

H.B.FINE

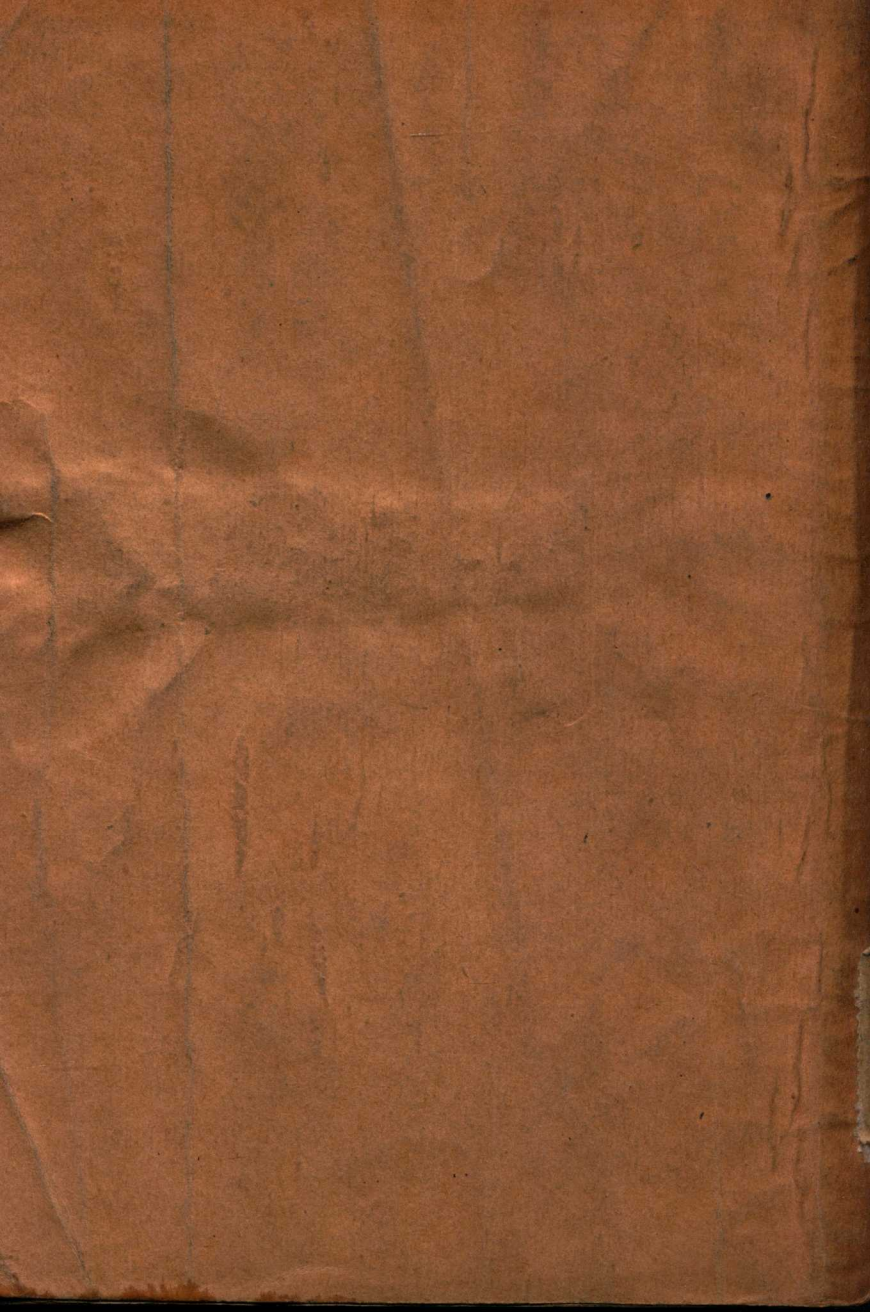
范氏大代数题解

莊 用 舟 編 演

995

$$x + y = 0$$

世界書局發行



中華民國三十五年七月新六版

范大代數題解

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編演者 莊用舟

發行人 李煜瀛

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例 言

一 本書係根據范氏大代數 (College Algebra By H. B. Fine) 教本習題所編之題解，每一習題均予以精確詳細之解答。

二 本書專供教師及學生於教授或演算時之參考。凡家居自修，預備應試，均可以本題解作為幫助，但僅為一般學生在演算困難，思索不得之際，作為指導，倘完全照書直抄，以此依賴，放棄練習演算，實非編者向願。

三 本書每一習題，重提示，詳簡不一，凡採用世界版漢譯本及其他書局出版之漢譯本，本題解一概適用。

四 本書在每習題前，均註明原書頁碼，使讀者便於查閱（世界版譯本與原書頁碼同），其他如排式醒目，印刷清晰，校對謹嚴，務期臻於完善，惟編印匆促，脫誤之處，尤恐難免，希國內各專家和讀者教正。



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范氏大代數題解

I. 緒 論

習 題 I

原本第 89 頁

- 解：函數 $x^2yz^3 + 2x^5y^4z^6 + 3x^7y^2z^8$ 中， x 的方次為 7， y 的方次為 4， z 的方次為 8。 y 與 z 的方次共為 10。 x, y, z 的方次共為 17。
- 解： $(x+1)(2x^2+3)(x^4-7)$ 為 7 次式。
- 解： $n=7$, $a_0=3$, $a_1=1$, $a_2=0$, $a_3=-4$, $a_4=1$, $a_5=0$, $a_6=0$, $a_7=-12$ 。
- 解： $f(0)=2(0)^3-(0)^2+3=3$ 。
 $f(-1)=2(-1)^3-(-1)^2+3=-2-1+3=0$ 。
 $f(3)=2 \times 3^3-3^2+3=54-9+3=48$ 。
 $f(8)=2 \times 8^3-8^2+3=1024-64+3=963$ 。
- 解： $f(0)=(0^2-3 \times 0+2)/(2 \times 0+5)=2/5$ 。
 $f(-2)=[(-2)^2-3(-2)+2]/\{2(-2)+5\}$
 $=\{4+6+2\}/\{-4+5\}=12$ 。
 $f(6)=(6^2-3 \times 6+2)/(2 \times 6+5)$
 $=(36-18+2)/(12+5)=20/17$ 。
- 解： $f(1)=1+\sqrt{1}+3=5$ 。
 $f(4)=4+\sqrt{4}+3=9$ 。
 $f(5)=5+\sqrt{5}+3=8+\sqrt{5}$ 。
- 解： $f(x-2)=2(x-2)+3=2x-4+3=2x-1$ 。
 $f(x^2+1)=2(x^2+1)+3=2x^2+2+3=2x^2+5$ 。
- 解： $f(0, 0)=0^3+0-0+8=8$ 。

$$f(1, 0) = 1^3 + 1 - 0 + 8 = 10.$$

$$f(0, 1) = 0^3 + 0 - 1 + 8 = 7.$$

$$f(1, 1) = 1^3 + 1 - 1 + 8 = 9.$$

$$\begin{aligned} f(-2, -3) &= (-2)^3 + (-2) - (-3) + 8 \\ &= -8 - 2 + 3 + 8 = 1. \end{aligned}$$

II. 基本演算

習題 II

原本第 97 頁

- 解: $4ax^2y + (-6ax^2y) + 5bx^2y + (-3bx^2y)$
 $= 4ax^2y - 6ax^2y + 5bx^2y - 3bx^2y$
 $= -2ax^2y + 2bx^2y = 2x^2y(b - a).$
- 解: $(7a^2 + 2a - b^2) + (3a + b^2 - 2a^2) + (b^2 - 4a - 4a^2)$
 $= (7 - 2 - 4)a^2 + (2 + 3 - 4)a + (-1 + 1 + 1)b^2$
 $= a^2 + a + b^2.$
- 解: $(3x^2 - 5x + 6) + (x^2 + 2x - 8) + (-4x^2 + 3x - 7)$
 $= (3 + 1 - 4)x^2 + (-5 + 2 + 3)x + (6 - 8 - 7) = -9.$
- 解: $(4a^3 + a^2b - 5b^3) + \left(\frac{5}{3}a^3 - 6ab^2 - a^2b\right) + \left(\frac{1}{3}a^3 + 10b^3\right)$
 $+ (6b^3 - 15ab^2 - 4a^2b - 10a^3)$
 $= \left(4 + \frac{5}{3} + \frac{1}{3} - 10\right)a^3 + (1 - 1 - 4)a^2b + (-6 - 15)ab^2$
 $+ (-5 + 10 + 6)b^3 = -4a^3 - 4a^2b - 21ab^2 + 11b^3.$
- 解: $(3a + b - c) - (4a - 2b + 6c) = 3a + b - c - 4a + 2b - 6c$
 $= -a + 3b - 7c.$
- 解: $(x^3 + 6x^2 + 5) - (2x^3 - 5x + 7) = x^3 + 6x^2 + 5 - 2x^3 + 5x - 7$
 $= x^3 + 4x^2 + 5x - 2.$
- 解: $(a^3 + b^3) - (a^3 + 5a^2b) = a^3 + b^3 - a^3 - 5a^2b = b(b^2 - 5a^2).$
- 解: $(x^3 + y^3 - 6x + 5y) - \{(-2x^2 - 6x + 7y - 8)$
 $+ (x^3 + 2x^2 - 5y + 9)\}$
 $= x^3 + y^3 - 6x + 5y - \{x^3 - 6x + 2y + 1\}$
 $= x^3 + y^3 - 6x + 5y - x^3 + 6x - 2y - 1 = y^3 + 3y - 1.$

9. 解: $-(a+b) + \{-a - (2a-b)\} - 6(a-4b)$
 $= -a - b - a - 2a + b - 6a + 24b = -10a + 24b.$
10. 解: $6x - \{4x + [2x - (3x + 5x + 7 - 1) + 3] - 8\}$
 $= 6x - 4x - [2x - (3x + 5x + 7 - 1) + 3] + 8$
 $= 2x - 2x + (3x + 5x + 7 - 1) - 3 + 8$
 $= 3x + 5x + 7 - 1 - 3 + 8 = 8x + 11.$
11. 解: $2a - [4a - c + \{3a - (4b - c) - (b + 3c)\} - 6c]$
 $= 2a - [4a - c + \{3a - 4b + c - b - 3c\} - 6c]$
 $= 2a - [4a - c + 3a - 4b + c - b - 3c - 6c]$
 $= 2a - 4a + c - 3a + 4b - c + b + 3c + 6c = -5a + 5b + 9c.$
12. 解: $z - [3x + (y + 5z)] - [x - (3y + 2z)]$
 $= z - 3x - y - 5z - x + 3y + 2z = 2y - 4x - 2z.$
13. 解: $x^3 - 7 - (x^2 + 8x + 5) = x^3 - 7 - x^2 - 8x - 5$
 $= x^3 - x^2 - 8x - 12.$
14. 解: $y^2 + x - 7 - (x^4 - 9x^2 + 3y) = y^2 + x - 7 - x^4 + 9x^2 - 3y$
 $= y^2 - 3y - x^4 + 9x^2 + x - 7.$

習 題 III

原本第 106 頁

1. 解: $3 - 2 - 1 + 7 - 6 + 5$
 $2 - 3 + 1$

 $6 - 4 - 2 + 14 - 12 + 10$
 $- 9 + 6 + 3 - 21 + 18 - 15$
 $3 - 2 - 1 + 7 - 6 + 5$

 $6 - 13 + 7 + 15 - 34 + 35 - 21 + 5$
- 答: $6x^7 - 13x^6 + 7x^5 + 15x^4 - 34x^3 + 35x^2 - 21x + 5.$
2. 解: $5 - 3 + 2 + 1$
 $3 - 1 - 2$

 $15 - 9 + 6 + 3$
 $- 5 + 3 - 2 - 1$
 $- 10 + 6 - 4 - 2$

 $15 - 14 - 1 + 7 - 5 - 2$

答: $15x^5 - 14ax^4 - a^2x^3 + 7a^3x^2 - 5a^4x - 2a^5$.

3. 解:
$$\begin{array}{r} 1-1+1-1+1-1 \\ 1+1 \\ \hline 1-1+1-1+1-1 \\ 1-1+1-1+1-1 \\ \hline 1+0+0+0+0+0-1 \end{array}$$

答: $x^6 - y^6$.

4. 解:
$$\begin{array}{r} 3-2+0+7 \\ 2+0-3+5 \\ \hline 6-4+0+14 \\ -9+6+0-21 \\ \hline 15-10+0+35 \\ \hline 6-4-9+35-10-21+35 \end{array}$$

答: $6x^6 - 4x^5 - 9x^4 + 35x^3 - 10x^2 - 21x + 35$.

5. 解: $(7x-2y)(4x-5y) = 28x^2 - 43xy + 10y^2$.

6. 解: $(a^2 - ax + bx - x^2)(b+x)$
 $= (a^2 - ax)(b+x) + (bx - x^2)(b+x)$
 $= a^2b - abx + a^2x - ax^2 + b^2x - x^3$
 $= a^2b + (a^2 - ab + b^2)x - ax^2 - x^3$.

7. 解:
$$\begin{array}{r} 1-1+5-2 \\ 1-1+3 \\ \hline 1-1+5-2 \\ -1+1-5+2 \\ \hline 3-3+15-6 \\ \hline 1-2+9-10+17-6 \end{array}$$

答: $x^6 - 2x^5 + 9x^4 - 10x^3 + 17x^2 - 6x$.

8. 解:
$$\begin{array}{r} 2+0-3+5 \\ 1-1 \\ \hline 2+0-3+5 \\ -2+0+3-5 \\ \hline 2-2-3+8-5 \end{array}$$

答: $2x^{2n-2} - 2x^{2n-3} - 3x^{2n-4} + 3x^{2n-5} - 5x^{2n-6}$.

9. 解: $(a^2 - ab + 3b^2)(a^2 + ab - 3b^2) = a^4 - (ab - 3b^2)^2$
 $= a^4 - a^2b^2 + 6ab^3 - 9b^4.$
10. 解: $(x + 3y - 2z)(x - 3y + 2z) = x^2 - (3y - 2z)^2$
 $= x^2 - 9y^2 + 12yz - 4z^2.$
11. 解: $(x^2 + xy + y^2 + x - y + 1)(x - y - 1)$
 $= [(x^2 + xy + y^2) + (x - y) + 1][(x - y) - 1]$
 $= (x^2 + xy + y^2)(x - y) - (x^2 + xy + y^2) + (x - y)^2 - 1$
 $= x^3 - y^3 - x^2 - xy - y^2 + x^2 - 2xy + y^2 - 1$
 $= x^3 - y^3 - 3xy - 1.$
12. 解: $(a^2 + b^2 + c^2 + bc + ca - ab)(a + b - c)$
 $= [a(a - b + c) + (b^2 + bc + c^2)][a + (b - c)]$
 $= a[a^2 - (b - c)^2] + a(b^2 + bc + c^2) + (b^2 + bc + c^2)(b - c)$
 $= a^3 - ab^2 + 2abc - ac^2 + ab^2 + abc + ac^2 + b^3 - c^3$
 $= a^3 + b^3 - c^3 + 3abc.$
13. 解: $(3x - 2y + 5)(x - 4y + 6)$
 $= 3x^2 - (3 \times 4 + 2 \times 1)xy + 8y^2 + (3 \times 6 + 5 \times 1)x$
 $- (2 \times 6 + 5 \times 4)y + 30$
 $= 3x^2 - 14xy + 8y^2 + 23x - 32y + 30.$
14. 解: $(x + 7y - 3z)(2x + y - 8z)$
 $= 2x^2 + (1 + 14)xy + 7y^2 - (6 + 8)xz - (3 + 56)yz + 24z^2$
 $= 2x^2 + 15xy + 7y^2 - 14xz - 59yz + 24z^2.$
15. 解: $(b + x)(b - x)(b^2 + x^2) = (b^2 - x^2)(b^2 + x^2) = b^4 - x^4.$
16. 解: $(x^2 + x + 1)(x^2 - x + 1)(x^4 - x^2 + 1)$
 $= [(x^4 + 1)^2 - x^2][x^4 - x^2 + 1] = [x^4 + x^2 + 1][x^4 - x^2 + 1]$
 $= [(x^4 + 1)^2 - x^4] = x^8 + x^4 + 1.$
17. 解: $(x + y + z)(-x + y + z)(x - y + z)(x + y - z)$
 $= [(y + z)^2 - x^2][x^2 - (y - z)^2]$
 $= -x^4 + [(y + z)^2 + (y - z)^2]x^2 - (y + z)^2(y - z)^2$
 $= -x^4 + (2y^2 + 2z^2)x^2 - (y^2 - z^2)^2$
 $= -x^4 + 2y^2x^2 + 2z^2x^2 - y^4 + 2y^2z^2 - z^4$
 $= 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4.$
18. 解: $x^2 + x + 1$ 前四幂之係數爲: $1 + 1 + 1, 1 + 2 + 3 + 2 + 1,$
 $1 + 3 + 6 + 7 + 6 + 3 + 1$ 及 $1 + 4 + 10 + 16 + 19 + 16 + 10$
 $+ 4 + 1.$

19. 解：依 § 312 之方法，得：

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & & 1 & 1 \\
 & & & & & & & 1 & 2 & 1 \\
 & & & & & & & 1 & 3 & 3 & 1 \\
 & & & & & & & 1 & 4 & 6 & 4 & 1 \\
 & & & & & & & 1 & 5 & 10 & 10 & 5 & 1 \\
 & & & & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 & & & & & & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
 & & & & & & & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
 & & & & & & & 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
 & & & & & & & 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1
 \end{array}$$

20. 解： $(4x-3y)^2 = (4x)^2 + 2(4x)(-3y) + (-3y)^2$
 $= 16x^2 - 24xy + 9y^2.$

$$\begin{aligned}
 & (4x-3y)^3 \\
 &= (4x)^3 + 3(4x)^2(-3y) + 3(4x)(-3y)^2 + (-3y)^3 \\
 &= 64x^3 - 144x^2y + 108xy^2 - 27y^3.
 \end{aligned}$$

21. 解： $(x+2y+3z-4u)^2 = [(x+2y) + (3z-4u)]^2$
 $= (x+2y)^2 + 2(x+2y)(3z-4u) + (3z-4u)^2$
 $= x^2 + 4xy + 4y^2 + 6xz + 12yz - 8xu - 16yu$
 $+ 9z^2 - 24zu + 16u^2.$

22. 解： $(x+2y+3z)^3 = [(x+2y) + 3z]^3$
 $= (x+2y)^3 + 3(x+2y)^2(3z) + 3(x+2y)(3z)^2 + (3z)^3$
 $= x^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3 + 3(x^2 + 4xy + 4y^2)(3z)$
 $+ 3(x+2y)(9z^2) + 27z^3$
 $= x^3 + 6x^2y + 12xy^2 + 8y^3 + 9x^2z + 36xyz + 36y^2z$
 $+ 27xz^2 + 54yz^2 + 27z^3.$
 $(x+2y-3z)^3 = [(x+2y) - 3z]^3$
 $= (x+2y)^3 - 3(x+2y)^2(3z) + 3(x+2y)(3z)^2 - (3z)^3$
 $= x^3 + 6x^2y + 12xy^2 + 8y^3 - 9x^2z - 36xyz - 36y^2z$
 $+ 27xz^2 + 54yz^2 - 27z^3.$

23. 解： $(a+2b)^2(a-2b)^2 = \{a^2-4b^2\}^2 = a^4 - 8a^2b^2 + 16b^4.$

24. 解：此乘積之方次為 $27+19=46$ 及 $46-29=17.$

故 x^{29} 之係數為 $a_0b_{17} + a_1b_{16} + \dots + a_{16}b_1 + a_{17}b_0.$

又 $46 - 15 = 31$ 及 $31 - 19 = 12$.

故 x^{15} 之係數為 $a_{12}b_{19} + a_{13}b_{18} + \dots + a_{27}b_1$.

25. 解: x^6 之係數為 $2(-8) + 3(-3) + 4 \times 2 + (-7)(-1)$
 $+ 2 \times 3 + (-5)0 = -16 - 9 + 8 + 7 + 6 + 0 = -4.$

x^8 之係數為 $0(-8) + 0 \times 3 + 2 \times 2 + (-3)(-1)$
 $+ 4 \times 0 + (-7)3 + 2 \times 0 + (-5)0 = 0 + 0 + 4 + 3 + 0$
 $- 21 + 0 + 0 = -14.$

x^4 之係數為 $4(-8) + (-7)3 + 0 \times 2 + 2(-1)$
 $+ (-5) \times 0 = -32 - 21 + 0 - 2 + 0 = -55.$

26. 解: 1. $(x+y+z)^3 - (x^3+y^3+z^3)$
 $= (x+y)^3 + 3z(x+y)^2 + 3z^2(x+y) + z^3 - x^3 - y^3 - z^3$
 $= 3xy(x+y) + 3z(x+y)^2 + 3z^2(x+y)$
 $= 3(x+y)[xy + z(x+y) + z^2]$
 $= 3(x+y)(y+z)(z+x).$

2. $(a^2+b^2)(x^2+y^2) = a^2x^2 + b^2y^2 + b^2x^2 + a^2y^2$
 $= a^2x^2 + 2abxy + b^2y^2 + b^2x^2 - 2abxy + a^2y^2$
 $= (ax+by)^2 + (bx-ay)^2.$

3. $(a^2-b^2)(x^2-y^2) = a^2x^2 + b^2y^2 - b^2x^2 - a^2y^2$
 $= a^2x^2 + 2abxy + b^2y^2 - b^2x^2 - 2abxy - a^2y^2$
 $= (ax+by)^2 - (bx+ay)^2.$

4. $(a+b+c)^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$
 $= a^3 + 3a^2(b+c) + 3ab^2 + 6abc + 3ac^2 + b^3$
 $+ 3b^2c + 3bc^2 + c^3$
 $= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c)$
 $+ 3c^2(a+b) + 6abc.$

27. 解: $(2a^2x^3y^7)^5 = 32a^{10}x^{15}y^{35}.$

$(-x^5y^8z^9)^7 = -x^{35}y^{56}z^{63}.$

$(a^2b^m c^3)^{2n} = a^{4n}b^{2mn}c^{6n}.$

$(a^m b^n c^{2n})^n = a^{mn}b^{n^2}c^{2n^2}.$

23. 解: $(-ab^2c^3)(a^3b)^2(-ac^8)^5 = (-ab^2c^3)(a^6b^2)(-a^5c^{15})$
 $= a^{12}b^4c^{18}.$

$(-2x^2y^4)^3(ax^5y^{11})^2 = (-8x^6y^{12})(a^2x^{10}y^{22})$
 $= -8a^2x^{16}y^{34}.$

原本第 110 頁

1. 解: $15a^3bc^3/10ab^2c^3=3a^2/2b.$
2. 解: $75x^2y^4z^{10}/-100ax^7z^9=-3y^4z/4ax^5.$
3. 解: $-35x^{2m}y^m/28x^my^{m+n}=-5x^m/4y^n.$
4. 解: $-54\{(ab^2)^2c\}^5/-18\{a(b^2c^2)\}^3$
 $=-54a^{10}b^{20}c^5/-18a^6b^{12}c^6=3a^7b^8/c.$
5. 解: $\frac{x^2y-xy^2}{x^2-y^2}=\frac{xy(x-y)}{(x-y)(x+y)}=\frac{xy}{x+y}.$
6. 解: $\frac{(x^2-y^2)(x^3+y^3)}{(x-y)(x^2-xy+y^2)}$
 $=\frac{(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)}{(x-y)(x^2-xy+y^2)}$
 $= (x+y)(x^2+xy+y^2).$
7. 解: $\frac{(a-b)^2(b-c)^3(c-a)^4}{(b-a)(c-b)^2(a-c)^3}$
 $=\frac{(a-b)^2(b-c)^3(c-a)^4}{-(a-b)(b-c)^2[-(c-a)^3]}$
 $= (a-b)(b-c)(c-a).$
8. 解: $\frac{30a^2b^3c^4-25a^3b^2c^5+20a^4b^4c^7}{-5ab^2c^3}$
 $= -6abc+5a^2c^2-4a^3b^2c^4.$
9. 解: $\frac{3(y-x)^4-2(x-y)^3+5(x-y)^2}{(y-x)^2}$
 $=\frac{(x-y)^2[3(x-y)^3-2(x-y)+5]}{(x-y)^2}$
 $= 3(x-y)^3-2(x-y)+5.$
10. 解: $4a^7 \times (3ab^3c^2)^2 \div (abc)^2 \div 6bc$
 $= 4a^7 \times 9a^2b^6c^4 \div a^2b^2c^2 \div 6bc = 6a^7b^3c.$
11. 解: (1) $a^7 \div \{a^5 \div (a^4 \div x^2 \times a) \times (a^3 \times a \div a^2)\}$
 $= a^7 \div \{a^5 \div a^3 \times a^2\}$
 $= a^7 \div a^4 = a^3.$
 (2) $a^7 \div \{a^5 \div (a^4 \div a^2 \times a) \times (a^3 \times a \div a^2)\}$
 $= a^7 \div \{a^5 \div a^4 \times a^2 \div a \times a^2 \times a \div a^3\}$

$$= a^7 \div a^5 \times a^4 \div a^2 \times a \div a^3 \div a \times a^2$$

$$= a^8.$$

12. 解: $\frac{-4a^2(x^2y^3)^2}{2a(x^2y^3)^2} = \frac{-4a^2x^6y^6}{2ax^4y^6} = -\frac{2ax^2}{y^2}.$

III. 一元一次方程

習題 V

原本第 119 頁

1. 解: $15 - 7 + 5x = 2x + 5 - 3x$

$$8 + 5x = 5 - x$$

$$6x = -3$$

$$\therefore x = -\frac{1}{2}.$$

2. 解: $x^2 + 3x - 4x^2 + 20x = 15x - 3x^2 - 16$

$$-3x^2 + 23x = 15x - 3x^2 - 16$$

$$8x = -16$$

$$\therefore x = -2.$$

3. 解: $x^2 + 3x + 2 - x^2 - 7x - 12 = 0$

$$-4x = 10$$

$$\therefore x = -\frac{10}{4} = -\frac{5}{2}.$$

4. 解: $x - \frac{x}{2} - \frac{x}{4} - \frac{x}{8} - \frac{x}{16} = 1$

$$\frac{x}{16} = 1$$

$$\therefore x = 16.$$

5. 解: $x - 2[x - 3x - 12 - 5] = 3\{2x - [x - 8x + 32]\} - 2$

$$x - 2x + 6x + 24 + 10 = 6x - 3x + 24x - 96 - 2$$

$$22x = 132$$

$$\therefore x = 6.$$

6. 解: $2\{3[20x - 4 - 8] - 20\} - 7 = 1$

$$2\{60x - 36 - 20\} - 7 = 1$$

$$120x - 112 - 7 = 1$$

$$120x = 120$$

$$\therefore x = 1.$$

$$7. \text{ 解: } \frac{1}{2} \left\{ \frac{1}{3} \left[\frac{1}{20}x - \frac{1}{4} - 6 \right] + 4 \right\} = 1$$

$$\frac{1}{2} \left\{ \frac{1}{60}x - \frac{1}{12} - 2 + 4 \right\} = 1$$

$$\frac{1}{120}x - \frac{1}{24} + 1 = 1$$

$$\frac{1}{120}x = \frac{1}{24}$$

$$\therefore x = 5.$$

$$8. \text{ 解: } \frac{15 - 5 + 2x}{5} = \frac{40 - 4 + 7x + 5x + 10}{10}$$

$$30 - 10 + 4x = 40 - 4 + 7x + 5x + 10$$

$$8x = -26$$

$$\therefore x = -\frac{13}{4}.$$

$$9. \text{ 解: } \frac{3x - 1 + 9}{3} = \frac{-2x + 8 + 9x + 15 - 30}{12}$$

$$12x + 32 = 7x - 7$$

$$5x = -39$$

$$\therefore x = -\frac{39}{5}.$$

$$10. \text{ 解: } \frac{10x - .8 + .39x - .015}{.6} = \frac{13.95 - 8x}{1.2}$$

$$20x - 1.6 + .78x - .03 = 13.95 - 8x$$

$$28.78x = 15.58$$

$$\therefore x = \frac{1558}{2878} = \frac{779}{1439}.$$

$$11. \text{ 解: } 3cx - 5a + b - 2c = 6b - a - 3bx - 2c$$

$$3cx + 3bx = 5b + 4a$$

$$\therefore x = \frac{5b + 4a}{3(c + b)}.$$

$$12. \text{ 解: 原式} = ab - a^2 - bx + cx + bc - ab - cx + ax + ac - bc - ax + bx = 1 - x$$

$$1-x=0$$

$$\therefore x=1.$$

$$13. \text{ 答: } x=a.$$

$$14. \text{ 解: } \begin{aligned} (x+1)(a-b) + (x-1)(a+b) &= 2a \\ ax+a-bx-b + ax-a+bx-b &= 2a \\ 2ax-2b &= 2a \end{aligned}$$

$$\therefore x = \frac{a+b}{a}.$$

$$15. \text{ 解: } \left(\frac{m^2+n^2}{mn}\right)x = \frac{m^2-n^2}{mn} - \frac{2mnx}{mn}$$

$$x\left(\frac{m^2+n^2+2mn}{mn}\right) = \frac{m^2-n^2}{mn}$$

$$x(m+n)^2 = (m+n)(m-n)$$

$$\therefore x = \frac{m-n}{m+n}.$$

$$16. \text{ 解: } 2x-1=0, 3x-1=0, 4x+1=0, 5x+2=0$$

$$\therefore x = \frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, -\frac{2}{5}.$$

$$17. \text{ 解: } (x^2-x)(2x-5-x-9)=0$$

$$x(x-1)(x-14)=0$$

$$\therefore x=0, 1, 14.$$

$$18. \text{ 解: } x^3+6x^2+12x+8-x^3+6x^2-12x+8=32x+16$$

$$12x^2-32x=0$$

$$x(12x-32)=0$$

$$\therefore x=0, 2\frac{2}{3}.$$

$$19. \text{ 解: } [ax+bx-c]^2 = [ax-bx+c]^2$$

$$ax+bx-c = \pm(ax-bx+c)$$

$$ax+bx-c = ax-bx+c,$$

$$ax+bx-c = -ax+bx-c$$

$$\therefore x = \frac{c}{b}, x=0.$$

$$20. \text{ 解: } (x-1)^4 - (x-1)^2(x-3)^2 = 0$$

$$(x-1)^2[(x-1)^2 - (x-3)^2] = 0$$

$$(x-1)^2(x^2-2x+1-x^2+6x-9)=0$$

$$(x-1)^2=0, 4x-8=0$$

$$\therefore x=1, 1, 2.$$

習 題 VI

原本第 124 頁

1. 解：設 x 爲十位數字， $14-x$ 爲個位數字。

$$10x+(14-x)=10(14-x)+x-18$$

$$10x+14-x=140-10x+x-18$$

$$18x=108$$

$$\therefore x=6,$$

$$14-x=8.$$

答：某數爲 68.

2. 解：設 x 爲除數。

$$156=11x+2$$

$$11x=154$$

$$\therefore x=14.$$

3. 解：設 x 爲小數， $x+298$ 爲大數。

$$x+298=12x+12$$

$$11x=286$$

$$\therefore x=26,$$

$$x+298=324.$$

4. 解：設 x 爲個位數字， $2x$ 爲十位數字。

$$(1+2x)10+(5+x)=3[10(x-1)+(2x-5)]$$

$$10+20x+5+x=30x-30+6x-15$$

$$15x=60$$

$$\therefore x=4,$$

$$2x=8.$$

答：該數爲 84.

5. 解：設 x 爲某數。

$$4(x-2)=2x+\frac{1}{2}(x-1)$$

$$\therefore x=5.$$

6. 解：設 x 爲子現年， $4x$ 爲父現年。

$$4x + 20 = 2(x + 20)$$

$$\therefore x = 10 \text{ (子現年),}$$

$$4x = 40 \text{ (父現年).}$$

設 y 年後父年爲子年之三倍。

$$40 + y = 3(10 + y)$$

$$\therefore y = 5 \text{ 年.}$$

答：5 年後，父年爲子年之三倍。

7. 解：設 x 小時後，池中水方能洩盡，故一小時內放去池水 $\frac{1}{x}$ 。

$$\frac{1}{x} = \left(\frac{1}{2} + \frac{1}{4}\right) - \frac{1}{3} = \frac{5}{12}$$

$$\therefore x = \frac{12}{5} = 2.4 \text{ 小時, 即二時二十四分.}$$

8. 解：設 x 與 y 各爲 A 與 B 獨作所需之日數，故 $\frac{1}{x}$ 與 $\frac{1}{y}$ 各爲 A 與 B 每日所作之工程。

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{10} \dots\dots\dots(1)$$

$$\left(\frac{1}{x} + \frac{1}{y}\right)7 + \frac{5}{y} = 1 \dots\dots\dots(2)$$

$$\therefore x = 25 \text{ 日, } y = \frac{50}{3} = 16\frac{2}{3} \text{ 日.}$$

9. 解：設 x 爲二針在同一方向時之分數， y 爲二針成相反方向時之分數。

$$x = 40 + \frac{x}{12}, y = 40 + \frac{y}{12} - 30$$

$$\therefore x = 43\frac{7}{11} \text{ 分, } y = 10\frac{10}{11} \text{ 分.}$$

答：8 時 $43\frac{7}{11}$ 分二針成同一方向，8 時 $10\frac{10}{11}$ 分二針成相反方向。

10. 解：設 x 爲所求之分數。

$$x = 20 + \frac{x}{12} \pm 15$$

$$\therefore x = 5\frac{5}{11} \text{ 分 及 } 38\frac{2}{11} \text{ 分.}$$

答：4時5 $\frac{5}{11}$ 分及4時38 $\frac{2}{11}$ 分時二針成直角。

11. 解：設 x 爲時針與分針連續相重二次間之正確分數。

$$x = 60 + \frac{x}{12}$$

$$\therefore x = 65 \text{ 分 } 27 \text{ 秒.}$$

答：每小時之錯誤爲

$$\frac{(66' - 65'27'')60'}{65'27''} = 30 \text{ 秒.}$$

12. 解：設 x 爲 A 所得之洋數。

$$x + \frac{2}{3}x + \frac{2}{3} \cdot \frac{2}{3}x + \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}x = 1300$$

$$\therefore x = 540 \text{ 元 (A 所得).}$$

答： A, B, C, D 所得各爲 540 元, 360 元, 240 元, 160 元。

13. 解：設 x 爲財產總數。

$$\left(\frac{x}{2} + 1000\right) + \left[\frac{x - \left(\frac{x}{2} + 1000\right)}{2} + 1000\right] \\ - \left[\frac{x - \left\{\left(\frac{x}{2} + 1000\right) + \left[\frac{x - \left(\frac{x}{2} + 1000\right)}{2} + 1000\right]\right\}}{2} + 1000\right] \\ + 3500 = x$$

$$\therefore x = 42000 \text{ 元.}$$

14. 解：設 x 爲正方形之邊長。

$$x^2 + 100 = (x + 2)^2 = x^2 + 4x + 4$$

$$\therefore x = 24 \text{ 呎, } x^2 = 576 \text{ 方呎.}$$

答：面積爲 576 方呎。

15. 解：設 x 爲竿長。

$$(x + 2)^2 = x^2 + 18^2 \quad \therefore x = 80 \text{ 呎,}$$

16. 解：設 x 爲金圖之數。

$$x + 0.5(2x) + 0.1(3x) = 11.5$$

$$\therefore x = 5.$$

答：袋中有一圓者五個，半圓者十個，一角者十五個。

17. 解：設利率 6% 之存款為 x 元。

$$\frac{6}{100}x + \frac{4}{100}(5000 - x) = \frac{5.5}{100} \times 5000$$

$$\therefore x = 3750 \text{ 元}, 5000 - x = 1250 \text{ 元}.$$

答：利率 4% 之存款為 1250 元，6% 之存款為 3750 元。

18. 解：設 $x:1$ 為所求之比例。

$$20x + 30 = 26(1 + x)$$

$$\therefore x = \frac{2}{3}.$$

答：二種咖啡以 2:3 混合始成每磅價 26 分。

19. 解：設 x 為所加之銅量。

$$\frac{3}{5} + x = \frac{7(x+1)}{3+7}$$

$$\therefore x = \frac{1}{3} \text{ 磅}.$$

20. 解：設 x 為第一次所加之水量。

$$\frac{30}{100}(x+1) = \frac{20}{100}(2x+1)$$

$$\therefore x = 1, \frac{30}{100}(x+1) = 60\%.$$

答：第一次加水一磅，第二次加水二磅，酒精在原液體中佔 60%。

21. 解：設一火車自 A 站開出 x 小時後，兩車相遇。

$$45x + 50\left(x - \frac{30}{60}\right) = 90$$

$$\therefore x = 1\frac{4}{19} \text{ 時} = 1 \text{ 時 } 12\frac{12}{19} \text{ 分},$$

$$50\left(x - \frac{30}{60}\right) = 35\frac{10}{19} \text{ 哩}.$$

答：11 時 12 $\frac{12}{19}$ 分兩車相遇，距 B 站 35 $\frac{10}{19}$ 哩。

22. 解：設 x 爲快車之速率。

$$\frac{45}{\frac{3}{5}x} = \frac{45}{x} + 0.5$$

$$\therefore x = 60 \text{ 哩/時}, \frac{3}{5}x = 36 \text{ 哩/時},$$

$$\frac{45}{60} + 0.5 + 10 = 11\frac{15}{60} \text{ 時}.$$

答：快車之速率爲每小時 60 哩；慢車 36 哩；二車相遇之時間爲 11 時 15 分。

23. 解：設兔再行 x 步後爲狐追及。

$$x + 50 = \frac{4}{5}x \times \frac{3}{2} = \frac{6}{5}x$$

$$\therefore x = 250 \text{ 步}.$$

24. 解：設合金內含金 x 兩，故含銀 $(387 - x)$ 兩。

$$x \times \frac{19 - 18}{19} + (387 - x) \times \frac{10 - 9}{10} = 387 - 351$$

$$\therefore x = 57 \text{ 兩}, 387 - x = 330 \text{ 兩(銀)}.$$

25. 解：設 x 爲此人所帶之洋數。

$$x = \left(\frac{x}{2} + 2 \right) + \left\{ \frac{x - \left(\frac{x}{2} + 2 \right)}{2} + 2 \right\}$$

$$+ \left\{ \frac{x - \left[\left(\frac{x}{2} + 2 \right) + \left(\frac{x - \left(\frac{x}{2} + 2 \right)}{2} + 2 \right) \right]}{2} + 2 \right\}$$

$$\therefore x = 28 \text{ 元}.$$

26. 解：設 x 爲底邊或高之長。

$$4\left(\frac{x^2}{2}\right) + x^2 + 117 = 4\left[\frac{(x+3)^2}{2}\right] + (x+3)^2$$

$$\therefore x = 5 \text{ 吋},$$

$$4\left(\frac{x^2}{2}\right) + x^2 = 75 \text{ 方吋}.$$

27. 解：設 x 爲個位數字， $a - x$ 爲十位數字，則

$$10(a - x) + x = 10x + (a - x) + b$$

$$\therefore x = \frac{9a+b}{18}, a-x = \frac{9a-b}{18}.$$

因數字必須爲正整數，故 $9a$ 須 $\geq b$ ，且 $9a+b$ 及 $9a-b$ 必須被 18 除盡。

28. 解：設距今 x 年時， A 年爲 B 年之 c 倍。

$$a+x=c(b+x)=cb+cx$$

$$\therefore x = \frac{a-cb}{c-1}.$$

1. 如 $a > b$ ，則 c 必須大於 1，否則 $x = \frac{a-cb}{c-1} < \frac{a-ca}{c-1} = -a < -b$ ，此時在 A, B 未生以前，故不合理。

2. 如 $a < b$ ，則 c 必須小於 1，否則 $x = \frac{a-cb}{c-1} < \frac{b-cb}{c-1} = -b < -a$ ，此時也在 A, B 未生以前，故不合理。

3. 如 $a = b$ ，則 c 必須等於 1，否則 $x < -a = -b$ ，故不可能。

IV. 聯立一次方程系

習 題 VII

原本第 134 頁

$$1. \text{ 解: } \begin{cases} x+y=62 & \dots\dots\dots(1) \\ x-y=12 & \dots\dots\dots(2) \end{cases}$$

$$(1)+(2), \text{ 得 } \quad 2x=74$$

$$\therefore x=37.$$

$$\therefore y=62-x=25.$$

$$2. \text{ 解: } \begin{cases} 6x-5y=25 & \dots\dots\dots(1) \\ 4x-3y=19 & \dots\dots\dots(2) \end{cases}$$

$$2 \times (1), \quad 12x-10y=50 \dots\dots\dots(3)$$

$$3 \times (2), \quad 12x-9y=57 \dots\dots\dots(4)$$

$$(4)-(3), \quad y=7.$$

$$\therefore x = \frac{19+3y}{4} = 10$$

$$\begin{aligned}
 3. \text{ 解: } & \begin{cases} 45x - 13y = 161 & \dots\dots\dots(1) \\ 18x + 11y = 32 & \dots\dots\dots(2) \end{cases} \\
 & 2 \times (1), \quad 90x - 26y = 322 & \dots\dots\dots(3) \\
 & 5 \times (2), \quad 90x + 55y = 160 & \dots\dots\dots(4) \\
 & (3) - (4), \quad -81y = 162 \\
 & \quad \quad \quad \therefore y = -2. \\
 & \quad \quad \quad \therefore x = \frac{32 - 11y}{18} = 3.
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ 解: } & \begin{cases} x - 3 = 7 - x & \dots\dots\dots(1) \\ 8x - 3y - 61 = 0 & \dots\dots\dots(2) \end{cases} \\
 & \text{從(1)式,} \quad 2y = 10 \quad \therefore x = 5. \\
 & \text{代入(2)式,} \quad 40 - 3y - 61 = 0 \\
 & \quad \quad \quad 3y = -21 \quad \therefore y = -7.
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ 解: } & \begin{cases} 12x = 9 - 10y & \dots\dots\dots(1) \\ 3y = 7 - 9x & \dots\dots\dots(2) \end{cases} \\
 & \text{從(1)式,} \quad x = \frac{9 - 10y}{12} \\
 & \text{代入(2)式,} \quad 8y = 7 - 9 \times \frac{9 - 10y}{12} = \frac{28 - 27 + 30y}{4} \\
 & \quad \quad \quad 32y = 1 + 30y \\
 & \quad \quad \quad 2y = 1 \quad \therefore y = \frac{1}{2}. \\
 & \quad \quad \quad \therefore x = \frac{9 - 10y}{12} = \frac{1}{3}.
 \end{aligned}$$

$$\begin{aligned}
 6. \text{ 解: } & \begin{cases} 2y - 3x = 0 & \dots\dots\dots(1) \\ 5x - 3y - 2 = 0 & \dots\dots\dots(2) \end{cases} \\
 & \text{從(1)式,} \quad y = \frac{3x}{2} \\
 & \text{代入(2)式,} \quad 5x - 3 \times \frac{3x}{2} = 2 \\
 & \quad \quad \quad \therefore x = 4. \\
 & \quad \quad \quad \therefore y = \frac{3x}{2} = 6.
 \end{aligned}$$

$$7. \text{ 解: } \begin{cases} x/0.3 + 53 = 31 & \dots\dots\dots(1) \\ 5x + 3y = 1.65 & \dots\dots\dots(2) \end{cases}$$

$$1.5 \times (1), \text{ 得 } 5x + 7.5y = 5.25 \dots\dots\dots(3)$$

$$(3) - (2), \quad 4.5y = 3.6 \\ \therefore y = 0.8.$$

$$\text{從 (2) 式,} \quad x = \frac{1.65 - 3y}{5} = -0.15.$$

$$3. \text{ 解: } \begin{cases} 2(2x + 3y) = 3(2x - 3y) + 10 & \dots\dots\dots(1) \\ 4x - 3y = 4(6y - 2x) + 3 & \dots\dots\dots(2) \end{cases}$$

$$\text{從 (1) 式,} \quad 4x + 6y = 6x - 9y + 10 \\ 2x - 15y = -10 \dots\dots\dots(3)$$

$$\text{從 (2) 式,} \quad 4x - 3y = 24y - 8x + 3 \\ 4x - 9y = 1 \dots\dots\dots(4)$$

$$2 \times (3) - (4), \quad -21y = -21 \quad \therefore y = 1.$$

$$\text{代入 (3) 式,} \quad 2x = 15y - 10 = 5 \quad \therefore x = \frac{5}{2}.$$

$$9. \text{ 解: } \begin{cases} (x+2)(y+1) = (x-5)(y-1) & \dots\dots\dots(1) \\ x(4+y) = -y(8-x) & \dots\dots\dots(2) \end{cases}$$

$$\text{從 (1) 式,} \quad xy + 2y + x + 2 = xy - 5y - x + 5 \\ 2x + 7y = 3 \dots\dots\dots(3)$$

$$\text{從 (2) 式,} \quad 4x + xy = -8y + xy \\ x = -2y \dots\dots\dots(4)$$

$$\text{代入 (3) 式,} \quad -4y + 7y = 3 \\ 3y = 3 \quad \therefore y = 1.$$

$$\therefore x = -2y = -2.$$

$$10. \text{ 解: } \begin{cases} ax + by = a^2 + 2a + b^2 & \dots\dots\dots(1) \\ bx + ay = a^2 + 2b + b^2 & \dots\dots\dots(2) \end{cases}$$

$$(1) - (2), \quad ax - bx + by - ay = 2(a - b)$$

$$(a - b)(x - y) = 2(a - b)$$

$$x - y = 2$$

$$x = 2 + y \dots\dots\dots(3)$$

$$\text{代入 (1) 式,} \quad a(2 + y) + by = a^2 + 2a + b^2$$

$$2a + y(a + b) = a^2 + 2a + b^2$$

$$\therefore y = \frac{a^2 + b^2}{a + b}$$

$$\therefore x = 2 + y = 2 + \frac{a^2 + b^2}{a + b}$$

11. 解: $\begin{cases} ax + by = c & \dots\dots\dots(1) \\ px = qy & \dots\dots\dots(2) \end{cases}$

從(2)式, $x = \frac{qy}{p}$

代入(1)式, $a\left(\frac{qy}{p}\right) + by = c$

$$aqy + bpy = pc$$

$$\therefore y = \frac{pc}{aq + bp}$$

$$\therefore x = \frac{qy}{p} = \frac{qc}{aq + bp}$$

12. 解: $\begin{cases} (a-b)x + (a+b)y = 2(a^2 - b^2) & \dots\dots\dots(1) \\ (a+b)x + (a-b)y = 2(a^2 + b^2) & \dots\dots\dots(2) \end{cases}$

(1) - (2), $-2bx + 2by = -4b^2$

$$x - y = 2b \dots\dots\dots(3)$$

(1) + (2), $2ax + 2ay = 4a^2$

$$x + y = 2a \dots\dots\dots(4)$$

(3) + (4), $2x = 2a + 2b$

$$\therefore x = a + b.$$

$$\therefore y = x - 2b = a - b.$$

13. 解: $\begin{cases} \frac{x+y}{3} + \frac{y-x}{2} = 5 & \dots\dots\dots(1) \\ \frac{x}{2} + \frac{x+y}{9} = 7 & \dots\dots\dots(2) \end{cases}$

從(1)式, $2x + 2y + 3y - 3x = 30 \dots\dots\dots(3)$

從(2)式, $9x + 2x + 2y = 126 \dots\dots\dots(4)$

從(3)式, $5y - x = 30 \dots\dots\dots(5)$

從(4)式, $11x + 2y = 126 \dots\dots\dots(6)$

11 × (5), $55y - 11x = 330 \dots\dots\dots(7)$

(6) + (7), $57y = 456$

$$\therefore y = 8.$$

$$\therefore x = 5y - 30 = 10.$$

$$14. \text{ 解: } \begin{cases} \frac{x-y}{4} - \frac{x+2y-5}{6} = \frac{y-3}{4} - \frac{y+2x-5}{6} & \dots\dots\dots(1) \\ 5x-2y+6=0 & \dots\dots\dots(2) \end{cases}$$

$$\text{從 (1) 式, } 3x-3y-2x-4y+10=3y-9-2y-4x+10$$

$$5x-8y+9=0 \dots\dots\dots(3)$$

$$(2)-(3), \quad 6y-3=0$$

$$\therefore y = \frac{1}{2}.$$

$$\therefore x = \frac{-6+2y}{5} = -1.$$

$$15. \text{ 解: } \begin{cases} \frac{x}{a} + \frac{y}{b} = \frac{1}{c} & \dots\dots\dots(1) \\ \frac{x}{a'} - \frac{y}{b'} = \frac{1}{c} & \dots\dots\dots(2) \end{cases}$$

$$\frac{1}{b'} \times (1), \quad \frac{x}{ab'} + \frac{y}{bb'} = -\frac{1}{b'c} \dots\dots\dots(3)$$

$$\frac{1}{b} \times (2), \quad \frac{x}{a'b} - \frac{y}{bb'} = \frac{1}{bc'} \dots\dots\dots(4)$$

$$(3) + (4): \quad x \left[\frac{a'b + ab'}{aa'bb'} \right] = \frac{bc' + b'c}{bb'cc'}$$

$$\therefore x = \frac{(bc' + b'c)aa'}{(a'b + ab')cc'}$$

$$\frac{1}{a'} \times (1), \quad \frac{x}{aa'} + \frac{y}{a'b} = \frac{1}{a'c} \dots\dots\dots(5)$$

$$\frac{1}{a} \times (2), \quad \frac{x}{aa'} - \frac{y}{ab'} = \frac{1}{ac'} \dots\dots\dots(6)$$

$$(5) - (6), \quad y \left[\frac{ab' + a'b}{aa'bb'} \right] = \frac{ac' - a'c}{aa'cc'}$$

$$\therefore y = \frac{(ac' - a'c)bb'}{(ab' + a'b)cc'}$$

$$16. \text{ 解: } \begin{cases} \frac{x}{a} + \frac{y}{b} = 1 + x & \dots\dots\dots (1) \\ \frac{x}{b} + \frac{y}{a} = 1 + y & \dots\dots\dots (2) \end{cases}$$

$$(1) - (2), \quad \left[\frac{1}{a} - \frac{1}{b} \right] x - \left[\frac{1}{a} - \frac{1}{b} \right] y = x - y.$$

$$(x - y) \left[\frac{1}{a} - \frac{1}{b} \right] = x - y$$

$$(x - y) \left[\frac{1}{a} - \frac{1}{b} - 1 \right] = 0$$

$$x = y \dots\dots\dots (3)$$

$$\text{代入 (1) 式,} \quad \frac{x}{a} + \frac{x}{b} = 1 + x$$

$$x \left[\frac{1}{a} + \frac{1}{b} - 1 \right] = 1$$

$$\therefore x = \frac{1}{\frac{1}{a} + \frac{1}{b} - 1} = \frac{ab}{a + b - ab}$$

$$\text{從 (3) 式,} \quad y = \frac{ab}{a + b - ab}$$

$$17. \text{ 解: } \begin{cases} 1\frac{1}{2}x - 2\frac{1}{2}y = 10 & \dots\dots\dots (1) \\ 6x - 10y = 15 & \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式,} \quad 3x - 5y = 20 \dots\dots\dots (3)$$

$$2 \times (3), \quad 6x - 10y = 40 \dots\dots\dots (4)$$

$$(4) - (2), \text{ 得} \quad 0 = 25$$

因 $0 \neq 25$, 故方程式不合理。

$$18. \text{ 解: (1) 設 } a, b, c, a', b', c' \text{ 各爲 } 2, 3, 5, 6, -9, 7; \text{ 則二式爲:}$$

$$\frac{x}{2} + \frac{y}{3} = \frac{1}{5} \dots\dots\dots (1)$$

$$\frac{x}{6} + \frac{y}{9} = \frac{1}{7} \dots\dots\dots (2)$$

$$\frac{1}{3} \times (1), \text{ 得} \quad \frac{x}{6} + \frac{y}{9} = \frac{1}{15} \dots\dots\dots (3)$$

$$(2) - (3), \quad \frac{1}{7} = \frac{1}{15}$$

$$7 = 15.$$

因 $7 \neq 15$, 故不合理。

(2) 設 a, b, c, a', b', c' 各爲 2, 3, 5, 6, -9, 15; 則二式化爲

$$\frac{x}{2} + \frac{y}{3} = \frac{1}{5} \dots\dots\dots(1)$$

$$\frac{x}{6} + \frac{y}{9} = \frac{1}{15} \dots\dots\dots(2)$$

$$3 \times (2), \text{ 得 } \frac{x}{2} + \frac{y}{3} = \frac{1}{5} \dots\dots\dots(3)$$

$$(3) - (1), \text{ 得 } 0 = 0.$$

故二方程式不相因。

習 題 VIII

原本第 135 頁

$$1. \text{ 解: } \begin{cases} \frac{7}{2x} + \frac{1}{3y} = 0 \dots\dots\dots(1) \\ \frac{3}{x} + \frac{14}{y} + 3 = 0 \dots\dots\dots(2) \end{cases}$$

$$42 \times (1), \quad \frac{147}{x} + \frac{14}{y} = 0 \dots\dots\dots(3)$$

$$(3) - (2), \quad \frac{144}{x} - 3 = 0$$

$$\therefore x = \frac{144}{3} = 48.$$

$$\text{從 (1) 式, } 21y + 2x = 0$$

$$\therefore y = -\frac{2x}{21} = -\frac{32}{7}$$

$$2. \text{ 解: } \begin{cases} 10x + \frac{6}{y} = 5 \dots\dots\dots(1) \\ 15x + \frac{10}{y} = 8 \dots\dots\dots(2) \end{cases}$$

$$3 \times (1), \quad 30x + \frac{18}{y} = 15 \dots\dots\dots (3)$$

$$2 \times (2), \quad 30x + \frac{20}{y} = 16 \dots\dots\dots (4)$$

$$(4) - (3), \quad \therefore \frac{2}{y} = 1$$

$$\therefore y = 2.$$

$$\text{代入 (1) 式, } 10x + 3 = 5$$

$$\therefore x = \frac{5-3}{10} = \frac{1}{5}.$$

$$3. \text{ 解: } \begin{cases} \frac{y}{x} = \frac{2(3-y)}{x} + \frac{3}{2} \dots\dots\dots (1) \\ \frac{y+3}{x} = \frac{3y-5}{x} + 1 \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式, } 2y = 12 - 4y + 3x$$

$$2y - x = 4 \dots\dots\dots (3)$$

$$\text{從 (2) 式, } y + 3 = 3y - 5 + x$$

$$2y + x = 8 \dots\dots\dots (4)$$

$$(4) - (3), \quad 2x = 4$$

$$\therefore x = 2.$$

$$(4) + (3), \quad 4y = 12$$

$$\therefore y = 3.$$

$$4. \text{ 解: } \begin{cases} xy = 0 \dots\dots\dots (1) \\ (x+2y-1)(3x-y+2) = 0 \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式, } x = 0 \text{ 或 } y = 0.$$

$$\text{從 (2) 式, } x + 2y - 1 = 0 \text{ 或 } 3x - y + 2 = 0$$

$$\text{若 } x = 0, \text{ 則 } 2y - 1 = 0 \text{ 或 } -y + 2 = 0$$

$$\therefore y = \frac{1}{2} \text{ 或 } y = 2.$$

$$\text{若 } y = 0, \text{ 則 } x - 1 = 0 \text{ 或 } 3x + 2 = 0$$

$$\therefore x = 1 \text{ 或 } x = -\frac{2}{3}.$$

$$5. \text{ 解: } \begin{cases} xy - y = 0 \dots\dots\dots (1) \\ 3x - 8y + 5 = 0 \dots\dots\dots (2) \end{cases}$$

從(1)式, $y(x-1)=0$
 $\therefore y=0$ 或 $x=1$.

代入(2)式:

若 $y=0$, 則 $3x+5=0 \therefore x=-\frac{5}{3}$.

若 $x=1$, 則 $3-8y+5=0 \therefore y=1$.

6. 解: $\begin{cases} x(x-y)(x+y)=0 & \dots\dots\dots(1) \\ x+2y-5=0 & \dots\dots\dots(2) \end{cases}$

從(1)式, $x=0$ 或 $x=y$ 或 $x=-y$.

各值代入(2)式:

若 $x=0$, 則 $2y-5=0 \therefore y=\frac{5}{2}$.

$x=y$, 則 $y+2y-5=0 \therefore y=\frac{5}{3}=x$.

$x=-y$, 則 $-y+2y-5=0 \therefore y=5, x=-5$.

7. 解: $\begin{cases} (x-1)(y-2)=0 & \dots\dots\dots(1) \\ (x-2)(y-3)=0 & \dots\dots\dots(2) \end{cases}$

從(1)式, $x=1$ 或 $y=2$.

代入(2)式, $y=3$ 或 $x=2$.

答: $x=1, y=3; x=2, y=2$.

8. 解: $\begin{cases} y^2=(x-1)^2 & \dots\dots\dots(1) \\ 2x+3y-7=0 & \dots\dots\dots(2) \end{cases}$

從(1)式, $y=\pm(x-1) \dots\dots\dots(3)$

代入(2)式, $2x+3(x-1)-7=0 \quad 5x-10=0 \therefore x=2$.

$2x-3(x-1)-7=0 \quad -x-4=0 \therefore x=-4$.

代入(2)式:

若 $x=2$, 則 $4+3y-7=0 \therefore y=1$.

$x=-4$, 則 $-8+3y-7=0 \therefore y=5$.

9. 解: $\begin{cases} (2x+y)^2=(x-3y+5)^2 & \dots\dots\dots(1) \\ (x+y)^2=1 & \dots\dots\dots(2) \end{cases}$

從(1)式, 得二式:

$2x+y=x-3y+5 \dots\dots\dots(3)$

$2x+y=-(x-3y+5) \dots\dots\dots(4)$

從 (2) 式, 得二式

$$x+y=1 \dots\dots\dots (5)$$

$$x+y=-1 \dots\dots\dots (6)$$

從 (3) 與 (5), 得 $x=-1/3, y=4/3.$

(3) 與 (6), 得 $x=-3, y=2.$

(4) 與 (5), 得 $x=-3/5, y=8/5.$

(5) 與 (6), 得 $x=-7/5, y=2/5.$

10. 解: $\begin{cases} (x-5y+8)(x+3y+5)=0 \dots\dots\dots (1) \\ (2x+y+5)(5x+2y-14)=0 \dots\dots\dots (2) \end{cases}$

從 (1) 式, $x-5y+8=0 \dots\dots\dots (3)$

$$x+3y+5=0 \dots\dots\dots (4)$$

從 (2) 式, $2x+y+5=0 \dots\dots\dots (5)$

$$5x+2y-14=0 \dots\dots\dots (6)$$

從 (3) 與 (5), 得 $x=-3, y=1.$

(3) 與 (6), 得 $x=2, y=2.$

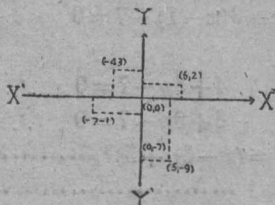
(4) 與 (5), 得 $x=-2, y=-1.$

(4) 與 (6), 得 $x=4, y=-3.$

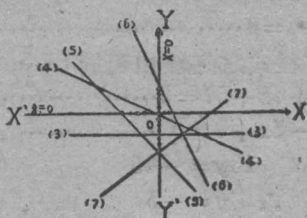
習 題 IX

原本第 143 頁

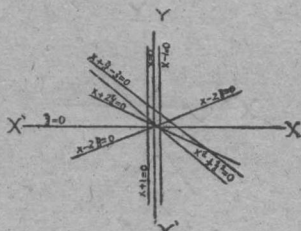
1. 解: $(0, 0), (6, 0), (0, -7), (6, 2), (-7, -1), (-4, 3), (5, -9).$



2. 解: (1) $x=0,$ (2) $y=0,$ (3) $2y+7=0,$
 (4) $3y+x=0,$ (5) $x+y+5=0,$ (6) $7x+3y-18=0,$
 (7) $3x-4y=24.$



3. 解: (1) $xy=0$, (2) $(x+y-3)(x-2y)=0$, (3) $x^2-1=0$,
 (4) $x^2=4y^2$, (5) $x^2+y^2=0$.



4. 解: (1) $\begin{cases} x+y-3=0 & \dots\dots\dots(1) \\ x-2y=0 & \dots\dots\dots(2) \end{cases}$

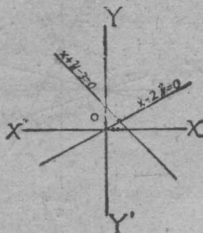
作二方程式之圖形

從圖,知二線相交於 (2, 1).

$$(1) - (2), \quad 2y + y - 3 = 0$$

$$\therefore y = 1.$$

$$\therefore x = 2.$$



$$(2) \quad \begin{cases} 3y + 2x + 19 = 0 & \dots\dots\dots(1) \\ 2y - 3x + 4 = 0 & \dots\dots\dots(2) \end{cases}$$

作圖，知二方程式之直線相交於 $(-5, -2)$ 。

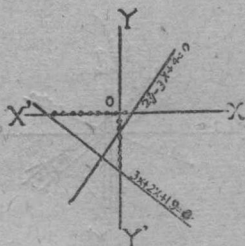
$$2 \times (1), \quad 9y + 6x + 57 = 0 \dots\dots\dots(3)$$

$$2 \times (2), \quad 4y - 6x + 8 = 0 \dots\dots\dots(4)$$

$$(3) + (4), \quad 13y = -65$$

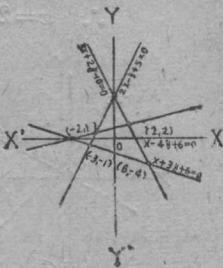
$$\therefore y = -5.$$

$$\therefore x = \frac{2y + 4}{3} = -2.$$



$$5. \text{ 解: } (1) \quad \begin{cases} (x - 4y + 6)(x + 3y + 6) = 0 & \dots\dots\dots(1) \\ (3x + 2y - 10)(2x - y + 5) = 0 & \dots\dots\dots(2) \end{cases}$$

作圖，知諸一次方程式之直線之交點為 $(2, 2)$, $(-2, 1)$, $(6, -4)$, $(-3, -1)$ 。



$$\text{從 } \begin{cases} x-4y+6=0 \\ 3x+2y-10=0 \end{cases} \text{ 得} \\ x=2, \quad y=2.$$

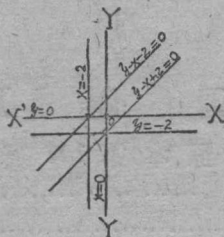
$$\text{從 } \begin{cases} x-4y+6=0 \\ 2x-y+5=0 \end{cases} \text{ 得} \\ x=-2, \quad y=1.$$

$$\text{從 } \begin{cases} 3x+2y-10=0 \\ x+3y+6=0 \end{cases} \text{ 得} \\ x=6, \quad y=-4.$$

$$\text{從 } \begin{cases} x+3y+6=0 \\ 2x-y+5=0 \end{cases} \text{ 得} \\ x=-3, \quad y=-1.$$

$$(2) \quad \begin{cases} (y-x-2)x=0 & \dots\dots\dots(1) \\ (y-x+2)y=0 & \dots\dots\dots(2) \end{cases}$$

作圖，知諸方程式直線之交點爲 $(0, 0)$, $(0, -2)$ 及 $(-2, 0)$ ，餘一組爲二平行線無交點。



$$\text{從 } \begin{cases} y-x-2=0 \\ y-x+2=0 \end{cases} \text{ 無解答。}$$

$$\text{從 } y-x+2=0 \text{ 及 } x=0, \text{ 得} \\ x=0, \quad y=-2.$$

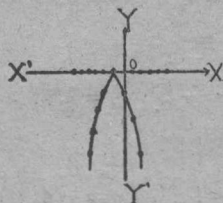
$$\text{從 } x=0 \text{ 及 } y=0, \text{ 得} \\ x=0, \quad y=0.$$

$$\text{從 } y-x-2=0 \text{ 及 } y=0, \text{ 得} \\ y=0, \quad x=-2.$$

6. 解: (1) $y = -(x+1)^2$

$$x = 0, 1, 2, -1, -2, -3, -4.$$

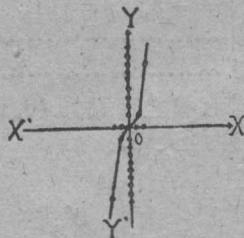
$$y = -1, -4, -9, 0, -1, -4, -9.$$



(2) $y = x^3$

$$x = 0, 1, 2, -1, -2.$$

$$y = 0, 1, 8, -1, -8.$$



習題 X

原本第 147 頁

1. 解:
$$\begin{cases} x+y=11 & \dots\dots\dots(1) \\ y+z=13 & \dots\dots\dots(2) \\ z+x=12 & \dots\dots\dots(3) \end{cases}$$

$$(1) + (2) + (3), \quad 2(x+y+z) = 36$$

$$x+y+z = 18 \quad \dots\dots\dots(4)$$

$$(4) - (2), \text{ 得} \quad x = 5.$$

$$(4) - (3), \text{ 得} \quad y = 6.$$

$$(4) - (1), \text{ 得} \quad z = 7.$$

2. 解:
$$\begin{cases} x+y+z=1 & \dots\dots\dots(1) \\ x+2y+3z=4 & \dots\dots\dots(2) \\ x+3y+7z=13 & \dots\dots\dots(3) \end{cases}$$

(2) - (1), $y + 2z = 3$ (4)

(3) - (2), $y + 4z = 9$ (5)

(5) - (4), $2z = 6$

$\therefore z = 3.$

代入 (4) 式, $y = 3 - 2z = -3.$

代入 (1) 式, $x = 1 - y - z = 1.$

3. 解: $\begin{cases} x + 2y - 3z = 3 \dots\dots\dots(1) \\ 3x - 5y + 7z = 19 \dots\dots\dots(2) \\ 5x - 8y - 11z = 13 \dots\dots\dots(3) \end{cases}$

從 (1) 式, $x = 3 - 2y + 3z$

代入 (2) 及 (3) 式, $16z - 11y = 10$ (4)

$4z - 18y = -28$ (5)

$4 \times (5), 16z - 72y = -112$ (6)

(4) - (6), $61y = 122$

$\therefore y = 2.$

代入 (5) 式, $4z - 36 = -28$

$\therefore z = \frac{8}{4} = 2.$

代入 (1) 式, $x = 5.$

4. 解: $\begin{cases} 5x - 2y = -33 \dots\dots\dots(1) \\ x + y - 7z = 13 \dots\dots\dots(2) \\ x + 3y = -10 \dots\dots\dots(3) \end{cases}$

$5 \times (3), 5x + 15y = -50$ (4)

(4) - (1), $17y = -17$

$\therefore y = -1.$

代入 (3) 式, $x = -10 - 3y = -7.$

代入 (2) 式, $z = -3.$

5. 解: $\begin{cases} x + 2y - 4z = 14 \dots\dots\dots(1) \\ 2x - 3y = 0 \dots\dots\dots(2) \\ y - 4z = 0 \dots\dots\dots(3) \end{cases}$

從 (3) 式, $y = 4z$ (4)

代入 (2) 式, $x = 6z$ (5)

代入 (1) 式, $6z + 8z - 4z = 11$

$$\therefore z = \frac{11}{10}.$$

代入(4)式, $y = 4z = \frac{22}{5}.$

代入(5)式, $x = 6z = \frac{33}{5}.$

6. 解:
$$\begin{cases} 3x - 5 = 2(x - 2) \dots\dots\dots(1) \\ (x + 1)(y - 1) = (x + 2)(y - 2) + 5 \dots\dots\dots(2) \\ 2x + 3y + z = 6 \dots\dots\dots(3) \end{cases}$$

從(1)式,得 $x = 1.$

代入(2)式, $2y - 2 = 3y - 6 + 5$

$$\therefore y = -1.$$

代入(3)式, $2 - 3 + z = 6$

$$\therefore z = 7.$$

7. 解:
$$\begin{cases} \frac{1}{x} + \frac{1}{y} - \frac{6}{z} = 9 \dots\dots\dots(1) \\ \frac{1}{x} - \frac{1}{y} + \frac{4}{z} = 5 \dots\dots\dots(2) \\ \frac{3}{y} - \frac{2}{x} - \frac{1}{z} = 4 \dots\dots\dots(3) \end{cases}$$

$$\frac{(1)-(2)}{2}, \quad \frac{1}{y} - \frac{5}{z} = 2 \dots\dots\dots(4)$$

$$\frac{(1)+(2)+(3)}{3}, \quad \frac{1}{y} - \frac{1}{z} = 6 \dots\dots\dots(5)$$

$$(5) - (4), \quad \frac{4}{z} = 4$$

$$\therefore z = 1.$$

代入(5)式, $\frac{1}{y} - 1 = 6$

$$\therefore y = \frac{1}{7}.$$

代入(2)式, $\frac{1}{x} - 7 + 4 = 5$

$$\therefore x = \frac{1}{8}.$$

$$8. \text{ 解: } \begin{cases} y+z+u=4 \dots\dots\dots(1) \\ x+z+u=3 \dots\dots\dots(2) \\ x+y+u=1 \dots\dots\dots(3) \\ x+y+z=10 \dots\dots\dots(4) \end{cases}$$

$$\text{四式相加, 得 } x+y+z+u=6 \dots\dots\dots(5)$$

$$(5)-(1), \quad x=2.$$

$$(5)-(2), \quad y=3.$$

$$(5)-(3), \quad z=5.$$

$$(5)-(4), \quad u=-4.$$

$$9. \text{ 解: } \begin{cases} 4x-3z+u=9 \dots\dots\dots(1) \\ 5y+z-4u=17 \dots\dots\dots(2) \\ 3y+u=12 \dots\dots\dots(3) \\ x+2y+3u=8 \dots\dots\dots(4) \end{cases}$$

$$\text{從 (3) 式, } u=12-3y \dots\dots\dots(5)$$

$$\text{代入 (4) 式, } x=8-2y-3(12-3y)=7y-23 \dots\dots\dots(6)$$

$$\text{代入 (2) 式, } z=17-5y+4(12-3y)=65-17y \dots\dots\dots(7)$$

$$\text{代入 (1) 式, } 4(7y-23)-3(65-17y)+12-3y=9 \dots\dots\dots(8)$$

$$76y-295=9$$

$$\therefore y=4.$$

$$\therefore u=12-12=0.$$

$$\therefore x=28-28=0.$$

$$\therefore z=65-68=-3.$$

$$10. \text{ 解: } \begin{cases} cx+by=l \dots\dots\dots(1) \\ by+az=m \dots\dots\dots(2) \\ az+cx=n \dots\dots\dots(3) \end{cases}$$

$$\text{三式相加, } cx+by+az = \frac{l+m+n}{2} \dots\dots\dots(4)$$

$$(4)-(2), \quad x = \frac{l-m+n}{2c}.$$

$$(4)-(3), \quad y = \frac{l+m-n}{2b}.$$

$$(4)-(1), \quad z = \frac{-l+m+n}{2a}$$

$$11. \text{ 解: } \begin{cases} lx=my=nz \dots\dots\dots(1) \\ ax+by+cz=d \dots\dots\dots(2) \end{cases}$$

$$\text{從(1)式,} \quad y = \frac{lx}{m}, \quad z = \frac{lx}{n}$$

$$\text{代入(2)式,} \quad ax + b\frac{lx}{m} + c\frac{lx}{n} = d$$

$$x(amn + bln + clm) = mnd$$

$$\therefore x = \frac{mnd}{amn + bln + clm}$$

$$\therefore y = \frac{l}{m}x = \frac{lnd}{amn + bln + clm}$$

$$\therefore z = \frac{l}{n}x = \frac{lmd}{amn + bln + clm}$$

$$12. \text{ 解: } \begin{cases} 2x=3y=6z \dots\dots\dots(1) \\ (x+2y+z-16)(3x-2y+20)=0 \dots\dots\dots(2) \end{cases}$$

$$\text{從(1)式,} \quad x=3z, \quad y=2z \dots\dots\dots(3)$$

$$\text{代入(2)式,} \quad (3z+4z+z-16)(9z-4z+20)=0$$

$$\therefore z=2 \quad \text{或} \quad -4.$$

$$\therefore x=6 \quad \text{或} \quad -12.$$

$$\therefore y=4 \quad \text{或} \quad -8.$$

$$13. \text{ 解: } \begin{cases} x-y=3 \dots\dots\dots(1) \\ y-z=-5 \dots\dots\dots(2) \\ z-x=2 \dots\dots\dots(3) \end{cases}$$

$$x-y-3=0$$

$$y-z+5=0$$

$$z-x-2=0.$$

此三式相加爲零,故知該組方程式爲不相因者,連絡各組方程式之恆等式爲

$$(x-y-3) + (y-z+5) + (z-x-2) = 0.$$

$$14. \text{ 解: } \begin{cases} 3x-8y+9z=10 \dots\dots\dots(1) \\ 2x+5y-3z=12 \dots\dots\dots(2) \\ 16x+9y-z=80 \dots\dots\dots(3) \end{cases}$$

以 $[5 \times (2) + 2 \times (1)]$ 即得第三方程式。故此組方程式為不相因者，其恆等式為：

$$2(3x - 8y + 9z - 10) + 5(2x + 5y - 3z - 12) - (16x + 9y - z - 80) = 0.$$

習題 XI

原本第 150 頁

1. 解：設 x, y, z 為三數。

$$\begin{cases} x + y + z = 20 & \dots\dots\dots(1) \\ x + 2y + 3z = 44 & \dots\dots\dots(2) \\ 2(x + y) - 4z = -14 & \dots\dots\dots(3) \end{cases}$$

$$(3) \div 2, \quad x + y - 2z = -7 \quad \dots\dots\dots(4)$$

$$(1) - (4), \quad 3z = 27 \quad \therefore z = 9.$$

$$(2) - (1), \quad y + 2z = 24$$

$$\therefore y = 6.$$

$$\text{代入 (1) 式,} \quad x = 5.$$

2. 解：設 x, y, z 為三數。

$$\begin{cases} x + y + z = 51 & \dots\dots\dots(1) \\ x = 2y + 5 & \dots\dots\dots(2) \\ y = 3z + 2 & \dots\dots\dots(3) \end{cases}$$

$$(3) \text{ 代入 (2) 式,} \quad x = 2(3z + 2) + 5 = 6z + 9 \quad \dots\dots\dots(4)$$

$$(3) \text{ 及 (4) 代入 (1) 式,} \quad 6z + 9 + 3z + 2 + z = 51$$

$$\therefore z = 4.$$

$$\text{代入 (3) 式,} \quad y = 3z + 2 = 14.$$

$$\text{代入 (4) 式,} \quad x = 33.$$

3. 解：設 x 為十位數字， y 為個位數字。

$$\begin{cases} 2x + 3y = 37 & \dots\dots\dots(1) \\ 10y + x = 10x + y - 9 & \dots\dots\dots(2) \end{cases}$$

$$\text{從 (2) 式,} \quad x - y = 1 \quad \dots\dots\dots(3)$$

$$2 \times (3), \quad 2x - 2y = 2 \quad \dots\dots\dots(4)$$

$$(1) - (4), \quad 5y = 35$$

$$\therefore y = 7.$$

$$\therefore x = 1 + y = 8.$$

答：此數為 87.

