x^2 - 4pq \geq 0$$

$$\text{故 } |x| \geq 2\sqrt{pq} \quad x \text{ 之極小值爲 } 2\sqrt{pq}$$

$$\text{則 } \tan \theta = \frac{x}{2q} = \sqrt{\frac{p}{q}}$$

$$\text{b. 設 } x = 3 - 2 \cos \theta + \cos^2 \theta$$

$$\text{則 } \cos^2 \theta - 2 \cos \theta + 3 - x = 0$$

$$\therefore \cos \theta = \frac{2 \pm \sqrt{4 - 4(3-x)}}{2} = 1 \pm \sqrt{x-2}$$

$$\therefore \Delta \geq 0 \quad \text{則 } x - 2 \geq 0 \quad \therefore x \geq 2$$

$$\text{故 } x \text{ 之極小值爲 } 2, \text{ 則 } \cos \theta = 1, \text{ 故 } \theta = 2n\pi.$$

$$97. \text{ a. 設 } x = \sin \theta + \cos \theta$$

$$\text{則 } (x - \sin \theta)^2 = \cos^2 \theta = 1 - \sin^2 \theta$$

$$\text{即 } 2 \sin^2 \theta - 2x \sin \theta + x^2 - 1 = 0$$

因  $\sin \theta$  爲實數

$$\text{故 } 4x^2 - 8(x^2 - 1) \geq 0 \quad \text{即 } x^2 \leq 2$$

$$\text{故 } x \text{ 之極大值爲 } \sqrt{2}$$

$$\text{則 } 2 \sin^2 \theta - 2\sqrt{2} \sin \theta + 1 = 0$$

$$\therefore \sin \theta = \frac{2\sqrt{2} \pm \sqrt{8-8}}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = n\pi + (-1)^n \cdot \frac{\pi}{4}$$

$$\text{b. 設 } x = \sin \theta \cos(\alpha - \theta)$$

$$\text{則 } 2x = \sin \alpha + \sin(2\theta - \alpha)$$

設  $x$  爲極大，則  $\sin(2\theta - \alpha)$  亦爲極大，而  $\sin(2\theta - \alpha)$  之極大值爲 1

$$\text{即 } 2\theta - \alpha = n\pi + (-1)^n \cdot \frac{\pi}{2}$$

$$\therefore \theta = n \cdot \frac{\pi}{2} + (-1)^n \cdot \frac{\pi}{4} + \frac{\alpha}{2}$$

代入原式得極大值爲  $x = \frac{1}{2}(\sin \alpha + 1)$

98. a. 設原式爲  $x$ ，則  $4 \cos^2 \theta - 2\sqrt{3}x \cos \theta + 3 = 0$

$$\therefore \cos \theta \text{ 爲實數，則 } 12x^2 - 12 \cdot 4 \geq 0$$

$$\text{故 } x \geq 2 \text{ 或 } x \leq -2$$

故極小值爲 2，極大值爲 -2。

b. 設原式爲  $x$

$$\text{則 } (x-1)\tan^2 \theta + (x+1)\tan \theta + (x-1) = 0$$

今  $\tan \theta$  爲實數

$$\text{故 } (1+x)^2 - 4(x-1)^2 > 0$$

$$\text{即 } 3x^2 - 10x + 3 < 0 \quad \text{即 } (3x-1)(x-3) < 0$$

$$\therefore 3 > x > \frac{1}{3} \quad \text{故極大值爲 3，極小值爲 } \frac{1}{3}.$$

99. 照第一列展開

$$1 + 2\Pi \cos \alpha - \Sigma \cos^2 \alpha - \cos^2 \theta (1 - \cos^2 \gamma) = 0$$

$$\text{即 } \sin^2 \gamma \sin^2 \theta = \cos^2 \alpha + \cos^2 \beta - 2\Pi \cos \alpha$$

$$\text{則 } \sin \theta = \pm \frac{\sqrt{\cos^2 \alpha + \cos^2 \beta - 2\Pi \cos \alpha}}{\sin \gamma}$$

$$\text{故 } \theta = n\pi \pm \sin^{-1} \frac{\sqrt{\cos^2 \alpha + \cos^2 \beta - 2\cos \alpha \cos \beta}}{\sin \gamma}$$

100. 設此為圓錐曲線之特例時(即退化曲線 degenerate curve), 則

$$\Theta = \frac{1}{2} \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 0$$

$$\text{即 } \begin{vmatrix} 2 & 2 \sin \alpha & 2 \sin \alpha \\ 2 \sin \alpha & 2 & 2 \sin \alpha \\ 2 \sin \alpha & 2 \sin \alpha & 2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & \sin \alpha & \sin \alpha \\ \sin \alpha & 1 & \sin \alpha \\ \sin \alpha & \sin \alpha & 1 \end{vmatrix} = 0$$

$$\text{即 } 2 \sin^3 \alpha - 3 \sin^2 \alpha + 1 = 0$$

$$\text{即 } (\sin \alpha - 1)^2 (2 \sin \alpha + 1) = 0$$

$$\therefore \sin \alpha = 1, 1, -\frac{1}{2}$$

$$\therefore \alpha = n\pi + (-1)^n \cdot \frac{\pi}{2} \text{ 或 } n\pi - (-1)^n \cdot \frac{\pi}{6}$$

## 習題十八 (226—228 頁)

1. 設第一, 第二角為  $\alpha, \beta$  則  $\alpha + \beta = \pi/4$

$$\therefore \tan(\alpha + \beta) = 1$$

$$\text{即 } \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = 1$$

$$\text{即 } \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{(x-1)(x+1)}{(x-2)(x+2)}} = 1$$

$$\text{即 } \frac{(x-1)(x+2) + (x+1)(x-2)}{(x^2-4) - (x^2-1)} = 1$$

$$\text{即 } 2x^2 = 1 \quad \therefore x = \pm \frac{\sqrt{2}}{2}$$

2. 設第一, 二, 三角依次爲  $\alpha, \beta, \gamma$

$$\text{則 } \alpha + \beta = \gamma \quad \text{即 } \cot(\alpha + \beta) = \cot \gamma$$

$$\text{即 } \frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha} = a - 1$$

$$\text{即 } \frac{x(a^2 - x + 1) - 1}{x + (a^2 - x + 1)} = a - 1$$

$$\text{即 } (x - a)[x - (a^2 - a + 1)] = 0$$

$$\therefore x = a, \quad a^2 - a + 1$$

3. 設四個角依次爲  $\alpha, \beta, \gamma, \delta$

$$\text{則 } \tan \alpha = x + 1, \quad \tan \beta = 1/(x - 1)$$

$$\text{又 } \sin \gamma = \frac{4}{5} \quad \therefore \cos \gamma = \frac{3}{5}$$

$$\text{則 } \tan \gamma = \frac{4}{3} \quad \tan \delta = \frac{4}{3}$$

$$\text{今 } \alpha + \beta = \gamma + \delta \quad \therefore \tan(\alpha + \beta) = \tan(\gamma + \delta)$$

$$\text{即 } \frac{x+1 + \frac{1}{x-1}}{1 - \frac{x+1}{x-1}} = \frac{\frac{4}{3} + \frac{4}{3}}{1 - \frac{4}{3} \cdot \frac{4}{3}}$$

$$\text{即 } -\frac{x^2}{2} = -\frac{24}{7} \quad \therefore 7x^2 = 48$$

$$\therefore x = \pm \sqrt{\frac{48}{7}} = \pm \frac{4}{7} \sqrt{21}$$

$$4. \text{ 兩邊取正切, 則 } \lambda + 1 = \frac{3(\lambda - 1) - (\lambda - 1)^3}{1 - 3(\lambda - 1)^2}$$

$$\text{則 } \lambda^3 - 2\lambda = 0 \quad \text{即 } \lambda(\lambda^2 - 2) = 0$$

$$\therefore \lambda = 0, \pm \sqrt{2}$$

$$5. \text{ 設四角依次爲 } \alpha, \beta, \gamma, \delta \quad \text{則 } \alpha + \beta = \delta - \gamma$$

$$\text{即 } \tan(\alpha + \beta) = \tan(\delta - \gamma)$$

$$\text{即 } \frac{x+1+x}{1-x(x+1)} = \frac{3x-(x-1)}{1+3x(x-1)}$$

$$\text{即 } 2x(4x^2 - 1) = 0 \quad \therefore 2x(2x-1)(2x+1) = 0$$

$$\therefore x = 0, \pm \frac{1}{2}$$

$$6. \text{ 設三角依次爲 } \alpha, \beta, \gamma$$

$$\text{則 } \cos \alpha = x, \quad \cos \beta = \sqrt{1-x^2}, \quad \cos \gamma = x\sqrt{3}$$

$$\therefore \sin \alpha = \sqrt{1-x^2}, \quad \sin \beta = x \quad \text{兩邊取餘弦}$$

$$\text{則 } x\sqrt{1-x^2} + x\sqrt{1-x^2} = 2\sqrt{3} \quad \therefore 4x^2(1-x^2) = 3x^2$$

$$\text{即 } x^2(4x^2 - 1) = 0 \quad \therefore x = 0, \pm \frac{1}{2}$$

$$7. \text{ 設兩角爲 } \alpha, \beta \quad \text{則 } \sin \alpha = 5/x, \quad \sin \beta = 12/x$$

$$\text{今 } \alpha = \pi/2 - \beta \quad \text{則 } \sin \alpha = \cos \beta$$

$$\text{即 } \frac{5}{x} = \sqrt{1 - \frac{144}{x^2}} \quad \text{即 } 25 = x^2 - 144$$

$$\therefore x^2 = 169 \quad \therefore x = 13 \quad (\text{負號不合因假定爲銳角也})$$

8. 設第一, 第二角各爲  $\alpha, \beta$

$$\text{則 } \sin \alpha = \frac{2x}{1+x^2}, \quad \cos \beta = \frac{1-x^2}{1+x^2}$$

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \frac{1-x^2}{1+x^2} = \cos \beta$$

$$\therefore \alpha = \beta \quad \therefore 2\alpha = \frac{\pi}{4} \quad \text{即 } \tan \frac{\alpha}{2} = \tan \frac{\pi}{16}$$

$$\text{但 } \tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = x$$

$$\therefore x = \tan \frac{\pi}{16}$$

9. 設三個角各爲  $\alpha, \beta, \gamma$

$$\text{則 } \csc \alpha = \lambda, \quad \sin \alpha = \frac{1}{\lambda}, \quad \sin \beta = \frac{1}{a},$$

$$\cos \beta = \frac{\sqrt{a^2 - 1}}{a}, \quad \sin \gamma = \frac{1}{b}, \quad \cos \gamma = \frac{\sqrt{b^2 - 1}}{b}$$

兩邊取正弦, 則

$$\frac{1}{\lambda} = \frac{\sqrt{a^2 - 1}}{ab} + \frac{\sqrt{b^2 - 1}}{ab} = \frac{\sqrt{a^2 - 1} + \sqrt{b^2 - 1}}{ab}$$

$$\therefore \lambda = \frac{ab}{\sqrt{a^2 - 1} + \sqrt{b^2 - 1}} = \frac{ab}{a^2 - b^2} (\sqrt{a^2 - 1} - \sqrt{b^2 - 1})$$

$$10. \text{ 今 } 3 \sin^{-1} \lambda - 2 \cos^{-1} \lambda = \frac{2}{3} \pi \quad (1)$$

$$\text{由公式(39)} \quad \sin^{-1} \lambda + \cos^{-1} \lambda = \frac{\pi}{2} \quad (2)$$

$$3(2) - (1) \quad 5 \cos^{-1} \lambda = \frac{5}{6} \pi \quad \therefore \cos^{-1} \lambda = \frac{\pi}{6}$$



則 
$$\lambda = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

11. 今 
$$\tan^{-1}x + m \cot^{-1}x = 135^\circ$$

由公式(39) 
$$\tan^{-1}x + \cot^{-1}x = 90^\circ$$

相減得 
$$(m-1)\cot^{-1}x = 45^\circ$$

$$\therefore \cot^{-1}x = \frac{45^\circ}{m-1} \quad \text{故} \quad x = \cot \frac{45^\circ}{m-1}$$

12. 設  $\tan^{-1}\frac{1}{4}$ ,  $\tan^{-1}\frac{1}{5}$ ,  $\tan^{-1}\frac{1}{6}$ ,  $\tan^{-1}\frac{1}{x}$  各爲  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$

則  $\tan \alpha = \frac{1}{4}$ ,  $\tan \beta = \frac{1}{5}$ ,  $\tan \gamma = \frac{1}{6}$ ,  $\tan \delta = \frac{1}{x}$

又  $\tan 2\beta = \frac{5}{12}$ ,  $\tan(\alpha + \gamma) = \frac{10}{23}$

今  $\alpha + 2\beta + \gamma = \frac{\pi}{4} - \delta$  兩邊取正切, 則

$$\frac{\frac{10}{23} + \frac{5}{12}}{1 - \frac{10}{23} \cdot \frac{5}{12}} = \frac{1 - \frac{1}{x}}{1 + \frac{1}{x}} \quad \text{即} \quad \frac{x-1}{x+1} = \frac{235}{226}$$

$$\therefore 235(x+1) = 226(x-1) \quad \therefore x = -461/9$$

13. 設角依次爲  $\alpha, \beta, \gamma, \delta$  則  $\alpha + \beta = \frac{\pi}{2} - (\gamma + \delta)$

即 
$$\cot(\alpha + \beta) = \tan(\gamma + \delta)$$

今 
$$\cot(\alpha + \beta) = \dots = \frac{x^2 - ab}{(a+b)x}$$

$$\tan(\gamma + \delta) = \dots = \frac{(c+d)x}{x^2 - cd}$$

代入去分母得  $(x^2 - ab)(x^2 - cd) = (a+b)(c+d)x^2$

即  $x^4 - (\Sigma ab)x^2 + abcd = 0$  此方程之根即為答案。

14. 設  $\tan^{-1}x = \alpha$  又右邊兩角各為  $\beta, \gamma$

則  $\tan \alpha = x, \quad \cos \beta = \frac{1 - \alpha^2}{1 + \alpha^2}$

$\therefore \tan \frac{\beta}{2} = a$  同理  $\tan \frac{\gamma}{2} = b$  (見題 8)

今  $2\alpha = \beta - \gamma$

即  $\alpha = \frac{\beta}{2} - \frac{\gamma}{2} \quad \therefore \tan \alpha = \tan\left(\frac{\beta}{2} - \frac{\gamma}{2}\right)$

$$\therefore x = \frac{\tan \frac{1}{2}\beta - \tan \frac{1}{2}\gamma}{1 + \tan \frac{1}{2}\beta \tan \frac{1}{2}\gamma} = \frac{a - b}{1 + ab}$$

15. 設兩角為  $\alpha, \beta$  則  $\cos \alpha = \frac{x^2 - 1}{x^2 + 1}$

$\therefore \tan \frac{\alpha}{2} = \frac{1}{x}$  (見題 8) 又  $\tan \beta = \frac{2x}{x^2 - 1}$

但  $\tan \alpha = \frac{2x}{x^2 - 1} \quad \therefore \alpha = \beta$  (設均為銳角)

今  $\alpha + \beta = \frac{2\pi}{3} \quad \therefore 2\alpha = \frac{2\pi}{3} \quad \text{即} \quad \frac{\alpha}{2} = \frac{\pi}{6}$

即  $\tan \frac{\alpha}{2} = \tan \frac{\pi}{6} \quad \therefore \frac{1}{x} = \frac{1}{\sqrt{3}} \quad \therefore x = \sqrt{3}$

若  $\alpha, \beta$  並非均為銳角, 但皆小於  $360^\circ$ , 則由  $\tan \alpha = \tan \beta$  更可得  $\alpha = \pi + \beta$

即  $\pi + \beta + \beta = \frac{2\pi}{3} \quad \therefore \frac{\beta}{2} = -\frac{\pi}{12}$

$$\therefore \tan \frac{\beta}{2} = \tan \left( -\frac{\pi}{12} \right) = -\tan \frac{\pi}{12}$$

$$\therefore x = 2 - \sqrt{3} \quad (102 \text{ 頁表 } 13)$$

16. 設  $\arctan x = \alpha$ , 則  $\tan \alpha = x$ ,  $\tan 2\alpha = 2x/(1-x^2)$

又設  $\frac{1}{2} \arcsin 5x = \beta$

則  $\sec 2\beta = 5x$ ,  $\tan 2\beta = \sqrt{25x^2 - 1}$

今  $\alpha + \beta = \frac{\pi}{4}$   $\therefore 2\beta = \frac{\pi}{2} - 2\alpha$

則  $\tan 2\beta = \cot 2\alpha$  即  $\sqrt{25x^2 - 1} = (1-x^2)/2x$

即  $99x^4 - 2x^2 - 1 = 0$  即  $(9x^2 - 1)(11x^2 + 1) = 0$

$$\therefore x = \frac{1}{3} \quad \left( -\frac{1}{3} \text{ 不合} \right)$$

17. 設  $\text{vers}^{-1}(1+x) = \alpha$ ,  $\text{vers}^{-1}(1-x) = \beta$

則  $\cos \alpha = 1 - (1+x) = -x$ ,  $\tan \alpha = \pm \frac{\sqrt{1-x^2}}{-x}$ ,

$\cos \beta = 1 - (1-x) = x$ ,  $\tan \beta = \pm \frac{\sqrt{1-x^2}}{x}$

(1) 取正號, 則  $\tan(\alpha - \beta) = 2\sqrt{1-x^2}$

$$\therefore \frac{2x\sqrt{1-x^2}}{2x^2-1} = 2\sqrt{1-x^2}$$

$$\therefore 2\sqrt{1-x^2}(2x^2-x-1) = 0$$

$$\therefore 2\sqrt{1-x^2}(2x+1)(x-1) = 0$$

$$\therefore x = \pm 1, -\frac{1}{2}, 1$$

(2) 取負號，同理可得  $x$  之四根為  $\pm 1, \frac{1}{2}, -1$

故  $x = \pm 1, \pm \frac{1}{2}$  (代入均合)

18. 今  $\tan \cos^{-1}\theta = \sin \cos^{-1}\frac{1}{2}$  令  $\cos^{-1}\theta = \alpha$

則  $\cos \alpha = \theta$  故  $\tan \alpha = \frac{\sqrt{1-\theta^2}}{\theta}$

又令  $\cos^{-1}\frac{1}{2} = \beta$  則  $\cos \beta = \frac{1}{2}$

故  $\sin \beta = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

今  $\tan \alpha = \sin \beta \quad \therefore \frac{\sqrt{1-\theta^2}}{\theta} = \frac{\sqrt{3}}{2}$

即  $\frac{1-\theta^2}{\theta^2} = \frac{3}{4} \quad \therefore 7\theta^2 = 4$

$\therefore \theta = \frac{2}{7}\sqrt{7}$  (負號不取)

19. 令第一角為  $A$ ，第二角為  $2B$

則  $\cot A = \frac{1}{2} \cot^2 x \quad \therefore \tan A = 2 \tan^2 x$

又  $\sin 2B = \frac{3 \sin 2x}{5 + 4 \cos 2x} = \frac{6 \sin x \cos x}{9 \cos^2 x + \sin^2 x} = \frac{6 \tan x}{9 + \tan^2 x}$

$\therefore \cos 2B = \pm \sqrt{1 - \frac{36 \tan^2 x}{(9 + \tan^2 x)^2}} = \frac{9 - \tan^2 x}{9 + \tan^2 x}$  (取正號)

$\therefore \tan B = \frac{1 - \cos 2B}{\sin 2B} = \frac{9 + \tan^2 x - 9 + \tan^2 x}{6 \tan x} = \frac{1}{3} \tan x$

今  $x = A - B$

$\therefore \tan x = \tan(A - B) = \frac{2 \tan^2 x - \frac{1}{3} \tan x}{1 + \frac{2}{3} \tan^2 x}$   
 $= \frac{6 \tan^2 x - \tan x}{3 + 2 \tan^2 x}$

$$\text{即} \quad 3 \tan x + 2 \tan^4 x = 6 \tan^2 x - \tan x$$

$$\text{即} \quad 2 \tan x (\tan^3 x - 3 \tan x + 2) = 0$$

$$\text{即} \quad 2 \tan x (\tan x - 1)^2 (\tan x + 2) = 0$$

$$\therefore \tan x = 0, 1, -2$$

20. 令此式爲  $x$ ，又第一，第二兩角各爲  $\alpha, \beta$

$$\text{則} \quad x = \alpha - \beta \quad \therefore \frac{x}{2} = \frac{\alpha}{2} - \frac{\beta}{2}$$

$$\begin{aligned} \text{今} \quad \cos \alpha &= \frac{1 - a^2 \cos 2c - 2a \sin c}{1 + a^2 - 2a \sin c} \\ &= \frac{1 + a^2 - 2a \sin c - 2a^2 \cos^2 c}{1 + a^2 - 2a \sin c} \end{aligned}$$

$$\text{即} \quad 1 - 2 \sin^2 \frac{\alpha}{2} = 1 - \frac{2a^2 \cos^2 c}{1 + a^2 - 2a \sin c}$$

$$\therefore \sin \frac{\alpha}{2} = \frac{a \cos c}{\sqrt{1 + a^2 - 2a \sin c}}$$

$$\begin{aligned} \cos^2 \frac{\alpha}{2} &= 1 - \frac{a^2 \cos^2 c}{1 + a^2 - 2a \sin c} \\ &= \frac{1 + a^2(1 - \cos^2 c) - 2a \sin c}{1 + a^2 - 2a \sin c} \end{aligned}$$

$$= \frac{1 + a^2 \sin^2 c - 2a \sin c}{1 + a^2 - 2a \sin c}$$

$$\therefore \cos \frac{\alpha}{2} = \frac{1 - a \sin c}{\sqrt{1 + a^2 - 2a \sin c}}$$

$$\begin{aligned} \text{又} \quad \cos \beta &= \frac{\cos c + 2a \sin c - a^3}{1 + a^2 - 2a \sin c} \\ &= \dots = \frac{2 \cos^2 c}{1 + a^2 - 2a \sin c} - 1 \end{aligned}$$

$$\therefore \cos \frac{\beta}{2} = \frac{\cos c}{\sqrt{1+a^2-2a \sin c}}$$

$$\sin \frac{\beta}{2} = \frac{a - \sin c}{\sqrt{1+a^2-2a \sin c}}$$

$$\begin{aligned} \text{今 } \cos \frac{x}{2} &= \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \\ &= \frac{\cos c(1-a \sin c) + a \cos c(a - \sin c)}{1+a^2-2a \sin c} \\ &= \frac{\cos c(1+a^2-2a \sin c)}{1+a^2-2a \sin c} = \cos c \end{aligned}$$

$$\therefore \frac{x}{2} = c \quad \therefore x = 2c \quad \text{故如題云。}$$

21. 設  $\tan^{-1}\alpha = A$ ,  $\tan^{-1}\beta = B$ ,  $\tan^{-1}\gamma = C$ ,  $\tan^{-1}\delta = D$   
則  $\alpha = \tan A$ ,  $\beta = \tan B$ ,  $\gamma = \tan C$ ,  $\delta = \tan D$

今從根與係數關係得

$$S_1 = \Sigma \alpha = \Sigma \tan A = \sin 2\lambda$$

$$S_2 = \Sigma \alpha\beta = \Sigma \tan A \tan B = \cos 2\lambda$$

$$S_3 = \Sigma \alpha\beta\gamma = \Sigma \tan A \tan B \tan C = \cos \lambda$$

$$S_4 = \alpha\beta\gamma\delta = \tan A \tan B \tan C \tan D = -\sin \lambda$$

$$\begin{aligned} \therefore \tan(A+B+C+D) &= \frac{S_1 - S_3}{1 - S_2 + S_4} = \frac{\sin 2\lambda - \cos \lambda}{1 - \cos 2\lambda - \sin \lambda} \\ &= \frac{\cos \lambda(2 \sin \lambda - 1)}{\sin \lambda(2 \sin \lambda - 1)} \\ &= \cot \lambda = \tan\left(\frac{\pi}{2} - \lambda\right) \end{aligned}$$

$$\therefore \Sigma A = n\pi + \frac{\pi}{2} - \lambda \quad \therefore \Sigma \tan^{-1}\alpha = n\pi + \frac{1}{2}\pi - \lambda$$

## 習題十九 (236—238 頁)

1. 從(1),  $x = 1 - \cos \phi$

代入(2),  $1 - \sin^2 \phi = 1 - 2 \cos \phi + \cos^2 \phi$

即  $2 \cos \phi = 1$                       即  $\cos \phi = \frac{1}{2}$

$\therefore \phi = \frac{\pi}{3}, \quad x = \frac{1}{2}$

2. 從十字法得

$$\left. \begin{array}{l} \sin x = \frac{b'c + bc'}{ab' + a'b} \\ \sin y = \frac{a'c - ac'}{ab' + a'b} \end{array} \right\} \text{即} \left. \begin{array}{l} x = \sin^{-1} \frac{b'c + bc'}{ab' + a'b} \\ y = \sin^{-1} \frac{a'c - ac'}{ab' + a'b} \end{array} \right\}$$

3. (1)/(2),  $\cos x = \frac{\sqrt{2}}{\sqrt{3}} \cos y$  (3)

(1)<sup>2</sup> + (3)<sup>2</sup>,  $1 = 2 \sin^2 y + \frac{2}{3} \cos^2 y$

即  $\cos^2 y = \frac{3}{4}$  }  
 以  $y$  值代入(3),  $\cos^2 x = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$  }

即  $\left\{ \begin{array}{l} y = \cos^{-1} \left( \pm \frac{\sqrt{3}}{2} \right) = n\pi \pm \frac{\pi}{6} \\ x = \cos^{-1} \left( \pm \frac{\sqrt{2}}{2} \right) = n\pi \pm \frac{\pi}{4} \end{array} \right.$

故一組主值為  $\frac{\pi}{4}, \frac{\pi}{6}$ .

$$4. \text{ 從(1), } 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = \sin \alpha \quad (3)$$

$$\text{從(2), } 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = 1 + \cos \alpha \quad (4)$$

$$(3)/(4), \tan \frac{1}{2}(x+y) = \tan \frac{1}{2}\alpha \quad \therefore x+y = \alpha$$

$$\text{代入(3), } \cos \frac{1}{2}(x-y) = \frac{\sin \alpha}{2 \sin \frac{1}{2}\alpha} = \frac{2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}{2 \sin \frac{1}{2}\alpha}$$

$$= \cos \frac{1}{2}\alpha$$

$$\therefore x-y = \pm \alpha \quad \text{二組主值爲 } \alpha, 0; 0, \alpha.$$

$$5. (1)^2 + (2)^2, 2 + 2(\sin \theta \cos \phi + \sin \phi \cos \theta) = a^2 + b^2$$

$$\text{即} \quad \sin(\theta + \phi) = \frac{a^2 + b^2 - 2}{2}$$

$$\therefore \theta + \phi = \sin^{-1} \frac{a^2 + b^2 - 2}{2}$$

$$(1) \cdot (2), \frac{1}{2}(\sin 2\theta + \sin 2\phi) + \cos(\theta - \phi) = ab$$

$$\text{即} \quad \sin(\theta + \phi) \cos(\theta - \phi) + \cos(\theta - \phi) = ab$$

$$\cos(\theta - \phi) [\sin(\theta + \phi) + 1] = ab$$

$$\text{即} \quad \cos(\theta - \phi) = \frac{2ab}{a^2 + b^2} \quad \therefore \theta - \phi = \cos^{-1} \frac{2ab}{a^2 + b^2}$$

$$\therefore \theta = \frac{1}{2} \left[ \sin^{-1} \frac{a^2 + b^2 - 2}{2} + \cos^{-1} \frac{2ab}{a^2 + b^2} \right]$$

$$\phi = \frac{1}{2} \left[ \sin^{-1} \frac{a^2 + b^2 - 2}{2} - \cos^{-1} \frac{2ab}{a^2 + b^2} \right]$$

$$6. \text{ 從(1), } 1 - m \cos \theta = n \cos \phi \quad (3)$$

$$\text{從(2), } -1 + m \sin \theta = n \sin \phi \quad (4)$$

$$(3)^2 + (4)^2, 2 - 2m \cos \theta - 2m \sin \theta + m^2 = n^2$$

$$\text{即} \quad 2m \cos \theta + 2m \sin \theta = m^2 - n^2 + 2$$



$$\therefore \sin \frac{\pi}{4} \cos \theta + \cos \frac{\pi}{4} \sin \theta = \frac{m^2 - n^2 + 2}{2m\sqrt{2}}$$

$$\text{即} \quad \sin\left(\frac{\pi}{4} + \theta\right) = \frac{m^2 - n^2 + 2}{2m\sqrt{2}}$$

$$\text{設令} \quad \frac{m^2 - n^2 + 2}{2m\sqrt{2}} = k$$

$$\text{則} \quad \frac{\pi}{4} + \theta = \sin^{-1}k \quad \text{即} \quad \theta = \sin^{-1}k - \frac{\pi}{4}$$

從(1),(2)兩式移  $\phi$  之函數至右邊,兩邊平方再加,得

$$2n \cos \phi - 2n \sin \phi = n^2 - m^2 + 2$$

$$\text{即} \quad \cos \frac{\pi}{4} \cos \phi - \sin \frac{\pi}{4} \sin \phi = \frac{n^2 - m^2 + 2}{2n\sqrt{2}}$$

$$\text{即} \quad \cos\left(\frac{\pi}{4} + \phi\right) = \frac{n^2 - m^2 + 2}{2n\sqrt{2}}$$

$$\text{令} \quad \frac{n^2 - m^2 + 2}{2n\sqrt{2}} = k'$$

$$\text{則} \quad \frac{\pi}{4} + \phi = \cos^{-1}k' \quad \therefore \quad \phi = \cos^{-1}k' - \frac{\pi}{4}$$

$$7. \text{ 即} \quad \left. \begin{aligned} 2\theta + 3\phi &= \cos^{-1} \frac{1}{2} = \frac{\pi}{3} & (1) \\ 3\theta + 2\phi &= \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6} & (2) \end{aligned} \right\} \text{主角}$$

$$\left. \begin{aligned} 2(1) - 3(2), \quad 5\theta &= -\frac{1}{6}\pi \quad \therefore \quad \theta = -\frac{1}{30}\pi \\ \text{代入(1),} \quad \phi &= \frac{2}{15}\pi \end{aligned} \right\} \text{一組主值}$$

$$8. \text{ 從(1)得 } a(\sec^2\theta - 2) = b \cos \phi \quad (3)$$

$$\text{從(2)得 } b(\cos^2\theta - 2) = a \sec \phi \quad (4)$$

$$(3) \cdot (4), \quad (\sec^2\theta - 2)(\cos^2\theta - 2) = 1$$

$$\text{即 } \cos^4\theta - 2\cos^2\theta + 1 = 0$$

$$\text{即 } \cos^2\theta = 1 \quad \therefore \cos \theta = \pm 1 \quad \therefore \theta = n\pi$$

$$\text{代入(1) } \phi = \cos^{-1}\left(-\frac{a}{b}\right)$$

$$9. \text{ 今 } \sin(x - 60^\circ) = \cos(x - 30^\circ)$$

$$\text{即 } \frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$$

$$\therefore \cos x = 0$$

$$\therefore x = 2m\pi \pm \frac{\pi}{2} = \frac{1}{2}(4m \pm 1)\pi = \frac{1}{2}(2n + 1)\pi$$

$$\text{代入(1)得 } \rho = 2 / \sin\left(\frac{2n+1}{2}\pi - \frac{\pi}{3}\right) = 2 \cos \frac{\pi}{3} = 4$$

一組主值爲  $4, 90^\circ$  註：本題亦可從例四法解。

$$10. \text{ 從(2)化得 } \cos^2x + \cos^2y = (b+2)/2 \quad (3)$$

$$(1)^2 - (3), \quad 2 \cos x \cos y = a^2 - (b+2)/2 \quad (4)$$

$$\text{從(3), (4), } \cos x - \cos y = \pm \sqrt{b+2-a^2} = \pm k$$

$$\therefore \begin{cases} \cos x = \frac{1}{2}(a \pm k) \\ \cos y = \frac{1}{2}(a \mp k) \end{cases} \quad \text{故} \quad \begin{cases} x = \cos^{-1} \frac{1}{2}(a \pm k) \\ y = \cos^{-1} \frac{1}{2}(a \mp k) \end{cases}$$

$$11. \text{ 由(1) } a \sin \phi \cos \theta - b \sin \theta \cos \phi = 0 \quad (3)$$

$$\text{由(2) } \sin \phi \cos \theta + \sin \theta \cos \phi = a + b \quad (4)$$

$$\text{從(3), (4), } \sin \phi \cos \theta = b, \quad \sin \theta \cos \phi = a$$

$$\text{故} \quad \sin \theta \cos \phi - \cos \theta \sin \phi = a - b$$

$$\therefore \sin(\theta - \phi) = a - b \quad \therefore \theta - \phi = \sin^{-1}(a - b)$$

$$\text{從(2)} \quad \theta + \phi = \sin^{-1}(a + b)$$

$$\therefore \theta = \frac{1}{2}[\sin^{-1}(a + b) + \sin^{-1}(a - b)]$$

$$\phi = \frac{1}{2}[\sin^{-1}(a + b) - \sin^{-1}(a - b)]$$

$$12. \text{ 舊版題爲: } n \sin x - m \cos x = 2m \sin y \quad (1)$$

$$n \sin 2x - m \cos 2y = m \quad (2)$$

$$\text{從(2)得} \quad n \sin 2x - 2m \cos^2 y = 0 \quad (3)$$

$$2 \cos x \cdot (1) - (2), \quad 2m(\cos^2 y - \cos^2 x) = 4m \cos x \cos y$$

$$\text{即} \quad (\cos x + \cos y)^2 = 1$$

$$\therefore \sin y = 1 - \cos x \quad (\text{取正號}) \quad (4)$$

$$\text{以(4)代入(1)} \quad n \sin x + m \cos x - 2m = 0$$

$$\text{平方化簡得} \quad (m^2 + n^2)\cos^2 x - 4m^2 \cos x + 4m^2 - n^2 = 0$$

$$\therefore \cos x = \frac{2m^2 \pm \sqrt{4m^4 - (m^2 + n^2)(4m^2 - n^2)}}{m^2 + n^2}$$

$$= (2m^2 \pm n\sqrt{n^2 - 3m^2}) / (m^2 + n^2) = A \quad (5)$$

$$\text{以(5)代入(4)} \quad \sin y = 1 - \frac{2m^2 \pm n\sqrt{n^2 - 3m^2}}{m^2 + n^2}$$

$$= \frac{n^2 - m^2 \mp n\sqrt{n^2 - 3m^2}}{m^2 + n^2} = B$$

$$\therefore x = \cos^{-1} A, \quad y = \sin^{-1} B$$

$$\text{即得} \quad x = \cos^{-1} \frac{2 \pm 2}{5} = \cos^{-1} \frac{4}{5} \quad (\text{或 } \tan^{-1} \frac{3}{4}),$$

$$\cos^{-1} 0 \quad (\text{或 } 2n\pi \pm \frac{\pi}{2})$$

$$y = \sin^{-1} \frac{3 \mp 2}{5} = \sin^{-1} \frac{1}{5}, \quad \sin^{-1} 1 \left( n\pi + (-1)^n \cdot \frac{\pi}{2} \right)$$

新版之 12 題則可照下法做

$$\text{從(2)} \quad 4 \sin x \cos x = 2 \cos^2 y \quad (3)$$

$$2(3) + (1)^2, \quad 4 \sin^2 x + 4 \sin x \cos x + \cos^2 x = 4$$

$$\text{即} \quad \cos x (4 \sin x - 3 \cos x) = 0$$

$$\therefore \cos x = 0, \quad \tan x = \frac{3}{4} \dots\dots\dots$$

$$13. \text{ 從(2)} \quad \cot(x+y) = \frac{1}{\tan(x+y)} = \frac{1 - \tan x \tan y}{\tan x + \tan y} = b$$

$$\text{以(1)之值代入得} \quad \tan x \tan y = 1 - ab \quad (3)$$

$$\sqrt{(1)^2 - 4(3)}, \quad \tan x - \tan y = \pm \sqrt{a^2 + 4ab - 4} = \pm k$$

$$\therefore \begin{cases} \tan x = \frac{1}{2}(a \pm k) \\ \tan y = \frac{1}{2}(a \mp k) \end{cases} \quad \text{故} \quad \begin{cases} x = \tan^{-1} \frac{1}{2}(a \pm k) \\ y = \tan^{-1} \frac{1}{2}(a \mp k) \end{cases}$$

$$14. \cot x \cdot (1) - \cot y \cdot (2), \quad \cot x = \cot y$$

$$\text{代入(1)} \quad \tan x + \cot x = 2, \quad \text{即} \quad (\tan x - 1)^2 = 0$$

$$\therefore \tan x = 1 \quad \text{及} \quad \tan y = 1$$

$$\text{故} \quad x = n\pi + \frac{1}{4}\pi, \quad y = n\pi + \frac{1}{4}\pi$$

$$\text{一組主值爲} \quad \frac{\pi}{4}, \quad \frac{\pi}{4}$$

$$15. \text{ 由(1), (2)得} \quad \sin(x+y) = \frac{\sqrt{2}}{2}$$

$$\text{故} \quad \cos(x+y) = \frac{\sqrt{2}}{2} \quad (3)$$

$$\text{以(2)代入(3)} \quad \sin x \sin y = 0$$

$$\therefore \tan x \tan y = 0 \quad (4)$$

$$\sqrt{(1)^2 - 4(4)}, \quad \tan x - \tan y = \pm 1$$

$$\therefore \left. \begin{array}{l} \tan x = 1 \\ \tan y = 0 \end{array} \right\} \left. \begin{array}{l} 0 \\ 1 \end{array} \right\} \text{故} \quad \left. \begin{array}{l} x = n\pi + \frac{1}{4}\pi \\ y = m\pi \end{array} \right\} \left. \begin{array}{l} m\pi \\ n\pi + \frac{1}{4}\pi \end{array} \right\}$$

$$16. \text{ 從(1)} \quad 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = 1 \quad (3)$$

$$\text{從(2)} \quad \cos(x+y) + \cos(x-y) = -\frac{3}{2}$$

$$\text{即} \quad \cos^2 \frac{1}{2}(x-y) - \sin^2 \frac{1}{2}(x+y) = -\frac{3}{4} \quad (4)$$

以  $\sin \frac{1}{2}(x+y) = z$ ,  $\cos \frac{1}{2}(x-y) = t$  代入(3), (4)得

$$t^2 - z^2 = -\frac{3}{4}, \quad 2tz = 1$$

$$\text{解之得} \quad z = \pm 1, \quad t = \pm \frac{1}{2}$$

$$\therefore \left. \begin{array}{l} x - y = 2 \cos^{-1}(\pm \frac{1}{2}) = 2(2m\pi \pm \frac{1}{3}\pi) \\ x + y = 2 \sin^{-1}(\pm 1) = 2 \cos^{-1} 0 = 2(2n\pi \pm \frac{1}{2}\pi) \end{array} \right\} \text{(第一組)}$$

$$\text{或} \quad \left. \begin{array}{l} 2(2m\pi \pm \frac{2}{3}\pi) \\ 2(2n\pi \pm \frac{3}{2}\pi) \end{array} \right\} \text{(第二組)}$$

$$\therefore \left. \begin{array}{l} x = 2k\pi \pm \frac{5\pi}{6} \\ y = 2k'\pi \pm \frac{\pi}{6} \end{array} \right\} \left. \begin{array}{l} 2k\pi \pm \frac{\pi}{6} \\ 2k'\pi \pm \frac{5\pi}{6} \end{array} \right\} \text{主值} \quad \left. \begin{array}{l} \frac{5\pi}{6} \\ \frac{\pi}{6} \end{array} \right\} \left. \begin{array}{l} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{array} \right\}$$

$$\text{第二組爲} \quad 2\left[(2m'+1)\pi \pm \frac{\pi}{3}\right], \quad 2\left[(2n'+1)\pi \pm \frac{1}{2}\pi\right]$$

所得  $x, y$  之解與第一組相同。

$$17. \text{ 從(1)得} \quad 3 \cos^2 2A - \cos^2 2B - 2 = 0 \quad (3)$$

$$\text{從(2)得} \quad \sqrt{3} \cos 2A + \cos 2B - 2 = 0 \quad (4)$$

$$\text{以(4)代入(3), } (2 - \cos 2B)^2 - \cos^2 2B - 2 = 0$$

$$\text{即} \quad \cos 2B = 1/2 \quad \text{代入(4)得} \quad \cos 2A = \sqrt{3}/2$$

$$\therefore \left. \begin{array}{l} 2A = 2n\pi \pm \pi/6 \\ 2B = 2n\pi \pm \pi/3 \end{array} \right\} \text{故} \quad \left. \begin{array}{l} A = n\pi \pm \pi/12 \\ B = n\pi \pm \pi/6 \end{array} \right\}$$

$$\text{主值爲} \quad \left. \begin{array}{l} 15^\circ \\ 30^\circ \end{array} \right\} \text{(上書答數誤)}$$

18. 設  $x = \tan \theta$ ,  $y = \tan \phi$ , 代入原式得

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} = 2 \quad (1), \quad x + y = 4 \quad (2)$$

$$\text{從(1)} \quad 3x^2y^2(x+y) - 18x^2y^2 - (x^3+y^3) - 9xy(x+y) \\ + 6(x^2+y^2) + 3(x+y) - 2 = 0$$

$$\therefore xy = -7, 1 \quad (3)$$

$$\text{從(2), (3)得} \quad \left. \begin{array}{l} x = 2 \pm \sqrt{11} \\ y = 2 \mp \sqrt{11} \end{array} \right\} \quad \left. \begin{array}{l} 2 \pm \sqrt{3} \\ 2 \mp \sqrt{3} \end{array} \right\}$$

$$\text{即} \quad \left. \begin{array}{l} \theta = \tan^{-1}(2 \pm \sqrt{11}) \\ \phi = \tan^{-1}(2 \mp \sqrt{11}) \end{array} \right\} \quad \left. \begin{array}{l} \frac{\pi}{12} \\ \frac{5\pi}{12} \end{array} \right\} \\ \left. \begin{array}{l} \frac{5\pi}{12} \\ \frac{\pi}{12} \end{array} \right\}$$

19. 設  $\tan^{-1}x = \alpha$ ,  $\tan^{-1}y = \beta$ ,  $\sin^{-1}x = \gamma$ ,  $\cos^{-1}y = \delta$

$$\text{則} \quad \cos \gamma = \sqrt{1-x^2}, \quad \sin \delta = \sqrt{1-y^2}$$

$$(1) \text{式兩邊取正切得} \quad \frac{x+y}{1-xy} = 1$$

$$\text{即} \quad x+y = 1-xy \quad (3)$$

$$(2) \text{式兩邊取正弦得} \quad xy + \sqrt{(1-x^2)(1-y^2)} = 0$$

$$\text{即} \quad x^2 + y^2 = 1 \quad (4)$$

$$(4) + 2(3), \quad (x+y)^2 + 2(x+y) - 3 = 0$$

$$\text{即} \quad (x+y-1)(x+y+3) = 0 \quad (5)$$

$$\text{從(3), (5)得} \quad \left. \begin{array}{l} x+y=1 \\ xy=0 \end{array} \right\} \quad \left. \begin{array}{l} -3 \\ 4 \end{array} \right\}$$

$$\therefore \left. \begin{array}{l} x=1 \\ y=0 \end{array} \right\} \quad \left. \begin{array}{l} 0 \\ 1 \end{array} \right\} \quad \left. \begin{array}{l} \frac{1}{2}(-3 \pm i\sqrt{7}) \\ \frac{1}{2}(-3 \mp i\sqrt{7}) \end{array} \right\}$$

只有一組值 1, 0, 第二解代入不合, 第三解爲虛數均去之。

$$20. \text{ 設} \quad x = \tan \alpha, \quad y = \tan \beta$$

$$\text{由(1)及(2)得} \quad \tan(\alpha + \beta) + \cot(\alpha - \beta) = a \quad (3)$$

$$\tan(\alpha - \beta) + \cot(\alpha + \beta) = b \quad (4)$$

$$\text{從(3)得} \quad \frac{\cos 2\beta}{\cos(\alpha + \beta)\sin(\alpha - \beta)} = a$$

$$\text{即} \quad \frac{2 \cos 2\beta}{\sin 2\alpha - \sin 2\beta} = a$$

$$\text{即} \quad \frac{\sin 2\alpha}{\cos 2\beta} - \tan 2\beta = \frac{2}{a}$$

$$\text{同理化(4)爲} \quad \frac{\sin 2\alpha}{\cos 2\beta} + \tan 2\beta = \frac{2}{b}$$

$$\text{故} \quad \tan 2\beta = \frac{a-b}{ab} \quad \therefore \beta = \frac{1}{2} \tan^{-1} \frac{a-b}{ab}$$

$$\text{則} \quad \cos 2\beta = \frac{ab}{\sqrt{a^2b^2 + (a-b)^2}}$$

$$\therefore \sin 2\alpha = \frac{a+b}{\sqrt{a^2b^2 + (a-b)^2}}$$

$$\alpha = \frac{1}{2} \sin^{-1} \frac{a+b}{\sqrt{a^2b^2+(a-b)^2}}$$

$\tan \alpha$  及  $\tan \beta$  即為  $x$  及  $y$  之值。

或用下法求  $\alpha$  及  $\beta$  亦可，從(3)，(4)消去  $(\alpha - \beta)$  得

$$\tan(\alpha + \beta) = A/2b \quad (A = ab \pm \sqrt{a^2b^2 - 4ab})$$

再從(3)，(4)消去  $(\alpha + \beta)$ ，得  $\tan(\alpha - \beta) = A/2a$

$$\text{故} \quad \alpha = \frac{1}{2} [\tan^{-1}(A/2b) + \tan^{-1}(A/2a)]$$

$$\beta = \frac{1}{2} [\tan^{-1}(A/2b) - \tan^{-1}(A/2a)]$$

21. 設  $a, b$  為實數，又  $|x| < 1, |y| < 1$

$$\text{令} \quad x = \sin^2 \theta, \quad y = \sin^2 \phi$$

$$\text{則} \quad \sin \theta \cos \phi + \sin \phi \cos \theta = a,$$

$$\sin \theta \sin \phi + \cos \theta \cos \phi = b$$

$$\text{即} \quad \begin{cases} \sin(\theta + \phi) = a & \therefore \theta + \phi = \sin^{-1} a \\ \cos(\theta - \phi) = b & \theta - \phi = \cos^{-1} b \end{cases}$$

$$\therefore \theta = \frac{1}{2} (\sin^{-1} a + \cos^{-1} b), \quad \phi = \frac{1}{2} (\sin^{-1} a - \cos^{-1} b)$$

$$\therefore \begin{cases} x = \sin^2 \theta = \sin^2 \frac{1}{2} (\sin^{-1} a + \cos^{-1} b) \\ y = \sin^2 \phi = \sin^2 \frac{1}{2} (\sin^{-1} a - \cos^{-1} b) \end{cases}$$

22. 設  $|z| < 1$ ，而令  $z = \sin \theta$ ，則

$$x = a \cos \theta, \quad y = b \cos \theta$$

$$\text{代入(3)，得} \quad \sin^2 \theta = \sqrt{(1 - a^2 \cos^2 \theta)(1 - b^2 \cos^2 \theta)}$$

平方之，以  $1 - \cos^2 \theta$  代  $\sin^2 \theta$ ，則得

$$\cos^2 \theta [a^2 b^2 \cos^2 \theta - (a^2 + b^2 - 1)] = 0$$

$$\therefore \cos \theta = \sqrt{a^2 + b^2 - 1} / ab, \quad 0$$



故求得  $x, y, z$  之值爲  $\sqrt{a^2+b^2-1}/b, \sqrt{a^2+b^2-1}/a,$   
 $\sqrt{(a^2-1)(b^2-1)}/ab, 0, 0, 1.$

23. 令(2)式之比值爲  $t$

$$\text{則 } \tan x = t, \quad \tan y = 2t, \quad \tan z = 3t \quad (3)$$

$$\text{又因 } x + y + z = \pi \quad \text{則 } \Sigma \tan x = \Pi \tan x$$

$$\text{故 } t + 2t + 3t = 6t^3 \quad \text{即 } t(t^2 - 1) = 0$$

$$\therefore t = 0, \pm 1$$

$$\text{故 } \left. \begin{array}{l} \tan x = 0 \\ \tan y = 0 \\ \tan z = 0 \end{array} \right\} \begin{array}{l} 1) \\ 2) \\ 3) \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} (t \text{ 不取負號})$$

$$\therefore \left. \begin{array}{l} x = \pi \\ y = 0 \\ z = 0 \end{array} \right\} \begin{array}{l} 0) \\ \pi) \\ 0) \end{array} \left. \begin{array}{l} 0) \\ 0) \\ \pi) \end{array} \right\} \begin{array}{l} \tan^{-1} 1) \\ \tan^{-1} 2) \\ \tan^{-1} 3) \end{array}$$

$$24. \text{ 從(1) } \frac{\sin(\theta - \beta) \cos \alpha}{\cos(\theta + \alpha) \sin \beta} = -\frac{\sin(\phi - \alpha) \cos \beta}{\cos(\phi - \beta) \sin \alpha}$$

$$\text{即 } \frac{\sin \theta \cos \beta \cos \alpha - \cos \theta \sin \beta \cos \alpha}{\cos \theta \cos \alpha \sin \beta - \sin \theta \sin \alpha \sin \beta}$$

$$= \frac{\cos \phi \sin \alpha \cos \beta - \sin \phi \cos \alpha \cos \beta}{\cos \phi \cos \beta \sin \alpha + \sin \alpha \sin \beta \sin \phi}$$

$$\text{即 } \frac{\tan \theta - \tan \beta}{\tan \beta - \tan \alpha \tan \beta \tan \theta} = \frac{\tan \alpha - \tan \phi}{\tan \alpha + \tan \alpha \tan \beta \tan \phi}$$

$$\text{即 } (\tan \alpha \tan \beta + 1) \tan \alpha \tan \theta$$

$$- (\tan \alpha \tan \beta - 1) \tan \beta \tan \phi$$

$$- 2 \tan \alpha \tan \beta = 0 \quad (3)$$

$$\text{從(2)} \quad \frac{\tan \theta \tan \alpha}{\tan \phi \tan \beta} = -\frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

$$\text{即} \quad \frac{\tan \theta \tan \alpha}{\tan \phi \tan \beta} = \frac{1 + \tan \alpha \tan \beta}{\tan \alpha \tan \beta - 1}$$

$$\text{即} \quad (\tan \alpha \tan \beta - 1) \tan \alpha \tan \theta - (\tan \alpha \tan \beta + 1) \tan \beta \tan \phi = 0 \quad (4)$$

(3) + (4), 化簡得

$$\tan \alpha \tan \theta - \tan \beta \tan \phi - 1 = 0 \quad (5)$$

(3) - (4), 化簡得

$$\tan \alpha \tan \theta + \tan \beta \tan \phi - \tan \alpha \tan \beta = 0 \quad (6)$$

$$(5) + (6), \quad 2 \tan \alpha \tan \theta = 1 + \tan \alpha \tan \beta$$

$$\therefore \tan \theta = \frac{1}{2}(\tan \beta + \cot \alpha)$$

$$(6) - (5), \quad 2 \tan \beta \tan \phi = \tan \alpha \tan \beta - 1$$

$$\therefore \tan \phi = \frac{1}{2}(\tan \alpha - \cot \beta)$$

$$\begin{aligned} 25. \quad x &= \begin{vmatrix} \cos(\beta - \gamma) & \cos \alpha & \sin \alpha \\ \cos(\gamma - \alpha) & \cos \beta & \sin \beta \\ \cos(\alpha - \beta) & \cos \gamma & \sin \gamma \end{vmatrix} \div \begin{vmatrix} 1 & \cos \alpha & \sin \alpha \\ 1 & \cos \beta & \sin \beta \\ 1 & \cos \gamma & \sin \gamma \end{vmatrix} \\ &= \frac{\Sigma \cos(\beta - \gamma) \sin(\beta - \gamma)}{\Sigma \sin(\beta - \gamma)} = \frac{\Sigma \sin 2(\beta - \gamma)}{2 \Sigma \sin(\beta - \gamma)} \end{aligned}$$

$$\text{但} \quad \Sigma \sin(\beta - \gamma) = -4\Pi \sin \frac{1}{2}(\beta - \gamma)$$

$$\text{又} \quad \Sigma \sin 2(\beta - \gamma) = -4\Pi \sin(\beta - \gamma)$$

$$= -32[\Pi \sin \frac{1}{2}(\beta - \gamma) \Pi \cos \frac{1}{2}(\beta - \gamma)]$$

$$\text{故} \quad x = 4\Pi \cos \frac{1}{2}(\beta - \gamma)$$

## 習題二十 (249—250 頁)

$$1. \quad B = 180^\circ - (A + C) = 180^\circ - 73^\circ 45' = 106^\circ 15'$$

$$a = \frac{b \sin A}{\sin B}$$

$$\log b = 3.21801$$

$$\log \sin A = 9.64953 - 10$$

$$\text{colog } \sin B = 0.01771$$

$$\log a = 2.88525$$

$$\therefore a = 767.8$$

$$c = \frac{b \sin C}{\sin B}$$

$$\log b = 3.21801$$

$$\log \sin C = 9.86589 - 10$$

$$\text{colog } \sin B = 0.01771$$

$$\log c = 3.10161$$

$$\therefore c = 1263.6$$

$$2. \quad C = 180^\circ - (A + B) = 180^\circ - (75^\circ + 30^\circ) = 75^\circ$$

$$\therefore A = C = 75^\circ$$

$$\therefore a = c$$

$$a = \frac{b \sin A}{\sin B} = \frac{\sqrt{8}(\sqrt{6} + \sqrt{2})/4}{1/2} = 2(\sqrt{3} + 1)$$

$$c = 2(\sqrt{3} + 1)$$

$$3. \quad \text{今 } \tan \frac{A}{2} = \frac{r}{s-a}, \quad \tan \frac{B}{2} = \frac{r}{s-b}, \quad \tan \frac{C}{2} = \frac{r}{s-c}$$

$$\left( r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \right)$$

$$a = 3.41$$

$$b = 2.605$$

$$c = 1.58$$

$$2s = 7.595$$

$$s = 3.7975$$

$$s - a = 0.3875$$

$$s - b = 1.1925$$

$$s - c = 2.2175$$

$$s = 3.7975 (\text{驗})$$

$$\operatorname{colog} s = 9.42050 - 10$$

$$\log(s-a) = 9.58827 - 10$$

$$\log(s-b) = 0.07644$$

$$\log(s-c) = 0.34587$$

---


$$2 \log r = 19.43108 - 20$$

$$\log r = 9.71554 - 10$$

$$\log \tan \frac{A}{2} = 0.12727$$

$$\log \tan \frac{B}{2} = 9.63910 - 10$$

$$\log \tan \frac{C}{2} = 9.36967 - 10$$

---


$$\frac{A}{2} = 53^\circ 16' 39'', \quad A = 106^\circ 33' 18''$$

$$\frac{B}{2} = 23^\circ 32' 19'', \quad B = 47^\circ 4' 38''$$

$$\frac{C}{2} = 13^\circ 11' 2'', \quad C = 26^\circ 22' 4''$$

---


$$\text{驗} \quad A+B+C = 180^\circ$$

(\*求  $\log \tan \frac{A}{2}$  時即以  $\log r$  減  $\log(s-a)$ , 不必另行寫出, 餘以此類推)。

$$\begin{aligned} 4. \text{ 今 } a:b:c &= \frac{1}{4}(\sqrt{6} - \sqrt{2}) : \frac{1}{2}\sqrt{2} : \frac{1}{2}\sqrt{3} \\ &= \sqrt{3} - 1 : 2 : \sqrt{6} \end{aligned}$$

$$\therefore a = k(\sqrt{3} - 1), \quad b = 2k, \quad c = k\sqrt{6}$$

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4 - 2\sqrt{3} + 4 - 6}{4(\sqrt{3} - 1)} = -\frac{1}{2}$$

$$\therefore C = 120^\circ$$

$$\cos B = \frac{6+4-2\sqrt{3}-4}{2\sqrt{6}(\sqrt{3}-1)} = \frac{2(3-\sqrt{3})}{2\sqrt{2}(3-\sqrt{3})} = \frac{1}{\sqrt{2}}$$

$$\therefore B = 45^\circ, \quad A = 15^\circ$$

5.	$a = 681$		$B = 50^\circ 42'$
	$c = 243$		$A + C = 129^\circ 18'$
	$a - c = 438$		$\frac{1}{2}(A + C) = 64^\circ 39'$
	$a + c = 924$		

$$\log(a - c) = 2.64147$$

$$\text{colog}(a + c) = 7.03433 - 10$$

$$\log \tan \frac{A + C}{2} = 10.32444 \quad (+)$$

$$\log \tan \frac{A - C}{2} = 10.00024$$

$$\therefore \frac{1}{2}(A - C) = 45^\circ 0' 57''$$

$$\therefore A = 109^\circ 39' 57'', \quad C = 19^\circ 38' 3''$$

$$b = \frac{c \sin B}{\sin C}$$

$$\log c = 2.38561$$

$$\log \sin B = 9.88865 \quad (+)$$

$$12.27426$$

$$\log \sin C = 9.52636 \quad (-)$$

$$\log b = 2.74790$$

$$\therefore b = 559.68$$

6. 今  $a < b$ ,  $C < 90^\circ$ , 而  $b \sin C = \frac{a}{2} > c$ , 故無解。

7. 今  $a < b$ ,  $A < 90^\circ$ , 而  $b > a > b \sin A$ , 故有兩解。

$$\therefore \sin B = \frac{b \sin A}{a} = \frac{1}{2} \sqrt{3} \quad \therefore B = 60^\circ \text{ 或 } 120^\circ$$

$$C = 180^\circ - (A + B) = 90^\circ \text{ 或 } 30^\circ$$

$$c = \frac{a \sin C}{\sin A} = 2 \text{ 或 } 1$$

$$\text{解爲 } \begin{cases} B = 60^\circ, & C = 90^\circ, & c = 2; \\ B' = 120^\circ, & C' = 30^\circ, & c' = 1. \end{cases}$$

8. 今  $c < b$ , 則  $C < B$ , 但  $C = 160^\circ > 90^\circ$ , 故不可解。

$$9. \text{ 今 } \sin C = \frac{c \sin A}{a} = \frac{\sqrt{3} + 1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

$$\therefore C = 105^\circ, \quad C' = 75^\circ \quad \text{故 } B = 30^\circ, \quad B' = 60^\circ$$

$$b = a \sin B / \sin A \quad \text{故 } b = \sqrt{2}, \quad b' = \sqrt{6}$$

$$10. \text{ 今 } \cos A = \frac{1}{4} \sqrt{7} \quad \therefore \sin A = \frac{3}{4} \quad (\text{負號不合因 } A < 180^\circ)$$

$$\sin B = \frac{b \sin A}{a} = \frac{1}{5} \left[ 7 \left( \frac{3}{4} \right) \right] = \frac{21}{20} > 1 \quad \text{故此題無解。}$$

$$\text{或從 } a < b, \quad a = 5 < b \sin A \left( = \frac{21}{4} \right) \quad \text{亦可知其無解。}$$

$$11. \quad a = 767, \quad b = 242, \quad A = 36^\circ 53' 2''$$

$$\sin B = \frac{b \sin A}{a}$$

$$\log 242 = 2.38382$$

$$\log \sin A = 9.77830 - 10$$

$$\text{colog } 767 = 7.11520 - 10$$

$$\hline \log \sin B = 19.27732 - 20$$

$$\therefore B = 10^\circ 54' 58''$$

$$C = 132^\circ 12'$$

$$c = \frac{a \sin C}{\sin A}$$

$$\log 767 = 2.88480$$

$$\log \sin C = 9.86970 - 10$$

$$\text{colog } \sin A = 0.22170$$

$$\hline \log c = 12.97620 - 10$$

$$\therefore c = 946.68$$

$$12. a=117.48, \quad b=726.3, \quad A=80^{\circ}0'50''$$

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 2.86112$$

$$\log \sin A = 9.99837 - 10$$

$$\text{colog } a = 7.93003 - 10 \quad (+)$$

$$\log \sin B = 0.78452$$

$\therefore \sin B > 1$  故此題不能解。

$$13. a=177.01, \quad b=216.45, \quad A=35^{\circ}36'20''$$

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 2.33536$$

$$\log \sin A = 9.76507 - 10$$

$$\text{colog } a = 7.75200 - 10 \quad (+)$$

$$\log \sin B = 9.85243 - 10$$

今  $a < b$ , 而  $\log \sin B < 0$ , 故有兩解。

$$B = 45^{\circ}23'28''$$

$$B' = 134^{\circ}36'32''$$

$$C = 99^{\circ}0'12''$$

$$C' = 9^{\circ}47'8''$$

$$\log a = 2.24800$$

$$\log a' = 2.24800$$

$$\log \sin C = 9.99462 - 10$$

$$\log \sin C' = 9.23035 - 10$$

$$\text{colog } \sin A = 0.23493$$

$$\text{colog } \sin A' = 0.23493$$

$$\log c = 2.47755$$

$$\log c' = 1.71828$$

$$\therefore c = 300.29$$

$$c' = 51.675$$

$$14. a=13.2, \quad b=15.7, \quad A=57^{\circ}13'15'' < 90^{\circ}, \quad b < a$$

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 1.19590$$

$$\log \sin A = 9.92467 - 10$$

$$\text{colog } a = 8.87943 - 10$$

$$\log \sin B = 0$$

$$\therefore B = 90^\circ$$

$$C = 32^\circ 46' 45''$$

$$c = b \sin C$$

$$\log b = 1.19590$$

$$\log \sin C = 9.73352 - 10$$

$$\log c = 0.92942$$

$$\therefore c = 8.5$$

15. 設  $a = 2x + 3$ ,  $b = x^2 + 3x + 3$ ,  $c = x^2 + 2x$

今  $b - a = x^2 + x > 0$ ,  $b - c = x + 3 > 0$  ( $\because x > 0$ )

故  $b$  為最大邊, 故知對  $b$  之角  $B$  為最大.

$$\begin{aligned} \text{今 } \cos B &= \frac{(2x+3)^2 + (x^2+2x)^2 - (x^2+3x+3)^2}{2(2x+3)(x^2+2x)} \\ &= \frac{-2x^3 - 7x^2 - 6x}{2(2x^3 + 7x^2 + 6x)} = -\frac{1}{2} \end{aligned}$$

故  $B = 120^\circ$  此為最大角.