4. 解：設 x 爲 A 所有款數， y 爲 B 所有款數。

$$\begin{cases} x + \frac{2}{3}y = 5000 & \dots\dots\dots(1) \end{cases}$$

$$\begin{cases} y + \frac{1}{2}x + 100 = 3000 & \dots\dots\dots(2) \end{cases}$$

$$\frac{3}{2} \times (1), \quad \frac{3}{2}x + y = 7500 \quad \dots\dots\dots(3)$$

$$(3) - (2), \quad x = 4600 \text{ 元.}$$

$$\text{代入 (2) 式,} \quad y = 600 \text{ 元.}$$

5. 解：設 x, y, z 各爲 A, B, C 之財富。

$$\begin{cases} x + y = p & \dots\dots\dots(1) \end{cases}$$

$$\begin{cases} y + z = q & \dots\dots\dots(2) \end{cases}$$

$$\begin{cases} x + z = r & \dots\dots\dots(3) \end{cases}$$

$$\text{三式相加,} \quad x + y + z = \frac{1}{2}(p + q + r) \quad \dots\dots\dots(4)$$

$$(4) - (2), \quad x = \frac{1}{2}(p + r - q).$$

$$(4) - (3), \quad y = \frac{1}{2}(p + q - r).$$

$$(4) - (1), \quad z = \frac{1}{2}(q + r - p).$$

答：其必須條件爲 $p + r > q, p + q > r, q + r > p$.

6. 解：設 x 爲本錢， y 爲利率。

$$\begin{cases} x + 2xy = 2546.05 & \dots\dots\dots(1) \end{cases}$$

$$\begin{cases} x + 4xy = 2767.10 & \dots\dots\dots(2) \end{cases}$$

$$(2) - (1), \quad 2xy = 211.05 \quad \dots\dots\dots(3)$$

$$(1) - (3), \quad x = 2345 \text{ 元.}$$

$$\text{代入 (3) 式,} \quad y = 4\frac{1}{2}\%.$$

7. 解：設 x 爲利率 4% 之公債票之銀數，

y 爲利率 5% 之公債票之銀數，

$x + y$ 爲所求之總銀數。

$$x \times \frac{4}{100} + y \times \frac{100}{110} \times \frac{5}{100} = 650 \quad \dots\dots\dots(1)$$

$$x \times \frac{100}{80} \times \frac{4}{100} + y \times \frac{100}{110} \times \frac{5}{100} = 750 \quad \dots\dots\dots(2)$$

$$(2) - (1), \quad \frac{x}{20} - \frac{x}{25} = 100$$

$$\therefore x=10000 \text{ 元.}$$

代入 (1) 式, $y=5500 \text{ 元.}$

答: 此人共投資 $10000+5500=15500 \text{ 元.}$

8. 解: 設 x 為長方形之長, y 為寬.

$$\begin{cases} (x+6) = \frac{3}{2}(y+6) \dots\dots\dots(1) \\ (x+6)(y+6) = xy+84 \dots\dots\dots(2) \end{cases}$$

從 (1) 式, $2x-3y=6$

從 (2) 式, $x+y=8$

$2 \times (2) - (1)$, $5y=10$

$$\therefore y=2 \text{ 吋.}$$

$$\therefore x=8-y=6 \text{ 吋.}$$

答: 長方形之面積為 $xy=12 \text{ 方吋.}$

9. 解: 設 x 為 A 原有洋數, y 為 B 原有洋數.

$$\begin{cases} x-y+(x-y)-[y+y-(x-y)] = 16 \dots\dots\dots(1) \\ y+y-(x-y)+[y+y-(x-y)] = 24 \dots\dots\dots(2) \end{cases}$$

$$(1)+(2), \quad x+y=16+24=40$$

代入 (1) 式, $8y=104$

$$\therefore y=13 \text{ 元.}$$

$$\therefore x=27 \text{ 元.}$$

10. 解: 設 x, y, z 各為 A, B, C 獨作所須之日數.

$$\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{7}{36} \dots\dots\dots(1) \\ \frac{1}{x} + \frac{1}{z} = \frac{5}{24} \dots\dots\dots(2) \\ 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \frac{26}{17}\left(\frac{1}{y} + \frac{1}{z}\right) = 1 \dots\dots\dots(3) \end{cases}$$

$$(3)-(1)-(2), \quad \frac{1}{y} + \frac{1}{z} + \frac{26}{17}\left(\frac{1}{y} + \frac{1}{z}\right) = 1 - \frac{7}{36} - \frac{5}{24} = \frac{43}{72}$$

$$\frac{1}{y} + \frac{1}{z} = \frac{17}{72} \dots\dots\dots(4)$$

$$\frac{(1)+(2)+(4)}{2}, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{23}{72} \dots\dots\dots(5)$$

$$(5)-(1), \quad \frac{1}{z} = \frac{9}{72} = \frac{1}{8} \quad \therefore z=8 \text{ 日.}$$

$$(5)-(2), \quad \frac{1}{y} = \frac{8}{72} = \frac{1}{9} \quad \therefore y = 9 \text{ 日.}$$

$$(5)-(4), \quad \frac{1}{x} = \frac{6}{72} = \frac{1}{12} \quad \therefore x = 12 \text{ 日.}$$

11. 解：設 x, y 各為二點之速率。

$$\begin{cases} 5x + 5y = 150 & \dots\dots\dots (1) \\ 25x - 25y = 150 & \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式,} \quad x + y = 30 \dots\dots\dots (3)$$

$$\text{從 (2) 式,} \quad x - y = 6 \dots\dots\dots (4)$$

$$(3) + (4), \quad 2x = 36.$$

$$\therefore x = 18 \text{ 呎.}$$

$$\text{代入 (4) 式,} \quad y = 12 \text{ 呎.}$$

答：一點之速率為每秒 18 呎，一為每秒 12 呎。

12. 解：設 x, y 各為快車，慢車每小時所行之哩數。

$$\begin{cases} \frac{25}{3600}(x+y) = \frac{240+200}{1760} & \dots\dots\dots (1) \\ \frac{15}{60}(x-y) = \frac{240+200}{1760} & \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式,} \quad x + y = 36 \dots\dots\dots (3)$$

$$\text{從 (2) 式,} \quad x - y = 4 \dots\dots\dots (4)$$

$$(3) + (4), \quad x = 20 \text{ 哩/時.}$$

$$(3) - (4), \quad y = 16 \text{ 哩/時.}$$

13. 解：設 x, y 各為 A, B 兩輪之速率。

$$\begin{cases} 2x = 3y & \dots\dots\dots (1) \\ \frac{200+10}{x} + 4 = \frac{200-10}{y} - 1 & \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式,} \quad \frac{210}{x} = \frac{140}{y} \dots\dots\dots (3)$$

$$(2) - (3), \quad 4 = \frac{190-140}{y} - 1.$$

$$\therefore y = 10 \text{ 哩/時.}$$

$$\text{代入 (1) 式,} \quad x = \frac{3}{2}y = 15 \text{ 哩/時.}$$

14. 解：設 x 為 B 勝 C 之碼數。

$$(880 - 20) : 880 = 10 : x$$

$$\therefore x = \frac{8800}{860} = 10\frac{10}{45} \text{ 碼.}$$

15. 解：設 x, y 各為 A, B 之速率。

$$\begin{cases} \frac{440}{x} = \frac{440-20}{y} - 2 \dots\dots\dots(1) \end{cases}$$

$$\begin{cases} \frac{440}{x} = \frac{440-6}{y} - 4 \dots\dots\dots(2) \end{cases}$$

$$(1) - (2), \quad \frac{420}{y} - \frac{434}{y} + 2 = 0$$

$$\therefore y = 7 \text{ 碼/秒.}$$

$$\text{代入 (1) 式, } x = \frac{440}{58} = 17\frac{17}{29} \text{ 碼/秒.}$$

16. 解：設 x 為允許攜帶之重量。

$$\frac{1.25 + 1.75}{500 - 2x} = \frac{4}{500 - x}$$

$$2000 - 8x = 1500 - 3x$$

$$5x = 500$$

$$\therefore x = 100 \text{ 磅.}$$

17. 解：設 x, y, z 各為三種合金應融之重量。

$$\begin{cases} x + y + z = 9 \dots\dots\dots(1) \end{cases}$$

$$\begin{cases} \frac{5}{8}x + \frac{2}{8}y + \frac{3}{8}z = 3 \dots\dots\dots(2) \end{cases}$$

$$\begin{cases} \frac{2}{8}x + \frac{5}{8}y + \frac{1}{8}z = 3 \dots\dots\dots(3) \end{cases}$$

$$\begin{cases} \frac{1}{8}x + \frac{1}{8}y + \frac{4}{8}z = 3 \dots\dots\dots(4) \end{cases}$$

$$8 \times (4) - (1), \quad 3z = 15 \quad \therefore z = 5 \text{ 兩.}$$

$$\text{代入 (1) 式, } x + y = 4 \dots\dots\dots(5)$$

$$\text{代入 (3) 式, } 2x + 5y = 19 \dots\dots\dots(6)$$

$$2 \times (5), \quad 2x + 2y = 8 \dots\dots\dots(7)$$

$$(6) - (7), \quad 3y = 11 \quad \therefore y = \frac{11}{3} \text{ 兩.}$$

$$\text{代入 (5) 式, } x = \frac{1}{3} \text{ 兩.}$$

18. 解：設 x, y 各為 A 與 B 含銀之百分數。

$$\begin{cases} \frac{5}{8}x + \frac{3}{8}y = 52 \dots\dots\dots (1) \\ \frac{5}{16}x + \frac{11}{16}y = 42 \dots\dots\dots (2) \end{cases}$$

$$2 \times (2) - (1), \quad y = 32\%$$

$$\therefore x = 64\%$$

19. 解：設 x, y 各爲音，子彈之速度。

$$\begin{cases} \frac{500}{x} + \frac{500}{y} = 2\frac{2}{5} \dots\dots\dots (1) \\ \frac{500}{y} + \frac{600}{x} - \frac{210}{x} = 2\frac{1}{10} \dots\dots\dots (2) \end{cases}$$

$$(1) - (2), \quad \frac{110}{x} = \frac{3}{10}$$

$$\therefore x = 366\frac{2}{3} \text{ 碼/秒.}$$

$$\text{代入 (1) 式, } y = 482\frac{26}{57} \text{ 碼/秒.}$$

20. 解：設 x, y, z 各爲每分鐘通過 A, B, C 三管之水量； V 爲油之容積。

$$\begin{cases} x = 100 + y \dots\dots\dots (1) \\ 3 \times 60(x + y - z) + V = 0 \dots\dots\dots (2) \\ 60(x - z) + V = 0 \dots\dots\dots (3) \\ 45(y - z) + V = 0 \dots\dots\dots (4) \end{cases}$$

$$\frac{(2) - (3)}{60}, \quad 2x - 2z + 3y = 0 \dots\dots\dots (5)$$

$$\frac{(3) - (4)}{15}, \quad 4x - 3y - z = 0 \dots\dots\dots (6)$$

$$2 \times (6) - (5), \quad 6x - 9y = 0 \quad x = \frac{3}{2}y \dots\dots\dots (7)$$

$$\text{代入 (1) 式, } \frac{3}{2}y = 100 + y \quad \therefore y = 200 \text{ 嚮/分.}$$

$$\text{代入 (7) 式, } x = 300 \text{ 嚮/分.}$$

$$\text{代入 (6) 式, } z = 1200 - 600 = 600 \text{ 嚮/分.}$$

$$\text{代入 (4) 式, } V = 45(600 - 200) = 18000 \text{ 嚮.}$$

習 題 XII

原本第 154 頁

1. 解: 設 $3x^3 - x^2 + 2x - 5 = a(x-2)^3 - b(x-2)^2 + c(x-2) - d$.
右式 $= ax^3 - (6a+b)x^2 + (12a+4b+c)x - (8a-4b+2c+d)$.

$$\therefore a=3.$$

$$6a+b=1 \quad \therefore b=-17.$$

$$12a+4b+c=2 \quad \therefore c=34.$$

$$8a-4b+2c+d=5 \quad \therefore d=-19.$$

答: $3x^3 - x^2 + 2x - 5 = 3(x-2)^3 + 17(x-2)^2 + 34(x-2) + 19.$

2. 解: 設 $4x^2 + 8x + 7 = a(2x+3)^2 + b(2x+3) + c$
 $= 4ax^2 + (12a+2b)x + (9a+3b+c).$

$$4a=4 \quad \therefore a=1.$$

$$12a+2b=8 \quad \therefore b=-2.$$

$$9a+3b+c=7 \quad \therefore c=4.$$

答: $4x^2 + 8x + 7 = (2x+3)^2 - 2(2x+3) + 4.$

3. 解: $a-b+c=11$ (1)
 $a+b+c=-5$ (2)
 $25a+5b+c=6$ (3)

$$\therefore a = \frac{43}{24}, b = -8, c = \frac{29}{24}.$$

答: $f(x) = \frac{43}{24}x^2 - 8x + \frac{29}{24}.$

4. 解: $d=5$ (1)
 $-a+b-c+d=1$ (2)
 $a+b+c+d=9$ (3)
 $8a+4b+2c+d=31$ (4)

$$\therefore a=3, b=0, c=1, d=5.$$

答: $f(x) = 3x^3 + x + 5.$

5. 解: $c=4$ (1)
 $4a+4b+c=0$ (2)
 $a+c=6$ (3)

$$\therefore a=2, b=-3, c=4.$$

答: $f(x, y) = 2x - 3y + 4.$

6. 解: $3a + b + 1 = 0 \dots\dots\dots(1)$

$$4a - b + 1 = 0 \dots\dots\dots(2)$$

$$\therefore a = -\frac{2}{7}, b = -\frac{1}{7}.$$

$$\therefore -\frac{2}{7}x - \frac{1}{7}y + 1 = 0.$$

答: $2x + y - 7 = 0.$

7. 解: $3a + b + c = 0 \dots\dots\dots(1)$

$$4a - b + c = 0 \dots\dots\dots(2)$$

$$a + b + c = 0 \dots\dots\dots(3)$$

(1) - (3), $2a = 0$

$$\therefore a = 0 \dots\dots\dots(4)$$

(2) - (3), $3a - 2b = 0 \dots\dots\dots(5)$

將(4)代入(5)式, $-2b = 0$

$$\therefore b = 0 \dots\dots\dots(6)$$

將(4)與(6)代入(3)式, $c = 0.$

答: 此簡單方程式不能求得。

8. 解: 設直線之方程式為 $x + by + c = 0.$

$$2 + 3b + c = 0 \dots\dots\dots(1)$$

$$-4 + 5b + c = 0 \dots\dots\dots(2)$$

$$\therefore b = -\frac{1}{3}, c = -\frac{11}{3}.$$

答: 直線之方程式為 $x + 3y - 11 = 0.$

9. 解: $-6 + 3 + c = 0$

$$\therefore c = 3.$$

10. 解: (1) 由 $x=2, y=3$, 得 $2a + 3b + 1 = 0 \dots\dots\dots(1)$

$$x=7, y=5, \text{ 得 } 7a + 5b + 1 = 0 \dots\dots\dots(2)$$

從(1)及(2), 得 $a = \frac{2}{11}, b = -\frac{5}{11}.$

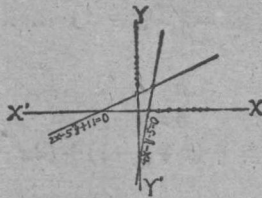
答: 方程式為 $2x - 5y + 11 = 0.$

(2) 由 $x=2, y=3$, 得 $2a'+3b'+1=0$ (1)

$x=3, y=7$, 得 $3a'+7b'+1=0$ (2)

從 (1) 及 (2), 得 $a' = -\frac{4}{5}, b' = \frac{1}{5}$.

答: 方程式為 $4x - y - 5 = 0$.



11. 解: $-8 + 4b - 2c + d = 0$ (1)

$1 + b + c + d = 0$ (2)

$27 + 9b + 3c + d = 0$ (3)

(2) - (1), $9 - 3b + 3c = 0$ (4)

(3) - (2), $26 + 8b + 2c = 0$ (5)

$2 \times (4) - 3 \times (5), 30b = -60$

$\therefore b = -2.$

代入 (4) 式, $c = -5.$

代入 (2) 式, $d = 6.$

答: 方程式為 $x^2 - 2x^2 - 5x + 6 = 0$.

12. 解: 由 $x=1, y=0$, 得 $1+c=0 \therefore c=-1.$

$x=2, y=1$, 得 $4+2b+2c+d=0$ (1)

$x=-2, y=1$, 得 $4-2b-2c+d=0$ (2)

(1) + (2), $2d = -8$

$\therefore d = -4.$

將 d 之值代入 (1) 式, $4+2b-2-4=0.$

$\therefore b = 1.$

答: 所求之方程式為 $x^2 + xy - x - 4y = 0$.

13. 解: $a(x+y-1) + b(2x-y+2) + c(x+2y-3)$

$$= (a+2b+c)x + (a-b+2c)y - (a-2b+3c) = 3x+2y-3$$

$$a+2b+c=3 \dots\dots\dots(1)$$

$$a-b+2c=2 \dots\dots\dots(2)$$

$$a-2b+3c=3 \dots\dots\dots(3)$$

$$\frac{(1)+(3)}{2}, \quad a+2c=3 \dots\dots\dots(4)$$

$$(3)-2 \times (2), \quad -a-c=-1 \dots\dots\dots(5)$$

$$(4)+(5), \quad c=2.$$

$$\text{代入 (4) 式,} \quad a=-1.$$

$$\text{代入 (1) 式,} \quad b=1.$$

$$\text{答: } 3x+2y-3 = -(x+y-1) + (2x-y+2) + 2(x+2y-3).$$

V. 除法變形

習題 XIII

原本第 165 頁

1. 解:

$$\begin{array}{r|l} 6-7-3+24-20 & 3+1-6 \\ 6+2-12 & 2-3+4 \\ \hline -9+9+24-20 & \\ -9-3+18 & \\ \hline 12+6-20 & \\ 12+4-24 & \\ \hline 2+4 & \end{array}$$

答: 商 $= Q = 2x^2 - 3x + 4$, 餘數 $= R = 2x + 4$.

2. 解:

$$\begin{array}{r|l} 3-2-32+66-35 & 1+2-7 \\ 3+6-21 & 3-8+5 \\ \hline -8-11+66-35 & \\ -8-16+56 & \\ \hline 5+10-35 & \\ 5+10-35 & \\ \hline 0 & \end{array}$$

答: $Q = 3x^2 - 8x + 5$, $R = 0$.

$$\begin{array}{r|l}
 3. \text{ 解:} & 2-5+13-15+22 \quad | \quad 1-2+4 \\
 & \underline{2-4+8} \quad | \quad \underline{2-1+3-5} \\
 & -1+5-15 \\
 & \underline{-1+2-4} \\
 & 3-11+22 \\
 & \underline{3-6+12} \\
 & -5+10 \\
 & \underline{-5+10-20} \\
 & 20
 \end{array}$$

答: $Q = 2x^3 - x^2 + 3x - 5, R = 20.$

$$\begin{array}{r|l}
 4. \text{ 解:} & 4+0-3+19+2+4-4+7 \quad | \quad 1+0-1+5 \\
 & \underline{4+0-4+20} \quad | \quad \underline{4+0+1-1+3} \\
 & 1-1+2+4 \\
 & \underline{1+0-1+5} \\
 & -1+3-1-4 \\
 & \underline{-1+0+1-5} \\
 & 3-2+1+7 \\
 & \underline{3+0-3+15} \\
 & -2+4-8
 \end{array}$$

答: $Q = 4x^4 + x^2 - x + 3, R = -2x^2 + 4x - 8.$

$$\begin{array}{l}
 5. \text{ 解: 設 } Q = c_0x + c_1 \text{ 及 } R = d_0x + d_1 \\
 2x^3 - 3x^2 + x - 5 \equiv (c_0x + c_1)(x^2 - 3x + 2) + d_0x + d_1 \\
 \equiv c_0x^3 - 3c_0 \quad \left| \begin{array}{l} x^2 + 2c_0 \\ x + 2c_1 \end{array} \right. \\
 \quad \quad \quad + c_1 \quad \left| \begin{array}{l} -3c_1 \\ + d_1 \end{array} \right. \\
 \quad \quad \quad \quad \quad \quad + d_0
 \end{array}$$

$$\therefore c_0 = 2.$$

$$-3c_0 + c_1 = -3 \quad \therefore c_1 = 3.$$

$$2c_0 - 3c_1 + d_0 = 1 \quad \therefore d_0 = 6.$$

$$2c_1 + d_1 = -5 \quad \therefore d_1 = -11.$$

答: $Q = 2x + 3, R = 6x - 11.$

$$\begin{array}{l}
 6. \text{ 解: 設 } Q = c_0x^2 + c_1x + c_2, R = d_0x^2 + d_1x + d_2 \\
 2x^5 - 3x^4 + x^2 - 5 \\
 \equiv (c_0x^2 + c_1x + c_2)(x^3 - 3x + 2) + d_0x^2 + d_1x + d_2 \\
 \equiv c_0x^5 + c_1x^4 + c_2 \quad \left| \begin{array}{l} x^3 + 2c_0 \\ x^2 + 2c_1 \\ x + 2c_2 \end{array} \right. \\
 \quad \quad \quad -3c_0 \quad \left| \begin{array}{l} -3c_1 \\ -3c_2 \\ + d_2 \end{array} \right. \\
 \quad \quad \quad + d_0 \quad \left| \begin{array}{l} + d_1 \end{array} \right.
 \end{array}$$

$$\therefore c_0 = 2, c_1 = -3.$$

$$-3c_0 + c_2 = 0 \quad \therefore c_2 = 6.$$

$$2c_0 - 3c_1 + d_0 = 1 \quad \therefore d_0 = -12.$$

$$2c_1 - 3c_2 + d_1 = 0 \quad \therefore d_1 = 24.$$

$$2c_2 + d_2 = -5 \quad \therefore d_2 = -17.$$

答: $Q = 2x^2 - 3x + 6, R = -12x^2 + 24x - 17.$

7. 解:

$$\begin{array}{r|l} 3-5-7+12 & 3+1-5 \\ 3+1-5 & 1-2 \\ \hline -6-2+12 & \\ -6-2+10 & \\ \hline & 2 \end{array}$$

答: $3x^3 - 5x^2 - 7x + 12 = (x-2)(3x^2 + x - 5) + 2,$

$$\frac{3x^3 - 5x^2 - 7x + 12}{3x^2 + x - 5} = (x-2) + \frac{2}{3x^2 + x - 5}.$$

8. 解:

$$\begin{array}{r|l} x^4 + ax^3 + x^2 + bx + 1 & x^2 - 2x + 1 \\ x^4 - 2x^3 + x^2 & x^2 + (a+2)x + 2(a+2) \\ \hline (a+2)x^3 + 0 & + bx + 1 \\ (a+2)x^3 - 2(a+2)x^2 + (a+2)x & \\ \hline & 2(a+2)x + (b-a-2)x + 1 \\ & 2(a+2)x^2 - 4(a+2)x + 2(a+2) \\ \hline & (3a+b+6)x - (2a+3) \end{array}$$

若 $x^4 + ax^3 + x^2 + bx + 1$ 能被 $x^2 - 2x + 1$ 除盡, 則

$$3a+b+6=0, \quad 2a+3=0$$

答: $a = -\frac{3}{2}, b = -\frac{3}{2}.$

9. 解:

$$\begin{array}{r|l} x^4 + 2x^3 + 3x^2 + ax + b & x^2 + 3x + 5 \\ x^4 + 3x^3 + 5x^2 & x^2 - x + 1 \\ \hline -x^3 - 2x^2 + ax & \\ -x^3 - 3x^2 - 5x & \\ \hline & x^2 + (a+5)x + b \\ & x^2 + 3x + 5 \\ \hline & (a+2)x + (b-5) \end{array}$$

如原式能化成正整式, 則

$$a+2=0, \quad b-5=0.$$

答: $a = -2, b = 5.$

18. 解：第一以 x 爲“主元”，第二，以 y 爲“主元”。

$$\begin{array}{r|l}
 (1) & 2x^3 + 0 + xy^2 + y^3 \\
 & 2x^3 + x^2y \\
 \hline
 & -x^2y + xy^2 + y^3 \\
 & -x^2y - \frac{xy^2}{2} \\
 \hline
 & \frac{3xy^2}{2} + y^3 \\
 & \frac{3xy^2}{2} + \frac{3}{4}y^3 \\
 \hline
 & \frac{1}{4}y^3
 \end{array}$$

答： $Q = x^2 - \frac{xy}{2} + \frac{3y^2}{4}$, $R = \frac{1}{4}y^3$.

$$\begin{array}{r|l}
 (2) & y^3 + y^2x + 0 + 2x^3 \\
 & y^3 + 2y^2x \\
 \hline
 & -y^2x + 0 + 2x^3 \\
 & -y^2x - 2yx^2 \\
 \hline
 & 2yx^2 + 2x^3 \\
 & 2yx^2 + 4x^3 \\
 \hline
 & -2x^3
 \end{array}$$

答： $Q = y^2 - yx + 2x^2$, $R = -2x^3$.

19. 解：

$$\begin{array}{r|l}
 1 - 3x + 5x^2 & 1 + x + 3x^2 \\
 1 + x + 3x^2 & 1 - 4x + 6x^2 \\
 \hline
 & -4x + 2x^2 \\
 & -4x - 4x^2 - 12x^3 \\
 \hline
 & 6x^2 + 12x^3 \\
 & 6x^2 + 6x^3 + 18x^4 \\
 \hline
 & 6x^3 - 18x^4
 \end{array}$$

答： $Q = 1 - 4x + 6x^2$, $R = 6x^3 - 18x^4$.

$$\begin{array}{r}
 20. \text{ 解: } \quad \begin{array}{r} 1+x+3x^2 \mid 1-3x+5x^2 \\ 1-3x+5x^2 \mid 1+4x+10x^2 \\ \hline 4x-2x^2 \\ 4x-12x^2+20x^3 \\ \hline 10x^2-20x^3 \\ 10x^2-30x^3+50x^4 \\ \hline 10x^3-50x^4 \end{array}
 \end{array}$$

$$\text{答: } Q=1+4x+10x^2, R=10x^3-50x^4.$$

$$\begin{aligned}
 21. \text{ 解: 設 } \quad \frac{1}{1-2x} &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \\
 1 &= (1-2x)(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) \\
 &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \\
 &\quad - 2a_0x - 2a_1x^2 - 2a_2x^3 - \dots \\
 &\quad \therefore a_0 = 1.
 \end{aligned}$$

$$a_1 - 2a_0 = 0 \quad \therefore a_1 = 2.$$

$$a_2 - 2a_1 = 0 \quad \therefore a_2 = 4.$$

$$a_3 - 2a_2 = 0 \quad \therefore a_3 = 8.$$

$$\text{答: } \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \dots$$

$$\begin{aligned}
 22. \text{ 解: 設 } \quad \frac{2+3x+4x^2}{1-x+2x^2} &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \\
 2+3x+4x^2 &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \\
 &\quad - a_0x - a_1x^2 - a_2x^3 - \dots \\
 &\quad + 2a_0x^2 + 2a_1x^3 + \dots \\
 &\quad \therefore a_0 = 2.
 \end{aligned}$$

$$a_1 - a_0 = 3 \quad \therefore a_1 = 5.$$

$$a_2 - a_1 + 2a_0 = 4 \quad \therefore a_2 = 5.$$

$$a_3 - a_2 + 2a_1 = 0 \quad \therefore a_3 = -5.$$

$$\text{答: } \frac{2+3x+4x^2}{1-x+2x^2} = 2 + 5x + 5x^2 - 5x^3 + \dots$$

習 題 XIV

原本第 173 頁

$$\begin{array}{r}
 1. \text{ 解: } \quad \begin{array}{r} 1-3-1-11-4 \mid 4 \\ 4+4+12 \mid 4 \\ \hline 1+1+3+1, 0 \end{array}
 \end{array}$$

答: $Q = x^3 + x^2 + 3x + 1, R = 0.$

2. 解:
$$\begin{array}{r} 5 - 6 - 8 + 7 + 6 + 3 \mid 3 \\ \underline{15 + 27 + 57 + 192 + 594} \\ 5 + 9 + 19 + 64 + 198, 597 \end{array}$$

答: $Q = 5x^4 + 9x^3 + 19x^2 + 64x + 198, R = 597.$

3. 解:
$$\begin{array}{r} 3 + 0 + 1 + 0 - 1 \mid -2 \\ \underline{-6 + 12 - 26 + 52} \\ 3 - 6 + 13 - 26, 51 \end{array}$$

答: $Q = 3x^3 - 6x^2 + 13x - 26, R = 51.$

4. 解:
$$\begin{array}{r} 3 + 16 - 13 - 6 \mid -\frac{1}{3} \\ \underline{-1 - 5 + 6} \\ 3 \mid \underline{3 + 15 - 18, 0} \\ 1 + 5 - 6, 0 \end{array}$$

答: $Q = x^2 + 5x - 6, R = 0.$

5. 解:
$$\begin{array}{r} 3 - 6 + 1 + 2 \mid \frac{1}{3} \\ \underline{1 - \frac{5}{3} - \frac{2}{9}} \\ 3 \mid \underline{3 - 5 - \frac{2}{3}, \frac{16}{9}} \\ 1 - \frac{5}{3} - \frac{2}{9}, \end{array}$$

答: $Q = x^2 - \frac{5}{3}x - \frac{2}{9}, R = \frac{16}{9}.$

6. 解:
$$\begin{array}{r} 1 - (a+b+c) + (ab+ac+bc) - abc \mid a \\ \underline{a \quad \quad \quad - (ab+ac) \quad \quad + abc} \\ 1 - (b+c) + \quad \quad \quad bc, \quad 0 \end{array}$$

答: $Q = x^2 - (b+c)x + bc, R = 0.$

7. 解:
$$\begin{array}{r} 2 - 1 - 7 + 7 - 10 \mid 2 \\ \underline{4 + 6 - 2 + 10} \\ 2 + 3 - 1 + 5, 0 \end{array}$$

答: $Q = 2x^3 + 3x^2y - xy^2 + 5y^3, R = 0.$

8. 解: (1) $f(1) = 2 - 5 + 3 = 0.$

$$(2) \quad f(2) = 16 - 10 + 3 = 9.$$

$$(3) \quad \begin{array}{r} 2 + 0 - 5 + 3 \quad | \quad 5 \\ \hline 2 + 10 + 50 + 225 \\ \hline 2 + 10 + 45, \quad 228 \end{array}$$

$$\therefore f(5) = 228.$$

$$(4) \quad f(-1) = -2 + 5 + 3 = 6.$$

$$(5) \quad f(-3) = -54 + 15 + 3 = -36.$$

$$(6) \quad \begin{array}{r} 2 + 0 - 5 + 3 \quad | \quad -6 \\ \hline -12 + 72 - 402 \\ \hline 2 - 12 + 67 - 399 \end{array}$$

$$\therefore f(-6) = -399.$$

$$9. \text{ 解: } \begin{array}{r} 1 + m \quad -20 \quad + \quad 6 \quad | \quad 3 \\ \quad 3 \quad + (9 + 3m) \quad + (9m - 33) \\ \hline 1 + (m + 3) + (3m - 11) + (9m - 27) \end{array}$$

若能除盡，則 $9m - 27 = 0$

$$\therefore m = 3.$$

$$10. \text{ 解: } \begin{array}{r} 2 - 1 + \quad l + m \quad | \quad -2 \\ \quad -4 + 10 - (2l + 20) \\ \hline 2 - 5 + (10 + l), (m - 2l - 20) = R \\ 2 - 1 + \quad l + m \quad | \quad 4 \\ \quad 8 + 28 + (4l + 112) \\ \hline 2 + 7 + (l + 28), (m + 4l + 112) = R \end{array}$$

依題意，則 $m - 2l = 20 \dots\dots\dots(1)$

$m + 4l = -112 \dots\dots\dots(2)$

$$(1) - (2), \quad 6l = -132$$

$$\therefore l = -22.$$

$$\therefore m = -24.$$

11. 解：以 $2n$ 代 m 於已知方程式中，得

$$3b(2n) + a(2n) - 2an - 6bn = 0$$

故原式必能被 $m - 2n$ 除盡。

同理於原式中以 $-3b$ 代 a ，則原式亦等於零，故 $a + 3b$ 亦能除盡原式 (§ 416)。

12. 解：以 b 代 a 於已知方程式，得

$$b(b-c)^3 + b(c-b)^3 + c(a-b)^3 = b(b-c)^3 - b(b-c)^3 + 0 = 0$$

故此已知方程式必能被 $a-b$ 除盡。

同理證 $b-c, c-a$ 亦能除盡原式。

故 $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$ 能被 $(a-b)(b-c)(c-a)$ 除盡 (§ 417)。

$$\begin{aligned} 13. \text{ 解: 設 } f(x) &= a_0(x-1)(x-4)(x+2) \\ -16 &= a_0(2-1)(2-4)(2+2) \\ -16 &= a_0(-8) \end{aligned}$$

$$\begin{aligned} \therefore a_0 &= 2, \\ \therefore f(x) &= 2(x-1)(x-4)(x+2) \\ &= 2x^3 - 6x^2 - 42x + 16. \end{aligned}$$

$$\begin{aligned} 14. \text{ 解: 設 } f(x) &= (a_0x + a_1)(x-2)(x-3) \\ 6 &= (0 \times a_0 + a_1)(0-2)(0-3) \end{aligned}$$

$$12 = (a_0 + a_1)(1-2)(1-3)$$

$$\therefore a_1 = 1, a_0 = 5.$$

$$\begin{aligned} \therefore f(x) &= (5x+1)(x-2)(x-3) \\ &= 5x^3 - 24x^2 + 25x + 6. \end{aligned}$$

15. 解: 如以上二式有 x 之四值使其相等, 則其相當係數必相等, 然此二式之係數不等, 故知其 x 不能有四個相同之值。

習 題 XV

原本第 176 頁

$$\begin{array}{r|l} 1. \text{ 解: } 1+1+0+0-1 & 1+0+1 \\ \hline 1+0+1 & 1+1-1 \quad 1+0+1 \\ \hline 1-1+0-1 & 1+0+1 \quad 1 \therefore Q_1=1 \\ \hline 1+0+1 & 1-2 \therefore R_1=x-2 \\ \hline -1-1-1 & \\ \hline -1-0-1 & \\ \hline -1 & R=-x \end{array}$$

$$\text{答: } x^4 + x^3 - 1 \equiv (x^2 + 1)^2 + (x-2)(x^2 + 1) - x.$$

$$\begin{array}{r|l} 2. \text{ 解: } 4+2+4+1+6 & 2+0+1 \\ \hline 4+0+2 & 2+1+1 \quad 2+0+1 \\ \hline 2+2+1+6 & 2+0+1 \quad 1 \therefore Q_1=1 \\ \hline 2+0+1 & 1 \therefore R_1=x \\ \hline 2+0+6 & \\ \hline 2+0+1 & \\ \hline 5 & \therefore R=5 \end{array}$$

答: $4x^4 + 2x^3 + 4x^2 + x + 6 \equiv (2x^2 + 1)^2 + x(2x^2 + 1) + 5.$

3. 解:

$$\begin{array}{r|l}
 2-3+2+5+0-1+0+6 & 1-1+1+3 \\
 \hline
 2-2+2+6 & 2-1-1-1+3 \\
 \hline
 -1+0-1+0 & 2-2+2+6 \\
 \hline
 -1+1-1-3 & 1-3-7+3 \\
 \hline
 -1+0+3-1 & 1-1+1+3 \\
 \hline
 -1+1-1-3 & -2-8+0 \\
 \hline
 -1+4+2+0 & \therefore R_1 = -2x^2 - 8x \\
 \hline
 -1+1-1-3 & \\
 \hline
 3+3+3+6 & \\
 \hline
 3-3+3+9 & \\
 \hline
 6+0-3 & \therefore R = 6x^2 - 3
 \end{array}$$

答: $2x^7 - 3x^6 + 2x^5 + 5x^4 - x^2 + 6 \equiv (2x+1)(x^3 - x^2 + x + 3)^2 - (2x^2 + 8x)(x^3 - x^2 + x + 3) + (6x^2 - 3).$

4. 解:

$$\begin{array}{r|l}
 1+0+1+1+0+1 & 1+1+1 \\
 \hline
 1+1+1 & 1-1+1+1 \\
 \hline
 -1+0+1 & 1+1+1 \\
 \hline
 -1-1-1 & -2+0+1 \\
 \hline
 1+2+0 & -2-2-2 \\
 \hline
 1+1+1 & 2+3 \\
 \hline
 1-1+1 & \therefore R_1 = 2xy^2 + 3y^3 \\
 \hline
 1+1+1 & = (2x+3y)y^2 \\
 \hline
 -2 & \therefore R = -2xy^4
 \end{array}$$

答: $x^5 + x^3y^2 + x^2y^3 + y^5 \equiv (x-2y)(x^2+xy+y^2)^2 + (2x+3y)y^2(x^2+xy+y^2) - 2xy^4.$

5. 解: $2-8+1+6 \mid 3$

$$\begin{array}{r|l}
 6-6-15 & \\
 \hline
 2-2-5-9 & \therefore R = -9 \\
 \hline
 6+12 & \\
 \hline
 2+4+7 & \therefore R_1 = 7 \\
 \hline
 6 &
 \end{array}$$

$$2+10 \therefore R_2 = 10, Q_2 = 2$$

答: $2x^3 - 8x^2 + x + 6 \equiv 2(x-3)^3 + 10(x-3)^2 + 7(x-3) - 9.$

$$\begin{array}{r}
 6. \text{ 解: } \quad 1+3-6+2-3+7 \mid -2 \\
 \quad \quad \quad -2-2+16-36+78 \\
 \hline
 \quad \quad \quad 1+1-8+18-39+85 \quad \therefore R=85 \\
 \quad \quad \quad -2+2+12-60 \\
 \hline
 \quad \quad \quad 1-1-6+30-99 \quad \therefore R_1=-99 \\
 \quad \quad \quad -2+6+0 \\
 \hline
 \quad \quad \quad 1-3+0+30 \quad \therefore R_2=30 \\
 \quad \quad \quad -2+10 \\
 \hline
 \quad \quad \quad 1-5+10 \quad \therefore R_3=10 \\
 \quad \quad \quad -2 \\
 \hline
 \quad \quad \quad 1-7 \quad \therefore R_4=-7, Q_4=1.
 \end{array}$$

$$\text{答: } x^5+3x^4-6x^3+2x^2-3x+7 \equiv (x+2)^5-7(x+2)^4 \\
 +10(x+2)^3+30(x+2)^2-99(x+2)+85.$$

$$\begin{array}{r}
 7. \text{ 解: } \quad 1+9+27+0 \mid -3 \\
 \quad \quad \quad -3-18-27 \\
 \hline
 \quad \quad \quad 1+6+9-27 \quad \therefore R=-27 \\
 \quad \quad \quad -3-9 \\
 \hline
 \quad \quad \quad 1+3+0 \quad \therefore R_1=0 \\
 \quad \quad \quad -3 \\
 \hline
 \quad \quad \quad 1+0 \quad \therefore R_2=0, Q_2=1.
 \end{array}$$

$$\text{答: } x^3+9x^2+27x = (x+3)^2-27.$$

$$\begin{array}{r}
 8. \text{ 解: } \quad 1+3+1-1 \mid -1 \\
 \quad \quad \quad -1-2+1 \\
 \hline
 \quad \quad \quad 1+2-1+0 \quad \therefore R=0 \\
 \quad \quad \quad -1-1 \\
 \hline
 \quad \quad \quad 1+1-2 \quad \therefore R_1=-2 \\
 \quad \quad \quad -1 \\
 \hline
 \quad \quad \quad 1+0 \quad \therefore R_2=0, Q_2=1
 \end{array}$$

$$\text{答: } x^3+3x^2+x-1 \equiv (x+1)^2-2(x+1).$$

VI. 有理整式之因式

習 題 XVI

原本第 180 頁

$$1. \text{ 解: } 6x^4y^3z^2-12x^2y^4z+8x^2y^3 = 2x^2y^3(3x^2z^2-6yz+4).$$

2. 解: $2n^2 + (n-3)n = 2n^2 + n^2 - 3n = 3n^2 - 3n = 3n(n-1)$.
3. 解: $ab - a + b - 1 = a(b-1) + (b-1) = (a+1)(b-1)$.
4. 解: $mx - nx - mn + n^2 = x(m-n) - n(m-n)$
 $= (x-n)(m-n)$.
5. 解: $3xy - 2x - 12y + 8 = x(3y-2) - 4(3y-2)$
 $= (x-4)(3y-2)$.
6. 解: $10xy + 5y^2 + 6x + 3y = 5y(2x+y) + 3(2x+y)$
 $= (5y+3)(2x+y)$.
7. 解: $x^3y^2 - x^2y^3 + 2x^2y - 2xy^2 = x^2y^2(x-y) + 2xy(x-y)$
 $= (x^2y^2 + 2xy)(x-y) = xy(xy+2)(x-y)$.
8. 解: $x^4 + x^3 + x^2 + x = x^2(x^2+1) + x(x^2+1)$
 $= (x^2+x)(x^2+1) = x(x+1)(x^2+1)$.
9. 解: $ac + bd - (bc + ad) = c(a-b) - d(a-b) = (c-d)(a-b)$.
10. 解: $a^2c - abd - abc + a^2d = a^2(c+d) - ab(c+d)$
 $= (a^2 - ab)(c+d) = a(a-b)(c+d)$.
11. 解: $ad + ce + bd + ae + cd + be = d(a+b+c) + e(a+b+c)$
 $= (d+e)(a+b+c)$.
12. 解: $a^2 + cd - ab - bd + ac + ad = a(a-b+c) + d(a-b+c)$
 $= (a+d)(a-b+c)$.

習題 XVII

原本第 184 頁

1. 解: $4x^2y - 20x^2y^2 + 25xy^3 = xy(4x^2 - 20xy + 25y^2)$
 $= xy(2x - 5y)^2$.
2. 解: $28tx^2 - 63ty^2 = 7t(4x^2 - 9y^2) = 7t(2x+3y)(2x-3y)$.
3. 解: $x^2 + 4y^2 + 9z^2 - 4xy - 12yz + 6zx$
 $= (x-2y)^2 + 2[(3z)(x-2y)] + (3z)^2 = (x-2y+3z)^2$.
4. 解: $(7a^2 + 2b^2)^2 - (2a^2 + 7b^2)^2$
 $= (7a^2 + 2b^2 + 2a^2 + 7b^2)(7a^2 + 2b^2 - 2a^2 - 7b^2)$
 $= 9(a^2 + b^2)5(a+b)(a-b)$
 $= 45(a^2 + b^2)(a+b)(a-b)$.
5. 解: $(7x^2 + 4x - 3)^2 - (x^2 + 4x + 3)^2$
 $= (7x^2 + 4x - 3 + x^2 + 4x + 3)(7x^2 + 4x - 3 - x^2 - 4x - 3)$
 $= 8x(x+1)6(x+1)(x-1) = 48(x+1)^2(x-1)$.

6. 解: $4(1-b^2-ab)-a^2=4-4b^2-4ab-a^2$
 $=4-(2b+a)^2=(2+2b+a)(2-2b-a).$
7. 解: $x^4+x^2+1=x^4+2x^2+1-x^2$
 $=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1).$
8. 解: $a^4-6a^2b^2+b^4=a^4-2a^2b^2+b^4-4a^2b^2$
 $=(a^2-b^2)^2-(2ab)^2=(a^2+2ab-b^2)(a^2-2ab-b^2).$
9. 解: $a^4+4a^2+16=a^4+8a^2+16-4a^2$
 $=(a^2+4)^2-(2a)^2=(a^2+2a+4)(a^2-2a+4).$
10. 解: $9x^4+15x^2y^2+16y^4=9x^4+24x^2y^2+16y^4-9x^2y^2$
 $=(3x^2+4y^2)^2-(3xy)^2$
 $=(3x^2+3xy+4y^2)(3x^2-3xy+4y^2).$
11. 解: $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$
 $=(2ab+2cd+a^2+b^2-c^2-d^2)(2ad+2cd-a^2-b^2+c^2+d^2)$
 $=[(a+b)^2-(c-d)^2][-(a-b)^2+(c+d)^2]$
 $=(a+b-c+d)(a+b+c-d)(c+d-a+b)(c+d+a-b).$
12. 解: $576x^6y^3-9y^{15}=9y^3(64x^6-y^{12})$
 $=9y^3(8x^3-y^6)(8x^3+y^6)$
 $=9y^3(2x-y^2)(2x+y^2)(4x^2+2xy^2+y^4)(4x^2-2xy^2+y^4).$
13. 解: $x^9-y^9=(x^3-y^3)(x^6+x^3y^3+y^6)$
 $=(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6).$
14. 解: $x^{12}-y^{12}=(x^6-y^6)(x^6+y^6)$
 $=(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)(x^2+y^2)$
 $\times (x^4-x^2y^2+y^4).$
15. 解: $x^{10}+y^{10}=(x^2)^5+(y^2)^5$
 $=(x^2+y^2)(x^8-x^6y^2+x^4y^4-x^2y^6+y^8).$
16. 解: $x^5-32=x^5-2^5=(x-2)(x^4+2x^3+4x^2+8x+16).$
17. 解: $x^7+y^{14}=x^7+(y^2)^7$
 $=(x+y^2)(x^6-x^5y^2+x^4y^4-x^3y^6+x^2y^8-xy^{10}+y^{12}).$

習 題 XVIII

原本第 185 頁

1. 解: $x^4-x^3+x-1=x^3(x-1)+(x-1)=(x-1)(x^3+1)$
 $=(x-1)(x+1)(x^2-x+1).$

2. 解: $x^5 - x^3 - 8x^2 + 8 = x^3(x^2 - 1) - 8(x^2 - 1)$
 $= (x^2 - 1)(x^3 - 8) = (x + 1)(x - 1)(x - 2)(x^2 + 2x + 4).$
3. 解: $x^4 - 2x^3 + 2x - 1 = (x^2 + 1)(x^2 - 1) - 2x(x^2 - 1)$
 $= (x^2 - 1)(x^2 - 2x + 1) = (x + 1)(x - 1)^2.$
4. 解: $x^3 - 7x^2 - 4x + 28 = x(x^2 - 4) - 7(x^2 - 4)$
 $= (x - 7)(x^2 - 4) = (x - 7)(x + 2)(x - 2).$
5. 解: $x^6 - x^4y^2 - x^2y^4 + y^6 = x^4(x^2 - y^2) - y^4(x^2 - y^2)$
 $= (x^4 - y^4)(x^2 - y^2)$
 $= (x^2 + y^2)(x - y)^2(x + y)^2.$
6. 解: $x^3 + 2x^2 + 3x + 2 = x^3 + 2x^2 + x + 2x + 2$
 $= x(x + 1)^2 + 2(x + 1) = (x + 1)(x^2 + x + 2).$
7. 解: $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1$
 $= x^4(x + 1) + x^3(x + 1) + 2x^2(x + 1) + x(x + 1) + (x + 1)$
 $= (x + 1)(x^4 + x^3 + 2x^2 + x + 1) = (x + 1)(x^2 + 1)(x^2 + x + 1)$
8. 解: $x^4 + 4x^3 + 10x^2 + 12x + 9$
 $= x^4 + 6x^2 + 9 + 4x^3 + 12x + 4x^2$
 $= (x^2 + 3)^2 + 4x(x^2 + 3) + 4x^2 = (x^2 + 2x + 3)^2.$

習題 XIX

原本第 190 頁

1. 解: $x^2 - 14x + 48 = (x - 6)(x - 8).$
2. 解: $x^2 - 21x - 120$
 $= \left[x - \frac{21}{2} - \frac{\sqrt{441 + 480}}{2} \right] \left[x - \frac{21}{2} + \frac{\sqrt{441 + 480}}{2} \right]$
 $= [x - (21 + \sqrt{921})/2] [x - (21 - \sqrt{921})/2].$
3. 解: $5x^2 - 53x - 22 = (x - 11)(5x + 2).$
4. 解: $16x^2 + 64x + 63 = (4x + 7)(4x + 9).$
5. 解: $54x^2 - 21x + 2 = (6x - 1)(9x - 2).$
6. 解: $12x^2 + 20xy - 8y^2 = 4(3x^2 + 5xy - 2y^2)$
 $= 4(3x - y)(x + 2y).$
7. 解: $x^4 - 13x^2 + 36 = (x^2 - 9)(x^2 - 4)$
 $= (x + 3)(x - 3)(x + 2)(x - 2).$

$$8. \text{ 解: } x^3y - 3x^2y^2 - 18xy^3 = xy(x^2 - 3xy - 18y^2) \\ = xy(x+3y)(x-6y).$$

$$9. \text{ 解: } x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} \\ = [x - (3 + \sqrt{3}i)/2][x - (3 - \sqrt{3}i)/2].$$

$$10. \text{ 解: } 3x^2 + 2x - 3 = 3\left[\left(x + \frac{1}{3}\right)^2 - \frac{10}{9}\right] \\ = 3\left[x + \frac{1}{3} + \frac{\sqrt{10}}{3}\right]\left[x + \frac{1}{3} - \frac{\sqrt{10}}{3}\right] \\ = 3\left[x + (1 + \sqrt{10})/3\right]\left[x + (1 - \sqrt{10})/3\right].$$

$$11. \text{ 解: } x^2 - 4xy - 2y^2 = x^2 - 4xy + 4y^2 - 4y^2 - 2y^2 \\ = (x - 2y)^2 - 6y^2 = [x - (2 + \sqrt{6})y][x - (2 - \sqrt{6})y].$$

$$12. \text{ 解: } x^2 - 6ax - 9b^2 - 18ab = x^2 - (3b)^2 - 6a(x + 3b) \\ = (x + 3b)(x - 3b) - 6a(x + 3b) \\ = (x + 3b)(x - 6a - 3b).$$

$$13. \text{ 解: } abx^2 - (a^2 + b^2)x - (a^2 - b^2) \\ = abx^2 - (a^2 + b^2)x - [(a + b)(a - b)] \\ = (ax + a - b)(bx - a + b) \\ = (ax + a - b)(bx - a - b).$$

$$14. \text{ 解: } x^2 + bd + dx + bx + cx^2 + cdx \\ = x(x + d) + b(x + d) + cx(x + d) = (cx + x + b)(x + d).$$

$$15. \text{ 解: } x^2 - 8xy + 15y^2 + 2x - 4y - 3 \\ \equiv (x - 5y + l)(x - 3y + m) \\ \equiv x^2 - 8xy + mx + 15y^2 - 5my + lx - 3ly + lm \\ \equiv x^2 - 8xy + 15y^2 + (m + l)x - (5m + 3l)y + lm. \\ m + l = 2, \quad 5m + 3l = 4, \quad lm = -3.$$

$$\therefore l = 3, \quad m = -1.$$

$$\text{答: } x^2 - 8xy + 15y^2 + 2x - 4y - 3 = (x - 5y + 3)(x - 3y - 1).$$

$$16. \text{ 解: } x^2 + 3xy + 2y^2 + 3zx + 5yz + 2z^2 \\ \equiv (x + 2y + lz)(x + y + mz) \\ \equiv x^2 + 3xy + 2y^2 + (l + m)xz + (l + 2m)yz + lmz^2. \\ l + m = 3, \quad l + 2m = 5, \quad lm = 2.$$

$$\therefore m = 2, \quad l = 1.$$

$$\text{答: } x^2 + 3xy + 2y^2 + 3zx + 5yz + 2z^2 = (x+y+2z)(x+2y+z).$$

習 題 XX

原本第 194 頁

1. 解:

$$\begin{array}{r} 1-0-7+6 \mid 1 \\ \underline{+1-6} \\ 1+1-6+0 \mid 2 \\ \underline{+2+6} \\ 1+3+0 \mid -3 \\ \underline{-3} \\ 1+0 \end{array}$$

$$\text{答: } x^3 - 7x + 6 = (x-1)(x-2)(x+3).$$

2. 解:

$$\begin{array}{r} 1+6+11+6 \mid -2 \\ \underline{-2-8-6} \\ 1+4+3+0 \mid -3 \\ \underline{-3-3} \\ 1+1+0 \mid -1 \\ \underline{-1} \\ 1+0 \end{array}$$

$$\text{答: } x^3 + 6x^2 + 11x + 6 = (x+2)(x+3)(x+1).$$

3. 解:

$$\begin{array}{r} 1-10+35-50+24 \mid 1 \\ \underline{-9+26-24} \\ 1-9+26-24+0 \mid 2 \\ \underline{+2-14+24} \\ 1-7+12+0 \mid 3 \\ \underline{+3-12} \\ 1-4+0 \mid 4 \\ \underline{+4} \\ 1+0 \end{array}$$

$$\text{答: } x^4 - 10x^3 + 35x^2 - 50x + 24 = (x-1)(x-2)(x-3)(x-4).$$

4. 解:

$$\begin{array}{r} 1-0-2+3-2 \mid 1 \\ \underline{+1-1+2} \\ 1+1-1+2+0 \mid -2 \\ \underline{-2+2-2} \\ 1-1+1+0 \end{array}$$

$$\text{答: } x^4 - 2x^2 + 3x - 2 = (x-1)(x+2)(x^2 - x + 1).$$

$$\begin{array}{r|l}
 5. \text{ 解:} & 6-13-14-3 \quad | \quad 3 \\
 & 6+5+1+0 \quad | \quad -\frac{1}{2} \\
 & 6+2+0 \quad | \quad -\frac{1}{3} \\
 & 6+0
 \end{array}$$

$$\text{答: } 6x^3 - 13x^2 - 14x - 3 = (x-3)(2x+1)(3x+1).$$

$$\begin{array}{r|l}
 6. \text{ 解:} & 2-5-2+2 \quad | \quad \frac{1}{2} \\
 & 2-4-4+0
 \end{array}$$

$$\begin{aligned}
 \text{答: } 2x^3 - 5x^2y - 2xy^2 + 2y^3 &= (2x-y)(x^2 - 2xy - 2y^2) \\
 &= (2x-y)[x - (1+\sqrt{3})y][x - (1-\sqrt{3})y].
 \end{aligned}$$

$$\begin{array}{r|l}
 7. \text{ 解:} & 2-1-9+13-5 \quad | \quad 1 \\
 & 2+1-8+5 \quad | \quad 1 \\
 \hline
 & 2+1-8+5+0 \quad | \quad 1 \\
 & 2+3-5 \\
 \hline
 & 2+3-5+0 \quad | \quad 1 \\
 & 2+5 \\
 \hline
 & 2+5+0
 \end{array}$$

$$\text{答: } 2x^4 - x^3 - 9x^2 + 13x - 5 = (x-1)^3(2x+5).$$

$$\begin{array}{r|l}
 8. \text{ 解:} & 4+0-41+0+46+0-9 \quad | \quad -1 \\
 & -4+4+37-37-9+9 \\
 \hline
 & 4-4-37+37+9-9+0 \quad | \quad -1 \\
 & 4+0-37+0+9 \\
 \hline
 & 4+0-37+0+9+0 \quad | \quad -3 \\
 & -12+36+3-9 \\
 \hline
 & 4-12-1+3+0 \quad | \quad 3 \\
 & 12+0-3 \\
 \hline
 & 4+0-1+0 \quad | \quad -\frac{1}{2} \\
 & -2+1 \\
 \hline
 2 & \overline{4-2+0} \\
 & 2-1 \quad | \quad \frac{1}{2} \\
 & 1 \\
 \hline
 2 & \overline{2+0} \\
 & 1
 \end{array}$$

$$\begin{aligned}
 \text{答: } 4x^6 - 41x^4 + 46x^2 - 9 \\
 = (x+1)(x-1)(x+3)(x-3)(2x+1)(2x-1)
 \end{aligned}$$

$$\begin{array}{r}
 9. \text{ 解: } \quad 6+19+22+23+16+4 \quad | -2 \\
 \quad \quad \quad -12-14-16-14-4 \\
 \hline
 \quad \quad \quad 6+7+8+7+2+0 \quad | -1/2 \\
 \quad \quad \quad -3-2-3-2 \\
 \hline
 \quad \quad \quad 6+4+6+4+0 \\
 \quad \quad \quad 3+2+3+2 \quad | -2/3 \\
 \quad \quad \quad -2+0-2 \\
 \hline
 \quad \quad \quad 3+0+3+0 \\
 \quad \quad \quad 1+0+1
 \end{array}$$

$$\begin{aligned}
 \text{答: } & 6x^5 + 19x^4 + 22x^3 + 23x^2 + 16x + 4 \\
 & = (x+2)(2x+1)(3x+2)(x^2+1).
 \end{aligned}$$

$$\begin{array}{r}
 10. \text{ 解: } \quad 5-7-8-1+7+8-4 \quad | -1 \\
 \quad \quad \quad -5+12-4+5-12+4 \\
 \hline
 \quad \quad \quad 5-12+4-5+12-4+0 \quad | 1 \\
 \quad \quad \quad 5-7-3-8+4 \\
 \hline
 \quad \quad \quad 5-7-3-8+4+0 \quad | 2 \\
 \quad \quad \quad 10+6+6-4 \\
 \hline
 \quad \quad \quad 5+3+3-2+0 \quad | 2/5 \\
 \quad \quad \quad 2+2+2 \\
 \hline
 \quad \quad \quad 5+5+5+0 \\
 \quad \quad \quad 1+1+1
 \end{array}$$

$$\begin{aligned}
 \text{答: } & 5x^6 - 7x^5 - 8x^4 - x^3 + 7x^2 + 8x - 4 \\
 & = (x+1)(x-1)(x-2)(5x-2)(x^2+x+1).
 \end{aligned}$$

$$\begin{aligned}
 11. \text{ 解: } & x^2 - 4x - 12 = 0. \quad (x+2)(x-6) = 0 \\
 & \therefore x = -2 \text{ 及 } 6.
 \end{aligned}$$

$$\begin{aligned}
 12. \text{ 解: } & 6x^2 - 7x + 2 = 0. \quad (3x-2)(2x-1) = 0 \\
 & \therefore x = \frac{2}{3} \text{ 及 } \frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 13. \text{ 解: } & x^2 - 5x = 14. \\
 & (x-7)(x+2) = 0 \\
 & \therefore x = 7 \text{ 及 } -2.
 \end{aligned}$$

$$\begin{aligned}
 14. \text{ 解: } & x^2 + 6x = 2. \\
 & x^2 + 6x + 9 - 11 = 0 \\
 & (x+3)^2 - 11 = 0 \\
 & (x + \sqrt{11} + 3)(x - \sqrt{11} + 3) = 0
 \end{aligned}$$

$$\therefore x = -3 - \sqrt{11} \text{ 及 } -3 + \sqrt{11}.$$

15. 解: $x^3 - 9x^2 + 26x = 24.$

$$\begin{array}{r|l} 1-9+26-24 & 2 \\ 1-7+12+0 & 3 \\ 1-4+0 & 4 \\ 1+0 & \end{array}$$

$$\therefore x = 2, 3 \text{ 及 } 4.$$

16. 解: $x^4 + 2x^3 - 4x^2 - 2x + 3 = 0.$

$$\begin{array}{r|l} 1+2-4-2+3 & 1 \\ 1+3-1-3+0 & 1 \\ 1+4+3+0 & -1 \\ 1+3+0 & -3 \\ 1+0 & \end{array}$$

$$\therefore x = 1, 1, -1, -3.$$

17. 解: $x^3 - 1 = 0.$

$$(x-1)(x^2+x+1) = 0$$

$$(x-1)\left(x + \frac{1-\sqrt{3}i}{2}\right)\left(x + \frac{1+\sqrt{3}i}{2}\right) = 0$$

$$\therefore x = 1, \frac{-1 \pm \sqrt{3}i}{2}.$$

18. 解: $10x^3 - 9x^2 - 3x + 2 = 0.$

$$\begin{array}{r|l} 10-9-3+2 & 1 \\ 10+1-2+0 & -\frac{1}{2} \\ 10-4+0 & \frac{2}{5} \\ 10+0 & \end{array}$$

$$\therefore x = 1, -\frac{1}{2}, \frac{2}{5}.$$

習 題 XXI

原本第 195 頁

1. 解: $6xy + 15x - 4y - 10 = 3x(2y+5) - 2(2y+5)$
 $= (3x-2)(2y+5).$

2. 解: $a^2bc - ac^2d - ab^2d + bcd^2 = ab(ac-bd) - cd(ac-bd)$
 $= (ab-cd)(ac-bd).$

3. 解: $a^3(a-b) + b^3(b-a) = a^3(a-b) - b^3(a-b)$
 $= (a-b)(a^3 - b^3) = (a-b)^2(a^2 + ab + b^2).$
4. 解: $a^5 - 81ab^4 = a(a^4 - 81b^4) = a(a^2 - 9b^2)(a^2 + 9b^2)$
 $= a(a+3b)(a-3b)(a^2 + 9b^2).$
5. 解: $a^4b - a^2b^3 + a^3b^2 - ab^4 = ab(a^3 - ab^2 + a^2b - b^3)$
 $= ab(a+b)(a^2 - b^2) = ab(a-b)(a+b)^2.$
6. 解: $3abx^2 - 6axy + bxy - 2y^2 = 3ax(bx - 2y) + y(bx - 2y)$
 $= (3ax + y)(bx - 2y).$
7. 解: $3x^6 - 192y^6 = 3(x^6 - 64y^6) = 3(x^3 + 8y^3)(x^3 - 8y^3)$
 $= 3(x+2y)(x^2 - 2xy + 4y^2)(x-2y)(x^2 + 2xy + 4y^2).$
8. 解: $(x^2 + x)^3 - 8 = (x^2 + x - 2)[(x^2 + x)^2 + 2(x^2 + x) + 4]$
 $= (x-1)(x+2)(x^4 + 2x^3 + 3x^2 + 2x + 4).$
9. 解: $64x^6y^3 - y^{15} = y^3(64x^6 - y^{12}) = y^2(8x^3 + y^6)(8x^3 - y^6)$
 $= y^3(2x + y^2)(4x^2 - 2xy^2 + y^4)(2x - y^2)(4x^2 + 2xy^2 + y^4).$
10. 解: $x^2 - (a-b)x - ab = (x-a)(x+b).$
11. 解: $x^{2n} - 3x^n - 18 = (x^n - 6)(x^n + 3).$
12. 解: $x - x^2 + 42 = 42 + x - x^2 = (7-x)(6+x).$
13. 解: $3x^4 + 3x^3 - 24x - 24 = 3x^3(x+1) - 24(x+1)$
 $= 3(x+1)(x^3 - 8) = 3(x+1)(x-2)(x^2 + 2x + 4).$
14. 解: $x^5 - 9x^3 + 8x^2 - 72 = x^3(x^2 - 9) + 8(x^2 - 9)$
 $= (x^3 + 8)(x^2 - 9) = (x+2)(x^2 - 2x + 4)(x+3)(x-3).$
15. 解: $2xc - a^2 + x^2 - 2ab + c^2 - b^2$
 $= (c^2 + 2xc + x^2) - (a^2 + 2ab + b^2)$
 $= (c+x)^2 - (a+b)^2$
 $= (c+x+a+b)(c+x-a-b).$
16. 解: $x^2(x^2 - 20) + 64 = x^4 - 20x^2 + 64$
 $= (x^2 - 16)(x^2 - 4)$
 $= (x+4)(x-4)(x+2)(x-2).$
17. 解: $a^2 - 2ab + b^2 - 5a + 5b + 6 = (a-b)^2 - 5(a-b) + 6$
 $= (a-b-3)(a-b-2).$
18. 解: $x^4 - 10x^2y^2 + 9y^4 = (x^2 - y^2)(x^2 - 9y^2)$
 $= (x+y)(x-y)(x+3y)(x-3y).$
19. 解: $6x^2 - 7xy - 5y^2 - 4x - 2y$
 $= (2x+y)(3x-5y) - 2(2x+y) = (3x-5y-2)(2x+y).$

20. 解: $x^4 - (a^2 + b^2)x^2 + a^2b^2 = (x^2 - a^2)(x^2 - b^2)$
 $= (x+a)(x-a)(x+b)(x-b).$
21. 解: $4(xz + uy)^2 - (x^2 - y^2 + z^2 - u^2)^2$
 $= (2xz + 2uy + x^2 - y^2 + z^2 - u^2)$
 $\times (2xz + 2uy - x^2 + y^2 - z^2 + u^2)$
 $= [(x+z)^2 - (y-u)^2][-(x-z)^2 + (y+u)^2]$
 $= (x+y+z-u)(x-y+z+u)(x+y-z+u)$
 $\times (-x+y+z+u).$
22. 解: $14x^2 + 19x - 3 = (2x+3)(7x-1).$
23. 解: $1 + 19y - 66y^2 = (1-3y)(1+22y).$
24. 解: $xy^3 + 55x^2y^2 + 204x^3y = xy(y^2 + 55xy + 204x^2)$
 $= xy(y+4x)(y+51x).$
25. 解: $a^4 - 18a^2b^2c^2 + 81b^4c^4 = (a^2 - 9b^2c^2)^2$
 $= (a+3bc)^2(a-3bc)^2.$
26. 解: $(x^2 - 7x)^2 + 6x^2 - 42x = (x^2 - 7x)^2 + 6(x^2 - 7x)$
 $= x(x-7)(x^2 - 7x + 6) = x(x-7)(x-6)(x-1).$
27. 解: $8(x+y)^3 - 27(x-y)^3 = [2(x+y) - 3(x-y)][4(x+y)^2$
 $+ 6(x+y)(x-y) + 9(x-y)^2]$
 $= (5y-x)(7y^2 - 10xy + 19x^2).$
28. 解: $(x-2y)x^3 - (y-2x)y^3 = x^4 - 2x^3y - y^4 + 2xy^3$
 $= (x^4 - y^4) - 2xy(x^2 - y^2)$
 $= (x^2 - y^2)(x^2 + y^2) - 2xy(x^2 - y^2)$
 $= (x+y)(x-y)(x^2 - 2xy + y^2)$
 $= (x+y)(x-y)^3.$
29. 解: $x^2 + a^2 - bx - ab + 2ax = (x+a)^2 - b(x+a)$
 $= (x+a-b)(x+a).$
30. 解: $x^6 - y^6 - (x-y)^6 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$
 $- (x-y)(x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4)$
 $= (x-y)(5x^3y - 5x^2y^2 + 5xy^3)$
 $= 5xy(x-y)(x^2 - xy + y^2).$
31. 解: $x^5 - x^4 - 2x^3 + 2x^2 + x - 1$
 $= x^4(x-1) - 2x^2(x-1) + (x-1)$
 $= (x-1)(x^4 - 2x^2 + 1) = (x-1)(x^2 - 1)^2$
 $= (x+1)^2(x-1)^3.$

32. 解: $b^4 + b^2 + 1 = b^4 + 2b^2 + 1 - b^2 = (b^2 + 1)^2 - b^2$
 $= (b^2 + b + 1)(b^2 - b - 1).$
33. 解: $2x^2 + 7xy + 3y^2 + 9x + 2y - 5 = (x + 3y + l)(2x + y - m)$
 $= 2x^2 + 7xy + 3y^2 + (2l - m)x + (l - 3m)y - lm.$
 $2l - m = 9, \quad l - 3m = 2, \quad lm = 5$
 $\therefore m = 1, \quad l = 5.$
 答: $2x^2 + 7xy + 3y^2 + 9x + 2y - 5 = (x + 3y + 5)(2x + y - 1).$
34. 解: $a^4 + 4 = a^4 + 4a^2 + 4 - 4a^2 = (a^2 + 2)^2 - (2a)^2$
 $= (a^2 + 2a + 2)(a^2 - 2a + 2).$
35. 解: $x^2 - xy - 2y^2 + 4xz - 5yz + 3z^2$
 $= (x - 2y + lz)(x + y + mz)$
 $= x^2 - xy - 2y^2 + (m + l)xz - (2m - l)yz + lmz^2.$
 $m + l = 4, \quad 2m - l = 5, \quad lm = 3.$
 $\therefore m = 3, \quad l = 1.$
 答: $x^2 - xy - 2y^2 + 4xz - 5yz + 3z^2 = (x - 2y + z)(x + y + 3z).$
36. 解: $4a^4 + 3a^2b^2 + 9b^4 = 4a^4 + 12a^2b^2 + 9b^4 - 9a^2b^2$
 $= (2a^2 + 3b^2)^2 - (3ab^2)$
 $= (2a^2 + 3ab + 3b^2)(2a^2 - 3ab + 3b^2).$
37. 解: $x^2 - 8ax - 40ab - 25b^2 = x^2 - 25b^2 - 8ax - 40ab$
 $= (x + 5b)(x - 5b) - 8a(x + 5b)$
 $= (x + 5b)(x - 8a - 5b).$
38. 解: $x^8 + x^4 + 1 = (x^4 + 1)^2 - x^4 = (x^4 + x^2 + 1)(x^4 - x^2 + 1)$
 $= (x^2 + x + 1)(x^2 - x + 1)(x^4 - x^2 + 1).$
39. 解: $(x^2 + 2x - 1)^2 - (x^2 - 2x + 1)^2 = [(x^2 + 2x - 1)$
 $+ (x^2 - 2x + 1)][(x^2 + 2x - 1) - (x^2 - 2x + 1)]$
 $= 4x^2(2x - 1).$
40. 解: $(ax + by)^2 - (bx + ay)^2$
 $= (ax + by - bx - ay)(ax + by + bx + ay)$
 $= [a(x - y) - b(x - y)][a(x + y) + b(x + y)]$
 $= (a - b)(x - y)(a + b)(x + y).$
41. 解: $x^3 - ax^2 - b^2x + ab^2 = x(x^2 - b^2) - a(x^2 - b^2)$
 $= (x - a)(x + b)(x - b).$
42. 解: $x^4 + bx^3 - a^3x - a^3b = x(x^3 - a^3) + b(x^3 - a^3)$
 $= (x + b)(x - a)(x^2 + ax + a^2).$

$$\begin{aligned}
 53. \text{ 解: } & (a-b)x^2 + 2ax + (a+b) = ax^2 - bx^2 + 2ax + a + b \\
 & = ax^2 + 2ax + a - bx^2 + b = a(x+1)^2 - b(x^2-1) \\
 & = (x+1)(ax+a-bx+b) = (x+1)[x(a-b) + (a+b)].
 \end{aligned}$$

$$\begin{aligned}
 54. \text{ 解: } & x^{15} - y^{15} = (x^5)^3 - (y^5)^3 = (x^5 - y^5)(x^{10} + x^5y^5 + y^{10}) \\
 & = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x^2 + xy + y^2)(x^8 - x^7y \\
 & \quad + x^6y^2 - x^4y^4 + x^3y^5 - xy^7 + y^8).
 \end{aligned}$$

$$\begin{array}{r|l}
 55. \text{ 解: } & \begin{array}{r} 1-6+7+6-8 \\ 1-5+2+8+0 \\ 1-6+8+0 \\ 1-4+0 \\ 1+0 \end{array} & \begin{array}{l} 1 \\ -1 \\ 2 \\ 4 \end{array}
 \end{array}$$

$$\text{答: } x^4 - 6x^3 + 7x^2 + 6x - 8 = (x-1)(x+1)(x-2)(x-4).$$

$$56. \text{ 解: } 4x^3 - 3x - 1 = (x-1)(4x^2 + 4x + 1) = (x-1)(2x+1)^2.$$

$$\begin{array}{r|l}
 57. \text{ 解: } & \begin{array}{r} 3-10-8-3+10+8 \\ 3-7-15-18-8 \\ 3-7-15-18-8+0 \\ 12+20+20+8 \\ 3+5+5+2+0 \\ -2-2-2 \\ 3+3+3+0 \\ 1+1+1 \end{array} & \begin{array}{l} 1 \\ 4 \\ 4 \\ -2/3 \end{array}
 \end{array}$$

$$\begin{aligned}
 \text{答: } & 3x^5 - 10x^4 - 8x^3 - 3x^2 + 10x + 8 \\
 & = (x-1)(x-4)(3x+2)(x^2+x+1).
 \end{aligned}$$

$$\begin{array}{r|l}
 58. \text{ 解: } & \begin{array}{r} 5+24-15-118+24 \\ -15-27+126-24 \\ 5+9-42+8+0 \\ 10+38-8 \\ 5+19-4+0 \\ -20+4 \\ 5-1+0 \end{array} & \begin{array}{l} -3 \\ 2 \\ -4 \end{array}
 \end{array}$$

$$\begin{aligned}
 \text{答: } & 5x^4 + 24x^3 - 15x^2 - 118x + 24 \\
 & = (x+3)(x-2)(x+4)(5x-1).
 \end{aligned}$$

$$\begin{aligned}
 59. \text{ 解: } & a^2bc + ac^2 + acd - abd - cd - d^2 \\
 & = ac(ab+c+d) - d(ab+c+d) \\
 & = (ac-d)(ab+c+d).
 \end{aligned}$$

$$\begin{aligned}
 60. \text{ 解: } & x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 \\
 & = (x^2 - 2yz - y^2 - z^2)(x^2 + 2yz - y^2 - z^2) \\
 & = (x + y + z)(x - y - z)(x + y - z)(x - y + z).
 \end{aligned}$$

VII. 最高公因式及最低公倍式

習題 XXII

原本第 204 頁

- 解: $10x^3y^2z^5, 4x^5yz^3, 6x^4y^3z^5$ 及 $x^4y^4z^4u$.
 $10x^3y^2z^5 = x^3yz^3(10yz^2)$
 $4x^5yz^3 = x^3yz^3(4x^2)$
 $6x^4y^3z^5 = x^3yz^3(6xy^2z^2)$
 $x^4y^4z^3u = x^3yz^3(xy^3zu)$
 $\therefore \text{H. C. F.} = x^3yz^3$.
- 解: $(a+b)^2(a-b), (a+b)(a-b)^2$ 及 $a^3b - ab^3$.
 $a^3b - ab^3 = ab(a+b)(a-b)$
 $\therefore \text{H. C. F.} = (a+b)(a-b)$.
- 解: $y^4 + y^2 + 1$ 及 $y^2 - y + 1$.
 $y^4 + y^2 + 1 = (y^2 + y + 1)(y^2 - y + 1)$
 $\therefore \text{H. C. F.} = y^2 - y + 1$.
- 解: $a^2 - 1, a^2 + 2a + 1$ 及 $a^3 + 1$.
 $a^2 - 1 = (a+1)(a-1)$
 $a^2 + 2a + 1 = (a+1)^2$
 $a^3 + 1 = (a+1)(a^2 - a + 1)$
 $\therefore \text{H. C. F.} = a + 1$.
- 解: $x^3 - 1$ 及 $x^3 + ax^2 - ax - 1$.
 $x^3 - 1 = (x-1)(x^2 + x + 1)$
 $x^3 + ax^2 - ax - 1 = (x-1)[x^2 + (a+1)x + 1]$
 $\therefore \text{H. C. F.} = x - 1$.
- 解: $x^4 - y^4, x^6 + y^6$ 及 $x^3 + x^2y + xy^2 + y^3$.
 $x^4 - y^4 = (x^2 + y^2)(x+y)(x-y)$
 $x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$
 $x^3 + x^2y + xy^2 + y^3 = (x^2 + y^2)(x+y)$
 $\therefore \text{H. C. F.} = x^2 + y^2$.

7. 解: x^2+5x+6, x^2+x-2 及 $x^2-14x-32$.
 $x^2+5x+6=(x+3)(x+2)$
 $x^2+x-2=(x+2)(x-1)$
 $x^2-14x-32=(x+2)(x-16)$
 \therefore H. C. F. $=x+2$.
8. 解: $(x-1)(x-2)$ 及 $5x-15x^3+8x^2+6x-4$.
 $5x-15x^3+8x^2+6x-4=(x-1)(x-2)(5x^2-2)$
 \therefore H. C. F. $=(x-1)(x-2)$.
9. 解: x^3-1 及 x^3-4x^2-4x-5 .
 $x^3-1=(x-1)(x^2+x+1)$
 $x^3-4x^2-4x-5=(x-5)(x^2+x+1)$
 \therefore H. C. F. $=x^2+x+1$.
10. 解: $(x^2-1)^2(x+1)^2$ 及 $(x^3+5x^2+7x+3)(x^2-6x-7)$.
 $(x^2-1)^2(x+1)^2=(x+1)^4(x-1)^2$
 $(x^3+5x^2+7x+3)(x^2-6x-7)=(x+1)^3(x+3)(x-7)$
 \therefore H. C. F. $=(x+1)^3$.
11. 解: $(x-1)^2(x-2)^2$ 及 $(x^2-3x+2)(2x^2-5x^2+5x-6)$.
 $(x^2-3x+2)(2x^3-5x^2+5x-6)$
 $=(x-1)(x-2)^2(2x^2-x+3)$
 \therefore H. C. F. $=(x-1)(x-2)^2$.
12. 解: $2x^3-3x^2-11x+6$ 及 $4x^3+3x^2-9x+2$.
 $2x^3-3x^2-11x+6=(x+2)(2x^2-7x+3)$
 $4x^3+3x^2-9x+2=(x+2)(4x^2-5x+1)$
 \therefore H. C. F. $=x+2$.
13. 解: x^3-2x^2-2x-3 及 $2x^3+x^2+x-1$.

$$\begin{array}{r|l} 1 & \begin{array}{ccc|ccc} 1 & -2 & -2 & -3 & 2 & +1 & +1 & -1 & 2 \\ & 1 & +1 & +1 & & 2 & -4 & -4 & -6 \\ -3 & & -3 & -3 & -3 & & 5 & +5 & +5 \\ & & -3 & -3 & -3 & & 1 & +1 & +1 \\ \hline & & & & 0 & & & & \end{array} \\ \hline & 0 \end{array}$$

 \therefore H. C. F. $=x^2+x+1$.
14. 解: $3x^3+2x^2-19x+6$ 及 $2x^3+x^2-13x+6$.

3	3+2-19+6	2+1-13+6	2
	6+4-38+12	2+2-12	
	6+3-39+18	-1-1+6	-1
	1+1-6	-1-1+6	
		0	

$$\therefore \text{H. C. F.} = x^2 + x - 6.$$

15. 解: $x^4 - x^3 - 3x^2 + x + 2$ 及 $2x^4 + 3x^3 - x^2 - 3x - 1$.

1-2	1-1-3+1+2	2+3-1-3-1	2
	1+1-1-1	2-2-6+2+4	
	-2-2+2+2	5+5-5-5	1-1
	-2-2+2+2	1+1-1-1	
	0		

$$\therefore \text{H. C. F.} = x^3 + x^2 - x - 1.$$

16. 解: $3x^3 - 13x^2 + 23x - 21$ 及 $6x^3 + x^2 - 44x + 21$.

1-1	3-13+23-21	6+1-44+21	2
	3-10+7	6-26+46-42	
	-3+16-21	27-90+63	1-1
	-3+10-7	3-10+7	
	6-14	3-7	
	3-7	-3+7	
		-3+7	
		0	

$$\therefore \text{H. C. F.} = 3x - 7.$$

17. 解: $3x^3 + 8x^2 - 4x - 15$ 及 $6x^4 + 10x^3 - 3x^2 - 2x + 5$.

1	3+8-4-15	6+10-3-2+5	2-2
	21+56-28-105	6+16-8-30	
	21+20-25	-6+5+28+5	21
4	36-3-105	-6-16+8+30	
	84-7-245	21+20-25	
	84+80-100	609+580-725	
	-87-145	609+1015	
	3+5	-435-725	-145
		-435-725	
		0	

$$\therefore \text{H. C. F.} = 3x + 5.$$

18. 解: $6x^5 + 7x^4 - 9x^3 - 7x^2 + 3x$ 及 $6x^5 + 7x^4 + 3x^2 + 7x^2 - 3x$.

1-1	6+7-9-7+3	6+7+3+7-3	1
	6+7-3	6+7-9-7+3	
	-6-7+3	12+14-6	
	-6-7+3	6+7-3	
	0		

$$\therefore \text{H.C.F.} = 3x^3 + 7x^2 - 3x = x(6x^2 + 7x - 3).$$

19. 解: $6x^4 - 3x^3 + 7x^2 + x - 3$ 及 $2x^4 + 3x^3 + 7x^2 + 3x + 9$.

3	6-3+7+1-3	2+3+7+3+9	
	6+9+21+9+27	6+9+21+9+27	1
	-12-14-8-30	6+7+4+15	
3	6+7+4+15	2+17-6+27	1+9
	6+51-18+81	2-1+3	
	-44+22-66	18-9+27	
	2-1+3	18-9+27	
		0	

$$\therefore \text{H.C.F.} = 2x^2 - x + 3.$$

20. 解: $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ 及 $4x^4 + 2x^3 - 18x^2 + 3x - 5$.

3	6-4-11-3-3-1	4+2-18+3-5	2
	12-8-22-6-6-2	4-8+2-2	
	12+6-54+9-15	10-20+5-5	5
	-14+32-15+9-2	10-20+5-5	
7	28-64+30-18+4	0	
	28+14-126+21-35		
	-78+156-39+39		
	2-4+1-1		

$$\therefore \text{H.C.F.} = 2x^3 - 4x^2 + x - 1.$$

21. 解: $x^5 - x^3 - 4x^2 - 3x - 2$ 及 $5x^4 - 3x^2 - 8x - 3$.

5	5-0-3-8-3	1-0-1-4-3-2	
	5+30+30+25	5-0-5-20-15-10	1
	-30-33-33-3	5-0-3-8-3	
10	10+11+11+1	-2-12-12-10	
	10+60+60+50	1+6+6+5	1+5
	-49-49-49	1+1+1	
	1+1+1	5+5+5	
		5+5+5	
		0	

$$\therefore \text{H. C. F.} = x^2 + x + 1.$$

22. 解: $3x^3 - x^2 - 12x + 4$, $x^3 - 2x^2 - 5x + 6$ 及 $7x^3 + 19x^2 + 8x - 4$.

3	3 - 1 - 12 + 4	1 - 2 - 5 + 6	
	3 - 6 - 15 + 18	5 - 10 - 25 + 30	1
5 - 7	5 + 3 - 14	5 + 3 - 14	
	5 + 10	- 13 - 11 + 30	
	- 7 - 14	- 65 - 55 + 150	- 13
	- 7 - 14	- 65 - 39 + 182	
	0	- 16 - 32	
		1 + 2	
1 + 2	7 + 19 + 8 - 4	7	
	7 + 14		
	5 + 8	5	
	5 + 10		
	- 2 - 4	- 2	
	- 2 - 4		
	0		

$$\therefore \text{H. C. F.} = x + 2.$$

23. 解: $x^3 + ax^2 - 3x - 3a$, $x^3 - x^2 - 3x + 3$ 及 $x^3 + x^2 - 3x - 3$.

$$x^3 + ax^2 - 3x - 3a = x(x^2 - 3) + a(x^2 - 3) = (x + a)(x^2 - 3)$$

$$x^3 - x^2 - 3x + 3 = x(x^2 - 3) - (x^2 - 3) = (x - 1)(x^2 - 3)$$

$$x^3 + x^2 - 3x - 3 = x(x^2 - 3) + (x^2 - 3) = (x + 1)(x^2 - 3)$$

$$\therefore \text{H. C. F.} = x^2 - 3.$$

24. 解: $7x^4y - 6x^3y^2 - 18x^2y^3 + 4xy^4$ 及 $14x^3y - 19x^2y^2 - 32xy^3 + 28y^4$.

- 1	7 - 6 - 18 + 4	14 - 19 - 32 + 28	2
	7 - 4 - 20	14 - 12 - 36 + 8	
	- 2 + 2 + 4	- 7 + 4 + 20	- 7
1 + 1	1 - 1 - 2	- 7 + 7 + 14	
	1 - 2	- 3 + 6	
	1 - 2	1 - 2	
	1 - 2		
	0		

$$\therefore \text{H. C. F.} = xy - 2y^2 = y(x - 2y).$$

25. 解: $x(x-1)(x^3+4x^2+4x+3)$ 及 $(x-1)(x+3)(12x^3+x^2+x-1)$.
 $x(x-1)(x^3+4x^2+4x+3) = x(x-1)(x+3)(x^2+x+1)$
 \therefore H. C. F. $= (x-1)(x+3)$.
26. 解: $4x^3-8x^2-3x+9$ 及 $(2x^2-x-3)(2x^2-7x+6)$.
 $4x^3-8x^2-3x+9 = (x+1)(2x-3)^2$
 $(2x^2-x-3)(2x^2-7x+6)$
 $= (x+1)(2x-3)(2x-3)(x-2) = (x+1)(2x-3)^2(x-2)$
 \therefore H. C. F. $= (x+1)(2x-3)^2$.

習題 XXIII

原本第 207 頁

1. 解: $3x-1$, $9x^2-1$ 及 $9x^2+1$.
 \therefore L. C. M. $= (9x^2-1)(9x^2+1)$.
2. 解: $(a+b)(a^5-b^5)$ 及 $(a-b)(a^5+b^5)$.
 $(a+b)(a^5-b^5)$
 $= (a+b)(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$
 $(a-b)(a^5+b^5)$
 $= (a-b)(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$
 \therefore L. C. M. $= (a^5-b^5)(a^5+b^5)$.
3. 解: a^5+a^2+a , a^5-a^3 及 a^6-a^3 .
 $a^3+a^2+a = a(a^2+a+1)$
 $a^5-a^3 = a^3(a^2-1) = a^3(a+1)(a-1)$
 $a^6-a^3 = a^3(a^3-1) = a^3(a-1)(a^2+a+1)$
 \therefore L. C. M. $= a^3(a^3-1)(a+1)$.
4. 解: $(x^3-y^3)(x-y)^3$, $(x^4-y^4)(x-y)^2$ 及 $(x^2-y^2)^2$
 $(x^3-y^3)(x-y)^3 = (x-y)^3(x^2+xy+y^2)(x-y)^3$
 $(x^4-y^4)(x-y)^2 = (x^2+y^2)(x-y)^2(x+y)$
 $(x^2-y^2)^2 = (x-y)^2(x+y)^2$
 \therefore L. C. M. $= (x-y)^4(x+y)^2(x^2+xy+y^2)(x^2+y^2)$.
5. 解: x^2-3x+2 , x^2-5x+6 及 x^2-4x+3 .
 $x^2-3x+2 = (x-1)(x-2)$
 $x^2-5x+6 = (x-2)(x-3)$
 $x^2-4x+3 = (x-1)(x-3)$
 \therefore L. C. M. $= (x-1)(x-2)(x-3)$.

6. 解: $x^2 - (y+z)^2$, $y^2 - (z+x)^2$ 及 $z^2 - (x+y)^2$.

$$x^2 - (y+z)^2 = (x+y+z)(x-y-z)$$

$$y^2 - (z+x)^2 = (y+z+x)(y-z-x)$$

$$z^2 - (x+y)^2 = (z+x+y)(z-x-y)$$

$$\therefore \text{L. C. M.} = (x+y+z)(x-y-z)(-x+y-z)(-x-y+z).$$

7. 解: $2x^2 + 3xy - 9y^2$, $3x^2 + 8xy - 3y^2$ 及 $6x^2 - 11y + 3y^2$.

$$2x^2 + 3xy - 9y^2 = (2x-3y)(x+3y)$$

$$3x^2 + 8xy - 3y^2 = (3x-y)(x+3y)$$

$$6x^2 - 11xy + 3y^2 = (2x-3y)(3x-y)$$

$$\therefore \text{L. C. M.} = (2x-3y)(x+3y)(3x-y).$$

8. 解: $x^3 + x^2 + x + 1$ 及 $x^3 - x^2 + x - 1$.

$$x^3 + x^2 + x + 1 = x^2(x+1) + (x+1) = (x^2+1)(x+1)$$

$$x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = (x^2+1)(x-1)$$

$$\therefore \text{L. C. M.} = (x^2+1)(x+1)(x-1).$$

9. 解: $2a^2x + 2x^2y + 3y^2x + 3a^2y$ 及 $(2x^2 - 3a^2)y + (2a^2 - 3y^2)x$.

$$2a^2x + 2x^2y + 3y^2x + 3a^2y = a^2(2x+3y) + xy(2x+3y) \\ = (a^2+xy)(2x+3y)$$

$$(2x^2 - 3a^2)y + (2a^2 - 3y^2)x = 2x^2y - 3a^2y + 2a^2x - 3xy^2 \\ = xy(2x-3y) + a^2(2x-3y) = (a^2+xy)(2x-3y).$$

$$\therefore \text{L. C. M.} = (a^2+xy)(2x+3y)(2x-3y).$$

10. 解: $8x^3 - 18xy^2$, $8x^3 + 8x^2y - 6xy^2$ 及 $8x^2 - 2xy - 15y^2$.

$$8x^3 - 18xy^2 = 2x(4x^2 - 9y^2) = 2x(2x+3y)(2x-3y)$$

$$8x^3 + 8x^2y - 6xy^2 = 2x(4x^2 + 4xy - 3y^2)$$

$$= 2x(2x-y)(2x+3y)$$

$$8x^2 - 2xy - 15y^2 = (2x-3y)(4x+5y)$$

$$\therefore \text{L. C. M.} = 2x(2x+3y)(2x-3y)(2x-y)(4x+5y).$$

11. 解: $x^3 + y^3$, $x^3 - y^3$ 及 $x^4 + x^2y^2 + y^4$.

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - (xy)^2 \\ = (x^2 + xy + y^2)(x^2 - xy + y^2).$$

$$\therefore \text{L. C. M.} = (x^3 + y^3)(x^3 - y^3).$$

12. 解: $x^6 - 1$, $3x^3 - 5x^2 - 3x + 5$ 及 $x^4 - 1$.

$$x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1)$$

$$3x^3 - 5x^2 - 3x + 5 = 3x(x^2 - 1) - 5(x^2 - 1)$$

$$= (3x - 5)(x^2 - 1)$$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x^2 + 1)(x - 1)(x + 1)$$

$$\therefore \text{L. C. M.} = (x^2 - 1)(x^4 + x^2 + 1)(3x - 5)(x^2 + 1).$$

13. 解: $8x^3 + 27, 16x^4 + 36x^2 + 81$ 及 $6x^2 + 3x - 6$.

$$8x^3 + 27 = (2x)^3 + 3^3 = (2x + 3)(4x^2 - 6x + 9).$$

$$16x^4 + 36x^2 + 81 = 16x^4 + 72x^2 + 81 - 36x^2$$

$$= (4x^2 + 6x + 9)(4x^2 - 6x + 9).$$

$$6x^2 + 3x - 6 = (2x + 3)(3x - 2)$$

$$\therefore \text{L. C. M.} = (2x + 3)(4x^2 - 6x + 9)(4x^2 + 6x + 9)(3x - 2)$$

14. 解: $x^2 - 4a^2, x^3 + 2ax^2 + 4a^2x + 8a^3$ 及 $x^3 - 2ax^2 + 4a^2x - 8a^3$.

$$x^2 - 4a^2 = (x - 2a)(x + 2a)$$

$$x^3 + 2ax^2 + 4a^2x + 8a^3 = x^2(x + 2a) + 4a^2(x + 2a)$$

$$= (x^2 + 4a^2)(x + 2a)$$

$$x^3 - 2ax^2 + 4a^2x - 8a^3 = x^2(x - 2a) + 4a^2(x - 2a)$$

$$= (x^2 + 4a^2)(x - 2a)$$

$$\therefore \text{L. C. M.} = (x - 2a)(x + 2a)(x^2 + 4a^2).$$

15. 解: $x^2 + 2x, x^2 + bx + 2x + 2b$ 及 $x^3 + ax^2 - b^2x - ab^2$.

$$x^2 + 2x = x(x + 2)$$

$$x^2 + bx + 2x + 2b = x(x + b) + 2(x + b) = (x + 2)(x + b)$$

$$x^3 + ax^2 - b^2x - ab^2 = x^2(x + a) - b^2(x + a)$$

$$= (x^2 - b^2)(x + a) = (x - b)(x + b)(x + a)$$

$$\therefore \text{L. C. M.} = x(x + 2)(x + b)(x - b)(x + a).$$

16. 解: $(x^2 + 3x + 2)(x^2 + 7x + 12)$ 及 $(x^2 + 5x + 6)(2x^2 - 3x - 5)$

$$(x^2 + 3x + 2)(x^2 + 7x + 12) = (x + 1)(x + 2)(x + 3)(x + 4)$$

$$(x^2 + 5x + 6)(2x^2 - 3x - 5) = (x + 2)(x + 3)(x + 1)(2x - 5)$$

$$\therefore \text{L. C. M.} = (x + 1)(x + 2)(x + 3)(x + 4)(2x - 5).$$

17. 解: $(x^3 - 8)(27x^3 + 1)$ 及 $(2x^3 + 5x^2 + 10x + 4)(x^3 - x^2 - x - 2)$

$$(x^3 - 8)(27x^3 + 1)$$

$$= (x - 2)(x^2 + 2x + 4)(3x + 1)(9x^2 - 3x + 1)$$

$$(2x^3 + 5x^2 + 10x + 4)(x^3 - x^2 - x - 2)$$

$$= (2x + 1)(x^2 + 2x + 4)(x - 2)(x^2 + x + 1)$$

$$\therefore \text{L. C. M.} = (x - 2)(x^2 + 2x + 4)(3x + 1)(9x^2$$

$$\times (2x+1)(x^2+x+1).$$

18. 解: $x^3-6x^2+11x-6$, $2x^3-7x^2+7x-2$ 及 $2x^3+x^2-13x+6$.

1	1-6+11-6	2-7+7-2	2
、	1-3-2	2-12+22-12	
-3	-3+9-6	5-15+10	
	-3+9-6	1-3+2	
	0		

故此二式之 H. C. F. 爲 x^2-3x+2 .

∴ 二式之 L. C. M. 爲 $(x^3-6x^2+11x-6)(2x^3-7x^2+7x-2) \div (x^2-3x+2) = 2x^4-13x^3+28x^2-23x+6$.

1-7	2-13+28-23+6	2+1-13+6	1+3
	2+1-13+6	2-5+2	
	-14+41-29+6	6-15+6	
	-14-7+91-42	6-15+6	
	48-120+48	0	
	2-5+2		

答: 此三式之 L. C. M. 爲 $(2x^4-13x^3+28x^2-23x+6)$

$$\times (2x^3+x^2-13x+6) \div (2x^2-5x+2) = 2x^5-7x^4-11x^3+61x^2-63x+18 = (x-1)(x-2)(x-3)(x+3)(2x-1).$$

19. 解: x^4+5x^2+4x+5 , $2x^4-x^3+10x^2+4x+5$ 及 $2x^4+x^3+7x^2+3x+3$.

-1	1-0+5+4+5	2-1+10+4+5	2
	1-0+4+5	2+0+10+8+10	
	1-1+5	-1+0-4-5	-1
		-1+1-5	
		-1+1-5	-1
		-1+1-5	
		0	

故 (1) 與 (2) 二式之 H. C. F. 爲 x^2-x+5 .

∴ 二式之 L. C. M. 爲 $(x^4+5x^2+4x+5)(2x^4-x^3+10x^2+4x+5) \div (x^2-x+5) = (x^2+x+1)(2x^4-x^3+10x^2+4x+5)$.

-1-2	2+1+7+3+3	2-1+10+4+5	1
	2-3-1-2	2+1+7+3+3	
	4+8+5+3	-2+3+1+2	-1
	4-6-2-4	-2-1-1	
	14+7+7	4+2+2	2
	2+1+1	4+2+2	
		0	

故(2),(3)式之 L. C. M. 爲 $(x^2+3)(2x^4-x^3+10x^2+4x+5)$

答: 此三式之 L. C. M. 爲 $(2x^4-x^3+10x^2+4x+5)$
 $\times (x^2+x+1)(x^2+3)$.

20. 解: $2x^4-x^3+2x^2+3x-2$, $2x^4+3x^3-4x^2+13x-6$ 及
 $x^4+3x^3+x^2+5x+6$.

2-1+2+3-2	2+3-4+13-6	1
2-3+5-2	2-1+2+3-2	
2-3+5-2	4-6+10-4	1+1
2-3+5-2	2-3+5-2	
0		

故(1),(2)式之 L. C. M. 爲 $(x+1)(2x^4+3x^3-4x^2+13x-6)$

2+3-4+13-6	1+3+1+5+6	2+1
2+6+2+10+12	1+2-1+6	
-3-6+3-18	1+2-1+6	1
1+2-1+6	1+2-1+6	
	0	

故(2),(3)式之 L. C. M. 爲 $(x+1)(2x^4+3x^3-4x^2+13x-6)$

答: 此三式之 L. C. M. 爲 $(x+1)(2x^4+3x^3-4x^2+13x-6)$

VIII. 有理分式

習題 XXIV

原本第 215 頁

1. 解:
$$\frac{x^5y^3-4x^3y^5}{x^3y^2-2x^2y^3} = \frac{x^3y^3(x^2-4y^2)}{x^2y^2(x-2y)} = \frac{x^3y^3(x-2y)(x+2y)}{x^2y^2(x-2y)}$$

$$= xy(x+2y).$$

- 2 解:
$$\frac{(x^6 - y^6)(x + y)}{(x^3 + y^3)(x^4 - y^4)} = \frac{(x^3 + y^3)(x^3 - y^3)(x + y)}{(x^3 + y^3)(x^2 + y^2)(x^2 - y^2)}$$
$$= \frac{x^2 + xy + y^2}{x^2 + y^2}.$$
- 3 解:
$$\frac{x^2 - 4x - 21}{x^2 + 2x - 63} = \frac{(x - 7)(x + 3)}{(x + 9)(x - 7)} = \frac{x + 3}{x + 9}.$$
- 4 解:
$$\frac{3x^2 - 8x - 3}{3x^2 + 7x + 2} = \frac{(3x + 1)(x - 3)}{(3x + 1)(x + 2)} = \frac{x - 3}{x + 2}.$$
- 5 解:
$$\frac{3x^2 - 18bx + 27b^2}{2x^2 - 18b^2} = \frac{3(x - 3b)^2}{2(x - 3b)(x + 3b)} = \frac{3(x - 3b)}{2(x + 3b)}.$$
- 6 解:
$$\frac{5x^2 + 6ax + a^2}{5x^2 + 2ax - 3a^2} = \frac{(5x + a)(x + a)}{(5x - 3a)(x + a)} = \frac{5x + a}{5x - 3a}.$$
- 7 解:
$$\frac{(x^2 - 25)(x^2 - 8x + 15)}{(x^2 - 9)(x^2 - 7x + 10)} = \frac{(x - 5)^2(x + 5)(x - 3)}{(x - 3)(x + 3)(x - 2)(x - 5)}$$
$$= \frac{(x - 5)(x + 5)}{(x + 3)(x - 2)}.$$
- 8 解:
$$\frac{15x^2 - 46x + 35}{10x^2 - 29x + 21} = \frac{(5x - 7)(3x - 5)}{(5x - 7)(2x - 3)} = \frac{3x - 5}{2x - 3}.$$
- 9 解:
$$\frac{x^4 + x^2y^2 + y^4}{(x^3 + y^3)(x^3 - y^3)} = \frac{x^4 + x^2y^2 + y^4}{x^6 - y^6}$$
$$= \frac{x^4 + x^2y^2 + y^4}{(x^2 - y^2)(x^4 + x^2y^2 + y^4)} = \frac{1}{x^2 - y^2}.$$
- 10 解:
$$\frac{x^2 - y^2 + z^2 + 2xz}{x^2 + y^2 - z^2 + 2xy} = \frac{(x + z)^2 - y^2}{(x + y)^2 - z^2}$$
$$= \frac{(x - y + z)(x + y + z)}{(x + y - z)(x + y + z)} = \frac{x - y + z}{x + y - z}.$$
- 11 解:
$$\frac{(1 + xy)^2 - (x + y)^2}{1 - x^2} = \frac{1 + 2xy + x^2y^2 - x^2 - 2xy + y^2}{1 - x^2}$$
$$= \frac{(1 - y^2)(1 - x^2)}{1 - x^2} = 1 - y^2.$$
- 12 解:
$$\frac{2mx - my - 12nx + 6ny}{6mx - 3my - 2nx + ny} = \frac{m(2x - y) - 6n(2x - y)}{3m(2x - y) - n(2x - y)}$$
$$= \frac{(m - 6n)(2x - y)}{(3m - n)(2x - y)} = \frac{m - 6n}{3m - n}.$$

$$13. \text{ 解: } \frac{2x^3+7x^2-7x-12}{2x^3+3x^2-14x-15} = \frac{(x+1)(2x^2+5x-12)}{(x+1)(2x^2+x-15)} \\ = \frac{2x^2+5x-12}{2x^2+x-15}$$

$$14. \text{ 解: } \frac{x^3-8x^2+19x-12}{2x^3-13x^2+17x+12} = \frac{(x-1)(x^2-7x+1)}{(2x+1)(x^2-7x+1)} = \frac{x-1}{2x+1}$$

$$15. \text{ 解: } \frac{x^4+x^3+5x^2+4x+4}{4x^4+2x^3+14x^2+12x+12} = \frac{(x^2+4)(x^2+x+1)}{2(x^2+6)(x^2+x+1)} \\ = \frac{x^2+4}{2(x^2+6)}$$

$$16. \text{ 解: } \frac{x^3-2x^2-x-6}{x^4+3x^3+8x^2+8x+8} = \frac{(x-3)(x^2+x+2)}{(x^2+2x+4)(x^2+x+2)} \\ = \frac{x-3}{x^2+2x+4}$$

$$17. \text{ 解: } \frac{(x^2+c^2)^2-4b^2x^2}{x^4+4bx^3+4b^2x^2-c^4} = \frac{(x^2+2bx+c^2)(x^2-2bx+c^2)}{(x^2+2bx+c^2)(x^2+2bx-c^2)} \\ = \frac{x^2-2bx+c^2}{x^2+2bx-c^2}$$

$$18. \text{ 解: } \frac{(a-b)^3+(b-c)^3+(c-a)^3}{(a-b)(b-c)(c-a)}$$

此式中分子，分母僅能有之公因子爲 $a-b$ 、 $b-c$ 及 $c-a$ ，今於分子中設 $a=b$ ，得分子爲 0。故知分子能以 $a-b$ 除盡，同樣， $b-c$ 及 $c-a$ 亦能將分子除盡。

故此分式中，分子可爲分母除盡，因分子，分母皆爲三次，故知其商必爲一數字，又當以 a 之多項式排列時，二者中含 a^2 之項爲 $-3a^2(b-c)$ 及 $-a^2(b-c)$ ，故此數字爲 3，即其商爲 3。

習題 XXV

原本第 223 頁

$$1. \text{ 解: } \frac{1}{2a-3b} + \frac{1}{2a+3b} - \frac{6b}{4a^2-9b^2} = \frac{2a+3b+2a-3b-6b}{(2a-3b)(2a+3b)}$$

$$= \frac{2(2a-3b)}{(2a-3b)(2a+3b)} = \frac{2}{2a+3b}$$

2. 解: $\frac{1}{x+1} + \frac{1}{x^2-1} + \frac{1}{x^3+1}$

$$= \frac{1}{x+1} + \frac{1}{(x+1)(x-1)} + \frac{1}{(x+1)(x^2-x+1)}$$

$$= \frac{(x-1)(x^2-x+1) + (x^2-x+1) + (x-1)}{(x+1)(x-1)(x^2-x+1)}$$

$$= \frac{x^3 - x^2 + 2x - 1}{(x^3+1)(x-1)}$$

3. 解: $\frac{1}{x^2-3x+2} + \frac{1}{x^2-5x+6} + \frac{1}{x^2-4x+3}$

$$= \frac{1}{(x-1)(x-2)} + \frac{1}{(x-2)(x-3)} + \frac{1}{(x-1)(x-3)}$$

$$= \frac{(x-3) + (x-1) + (x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{3(x-2)}{(x-1)(x-2)(x-3)} = \frac{3}{(x-1)(x-3)}$$

4. 解: $\frac{x+1}{(x-1)(x-2)} - \frac{x+2}{(2-x)(x-3)} + \frac{x+3}{(x-1)(x-3)}$

$$= \frac{x+1}{(x-1)(x-2)} + \frac{x+2}{(x-2)(x-3)} + \frac{x+3}{(x-1)(x-3)}$$

$$= \frac{(x+1)(x-3) + (x+2)(x-1) + (x+3)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{x^2 - 2x - 3 + x^2 + x - 2 + x^2 + x - 6}{(x-1)(x-2)(x-3)}$$

$$= \frac{3x^2 - 11}{(x-1)(x-2)(x-3)}$$

5. 解: $\frac{1}{x+b} - \frac{1}{x+c} + \frac{1}{x-b} - \frac{1}{x-c}$

$$= \frac{(x-b)(x^2-c^2) - (x^2-b^2)(x-c) + (x+b)(x^2-c^2) - (x+c)(x^2-b^2)}{(x^2-b^2)(x^2-c^2)}$$

$$= \frac{2b^2x - 2c^2x}{(x^2-b^2)(x^2-c^2)} = \frac{2x(b^2-c^2)}{(x^2-b^2)(x^2-c^2)}$$

$$\begin{aligned}
 6. \text{ 解: } & \frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)} \\
 &= \frac{a}{(a-b)(a-c)} - \frac{b}{(b-c)(a-b)} + \frac{c}{(a-c)(b-c)} \\
 &= \frac{ab-ac-ab+bc+ac-bc}{(a-b)(b-c)(a-c)} = 0.
 \end{aligned}$$

$$\begin{aligned}
 7. \text{ 解: } & \frac{yz(x+a)}{(x-y)(x-z)} + \frac{zx(y+a)}{(y-z)(y-x)} + \frac{xy(z+a)}{(z-x)(z-y)} \\
 &= \frac{yz(x+a)}{(x-y)(x-z)} - \frac{zx(y+a)}{(x-y)(y-z)} + \frac{xy(z+a)}{(x-z)(y-z)} \\
 &= \frac{yz(x+a)(y-z) - zx(y+a)(x-z) + xy(z+a)(x-y)}{(x-y)(x-z)(y-z)} \\
 &= \frac{a\{z^2(x-y) + y^2(x-z) + x^2(y-z)\}}{(x-y)(x-z)(y-z)} = a.
 \end{aligned}$$

$$\begin{aligned}
 8. \text{ 解: } & x + \frac{1}{3-2x} - \frac{8x^4-33x}{8x^3-27} - \frac{2x+6}{4x^2+6x+9} \\
 &= x - \frac{1}{2x-3} - \frac{8x^4-33x}{8x^3-27} - \frac{2x+6}{4x^2+6x+9} = -\frac{8x^2+6x-9}{8x^3-27}
 \end{aligned}$$

$$\begin{aligned}
 9. \text{ 解: } & \left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 + \left(xy + \frac{1}{xy}\right)^2 \\
 & - \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(xy + \frac{1}{xy}\right) \\
 &= x^2 + 2 + \frac{1}{x^2} + y^2 + 2 + \frac{1}{y^2} + x^2y^2 + 2 + \frac{1}{x^2y^2} - x^2y^2 - y^2 \\
 & - x^2 - 1 - 1 - \frac{1}{x^2} - \frac{1}{y^2} - \frac{1}{x^2y^2} = 4.
 \end{aligned}$$

$$\begin{aligned}
 10. \text{ 解: } & \frac{(a+b)^3-c^3}{a+b-c} + \frac{(b+c)^3-a^3}{b+c-a} + \frac{(c+a)^3-b^3}{c+a-b} \\
 &= \frac{(a+b-c)[(a+b)^2+(a+b)c+c^2]}{a+b-c} \\
 & + \frac{(b+c-a)[(b+c)^2+(b+c)a+a^2]}{b+c-a} \\
 & + \frac{(c+a-b)[(c+a)^2+(c+a)b+b^2]}{c+a-b}
 \end{aligned}$$

$$\begin{aligned}
 &= a^2 + 2ab + b^2 + ac + bc + c^2 + b^2 + 2bc + c^2 + ab + ac \\
 &\quad + a^2 + c^2 + 2ac + a^2 + bc + ab + b^2 \\
 &= 3a^2 + 3b^2 + 3c^2 + 4ab + 4ac + 4bc.
 \end{aligned}$$

$$\begin{aligned}
 11. \text{ 解: } & \frac{x^2-4}{x^3-3x^2-x+6} - \frac{3x^2-14x-5}{3x^3-2x^2-10x-3} \\
 &= \frac{(x-2)(x+2)}{(x-2)(x^2-x-3)} - \frac{(3x+1)(x-5)}{(3x+1)(x^2-x-3)} \\
 &= \frac{x+2-x+5}{x^2-x-3} = \frac{7}{x^2-x-3}.
 \end{aligned}$$

$$\begin{aligned}
 12. \text{ 解: } & \frac{1}{x^4-4x^2-x+2} + \frac{1}{2x^4-3x^3-5x^2+7x-2} \\
 & \quad + \frac{1}{2x^4+3x^3-2x^2-2x+1} \\
 &= \frac{1}{(x+1)(x-2)(x^2+x-1)} + \frac{1}{(x-2)(2x-1)(x^2+x-1)} \\
 & \quad + \frac{1}{(x+1)(2x-1)(x^2+x-1)} \\
 &= \frac{2x-1+x+1+x-2}{(x+1)(x-2)(x^2+x-1)(2x-1)} \\
 &= \frac{2(2x-1)}{(x+1)(x-2)(x^2+x-1)(2x-1)} = \frac{2}{x^4-4x^2-x+2}.
 \end{aligned}$$

$$\begin{aligned}
 13. \text{ 解: } & \left(a^4 - \frac{1}{a^4}\right) \div \left(a - \frac{1}{a}\right) \\
 &= \frac{a^8-1}{a^4} \times \frac{a}{a^2-1} = \frac{(a^2-1)(a^6+a^4+a^2+1)a}{a^4(a^2-1)} \\
 &= \frac{a^6+a^4+a^2+1}{a^3}.
 \end{aligned}$$

$$\begin{aligned}
 14. \text{ 解: } & \left(\frac{1}{a^3} - \frac{1}{a^2} + \frac{1}{a}\right)(a^4+a^3) = \frac{1-a+a^2}{a^3} \times a^3(a+1) \\
 &= a^3+1.
 \end{aligned}$$

$$\begin{aligned}
 15. \text{ 解: } & \frac{x^2-5x+6}{x^2+3x-4} \cdot \frac{x^2+7x+12}{x^2-8x+15} \cdot \frac{x^2+x-6}{x^2-4x-5} \\
 &= \frac{(x-3)(x-2)}{(x+4)(x-1)} \times \frac{(x+3)(x+4)}{(x-3)(x-5)} \times \frac{(x-5)(x+1)}{(x+3)(x-2)} = \frac{x+1}{x-1}.
 \end{aligned}$$

$$\begin{aligned}
 16. \text{ 解: } & \frac{1}{x} - \left\{ 1 - \left[\frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x+1} - \frac{(x-2)(x-3)}{x(x+1)} \right) \right] \right\} \\
 &= \frac{1}{x} - \left\{ 1 - \left[\frac{x-1}{x} + \frac{4x-6}{2x(x+1)} \right] \right\} \\
 &= \frac{1}{x} - \left\{ 1 - \frac{2x^2+4x-8}{2x(x+1)} \right\} \\
 &= \frac{1}{x} - \frac{-2x+8}{2x(x+1)} = \frac{4x-6}{2x(x+1)} = \frac{2x-3}{x(x+1)}
 \end{aligned}$$

$$17. \text{ 解: } \frac{ax+x^2}{2b-cx} \cdot \frac{2bx^2-cx^3}{(a+x)^2} = \frac{x(a+x)}{2b-cx} \cdot \frac{x^2(2b-cx)}{(a+x)^2} = \frac{x^3}{a+x}$$

$$\begin{aligned}
 18. \text{ 解: } & (x^2-y^2-z^2+2yz) \div \frac{x-y+z}{x-y-z} \\
 &= (x+y-z)(x-y+z) \cdot \frac{x-y-z}{x-y+z} \\
 &= (x+y-z)(x-y-z) \\
 &= x^2-y^2+z^2-2xz.
 \end{aligned}$$

$$\begin{aligned}
 19. \text{ 解: } & \left(\frac{a+b}{a-b} - \frac{a^3+b^3}{a^3-b^3} \right) \left(\frac{a+b}{a-b} + \frac{a^2+b^2}{a^2-b^2} \right) \\
 &= \frac{(a+b)(a^2+ab+b^2) - a^3 - b^3}{a^3-b^3} \cdot \frac{a^2+2ab+b^2+a^2+b^2}{a^2-b^2} \\
 &= \frac{2ab(a+b) \cdot 2(a^2+ab+b^2)}{(a^3-b^3)(a+b)(a-b)} = \frac{4ab(a^2+ab+b^2)}{(a-b)^2(a^2+ab+b^2)} \\
 &= \frac{4ab}{(a-b)^2}
 \end{aligned}$$

$$\begin{aligned}
 20. \text{ 解: } & \frac{\frac{1}{x} - \frac{1}{y+z}}{\frac{1}{x} + \frac{1}{y+z}} \div \frac{\frac{1}{y} - \frac{1}{x+z}}{\frac{1}{y} + \frac{1}{x+z}} = \frac{y+z-x}{x(y+z)} \div \frac{y(x+z)}{x+z+y} \\
 &= \frac{y+z-x}{y+z+x} \cdot \frac{x+z+y}{x+z-y} = \frac{y+z-x}{x+z-y}
 \end{aligned}$$

$$\begin{aligned}
 21. \text{ 解: } & \frac{\frac{a}{b} + \frac{b}{a}}{\frac{a}{b} - \frac{b}{a}} \div \frac{\frac{1}{a^4} - \frac{1}{b^4}}{\left(\frac{1}{a} + \frac{1}{b} \right)^2} = \frac{a^2+b^2}{ab} \div \frac{b^4-a^4}{a^4b^4} \\
 &= \frac{a^2+b^2}{ab} \cdot \frac{a^4b^4}{b^4-a^4} = \frac{a^2+b^2}{ab} \cdot \frac{a^4b^4}{(b^2+a^2)(b^2-a^2)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{a^2+b^2}{a^2-b^2} \cdot \frac{(b^2-a^2)(b^2+a^2)}{a^2b^2(b^2+2ab+a^2)} \\
 &= \frac{a^2+b^2}{a^2-b^2} \cdot \frac{a^2b^2(a+b)^2}{(b^2-a^2)(b^2+a^2)} \\
 &= \frac{a^2b^2(a+b)^2}{(a+b)(a-b)(a+b)(-a+b)} = -\frac{a^2b^2}{(a-b)^2}
 \end{aligned}$$

22. 解: $\frac{x-2}{x-\frac{x-1}{x-2}} = \frac{x-2}{x-\frac{x}{x-3x+1}} = \frac{x-2}{x-2 \cdot \frac{x^2-2x}{x^2-3x+1}}$

$$\begin{aligned}
 &= \frac{x-2}{x-\frac{2x^2-6x+2+x^2-2x}{x^2-3x+1}} = \frac{x-2}{x-\frac{x^3-3x^2+x-3x^2+8x-2}{x^2-3x+1}} \\
 &= \frac{(x-2)(x^2-3x+1)}{(x-2)(x^2-4x+1)} = \frac{x^2-3x+1}{x^2-4x+1}
 \end{aligned}$$

23. 解: $x + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} = x + \frac{1}{x} + \frac{x}{x^2+1} = x + \frac{x^2+1}{x^3+2x}$

$$= \frac{x^4+2x^2+x^2+1}{x^3+2x} = \frac{x^4+3x^2+1}{x^3+2x}$$

習題 XXVI

原本第 230 頁

1. 解: $\lim_{x \rightarrow 2} \frac{x^2-5x+6}{x^2-6x+8} = \lim_{x \rightarrow 2} \frac{(x-2)(x-3)}{(x-2)(x-4)}$

$$= \lim_{x \rightarrow 2} \frac{x-3}{x-4} = \frac{1}{2}$$

2. 解: $\lim_{x \rightarrow 1} \frac{x^3-3x^2+2}{x^3-2x+1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2-2x-2)}{(x-1)(x^2+x-1)}$

$$= \lim_{x \rightarrow 1} \frac{x^2-2x-2}{x^2+x-1} = -3$$

3. 解: $\lim_{x \rightarrow 1} \frac{x^2-1}{x^2-2x+1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)^2}$

$$= \lim_{x \rightarrow 1} \frac{x+1}{x-1} = \infty.$$

$$\begin{aligned} 4. \text{ 解: } \lim_{x \rightarrow a} \frac{x^2 - 2ax + a^2}{x^2 - (a+b)x + ab} &= \lim_{x \rightarrow a} \frac{(x-a)^2}{(x-a)(x-b)} \\ &= \lim_{x \rightarrow a} \frac{x-a}{x-b} = 0. \end{aligned}$$

$$\begin{aligned} 5. \text{ 解: } \lim_{x \rightarrow -2} \frac{(3x+1)(x+2)^2}{(x^2-4)(x^2+3x+2)} &= \lim_{x \rightarrow -2} \frac{(3x+1)(x+2)^2}{(x-2)(x+2)(x+1)(x+2)} \\ &= \lim_{x \rightarrow -2} \frac{3x+1}{(x-2)(x+1)} = -\frac{5}{4}. \end{aligned}$$

$$\begin{aligned} 6. \text{ 解: } \lim_{x \rightarrow 1} \frac{x^3 - x^2 - x + 1}{x^3 - 3x^2 + 3x - 1} &= \lim_{x \rightarrow 1} \frac{(x+1)(x-1)^2}{(x-1)^3} \\ &= \lim_{x \rightarrow 1} \frac{x+1}{x-1} = \infty. \end{aligned}$$

$$7. \text{ 解: } \lim_{x \rightarrow \infty} \frac{3x^2 - x + 5}{2x^2 + 6x - 7} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x} + \frac{5}{x^2}}{2 + \frac{6}{x} - \frac{7}{x^2}} = \frac{3}{2},$$

$$\lim_{x \rightarrow \infty} \frac{x^2 + 1}{x} = \lim_{x \rightarrow \infty} \frac{x + \frac{1}{x}}{\frac{1}{x}} = \infty,$$

$$\lim_{x \rightarrow \infty} \frac{3x}{x^2 + 1} = \lim_{x \rightarrow \infty} \frac{\frac{3}{x}}{1 + \frac{1}{x^2}} = \frac{0}{1} = 0,$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(2x^2+1)(x^3-5)}{(x^4+1)(x-6)} &= \lim_{x \rightarrow \infty} \frac{2x^5 + x^3 - 10x^2 - 5}{x^5 - 6x^4 + x - 6} \\ &= \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x^2} - \frac{10}{x^3} - \frac{5}{x^5}}{1 - \frac{6}{x} + \frac{1}{x^4} - \frac{6}{x^5}} = 2. \end{aligned}$$

$$8. \text{ 解: } \lim_{x \rightarrow 3} \left[\frac{x-1}{x^2-9} - \frac{x-2}{x(x-3)} \right] = \lim_{x \rightarrow 3} \frac{-2(x-3)}{x(x^2-9)}$$

$$= \lim_{x \rightarrow 3} \frac{-2}{x(x+3)} = -\frac{2}{18} = -\frac{1}{9}.$$

9. 解: $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} + \frac{2}{x(x-1)} \right] = \lim_{x \rightarrow 1} \left[\frac{x+2}{x(x-1)} \right] = \frac{3}{0} = \infty.$

10. 解: $\lim_{x \rightarrow 2} \frac{x^2 + \frac{x+1}{x-2}}{x^2 + \frac{x-1}{x-2}} = \lim_{x \rightarrow 2} \frac{\frac{x^2(x-2) + (x+1)}{x-2}}{\frac{x^2(x-2) + (x-1)}{x-2}}$

$$= \lim_{x \rightarrow 2} \frac{x^3 - 2x^2 + x + 1}{x^3 - 2x^2 + x - 1} = 3.$$

11. 解: $\lim_{x \rightarrow \infty} \frac{\frac{x}{x-1} - \frac{x}{x+1}}{\frac{3x+1}{x^2+1}} = \lim_{x \rightarrow \infty} \frac{\frac{x^2+x-x^2+x}{x^2-1}}{\frac{3x+1}{x^2+1}}$

$$= \lim_{x \rightarrow \infty} \frac{2x^3+2x}{3x^3+x^2-3x-1} = \lim_{x \rightarrow \infty} \frac{2 + \frac{2}{x^2}}{3 + \frac{1}{x} - \frac{3}{x^2} - \frac{1}{x^3}} = \frac{2}{3}$$

習 題 XXVII

原本第 235 頁

1. 解: $\frac{6x-1}{3x+2} - \frac{4x-7}{2x-5} = 0$

$$12x^2 - 30x - 2x + 5 - 12x^2 + 21x - 8x + 14 = 0$$

$$19x = 19 \quad \therefore x = 1.$$

2. 解: $\frac{6x}{5x-1} + \frac{8}{3-15x} = \frac{1}{6}$

$$\frac{6x}{5x-1} + \frac{8}{3(1-5x)} = \frac{1}{6}$$

$$\frac{6x}{5x-1} - \frac{8}{3(1-5x)} = \frac{1}{6}$$

$$36x - 16 = 5x - 1$$

$$31x = 15$$

$$\therefore x = \frac{15}{31}$$

$$3. \text{ 解: } \frac{4}{x-2} - \frac{1}{x-4} = \frac{4}{x^2-6x+8}$$

$$\frac{4x-16-x+2}{x^2-6x+8} = \frac{4}{x^2-6x+8}$$

$$3x-14=4$$

$$3x=18$$

$$\therefore x=6.$$

$$4. \text{ 解: } \frac{3}{2x+3} + \frac{1}{x-5} - \frac{8}{2x-7x-15} = 0$$

$$3x-15+2x+3-8=0$$

$$5x=20 \quad \therefore x=4.$$

$$5. \text{ 解: } \frac{1}{(x+1)(x-3)} + \frac{2}{(x-3)(x+2)} + \frac{3}{(x+2)(x+1)} = 0$$

$$(x+2)+2(x+1)+3(x-3)=0$$

$$x+2+2x+2+3x-9=0$$

$$6x=5$$

$$\therefore x = \frac{5}{6}.$$

$$6. \text{ 解: } \frac{2}{x^2-1} - \frac{2}{x^2+4x-5} + \frac{3}{x^2+6x+5} = 0$$

$$\frac{2}{(x+1)(x-1)} - \frac{2}{(x-1)(x+5)} + \frac{3}{(x+1)(x+5)} = 0$$

$$2(x+5)-2(x+1)+3(x-1)=0$$

$$3x=-5 \quad \therefore x = -\frac{5}{3}.$$

$$7. \text{ 解: } \frac{x+1}{3x+1} + \frac{2x}{5-6x} = \frac{5}{5+9x-18x^2}$$

$$\frac{(x+1)(5-6x)+2x(3x+1)}{5+9x-18x^2} = \frac{5}{5+9x-18x^2}$$

$$-6x^2-x+5+6x^2+2x=5$$

$$\therefore x=0.$$

$$\begin{aligned}
 8. \text{ 解: } & \frac{x+a}{b(x+b)} + \frac{x+b}{a(x+a)} = \frac{a+b}{ab} \\
 & \frac{a(x+a)^2 + b(x+b)^2}{ab(x+a)(x+b)} = \frac{a+b}{ab} \\
 & ax^2 + 2a^2x + a^3 + bx^2 + 2b^2x + b^3 - ax^2 - 2abx - bx^2 \\
 & - a^2x - b^2x - a^3b - ab^2 = 0 \\
 & x(a^2 + b^2 - 2ab) = -a^2 - b^2 + a^2b + ab^2 \\
 \therefore x &= -\frac{(a+b)(a-b)^2}{(a-b)^2} = -(a+b).
 \end{aligned}$$

$$\begin{aligned}
 9. \text{ 解: } & \frac{x^3+1}{x+1} - \frac{x^3-1}{x-1} = 20 \\
 & \frac{(x+1)(x^2-x+1)}{x+1} - \frac{(x-1)(x^2+x+1)}{x-1} = 20 \\
 & x^2 - x + 1 - x^2 - x - 1 = 20 \\
 \therefore x &= -10.
 \end{aligned}$$

$$\begin{aligned}
 10. \text{ 解: } & \frac{x^2+2x+1}{x^2+5x+4} + \frac{x-1}{x^2+3x-4} = 0 \\
 & \frac{(x+1)^2}{(x+1)(x+4)} + \frac{x-1}{(x-1)(x+4)} = 0 \\
 & x+1+1=0, \quad \therefore x=-2.
 \end{aligned}$$

$$\begin{aligned}
 11. \text{ 解: } & \frac{x-8}{x-3} - \frac{x-9}{x-4} = \frac{x+7}{x+8} - \frac{x+2}{x+3} \\
 & 1 + \frac{-5}{x-3} - 1 + \frac{5}{x-4} = 1 + \frac{-1}{x+8} - 1 + \frac{1}{x+3} \\
 & \frac{5}{(x-3)(x-4)} = \frac{5}{(x+8)(x+3)} \\
 & (x-3)(x-4) = (x+8)(x+3) \\
 & x^2 - 7x + 12 = x^2 + 11x + 24 \\
 & 18x = -12 \\
 \therefore x &= -\frac{2}{3}.
 \end{aligned}$$

$$12. \text{ 解: } \frac{x+7}{x+6} + \frac{x+9}{x+8} = \frac{x+10}{x+9} + \frac{x+6}{x+5}$$

$$1 + \frac{1}{x+6} + 1 + \frac{1}{x+8} = 1 + \frac{1}{x+9} + 1 + \frac{1}{x+5}$$

$$\frac{1}{x+6} - \frac{1}{x+9} = \frac{1}{x+5} - \frac{1}{x+8}$$

$$\frac{3}{(x+6)(x+9)} = \frac{3}{(x+5)(x+8)}$$

$$(x+6)(x+9) = (x+5)(x+8)$$

$$2x = -14$$

$$\therefore x = -7.$$

13. 解: $\frac{x^3+2}{x-2} - \frac{x^3-2}{x+2} - \frac{15}{x^2-4} = 4x.$

$$(x^3+2)(x+2) - (x^3-2)(x-2) - 15 = 4x(x^2-4)$$

$$x^4 + 2x^3 + 2x + 4 - x^4 + 2x^3 + 2x - 4 = 4x^3 - 16x + 15$$

$$20x = 15$$

$$\therefore x = \frac{3}{4}.$$

14. 解: $\frac{1}{x-1} - \frac{x-2}{x^2-1} + \frac{3x^2+x}{1-x^4} = 0$

$$\frac{1}{x-1} - \frac{x-2}{x^2-1} - \frac{3x^2+x}{x^4-1} = 0$$

$$(x+1)(x^2+1) - (x-2)(x^2+1) - (3x^2+x) = 0$$

$$x^3 + x^2 + x + 1 - x^3 + 2x^2 - x + 2 - 3x^2 - x = 0$$

$$\therefore x = 3.$$

15. 解: $\frac{3}{x^3-8} + \frac{2x+5}{2x^2+4x+8} - \frac{1}{x-2} = 0$

$$\frac{3 \times 2 + (2x+5)(x+2) - 2(x^2+2x+4)}{2(x^3-8)} = 0$$

$$6 + 2x^2 + x - 10 - 2x^2 - 4x - 8 = 0$$

$$3x = -12 \quad \therefore x = -4.$$

16. 解: $\frac{ax+c}{x-p} + \frac{bx+d}{x-q} = a+b$

$$\frac{ax+c}{x-p} - a + \frac{bx+d}{x-q} - b = 0$$

$$\frac{ap+c}{x-p} + \frac{bq+d}{x-q} = 0$$

$$(ap+c)(x-q) + (bq+d)(x-p) = 0$$

$$apx+cx-apq-cq+bqx+dx-bpq-dp=0$$

$$x(c+d+ap+bq) = cq+dp+apq+bpq$$

$$\therefore x = \frac{cq+dp+(a+b)pq}{c+d+ap+bq}$$

$$17. \text{ 解: } \frac{x^2+7x-8}{x-1} + \frac{x^2+x+3}{x+2} + \frac{2x^2-x+7}{x+3} = 4x$$

$$x+8+x-1 + \frac{5}{x+2} + 2x-7 + \frac{28}{x+3} = 4x$$

$$\frac{5}{x+2} + \frac{28}{x+2} = 0$$

$$5x+15+28x+56=0$$

$$33x = -71$$

$$\therefore x = -\frac{71}{33}$$

$$18. \text{ 解: } \frac{x^2-ax+2bx-2ab}{x-a} + \frac{b^2-x^2}{x-2b} + \frac{3c^2}{x-2c} = 0$$

$$\frac{(x-a)(x+2b)}{x-a} - \frac{x^2-b^2}{x-2b} + \frac{3c^2}{x-2c} = 0$$

$$x+2b - (x+2b) - \frac{3b^2}{x-2b} + \frac{3c^2}{x-2c} = 0$$

$$-\frac{3b^2}{x-2b} + \frac{3c^2}{x-2c} = 0$$

$$-b^2x+2b^2c+c^2x-2bc^2=0$$

$$(c^2-b^2)x = 2bc(c-b)$$

$$(c+b)x = 2bc$$

$$\therefore x = \frac{2bc}{b+c}$$

$$19. \text{ 解: } \frac{(x-a)^2}{(x-b)(x-c)} + \frac{(x-b)^2}{(x-c)(x-a)} + \frac{(x-c)^2}{(x-a)(x-b)} = 3$$

$$(x-a)^3 + (x-b)^3 + (x-c)^3 = 3(x-a)(x-b)(x-c)$$

$$3x(a^2+b^2+c^2-ab-ac-bc) = a^3+b^3+c^3-3abc$$

$$\therefore x = \frac{a^3+b^3+c^3-3abc}{3(a^2+b^2+c^2-ab-ac-bc)} = \frac{a+b+c}{3}$$

20. 解: $\frac{3x+2}{x^2+x} - \frac{x-5}{x^2-1} - \frac{x-2}{x^2-x} = 0$

$$\frac{3x+2}{x(x+1)} - \frac{x-5}{(x-1)(x+1)} - \frac{x-5}{x(x-1)} = 0$$

$$(3x+2)(x-1) - (x-5)x - (x-2)(x+1) = 0$$

$$3x^2 - x - 2 - x^2 + 5x - x^2 + x + 2 = 0$$

$$x^2 + 5x = 0$$

$$\therefore x = -5.$$

21. 解: $\frac{a}{x+2} + \frac{2}{x-2} - \frac{x+6}{x^2-4} = 0$

$$a(x-2) + 2(x+2) - x - 6 = 0$$

$$(a+1)x - 2(a+1) = 0$$

$$(a+1)x = 2(a+1)$$

$$\therefore x = 2.$$

惟因 2 使分式中之分母爲 0, 故不得爲根。

22. 解:
$$\begin{cases} \frac{3x+y-1}{x-y+3} = \frac{6}{7} \dots\dots\dots (1) \\ \frac{x+9}{y+4} = \frac{x+3}{y+3} \dots\dots\dots (2) \end{cases}$$

從 (1) 式, $21x+7y-7=6x-6y+12$

即 $15x+13y-19=0 \dots\dots\dots (3)$

從 (2) 式, $xy+9y+3x+27=xy+3y+4x+12$

即 $6y-x+15=0 \dots\dots\dots (4)$

$15 \times (4) + (3), \quad 103y = -206$

$$\therefore y = -2.$$

代入 (4) 式, 得 $x = 3.$

23. 解:
$$\begin{cases} \frac{y-2}{x-3} + \frac{x-y}{x^2-9} = \frac{y-4}{x+3} \dots\dots\dots (1) \end{cases}$$

$$\begin{cases} \frac{2}{x^2-2x} + \frac{3}{xy-2y} + \frac{9}{xy} = 0 \dots\dots\dots (2) \end{cases}$$

從 (1) 式, $xy - 2x + 3y - 6 + x - y - xy + 4x + 3y - 12 = 0$
 $3x + 5y - 18 = 0 \dots\dots\dots (3)$

從 (2) 式, $6x + y - 9 = 0 \dots\dots\dots (4)$

$2 \times (3) - (4), \quad 9y = 27$

$\therefore y = 3.$

代入 (4) 式, 得 $x = 1.$

24. 解: $\begin{cases} \frac{xy}{x+y} = a \dots\dots\dots (1) \\ \frac{yz}{y+z} = b \dots\dots\dots (2) \\ \frac{zx}{z+x} = c \dots\dots\dots (3) \end{cases}$

從 (1) 式, $\frac{1}{x} + \frac{1}{y} = \frac{1}{a} \dots\dots\dots (4)$

從 (2) 式, $\frac{1}{y} + \frac{1}{z} = \frac{1}{b} \dots\dots\dots (5)$

從 (3) 式, $\frac{1}{z} + \frac{1}{x} = \frac{1}{c} \dots\dots\dots (6)$

(4) + (5) + (6), $2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$
 $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{bc + ca + ab}{2abc} \dots\dots\dots (7)$

(7) - (5), $\frac{1}{x} = \frac{bc - ca + ab}{2abc}$
 $\therefore x = 2abc / (ab + bc - ca).$

(7) - (6), $\frac{1}{y} = \frac{bc + ca - ab}{2abc}$
 $\therefore y = 2abc / (-ab + bc + ca).$

(7) - (4), $\frac{1}{z} = \frac{-bc + ca + ab}{2abc}$
 $\therefore z = 2abc / (ab - bc + ca).$

$$25. \text{ 解: } \begin{cases} \frac{2}{x+2y} + 2y + 2z = 3 \dots\dots\dots (1) \\ \frac{y+z}{2} - \frac{5}{z-3x} = \frac{7}{2} \dots\dots\dots (2) \\ \frac{4}{z-3x} - \frac{3}{x+2y} = -1 \dots\dots\dots (3) \end{cases}$$

$$(1) + (3), \quad y + z + \frac{2}{z-3x} = 1 \dots\dots\dots (4)$$

$$(4) \times 10, \quad 10(y+z) + \frac{20}{z-3x} = 10 \dots\dots\dots (5)$$

$$(2) \times 4, \quad 2(y+z) - \frac{20}{z-3x} = 14 \dots\dots\dots (6)$$

$$(5) + (6), \quad 12(y+z) = 24 \\ y + z = 2 \\ z = 2 - y \dots\dots\dots (7)$$

將 (7) 代入 (1) 式, 得 $\frac{2}{x+2y} + 2y + 2(2-y) = 3$

即 $x + 2y = -2 \dots\dots\dots (8)$

將 (7) 代入 (3) 式, 得 $\frac{4}{2-y-3x} - \frac{3}{x+2y} = -1 \dots\dots\dots (9)$

從 (8) 式, $x = -2(y+1)$

代入 (9) 式, 得 $\frac{4}{5y+8} - \frac{2}{-2} = -1$

$$-10y - 16 = 4 \\ \therefore y = -2.$$

代入 (7) 式, 得 $z = 4.$

代入 (8) 式, 得 $x = 2.$

習 題 XXVIII

原本第 244 頁

1. 解: $\frac{2x+11}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

$$2x+11=A(x+3)+B(x-2)$$

設 $x=2$ 則 $15=5A$

$$\therefore A=3.$$

設 $x=-3$ 則 $5=-5B$

$$\therefore B=-1.$$

答: $\frac{2x+11}{(2x+1)(3x-1)} = \frac{3}{x-2} - \frac{1}{x+3}$

2. 解: $\frac{6x-1}{(2x+1)(3x-1)} = \frac{A}{2x+1} + \frac{B}{3x-1}$

$$6x-1=A(3x-1)+B(2x+1)$$

$$=x(3A+2B)-(A-B)$$

$$3A+2B=6 \quad A-B=1$$

$$\therefore A=\frac{8}{5}, \quad B=\frac{3}{5}.$$

答: $\frac{6x-1}{(2x+1)(3x-1)} = \frac{8}{5(2x+1)} + \frac{3}{5(3x-1)}$

3. 解: $\frac{4x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$$4x=A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$$

設 $x=-1$ $-4=2A$ $\therefore A=-2.$

$x=-2$ $-8=-B$ $\therefore B=8.$

$x=-3$ $-12=2C$ $\therefore C=-6.$

答: $\frac{4x}{(x+1)(x+2)(x+3)} = -\frac{2}{x+1} + \frac{8}{x+2} - \frac{6}{x+3}$

4. 解: $\frac{x^2+2x+3}{(x-1)(x-2)(x-3)(x-4)}$

$$= \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x-4}$$

$$x^2+2x+3=A(x-2)(x-3)(x-4)$$

$$+B(x-1)(x-3)(x-4)$$

$$+C(x-1)(x-2)(x-4)$$

$$+D(x-1)(x-2)(x-3)$$

設 $x=1$ $6=-6A$ $\therefore A=-1.$

$$x=2 \quad 11=2B \quad \therefore B=\frac{11}{2}$$

$$x=3 \quad 18=-2C \quad \therefore C=-9$$

$$x=4 \quad 27=6D \quad \therefore D=\frac{9}{2}$$

$$\text{答: } \frac{x^2+2x+3}{(x-1)(x-2)(x-3)(x-4)}$$

$$= -\frac{1}{x-1} + \frac{11}{2(x-2)} - \frac{9}{x-3} + \frac{9}{2(x-4)}$$

$$5. \text{ 解: } \frac{x^2+2}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}$$

$$x^2+2 = A(1-x+x^2) + (Bx+C)(1+x)$$

$$\text{設 } x=-1 \quad 3=3A \quad \therefore A=1$$

$$x^2+2 = 1-x+x^2 + (Bx+C)(x+1)$$

$$x+1 = (Bx+C)(x+1) \quad Bx+C=1$$

$$\therefore B=0, \quad C=1$$

$$\text{答: } \frac{x^2+2}{1+x^3} = \frac{1}{1+x} + \frac{1}{1-x+x^2}$$

$$6. \text{ 解: } \frac{8x+2}{x-x^3} = \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x}$$

$$8x+2 = A(1+x)(1-x) + Bx(1-x) + Cx(1+x)$$

$$\text{設 } x=0 \quad 2=A \quad \therefore A=2$$

$$x=-1 \quad -2B=-6 \quad \therefore B=3$$

$$x=1 \quad 2C=10 \quad \therefore C=5$$

$$\text{答: } \frac{8x+2}{x-x^3} = \frac{2}{x} + \frac{3}{1+x} + \frac{5}{1-x}$$

$$7. \text{ 解: } \frac{x^3-x^2-5x+4}{x^2-3x+2}$$

$$= x+2 + \frac{-x}{(x-1)(x-2)} = x+2 + \frac{A}{x-1} + \frac{B}{x-2}$$

$$-x = A(x-2) + B(x-1)$$

$$\text{設 } x=1 \quad -1=-A \quad \therefore A=1$$

$$x=2 \quad -2=B \quad \therefore B=-2$$

答: $\frac{x^3 - x^2 - 5x + 4}{x^2 - 3x + 2} = x + 2 + \frac{1}{x-1} - \frac{2}{x-2}$

8. 解: $\frac{2x^3 - x^2 + 1}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4}$
 $2x^3 - x^2 + 1 = A(x-2)^3 + B(x-2)^2 + C(x-2) + D$

設 $x=2$ $\therefore D=13$

$$2x^3 - x^2 - 12 = A(x-2)^3 + B(x-2)^2 + C(x-2)$$

$$2x^2 + 3x + 6 = A(x-2)^2 + B(x-2) + C$$

設 $x=2$ $\therefore C=20$

$$2x^2 + 3x - 14 = A(x-2)^2 + B(x-2)$$

$$2x + 7 = A(x-2) + B$$

設 $x=2$ $\therefore B=11$

$$\therefore A=2$$

答: $\frac{2x^3 - x^2 + 1}{(x-2)^4} = \frac{2}{x-2} + \frac{11}{(x-2)^2} + \frac{20}{(x-2)^3} + \frac{13}{(x-2)^4}$