16. 設  $a = 242$ ,  $b = 188$ ,  $c = 270$ , 故  $C$  角最大

$$\therefore s = \frac{1}{2}(a+b+c) = 350, \quad s-a = 108,$$

$$s-b = 162, \quad s-c = 80$$

$$\text{今 } \tan \frac{1}{2}C = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\frac{162 \cdot 108}{350 \cdot 80}} = \sqrt{\frac{2 \cdot 3^7}{7 \cdot 10^3}}$$

$$\begin{aligned} \therefore \log \tan \frac{1}{2}C &= \frac{1}{2}(\log 2 + 7 \log 3 - \log 7 - 3) \\ &= \frac{1}{2}(.30103 + 3.33984 - .84510 - 3) \\ &= -.10211 = 9.89789 - 10 \end{aligned}$$

$$\therefore \frac{1}{2}C = 38^\circ 19' 32.3'' \quad \therefore C = 76^\circ 39' 5''$$



$$\begin{aligned}
 17. \text{ 今 } c^2 &= (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 \\
 &\quad - 2(\sqrt{3}+1)(\sqrt{3}-1)\cos 60^\circ \\
 &= 3+2\sqrt{3}+1+3-2\sqrt{3}+1-2=6 \\
 \therefore c &= \sqrt{6}
 \end{aligned}$$

18. 設比值為  $k$ , 則

$$\begin{aligned}
 a &= (m+n)k, \quad b = (m-n)k, \quad c = \sqrt{2(m^2+n^2)}k \\
 \therefore \cos C &= \frac{(m+n)^2k^2 + (m-n)^2k^2 - 2(m^2+n^2)k^2}{2(m+n)(m-n)k^2} = 0 \\
 \therefore C &= 90^\circ
 \end{aligned}$$

## 習題二十一 (261—263 頁)

1. 設比值為  $k$ , 則  $a=5k$ ,  $b=7k$ ,  $c=8k$ , 今  $A$  最小

$$\cos A = \frac{49+64-25}{2 \times 7 \times 8} = \frac{11}{14} = .7857 \quad \therefore A = 38^\circ 12'$$

2. 今邊順次為  $a, b, c$ , 則  $s = \frac{1}{4}(2\sqrt{3}+3\sqrt{2}+\sqrt{6})$ ,

$$s-a = \frac{1}{4}(3\sqrt{2}+\sqrt{6}-2\sqrt{3}),$$

$$s-b = \frac{1}{4}(\sqrt{6}+2\sqrt{3}-\sqrt{2}),$$

$$s-c = \frac{1}{4}(2\sqrt{3}+\sqrt{2}-\sqrt{6}).$$

$$\Delta = \frac{1}{16} \times$$

$$\sqrt{(2\sqrt{3}+3\sqrt{2}+\sqrt{6})(3\sqrt{2}+\sqrt{6}-2\sqrt{3})(\sqrt{6}+2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2}-\sqrt{6})}$$

$$= \frac{1}{16} \sqrt{(3\sqrt{2}+\sqrt{6})^2 - 12)(12 - (\sqrt{2}-\sqrt{6})^2)} = \frac{1}{4} \sqrt{(3+3\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{1}{4}(1+\sqrt{3})\sqrt{3} = \frac{1}{4}(3+\sqrt{3}) = \frac{1}{4}(4.7320) = 1.183$$

$$3. \text{ 今 } s = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \quad s-a = \frac{z}{x}, \quad s-b = \frac{x}{y}, \quad s-c = \frac{y}{z}$$

$$\begin{aligned} \therefore \Delta &= \sqrt{\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \left(\frac{x}{y}\right) \left(\frac{y}{z}\right) \left(\frac{z}{x}\right)} \\ &= \sqrt{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}} \end{aligned}$$

$$4. \text{ 今 } \Delta = \frac{a^2 \sin B \sin C}{2 \sin(B+C)} = \frac{4(\sqrt{3}+1)^2 \sin 45^\circ \sin 60^\circ}{2 \sin(45^\circ+60^\circ)}$$

$$= \frac{4(2+\sqrt{3})\sqrt{6}}{\sqrt{6}+\sqrt{2}}$$

$$(\because \sin 75^\circ = \frac{1}{4}(\sqrt{6}+\sqrt{2}))$$

$$= (2+\sqrt{3})(6-2\sqrt{3}) = 6+2\sqrt{3}$$

$$5. \text{ 從 } S = \frac{1}{2}bc \sin A \quad \text{即} \quad \frac{\sqrt{3}}{4}bc = 10\sqrt{3}$$

$$\therefore bc = 40 \quad (1)$$

$$\text{又從 } \cos 60^\circ = \frac{b^2+c^2-a^2}{2bc} \quad \text{及} \quad a+b+c=20$$

$$\text{即} \quad b^2+c^2 - [20 - (b+c)]^2 = bc$$

$$\text{即} \quad 40(b+c) - 400 = 3bc = 120$$

$$\therefore b+c = 13 \quad (2)$$

$$\text{解(1), (2)得兩邊爲 } 8, 5 \quad \text{則} \quad a=7$$

故三邊爲 8 尺, 5 尺, 7 尺。

$$6. \text{ 設三角爲 } A=B-y, \quad B, \quad C=B+y$$

$$\text{又三邊爲 } a=16 \text{ 尺, } b, \quad c=24 \text{ 尺}$$

$$\text{今 } A+B+C=180^\circ \quad \therefore 3B=180^\circ \text{ 或 } B=60^\circ$$

$$\text{又 } b^2=a^2+c^2-2ac \cos B=256+576-2 \cdot 16 \cdot 24 \cdot \frac{1}{2}$$

$$\therefore b^2=448 \quad \text{故 } b=8\sqrt{7} \text{ 尺}$$

$$\therefore \frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(C-A)}{\tan \frac{1}{2}(C+A)} = \frac{\tan y}{\tan 60^\circ}$$

$$\therefore \tan y = \frac{8}{40} \sqrt{3} = \frac{2\sqrt{3}}{10} = \frac{\sqrt{3}}{5}$$

$$\log \tan y = \log 2 + \frac{1}{2} \log 3 - 1 = .53959 - 1$$

$$= 9.53959 - 10 \quad \therefore y = 19^\circ 6' 24''$$

故三角爲  $40^\circ 53' 36''$ ,  $60^\circ$ ,  $79^\circ 6' 24''$ , 第三邊則爲  $8\sqrt{7}$  尺。

$$7. \therefore \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\text{今 } s = \frac{1}{2}(a+91+125) = \frac{a}{2} + 108$$

$$s-a = 108 - \frac{a}{2}, \quad s-b = \frac{a}{2} + 17, \quad s-c = \frac{a}{2} - 17$$

$$\therefore \left(\frac{17}{6}\right)^2 = \frac{\left(\frac{a}{2}\right)^2 - 289}{-\left(\frac{a}{2}\right)^2 + 108^2} \quad \text{解之得 } a=204$$

$$8. \text{ 因 } \cot \frac{C}{2} = \tan \frac{1}{2}(A+B) = \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5 \cdot 20}{6 \cdot 37}} = \frac{305}{122} = \frac{5}{2}$$

$$\therefore \tan \frac{C}{2} = \frac{2}{5}$$

$$\begin{aligned} \therefore \frac{b-c}{b+c} &= \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{\left(\frac{20}{37} - \frac{2}{5}\right) / \left(1 + \frac{20}{37} \cdot \frac{2}{5}\right)}{\cot \frac{1}{2}A} \\ &= \frac{26}{\frac{225}{6}} = \frac{13}{135} \\ &= \frac{26}{5} \end{aligned}$$

$$\begin{aligned} \text{又} \quad \frac{a-b}{a+b} &= \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{\left(\frac{5}{6} - \frac{20}{37}\right) / \left(1 + \frac{5}{6} \cdot \frac{20}{37}\right)}{\cot \frac{1}{2}C} \\ &= \frac{65}{\frac{322}{5}} = \frac{13}{161} \\ &= \frac{65}{2} \end{aligned}$$

從上兩式推得  $c = \frac{61}{74}b$ ,  $a = \frac{87}{74}b$

$$\therefore a+c = \frac{87}{74}b + \frac{61}{74}b = 2b$$

9. 今  $\frac{A}{1} = \frac{B}{2} = \frac{C}{3} = \frac{A+B+C}{6} = \frac{180^\circ}{6} = 30^\circ$

$$\therefore A=30^\circ, B=60^\circ, C=90^\circ$$

從正弦定律得  $a:b:c = \sin 30^\circ : \sin 60^\circ : \sin 90^\circ$

$$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1 = 1 : \sqrt{3} : 2$$

10. 今  $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{(a+b+c)(b+c-a) - 2bc}{2bc}$

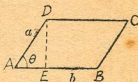
但  $(a+b+c)(b+c-a) = 3bc$

$$\therefore \cos A = \frac{1}{2} \quad \text{故} \quad A = 60^\circ$$

$$\begin{aligned} \text{或從 } \cos A &= 2 \cos^2 \frac{A}{2} - 1 = \frac{2s(s-a)}{bc} - 1 \\ &= \frac{3}{2} - 1 = \frac{1}{2} \quad (\because 4s(s-a) = 3bc) \\ \therefore A &= 60^\circ \end{aligned}$$

11. 作  $DE \perp AB$  則  $DE = a \sin \theta$

$$\begin{aligned} \text{今 } \square \text{面積} &= \overline{DE} \cdot \overline{AB} = a \sin \theta \cdot b \\ &= ab \sin \theta \end{aligned}$$



12. 如圖設  $BE \perp AC$ ,  $DF \perp AC$

$$AC = d_1, \quad BD = d_2$$

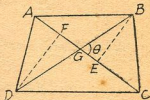
$$\text{則 } BE = BG \sin \theta,$$

$$DF = DG \sin \theta$$

$$\begin{aligned} \text{今 } \triangle ABC &= \frac{1}{2} \overline{AC} \cdot \overline{BE} \\ &= \frac{1}{2} d_1 \overline{BG} \sin \theta \end{aligned}$$

$$\triangle ADC = \frac{1}{2} \overline{AC} \cdot \overline{DF} = \frac{1}{2} d_1 \overline{DG} \sin \theta$$

$$\text{故 } ABCD = \frac{1}{2} d_1 \sin \theta (\overline{BG} + \overline{DG}) = \frac{1}{2} d_1 d_2 \sin \theta$$



13. 設  $\triangle ABC$  之三邊為  $a', b', c'$

又周界之半為  $s$ , 則  $s = 3a_0$

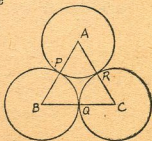
$$s - a' = s - b' = s - c' = a$$

$$\text{故 } \triangle = a^2 \sqrt{3}$$

$$\text{又 } A = B = C = 60^\circ$$

故扇形  $APR = BPQ = CQR$

$$= \frac{1}{6} \text{圓} = \frac{1}{6} \pi a^2$$



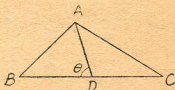
故空隙面積 =  $\triangle - 3(\text{扇形 } APR)$

$$= a^2\sqrt{3} - \frac{1}{2}\pi a^2 = (\sqrt{3} - \frac{1}{2}\pi)a^2$$

14. 如圖在  $\triangle ABD, ACD$  中, 設  $\angle ADB = \theta$

$$\text{則 } \overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2\overline{AD} \cdot \overline{BD} \cos \theta$$

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2 + 2\overline{AD} \cdot \overline{DC} \cos \theta$$



因  $BD = DC = \frac{1}{2}BC$

代入上兩式再相加之, 又兩邊乘以 2

$$\text{則 } 2\overline{AB}^2 + 2\overline{AC}^2 = \overline{BC}^2 + 4\overline{AD}^2$$

15. 即  $(a^2 + b^2)^2 - 2(a^2 + b^2)c^2 + c^4 = a^2b^2$

$$\text{即 } (a^2 + b^2 - c^2)^2 = a^2b^2 \quad \therefore a^2 + b^2 - c^2 = \pm ab$$

$$\text{從 } \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \pm \frac{1}{2}$$

$$\therefore C = 60^\circ \text{ 或 } 120^\circ$$

$$16. \therefore \tan B = \frac{\cos(C-B)}{\sin(C+B) + \sin(C-B)}$$

$$= \frac{\cos C \cos B + \sin C \sin B}{2 \sin C \cos B}$$

$$= \frac{1}{2}(\tan B + \cot C)$$

$$\therefore \tan B = \cot C = \tan\left(\frac{\pi}{2} - C\right)$$

$$\therefore B = \frac{\pi}{2} - C \quad \therefore B + C = \frac{\pi}{2} \quad \text{即} \quad A = \frac{\pi}{2}$$

故如題云。

17. 即  $\sin(B+C) = 2 \sin B \cos C$

即  $\sin(B-C) = 0 \quad \therefore B=C$

18. 今  $c(a+b) \sqrt{\frac{s(s-b)}{ac}} = b(c+a) \sqrt{\frac{s(s-c)}{ab}}$

即  $\frac{c(a+b)}{b(c+a)} = \sqrt{\frac{c(s-c)}{b(s-b)}} \quad \text{即} \quad \frac{c(a+b)^2}{b(c+a)^2} = \frac{a+b-c}{a-b+c}$

去分母整理之：

$$ca^3 + c^2a^2 - cb^3 = 3ab^2c - 3abc^2 - bc^3 + ba^3 + b^2a^2$$

即  $a^3(c-b) + a^2(c-b)(c+b) + bc(c-b)(c+b) + 3abc(c-b) = 0$

即  $(c-b)[a^3 + a^2(c+b) + bc(c+b) + 3abc] = 0$

$$\therefore c-b=0 \quad \therefore b=c$$

19. 即  $3b^2 - 6bc + 3c^2 = b^2 + c^2 - 2bc \cos 60^\circ = b^2 + c^2 - bc$

即  $2b^2 - 5bc + 2c^2 = 0 \quad \text{即} \quad (2b-c)(b-2c) = 0$

$\therefore b-c$  爲正 即  $b > c$  故  $b=2c$

故  $a = (2c-c)\sqrt{3} = c\sqrt{3}$

今  $a^2 + c^2 = 3c^2 + c^2 = 4c^2 = b^2$

故  $B=90^\circ \quad \text{則} \quad C=30^\circ$

【此題亦可用正弦定律從角着手】

$$20. \text{ 即 右邊} = \frac{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}{2 \cos \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}$$

$$= \frac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}(B+C)} = \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A}$$

$$\text{即 } \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A} = \sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$$

( $\cos \frac{1}{2}A = 0$  無意義)

$$\therefore \sin^2 \frac{A}{2} = \frac{1}{2} \qquad \therefore \sin \frac{A}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{A}{2} = 45^\circ \qquad \therefore A = 90^\circ \quad \text{故如題云。}$$

$$21. \text{ 即 } (1 - \cos^2 A) + (1 - \cos^2 B) + (1 - \cos^2 C) = 2$$

$$\text{即 } \Sigma \cos^2 A = 1 \qquad \text{即 } \Sigma \cos 2A = -1$$

$$\text{但因 } A + B + C = \pi$$

$$\text{故從 142 頁例六知 } \Sigma \cos 2A = -1 - 4\Pi \cos A$$

$$\text{代入上式得 } \cos A \cos B \cos C = 0$$

中間任一個因式為 0, 如  $\cos A = 0$ , 則  $A = 90^\circ$

故如題云。

註：此題化至  $\Sigma \cos^2 A = 1$  時, 如用餘弦定律, 則得

$$\Pi(a^2 + b^2 - c^2) = 0 \quad \text{惟甚繁也。}$$

$$22. \text{ 即 } a(\tan A - \cot \frac{1}{2}C) = b(\cot \frac{1}{2}C - \tan B)$$

$$(\because \tan \frac{1}{2}(A+B) = \cot \frac{1}{2}C)$$

$$\text{即 } a\left(\frac{\sin A}{\cos A} - \frac{1 + \cos C}{\sin C}\right) = b\left(\frac{1 + \cos C}{\sin C} - \frac{\sin B}{\cos B}\right)$$



$$\begin{aligned} \text{即 } \frac{a}{\cos A} [-\cos(A+C) - \cos A] & \quad \text{【此題用正弦} \\ & \quad \text{定律便甚繁】} \\ & = \frac{b}{\cos B} [\cos(B+C) + \cos B] \end{aligned}$$

$$\text{即 } a \cos B (\cos B - \cos A) = b \cos A (\cos B - \cos A)$$

$$\text{若 } \cos B - \cos A = 0 \quad \text{則 } \cos A = \cos B$$

$$\therefore A = B \quad \text{又 } a \cos B = b \cos A$$

$$\text{即 } \frac{a^2 + c^2 - b^2}{2c} = \frac{b^2 + c^2 - a^2}{2c} \quad \therefore a^2 = b^2 \quad \text{即 } a = b$$

故此題必爲一等腰三角形。

$$\begin{aligned} \text{23. 因 } \frac{\sin A \cos B}{\cos A \sin B} &= \frac{\sin A}{\sin B} & \text{即 } \cos B &= \cos A \\ & \therefore A = B & \text{故如題云。} \end{aligned}$$

$$\begin{aligned} \text{24. 從合分之理得 } \frac{\tan A}{\tan B} &= \frac{2c-b}{b} \\ \text{即 } \frac{\sin A \cos B}{\cos A \sin B} &= \frac{2c-b}{b} \quad \text{以 } \sin A = \frac{a \sin B}{b} \text{ 代入} \end{aligned}$$

$$\text{則得 } a \cos B = 2c \cos A - b \cos A$$

$$\text{但 } a \cos B + b \cos A = c \quad \text{【公式(41A)]}$$

$$\text{故 } \cos A = \frac{1}{2} \quad \therefore A = 60^\circ$$

$$\begin{aligned} \text{25. 即 } \cos A \sin C + 2 \cos C \sin C \\ &= \sin B \cos A + 2 \cos B \sin B \\ \text{即 } \cos A (\sin C - \sin B) + \sin 2C - \sin 2B &= 0 \\ \text{即 } \cos A (\sin C - \sin B) - 2 \sin(C-B) \cos A &= 0 \\ \text{即 } \cos A [\sin C - \sin B - 2 \sin(C-B)] &= 0 \end{aligned}$$

$$\text{即 } \cos A \left[ 2 \sin \frac{C-B}{2} \cos \frac{C+B}{2} - 4 \sin \frac{C-B}{2} \cos \frac{C-B}{2} \right] = 0$$

$$\text{即 } 2 \cos A \sin \frac{C-B}{2} \left[ \cos \frac{C+B}{2} - 2 \cos \frac{C-B}{2} \right] = 0$$

$$\therefore \cos A = 0 \quad \text{則 } A = 90^\circ \quad (\text{此爲一直角三角形})$$

$$\text{或 } \sin \frac{C-B}{2} = 0 \quad \text{則 } \frac{C-B}{2} = 0$$

$$\therefore C = B \quad (\text{此爲一等腰三角形})$$

$$26. \text{ 今 } \tan \frac{A}{2} \tan \frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)} \cdot \frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \frac{c+a-b}{a+b+c} = \frac{b}{3b} = \frac{1}{3}$$

$$(\because a+b=2b)$$

註：此題如用正弦定律做便甚繁。

$$27. \text{ 設 } A = 8\theta, \quad B = 4\theta, \quad C = 2\theta$$

$$\text{則 } 14\theta = 180^\circ \quad \therefore 7\theta = 90^\circ$$

$$\text{今 } \frac{a}{\sin 8\theta} = \frac{b}{\sin 4\theta} = \frac{c}{\sin 2\theta} = \frac{a+b+c}{\sin 8\theta + \sin 4\theta + \sin 2\theta}$$

$$\text{但 } \sin 8\theta = 2 \sin 4\theta \cos 4\theta = \dots\dots\dots$$

$$= 8 \sin \theta \cos \theta \cos 2\theta \cos 4\theta$$

$$\text{又 } \sin 8\theta + \sin 4\theta + \sin 2\theta = 4 \cos 4\theta \cos 2\theta \cos \theta$$

(見附錄一公式 50A)

$$\therefore \frac{a}{a+b+c} = 2 \sin \theta = 2 \sin \frac{90^\circ}{7} : 1$$

$$28. \text{ 設三邊爲 } k-r, k, k+r, \text{ 即爲 } k(1-x), k, k(1+x)\text{之}$$

形式，則  $a:b:c=1-x:1:1+x$

$$\text{今 } s = \frac{3k}{2}, \quad s-a = \left(\frac{1}{2} + x\right)k, \quad \dots\dots$$

$$\begin{aligned} \text{故 } \cos \frac{\alpha}{2} &= \cos \frac{1}{2}(A-C) = \cos \frac{A}{2} \cos \frac{C}{2} + \sin \frac{A}{2} \sin \frac{C}{2} \\ &= \frac{s}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} + \frac{s-b}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} \\ &= \frac{2s-b}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} = 2\sqrt{\frac{\frac{1}{4}-x^2}{1-x^2}} \\ &= \sqrt{\frac{1-4x^2}{1-x^2}} \quad \therefore \cos \alpha = \frac{1-7x^2}{1-x^2} \end{aligned}$$

$$\text{即 } (7 - \cos \alpha)x^2 = 1 - \cos \alpha \quad \therefore x = \sqrt{\frac{1 - \cos \alpha}{7 - \cos \alpha}}$$

29. 設周界為  $p$ ，則  $p = 2\pi r$

$$\therefore r = \frac{p}{2\pi}, \quad S(\text{圓面積}) = \pi r^2 = \frac{p^2}{4\pi}$$

又設正  $n$  邊形之邊心距為  $d$ ，則因每邊長為  $\frac{p}{n}$ ，所對之

圓心角為  $\frac{2\pi}{n}$

$$\therefore \frac{\frac{p}{2n}}{d} = \tan \frac{\pi}{n} \quad \therefore d = \frac{\frac{p}{2n}}{\tan \frac{\pi}{n}}$$

$$\therefore S(\text{正 } n \text{ 邊形面積}) = \frac{\frac{p}{2n}}{\tan \frac{\pi}{n}} \cdot \frac{p}{2} = \frac{p^2}{4n \tan \frac{\pi}{n}}$$

$$\therefore S:S' = \frac{p^2}{4\pi} : \frac{p^2}{4n \tan \frac{\pi}{n}} = \tan \frac{\pi}{n} : \frac{\pi}{n}$$

30. 設  $\Omega$  為  $ABCD$  之面積

$$\text{則 } \Omega = \triangle ABD + \triangle BCD$$

$$\begin{aligned} \therefore \Omega &= \frac{1}{2}ab \sin A + \frac{1}{2}cd \sin A \\ &= \frac{1}{2}(ab+cd) \sin A \end{aligned}$$

今  $ABCD$  內接於一圓

$$\text{則 } A+C=180^\circ$$

$$\therefore \cos C = -\cos A \quad \text{故 } x^2 = c^2 + d^2 - 2cd \cos A$$

$$\text{又 } x^2 = a^2 + b^2 - 2ab \cos C = a^2 + b^2 + 2ab \cos A$$

$$\text{消去 } x \text{ 得 } \cos A = \frac{c^2 + d^2 - a^2 - b^2}{2(ab+cd)}$$

$$\begin{aligned} \text{則 } \sin^2 A &= 1 - \left[ \frac{c^2 + d^2 - a^2 - b^2}{2(ab+cd)} \right]^2 \\ &= \frac{4(ab+cd)^2 - (c^2 + d^2 - a^2 - b^2)^2}{4(ab+cd)^2} \end{aligned}$$

$$\text{分子爲 } (2ab + 2cd + c^2 + d^2 - a^2 - b^2)$$

$$\times (2ab + 2cd - c^2 - d^2 + a^2 + b^2)$$

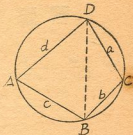
$$= [(c+d)^2 - (a-b)^2][(a+b)^2 - (c-d)^2]$$

$$= (-a+b+c+d)(a-b+c+d)$$

$$\times (a+b-c+d)(a+b+c-d)$$

$$\text{設 } s = \frac{1}{2}(a+b+c+d) \quad \text{則 } s-a = \frac{1}{2}(-a+b+c+d)$$

$$\therefore \sin A = \frac{2\sqrt{\Pi(s-a)}}{ab+cd}$$



$$\text{故 } \Omega = \frac{1}{2}(ab+cd) \cdot \frac{2\sqrt{\Pi(s-a)}}{ab+cd} = \sqrt{\Pi(s-a)}$$

## 習題二十二 (279—282 頁)

$$\begin{aligned} 1. \text{ 今 } \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{ma}{m \sin A} \\ &= \frac{ma+b+c}{m \sin A + \sin B + \sin C} \\ &= \frac{ma-b+c}{m \sin A - \sin B + \sin C} \end{aligned}$$

$$\therefore \frac{ma+b+c}{ma-b+c} = \frac{m \sin A + \sin B + \sin C}{m \sin A - \sin B + \sin C}$$

$$2. \text{ 今 } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\begin{aligned} \text{即 } \frac{ma^2}{m \sin^2 A} &= \frac{nab}{n \sin A \sin B} = \frac{pb^2}{p \sin^2 B} \\ &= \frac{ma^2 + nab + pb^2}{m \sin^2 A + n \sin A \sin B + p \sin^2 B} \\ &= \frac{ma^2 - nab + pb^2}{m \sin^2 A - n \sin A \sin B + p \sin^2 B} \end{aligned}$$

$$\text{或 } \frac{ma^2 + ncb + pb^2}{ma^2 - nab + pb^2} = \frac{m \sin^2 A + n \sin A \sin B + p \sin^2 B}{m \sin^2 A - n \sin A \sin B + p \sin^2 B}$$

$$\begin{aligned} \therefore \sqrt{\frac{ma^2 + nab + pb^2}{ma^2 - nab + pb^2}} \\ = \sqrt{\frac{m \sin^2 A + n \sin A \sin B + p \sin^2 B}{m \sin^2 A - n \sin A \sin B + p \sin^2 B}} \end{aligned}$$

$$3. \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{bc}{c \sin B} = \frac{bc}{b \sin C}$$

$$= \frac{bc}{\sqrt{bc \sin B \sin C}} = \frac{bc^2 + b^2c}{c^2 \sin B + b^2 \sin C}$$

故  $(b+c)\sqrt{bc \sin B \sin C} = b^2 \sin C + c^2 \sin B$

4.  $\therefore b = a \cos C + c \cos A, \quad c = a \cos B + b \cos A$

$\therefore b - c = a(\cos C - \cos B) + (c - b) \cos A$

即  $(b - c)(1 + \cos A) = a(\cos C - \cos B)$

即  $(b - c)\left(2 \cos^2 \frac{A}{2}\right) = a\left(2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}\right)$

$$\therefore (b - c) \cos \frac{A}{2} = a \sin \frac{B - C}{2}$$

【從正弦定理亦可證，參考 265 頁例二】

5. 從  $b = a \cos C + c \cos A, \quad c = b \cos A + a \cos B$

得  $\cos B = \frac{c - b \cos A}{a}, \quad \cos C = \frac{b - c \cos A}{a}$

故  $\cos B : \cos C = c - b \cos A : b - c \cos A$

【從正弦定律亦可做，正常方法是由右至左】

6. 做法同題 4 不過易  $b - c$  爲  $b + c$  (即易減爲加) 耳。

7.  $\therefore \frac{\sin B}{b} = \frac{\sin C}{c} \quad \therefore c \sin B = b \sin C$

$$\text{左邊} = 2(b^2 \sin C \cos C + c^2 \sin B \cos B)$$

$$= 2(b \cos C \cdot c \sin B + c \cos B \cdot b \sin C)$$

$$= 2bc(\cos C \sin B + \sin C \cos B)$$

$$= 2bc \sin(B + C) = 2bc \sin A$$

8. 從  $a = b \cos C + c \cos B$ ,  $b = c \cos A + a \cos C$

$$\therefore (a+b)(1 - \cos C) = c(\cos A + \cos B)$$

$$\therefore \cos A + \cos B = 2 \left( \frac{a+b}{c} \right) \sin^2 \frac{C}{2}$$

$$(\because 1 - \cos C = 2 \sin^2 \frac{C}{2} \text{ 也})$$

【由左邊 =  $2 \sin^2 \frac{B}{2} - 2 \sin^2 \frac{C}{2}$  做便繁】

9. 因  $a = \frac{c \sin A}{\sin C}$ ,  $b = \frac{c \sin B}{\sin C}$

$$\text{故左邊} = \frac{c \sin A \cos A}{\sin C} + \frac{c \sin B \cos B}{\sin C}$$

$$= \frac{c}{2 \sin C} (\sin 2A + \sin 2B)$$

$$= \frac{c}{\sin C} [\sin(A+B) \cos(A-B)] = c \cos(A-B)$$

$$(\because \sin(A+B) = \sin C \text{ 也})$$

10. 左邊 =  $\frac{1 - \cos A}{1 - \cos B} = \frac{\sin^2 \frac{A}{2}}{\sin^2 \frac{B}{2}} = \frac{(s-b)(s-c)}{bc} \cdot \frac{(s-a)(s-c)}{ac}$

$$= \frac{a(s-b)}{b(s-a)} = \frac{a(a-b+c)}{b(b+c-a)}$$

11. 從  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$$\therefore a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C$$

$$\begin{aligned} \text{故 } \Sigma(a-b) \sin C &= 2R \Sigma \sin C (\sin A - \sin B) \\ &= 2R \cdot 0 = 0 \end{aligned}$$

$$12. \text{ 右邊} = \frac{1}{2} \sin A (2R \sin B)(2R \sin C) = \frac{1}{2} bc \sin A = S$$