9. 解: $\frac{x-1}{2x^3 - 5x^2 - 12x} = \frac{x-1}{x(2x+3)(x-4)} = \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x-4}$
 $x-1 = A(2x+3)(x-4) + Bx(x-4) + Cx(2x+3)$

設 $x=0$ $-1 = -12A$ $\therefore A = \frac{1}{12}$

$$x = -\frac{3}{2} \quad -\frac{5}{2} = \frac{33}{4}B \quad \therefore B = -\frac{10}{33}$$

$$x = 4 \quad 3 = 44C \quad \therefore C = \frac{3}{44}$$

答: $\frac{x-1}{2x^3 - 5x^2 - 12x} = \frac{1}{12x} - \frac{10}{33(2x+3)} + \frac{3}{44(x-4)}$

10. 解: $\frac{6}{2x^4 - x^2 - 1} = \frac{Ax+B}{2x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$
 $6 = (Ax+B)(x+1)(x-1) + C(2x^2+1)(x-1) + D(2x^2+1)(x+1) \dots\dots\dots(1)$

設 $x=1$ $6 = 6D$ $\therefore D=1$

$$x = -1 \quad 6 = -6C \quad \therefore C = -1$$

將 C, D 的値代入 (1) 式, 得

$$6 = (Ax+B)(x+1)(x-1) - 2x^3 + 2x^2 - x + 1$$

$$+2x^3+2x^2+x+1$$

$$4x^2+4=Ax^3+Bx^2-Ax-B$$

$$\therefore A=0, \quad B=-4.$$

$$\text{答: } \frac{6}{2x^4-x^2-1} = -\frac{4}{2x^2+1} - \frac{1}{x+1} + \frac{1}{x-1}.$$

$$11. \text{ 解: } \frac{2x^3-3x^2+4x-5}{(x+3)^5}$$

$$= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} + \frac{E}{(x+3)^5}$$

$$2x^3-3x^2+4x-5 = A(x+3)^4 + B(x+3)^3 + C(x+3)^2 + D(x+3) + E$$

$$\text{設 } x=-3 \quad \therefore E=-98.$$

$$2x^2-9x+31 = A(x+3)^3 + B(x+3)^2 + C(x+3) + D$$

$$\text{設 } x=-3 \quad \therefore D=76.$$

$$2x-15 = A(x+3)^2 + B(x+3) + C$$

$$\text{設 } x=-3 \quad \therefore C=-21.$$

$$2 = A(x+3) + B$$

$$\text{設 } x=-3 \quad \therefore B=2.$$

$$0 = A(x+3) \quad \therefore A=0.$$

$$\text{答: } \frac{2x^3-3x^2+4x-5}{(x+3)^5}$$

$$= \frac{2}{(x+3)^2} - \frac{21}{(x+3)^3} + \frac{76}{(x+3)^4} - \frac{98}{(x+3)^5}.$$

$$12. \text{ 解: } \frac{x^2+x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$x^2+x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

$$A+C=0, \quad B+D=1,$$

$$2A+C=1, \quad 2B+D=1$$

$$\therefore A=1, \quad B=0, \quad C=-1, \quad D=1.$$

$$\text{答: } \frac{x^2+x+1}{(x^2+1)(x^2+2)} = \frac{x}{x^2+1} - \frac{x-1}{x^2+2}.$$

$$13. \text{ 解: } \frac{x^2+6x-1}{(x-3)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$x^2+6x-1 = A(x-3)^2 + B(x-1)(x-3) + C(x-1)$$

$$\text{設 } x=1 \quad 6=4A \quad \therefore A = \frac{3}{2}$$

$$x=3 \quad 26=2C \quad \therefore C=13$$

$$x=2 \quad 15 = \frac{3}{2} - B + 13 \quad \therefore B = -\frac{1}{2}$$

$$\text{答: } \frac{x^2+6x-1}{(x-3)^2(x-1)} = \frac{3}{2(x-1)} - \frac{1}{2(x-3)} + \frac{13}{(x-3)^2}$$

$$14. \text{ 解: } \frac{3x-1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$3x-1 = A(x^2+1) + (Bx+C)(x-2)$$

$$= (A+B)x^2 - (2B-C)x + (A-2C)$$

$$A+B=0, \quad -2B+C=3, \quad A-2C=-1$$

$$\therefore A=1, \quad B=-1, \quad C=1$$

$$\text{答: } \frac{3x-1}{(x-2)(x^2+1)} = \frac{1}{x-2} - \frac{x-1}{x^2+1}$$

$$15. \text{ 解: } \frac{2x^5-x+1}{(x^2+x+1)^3} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{Ex+F}{(x^2+x+1)^3}$$

$$2x^5-x+1 = (Ax+B)(x^2+x+1)^2 + (Cx+D)(x^2+x+1) + (Ex+F)$$

$$= Ax^5 + (2A+B)x^4 + (3A+2B+C)x^3$$

$$+ (2A+3B+C+D)x^2 + (A+2B+C+D+E)x + (B+D+F)$$

$$A=2, \quad 2A+B=0, \quad 3A+2B+C=0$$

$$2A+3B+C+D=0, \quad A+2B+C+D+E=-1$$

$$\text{及 } B+D+F=1$$

$$\therefore A=2, \quad B=-4, \quad C=2, \quad D=6,$$

$$E=-3, \quad F=-1$$

$$\text{答: } \frac{2x^5-x+1}{(x^2+x+1)^3} = \frac{2x-4}{x^2+x+1} + \frac{2x+6}{(x^2+x+1)^2} - \frac{3x+1}{(x^2+x+1)^3}$$

$$16. \text{ 解: } \frac{2x^2-x+1}{(x^2-x)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\begin{aligned}
 & 2x^2 - x + 1 \\
 &= A(x^3 - 2x^2 + x) + B(x^2 - 2x + 1) + C(x^3 - x^2) + Dx^2 \\
 &= (A+C)x^3 - (2A-B+C-D)x^2 + (A-2B)x + B \\
 & A+C=0, \quad 2A-B+C-D=-2, \\
 & A-B=-1, \quad B=1
 \end{aligned}$$

$$\therefore A=1, \quad B=1, \quad C=-1, \quad D=2.$$

$$\text{答: } \frac{2x^2 - x + 1}{(x^2 - x)^2} = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x-1} + \frac{2}{(x-1)^2}.$$

$$\begin{aligned}
 17. \text{ 解: } & \frac{3x^2 - x + 2}{(x^2 + 2)(x^2 - x - 2)} = \frac{Ax + B}{x^2 + 2} + \frac{C}{x-2} + \frac{D}{x+1} \\
 & 3x^2 - x + 2 = (Ax + B)(x^2 - x - 2) + C(x^2 + 2)(x+1) \\
 & \quad + D(x^2 + 2)(x-2) \dots\dots\dots (1)
 \end{aligned}$$

$$\text{設 } x=2, \quad C = \frac{2}{3}.$$

$$x=-1, \quad D = -\frac{2}{3}.$$

將 C, D 二值代入 (1) 式, 得

$$\begin{aligned}
 3x^2 - x + 2 &= (Ax + B)(x^2 - x - 2) + 2x^2 + 4 \\
 x^2 - x - 2 &= Ax^3 - (A-B)x^2 - (2A+B)x - 2B
 \end{aligned}$$

$$\therefore A=0, \quad B=1.$$

$$\text{答: } \frac{3x^2 - x + 2}{(x^2 + 2)(x^2 - x - 2)} = \frac{1}{x^2 + 2} + \frac{2}{3(x-2)} - \frac{2}{3(x+1)}.$$

$$\begin{aligned}
 18. \text{ 解: } & \frac{x^2 + px + q}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \\
 & x^2 + px + q = A(x-b)(x-c) + B(x-a)(x-c) \\
 & \quad + C(x-a)(x-b)
 \end{aligned}$$

$$\text{設 } x=a \quad a^2 + ap + q = A(a-b)(a-c)$$

$$\therefore A = \frac{a^2 + ap + q}{(a-b)(a-c)}.$$

$$\text{設 } x=b \quad b^2 + bp + q = B(b-a)(b-c)$$

$$\therefore B = \frac{b^2 + bp + q}{(b-a)(b-c)}.$$

$$\text{設 } x=c \quad c^2 + cp + q = C(c-a)(c-b)$$

$$\therefore C = \frac{c^2 + cp + q}{(c-a)(c-b)}$$

$$\begin{aligned} \text{答: } & \frac{x^2 + px + q}{(x-a)(x-b)(x-c)} \\ &= \frac{a^2 + ap + q}{(a-b)(a-c)} \cdot \frac{1}{x-a} + \frac{b^2 + bp + q}{(b-a)(b-c)} \cdot \frac{1}{x-b} \\ & \quad + \frac{c^2 + cp + q}{(c-a)(c-b)} \cdot \frac{1}{x-c} \end{aligned}$$

$$\begin{aligned} 19. \text{ 解: } & \frac{2x^2 - 3x - 2}{x(x-1)^2(x+3)^3} \\ &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{x+3} + \frac{E}{(x+3)^2} + \frac{F}{(x+3)^3} \\ 2x^2 - 3x - 2 &= A(x-1)^2(x+3)^3 + Bx(x-1)(x+3)^2 \\ & \quad + Cx(x-1)(x+3)^3 + Dx(x-1)^2(x+3)^2 \\ & \quad + Ex(x-1)^2(x+3) + Fx(x-1)^2 \\ &= (A+B+D)x^5 + (7A+8B+C+4D+E)x^4 \\ & \quad - (2A-18B-9C+2D+E+F)x^3 \\ & \quad - (18A-27C+12D+E+2F)x^2 \\ & \quad - (27A+27B-27C-9D-3E-F)x + 27A \\ A+B+D &= 0, \quad 7A+8B+C+4D+E=0, \\ 2A-18B-9C+2D+E+F &= 0, \\ 18A-27C+12D+E+2F &= -2, \\ 27A+27B-27C-9D-3E-F &= 3, \\ 27A &= -2 \end{aligned}$$

$$\therefore A = -\frac{2}{27}, \quad B = \frac{25}{256}, \quad C = -\frac{3}{64},$$

$$D = -\frac{163}{6912}, \quad E = -\frac{35}{288}, \quad F = -\frac{25}{48}.$$

$$\begin{aligned} \text{答: } & \frac{2x^2 - 3x - 2}{x(x-1)^2(x+3)^3} = -\frac{2}{27x} + \frac{25}{256(x-1)} - \frac{3}{64(x-1)^2} \\ & \quad - \frac{163}{6912(x+3)} - \frac{35}{288(x+3)^2} \\ & \quad - \frac{25}{48(x+3)^3} \end{aligned}$$

20. 解: $\frac{x^3+x+3}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$
 $x^3+x+3 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1)$
 $= (A+C)x^3 - (A-B-C-D)x^2$
 $+ (A-B+C+D)x + (B+D)$
 $A+C=1, A-B-C-D=0, A-B+C+D=1,$
 $B+D=3$
 $\therefore A=2, B=\frac{3}{2}, C=-1, D=\frac{3}{2}.$

答: $\frac{x^3+x+3}{x^4+x^2+1} = \frac{4x+3}{2(x^2+x+1)} - \frac{2x-3}{2(x^2-x+1)}.$

IX. 對稱函數

習題 XXIX

原本第 249 頁

1. 解: 於 $x^4 - 2y^4 + z^4 + 4(x^3 - y^3)(y^3 - z^3)(x^2 + z^2)$ 式中, 試述何字爲對稱者。

以 x 替 z , z 替 x , 則吾人可得一與已知式相等之方程式, 卽 $z^4 - 2y^4 + x^4 + 4(z^3 - y^3)(y^3 - x^3)(z^2 + x^2)$, 故知上式對 x 與 z 爲對稱者。

2. 解: 完全書出下列各式中 a, b, c 之對稱函數,

$$\Sigma a^2 b^2, \Sigma a^3 b^4, \Sigma a^2/b, \Sigma a^2 b^3 c^5, \Sigma a^2 b^2 c^4,$$

$$\Sigma(a+b)c, \Sigma(a+b^2)c^3, \Sigma(a+2b+3c).$$

$$\Sigma a^2 b^2 = a^2 b^2 + b^2 c^2 + c^2 a^2.$$

$$\Sigma a^3 b^4 = a^3 b^4 + b^3 a^4 + b^3 c^4 + c^3 b^4 + c^3 a^4 + a^3 c^4.$$

$$\Sigma a^2/b = a^2/b + b^2/a + b^2/c + c^2/b + c^2/a + a^2/c.$$

$$\Sigma a^2 b^3 c^5 = a^2 b^3 c^5 + a^2 c^3 b^5 + b^2 a^3 c^5 + b^2 c^3 a^5 + c^2 a^3 b^5 + c^2 b^3 a^5.$$

$$\Sigma a^2 b^2 c^4 = a^2 b^2 c^4 + a^2 c^2 b^4 + b^2 c^2 a^4.$$

$$\Sigma(a+b)c = (a+b)c + (b+c)a + (c+a)b.$$

$$\Sigma(a+b^2)c^3 = (a+b^2)c^3 + (b+a^2)c^3 + (b+c^2)a^3 + (c+b^2)a^3$$

$$+ (c+a^2)b^3 + (a+c^2)b^3.$$

$$\Sigma(a+2b+3c) = (a+2b+3c) + (a+2c+3b) + (b+2c+3a) \\ + (b+2a+3c) + (c+2a+3b) + (c+2b+3a).$$

3. 解：指示 $(a-b)(b-c)(c-a)$ 式中， a, b, c 爲輪換而非絕對對稱；又 $(a-b)^2(b-c)^2(c-a)^2$ 中爲對稱。

以 a 代 b, b 代 c, c 代 a 代入原式，則原式不變。如以 a 及 b 互換，則原式變爲 $(b-a)(a-c)(c-b)$ ，與原式差一符號，故 $(a-b)(b-c)(c-a)$ 爲輪換而非絕對對稱。

$(a-b)^2(b-c)^2(c-a)^2$ 式，以 a, b, c 之任二字互換，所得之新多項式，仍與原式相等，故知爲對稱者。

4. 解： $(a-b)^2(b-c)^2(c-d)^2(d-a)^2$ 式，是否對於 a, b, c, d 對稱？

不對稱，因吾人以 a, b, c, d 中之任二字互換，所得之方程式，皆不與已知者相等也。

5. 解： $y^2 - x^2, z^2 - y^2, x^2 - z^2; a^2bc, abd^2, ac^2d, b^2cd;$

$$(a-c)(b-a), (a-c)(c-b), (a-b)(b-c).$$

$$y^2 - x^2, z^2 - y^2, x^2 - z^2 = y^2 - x^2, x^2 - z^2, z^2 - y^2.$$

$$a^2bc, abd^2, ac^2d, b^2cd = a^2bc, b^2cd, c^2da, d^2ab.$$

$$(a-c)(b-a), (a-c)(c-b), (a-b)(b-c)$$

$$= (a-c)(b-a), (b-a)(c-b), (c-b)(a-c).$$

6. 解： $ab^3c^2; a(b-c), (b+2c)(a+d); a^2/(a-b)(a-c).$

$$ab^3c^2 + bc^3d^2 + cd^3a^2 + da^3b^2.$$

$$a(b-c) + b(c-d) + c(d-a) + d(a-b).$$

$$(b+2c)(a+d) + (c+2d)(b+a) + (d+2a)(c+b)$$

$$+ (a+2b)(d+c).$$

$$a^2/(a-b)(a-c) + b^2/(b-c)(b-d) + c^2/(c-d)(c-a)$$

$$+ d^2/(d-a)(d-b).$$

7. 解： $\Sigma a^3 \cdot \Sigma a = \Sigma a^4 + \Sigma a^3b; \Sigma ab \cdot \Sigma a = \Sigma a^2b + 3\Sigma abc.$

$$\Sigma a^3 \cdot \Sigma a = (a^3 + b^3 + c^3)(a + b + c)$$

$$= a^4 + b^4 + c^4 + a^3b + a^3c + b^3a + b^3c + c^3a + c^3b$$

$$= \Sigma a^4 + \Sigma a^3b.$$

$$\Sigma ab \cdot \Sigma a = (ab + bc + ca)(a + b + c)$$

$$= a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc,$$

$$\Sigma a^2b + 3\Sigma abc = a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc.$$

$$\therefore \Sigma ab \cdot \Sigma a = \Sigma a^2b + 3\Sigma abc.$$

習題 XXX

原本第 251 頁

1. 解: $x^2(y-z) + y^2(z-x) + z^2(x-y)$
 當 $x=y$ 或 $y=z$ 或 $z=x$ 時, 此函數皆為零, 故此函數能被 $(x-y)(y-z)(z-x)$ 除盡。
 故 $x^2(y-z) + y^2(z-x) + z^2(x-y) = k(y-z)(z-x)(x-y)$
 設 $x=2, y=1, z=0$, 得 $k=-1$ 。
 答: $x^2(y-z) + y^2(z-x) + z^2(x-y) = -(y-z)(z-x)(x-y)$ 。
2. 解: $yz(y-z) + zx(z-x) + xy(x-y)$
 當 $x=y$ 或 $y=z$ 或 $z=x$ 時, 此函數為 0, 故此函數能被 $(x-y)(y-z)(z-x)$ 除盡。
 故 $yz(y-z) + zx(z-x) + xy(x-y) = k(x-y)(y-z)(z-x)$
 設 $x=2, y=1, z=0$, 得 $k=-1$ 。
 答: $yz(y-z) + zx(z-x) + xy(x-y) = -(x-y)(y-z)(z-x)$ 。
3. 解: $(y-z)^3 + (z-x)^3 + (x-y)^3$
 當 $x=y$ 或 $y=z$ 或 $z=x$ 時, 此函數為 0, 故此函數能被 $(x-y)(y-z)(z-x)$ 除盡。
 故 $(y-z)^3 + (z-x)^3 + (x-y)^3 = k(x-y)(y-z)(z-x)$
 設 $x=2, y=1, z=0$, 得 $k=3$ 。
 答: $(y-z)^3 + (z-x)^3 + (x-y)^3 = 3(x-y)(y-z)(z-x)$ 。
4. 解: $x(y-z)^3 + y(z-x)^3 + z(x-y)^3$
 設 $x=y$ 或 $y=z$ 或 $z=x$ 時, 此函數為 0, 故此函數能為 $(x-y)(y-z)(z-x)$ 除盡, 因 $\Sigma x(y-z)^3$ 為輪換對稱及齊次的四次式, 而 $\Sigma(x-y)$ 為輪換對稱及齊次的三次式, 故

$$x(y-z)^3 + y(z-x)^3 + z(x-y)^3$$

$$= k(x-y)(y-z)(z-x)(x+y+z).$$
 設 $x=2, y=1, z=0$, 得 $k=1$ 。
 答: $x(y-z)^3 + y(z-x)^3 + z(x-y)^3$

$$= (x-y)(y-z)(z-x)(x+y+z).$$
5. 解: $x^2(y-z)^3 + y^2(z-x)^3 + z^2(x-y)^3$
 此函數能為 $(x-y)(y-z)(z-x)$ 除盡, 故

$$x^2(y-z)^3 + y^2(z-x)^3 + z^2(x-y)^3$$

$$= (x-y)(y-z)(z-x)[k(x^2+y^2+z^2)+l(xy+yz+zx)]$$

設 $x=1, y=-1, z=0$, 得 $-2=4k-2l$ (1)

設 $x=2, y=1, z=0$, 得 $-4=-10k-4l$ (2)

解 (1), (2) 式, 得 $k=0, l=1$.

答: $x^2(y-z)^3+y^2(z-x)^3+z^2(x-y)^3$

$$= (x-y)(y-z)(z-x)(xy+yz+zx).$$

6. 解: $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$

此函數能為 $(x^2-y^2)(y^2-z^2)(z^2-x^2)$ 除盡. 且兩式各為對稱及齊次的六次式, 故

$$x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$$

$$= k(x^2-y^2)(y^2-z^2)(z^2-x^2)$$

設 $x=2, y=1, z=0$, 得 $k=-1$.

答: $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$

$$= -(x^2-y^2)(y^2-z^2)(z^2-x^2)$$

$$= -(x-y)(x+y)(y-z)(y+z)(z-x)(z+x).$$

7. 解: $(x+y+z)^3-x^3-y^3-z^3$

此函數能為 $(x+y)(y+z)(z+x)$ 除盡. 故

$$(x+y+z)^3-x^3-y^3-z^3=k(x+y)(y+z)(z+x)$$

設 $x=2, y=1, z=0$, 得 $k=3$.

答: $(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)$.

8. 解: $(y-z)^5+(z-x)^5+(x-y)^5$

此函數能為 $(x-y)(y-z)(z-x)$ 除盡. 故

$$(y-z)^5+(z-x)^5+(x-y)^5$$

$$= (y-z)(z-x)(x-y)[k(x^2+y^2+z^2)+l(xy+yz+zx)]$$

設 $x=2, y=1, z=0$, $30=10k+4l$

$x=3, y=1, z=0$, $-210=-60k-18l$

$\therefore k=5, l=-5$.

答: $(y-z)^5+(z-x)^5+(x-y)^5$

$$= 5(y-z)(z-x)(x-y)(x^2+y^2+z^2-xy-yz-zx).$$

9. 解: $(x+y+z)^5-(y+z-x)^5-(z+x-y)^5-(x+y-z)^5$

此函數能為 x, y 與 z 除盡. 故

$$(x+y+z)^5-(y+z-x)^5-(z+x-y)^5-(x+y-z)^5$$

$$= xyz[k(x^2+y^2+z^2)+l(xy+yz+zx)]$$

設 $x=1, y=1, z=1$, $240=3k+3l$

$$x=1, y=1, z=-1, -240 = -3k+l$$

$$\therefore l=0, k=80.$$

$$\text{答: } (x+y+z)^5 - (y+z-x)^5 - (z+x-y)^5 - (x+y-z)^5 \\ = 80xyz(x^2+y^2+z^2).$$

10. 解: $(y-z)(y+z)^3 + (z-x)(z+x)^3 + (x-y)(x+y)^3$
此函數能為 $(x-y)(y-z)(z-x)$ 除盡. 故

$$(y-z)(y+z)^3 + (z-x)(z+x)^3 + (x-y)(x+y)^3 \\ = k(x-y)(y-z)(z-x)(x+y+z)$$

$$\text{設 } x=2, y=1, z=0, \text{ 得 } k=-2.$$

$$\text{答: } (y-z)(y+z)^3 + (z-x)(z+x)^3 + (x-y)(x+y)^3 \\ = -2(x-y)(y-z)(z-x)(x+y+z).$$

11. 解: $x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 4xyz$
此函數能為 $(x+y)(y+z)(z+x)$ 除盡. 故

$$x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 4xyz \\ = k(x+y)(y+z)(z+x)$$

$$\text{設 } x=2, y=1, z=0, \text{ 得 } k=1.$$

$$\text{答: } x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 4xyz \\ = (x+y)(y+z)(z+x).$$

12. 解: $x^5(y-z) + y^5(z-x) + z^5(x-y)$
此函數能為 $(x-y)(y-z)(z-x)$ 除盡. 故

$$x^5(y-z) + y^5(z-x) + z^5(x-y) \\ = (x-y)(y-z)(z-x)[k(x^3+y^3+z^3) \\ + l(x^2y+y^2x+x^2z+z^2x+y^2z+z^2y) + mxyz]$$

$$\text{設 } x=2, y=1, z=0, \quad 30 = -18k - 12l$$

$$x=3, y=1, z=0, \quad 240 = -6[28k+12l]$$

$$x=3, y=2, z=1, \quad 180 = -72k - 99l - 12m$$

$$\therefore k=-1, l=-1, m=-1.$$

$$\text{答: } x^5(y-z) + y^5(z-x) + z^5(x-y) \\ = -(x-y)(y-z)(z-x)(x^3+y^3+z^3+x^2y+y^2x+x^2z \\ +z^2x+y^2z+z^2y+xyz).$$

13. 解: $\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}$
 $= \frac{-a^4(b-c) - b^4(c-a) - c^4(a-b)}{(a-b)(b-c)(c-a)}$

因 $-a^4(b-c) - b^4(c-a) - c^4(a-b)$ 能為 $(a-b)(b-c)(c-a)$

除盡，故

$$\begin{aligned} & -a^4(b-c) - b^4(c-a) - c^4(a-b) \\ & = (b-c)(c-a)(a-b)[k(a^2+b^2+c^2) + l(ab+bc+ca)] \end{aligned}$$

設 $a=2, b=1, c=0, 14=10k+4l$

$a=3, b=1, c=0, 13=10k+3l$

$\therefore l=1, k=1.$

答：
$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}$$
$$= a^2 + b^2 + c^2 + ab + bc + ca.$$

14. 解：
$$\frac{x+a}{(a-b)(a-c)} + \frac{x+b}{(b-c)(b-a)} + \frac{x+c}{(c-a)(c-b)}$$

因 $-(x+a)(b-c) - (x+b)(c-a) - (x+c)(a-b)$ 能以 $(a-b)(b-c)(c-a)$ 除盡，故等於 $k(a-b)(b-c)(c-a)$

設 $a=2, b=1, c=0$ ，得 $k=0$ 。

答：
$$\frac{x+a}{(a-b)(a-c)} + \frac{x+b}{(b-c)(b-a)} + \frac{x+c}{(c-a)(c-b)} = 0.$$

15. 解：
$$\frac{a^2-bc}{(a-b)(a-c)} + \frac{b^2-ca}{(b-c)(b-a)} + \frac{c^2-ab}{(c-a)(c-b)}$$
$$= \frac{-(a^2-bc)(b-c) - (b^2-ca)(c-a) - (c^2-ab)(a-b)}{(a-b)(b-c)(c-a)}$$

因 $-(a^2-bc)(b-c) - (b^2-ca)(c-a) - (c^2-ab)(a-b)$ 能以 $(a-b)(b-c)(c-a)$ 除盡，故等於 $k(a-b)(b-c)(c-a)$

設 $a=2, b=1, c=0$ ，得 $k=0$ 。

答：
$$\frac{a^2-bc}{(a-b)(a-c)} + \frac{b^2-ca}{(b-c)(b-a)} + \frac{c^2-ab}{(c-a)(c-b)} = 0.$$

16. 解：
$$\frac{(b+c)^2}{(a-b)(a-c)} + \frac{(c+a)^2}{(b-c)(b-a)} + \frac{(a+b)^2}{(c-a)(c-b)}$$
$$= \frac{(b+c)^2(b-c) - (c+a)^2(c-a) - (a+b)^2(a-b)}{(a-b)(b-c)(c-a)}$$

因 $-(b+c)^2(b-c) - (c+a)^2(c-a) - (a+b)^2(a-b)$ 能以 $(a-b)(b-c)(c-a)$ 除盡，故等於 $k(a-b)(b-c)(c-a)$

設 $a=2, b=1, c=0$ ，得 $k=1$ 。

答：
$$\frac{(b+c)^2}{(a-b)(a-c)} + \frac{(c+a)^2}{(b-c)(b-a)} + \frac{(a+b)^2}{(c-a)(c-b)} = 1.$$

$$\begin{aligned}
 17. \text{ 解: } & \frac{a^2}{(a-b)(a-c)(x-a)} + \frac{b^2}{(b-c)(b-a)(x-b)} \\
 & + \frac{c^2}{(c-a)(c-b)(x-c)} \\
 = & \frac{-a^2(b-c)(x-b)(x-c) - b^2(c-a)(x-a)(x-c) - c^2(a-b)(x-a)(x-b)}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}
 \end{aligned}$$

因 $-a^2(b-c)(x-b)(x-c) - b^2(c-a)(x-a)(x-c) - c^2(a-b)(x-a)(x-b)$ 能以 $(a-b)(b-c)(c-a)$ 除盡，故等於 $k(b-c)(c-a)(a-b)$

設 $a=2, b=1, c=0$ ，得 $k=x^2$ 。

$$\begin{aligned}
 \text{答: } & \frac{a^2}{(a-b)(a-c)(x-a)} + \frac{b^2}{(b-c)(b-a)(x-b)} \\
 & + \frac{c^2}{(c-a)(c-b)(x-c)} \\
 = & \frac{x^2}{(x-a)(x-b)(x-c)}
 \end{aligned}$$

X. 二項式定理

習題 XXXI

原本第 259 頁

1. 解: $(3x+2y)^3$

$$\begin{aligned}
 & = 27x^3 + 3(3x)^2 \cdot 2y + \frac{3 \cdot 2}{1 \cdot 2} (3x) \cdot (2y)^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} (2y)^3 \\
 & = 27x^3 + 54x^2y + 36xy^2 + 8y^3.
 \end{aligned}$$

2. 解: $(a-b)^8 = a^8 - 8a^7b + \frac{8 \cdot 7}{1 \cdot 2} a^6b^2 - \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} a^5b^3$

$$+ \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} a^4b^4 - \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^3b^5$$

$$+ \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} a^2b^6 - \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} ab^7 + b^8$$

$$\begin{aligned}
 = & a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 \\
 & - 8ab^7 + b^8.
 \end{aligned}$$

$$\begin{aligned}
 3. \text{ 解: } (1+2x^2)^7 &= 1^7 + 7(2x^2) + \frac{7 \cdot 6}{1 \cdot 2}(2x^2)^2 + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}(2x^2)^3 \\
 &+ \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}(2x^2)^4 + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(2x^2)^5 \\
 &+ \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2x^2)^6 + (2x^2)^7 \\
 &= 1 + 14x^2 + 84x^4 + 280x^6 + 560x^8 + 672x^{10} + 448x^{12} \\
 &+ 128x^{14}.
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ 解: } \left(2 + \frac{1}{x}\right)^4 &= 2^4 + 4 \cdot 2^3 \cdot \frac{1}{x} + \frac{4 \cdot 3}{1 \cdot 2} \cdot 2^2 \left(\frac{1}{x}\right)^2 \\
 &+ \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 2 \left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 \\
 &= 16 + \frac{32}{x} + \frac{24}{x^2} + \frac{8}{x^3} + \frac{1}{x^4}.
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ 解: } \left(x - \frac{3}{x}\right)^6 &= x^6 - 6 \cdot x^5 \left(\frac{3}{x}\right) + \frac{6 \cdot 5}{1 \cdot 2} x^4 \left(\frac{3}{x}\right)^2 - \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} x^3 \left(\frac{3}{x}\right)^3 \\
 &+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} x^2 \left(\frac{3}{x}\right)^4 - \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x \left(\frac{3}{x}\right)^5 + \left(\frac{3}{x}\right)^6 \\
 &= x^6 - 18x^4 + 135x^2 - 540 + \frac{1215}{x^2} - \frac{1458}{x^4} + \frac{729}{x^6}.
 \end{aligned}$$

$$\begin{aligned}
 6. \text{ 解: } \left(\frac{x}{y} - \frac{y}{x}\right)^5 &= \left(\frac{x}{y}\right)^5 - 5 \left(\frac{x}{y}\right)^4 \left(\frac{y}{x}\right) + \frac{5 \cdot 4}{1 \cdot 2} \left(\frac{x}{y}\right)^3 \left(\frac{y}{x}\right)^2 \\
 &- \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \left(\frac{x}{y}\right)^2 \left(\frac{y}{x}\right)^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{x}{y}\right) \left(\frac{y}{x}\right)^4 - \left(\frac{y}{x}\right)^5 \\
 &= \frac{x^5}{y^5} - 5 \frac{x^4 y}{y^5} + 10 \frac{x^3 y^2}{y^5} - 10 \frac{x^2 y^3}{y^5} + 5 \frac{x y^4}{y^5} - \frac{y^5}{x^5}.
 \end{aligned}$$

$$\begin{aligned}
 7. \text{ 解: } (1-x+2x^2)^4 &= [1 - (x-2x^2)]^4 \\
 &= 1^4 - 4(x-2x^2) + \frac{4 \cdot 3}{1 \cdot 2}(x-2x^2)^2 \\
 &- \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(x-2x^2)^3 + (x-2x^2)^4 \\
 &= 1 - 4x + 8x^2 + 6x^2 - 24x^3 + 24x^4 - 4x^3 + 24x^4 - 48x^6 \\
 &+ 32x^6 + x^4 - 8x^5 + 24x^6 - 32x^7 + 16x^8.
 \end{aligned}$$

$$= 1 - 4x + 14x^2 - 28x^3 + 49x^4 - 56x^5 + 56x^6 - 32x^7 + 16x^8.$$

8. 解: $(a^2 + ax - x^2)^3 = (a^2)^3 + 3(a^2)^2(ax - x^2)$

$$+ \frac{3 \cdot 2}{1 \cdot 2} a^2(ax - x^2)^2 + (ax - x^2)^3$$

$$= a^6 + 3a^5x - 3a^4x^2 + 3a^4x^2 - 6a^3x^3 + 3a^2x^4 + a^2x^2 - 3a^2x^4 + 3a^2x^4 + 3ax^5 - x^6$$

$$= a^6 + 3a^5x - 5a^3x^3 + 3ax^5 - x^6.$$

9. 解: $(1+x/2)^{11}$

$$n=11, r+1=6 \quad \therefore r=5$$

答: 所求之項爲 $\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{x^5}{32} = \frac{231}{16} x^5.$

10. 解: $(3a-4b)^{12}$

$$n=12, r+1=8 \quad \therefore r=7$$

答: $-\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (3a)^5 (4b)^7$

$$= -3153199104a^5b^7.$$

11. 解: $(a^2-2bc)^{10}$

$$n=10, r+1=6 \quad \therefore r=5$$

答: $-\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (a^2)^5 (2bc)^5$

$$= -8064a^{10}b^5c^5.$$

12. 解: $(1-x)^9$

$$n=9, r+1=5 \quad \therefore r=4$$

答: $\frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} x^4 = 126x^4.$

$$n=9, r+1=6 \quad \therefore r=5$$

答: $-\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^5 = -126x^5.$

13. 解: $(1+x)^8$

$$n=8, r+1=6 \quad \therefore r=5$$

答: 所求之係數爲 $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 56.$

14. 解: $(3-2x)^7$

$$n=7, r=4$$

答：所求之係數為 $\frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 3^3 (-2)^4 = 15120$.

15. 解： $(1-a^2)^6$
 $n=6, r=4$ [因 $x^8 = (x^2)^4$]

答：所求之係數為 $\frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} = 15$.

16. 解： $(1+2x)^9 + (1-2x)^{11}$
 $n=9$ 及 $11, r=3$

答：所求之係數為 $\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \cdot 2^3 + \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} (-2)^3$
 $= 672 - 1320 = -648$.

17. 解： $\left(x + \frac{1}{x}\right)^{12}$

$$\begin{aligned} \text{通項} &= \frac{n(n-1)\cdots(n-r+1)}{r!} x^{12-r} \left(\frac{1}{x}\right)^r \\ &= \frac{n(n-1)\cdots(n-r+1)}{r!} x^{12-2r} \end{aligned}$$

$$2r = 12 \quad \therefore r = 6$$

答：常數項 = $\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 924$.

18. 解： $\left(2x - \frac{1}{x}\right)^{15}$

$$\begin{aligned} \text{通項} &= \frac{n(n-1)\cdots(n-r+1)}{r!} (2x)^{15-r} \left(-\frac{1}{x}\right)^r \\ &= \frac{n\cdots(n-r+1)}{r!} x (-1)^r 2^{15-r} x^{15-2r} \end{aligned}$$

$$15 - 2r = 7 \quad \therefore r = 4$$

答： x^7 之係數 = $\frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^{11} = 2795520$.

19. 解： $(x+2y)(x-3y)(x-5y) = x^3 - 6x^2y - xy^2 + 30y^3$.

20. 解： $(x+2)(x+3)(x-4)(x-5)$
 $= x^4 - 4x^3 - 19x^2 + 46x + 120$.

21. 解： $(a+b+c+d)(f+g+h)(k+l)(m+n+p+q)$ 乘積中項
 之總數為 $4 \times 3 \times 2 \times 4 = 96$.

22. 解: 1. $(1+x^2+x^3+x^4)^3$ 各項係數之總數爲
 $(1+1+1+1)^3 = 4^3 = 64$.
2. $(1+2x+x^2)^2(1+x+3x^3)^2$ 各項係數之總數爲
 $(1+2+1)^2(1+1+3)^2 = 4^2 \cdot 5^2 = 400$.
23. 解: 1. $\Sigma a^2 \cdot \Sigma a = (a^2+b^2+c^2+d^2)(a+b+c+d)$
其係數之和爲 $4 \times 4 = 16$.
2. $\Sigma a^3 \cdot \Sigma abc$
 $= (a^3+b^3+c^3+d^3)(abc+bcd+cda+dab)$
其係數之和爲 $4 \times 4 = 16$.
3. $\Sigma ab \cdot \Sigma abc = (ab+bc+cd+da+ac+bd)$
 $\times (abc+bcd+cda+dab)$
其係數之和爲 $6 \times 4 = 24$.
24. 解: $(a+b)^n = (a+b)(a+b)\dots$ 至 n 項。
答: 其係數之和爲 $2 \cdot 2 \cdot 2 \dots$ 至第 n 項 $= 2^n$.
25. 解: $(a-b)^n$ 展開式中各項之係數與 $(1-1)^n$ 展開式中各項相等。因 $(1-1)^n = 0$, $\therefore (a-b)^n$ 展開式中各項係數之和等於 0, 即其正負係數之和相等。

XI. 開 方

習 題 XXXII

原本第 269 頁

1. 解: $\sqrt[3]{-\frac{27x^6y^{15}}{125a^9z^{12}}} = -\frac{3x^2y^5}{5a^3z^4}$.
2. 解: $\sqrt{\frac{529a^4b^6}{625c^2d^8}} = \frac{23a^2b^3}{25cd^4}$.
3. 解: $\sqrt[6]{(x^4y^2-2x^3y^3+x^2y^4)^6} = \sqrt[6]{(x^2y-xy^2)^6} = x^2y-xy^2$
 $= xy(x-y)$.
4. 解: $x^4-2x^3+3x^2-2x+1 \equiv (x^2+px+q)^2$
 $\equiv x^4+2px^3+(p^2+2q)x^2+2pqx+q^2$
 $2p = -2 \quad \therefore p = -1$
 $p^2+2q = 3 \quad \therefore q = 1$

因 p, q 之值能適合 $2p = -2$ 及 $p^2 = 1$

答：其平方根爲 $x^2 - x + 1$ 。

5. 解： $x^2 - 2x^4 + 6x^3 - 6x + x^6 + 9 = x^6 - 2x^4 + 6x^3 + x^2 - 6x + 9$

$$\equiv (x^3 + px + q)^2 \equiv x^6 + 2px^4 + 2qx^3 + p^2x^2 + 2pqx + q^2$$

$$2p = -2 \quad \therefore p = -1$$

$$2q = 6 \quad \therefore q = 3$$

答：其平方根爲 $x^3 - x + 3$ 。

6. 解： $4x^6 + 12x^5y + 9x^4y^2 - 4x^3y^3 - 6x^2y^4 + y^6$

$$\equiv (2x^3 + px^2y + qy^3)^2$$

$$\equiv 4x^6 + 4px^5y + p^2x^4y^2 + 4qx^3y^3 + 2pqx^2y^4 + q^2y^6$$

$$4p = 12 \quad \therefore p = 3$$

$$4q = -4 \quad \therefore q = -1$$

答：其平方根爲 $2x^3 + 3x^2y - y^3$ 。

7. 解： $4x^2 - 20x + 13 + 30/x + 9/x^2 \equiv (2x + p + q/x)^2$

$$\equiv 4x^2 + 4px + (4q + p^2) + 2pq/x + q^2/x^2$$

$$4p = -20 \quad \therefore p = -5$$

$$4q + p^2 = 13 \quad \therefore q = -3$$

答：其平方根爲 $2x - 5 - 3/x$ 。

8. 解： $49 - 84x - 34x^2 + 60x^3 + 25x^4 \equiv (7 + px + qx^2)^2$

$$\equiv 49 + 14px + (14q + p^2)x^2 + 2pqx^3 + q^2x^4$$

$$14p = -84 \quad \therefore p = -6$$

$$14q + p^2 = -34 \quad \therefore q = -5$$

答：其平方根爲 $7 - 6x - 5x^2$ 。

9. 解： $a^8 + 2x^7 - x^6 - x^4 - 6x^3 + 5x^2 - 4x + 4$

$$a^8 + 2x^7 - x^6 - x^4 - 6x^3 + 5x^2 - 4x + 4 \quad \underline{x^4 + x^3 - x^2 + x - 2}$$

x^8

$$\begin{array}{r|l} 2x^4 + x^3 & 2x^7 - x^6 \\ & 2x^7 + x^6 \end{array}$$

$$\begin{array}{r|l} 2x^4 + 2x^3 - x^2 & -2x^6 - x^4 \\ & -2x^6 - 2x^5 + x^4 \end{array}$$

$$\begin{array}{r|l} 2x^4 + 2x^3 - 2x^2 + x & 2x^5 - 2x^4 - 6x^3 + 5x^2 \\ & 2x^5 + 2x^4 - 2x^3 + x^2 \end{array}$$

$$\begin{array}{r|l} 2x^4 + 2x^3 - 2x^2 + 2x - 2 & -4x^4 - 4x^3 + 4x^2 - 4x + 4 \\ & -4x^4 - 4x^3 + 4x^2 - 4x + 4 \end{array}$$

0

答：其平方根爲 $x^4 + x^3 - x^2 + x - 2$ 。

$$\begin{aligned}
 10. \text{ 解: } & (x^2+1)^2 - 4x(x^2-1) = x^4 + 2x^2 + 1 - 4x^3 + 4x \\
 & = x^4 - 4x^3 + 2x^2 + 4x + 1 \equiv (x^2 + px + q)^2 \\
 & \equiv x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2 \\
 & \quad 2p = -4 \quad \therefore p = -2 \\
 & \quad p^2 + 2q = 2 \quad \therefore q = -1
 \end{aligned}$$

答：其平方根爲 $x^2 - 2x - 1$ 。

$$\begin{aligned}
 11. \text{ 解: } & 4x^4 + 9x^2y^2 - 12x^3y + 16x^2 - 24xy + 16 \equiv (2x^2 + pxy + q)^2 \\
 & \equiv 4x^4 + 4px^3y + 4qx^2 + p^2x^2y^2 + 2pqxy + q^2 \\
 & \quad 4p = -12 \quad \therefore p = -3 \\
 & \quad 4q = 16 \quad \therefore q = 4
 \end{aligned}$$

答：其平方根爲 $2x^2 - 3xy + 4$ 。

$$\begin{aligned}
 12. \text{ 解: } & x^2/y^2 + y^2/x^2 + 2 + 2x^2 + 2y^2 + x^2y^2 \\
 & \equiv (x/y + pxy + qy/x)^2 \\
 & \equiv x^2/y^2 + 2px^2 + 2q + 2pqy^2 + p^2x^2y^2 + q^2y^2/x^2 \\
 & \quad 2p = 2 \quad \therefore p = 1 \\
 & \quad 2q = 2 \quad \therefore q = 1
 \end{aligned}$$

因 p, q 之值能適合 $2pq = 2, p^2 = 1, q^2 = 1$

答： $x/y + xy + y/x$ 爲其平方根。

$$13. \text{ 解: } 1 - 2x$$

$$\begin{array}{r}
 1 - 2x \qquad \qquad \qquad \boxed{1 - x - \frac{x^2}{2} - \frac{x^3}{2} + \dots} \\
 \hline
 1 \\
 \hline
 2 - x \quad \left| \begin{array}{l} -2x \\ -2x + x^2 \end{array} \right. \\
 \hline
 2 - 2x - \frac{x^2}{2} \quad \left| \begin{array}{l} -x^2 \\ -x^2 + x^3 + \frac{x^4}{4} \end{array} \right. \\
 \hline
 2 - 2x - x^2 - \frac{x^3}{4} \quad \left| \begin{array}{l} -x^3 - \frac{x^4}{4} \\ -x^3 + x^4 + \frac{x^5}{2} + \frac{x^6}{4} \\ -\frac{5x^4}{4} - \frac{x^5}{2} - \frac{x^6}{4} \end{array} \right.
 \end{array}$$

答：其平方根爲 $1-x-\frac{x^2}{2}-\frac{x^3}{2}+\dots\dots$

$$14. \text{ 解: } 4-x+3x^2 \equiv (2+px+qx^2+rx^3+\dots)^2 \\ \equiv 4+4px+(4q+p^2)x^2+(4r+2pq)x^3+\dots\dots$$

$$4p = -1 \quad \therefore p = -\frac{1}{4}$$

$$4q+p^2=3 \quad \therefore q = \frac{47}{64}$$

$$4r+2pq=0 \quad \therefore r = \frac{47}{512}$$

答：其平方根爲 $2-\frac{x}{4}+\frac{47x^2}{64}+\frac{47x^3}{512}+\dots\dots$

$$15. \text{ 解: } x^6+3x^5+6x^4+7x^3+6x^2+3x+1 \equiv (x^2+px+q)^3 \\ \equiv x^6+3px^5+3(p^2+q)x^4+(p^3+6pq)x^3+3(p^2q+q^2)x^2 \\ +3pqx+q^3$$

$$3p=3 \quad \therefore p=1$$

$$p^2+q=2 \quad \therefore q=1$$

因 p, q 之值能適合 $p^3+6pq=7, 3(p^2q+q^2)=6, 3pq=3, q^3=1$

答：其立方根爲 x^2+x+1 .

$$16. \text{ 解: } 27x^{12}+27x^{10}-18x^8-17x^6+6x^4+3x^2-1 \\ \equiv (3x^4+px^2+q)^3 \\ \equiv 27x^{12}+(18p+9p)x^{10}+(18q+3p^2+6p^2+9q)x^8+\dots\dots$$

$$27p=27 \quad \therefore p=1$$

$$27q+9p^2=-18 \quad \therefore q=-1$$

答：其立方根爲 $3x^4+x^2-1$.

$$17. \text{ 解: } 8x^6-36ax^5+90a^2x^4-135a^3x^3+135a^4x^2-81a^5x+27a^6 \\ \equiv (2x^2+pa^2x+qa^2)^3 \equiv 8x^6+12pa^2x^5+(12q+6p^2)x^4a^2 \\ +(12pq+p^3)a^3x^3+(6q^2+3p^2q)a^4x^2+3pq^2a^5x+q^3a^6$$

$$12p=-36 \quad \therefore p=-3$$

$$12q+6p^2=90 \quad \therefore q=3$$

答：其立方根爲 $2x^2-3ax^2+3a^2$.

$$18. \text{ 解: } \frac{x^3}{y^3}+\frac{y^3}{x^3}+3\frac{x^2}{y^2}+3\frac{y^2}{x^2}+6\frac{x}{y}+6\frac{y}{x}+7.$$

$$\frac{x^3}{y^3} + 3\frac{x^2}{y^2} + 6\frac{x}{y} + 7 + 6\frac{y}{x} + 3\frac{y^2}{x^2} + \frac{y^3}{x^3} \left| \frac{x}{y} + 1 + \frac{y}{x} \right.$$

$$\frac{x^3}{y^3}$$

$$\frac{3x^2}{y^2} + 3\frac{x}{y} + 1 \quad \left| \begin{array}{l} 3x^2 + 6\frac{x}{y} + 7 + 6\frac{y}{x} + 3\frac{y^2}{x^2} + \frac{y^3}{x^3} \\ 3x^2 + 3\frac{x}{y} + 1 \end{array} \right.$$

$$\frac{3x^2}{y^2} + 6\frac{x}{y} + 3 \quad \left| \begin{array}{l} 3\frac{x}{y} + 6 + 6\frac{y}{x} + 3\frac{y^2}{x^2} + \frac{y^3}{x^3} \\ 3\frac{x}{y} + 6 + 6\frac{y}{x} + 3\frac{y^2}{x^2} + \frac{y^3}{x^3} \end{array} \right.$$

$$+ 3\left(\frac{x}{y} + 1\right)\frac{y}{x} + \frac{y^2}{x^2}$$

$$\frac{3x^2}{y^2} + 6\frac{x}{y} + 6 + 3\frac{y}{x} + \frac{y^2}{x^2} \quad \left| \begin{array}{l} 0 \\ 0 \end{array} \right.$$

答：其立方根爲 $\frac{x}{y} + 1 + \frac{y}{x}$ 。

19. 解： $1 - x + x^2 \equiv (1 + px + qx^2 + \dots)^3$
 $\equiv 1 + 3px + (3q + 3p^2)x^2 + \dots$

$$3p = -1 \quad \therefore p = -\frac{1}{3}$$

$$3q + 3p^2 = 1 \quad \therefore q = \frac{2}{9}$$

答：其立方根爲 $1 - \frac{1}{3}x + \frac{2}{9}x^2 - \dots$ 。

20. 解： $x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1$
 $\equiv (x^2 + px + q)^4 \equiv x^8 + 4px^7 + (4q + 6p^2)x^6 + \dots$

$$4p = -4 \quad \therefore p = -1$$

$$4q + 6p^2 = 10 \quad \therefore q = 1$$

答：其四次根爲 $x^2 - x + 1$ 。

21. 解： $x^{10} + 5x^9 + 15x^8 + 30x^7 + 45x^6 + 51x^5 + 45x^4 + 30x^3$
 $+ 15x^2 + 5x + 1$

$$\equiv (x^2 + px + q)^5 \equiv x^{10} + 5px^9 + (5q + 10p^2)x^8 + \dots$$

$$5p = 5 \quad \therefore p = 1$$

$$5q + 10p^2 = 15 \quad \therefore q = 1$$

答：其五次根爲 $x^2 + x + 1$ 。

22. 解: $x^4 + 6x^3 + 11x^2 + ax + b \equiv (x^2 + px + q)^2$
 $\equiv x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2.$

$$2p = 6 \quad \therefore p = 3$$

$$p^2 + 2q = 11 \quad \therefore q = 1$$

又 $2pq = a \quad \therefore a = 6$

$$q^2 = b \quad \therefore b = 1$$

答: a 應為 6, b 應為 1.

23. 解: 27889.

$$\begin{array}{r} 2,78,89 \quad | \quad 167 \\ \underline{1} \\ 26 \quad | \quad 178 \\ \underline{156} \\ 327 \quad | \quad 2289 \\ \underline{2289} \\ 0 \end{array}$$

答: $\sqrt{27889} = 167.$

24. 解: 2313.61.

$$\begin{array}{r} 23,13.61 \quad | \quad 48.1 \\ \underline{16} \\ 88 \quad | \quad 713 \\ \underline{704} \\ 961 \quad | \quad 961 \\ \underline{961} \\ 0 \end{array}$$

答: $\sqrt{2313.61} = 48.1.$

25. 解: 583.2225.

$$\begin{array}{r} 5,83.22,25 \quad | \quad 24.15 \\ \underline{4} \\ 44 \quad | \quad 183 \\ \underline{176} \\ 481 \quad | \quad 722 \\ \underline{481} \\ 4825 \quad | \quad 24125 \\ \underline{24125} \\ 0 \end{array}$$

答: $\sqrt{583.2225} = 24.15,$

26. 解: 4149369.

$$\begin{array}{r}
 4,14,93,69 \mid 2037 \\
 \underline{4} \\
 403 \mid 1493 \\
 \underline{1209} \\
 4067 \mid 28469 \\
 \underline{28469} \\
 0
 \end{array}$$

答: $\sqrt{4149369} = 2037$.

27. 解: 0.00320356.

$$\begin{array}{r}
 0.00,32,03,56 \mid 0.0566 \\
 \underline{25} \\
 106 \mid 703 \\
 \underline{636} \\
 1126 \mid 6756 \\
 \underline{6756} \\
 0
 \end{array}$$

答: $\sqrt{0.00320356} = 0.0566$.

28. 解: 9.024016.

$$\begin{array}{r}
 9.02,40,16 \mid 3.004 \\
 \underline{9} \\
 6004 \mid 24016 \\
 \underline{24016} \\
 0
 \end{array}$$

答: $\sqrt{9.024016} = 3.004$.

29. 解: 2.

$$\begin{array}{r}
 2 \mid 1.414 \\
 \underline{1} \\
 24 \mid 100 \\
 \underline{96} \\
 281 \mid 400 \\
 \underline{281} \\
 2824 \mid 11900 \\
 \underline{11296} \\
 604
 \end{array}$$

答: $\sqrt{2} = 1.414$.

30. 解: 55.5.

$$\begin{array}{r}
 55.5 \mid 7.449 \\
 \underline{49} \\
 144 \mid 650 \\
 \underline{576} \\
 1484 \mid 7400 \\
 \underline{5936} \\
 14889 \mid 146400 \\
 \underline{134001} \\
 12399
 \end{array}$$

答: $\sqrt{55.5} = 7.449$.

31. 解: 234.561.

$$\begin{array}{r}
 2,34.56,10 \mid 15.315 \\
 \underline{1} \\
 25 \mid 134 \\
 \underline{125} \\
 303 \mid 956 \\
 \underline{909} \\
 3061 \mid 4710 \\
 \underline{3061} \\
 30625 \mid 164900 \\
 \underline{153125} \\
 11775
 \end{array}$$

答: $\sqrt{234.561} = 15.315$.

32. 解: 1860867

$$\begin{array}{r}
 1,860,867 \mid 123 \\
 \underline{1} \\
 300 \mid 860 \\
 \underline{60} \\
 4 \\
 364 \mid 728 \\
 \underline{43200} \mid 132867 \\
 1080 \\
 \underline{9} \\
 44289 \mid 132867 \\
 \underline{\hspace{1.5cm}} \\
 0
 \end{array}$$

答: $\sqrt[3]{1860867} = 123$.

33. 解: 167284.151.

$$\begin{array}{r|l}
 167,284.151 & 55.1 \\
 \hline
 125 & \\
 \hline
 7500 & 42284 \\
 750 & \\
 25 & \\
 \hline
 8275 & 41375 \\
 907500 & 909151 \\
 1650 & \\
 1 & \\
 \hline
 909151 & 909151 \\
 \hline
 & 0
 \end{array}$$

答: $\sqrt[3]{167284.151} = 55.1$.

34. 解: 1036.433728.

$$\begin{array}{r|l}
 1,036,433,728 & 10.12 \\
 \hline
 1 & \\
 \hline
 30000 & 36433 \\
 300 & \\
 1 & \\
 \hline
 30301 & 30301 \\
 3060300 & 6132728 \\
 6060 & \\
 4 & \\
 \hline
 3066364 & 6132728 \\
 \hline
 & 0
 \end{array}$$

答: $\sqrt[3]{1036.433728} = 10.12$.

XII. 無理函數, 根式與分指數

習題 XXXIII

原本第 274 頁

1. 解: $\sqrt{18} = \sqrt{3^2 \times 2} = 3\sqrt{2}$.
2. 解: $\sqrt{588} = \sqrt{14^2 \times 3} = 14\sqrt{3}$.
3. 解: $\sqrt[3]{-27^2} = \sqrt[3]{-(3^3)^2} = \sqrt[3]{-(3^2)^3} = -9$.
4. 解: $\sqrt[9]{-1000} = \sqrt[9]{-10^3} = \sqrt[3]{-10}$.

5. 解: $\sqrt{3/2} = \sqrt{3/2} \cdot \sqrt{2/2} = \frac{1}{2}\sqrt{6}$.

6. 解: $\sqrt[3]{3/2} = \sqrt[3]{\frac{3 \times 4}{2 \times 4}} = \frac{1}{2}\sqrt[3]{12}$.

7. 解: $\sqrt[3]{3/4} = \sqrt[3]{\frac{3 \times 2}{4 \times 2}} = \frac{1}{2}\sqrt[3]{6}$.

8. 解: $\sqrt[5]{3/16} = \sqrt[5]{6/32} = \frac{1}{2}\sqrt[5]{6}$.

9. 解: $\sqrt[5]{25a^5b^{10}c^{15}d^6} = ab^2c^3d\sqrt[5]{25d}$.

10. 解: $\sqrt[6]{128a^2b^4c^8} = 2c\sqrt[6]{2a^2b^4c^2}$.

11. 解: $\sqrt[12]{8x^6y^9z^{15}} = z\sqrt[12]{8x^6y^9z^3} = z\sqrt[4]{2x^2y^3z}$.

12. 解: $\sqrt[5]{25a^2b^4c^6} = \sqrt[5]{5ab^2c^3}$.

13. 解: $\sqrt[3n]{a^nb^{2n}c^{3n}} = c\sqrt[3]{ab^2}$.

14. 解: $\sqrt[n]{a^{2n+1}b^{3n+2}c^{4n}} = a^2b^3c^4\sqrt[n]{ab^3}$.

15. 解: $\sqrt{x^2y^2 - x^2z^2} = x\sqrt{y^2 - z^2}$.

16. 解: $\sqrt{(x^2 - y^2)(x + y)} = \sqrt{(x - y)(x + y)^2} = (x + y)\sqrt{x - y}$.

17. 解: $\sqrt[3]{x^6 - x^3y^3} = x\sqrt{x^3 - y^3}$.

18. 解: $\sqrt[4]{a^4b^4 - 2a^3b^5 + a^2b^6} = \sqrt[4]{a^2b^4(a^2 - 2ab + b^2)}$
 $= \sqrt[4]{a^2b^4(a - b)^2} = b\sqrt{a(a - b)}$.

19. 解: $\sqrt{\frac{a^3 + b^3}{32ab^2}} = \sqrt{\frac{2(a^3 + b^3)a^2b}{64a^3b^3}} = \frac{1}{4ab}\sqrt{2(a^3 + b^3)a^2b}$.

20. 解: $\sqrt{\frac{a+b}{a-b}} = \sqrt{\frac{(a+b)(a-b)}{(a-b)^2}} = \frac{1}{a-b}\sqrt{a^2 - b^2}$.

21. 解: $\sqrt[3]{\frac{x^2 - x + 1}{9(x+1)^2}} = \sqrt[3]{\frac{(x^2 - x + 1)3(x+1)}{27(x+1)^3}}$
 $= \frac{1}{3(x+1)}\sqrt[3]{3(x^3 + 1)}$.

22. 解: $\sqrt{1 - \frac{a^3}{b^3}} = \sqrt{\frac{b^3 - a^3}{b^3}} = \frac{1}{b}\sqrt[3]{b^3 - a^3}$.

23. 解: $\sqrt{\frac{c^{n+3}}{a^{3n}b^{3n+2}}} = \sqrt[3]{\frac{bc^{n+3}}{a^3b^{3n+3}}} = \frac{c}{a^3b^{n+1}}\sqrt[3]{bc^3}$.

24. 解: $\sqrt{\frac{a^2x^2}{b^3} - \frac{2ax}{b^2} + \frac{1}{b}} = \sqrt{\frac{b(a^2x^2 - 2abx + b^2)}{b^4}}$
 $= \sqrt{\frac{b(ax - b)^2}{b^4}} = \frac{ax - b}{b^2}\sqrt{b}$.

25. 解: $3a\sqrt{3a} = \sqrt{(3a)^2} \sqrt{3a} = \sqrt{27a^3}$.
26. 解: $\frac{a+b}{a-b} \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{(a-b)(a+b)^2}{(a+b)(a-b)^2}} = \sqrt{\frac{a+b}{a-b}}$.
27. 解: $3ax \sqrt[4]{\frac{1}{27a^3x^3}} = \sqrt[4]{(3ax)^4} = \sqrt[4]{81a^4x^4} = \sqrt[4]{27a^3x^3} = \sqrt[4]{3ax}$.
28. 解: $\sqrt{18} = \sqrt{2 \times 3^2} = 3\sqrt{2}$.
 $\sqrt{50} = \sqrt{2 \times 5^2} = 5\sqrt{2}$.
 $\sqrt{1/8} = \sqrt{2/16} = 1/4\sqrt{2}$.
29. 解: $\sqrt[3]{24} = \sqrt[3]{3 \times 2^3} = 2\sqrt[3]{3}$.
 $\sqrt[3]{192} = \sqrt[3]{3 \times 4^3} = 4\sqrt[3]{3}$.
 $\sqrt[3]{8/9} = \sqrt[3]{24/27} = 2/3\sqrt[3]{3}$.
30. 解: $\sqrt{(x^3-y^3)(x-y)} = \sqrt{(x-y)^2(x^2+xy+y^2)}$
 $= (x-y)\sqrt{x^2+xy+y^2}$.
 $\sqrt{x^4y^2+x^3y^3+x^2y^4} = \sqrt{x^2y^2(x^2+xy+y^2)}$
 $= xy\sqrt{x^2+xy+y^2}$.

習題 XXXIV

原本第 277 頁

1. 解: $\sqrt[6]{3} = \sqrt[30]{3^5} = \sqrt[30]{243}$.
 $\sqrt[10]{3} = \sqrt[30]{3^3} = \sqrt[30]{27}$.
 $\sqrt[15]{3} = \sqrt[30]{3^2} = \sqrt[30]{9}$.
2. 解: $\sqrt[3]{a^2} = \sqrt[12]{(a^2)^4} = \sqrt[12]{a^8}$.
 $\sqrt[4]{2a^8b^2} = \sqrt[12]{(2a^8b^2)^3} = \sqrt[12]{8a^9b^6}$.
 $\sqrt[6]{7b^5} = \sqrt[12]{(7b^5)^2} = \sqrt[12]{49b^{10}}$.
3. 解: $3\sqrt{2} = \sqrt{18} = \sqrt[6]{18^3} = \sqrt[6]{5832}$.
 $2\sqrt[3]{3} = \sqrt[3]{24} = \sqrt[6]{24^2} = \sqrt[6]{576}$.
 $\therefore 3\sqrt{2} > 2\sqrt[3]{3}$.
4. 解: $\sqrt{3} = \sqrt[12]{3^6} = \sqrt[12]{729}$.
 $\sqrt[3]{4} = \sqrt[12]{4^4} = \sqrt[12]{256}$.
 $\sqrt[4]{5} = \sqrt[12]{5^3} = \sqrt[12]{125}$.

$$\therefore \sqrt{3} > \sqrt[3]{4} > \sqrt[4]{5}.$$

5. 解: $\sqrt{35} \div \sqrt{7/5} = \sqrt{35} \cdot \sqrt{5/7} = \sqrt{5^2} = 5.$

6. 解: $10 \div \sqrt{5} = 10 \times 1/\sqrt{5} = 10 \times \frac{\sqrt{5}}{5} = 2\sqrt{5}.$

7. 解: $4 \div \sqrt[3]{2} = 4 \times \frac{1}{\sqrt[3]{2}} = 4 \times \frac{\sqrt[3]{4}}{2} = 2\sqrt[3]{4}.$

8. 解: $\sqrt{6} \cdot \sqrt{10} \cdot \sqrt{15} = \sqrt{900} = 30.$

9. 解: $\sqrt[3]{60} \cdot \sqrt[3]{90} \cdot \sqrt[3]{15} = \sqrt[3]{60 \cdot 90 \cdot 15}$
 $= \sqrt[3]{30 \cdot 2 \cdot 3 \cdot 30 \cdot 15} = 30\sqrt[3]{3}.$

10. 解: $2\sqrt{3} \div 3\sqrt{2} = \sqrt{12} \times \sqrt{1/18} = \sqrt{6/9} = \frac{1}{3}\sqrt{6}.$

11. 解: $\sqrt{2} \cdot \sqrt[3]{2} \cdot \sqrt[4]{2} = \sqrt[12]{2^6} \cdot \sqrt[12]{2^4} \cdot \sqrt[12]{2^3} = \sqrt[12]{2^{13}} = 2\sqrt[12]{2}.$

12. 解: $\sqrt[6]{3} \div \sqrt[4]{5} = \sqrt[12]{9} \cdot \sqrt[12]{1/5^3}$
 $= \sqrt[12]{9/5^3} = \frac{1}{5} \sqrt[12]{9 \cdot 5^9} = 1/5 \sqrt[12]{17578125}.$

13. 解: $2\sqrt{35} \cdot \sqrt{65} \div \sqrt{91} = 2\sqrt{35 \times 65/91}$
 $= 2\sqrt{5 \times 7 \times 5 \times 13/7 \times 13} = 10.$

14. 解: $\sqrt{a^3b^5c^7} \cdot \sqrt[3]{a^2b^4c^8} = ab^2c^3 \cdot \sqrt{abc} \times bc^2 \sqrt[3]{a^2bc^2}$
 $= ab^3c^5 \sqrt[6]{a^3b^3c^3} \times a^4b^2c^4 = a^2b^3c^6 \sqrt[6]{ab^5c^7}.$

15. 解: $\sqrt[2n]{a} \cdot \sqrt[3n]{a} = \sqrt[6n]{a^3} \cdot \sqrt[6n]{a^2} = \sqrt[6n]{a^5}.$

16. 解: $\sqrt{a^3b^3} \div \sqrt[6]{a^5b^5} = \sqrt[6]{a^9b^9} \cdot \sqrt[6]{1/a^5b^5} = \sqrt[6]{a^4b^4} = \sqrt[3]{a^2b^2}.$

17. 解: $\sqrt[3]{a^2bc^2} \cdot \sqrt[3]{ab^2c^4} = \sqrt[3]{a^3b^3c^6} = abc^2.$

18. 解: $\sqrt[3]{a} \cdot \sqrt[6n]{a} = \sqrt[6n]{a^{2n}} \cdot \sqrt[6n]{a} = \sqrt[6n]{a^{2n+1}}.$

19. 解: $\sqrt[6]{a/b} \div \sqrt[9]{a/b} = \sqrt[18]{a^3/b^3} \cdot \sqrt[18]{b^2/a^2}$
 $= \sqrt[18]{a/b} = 1/b \sqrt[18]{ab^{17}}.$

20. 解: $\sqrt[3]{ab^2} \cdot \sqrt[6]{ab^5} \div (\sqrt[10]{a^7b^9} \cdot \sqrt[15]{a^{12}b^{14}})$
 $= \sqrt[6]{a^2b^4} \cdot \sqrt[6]{ab^5} \div (\sqrt[30]{a^{21}b^{27}} \cdot \sqrt[30]{a^{24}b^{28}})$
 $= \sqrt[6]{a^3b^9} \div \sqrt[30]{a^{45}b^{55}} = \sqrt[6]{a^3b^9} \div \sqrt[3]{a^9b^{11}} = \sqrt[6]{a^3b^9/a^9b^{11}}$
 $= \frac{1}{ab} \sqrt[6]{b^4} = \frac{1}{ab} \sqrt[3]{b^2}.$

21. 解: $(\sqrt{12})^3 = 12\sqrt{12} = 24\sqrt{3}.$

22. 解: $(\sqrt[3]{a^2})^6 = (a^2)^2 = a^4.$

23. 解: $(2\sqrt[4]{xy^2z^5})^6 = 64\sqrt[4]{x^6y^{12}z^{18}} = 64xy^3z^4\sqrt[4]{x^2z^2}$
 $= 64xy^3z^4\sqrt{xz}.$

24. 解: $\sqrt[4]{\sqrt[3]{a^2}} = \sqrt[12]{a^2} = \sqrt[6]{a}$

25. 解: $\sqrt[3]{\sqrt{8}} = \sqrt[6]{8} = \sqrt{2}.$

26. 解: $\sqrt[6]{\sqrt[5]{a^3b^6/c^9}} = \sqrt[30]{a^3b^6/c^9} = \sqrt[10]{ab^2/c^3}$
 $= \sqrt[10]{ab^2c^7/c^{10}} = 1/c\sqrt[10]{ab^2c^7}.$

27. 解: $\sqrt[4]{\sqrt[3]{256}} = \sqrt[12]{256} = \sqrt[12]{4^4} = \sqrt[3]{4}.$

28. 解: $\sqrt{2}\sqrt{2} = \sqrt{\sqrt{8}} = \sqrt[4]{8}.$

29. 解: $\sqrt{2}\sqrt[3]{2} = \sqrt{\sqrt[3]{16}} = \sqrt[6]{16} = \sqrt[3]{4}.$

30. 解: $\sqrt{\sqrt{2} \cdot \sqrt[3]{2}} = \sqrt{\sqrt[6]{8} \cdot \sqrt[6]{4}} = \sqrt{\sqrt[6]{32}} = \sqrt[12]{32}.$

31. 解: $\sqrt[m]{\sqrt[n]{a^m}} = \sqrt[2mn]{a^m} = \sqrt[n]{a}.$

32. 解: $(\sqrt[m]{\sqrt[n]{a}})^{mnp} = \sqrt[2mn]{a^{mnp}} = \sqrt[n]{a^p}.$

33. 解: $\sqrt{12} + \sqrt{75} - \sqrt{48} + \sqrt{147}$
 $= 2\sqrt{3} + 5\sqrt{3} - 4\sqrt{3} + 7\sqrt{3} = 10\sqrt{3}.$

34. 解: $\sqrt{125} + \sqrt{175} - \sqrt{28} + \sqrt{1/20}$
 $= 5\sqrt{5} + 5\sqrt{7} - 2\sqrt{7} + \frac{1}{10}\sqrt{5}$
 $= \frac{51}{10}\sqrt{5} + 3\sqrt{7}.$

35. 解: $\sqrt[3]{500} - \sqrt[3]{108} + \sqrt[3]{1/2} = 5\sqrt[3]{4} - 3\sqrt[3]{4} + \frac{1}{2}\sqrt[3]{4}$
 $= \frac{5}{2}\sqrt[3]{4}.$

36. 解: $\sqrt{a/bc} + \sqrt{b/ca} + \sqrt{c/ab}$
 $= \sqrt{\frac{a^3bc}{a^2b^2c^2}} + \sqrt{\frac{ab^3c}{a^2b^2c^2}} + \sqrt{\frac{abc^3}{a^2b^2c^2}}$
 $= \frac{a}{abc}\sqrt{abc} + \frac{b}{abc}\sqrt{abc} + \frac{c}{abc}\sqrt{abc}$
 $= \frac{a+b+c}{abc}\sqrt{abc}.$

37. 解: $\sqrt{50} - \sqrt{4\frac{1}{2}} + \sqrt[3]{-24} + \sqrt[3]{7\frac{1}{2}}$

$$= 5\sqrt{2} - 3/2\sqrt{2} - 2\sqrt[3]{3} + 4/3\sqrt[3]{3} = 7/2\sqrt{2} - 2/3\sqrt[3]{3}.$$

33. 解: $\sqrt{(a+b)^2c} - \sqrt{a^2c} - \sqrt{b^2c}$
 $= (a+b)\sqrt{c} - a\sqrt{c} - b\sqrt{c} = 0.$

39. 解: $\sqrt{ax^3+6ax^2+9ax} - \sqrt{ax^3-4a^2x^2+4a^2x}$
 $= \sqrt{ax(x^2+6x+9)} - \sqrt{ax(x^2-4ax+4a^2)}$
 $= \sqrt{ax(x+3)^2} - \sqrt{ax(x-2a)^2}$
 $= (x+3)\sqrt{ax} - (x-2a)\sqrt{ax}$
 $= (3+2a)\sqrt{ax}.$

40. 解: $(x+y)\sqrt{\frac{x-y}{x+y}} - (x-y)\sqrt{\frac{x+y}{x-y}} + \sqrt{\frac{1}{x^2-y^2}}$
 $= (x+y)\sqrt{\frac{(x^2-y^2)}{(x+y)^2}} - (x-y)\sqrt{\frac{x^2-y^2}{(x-y)^2}} + \sqrt{\frac{x^2-y^2}{(x^2-y^2)^2}}$
 $= \frac{1}{x^2-y^2}\sqrt{x^2-y^2}.$

41. 解: $(\sqrt{2} + \sqrt{3} + \sqrt{6}) \cdot \sqrt{6} = 2\sqrt{3} + 3\sqrt{2} + 6.$

42. 解: $(\sqrt{6} + \sqrt{10} + \sqrt{14}) \div \sqrt{2} = \sqrt{3} + \sqrt{5} + \sqrt{7}.$

43. 解: $(\sqrt{6} + \sqrt{5})(\sqrt{2} + \sqrt{15}) = 2\sqrt{3} + 3\sqrt{10} + \sqrt{10} + 5\sqrt{3}$
 $= 7\sqrt{3} + 4\sqrt{10}.$

44. 解: $\sqrt{5+2\sqrt{2}} \cdot \sqrt{5-2\sqrt{2}} = \sqrt{5^2 - (2\sqrt{2})^2} = \sqrt{17}.$

45. 解: $(1 + \sqrt{3})^3 = 1 + 3\sqrt{3} + 9 + 3\sqrt{3} = 10 + 6\sqrt{3}.$

46. 解: $(\sqrt{a} + \sqrt[4]{a} + 1)(\sqrt{a} - \sqrt[4]{a} + 1)$
 $= (\sqrt{a} + 1 + \sqrt[4]{a})(\sqrt{a} + 1 - \sqrt[4]{a})$
 $= (\sqrt{a} + 1)^2 - (\sqrt[4]{a})^2 = a + 2\sqrt{a} + 1 - \sqrt{a} = a + \sqrt{a} + 1.$

習 題 XXXV

原本第 282 頁

1. 解: $\sqrt[12]{a^8} = a^{\frac{8}{12}} = a^{\frac{2}{3}}.$

2. 解: $\sqrt{c^{\frac{4}{3}}} = c^{\frac{4}{3} \times \frac{1}{2}} = c^{\frac{2}{3}}.$

3. 解: $\frac{a^{\frac{2}{5}}}{\sqrt[3]{a^{\frac{6}{5}}}} = \frac{a^{\frac{2}{5}}}{a^{\frac{2}{5}}} = a^{\frac{2}{5} - \frac{2}{5}} = a^0 = 1.$

4. 解: $b^{\sqrt[3]{b^4}} \cdot \sqrt[5]{b^5} = b \cdot b^{\frac{4}{3}} \cdot b^{\frac{5}{5}} = b^{1+\frac{4}{3}+\frac{5}{5}} = b^{\frac{19}{3}}$.
5. 解: $a^{\frac{14}{21}} = a^{\frac{2}{3}} = \sqrt[3]{a^2}$.
6. 解: $c^{-1.5} = c^{-\frac{3}{2}} = 1/c^{\frac{3}{2}} = \frac{1}{c\sqrt{c}} = \frac{1}{c^2} \sqrt{c}$.
7. 解: $(d^{\frac{2}{3}})^{-6} = \frac{1}{(d^{\frac{2}{3}})^6} = \frac{1}{d^4}$.
8. 解: $(e^{-3\frac{1}{2}})^{-\frac{1}{7}} = (e^{-\frac{7}{2}})^{-\frac{1}{7}} = e^{\frac{1}{2}} = \sqrt{e}$.
9. 解: $\frac{a^{-1}}{b^{-3}c^{-2}} = \frac{\frac{1}{a}}{\frac{1}{b^3} \cdot \frac{1}{c^2}} = \frac{b^3c^2}{a}$.
10. 解: $x^{-\frac{1}{2}} \sqrt{y^{-3}} = \frac{1}{x^{\frac{1}{2}} y^{\frac{3}{2}}}$.
11. 解: $\left(\frac{1}{\sqrt{x^{-5}}}\right)^{-4} = \left(\frac{1}{x^{-\frac{5}{2}}}\right)^{-4} = \frac{1}{x^{10}}$.
12. 解: $\frac{x^{-2} \sqrt{y^{-3}}}{y^{-2} \sqrt{x^{-3}}} = \frac{x^{-2} \cdot y^{-\frac{3}{2}}}{y^{-2} \cdot x^{-\frac{3}{2}}} = \frac{y^2 x^{\frac{3}{2}}}{x^2 y^{\frac{3}{2}}} = \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$.
13. 解: $\frac{a}{bc} - \frac{b^{-1}}{c^{-2}} - \frac{a^{-1}(b^{-1}+c^{-1})}{a^{-2}(b+c)} + \frac{b+c}{b^{-1}+c^{-1}} = \frac{a}{bc} - \frac{c^2}{b} - \frac{a}{bc} + bc$
 $= \frac{-c^3 + b^2c^2}{bc} = \frac{-c^2 + b^2c}{b} = (-c^2 + b^2c)b^{-1} = bc - c^2b^{-1}$.
14. 解: $\left(3\frac{1}{8}\right)^{\frac{3}{2}} = \left(\frac{25}{8}\right)^{\frac{3}{2}} = \left(\frac{5^2}{2^3}\right)^{\frac{3}{2}} = \frac{5^3}{16\sqrt{2}} = \frac{125 \cdot 2^{\frac{1}{2}}}{32}$.
15. 解: $(81)^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^3 = 27$.
16. 解: $(-27)^{\frac{2}{3}} = [(-3)^3]^{\frac{2}{3}} = (-3)^2 = 9$.
17. 解: $8^{-\frac{5}{2}} = \frac{1}{8^{\frac{5}{2}}} = \frac{1}{(2^3)^{\frac{5}{2}}} = \frac{1}{2^{\frac{15}{2}}} = \frac{1}{\sqrt{2^{15}}} = \frac{\sqrt{2}}{\sqrt{2^{16}}} = 2^{\frac{1}{2}}/2^8 = 2^{\frac{1}{2}}/256$

18. 解: $a^{\frac{2}{3}}a^{\frac{3}{4}}a^{\frac{5}{6}} = a^{\frac{2}{3} + \frac{3}{4} + \frac{5}{6}} = a^{\frac{9}{4}}$.