【參考上題】

13 從公式 41A 得

$$a + b + c = 2s = (a + b) \cos C + (b + c) \cos A + (a + c) \cos B$$

【此題如用 41B 做便甚繁】

$$14. \text{ 左邊} = \Sigma \left( 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc} \right) = \Sigma (b^2 + c^2 - a^2) = \Sigma a^2$$

$$15. \text{ 左邊} = \frac{b^2(b^2 + c^2 - a^2)}{2abc} + \frac{c^2(a^2 + c^2 - b^2)}{2abc} + \frac{a^2(a^2 + b^2 - c^2)}{2abc}$$

$$= \frac{\Sigma a^4}{2\Pi a}$$

$$16. \therefore a = b \cos C + c \cos B \quad (1)$$

$$b = c \cos A + a \cos C \quad (2)$$

$$a(1) - b(2), \quad a^2 - b^2 = c(a \cos B - b \cos A)$$

【用 41B 做亦不繁】

$$17. c^2 = a^2 + b^2 - 2ab \cos C$$

$$= a^2 (\sin^2 \frac{1}{2} C + \cos^2 \frac{1}{2} C) + b^2 (\sin^2 \frac{1}{2} C + \cos^2 \frac{1}{2} C)$$

$$- 2ab (\cos^2 \frac{1}{2} C - \sin^2 \frac{1}{2} C)$$

$$= (a^2 + 2ab + b^2) \sin^2 \frac{1}{2} C + (a^2 - 2ab + b^2) \cos^2 \frac{1}{2} C$$

$$= (a + b)^2 \sin^2 \frac{1}{2} C + (a - b)^2 \cos^2 \frac{1}{2} C$$

$$18. \text{ 從 } \tan \frac{A}{2} = \frac{r}{s - a}, \quad \tan \frac{B}{2} = \frac{r}{s - b}, \quad \tan \frac{C}{2} = \frac{r}{s - c}$$

$$\therefore (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2} (= r)$$



$$19. \text{ 從 } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\therefore \frac{\Pi a}{\Pi \sin A} = 8R^3 \quad \therefore R = \frac{1}{2} \left( \frac{\Pi a}{\Pi \sin A} \right)^{\frac{1}{3}}$$

$$20. \therefore \tan \frac{A}{2} = \frac{r}{s-a}, \quad \tan \frac{B}{2} = \frac{r}{s-b}, \quad \tan \frac{C}{2} = \frac{r}{s-c}$$

$$\therefore \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} = \frac{r^3}{(s-a)(s-b)(s-c)} = \frac{sr^3}{S^2}$$

$$(\because S^2 = s(s-a)(s-b)(s-c) \text{ 也})$$

$$21. \text{ 左邊} = \frac{s(s-b)(s-c) + s(s-a)(s-c)}{\Delta^2}$$

$$+ \frac{s(s-a)(s-b) - (s-a)(s-b)(s-c)}{\Delta^2}$$

$$= \frac{2s^3 - cs^2 - bs^2 - cs^2 + abc}{\Delta^2}$$

$$= \frac{s^2(a+b+c-a-b-c) + abc}{\Delta^2}$$

$$= \frac{abc}{\Delta^2} = \frac{abc}{\Delta} \cdot \frac{1}{\Delta} = \frac{4R}{\Delta}$$

$$22. \therefore \text{ 左邊} = \frac{2 - 2 \cos(A-B) \cos(A+B)}{2 - 2 \cos(A-C) \cos(A+C)}$$

$$= \frac{2 - (\cos 2A + \cos 2B)}{2 - (\cos 2A + \cos 2C)} = \frac{2 \sin^2 A + 2 \sin^2 B}{2 \sin^2 A + 2 \sin^2 C}$$

$$= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{a^2 + b^2}{a^2 + c^2}$$

$$(\because \frac{a^2}{\sin^2 A} = \frac{b^2}{\sin^2 B} = \frac{c^2}{\sin^2 C} = \frac{a^2 + b^2}{\sin^2 A + \sin^2 B}$$

$$= \frac{a^2 + c^2}{\sin^2 A + \sin^2 C} \text{ 也})$$

$$23. \text{ 令 } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\text{則 } a = k \sin A, \quad b = k \sin B, \quad c = k \sin C$$

$$\text{左邊} = k^2 \Sigma \sin A \sin (B - C)$$

$$= \frac{1}{2} k^2 \Sigma 2 \sin (B + C) \sin (B - C)$$

$$= \frac{1}{2} k^2 \Sigma (\cos 2C - \cos 2B) = \frac{1}{2} k^2 \cdot 0 = 0$$

$$\text{或化左邊} = k \Sigma a \sin (B - C) = k \cdot 0 = 0$$

(從 266 頁例三)

$$24. \text{ 今 } \frac{a^2 \sin (B - C)}{\sin B + \sin C} = \frac{k^2 \sin^2 A \sin (B - C)}{\sin B + \sin C}$$

$$= \frac{k^2 \sin A \sin (B + C) \sin (B - C)}{\sin B + \sin C}$$

$$= \frac{k^2 \sin A (\cos 2C - \cos 2B)}{2(\sin B + \sin C)}$$

$$= \frac{k^2 \sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C}$$

$$= k^2 \sin A (\sin B - \sin C)$$

$$\therefore \frac{a^2 \sin (B - C)}{\sin B + \sin C} = k^2 \Sigma \sin A (\sin B - \sin C) = k^2 \cdot 0 = 0$$

$$25. \text{ 左邊} = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B - \sin^2 A & \cot B - \cot A & 0 \\ \sin^2 C - \sin^2 A & \cot C - \cot A & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2 B - \sin^2 A & \cot B - \cot A \\ \sin^2 C - \sin^2 A & \cot C - \cot A \end{vmatrix}$$

$$= - \begin{vmatrix} \sin(A+B)\sin(A-B) & \frac{\sin(A-B)}{\sin A \sin B} \\ \sin(A+C)\sin(A-C) & \frac{\sin(A-C)}{\sin A \sin C} \end{vmatrix}^* \\ = - \frac{\sin(A-B)\sin(A-C)}{\sin A} \begin{vmatrix} \sin C & \frac{1}{\sin B} \\ \sin B & \frac{1}{\sin C} \end{vmatrix} = 0$$

或從  $\Sigma \sin^2 A (\cot B - \cot C) = \dots\dots\dots$

$$= \Sigma \sin^2 A \sin(C-B) / \sin A \sin B \sin C$$

$$= \frac{\Sigma (1 - \cos 2A)(\cos 2B - \cos 2C)}{4 \sin A \sin B \sin C} = \dots\dots = 0$$

26.

$$\text{左邊} = \begin{vmatrix} a & a^2 & \frac{s(s-a)a}{abc} \\ b & b^2 & \frac{s(s-b)b}{abc} \\ c & c^2 & \frac{s(s-c)c}{abc} \end{vmatrix} = \frac{s^2 bc}{abc^3} \begin{vmatrix} 1 & a & s-a \\ 1 & b & s-b \\ 1 & c & s-c \end{vmatrix}$$

$$= s \begin{vmatrix} 1 & a & s \\ 1 & b & s \\ 1 & c & s \end{vmatrix} = s^2 \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$$

27. 原式  $= \Sigma (a-b) \cdot \frac{s-c}{r}$  ( $\because \cot \frac{C}{2} = 1 \tan \frac{C}{2} = \frac{s-c}{r}$ )

$$= \frac{1}{r} \Sigma (a-b)(s-c) = \frac{1}{r} \Sigma [s(a-b) - c(a-b)] = 0$$

\* 因  $\sin^2 B - \sin^2 A = \frac{1}{2}[(1 - 2\sin^2 A) - (1 - 2\sin^2 B)]$

$$= \frac{1}{2}(\cos 2A - \cos 2B) = -\sin(A+B)\sin(A-B)$$

從原式  $= 2R \sum (\sin A - \sin B) \cot \frac{C}{2}$  做較繁，若以附錄一  
公式 43C 代入做更繁。

$$\begin{aligned}
 28. \text{ 左邊} &= \sum \frac{a \cos C + c \cos A - 2a \cos C}{a \sin C} = \sum \frac{c \cos A - a \cos C}{a \sin C} \\
 &= \sum \left( \frac{\sin C \cos A}{\sin A \sin C} - \frac{\cos C}{\sin C} \right) = \sum (\cot A - \cot C) = 0 \\
 &\quad \text{【或用公式 41B 可推到 } \frac{2R}{abc} \sum (c^2 - a^2) \text{】}
 \end{aligned}$$

$$\begin{aligned}
 29. \text{ 左邊} &= \sum \frac{a \cdot 2 \sin \frac{1}{2}(B-C) \sin \frac{1}{2}(B+C)}{\sin A} \\
 &= 2R \sum [2 \sin \frac{1}{2}(B-C) \sin \frac{1}{2}(B+C)] \\
 &= 2R \sum (\cos C - \cos B) = 2R \cdot 0 = 0
 \end{aligned}$$

$$\begin{aligned}
 30. \sum a \cos A &= \frac{1}{2abc} \sum a^2 (b^2 + c^2 - a^2) = \frac{1}{2abc} (2 \sum a^2 b^2 - \sum a^4) \\
 &= \frac{(a+b+c)(a-b+c)(a+b-c)(-a+b+c)}{2abc} \\
 &= \frac{16s(s-a)(s-b)(s-c)}{2abc} = \frac{16S^2}{2abc} = \frac{2S}{R}
 \end{aligned}$$

從公式 41A 做較繁且須用到  $\sum \cos A = 4R \sin \frac{1}{2} A + 1$

$$31. \sum a \sin A = \frac{1}{2R} \sum a \cdot 2R \sin A = \frac{1}{2R} \sum a \cdot a = \frac{\sum a^2}{2R}$$

$$32. \text{ 今 左邊} = \frac{2R \sum \sin A \sin A}{2R \sum \sin A \cos A} = \frac{\sum 2 \sin^2 A}{\sum \sin 2A}$$

$$\text{但 } \sum 2 \sin^2 A = 4 + 4 \cos A \cos B \cos C \quad (143 \text{ 頁例七})$$

$$\text{又 } \sum \sin 2A = 4 \sin A \sin B \sin C \quad (145 \text{ 頁例十一})$$

$$\begin{aligned} \therefore \text{左邊} &= \frac{4 + 4\Pi \cos A}{4\Pi \sin A} \\ &= \frac{1 + [\cos(A+B+C) + \Sigma \sin A \sin B \cos C]^*}{\Pi \sin A} \\ &= \frac{\Sigma \sin A \sin B \cos C}{\Pi \sin A} = \Sigma \cot A \end{aligned}$$

$$* \therefore \cos(A+B+C) = \Pi \cos A - \Sigma \sin A \sin B \cos C$$

(77 頁題  $n$  公式 15)

$$\begin{aligned} 33. \text{左邊} &= \Sigma \frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)}{ab \cdot 2bc \cdot 2ac} \\ &= \Sigma \frac{c^4 - (a^2 - b^2)^2}{4a^2b^2c^2} = \frac{\Sigma a^4 + 2\Sigma a^2b^2}{4a^2b^2c^2} \\ &= \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4a^2b^2c^2} \\ &= \frac{16s(s-a)(s-b)(s-c)}{4 \cdot 16R^2S^2} = \frac{4S^2}{16R^2S^2} = \frac{1}{4R^2} \\ &\quad \left( \because S = \frac{abc}{4R} \right) \end{aligned}$$

$$\begin{aligned} 34. \text{左邊} &= 2R\Sigma a^2 \sin(B+C) \cos(B-C) \\ &= R\Sigma a^2 (\sin 2B + \sin 2C) \\ &= 2R\Sigma a^2 (\sin B \cos B + \sin C \cos C) \\ &= \Sigma a^2 (b \cos B + c \cos C) \\ &= \Sigma a^2 \left[ \frac{b^2(c^2 + a^2 - b^2)}{2abc} + \frac{c^2(a^2 + b^2 - c^2)}{2abc} \right] \\ &= \frac{\Sigma [a^2(2b^2c^2 - b^4 - c^4) + a^4(b^2 + c^2)]}{2abc} \\ &= \frac{\Sigma (2a^2b^2c^2)}{2abc} = \frac{6a^2b^2c^2}{2abc} = 3abc \end{aligned}$$

$$\begin{aligned}
35. \text{ 今 } (a+b+c)^2(\sum \cot A) &= (\sum a)^2 \cdot \frac{\sum a \sin A}{\sum a \cos A} \quad (\text{見題 32}) \\
&= (\sum a)^2 \cdot \frac{(\sum a^2) \cdot R}{2R \cdot 2S} \quad (\text{見題 30 及 31}) \\
&= (\sum a)^2 \cdot \frac{\sum a^2}{4S} = (\sum a^2) \cdot \frac{(\sum a)^2}{4S} = (\sum a^2) \cdot \frac{s(\sum a)^2}{4s^2 r} \\
&= (\sum a^2) \cdot \frac{s}{r} = (\sum a^2) \cdot \frac{(s-a) + (s-b) + (s-c)}{r} \\
&= (\sum a^2) \left( \sum \cot \frac{A}{2} \right) \quad \text{故如題云。}
\end{aligned}$$

$$\begin{aligned}
36. \text{ 今 } \sum a^2 \cot A &= 2R \sum a \cos A = 2R \sum a \cdot \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{2R}{2abc} \sum (a^2 b^2 + a^2 c^2 - a^4) \\
&= \frac{2R}{2abc} (2 \sum a^2 b^2 - \sum a^4) \\
&= \frac{2R}{2abc} (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\
&= \frac{16R \Delta^2}{abc} = \frac{16R \Delta^2}{4R \Delta} = 4\Delta
\end{aligned}$$

$$\text{或原式} = \frac{a}{\sin A} \cdot \sum a \cos A = 2R(2\Delta/R) = 4\Delta$$

(見題 30)

$$\begin{aligned}
37. \text{ 右邊} &= \frac{abc}{S} \cdot \frac{S}{s} \cdot \sqrt{\frac{s^3(s-a)(s-b)(s-c)}{a^2 b^2 c^2}} \\
&= \sqrt{s(s-a)(s-b)(s-c)} = S \quad \text{故如題云。}
\end{aligned}$$

$$\text{或從 } S = sr = r \cdot \frac{a+b+c}{2}$$

$$\begin{aligned}
 &= \frac{r}{2}(2R \sin A + 2R \sin B + 2R \sin C) \\
 &= Rr(\Sigma \sin A) = Rr\left(4\Pi \cos \frac{A}{2}\right) \quad (\text{公式 } 50A) \\
 &= 4Rr\Pi \cos \frac{A}{2}
 \end{aligned}$$

$$38. \quad \Sigma a \cos A = 2R \Sigma \sin A \cos A = R \Sigma \sin 2A$$

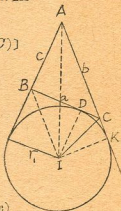
$$\begin{aligned}
 &= R[2 \sin A \cos A \\
 &\quad + 2 \sin(B+C) \cos(B-C)] \\
 &= R(2 \sin A)[\cos(B-C) \\
 &\quad - \cos(B+C)] \\
 &= 4R \sin A \sin B \sin C
 \end{aligned}$$

$$39. \quad \Delta = \Delta IAB + \Delta IAC - \Delta IBC$$

$$= \frac{1}{2}(r_1 b + r_1 c - r_1 a) = r_1(s-a)$$

$$\text{同理 } \Delta = r_2(s-b) = r_3(s-c)$$

$$\text{故 } r_1(s-a) = r_2(s-b) = r_3(s-c)$$



$$40. \quad \text{由前題 } r_1 = \frac{\Delta}{s-a}, \quad r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}, \quad r = \frac{\Delta}{s}$$

$$\begin{aligned}
 \therefore r_1 + r_2 + r_3 - r &= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s} \\
 &= \Delta \cdot \frac{\Sigma s(s-b)(s-c) - \Pi(s-a)}{s(s-a)(s-b)(s-c)} \\
 &= \frac{abc}{\Delta} = \frac{4R\Delta}{\Delta} = 4R \quad (\text{見題 } 21)
 \end{aligned}$$

$$41. \quad \therefore \Delta = r_1(s-a) \quad \therefore \frac{1}{r_1} = \frac{s-a}{\Delta}$$

$$\sum \frac{1}{r_1} = \frac{(s-a) + (s-b) + (s-c)}{\Delta} = \frac{s}{\Delta} = \frac{s}{sr} = \frac{1}{r}$$

或由題 39 之圖  $a = r_1 \cot ICD + r_1 \cot IBD$

$$= r_1 \left( \cot \frac{\pi - C}{2} + \cot \frac{\pi - B}{2} \right)$$

$$= r_1 \left( \tan \frac{C}{2} + \tan \frac{B}{2} \right)$$

$$= r_1 \left( \frac{r}{s-b} + \frac{r}{s-c} \right) = \frac{ar_1 r}{(s-b)(s-c)}$$

$$\therefore \frac{1}{r_1} = \frac{r}{(s-b)(s-c)}, \dots\dots\dots$$

$$\therefore \sum \frac{1}{r_1} = \sum \frac{r}{(s-b)(s-c)} = \frac{r(3s-a-b-c)}{\Pi(s-a)}$$

$$= \frac{rs}{\Pi(s-a)} = r \cdot \frac{1}{r^2} = \frac{1}{r}$$

42.

$$\therefore \frac{a^2}{\sin A} = a \cdot \frac{a}{\sin A} = a \cdot 2R$$

$$\therefore \sum \frac{a^2}{\sin A} = 2R \cdot \sum a = 4sR$$

$$\text{又} \quad \therefore \sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\therefore \Pi \sin \frac{1}{2}A = \frac{(s-a)(s-b)(s-c)}{abc}$$

$$\text{故} \quad \left( \sum \frac{a^2}{\sin A} \right) \Pi \sin \frac{1}{2}A = \frac{4s(s-a)(s-b)(s-c)R}{abc}$$

$$\therefore \sqrt{s(s-a)(s-b)(s-c)} = \Delta, \quad \frac{abc}{4R} = \Delta$$

$$\therefore \left( \sum \frac{a^2}{\sin A} \right) \Pi \sin \frac{1}{2}A = \Delta^2 \cdot \frac{1}{\Delta} = \Delta$$



$$\begin{aligned}
 43. \quad \Sigma(b^2 - c^2) \cot^2 \frac{A}{2} &= \Sigma(b^2 - c^2) \frac{(s-a)^2}{r^2} \\
 &= \frac{1}{r^2} \Sigma(b^2 - c^2)(-2as) * \\
 &= -\frac{2s}{r^2} \Sigma(b^2 - c^2)a = -\frac{2s}{r^2} \Pi(a-b) \\
 &= -\frac{2s^3 \Pi(a-b)}{s^2 r^2} = -\frac{2s^3 \Pi(a-b)}{\Delta^2}
 \end{aligned}$$

$$\therefore \Sigma(b^2 - c^2) \cot^2 \frac{A}{2} + \frac{2s^3 \Pi(a-b)}{\Delta^2} = 0$$

$$* \because \Sigma(b^2 - c^2)s^2 = 0, \quad \Sigma(b^2 - c^2)a^2 = 0$$

$$44. \quad \because \cot \gamma = \cot A + \cot B + \cot C \quad (\text{見 275 頁例 15})$$

$$\therefore \cot^2 \gamma = (\Sigma \cot A)^2$$

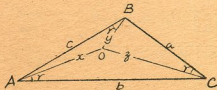
$$\text{故 } \csc^2 \gamma - 1 = \Sigma \cot^2 A + 2 \Sigma \cot A \cot B$$

$$= \Sigma \cot^2 A + 2 = \Sigma \csc^2 A - 3 + 2$$

$$\therefore \csc^2 \gamma = \Sigma \csc^2 A$$

$$\text{或從 } \frac{\Delta OAB}{\Delta ABC} = \frac{cy \sin \gamma}{ca \sin B} = \frac{\sin^2 \gamma}{\sin^2 B} = \frac{\csc^2 B}{\csc^2 \gamma}$$

$$\text{同理 } \frac{\Delta OBC}{\Delta ABC} = \frac{\csc^2 C}{\csc^2 \gamma}, \quad \frac{\Delta OCA}{\Delta ABC} = \frac{\csc^2 A}{\csc^2 \gamma}$$



$$\begin{aligned} \therefore \frac{\triangle OAB + \triangle OBC + \triangle OCA}{\triangle ABC} \\ = 1 = \frac{\csc^2 A + \csc^2 B + \csc^2 C}{\csc^2 \gamma} \quad \text{故如題云。} \end{aligned}$$

$$45. (1) \quad \cot \frac{A}{2} - \cot \frac{B}{2} = \frac{s-a}{r} - \frac{s-b}{r} = \frac{b-a}{r}$$

$$\cot \frac{B}{2} - \cot \frac{C}{2} = \frac{s-b}{r} - \frac{s-c}{r} = \frac{c-b}{r}$$

今  $a, b, c$  成  $A.P.$  則  $b-a=c-b$

$$\therefore \cot \frac{A}{2} - \cot \frac{B}{2} = \cot \frac{B}{2} - \cot \frac{C}{2} \quad \text{故如題云。}$$

$$\begin{aligned} (2) \quad \cos A \cot \frac{A}{2} - \cos B \cot \frac{B}{2} \\ = \left(1 - 2 \sin^2 \frac{A}{2}\right) \cot \frac{A}{2} - \left(1 - 2 \sin^2 \frac{B}{2}\right) \cot \frac{B}{2} \\ = \left(\cot \frac{A}{2} - \cot \frac{B}{2}\right) - \left(2 \sin \frac{A}{2} \cos \frac{A}{2} - 2 \sin \frac{B}{2} \cos \frac{B}{2}\right) \\ = \left(\cot \frac{A}{2} - \cot \frac{B}{2}\right) - (\sin A - \sin B) \end{aligned}$$

$$\begin{aligned} \text{同理} \quad \cos B \cot \frac{B}{2} - \cos C \cot \frac{C}{2} \\ = \left(\cot \frac{B}{2} - \cot \frac{C}{2}\right) - (\sin B - \sin C) \end{aligned}$$

今  $a, b, c$  成  $A.P.$  即  $b-a=c-b$

$$\therefore 2R(\sin A - \sin B) = 2R(\sin B - \sin C)$$

$$\text{即} \quad \sin A - \sin B = \sin B - \sin C$$

$$\text{且由(1)知} \quad \cot \frac{A}{2} - \cot \frac{B}{2} = \cot \frac{B}{2} - \cot \frac{C}{2}$$

$$\begin{aligned} \text{故} \quad \cos A \cot \frac{A}{2} - \cos B \cot \frac{B}{2} \\ = \cos B \cot \frac{B}{2} - \cos C \cot \frac{C}{2} \quad \text{故如題云.} \end{aligned}$$

$$\begin{aligned} 43. \quad \frac{1}{a \sec A} - \frac{1}{b \sec B} &= \frac{\cos A}{a} - \frac{\cos B}{b} \\ &= \frac{b^2 + c^2 - a^2 - c^2 - a^2 + b^2}{2abc} = \frac{b^2 - a^2}{abc} \end{aligned}$$

$$\text{同理} \quad \frac{1}{b \sec B} - \frac{1}{c \sec C} = \frac{c^2 - b^2}{abc}$$

$$\text{今 } a^2, b^2, c^2 \text{ 成 } A.P. \quad \therefore b^2 - a^2 = c^2 - b^2$$

$$\therefore \frac{1}{a \sec A} - \frac{1}{b \sec B} = \frac{1}{b \sec B} - \frac{1}{c \sec C} \quad \text{故如題云.}$$

$$\begin{aligned} 47. \quad \therefore \cos(A+B) + \cos(B+C) &= -\cos C - \cos A \\ \therefore \cos A \cos B - \sin A \sin B + \cos B \cos C - \sin B \sin C \\ &= -\cos C - \cos A \\ \therefore (1 + \cos B)(\cos C + \cos A) &= \sin B(\sin A + \sin C) \\ \therefore 2 \cos^2 \frac{B}{2} = \frac{\sin B(\sin A + \sin C)}{\cos A + \cos C} &= \frac{2 \sin A \sin C}{\cos A + \cos C} * \\ \therefore \cos \frac{B}{2} &= \sqrt{\frac{\sin A \sin C}{\cos A + \cos C}} \end{aligned}$$

$$* \left( \because b = \frac{2ac}{a+c} \quad \therefore \sin B = \frac{2 \sin A \sin C}{\sin A + \sin C} \right)$$

【此題從右邊推起較簡但不及上法有趣也】

$$48. \quad \therefore c = \frac{a-b}{\cos \theta} \quad \therefore \cos \theta = \frac{a-b}{c}, \quad \sin \theta = \frac{\sqrt{c^2 - (a-b)^2}}{c}$$

$$\therefore \tan \theta = \frac{\sqrt{c^2 - (a-b)^2}}{a-b}$$

$$\begin{aligned} \therefore (a-b)\tan \theta &= \sqrt{(c+a-b)(c-a+b)} \\ &= \sqrt{4(s-b)(s-a)} \\ &= 2\sqrt{\frac{(s-b)(s-a)}{ba}} \cdot \sqrt{ba} \\ &= 2\sqrt{ba} \sin \frac{C}{2} \end{aligned}$$

$$\therefore \sin \frac{C}{2} = \frac{(a-b)\tan \theta}{2\sqrt{ab}}$$

$$\begin{aligned} 49. \quad \Pi \sin \frac{A}{2} &= \frac{(s-a)(s-b)(s-c)}{abc} \\ &= \frac{(-a+b+c)(a-b+c)(a+b-c)}{8abc} \end{aligned}$$

從代數不等式  $(x+y)(y+z)(z+x) > 8xyz$

設  $x = -a+b+c$ ,  $y = a-b+c$ ,  $z = a+b-c$

則  $x+y=2c$ ,  $y+z=2a$ ,  $z+x=2b$

$\therefore 8abc > 8(-a+b+c)(a-b+c)(a+b-c)$

故  $\frac{(-a+b+c)(a-b+c)(a+b-c)}{8abc} < \frac{1}{8}$

即  $\Pi \sin \frac{A}{2} < \frac{1}{8}$

$$\begin{aligned} 50. \quad \Sigma \sin 2A &= 2 \sin A \cos A + 2 \sin(B+C) \cos(B-C) \\ &= 2 \sin A [\cos(B-C) - \cos(B+C)] \\ &= 4 \Pi \sin A \end{aligned} \quad (1)$$

設  $\sin^{-1} \alpha = A$ ,  $\sin^{-1} \beta = B$ ,  $\sin^{-1} \gamma = C$

$$\therefore \sin A = \alpha, \quad \sin B = \beta, \quad \sin C = \gamma,$$

$$\cos A = \sqrt{1 - \alpha^2}, \quad \cos B = \sqrt{1 - \beta^2}, \quad \cos C = \sqrt{1 - \gamma^2}$$

$$\text{今 } A + B + C = \pi \quad \therefore \sin^{-1} \alpha + \sin^{-1} \beta + \sin^{-1} \gamma = \pi$$

$$\text{從(1), 知 } \Sigma \sin A \cos A = 2\Pi \sin A$$

$$\therefore \Sigma \alpha \sqrt{1 - \alpha^2} = 2\alpha\beta\gamma$$

### 習題二十三 (307—317 頁)

1. 設湖長 =  $AB = x$  里,  $C$  為測點

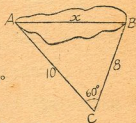
$$\text{則 } \angle ACB = 60^\circ,$$

$$CA = 10 \text{ 里}, \quad CB = 8 \text{ 里}$$

$$\therefore x^2 = 10^2 + 8^2 - 2 \cdot 10 \cdot 8 \cos 60^\circ$$

$$= 164 - 80 = 84$$

$$\therefore x = \sqrt{84} = 9.165 \text{ 里} \quad \text{故湖長 9.165 里.}$$



2. 設  $B, C$  為兩燈塔,  $x$  理為  $BC$

$$\text{之距離, 則 } AO = 5 \text{ 理}$$

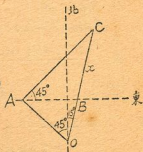
$$\text{今 } x = OC - OB$$

$$\text{又 } OC = \frac{5}{\cos AOC} = \frac{5}{\cos 60^\circ} = 10$$

$$OB = 5 \cdot \frac{\sin 45^\circ}{\sin 75^\circ} = 5 \times \frac{1}{\sqrt{2}}$$

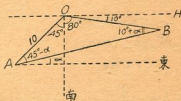
$$\times \frac{2\sqrt{2}}{\sqrt{3} + 1} = 5(\sqrt{3} - 1)$$

$$\therefore x = 10 - 5\sqrt{3} + 5 = 5(3 - \sqrt{3}) = 6.3395 \text{ 理}$$



3. 設  $O$  為港口,  $OB$  為敵船行程,  $B$  為敵船被捕之處,  $A$  為艦所在處, 設  $x$  哩為兵艦每時之速度, 其方向為東  $\alpha^\circ$  北, 今於  $3/2$  小時追及, 則

$$AB = \frac{3x}{2} \text{ 哩}, \quad OB = \frac{3}{2} \cdot 9 = 27/2 \text{ 哩}, \quad AO = 10 \text{ 哩}$$



$$\therefore \angle AOB = 45^\circ + 80^\circ = 125^\circ$$

$$\therefore \overline{AB}^2 = \overline{AO}^2 + \overline{OB}^2 - 2\overline{AO} \cdot \overline{OB} \cos 125^\circ$$

$$\text{即 } \frac{9}{4}x^2 = 100 + \left(\frac{27}{2}\right)^2 + 270 \cos 55^\circ$$

$$x^2 = \frac{1}{9}(400 + 27^2 + 1080 \sin 35^\circ)$$

$$= \frac{1}{9}(1129 + 619.488) = 194.3$$

$$\therefore x = \sqrt{194.3} = 13.9$$

$$\text{又 } \frac{AO}{\sin(10^\circ + \alpha)} = \frac{AB}{\sin 125^\circ} = \frac{3}{2}(13.9) \frac{1}{.8192}$$

$$\therefore \sin(10^\circ + \alpha) = \frac{2}{3} \left( \frac{.8192}{13.9} \right) 10 = 0.3918$$

$$\therefore 10^\circ + \alpha = 23^\circ 4' \quad \text{即 } \alpha = 13^\circ 4'$$

故速度為 13.9 哩/時, 其方向為東  $13^\circ 4'$  北(或北  $76^\circ 56'$  東)。

4. 設塔高  $AB$  爲  $h$  尺, 又  $C, D$  爲兩測點

在  $\triangle CBA$  中,  $AC = h \cot 45^\circ = h$

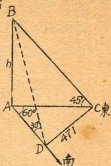
在  $\triangle BAD$  中,  $AD = h \cot 30^\circ$   
 $= \sqrt{3}h$

$$\text{今 } \overline{CD}^2 = \overline{AD}^2 + \overline{AC}^2 - 2\overline{AD} \cdot \overline{AC} \cos 60^\circ$$

$$\text{即 } 471^2 = 3h^2 + h^2 - \sqrt{3}h^2$$

$$= (4 - \sqrt{3})h^2$$

$$\text{故所求塔高 } h = \frac{471}{\sqrt{4 - \sqrt{3}}} = \frac{471}{1.5} = 314 \text{ 尺}$$



5. 設測點之高  $DA = x$  尺

山高  $EB = x + l = 6400$  尺

$$AC = .5(50000) = 25000 \text{ 尺}$$

$$l = 25000 \tan 13^\circ$$

$$= 25000(.2309)$$

$$= 5772 \text{ 尺}$$

$$\therefore x = 6400 - 5772 = 628 \text{ 尺}$$

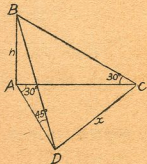
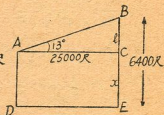
6. 設山高  $AB$  爲  $h$ ;  $x$  爲  $C, D$  兩點之距離, 則

$$\angle BCA = 30^\circ, \angle BDA = 45^\circ,$$

$$\text{及 } \angle CAD = 30^\circ$$

$$\therefore AD = h \tan 45^\circ = h,$$

$$AC = h\sqrt{3}$$



$$\begin{aligned}\therefore x^2 &= h^2 + 3h^2 - 2\sqrt{3}h^2 \cos 30^\circ \\ &= 4h^2 - 2\sqrt{3}h^2 \cdot \frac{\sqrt{3}}{2} = h^2\end{aligned}$$

$$\therefore x = h \quad \text{即 } AB = CD$$

7. 今  $\angle ACB = 15^\circ$ ,  $AB = 1$  里

設山高  $DC = h$  里

則在  $\triangle ABC$  中

$$AC = \frac{AB \sin 120^\circ}{\sin 15^\circ} = \frac{\frac{1}{2}\sqrt{3}}{\sin 15^\circ}$$

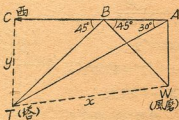


$$\text{又 } h = AC \sin 15^\circ = \frac{1}{2}\sqrt{3} = 0.866 \text{ 里}$$

8. 設塔與風磨之距離  $TW = x$  里, 塔與路之距離  $TC = y$  里

今  $\angle ABT = 135^\circ$ ,  $\angle ATB = 15^\circ$ ,  $AB = 1$  里

$$\text{在 } \triangle TAB \text{ 中, } \frac{BT}{\sin 30^\circ} = \frac{AB}{\sin 15^\circ}$$



$$\therefore BT = \frac{1}{2 \sin 15^\circ} = \frac{1}{.5176} = 1.9319$$

故塔與路之距  $y = BT \sin 45^\circ = 1.9319 \times .7071 = 1.37$  里

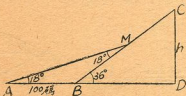
$$\text{又 } \angle TBW = 180^\circ - 45^\circ - 45^\circ = 90^\circ$$



$$\begin{aligned} \text{故塔與風磨之距 } x &= \sqrt{(BT)^2 + (WB)^2} \\ &= \sqrt{(1.932)^2 + (\sqrt{2})^2} = 2.39 \text{ 里} \end{aligned}$$

9. 設山高  $DC = h$  碼

$$\text{今 } MC = MB, \quad \angle AMB = 36^\circ - 18^\circ = 18^\circ = \angle MAB$$



$$\therefore BM = AB = 100 \text{ 碼}, \quad \text{則 } BC = 200 \text{ 碼}$$

$$\text{故 } h = BC \sin 36^\circ = 200 \times .5878 = 117.56$$

故所求山高為 117.6 碼。

10. 今  $\angle ASC = 135^\circ$ ,  $AS = 1000$  尺

設  $BC = h$  尺

$$\therefore \frac{AC}{\sin 135^\circ} = \frac{AS}{\sin 30^\circ}$$

$$\therefore AC = 1000\sqrt{2}$$

$$\therefore h = AC \sin 47^\circ = 1000\sqrt{2}$$

$$\times .781 = 1034 \text{ 尺}$$

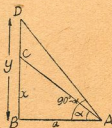
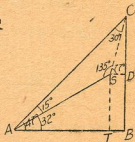
(參考 290 頁例十一)

11. 設塔高  $BC = x$  尺,

桿頂高  $BD = y$  尺

$$\text{則 } x = a \tan \alpha,$$

$$y = a \tan(90^\circ - \alpha) = \cot \alpha$$



$$\begin{aligned} \text{故桿長} &= y - x = a(\cot \alpha - \tan \alpha) = \frac{a(\cot \alpha - 1)}{\cot \alpha} \\ &= 2a \cot 2\alpha \text{ 尺} \end{aligned}$$

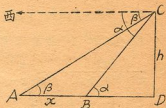
12. 設  $C$  爲峭壁之頂，又

$$AB = x$$

$$\text{則 } x = AD - BD$$

$$= h(\cot \beta - \cot \alpha)$$

$$\begin{aligned} \text{即 } x &= h \left( \frac{\cos \beta}{\sin \beta} - \frac{\cos \alpha}{\sin \alpha} \right) \\ &= h \sin(\alpha - \beta) \csc \alpha \csc \beta \end{aligned}$$

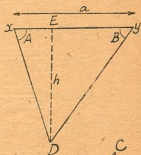


13. 今  $a = xE + Ey$

$$= h(\cot A + \cot B)$$

$$\therefore h = \frac{a}{\cot A + \cot B}$$

$$\begin{aligned} \text{或 } h &= a / \left( \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} \right) \\ &= \frac{a \sin A \sin B}{\sin(A+B)} \end{aligned}$$



14. 設塔高  $DC$  爲  $h$  尺

$$\text{今 } AD = 100 \text{ 尺}, BD = 100\sqrt{2} \text{ 尺}$$

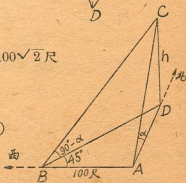
$$\text{又設 } \angle DAC = \alpha$$

$$\text{則 } \angle DBC = 90^\circ - \alpha$$

$$\text{今 } h = BD \tan(90^\circ - \alpha)$$

$$= 100\sqrt{2} \cot \alpha$$

$$\text{又 } h = 100 \tan \alpha$$



兩式相乘得  $h^2 = 10000\sqrt{2}$

$$\therefore h = 100\sqrt[4]{2} = 119.1 \text{ 尺}$$

15. 如圖  $BA$  爲窗高,  $DC = h$  爲桿高

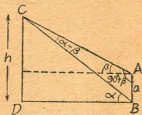
今  $\angle CAE = 90^\circ + \beta$

在  $\triangle ABC$  中(參考 49 頁例四)

$$\frac{BC}{\sin(90^\circ + \beta)} = \frac{a}{\sin(\alpha - \beta)}$$

$$\therefore BC = \frac{a \cos \beta}{\sin(\alpha - \beta)}$$

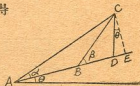
$$\text{故 } h = \overline{BC} \sin \alpha = \frac{a \sin \alpha \cos \beta}{\sin(\alpha - \beta)}$$



16. 設  $CD = h$ , 則由 288 頁例七得

$$\cos \theta = \frac{a \sin \alpha \sin \beta}{h \sin(\beta - \alpha)}$$

$$\therefore \theta = \cos^{-1} \frac{a \sin \alpha \sin \beta}{h \sin(\beta - \alpha)}$$



17. 設汽球高  $DC$  爲  $h$

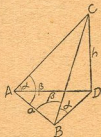
今  $\angle CAD = \angle CBD = \alpha$

$$\therefore AC = BC (= h \csc \alpha)$$

故  $\triangle ACB$  爲等腰三角形

$$\therefore AC = \frac{a}{2} \sec \beta$$

$$\text{今 } h = AC \sin \alpha = \frac{a}{2} \sin \alpha \sec \beta$$



18. 今各邊於  $P$  所張角  $APB = BPC = CPA = 120^\circ$

故兩鈍角  $\triangle AFC, BPC$  全相等 (SSA)

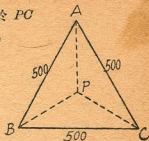
故  $PA=PB$ ，同理可證亦等於  $PC$

設此值為  $x$  尺，則在  $\triangle PAC$  中

$$\overline{AC}^2 = x^2 + x^2 - 2x^2 \cos 120^\circ$$

$$\text{即 } 3x^2 = 500^2$$

$$\therefore x = \frac{500}{3} \sqrt{3} = 288.67 \text{ 尺}$$



19. 設  $OA$  代 80 磅之力， $OB$  代 50 磅

則  $\square OABC$  中

$$\angle O = 120^\circ, \quad \angle A = 60^\circ$$

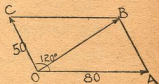
$OB$  表其合力之大小

則在  $\triangle OAB$  中

$$\overline{OB}^2 = \overline{OA}^2 + \overline{OC}^2 - 2\overline{OA} \cdot \overline{OC} \cos 60^\circ$$

$$= 6400 + 2500 - 2 \times 80 \times 50 \times \frac{1}{2} = 4900$$

$$\therefore OB = 70 \text{ (合力為 70 磅)} \quad \text{【參考 286 頁例四】}$$



20. 如圖  $\angle CBD = 120^\circ - 45^\circ = 75^\circ$ ,

$$\angle ACB = 15^\circ$$

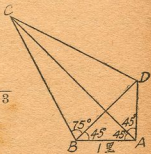
$$\therefore BC = \frac{AB \sin 45^\circ}{\sin 15^\circ}$$

$$= 3 - 4 \sin^2 15^\circ$$

$$= 1 + 2 \cos 30^\circ = 1 + \sqrt{3}$$

$$\text{又 } BD = \frac{\sin 90^\circ}{\sin 45^\circ} = \sqrt{2}$$

$$\text{今 } \overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 - 2\overline{BC} \cdot \overline{BD} \cos 75^\circ$$



$$= (1 + \sqrt{3})^2 + (\sqrt{2})^2 - 2(\sqrt{2}) \cdot \frac{\sin 45^\circ}{\sin 15^\circ} \cdot \sin 15^\circ$$

$$= (1 + \sqrt{3})^2 + 2 - 2 = (1 + \sqrt{3})^2$$

$$\therefore CD = 1 + \sqrt{3} = 2.7 \text{ 里}$$

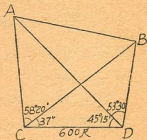
21. 今  $\angle CAD = 39^\circ 25'$

在  $\triangle ACD$ ,  $\triangle BCD$  中

$$AC = \frac{600 \sin 45^\circ 15'}{\sin 39^\circ 25'} = 671.1$$

(此應用對數做)

$$BC = \frac{600 \sin 98^\circ 45'}{\sin 44^\circ 15'} = 849.8$$



在  $\triangle ABC$  中今兩邊一夾角已知, 可用正切定律

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B) = \frac{178.7}{1520.9} \tan 60^\circ 50'$$

$$A - B = 23^\circ 46' \quad \therefore A = 72^\circ 43', \quad B = 48^\circ 57'$$

$$\text{再 } AB = \frac{AC \sin 58^\circ 20'}{\sin 48^\circ 57'} = \dots = 757.5 \text{ 尺}$$

22. 今  $AB = 500$ ,  $DA = EB = 100$ ,

$$DB = 560, \quad EA = 550$$

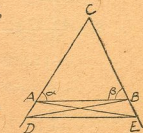
於  $\triangle ABD$  中

$$560^2 = 100^2 + 500^2$$

$$- 2 \cdot 100 \cdot 500 \cos(\pi - \alpha)$$

$$\therefore 1000 \cos \alpha = 536$$

$$\therefore \cos \alpha = 0.536$$



$$\text{故 } \alpha = 57^\circ 35'$$

$$\text{又 } 550^2 = 500^2 + 100^2 - 2 \cdot 500 \cdot 100 \cos(\pi - \beta)$$

$$\therefore \cos \beta = .425 \quad \text{故 } \beta = 64^\circ 51', \quad C = 57^\circ 34'$$

$$\text{於 } \triangle ABC \text{ 中, } \frac{AB}{\sin C} = \frac{AC}{\sin B}$$

$$\therefore AC = 500 \left( \frac{\sin 64^\circ 51'}{\sin 57^\circ 34'} \right) = 536.3$$

23. 如圖  $EA = 2a$  爲底之對角線之延長線

$$\text{今 } YE = AE \tan 45^\circ = AE = 2a$$

$$\text{即 } XD = ZF = 2a$$

$$AD = XD \cot 30^\circ = 2\sqrt{3}a$$

$$\text{又 } \angle DEA = 135^\circ$$

故在  $\triangle ADE$  中

$$\begin{aligned} \overline{AD}^2 &= \overline{DE}^2 + \overline{EA}^2 - 2\overline{DE} \\ &\quad \times \overline{EA} \cos 135^\circ \end{aligned}$$

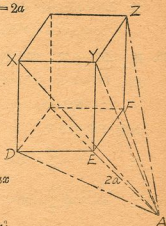
$$\text{即 } 12a^2 = x^2 + 4a^2 + 2\sqrt{2}ax$$

$$\text{(設 } DE = x)$$

$$\text{即 } x^2 + 2\sqrt{2}ax + 2a^2 = 10a^2$$

$$\text{即 } (x + a\sqrt{2})^2 = 10a^2 \quad \therefore x + a\sqrt{2} = a\sqrt{10}$$

$$\therefore x = a(\sqrt{10} - \sqrt{2}) \quad \text{(負根不合)}$$



24. 設峯高  $MP = h$ , 今從幾何知  $AM = BM$ , 則中線  $MC$

垂直於  $AB$ , 故

$$\overline{AM}^2 - \overline{MC}^2 = \overline{AC}^2 = a^2 \quad (1)$$

$$\text{但 } AM = h \cot \theta, \quad MC = h \cot \phi$$

$$\text{故 } h^2(\cot^2\theta - \cot^2\phi) = a^2$$

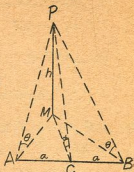
$$\therefore h^2 = a^2 \left( \frac{\cos^2\theta}{\sin^2\theta} - \frac{\cos^2\phi}{\sin^2\phi} \right)$$

$$= \frac{a^2 \sin^2\theta \sin^2\phi}{\cos^2\theta \sin^2\phi - \cos^2\phi \sin^2\theta}$$

$$= \frac{a^2 \sin^2\theta \sin^2\phi}{\sin(\theta + \phi) \sin(\phi - \theta)}$$

$$\text{故 } h = a \sin\theta \sin\phi$$

$$\times \sqrt{\csc(\phi + \theta) \csc(\phi - \theta)}$$



25. 設  $AC = y$  尺, 烟囪高  $AB = CD = x$  尺

$$\text{則 } PC = x \cot 60^\circ = x / \sqrt{3}$$

$$\therefore EC = x$$

如圖  $PE = 80$  尺

$$\text{故 } x^2 = \frac{x^2}{3} + 80^2$$

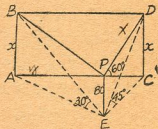
$$\therefore x = 80 \sqrt{\frac{3}{2}} = 40\sqrt{6} \text{ 尺}$$

$$\text{則 } PC = 80 \sqrt{\frac{3}{2}} \left( \frac{1}{\sqrt{3}} \right) = 40\sqrt{2} = 56.57 \text{ 尺}$$

$$\text{又 } AE = x \cot 30^\circ = 120\sqrt{2}$$

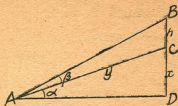
$$\therefore AP = \sqrt{AE^2 - PE^2} = 40\sqrt{14} = 149.64 \text{ 尺}$$

$$\text{故 } y = AC = AP + PC = 149.64 + 56.57 = 206.2 \text{ 尺}$$



26. 設塔高  $CD$  為  $x$  尺, 旗桿  $CB$  長  $h$  尺

又設  $AG = y$ , 今  $\angle ACD = 90^\circ - \alpha$ ,  $\angle B = 90^\circ - \alpha - \beta$



$$\therefore \frac{h}{\sin \beta} = \frac{y}{\sin[90^\circ - (\alpha + \beta)]}$$

$$\therefore y = h \csc \beta \cos(\alpha + \beta)$$

故  $x = y \sin \alpha = h \sin \alpha \csc \beta \cos(\alpha + \beta)$

27. 設  $EF = x$  尺

今塔高為  $DC$

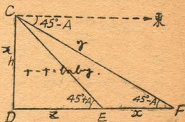
$$\therefore DF = h \cot(45^\circ - A)$$

$$DE = h \cot(45^\circ + A)$$

$$\therefore x = DF - DE$$

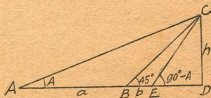
$$= h \left[ \frac{\cos(45^\circ - A)}{\sin(45^\circ - A)} - \frac{\cos(45^\circ + A)}{\sin(45^\circ + A)} \right]$$

$$= \frac{2h \sin 2A}{\cos 2A - \cos 90^\circ} = 2h \tan 2A \text{ 尺}$$



28. 如圖  $ED = BD - BE = h - b$

又  $ED = h \cot(90^\circ - A) = h \tan A$





$$\therefore \tan A = (h-b)/h \quad (1)$$

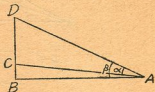
但  $\tan A = h/AD = h/(a+h) \quad (2)$

故  $(h-b)/h = h/(a+h) \quad \therefore h = ab/(a-b)$  尺

29. 今  $CD = \frac{3}{4}BD$

$\therefore BC = \frac{1}{4}BD$

如圖  $\tan \alpha = 0.6$ ,



$$AB = BC \cot(\beta - \alpha) = BD \cot \beta$$

$$\therefore \tan \beta = \frac{BD}{BC} \tan(\beta - \alpha), \quad \text{即} \quad \frac{1}{4} \tan \beta = \frac{\tan \beta - 0.6}{1 + 0.6 \tan \beta}$$

即  $\tan^2 \beta - 5 \tan \beta + 4 = 0$

即  $(\tan \beta - 1)(\tan \beta - 4) = 0 \quad \therefore \tan \beta = 1, 4$

30. 設  $CA = x$  尺,  $CB = y$  尺,

塔高  $CD = h$  尺

因  $\frac{h}{6} = \frac{x+24}{24}$

$\therefore x = 4h - 24$

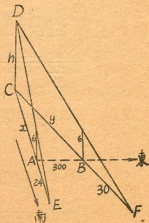
又  $\frac{h}{6} = \frac{y+30}{30}$

$\therefore y = 5h - 30$

惟  $x^2 + 300^2 = y^2$

( $\because \angle CAB = 90^\circ$ )

代入得  $16(h-6)^2 + 300^2$   
 $= 25(h-6)^2$



$$\text{即 } h^2 - 12h - 9964 = 0 \quad \text{即 } (h - 106)(h + 94) = 0$$

$$\therefore h = 106 \quad (\text{負值不合})$$

故燈塔之高度為 106 尺。

31. 如圖塔高  $OT$  為  $h$  尺，池半徑為  $r$  尺， $BN = 45$  尺，

$AE = 120$  尺，則

$$r = h \cot 60^\circ = h/\sqrt{3}$$

今  $\angle NOE = 90^\circ$

$$\therefore \overline{ON}^2 + \overline{OE}^2 = \overline{NE}^2$$

$$\text{即 } \left(45 + \frac{h}{\sqrt{3}}\right)^2$$

$$+ \left(120 + \frac{h}{\sqrt{3}}\right)^2 = 375^2$$

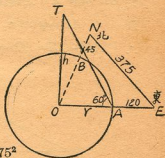
$$\text{即 } \frac{2h^2}{3} + \frac{330\sqrt{3}h}{3} = 124200$$

$$\therefore h^2 + 165\sqrt{3}h - 186300 = 0$$

$$\text{即 } (h - 180\sqrt{3})(h + 345\sqrt{3}) = 0$$

$$\therefore h = 180\sqrt{3} \text{ 尺} \quad \therefore r = 180 \text{ 尺}$$

故圓池直徑為 360 尺，塔高  $180\sqrt{3}$  尺。

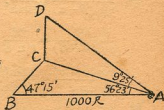


32. 設塔高  $CD$  為  $x$  尺

今  $\angle ACB = 76^\circ 22'$

$$\therefore \frac{1000}{\sin 76^\circ 22'} = \frac{AC}{\sin 47^\circ 15'}$$

$$\therefore AC = \frac{1000 \sin 47^\circ 15'}{\sin 76^\circ 22'}$$



$$\text{又 } h = AC \tan 9^\circ 25' = \frac{1000 \times .7343 \times .1658}{.9718} = 125.3 \text{ 尺}$$

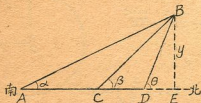
〔註〕 此題普通應用對數做

33.  $DB$  爲塔，傾斜向北  $\theta$  角， $CD = b$ ， $AD = a$

作  $BE$  垂直  $AD$  之延長線，令  $EB = y$

則  $AE = y \cot \alpha$ ，  $CE = y \cot \beta$

$$\therefore AE - CE = AC = a - b = y(\cot \alpha - \cot \beta)$$



$$\text{又 } DE = y \cot \theta \quad \therefore CE - DE = b = y(\cot \beta - \cot \theta)$$

$$\therefore \frac{a - b}{b} = \frac{\cot \alpha - \cot \beta}{\cot \beta - \cot \theta}$$

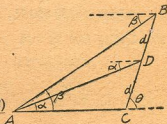
$$\therefore \cot \theta = \frac{b \cot \alpha - a \cot \beta}{b - a} \quad \text{故如題云。}$$

34. 設所求之傾斜角爲  $\theta$

$$\text{今 } \frac{AC}{\sin(\theta - \alpha)} = \frac{d}{\sin \alpha}$$

$$\frac{AC}{\sin(\theta - \beta)} = \frac{2d}{\sin \beta}$$

$$\therefore \frac{\sin(\theta - \alpha)}{\sin \alpha} = \frac{2 \sin(\theta - \beta)}{\sin \beta}$$



$$\text{即 } \sin \theta \cot \alpha - \cos \theta = 2 \sin \theta \cot \beta - 2 \cos \theta$$

$$\therefore \cot \theta = 2 \cot \beta - \cot \alpha$$

$$\therefore \theta = \cot^{-1}(2 \cot \beta - \cot \alpha)$$

35. 今  $C$  為測點,  $AC = h$  尺

作  $CF \parallel AB$

設雲高  $BD = x$  尺

則  $FD = x - h$

又以  $E$  為  $D$  在水中之影

則  $BE = x$ ,  $EF = x + h$

今  $AB = CF = FD \cot \alpha$ ,  $CF = EF \cot \beta$

故  $FD \cot \alpha = EF \cot \beta$

即  $(x - h) \cot \alpha = (x + h) \cot \beta$

$$\therefore x(\cot \alpha - \cot \beta) = h(\cot \alpha + \cot \beta)$$

$$\therefore x = \frac{h(\cos \alpha \sin \beta + \cos \beta \sin \alpha)}{\cos \alpha \sin \beta - \cos \beta \sin \alpha} = \frac{h \sin(\alpha + \beta)}{\sin(\beta - \alpha)}$$

故白雲離地之高為  $\frac{h \sin(\alpha + \beta)}{\sin(\beta - \alpha)}$ .

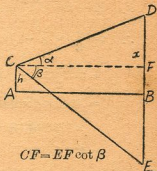
36. 設上升速為每分鐘  $v$  哩, 則兩分鐘內上升  $GF = 2v$  哩

又經水平距離  $EG = 2 \cdot 80/60 = \frac{8}{3}$  哩

今因  $\angle PE'F' = 90^\circ$ ,  $\angle E'PF' = 45^\circ$

則  $PE' = E'F' = \frac{8}{3}$  哩,  $PF' = \frac{8}{3} \sqrt{2}$  哩

今  $\tan 8^\circ = \frac{h}{PE'}$   $\therefore h = \frac{8}{3} \tan 8^\circ$





38. 作  $PQ \perp ABC$

且令  $PQ = x$

又令  $\angle BPQ = \theta$

及  $\tan \theta = t$

今  $AB = BC = 1$  里,

$\angle CPQ = 30^\circ + \theta$ ,

$\angle APQ = 45^\circ - \theta$

故 
$$\begin{cases} AB = QB + AQ = x \tan \theta + x \tan(45^\circ - \theta) \\ BC = QC - QB = x \tan(30^\circ + \theta) - x \tan \theta \end{cases}$$

即  $1 = x \left( t + \frac{1-t}{1+t} \right) = \left( \frac{1+t^2}{1+t} \right) x$  (1)

$1 = x \left( \frac{1+t\sqrt{3}}{\sqrt{3}-t} - t \right) = \left( \frac{1+t^2}{\sqrt{3}-t} \right) x$  (2)

從(1),(2)得  $1+t = \sqrt{3}-t \therefore t = \frac{1}{2}(\sqrt{3}-1)$  (3)

代入(1),得  $x = \frac{1+t}{1+t^2} = \frac{1+\sqrt{3}}{4-\sqrt{3}} = \frac{7+5\sqrt{3}}{13}$

39.  $\therefore \alpha + \beta + \gamma = 180^\circ$

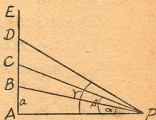
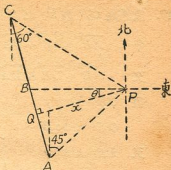
$\therefore \Sigma \tan \alpha = \Pi \tan \alpha$

(公式 50 C)

今  $\tan \alpha = \frac{a}{x}$ ,  $\tan \beta = \frac{b}{x}$ ,

$\tan \gamma = \frac{c}{x}$

$\therefore \frac{a+b+c}{x} = \frac{abc}{x^3} \quad x^2(a+b+c) = abc$



40. 設塔高  $CD$  爲  $h$ ,  $\angle ADB = \theta$

$$\text{今 } \overline{AB}^2 = \overline{CB}^2 + \overline{CA}^2 - 2\overline{CB} \cdot \overline{CA} \cos \alpha$$

$$\text{又 } \overline{AB}^2 = \overline{DB}^2 + \overline{DA}^2 - 2\overline{DB} \cdot \overline{DA} \cos \theta$$

$$\text{但 } CB = h \csc \gamma$$

$$CA = h \csc \beta$$

$$\text{又 } DB = h \cot \gamma$$

$$DA = h \cot \beta$$

$$\text{故 } h^2(\csc^2 \gamma + \csc^2 \beta)$$

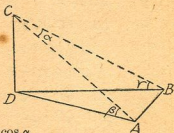
$$- 2h^2 \csc \gamma \csc \beta \cos \alpha$$

$$= h^2(\cot^2 \beta + \cot^2 \gamma) - 2h^2 \cot \beta \cot \gamma \cos \theta$$

$$\text{即 } 1 - \csc \beta \csc \gamma \cos \alpha = -\cot \beta \cot \gamma \cos \theta$$

$$\text{即 } \sin \beta \sin \gamma - \cos \alpha = -\cos \beta \cos \gamma \cos \theta$$

$$\text{故 } \cos \theta = (\cos \alpha - \sin \beta \sin \gamma) / \cos \beta \cos \gamma$$



41. 設山高  $DP$  爲  $h$  尺, 聯  $AC$

在  $\triangle ABC$  中,

$$AB = BC = 800 \text{ 尺}$$

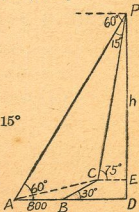
$$\text{故 } \angle CAB = \angle BCA = 15^\circ$$

$$\therefore AC = \frac{800 \sin 30^\circ}{\sin 15^\circ} = 400 \csc 15^\circ$$

在  $\triangle ACP$  中,

$$\angle CAP = 45^\circ, \angle APC = 15^\circ,$$

$$\angle ACP = 120^\circ$$



$$\therefore AP = \frac{AC \sin 120^\circ}{\sin 15^\circ} = 400 \sin 60^\circ \csc^2 15^\circ$$

$$\text{今 } h = AP \sin 60^\circ = 400 \times 3 \times 3.7321 = 4478.5 \text{ 尺}$$

42. 設  $QP = x$  碼, 又坡之斜度為  $\theta$

$$\text{在 } \triangle OAQ \text{ 中, } \frac{320}{\sin 25^\circ} = \frac{500}{\sin(\theta - 25^\circ)}$$

$$\therefore \sin(\theta - 25^\circ) = .6603$$

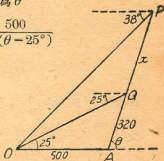
$$\therefore \theta - 25^\circ = 41^\circ 19' 33''$$

$$\text{故 } \theta = 66^\circ 19' 33''$$

又在  $\triangle OAP$  中,

$$\frac{x + 320}{\sin 38^\circ} = \frac{500}{\sin(\theta - 38^\circ)}$$

$$\therefore x + 320 = \frac{500 \times .6157}{.4748} = 649 \quad \therefore x = 329 \text{ 碼}$$



43. 設塔高  $DC$  為  $x$  尺, 又  $\angle CAD = \theta$

$$\text{則 } \tan \theta = x/b$$

$$\text{今 } \frac{x+a}{b} = \tan(\theta + \alpha) = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}$$

$$= \frac{x + b \tan \alpha}{b - x \tan \alpha}$$

$$\text{故 } x^2 \tan \alpha + ax \tan \alpha + b^2 \tan \alpha - ab = 0$$

$$\text{即 } x^2 + ax + b^2 - ab \cot \alpha = 0$$

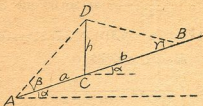
$$\therefore x = \frac{1}{2}(-a + \sqrt{a^2 + 4ab \cot \alpha - 4b^2}) \quad (\text{負值不合})$$



44. 設塔高  $CD$  為  $h$ , 則在  $\triangle ABD$  中

$$\frac{a+b}{\sin[\pi - (\beta + \gamma)]} = \frac{AD}{\sin \gamma}$$





$$\therefore AD = \frac{(a+b)\sin\gamma}{\sin(\beta+\gamma)}$$

且在  $\triangle ADC$  中,  $\angle ACD = 90^\circ + \alpha$

故 
$$\frac{AD}{\sin(90^\circ + \alpha)} = \frac{h}{\sin\beta}$$

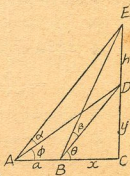
$$\therefore h = \frac{AD \sin\beta}{\cos\alpha} = \frac{(a+b)\sin\beta \sin\gamma}{\cos\alpha \sin(\beta+\gamma)} \quad (\text{塔高})$$

45. 設所求距離  $BC$  爲  $x$  尺, 柱高  $CD$  爲  $y$  尺

又設  $\angle DBC = \theta$ ,  $\angle DAC = \phi$

則  $\tan\theta = \frac{y}{x}$ ,  $\tan\phi = \frac{y}{x+a}$

故 
$$\begin{cases} \frac{y+h}{x} = \tan(\theta + \beta) = \frac{\frac{y}{x} + \tan\beta}{1 - \frac{y}{x}\tan\beta} \\ \frac{y+h}{x+a} = \tan(\phi + \alpha) = \frac{\frac{y}{x+a} + \tan\alpha}{1 - \frac{y}{x+a}\tan\alpha} \end{cases}$$



即 
$$\begin{cases} (y+h)(x - y \tan\beta) = x(y + x \tan\beta) \\ (y+h)(a + x - y \tan\alpha) = (x+a)(y + (a+x)\tan\alpha) \end{cases}$$

$$\text{即 } \begin{cases} x^2 + y^2 - hx \cot \beta + hy = 0 \\ x^2 + y^2 + (2a - h \cot \alpha)x + hy = ah \cot \alpha - a^2 \end{cases}$$

相減得  $x(h \cot \alpha - 2a - h \cot \beta) = a^2 - ah \cot \alpha$

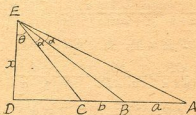
$$\therefore x = (a^2 - ah \cot \alpha) / (h \cot \alpha - 2a - h \cot \beta)$$