19. 解: $a^{\frac{2}{5}}a^{-\frac{1}{3}}a^{-\frac{1}{15}} = a^{\frac{2}{5} - \frac{1}{3} - \frac{1}{15}} = a^0 = 1$.

20. 解: $\frac{ab^{-2}}{a^{-3}b} = \frac{a}{b^2} \cdot \frac{a^3}{b} = \frac{a^4}{b^3} = a^4b^{-3}$.

21. 解: $(a^{\frac{3}{2}}b)^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{3}{4}} = a^{\frac{3}{4}}b^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{3}{4}} = a^{\frac{3}{4} + \frac{1}{2}}b^{\frac{1}{2} + \frac{3}{4}} = (ab)^{\frac{5}{4}}$.

22. 解: $(a^{\frac{3}{4}})^{\frac{2}{9}} = a^{\frac{6}{36}} = a^{\frac{1}{6}}$.

23. 解: $(a^{-1}b^{-2}c^3)^{-2} = a^2b^4c^{-6}$.

24. 解: $(-32a^{10})^{\frac{3}{5}} = (-2^5a^{10})^{\frac{3}{5}} = -2^3a^6 = -8a^6$.

25. 解: $(-a^6b^{-9})^{-\frac{2}{3}} = +a^{-4}b^6 = a^{-4}b^6$.

26. 解: $b^{-\frac{1}{3} \cdot 12} \sqrt{b^{-5}} \div b^{-1} \sqrt{b^{-1}} = b^{-\frac{1}{3}} \cdot b^{-\frac{5}{2}} \div b^{-1} \cdot b^{-\frac{1}{2}}$
 $= b^{-\frac{3}{4}} \div b^{-\frac{3}{2}} = b^{\frac{3}{4}}$.

27. 解: $(a^{-\frac{3}{2}} \sqrt{bc^5})^{\frac{2}{3}} = a^{-1}b^{\frac{1}{3}}c^{\frac{5}{3}}$.

28. 解: $\left(\frac{8a^{-15}}{\sqrt{125a^3}}\right)^{-\frac{2}{3}} = \left(\frac{2^3a^{-15}}{5^{\frac{3}{2}}a^{\frac{3}{2}}}\right)^{-\frac{2}{3}} = \frac{2^{-2}a^{10}}{5^{-1}a^{-1}} = 2^{-2} \cdot 5a^{11} = \frac{5}{4}a^{11}$.

29. 解: $\sqrt{a^{\frac{2}{3}}(bc^{-1})^{-2}} = a^{\frac{2}{6}}(bc^{-1})^{-\frac{2}{3}} = a^{\frac{1}{3}}bc$.

30. 解: $\sqrt[3]{a^{-1}\sqrt{a^3}} = a^{-\frac{1}{3}} \cdot a^{\frac{3}{12}} = a^{-\frac{1}{3}} \cdot a^{\frac{1}{4}} = a^{-\frac{1}{12}}$.

31. 解: $\frac{\sqrt[3]{a^{\frac{9}{2}}\sqrt{a^{-8}}}}{\sqrt{\sqrt[3]{a^{-7}} \cdot \sqrt[3]{a}}} = \frac{a^{\frac{3}{2}} \cdot a^{-\frac{1}{2}}}{a^{-\frac{7}{6}} \cdot a^{\frac{1}{6}}} = \frac{a}{a^{-1}} = a^2$.

32. 解: $[(x^x)^x]^x = [x^{x^2}]^x = x^{x^3}$.

33. 解: $(x^{x^2+xy}y^{y^2+xy})^{\frac{xy}{x+y}} = x^{x^2y} + y^{xy^2}$.

34. 解: $\frac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x^{-\frac{1}{4}} + y^{-\frac{1}{4}}} = \frac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{y^{\frac{1}{4}} + x^{\frac{1}{4}}} = (x^{\frac{1}{2}} - y^{\frac{1}{2}}) \left(\frac{x^{\frac{1}{4}}y^{\frac{1}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} \right)$

$$= (x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}}) \left(\frac{x^{\frac{1}{4}}y^{\frac{1}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} \right) = x^{\frac{1}{2}}y^{\frac{1}{4}} - x^{\frac{1}{4}}y^{\frac{1}{2}}$$

$$35. \text{ 解: } (x^{\frac{1}{4}} + x^{\frac{1}{3}}y^{\frac{1}{8}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - x^{\frac{1}{8}}y^{\frac{1}{8}} + y^{\frac{1}{4}}) = (x^{\frac{1}{4}} + y^{\frac{1}{4}})^2 - (x^{\frac{1}{8}}y^{\frac{1}{8}})^2 \\ = (x^{\frac{1}{4}})^2 + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + (y^{\frac{1}{4}})^2 - x^{\frac{1}{4}}y^{\frac{1}{4}} = x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}.$$

$$36. \text{ 解: } \begin{array}{r} a^{\frac{1}{3}} - b^{\frac{1}{2}} \\ \hline a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{2}} + ab + a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{1}{3}}b^2 + b^{\frac{5}{2}} \\ a^2 - b^3 \\ \hline a^2 - a^{\frac{5}{3}}b^{\frac{1}{2}} \\ \hline a^{\frac{5}{3}}b^{\frac{1}{2}} \\ \hline a^{\frac{5}{3}}b^{\frac{1}{2}} - a^{\frac{4}{3}}b \\ \hline a^{\frac{4}{3}}b \\ \hline a^{\frac{4}{3}}b - ab^{\frac{3}{2}} \\ \hline ab^{\frac{3}{2}} \\ \hline ab^{\frac{3}{2}} - a^{\frac{2}{3}}b^2 \\ \hline a^{\frac{2}{3}}b^2 \\ \hline a^{\frac{2}{3}}b^2 - a^{\frac{1}{3}}b^{\frac{5}{2}} \\ \hline a^{\frac{1}{3}}b^{\frac{5}{2}} - b^3 \\ \hline a^{\frac{1}{3}}b^{\frac{5}{2}} - b^3 \\ \hline 0 \end{array}$$

$$\text{答: } Q = a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{2}} + ab + a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{1}{3}}b^2 + b^{\frac{5}{2}}.$$

$$37. \text{ 解: } (x^{\frac{1}{2}} - y^{\frac{1}{4}}z^{\frac{1}{4}})^4 = (x^{\frac{1}{2}})^4 - 4(x^{\frac{1}{2}})^3(y^{\frac{1}{4}}z^{\frac{1}{4}}) + 6(x^{\frac{1}{2}})^2(y^{\frac{1}{4}}z^{\frac{1}{4}})^2 \\ - 4(x^{\frac{1}{2}})(y^{\frac{1}{4}}z^{\frac{1}{4}})^3 + (y^{\frac{1}{4}}z^{\frac{1}{4}})^4 \\ = x^2 - 4x^{\frac{3}{2}}y^{\frac{1}{4}}z^{\frac{1}{4}} + 6xy^{\frac{1}{2}}z^{\frac{1}{2}} - 4x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}} + yz.$$

$$38. \text{ 解: } [(e^x + e^{-x})^2 - 4]^{\frac{1}{2}} = [e^{2x} + 2 + e^{-2x} - 4]^{\frac{1}{2}} \\ = [e^{2x} - 2 + e^{-2x}]^{\frac{1}{2}} = [(e^x - e^{-x})^2]^{\frac{1}{2}} = e^x - e^{-x}.$$

$$39. \text{ 解: } \begin{array}{r} x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + 3x^{-\frac{1}{2}}y^{\frac{3}{2}} \\ \hline x^2 + 4x^{\frac{3}{2}}y^{\frac{1}{2}} + 4xy + 6x^{\frac{1}{2}}y^{\frac{3}{2}} + 12y^2 + 9x^{-1}y^3 \end{array}$$

$$\begin{array}{r|l}
 x^2 & \\
 \hline
 2x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} & 4x^{\frac{3}{2}}y^{\frac{1}{2}} + 4xy \\
 & 4x^{\frac{3}{2}}y^{\frac{1}{2}} + 4xy \\
 \hline
 2x + 4x^{\frac{1}{2}}y^{\frac{1}{2}} + 3x^{\frac{1}{2}}y^{\frac{3}{2}} & 6x^{\frac{1}{2}}y^{\frac{3}{2}} + 12y^2 + 9x^{-1}y^3 \\
 & 6x^{\frac{1}{2}}y^{\frac{3}{2}} + 12y^2 + 9x^{-1}y^3 \\
 \hline
 & 0
 \end{array}$$

答：其平方根爲 $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + 3x^{-\frac{1}{2}}y^{\frac{3}{2}}$ 。

40. 解： $x^3 + 3x^2 + 6x + 7 + 6x^{-1} + 3x^{-2} + x^{-3} = (x + p + qx^{-1})^3$
 $= x^3 + 3px^2 + \dots + qx^{-3}$ 。

$$3p = 6 \quad \therefore p = 1, \quad q = 1.$$

答：其立方根爲 $x + 1 + x^{-1}$ 。

習 題 XXXVI

原 本 第 284 頁

1. 解： $(1+x)^{\frac{1}{3}} = 1^{\frac{1}{3}} + \frac{1}{3}x + \frac{(\frac{1}{3})(-\frac{2}{3})}{1 \cdot 2}x^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{1 \cdot 2 \cdot 3}x^3 + \dots$

$$= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} + \dots$$

2. 解： $(a^{\frac{2}{3}} + x^{-\frac{2}{3}})^{-\frac{1}{2}} = (a^{\frac{2}{3}})^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(a^{\frac{2}{3}})^{-\frac{3}{2}}(x^{-\frac{2}{3}})$

$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2}(a^{\frac{2}{3}})^{-\frac{5}{2}}(x^{-\frac{2}{3}})^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1 \cdot 2 \cdot 3}(a^{\frac{2}{3}})^{-\frac{7}{2}}(x^{-\frac{2}{3}})^3$$

$$+ \dots$$

$$= a^{-\frac{1}{3}} - \frac{1}{2}a^{-1}x^{-\frac{2}{3}} + \frac{3}{8}a^{-\frac{5}{3}}x^{-\frac{4}{3}} - \frac{5}{16}a^{-\frac{7}{3}}x^{-2} + \dots$$

3. 解： $\sqrt[3]{(27-2x)^2} = (27-2x)^{\frac{2}{3}} = 27^{\frac{2}{3}} - \frac{2}{3} \cdot 27^{-\frac{1}{3}}(2x)$

$$+ \frac{\frac{2}{3}(-\frac{1}{3})}{1 \cdot 2}(27)^{-\frac{4}{3}}(2x)^2 - \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{1 \cdot 2 \cdot 3}(27)^{-\frac{7}{3}}(2x)^3 + \dots$$

$$= 9 - \frac{4}{3^2}x - \frac{4}{3^6}x^2 - \frac{32}{3^{11}}x^3 + \dots$$

$$\begin{aligned}
 4. \text{ 解: } (a^m + x)^{\frac{1}{m}} &= (a^m)^{\frac{1}{m}} + \frac{1}{m}(a^m)^{\frac{1}{m}-1}x \\
 &+ \frac{\frac{1}{m}\left(\frac{1-m}{m}\right)}{1 \cdot 2}(a^m)^{\frac{1}{m}-2}x^2 + \frac{\frac{1}{m}\left[\frac{1-m}{m}\right]\left[\frac{1-2m}{m}\right]}{1 \cdot 2 \cdot 3}(a^m)^{\frac{1}{m}-3}x^3 \\
 &+ \dots \\
 &= a + \frac{1}{m}a^{1-m}x + \frac{1-m}{2m^2}a^{1-2m}x^2 + \frac{(1-m)(1-2m)}{6m^3} \\
 &+ a^{1-3m}x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ 解: } (a^{-1} - b^{-\frac{1}{2}})^{-4} &= a^4 + 4a^5b^{-\frac{1}{2}} + \frac{(-4)(-5)}{2}a^6b^{-1} \\
 &- \frac{(-4)(-5)(-6)}{2 \cdot 3}a^7b^{-\frac{3}{2}} \\
 &= a^4 + 4a^5b^{-\frac{1}{2}} + 10a^6b^{-1} + 20a^7b^{-\frac{3}{2}} + \dots
 \end{aligned}$$

$$\begin{aligned}
 6. \text{ 解: } (\sqrt{x} + \sqrt[3]{y})^{-6} &= (x^{\frac{1}{2}} + y^{\frac{1}{3}})^{-6} = x^{-3} - 6x^{-\frac{7}{2}}y^{\frac{1}{3}} \\
 &+ \frac{(-6)(-7)}{2}x^{-4}(y^{\frac{1}{3}})^2 + \frac{(-6)(-7)(-8)}{2 \cdot 3}x^{-\frac{9}{2}}(y^{\frac{1}{3}})^3 \\
 &+ \dots \\
 &= x^{-3} - 6x^{-\frac{7}{2}}y^{\frac{1}{3}} + 21x^{-4}y^{\frac{2}{3}} - 56x^{-\frac{9}{2}}y + \dots
 \end{aligned}$$

$$\begin{aligned}
 7. \text{ 解: } \frac{1}{2+3x} &= (2+3x)^{-1} = 2^{-1} - 2^{-2}3x + \frac{(-1)(-2)}{2} \cdot 2^{-3}(3x)^2 \\
 &+ \frac{(-1)(-2)(-3)}{2 \cdot 3}2^{-4}(3x)^3 + \dots \\
 &= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 8. \text{ 解: } \frac{1}{\sqrt{1+x^2}} &= (1+x^2)^{-\frac{1}{2}} = 1 - \frac{2}{5}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}x^2 \\
 &+ \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{2 \cdot 3}x^3 + \dots \\
 &= 1 - \frac{2}{5}x + \frac{3}{25}x^2 - \frac{2}{125}x^3 + \dots
 \end{aligned}$$

$$9. \text{ 解: } \left(\frac{1}{\sqrt{1+3\sqrt{x}}} \right)^3 = (1+3\sqrt{x})^{-\frac{3}{2}}$$

$$= 1 - \frac{3}{2} 3x^{\frac{1}{2}} + \frac{(-\frac{3}{2})(-\frac{5}{2})}{2} (3x^{\frac{1}{2}})^2 + \frac{(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{2 \cdot 3} (3x^{\frac{1}{2}})^3 + \dots$$

$$= 1 - \frac{9}{2} x^{\frac{1}{2}} + \frac{135}{8} x - \frac{945}{16} x^{\frac{3}{2}} + \dots$$

10. 解: $n = -3, a = 1, b = x, r = 9.$

$$\frac{(-3)(-4)(-5)(-6)(-7)(-8)(-9)(-10)(-11)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} x^9$$

$$= -55x^9.$$

11. 解: $n = \frac{3}{4}, a = x^{-2}, b = 2y^{\frac{1}{3}}, r = 6.$

$$\frac{\frac{3}{4}(-\frac{1}{4})(-\frac{5}{4})(-\frac{9}{4})(-\frac{13}{4})(-\frac{17}{4})}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} a^{-\frac{21}{2}} (2y^{\frac{1}{3}})^6 = -\frac{663}{1024} x^{-\frac{21}{2}} y^2.$$

12. 解: $x^{\frac{9}{2}} = (-x^{\frac{1}{2}})^n = -x^{\frac{n}{2}} \quad \therefore n = 9.$

$$\frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})(-\frac{11}{4})(-\frac{15}{4})(-\frac{19}{4})(-\frac{23}{4})(-\frac{27}{4})(-\frac{31}{4})}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} x^{\frac{9}{2}}$$

$$= -\frac{19 \cdot 23 \cdot 31 \cdot 33}{2^{25}} x^{\frac{9}{2}} = -\frac{507051}{31554432} x^{\frac{9}{2}}.$$

13. 在 $x^{-\frac{3}{2}}(2+x^{-\frac{1}{6}})^{-3}$ 展開式中, x^{-2} 項即在 $(2+x^{-\frac{1}{6}})^{-3}$ 展開式中 $x^{-\frac{1}{2}}$ 項. 設 t_r 為 $(2+x^{-\frac{1}{6}})^{-3}$ 展開式中第 $r+1$ 項, 則

$$t_r = \frac{-3(-3-1)\dots(-3-r+1)}{\underbrace{1 \cdot 2 \cdot 3 \dots r}} 2^{-3-r} (x^{-\frac{1}{6}})^r$$

$$= \frac{(-3)(-4)\dots(-3-r+1)}{\underbrace{1 \cdot 2 \cdot 3 \dots r}} 2^{-3-r} x^{-\frac{r}{6}}$$

設 $-\frac{r}{6} = -\frac{1}{2} \quad r = 3.$

$$t_3 = \frac{(-3)(-4)(-5)}{1 \cdot 2 \cdot 3} 2^{-6} x^{-\frac{1}{2}} = -\frac{5}{2^5} x^{-\frac{1}{2}}$$

答: 所求之項為 $-\frac{5}{2^5} x^{-2}.$

14. 解: 1. $\sqrt{99} = 3(9+2)^{\frac{1}{2}} = 3 \left[3 + \frac{1}{2} 9^{-\frac{1}{2}} \cdot 2 + \frac{1}{2} \left(-\frac{1}{2}\right) 9^{-\frac{3}{2}} \cdot 2^2 + \frac{1}{2} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \cdot 9^{-\frac{5}{2}} \cdot 2^3 \right] = 9 + 1 - \frac{4}{216} + \frac{8}{2888} = 9.9498.$

$$2. \sqrt[3]{62} = (64-2)^{\frac{1}{3}} = 4 - \frac{1}{3} \cdot 64^{-\frac{2}{3}} \cdot 2 + \frac{1}{3} \left(-\frac{2}{3}\right) \cdot 64^{-\frac{5}{3}} \cdot 2^2 \\ - \frac{1}{3} \left(-\frac{2}{3}\right) \left(-\frac{5}{3}\right) \cdot 64^{-\frac{8}{3}} \cdot 2^3 = 4 - \frac{2}{48} + \frac{1}{2304} - \frac{4}{9216} = 3.9578.$$

$$3. \sqrt[5]{31} = (32-1)^{\frac{1}{5}} = 2 - \frac{1}{5} \cdot 32^{-\frac{4}{5}} + \frac{1}{5} \left(-\frac{4}{5}\right) \cdot 32^{-\frac{9}{5}} \\ + \frac{1}{5} \left(-\frac{4}{5}\right) \left(-\frac{9}{5}\right) \cdot 32^{-\frac{14}{5}} = 2 - \frac{1}{80} - \frac{1}{64075} = 1.0873.$$

習題 XXXVII

原本第 287 頁

1. 解: $\sqrt[7]{a^5}$.

$$\therefore a^{\frac{5}{7}} a^{\frac{2}{7}} = a.$$

答: $\sqrt[7]{a^5}$ 之理化因子爲 $a^{\frac{2}{7}}$.

2. 解: $\sqrt[3]{a^2} \sqrt{b^3}$.

$$\therefore (a^{\frac{2}{3}} b^{\frac{3}{2}}) (a^{\frac{1}{3}} b^{\frac{1}{2}}) = ab^2.$$

答: $\sqrt[3]{a^2} \sqrt{b^3}$ 之理化因子爲 $a^{\frac{1}{3}} b^{\frac{1}{2}}$.

3. 解: $x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{7}{2}}$.

$$\therefore (x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{7}{2}}) x^{\frac{1}{2}} = x^2 + x^3 + x^4.$$

答: $x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{7}{2}}$ 之理化因子爲 $x^{\frac{1}{2}}$.

4. 解: $\sqrt{a} + \sqrt{bc}$.

$$(a^{\frac{1}{2}} + bc^{\frac{1}{2}}) (a^{\frac{1}{2}} - bc^{\frac{1}{2}}) = a - bc.$$

答: $\sqrt{a} + \sqrt{bc}$ 之理化因子爲 $\sqrt{a} - \sqrt{bc}$.

5. 解: $\sqrt{x} + \sqrt{y} + \sqrt{z}$.

$$(\sqrt{x} + \sqrt{y} + \sqrt{z}) (\sqrt{x} + \sqrt{y} - \sqrt{z}) \\ = x + y - z + 2\sqrt{xy}(x + y - z + 2\sqrt{xy}). \\ (\sqrt{x} + \sqrt{y} + \sqrt{z}) (\sqrt{x} + \sqrt{y} - \sqrt{z})$$

$$= (x+y-z)^2 - 4xy.$$

答: $\sqrt{x} + \sqrt{y} + \sqrt{z}$ 之理化因子爲 $(\sqrt{x} + \sqrt{y} - \sqrt{z})(x+y-z-2\sqrt{xy})$.

6. 解: $\sqrt{xy} + \sqrt{yz} + \sqrt{zx}$.

$$\begin{aligned} & (\sqrt{xy} + \sqrt{yz} + \sqrt{zx})(\sqrt{xy} + \sqrt{yz} - \sqrt{zx}) \\ &= xy + yz - zx + 2y\sqrt{xz}(xy + yz - zx + 2y\sqrt{xy}) \\ & \quad (xy + yz - zx + 2y\sqrt{xz})(xy + yz - zx - 2y\sqrt{xz}) \\ &= (xy + yz - zx)^2 - 4y^2xz. \end{aligned}$$

答: $\sqrt{xy} + \sqrt{yz} + \sqrt{zx}$ 之理化因子爲 $(\sqrt{xy} + \sqrt{yz} - \sqrt{zx})(xy + yz - zx - 2y\sqrt{xz})$.

7. 解: $\sqrt{x} + \sqrt{y} - \sqrt{z} - \sqrt{u}$.

$$\begin{aligned} & [(\sqrt{x} + \sqrt{y}) - (\sqrt{z} + \sqrt{u})][(\sqrt{x} + \sqrt{y}) \\ & \quad + (\sqrt{z} + \sqrt{u})] = [(\sqrt{x} + \sqrt{y})^2 - (\sqrt{z} + \sqrt{u})^2] \\ &= x + y - z - u + 2(\sqrt{xy} - \sqrt{zu}). \\ & [x + y - z - u + 2(\sqrt{xy} - \sqrt{zu})][x + y - z - u \\ & \quad - 2(\sqrt{xy} - \sqrt{zu})] = (x + y - z - u)^2 - 4(\sqrt{xy} - \sqrt{zu})^2 \\ &= x^2 + y^2 + z^2 + u^2 - 2xy - 2xz - 2xu - 2yz - 2yu - 2zu \\ & \quad + 8\sqrt{xyzu}. \\ & [x^2 + y^2 + z^2 + u^2 - 2xy - 2xz - 2xu - 2yz - 2yu - 2zu \\ & \quad + 8 + \sqrt{xyzu}][x^2 + y^2 + z^2 + u^2 - 2xy - 2xz - 2xu - 2yz \\ & \quad - 2yu - 2zu - 8\sqrt{xyzu}] \\ &= [x^2 + y^2 + z^2 + u^2 - 2xy - 2xz - 2xu - 2yz - 2yu - 2zu]^2 \\ & \quad - 64xyzu. \end{aligned}$$

答: $\sqrt{x} + \sqrt{y} - \sqrt{z} - \sqrt{u}$ 之理化因子爲 $(\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{u})[x + y - z - u - 2(\sqrt{xy} - \sqrt{zu})][x^2 + y^2 + z^2 + u^2 - 2xy - 2xz - 2xu - 2yz - 2yu - 2zu - 8\sqrt{xyzu}]$.

8. 解: $(\sqrt{x} + 1 + \sqrt[4]{x})(\sqrt{x} + 1 - \sqrt[4]{x})$

$$= x + 1 + 2\sqrt{x} - \sqrt{x} = x + 1 + \sqrt{x}.$$

$$(x + 1 + \sqrt{x})(x + 1 - \sqrt{x}) = (x + 1)^2 - x = x^2 + x + 1.$$

答: $\sqrt{x} + \sqrt[4]{x+1}$ 之理化因子爲 $(\sqrt{x} - \sqrt[4]{x+1}) \times (x - \sqrt{x+1})$.

9. 解: $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) = x + y$.

答: $x^{\frac{1}{3}} + y^{\frac{1}{3}}$ 之理化因子爲 $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

10. 解: $(a^{\frac{1}{3}} - b^{\frac{2}{3}}) = [a^{\frac{1}{3}} - (b^2)^{\frac{1}{3}}]$.

$$[a^{\frac{1}{3}} - (b^2)^{\frac{1}{3}}][a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}] = a - b^2$$

答: $\sqrt[3]{a} - \sqrt[3]{b^2}$ 之理化因子爲 $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

11. 解: $(x^{\frac{1}{4}} - y^{\frac{1}{4}})(x^{\frac{1}{4}} + y^{\frac{1}{4}}) = x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

$$(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x^{\frac{1}{4}} + y^{\frac{1}{4}}) = x - y$$

答: $x^{\frac{1}{4}} - y^{\frac{1}{4}}$ 之理化因子爲 $(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})$.

12. 解: $x^{\frac{3}{4}} + y^{\frac{2}{3}} = (x^9)^{\frac{1}{12}} + (y^8)^{\frac{1}{12}}$.

$$[(x^9)^{\frac{1}{12}} + (y^8)^{\frac{1}{12}}][(x^9)^{\frac{11}{12}} - (x^9)^{\frac{10}{12}}(x^8)^{\frac{1}{12}} + (x^9)^{\frac{9}{12}}(y^8)^{\frac{9}{12}} - \dots - (y^8)^{\frac{11}{12}}] = x^9 - y^8$$

答: $x^{\frac{3}{4}} + y^{\frac{2}{3}}$ 之理化因子爲 $[(x^9)^{\frac{11}{12}} - (x^9)^{\frac{10}{12}}(y^8)^{\frac{1}{12}} + (x^9)^{\frac{9}{12}}(y^8)^{\frac{2}{12}} - \dots - (y^8)^{\frac{11}{12}}]$.

13. 解: $(1 + x^{\frac{1}{2}}y^{\frac{1}{3}})(1 - x^{\frac{1}{2}}y^{\frac{1}{3}}) = 1 - xy^{\frac{2}{3}}$.

$$(1 - xy^{\frac{2}{3}})(1 + xy^{\frac{2}{3}} + x^2y^{\frac{4}{3}}) = 1 - x^3y^2$$

答: $1 + x^{\frac{1}{2}}y^{\frac{1}{3}}$ 之理化因子爲 $(1 - x^{\frac{1}{2}}y^{\frac{1}{3}})(1 + xy^{\frac{2}{3}} + x^2y^{\frac{4}{3}})$.

14. 解: $(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{1}{3}} - 1) = x - 1$.

答: $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ 之理化因子爲 $x^{\frac{1}{3}} - 1$.

15. 解: $(3 - \sqrt{5})(3 + \sqrt{5}) = 9 - 5 = 4$.

答: $3 - \sqrt{5}$ 之理化因子爲 $3 + \sqrt{5}$.

16. 解: $(1 + \sqrt{2} + \sqrt{3})(1 + \sqrt{2} - \sqrt{3}) = 1 + 2 - 3 + 2\sqrt{2} = 2\sqrt{2}$. $2\sqrt{2} \cdot \sqrt{2} = 4$.

答: $1 + \sqrt{2} + \sqrt{3}$ 之理化因子爲 $(1 + \sqrt{2} - \sqrt{3})\sqrt{2}$.

17. 解: $(1+2^{\frac{1}{3}})(1-2^{\frac{1}{3}}+2^{\frac{2}{3}})=1-2=-1$.

答: $1+\sqrt[3]{2}$ 之理化因子爲 $1-2^{\frac{1}{3}}+2^{\frac{2}{3}}$.

18. 解: $(\sqrt[3]{9}+\sqrt[3]{3}+1)(1-\sqrt[3]{3})=1-3$.

答: $\sqrt[3]{9}+\sqrt[3]{3}+1$ 之理化因子爲 $1-\sqrt[3]{3}$.

19. 解: $(\sqrt[3]{12}+\sqrt[3]{6}+\sqrt[3]{3})\sqrt[3]{3^2}=3\sqrt[3]{2^2}+3\sqrt[3]{2}+3$.

$(3\sqrt[3]{2^2}+3\sqrt[3]{2}+3)(1-\sqrt[3]{2})=3$.

答: $\sqrt[3]{12}+\sqrt[3]{6}+\sqrt[3]{3}$ 之理化因子爲 $\sqrt[3]{3^2}(1-\sqrt[3]{2})$.

20. 解: $\frac{1}{\sqrt{a}\sqrt[3]{b^2}} = \frac{\sqrt{a}\sqrt[5]{b^3}}{ab}$.

21. 解: $\frac{a+\sqrt{b}}{a-\sqrt{b}} = \frac{a^2+2a\sqrt{b}+b}{a^2-b}$.

22. 解: $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}+3\sqrt{3}} = \frac{\sqrt{3}-\sqrt{2}}{5\sqrt{3}} = \frac{3-\sqrt{6}}{15}$.

23. 解: $\frac{1}{b+\sqrt{b^2-a^2}} = \frac{b-\sqrt{b^2-a^2}}{a^2}$.

24. 解: $\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}} = \frac{2x+2\sqrt{x^2-y^2}}{2y} = \frac{x+\sqrt{x^2-y^2}}{y}$.

25. 解: $\frac{1+\sqrt{2}+\sqrt{3}}{1-\sqrt{2}+\sqrt{3}} = \frac{(1+\sqrt{2}+\sqrt{3})(1-\sqrt{2}-\sqrt{3})}{(1-\sqrt{2}+\sqrt{3})(1-\sqrt{2}-\sqrt{3})}$
 $= \frac{-4-2\sqrt{6}}{-2\sqrt{2}} = \frac{2+\sqrt{6}}{\sqrt{2}} = \frac{2\sqrt{2}+\sqrt{12}}{2} = \sqrt{2}+\sqrt{3}$.

26. 解: $\frac{1}{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}$
 $= \frac{(1+\sqrt{2})-(\sqrt{3}+\sqrt{6})}{(1+\sqrt{2})^2-(\sqrt{3}+\sqrt{6})^2}$
 $= \frac{1+\sqrt{2}-\sqrt{3}-\sqrt{6}}{-6-4\sqrt{2}}$
 $= \frac{(1+\sqrt{2}-\sqrt{3}-\sqrt{6})(6-4\sqrt{2})}{36-32}$

$$= \frac{2 - 2\sqrt{2} - 2\sqrt{3} + 2\sqrt{6}}{4}$$

$$= \frac{1 - \sqrt{2} - \sqrt{3} + \sqrt{6}}{2}$$

27. 解:
$$\frac{x\sqrt{y} + y\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{x+y}}$$

$$= \frac{(x\sqrt{y} + y\sqrt{x})(\sqrt{x} + \sqrt{y} - \sqrt{x+y})}{x+y-x-y+2\sqrt{xy}}$$

$$= \frac{(\sqrt{x} + \sqrt{y} - \sqrt{x+y})(x\sqrt{y} + y\sqrt{x})\sqrt{xy}}{2xy}$$

$$= \frac{(\sqrt{x} + \sqrt{y} - \sqrt{x+y})(xy\sqrt{x} + xy\sqrt{y})}{2xy}$$

$$= \frac{(\sqrt{x} + \sqrt{y} - \sqrt{x+y})(\sqrt{x} + \sqrt{y})}{2}$$

28. 解:
$$\frac{1}{\sqrt[3]{3}-1} + \frac{1}{\sqrt[3]{3}+1} = \frac{\sqrt[3]{3}+1 + \sqrt[3]{3}-1}{\sqrt[3]{3^2}-1}$$

$$= \frac{2\sqrt[3]{3}}{\sqrt[3]{3^2}-1} = \frac{2\sqrt[3]{3}(\sqrt[3]{3^4} + \sqrt[3]{3^2} + 1)}{(\sqrt[3]{3^2}-1)(\sqrt[3]{3^4} + \sqrt[3]{3^2} + 1)}$$

$$= \frac{2\sqrt[3]{3}(\sqrt[3]{3^4} + \sqrt[3]{3^2} + 1)}{8}$$

$$= \frac{\sqrt[3]{3}(\sqrt[3]{3^4} + \sqrt[3]{3^2} + 1)}{4}$$

29. 解:
$$\frac{5}{\sqrt{125}} = \frac{5}{5\sqrt{5}} = \frac{\sqrt{5}}{5} = 0.447.$$

30. 解:
$$\frac{2 + \sqrt{28}}{\sqrt{7}} = \frac{2\sqrt{7} + \sqrt{196}}{7} = \frac{2\sqrt{7} + 14}{7} = 2.756.$$

31. 解:
$$\frac{3 + \sqrt{6}}{\sqrt{2} + \sqrt{3}} = \frac{-\sqrt{3}}{2-3} = \sqrt{3} = 1.732.$$

習題 XXXVIII

原本第 290 頁

1. 解: $x^{\frac{1}{4}} = 4.$

$$\therefore x = 4^4 = 256.$$

2. 解: $x^{-\frac{1}{2}} = 3.$

$$\frac{1}{x^{\frac{1}{2}}} = 3 \quad \text{或} \quad \frac{1}{x} = 3^2 = 9.$$

$$\therefore x = \frac{1}{9}.$$

3. 解: $x^{\frac{2}{3}} = 8.$

$$\therefore x = 8^{\frac{3}{2}} = \sqrt[3]{(2^3)^2} = 4.$$

4. 解: $(\sqrt{2x-1})^{\frac{1}{3}} = \sqrt{3}.$

$$(2x-1)^{\frac{1}{3}} = 3.$$

$$2x-1 = 3^3 = 27.$$

$$\therefore x = 14.$$

5. 解: $\sqrt{2+\sqrt{3+\sqrt{x}}} = 2.$

$$[2+(3+x^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}} = 2.$$

$$2+3+x^{\frac{1}{2}} = 4.$$

$$(3+x^{\frac{1}{2}})^{\frac{1}{2}} = 2.$$

$$3+x^{\frac{1}{2}} = 4.$$

$$x^{\frac{1}{2}} = 1. \quad \therefore x = 1.$$

6. 解: $\sqrt{ax} + \sqrt{bx} + \sqrt{cx} = d.$

$$\sqrt{x}(\sqrt{a} + \sqrt{b} + \sqrt{c}) = d.$$

$$\sqrt{x} = \frac{d}{\sqrt{a} + \sqrt{b} + \sqrt{c}}.$$

$$\therefore x = \left[\frac{d}{\sqrt{a} + \sqrt{b} + \sqrt{c}} \right]^2.$$

7. 解: $\sqrt{4x^2+x+10} = 2x+1.$

$$4x^2+x+10 = 4x^2+4x+1.$$

$$3x = 9.$$

$$\therefore x = 3.$$

8. 解: $\sqrt{x+4} + \sqrt{x+11} = 7.$

$$\sqrt{x+4} = 7 - \sqrt{x+11}.$$

$$x+4 = 49 - 14\sqrt{x+11} + x+11.$$

$$14\sqrt{x+11} = 56.$$

$$\sqrt{x+11} = 4.$$

$$x+11 = 16.$$

$$\therefore x = 5.$$

9. 解: $\sqrt{4x+5} + \sqrt{x+1} - \sqrt{9x+10} = 0.$

$$\sqrt{4x+5} + \sqrt{x+1} = \sqrt{9x+10}.$$

$$4x+5+x+1+2\sqrt{4x^2+9x+5} = 9x+10.$$

$$\sqrt{4x^2+9x+5} = 2x+2.$$

$$4x^2+9x+5 = 4x^2+8x+4.$$

$$\therefore x = -1.$$

10. 解: $\sqrt{x+1} + \frac{x-6}{\sqrt{x+2}} = 0.$

$$\sqrt{x+1} = -\frac{x-6}{\sqrt{x+2}}.$$

$$x+1 = \frac{(x-6)^2}{x+2}.$$

$$x^2+3x+2 = x^2-12x+36.$$

$$15x = 34.$$

$$\therefore x = \frac{34}{15}.$$

11. 解: $\sqrt{x^2+3x-1} - \sqrt{x^2-x-1} = 2.$

$$\sqrt{x^2+3x-1} = 2 + \sqrt{x^2-x-1}.$$

$$x^2+3x-1 = 4 + 4\sqrt{x^2-x-1} + x^2-x-1.$$

$$x-1 = \sqrt{x^2-x-1}.$$

$$x^2-2x+1 = x^2-x-1.$$

$$\therefore x = 2.$$

12. 解: $\sqrt{x+7} + \sqrt{x-2} = \sqrt{x+2} + \sqrt{x-1}.$

$$x+7+x-2+2\sqrt{x^2+5x-14}$$

$$= x+2+x-1+2\sqrt{x^2+x-2}.$$

$$2 + \sqrt{x^2+5x-14} = \sqrt{x^2+x-2}.$$

$$4 + 4\sqrt{x^2+5x-14} + x^2+5x-14 = x^2+x-2.$$

$$\sqrt{x^2+5x-14}=2-x.$$

$$x^2+5x-14=4-4x+x^2.$$

$$9x=18.$$

$$\therefore x=2.$$

$$13. \text{ 解: } \frac{\sqrt{x+3}+\sqrt{x-5}}{\sqrt{x+3}-\sqrt{x-5}}=2.$$

$$\sqrt{x+3}+\sqrt{x-5}=2\sqrt{x-3}-2\sqrt{x-5}.$$

$$3\sqrt{x-5}=\sqrt{x+3}.$$

$$9x-45=x+3.$$

$$8x=48.$$

$$\therefore x=6.$$

$$14. \text{ 解: } \frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x-1}}+\frac{1}{\sqrt{x^2-1}}=0.$$

$$\sqrt{x-1}-\sqrt{x+1}+1=0.$$

$$\sqrt{x-1}+1=\sqrt{x+1}.$$

$$x-1+1+2\sqrt{x-1}=x+1.$$

$$4x-4=1.$$

$$4x=5.$$

$$\therefore x=\frac{5}{4}.$$

$$15. \text{ 解: } \begin{cases} \sqrt{x+17}+\sqrt{y-2}=\sqrt{x+5}+\sqrt{y+6} \dots\dots\dots (1) \\ \sqrt{y-x}=\sqrt{3-x}+\sqrt{y-3} \dots\dots\dots (2) \end{cases}$$

$$\text{從 (2) 式, } y-x=3-x+y-3+2\sqrt{(3-x)(y-3)}$$

$$\sqrt{(3-x)(y-3)}=0$$

$$3-x=0 \text{ 及 } y-3=0$$

$$\therefore x=3 \text{ 及 } y=3.$$

將此二值各代入 (1) 式; 當 $x=3$ 時, 則

$$\sqrt{20}+\sqrt{y-2}=\sqrt{8}+\sqrt{y+6}$$

$$\sqrt{20}-\sqrt{8}=\sqrt{y+6}-\sqrt{y-2}$$

$$28-2\sqrt{160}=2y+4-2\sqrt{(y+6)(y-2)}$$

$$12-4\sqrt{10}-y=\sqrt{(y+6)(y-2)}$$

$$(2-4\sqrt{10})^2-2(12-4\sqrt{10})y+y^2=y^2+4y-12$$

$$4^2(3-\sqrt{10})^2-2\times 4(3-\sqrt{10})y=4(y-3)$$

$$4(9-6\sqrt{10}+10)-6y+2\sqrt{10}y=y-3$$

$$7y-2\sqrt{10}y=79-24\sqrt{10}.$$

$$\begin{aligned}\therefore y &= \frac{79-24\sqrt{10}}{7-2\sqrt{10}} = \frac{73-10\sqrt{10}}{49-40} \\ &= \frac{73-10\sqrt{10}}{9}.\end{aligned}$$

當 $y=3$ 時, 則

$$\sqrt{x+17}+\sqrt{1}=\sqrt{x+5}+\sqrt{9}$$

$$\sqrt{x+17}=\sqrt{x+5}+2$$

兩邊平方, 化簡: $2=\sqrt{x+5}$

$$\therefore x=-1.$$

答: $x=-1, y=3; x=3, y=(73-10\sqrt{10})/9.$

16. 解: $\begin{cases} 3\sqrt{x-2y}-\sqrt{x+y-4}=3 \dots\dots\dots(1) \\ \sqrt{x-2y}+2\sqrt{x+y-4}=8 \dots\dots\dots(2) \end{cases}$

(1) $\times 2 + (2)$, $7\sqrt{x-2y}=14.$

$$x-2y=4 \dots\dots\dots(3)$$

(2) $\times 3 - (1)$, $7\sqrt{x+y-4}=21$

$$x+y=13 \dots\dots\dots(4)$$

(4) $\times 2 + (3)$, $3x=30$

$$\therefore x=10.$$

代入 (4) 式, $y=3.$

17. 解: $\sqrt{x+a}+\sqrt{x+b}+\sqrt{x+c}+\sqrt{x+d}=0$

$$\sqrt{x+a}+\sqrt{x+b}=-\left(\sqrt{x+c}+\sqrt{x+d}\right)$$

$$x+a+x+b+2\sqrt{(x+a)(x+b)}$$

$$=x+c+x+d+2\sqrt{(x+c)(x+d)}.$$

於此間 $2x+a+b=2x+c+d$

及 $\sqrt{(x+a)(x+b)}=\sqrt{(x+c)(x+d)}$

$$x^2+(a+b)x+ab=x^2+(c+d)x+cd$$

$$(a+b-c-d)x+ab-cd=0.$$

$$\begin{aligned}
 18 \text{ 解: } & \sqrt{ax+b} + \sqrt{cx+d} - \sqrt{ex+f} = 0 \\
 & \sqrt{ax+b} + \sqrt{cx+d} = \sqrt{ex+f} \\
 & ax+b+cx+d+2\sqrt{acx^2+(ad+bc)x+bd} = ex+f \\
 & (a+c-e)x+b+d-f = -2\sqrt{acx^2+(ad+bc)x+bd} \\
 & (a+c-e)^2x^2+(b+d-f)^2+2(a+c-e)(b+d-f)x \\
 & = 4[acx^2+(ad+bc)x+bd] \dots\dots\dots(1)
 \end{aligned}$$

若 $\sqrt{a} + \sqrt{c} - \sqrt{e} = 0$
 則 $(\sqrt{a} + \sqrt{c})^2 = e$
 $a+c-e = -2\sqrt{ac}$
 $\therefore (a+c-e)^2 - 4ac = 0$

代入 (1) 答且化簡之, 得一一次有理方程:

$$2[(a+c-e)(b+d-f) - 2(ad+bc)]x + (b+d-f)^2 - 4bd = 0.$$

習 題 XXXIX

原本第 293 頁

- 解: $9 + \sqrt{56} = 9 + 2\sqrt{14}$
 但 $14 = 2 \times 7$ 及 $9 = 2 + 7$.
 $\therefore \sqrt{9 + \sqrt{56}} = \sqrt{2} + \sqrt{7}$.
- 解: $20 + 2\sqrt{96} = 20 + 8\sqrt{6} = 4(5 + 2\sqrt{6})$
 但 $6 = 3 \times 2$ 及 $5 = 3 + 2$.
 $\therefore \sqrt{20 + 2\sqrt{96}} = 2(\sqrt{2} + \sqrt{3})$.
- 解: $32 - 2\sqrt{175}$
 因 $175 = 25 \times 7$ 及 $32 = 25 + 7$
 $\therefore \sqrt{32 - 2\sqrt{175}} = \sqrt{25} - \sqrt{7} = 5 - \sqrt{7}$.
- 解: $\sqrt{1 + \frac{2\sqrt{6}}{5}} = \sqrt{\frac{5 + 2\sqrt{6}}{5}} = \sqrt{\frac{3 + 2 + 2\sqrt{3 \times 2}}{5}}$
 $= \frac{\sqrt{3} + \sqrt{2}}{\sqrt{5}} = \frac{\sqrt{15} + \sqrt{10}}{5}$.
- 解: $\sqrt{7 - 3\sqrt{5}} = \sqrt{7 - \sqrt{45}} = \sqrt{\frac{28 - 2\sqrt{180}}{4}}$

$$= \sqrt{\frac{10+18-2\sqrt{10 \times 18}}{4}} = \frac{3\sqrt{2}-\sqrt{10}}{2}$$

6. 解: $\sqrt{8\sqrt{2}+2\sqrt{30}} = \sqrt{\sqrt{2}(8+2\sqrt{15})}$
 $= \sqrt[4]{2}(\sqrt{5}+\sqrt{3})$.
7. 解: $\sqrt{2(a+\sqrt{a^2-b^2})} = \sqrt{2a+2\sqrt{a^2-b^2}}$
 但 $a^2-b^2=(a+b)(a-b)$ 及 $2a=(a+b)(a-b)$
 $\therefore \sqrt{2(a+\sqrt{a^2-b^2})} = \sqrt{a+b} + \sqrt{a-b}$.
8. 解: $\sqrt{b-2\sqrt{ab}-a^2} = \sqrt{b-2\sqrt{a(b-a)}} = \sqrt{a} - \sqrt{b-a}$.
9. 解: $\sqrt[4]{17+12\sqrt{2}} = \sqrt[4]{17+2\sqrt{72}} = \sqrt{\sqrt{8}+\sqrt{9}}$
 $= \sqrt{3+2\sqrt{2}} = \sqrt{1} + \sqrt{2} = 1 + \sqrt{2}$.
10. 解: $\sqrt{9+4\sqrt{4}+2\sqrt{3}} = \sqrt{9+4(1+\sqrt{3})}$
 $= \sqrt{9+4+4\sqrt{3}} = \sqrt{13+2\sqrt{12}} = \sqrt{1} + \sqrt{12}$
 $= 1+2\sqrt{3}$.

習題 XL

原本第 297 頁

1. 解: $\sqrt{-49} = \sqrt{49i} = 7i$.
2. 解: $\sqrt{-18} = \sqrt{18i} = 3\sqrt{2}i$.
3. 解: $\sqrt{-8} \cdot \sqrt{-12} = \sqrt{8} \cdot \sqrt{12} \cdot i^2 = \sqrt{96i^2}$
 $= 4\sqrt{6i^2} = -4\sqrt{6}$.
4. 解: $\sqrt{-2^2} = \sqrt{2^2i} = 2i$.
5. 解: $(\sqrt{-2})^2 = (\sqrt{2}i)^2 = 2i^2 = -2$.
6. 解: $i^{12} = i^4 \times 3 = 1^3 = 1$.
7. 解: $i^{-7} = i^{-7} \times i^8 = i$.
8. 解: $i^{15} = (i^4)^3 \cdot i^3 = 1 \cdot (-1)i = -i$.
9. 解: $\sqrt{x-y} \cdot \sqrt{y-x} = \sqrt{x-y}\sqrt{x-y}i = (x-y)i$.
10. 解: $(2+\sqrt{-3})(1+\sqrt{-2}) = (2+\sqrt{3}i)(1+\sqrt{2}i)$
 $= 2 + \sqrt{3}i + 2\sqrt{2}i + \sqrt{6}i^2 = 2 - \sqrt{6} + (\sqrt{3} + 2\sqrt{2})i$.
11. 解: $(\sqrt{-2})^7(\sqrt{-3})^9 = (\sqrt{2}i)^7(\sqrt{3}i)^9$

$$= 8\sqrt{2} \times 81\sqrt{3}i^{16} = 648\sqrt{6}.$$

$$\begin{aligned} 12. \text{ 解: } & (1+2i)^3 + (1-2i)^3 \\ & = 1+6i+12i^2+8i^3+1-6i+12i^2-8i^3 \\ & = 1-12+1-12 = -22. \end{aligned}$$

$$13. \text{ 解: } \frac{a}{\sqrt{-a^2}} - \frac{b}{i\sqrt{b^2}} = \frac{a}{i\sqrt{a^2}} - \frac{b}{i\sqrt{b^2}} = \frac{1}{i} - \frac{1}{i} = 0.$$

$$14. \text{ 解: } \frac{4+6i}{1+i} + \frac{4-6i}{1-i} = \frac{4+2i-6i^2+4-2i-6i^2}{1-i^2} = \frac{20}{2} = 10.$$

$$\begin{aligned} 15. \text{ 解: } & (\sqrt{3+4i} + \sqrt{3-4i})^2 = 3+4i+3-4i+2\sqrt{9-16i^2} \\ & = 6+2\sqrt{25} = 6+10 = 16. \end{aligned}$$

$$16. \text{ 解: } \frac{1+i^3}{1+i} = 1-i+i^2 = 1-i-1 = -i.$$

$$17. \text{ 解: } \frac{a+bi}{a-bi} = \frac{(a+bi)^2}{a^2-b^2i^2} = \frac{a^2+2abi+bi^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i.$$

$$\begin{aligned} 18. \text{ 解: } & \frac{9+3\sqrt{2}i}{(3+\sqrt{2}i)(1+\sqrt{2}i)} = \frac{3(3+\sqrt{2}i)}{(3+\sqrt{2}i)(1+\sqrt{2}i)} \\ & = \frac{3}{1+\sqrt{2}i} = \frac{3(1-\sqrt{2}i)}{1-2i^2} = \frac{3(1-\sqrt{2}i)}{3} = 1-\sqrt{2}i. \end{aligned}$$

$$19. \text{ 解: } \frac{4}{1+\sqrt{3}i} = \frac{4(1-\sqrt{3}i)}{1-(\sqrt{3}i)^2} = \frac{4(1-\sqrt{3}i)}{4} = 1-\sqrt{3}i.$$

$$\begin{aligned} 20. \text{ 解: } & \sqrt[4]{-16} = \sqrt{\sqrt{-16}} = \sqrt{4i} = \sqrt{2}\sqrt{2i} = \sqrt{2}\sqrt{1+2i+i^2} \\ & = \sqrt{2}(1+i). \end{aligned}$$

$$\begin{aligned} 21. \text{ 解: } & \left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{-1+3\sqrt{3}i-9i^2+3\sqrt{3}i^3}{8} \\ & = \frac{-1+3\sqrt{3}i+9-3\sqrt{3}i}{8} = \frac{8}{8} = 1. \end{aligned}$$

答: $\frac{-1+\sqrt{3}i}{2}$ 爲 1 之立方根.

$$\begin{aligned} 22. \text{ 解: } & \left(\frac{1+i}{\sqrt{2}}\right)^4 = \frac{1+4i+6i^2+4i^3+i^4}{4} = \frac{1+4i-6-4i+1}{4} \\ & = \frac{2-6}{4} = -1. \end{aligned}$$

$$23. \text{ 解: } 3+2i+x(i-1)+2yi = (3i+4)(x+y).$$

$$3 + 2i + xi - x + 2yi = 3xi + 3yi + 4x + 4y.$$

$$3 - x + (2 + x + 2y)i = 4x + 4y + (3x + 3y)i.$$

於此間 $3 - x = 4x + 4y \dots\dots\dots (1)$

及 $2 + x + 2y = 3x + 3y \dots\dots\dots (2)$

由 (1) 式, $5x + 4y = 3 \dots\dots\dots (3)$

由 (2) 式, $2x + y = 2 \dots\dots\dots (4)$

解 (1), (2), 得 $x = \frac{5}{3}$ 及 $y = -\frac{4}{3}$.

24. 解: $\sqrt{5+12i}$.

$$a=5 \text{ 及 } \sqrt{a^2+b^2} = \sqrt{5^2+12^2} = \sqrt{169} = 13.$$

$$\therefore x = \frac{5+13}{2} = 9, \quad y = \frac{-5+13}{2} = 4.$$

$$\therefore \sqrt{5+12i} = 3+2i.$$

25. 解: $\sqrt{2i}$.

$$a=0 \text{ 及 } \sqrt{a^2+b^2} = \sqrt{0+2^2} = 2.$$

$$\therefore x = \frac{2}{2} = 1, \quad y = \frac{2}{2} = 1,$$

$$\therefore \sqrt{2i} = 1+i.$$

26. 解: $\sqrt{4ab+2(a^2-b^2)i}$.

設 $\sqrt{4ab+2(a^2-b^2)i} = \sqrt{x} + \sqrt{yi}$.

$$4ab + 2(a^2 - b^2)i = x - y + 2\sqrt{xy}i.$$

$$x - y = 4ab \dots\dots\dots (1)$$

$$\sqrt{xy} = a^2 - b^2 \dots\dots\dots (2)$$

由 (2), $xy = a^4 - 2a^2b^2 + b^4 \dots\dots\dots (3)$

由 (1), $x^2 - 2xy + y^2 = 16a^2b^2 \dots\dots\dots (4)$

$$(4) + 4 \times (3), \quad (x+y)^2 = 4(a^4 + 2a^2b^2 + b^4).$$

$$x+y = 2(a^2+b^2) \dots\dots\dots (5)$$

$$\frac{(5)+(1)}{2}, \quad x = (a+b)^2.$$

$$\frac{(5)-(1)}{2}, \quad y = (a-b)^2.$$

$$\therefore \sqrt{4ab+2(a^2-b^2)i} = (a+b) + (a-b)i.$$

XIII. 二次方程

習 題 XLI

原本第 301 頁

1. 解: $x^2 + 2x = 35.$

$$(x-5)(x+7) = 0.$$

$$\therefore x = 5 \text{ 及 } -7.$$

2. 解: $4x^2 - 4x = 3.$

$$(2x-3)(2x+1) = 0.$$

$$\therefore x = \frac{3}{2} \text{ 及 } -\frac{1}{2}.$$

3. 解: $x^2 = 10x - 18.$

$$x^2 - 10x + 18 = 0.$$

$$\therefore x = 5 \pm \sqrt{5^2 - 18} = 5 \pm \sqrt{7}.$$

4. 解: $9x^2 + 6x + 5 = 0.$

$$x = \frac{-3 \pm \sqrt{9 - 5 \times 9}}{9} = \frac{-3 \pm \sqrt{36i}}{9} = \frac{-1 \pm 2i}{3}.$$

5. 解: $2x^2 + 3x - 4 = 0.$

$$x = \frac{-3 \pm \sqrt{9 + 32}}{4} = \frac{-3 \pm \sqrt{41}}{4}.$$

6. $(2x-3)^2 = 8x.$

$$4x^2 - 12x + 9 - 8x = 0.$$

$$4x^2 - 20x + 9 = 0.$$

$$(2x-9)(2x-1) = 0.$$

$$\therefore x = \frac{9}{2} \text{ 及 } \frac{1}{2}.$$

7. 解: $x^2 + 9x - 252 = 0.$

$$(x+21)(x-12) = 0.$$

$$\therefore x = -21 \text{ 及 } 12.$$

8. 解: $12x^2 + 56x - 255 = 0.$

$$(6x-17)(2x+15) = 0.$$

$$\therefore x = \frac{17}{6} \quad \text{及} \quad -\frac{15}{2}$$

9. 解: $8x^2 - 82x + 207 = 0.$

$$(4x - 23)(2x - 9) = 0.$$

$$\therefore x = \frac{23}{4} \quad \text{及} \quad \frac{9}{2}$$

10. 解: $15x^2 - 86x - 64 = 0.$

$$(5x - 32)(3x + 2) = 0.$$

$$\therefore x = \frac{32}{5} \quad \text{及} \quad -\frac{2}{3}$$

11. 解: $x^2 - 3x - 1 + \sqrt{3} = 0.$

$$x = \frac{3 \pm \sqrt{9 + 4 - 4\sqrt{3}}}{2} = \frac{3 \pm \sqrt{13 - 2\sqrt{12}}}{2}$$

$$= \frac{3 \pm (1 - \sqrt{12})}{2}$$

$$\therefore x = \frac{3 + 1 - \sqrt{12}}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3},$$

$$x = \frac{3 - 2 + \sqrt{12}}{2} = \frac{2 + 2\sqrt{3}}{2} = 1 + \sqrt{3}.$$

12. 解: $x^2 - (6+i)x + 8 + 2i = 0.$

$$x^2 - (6+i)x + 2(4+i) = 0.$$

$$(x-2)[x-(4+i)] = 0.$$

$$\therefore x = 2 \quad \text{及} \quad 4+i.$$

13. 解: $(x-2)^2(x-7) = (x+2)(x-3)(x-6).$

$$x^3 - 11x^2 + 32x - 28 = x^3 - 7x^2 + 36.$$

$$4x^2 - 32x + 64 = 0.$$

$$x^2 - 8x + 16 = 0.$$

$$(x-4)^2 = 0.$$

$$\therefore x = 4, \quad 4.$$

14. 解: $\frac{2x}{x+2} + \frac{x+2}{2x} = 2.$

$$1 + \left(\frac{x+2}{2x}\right)^2 - 2\left(\frac{x+2}{2x}\right) = 0.$$

$$\left(\frac{x+2}{2x}-1\right)^2=0.$$

$$\frac{x+2}{2x}=1.$$

$$x+2=2x.$$

$$\therefore x=2, 2.$$

15. 解: $\frac{x+1}{x}+1=\frac{x}{x-1}.$

$$2x^2-x-1=x^2.$$

$$x^2-x-1=0.$$

$$\therefore x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}.$$

16. 解: $\frac{3}{2(x^2-1)}+\frac{x}{4x+4}=\frac{3}{8}.$

$$x^2+2x-15=0.$$

$$(x+5)(x-3)=0.$$

$$\therefore x=-5 \text{ 及 } 3.$$

17. 解: $\frac{3}{2x+1}-\frac{1}{4x-2}-\frac{2x}{1-4x^2}=\frac{7}{8}.$

$$28x^2-56x+21=0.$$

$$4x^2-8x+3=0.$$

$$(2x-3)(2x-1)=0.$$

$$\therefore x=\frac{3}{2} \text{ 及 } \frac{1}{2}.$$

18. 解: $\frac{2x-1}{x-2}+\frac{3x+1}{x-3}=\frac{5x-14}{x-4}.$

$$2+\frac{3}{x-2}+3+\frac{10}{x-3}=5+\frac{6}{x-4}.$$

$$\frac{3}{x-2}+\frac{10}{x-3}-\frac{6}{x-4}=0.$$

$$3(x-3)(x-4)+10(x-2)(x-4)-6(x-2)(x-3)=0.$$

$$7x^2-51x+80=0.$$

$$(7x-16)(x-5)=0.$$

$$\therefore x = \frac{16}{7} \text{ 及 } 5.$$

19. 解: $\frac{x+1}{x(x-2)} - \frac{1}{2x-2} + \frac{1}{2x} = 0.$
 $2(x+1)(x-1) - x(x-2) + (x-2)(x-1) = 0.$
 $2x^2 - x = 0. \quad x(2x-1) = 0.$
 $\therefore x = \frac{1}{2} \text{ 或 } 0.$

惟 0 能使分母爲 0, 故不得爲根.

20. 解: $\frac{4}{x-1} - \frac{1}{4-x} = \frac{3}{x-2} - \frac{2}{3-x}.$
 $\frac{4}{x-1} - \frac{3}{x-2} = \frac{2}{x-3} - \frac{1}{x-4}.$
 $\frac{x-5}{(x-1)(x-2)} = \frac{x-5}{(x-3)(x-4)}.$
 $(x-1)(x-2) = (x-3)(x-4).$
 $x^2 - 3x + 2 = x^2 - 7x + 12.$
 $4x = 10.$
 $\therefore x = \frac{5}{2}.$

又 $x - 5 = 0.$
 $\therefore x = 5.$

21. 解: $\frac{x+3}{4(x+2)(3x-1)} + \frac{2x+1}{3(3x-1)(x+4)} - \frac{17x+7}{6(x+4)(x+2)} = 0.$
 $3(x+3)(x+4) + 4(2x+1)(x+2) - 2(17x+7)(3x-1) = 0.$
 $91x^2 - 33x - 58 = 0.$
 $(x-1)(91x+58) = 0.$
 $\therefore x = 1 \text{ 及 } -\frac{58}{91}.$

22. 解: $\frac{x+7}{2x^2-7x+3} + \frac{x}{x^2-2x-3} + \frac{x+3}{2x^2+x-1} = 0.$
 $\frac{x+7}{(2x-1)(x-3)} + \frac{x}{(x+1)(x-3)} + \frac{x+3}{(2x-1)(x+1)} = 0.$
 $(x+7)(x+1) + x(2x-1) + (x+3)(x-3) = 0.$

$$4x^2 + 7x - 2 = 0.$$

$$(4x-1)(x+2) = 0.$$

$$\therefore x = \frac{1}{4} \quad \text{及} \quad -2.$$

23. 解: $3x^2 + (9a-1)x - 3a = 0.$

$$(3x-1)(x+3a) = 0.$$

$$\therefore x = \frac{1}{3} \quad \text{及} \quad -3a.$$

24. 解: $x^2 - 2ax + a^2 - b^2 = 0.$

$$(x-a)^2 - b^2 = 0.$$

$$(x-a+b)(x-a-b) = 0.$$

$$\therefore x = a-b \quad \text{及} \quad a+b.$$

25. 解: $c^2x^2 + c(a-b)x - ab = 0.$

$$(cx+a)(cx-b) = 0.$$

$$\therefore x = -\frac{a}{c} \quad \text{及} \quad \frac{b}{c}.$$

26. 解: $x^2 - 4ax + 4a^2 - b^2 = 0.$

$$(x-2a)^2 - b^2 = 0.$$

$$(x-2a+b)(x-2a-b) = 0.$$

$$\therefore x = 2a-b \quad \text{及} \quad 2a+b.$$

27. 解: $x^2 - 6acx + a^2(9c^2 - 4b^2) = 0.$

$$x^2 - 6acx + 9a^2c^2 - 4a^2b^2 = 0.$$

$$(x-3ac)^2 - (2ab)^2 = 0.$$

$$(x-3ac+2ab)(x-3ac-2ab) = 0.$$

$$\therefore x = a(3c-2b) \quad \text{及} \quad a(3c+2b).$$

28. 解: $(a^2 - b^2)x^2 - 2(a^2 + b^2)x + a^2 - b^2 = 0.$

$$\{(a-b)x - (a+b)\} \{(a+b)x - (a-b)\} = 0.$$

$$\therefore x = \frac{a+b}{a-b} \quad \text{及} \quad \frac{a-b}{a+b}.$$

29. 解: $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0.$

$$(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0.$$

$$3x^2 - (2a+2b+2c)x + ca + bc + ab = 0.$$

$$\therefore x = \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}$$

30. 解: $\frac{(x-a)^2 - (x-b)^2}{(x-a)(x-b)} + \frac{4ab}{a^2-b^2} = 0.$

$$[(x-a)^2 - (x-b)^2](a^2-b^2) + 4ab(x-a)(x-b) = 0.$$

$$4abx^2 - 2(a^2+b^2)(a+b)x + (a^2+b^2)^2 = 0.$$

$$[2ax - (a^2+b^2)][2bx - (a^2+b^2)] = 0.$$

$$\therefore x = \frac{a^2+b^2}{2a} \quad \text{及} \quad \frac{a^2+b^2}{2b}.$$

習 題 XLII

原本第 302 頁

1. 解: 設 $x, x+1$ 爲二連續數.

$$x(x+1) = 506$$

$$x^2 + x - 506 = 0$$

$$(x-22)(x+23) = 0$$

$$\therefore x = 22 \quad \text{或} \quad -23,$$

$$x+1 = 23 \quad \text{或} \quad -22.$$

答: 此二數爲 22 及 23 或 -23 及 -22.

2. 解: 設 $x, x+1$ 爲所求之二數.

$$x^2 + (x+1)^2 = 481$$

$$x^2 + x - 240 = 0$$

$$(x-15)(x+16) = 0$$

$$\therefore x = 15 \quad \text{或} \quad -16,$$

$$x+1 = 16 \quad \text{或} \quad -15.$$

答: 此二數爲 15 及 16, 或 -16 及 -15.

3. 解: 設 $x, x+1$ 爲所求之二數.

$$(x+1)^3 - x^3 = 91$$

$$x^2 + x - 30 = 0$$

$$(x+6)(x-5) = 0$$

$$\therefore x = 5 \quad \text{或} \quad -6,$$

$$x+1 = 6 \quad \text{或} \quad -5.$$

答: 此二數爲 5 及 6, 或 -6 及 -5.

4. 解：設 $x-1, x, x+1$ 爲三連續數。

$$(x-1)x+x(x+1)+(x+1)(x-1)=587$$

$$3x^2=588 \quad x^2=196.$$

$$\therefore x=\pm 14.$$

答：13, 14, 15 或 -15, -14, -13 爲三連續數。

5. 解：設 x 爲十位數字， $\frac{48}{x}$ 爲個位數字。

$$10x + \frac{48}{x} - 18 = 10 \cdot \frac{48}{x} + x$$

$$9x^2 - 18x - 432 = 0$$

$$x^2 - 2x - 48 = 0$$

$$(x-8)(x+6) = 0$$

$$\therefore x = 8,$$

$$\frac{48}{x} = 6.$$

答：此數爲 86.

6. 解：設 x 爲其分母。

$$\frac{x+2}{x} - \frac{x}{x+2} = \frac{24}{35}$$

$$6x^2 - 23x - 35 = 0$$

$$(x-5)(6x+7) = 0$$

$$\therefore x = 5.$$

答：此分數爲 $\frac{x+2}{x}$ 即 $\frac{7}{5}$.

7. 解：設 x 爲此人所購牛之頭數。

$$(x-4)\left[\frac{1260}{x}+1\right]-1260=260$$

$$x^2-30x-504=0$$

$$(x+12)(x-42)=0$$

$$\therefore x=42.$$

答：此人原購牛四十二頭。

8. 解：設 x 爲該物之原價。

$$x + \frac{x}{100} = 48$$

$$x^2 + 200x - 9600 = 0$$

$$(x + 240)(x - 40) = 0$$

$$\therefore x = 40.$$

答：該物原值銀 40 元。

9. 解：設 $x\%$ 為利率。

$$4000 \left[1 + \frac{x}{100} \right]^2 = 4410$$

$$x^2 + 200x - 1025 = 0$$

$$(x - 5)(x + 205) = 0$$

$$\therefore x = 5.$$

答：利率為 5%。

10. 解：設 x 為承繼遺產稅率。

$$25000 \left[\frac{x}{100} + \frac{x+1}{100} \right] = 25000 - 22890$$

$$500x = 1950$$

$$\therefore x = 4.$$

答：承繼遺產稅率為 4%。

11. 解：設 x 為股票數目。

$$3 \times \frac{50 - \frac{4500}{x}}{50} = \frac{\frac{5850}{x-10} - 50}{50}$$

$$\frac{3x - 270}{x} = \frac{127 - x}{x - 10}$$

$$4x^2 - 427x + 2700 = 0$$

$$(x - 100)(4x - 27) = 0$$

$$\therefore x = 100 \text{ 及 } \frac{27}{4} \text{ (因張數不得為分數, 故不合理).}$$

答：此人共購股票數一百張。

12. 解：設 x 為前輪周長， $x+8$ 為後輪周長。

$$1 \text{ 哩} = 63360 \text{ 呎.}$$

$$\frac{63360}{x} - \frac{63360}{x+8} = 88$$

$$x^2 + 8x - 5760 = 0$$

$$(x + 80)(x - 72) = 0$$

$$\begin{aligned}\therefore x &= 72 \text{ 吋即 } 6 \text{ 呎,} \\ x+8 &= 80 \text{ 吋即 } 6\frac{2}{3} \text{ 呎.}\end{aligned}$$

答：前後二輪之圓周長度各為 6 呎及 $6\frac{2}{3}$ 呎。

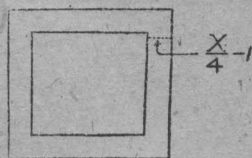
13. 解：設 x 為正方形之邊長， $\frac{x}{4}-1$ 為其鑲邊之寬。

$$\begin{aligned}\left[x+2\left(\frac{x}{4}-1\right)\right]^2-x^2 &= 4x+64 \\ 4x\left(\frac{x}{4}-1\right)+4\left(\frac{x}{4}-1\right)^2 & \\ = 4x+64 &\end{aligned}$$

$$\begin{aligned}x^2-8x-48 &= 0 \\ (x-12)(x+4) &= 0 \\ x &= 12 \text{ 吋}\end{aligned}$$

$$\therefore x^2 = 144 \text{ 方吋,}$$

$$\left[x+2\left(\frac{x}{4}-1\right)\right]^2-x^2 = 112 \text{ 方吋.}$$



答：正方形及鑲邊之面積各為 144 方吋及 112 方吋。

14. 解：設 x 為所求之邊長。

$$x^2 = \overline{AC}^2 + \overline{BA}^2 = 2\overline{AB}^2 = 2\left[\frac{2-x}{2}\right]^2$$

$$x^2 + 4x - 4 = 0$$

$$\begin{aligned}\therefore x &= -2 \pm \sqrt{4+4} = -2 \pm 2\sqrt{2} \\ &= 2(\pm\sqrt{2}-1).\end{aligned}$$

答：八邊形之邊長為 $2(\sqrt{2}-1)$ 。

15. 解：設 x 為每次所取出之骰數。

$$63-x-\frac{63-x}{63}x=28$$

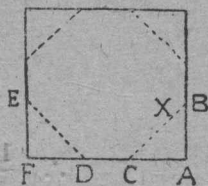
$$x^2-126x+2205=0$$

$$(x-105)(x-21)=0$$

$$\therefore x=21 \text{ 骰.}$$

答：此人每次取出二十一骰。

16. 解：設 x 為 A 車之速率， $x+5$ 為 B 車之速率。



$$\frac{50}{x} + \frac{1}{12} + \frac{50}{x+5} = 2\frac{4}{9}$$

$$17x^2 - 635x - 1800 = 0$$

$$(17x+45)(x-40) = 0$$

$$\therefore x = 40 \text{ 哩/時,}$$

$$x+5 = 45 \text{ 哩/時.}$$

答：A, B 二車之速率各為每小時 40 哩及 45 哩。

17. 解：設 x 為步行者之速率， $\frac{6}{x}$ 為所求之時間。

$$\left(\frac{6}{x} - \frac{1}{2}\right)(x+2) = 6$$

$$x^2 - 2x - 24 = 0$$

$$(x+6)(x-4) = 0$$

$$\therefore x = 4 \text{ 哩/時,}$$

$$\frac{6}{x} = 1\frac{1}{2} \text{ 時.}$$

答：所求之時間及速率為 $1\frac{1}{2}$ 小時及每小時 4 哩。

18. 解：設 x 為原速率， $\frac{12}{x} + \frac{6}{x+\frac{1}{2}}$ 為所求之時間。

$$\frac{12}{x} + \frac{6}{x+\frac{1}{2}} - \frac{18}{x+\frac{1}{2}} = \frac{20}{60}$$

$$2x^2 + x - 36 = 0$$

$$(2x+9)(x-4) = 0$$

$$x = 4 \text{ 哩/時}$$

$$\therefore \frac{12}{x} + \frac{6}{x+\frac{1}{2}} = 3 + \frac{4}{3} = 4\frac{1}{3} \text{ 小時} = 4 \text{ 小時 } 20 \text{ 分.}$$

答：所求之時間為 4 小時 20 分。

19. 解：設 x 為所求之時間。

$$(3x)^2 + (4x)^2 = 30^2$$

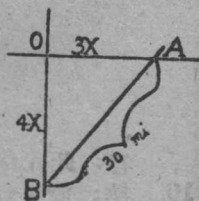
$$9x^2 + 16x^2 = 900$$

$$25x^2 = 900$$

$$5x = 30$$

$$\therefore x = 6 \text{ 小時.}$$

答：所須之時間為 6 小時。



20. 解：設 x 為所求之時間。

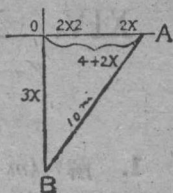
$$(2x + 2 \cdot 2)^2 + (3x)^2 = 10^2$$

$$13x^2 + 16x - 84 = 0$$

$$(x-2)(13x+42) = 0$$

$$\therefore x = 2 \text{ 小時.}$$

答：須經 2 小時後，二人方能相距 10 哩。



21. 解：(1)

$$7 = 0 + 32t - 16t^2$$

$$16t^2 - 32t + 7 = 0$$

$$(4t-7)(4t-1) = 0$$

$$\therefore t = \frac{1}{4} \text{ 秒或 } \frac{7}{4} \text{ 秒.}$$

$$16 = 0 + 32t - 16t^2$$

$$16t^2 - 32t + 16 = 0$$

$$(16t-16)(t-1) = 0$$

$$\therefore t = 1 \text{ 秒.}$$

$$17 = 0 + 32t - 16t^2$$

$$16t^2 - 32t + 17 = 0$$

$$\therefore t = \frac{16 \pm \sqrt{256 - 272}}{16} = \frac{16 \pm \sqrt{-16}}{16} \text{ (不可能).}$$

(2)

$$36 = 64 - 48t - 16t^2$$

$$10t^2 + 48t - 28 = 0$$

$$4t^2 + 12t - 7 = 0$$

$$(2t-1)(2t+7) = 0$$

$$\therefore t = \frac{1}{2} \text{ 秒.}$$

(3)

$$16t^2 = 36 \quad 4t = \pm 6$$

$$\therefore t = \frac{3}{2} \text{ 秒.}$$

答：所求之時間為 (1) $\frac{1}{4}$ 秒及 $\frac{7}{4}$ 秒，1 秒，不可能；(2) $\frac{1}{2}$

秒；(3) $\frac{3}{2}$ 秒。

XIV. 二次方程式之討論, 極大與極小

習題 XLIII

原本第 303 頁

1. 解: $(m+2)x^2 - 2mx + 1 = 0$

$$B^2 - 4AC = (2m)^2 - 4(m+2) = 4m^2 - 4m - 8$$

$$= 4(m-2)(m+1) = 0$$

$$\therefore m = 2 \text{ 或 } -1.$$

2. 解: $(m^2+m)x^2 + 3mx - 2 = 0$

當 $m = -1$ 時, $(1-1)x^2 - 3x - 2 = 0$

$0 \cdot x^2 - 3x - 2 = 0$

$(0 \cdot x - 1)(3x + 2) = 0$

$$\therefore x = \infty \text{ 及 } -\frac{2}{3}.$$

當 $m = 0$ 時, $0 \cdot x^2 + 0 \cdot x - 2 = 0$

$(0 \cdot x - 2)(0 \cdot x + 1) = 0$

$$\therefore x = \infty, \infty.$$

3. 解: $3x^2 + (5y-5)x - (2y^2 - 4y + 2) = 0$

$$x = \frac{-(5y-5) \pm \sqrt{(5y-5)^2 + 12(2y^2 - 4y + 2)}}{6}$$

$$= \frac{-(5y-5) \pm (7y-7)}{6} = \frac{y-1}{3} \text{ 或 } -2y+2$$

$$\therefore 3x^2 + 5xy - 2y^2 - 5x + 4y - 2 = (3x - y + 1)(x + 2y - 2).$$

4. 解: $x^2 - y^2 + mx + 5y - 6 = (x+y-\alpha)(x-y+\beta)$

$$= x^2 - y^2 + (\beta - \alpha)x + (\alpha + \beta)y - \alpha\beta$$

$$\beta - \alpha = m \dots\dots\dots (1)$$

$$\alpha + \beta = 5 \dots\dots\dots (2)$$

$$\alpha\beta = 6 \dots\dots\dots (3)$$

$\therefore m = \pm 1$ 時, 上式能分解.

5. 解: $x^2 + px + q = 0$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 - 4q$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$$

$$= (p^2 - 2q)^2 - 2q^2 = p^4 - 4p^2q + 2q^2$$

$$a/\beta + \beta/a = \frac{a^2 + \beta^2}{a\beta} = \frac{(a+\beta)^2 - 2a\beta}{a\beta} = \frac{p^2 - 2q}{q}$$

6. 解: $2x^2 - 3x + 4 = 0$

$$\frac{a}{\beta^2} + \frac{\beta}{a^2} = \frac{a^3 + \beta^3}{a^2\beta^2} = \frac{(a+\beta)^3 - 3a\beta(a+\beta)}{(a\beta)^2}$$

$$= \frac{\left(\frac{3}{2}\right)^3 - \frac{3}{2}\left(\frac{3}{2}\right) \cdot 3}{2^2} = -\frac{45}{32}$$

$$a^3\beta + a\beta^3 = a\beta(a^2 + \beta^2) = a\beta[(a+\beta)^2 - 2a\beta]$$

$$= 2\left[\left(\frac{3}{2}\right)^2 - 4\right] = 2\left[\frac{-7}{4}\right] = -\frac{7}{2}$$

7. 解: $x^2 + x + 2 = 0$

$$a + \beta = -1, \quad a \cdot \beta = 2$$

故所求之方程式爲:

(i) $x^2 - (-a - \beta)x + (-a)(-\beta) = x^2 - x + 2 = 0.$

(ii) $x^2 - \left(\frac{1}{a} + \frac{1}{\beta}\right)x + \left(\frac{1}{a}\right)\left(\frac{1}{\beta}\right) = x^2 - \frac{a+\beta}{a\beta}x + \frac{1}{a\beta}$

$$= x^2 + \frac{1}{2}x + \frac{1}{2} = 0$$

即 $2x^2 + x + 1 = 0.$

(iii) $x^2 - (2a + 2\beta)x + (2a)(2\beta) = x^2 - 2(a + \beta)x + 4a\beta = 0$

即 $x^2 + 2x + 8 = 0.$

(iv) $x^2 - (a+1 + \beta+1)x + (a+1)(\beta+1) = x^2 - [(a+\beta) + 2]x + [a\beta + (a+\beta) + 1] = x^2 - x + 2 = 0.$

8. 解: 1. $x^2 - 8x + 3$

設 $y = x^2 - 8x + 3 + 13 - 13 = (x-4)^2 - 13$

答: 當 $x=4$ 時, y 有一極小值 -13 .

2. $2x^2 - x + 4$

設 $y = 2x^2 - x + 4 = 2\left[x^2 - \frac{1}{2}x + \frac{1}{16}\right] + \frac{31}{8}$

$$= 2\left(x - \frac{1}{4}\right)^2 = \frac{31}{8}$$

答: 當 $x = \frac{1}{4}$ 時, y 有一極小值 $\frac{31}{8}$.

3. $1 + 4x - x^2$

設 $y = 1 + 4x - x^2 = 5 - (x-2)^2$

答：當 $x=2$ 時， y 有一極大值 5。

4. $x/(x^2+1)$

設 $y = x/(x^2+1)$ ，即 $x^2y - x + y = 0$

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y} = \frac{1 \pm \sqrt{(1+2y)(1-2y)}}{2y}$$

$$\therefore (1-2y)(1+2y) \geq 0$$

$$\therefore y \leq \frac{1}{2} \text{ 及 } y \geq -\frac{1}{2}$$

答： $x=1$ 時， y 有一極大值 $\frac{1}{2}$ ；

$x=-1$ 時， y 有一極小值 $-\frac{1}{2}$ 。

5. $1/x + 1/(1-x)$

設 $y = \frac{1}{x} + \frac{1}{1-x}$ ，即 $x^2y - xy + 1 = 0$

$$x = \frac{y \pm \sqrt{y^2 - 4y}}{2y} = \frac{y \pm \sqrt{y(y-4)}}{2y}$$

$$\therefore y(y-4) \geq 0$$

$$\therefore y \leq 0 \text{ 及 } y \geq 4$$

答： $x = \frac{1}{2}$ 時， y 有一極小值 4。

6. $(x+1)/(2x^2-1)$

設 $y = (x+1)/(2x^2-1)$ ，即 $2yx^2 - x - (y+1) = 0$

$$\therefore x = \frac{1 \pm \sqrt{1+8y(y+1)}}{4y} = \frac{1 \pm \sqrt{8y^2+8y+1}}{4y}$$

設 $8y^2+8y+1=0$

$$\therefore y = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 8 \cdot 1}}{2 \cdot 8} = \frac{-8 \pm 4\sqrt{2}}{16} = \frac{-2 \pm \sqrt{2}}{4}$$

因根式內 y^2 之係數為正。

答： y 之極大值為 $\frac{-2+\sqrt{2}}{4}$ ，極小值為 $\frac{-2-\sqrt{2}}{4}$ 。

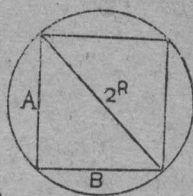
9. 解：求一圓內面積最大之內接矩形及周長最大之內接矩

形。

設 x 為矩形一邊之長， d 為圓之直徑之長，則 $\sqrt{d^2 - x^2}$ 為矩形另一邊之長。

設 y 示矩形之面積，則

$$y = x\sqrt{d^2 - x^2}$$



平方，得

$$y^2 = x^2(d^2 - x^2) \quad x^4 - d^2x^2 + y^2 = 0$$

$$\therefore x^2 = \frac{d^2 \pm \sqrt{d^4 - 4y^2}}{2} = \frac{d^2 \pm \sqrt{(d^2 - 2y)(d^2 + 2y)}}{2}$$

答： $x^2 = \frac{d^2}{2}$ ，即 $x = \frac{\sqrt{2}}{2}d$ 時， y 有一極大值 $\frac{d^2}{2}$ ；

即 圓之內接正方形之面積為最大。

設 P 示矩形之周長，則

$$P = 2x + 2\sqrt{d^2 - x^2}$$

化簡，得 $8x^2 - 4Px + (P^2 - 4d^2) = 0$

$$\therefore x = \frac{P \pm \sqrt{8d^2 - P^2}}{4} = \frac{P \pm \sqrt{(2\sqrt{2}d - P)(2\sqrt{2}d + P)}}{4}$$

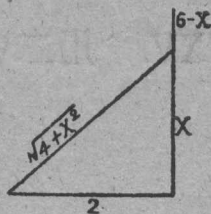
答： $x = \frac{P}{4} = \frac{\sqrt{2}}{2}d$ 時， P 有一極大值 $2\sqrt{2}d$ ；

即 內接矩形中正方形之周長最大。

10. 解： 設 x 為所求點至最近點之距離， y 為某人划行及步行共需之時間。

$$y = \frac{6-x}{5} + \frac{\sqrt{4+x^2}}{4}$$

$$400y^2 - 960y + 160xy + 476 - 192x - 9x^2 = 0$$



$$\therefore x = \frac{\frac{160y - 192}{9} \pm \sqrt{\frac{(192 - 160y)^2}{81} + \frac{1500y^2 - 3840y + 1904}{9}}}{2}$$

$$= \frac{160y - 192}{9 \cdot 2} \pm \frac{1}{2} \sqrt{\frac{40000y^2 - 96000y + 54000}{81}} \dots\dots\dots(1)$$

設 $40000y^2 - 96000y + 54000 = 0$

即 $20y^2 - 48y + 27 = 0$

$$(2y - 3)(10y - 9) = 0$$

$$\therefore y = \frac{3}{2} \quad \text{或} \quad \frac{9}{10}$$

因根式內 y^2 之係數大於 0, 故 $\frac{3}{2}$ 為 y 之極小值。

代入 (1) 式, 得

$$x = \frac{160 \times \frac{3}{2} - 192}{18} = 2\frac{2}{3} \text{ 哩.}$$

答: 須划至距最近點 $2\frac{2}{3}$ 哩之點, 方便行程所需之時間為最短。

11. 解: $h = a + bt - 16t^2$
 $= 0 + 48t - 16t^2$
 $= 36 - (36 - 48t + 16t^2)$
 $= 36 - (3 - 4t)^2$

$$\therefore t = \frac{3}{2} \text{ 秒時, } h \text{ 有一最大值 } 36.$$

答: 所求之高度為 36 呎, 時間為 $\frac{3}{2}$ 秒。

XV. 用二次方程式可解之高次方程式

習 題 XLIV

原本第 316 頁

1. 解: $4x^4 - 17x^2 + 18 = 0$
 $(4x^2 - 9)(x^2 - 2) = 0$

$$\therefore x = \pm \frac{3}{2} \text{ 及 } \pm \sqrt{2}.$$

2. 解: $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 7$

$$(3x^{\frac{3}{4}} - 7)(x^{\frac{3}{4}} + 1) = 0$$

$$x^{\frac{3}{4}} = \frac{7}{3} \quad \text{及} \quad -1$$

$$\therefore x = \sqrt[3]{\left(\frac{7}{3}\right)^4} = \frac{7}{3} \sqrt[3]{\frac{7}{3}} = \frac{7}{9} \sqrt[3]{63}$$

及 $x = \sqrt[3]{(-1)^4} = 1.$

3. 解: $(x^2-4)(x^2-9) = 7x^2$

$$x^4 - 20x^2 + 36 = 0$$

$$(x^2-18)(x^2-2) = 0$$

$$x^2 = 18 \quad \text{及} \quad 2$$

$$\therefore x = \pm 3\sqrt{2} \quad \text{及} \quad \pm\sqrt{2}.$$

4. 解: $(2x^2-x-3)(3x^2+x-2)^2 = 0$

$$(2x-3)(x+1)(3x-2)^2(x+1)^2 = 0$$

$$\therefore x = \frac{3}{2}, -1, \frac{2}{3}, \frac{2}{3}, -1, -1.$$

5. 解: $x^4 - x^3 + x^2 + 3x - 6 = 0$

$$(x-1)(x^3+3)(x+2) = 0$$

$$\therefore x = 1, \pm\sqrt{3}i, -2.$$

6. 解: $x^4 - 2x^3 + x^2 + 2x - 2 = 0$

$$(x+1)(x-1)(x^2-2x+2) = 0$$

$$\therefore x = -1, 1, 1 \pm i.$$

7. 解: $(3x^2-2x+1)(3x^2-2x-7)+12=0$

$$(3x^2-2x)^2 - 6(3x^2-2x) + 5 = 0$$

$$(3x^2-2x-5)(3x^2-2x-1) = 0$$

$$(3x-5)(x+1)(3x+1)(x-1) = 0$$

$$\therefore x = \frac{5}{3}, -1, -\frac{1}{3}, 1.$$

8. 解: $x^4 - 12x^3 + 33x^2 + 18x - 28 = 0$

$$(x^4 - 12x^3 + 36x^2) - 3x^2 + 18x - 28 = 0$$

$$(x^2 - 6x)^2 - 3(x^2 - 6x) - 28 = 0$$

$$(x^2 - 6x - 7)(x^2 - 6x + 4) = 0$$

$$(x+1)(x-7)(x^2 - 6x + 4) = 0$$

$$\therefore x = -1, 7, 3 \pm \sqrt{5}.$$

9. 解: $4x^4 + 4x^3 - x^2 - x - 2 = 0$

$$(2x^2+x)^2 - (2x^2+x) - 2 = 0$$

$$(2x^2+x+1)(2x^2+x-2) = 0$$

$$2x^2+x+1=0$$

$$\therefore x = \frac{-1 \pm \sqrt{1-8}}{4} = \frac{-1 \pm \sqrt{7}i}{4}$$

$$2x^2+x-2=0$$

$$\therefore x = \frac{-1 \pm \sqrt{1+16}}{4} = \frac{-1 \pm \sqrt{17}}{4}$$

10. 解: $x^4 - 2x^3 + 2x^2 - 2x + 1 = 0$

$$x^2(x^2 - 2x + 1) + (x^2 - 2x + 1) = 0$$

$$(x^2 + 1)(x^2 - 2x + 1) = 0$$

$$(x^2 + 1)(x - 1)^2 = 0$$

$$\therefore x = \pm i, 1, 1.$$

11. 解: $x^4 + x^3 + 2x^2 + x + 1 = 0$

$$x^2 + x + 2 + \frac{1}{x} + \frac{1}{x^2} = 0$$

$$\left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 0$$

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 0$$

$$\left(x + \frac{1}{x}\right)\left(x + \frac{1}{x} + 1\right) = 0$$

$$x + \frac{1}{x} = 0$$

$$x^2 + 1 = 0$$

$$\therefore x = \pm i.$$

$$x + \frac{1}{x} + 1 = 0$$

$$x^2 + x + 1 = 0$$

$$\therefore x = \frac{-1 \pm \sqrt{3}i}{2}$$

12. 解: $x^5 - 11x^4 + 36x^3 - 36x^2 + 11x - 1 = 0$

$$(x^5 - 1) - 11(x^4 - x) + 36(x^3 - x^2) = 0$$

$$(x - 1)[x^4 - 10x^3 + 26x^2 - 10x + 1] = 0$$

$$(x-1)\left[x^2-10x+26-\frac{10}{x}+\frac{1}{x^2}\right]=0$$

$$(x-1)\left[\left(x+\frac{1}{x}\right)^2-10\left(x+\frac{1}{x}\right)+24\right]=0$$

$$(x-1)\left(x+\frac{1}{x}-4\right)\left(x+\frac{1}{x}-6\right)=0$$

$$(x-1)(x^2-4x+1)(x^2-6x+1)=0$$

$$\therefore x=1, 2\pm\sqrt{3}, 3\pm 2\sqrt{2}.$$

13. 解: $x^5-243=0$

設

$$x=3y$$

代入上式, 得

$$y^5-1=0$$

$$(y-1)(y^4+y^3+y^2+y+1)=0$$

$$(y-1)\left(y^2+y+1+\frac{1}{y}+\frac{1}{y^2}\right)=0$$

$$(y-1)\left[\left(y+\frac{1}{y}\right)^2+\left(y+\frac{1}{y}\right)-1\right]=0$$

$$y=1, \quad \left(y+\frac{1}{y}\right)^2+\left(y+\frac{1}{y}\right)-1=0$$

$$y+\frac{1}{y}=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$$

$$y^2-\left[\frac{-1\pm\sqrt{5}}{2}\right]y+1=0$$

$$\therefore y=\left[\frac{-1\pm\sqrt{5}}{4}\right]\pm\sqrt{\left[\frac{-1\pm\sqrt{5}}{4}\right]^2-1}$$

$$=\left[\frac{-1\pm\sqrt{5}}{4}\right]\pm\sqrt{\frac{[-1\pm\sqrt{5}]^2-16}{16}}$$

$$=\frac{1}{4}[-1\pm\sqrt{5}\pm i\sqrt{10\pm 2\sqrt{5}}].$$

$$\therefore x=3 \text{ 及 } \frac{3[-1\pm\sqrt{5}\pm i\sqrt{10\pm 2\sqrt{5}}]}{4}.$$

14. 解: $(2x-1)^3=1$

設

$$2x-1=y$$

$$y^3-1=0$$

$$(y-1)(y^2+y+1)=0 \quad \therefore y=1.$$

$$2x-1=1 \quad \therefore x=1.$$

$$y^2+y+1=0 \quad \therefore y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore x = \frac{1 \pm \sqrt{3}i}{4}.$$

15. 解: $(1+x)^3 = (1-x)^3$

$$(1+x)^3 - (1-x)^3$$

$$= (1+x-1+x)[(1+x)^2 + (1+x)(1-x) + (1-x)^2]$$

$$= 2x[x^2+3]=0$$

$$2x=0 \quad \therefore x=0.$$

$$x^2+3=0 \quad \therefore x = \pm\sqrt{3}i.$$

16. 解: $(x-2)^4 - 81 = 0$

$$[(x-2)^2-9][(x-2)^2+9]=0$$

$$(x-5)(x+1)(x^2-4x+13)=0$$

$$\therefore x=5, -1, 2 \pm 3i.$$

17. 解: $(a+x)^3 + (b+x)^3 = (a+b+2x)^3$

$$(a+x)^3 + (b+x)^3 - [(a+x) + (b+x)]^3$$

$$= (a+b+2x)[(a+x)^2 - (a+x)(b+x) + (b+x)^2$$

$$- (a+x)^2 - 2(a+x)(b+x) - (b+x)^2] = 0$$

$$(a+b+2x)(a+x)(b+x) = 0$$

$$\therefore x = -a, -b, -\frac{a+b}{2}.$$

18. 解: $(a-x)^4 - (b-x)^4 = (a-b)(a+b-2x)$

$$[(a-x)^2 + (b-x)^2][(a-x)^2 - (b-x)^2]$$

$$= (a-b)(a+b-2x)$$

$$[2x^2 - 2(a+b)x + a^2 + b^2][(a+b-2x)(a-b)]$$

$$= (a-b)(a+b-2x)$$

$$2x^2 - 2(a+b)x + (a^2 + b^2 - 1) = 0$$

$$\therefore x = \frac{a+b \pm \sqrt{(a+b)^2 - 2(a^2 + b^2 - 1)}}{2}$$

$$= \frac{a+b \pm \sqrt{2 - (-b)^2}}{2}.$$

$$\text{又} \quad a+b-2x=0$$

$$\therefore x = \frac{a+b}{2}.$$

$$19. \text{ 解: } \frac{x^2+3x+1}{4x^2+6x-1} - 3 \frac{4x^2+6x-1}{x^2+3x+1} - 2 = 0$$

$$\left[\frac{x^2+3x+1}{4x^2+6x-1} \right]^2 - 2 \left[\frac{x^2+3x+1}{4x^2+6x-1} \right] - 3 = 0$$

$$\left[\frac{x^2+3x+1}{4x^2+6x-1} + 1 \right] \left[\frac{x^2+3x+1}{4x^2+6x-1} - 3 \right] = 0$$

$$\frac{x^2+3x+1}{4x^2+6x-1} + 1 = 0$$

$$x(5x+9) = 0$$

$$\therefore x = 0 \quad \text{及} \quad -\frac{9}{5}.$$

$$\text{又} \quad \frac{x^2+3x+1}{4x^2+6x-1} - 3 = 0$$

$$11x^2+15x-4=0$$

$$\therefore x = \frac{-15 \pm \sqrt{225+176}}{22} = \frac{-15 \pm \sqrt{401}}{22}.$$

$$20. \text{ 解: } x^2 + \frac{1}{x^2} = a^2 + \frac{1}{a^2}$$

$$(x^2 - a^2) - \frac{(x^2 - a^2)}{a^2 x^2} = 0$$

$$(x^2 - a^2) \left\{ 1 - \frac{1}{a^2 x^2} \right\} = 0$$

$$\therefore x = \pm a, \quad \pm \frac{1}{a}.$$

$$21. \text{ 解: } 3x^2 - 2x - 5\sqrt{3x^2 - 2x + 3} + 9 = 0$$

$$(3x^2 - 2x + 3) - 5\sqrt{3x^2 - 2x + 3} + 6 = 0$$

$$[\sqrt{3x^2 - 2x + 3} - 2][\sqrt{3x^2 - 2x + 3} - 3] = 0$$

$$\sqrt{3x^2 - 2x + 3} = 2$$

$$3x^2 - 2x - 1 = 0$$

$$\therefore x = -\frac{1}{3}, \quad 1.$$

$$\begin{aligned} \text{又} \quad & \sqrt{3x^2-2x+3}=3 \\ & 3x^2-2x-6=0 \\ \therefore x &= \frac{1 \pm \sqrt{1+18}}{3} = \frac{1 \pm \sqrt{19}}{3}. \end{aligned}$$

$$\begin{aligned} 22. \text{ 解: } & 4x^2-2x-1=\sqrt{2x^2-x} \\ & 2(2x^2-x)-\sqrt{2x^2-x}-1=0 \\ & (2\sqrt{2x^2-x}+1)(\sqrt{2x^2-x}-1)=0 \\ & 2\sqrt{2x^2-x}=-1 \\ & 8x^2-4x-1=0 \\ \therefore x &= \frac{2 \pm \sqrt{4+8}}{8} = \frac{2 \pm 2\sqrt{3}}{8} = \frac{1 \pm \sqrt{3}}{4}. \end{aligned}$$

$$\begin{aligned} \text{又} \quad & \sqrt{2x^2-x}=1 \\ & 2x^2-x-1=0 \\ & (2x+1)(x-1)=0 \\ \therefore x &= -\frac{1}{2}, 1. \end{aligned}$$

$$\begin{aligned} 23. \text{ 解: } & \sqrt{3-x}+\sqrt{2-x}=\sqrt{5-2x} \\ & 3-x+2-x+2\sqrt{(3-x)(2-x)}=5-2x \\ & 2\sqrt{(3-x)(2-x)}=0 \\ \therefore x &= 3, 2. \end{aligned}$$

$$\begin{aligned} 24. \text{ 解: } & \sqrt{2x+3}+\sqrt{3x-5}-\sqrt{x+1}-\sqrt{4x-3}=0 \\ & \sqrt{2x+3}+\sqrt{3x-5}=\sqrt{x+1}+\sqrt{4x-3} \\ & 2\sqrt{(2x+3)(3x-5)}=2\sqrt{(x+1)(4x-3)} \\ & 6x^2-x-15=4x^2+x-3 \\ & x^2-x-6=0 \\ & (x-3)(x+2)=0 \\ \therefore x &= 3, -2. \end{aligned}$$

$$\begin{aligned} 25. \text{ 解: } & \frac{x^2-x+1}{x-1}-x=\sqrt{\frac{6}{x}} \\ & \frac{1}{x-1}=\sqrt{\frac{6}{x}} \\ & 6x^2-13x+6=0 \end{aligned}$$

$$(2x-3)(3x-2)=0$$

$$\therefore x = \frac{3}{2}, \frac{2}{3}.$$

26. 解: $\sqrt{x} + \sqrt{x-1} - x = 1$

$$\sqrt{x-1} = 1 - \sqrt{x}$$

$$x - \sqrt{x-1} = 1 + x - 2\sqrt{x}$$

$$4\sqrt{x} = 5x$$

$$16x = 25x^2$$

$$x(25x-16)=0$$

$$\therefore x = 0, \frac{16}{25}.$$

27. 解: $\sqrt{x+3} - \sqrt{x^2+3x} = 0$

$$\sqrt{x+3} = \sqrt{x^2+3x}$$

$$x+3 = x(x+3)$$

$$\therefore x = 1, -3.$$

28. 解: $\sqrt[4]{x^8} - 5\sqrt{x} + 6\sqrt[4]{x} = 0$

$$x^2 - 5x^{\frac{1}{2}} + 6x^{\frac{1}{4}} = 0$$

$$x^{\frac{1}{4}}(x^{\frac{3}{4}} - 5x^{\frac{1}{4}} + 6) = 0$$

$$\therefore x = 0.$$

$$x^{\frac{3}{4}} - 5x^{\frac{1}{4}} + 6 = 0$$

$$(x^{\frac{1}{4}} - 3)(x^{\frac{1}{4}} - 2) = 0$$

$$x^{\frac{1}{4}} = 3, 2$$

$$\therefore x = 81, 16.$$

29. 解: $\sqrt{\frac{2x-5}{x-2}} - 3\sqrt{\frac{x-2}{2x-5}} + 2 = 0$

$$\frac{2x-5}{x-2} + 2\sqrt{\frac{2x-5}{x-2}} - 3 = 0$$

$$\left(\sqrt{\frac{2x-5}{x-2}} + 3\right)\left(\sqrt{\frac{2x-5}{x-2}} - 1\right) = 0$$

$$2x-5-x+2=0 \quad \therefore x=3.$$

$$\begin{aligned}
 30. \text{ 解: } & \frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}=x-3 \\
 & \frac{(\sqrt{x-1}-\sqrt{x+1})^2}{(x-1)-(x+1)}=x-3 \\
 & -x+\sqrt{x^2-1}=x-3 \\
 & \sqrt{x^2-1}=2x-3 \\
 & x^2-1=4x^2-12x+9 \\
 & 3x^2-12x+10=0 \\
 \therefore x & =\frac{6\pm\sqrt{36-30}}{3}=\frac{6\pm\sqrt{6}}{3}
 \end{aligned}$$

$$\begin{aligned}
 31. \text{ 解: } & \sqrt{5x^2-6x+1}-\sqrt{5x^2+9x-2}=5x-1 \\
 & \sqrt{(5x-1)(x-1)}-\sqrt{(5x-1)(x+2)}=(\sqrt{5x-1})^2 \\
 & \sqrt{5x-1}=0 \quad \therefore x=\frac{1}{5} \\
 & \sqrt{x-1}-\sqrt{x+2}=\sqrt{5x-1} \\
 & 5x^2-16x+12=0 \\
 & (5x-6)(x-2)=0 \\
 \therefore x & =\frac{6}{5}, 2.
 \end{aligned}$$

但此二值非原方程式之二根，故當略去。

$$\begin{aligned}
 32. \text{ 解: } & \frac{\sqrt{2x-1}+\sqrt{3x}}{\sqrt{2x-1}-\sqrt{3x}}+3=0 \\
 & \frac{2x-1+3x+2\sqrt{2x-1}\cdot\sqrt{3x}}{2x-1-3x}+3=0 \\
 & 5x-1+2\sqrt{2x-1}\cdot\sqrt{3x}-3x-3=0 \\
 & \sqrt{2x-1}\cdot\sqrt{3x}=2-x \\
 & 5x^2+x-4=0 \\
 & (5x-4)(x+1)=0 \\
 \therefore x & =\frac{4}{5}, -1.
 \end{aligned}$$

但-1非原方程式之根，故當略去。

$$33. \text{ 解: } \sqrt[3]{x}+\sqrt[3]{2-x}=2$$

$$2-x=(2-\sqrt[3]{x})^3=8-12\sqrt[3]{x}+6\sqrt[3]{x^2}-x$$

$$6x^{\frac{2}{3}}-12x^{\frac{1}{3}}+6=0$$

$$6(x^{\frac{1}{3}}-1)^2=0$$

$$x^{\frac{1}{3}}=1$$

$$\therefore x=1.$$

34. 解: $(x+a)^{\frac{1}{3}}+(x+b)^{\frac{1}{3}}+(x+c)^{\frac{1}{3}}=0$

$$(x+a)^{\frac{1}{3}}+(x+b)^{\frac{1}{3}}=-(x+c)^{\frac{1}{3}}$$

$$x+a+3(x+a)^{\frac{2}{3}}(x+b)^{\frac{1}{3}}+3(x+a)^{\frac{1}{3}}(x+b)^{\frac{2}{3}}+x+b \\ =-(x+c)$$

$$3x+a+b+c=-3(x+a)^{\frac{1}{3}}(x+b)^{\frac{1}{3}}\{(x+a)^{\frac{1}{3}}+(x+b)^{\frac{1}{3}}\}$$

$$3x+(a+b+c)=-3(x+a)^{\frac{1}{3}}(x+b)^{\frac{1}{3}}\{-(x+c)^{\frac{1}{3}}\}$$

$$27x^3+27x^2(a+b+c)+9x(a+b+c)^2+(a+b+c)^2 \\ =27(x+a)(x+b)(x+c)$$

$$27x^3+27x^2(a+b+c)+9x(a+b+c)^2+(a+b+c)^3 \\ =27x^3+27x^2(a+b+c)+27x(ab+ca+bc)+27abc$$

$$\{9(a+b+c)^2-27(ab+ca+bc)\}x=27abc-(a+b+c)^3$$

$$\{9(a^2+b^2+c^2)-9(ab+bc+ca)\}x=27abc-(a+b+c)^3$$

$$\therefore x=\frac{27abc-(a+b+c)^3}{9(a^2+b^2+c^2-ab-bc-ca)}$$

35. 解: $x(x-1)(x-2)(x-3)=6\cdot 5\cdot 4\cdot 3$

$$(x^2-3x+2)(x^2-3x)=18\cdot 20$$

$$(x^2-3x)^2+2(x^2-3x)-18\cdot 20=0$$

$$(x^2-3x-18)(x^2-3x+20)=0$$

$$(x-6)(x+3)(x^2-3x+20)=0$$

$$\therefore x=6, -3.$$

$$x^2-3x+20=0$$

$$\therefore x=\frac{3\pm\sqrt{9-80}}{2}=\frac{3\pm\sqrt{71}i}{2}$$

36. 解: $(x+a)^2+4(x+a)\sqrt{x}=a^2-4a\sqrt{x}$

$$(x+a)^2-a^2=-4\sqrt{x}(x+2a)$$

$$x(x+2a) = -4\sqrt{x(x+2a)}$$

$$\sqrt{x}(\sqrt{x+4})(x+2a) = 0$$

$$\therefore x=0, -2a.$$

37. 解: $\sqrt[3]{1+\left(\frac{2x}{x^2-1}\right)^2} + \sqrt[3]{1+\frac{2}{x^2-1}} = 6$

$$\sqrt[3]{\frac{x^4+2x^2+1}{(x^2-1)^2}} + \sqrt[3]{\frac{x^2+1}{x^2-1}} = 6$$

$$\left[\frac{x^2+1}{x^2-1}\right]^{\frac{2}{3}} + \left[\frac{x^2+1}{x^2-1}\right]^{\frac{1}{3}} - 6 = 0$$

$$\left\{\left[\frac{x^2+1}{x^2-1}\right]^{\frac{1}{3}} + 3\right\} \left\{\left[\frac{x^2+1}{x^2-1}\right]^{\frac{1}{3}} - 2\right\} = 0$$

$$\frac{x^2+1}{x^2-1} = (-3)^3 = -27$$

$$x^2+1 = -27x^2+27$$

$$28x^2 = 26$$

$$x = \pm \sqrt{\frac{26}{28}} = \pm \sqrt{\frac{13}{14}} = \pm \frac{1}{14} \sqrt{182}.$$

$$\frac{x^2+1}{x^2-1} = 2^3 = 8$$

$$x^2+1 = 8x^2-8$$

$$7x^2 = 9$$

$$\therefore x = \pm \sqrt{\frac{9}{7}} = \pm \frac{1}{7} \sqrt{63} = \pm \frac{3}{7} \sqrt{7}.$$

XVI. 聯立方程式之能以二次方程解之者

習題 XLV

原本第 320 頁

1. 解: $\begin{cases} 7x^2 - 6xy = 8 & \dots\dots\dots (1) \\ 2x - 3y = 8 & \dots\dots\dots (2) \end{cases}$

從 (2) 式, $y = \frac{2x-5}{3}$

代入(1)式,得 $3x^2+10x-8=0$
 $(3x-2)(x+4)=0$

$$\therefore x = \frac{2}{3} \quad \text{及} \quad -4,$$

$$y = -\frac{11}{9} \quad \text{及} \quad \frac{13}{3}.$$

答: 其根爲 $\frac{2}{3}, -\frac{11}{9}, -4, -\frac{13}{3}$.

2. 解: $\begin{cases} xy=1 \dots\dots\dots(1) \\ 3x-5y=2 \dots\dots\dots(2) \end{cases}$

從(1)式, $x = \frac{1}{y}$

代入(2)式,得 $5y^2+2y-3=0$
 $(5y-3)(y+1)=0$

$$\therefore y = \frac{3}{5} \quad \text{及} \quad -1,$$

$$x = \frac{5}{3} \quad \text{及} \quad -1.$$

答: 其根爲 $\frac{5}{3}, \frac{3}{5}, -1, -1$.

3. 解: $\begin{cases} x^2+x=4y^2 \dots\dots\dots(1) \\ 3x+6y=1 \dots\dots\dots(2) \end{cases}$

從(2)式, $y = \frac{1-3x}{6}$

代入(1)式,得 $x^2+x=4\left[\frac{1-3x}{6}\right]^2$

$$15x=1$$

$$\therefore x = \frac{1}{15}, y = \frac{2}{15}.$$

答: 其根爲 $\frac{1}{15}, \frac{2}{15}$; 一無根。

4. 解: $\begin{cases} 3x^2-3xy-y^2-4x-8y+3=0 \dots\dots\dots(1) \\ 3x-y-8=0 \dots\dots\dots(2) \end{cases}$

從(2)式, $3x-8$

代入 (1) 式, 得 $15x^2 - 44x - 3 = 0$
 $(15x+1)(x-3) = 0$

$$\therefore x = -\frac{1}{15} \text{ 及 } 3,$$

$$y = -\frac{41}{5} \text{ 及 } 1.$$

答: 其根爲 $-\frac{1}{15}, -\frac{41}{5}; 3, 1.$

5. 解: $\begin{cases} x^2 + 5y^2 - 8x - 7y = 0 \dots\dots\dots(1) \\ x + 3y = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $x = -3y$

代入 (1) 式, 得 $14y^2 + 17y = 0$
 $y(14y + 17) = 0$

$$\therefore y = 0 \text{ 及 } -\frac{17}{14},$$

$$x = 0 \text{ 及 } \frac{51}{14}.$$

答: 其根爲 $0, 0; \frac{51}{14}, -\frac{17}{14}.$

6. 解: $\begin{cases} 2x^2 - xy - 3y = 0 \dots\dots\dots(1) \\ 7x - 6y - 4 = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $y = \frac{7x-4}{6}$

代入 (1) 式, 得 $5x^2 - 17x + 12 = 0$
 $(x-1)(5x-12) = 0$

$$\therefore x = 1 \text{ 及 } \frac{12}{5},$$

$$y = \frac{1}{2} \text{ 及 } \frac{32}{15}.$$

答: 其根爲 $1, \frac{1}{2}; \frac{12}{5}, \frac{32}{15}.$

7. 解: $\begin{cases} x^2 + 3xy + 2y^2 - 1 = 0 \dots\dots\dots(1) \\ x + y = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $y = -x$

代入 (1) 式, 得 $x^2 - 3x^2 + 2x^2 - 1 = 0$

但 $-1 \neq 0$

答: 此組方程式有兩對無限根。

$$8. \text{ 解: } \begin{cases} 2x + 3y = 37 \dots\dots\dots (1) \\ \frac{1}{x} + \frac{1}{y} = \frac{14}{45} \dots\dots\dots (2) \end{cases}$$

從 (2) 式, $45y + 45x - 14xy = 0$

$$y = \frac{45x}{14x - 45}$$

代入 (1) 式, 得 $28x^2 - 473x + 1665 = 0$

$$\therefore x = \frac{473 \pm \sqrt{223729 - 186480}}{56}$$

$$= \frac{473 \pm 193}{56}$$

$$= \frac{333}{28} \quad \text{及} \quad 5,$$

$$y = \frac{185}{42} \quad \text{及} \quad 9.$$

答: 其根爲 $\frac{333}{28}, \frac{185}{42}; 5, 9.$

$$9. \text{ 解: } \begin{cases} \frac{1}{y} - \frac{3}{x} = 1 \dots\dots\dots (1) \\ \frac{7}{xy} - \frac{1}{y^2} = 12 \dots\dots\dots (2) \end{cases}$$

從 (1) 式, $\frac{1}{x} = \frac{1}{3} \left(\frac{1}{y} - 1 \right)$

代入 (2) 式, 得 $\frac{7}{3} \cdot \frac{1}{y^2} - \frac{7}{3} \cdot \frac{1}{y} - \frac{1}{y^2} = 12$

$$36y^2 + 7y - 4 = 0$$

$$(4y - 1)(9y + 4) = 0$$

$$\therefore y = -\frac{4}{9} \quad \text{及} \quad \frac{1}{4},$$

$$x = -\frac{12}{13} \quad \text{及} \quad 1.$$

答：其根爲 $-\frac{12}{13}, -\frac{4}{9}; 1, \frac{1}{4}$.

10. 解： $\begin{cases} x^2 + xy + 2 = 0 \dots\dots\dots(1) \\ (3x + y)(2x + y - 1) = 0 \dots\dots\dots(2) \end{cases}$

從(2)式, $\begin{cases} 3x + y = 0 & \text{及} & 2x + y - 1 = 0 \\ y = -3x & \text{及} & -2x + 1 = 0 \dots\dots\dots(3) \end{cases}$

分別代入(1)式, 得 $2x^2 = 2 \therefore x = \pm 1$.

及 $x^2 - x - 2 = 0 \therefore x = -1, 2$.

將 x 之值分別代入(3)式, 得

$$y = \mp 3, 3, -3.$$

答：其根爲 $1, -3; -1, 3; -1, 3; 2, -3$.

11. 解： $\begin{cases} x^2 + y^2 - 8 = 0 \dots\dots\dots(1) \\ (x + 1)^2 = (y - 1)^2 \dots\dots\dots(2) \end{cases}$

從(2)式, $\begin{cases} (x - y + 2)(x + y) = 0 \\ x - y + 2 = 0 \dots\dots\dots(3) \\ x + y = 0 \dots\dots\dots(4) \end{cases}$

解(1), (3); 得 $x = -1 \pm \sqrt{3}$,

$$y = 1 \pm \sqrt{3}.$$

解(1), (4); 得 $x = \pm 2$,

$$y = \mp 2.$$

答：其根爲 $-1 + \sqrt{3}, 1 + \sqrt{3}; -1 - \sqrt{3}, 1 - \sqrt{3}; 2, -2; -2, 2$.

12. 解： $\begin{cases} x^2 - xy - 2y^2 + y = 0 \dots\dots\dots(1) \\ (x - 2y)(x + y - 3) = 0 \dots\dots\dots(2) \end{cases}$

從(1)式, $(x + y)(x - 2y) + y = 0 \dots\dots\dots(3)$

從(2)式, $\begin{cases} x - 2y = 0 \dots\dots\dots(4) \\ x + y - 3 = 0 \dots\dots\dots(5) \end{cases}$

解(3), (4); 得 $x = 0, y = 0$; 及一無限解。

解(3), (5); 得 $x = \frac{15}{8}, y = \frac{9}{8}$; 及一無限解。

答：其根爲 $0, 0; \frac{15}{8}, \frac{9}{8}$; 及二無限解。

13. 解： $\begin{cases} y^2 + 4x + 4 = 0 \dots\dots\dots(1) \\ y = mx \dots\dots\dots(2) \end{cases}$

將 (2) 式代入 (1) 式, 得

$$m^2x^2 + 4x + 4 = 0$$

若二解答相等, 則

$$b^2 - 4ac = 0$$

即

$$16 - 16m^2 = 0$$

$$m^2 = 1$$

$$\therefore m = \pm 1.$$

14. 解: $\begin{cases} x^2 + xy - 2y^2 + x = 0 & \dots\dots\dots(1) \\ y = mx + c & \dots\dots\dots(2) \end{cases}$

將 (2) 代入 (1) 式, 得

$$(1 + m - 2m^2)x^2 + (c - 4mc + 1)x - 2c^2 = 0$$

若兩解答爲無限, 則

$$1 + m - 2m^2 = 0 \dots\dots\dots(3)$$

$$c - 4mc + 1 = 0 \dots\dots\dots(4)$$

從 (3) 式,

$$(1 + 2m)(1 - m) = 0$$

$$\therefore m = -\frac{1}{2} \quad \text{及} \quad 1.$$

代入 (4) 式, 得

$$c = -\frac{1}{3} \quad \text{及} \quad \frac{1}{3}.$$

答: m, c 之值爲 $-\frac{1}{2}, -\frac{1}{3}; 1, \frac{1}{3}$.

15. 解: $\begin{cases} 2x - y + 4 = 0 & \dots\dots\dots(1) \\ 2x^2 + xy - y^2 + 10x + y + 12 = 0 & \dots\dots\dots(2) \end{cases}$

從 (1) 式,

$$y = 2x + 4$$

代入 (2) 式, 得 $4x^2 - 4x^2 + 16x - 16x + 16 - 16 = 0$

但此方程式爲一恆等式, 故 x 之任何值皆能滿足之, 是以 $2x - y + 4$ 爲 $2x^2 + xy - y^2 + 10x + y + 12$ 之因子。

16. 解: $\begin{cases} xy = 1 & \dots\dots\dots(1) \\ xy + x + y = 0 & \dots\dots\dots(2) \end{cases}$

從 (1) 式,

$$x = \frac{1}{y}$$

代入 (2) 式, 得

$$1 + \frac{1}{y} + y = 0$$

即 $y^2 + y + 1 = 0$

此方程式為二次方程式，故僅能有二有限解答。再 (1), (2) 兩式中最高次的項均有 x 及 y 兩因式已示 (1), (2) 兩式有二無限解，故 (1), (2) 二式僅能有 $4 - 2$ 即 2 有限解答。

$$\begin{cases} x^2y + xy = 1 \dots\dots\dots (3) \\ x^2 + y^2 = 2 \dots\dots\dots (4) \end{cases}$$

從 (3) 式， $y = \frac{1}{y^2 + x}$

代入 (4) 式，得 $x^4 + 3x^3 + 2x^2 - 1 = 0$

此方程式為四次方程式，故僅能有四有限解。再對應於 (3), (4) 兩式中三次項的公因式 x^2 示 (3), (4) 二式有二無限解。又對應於 (3), (4) 兩式中一, 二, 三次項的公因式 y 示 (3), (4) 兩式又有 3 無限解，故 (3), (4) 兩式有 5 無限解及 4 有限解。

習 題 XLVI.

原本第 324 頁

1. 解: $\begin{cases} x^2 + 3y^2 = 31 \dots\dots\dots (1) \\ 7x^2 - 2y^2 = 10 \dots\dots\dots (2) \end{cases}$

(1) $\times 2 +$ (2) $\times 3$, $23x^2 = 92$

$x^2 = 4 \therefore x = \pm 2.$

代入 (1) 式，得

$y^2 = 9 \therefore y = \pm 3.$

答：方程式 (1), (2) 之四解為 $2, 3; 2, -3; -2, 3; -2, -3.$

2. 解: $\begin{cases} \frac{36}{x^2} + \frac{1}{y^2} = 18 \dots\dots\dots (1) \\ \frac{1}{y^2} - \frac{4}{x^2} = 8 \dots\dots\dots (2) \end{cases}$

(1) $-$ (2), $\frac{40}{x^2} = 10 \therefore x^2 = 4$

$\therefore x = \pm 2.$

將 x 之各值代入 (2) 式：

當 $x=2$ 時, $y = \pm \frac{1}{3}$.

當 $x=-2$ 時, $y = \pm \frac{1}{3}$.

答: 其根爲 $2, \frac{1}{3}; 2, -\frac{1}{3}; -2, \frac{1}{3}; -2, -\frac{1}{3}$.

3. 解: $\begin{cases} y^2 + xy + 6 = 0 \dots\dots\dots(1) \\ y^2 - y - 2 = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $(y+1)(y-2) = 0$

$\therefore y = -1$ 及 2 .

代入 (1) 式, 得 $x = 7$ 及 -5 .

答: 其根爲 $7, -1; -5, 2$; 及二無限根對應於 (1), (2) 中一, 二次項的公因式 y .

4. 解: $\begin{cases} x^2 + y^2 - 3x + 2y - 39 = 0 \dots\dots\dots(1) \\ 3x^2 - 17xy + 10y^2 = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $(3x-2y)(x-5y) = 0$

$x = \frac{2y}{3}$ 及 $5y$

將 $x = \frac{2y}{3}$ 代入 (1) 式, 得

$y^2 = 27$

$\therefore y = \pm 3\sqrt{3}$,

$x = \pm 2\sqrt{3}$.

將 $x = 5y$ 代入 (1) 式, 得

$2y^2 - y - 3 = 0$

$(2y-3)(y+1) = 0$

$\therefore y = \frac{3}{2}$ 及 -1 ,

$x = \frac{15}{2}$ 及 -5 .

答: 其四解爲 $2\sqrt{3}, 3\sqrt{3}; -2\sqrt{3}, -3\sqrt{3}; \frac{15}{2}, \frac{3}{2}; -5, -1$.

5. 解: $\begin{cases} y^2 - x^2 - 5 = 0 \dots\dots\dots(1) \\ 4x^2 + 4xy + y^2 + 4x + 2y = 3 \dots\dots\dots(2) \end{cases}$

分解(2)式,得 $(2x+y+3)(2x+y-1) = 0$

$y = -(2x+3) \dots\dots\dots(3)$

及 $y = 1 - 2x \dots\dots\dots(4)$

將(3)式代入(1)式,得

$$3x^2 + 12x + 4 = 0$$

$$\therefore x = \frac{-6 \pm \sqrt{36 - 12}}{3} = \frac{-6 \pm 2\sqrt{6}}{3}$$

代入(3)式,得

$$y = \frac{3 \mp 4\sqrt{6}}{3}$$

將(4)式代入(1)式,得

$$3x^2 - 4x - 4 = 0$$

$$(3x+2)(x-2) = 0$$

$$\therefore x = -\frac{2}{3} \quad \text{及} \quad 2$$

代入(4)式,得 $y = \frac{7}{3}$ 及 -3 .

答: 其根爲 $2, -3; -\frac{2}{3}, \frac{7}{3}; \frac{-6 \pm 2\sqrt{6}}{3}, \frac{3 \mp 4\sqrt{6}}{3}$.

6. 解: $\begin{cases} x^2 + 5xy - 2x + 3y + 1 = 0 \dots\dots\dots(1) \\ 3x^2 + 15xy - 7x + 8y + 4 = 0 \dots\dots\dots(2) \end{cases}$

(1) $\times 3 - (2)$, $x + y = 1$

$$y = 1 - x$$

代入(1)式,得 $4x^2 - 4 = 0$

$$\therefore x = \pm 1,$$

$$y = 0 \quad \text{及} \quad 2.$$

答: 其根爲 $1, 0; -1, 2$; 及二無限根對應於(1), (2)兩式中最高次項的公因式 x 及 $x+5y$.

7. 解: $\begin{cases} x^2 - 15xy - 3y^2 + 2x + 9y = 98 \dots\dots\dots(1) \\ 5xy + y^2 - 3y = -21 \dots\dots\dots(2) \end{cases}$

(1) $+ (2) \times 3$, $x^2 + 2x - 35 = 0$

$$(x+7)(x-5)=0$$

$$\therefore x=-7 \text{ 及 } 5.$$

將 $x=-7$ 代入 (2) 式, 得

$$y^2-38y+21=0$$

$$\therefore y=19 \pm \sqrt{361-21}=19 \pm 2\sqrt{85}.$$

將 $x=5$ 代入 (2) 式, 得

$$y^2+22y+21=0$$

$$(y+1)(y+21)=0$$

$$\therefore y=-1, -21.$$

答: 其根爲 $-7, 19+2\sqrt{85}; -7, 19-2\sqrt{85}; 5, -1; 5, -21.$

$$8. \text{ 解: } \begin{cases} 2x^2+3y-4y^2=25 & \dots\dots\dots(1) \\ 15x^2+24xy-31y^2=200 & \dots\dots\dots(2) \end{cases}$$

$$(1) \times 8 - (2), \quad x^2 - y^2 = 0$$

$$(x+y)(x-y)=0$$

$$x=-y \text{ 及 } x=y$$

將 $x=-y$ 代入 (1) 式, 得

$$y^2 = -5$$

$$\therefore y = \pm \sqrt{5}i,$$

$$x = \mp \sqrt{5}i.$$

將 $x=y$ 代入 (1) 式, 得

$$y^2 = 25$$

$$\therefore y = \pm 5,$$

$$x = \pm 5.$$

答: 其根爲 $5, 5; -5, -5; \sqrt{5}i, -\sqrt{5}i; -\sqrt{5}i, \sqrt{5}i.$

$$9. \text{ 解: } \begin{cases} x(x+3y)=18 & \dots\dots\dots(1) \\ x^2-5y^2=4 & \dots\dots\dots(2) \end{cases}$$

$$2 \times (1) - 9 \times (2), \quad 45y^2 + 6xy - 7x^2 = 0$$

$$(15y+7x)(3y-x)=0$$

$$y = -\frac{7x}{15} \text{ 及 } y = \frac{x}{3}$$

將 $y = -\frac{7x}{15}$ 代入 (2) 式, 得

$$x^2 = -45$$

$$\therefore x = \pm \sqrt{45i} = \pm 3\sqrt{5i},$$

$$y = \mp \frac{7\sqrt{5i}}{5}.$$

將 $y = \frac{x}{3}$ 代入 (2) 式, 得

$$x^2 = 9$$

$$\therefore x = \pm 3,$$

$$y = \pm 1.$$

答: 其根爲 $3\sqrt{5i}, \frac{-7\sqrt{5i}}{5}; -3\sqrt{5i}, \frac{7\sqrt{5i}}{5}; 3, 1; -3, -1.$

10. 解: $\begin{cases} x^2 - 3xy + 3y^2 = x^2y^2 \dots\dots\dots(1) \\ 7x^2 - 10xy + 7y^2 = 12x^2y^2 \dots\dots\dots(2) \end{cases}$

$$(1) \times 12 - (2), \quad 5x^2 - 26xy + 32y^2 = 0$$

$$(5x - 16y)(x - 2y) = 0$$

$$x = \frac{16}{5}y \quad \text{及} \quad x = 2y$$

將 $x = \frac{16}{5}y$ 代入 (1) 式, 得

$$y^2(256y^2 - 91) = 0$$

$$\therefore y = 0, 0, \pm \frac{1}{16}\sqrt{91};$$

$$x = 0, 0, \pm \frac{1}{5}\sqrt{91}.$$

將 $x = 2y$ 代入 (1) 式, 得

$$y^2(4y^2 - 1) = 0$$

$$\therefore y = 0, 0, \pm \frac{1}{2};$$

$$x = 0, 0, \pm 1.$$

答: 其根爲 $0, 0; 0, 0; 0, 0; 0, 0; \frac{1}{16}\sqrt{91}, \frac{1}{5}\sqrt{91}; -\frac{1}{16}\sqrt{91}, -\frac{1}{5}\sqrt{91}; \frac{1}{2}, 1; -\frac{1}{2}; -1;$ 及 8 無限解對應於 (1),

(2) 兩式中三次項及四次項的公因式 $x, x, y, y.$

11. 解: $\begin{cases} x^2 + xy + y^2 = 38 & \dots\dots\dots(1) \\ x^2 - xy + y^2 = 14 & \dots\dots\dots(2) \end{cases}$

(1) + (2), $x^2 + y^2 = 26$ $\dots\dots\dots(3)$

(1) - (2), $2xy = 24$ $\dots\dots\dots(4)$

(3) + (4), $(x+y)^2 = 50$

$x+y = \pm 5\sqrt{2}$ $\dots\dots\dots(5)$

(3) - (4), $(x-y)^2 = 2$

$x-y = \pm\sqrt{2}$ $\dots\dots\dots(6)$

(5) + (6), $x = \pm 3\sqrt{2}, \pm 2\sqrt{2}.$

(5) - (6), $y = \pm 2\sqrt{2}, \pm 3\sqrt{2}.$

答: 其根爲 $3\sqrt{2}, 2\sqrt{2}; -3\sqrt{2}, -2\sqrt{2}; 2\sqrt{2}, 3\sqrt{2}; -2\sqrt{2}, -3\sqrt{2}.$

12. 解: $\begin{cases} x^2 - xy + y^2 = 21(x-y) & \dots\dots\dots(1) \\ xy = 20 & \dots\dots\dots(2) \end{cases}$

(1) - (2), $x^2 - 2xy + y^2 = 21(x-y) - 20$

$(x-y)^2 - 21(x-y) + 20 = 0$

$(x-y-1)(x-y-20) = 0$

$x = 1 + y$ $\dots\dots\dots(3)$

$x = 20 + y$ $\dots\dots\dots(4)$

將 (3) 式代入 (2) 式, 得

$$y^2 + y - 20 = 0$$

$$(y-4)(y+5) = 0$$

$$\therefore y = 4 \quad \text{及} \quad -5,$$

$$x = 5 \quad \text{及} \quad -4.$$

將 (4) 式代入 (2) 式, 得

$$y^2 + 20y - 20 = 0$$

$$\therefore y = -10 \pm \sqrt{100 + 20} = -10 \pm 2\sqrt{30},$$

$$x = 10 \pm 2\sqrt{30}.$$

答: 其根爲 $5, 4; -4, -5; 10 \pm 2\sqrt{30}, -10 \pm 2\sqrt{30}.$

13. 解: $\begin{cases} x^2 + y - 8 = 0 & \dots\dots\dots(1) \\ y^2 + 15x - 46 = 0 & \dots\dots\dots(2) \end{cases}$

從 (1) 式, $y = 8 - x^2$

代入(2)式,得 $x^4 - 16x^2 + 15x + 18 = 0$
 $(x-2)(x-3)(x^2+5x+3) = 0$

$$\therefore x = 2, 3, \frac{-5 \pm \sqrt{13}}{2};$$

$$y = 4, -1, \frac{-3 \pm 5\sqrt{13}}{2}.$$

答: 其根爲 2, 4; 3, -1; $\frac{-5 \pm \sqrt{13}}{2}$, $\frac{-3 \pm 5\sqrt{13}}{2}$.

習題 XLVII

原本第 325 頁

1. 解: $\begin{cases} x^3 - y^3 = 63 & \dots\dots\dots(1) \\ x - y = 3 & \dots\dots\dots(2) \end{cases}$

(1) \div (2), $x^2 + xy + y^2 = 21$ $\dots\dots\dots(3)$

(2) 平方, $x^2 - 2xy + y^2 = 9$ $\dots\dots\dots(4)$

(3) $-$ (4), $3xy = 12$

$xy = 4$ $\dots\dots\dots(5)$

(3) $+$ (5), $x^2 + 2xy + y^2 = 25$

$x + y = \pm 5$ $\dots\dots\dots(6)$

(2) $+$ (6), $x = 4, -1.$

代入(2)式,得 $y = 1, -4.$

答: 其根爲 4, 1; -1, -1; 及一無限根對應於(1), (2)兩式
 中最高次項的公因式 $x - y$.

2. 解: $\begin{cases} x + y = 98 & \dots\dots\dots(1) \\ \sqrt[3]{x} + \sqrt[3]{y} = 2 & \dots\dots\dots(2) \end{cases}$

(1) \div (2), $\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2} = 49$ $\dots\dots\dots(3)$

從(2)式, $\sqrt[3]{x} = 2 - \sqrt[3]{y}$

(2) 平方, $\sqrt[3]{x^2} + 2\sqrt[3]{x}\sqrt[3]{y} + \sqrt[3]{y^2} = 4$ $\dots\dots\dots(4)$

(4) $-$ (3), $3\sqrt[3]{x}\sqrt[3]{y} = -45$

$\sqrt[3]{xy} = -15$ $\dots\dots\dots(5)$

(3) $-$ (5), $(\sqrt[3]{x} - \sqrt[3]{y})^2 = 64$

$\sqrt[3]{x} - \sqrt[3]{y} = \pm 8$ $\dots\dots\dots(6)$

$$(2)+(6), \quad 2\sqrt[3]{x}=10 \text{ 或 } -6$$

$$\therefore x=125 \text{ 或 } -27.$$

代入 (1) 式, 得 $y=-27$ 或 125 .

答: 其根爲 $-27, 125; 125, -27$; 及一無限根.

$$3. \text{ 解: } \begin{cases} x^4+x^2y^2+y^4=931 \dots\dots\dots(1) \\ x^2+xy+y^2=49 \dots\dots\dots(2) \end{cases}$$

$$(1) \div (2), \quad x^2-xy+y^2=19 \dots\dots\dots(3)$$

$$(2)-(3), \quad 2xy=30$$

$$xy=15 \dots\dots\dots(4)$$

$$(2)+(4), \quad (x+y)^2=64$$

$$x+y=\pm 8 \dots\dots\dots(5)$$

$$(3)-(4), \quad (x-y)^2=4$$

$$x-y=\pm 2 \dots\dots\dots(6)$$

$$\frac{(5)+(6)}{2}, \quad x=5, -5, 3, -3.$$

代入 (4) 式, 得 $y=3, -3, 5, -5$.

答: 其各根爲 $5, 3; -5, -3; 3, 5; -3, -5$; 及四無限解對
應於 (1), (2) 兩式中最高次項的公因式 x^2+xy+y^2 .

$$4. \text{ 解: } \begin{cases} (x+y)(x^2-2y^2)=-70 \dots\dots\dots(1) \\ (x-y)(x^2-2y^2)=14 \dots\dots\dots(2) \end{cases}$$

$$(1) \div (2), \quad x+y=-5(x-y).$$

$$x=\frac{2}{3}y.$$

$$\text{代入 (2) 式, 得 } \left[\frac{2}{3}y-y\right]\left[\frac{4}{9}y^2-2y^2\right]=14$$

$$y^3-27=0$$

$$(y-3)(y^2+3y+9)=0$$

$$\therefore y=3, 3(-1 \pm \sqrt{3}i)/2;$$

$$x=2, -1 \pm \sqrt{3}i.$$

答: 其根爲 $2, 3; -1 \pm \sqrt{3}i, 3(-1 \pm \sqrt{3}i)/2$; 及六無限根.

$$5. \text{ 解: } \begin{cases} (x+y)^2(x-y)=3xy+6y \dots\dots\dots(1) \\ x^2-y^2=x+2 \dots\dots\dots(2) \end{cases}$$

$$(1) \div (2), \quad x+y=3y$$

$$x = 2y \dots\dots\dots (3)$$

又
及

$$x = -2 \dots\dots\dots (4)$$

$$x^2 = y^2$$

$$x = \pm y \dots\dots\dots (5)$$

代入 (2) 式:

$$\text{當 } x = -y, \quad y = 2 \quad \therefore x = -2.$$

$$\text{當 } x = y, \quad y = -2 \quad \therefore x = -2.$$

$$\text{當 } x = -2, \quad y = \pm 2$$

$$\text{當 } x = 2y, \quad y = (1 \pm \sqrt{7})/3, \quad x = (2 \pm 2\sqrt{7})/3.$$

答: 其根爲 $-2, 2; -2, -2; (2 \pm 2\sqrt{7})/3, (1 \pm \sqrt{7})/3$; 及二無限根.

$$6. \text{ 解: } \begin{cases} x^2 - 3xy + 2y^2 = 6x \dots\dots\dots (1) \\ x^2 - y^2 = -5y \dots\dots\dots (2) \end{cases}$$

$$(1) \div (2), \quad \frac{x-2y}{x+y} = -\frac{6x}{5y}$$

$$6x^2 + 11xy - 10y^2 = 0 \quad (3x-2y)(2x+5y) = 0$$

$$x = \frac{2}{3}y \quad \text{及} \quad x = \frac{5}{2}y$$

$$\text{代入 (2) 式, 得 } y^2 - 9y = 0 \dots\dots\dots (3)$$

$$\text{及 } 21y^2 + 20y = 0 \dots\dots\dots (4)$$

$$\text{解 (3) 式, 得 } y = 0 \text{ 及 } 9,$$

$$x = \frac{2}{3}y = 0, \text{ 及 } 6.$$

$$\text{解 (4) 式, 得 } y = 0 \text{ 及 } -\frac{20}{21},$$

$$x = -\frac{5}{2}y = 0 \text{ 及 } -\frac{50}{21}.$$

答: 其根爲 $0, 0; 6, 9; \frac{50}{21}, -\frac{20}{21}$; 及一無限根.

習題 XLVIII

原本第 328 頁

$$1. \text{ 解: } \begin{cases} x+y=5 \dots\dots\dots (1) \\ xy+36=0 \dots\dots\dots (2) \end{cases}$$

從 (1) 式, $x = 5 - y$
 代入 (2) 式, 得 $y^2 - 5y - 36 = 0$
 $(y - 9)(y + 4) = 0$
 $\therefore y = 9$ 及 $-4,$
 $x = -4$ 及 $9.$

2. 解: $\begin{cases} x^2 + y^2 = 200 & \dots\dots\dots(1) \\ x + y = 12 & \dots\dots\dots(2) \end{cases}$
 (2) 平方, $x^2 + 2xy + y^2 = 144 \dots\dots\dots(3)$
 $2 \times (1) - (3),$ $x^2 - 2xy + y^2 = 256$
 $x - y = \pm 16 \dots\dots\dots(4)$
 $\frac{(2) + (4)}{2},$ $x = 14$ 及 $-2,$
 $y = -2$ 及 $14.$

3. 解: $\begin{cases} x^2 + y^2 = 293 & \dots\dots\dots(1) \\ xy = 34 & \dots\dots\dots(2) \end{cases}$
 $(1) + 2 \times (2),$ $(x + y)^2 = 361$
 $x + y = \pm 19 \dots\dots\dots(3)$
 $(1) - 2 \times (2),$ $(x - y)^2 = 225$
 $x - y = \pm 15 \dots\dots\dots(4)$
 從 (3) 及 (4), $x = \pm 17, \pm 2.$
 代入 (2) 式, $y = \pm 2, \pm 17.$

4. 解: $\begin{cases} x^2 + y^2 = 85 & \dots\dots\dots(1) \\ x - y = 7 & \dots\dots\dots(2) \end{cases}$
 從 (1) 式, $x = 7 + y$
 代入 (1) 式, 得 $y^2 + 7y - 18 = 0$
 $(y + 9)(y - 2) = 0$
 $\therefore y = -9$ 及 $2,$
 $x = -2$ 及 $9.$

5. 解: $\begin{cases} x^2 + y^2 = 513 & \dots\dots\dots(1) \\ x + y = 9 & \dots\dots\dots(2) \end{cases}$
 $(1) \div (2),$ $x^2 - xy + y^2 = 57 \dots\dots\dots(3)$
 從 (2) 式, $y = 9 - x$
 代入 (3) 式, 得 $x^2 - 9x + 8 = 0$
 $(x - 8)(x - 1) = 0$

$$\therefore x=8, 1;$$

$y=1, 8$; 及一無限解對應於 (1), (2) 中最高次項的公因式 $x+y$.

$$6. \text{ 解: } \begin{cases} x^3+y^3=468 & \dots\dots\dots(1) \\ x^2y+xy^2=420 & \dots\dots\dots(2) \end{cases}$$

對應於 (1), (2) 兩式中一次, 二次, 三項的公因式 $x+y$ 有三無限解.

$$(1)+(2) \times 3, \quad x^3+3x^2y+3xy^2+y^3=1728$$

$$(x+y)^3=12^3$$

$$[(x+y)-12][(x+y)^2+12(x+y)+144]=0.$$

$$x+y=12 \quad \dots\dots\dots(3)$$

$$(x+y)^2+12(x+y)+144=0.$$

$$x+y=-6 \pm \sqrt{36-144} = -6 \pm 6\sqrt{3}i \quad \dots\dots\dots(4)$$

解 (2) 和 (3) 及 (2) 和 (4), 即得 (1), (2) 兩式之各有限解.