46. 如圖設塔高  $DE = x$

則因  $EC = x / \cos \theta$ ,  $EA = x / \cos(\theta + 2\alpha)$

但從幾何知  $EC : b = EA : a$

$$\text{即 } b \cos \theta : x = a \cos(\theta + 2\alpha) : x$$



$$\therefore b \cos \theta = a(\cos \theta \cos 2\alpha - \sin \theta \sin 2\alpha)$$

$$\therefore \tan \theta = (a \cos 2\alpha - b) / a \sin 2\alpha$$

$$\therefore \sin \theta = \frac{a \cos 2\alpha - b}{\sqrt{a^2 + b^2 - 2ab \cos 2\alpha}}$$

$$\cos \theta = \frac{a \sin 2\alpha}{\sqrt{a^2 + b^2 - 2ab \cos 2\alpha}}$$

$$\text{今 } \frac{b}{\sin \alpha} = \frac{EC}{\sin(90^\circ - \theta - \alpha)} = \frac{EC}{\cos(\theta + \alpha)}$$

$$\therefore EC = \frac{b}{\sin \alpha} (\cos \theta \cos \alpha - \sin \theta \sin \alpha),$$

$$DE = \frac{b}{\sin \alpha} \left[ \cos \alpha \cdot \frac{a^2 \sin^2 2\alpha}{a^2 + b^2 - 2ab \cos 2\alpha} - \frac{a \sin \alpha \sin 2\alpha (a \cos 2\alpha - b)}{a^2 + b^2 - 2ab \cos 2\alpha} \right]$$

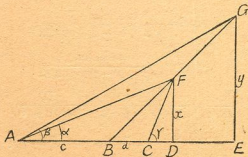
$$\begin{aligned} \text{即 } x &= \frac{b}{\sin \alpha} \left[ \frac{ab \sin \alpha \sin 2\alpha + a^2 \sin 2\alpha (\sin \alpha)}{a^2 + b^2 - 2ab \cos 2\alpha} \right] \\ &= \frac{ab(a+b) \sin 2\alpha}{a^2 + b^2 - 2ab \cos 2\alpha} = \frac{2ab(a+b) \sin \alpha \cos \alpha}{a^2 + b^2 - 2ab(2 \cos^2 \alpha - 1)} \\ &= \frac{2ab(a+b) \sin \alpha \cos \alpha}{(a+b)^2 - 4ab \cos^2 \alpha} = \frac{2ab(a+b) \tan \alpha}{(a+b)^2 \sec^2 \alpha - 4ab} \\ &= \frac{2ab(a+b) \tan \alpha}{(a+b)^2 (1 + \tan^2 \alpha) - 4ab} = \frac{2ab(a+b) \tan \alpha}{(a-b)^2 + (a+b)^2 \tan^2 \alpha} \end{aligned}$$

47. 設低山  $DF$  高  $x$  里，高山  $EG$  高  $y$  里

$$\text{則 } \frac{CF}{\sin \alpha} = \frac{AC}{\sin AFC} = \frac{c+d}{\sin(\gamma-\alpha)}$$

$$\therefore CF = \frac{(c+d) \sin \alpha}{\sin(\gamma-\alpha)} \quad (\text{參考 } 287 \text{ 頁例六第一法})$$

$$\therefore x = CF \sin \gamma = \frac{(c+d) \sin \alpha \sin \gamma}{\sin(\gamma-\alpha)} \quad (\text{低山之高})$$



$$\begin{aligned} \text{又 } y:x &= BE:BD = (AE - AB):(AD - AB) \\ &= y \cot \beta - c : x \cot \alpha - c \end{aligned}$$

$$\begin{aligned} \therefore y &= \frac{cx}{c + x(\cot \beta - \cot \alpha)} \\ &= \frac{c(c+d)}{c(\cot \alpha - \cot \gamma) + (c+d)(\cot \beta - \cot \alpha)} \end{aligned}$$

$$\text{或 } y = \frac{c(c+d) \sin \alpha \sin \beta \sin \gamma}{c \sin \beta \sin(\gamma - \alpha) + (c+d) \sin \gamma \sin(\alpha - \beta)}$$

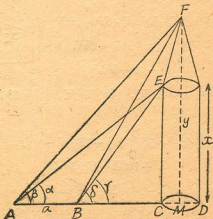
(高山之高)

48. 設塔高  $CE = x$  尺, 尖頂離地  $MF$  為  $y$  尺

$$\text{則 } a = AB = AM - BM = y \cot \beta - y \cot \delta$$

$$\begin{aligned} \therefore y &= a / (\cot \beta - \cot \delta) \\ &= a \sin \beta \sin \delta \csc(\delta - \beta) \quad (\text{尖頂高}) \end{aligned}$$

$$\text{又 } a = AC - BC = x \cot \alpha - x \cot \gamma$$



$$\begin{aligned} \therefore x &= a / (\cot \alpha - \cot \gamma) \\ &= a \sin \alpha \sin \gamma \csc(\gamma - \alpha) \quad (\text{塔高}) \end{aligned}$$

$$\begin{aligned} \text{塔之直徑} &= CD = 2CM = 2(BM - BC) \\ &= 2(y \cot \delta - x \cot \gamma) \\ &= 2a[\sin \beta \cos \delta \csc(\delta - \beta) \\ &\quad - \sin \alpha \cos \gamma \csc(\gamma - \alpha)] \end{aligned}$$

從  $2(AM - AC)$ , 直徑亦可為

$$2a[\sin \delta \cos \beta \csc(\delta - \beta) - \sin \gamma \cos \alpha \csc(\gamma - \alpha)]$$

49. 設  $AD = h$ , 今  $\angle EDC = \angle BDC$

$$\therefore DE : DB = CE : BC$$

$$\text{即 } \overline{DE}^2 : \overline{DB}^2 = \overline{CE}^2 : \overline{BC}^2$$

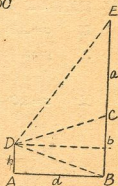
$$\text{但 } \overline{DE}^2 = d^2 + (a + b - h)^2,$$

$$\overline{DB}^2 = d^2 + h^2$$

$$\therefore d^2 + (a + b - h)^2 : d^2 + h^2 = a^2 : b^2$$

$$\text{即 } (a - b)d^2 = (a + b)b^2 - 2b^2h$$

$$- (a - b)h^2$$



50. 在上題中,  $a = 48$  尺,  $b = 30$  尺,  $d = 48$  尺

代入上式, 則

$$18(48)^2 = 78(900) - 1800h - 18h^2$$

$$\text{即 } h^2 + 100h - 1596 = 0$$

$$\text{即 } (h - 14)(h + 114) = 0$$

$$\therefore h = 14 \text{ 尺} \quad (\text{負值不合})$$

51. 設峭壁高  $y$  尺, 塔高  $x$  尺

因  $\angle CBD = \angle CAD = \beta$

故  $A, B, C, D$  在一圓周上

$\therefore EC \cdot ED = EB \cdot EA$

即  $y(x+y) = ab$  (1)

設  $\tan \phi = y/a$

則  $\tan(\phi + \beta) = (x+y)/a$

即  $\frac{y/a + \tan \beta}{1 - y \tan \beta/a} = \frac{x+y}{a} = \frac{b}{y}$

即  $y^2 + (a+b) \tan \beta \cdot y - ab = 0$  (2)

(1) - (2),  $xy = (a+b) \tan \beta \cdot y$

即  $x = (a+b) \tan \beta$

52. 在上題中設  $\tan \beta = 1/10$ ,  $AE = 11$ ,  $BE = 9$ , 山高為  $y$

從上題(2)  $y^2 + 2y - 99 = 0$

即  $(y-9)(y+11) = 0 \quad \therefore$  山高  $y = 9$

53. 參考 297 頁例十七, 知  $OP$  為圓  $APB$  之切線

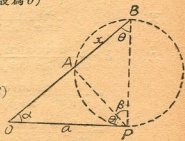
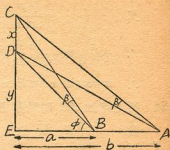
故知  $\angle B = \angle OPA$  (設為  $\theta$ )

今  $\theta + \beta + \theta + \alpha = \pi$

$\therefore \theta = \frac{\pi}{2} - \frac{1}{2}(\alpha + \beta)$

$\theta + \beta = \frac{\pi}{2} - \frac{1}{2}(\alpha - \beta)$

今  $\frac{x}{\sin \beta} = \frac{AP}{\sin \theta}$



$$\text{又} \quad \frac{AP}{\sin \alpha} = \frac{a}{\sin(\theta + \beta)}$$

$$\therefore \frac{x}{\sin \beta} = \frac{a \sin \alpha}{\sin \theta \sin(\theta + \beta)} = \frac{a \sin \alpha}{\cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)}$$

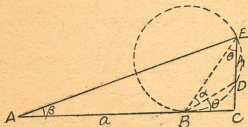
$$\therefore x = a \sin \alpha \sin \beta \sec \frac{1}{2}(\alpha + \beta) \sec \frac{1}{2}(\alpha - \beta)$$

54. 從上題知  $AB$  切於圓  $BDE$

設  $DE = h$ ;  $\angle DBC = \theta = \angle BED$

則  $\angle AEB = \theta + \alpha - \beta$ , 又  $\theta + \alpha = 90^\circ - \theta$

$$\therefore \frac{BE}{\sin \beta} = \frac{a}{\sin(\theta + \alpha - \beta)}$$



$$\therefore BE = \frac{a \sin \beta}{\sin(90^\circ - \theta - \beta)} = \frac{a \sin \beta}{\cos(\theta + \beta)}$$

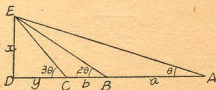
$$\text{又} \quad \frac{h}{\sin \alpha} = \frac{BE}{\sin(\theta + \alpha)}$$

$$\begin{aligned} \therefore h &= \frac{a \sin \alpha \sin \beta}{\sin(\theta + \alpha) \cos(\theta + \beta)} \\ &= \frac{2a \sin \alpha \sin \beta}{\sin(2\theta + \alpha + \beta) + \sin(\alpha - \beta)} \\ &= \frac{2a \sin \alpha \sin \beta}{\cos \beta + \sin(\alpha - \beta)} \end{aligned}$$

55. 設  $DE=x$ ,  $DC=y$ , 則  $\tan 3\theta = x/y$ ,  
 $\tan 2\theta = x/(b+y)$ ,  $\tan \theta = x/(a+b+y)$

今 
$$\tan 3\theta = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta}$$

$$\therefore \frac{x}{y} = \frac{\frac{x}{b+y} + \frac{x}{a+b+y}}{1 - \frac{x}{b+y} \cdot \frac{x}{a+b+y}} = \frac{x(a+b+y) + x(b+y)}{(b+y)(a+b+y) - x^2}$$



即  $(b+y)(a+b+y) - x^2 = (a+2b+2y)y$

即  $x^2 + y^2 - b(a+b) = 0$  (1)

又 
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\therefore \frac{x}{b+y} = \frac{\frac{2x}{a+b+y}}{1 - \frac{x^2}{(a+b+y)^2}} = \frac{2x(a+b+y)}{(a+b+y)^2 - x^2}$$

即  $(a+b+y)^2 - x^2 = 2(b+y)(a+b+y)$

即  $x^2 + y^2 + 2by + (a+b)(b-a) = 0$  (2)

(2) - (1),  $2by + (a+b)(2b-a) = 0$

$$y = \frac{(a+b)(a-2b)}{2b}$$



$$\begin{aligned}
 \text{代入(1)得 } x^2 &= b(a+b) - \frac{(a+b)^2(a-2b)^2}{4b^2} \\
 &= \frac{(a+b)(4b^3 - a^3 + 3a^2b - 4b^3)}{4b^2} \\
 &= \frac{a^2(a+b)(3b-a)}{4b^2} \\
 \therefore x &= \frac{a\sqrt{(a+b)(3b-a)}}{2b}
 \end{aligned}$$

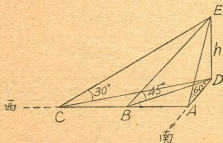
今如  $a=50, b=20$

則  $x = \frac{50\sqrt{70(10)}}{2(20)} = \frac{25}{2}\sqrt{7} = 33.07$

56. 設  $DE$  爲塔高，且等於  $h$

則  $AD = h \cot 60^\circ = h/\sqrt{3}, \quad BD = h \cot 45^\circ = h,$

$CD = h \cot 30^\circ = \sqrt{3}h$



$\therefore \angle DAB = \angle DAC = 90^\circ$

$\therefore \overline{BA}^2 + \overline{AD}^2 = \overline{BD}^2 \quad \text{即} \quad \overline{BA}^2 = h^2 - \frac{h^2}{3} = \frac{2h^2}{3}$

又  $\overline{CA}^2 = \overline{CD}^2 - \overline{AD}^2 = 3h^2 - \frac{h^2}{3} = \frac{8h^2}{3}$

$$\therefore CA = 2 \cdot \sqrt{\frac{2h^2}{3}} = 2BA = BA + CB$$

$$\therefore BA = CB \quad \text{故如題云。}$$

57. 如在上題中  $AB = \frac{1}{2} \quad \therefore \frac{2h^2}{3} = \frac{1}{4} \quad \therefore h^2 = \frac{3}{8}$

$$\therefore h = \frac{1}{4} \sqrt{6} \text{ 哩} = 440 \sqrt{6} \text{ 碼}$$

58. 見題 24.

59. 設塔高  $EF$  爲  $h$ ,  $\angle EBC = \alpha$

則  $\angle ABE = 90^\circ - \alpha$

故  $CE = h \cot 60^\circ = \frac{h}{\sqrt{3}}$ ,

$$AE = h \cot 30^\circ = \sqrt{3}h,$$

$$BE = \sqrt{3}h$$

今 
$$\begin{cases} 3h^2 = a^2 + 3h^2 - 2\sqrt{3}ha \sin \alpha \\ \frac{h^2}{3} = \frac{25}{9}a^2 + 3h^2 - \frac{10a\sqrt{3}}{3}h \cos \alpha \end{cases}$$

即 
$$\begin{cases} \sin \alpha = \frac{a}{2\sqrt{3}h} & (1) \end{cases}$$

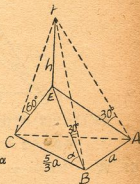
$$\begin{cases} \cos \alpha = \frac{24h^2 + 25a^2}{30\sqrt{3}ah} & (2) \end{cases}$$

$$\therefore \frac{a^2}{12h^2} + \frac{576h^4 + 1200a^2h^2 + 625a^4}{2700a^2h^2} = 1$$

即 
$$288h^4 - 750a^2h^2 + 425a^4 = 0$$

即 
$$(6h^2 - 5a^2)(48h^2 - 85a^2) = 0$$

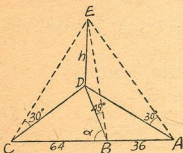
$$\therefore h = \sqrt{\frac{5}{6}}a = \frac{a}{6}\sqrt{30} \quad \text{或} \quad h = \sqrt{\frac{85}{48}}a = \frac{a}{12}\sqrt{255}$$



60. 設塔高  $DE=h$  碼,  $\angle CBD=\alpha$

則  $AD=h \cot 30^\circ = h\sqrt{3}$ ,  $DC=h\sqrt{3}$ ,  $BD=h$

$$\text{今 } \begin{cases} 3h^2 = h^2 + 36^2 + 72h \cos \alpha & (1) \\ 3h^2 = h^2 + 64^2 - 128h \cos \alpha & (2) \end{cases}$$



16(1) + 9(2), 并化簡之得  $h^2 = 9 \times 128$

$\therefore h = 24\sqrt{2}$  即塔高  $24\sqrt{2}$  碼

61. 設  $\tan \phi = a$ ,  $\tan \theta = \beta$ ,  $DE = h$

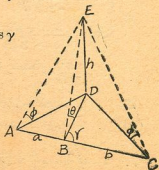
則  $AD = h/a$ ,  $BD = h/\beta$ ,  $CD = h/a$

$$\text{今 } \begin{cases} \frac{h^2}{a^2} = \frac{h^2}{\beta^2} + a^2 + 2a\left(\frac{h}{\beta}\right)\cos \gamma \\ \frac{h^2}{a^2} = \frac{h^2}{\beta^2} + b^2 - 2b\left(\frac{h}{\beta}\right)\cos \gamma \end{cases}$$

$$\therefore (a+b)\frac{h^2}{a^2} = \frac{h^2}{\beta^2}(a+b) + a^2b + b^2a$$

$$\text{即 } \frac{h^2}{a^2} - \frac{h^2}{\beta^2} = ab$$

$$\text{故 } h = a\beta\sqrt{\frac{ab}{\beta^2 - a^2}}$$



62. 見 298 頁例十九之圖，設  $\tan \alpha = \sqrt{2}$ ， $\tan \beta = \sqrt{3} + 1$   
 則  $\cot \alpha = 1/\sqrt{2}$ ， $\cot \beta = 1/(\sqrt{3} + 1) = (\sqrt{3} - 1)/2$   
 設塔高為  $x$  尺，則因  $a = 80$  尺，故從 299 頁

$$(\cot^2 \alpha - \cot^2 \beta)x^2 + a\sqrt{3}x \cot \beta - a^2 = 0$$

$$\text{即 } \left( \frac{1}{2} - \frac{2 - \sqrt{3}}{2} \right) x^2 + 40\sqrt{3}(\sqrt{3} - 1)x - 6400 = 0$$

$$\text{即 } x^2 + 80\sqrt{3}x - 6400(\sqrt{3} + 1) = 0$$

$$\text{即 } (x - 80)[x + 80(\sqrt{3} + 1)] = 0 \quad \therefore x = 80 \text{ 尺}$$

63. 今因  $A, B, C$  對於桿高之仰角相同

$$\therefore AD = BD = CD = 100 \cot 60^\circ = 100/\sqrt{3}$$

故  $D$  為  $\triangle ABC$  之外心

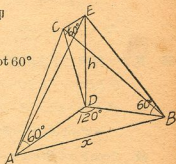
又因各邊相等

故  $D$  亦為其內心，重心

$$\text{故 } \angle ADB = 120^\circ$$

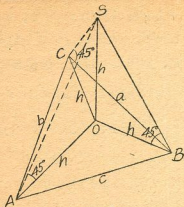
$$\text{今 } \frac{AB}{\sin 120^\circ} = \frac{AD}{\sin 30^\circ}$$

$$\text{即 } x = \frac{\sqrt{3}}{2} \left( \frac{100}{\sqrt{3}} \right) 2 = 100 \text{ 尺}$$



64. 設氣球高  $OS = h$ ，則  $AO = BO = CO = h \tan 45^\circ = h$   
 故  $AO$  為  $\triangle ABC$  之外接圓半徑

$$\text{從公式 47 得 } \Delta = \frac{abc}{4R}$$



$$\therefore AO = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

55. 見 299 頁例二十之圖，今因  $A, B, C, D$  在同一圓周上

$$\therefore \angle BDC = \angle BAC = \theta, \quad \angle BDA = \angle BCA = \theta$$

$$\begin{aligned} \text{從 } x^2 \cot^2 \gamma + x^2 \cot^2 \beta - 2x^2 \cot \gamma \cot \beta \cos \theta \\ = x^2 \cot^2 \alpha + x^2 \cot^2 \beta - 2x^2 \cot \alpha \cot \beta \cos \theta \end{aligned}$$

$$\text{故 } 2 \cos \theta \cot \beta (\cot \alpha - \cot \gamma) = \cot^2 \alpha - \cot^2 \gamma$$

$$\text{即 } 2 \cos \theta \cot \beta = \cot \alpha + \cot \gamma$$

66. 設  $AB = x$ ，今  $\angle DAC = \pi - (\alpha + \gamma)$

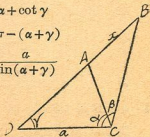
$$\therefore \frac{AC}{\sin \gamma} = \frac{a}{\sin(\pi - \alpha - \gamma)} = \frac{a}{\sin(\alpha + \gamma)}$$

$$\therefore AC = a \sin \gamma / \sin(\alpha + \gamma)$$

$$\text{又 } \angle ABC = \pi - (\beta + \gamma)$$

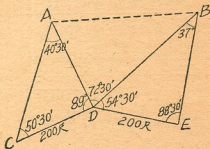
故在  $\triangle ABC$  中

$$x = \frac{AC \sin(\beta - \alpha)}{\sin(\pi - \beta - \gamma)} = \frac{a \sin \gamma \sin(\beta - \alpha)}{\sin(\alpha + \gamma) \sin(\beta + \gamma)}$$



$$67. \text{ 今 } \overline{BD} = \frac{200 \sin 88^\circ 30'}{\sin 37^\circ} \quad \text{又 } \overline{AD} = \frac{200 \sin 50^\circ 30'}{\sin 40^\circ 30'}$$

$$\therefore \overline{AB} = \sqrt{\overline{BD}^2 + \overline{AD}^2 - 2 \cdot \overline{AD} \cdot \overline{BD} \cos 72^\circ 30'}$$



今

$$\log 200 = 2.30103$$

$$\log \sin 88^\circ 30' = 9.99985 - 10$$

$$\text{colog } \sin 37^\circ = 0.22054$$

---


$$\log \overline{BD} = 2.52142$$

$$2 \log \overline{BD} = 5.04284$$

$$\therefore \overline{BD}^2 = 110367$$

$$\log 200 = 2.30103$$

$$\log \sin 50^\circ 30' = 9.88741 - 10$$

$$\text{colog } \sin 40^\circ 30' = 0.18746$$

---


$$\log \overline{AD} = 2.37590$$

$$2 \log \overline{AD} = 4.75180$$

$$\overline{AD}^2 = 56468$$


---

$$\log 2 = 0.30103$$

$$\log \overline{AD} = 2.37590$$

$$\log \overline{BD} = 2.52142$$

$$\log \cos 72^\circ 30' = 9.47814 - 10$$

$$\log 2 \cdot \overline{AD} \cdot \overline{BD} \cos 72^\circ 30' = 4.67649$$

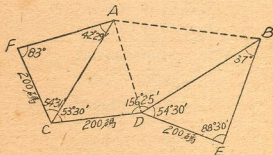
$$\therefore 2 \overline{AD} \cdot \overline{BD} \cos 72^\circ 30' = 47478$$

$$\therefore \overline{AB} = \sqrt{119357}$$

$$\text{今 } \log 119357 = 5.07685, \quad \frac{1}{2} \log 119357 = 2.53843$$

$$\therefore \overline{AB} = 345.48 \text{ 尺} \quad \text{故 } AB \text{ 長爲 } 345 \text{ 尺.}$$

68.



$$\text{今 } \overline{AC} = \frac{200 \sin 83^\circ}{\sin 42^\circ 29'}$$

$$\therefore \overline{AD} = \sqrt{\overline{AC}^2 + \overline{CD}^2 - 2 \overline{AC} \cdot \overline{CD} \cdot \cos 53^\circ 30'}$$

$$\therefore \overline{AB} = \sqrt{\overline{AD}^2 + \overline{BD}^2 - 2 \overline{AD} \cdot \overline{BD} \cdot \cos \angle ADB}$$

今

$$\log 200 = 2.30103$$

$$\log \sin 83^\circ = 9.99675 - 10$$

$$\text{colog} \sin 42^\circ 29' = 0.17045$$


---

$$\log \overline{AC} = 2.46823$$

$$2 \log \overline{AC} = 4.93646$$

$$\therefore \overline{AC}^2 = 86390$$

$$\text{及 } \overline{CD}^2 = 40000$$


---

$$\log 2 = 0.30103$$

$$\log \overline{AC} = 2.46823$$

$$\log \overline{CD} = 2.30103$$

$$\log \cos 53^\circ 30' = 9.77439 - 10$$


---

$$\log 2\overline{AC} \cdot \overline{CD} \cos 53^\circ 30' = 4.84468$$

$$\therefore 2\overline{AC} \cdot \overline{CD} \cos 53^\circ 30' = 69933$$


---

$$\therefore \overline{AD} = \sqrt{56457}$$

$$\log 56457 = 4.75172$$

$$\log \overline{AD} = 2.37586$$


---

$$\sin ADC = \frac{\overline{AC} \sin 53^\circ 30'}{\overline{AD}}$$

$$\log \overline{AC} = 2.46823$$

$$\log \sin 53^\circ 30' = 9.90518 - 10$$

$$\text{colog} \overline{AD} = 7.62414 - 10$$


---

$$\log \sin ADC = 19.99755 - 20$$

$$\therefore \angle ADC = 83^\circ 55'$$



$$\therefore \angle ADB = 72^\circ 30'$$

$$\log 2 = 0.30103$$

$$\log \overline{AD} = 2.37586 \quad (\text{見前})$$

$$\log \overline{BD} = 2.52142 \quad (\text{學者自求})$$

$$\log \cos 72^\circ 30' = 9.47814 - 10$$

$$\log 2\overline{AD} \cdot \overline{BD} \cos 72^\circ 30' = 4.67645$$

$$\overline{2AD} \cdot \overline{BD} \cos 72^\circ 30' = 47473$$

$$\therefore \overline{AB} = \sqrt{56457 + 110367 - 47473} = \sqrt{119351}$$

$$\text{今 } \log 119351 = 5.07682, \quad \log \overline{AB} = 2.53841$$

$$\therefore \overline{AB} = 345.47 \text{ 碼} \quad \text{故 } AB \text{ 長為 } 345 \text{ 碼。}$$

69. 過  $\overline{BD}$  作平面  $\perp VA$  交  $VA$  於  $K$

則  $\angle BKD$  即為所求之角

今  $FK \perp VA$

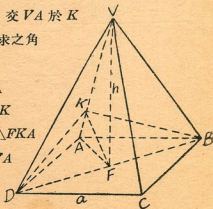
故  $\triangle FKA$  為  $rt.\triangle$

又  $\angle VAF = \angle FAK$

$\therefore rt.\triangle VFA \sim rt.\triangle FKA$

$\therefore FK:FA = VF:VA$

但  $FA = \frac{\sqrt{2}}{2}a,$



$$VF = h, \quad VA = \sqrt{\frac{1}{2}a^2 + h^2} = \frac{\sqrt{2a^2 + 4h^2}}{2}$$

$$\therefore FK = \frac{\frac{\sqrt{2}}{2}ah}{\frac{1}{2}\sqrt{2a^2+4h^2}} = \frac{ah}{\sqrt{a^2+2h^2}}$$

$$\text{又 } BF = FA = \frac{\sqrt{2}}{2}a$$

故在  $rt. \triangle KFB$  中 ( $\angle KFB = 90^\circ$ )

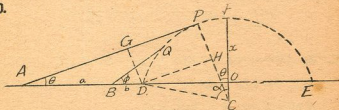
$$BK = \sqrt{\frac{a^2h^2}{a^2+2h^2} + \frac{1}{2}a^2} = \sqrt{\frac{4a^2h^2+a^4}{2a^2+4h^2}} = a\sqrt{\frac{a^2+4h^2}{2a^2+4h^2}}$$

$$\therefore \sin BKF = \frac{BF}{BK} = \frac{\frac{\sqrt{2}}{2}a}{a\sqrt{\frac{a^2+4h^2}{2a^2+4h^2}}} = \sqrt{\frac{a^2+2h^2}{a^2+4h^2}}$$

$$\cos BKF = \sqrt{\frac{2h^2}{a^2+4h^2}} = h\sqrt{\frac{2}{a^2+4h^2}}$$

$$\text{故 } \sin 2BKF = \sin BKD = 2h\sqrt{\frac{2a^2+4h^2}{a^2+4h^2}}$$

70.