解 (2) 和 (3):

$$(2) \div (3), \quad xy=35 \quad \dots\dots\dots(5)$$

解 (5) 和 (3), 得 $x=7, 5$;

$$y=5, 7.$$

解 (2) 和 (4):

$$(2) \div (4), \quad xy = \frac{70}{-1 \pm \sqrt{3}i} = \frac{35}{2}(-1 \mp \sqrt{3}i) \quad \dots\dots\dots(6)$$

$$(4)^2 - 4(6), \quad (x-y)^2 = -2(1 \pm \sqrt{3}i)$$

$$x-y = \pm \sqrt{-2 \mp 2\sqrt{3}i}$$

$$= \pm(\sqrt{3}i \mp 1) \quad \dots\dots\dots(7)$$

$$(4)+(7), \quad 2x = (-6+6\sqrt{3}i) \pm (\sqrt{3}i-1) \quad \dots\dots\dots(8)$$

$$\text{及} \quad 2x = (-6-6\sqrt{3}i) \pm (\sqrt{3}i+1) \quad \dots\dots\dots(9)$$

$$\text{從 (8) 式,} \quad x = \frac{-7+7\sqrt{3}i}{2}, \quad \frac{-5+5\sqrt{3}i}{2}.$$

代入 $x+y = -6+6\sqrt{3}i$, 得

$$y = \frac{-5+5\sqrt{3}i}{2}, \quad \frac{-7+7\sqrt{3}i}{2}.$$

$$\text{從 (9) 式,} \quad x = \frac{-7-7\sqrt{3}i}{2}, \quad \frac{-5-5\sqrt{3}i}{2}.$$

代入 $x+y = -6 - 6\sqrt{3}i$, 得

$$y = \frac{-5 - 5\sqrt{3}i}{2}, \frac{-7 - 7\sqrt{3}i}{2}$$

答: 其有限解爲 $7, 5; 5, 7; \frac{7}{2}(-1 + \sqrt{3}i), \frac{5}{2}(-1 + \sqrt{3}i);$

$$\frac{5}{2}(-1 - \sqrt{3}i), \frac{7}{2}(-1 + \sqrt{3}i); \frac{-5}{2}(1 + \sqrt{3}i),$$

$$\frac{-7}{2}(1 + \sqrt{3}i); \frac{-7}{2}(1 + \sqrt{3}i), \frac{-5}{2}(1 + \sqrt{3}i).$$

7. 解: $\begin{cases} 27x^2 + 64y^2 = 65 \dots\dots\dots(1) \\ 3x + 4y = 5 \dots\dots\dots(2) \end{cases}$

對應於最高次項中的公因式 $6x + 4y$ 有一無限解。

(1) \div (2), $9x^2 - 12xy + 16y^2 = 13 \dots\dots\dots(3)$

從 (2) 式, $y = \frac{5 - 3x}{4}$

代入 (3) 式, 得 $27x^2 - 45x + 12 = 0$

$$(3x - 4)(9x - 3) = 0$$

$$\therefore x = \frac{4}{3} \text{ 及 } \frac{1}{3},$$

$$y = \frac{1}{4} \text{ 及 } 1.$$

8. 解: $\begin{cases} x^4 + y^4 = 82 \dots\dots\dots(1) \\ x - y = 2 \dots\dots\dots(2) \end{cases}$

設 $x = u + v, y = u - v$

從 (1) 式, $(u + v)^4 + (u - v)^4 = 82 \dots\dots\dots(3)$

從 (2) 式, $2v = 2$

代入 (3) 式, $u^4 + 6u^2 - 40 = 0$

$$(u^2 + 10)(u^2 - 4) = 0$$

$$u^2 = -10 \text{ 及 } 4$$

$$\therefore u = \pm\sqrt{10}i \text{ 及 } \pm 2;$$

$$x = \pm\sqrt{10}i + 1, 3, -1;$$

$$y = \pm\sqrt{10}i - 1, 1, -3.$$

9. 解: $\begin{cases} x^5 + y^5 = 32 \dots\dots\dots(1) \\ x + y = 2 \dots\dots\dots(2) \end{cases}$

對應於二式中最高次項的公因式 $x+y$ 有一無限解。

設 $x=u+v, y=u-v$

從 (2) 式, $2u=2 \therefore u=1.$

從 (1) 式, $v^4+2v^2-3=0$

$$(v^2+3)(v^2-1)=0$$

$$v = \pm i\sqrt{3} \text{ 及 } \pm 1;$$

$$x = 1 \pm i\sqrt{3}, 2, 0;$$

$$y = 1 \mp i\sqrt{3}, 0, 2.$$

10. 解: $\begin{cases} x+y = \frac{1}{2} \dots\dots\dots(1) \\ 56\left(\frac{x}{y} + \frac{y}{x}\right) + 113 = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $56x^2 + 113xy + 56y^2 = 0$

$$(7x+8y)(8x+7y) = 0$$

$$y = -\frac{7}{8}x \text{ 及 } y = -\frac{8}{7}x.$$

代入 (1) 式, $x=4$ 及 $-\frac{7}{2},$

$$y = -\frac{7}{2} \text{ 及 } 4.$$

11. 解: $\begin{cases} xy + x + y = -19 \dots\dots\dots(1) \\ x^2y + xy^2 + 20 = 0 \dots\dots\dots(2) \end{cases}$

對應於二式中最高次項的公因式 x 及 y 有二無限解。

從 (2) 式, $xy = -\frac{20}{x+y}$

代入 (1) 式, 得 $(x+y)^2 + 19(x+y) - 20 = 0$

$$(x+y+20)(x+y-1) = 0$$

$$x = -20 - y \text{ 及 } x = 1 - y$$

代入 (1) 式, 得 $y^2 + 20y + 1 = 0 \dots\dots\dots(3)$

及 $y^2 - y - 20 = 0 \dots\dots\dots(4)$

從 (3) 式, $y = -10 \pm 3\sqrt{11},$

$$x = -10 \mp 3\sqrt{11}.$$

從 (4) 式, $y = 5$ 及 $-4,$

$$x = -4 \text{ 及 } 5.$$

$$12. \text{ 解: } \begin{cases} x^4 + y^4 - (x^2 + y^2) = 72 & \dots\dots\dots(1) \\ x^2 + x^2y^2 + y^2 = 19 & \dots\dots\dots(2) \end{cases}$$

$$(1) + 2 \times (2), \quad \begin{aligned} x^4 + 2x^2y^2 + y^4 + x^2 + y^2 &= 110 \\ (x^2 + y^2 + 11)(x^2 + y^2 - 10) &= 0 \\ y^2 &= 10 - x^2 \text{ 及 } y^2 = -(11 + x^2) \end{aligned}$$

$$\text{代入 (2) 式, 得 } x^4 - 10x^2 + 9 = 0 \dots\dots\dots(3)$$

$$\text{及 } x^4 + 11x^2 + 30 = 0 \dots\dots\dots(4)$$

$$\text{從 (3) 式, } x = \pm 1 \text{ 及 } \pm 3,$$

$$y = \pm 3 \text{ 及 } \pm 1.$$

$$\text{從 (4) 式, } x = \pm \sqrt{6}i \text{ 及 } \pm \sqrt{5}i,$$

$$y = \pm \sqrt{5}i \text{ 及 } \pm \sqrt{6}i.$$

答: 其解爲 1, ± 3 ; $-1, \pm 3$; 3, ± 1 ; $-3, \pm 1$, ... 等共十六個。

$$13. \text{ 解: } \begin{cases} x^2y + xy^2 = 30 & \dots\dots\dots(1) \\ \frac{1}{x} + \frac{1}{y} = \frac{3}{10} & \dots\dots\dots(2) \end{cases}$$

對應於兩式中最高次項的公因式 x 及 y 有二無限解。

$$\text{從 (1) 式, } xy(x+y) = 30 \dots\dots\dots(3)$$

$$\text{從 (2) 式, } \frac{x+y}{xy} = \frac{3}{10} \dots\dots\dots(4)$$

$$\sqrt{(3) \times (4)}, \quad x+y = \pm 3$$

$$x = 3 - y \text{ 及 } x = -3 - y$$

$$\text{代入 (4) 式, 得 } y^2 - 3y + 10 = 0 \dots\dots\dots(5)$$

$$\text{及 } y^2 + 3y - 10 = 0 \dots\dots\dots(6)$$

$$\text{從 (5) 式, } y = \frac{3 \pm \sqrt{31}i}{2},$$

$$x = \frac{3 \mp \sqrt{31}i}{2}$$

$$\text{從 (6) 式, } y = -5 \text{ 及 } 2,$$

$$x = 2 \text{ 及 } -5.$$

$$14. \text{ 解: } \begin{cases} x^2 + 3xy + y^2 + 2x + 2y = 8 & \dots\dots\dots(1) \\ 2x^3 + 2y^3 + 3x + 3y = 14 & \dots\dots\dots(2) \end{cases}$$

從 (1) 式, $(x+y)^2 + xy + 2(x+y) - 8 = 0 \dots\dots\dots(3)$

從 (2) 式, $2(x+y)^2 - 4xy + 3(x+y) - 14 = 0 \dots\dots(4)$

$4 \times (3) + (4), \quad 6(x+y)^2 + 11(x+y) - 46 = 0$

$[6(x+y) + 23][(x+y) - 2] = 0$

$x+y = -\frac{23}{6}$ 及 $x+y = 2 \dots\dots\dots(5)$

代入 (3) 式, 得 $xy = \frac{35}{36}$ 及 $xy = 0 \dots\dots\dots(6)$

從 (5) 及 (6) 式, 得 $36y^2 + 138y + 35 = 0 \dots\dots\dots(7)$

及 $(2-y)y = 0 \dots\dots\dots(8)$

從 (7) 式, $y = (-23 \pm \sqrt{389})/12,$

$x = (-23 \mp \sqrt{389})/12.$

從 (8) 式, $y = 0$ 及 $2,$

$x = 2$ 及 $0.$

答: 其解爲 $0, 2; 2, 0; (-32 \pm \sqrt{389})/12, (-3 \mp \sqrt{389})/12.$

15. 解: $\begin{cases} x^3 = 5y & \dots\dots\dots(1) \\ y^3 = 5x & \dots\dots\dots(2) \end{cases}$

(1) + (2), $x^3 + y^3 = 5(x+y)$

$(x+y)(x^2 - xy + y^2) = 5(x+y)$

$x+y = 0 \dots\dots\dots(3)$

$x^2 - xy + y^2 - 5 = 0 \dots\dots\dots(4)$

(1) - (2), $x^3 - y^3 = -5(x-y)$

$(x-y)(x^2 + xy + y^2) = -5(x-y)$

$x-y = 0 \dots\dots\dots(5)$

$x^2 + xy + y^2 + 5 = 0 \dots\dots\dots(6)$

從 (3) 及 (5), 得 $x = 0, \quad y = 0.$

從 (3) 及 (6), 得 $x = \pm \sqrt{5}i, \quad y = \mp \sqrt{5}i.$

從 (4) 及 (5), 得 $x = \pm \sqrt{5}i, \quad y = \mp \sqrt{5}i.$

從 (4) 及 (6), 得 $x = \pm \sqrt{10}(1+i)/2, \quad \pm \sqrt{10}(1-i)/2;$

$y = \mp \sqrt{10}(1-i)/2, \quad \mp \sqrt{10}(1+i)/2.$

習 題 XLIX

原本第 329 頁

$$1. \text{ 解: } \begin{cases} x+y=3 & \dots\dots\dots (1) \\ y+z=2 & \dots\dots\dots (2) \\ x^2-yz=19 & \dots\dots\dots (3) \end{cases}$$

$$\text{從 (1) 式, } x=3-y$$

$$\text{從 (2) 式, } z=2-y$$

$$\text{代入 (3) 式, 得 } 9-6y+y^2-2y+y^2=19$$

$$y^2-4y-5=0$$

$$\therefore y=5 \text{ 及 } -1,$$

$$x=-2 \text{ 及 } 4,$$

$$z=-3 \text{ 及 } 3.$$

$$2. \text{ 解: } \begin{cases} x(y+z)=12 & \dots\dots\dots (1) \\ y(z+x)=6 & \dots\dots\dots (2) \\ z(x+y)=10 & \dots\dots\dots (3) \end{cases}$$

$$(1)+(2)-(3), \quad xy=4 \dots\dots\dots (4)$$

$$(1)+(3)-(2), \quad xz=8 \dots\dots\dots (5)$$

$$(2)+(3)-(1), \quad yz=2 \dots\dots\dots (6)$$

$$(4) \times (5) \div (6), \quad x^2=16$$

$$\therefore x=4 \text{ 或 } -4,$$

$$y=1 \text{ 或 } -1,$$

$$z=2 \text{ 或 } -2.$$

$$3. \text{ 解: } \begin{cases} (y+b)(z+c)=a^2 & \dots\dots\dots (1) \\ (z+c)(x+a)=b^2 & \dots\dots\dots (2) \\ (x+a)(y+b)=c^2 & \dots\dots\dots (3) \end{cases}$$

$$(1) \times (2) \div (3), \quad \frac{(y+b)(z+c)(z+c)(x+a)}{(x+a)(y+b)} = \frac{a^2 b^2}{c^2}$$

$$(z+c)^2 = \frac{a^2 b^2}{c^2}$$

$$z+c = \pm \frac{ab}{c}$$

$$\begin{aligned}\therefore z &= \frac{-c^2 \pm ab}{c}, \\ y &= \frac{-b^2 \pm ca}{b}, \\ x &= \frac{-a^2 \pm bc}{a}.\end{aligned}$$

習題 L

原本第 33 頁

1. 解: $\begin{cases} 7x^2 - 6xy = 8 & \dots\dots\dots(1) \\ 2x - 3y = 5 & \dots\dots\dots(2) \end{cases}$

從 (2) 式, $3y = 2x - 5$
 代入 (1) 式, 得 $3x^2 + 10x - 8 = 0$
 $(3x - 2)(x + 4) = 0$

$$\therefore x = \frac{2}{3} \text{ 及 } -4,$$

$$y = -\frac{11}{9} \text{ 及 } -\frac{13}{3}.$$

2. 解: $\begin{cases} x^2 + y^2 = 25 & \dots\dots\dots(1) \\ x - y = 1 & \dots\dots\dots(2) \end{cases}$

從 (2) 式, $y = x - 1$
 代入 (1) 式, 得 $x^2 - x - 12 = 0$
 $(x - 4)(x + 3) = 0$

$$\therefore x = 4 \text{ 及 } -3,$$

$$y = 3 \text{ 及 } -4.$$

3. 解: $\begin{cases} x - y = a & \dots\dots\dots(1) \\ xy = (b^2 - a^2)/4 & \dots\dots\dots(2) \end{cases}$

(1) 平方, $x^2 - 2xy + y^2 = a^2 \dots\dots\dots(3)$

(3) + 4 × (2), $(x + y)^2 = b^2$

$$x + y = \pm b \dots\dots\dots(4)$$

$$\frac{(1) + (4)}{2}, \quad x = \frac{a + b}{2}, \frac{a - b}{2}.$$

代入 (4) 式, 得 $y = \frac{-a+b}{2}, \frac{-a-b}{2}$

4. 解: $\begin{cases} \frac{a}{x^2} + \frac{b}{y^2} = a^2 + b^2 \dots\dots\dots(1) \\ x^2 + y^2 = 0 \dots\dots\dots(2) \end{cases}$

從 (2) 式, $y^2 = -x^2 \dots\dots\dots(3)$

代入 (1) 式, 得 $\frac{a}{x^2} - \frac{b}{x^2} = a^2 + b^2$

$$x^2 = \frac{a-b}{a^2+b^2}$$

$$\therefore x = \pm \sqrt{\frac{a-b}{a^2+b^2}}$$

當 $x = \sqrt{\frac{a-b}{a^2+b^2}}, y = \pm \sqrt{\frac{b-a}{a^2+b^2}}$

當 $x = -\sqrt{\frac{a-b}{a^2+b^2}}, y = \pm \sqrt{\frac{b-a}{a^2+b^2}}$

5. 解: $\begin{cases} \frac{1}{y} - \frac{3}{x} = 1 \dots\dots\dots(1) \\ \frac{7}{xy} - \frac{1}{y^2} = 12 \dots\dots\dots(2) \end{cases}$

從 (1) 式, $\frac{1}{y} = 1 + \frac{3}{x}$

代入 (2) 式, 得 $13x^2 - x - 12 = 0$
 $(13x+12)(x-1) = 0$

$$\therefore x = -\frac{12}{13} \text{ 及 } 1,$$

$$y = -\frac{4}{9} \text{ 及 } \frac{1}{4}$$

6. 解: $\begin{cases} x+y = a+b \dots\dots\dots(1) \\ \frac{a}{x+b} + \frac{b}{y+a} = 1 \dots\dots\dots(2) \end{cases}$

從 (1) 式, $y = a+b-x$

代入 (2) 式, 得 $x^2 - (3a-b)x - (ab-2a^2) = 0$
 $(x-a)(x-2a+b) = 0$

$$\begin{aligned} \therefore x &= a \text{ 及 } 2a - b, \\ y &= b \text{ 及 } 2b - a. \end{aligned}$$

$$7. \text{ 解: } \begin{cases} \frac{1}{x^3} + \frac{1}{y^3} = \frac{1001}{125} & \dots\dots\dots (1) \\ \frac{1}{x} + \frac{1}{y} = \frac{11}{5} & \dots\dots\dots (2) \end{cases}$$

$$(1) \div (2), \quad \frac{1}{y^2} - \frac{1}{xy} + \frac{1}{x^2} - \frac{91}{25} = 0 \quad \dots\dots\dots (3)$$

$$\text{從 (2) 式, } \frac{1}{y} = \frac{11}{5} - \frac{1}{x}$$

$$\text{代入 (3) 式, 得 } \begin{aligned} 2x^2 - 11x + 5 &= 0 \\ (2x - 1)(x - 5) &= 0 \end{aligned}$$

$$\therefore x = \frac{1}{2} \text{ 及 } 5,$$

$$y = 5 \text{ 及 } \frac{1}{2}.$$

$$8. \text{ 解: } \begin{cases} \frac{a^2}{x^2} + \frac{b^2}{y^2} = 5 & \dots\dots\dots (1) \\ \frac{ab}{xy} = 2 & \dots\dots\dots (2) \end{cases}$$

$$\text{從 (2) 式, } \frac{b}{y} = \frac{2x}{a}$$

$$\text{代入 (1) 式, 得 } \begin{aligned} 4x^4 - 5a^2x^2 + a^4 &= 0 \\ (4x^2 - a^2)(x^2 - a^2) &= 0 \end{aligned}$$

$$\therefore x = \pm \frac{a}{2} \text{ 及 } \pm a,$$

$$y = \pm b \text{ 及 } \pm \frac{b}{2}.$$

$$9. \text{ 解: } \begin{cases} x^2 + y^2 = \frac{17}{4}xy & \dots\dots\dots (1) \\ x - y = \frac{3}{4}xy & \dots\dots\dots (2) \end{cases}$$

$$\text{從 (2) 式, } y = \frac{4x}{3x + 4}$$

代入 (1) 式, 得 $x^4 - 3x^3 - 4x^2 = 0$
 $\therefore x=0, 0; y=0, 0.$
 $x^2 - 3x - 4 = 0$
 $(x-4)(x+1) = 0$
 $\therefore x=4, -1; y=1, -4.$

10. 解: $a(x+y) = b(x-y) = xy$

$$y = \frac{ax}{x-a}$$

$$bx - \frac{abx}{x-a} = \frac{ax^2}{x-a}$$

$$bx^2 - abx - abx - ax^2 = 0$$

$$\therefore x=0, y=0.$$

$$bx - 2ab - ax = 0$$

$$\therefore x = \frac{2ab}{b-a}, y = \frac{2ab}{b+a}$$

11. 解: $40xy = 21(x^2 - y^2) = 210(x+y)$

$$4xy = 21(x+y) \dots\dots\dots(1)$$

$$x^2 - y^2 - 10(x+y) = 0 \dots\dots\dots(2)$$

從 (2) 式, $(x-y-10)(x+y) = 0$
 $y = x-10$ 及 $y = -x$

代入 (1) 式, 得 $2x^2 - 41x + 105 = 0$
 $(x-3)(2x-35) = 0$

$$\therefore x=3 \quad \text{及} \quad \frac{35}{2},$$

$$y = -7 \quad \text{及} \quad \frac{15}{2}.$$

又

$$4x(-x) = 21(x^2 - x^2)$$

$$x^2 = 0$$

$$\therefore x=0, 0; y=0, 0.$$

12. 解: $\begin{cases} 4x^2 - 25y^2 = 0 \dots\dots\dots(1) \\ 2x^2 - 10y^2 - 3y = 4 \dots\dots\dots(2) \end{cases}$

從 (1) 式, $(2x-5y)(2x+5y) = 0$

$$y = \frac{2}{5}x \quad \text{及} \quad -\frac{2}{5}x$$

代入 (2) 式, 得 $x^2 - 3x - 10 = 0 \dots\dots\dots (3)$

及 $x^2 + 3x - 10 = 0 \dots\dots\dots (4)$

從 (3) 式, $(x-5)(x+2) = 0$

$\therefore x = 5$ 及 $-2,$
 $y = 2$ 及 $-\frac{4}{5}.$

從 (4) 式, $(x+5)(x-2) = 0$

$\therefore x = -5$ 及 $-2,$
 $y = 2$ 及 $-\frac{4}{5}.$

13. 解: $\begin{cases} x^2 + 3xy - 9y^2 = 9 \dots\dots\dots (1) \\ x^2 - 13xy + 21y^2 = -9 \dots\dots\dots (2) \end{cases}$

(1) + (2), $2x^2 - 10xy + 12y^2 = 0$

$2(x-3y)(x-2y) = 0$

$y = \frac{x}{3}$ 及 $\frac{x}{2}$

代入 (1) 式, 得 $x^2 + x^2 - x^2 = 9$

$\therefore x = \pm 3, y = \pm 1.$

又 $4x^2 + 6x^2 - 9x^2 = 36$

$\therefore x = \pm 6, y = \pm 3.$

14. 解: $\begin{cases} x^2 - 7y^2 - 29 = 0 \dots\dots\dots (1) \\ x^2 - 6xy + 9y^2 - 2x + 6y = 3 \dots\dots\dots (2) \end{cases}$

從 (2) 式, $(x-3y)^2 - 2(x-3y) - 3 = 0$

$(x-3y-3)(x-3y+1) = 0$

$x = 3y + 3$ 及 $x = 3y - 1.$

代入 (1) 式, 得 $y^2 + 9y - 10 = 0$

$(y+10)(y-1) = 0$

$\therefore y = -10$ 及 $1,$

$x = -27$ 及 $6.$

又 $y^2 - 3y - 14 = 0$

$\therefore y = \frac{3 \pm \sqrt{9+56}}{2} = \frac{3 \pm \sqrt{65}}{2},$

$x = \frac{9 \pm 3\sqrt{65} - 2}{2} = \frac{7 \pm 3\sqrt{65}}{2}.$

15. 解: $\begin{cases} \frac{x}{y} + \frac{y}{x} = \frac{65}{28} \dots\dots\dots(1) \\ 2(x^2 + y^2) + (x - y) = 34 \dots\dots\dots(2) \end{cases}$

從 (1) 式, $\begin{cases} 28(x^2 + y^2) = 65xy \\ 28(x - y)^2 - 9xy = 0 \dots\dots\dots(3) \end{cases}$

從 (2) 式, $\begin{cases} 2(x - y)^2 + (x - y) + 4xy = 34 \dots\dots\dots(4) \\ 4(3) + 9(4), \quad 130(x - y)^2 + 9(x - y) - 306 = 0 \end{cases}$

$$[65(x - y) + 102][2(x - y) - 3] = 0$$

$$x = -\frac{102}{65} + y \quad \text{及} \quad x = \frac{3}{2} + y$$

代入 (3) 式, 得 $(65y - 238)(65y + 136) = 0$

及 $2y^2 + 3y - 14 = 0$

$$\therefore y = \frac{238}{65}, -\frac{136}{65}, 2, -\frac{7}{2};$$

$$x = \frac{136}{65}, -\frac{238}{65}, \frac{7}{2}, -2.$$

16. 解: $\begin{cases} x^2y = a \dots\dots\dots(1) \\ xy^2 = b \dots\dots\dots(2) \end{cases}$

對應於 (1), (2) 二式中一次項, 二次項, 三次項的公因式 x 及 y 有 6 無限解.

(1) \div (2), $\frac{x}{y} = \frac{a}{b} \quad x = \frac{a}{b}y$

代入 (2) 式, $y^3 = \frac{b^3}{a}$

$$\therefore y = \sqrt[3]{\frac{b^2}{a}}, \sqrt[3]{\frac{b^2}{a}}\omega, \sqrt[3]{\frac{b^2}{a}}\omega^2;$$

$$x = \sqrt[3]{\frac{a^2}{b}}, \sqrt[3]{\frac{a^2}{b}}\omega, \sqrt[3]{\frac{a^2}{b}}\omega^2.$$

其中 ω 示 $(-1 + \sqrt{3}i)/2$.

17. 解: $\begin{cases} x^2y + xy^2 = a \dots\dots\dots(1) \\ x^2y - xy^2 = b \dots\dots\dots(2) \end{cases}$

對應於 (1), (2) 式中三次項, 二次項及一次項的公因式 x 及 y 有 6 無限解.

$$(1) + (2), \quad 2x^2y = a + b \dots\dots\dots(3)$$

$$(1) - (2), \quad 2xy^2 = a - b \dots\dots\dots(4)$$

$$(3) \div (4), \quad \frac{x}{y} = \frac{a+b}{a-b}$$

$$x = \frac{a+b}{a-b}y$$

代入 (4) 式, $y^3 = \frac{(a-b)^2}{2(a+b)}$

$$\therefore y = \sqrt[3]{\frac{(a-b)^2}{2(a+b)}}, \quad \sqrt[3]{\frac{(a-b)^2}{2(a+b)}}\omega, \quad \sqrt[3]{\frac{(a-b)^2}{2(a+b)}}\omega^2;$$

$$x = \sqrt[3]{\frac{(a+b)^2}{2(a-b)}}, \quad \sqrt[3]{\frac{(a+b)^2}{2(a-b)}}\omega, \quad \sqrt[3]{\frac{(a+b)^2}{2(a-b)}}\omega^2.$$

其中 ω 示 $(-1 + \sqrt{3}i)/2$.

18. 解: $\begin{cases} x = a(x^2 + y^2) \dots\dots\dots(1) \\ y = b(x^2 + y^2) \dots\dots\dots(2) \end{cases}$

$$(1) \div (2), \quad x = \frac{a}{b}y$$

代入 (2) 式, $y\left[\frac{a^2+b^2}{b}y - 1\right] = 0$

$$\therefore y = 0 \text{ 及 } \frac{b}{a^2+b^2},$$

$$x = 0 \text{ 及 } \frac{a}{a^2+b^2}.$$

19. 解: $\begin{cases} \frac{x+y}{x-y} = \frac{5}{3} \dots\dots\dots(1) \\ (2x+3y)(3x-2y) = 110a^2 \dots\dots\dots(2) \end{cases}$

從 (1) 式, $x = 4y$

代入 (2) 式, $(8y+3y)(12y-2y) = 110a^2$

$$y^2 = a^2$$

$$\therefore y = \pm a,$$

$$x = \pm 4a.$$

20. 解: $\begin{cases} 3(x^3 - y^3) = 13xy \dots\dots\dots(1) \\ x - y = 1 \dots\dots\dots(2) \end{cases}$

對應於 (1), (2) 式中最高次項的公因式 $x-y$ 有一無限解。

$$(1) \div (2), \quad 3x^2 - 10xy + 3y^2 = 0$$

$$(3x-y)(x-3y) = 0$$

$$x = \frac{1}{3}y \quad \text{及} \quad x = 3y$$

代入 (2) 式, $y = \frac{1}{2}$ 及 $-\frac{3}{2}$,

$$x = \frac{3}{2} \quad \text{及} \quad -\frac{1}{2}$$

21. 解: $\begin{cases} x^4 + y^4 = a^4 & \dots\dots\dots(1) \\ x + y = a & \dots\dots\dots(2) \end{cases}$

設 $x = u + v$ 及 $y = u - v$

從 (2) 式, $2u = a$

$$u = \frac{a}{2}$$

代入 (1) 式, $\left[\frac{a}{2} + v\right]^4 + \left[\frac{a}{2} - v\right]^4 = a^4$

$$16v^4 + 24a^2v^2 - 7a^4 = 0$$

$$(4v^2 + 7a^2)(4v^2 - a^2) = 0$$

$$v^2 = -\frac{7}{4}a^2 \quad \text{及} \quad v^2 = \frac{1}{4}a^2$$

$$\therefore v = \pm \frac{a}{2}\sqrt{7}i \quad \text{及} \quad \pm \frac{a}{2}$$

$$x = \frac{a(1 \pm \sqrt{7}i)}{2}, \quad a, \quad 0;$$

$$y = \frac{a(1 \mp \sqrt{7}i)}{2}, \quad 0, \quad a.$$

22. 解: $\begin{cases} 21(x+y) = 10xy & \dots\dots\dots(1) \\ x+y+x^2+y^2 = 68 & \dots\dots\dots(2) \end{cases}$

從 (2) 式, $(x+y)^2 + (x+y) - 68 = 2xy \dots\dots\dots(3)$

5(3) - (1), $5(x+y)^2 - 16(x+y) - 340 = 0$

$$[5(x+y) + 34][x+y - 10] = 0$$

$$x+y = -\frac{34}{5} \quad \text{及} \quad x+y = 10 \dots\dots\dots(4)$$

解 (1) 及 (4), 得 $x=7, 3, (-17 \pm \sqrt{646})/5$;
 $y=3, 7, (-17 \mp \sqrt{646})/5$.

23. 解: $x^2 + y^2 = xy = x + y$
 $xy = x + y \dots\dots\dots(1)$

$$(x+y)^2 - 2xy = xy \dots\dots\dots(2)$$

從 (1) 及 (2), $(x+y)(x+y-3) = 0$

$$x = -y \dots\dots\dots(3)$$

$$x = 3 - y \dots\dots\dots(4)$$

代入 (1) 式, 得 $y=0, 0; x=0, 0$.

又 $y^2 - 3y + 3 = 0 \dots\dots\dots(5)$

$$\therefore y = \frac{3 \pm \sqrt{3}i}{2},$$

$$x = 3 - \frac{3 \pm \sqrt{3}i}{2} = \frac{3 \mp \sqrt{3}i}{2}.$$

24. 解: $x^2 - xy + y^2 = 3a^2 = x^2 - y^2$

$$y(x - 2y) = 0$$

$$y = 0 \text{ 及 } x = 2y$$

將此二值各代入 $x^2 - y^2 = 3a^2$ 中, 得下列二式:

$$x^2 = 3a^2 \dots\dots\dots(1)$$

$$3y^2 = 3a^2 \dots\dots\dots(2)$$

從 (1) 式, $x = \pm a\sqrt{3}, y = 0$.

從 (2) 式, $y = \pm a, x = \pm 2a$.

答: 其根爲 $a\sqrt{3}, 0; -a\sqrt{3}, 0; 2a, a; -2a; -a$.

25. 解: $\begin{cases} x^2 + xy + y^2 = 21 \dots\dots\dots(1) \\ x + \sqrt{xy} + y = 7 \dots\dots\dots(2) \end{cases}$

對應於 (1), (2) 式中最高次項的公因式 $x + \sqrt{xy} + y$ 有無限解。

$$(1) \div (2), \quad x - \sqrt{xy} + y = 3 \dots\dots\dots(3)$$

$$(2) + (3), \quad 2x + 2y = 10$$

$$x + y = 5$$

$$y = 5 - x$$

代入 (1) 式, 得 $x^2 - 5x - 4 = 0$

$$(x-1)(x-4) = 0$$

$$\therefore x=1 \text{ 及 } 4,$$

$$y=4 \text{ 及 } 1.$$

26. 解: $\begin{cases} 4x^2-3y^2=12(x-y) \dots\dots\dots(1) \\ xy=0 \dots\dots\dots(1) \end{cases}$

從 (2) 式, $x=0, y=0.$

將 $x=0$ 代入 (1) 式, 得 $y^2-4y=0$

$$\therefore y=0 \text{ 及 } 4.$$

將 $y=0$ 代入 (1) 式, 得 $x^2-3x=0$

$$\therefore x=0 \text{ 及 } 3.$$

答: 其根爲 $0, 0; 0, 4; 0, 0; 3, 0.$

27. 解: $\begin{cases} x^2+y^2=x+y+20 \dots\dots\dots(1) \\ xy+10=2(x+y) \dots\dots\dots(2) \end{cases}$

從 (1) 式, $(x+y)^2-(x+y)-2xy-20=0 \dots\dots\dots(3)$

從 (2) 式, $xy=2(x+y)-10$

代入 (3) 式, $(x+y)^2-5(x+y)=0$

$$(x+y)(x+y-5)=0$$

$$y=-x \text{ 及 } y=5-x$$

將 $x=-x$ 代入 (2) 式, 得 $x^2=10$

$$\therefore x=\pm\sqrt{10}, y=\mp\sqrt{10}.$$

將 $y=5-x$ 代入 (2) 式, 得 $x(5-x)=0$

$$\therefore x=0 \text{ 及 } 5, y=5 \text{ 及 } 0.$$

28. 解: $\begin{cases} x^2+4x-3y=0 \dots\dots\dots(1) \\ y^2+10x-9y=0 \dots\dots\dots(2) \end{cases}$

(1) - (2), $x^2-y^2-6x+6y=0$

$$(x-y)[(x+y)-6]=0$$

$$x=y \text{ 及 } x=6-y$$

代入 (1) 式, 得 $x^2+x=0$

$$\therefore x=0 \text{ 及 } -1,$$

$$y=0 \text{ 及 } -1.$$

又 $y^2-19y+60=0$

$$(y-4)(y-15)=0$$

$$\therefore y=4 \text{ 及 } 15,$$

$$x=2 \text{ 及 } -9.$$

$$29. \text{ 解: } \begin{cases} 28(x^5+y^2)=61(x^3+y^3) \dots\dots\dots(1) \\ x+y=2 \dots\dots\dots(2) \end{cases}$$

對應於最高次項的公因式 $x+y$ 有一無限解。

設 $x=u+v, y=u-v$

從 (2) 式, $u=1.$

代入 (1) 式, $28[(1+v)^5+(1-v)^5]=61[(1+v)^3+(1-v)^3]$

$$140v^4+97v^2+33=0$$

$$(4v^2-1)(35v^2+33)=0$$

$$\therefore v = \pm \frac{1}{2} \text{ 及 } \pm \frac{i\sqrt{1155}}{35};$$

$$x = \frac{3}{2}, \frac{1}{2}, 1 \pm \frac{i\sqrt{1155}}{35};$$

$$y = \frac{1}{2}, \frac{3}{2}, 1 \mp \frac{i\sqrt{1155}}{35}.$$

$$30. \text{ 解: } \begin{cases} xy - \frac{x}{y} = a \dots\dots\dots(1) \\ xy - \frac{y}{x} = \frac{1}{a} \dots\dots\dots(2) \end{cases}$$

$$(1)-(2), \quad \frac{y}{x} - \frac{x}{y} = a - \frac{1}{a}$$

$$y^2 - \left(a - \frac{1}{a}\right)xy - x^2 = 0$$

$$(y-ax)\left(y + \frac{x}{a}\right) = 0$$

$$y = ax \text{ 及 } y = -\frac{x}{a}$$

代入 (1) 式, $ax^2 - \frac{1}{a} = a$

$$\therefore x = \frac{\sqrt{a^2+1}}{a} \text{ 或 } -\frac{\sqrt{a^2+1}}{a},$$

$$y = \sqrt{a^2+1} \text{ 或 } -\sqrt{a^2+1}.$$

又 $-\frac{x^2}{a} + a = a$

$\therefore x=0, 0$ [此為不合理, 因其能使 (2) 中分母為 0].

答: (1), (2) 兩式之有限解僅有

$$\frac{\sqrt{a^2+1}}{a}, \sqrt{a^2+1}; -\frac{\sqrt{a^2+1}}{a}, -\sqrt{a^2+1}.$$

31. 解: $\begin{cases} (x+1)^3 + (y-2)^3 = 19 & \dots\dots\dots(1) \\ x+y=2 & \dots\dots\dots(2) \end{cases}$

對應於 (1), (2) 兩式中最高次項的公因式 $x+y$ 有一無限解.

從 (2) 式, $y=2-x$

代入 (1) 式, 得 $x^2+x-6=0$

$$(x+3)(x-2)=0$$

$$\therefore x=-3 \text{ 及 } 2,$$

$$y=5 \text{ 及 } 0.$$

32. 解: $\begin{cases} x^2+y=\frac{8}{3} & \dots\dots\dots(1) \\ x+y^2=\frac{34}{9} & \dots\dots\dots(2) \end{cases}$

從 (1) 式, $y=\frac{8}{3}-x^2$

代入 (2) 式, 得 $3x^4-16x^2+3x+10=0$

$$(x-1)(x-2)(3x^2+9x+5)=0$$

$$\therefore x=1, 2 \text{ 或 } \frac{-9 \pm \sqrt{21}}{6};$$

$$y=\frac{5}{3}, -\frac{4}{3} \text{ 或 } \frac{-1 \pm 3\sqrt{21}}{6}.$$

33. 解: $\begin{cases} y^2-xy-yz=3 & \dots\dots\dots(1) \\ x+4y+z=14 & \dots\dots\dots(2) \\ x-y+2z=0 & \dots\dots\dots(3) \end{cases}$

(2)-(3), $5y-z=14$ $z=5y-14$

代入 (2) 式, 得 $x+9y-14=14$ $x=28-9y$

將 x, z 之值代入 (1) 式, 得下列方程式:

$$5y^2-14y-3=0$$

$$(5y+1)(y-3)=0$$

$$y = -\frac{1}{5} \text{ 及 } 3,$$

$$z = -15 \text{ 及 } 1,$$

$$x = \frac{149}{5} \text{ 及 } 1.$$

$$34. \text{ 解: } \begin{cases} x+y+z+u=0 & \dots\dots\dots(1) \\ 3x+z+u=0 & \dots\dots\dots(2) \\ 3y+2z=0 & \dots\dots\dots(3) \\ x^2+y^2+zu=5 & \dots\dots\dots(4) \end{cases}$$

從 (2) 及 (3), $x = \frac{-z-u}{3}$ 及 $y = -\frac{2z}{3}$

代入 (1) 式, 得 $-u-z-2z+3z+3u=0 \quad \therefore u=0.$

$$x = -\frac{z}{3}$$

代入 (4) 式, 得 $z^2+4z^2=45 \quad \therefore z=\pm 3.$

$$\therefore x=\mp 1, \quad y=\mp 2.$$

答: $x, y, z, u=1, 2, -3, 0; -1, -2, 3, 0.$

$$35. \text{ 解: } \begin{cases} (y+z)(x+y+z)=10 & \dots\dots\dots(1) \\ (z+x)(x+y+z)=20 & \dots\dots\dots(2) \\ (x+y)(x+y+z)=20 & \dots\dots\dots(3) \end{cases}$$

(1)+(2)+(3), $(x+y+z)^2=25$

$$x+y+z=\pm 5.$$

代入 (1), (2) 及 (3) 式, 得

$$y+z=\pm 2 \dots\dots\dots(4)$$

$$z+x=\pm 4 \dots\dots\dots(5)$$

$$x+y=\pm 5 \dots\dots\dots(6)$$

(4)+(5)-(6), $2z=\pm 6 \mp 4 = \pm 2$

$$\therefore z=\pm 1.$$

代入 (5) 式, 得 $x=\pm 3.$

代入 (6) 式, 得 $y=\pm 1.$

答: $x, y, z=3, 1, 1; -3, -1, -1.$

$$36. \text{ 解: } \begin{cases} x^2+y^2+z^2=6 & \dots\dots\dots(1) \\ xy+yz+zx=-1 & \dots\dots\dots(2) \\ 2x+y-2z=-3 & \dots\dots\dots(3) \end{cases}$$

$$(1) + 2 \times (2), \quad (x+y+z)^2 = 4$$

$$x+y+z = 2 \quad \dots\dots\dots(4)$$

$$x+y+z = -2 \quad \dots\dots\dots(5)$$

解 (3) 及 (4), 得

$$x = 3z - 5 \quad \text{及} \quad y = 7 - 4z$$

代入 (1) 式, 得 $13z^2 - 43z + 34 = 0$

$$(13z - 17)(z - 2) = 0$$

$$\therefore z = \frac{17}{13} \quad \text{及} \quad 2,$$

$$y = \frac{23}{13} \quad \text{及} \quad -1,$$

$$x = -\frac{14}{13} \quad \text{及} \quad 1.$$

解 (3) 及 (5), 得

$$x = 3z - 1 \quad \text{及} \quad y = -4z - 1$$

代入 (1) 式, 得 $13z^2 + z - 2 = 0$

$$\therefore z = \frac{-1 \pm \sqrt{105}}{26},$$

$$y = \frac{-11 \mp 2\sqrt{105}}{13},$$

$$x = \frac{-29 \pm 3\sqrt{105}}{26}.$$

答: $x, y, z = 1, -1, 2; -\frac{14}{13}, \frac{23}{13}, \frac{17}{13};$

$$\frac{-29 \pm 3\sqrt{105}}{26}, \frac{-11 \mp 2\sqrt{105}}{13}, \frac{-1 \pm \sqrt{105}}{26}.$$

習 題 LI

原本第 331 頁

1. 解: 設 x 爲大數, y 爲小數.

$$\begin{cases} x^2 - y^2 = 218 \quad \dots\dots\dots(1) \\ (x - y)^2 = 8 \quad \dots\dots\dots(2) \end{cases}$$

$$\text{從 (2) 式, } x - y = 2 \dots\dots\dots (3)$$

$$(1) \div (3), \quad x^2 + xy + y^2 = 109 \dots\dots\dots (4)$$

$$\text{以 (3) 代入 (4), 得 } y^2 + 2y - 35 = 0$$

$$(y - 5)(y + 7) = 0$$

$$\therefore y = 5 \text{ 及 } -7,$$

$$x = 7 \text{ 及 } -5.$$

2. 解：設 x 爲大數， y 爲小數。

$$\begin{cases} (x+y)^2 - xy = 63 \dots\dots\dots (1) \\ x^3 - y^3 = 189 \dots\dots\dots (2) \end{cases}$$

$$\text{從 (1) 式, } x^2 + xy + y^2 = 63 \dots\dots\dots (3)$$

$$(2) \div (3), \quad x - y = 3 \quad x = 3 + y$$

$$\text{代入 (3) 式, 得 } y^2 + 3y - 18 = 0$$

$$(y - 3)(y + 6) = 0$$

$$\therefore y = 3 \text{ 及 } -6,$$

$$x = 6 \text{ 及 } -3.$$

3. 解：設 x 爲分子， y 爲分母。

$$\begin{cases} x + y = 11 \dots\dots\dots (1) \\ \frac{x}{y} \cdot \frac{x+3}{y+4} = \frac{2}{3} \dots\dots\dots (2) \end{cases}$$

$$\text{從 (2) 式, } 3x^2 - 2y^2 + 9x - 8y = 0 \dots\dots\dots (3)$$

$$\text{從 (1) 式, } x = 11 - y$$

$$\text{代入 (3) 式, 得 } y^2 - 83y + 462 = 0$$

$$(y - 6)(y - 77) = 0$$

$$\therefore y = 6 \text{ 或 } 77,$$

$$x = 5 \text{ 或 } -66.$$

答：所求之分數爲 $\frac{5}{6}$ 。

4. 解：設 x, y 及 z 爲所求之三部。

$$\begin{cases} x + y + z = 37 \dots\dots\dots (1) \\ x \cdot y \cdot z = 1440 \dots\dots\dots (2) \\ xy - 3z = 12 \dots\dots\dots (3) \end{cases}$$

$$\text{從 (3) 式, } xy = 12 + 3z$$

$$\text{代入 (2) 式, 得 } 3z^2 + 12z - 1440 = 0$$

$$(3z+72)(z-20)=0$$

$$\therefore z=20.$$

$$xy=12+3 \cdot 20=72$$

$$x=\frac{72}{y}$$

代入 (1) 式, 得 $y^2-17y+72=0$

$$(y-8)(y-9)=0$$

$$\therefore y=8 \text{ 及 } 9,$$

$$x=9 \text{ 及 } 8.$$

答: 此三部爲 8, 9 及 20.

5. 解: 設 x 與 y 爲長方形之邊長.

$$\begin{cases} x^2+y^2=13^2 & \dots\dots\dots(1) \\ (x+2)(y+2)=38+x \cdot y & \dots\dots\dots(2) \end{cases}$$

$$\begin{cases} x^2+y^2=13^2 & \dots\dots\dots(1) \\ (x+2)(y+2)=38+x \cdot y & \dots\dots\dots(2) \end{cases}$$

從 (2) 式, $x+y=17$ $y=17-x$

代入 (1) 式, 得 $x^2-17x+60=0$

$$(x-5)(x-12)=0$$

$$\therefore x=5 \text{ 及 } 12,$$

$$y=12 \text{ 及 } 5.$$

答: 其邊長爲 5 呎與 12 呎.

6. 解: 設 x, y 爲直角三角形勾與股之長度.

$$\begin{cases} x+y+\sqrt{x^2+y^2}=36 & \dots\dots\dots(1) \\ \frac{1}{2}xy=54 & \dots\dots\dots(2) \end{cases}$$

$$\begin{cases} x+y+\sqrt{x^2+y^2}=36 & \dots\dots\dots(1) \\ \frac{1}{2}xy=54 & \dots\dots\dots(2) \end{cases}$$

從 (2) 式, $xy=180$ $y=180/x$

代入 (1) 式, 得 $x^2-21x+108=0$

$$(x-12)(x-9)=0$$

$$\therefore x=12 \text{ 及 } 9,$$

$$y=9 \text{ 及 } 12,$$

$$\sqrt{x^2+y^2}=15.$$

答: 其三邊之長度爲 9 吋, 12 吋及 15 吋.

7. 解: 設 $x, x-3, x-24$ 爲弦及垂直二邊之長度.

$$(x-3)^2+(x-24)^2=x^2$$

$$x^2-54x+585=0$$

$$(x-39)(x-15)=0$$

∴ $x=39$ 吋或 15 吋(不合理),

$$x-3=36 \text{ 吋}$$

$$x-24=15 \text{ 吋.}$$

答: 其三邊之長度為 39 吋, 36 吋及 15 吋.

8. 解: 設 x, y 與 z 為所求之長, 寬, 高

$$\begin{cases} x \cdot y = 224 & \dots\dots\dots(1) \\ y \cdot z = 144 & \dots\dots\dots(2) \\ z \cdot x = 126 & \dots\dots\dots(3) \end{cases}$$

$$(1) \times (3) \div (2), \quad \frac{xy \cdot zx}{yz} = \frac{224 \times 126}{144}$$

$$x^2 = 14^2$$

$$\therefore x = 14 \text{ 呎.}$$

代入 (1) 及 (3) 式, 得 $y = 16$ 呎,

$$z = 9 \text{ 呎.}$$

答: 此室之長, 寬, 高為 14 呎, 16 呎及 9 呎.

9. 解: 設 x, y 為長方形之長及寬.

$$\begin{cases} x \cdot y = 168 & \dots\dots\dots(1) \\ (x+10)(y+10) = 360 + 168 & \dots\dots\dots(2) \end{cases}$$

$$\text{從 (2) 式,} \quad xy + 10x + 10y - 428 = 0 \dots\dots\dots(3)$$

$$\text{從 (1) 式,} \quad y = \frac{168}{x}$$

$$\text{代入 (3) 式, 得} \quad x^2 - 26x + 168 = 0$$

$$(x-14)(x-12) = 0$$

$$\therefore x = 14 \text{ 及 } 12.$$

$$y = 12 \text{ 及 } 14.$$

答: 長方形之長及寬為 14 吋及 12 吋.

10. 解: 設 x, y 各為二人購煤一噸之價.

$$\begin{cases} \frac{135}{x} = \frac{135}{y} + 3 & \dots\dots\dots(1) \\ 5y = 4x + 7 & \dots\dots\dots(2) \end{cases}$$

$$\text{從 (2) 式,} \quad y = \frac{4x+7}{5}$$

代入 (1) 式, 得 $4x^2 + 52x - 315 = 0$
 $(2x - 9)(2x + 35) = 0$

$$\therefore x = \frac{9}{2} = 4.5 \text{ 元,}$$

$$y = 5 \text{ 元.}$$

答: A 購煤每噸 4.5 元; B, 5 元.

11. 解: 設 x 為本金, y 為利率.

$$\begin{cases} x(1+y) = 1248 & \dots\dots\dots(1) \\ (x+100)\left(1+y \times \frac{3}{2} \times 2\right) = 1456 & \dots\dots\dots(2) \end{cases}$$

$$\begin{cases} x(1+y) = 1248 & \dots\dots\dots(1) \\ (x+100)\left(1+y \times \frac{3}{2} \times 2\right) = 1456 & \dots\dots\dots(2) \end{cases}$$

從 (1) 式, $x = \frac{1248}{1+y}$

代入 (2) 式, 得下列方程式:

$$300y^2 + 2688y - 108 = 0$$

$$(100y - 4)(3y + 27) = 0$$

$$\therefore y = 4\%,$$

$$x = \frac{1248}{1 + \frac{4}{100}} = \frac{124800}{104} = 1200 \text{ 元.}$$

答: 本金為 1200 元, 利率為 4%.

12. 解: 設 x 為其子數, 則 $7-x$ 為其孫數.

$$\frac{20000}{x} - \frac{40000}{7-x} = 2000$$

$$x^2 - 37x + 70 = 0$$

$$(x-2)(x-35) = 0$$

$$\therefore x = 2 \text{ 人(子數),}$$

$$7-x = 5 \text{ 人(孫數),}$$

$$\frac{20000}{x} = 10000 \text{ 元(子所得),}$$

$$\frac{40000}{7-x} = 8000 \text{ 元(孫所得).}$$

答: 其子二人, 其孫五人; 子每人所得為 10000 元, 孫每人所得為 8000 元.

13. 解: 設 x 為水流速率, y 為靜水中划行速率.

$$\begin{cases} \frac{15}{y-x} - \frac{15}{y+x} = 5 \dots\dots\dots(1) \\ \frac{15}{2y-x} - \frac{15}{2y+x} = 1 \dots\dots\dots(2) \end{cases}$$

從(1)式, $x^2 + 6x - y^2 = 0 \dots\dots\dots(3)$

從(2)式, $x^2 + 30x - 4y^2 = 0 \dots\dots\dots(4)$

5(3) - (4), $4x^2 = y^2$

代入(3)式, 得 $3x^2 - 6x = 0$

$$3x(x-2) = 0$$

$$\therefore x = 2 \text{ 哩/時,}$$

$$y = \sqrt{4x^2} = 4 \text{ 哩/時.}$$

答: 水流速率每小時 2 哩, 靜水中划行速率每小時 4 哩。

14. 解: 設 x, y, z 各為 A, B, C 單獨為之所須之時間。

$$\begin{cases} 1 \div \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = 1\frac{1}{3} \dots\dots\dots(1) \\ z = 2x \dots\dots\dots(2) \\ z = 2 + y \dots\dots\dots(3) \end{cases}$$

從(2), (3)式, $y = 2x - 2 \dots\dots\dots(4)$

將(2), (4)代入(1)式, 得 $3x^2 - 11x + 6 = 0$

$$(3x-2)(x-3) = 0$$

$$\therefore x = 3 \text{ 小時,}$$

$$y = 2x - 2 = 4 \text{ 小時,}$$

$$z = 2x = 6 \text{ 小時.}$$

答: A 單獨為之須 3 小時; B , 4 小時; C , 6 小時。

15. 解: 設 x 與 y 各為 A 與 B 之速率。

$$\begin{cases} \frac{20}{y} - \frac{20}{x} = 2 \dots\dots\dots(1) \\ \frac{20}{x-y} = 60 \dots\dots\dots(2) \end{cases}$$

從(1)式, $10x - 10y = xy$

$$x = \frac{10y}{10-y}$$

代入(2)式, 得 $3y^2 + y - 10 = 0$

$$(3y-2)(y+2) = 0$$

$$\therefore y = \frac{5}{3} \text{ 呎或} = 20 \text{ 吋,}$$

$$x = 2 \text{ 呎或} = 24 \text{ 吋.}$$

答: A 之速率爲每秒 24 吋; B , 每秒 20 吋。

16. 解: 設 x 與 y 各爲 A 與 B 之速率。

$$\begin{cases} (28-2x)^2 + (9-2y)^2 = 13^2 & \dots\dots\dots(1) \\ (28-3x)^2 + (9-3y)^2 = 5^2 & \dots\dots\dots(2) \end{cases}$$

從 (1) 式, $x^2 + y^2 - 28x - 9y + 174 = 0 \dots\dots\dots(3)$

從 (2) 式, $3x^2 + 3y^2 - 56x - 18y + 280 = 0 \dots\dots\dots(4)$

$3 \times (3) - (4),$ $28x + 9y - 242 = 0$

$$y = \frac{242 - 28x}{9}$$

代入 (3) 式, 得 $865x^2 - 13552x + 53056 = 0$

$$(865x - 6632)(x - 8) = 0$$

$$\therefore x = 8 \text{ 吋/秒 或 } \frac{6632}{865} \text{ 吋/秒,}$$

$$y = 2 \text{ 吋/秒 或 } \frac{23634}{7785} \text{ 吋/秒.}$$

答: A 之速率每秒 8 吋; B , 每秒 2 吋。

17. 解: 設 x 爲距離, y 爲 B 之速率。

$$\frac{x}{y} - \frac{x}{4\frac{1}{2}} = 2 \dots\dots\dots(1)$$

$$\frac{x}{y-1} - \frac{x}{y} = 3 \dots\dots\dots(2)$$

從 (1) 式, $9x - 2xy = 18y \dots\dots\dots(3)$

從 (2) 式, $3y^2 - 3y - x = 0$

$$x = 3y^2 - 3y$$

代入 (3) 式, 得 $2y^2 - 11y + 15 = 0$

$$(2y-5)(y-3) = 0$$

$$y = 3 \text{ 哩/時或 } \frac{5}{2} \text{ 哩/時}$$

$$\therefore x = 18 \text{ 哩 或 } \frac{45}{4} \text{ 哩.}$$

答：其距離為 18 哩或 $\frac{45}{4}$ 哩。

18. 解：設二人相遇時 B 行 x 哩， A 行 $x+12$ 哩，故 P, Q 間之距離為 $2x+12$ 。

$$\frac{x+12}{x} = \frac{x}{x+12}$$

$$\frac{4\frac{2}{3}}{7\frac{5}{7}}$$

$$\frac{14(x+12)}{3x} = \frac{54x}{6(x+12)}$$

$$\frac{7(x+12)}{3x} = \frac{27x}{7(x+12)}$$

$$[7(x+12)]^2 = (9x)^2$$

$$7(x+12) = 9x$$

$$2x = 84$$

$$\therefore 2x+12 = 96 \text{ 哩。}$$

答： P, Q 間之距離為 96 哩。

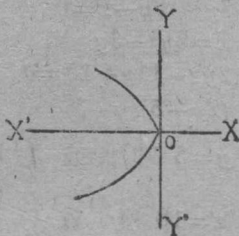
習題 LII

原本第 339 頁

1. 解： $y^2 = -8x$

$$y = \pm 2\sqrt{-2x}$$

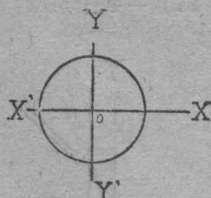
x	0, -1, -2, -3
y	0, $\pm 2\sqrt{2}$, ± 4 , $\pm 2\sqrt{6}$



2. 解: $x^2 + y^2 = 9$

$$y = \pm \sqrt{(3-x)(3+x)}$$

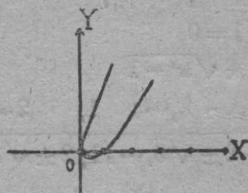
x	-3,	-2,	-1,	0,	1, 2,	3
y	0, $\pm\sqrt{5}$, $\pm 2\sqrt{2}$, ± 3 , $\pm 2\sqrt{2}$, $\pm\sqrt{5}$, 0					



3. 解: $(y-x)^2 = x$

$$y = x \pm \sqrt{x}$$

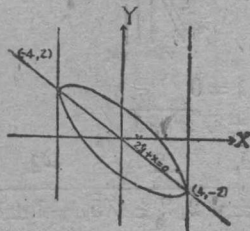
x	0;	1;	2;	3;	4
y	0; 1 ± 1 ; $2 \pm \sqrt{2}$; $3 \pm \sqrt{3}$; 4 ± 2				



4. 解: $x^2 + 2xy + 2y^2 = 8$

$$y = \frac{-x \pm \sqrt{16 - x^2}}{2}$$

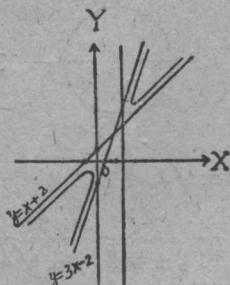
x	-4,	-2,	0,	2,	4
y	2, $1 \pm \sqrt{3}$, ± 2 , $-1 \pm \sqrt{3}$, -2				



5. 解: $y^2 - 4xy + 3x^2 + 4x = 0$

$$y = 2x \pm \sqrt{x(x-4)}$$

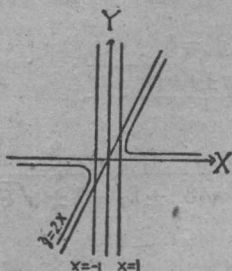
x	-3;	-2;	-1;
	0;	4;	5
y	$-6 \pm \sqrt{21}$;	$-4 \pm \sqrt{12}$;	$2 \pm \sqrt{5}$;
	0;	8;	$10 \pm \sqrt{5}$



6. 解: $y^2 - 2xy + 1 = 0$

$$y = x \pm \sqrt{x^2 - 1}$$

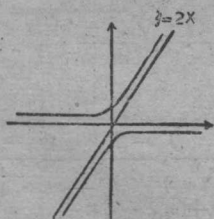
x	-1,	-2, ...;	1,	2...
y	-1,	$-2 \pm \sqrt{3}, \dots$;	$1, 2 \pm \sqrt{3}, \dots$	



7. 解: $y^2 - 2xy - 1 = 0$

$$y = x \pm \sqrt{x^2 + 1}$$

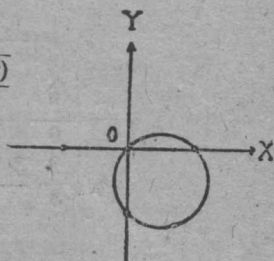
x	0,	1,	2,	-1
y	$\pm 1, 1 \pm \sqrt{2}, 2 \pm \sqrt{5}, -1 \pm \sqrt{2}$			



8. 解: $2x^2 + 3y^2 - 4x + 6y = 0$

$$y = \frac{-3 \pm \sqrt{3(3+4x-2x^2)}}{3}$$

x	0;	1;	2
y	0, -2;	$\frac{-3 \pm \sqrt{15}}{3}$;	0, -2



9. 解: $y^2 - x^2 - 3x + y - 2 = 0$

$$(y+x+2)(y-x-1) = 0$$

故此方程式為二直線:

$$y+x+2=0 \dots\dots\dots (1)$$

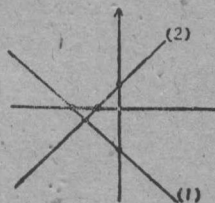
$$y-x-1=0 \dots\dots\dots (2)$$

(1)

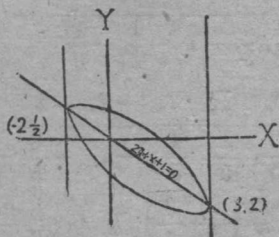
x	0,	1,	-1
y	-2,	-3,	-1

(2)

x	0,	1,	-1
y	1,	2,	0



10. 解: $2x^2 + 4xy + 4y^2 + x + 4y - 5 = 0$



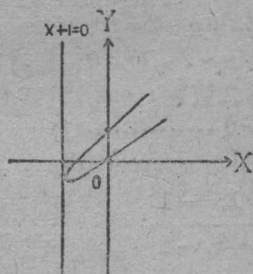
$$y = \frac{-(x+1) \pm \sqrt{(3-x)(2+x)}}{2}$$

x	0,	1,	2,	3,	-1,	-3
y	$\frac{-1 \pm \sqrt{6}}{2}$,	$\frac{-2 \pm \sqrt{6}}{2}$,	$\frac{-3 \pm 4}{2}$,	-2,	± 1 ,	$\frac{1}{2}$

11. 解: $4x^2 - 12xy + 9y^2 + 3x - 6y = 0$

$$y = \frac{(2x+1) \pm \sqrt{x+1}}{3}$$

x	0;	1;	2;	3;	-1
y	$\frac{2}{3}$;	$\frac{3 \pm \sqrt{2}}{3}$;	$\frac{5 \pm \sqrt{3}}{3}$;	$\frac{5}{3}$;	$-\frac{1}{3}$



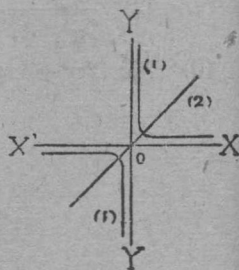
12. 解: $\begin{cases} xy = 1 \dots\dots\dots(1) \\ 3x - 5y = 2 \dots\dots\dots(2) \end{cases}$

(1)

x	1, 2, 3, -1, -2, -3
y	$1, \frac{1}{2}, \frac{1}{3}, -1, -\frac{1}{2}, -\frac{1}{3}$

(2)

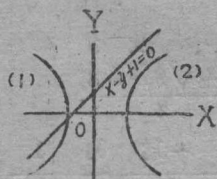
x	0, 1, 2, -1, -2
y	$-\frac{2}{5}, \frac{1}{5}, \frac{4}{5}, -1, -\frac{8}{5}$



答: 其交點為 $(\frac{5}{3}, \frac{3}{5})$ 及 $(-1, -1)$.

13. 解: $\begin{cases} x^2 - y^2 = 1 \dots\dots\dots(1) \\ x^2 - xy + x = 0 \dots\dots\dots(2) \end{cases}$

(1)
$$\begin{array}{c|c} x & \pm 1, \pm 2, \pm 3 \\ \hline y & 0, \pm\sqrt{3}, \pm 2\sqrt{3} \end{array}$$



(2) $x(x - y + 1) = 0$

故 (2) 式爲二直線:

$x = 0 \dots\dots\dots(3)$

$x - y + 1 = 0 \dots\dots\dots(4)$

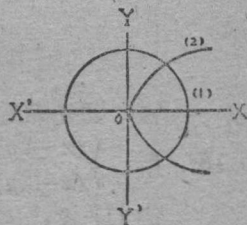
(1)
$$\begin{array}{c|c} x & 0, 1, 2, -1, -2 \\ \hline y & 1, 2, 3, 0, -1 \end{array}$$

答: 其交點爲 $(-1, 0)$.

14. 解: $\begin{cases} x^2 + y^2 = 3 \dots\dots\dots(1) \\ y^2 = 2x \dots\dots\dots(1) \end{cases}$

(1)
$$\begin{array}{c|c} x & \pm\sqrt{3}, \pm\sqrt{2}, 0, \pm 1 \\ \hline y & 0, \pm 1, \pm\sqrt{3}, \pm\sqrt{2} \end{array}$$

(2)
$$\begin{array}{c|c} x & 0, 1, 2, 3, 4 \\ \hline y & 0, \pm\sqrt{2}, \pm 2, \pm\sqrt{6}, \pm 2\sqrt{2} \end{array}$$



答: 其交點爲 $(1, \sqrt{2}), (1, -\sqrt{2})$.

15. 解: $\begin{cases} y^2 - xy - 2x^2 - 2x - 2y - 2 = 0 \dots\dots\dots(1) \\ y^2 - xy - 2x^2 + 2 = 0 \dots\dots\dots(2) \end{cases}$

從 (1) 式, $(y+x)(y-2x-2) - 2 = 0 \dots\dots\dots(3)$

故其漸近線爲 $y+x=0$ 及 $y-2x-2=0$.

從 (3) 式, $y = \frac{(x+2) \pm \sqrt{3 \cdot 3x^2 + 4x + 4}}{2}$

x	$-2; -1; 0; 1$
y	$\pm 2\sqrt{6}; \frac{-1}{2}; 1 \pm \sqrt{3}; \frac{3 \pm \sqrt{33}}{2}$

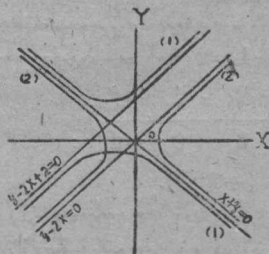
從(2)式, $(y+x)(y-2x)+2=0$ (4)

故其漸近線為

$$y+x=0 \quad \text{及} \quad y-2x=0.$$

從(4)式, $y = \frac{x \pm \sqrt{9x^2 - 8}}{2}$.

x	-2	$-1; 0; 1$
y	$-1 \pm \sqrt{7}; \frac{0}{-1};$	虛數; $\frac{1}{0}$



答: 其交點為 $(-1, -1)$.

16. 解: $\begin{cases} (x-2y)(x+y)+x-3y=0 \dots\dots\dots (1) \\ (x-2y)(x-y)+2x-6y=0 \dots\dots\dots (2) \end{cases}$

從(1)式, $2y^2+(x+3)y-(x^2+x)=0$

$$\therefore y = \frac{-(x+3) \pm \sqrt{9x^2+14x+9}}{4}$$

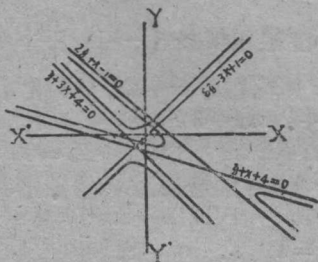
x	$0; 1; 2; -1; -2; -3; -4$
y	$\frac{0}{-2}, -1 \pm \sqrt{2}; \frac{-5 \pm \sqrt{73}}{4}, 0, \frac{-1 \pm \sqrt{17}}{4}; \pm \sqrt{3}; \frac{1 \pm \sqrt{97}}{4}$

故其漸近線為 $6y-3x+1=0$ 與 $y+3x+4=0$

從(2)式, $2y^2-(3x+6)y+(x^2+2x)=0$

$$\begin{aligned} \therefore y &= \frac{(3x+6) \pm \sqrt{x^2+20x+36}}{4} \\ &= \frac{(3x+6) \pm \sqrt{(x+2)(x+18)}}{4} \end{aligned}$$

x	0;	1;	2;	-1
y	3;	$\frac{9 \pm \sqrt{51}}{4}$;	$3 \pm \sqrt{5}$;	$\frac{3 \pm \sqrt{17}}{4}$
x	-2;	-3;	-18;	19
y	0;	虛數;	-12;	$\frac{-51 \pm \sqrt{17}}{4}$

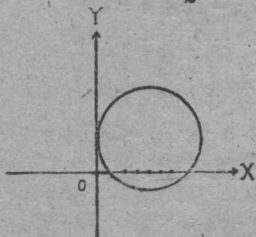


故其漸近線為 $2y+x-1=0$ 與 $y+x+4=0$ 。

答：其交點為 $(0, 0)$; $(-\frac{9}{5}, \frac{3}{5})$ 。

17. 解： $x^2+y^2-6x-2y+1=0$

x	0,	1,	2,	3,	$2 \pm 2\sqrt{2}$
y	1,	$1 \pm \sqrt{5}$,	$1 \pm \sqrt{8}$,	$-\frac{4}{2}$,	0



以 $x=0$ 代入原方程式中, 得 y 軸之交點 $(0, 1)$.

以 $y=0$ 代入原方程式中, 得 x 軸之交點 $(3 \pm \sqrt{8}, 0)$.

18. 解: $(x-y)^2 - 2(x+y) + 1 = 0$

以 $x=0$ 代入原方程式中, 得 $y=1$; 以 $y=0$ 代入原方程式中, 得 $x=1$, 故知此圖形與 x, y 軸相交.

19. 解: 將 $y=3x+5$ 代入 $16x^2 + y^2 - 16 = 0$ 中, 得
 $25x^2 + 30x + 0 = 0$

$$(5x+3)^2 = 0 \quad \therefore x = -\frac{3}{5}$$

代入 $y=3x+5$, 得 $y = \frac{16}{5}$.

答: 此圖形與直線相切於 $(-\frac{3}{5}, \frac{16}{5})$ 點.

20. 解: 將 $y=mx+3$ 代入 $x^2 + 2y^2 = 6$ 式中, 得

$$x^2 + 2(mx+3)^2 = 6$$

$$x^2(1+2m^2) + 12mx + 12 = 0$$

若 $B^2 - 4AC = 0$ 時, 則此直線與該圖形相切, 即

$$144m^2 - 4 \cdot 12(1+2m^2) = 0$$

$$m^2 = 1 \quad \therefore m = \pm 1.$$

答: 當 $m = \pm 1$ 時, 則 $y = mx + 3$ 與 $x^2 + 2y^2 = 6$ 相切.

21. 解: $\begin{cases} 7x - 4y + c = 0 \dots\dots\dots (1) \\ 3x^2 - y^2 + x = 0 \dots\dots\dots (2) \end{cases}$

從 (1) 式, $y = \frac{7x+c}{4}$

代入 (2) 式, 得 $x^2 + 14cx - 16x + c^2 = 0$

$$B^2 - 4AC = (14-16)^2 - 4c^2 = 0$$

$$3c^2 - 7c + 4 = 0$$

$$(3c-4)(c-1) = 0$$

$$\therefore c = \frac{4}{3} \text{ 及 } 1.$$

22. 解: 設 $y = mx + b$ 為 $xy - 2y^2 + y + 6 = 0$ 之漸近線, 將 y 之值代入此方程式中, 得

$$(m-2m^2)x^2 + (b-4bm+m)x + (2b^2+b+6) = 0$$

若 x^2 及 x 之係數為零時, 則可求得其漸近線之方程式.

$$m(1-2m)=0 \quad \therefore m=0 \text{ 及 } \frac{1}{2}$$

$$b-4bm+m=0 \quad \therefore b=0 \text{ 及 } \frac{1}{2}$$

將 m 及 b 之值代入 $y=mx+b$ 式中, 得

$$y=0 \text{ 及 } x-2y+1=0$$

答: $y=0$ 及 $x-2y+1=0$ 為 $xy-2y^2+y+6=0$ 之漸近線。

23. 解: 設 $y=mx+b$ 為其漸近線, 代入 $2x^2+3xy-2y^2+x+2y+2=0$ 式中, 得

$$(2+3m-2m^2)x^2+(3b-4mb+1+2m)x+(2b-7b^2+2)=0$$

$$2+3m-2m^2=0 \quad \therefore m=2 \text{ 及 } -\frac{1}{2}$$

$$3b-4mb+1+2m=0 \quad \therefore b=1 \text{ 及 } 0.$$

將 m 及 b 之值代入 $y=mx+b$ 式中, 得

$$y=2x+1 \text{ 及 } x+2y=0$$

即所求之漸近線方程。

24. 解: $x^2+\lambda xy+y^2-x=0$

$$a=1, 2h=\lambda, b=1$$

當其為橢圓時, $h^2-ab < 0$

$$\left(\frac{\lambda}{2}\right)^2 - 1 < 0$$

$$\frac{1}{4}\lambda^2 < 1$$

$$\lambda^2 < 4$$

$$|\lambda| < 2$$

故當 $|\lambda| < 2$ 時, 此方程式之圖形為一橢圓。

當其為拋物線時, $h^2-ab=0$

$$\left(\frac{\lambda}{2}\right)^2 - 1 = 0$$

$$\frac{1}{4}\lambda^2 = 1$$

$$\lambda^2 = 4$$

$$|\lambda| = 2$$

故當 $\lambda = 2$ 時，此方程式之圖形表一拋物線。

當其為雙曲線時， $h^2 - ab > 0$

$$\left(\frac{\lambda}{2}\right)^2 - 1 > 0$$

$$\frac{1}{4}\lambda^2 > 1$$

$$\lambda^2 > 4$$

$$|\lambda| > 2$$

故當 $|\lambda| > 2$ 時，此方程式之圖形表一雙曲線。

XVII 不 等 式

習 題 LIII

原本第 341 頁

1. 求證： $\frac{a}{b} + \frac{b}{a} > 2$

$$(a-b)^2 > 0$$

$$a^2 - 2ab + b^2 > 0$$

$$a^2 + b^2 > 2ab$$

$$\therefore \frac{a}{b} + \frac{b}{a} > 2.$$

2. 求證： $(a+b)(a^3+b^3) > (a^2+b^2)^2$

$$(a-b)^2 > 0$$

$$a^2 + b^2 > 2ab$$

$$ab(a^2 + b^2) > 2a^2b^2$$

$$a^3b + ab^3 > 2a^2b^2$$

$$a^4 + a^3b + ab^3 + b^4 > a^4 + 2a^2b^2 + b^4$$

$$\therefore (a+b)(a^3+b^3) > (a^2+b^2)^2.$$

3. 求證： $a^3 + b^3 > a^2b + ab^2$

$$(a+b)(a-b)^2 > 0$$

$$a^3 + b^3 - a^2b - ab^2 > 0$$

$$\therefore a^3 + b^3 > a^2b + ab^2.$$

4. 求證: $a^2b + b^2a + b^2c + c^2b + c^2a + a^2c > 6abc$
 $(a-b)^2 > 0$

$$a^2 + b^2 > 2ab$$

$$a^2c + b^2c > 2abc$$

同理 $b^2a + c^2a > 2abc$

$$c^2a + c^2b > 2abc$$

$$\therefore a^2b + b^2a + b^2c + c^2b + c^2a + a^2c > 6abc.$$

5. 求證: $a^3 + b^3 + c^3 > 3abc$

$$a^3 + b^3 > a^2b + ab^2 \quad (\text{從第 3 題})$$

$$b^3 + c^3 > b^2c + bc^2$$

$$c^3 + a^3 > c^2a + ca^2$$

$$2(a^3 + b^3 + c^3) > a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$$

但 $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 > 6abc$

$$2(a^3 + b^3 + c^3) > 6abc$$

$$\therefore a^3 + b^3 + c^3 > 3abc$$

6. 解: $x+7 > \frac{3x}{2} - 8$

$$2x+14 > 3x-16$$

$$x-30 > 0$$

$$\therefore x > 30.$$

7. 解: $2x^2 + 4x > x^2 + 6x + 8$

$$x^2 - 2x - 8 > 0$$

$$(x-4)(x+2) > 0$$

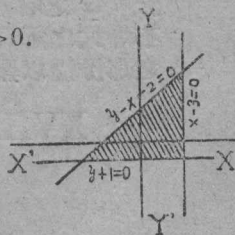
$$\therefore x > 4 \text{ 或 } x < -2.$$

8. 解: $(x+1)(x-3)(x-6) > 0$

左邊三因式之值全為正式, 保有二因式之值為負, 此不等式能滿足, 故得 $-1 < x < 3$ 或 $x > 6$.