如圖設測點  $A, B$  與  $AP, BQ, D, E$  同在一垂直平面

今  $AD = a, BD = b, AP, BQ$  均為球山之切線

設  $C$  為球心, 半徑  $CF \perp DE$ , 則  $OF$  為所求之山高, 設為  $x$

今作  $CP$ , 則  $CP \perp AP$ , 又作  $DH \perp CP$ , 則  $\angle PCF = \theta$

又連  $CD$ , 設  $\angle DCF$  為  $\alpha$ , 則  $\angle DCP = \alpha - \theta$

$$\begin{aligned} \therefore a \sin \theta &= \overline{DG} = \overline{HP} = \overline{CP} - \overline{CH} = r - r \cos(\alpha - \theta) \\ &= r[1 - \cos(\alpha - \theta)] = 2r \sin^2 \frac{1}{2}(\alpha - \theta) \end{aligned}$$

$$\therefore 2a \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta = 2r \sin^2 \frac{1}{2}(\alpha - \theta)$$

兩邊以  $2 \sin^2 \frac{1}{2} \theta$  除之並開平方，則

$$\sqrt{a \cot \frac{1}{2} \theta} = \sqrt{r} (\sin \frac{1}{2} \alpha \cot \frac{1}{2} \theta - \cos \frac{1}{2} \alpha) \quad (1)$$

依同理  $\sqrt{b \cot \frac{1}{2} \phi} = \sqrt{r} (\sin \frac{1}{2} \alpha \cot \frac{1}{2} \phi - \cos \frac{1}{2} \alpha) \quad (2)$

$$(2) - (1), \sqrt{b \cot \frac{1}{2} \phi} - \sqrt{a \cot \frac{1}{2} \theta}$$

$$= \sqrt{r} \sin \frac{1}{2} \alpha (\cot \frac{1}{2} \phi - \cot \frac{1}{2} \theta)$$

$$\therefore \sqrt{r} \sin \frac{1}{2} \alpha = \frac{\sqrt{b \cot \frac{1}{2} \phi} - \sqrt{a \cot \frac{1}{2} \theta}}{\cot \frac{1}{2} \phi - \cot \frac{1}{2} \theta} \quad (3)$$

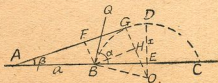
但  $x = \overline{OF} = \overline{CF} - \overline{CO} = r - r \cos \alpha = r(1 - \cos \alpha)$

$$= 2r \sin^2 \frac{1}{2} \alpha = 2(\sqrt{r} \sin \frac{1}{2} \alpha)^2 \quad (4)$$

以(3)代入(4)，則得

$$x = 2 \left( \frac{\sqrt{b \cot \frac{1}{2} \phi} - \sqrt{a \cot \frac{1}{2} \theta}}{\cot \frac{1}{2} \phi - \cot \frac{1}{2} \theta} \right)^2$$

71. 設  $AB = a$ ,  $BQ$ ,  $AG$  切球山，而  $\angle GAB = \beta$ ,  $\angle QBE = \alpha$   
半徑  $OD \perp BC$ ，聯  $OG$ ，作  $BH \perp OG$ ，並設  $DE$  為  $x$   
又球山之半徑為  $r$ ，球心為  $O$



$$\begin{aligned} \text{則 } x &= r - \overline{OE} = r - r \cos \alpha = r(1 - \cos \alpha) \\ &= 2r \sin^2 \frac{1}{2} \alpha \end{aligned} \quad (1)$$

$$(\because OB \perp BQ, OD \perp BC \quad \therefore \angle BOD = \angle QBE = \alpha)$$

$$\begin{aligned} \text{又 } \overline{GH} &= r - \overline{OH} = r - r \cos(\alpha - \beta) \\ &= r[1 - \cos(\alpha - \beta)] = 2r \sin^2 \frac{1}{2}(\alpha - \beta) \end{aligned}$$

$$\begin{aligned} (\because OG \perp AG, OD \perp BC \quad \therefore \angle GOD = \angle GAB = \beta) \\ \therefore \angle BOG = \angle BOD - \angle GOD = \alpha - \beta \end{aligned}$$

$$\begin{aligned} \text{但 } \overline{GH} = \overline{BF} = a \sin \beta \quad \therefore a \sin \beta = 2r \sin^2 \frac{1}{2}(\alpha - \beta) \\ \therefore 2r = a \sin \beta \csc^2 \frac{1}{2}(\alpha - \beta) \end{aligned} \quad (2)$$

$$\text{以(2)代入(1), 則 } x = a \sin \beta \sin^2 \frac{1}{2} \alpha \csc^2 \frac{1}{2}(\alpha - \beta)$$

72. 由 A 作切線 AC, 由 B 作切線 BD

$$\text{則 } \angle COA = \alpha, \angle DOB = \beta$$

$$\text{設半徑 } OC = x, \text{ 則 } \widehat{EF} = d$$

$$\therefore \angle EOF = d/x$$

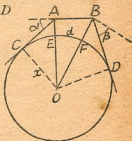
$$\text{今 } AO = OC / \cos \alpha = x / \cos \alpha$$

$$BO = OD / \cos \beta = x / \cos \beta$$

$$\therefore \frac{AO}{BO} = \frac{\cos \beta}{\cos \alpha} \quad \text{但 } \frac{AO}{BO} = \cos \frac{d}{x}$$

$$\therefore \frac{\cos \beta}{\cos \alpha} = \cos \frac{d}{x} \quad \therefore \frac{d}{x} = \cos^{-1}(\cos \beta \sec \alpha)$$

$$\therefore x = d / \cos^{-1}(\cos \beta \sec \alpha) \quad (\text{地球半徑})$$



73. 設 C 為燈所在處, A, B 為二測點

$$\text{今 } EA = 64 \text{ 呎, } FB = 16 \text{ 呎, } OE = 4000 \text{ 哩}$$

船自  $E$  航至  $F$  費時 30 分鐘

因  $\widehat{EF}$  比全圓周為甚小

故  $AB \doteq \widehat{EF}$

$$\overline{AC}^2 = \overline{OA}^2 - \overline{OC}^2$$

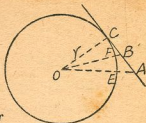
$$= (r+64)^2 - r^2 \doteq 128r$$

$$\therefore AC = 8\sqrt{2r} \quad \text{同理} \quad BC = 4\sqrt{2r}$$

$$\therefore AB = AC - BC = 4\sqrt{2r} = 4\sqrt{8000(5280)} \text{ 呎}$$

$$\text{故每時速} \quad \frac{6400\sqrt{66}}{6080} = 8.62 \text{ 哩} \quad (1 \text{ 哩} = 6080.27 \text{ 呎})$$

(參考 305 頁例二十七)



74. 設第  $p$  及  $q$  根電桿 ( $A$  處一根不算) 各在  $P$

及  $Q$  點,  $C$  為圓心,  $x$  為

半徑, 又連續兩桿對於曲

道中心之水平角為  $2\phi$

$$\text{則} \quad 2\phi = \frac{\pi}{2(n-1)}$$

$$\text{又} \quad \angle PCO = 2p\phi, \quad \angle QCO = 2q\phi$$

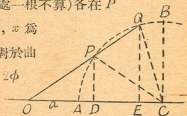
$$\therefore \tan POD = \frac{\overline{PD}}{\overline{OD}} = \frac{\overline{QE}}{\overline{OE}}$$

$$\text{今} \quad \overline{PD} = \overline{PC} \sin 2p\phi = x \sin 2p\phi$$

$$\overline{OD} = \overline{OC} - \overline{DC} = a + x - x \cos 2p\phi$$

$$\overline{QE} = \overline{QC} \sin 2q\phi = x \sin 2q\phi$$

$$\overline{OE} = \overline{OC} - \overline{EC} = a + x - x \cos 2q\phi$$



$$\therefore \frac{x \sin 2p\phi}{a+x-x \cos 2p\phi} = \frac{x \sin 2q\phi}{a+x-x \cos 2q\phi}$$

$$\text{即 } \frac{\sin 2p\phi}{\sin 2q\phi} = \frac{a+x(1-\cos 2p\phi)}{a+x(1-\cos 2q\phi)} = \frac{a+2x \sin^2 p\phi}{a+2x \sin^2 q\phi}$$

$$\text{即 } 2x(\sin^2 p\phi \sin 2q\phi - \sin^2 q\phi \sin 2p\phi)$$

$$= a(\sin 2p\phi - \sin 2q\phi)$$

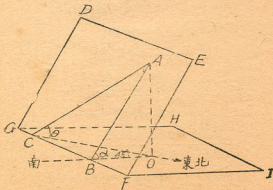
$$= 2a \sin(p-q)\phi \cos(p+q)\phi$$

$$\therefore x = \frac{a \sin(p-q)\phi \cos(p+q)\phi}{2 \sin p\phi \sin q\phi (\sin p\phi \cos q\phi - \cos p\phi \sin q\phi)}$$

$$= \frac{a \sin(p-q)\phi \cos(p+q)\phi}{2 \sin p\phi \sin q\phi \sin(p-q)\phi}$$

$$= \frac{a \cos(p+q)\phi}{2 \sin p\phi \sin q\phi}$$

75. 設平面  $DEFG$  代表山之傾斜面，平面  $HIFG$  代表地平面，作  $AB \perp GF$ ， $AO \perp$  平面  $HIFG$ ，連  $OB$ ，則  $OB \perp GF$  (三垂線定理)，又作  $AC$ ，使成東北向，連  $OC$ ，則由題意知，設  $\angle COB = \beta$ ，則得  $\beta = 45^\circ$ ，



設  $\angle ACO = \theta$ , 則  $\sin \theta$  即為所求者, 設  $\angle ABO = \alpha$ ,  
則  $\sin \alpha = 1:5$

今  $\triangle AOB$ ,  $\triangle AOC$ ,  $\triangle OBC$  均為直角三角形

故  $\overline{AO} = \overline{CO} \tan \theta = \overline{BO} \tan \alpha = \overline{CO} \cos \beta \tan \alpha$

$\therefore \tan \theta = \cos \beta \tan \alpha$  (參考 302 頁例二十四)

但  $\sin \alpha = \frac{1}{5}$   $\therefore \tan \alpha = \frac{1}{2\sqrt{6}}$

又  $\cos \beta = \cos 45^\circ = \frac{\sqrt{2}}{2}$   $\therefore \tan \theta = \frac{1}{4\sqrt{3}}$

即  $\sin \theta = \frac{1}{7}$  故如題云。

76. 由 302 頁例二十四得  $\tan \theta = \cos \beta \tan \alpha$

a. 今  $\alpha = 15^\circ$ ,  $\beta = 45^\circ$

$\therefore \tan \theta = \cos 45^\circ \tan 15^\circ$

$\log \cos \beta = 9.84949 - 10$

$\log \tan \alpha = 9.42805 - 10$

---

$\log \tan \theta = 9.27754 - 10$

$\therefore \theta = 10^\circ 44'$

b. 今  $\theta = 5^\circ$ ,  $\alpha = 15^\circ$

$\therefore \cos \beta = \frac{\tan \theta}{\tan \alpha} = \frac{\tan 5^\circ}{\tan 15^\circ}$

$\log \tan \theta = 8.94195 - 10$

$\operatorname{colog} \tan \alpha = 0.57195$

---

$\log \cos \beta = 9.51390 - 10$

$\therefore \beta = N19^\circ 3' E$

## 習題二十四 (329—332 頁)

$$1. (1)^2 + (2)^2, \quad x^2 + y^2 = \rho^2$$

$$2. \text{從(1)} \quad m \sin \phi = 1 - \cos \phi \quad (3)$$

$$\text{從(2)} \quad n \sin \phi = 1 + \cos \phi \quad (4)$$

$$(3), (4) \text{相乘得} \quad mn = 1$$

$$3. \text{自(1)} \quad \cot^2 t = \frac{x^2}{y^2} \quad (3)$$

$$\text{自(2)} \quad \csc^2 t = \frac{(a-x)^2}{y^2} \quad (4)$$

$$(4) - (3), \quad \frac{a^2 - 2ax}{y^2} = 1$$

$$\text{即} \quad y^2 = a^2 - 2ax \quad (\text{拋物線})$$

$$4. \text{從(2)} \quad \cos \theta = \frac{a-y}{b} \quad \therefore \theta = \cos^{-1} \frac{a-y}{b}$$

$$\text{及} \quad \sin \theta = \frac{\sqrt{b^2 - (a-y)^2}}{b}$$

$$\text{代入(1)得} \quad x = a \cos^{-1} \frac{a-y}{b} - \sqrt{b^2 - (a-y)^2}$$

$$5. \text{今} \quad 2 \tan 2\psi = m+n, \quad 2 \sin \psi = m-n$$

$$\text{即} \quad \cot 2\psi = \frac{2}{m+n}, \quad \csc 2\psi = \frac{2}{m-n}$$

$$\text{故得} \quad \frac{4}{(m-n)^2} - \frac{4}{(m+n)^2} = 1$$

$$\text{即} \quad 16mn = (m^2 - n^2)^2$$



$$6. \text{ 由(2) } \sin \theta = \frac{n}{\sqrt{m^2+n^2}}, \quad \cos \theta = \frac{m}{\sqrt{m^2+n^2}}$$

$$\text{代入(1) } \frac{ax}{m} - \frac{by}{n} = \frac{c^2}{\sqrt{m^2+n^2}}$$

$$7. \text{ 即 } x \sin 2\phi = 1 - \cos 2\phi, \quad y \sin 2\phi = 1 + \cos 2\phi$$

$$\text{相乘得 } xy = 1 \quad (\text{等邊雙曲線})$$

$$8. \text{ 從(1) } a = \frac{1}{\sin \theta} - \sin \theta = \frac{\cos^2 \theta}{\sin \theta} \quad (3)$$

$$\text{從(2) } b = \frac{1}{\cos \theta} - \cos \theta = \frac{\sin^2 \theta}{\cos \theta} \quad (4)$$

$$\text{即 } a^2 b = \cos^3 \theta, \quad ab^2 = \sin^3 \theta$$

$$\therefore (a^2 b)^{\frac{2}{3}} + (ab^2)^{\frac{2}{3}} = 1$$

$$9. \text{ 從(1), } \frac{x}{a} = 2 \cos \frac{3\theta}{2} \cos \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} \left( 4 \cos^2 \frac{\theta}{2} - 3 \right) \quad (3)$$

$$(1)^2 + (2)^2, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2(1 + \cos \theta) = 4 \cos^2 \frac{\theta}{2} \quad (4)$$

(4)中  $\cos^2 \frac{\theta}{2}$  代入(3)得

$$\frac{2x}{a} = \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 3 \right)$$

$$\text{別解: } (1)^2 + (2)^2, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 + 2 \cos \theta$$

$$\text{即 } \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 1 + 2 \cos \theta \quad (5)$$

$$\text{又從(1) } \frac{x}{a} + 1 = \cos \theta + 2 \cos^2 \theta = \cos \theta (1 + 2 \cos \theta)$$

$$\text{從(2) } \frac{y}{b} = \sin \theta (1 + 2 \cos \theta)$$

$$\therefore \left(\frac{x}{a} + 1\right)^2 + \frac{y^2}{b^2} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)^2$$

10.  $(1)^2 + (2)^2, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  (橢圓)

11. 即  $a \sec \theta = y + x \tan \theta, \quad b \sec \theta = -y \tan \theta + x$

上兩式自乘相加  $(a^2 + b^2) \sec^2 \theta = (x^2 + y^2)(1 + \tan^2 \theta)$

即  $x^2 + y^2 = a^2 + b^2$  (圓)

12. 從(1),  $x + y = 3 - (1 - 2 \sin^2 2\theta) = 2 + 2 \sin^2 2\theta$  (3)

從(3) + (2),  $x = (1 + \sin 2\theta)^2$

$$\therefore x^{\frac{1}{2}} = 1 + \sin 2\theta$$
 (4)

從(3) - (2),  $y = (1 - \sin 2\theta)^2$

$$\therefore y^{\frac{1}{2}} = 1 - \sin 2\theta$$
 (5)

$$\therefore x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$$
 (拋物線)

別解：用他法消去則得  $(x - y)^2 = 8(x + y - 2)$

即  $(x + y - 4)^2 = 4xy$

即  $x + y - 4 = -2\sqrt{xy}$  (取負號)

即  $x + y + 2\sqrt{xy} = 4$

即  $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$

13. 從(1),  $x = \cot \phi + \frac{1}{\cot \phi} = \frac{1 + \cot^2 \phi}{\cot \phi} = \frac{\csc^2 \phi}{\cot \phi}$

從(2),  $y = \csc \phi - \frac{1}{\csc \phi} = \frac{\csc^2 \phi - 1}{\csc \phi} = \frac{\cot^2 \phi}{\csc \phi}$

$\therefore x^2 y = \csc^3 \phi, \quad xy^2 = \cot^3 \phi$  (參考 320 頁例五)

故  $(x^2 y)^{\frac{2}{3}} - (xy^2)^{\frac{2}{3}} = 1$  即  $x^{\frac{4}{3}} y^{\frac{2}{3}} - x^{\frac{2}{3}} y^{\frac{4}{3}} = 1$

$$14. \text{ 從(1)移項得 } x \sec^2 \theta = a \tan^2 \theta \quad \therefore x = a \left( \frac{\tan^2 \theta}{\sec^2 \theta} \right)$$

$$\text{同理從(2)} \quad y \tan^2 \theta = a \sec^2 \theta \quad \therefore y = a \left( \frac{\sec^2 \theta}{\tan^2 \theta} \right)$$

$$\therefore x^2 y^3 = a^6 \sec^6 \theta, \quad x^5 y^2 = a^6 \tan^6 \theta$$

$$\therefore x^{\frac{4}{5}} y^{\frac{6}{5}} - x^{\frac{6}{5}} y^{\frac{4}{5}} = a^2$$

$$15. \text{ 從(1)} \quad x = a \cos \theta (4 \cos^2 \theta - 3) = a \cos 3\theta$$

$$\text{從(2)} \quad y = b \sin \theta (3 - 4 \sin^2 \theta) = b \sin 3\theta$$

$$\text{故} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{橢圓})$$

$$16. \text{ 今} \quad \begin{cases} x = \cos(\theta - \alpha) = \cos \theta \cos \alpha + \sin \theta \sin \alpha \\ y = \sin(\theta - \beta) = \sin \theta \cos \beta - \cos \theta \sin \beta \end{cases}$$

$$\text{即} \quad \begin{cases} \sin \alpha \cdot \sin \theta + \cos \alpha \cdot \cos \theta - x = 0 \\ \cos \beta \cdot \sin \theta - \sin \beta \cdot \cos \theta - y = 0 \end{cases}$$

$$\therefore \frac{\sin \theta}{-y \cos \alpha - x \sin \beta} = \frac{\cos \theta}{-x \cos \beta + y \sin \alpha} \\ = \frac{1}{-\cos(\alpha - \beta)}$$

$$\text{故} \quad (x \sin \beta + y \cos \alpha)^2 + (x \cos \beta - y \sin \alpha)^2 \\ = \cos^2(\alpha - \beta)$$

$$\therefore x^2 - 2xy \sin(\alpha - \beta) + y^2 = \cos^2(\alpha - \beta)$$

今因  $\Delta = 4 \sin^2(\alpha - \beta) - 4 \leq 0$ , 故此為橢圓或拋物線。

$$17. \text{ 即} \quad x \cos \theta + y \sin \theta = 1, \quad x' \cos \theta + y' \sin \theta = 1$$

$$\therefore \cos \theta = \frac{\begin{vmatrix} 1 & y \\ x & y' \end{vmatrix}}{\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}} = \frac{y' - y}{xy' - x'y},$$

$$\sin \theta = \frac{\begin{vmatrix} x & 1 \\ x' & 1 \end{vmatrix}}{\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}} = \frac{x - x'}{xy' - x'y}$$

$$\therefore \left( \frac{y' - y}{xy' - x'y} \right)^2 + \left( \frac{x - x'}{xy' - x'y} \right)^2 = 1$$

即  $(x - x')^2 + (y' - y)^2 = (xy' - x'y)^2$

18. 從(1), (2)  $a = \sin \theta(1 + 2 \cos \theta),$

$$b + 1 = \cos \theta(1 + 2 \cos \theta)$$

$$\therefore a^2 + (b + 1)^2 = (1 + 2 \cos \theta)^2 \quad (3)$$

$$(1)^2 + (2)^2, \quad 2 + 2 \cos \theta = a^2 + b^2$$

$$\therefore 1 + 2 \cos \theta = a^2 + b^2 - 1 \quad (4)$$

從(3), (4)  $a^2 + (b + 1)^2 = (a^2 + b^2 - 1)^2$

即  $(a^2 + b^2)^2 - 3(a^2 + b^2) - 2b = 0$

19. (1)<sup>2</sup>,  $x^2 \sin^2 \theta - 2xy \sin \theta \cos \theta + y^2 \cos^2 \theta = x^2 + y^2$

即  $x^2(1 - \sin^2 \theta) + 2xy \sin \theta \cos \theta + y^2(1 - \cos^2 \theta) = 0$

即  $x^2 \cos^2 \theta + 2xy \sin \theta \cos \theta + y^2 \sin^2 \theta = 0$

即  $(x \cos \theta + y \sin \theta)^2 = 0$  即  $x \cos \theta + y \sin \theta = 0$

$$\therefore \tan^2 \theta = \frac{x^2}{y^2}, \quad \sin^2 \theta = \frac{x^2}{x^2 + y^2}, \quad \cos^2 \theta = \frac{y^2}{x^2 + y^2}$$

代入(2)化簡得  $x^2/a^2 + y^2/b^2 = 1$  (橢圓)

$$20. \text{ 從(1) } \quad x^{\frac{1}{m}} = a^{\frac{1}{m}} \cos \theta \cos \phi \quad (4)$$

$$\text{從(2) } \quad y^{\frac{1}{m}} = b^{\frac{1}{m}} \cos \theta \sin \phi \quad (5)$$

$$\text{從(3) } \quad z^{\frac{1}{m}} = c^{\frac{1}{m}} \sin \theta \quad (6)$$

$$(4)^2 + (5)^2 + (6)^2, \quad \left(\frac{x}{a}\right)^{\frac{2}{m}} + \left(\frac{y}{b}\right)^{\frac{2}{m}} + \left(\frac{z}{c}\right)^{\frac{2}{m}} = 1$$

(參考 322 頁例九)

21. 舊版之題同 321 頁例七，今則改爲

$$\begin{cases} x = r \cos \theta + r\theta \sin \theta & (1) \\ y = r \sin \theta - r\theta \cos \theta & (2) \end{cases}$$

$$(1)^2 + (2)^2, \quad x^2 + y^2 = r^2 + r^2\theta^2 \quad \therefore \theta = \sqrt{x^2 + y^2 - r^2}/r$$

$$\text{代入(1)得 } \quad x = r \cos(\sqrt{x^2 + y^2 - r^2}/r) \\ + \sqrt{x^2 + y^2 - r^2} \sin(\sqrt{x^2 + y^2 - r^2}/r)$$

$$22. \text{ 從(2) } \quad a \sec \theta + b \tan \theta = y \sec \theta \tan \theta \quad (3)$$

$$(1) + (3), \quad (a+b)(\tan \theta + \sec \theta) = y \sec \theta \tan \theta + x \quad (4)$$

$$(1) - (3), \quad (a-b)(\tan \theta - \sec \theta) = x - y \sec \theta \tan \theta \quad (5)$$

$$(3)^2 - (1)^2, \quad a^2 - b^2 = y^2 \sec^2 \theta \tan^2 \theta - x^2 \\ \therefore \sec^2 \theta \tan^2 \theta = \frac{a^2 - b^2 + x^2}{y^2} \quad (6)$$

$$\text{又從(4), (5) } \quad (\tan \theta + \sec \theta)^2 - (\tan \theta - \sec \theta)^2 \\ = \left(\frac{y \sec \theta \tan \theta + x}{a+b}\right)^2 - \left(\frac{x - y \sec \theta \tan \theta}{a-b}\right)^2$$

$$\text{即 } \quad (a^2 - b^2)^2 \tan \theta \sec \theta = -ab(y^2 \tan^2 \theta \sec^2 \theta + x^2) \\ + xy(a^2 + b^2) \tan \theta \sec \theta$$

以(6)之值代入且移項而平方之得

$$\begin{aligned} & a^2b^2(a^2 - b^2 + 2x^2)^2 \\ & = [xy(a^2 + b^2) - (a^2 - b^2)^2]^2 \tan^2\theta \sec^2\theta \end{aligned}$$

$$\begin{aligned} \text{即 } & a^2b^2y^2(a^2 - b^2 + 2x^2)^2 \\ & = [xy(a^2 + b^2) - (a^2 - b^2)^2]^2(a^2 - b^2 + x^2) \quad (7) \end{aligned}$$

註：若從(1)·(3)得

$$\begin{aligned} & ab(\tan^2\theta + \sec^2\theta) + (a^2 + b^2)\tan\theta \sec\theta \\ & = xy \tan\theta \sec\theta \end{aligned}$$

$$\text{即 } (a^2 + b^2 - xy)^2 \tan^2\theta \sec^2\theta = a^2b^2(1 + 4 \tan^2\theta \sec^2\theta)$$

$$\text{即 } (a^2 - b^2 + x^2)[(a^2 + b^2 - xy)^2 - 4a^2b^2] = a^2b^2y^2 \quad (8)$$

(8)與(7)相差爲(7)中一因式  $a^2 - b^2$  未曾括出。

$$23. (1)^2 + (2)^2, \quad x^2 + y^2 = a^2 + b^2 + 2ab \cos \theta \quad (3)$$

$$\text{從(1)} \quad x + b = \cos \theta (a + 2b \cos \theta)$$

$$\text{從(2)} \quad y = \sin \theta (a + 2b \cos \theta)$$

$$\therefore (x + b)^2 + y^2 = (a + 2b \cos \theta)^2$$

$$\text{即 } (x + b)^2 + y^2 = \left( \frac{x^2 + y^2 - b^2}{a} \right)^2$$

$$\text{故 } (x^2 + y^2 - b^2)^2 = a^2[(x + b)^2 + y^2]$$

$$\begin{aligned} 24. \quad & x = a \cos \theta \cos 2\theta + 2a \sin \theta \sin 2\theta \\ & = a \cos \theta (1 - 2 \sin^2\theta) + 4a \sin^2\theta \cos \theta \\ & = a \cos \theta (1 + 2 \sin^2\theta) \\ & y = 2a \cos \theta \sin 2\theta - a \sin \theta \cos 2\theta \\ & = 4a \sin \theta \cos^2\theta - a \sin \theta (2 \cos^2\theta - 1) \end{aligned}$$

$$= a \sin \theta (1 + 2 \cos^2 \theta)$$

$$\begin{aligned} \therefore x + y &= a(\cos \theta + \sin \theta) + 2a \sin \theta \cos \theta (\cos \theta + \sin \theta) \\ &= a(\cos \theta + \sin \theta)(1 + 2 \sin \theta \cos \theta) \\ &= a(\cos \theta + \sin \theta)^3 \quad \because 1 = \cos^2 \theta + \sin^2 \theta \end{aligned}$$

同理  $x - y = a(\cos \theta - \sin \theta)^3$

$$\begin{aligned} \therefore (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} &= a^{\frac{2}{3}} [(\cos \theta + \sin \theta)^2 \\ &\quad + (\cos \theta - \sin \theta)^2] = 2a^{\frac{2}{3}} \end{aligned}$$

25. 即 
$$\begin{cases} x \cos \theta \cos \alpha - x \sin \theta \sin \alpha + y \sin \theta \cos \alpha \\ \quad + y \cos \theta \sin \alpha - a \sin 2\theta = 0 \\ x \sin \theta \cos \alpha + x \cos \theta \sin \alpha - y \cos \theta \cos \alpha \\ \quad + y \sin \theta \sin \alpha + 2a \cos 2\theta = 0 \end{cases}$$

設  $A = x \cos \alpha + y \sin \alpha$ ,  $B = x \sin \alpha - y \cos \alpha$

則為 
$$\begin{cases} A \cos \theta - B \sin \theta = a \sin 2\theta \\ A \sin \theta + B \cos \theta = -2a \cos 2\theta \end{cases}$$

用十字法得 
$$\begin{aligned} A &= a(\cos \theta \sin 2\theta) - 2a(\cos 2\theta \sin \theta) \\ &= a \sin \theta - a \sin \theta \cos 2\theta \\ &= a \sin \theta (2 \sin^2 \theta) = 2a \sin^3 \theta \end{aligned}$$

又 
$$\begin{aligned} B &= -a \cos \theta - a \cos 2\theta \cos \theta = -2a \cos^3 \theta \\ \therefore A^{\frac{2}{3}} + B^{\frac{2}{3}} &= (2a)^{\frac{2}{3}} \end{aligned}$$

即  $(x \cos \alpha + y \sin \alpha)^{\frac{2}{3}} + (x \sin \alpha - y \cos \alpha)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$

26. 今 
$$\begin{cases} x + y = (\sin \theta + \cos \theta)(1 + \sin 2\theta) = (\sin \theta + \cos \theta)^3 \\ x - y = (\sin \theta - \cos \theta)(1 - \sin 2\theta) = (\sin \theta - \cos \theta)^3 \end{cases}$$

$$\therefore (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2$$

$$\begin{aligned}
 27. \quad a &= 4(\cos^3\theta - \sin^3\theta) - 3(\cos\theta - \sin\theta) \\
 &= (\cos\theta - \sin\theta)(4 + 4\cos\theta\sin\theta - 3) \\
 &= b(1 + 4\cos\theta\sin\theta) \quad (3)
 \end{aligned}$$

$$\text{又從(1)} \quad b^2 = 1 - 2\cos\theta\sin\theta \quad (4)$$

$$\text{比較(3), (4)} \quad \therefore 2(1 - b^2)b = a - b$$

$$\text{即} \quad 2b^3 - 3b + a = 0$$

$$28. \text{ 從(1)} \quad \frac{a}{b} = \frac{\tan(\theta + \alpha) + \tan(\theta - \alpha)}{\tan(\theta + \alpha) - \tan(\theta - \alpha)} = \frac{\sin 2\theta}{\sin 2\alpha}$$

$$\text{故} \quad \sin^2 2\theta = \frac{a^2}{b^2} \sin^2 2\alpha \quad (1)$$

$$\text{從(2)} \quad \cos 2\theta = \frac{c - a \cos 2\alpha}{b}$$

$$\therefore \cos^2 2\theta = \frac{(c - a \cos 2\alpha)^2}{b^2} \quad (2)$$

$$\therefore \frac{a^2 \sin^2 2\alpha}{b^2} + \frac{c^2 - 2ac \cos 2\alpha + a^2 \cos^2 2\alpha}{b^2} = 1$$

$$\text{即} \quad a^2 - 2ac \cos 2\alpha + c^2 = b^2$$

$$29. \text{ (1) + (2), } 2(ax + by) = (\sin\theta + \cos\theta)(1 + 2\sin\theta\cos\theta) \\ = (\sin\theta + \cos\theta)^3$$

$$\text{(1) - (2), } 2(bx + ay) = (\cos\theta - \sin\theta)(1 - 2\sin\theta\cos\theta) \\ = (\cos\theta - \sin\theta)^3$$

$$\therefore 2^{\frac{2}{3}}(ax + by)^{\frac{2}{3}} + 2^{\frac{2}{3}}(bx + ay)^{\frac{2}{3}} = 2$$

$$\text{即} \quad (ax + by)^{\frac{2}{3}} + (bx + ay)^{\frac{2}{3}} = 2^{\frac{1}{3}}$$

$$30. \text{ 從(1)} \quad x = \csc\theta \tan^3\theta (\cot^2\theta + 2) + \sec\theta$$



$$\begin{aligned}
 &= \csc \theta \tan \theta + 2 \csc \theta \tan^3 \theta + \sec \theta \\
 &= \sec \theta + 2 \sec \theta (\sec^2 \theta - 1) + \sec \theta = 2 \sec^3 \theta
 \end{aligned}$$

從(2)  $y = \tan^3 \theta (\cot^2 \theta + 2) - \tan \theta$

$$= \tan \theta + 2 \tan^3 \theta - \tan \theta = 2 \tan^3 \theta$$

$$\therefore x^{\frac{2}{3}} - y^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

31. 從(1)  $p^2 \sin^2 \theta = (\sin^2 \theta + 1)^2$

即  $p^2(1 - \cos^2 \theta) = (2 - \cos^2 \theta)^2$

即  $\cos^4 \theta + (p^2 - 4)\cos^2 \theta + 4 - p^2 = 0$  (3)

從(2)  $q^2 \cos^2 \theta = \cos^4 \theta + 2 \cos^2 \theta + 1$

即  $\cos^4 \theta + (2 - q^2)\cos^2 \theta + 1 = 0$  (4)

自(3), (4)  $\frac{\cos^4 \theta}{p^2 - 4 - (p^2 - 4)(q^2 - 2)} = \frac{\cos^2 \theta}{4 - p^2 - 1}$

$$= \frac{1}{2 - q^2 - p^2 + 4}$$

故  $(p^2 - 4)(-3 + q^2)(-6 + p^2 + q^2) = (p^2 - 3)^2$  (5)

又如(1)從  $\sin^4 \theta$ ,  $\sin^2 \theta$  着手則

$$(q^2 - 4)(p^2 - 3)(p^2 + q^2 - 6) = (q^2 - 3)^2$$
 (6)

(5)(6)實為一式, 因均等於

$$\begin{aligned}
 &p^2 q^2 (p^2 + q^2) - 4(p^4 + q^4) + 36(p^2 + q^2) \\
 &\quad - 81 - 13p^2 q^2 = 0 \text{ 也.}
 \end{aligned}$$

32. 用十字法照  $1/a$ ,  $1/b$ ,  $1/c$ , 解之得

$$\frac{1/a}{\frac{\cos 2\theta}{2} - \cos^2 \theta} = \frac{1/b}{\cos 2\theta \cos \theta - \frac{\cos 3\theta}{2}} = \frac{1/c}{\cos 3\theta \cos \theta - \cos^2 2\theta}$$

$$\text{即} \quad \frac{1/a}{-1/2} = \frac{1/b}{\cos \theta / 2} = \frac{1/c}{\cos 2\theta - 1 / 2}$$

$$\text{即} \quad -\frac{1}{a} = \frac{1}{b \cos \theta} = -\frac{1}{2c \sin^2 \theta}$$

$$\therefore \sin^2 \theta = \frac{a}{2c}, \quad \cos^2 \theta = \frac{a^2}{b^2}$$

$$\therefore \frac{a}{2c} + \frac{a^2}{b^2} = 1 \quad \text{即} \quad ab^2 + 2c(a^2 - b^2) = 0$$

$$33. \quad \frac{\sin(\theta - \alpha)}{\sin(\theta - \beta)} = \frac{a}{b} \quad (1), \quad \frac{\cos(\theta - \alpha)}{\cos(\theta - \beta)} = \frac{a'}{b'} \quad (2)$$

$$\begin{aligned} \text{從(1)} \quad & b(\sin \theta \cos \alpha - \cos \theta \sin \alpha) \\ & = a(\sin \theta \cos \beta - \cos \theta \sin \beta) \end{aligned}$$

$$\text{即} \quad b(\tan \theta \cos \alpha - \sin \alpha) = a(\tan \theta \cos \beta - \sin \beta)$$

$$\therefore \tan \theta = \frac{a \sin \beta - b \sin \alpha}{a \cos \beta - b \cos \alpha} \quad (3)$$

$$\text{同理從(2)} \quad \tan \theta = \frac{-(a' \cos \beta - b' \cos \alpha)}{a' \sin \beta - b' \sin \alpha} \quad (4)$$

$$\begin{aligned} \text{從(3), (4)} \quad & (a \sin \beta - b \sin \alpha)(a' \sin \beta - b' \sin \alpha) \\ & + (a \cos \beta - b \cos \alpha)(a' \cos \beta - b' \cos \alpha) = 0 \end{aligned}$$

$$\therefore \cos(\alpha - \beta) = (aa' + bb') / (ab' + a'b)$$

$$\text{或從} \quad \frac{\sin 2(\theta - \alpha)}{\sin 2(\theta - \beta)} = \frac{aa'}{bb'}$$

$$\text{即} \quad \frac{2 \sin(2\theta - \alpha - \beta) \cos(\alpha - \beta)}{\sin 2(\theta - \beta)} = \frac{aa' + bb'}{bb'} \quad (3)$$

$$\text{又(1)+(2),} \quad \frac{\sin(2\theta - \alpha - \beta)}{\sin(\theta - \beta) \cos(\theta - \beta)} = \frac{ab' + a'b}{bb'} \quad (4)$$

$$(3) \div (4), \quad \cos(\alpha - \beta) = (aa' + bb') / (ab' + a'b)$$

$$34. \text{ 從(1) } \sin \theta \cos \alpha + \cos \theta \sin \alpha = 2a \sin \theta \cos \theta$$

$$\text{即 } \tan \theta \cos \alpha - 2a \sin \theta + \sin \alpha = 0 \quad (3)$$

$$\text{同理從(2) } \tan \theta \cos \beta - 2a \sin \theta + \sin \beta = 0 \quad (4)$$

$$\therefore \frac{\tan \theta}{2a(\sin \alpha - \sin \beta)} = \frac{\sin \theta}{\sin(-\beta + \alpha)} = \frac{1}{2a(\cos \beta - \cos \alpha)}$$

$$\text{即 } \frac{\tan \theta}{2a \cos \frac{1}{2}(\alpha + \beta)} = \frac{\sin \theta}{\cos \frac{1}{2}(\alpha - \beta)} = \frac{1}{2a \sin \frac{1}{2}(\alpha + \beta)}$$

$$\text{即 } \cot \theta = \tan \frac{1}{2}(\alpha + \beta), \quad \csc \theta = \frac{2a \sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}$$

$$\therefore 1 + \tan^2 \frac{1}{2}(\alpha + \beta) = \frac{4a^2 \sin^2 \frac{1}{2}(\alpha + \beta)}{\cos^2 \frac{1}{2}(\alpha - \beta)}$$

$$\text{即 } \frac{1}{\cos^2 \frac{1}{2}(\alpha + \beta)} = \frac{4a^2 \sin^2 \frac{1}{2}(\alpha + \beta)}{\cos^2 \frac{1}{2}(\alpha - \beta)}$$

$$\therefore \cos^2 \frac{1}{2}(\alpha - \beta) = 4a^2 \sin^2 \frac{1}{2}(\alpha + \beta) \cos^2 \frac{1}{2}(\alpha + \beta) \\ = a^2 \sin^2(\alpha + \beta)$$

$$35. (1) + (2), \quad a(\tan \theta + \cot \theta) + b(\cot 2\theta - \tan 2\theta) = 2c$$

$$\text{即 } \frac{a}{\sin \theta \cos \theta} + \frac{b \cos 4\theta}{\cos 2\theta \sin 2\theta} = 2c$$

$$\text{即 } \frac{a}{\sin 2\theta} + \frac{b \cos 4\theta}{\sin 4\theta} = c$$

$$\text{即 } a \csc 2\theta + b \cot 4\theta = c \quad (3)$$

$$(2) - (1), \quad a(\cot \theta - \tan \theta) = b(\cot 2\theta + \tan 2\theta)$$

$$\text{即 } 2a \cot 2\theta = b(\cot 2\theta + \tan 2\theta)$$

$$\text{即 } b \tan 2\theta = (2a - b) \cot 2\theta$$

$$\therefore \tan^2 2\theta = \frac{2a-b}{b}$$

則  $\cot^2 2\theta = \frac{b}{2a-b}, \quad \csc^2 2\theta = \frac{2a}{2a-b}$

又  $\cot 4\theta = \frac{\cot^2 2\theta - 1}{2 \cot 2\theta} = \frac{b-a}{\sqrt{b(2a-b)}}$

代入(3)  $a\sqrt{\frac{2a}{2a-b}} + (b-a)\sqrt{\frac{b}{2a-b}} = c$

$$\therefore c\sqrt{2a-b} = a\sqrt{2a} - (a-b)\sqrt{b}$$

36. 從(2)  $\tan^2 \frac{\theta}{2} = \tan^{\frac{2}{3}} \alpha$  即  $\frac{1-\cos \theta}{1+\cos \theta} = \frac{\sin^{\frac{2}{3}} \alpha}{\cos^{\frac{2}{3}} \alpha}$

$$\therefore \frac{\cos \theta}{1} = \frac{\cos^{\frac{3}{2}} \alpha - \sin^{\frac{2}{3}} \alpha}{\cos^{\frac{3}{2}} \alpha + \sin^{\frac{2}{3}} \alpha}$$

即  $\frac{m^2 - 1}{3} = \frac{(\cos^{\frac{3}{2}} \alpha - \sin^{\frac{2}{3}} \alpha)^2}{(\cos^{\frac{3}{2}} \alpha + \sin^{\frac{2}{3}} \alpha)^2}$

即  $m^2(\cos^{\frac{3}{2}} \alpha + \sin^{\frac{2}{3}} \alpha)^2 = 4(\cos^{\frac{4}{3}} \alpha - \cos^{\frac{3}{2}} \alpha \sin^{\frac{2}{3}} \alpha + \sin^{\frac{4}{3}} \alpha)$

即  $m^2(\cos^{\frac{3}{2}} \alpha + \sin^{\frac{2}{3}} \alpha)^3 = 4[(\cos^{\frac{3}{2}} \alpha)^3 + (\sin^{\frac{2}{3}} \alpha)^3] = 4$

$$\therefore \cos^{\frac{3}{2}} \alpha + \sin^{\frac{2}{3}} \alpha = (2/m)^{\frac{2}{3}}$$

或自(2)  $\tan^2 \frac{\theta}{2} = \tan^{\frac{2}{3}} \alpha = \frac{1-\cos \theta}{1+\cos \theta}$

$$\therefore \frac{\sin^{\frac{2}{3}} \alpha}{1-\cos \theta} = \frac{\cos^{\frac{2}{3}} \alpha}{1+\cos \theta} = \frac{\sin^2 \alpha + \cos^{\frac{2}{3}} \alpha}{2} \quad (3)$$

又(3)<sup>3</sup>,  $\frac{\sin^2 \alpha}{(1-\cos \theta)^3} = \frac{\cos^2 \alpha}{(1+\cos \theta)^3} = \frac{1}{2+6 \cos^2 \theta} = \frac{1}{2m^2}$

$$\text{即 } \frac{\sin^{\frac{2}{3}}\alpha}{1 - \cos \theta} = \left(\frac{1}{2m^2}\right)^{\frac{1}{3}} \quad \therefore \sin^{\frac{2}{3}}\alpha + \cos^{\frac{2}{3}}\alpha = \left(\frac{2}{m}\right)^{\frac{2}{3}}$$

$$37. (1) + (2), \quad \sqrt{2}x(\sin \theta + \cos \theta)$$

$$= \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}(\sin 2\theta + \cos 2\theta)$$

$$\text{即 } 2x(\sin \theta + \cos \theta) = a + a(\sin 2\theta + \cos 2\theta)$$

$$= a(1 + 2 \sin \theta \cos \theta) + a \cos 2\theta$$

$$= a(\sin \theta + \cos \theta)^2 + a(\cos^2 \theta - \sin^2 \theta)$$

$$\text{即 } 2x = a(\sin \theta + \cos \theta) + a(\cos \theta - \sin \theta)$$

$$\text{即 } x = a \cos \theta \quad (3)$$

$$(1) - (2), \quad \sqrt{2}y(\sin \theta - \cos \theta)$$

$$= \frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}}(\sin 2\theta + \cos 2\theta)$$

$$\text{即 } 2y(\sin \theta - \cos \theta) = a(1 - \sin 2\theta) - a \cos 2\theta$$

$$\text{即 } 2y = a(\sin \theta - \cos \theta) + a(\cos \theta + \sin \theta)$$

$$\therefore y = a \sin \theta \quad (4)$$

$$(3)^2 + (4)^2, \quad x^2 + y^2 = a^2$$

$$38. (1) + (2), \quad (a+b)(\sin 3\theta + \cos 3\theta)$$

$$= 2c \cos\left(\theta + \frac{\pi}{12}\right) \cos \frac{\pi}{12}$$

$$\text{即 } \frac{1}{\sqrt{2}}(a+b)(\sin 3\theta + \cos 3\theta)$$

$$= \sqrt{2} \cdot c \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \cos\left(\theta + \frac{\pi}{12}\right)$$

$$(a+b)\sin\left(3\theta + \frac{\pi}{4}\right) = c\left(\frac{\sqrt{3}+1}{2}\right)\cos\left(\theta + \frac{\pi}{12}\right)$$

$$\text{設 } \theta + \frac{\pi}{12} = \phi, \quad 3\theta + \frac{\pi}{4} = 3\phi, \quad h = \frac{c(\sqrt{3}+1)}{2(a+b)}$$

$$\text{則 } \sin 3\phi = h \cos \phi \quad (3)$$

$$(1) - (2), \quad (a-b)(\cos 2\theta - \sin 3\theta)$$

$$= 2c \sin\left(\theta + \frac{\pi}{12}\right) \sin \frac{\pi}{12}$$

$$(a-b)\cos\left(3\theta + \frac{\pi}{4}\right) = c\left(\frac{\sqrt{3}-1}{2}\right)\sin\left(\theta + \frac{\pi}{12}\right)$$

$$\text{設 } k = \frac{c(\sqrt{3}-1)}{2(a-b)} \quad \text{則 } \cos 3\phi = k \sin \phi \quad (4)$$

$$(3) \div (4), \quad h \cos^2 \phi (4 \cos^2 \phi - 3)$$

$$= k \sin^2 \phi (3 - 4 \sin^2 \phi) \quad (5)$$

$$\text{又 } (3)^2 + (4)^2, \quad 1 = h^2 \cos^2 \phi + k^2 \sin^2 \phi$$

$$\begin{aligned} \text{即 } & 1 - k^2 = (h^2 - k^2) \cos^2 \phi \\ \text{或 } & 1 - h^2 = (k^2 - h^2) \sin^2 \phi \end{aligned} \quad (6)$$

$$\begin{aligned} \text{以(6)代入(5)} \quad & h \left( \frac{1-k^2}{h^2-k^2} \right) \left[ 4 \left( \frac{1-k^2}{h^2-k^2} \right) - 3 \right] \\ & = k \left( \frac{1-h^2}{k^2-h^2} \right) \left[ 3 - 4 \left( \frac{1-h^2}{k^2-h^2} \right) \right] \end{aligned}$$

$$h(1-k^3)(4-k^2-2h^2) = k(1-h^2)(3k^2+h^2-4)$$

$$\text{化簡之可得 } (h-k)^2(3+hk) = 4(1-hk)^2$$

$$39. \quad \therefore \theta = \frac{\pi}{2} + \phi \quad \therefore \sin \theta = \cos \phi \quad \text{及} \quad \cos \theta = -\sin \phi$$

$$\text{故原兩式爲} \begin{cases} \frac{ax}{\sin \phi} + \frac{by}{\cos \phi} + (a^2 - b^2) = 0 \\ \frac{by}{\sin \phi} - \frac{ax}{\cos \phi} + (a^2 - b^2) = 0 \end{cases}$$

$$\therefore \frac{1}{(a^2 - b^2)(by + ax)} = \frac{1}{(a^2 - b^2)(by - ax)} = -\frac{1}{a^2x^2 + b^2y^2}$$

$$\therefore \left[ \frac{a^2x^2 + b^2y^2}{(a^2 - b^2)(by + ax)} \right]^2 + \left[ \frac{a^2x^2 + b^2y^2}{(a^2 - b^2)(by - ax)} \right]^2 = 1$$

$$\left[ \frac{a^2x^2 + b^2y^2}{a^2 - b^2} \right]^2 \left[ \frac{1}{(by + ax)^2} + \frac{1}{(by - ax)^2} \right] = 1$$

$$\text{即 } 2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2$$

$$40. \text{ 從(2) } \tan \theta + \tan \phi = b \tan \theta \tan \phi$$

$$\text{以(1)代入 } \tan \theta \tan \phi = a/b$$

$$\text{今 } \tan \alpha = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{a}{1 - \frac{a}{b}} = \frac{ab}{b - a}$$

$$\text{故 } ab + (a - b) \tan \alpha = 0$$

$$41. (1)^2 + (2)^2, \quad 2 + 2 \cos(\theta - \phi) = a^2 + b^2$$

$$\text{故 } a^2 + b^2 = 2(1 + \cos \alpha)$$

$$42. \text{ 從(1) } \frac{x}{\sin \phi - \sin \theta} = \frac{y}{\cos \theta - \cos \phi} = \frac{1}{\sin(\phi - \theta)}$$

$$\therefore \frac{x}{a} = \frac{\cos \frac{1}{2}(\theta + \phi)}{\cos \frac{1}{2}(\phi - \theta)}, \quad \frac{y}{b} = \frac{\sin \frac{1}{2}(\theta + \phi)}{\cos \frac{1}{2}(\phi - \theta)}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2}{1 + \cos(\theta - \phi)} = \frac{2}{1 + \cos 2\alpha}$$

故 
$$\frac{x^2 + y^2}{a^2 + b^2} = \frac{2}{1 + \cos 2\alpha}$$

43.  $(3)^2 - (2)^2, 2 + \frac{2}{\sin \theta \sin \phi} - \frac{2 \cos \theta \cos \phi}{\sin \theta \sin \phi} = c^2 - b^2$

$$\therefore \frac{2 - 2 \cos(\theta + \phi)}{\sin \theta \sin \phi} = c^2 - b^2$$

$$2[1 - \cos(\theta + \phi)] = (c^2 - b^2) \sin \theta \sin \phi \quad (4)$$

從(3)  $\sin \theta + \sin \phi = c \sin \theta \sin \phi \quad (5)$

從(2)  $\sin(\theta + \phi) = b \sin \theta \sin \phi \quad (6)$

(4)/(5),  $2c \sin \frac{\theta + \phi}{2} = (c^2 - b^2) \cos \frac{\theta - \phi}{2} \quad (7)$

(5)/(6),  $b \cos \frac{\theta - \phi}{2} = c \cos \frac{\theta + \phi}{2} \quad (8)$

4(8)<sup>2</sup> + (7)<sup>2</sup>,  $4c^2 = [4b^2 + (b^2 - c^2)^2] \cos^2 \frac{\theta - \phi}{2} \quad (9)$

又從(1)  $2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} = a$

以(8)代入  $2b \cos^2 \frac{\theta - \phi}{2} = ac \quad (10)$

從(9), (10)  $8bc^2 = ac [4b^2 + (b^2 - c^2)^2]$

即  $8bc = a [4b^2 + (b^2 - c^2)^2]$

44.  $(1)^2 + (2)^2, a^2 + b^2 + 2ab \cos(\theta + \phi) = c^2$

即  $\cos(\theta + \phi) = \frac{c^2 - a^2 - b^2}{2ab}$

即  $\sec^2(\theta + \phi) = \frac{4a^2b^2}{(c^2 - a^2 - b^2)^2}$

從(3)  $\tan^2(\theta + \phi) = x^2/y^2$



$$\begin{aligned} \text{則} \quad \frac{x^2}{y^2} &= \frac{4a^2b^2}{(c^2 - a^2 - b^2)^2} - 1 \\ &= \frac{(2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2)}{(c^2 - a^2 - b^2)^2} \end{aligned}$$

$$\begin{aligned} \text{今分子} &= [(a+b)^2 - c^2][c^2 - (a-b)^2] \\ &= (a+b+c)(a+b-c)(c-a+b)(c+a-b) \\ &= 16s(s-a)(s-b)(s-c) \\ &\quad (\text{設 } a+b+c=2s) \end{aligned}$$

代入上式，去分母又開平方，則得

$$x(c^2 - a^2 - b^2) = 4y\sqrt{s(s-a)(s-b)(s-c)}$$

或從(1),(2)消去  $\theta$  得  $\cos \phi = \frac{b^2 + c^2 - a^2}{2bc}$

$$\text{則} \quad \cos \theta = \frac{c^2 + a^2 - b^2}{2ca}$$

設法化入(3)亦可求得，惟較此為繁耳。

$$45. (1) + (2), \quad a + b = m \cos^2 \phi + n \sin^2 \phi$$

$$\text{即} \quad (a+b)(\sin^2 \phi + \cos^2 \phi) = m \cos^2 \phi + n \sin^2 \phi$$

$$\therefore \tan^2 \phi = (m - a - b) / (a + b - n) \quad (4)$$

$$\begin{aligned} m(2) + n(1), \quad (ma + nb)\sin^2 \theta + (mb + na)\cos^2 \theta \\ = mn(\sin^2 \theta + \cos^2 \theta) \end{aligned}$$

$$\therefore \tan^2 \theta = (mn - na - mb) / (ma + nb - mn)$$

$$\text{代入(3)} \quad \frac{m(mn - na - mb)}{ma + nb - mn} - \frac{n(m - a - b)}{a + b - n} = 0$$

$$\text{化簡得} \quad (a+b)[(a+b)(m+n) - 2mn] = 0$$

$$\therefore a+b=0 \quad \text{或} \quad a+b=2mn/(m+n)$$

$$46 \quad (1)^3, \quad (\cos \theta + \cos \phi)(\cos^2 \theta + \cos^2 \phi + 2 \cos \theta \cos \phi) = a^3 \quad (4)$$

$$\text{自(3)} \quad 4(\cos^3 \theta + \cos^3 \phi) - 3(\cos \theta + \cos \phi) = c$$

$$\therefore (\cos \theta + \cos \phi)(4 \cos^2 \theta - 4 \cos \theta \cos \phi + 4 \cos^2 \phi - 3) = c \quad (5)$$

$$\begin{aligned} \text{從(4), (5)} \quad 2a^3 + c &= (\cos \theta + \cos \phi)(6 \cos^2 \theta + 6 \cos^2 \phi - 3) \\ &= 3a(2 \cos^2 \theta + 2 \cos^2 \phi - 2 + 1) \\ &= 3a(\cos 2\theta + \cos 2\phi + 1) \end{aligned}$$

$$\text{即} \quad 2a^3 + c = 3a(b + 1) \quad [\text{以(2)值代入}]$$

$$47. \quad (1) \cdot (2), \quad \sin 2\theta + \sin 2\phi + 2 \sin(\theta + \phi) = 2ab$$

$$\text{即} \quad \sin(\theta + \phi)[\cos(\theta - \phi) + 1] = ab \quad (4)$$

$$(1)^2 + (2)^2, \quad 2 + 2 \cos(\theta - \phi) = a^2 + b^2$$

$$\therefore \cos(\theta - \phi) = \frac{a^2 + b^2 - 2}{2}$$

$$\text{代入(4)} \quad \sin(\theta + \phi) = \frac{2ab}{a^2 + b^2} \quad (5)$$

$$(2)^2 - (1)^2, \quad \cos 2\phi + \cos 2\theta + 2 \cos(\phi + \theta) = b^2 - a^2$$

$$\text{即} \quad 2 \cos(\theta + \phi)[\cos(\theta - \phi) + 1] = b^2 - a^2$$

$$\therefore \cos(\theta + \phi) = \frac{b^2 - a^2}{a^2 + b^2}$$

$$\text{今} \quad \cos(\theta + \phi) + \cos(\theta - \phi) = \frac{b^2 - a^2}{a^2 + b^2} + \frac{a^2 + b^2 - 2}{2}$$

$$\text{即} \quad \cos \theta \cos \phi = [(a^2 + b^2)^2 - 4a^2] / 4(a^2 + b^2) \quad (6)$$

$$\text{從(3)} \quad \sin(\theta + \phi) = c \cos \theta \cos \phi$$

$$\text{以(5),(6)代入,得} \quad \frac{2ab}{a^2 + b^2} = c \frac{(a^2 + b^2)^2 - 4a^2}{4(a^2 + b^2)}$$

$$\text{即} \quad \frac{2ab}{c} = (a^2 + b^2)^2 - 4a^2$$

$$\begin{aligned} 48. \quad (1) - (2), \quad & \frac{x}{a} \cdot 2 \sin \frac{\theta + \phi}{2} \sin \frac{\phi - \theta}{2} \\ & + \frac{y}{b} \cdot 2 \sin \frac{\theta - \phi}{2} \cos \frac{\theta + \phi}{2} = 0 \end{aligned}$$

$$\text{即} \quad \frac{x}{a} \sin \frac{\theta + \phi}{2} - \frac{y}{b} \cos \frac{\theta + \phi}{2} = 0$$

$$\therefore \frac{\sin \frac{\theta + \phi}{2}}{ay} = \frac{\cos \frac{\theta + \phi}{2}}{bx} = \frac{1}{\sqrt{a^2 y^2 + b^2 x^2}} \quad (4)$$

$$\text{從(1),(2)得} \quad \sin \theta \left(1 - \frac{x}{a} \cos \phi\right) = \sin \phi \left(1 - \frac{x}{a} \cos \theta\right)$$

$$\text{即} \quad \frac{x}{a} \sin(\phi - \theta) = \sin \phi - \sin \theta$$

$$= 2 \sin \frac{\phi - \theta}{2} \cos \frac{\phi + \theta}{2}$$

$$\therefore \cos \frac{\theta - \phi}{2} = \frac{a}{x} \cos \frac{\theta + \phi}{2} = \frac{ab}{\sqrt{b^2 x^2 + a^2 y^2}} \quad (5)$$

$$\begin{aligned} \text{從(3)} \quad & a^2 \left( \cos \frac{\theta - \phi}{2} - \cos \frac{\theta + \phi}{2} \right) \\ & + b^2 \left( \cos \frac{\theta - \phi}{2} + \cos \frac{\theta + \phi}{2} \right) = 2c^2 \end{aligned}$$

$$\text{即} \quad (a^2 + b^2) \cos \frac{\theta - \phi}{2} - (a^2 - b^2) \cos \frac{\theta + \phi}{2} = 2c^2$$

以(4), (5)之值代入, 化簡之得

$$b(x(b^2 - a^2) + a(a^2 + b^2)) = 2c^2 \sqrt{b^2 x^2 + a^2 y^2}$$

或法: 從(3), 設  $\tan \frac{\theta}{2} = t$ ,  $\tan \frac{\phi}{2} = t'$ , 則為

$$\begin{aligned} (a^2 t t' + b^2)^2 &= c^4 \sec^2 \frac{\theta}{2} \sec^2 \frac{\phi}{2} \\ &= c^4 (1 + t^2)(1 + t'^2) \\ &= c^4 [1 + (t + t')^2 - 2tt' + t^2 t'^2] \end{aligned} \quad (4)$$

$$\text{今 } \sin \theta = \frac{2t}{1+t^2}, \quad \cos \theta = \frac{1-t^2}{1+t^2}$$

代入(1),  $b(a+x)t^2 - 2ayt + b(a-x) = 0$

同理可得  $b(a+x)t'^2 - 2ayt' + b(a-x) = 0$

故  $t, t'$  為下式  $T$  之根

$$b(a+x)T^2 - 2ayT + b(a-x) = 0$$

$$\therefore t + t' = 2ay/b(a+x), \quad tt' = (a-x)/(a+x)$$

代入(4),  $b^2[x(b^2 - a^2) + a(a^2 + b^2)]^2 = 4c^4(b^2 x^2 + a^2 y^2)$

再法: 從(3), 設  $r = \cos \theta$ ,  $r' = \cos \phi$ , 則

$$a^2 \sqrt{(1-r)(1-r')} + b^2 \sqrt{(1+r)(1+r')} = 2c^2 \quad (5)$$

從(1),  $\frac{y}{b} \sin \theta = 1 - \frac{x}{a} \cos \theta$

即  $a^2 y^2 (1 - r^2) = b^2 (a - rx)^2$

即  $(b^2 x^2 + a^2 y^2) r^2 - 2ab^2 x r + a^2 (b^2 - y^2) = 0$

同理從(2),  $(b^2 x^2 + a^2 y^2) r'^2 - 2ab^2 x r' + a^2 (b^2 - y^2) = 0$

故  $r, r'$  為下式  $A$  之根

$$(b^2x^2 + a^2y^2)A^2 - 2ab^2xA + a^2(b^2 - y^2) = 0$$

$$\therefore r + r' = \frac{2ab^2x}{b^2x^2 + a^2y^2}, \quad rr' = \frac{a^2(b^2 - y^2)}{b^2x^2 + a^2y^2}$$

$$\text{今 } (1-r)(1-r') = 1 - (r+r') + rr'$$

$$= \frac{b^2(a-x)^2}{b^2x^2 + a^2y^2},$$

$$(1+r)(1+r') = 1 + (r+r') + rr'$$

$$= \frac{b^2(a+x)^2}{b^2x^2 + a^2y^2}.$$

$$\text{代入(5)得 } b^2[x(b^2 - a^2) + a(a^2 + b^2)]^2 = 4c^4(b^2x^2 + a^2y^2)$$

$$49. (3)^2, \quad 1 - \sin(\theta + \phi) = 2(x^2 - y^2)^2 \quad (4)$$

$$(1)^2 + (2)^2, \quad 2 + 2\sin(\theta + \phi) = 16x^2y^2 + 4(x^2 - y^2)^2 \\ = 4(x^2 + y^2)^2$$

$$\therefore 1 + \sin(\theta + \phi) = 2(x^2 + y^2)^2 \quad (5)$$

$$(4) + (5), \quad 2 = 2(x^2 - y^2)^2 + 2(x^2 + y^2)^2$$

$$\therefore x^4 + y^4 = \frac{1}{2}$$

$$50. \text{ 即 } \begin{cases} x - y \cos \gamma - z \cos \beta = 0 \\ x \cos \gamma - y + z \cos \alpha = 0 \\ x \cos \beta + y \cos \alpha - z = 0 \end{cases}$$

今設三式中  $x, y, z$  有共同之解, 則

$$\Delta = \begin{vmatrix} 1 & -\cos \gamma & -\cos \beta \\ \cos \gamma & -1 & \cos \alpha \\ \cos \beta & \cos \alpha & -1 \end{vmatrix} = 0$$

$$\text{即 } 1 - 2\cos \alpha - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma = 0$$

$$-2 - 2\Pi \cos \alpha + \Sigma \sin^2 \alpha = 0$$

$$\therefore \Sigma \sin^2 \alpha = 2(1 + \Pi \cos \alpha)$$

$$51. \text{ 即 } bx \cos \theta + ay \sin \theta = ab \quad (1)$$

$$bx \cos \phi + ay \sin \phi = ab \quad (2)$$

$$\text{從(1)} \quad b^2 x^2 \cos^2 \theta = a^2 (b - y \sin \theta)^2$$

化去  $\cos \theta$  且合併之

$$(b^2 x^2 + a^2 y^2) \sin^2 \theta - 2a^2 b y \sin \theta + b^2 (a^2 - x^2) = 0$$

同樣化(2)可得  $\sin \phi$  之同一形式, 故知  $\sin \theta, \sin \phi$  爲

$$(b^2 x^2 + a^2 y^2) \lambda^2 - 2a^2 b y \lambda + b^2 (a^2 - x^2) = 0 \text{ 之根.}$$

故從根與係數關係得

$$\sin \theta + \sin \phi = 2a^2 b y / (b^2 x^2 + a^2 y^2) \quad (4)$$

同理從(1), (2), 各化去正弦可知  $\cos \theta, \cos \phi$  爲

$$(b^2 x^2 + a^2 y^2) \mu^2 - 2ab^2 x \mu - a^2 (b^2 - y^2) = 0 \text{ 之根.}$$

$$\text{故 } \cos \theta + \cos \phi = 2ab^2 x / (b^2 x^2 + a^2 y^2) \quad (5)$$

$$\text{以(4), (5)代入(3)} \quad \frac{2a^2 b^2}{b^2 x^2 + a^2 y^2} + \frac{2a^2 b^2}{b^2 x^2 + a^2 y^2} = 4$$

$$\text{即 } b^2 x^2 + a^2 y^2 = a^2 b^2$$

$$\text{即 } x^2/a^2 + y^2/b^2 = 1 \quad (\text{橢圓})$$

$$52. \text{ 化第一式爲 } y \sin \theta = 2a - x \cos \theta \quad (1)$$

$$y \sin \phi = 2a - x \cos \phi \quad (2)$$

$$\text{從(1)} \quad y^2 (1 - \cos^2 \theta) = 4a^2 - 4ax \cos \theta + x^2 \cos^2 \theta$$

$$\text{即 } (x^2 + y^2) \cos^2 \theta - 4ax \cos \theta + (4a^2 - y^2) = 0$$

$$\text{同理化(2)} \quad (x^2 + y^2) \cos^2 \phi - 4ax \cos \phi + (4a^2 - y^2) = 0$$

故知  $\cos \theta, \cos \phi$  爲下式  $\lambda$  之根

$$(x^2 + y^2)\lambda^2 - 4ax\lambda + (4a^2 - y^2) = 0$$

故  $\cos \theta + \cos \phi = 4ax / (x^2 + y^2),$

$$\cos \theta \cos \phi = (4a^2 - y^2) / (x^2 + y^2)$$

化(3)  $4 \cos^2 \frac{1}{2} \theta \cos^2 \frac{1}{2} \phi = 1$

即  $(1 + \cos \theta)(1 + \cos \phi) = 1$

即  $(\cos \theta + \cos \phi) + \cos \theta \cos \phi = 0$

代入得  $\frac{4ax}{x^2 + y^2} + \frac{4a^2 - y^2}{x^2 + y^2} = 0$  故  $y^2 = 4a(x + a)$

或從  $x \cos \theta + y \sin \theta = 2a, \quad x \cos \phi + y \sin \phi = 2a$

得  $\frac{x}{\sin \phi - \sin \theta} = \frac{y}{\cos \theta - \cos \phi} = \frac{2a}{\sin(\phi - \theta)}$

即  $\frac{x}{\cos \frac{1}{2}(\theta + \phi)} = \frac{y}{\sin \frac{1}{2}(\theta + \phi)} = \frac{2a}{\cos \frac{1}{2}(\theta - \phi)}$   
 $= \frac{x + 2a}{\cos \frac{1}{2}(\theta + \phi) + \cos \frac{1}{2}(\theta - \phi)}$

即  $\frac{x^2 + y^2}{\cos^2 \frac{1}{2}(\theta + \phi) + \sin^2 \frac{1}{2}(\theta + \phi)} = \left( \frac{x + 2a}{2 \cos \frac{1}{2} \theta \cos \frac{1}{2} \phi} \right)^2$

即  $\frac{x^2 + y^2}{1} = \left( \frac{x + 2a}{1} \right)^2$

即  $y^2 = 4a(x + a)$  (拋物線)



中學叢書

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