9. 解: $y-x-2 < 0$, $x-3 < 0$ 及 $y+1 > 0$.

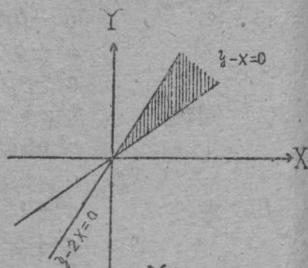
適合不等式 $y-x-2 < 0$ 之各點位於直線 $y-x-2=0$ 之下; 適合不等式 $x-3 < 0$ 之各點位於直線 $x-3=0$ 之左; 適合不等式 $y+1 > 0$ 之各點位於直線 $y+1=0$ 之上. 故適合三不等式之各點, 其圖形位於 $y-x$



$-2=0, x-3=0, y+1=0$ 三線所成之三角形間。

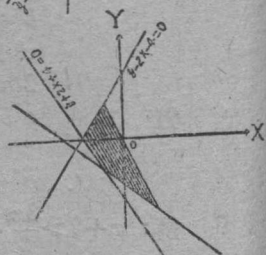
10. 解: $y-x>0, y-2x<0$.

不等式 $y-x>0$ 之圖形位在直線 $y-x=0$ 之上, $y-2x<0$ 之圖形位在直線 $y-2x=0$ 之下。故適合兩不等式之各點, 其圖形位在第一象限內 $y-x=0$ 及 $y-2x=0$ 二直線之間。



11. 解: $x+y+3>0, y-2x-4<0, y+2x+4>0$.

適合不等式 $x+y+3>0, y-2x-4<0$ 及 $y+2x+4>0$ 之各對 x, y 值, 其圖形位於直線 $x+y+3=0$ 之上, $y-2x-4=0$ 之下及 $y+2x+4=0$ 之上, 即圖中繪平行線之部份。



12. 證明 x 為任何值時, $x^2+2x+5>0$ 皆正確不誤。

$$x^2+2x+5>0$$

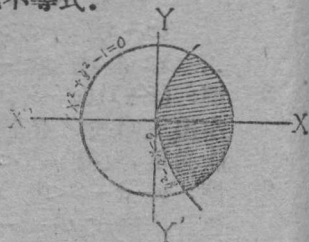
$$x^2+2x+1+4>0$$

$$\therefore (x+1)^2+4>0.$$

答: 無論 x 為何值, 均能適合此不等式。

13. 解: $x^2+y^2-1<0, y^2-4x<0$.

不等式 $x^2+y^2<1$ 之圖形為圓 $x^2+y^2=1$ 之內部又不等式 $y^2<4x$ 之圖形為拋物線 $y^2=4x$ 之右面部份故適合二不等式之解為圓及拋物線間之部份塗以黑線。



XVIII. 一次不定方程式

習題 LIV

原本第 346 頁

1. 解: $6x - 17y = 18$

$$x = 3 + 2y + \frac{5}{6}y$$

若 x 均為整數, 則 $\frac{5}{6}y$ 必為整數。

設 $\frac{5}{6}y = u$, 即 $5y - 6u = 0$

$$y = u + \frac{u}{5}$$

同理 $\frac{u}{5}$ 必為整數, 命其為 v , 故

$$\frac{u}{5} = v, \text{ 即 } u = 5v$$

當 $v = 1$ 時, $u = 5$,
 $y = 6, x = 20$.

答: 其普通整數解為 $x = 20 + 17t, y = 6 + 6t; t$ 為 $-1, 0, 1, 2, \dots$ 時, x 及 y 之值皆為正整數解。

2. 解: $43x - 12y = 158$

$$y = -13 + 3x + \frac{-2 + 7x}{12}$$

設 $\frac{-2 + 7x}{12} = u$ $7x - 12u - 2 = 0$

$$x = u + \frac{2 + 5u}{7}$$

當 $u = 1$ 時, $x = 2, y = -6$.

答: 其普通整數解為 $x = 2 - 12t, y = -6 - 43t; t$ 為 $-1, -2, \dots$ 時, x, y 之值皆為正整數解。

3. 解: $16x + 39y = 1$

$$x = -2y + \frac{1 - 7y}{16}$$

設 $\frac{1 - 7y}{16} = u$ $16u + 7y = 1$

$$y = -2u + \frac{1 - 2u}{7}$$

當 $u = -3$ 時, $y = 7, x = -17$.

答：其普通整數解爲 $x = -17 + 39t$, $y = 7 - 16t$, 無正整數解。

4. 解： $72x + 23y = 845$

$$y = 36 - 3x + \frac{17 - 3x}{23}$$

設 $\frac{17 - 3x}{23} = u$ $3x + 23u = 17$

$$x = 5 - 7u + \frac{2 - 2u}{3}$$

當 $u = 1$ 時, $x = -2$ $y = 43$.

答：其普通整數解爲 $x = -2 + 23t$, $y = 43 - 72t$, 無正整數解。

5. 解： $49x - 27y = 28$

$$y = x - 1 + \frac{22x - 1}{27}$$

設 $\frac{22x - 1}{27} = u$ $22x - 27u = 1$

$$x = u + \frac{5u + 1}{22}$$

設 $\frac{5u + 1}{22} = v$ $22v - 5u = 1$

$$u = \frac{22v - 1}{5} = 4v + \frac{2v - 1}{5}$$

當 $v = 3$ 時, $u = 13$, $x = 16$, $y = 28$.

答：其普通整數解爲 $x = 16 - 27t$, $y = 28 - 49t$, t 爲 $0, -1, -2, \dots$ 時, x, y 之值爲正整數解。

6. 解： $47x - 97y = 501$

$$x = 10 + 2y + \frac{31 + 3y}{47}$$

設 $31 + 3y = 47u$

$$y = 15u - 10 + \frac{2u - 1}{3}$$

當 $u = 2$ 時, $y = 21$, $x = 54$.

答：其普通整數解爲 $x = 54 - 97t$, $y = 21 - 47t$, t 爲 $0, -1, -2, \dots$ 時, x, y 之值爲正整數解。

$$7. \text{ 解: } \begin{cases} 2x+5y-8z=27 & \dots\dots\dots(1) \\ 3x+2y+z=11 & \dots\dots\dots(2) \end{cases}$$

$$8 \times (2) + (1), \quad 26x+21y=115$$

$$y=5-x+\frac{10-5x}{21}$$

設

$$\frac{10-5x}{21}=u$$

$$x=2-4u-\frac{u}{5}$$

當 $u=0$ 時,

$$x=2, y=3.$$

故

$$x=2+21t, y=3-26t.$$

將 x, y 之值代入 (2) 式, 得 $z=-11t-1$

答: 其普通整數解爲 $x=2+21t, y=3-26t, z=-1-11t$,
無正整數解。

$$8. \text{ 解: } \begin{cases} 5x+2y=42 & \dots\dots\dots(1) \\ 3y-7z=2 & \dots\dots\dots(2) \end{cases}$$

$$3 \times (1) - 2 \times (2), \quad 15x+14z=122$$

$$z=-x+8+\frac{10-x}{14}$$

$$\therefore x=10+14t, z=-2-15t$$

代入 (2) 式, 得 $y=-4-35t$.

答: 其普通整數解爲 $x=10+14t, y=-4-35t, z=-2-15t$,
無正整數解。

$$9. \text{ 解: } 4x+3y=2z+3$$

$$z=2x+y-1+\frac{y-1}{2}$$

$$y-1+2u$$

$$\therefore y=1+2u,$$

$$z=2x+3u,$$

$$x=x.$$

答: 其普通整數解爲 $x=x, y=2u+1, z=2x+3u$, 當 x, u
爲正值時, x, y, z 之值皆爲正整數解。

$$10. \text{ 解: } 2x+3y+4z=17$$

$$x = 8 - y - 2z - \frac{y-1}{2} \dots\dots\dots(1)$$

$$y - 1 = 2u$$

$$y = 1 + 2u \dots\dots\dots(2)$$

將 (2) 代入 (1) 式, 得 $x = 7 - 3u - 2z \dots\dots\dots(3)$

再設 $z = v$ 代入 (3) 式, 得

$$x = 7 - 3u - 2v$$

故其普通整數解爲 $x = 7 - 3u - 2v$, $y = 1 + 2u$, $z = v$.

$$\text{當 } u=0, \quad v=0; \quad x=7, \quad y=1, \quad z=0.$$

$$u=0, \quad v=1; \quad x=5, \quad y=1, \quad z=1.$$

$$u=0, \quad v=2; \quad x=3, \quad y=1, \quad z=2.$$

$$u=0, \quad v=3; \quad x=1, \quad y=1, \quad z=3.$$

$$u=1, \quad v=0; \quad x=4, \quad y=3, \quad z=0.$$

$$u=2, \quad v=0; \quad x=1, \quad y=5, \quad z=0.$$

$$u=1, \quad v=1; \quad x=2, \quad y=3, \quad z=1.$$

$$u=1, \quad v=2; \quad x=0, \quad y=3, \quad z=2.$$

答: 其正整數解爲 7, 1, 0; 5, 1, 1; 3, 1, 2; 1, 1, 3; 4, 3, 0;
1, 5, 0; 2, 3, 1; 0, 3, 2.

11. 解: $3x + 7y = 1043$

$$x = 347 - 2y + \frac{2-y}{3}$$

$$2 - y = 3u$$

$$\therefore y = 2 - 3u,$$

$$x = 343 + 7u.$$

當 $u = 0, -1, -2, \dots, -49$ 時, x, y 之值爲正整數。

答: 此方程式有 50 正解。

12. 解: 設 x 與 y 爲二正分式之分子。

$$\frac{x}{5} + \frac{y}{7} = \frac{41}{35}, \text{ 即 } 7x + 5y = 41$$

$$\therefore y = 8 - x + \frac{1-2x}{5},$$

$$x = -2u + \frac{1-u}{2}.$$

當 $u = -1$ 時, $x = 3, y = 4$.

故其普通整數解爲 $x=3+5t, y=4-7t$, 當 t 爲 0 時, 其正整數解爲 $x=3, y=4$.

答: $\frac{41}{35} = \frac{3}{5} + \frac{4}{7}$.

13. 解: 設 x 與 y 各爲牛與羊之頭數.

$$7x+6y=110$$

$$y=18-x+\frac{2-x}{6}$$

$$2-x=6u$$

$$x=2-6u$$

當 $u=0$ 時, $x=2, y=16$.

故其普通整數解爲 $x=2+6t, y=16-7t$, 當 t 爲 0, 1, 2 時, x, y 之值爲正整數解, 即 $x, y=2, 16; 8, 9; 14, 2$.

答: 此人購牛 2 頭, 羊 16 頭; 或牛 8 頭, 羊 9 頭; 或牛 14 頭, 羊 2 頭.

14. 解: 設 x, y 與 z 爲所求之三部.

$$\begin{cases} x+y+z=23 & \dots\dots\dots(1) \\ 3x+2y+5z=79 & \dots\dots\dots(2) \end{cases}$$

$$\begin{cases} x+y+z=23 & \dots\dots\dots(1) \\ 3x+2y+5z=79 & \dots\dots\dots(2) \end{cases}$$

$$5 \times (1) - (2), \quad 2x+3y=36$$

$$x=18-y-\frac{y}{2}$$

設 $y=2u$ 時, $x=18-3u$

代入 (1) 式, 得 $z=5+u$

當 u 爲 0, 1, 2, 3, 4, 5, 6 時, 則 x, y, z 之值爲正整數解.

答: 所求之三部爲 18, 0, 5; 15, 2, 6; 12, 4, 7; 9, 6, 8; 6, 8, 9; 3, 10, 10; 及 0, 12, 11.

15. 解: 設 x, y, z 爲此最小數值被 5, 7, 9 所除得之各商.

$$5x+4=7y+6=9z+8$$

$$5x-7y=2 \dots\dots\dots(1)$$

$$5x-9z=4 \dots\dots\dots(2)$$

$$(2)-(1), \quad 7y-9z=2$$

$$y=z+\frac{2z+2}{7}$$

$$z = 3u - 1 + \frac{u}{2}$$

當 $u=2$ 時, $z=6, y=8$.

故其普通整數解爲 $y=8-9t, z=6-7t$.

代入 (1) 式, 得 $5x+63t=58$

$$x = 11 - 12t + \frac{3-3t}{5}$$

$$t = 1 - u - \frac{2u}{3}$$

$$u = v + \frac{v}{2}$$

當 $v=2$ 時, $u=3, t=-4$.

$$\therefore x=62, y=44, z=34.$$

答: 所求之數爲 $62 \times 5 + 4 = 314$.

16. 解: 設二桿之長爲 l , 從零端至其等份最近之處, 一爲 x 等

份, 一爲 y 等份. 故其距離一爲 $\frac{x}{250}l$, 一爲 $\frac{y}{253}l$, 是

以此部份間之距離爲 $\left(\frac{x}{250} - \frac{y}{253}\right)l$ 即 $\frac{253x-250y}{250 \cdot 253}l$. 因

分子不能爲零, 故此分數於 $253x-250y = \pm 1$ 時最小:

$253x-250y = \pm 1$ 之普通整數解爲 $x=167-250t, y$

$=169-253t$.

答: 所求之值爲 $x=167, y=169$, 即一桿之 167 份與另一桿之 169 份相處最近.

XIX. 比及比例 變數法

習題 LV

原本第 350 頁

1. 解: (a) $15:24=20:b$

$$\therefore b = \frac{24 \times 20}{15} = 32.$$

(b) $15:24=24:c$

$$\therefore c = \frac{24 \times 24}{15} = \frac{192}{5}$$

$$(c) \quad 5a^3b^2 : x = x : 20ab^2$$

$$x^2 = 20ab^2 \times 5a^3b^2 = 100a^4b^4$$

$$\therefore x = \pm 10a^2b^2$$

$$(d) \quad \sqrt{12} : x = x : \sqrt{75}$$

$$x^2 = 30$$

$$\therefore x = \pm \sqrt{30}$$

$$2. \text{ 解: } \quad 3x - 2y = x - 5y$$

$$2x = -3y$$

$$\therefore x : y = -3 : 2,$$

$$x + y : x - y = (-3 + 2) : (-3 - 2) = 1 : 5.$$

$$3. \text{ 解: } \quad 2x^2 - 5xy - 3y^2 = 0$$

$$(x - 3y)(2x + y) = 0$$

$$x = 3y \text{ 及 } 2x = -y$$

$$\therefore x : y = 3 : 1 \text{ 及 } -1 : 2,$$

$$y : x = 1 : 3 \text{ 及 } 2 : -1.$$

$$4. \text{ 解: } \quad ax + by = -cz \dots\dots\dots(1)$$

$$a'x + b'y = -c'z \dots\dots\dots(2)$$

$$c' \times (1) - c \times (2), \quad c'ax + bc'y = ca'x + b'cy$$

$$(bc' - b'c)y = (ca' - c'a)x$$

$$x : y = bc' - b'c : ca' - c'a.$$

$$\text{同理} \quad x : z = bc' - b'c : ab' - a'b.$$

$$\therefore x : y : z = bc' - b'c : ca' - c'a : ab' - a'b.$$

$$5. \text{ 解: } \quad a : b = c : d$$

$$\frac{a}{b} = \frac{c}{d} = r$$

$$a = br, \quad c = dr \dots\dots\dots(1)$$

$$\text{即} \quad b = \frac{a}{r}, \quad d = \frac{c}{r} \dots\dots\dots(2)$$

$$\text{將 (1) 代入 } ab + cd, \text{ 得 } ab + cd = b^2r + d^2r = (b^2 + d^2)r$$

$$\text{將 (2) 代入 } ab + cd, \text{ 得 } ab + cd = a\left(\frac{a}{r}\right) + c\left(\frac{c}{r}\right) = \frac{a^2 + c^2}{r}$$

$$\therefore (ab+cd)^2 = (b^2+d^2)(a^2+c^2)$$

即 $a^2+c^2 : ab+cd = ab+cd : b^2+d^2$.

答: $ab+cd$ 爲 a^2+c^2 與 b^2+d^2 之比例中項.

6. 解: $(a^2+b^2)cd = (c^2+d^2)ab$

$$\frac{a^2+b^2}{2ab} = \frac{c^2+d^2}{2cd}$$

$$\frac{a^2+b^2+2ab}{a^2+b^2-2ab} = \frac{c^2+d^2+2cd}{c^2+d^2-2cd}$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\therefore a : b = c : d.$$

$$a : b = c : d$$

$$a+b : b = c+d : d$$

$$\sqrt{a+b} : \sqrt{b} = \sqrt{c+d} : \sqrt{d}$$

又

$$\sqrt{a} : \sqrt{b} = \sqrt{c} : \sqrt{d}$$

$$\sqrt{a} + \sqrt{b} : \sqrt{b} = \sqrt{c} + \sqrt{d} : \sqrt{d}$$

$$\therefore \sqrt{a} + \sqrt{b} : \sqrt{a+b} = \sqrt{c} + \sqrt{d} : \sqrt{c+d}.$$

8. 解: $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = r$

$$x = ar, y = br, z = cr$$

按 § 987, $r = \frac{x+y+z}{a+b+c}$

$$\begin{aligned} \therefore \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} &= \frac{a^3 r^3}{a^3} + \frac{b^3 r^3}{b^3} + \frac{c^3 r^3}{c^3} \\ &= 3r^3 = 3 \frac{(x+y+z)^3}{(a+b+c)^3}. \end{aligned}$$

9. 解: 設 $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots$

且 $\frac{a_r}{b_r} =$ 最大值, $\frac{a_s}{b_s} =$ 最小值.

$$\frac{l_1 a_1}{l_1 b_1} < \frac{l_1 a_r}{l_1 b_r}, \quad l_1 a_1 < \frac{a_r}{b_r} l_1 b_1;$$

$$\frac{l_2 a_2}{l_2 b_2} < \frac{l_2 a_r}{l_2 b_r}, \quad l_2 a_2 < \frac{a_r}{b_r} l_2 b_2;$$

.....

$$\text{相加} \quad l_1 a_1 + l_2 a_2 + \dots < \frac{a_r}{b_r} (l_1 b_1 + l_2 b_2 + \dots)$$

$$\therefore l_1 a_1 + l_2 a_2 + \dots : l_1 b_1 + l_2 b_2 + \dots < a_r : b_r$$

$$\text{同理} \quad \frac{l_1 a_1}{l_1 b_1} > \frac{l_1 a_s}{l_1 b_s}, \quad l_1 a_1 > \frac{a_s}{b_s} l_1 b_1;$$

$$\frac{l_2 a_2}{l_2 b_2} > \frac{l_2 a_s}{l_2 b_s}, \quad l_2 a_2 > \frac{a_s}{b_s} l_2 b_2;$$

.....

$$\text{相加} \quad l_1 a_1 + l_2 a_2 + \dots > \frac{a_s}{b_s} (l_1 b_1 + l_2 b_2 + \dots)$$

$$\therefore l_1 a_1 + l_2 a_2 + \dots : l_1 b_1 + l_2 b_2 + \dots > a_s : b_s$$

答: $l_1 a_1 + l_2 a_2 + \dots : l_1 b_1 + l_2 b_2 + \dots + l_n b_n$ 介於 $a_s : b_s$ 及 $a_r : b_r$ 之間。

10. 解: 已知 $\frac{a-b}{k} = \frac{b-c}{l} = \frac{c-a}{m}$

分子分母各相加, 得

$$\frac{a-b}{k} = \frac{b-c}{l} = \frac{c-a}{m} = \frac{0}{k+l+m}$$

$$\therefore k+l+m=0.$$

11. 解: 已知 $\frac{x}{mz-ny} = \frac{y}{nx-lz} = \frac{z}{ly-mx}$

$$\frac{x^2}{(mz-ny)x} = \frac{y^2}{(nx-lz)y} = \frac{z^2}{(ly-mx)z} = \frac{x^2+y^2+z^2}{0}$$

$$\therefore x^2+y^2+z^2=0.$$

12. 解: $a_1 : b_1 = a_2 : b_2 = a_3 : b_3$

$$\frac{l_1 a_1^n}{l_1 b_1^n} = \frac{l_2 a_2^n}{l_2 b_2^n} = \frac{l_3 a_3^n}{l_3 b_3^n} = \frac{l_1 a_1^n + l_2 a_2^n + l_3 a_3^n}{l_1 b_1^n + l_2 b_2^n + l_3 b_3^n}$$

$$\therefore \frac{(l_1 a_1^n + l_2 a_2^n + l_3 a_3^n)^{\frac{1}{n}}}{(l_1 b_1^n + l_2 b_2^n + l_3 b_3^n)^{\frac{1}{n}}} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

13. 解: (1) $\frac{x^2+ax-a}{x^2-ax+a} = \frac{2x^2+a}{2x^2-a}$

按 § 686, 3, $2x^2 : 2a(x-1) = 4x^2 : 2a$

故 $x^2[2a(x-1) - a] = 0$

$$\therefore x=0, 0, \frac{3}{2}$$

$$(2) \quad \frac{2x^3 - 3x^2 + 2x + 2}{2x^3 - 3x^2 - 2x - 2} = \frac{3x^3 - x^2 + 10x - 26}{3x^3 - x^2 - 10x + 26}$$

按 § 686, 3, $4x^3 - 6x^2 : 4x + 4 = 6x^3 - 2x^2 : 20x - 52$

故 $x^2[(2x-3)(5x-13) - (x+1)(3x-1)] = 0$

$$x^2(7x^2 - 43x + 40) = 0$$

$$x^2(x-5)(7x-8) = 0$$

$$\therefore x=0, 0, 5, \frac{8}{7}$$

14. 解：設此三份爲 $2x, 3x, 5x$.

$$2x + 3x + 5x = 520$$

$$x = 52$$

$$\therefore 2x = 104,$$

$$3x = 156,$$

$$5x = 260.$$

答：所求之三份爲 104, 156, 260.

15. 解：設 x 與 y 爲自 A 與 B 桶中所取出之酒量。

$$\frac{3}{8}x + \frac{3}{10}y = 6 \dots\dots\dots(1)$$

$$\frac{5}{8}x + \frac{7}{10}y = 12 \dots\dots\dots(2)$$

$$(1) \times 5, \quad \frac{15}{8}x + \frac{15}{10}y = 30 \dots\dots\dots(3)$$

$$(2) \times 3, \quad \frac{15}{8}x + \frac{21}{10}y = 36 \dots\dots\dots(4)$$

$$(4) - (3), \quad \left[\frac{21}{10} - \frac{15}{10} \right] y = 6$$

$$\therefore y = 10 \text{ 釐}.$$

代入 (1) 式，得 $x = 8$ 釐。

答：須自 A 桶中取出 8 釐， B 桶中取出 10 釐。

習 題 LVI

1. 解：設 y 與 x 成正變， $x=5$ 時， $y=-2$ 。

$$y = cx$$

$$-2 = c \cdot 5$$

$$c = -\frac{2}{5}$$

$$y = -\frac{2}{5}x$$

$$= -\frac{2}{5} \times 7 = -\frac{14}{5}$$

2. 解：設 y 與 x^2 成反變， $x=2$ 時， $y=1$ 。

$$y = \frac{c}{x^2}$$

$$1 = \frac{c}{4}$$

$$c = 4 \text{ 及 } y = \frac{4}{x^2}$$

$$3 = \frac{4}{x^2}$$

$$x^2 = \frac{4}{3}$$

$$\therefore x = \pm \sqrt{\frac{4}{3}} = \pm \frac{2}{3} \sqrt{3}$$

3. 解：設 y 爲一常數與另一項之和，此項與 x^2 成正變。又當 $x=1$ 時， $y=1$ ； $x=2$ 時， $y=0$ 。

$$y = b + ax^2 \dots\dots\dots(1)$$

$$1 = b + a \dots\dots\dots(2)$$

及 $0 = b + 4a \dots\dots\dots(3)$

從 (2) 及 (3)，得 $b = \frac{4}{3}$ ， $a = -\frac{1}{3}$ 。

代入 (1) 式，得 $y = \frac{4}{3} - \frac{x^2}{3}$ 。

答：所求之方程式爲 $x^2 + 3y - 4 = 0$ 。

4. 解：設 y 與 x^2 成正變，與 z^3 成反變；又 $x=-1$ ， $z=2$ 時， $y=1$ 。

$$y = c \cdot \frac{x^3}{z^3}$$

$$1 = c \cdot \frac{1}{8}$$

$$c = 8, y = 8 \cdot \frac{x^3}{z^3}$$

當 $x=3, z=-1; y=8 \cdot 9 / -1 = -72$.

5. 解：設 y 與 x 成正變。

$$\begin{aligned} y &= cx \\ \frac{x^2 + y^2}{xy} &= \frac{x^2 + c^2x^2}{cx^2} = \frac{1 + c^2}{c} \end{aligned}$$

因 $\frac{1+c^2}{c}$ 為常數，故 $x^2 + y^2 \sim xy$ 。

6. 解：設 y 之平方與 z 之立方成正變，又 z 與 x 成反變。

$$y^2 = cz^3 \quad z = \frac{c'}{x}$$

$$y = \sqrt{cz^3} = \sqrt{c \left(\frac{c'}{x}\right)^3}$$

$$yx \cdot \sqrt{x} = \sqrt{c \cdot c'^3}$$

$$\therefore xy \sim \frac{1}{\sqrt{x}}$$

7. 解：設 z 示工資之元數， y 示人數， x 示週數；則

$$z = c \cdot xy$$

$$108 = c \cdot 3 \cdot 4$$

$$c = 9$$

$$z = 9xy$$

$$135 = 9 \cdot 5 \cdot x$$

$$\therefore x = 3.$$

答：須作工 3 星期。

8. 解：設 V 為圓盤之體積， r 為其半徑， h 為其厚度。

$$V = c \cdot hr^2$$

$$V_1 = c \cdot 3 \cdot 24^2 = 1728c$$

$$V_2 = c \cdot 2 \cdot 36^2 = 2592c$$

$$1728c + 2592c = c \cdot h \cdot 48^2$$

即 $4320 = 2304h$

$$\therefore h = \frac{15}{8}$$

答：其厚度為 $\frac{15}{8}$.

9. 解：設 R 及 r 為底與斷面之半徑， x 為所求之高度；則
 $x : a = r : R \dots\dots\dots(1)$

又 $\pi r^2 = \frac{1}{2} \pi R^2 \dots\dots\dots(2)$

$$r = \frac{\sqrt{2}}{2} R \dots\dots\dots(3)$$

代入 (1) 式， $x = \frac{\sqrt{2}}{2} a$

又如 $\frac{1}{3} \pi x r^2 = \frac{1}{2} \times \frac{1}{3} \pi a R^2$

則 $\frac{r^2}{R^2} = \frac{a}{2x} \dots\dots\dots(4)$

將 (1) 平方，得 $\frac{r^2}{R^2} = \frac{x^2}{a^2} \dots\dots\dots(5)$

從 (4) 及 (5)， $2x^3 = a^3$
 $\therefore x = \frac{\sqrt[3]{4}}{2} a$

答：斷面面積為底之半時距其頂端為 $\frac{\sqrt{2}}{2} a$ ，若將此圓錐體
 分為二等份時，則其高度為 $\frac{\sqrt[3]{4}}{2} a$.

XX. 等 差 級 數

習 題 LVII

原本第 356 頁

1. 解：(1) 3, 6, 9.....
 $a = 3, \quad d = 3, \quad n = 20$

$$\therefore l = a + (n-1)d = 3 + 19 \cdot 3 = 60,$$

$$S = \frac{n}{2}(a+l) = \frac{20}{2}(3+60) = 630.$$

$$(2) \quad -3, -1\frac{1}{2}, 0, \dots$$

$$a = -3, \quad d = \frac{3}{2}, \quad n = 20$$

$$\therefore l = -3 + (20-1) \times \frac{3}{2} = 25\frac{1}{2},$$

$$S = \frac{20}{2} \left(-3 + \frac{51}{2} \right) = 5 \times 45 = 225.$$

2. 解: (1) 1, 2, 3, \dots

$$S = \frac{n}{2} [1 + \{1 + (n-1) \cdot 1\}] = \frac{n}{2}(n+1).$$

(2) 1, 3, 5, \dots

$$S = \frac{n}{2} [1 + \{1 + (n-1) \cdot 2\}] = \frac{n}{2} \cdot 2n = n^2.$$

(3) 2, 4, 6, \dots

$$S = \frac{n}{2} [2 + \{2 + (n-1) \cdot 2\}] = n(n+1).$$

3. 解: $r = 0, 1, 2, 3, \dots$ 時,

$$6r+1 = 1, 7, 13, 19, \dots$$

$$S = \frac{n}{2} [1 + \{1 + (n-1)6\}]$$

$$= n[1 + (n-1)3]$$

$$= n(3n-2).$$

4. 解:

$$a+4d=1, \quad a+7d=2$$

$$a = -\frac{1}{3}, \quad d = \frac{1}{3}$$

$$\therefore l = -\frac{1}{3} + 9 \cdot \frac{1}{3} = \frac{8}{3}.$$

答: 所求之等差級數為 $-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}$

5. 解: $a = -1, l = 2, n = 5 + 2 = 7$

$$2 = -1 + 6d \quad \therefore d = \frac{1}{2}.$$

答：所求之等差中項爲 $-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$.

6. 解：設 $n=16, a=0, d=\frac{4}{3}$.

$$\therefore l = 0 + (16-1)\frac{4}{3} = 15 \times \frac{4}{3} = 20,$$

$$S = \frac{16}{2}(0+20) = 16 \cdot 10 = 160.$$

7. 解：設 $n=7, l=-7, d=-\frac{5}{3}$.

$$\therefore a = l - (n-1)d = -7 - (7-1)\left(-\frac{5}{3}\right) = -7 + 10 = 3,$$

$$S = \frac{7}{2}(3-7) = -14.$$

8. 解：設 $n=12, a=-\frac{5}{3}, l=31\frac{1}{3}$.

$$\therefore d = \frac{l-a}{n-1} = \frac{31\frac{1}{3} + \frac{5}{3}}{12-1} = \frac{33}{11} = 3,$$

$$S = \frac{12}{2}\left(-\frac{5}{3} + \frac{94}{3}\right) = 6 \times \frac{89}{3} = 178.$$

9. 解：設 $a=2, l=-23\frac{1}{2}, S=-559$.

$$\therefore n = \frac{2S}{a+l} = \frac{-1118}{-\frac{43}{2}} = 52,$$

$$d = \frac{l-a}{n-1} = \frac{-\frac{47}{2} - 2}{52-1} = -\frac{1}{2}.$$

10. 解：設 $n=7, a=\frac{3}{7}, S=45$.

$$\therefore l = \frac{2S}{n} - a = \frac{90}{7} - \frac{3}{7} = \frac{87}{7} = 12\frac{3}{7},$$

$$d = \frac{\frac{87}{7} - \frac{3}{7}}{6} = \frac{84}{42} = 2.$$

11. 解: 設 $a=4$, $d=\frac{1}{5}$, $l=9\frac{2}{5}$.

$$\therefore n = \frac{l-a+d}{d} = \frac{\frac{47}{5} - \frac{20}{5} + \frac{1}{5}}{\frac{1}{5}} = 28,$$

$$S = \frac{28}{2} \left(4 + \frac{47}{5} \right) = 14 \times \frac{67}{5} = 187\frac{3}{5}.$$

12. 解: 設 $n=9$, $a=-4$, $S=135$.

$$a+l = \frac{2S}{n} = \frac{2 \times 135}{9} = 30$$

$$l-a = (n-1)d = -32$$

$$\therefore l = \frac{30-32}{2} = -1,$$

$$a = \frac{30+32}{2} = 31.$$

13. 解: 設 $n=10$, $l=-2$, $S=115$.

$$\therefore a = \frac{2S}{n} - l = \frac{2 \times 115}{10} + 2 = 25,$$

$$d = \frac{-2-25}{9} = \frac{-27}{9} = -3.$$

14. 解: 設 $d=5$, $l=-47$, $S=-357$.

$$-47 = a + (n-1)5 = a + 5n - 5$$

$$a + 5n = -42 \dots\dots\dots(1)$$

$$-357 = \frac{n}{2}(a-47)$$

$$n(a-47) = -714 \dots\dots\dots(2)$$

從(1)及(2)式, $a = -72$, $n = 6$.

15. 解: 設 $a = -10$, $d = 7$, $S = 20$.

$$l = -10 + (n-1)7 = 7n - 17 \dots\dots\dots(1)$$

$$20 = \frac{n}{2}(-10 + l) = -5n + \frac{n}{2}l \dots\dots\dots(2)$$

將(1)代入(2)式,得 $7n^2 - 27n - 40 = 0$

$$(n-5)(7n+8) = 0$$

$$\therefore n = 5,$$

$$l = 35 - 17 = 18.$$

16. 證: $\because a^2, b^2, c^2$ 爲等差級數

$$\therefore b^2 - a^2 = c^2 - b^2$$

$$(b-a)(b+a) = (c-b)(c+b)$$

$$\frac{b-a}{b+c} = \frac{c-b}{a+b}$$

$$\frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)}$$

$$\frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\therefore \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

故 $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ 亦爲等差級數。

17. 證: $S = \frac{n}{2}(a+l) = \frac{n}{2}[2a + (n-1)d]$

$$\frac{S}{n} = \frac{1}{2}(a+l) = a + \frac{n-1}{2}d$$

若 n 爲奇數時,則 $n-1$ 爲偶數,故 $\frac{S}{n}$ 之商爲整數。故 n 個連續數之和能以 n 除盡。

18. 解: $S_1 = a + (a+d) + (a+2d)$
 $= 3a + 3d$

$$S_2 = (a+3d) + (a+4d) + (a+5d) + (a+6d)$$

$$= 4a + 18d$$

$$3a + 3d = \frac{1}{2}[4a + 18d]$$

$$= 2a + 9d$$

$$6d = a$$

$$\therefore d = \frac{a}{6} = \frac{1}{6}$$

答：所求之等差級數為 $1, 1\frac{1}{6}, 1\frac{2}{3}, 1\frac{1}{2}, 1\frac{2}{3}, 1\frac{5}{6}, 2, \dots$

19. 解：設三數為 $a, a+d$ 與 $a-d$ 。

$$a+a+d+a-d=15 \dots\dots\dots(1)$$

$$a^2+(a+d)^2+(a-d)^2=83 \dots\dots\dots(2)$$

從 (1) 式, $a=5$ 。

代入 (2) 式, 得 $25+25+10d+d^2+25-10d+d^2=83$

$$2d^2=8$$

$$\therefore d=2.$$

答：所求之三數為 3, 5, 7。

20. 解：9 之倍數中含有三位數字者必在於 108 與 999 之間，故 $a=108, l=999, d=9$ 。

$$999=108+9n-9=9n+99$$

$$n=111-11=100$$

$$\therefore S_n = \frac{100}{2}(108+999) = 50 \times 1107 = 55350.$$

21. 解：設 $a=130, d=130 \cdot \frac{4}{100}, n=11$ 。

$$l = 130 + (11-1)130 \cdot \frac{4}{100} = 182$$

$$\therefore S = \frac{11}{2}(130+182) = 11 \times 156 = \$1715.$$

22. 解：設 n 小時後二人相遇。

$$4n + \frac{n}{2} \left[2 + \left\{ 2 + (n-1) \frac{1}{2} \right\} \right] = 72$$

$$n^2 + 23n - 288 = 0$$

$$(n-9)(n+32) = 0$$

$$\therefore n = 9 \text{ 小時.}$$

答：二人於 9 小時後相遇，其相遇之點距 A 起行之處為 $4n$ 即 36 哩。

XXI. 等 比 級 數

習 題 LVIII

原 本 第 360 頁

1. 解: $a=2, r=-3, n=5$

$$\therefore l = ar^{n-1} = 2(-3)^4 = 2 \times 81 = 162,$$

$$S = \frac{a(1-r^n)}{1-r} = \frac{2(1+3^5)}{1+3} = 122.$$

2. 解: $a=4, r=\frac{3}{2}, n=4$

$$\therefore l = 4\left(\frac{3}{2}\right)^3 = 4 \times \frac{27}{8} = 13\frac{1}{2},$$

$$S = \frac{4\left[1 - \left(\frac{3}{2}\right)^4\right]}{1 - \frac{3}{2}} = \frac{4\left[1 - \frac{81}{16}\right]}{-\frac{1}{2}} = \frac{65}{2} = 32\frac{1}{2}.$$

3. 解: (a) $12 - 6 + 3 - \dots$

$$a=12, r=-\frac{1}{2}$$

$$\therefore S = \frac{a}{1-r} = \frac{12}{1+\frac{1}{2}} = 8.$$

(b) $1 - \frac{1}{2} + \frac{1}{4} - \dots$

$$a=1, r=-\frac{1}{2}$$

$$\therefore S = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}.$$

(c) $\frac{5}{3} + \frac{1}{3} + \frac{1}{15} + \dots$

6. 解：題設 $n=7$, $a=48$, $l=\frac{3}{4}$.

$$\frac{3}{4} = 48r^6 \quad \therefore r = \frac{1}{2},$$

$$S = \frac{48 \left[1 - \left(\frac{1}{2} \right)^7 \right]}{1 - \frac{1}{2}} = \frac{381}{4} = 95 \frac{1}{4}.$$

7. 解：題設 $a = \frac{1}{16}$, $r = 2$, $l = 8$.

$$8 = \frac{1}{16} \cdot 2^{n-1} \quad 128 = 2^{n-1}$$

$$2^7 = 2^{n-1} \quad \therefore n = 8,$$

$$S = \frac{\frac{1}{16} [1 - 2^8]}{1 - 2} = \frac{255}{16}.$$

8. 解：題設 $n=5$, $r=-3$, $l=81$.

$$81 = a(-3)^4 = 81a$$

$$\therefore a = 1,$$

$$S = \frac{1 \cdot [1 + 3^5]}{1 + 3} = \frac{244}{4} = 61.$$

9. 解：題設 $a=54$, $r=\frac{1}{3}$, $S=80\frac{2}{3}$.

$$80\frac{2}{3} = \frac{54 \left[1 - \left(\frac{1}{3} \right)^n \right]}{1 - \frac{1}{3}} = 81 \left[1 - \frac{1}{3^n} \right]$$

$$242 = 243 - 243 \cdot \frac{1}{3^n}$$

$$3^n = 243 = 3^5$$

$$\therefore n = 5,$$

$$l = 54 \times \frac{1}{3^4} = \frac{2}{3}.$$

10. 解：題設 $n=4$, $a=-3$, $S=-468$.

$$-468 = \frac{-3(1-r^4)}{1-r} = -3(1+r+r^2+r^3)$$

$$r^3+r^2+r-155=0$$

$$(r-5)(r^2+6r+31)=0$$

$$\therefore r=5,$$

$$l=-3 \cdot 5^3 = -375.$$

11. 解：題設 $a = -\frac{9}{16}$, $l = -\frac{16}{9}$, $S = -\frac{781}{144}$.

$$S = \frac{a-rl}{1-r}$$

$$-\frac{781}{144} = \frac{-\frac{9}{16} + \frac{16}{9}r}{1-r} = \frac{-81+256r}{144(1-r)}$$

$$525r = 700$$

$$\therefore r = \frac{4}{3}.$$

$$-\frac{16}{9} = -\frac{9}{16} \left(\frac{4}{3}\right)^{n-1}$$

$$\left(\frac{4}{3}\right)^{n-1} = \frac{256}{81} = \left(\frac{4}{3}\right)^4$$

$$\therefore n=5.$$

12. 解：題設 $n=6$, $r = -\frac{2}{3}$, $S = \frac{665}{216}$.

$$\frac{665}{216} = \frac{a \left[1 - \left(-\frac{2}{3}\right)^6 \right]}{1 + \frac{2}{3}} = \frac{a \left[1 - \frac{64}{729} \right]}{\frac{5}{3}} = \frac{133}{243} a$$

$$\therefore a = \frac{45}{8},$$

$$l = \frac{45}{8} \left(-\frac{2}{3}\right)^5 = -\frac{20}{27}.$$

13. 解：題設 $r = \frac{3}{2}$, $l = 30\frac{3}{8}$, $S = 83\frac{1}{8}$.

$$S = \frac{a - rl}{1 - r}$$

$$\frac{665}{8} = \frac{a - \frac{3}{2} \cdot \frac{243}{8}}{1 - \frac{3}{2}} = \frac{16a - 729}{-8}$$

$$-665 = 16a - 729$$

$$\therefore a = 4.$$

$$\frac{243}{8} = 4 \left(\frac{3}{2} \right)^{n-1}$$

$$n - 1 = 5$$

$$\therefore n = 6.$$

14. 解: 題設 $n = 4$, $l = \frac{54}{25}$, $S = \frac{544}{25}$.

$$\frac{54}{25} = ar^{4-1} = ar^3 \dots\dots\dots(1)$$

$$\frac{544}{25} = \frac{a(r^4 - 1)}{r - 1} \dots\dots\dots(2)$$

從 (2) 式, $\frac{544}{25} = a(r^3 + r^2 + r + 1) \dots\dots\dots(3)$

(3) \div (1), $272r^3 = 27(r^3 + r^2 + r + 1)$

$$245r^3 - 27r^2 - 27r - 27 = 0$$

$$(5r - 3)(49r^2 + 24r + 9) = 0$$

$$\therefore r = \frac{3}{5}.$$

$$\frac{54}{25} = a \left(\frac{3}{5} \right)^3 = \frac{27}{125} a$$

$$\therefore a = 10.$$

15. 解: 題設 $n = 5$, $l = 48$, $S = 93$.

$$48 = ar^{5-1} = ar^4 \dots\dots\dots(1)$$

$$93 = \frac{a(r^5 - 1)}{r - 1} \dots\dots\dots(2)$$

從 (2) 式, $93 = a(r^4 + r^3 + r^2 + r + 1) \dots\dots\dots(3)$

$$\begin{aligned}
 (3) \div (1), \quad & 93r^4 = 48(r^4 + r^3 + r^2 + r + 1) \\
 & 31r^4 = 16(r^4 + r^3 + r^2 + r + 1) \\
 & 15r^4 - 16r^3 - 16r^2 - 16r - 16 = 0 \\
 & (r-2)(15r^3 + 14r^2 + 12r + 8) = 0 \\
 & \therefore r = 2.
 \end{aligned}$$

代入 (1) 式, 得 $a = 3$.

16. 解: $\frac{a^3}{b}$ 與 $\frac{b^3}{a}$.

$$x = \sqrt{\frac{a^3}{b}} \cdot \sqrt{\frac{b^3}{a}} = \sqrt{a^2 b^2} = ab.$$

17. 解: $a = 5, l = 405, n = 3 + 2 = 5$

$$405 = 5r^4 \quad \therefore r = \sqrt[4]{81} = 3.$$

答: 所求之等比中項爲 15, 45, 135.

18. 解: $ar^3 = 3$ (1)

$$ar^6 = -\frac{3}{8}$$
 (2)

$$(2) \div (1), \quad r^3 = -\frac{1}{8} \quad \therefore r = -\frac{1}{2}.$$

代入 (1) 式, 得 $a \times \frac{1}{4} = 3 \quad \therefore a = 12$.

答: 其第七項爲 $12 \left(-\frac{1}{2}\right)^6 = \frac{12}{64} = \frac{3}{16}$.

19. 解: $\begin{cases} a + ar^3 = 133 & \dots\dots\dots (1) \\ ar + ar^2 = 70 & \dots\dots\dots (2) \end{cases}$

從 (1) 式, $a(1+r^3) = 133$ (3)

從 (2) 式, $ar(1+r) = 70$ (4)

$$(1) \div (2), \quad \frac{1-r+r^2}{r} = \frac{133}{70}$$

$$70r^2 - 203r + 70 = 0$$

$$(14r - 35)(5r - 2) = 0$$

$$\therefore r = \frac{5}{2}.$$

代入 (3) 式, 得 $a = 8$.

答: 所求之等比級數爲 8, 20, 50, 125.

20. 解: $\begin{cases} a(1+r+r^2)=7 & \dots\dots\dots(1) \\ a^2(1+r^2+r^4)=91 & \dots\dots\dots(2) \end{cases}$

(2) \div (1), $a(1-r+r^2)=13$(3)

(3) - (1), $ar = -3$

$$r = -\frac{3}{a}$$

代入 (1) 式, 得 $a^2 - 10a + 9 = 0$

$$(a-1)(a-9) = 0$$

$\therefore a = 1, r = -3.$

答: 所求之三數爲 1, -3, 9.

21. 解: $\begin{cases} a+a+d+a-d=36 & \dots\dots\dots(1) \\ \frac{a+4}{a-d+1} = \frac{a+d+43}{a+4} & \dots\dots\dots(2) \end{cases}$

從 (1) 式, $a = 12.$

代入 (2) 式, 得 $\frac{16}{13-d} = \frac{55+d}{16}$

$$d^2 + 42d - 459 = 0$$

$\therefore d = 9$ 或 $-51.$

答: 所求之三數爲 3, 12, 21; 或 63, 12, -39.

22. 解: 設此四數爲 $a, a+d, a+2d, \frac{(a+2d)^2}{a+d}.$

$$\begin{cases} a + \frac{(a+2d)^2}{a+d} = 16 & \dots\dots\dots(1) \\ a+d+a+2d = 8 & \dots\dots\dots(2) \end{cases}$$

從 (2) 式, $a = \frac{8-3d}{2}$

代入 (1) 式, 得 $\frac{8-3d}{2} + \frac{(a+2d)^2}{a+d} = 16$

$$4d^2 + 16d - 128 = 0$$

$$d^2 + 4d - 32 = 0$$

$$(d-4)(d+8) = 0$$

$\therefore d = 4$ 及 $-8,$

$$a = (8-12)/2 = -2.$$

答: 所求之數爲 -2, 2, 6, 18.

23. 解: $a=15$ 呎, $r=\frac{2}{3}$, $n=\infty$

$$15, 15\left(\frac{2}{3}\right), 15\left(\frac{2}{3}\right)^2, 15\left(\frac{2}{3}\right)^3 \dots\dots$$

設 S 爲所求之距離, 故

$$S=15+2\left[15\left(\frac{2}{3}\right)+15\left(\frac{2}{3}\right)^2+15\left(\frac{2}{3}\right)^3+\dots\dots\right]$$

$$=15+2 \cdot 15\left[\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\dots\dots\right]$$

$$=15+30\left[\frac{\frac{2}{3}}{1-\frac{2}{3}}\right]$$

$$=15+30 \times 2=75 \text{ 呎.}$$

答: 此球所經過之距離爲 75 呎.

XXII. 調和級數

習題 LIX

原本第 363 頁

1. 解: 此調和級數之相對等差級數爲 $\frac{5}{3}, \frac{7}{3}, 3$; 故

$$a=\frac{5}{3}, \quad d=\frac{2}{3}.$$

答: 所求之二項爲 $\frac{3}{11}$ 與 $\frac{3}{13}$.

2. 解: 設 x 爲所求之調和中項.

$$x=\frac{2ab}{a+b} \quad (\text{p. 362, } \S 707)$$

$$\therefore x=\frac{2 \cdot \frac{3}{4} \cdot 5}{\frac{3}{4}+5}=\frac{30}{23}.$$

3. 解：其相對之等差級數中之二項爲 $\frac{1}{10}$ 與 $\frac{1}{15}$ ；故

$$a = \frac{1}{10}, \quad l = \frac{1}{15}, \quad n = 4 + 2 = 6$$

$$\frac{1}{15} = \frac{1}{10} + 5d \quad \therefore d = -\frac{1}{150}$$

是以 $\frac{1}{10}$ 與 $\frac{1}{15}$ 之間各項爲 $\frac{14}{150}, \frac{13}{150}, \frac{12}{150}, \frac{11}{150}$

答：所求調和中項爲 $\frac{75}{7}, \frac{150}{13}, \frac{75}{6}, \frac{150}{11}$

4. 解：設所求之第三項爲 x ，則

$$\frac{1}{x} - \frac{5}{4} = -\frac{1}{4} - \frac{1}{x}$$

$$\therefore x = 2.$$

5. 解：設二數爲 a 與 b 。

$$4 = \frac{a+b}{2} \dots\dots\dots(1)$$

$$\frac{15}{4} = \frac{2ab}{a+b} \dots\dots\dots(2)$$

$$(1) \times (2), \quad 15 = ab \quad a = \frac{15}{b}$$

代入 (1) 式，得 $b^2 - 8b + 15 = 0$

$$(b-3)(b-5) = 0$$

$$\therefore b = 3, 5;$$

$$a = 5, 3$$

答：所求之二數爲 5 與 3。

6. 解：設二數爲 a 與 b 。

$$4 = \sqrt{ab} \dots\dots\dots(1)$$

$$\frac{16}{5} = \frac{2ab}{a+b} \dots\dots\dots(2)$$

$$\text{從 (1) 式,} \quad ab = 16 \dots\dots\dots(3)$$

代入 (2) 式, 得 $a+b=10$ (4)

$$a=10-b$$

代入 (3) 式, 得 $b^2-10b+16=0$

$$(b-8)(b-2)=0$$

$$\therefore b=8, 2;$$

$$a=2, 8.$$

答: 所求之二數為 2 與 8.

7. 證: $\because a, b, c$ 為調和級數.

$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ 為等差級數.

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{a+b+c}{b} - \frac{a+b+c}{a} = \frac{a+b+c}{c} - \frac{a+b+c}{b}$$

$$\frac{a+c}{b} + 1 - \frac{b+c}{a} - 1 = \frac{a+b}{c} + 1 - \frac{a+c}{b} - 1$$

$$\frac{a+c}{b} - \frac{b+c}{a} = \frac{a+b}{c} - \frac{a+c}{b}$$

故 $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ 為等差級數.

是以 $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ 為調和級數.

8. 證: 設 a, b, c 為該三數.

$$\frac{\frac{1}{b} - \frac{1}{a}}{\frac{1}{2}} = \frac{1}{c} \dots\dots\dots (1)$$

$$\frac{a - \frac{b}{2}}{\frac{a}{2}} = \frac{1}{c} \dots\dots\dots (2)$$

$$\frac{a - \frac{b}{2}}{\frac{b}{2}} = \frac{a}{c} \dots\dots\dots (3)$$

又 $\frac{1}{\frac{b}{2}} - \frac{1}{c} = \frac{1}{a} \dots\dots\dots (4)$

$$\frac{c - \frac{b}{2}}{\frac{b}{2}} = \frac{c}{a} \dots\dots\dots (5)$$

(5) \times (3), $\frac{a - \frac{b}{2}}{\frac{b}{2}} = \frac{\frac{b}{2}}{c - \frac{b}{2}}$

9. 證: 已知 $x = \frac{2ab}{a+b}$

$$\begin{aligned} \therefore \frac{1}{x-a} + \frac{1}{x-b} &= \frac{1}{\frac{2ab}{a+b} - a} + \frac{1}{\frac{2ab}{a+b} - b} \\ &= \frac{a+b}{ab-a^2} + \frac{a+b}{ab-b^2} \\ &= \frac{a+b}{a-b} \left(-\frac{1}{a} + \frac{1}{b} \right) \\ &= \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} \end{aligned}$$

10. 證: $\because AC : CB = AD : DB$

及 $AC : CB = AE : EB$

$\therefore AD : DB = AE : EB$

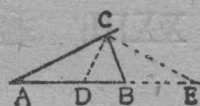
$AD \times EB = DB \times AE$

但 $EB = AE - AB$

$DB = AB - AD$

$\therefore AD(AE - AB) = AE(AB - AD)$

$AD \cdot AE - AD \cdot AB = AE \cdot AB - AE \cdot AD$

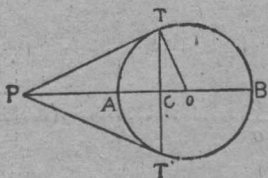


$$\therefore 2AD \cdot AE = AB(AE + AD)$$

$$\therefore AB = \frac{2AD \cdot AE}{AE + AD}$$

故 AD, AB, AE 爲調和級數。

11. 證：聯 TO 與 $T'O$ 。



$$\therefore PC \perp TT'$$

$$\begin{aligned} \therefore \overline{PC}^2 &= \overline{PT}^2 - \overline{TC}^2 \\ &= \overline{PT}^2 - (\overline{TO}^2 - \overline{OC}^2) \\ &= \overline{PT}^2 - \overline{TO}^2 + \overline{OC}^2 \end{aligned}$$

$$\therefore TO = \frac{PB - PA}{2}, \quad OC = PB - PC - \frac{PB - PA}{2}$$

$$\begin{aligned} \therefore \overline{PC}^2 &= \overline{PT}^2 - \left(\frac{PB - PA}{2}\right)^2 + \left(PB - PC - \frac{PB - PA}{2}\right)^2 \\ &= \overline{PT}^2 + PA \cdot PB + \overline{PC}^2 - PC(PA + PB) \end{aligned}$$

但 $\overline{PT}^2 = PA \cdot PB$

$$2PA \cdot PB = PC(PA + PB)$$

$$\therefore PC = \frac{2PA \cdot PB}{PA + PB}$$

故 PC 爲 PA 與 PB 之調和中項。

XXIII. 遞差法, 高階等差級數, 插入法

習題 LX

原本第 369 頁

1. 解: $a_1 = 1, d_1 = 1, d_2 = 1, d_3 = d_4 = \dots = 0, n = 20$

$$a_n = a_1 + (n-1)d_1 + \frac{(n-1)(n-2)}{1 \cdot 2} d_2 + \dots$$

$$+ \frac{(n-1) \dots (n-r)}{1 \cdot 2 \dots r} d_r$$

$$a_{20} = 1 + 19 \times 1 + \frac{19 \times 18}{1 \cdot 2} \times 1 = 191$$

$$S_n = na_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \dots + \frac{n(n-1) \dots (n-r)}{1 \cdot 2 \dots (r+1)} d_r$$

$$S_{20} = 20 + \frac{20 \times 19}{1 \cdot 2} + \frac{20 \times 19 \times 18}{1 \cdot 2 \cdot 3} = 1350.$$

2. 解: $a_1 = 3, d_1 = 5, d_2 = 2, d_3 = d_4 = \dots = 0, n = 80$

$$a_{80} = 3 + (80-1)5 + \frac{(80-1)(80-2)}{1 \cdot 2} \cdot 2$$

$$= 3 + 395 + 6162 = 6560$$

$$S_{80} = 80 \times 3 + \frac{80 \times 79}{2} \cdot 5 + \frac{80 \times 79 \times 78}{1 \cdot 2 \cdot 3} \cdot 2$$

$$= 240 + 15800 + 164320 = 180360.$$

3. 解: (1) 3, 0, -1, 0, 3, \dots

-3, -1, 1, 3, \dots

2, 2, 2, \dots

故此等差級數爲二次。

$$a_{18} = 3 + (18-1)(-3) + \frac{(18-1)(18-2)}{2!} \cdot 2 = 224.$$

(2) 10, 38, 88, 166, 278, 430, \dots

28, 50, 78, 112, 152, \dots

22, 28, 34, 40, \dots

6, 6, 6, \dots

故此等差級數爲三次。

$$a_{20} = 10 + (20-1)28 + \frac{(20-1)(20-2)}{2!} \cdot 22$$

$$+ \frac{(20-1)(20-2)(20-3)}{3!} \cdot 6 = 10118.$$

- (3) 285, 204, 140, 91, 55.....
 -81, -64, -49, -36.....
 17, 15, 13.....
 -2, -2.....

故此等差級數為三次。

$$a_{12} = 285 + (12-1)(-81) + \frac{(12-1)(12-2)}{2!} \cdot 17 \\ + \frac{(12-1)(12-2)(12-3)}{3!} (-2) = -1.$$

- (4) 2, 20, 90, 272, 650, 1332.....
 18, 70, 182, 378, 682.....
 52, 112, 196, 304.....
 60, 84, 108.....
 24, 24.....

故此等差級數為四次。

$$a_{10} = 2 + (10-1)18 + \frac{(10-1)(10-2)}{2!} \cdot 52 \\ + \frac{(10-1)(10-2)(10-3)}{3!} \cdot 60 \\ + \frac{(10-1)(10-2)(10-3)(10-4)}{4!} \cdot 24 = 10100.$$

4. 解: (a) 6, 24, 60, 120, 210.....
 18, 36, 60, 90.....
 18, 24, 30.....
 6, 6.....

故此數串為三次。

$$a = 6, d_1 = 18, d_2 = 18, d_3 = 6 \\ a_n = 6 + (n-1)18 + \frac{(n-1)(n-2)}{2} \cdot 18 + \frac{(n-1)(n-2)(n-3)}{6} \cdot 6 \\ = 6 + 18n - 18 + 9n^2 - 27n + 18 + n^3 - 6n^2 + 11n - 6$$

$$\begin{aligned}
 &= n^3 + 3n^2 + 2n = n(n+1)(n+2) \\
 S_n &= 6n + \frac{n(n-1)}{2} \cdot 18 + \frac{n(n-1)(n-2)}{6} \cdot 18 \\
 &\quad + \frac{6n(n-1)(n-2)(n-3)}{24} \\
 &= \frac{n^4 + 6n^3 + 11n^2 + 6n}{4} = \frac{n(n+1)(n+2)(n+3)}{4}.
 \end{aligned}$$

(b) 16, 108, 384, 1000, 2160, 4116.....
 92, 276, 616, 1160, 1956.....
 184, 340, 544, 796.....
 156, 201, 252.....
 48, 40.....

故此數串爲四次。

$$a = 16, d_1 = 92, d_2 = 184, d_3 = 156, d_4 = 48$$

$$a_n = 16 + (n-1)92 + \frac{(n-1)(n-2)}{2} \cdot 184$$

$$+ \frac{(n-1)(n-2)(n-3)}{6} \cdot 156$$

$$+ \frac{(n-1)(n-2)(n-3)(n-4)}{24} \cdot 48$$

$$= 2n^4 + 6n^3 + 6n^2 + 2n$$

$$= 2n(n+1)^3.$$

5. 解: $S_{14} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} = \frac{14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3} = 560,$

$$a_{14} = 1 + (14-1)2 + \frac{(14-2)(14-1)}{2!} = 105.$$

6. 解: $S_{15} - S_6 = \frac{15(15+1)(30+1)}{3!} - \frac{6(6+1)(12+1)}{3!} = 1149.$

7. 解: $S_{12} = \frac{12(12+1)(3 \cdot 5 + 2 \cdot 12 - 2)}{3!} = 962$ 粒。

8. 解: $253 = 1 + (n-1)2 + \frac{(n-1)(n-2)}{2!} = \frac{n^2 + n}{2}$

$$n^2 + n - 506 = 0$$

$$(n+23)(n-22) = 0$$

$$\therefore n = 22$$

$$S_{22} = \frac{22(22+1)(22+2)}{3!} = 2024 \text{ 粒.}$$

9. 解: $\frac{n(n+1)(n+2)}{3!} = \frac{4}{7} \times \frac{n(n+1)(2n+1)}{3!}$

$$7n+14=8n+4$$

$$\therefore n = 10$$

$$S_{10} = \frac{10(10+1)(10+2)}{3!} = 220$$

$$S'_{10} = \frac{10(10+1)(20+1)}{3!} = 385.$$

10. 解: $a_1 = 9, d_1 = 11, d_2 = 2, a_n = 240$

$$240 = 9 + (n-1)11 + \frac{(n-1)(n-2)}{2} \cdot 2$$

$$= 9 + 11n - 11 + n^2 - 3n + 2$$

$$n^2 + 8n - 240 = 0$$

$$(n-12)(n+20) = 0$$

$$\therefore n = 12$$

$$S_{12} = \frac{12(12+1)(27+24-2)}{3!} = 1274.$$

11. 證: $a_1 = 1, d_1 = 7, d_2 = 12, d_3 = 6$

$$S_n = n + \frac{n(n-1)}{2!} \cdot 7 + \frac{n(n-1)(n-2)}{3!} \cdot 12$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 6$$

$$= \frac{n^2 + 2n^3 + n^4}{4} = \frac{n^2(1 + 2n + n^2)}{4}$$

$$= \frac{n^2(n+1)^2}{4}.$$

但 $1+2+3+\dots+n = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

故 $1^3+2^3+\dots+n^3 = (1+2+\dots+n)^2$.

12. 證: $a_1 = 1, d_1 = 15, d_2 = 50, d_3 = 60, d_4 = 24.$

$$\begin{aligned}
 S_n &= n + \frac{n(n-1)}{2!} \cdot 15 + \frac{n(n-1)(n-2)}{3!} \cdot 50 \\
 &\quad + \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 60 \\
 &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} \cdot 24 \\
 &= \frac{n(6n^4 + 15n^3 + 10n^2 - 1)}{30} \\
 &= \frac{n}{30} (2n^2 + 3n + 1)(3n^2 + 3n - 2) \\
 &= \frac{n}{30} (n+1)(2n+1)(3n^2 + 3n - 1)
 \end{aligned}$$

故 $1^4 + 2^4 + \dots + n^4 = \frac{n}{30} (n+1)(2n+1)(3n^2 + 3n - 1)$.

13. 解: (1) $\phi(n) = n^2 - n + 1$
 $\phi_1(n) = (n+1)^2 - (n+1) + 1 - (n^2 - n + 1) = 2n$
 $\phi_2(n) = 2(n+1) - 2n = 2$

命 $n=1, 2, 3, 4$
 $n^2 - n + 1 = 1, 3, 7, 13 \dots\dots$
 $2n = 2, 4, 6, 8 \dots\dots$
 $2 = 2, 2, 2, 2 \dots\dots$

故此級數爲二次。

$$a_1 = 1, \quad d_1 = 2, \quad d_2 = 2$$

$$\begin{aligned}
 S_n &= n + \frac{n(n-1)}{2!} \cdot 2 + \frac{n(n-1)(n-2)}{3!} \cdot 2 \\
 &= \frac{n^3 + 2n}{3} = \frac{n(n^2 + 2)}{3}
 \end{aligned}$$

$$(2) \quad \phi(n) = \frac{n(n+1)(n+2)}{6}$$

$$\begin{aligned}
 \phi_1(n) &= \frac{(n+1)(n+2)(n+3)}{6} - \frac{n(n+1)(n+2)}{6} \\
 &= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}
 \end{aligned}$$

$$\phi_2(n) = \frac{(n+2)(n+3)}{2} - \frac{(n+1)(n+2)}{2} = n+2$$

$$\phi_3(n) = (n+3) - (n+2) = 1$$

命 $n=1, 2, 3, 4, 5$

$$\frac{n(n+1)(n+2)}{6} = 1, 4, 10, 20, 35 \dots$$

$$\frac{(n+1)(n+2)}{2} = 3, 6, 10, 15, 21 \dots$$

$$n+2 = 3, 4, 5, 6, 7 \dots$$

$$1 = 1, 1, 1, 1, 1 \dots$$

故此級數爲三次。

$$a_1 = 1, d_1 = 3, d_2 = 3, d_3 = 1$$

$$S_n = n + \frac{n(n-1)}{2!} \cdot 3 + \frac{n(n-1)(n-2)}{3!} \cdot 3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$= \frac{n^4 + 6n^3 + 11n^2 + 6n}{24} = \frac{n(n+1)(n+2)(n+3)}{24}$$

14. 證：1, 3, 6, 10...; 1, 4, 9, 16...; 1, 5, 12, 22...; 示此數串中之第 k 個之 n 項與前 n 項之和爲

$$\frac{n(kn-k+2)}{2} \text{ 與 } \frac{n(n+1)(kn-k+3)}{6}$$

第 k 數串之第二次差之各項皆爲 k ,

故其第一次差爲 $1, 1+k, 1+2k, 1+3k, \dots$

第 k 數串爲 $1, 2+k, 3+3k, 4+6k, \dots$

由此得知 $a_1 = 1, d_1 = 1+k, d_2 = k$

$$\therefore a_n = 1 + (n-1)(1+k) + \frac{(n-1)(n-2)}{2} \cdot k$$

$$= \frac{kn^2 - kn + 2n}{2} = \frac{n(kn - k + 2)}{2}$$

$$S_n = n + \frac{n(n-1)}{2!} (1+k) + \frac{n(n-1)(n-2)}{3!} \cdot k$$

$$= \frac{n}{6} \{6 + 3(n-1)(1+k) + (n-1)(n-2)k\}$$

$$= \frac{n}{6}(n+1)(kn-k+3).$$

15. 證：由 § 713, 知一 $(r+s)$ 次之等差級數可以 $x=1, 2, 3, \dots$ 各值代入一 $(r+s)$ 次之多項式, 示為 $f(x)$ 而得; 同理一 s 次之等差級數可以 $x=1, 2, 3, \dots$ 各值代入一 s 次之多項式, 設為 $\phi(x)$ 而得. 故該 $(r+s)$ 次之等差級數之各項加以上述 s 次等差級數之對應項所組成之等差級數可以 $x=1, 2, 3, \dots$ 各值代入 $f(x)+\phi(x)$ 而得, 因 $f(x)+\phi(x)$ 仍為 $(r+s)$ 次. 故按 § 714, $(r+s)$ 次之等差級數與一較低方次, 設為 s 次之等差級數之對應項相加而得之等差級數, 其方次仍為 $(r+s)$ 次.

16. 證：設 x 示一次等差級數之通項, 則按 § 713, 得

$$x=c+zd$$

其中 $z=1, 2, 3, \dots$

$$\therefore f(x)=f(c+zd) \equiv F(z)$$

因 $c+zd$ 為一次 $\therefore F(z)$ 為 n 次

因此按 § 714, 以一次等差級數之連續各項代 x 代入 $f(x)$ 中, 所得之級數為 n 次.

又設 x 示 r 次等差級數之通項, 則按 § 713, 得

$$x=c_0z^r+\dots+c_r$$

$$\therefore f(x) \equiv a_0x^n+\dots+a_n$$

$$=a_0(c_0z^r+\dots+c_r)^n+\dots+a_n$$

$$=a_0c_0z^{rn}+\dots$$

$$\equiv F(z).$$

因 $F(z)$ 為 rn 次, 故按 § 714, 以 r 次等差級數之連續各項代 x 代入 $f(x)$, 可得一 rn 次之等差級數.

習 題 LXI

原本第 374 頁

1. 解: $x = -3, -2, -1, 0$
 $y = -20, 6, 0, 4$
 第一次差 $26, -6, 4$
 第二次差 $-32, 10$

第三次差 42

$$y = -20 + (x+3)26 - 16(x+2)(x+3) + 7(x+1)(x+3)(x+2) \\ = 7x^3 + 26x^2 + 23x + 4$$

$$\text{當 } x = -\frac{5}{2} \text{ 時, } y = 7\left[-\frac{5}{2}\right]^3 + 26\left[-\frac{5}{2}\right]^2 + 23\left[-\frac{5}{2}\right] + 4 \\ = -\frac{3}{8}$$

$$\text{當 } x = -\frac{1}{2} \text{ 時, } y = 7\left[-\frac{1}{2}\right]^3 + 26\left[-\frac{1}{2}\right]^2 + 23\left[-\frac{1}{2}\right] + 4 \\ = -\frac{15}{8}$$

2. 解: 設 $f(x) = b_0 + b_1x + b_2x^2 + b_3x^3 \dots \dots \dots (1)$

$$f(4) = b_0 + 4b_1 + 16b_2 + 64b_3 = 10 \dots \dots \dots (2)$$

$$f(6) = b_0 + 6b_1 + 36b_2 + 216b_3 = -12 \dots \dots \dots (3)$$

$$f(7) = b_0 + 7b_1 + 49b_2 + 343b_3 = -20 \dots \dots \dots (4)$$

$$f(8) = b_0 + 8b_1 + 64b_2 + 512b_3 = -18 \dots \dots \dots (5)$$

從 (2), (3), (4) 與 (5) 各式, 得

$$b_0 = -90, b_1 = 73, b_2 = -16, b_3 = 1$$

代入 (1) 式, 得 $f(x) = -90 + 73x - 16x^2 + x^3$

$$f(12) = -90 + 73 \cdot 12 - 16 \cdot 144 + 1728 = 210.$$

3. 解: $x_1, x_2, x_3 = 25, 26, 27$

$$y_1, y_2, y_3 = 625, 676, 729$$

第一次差 51, 53

第二次差 2

$$y = 625 + (26.54 - 25)51 + \frac{(26.54 - 25)(26.54 - 26)}{2} \cdot 2$$

$$= 625 + 78.54 + 0.8316 = 704.3716.$$

4. 解: $x_1, x_2, x_3, x_4 = 2, 3, 4, 5$

$$y_1, y_2, y_3, y_4 = 8, 27, 64, 125$$

第一次差 19, 37, 61

第二次差 18, 24

第三次差 6

$$y = 8 + (4.8 - 2)19 + \frac{(4.8 - 2)(4.8 - 3)}{2!} \cdot 18$$

$$+ \frac{(4.8-2)(4.8-3)(4.8-4)}{3!} \cdot 6 = 110.592.$$

5. 解: $x_1, x_2, x_3, x_4 = 22, 23, 24, 25$

$$y_1, y_2, y_3, y_4 = 0.04546, 0.04348, 0.04167, 0.04$$

$$\text{第一次差} \quad -0.00198, -0.00181, -0.00167$$

$$\text{第二次差} \quad 0.00017, 0.00014$$

$$\text{第三次差} \quad -0.00003$$

$$y = 0.04546 + (23.6 - 22)(-0.00198) + \frac{(23.6 - 22)(23.6 - 23)}{2!}$$

$$\times (0.00017) + \frac{(23.6 - 22)(23.6 - 23)(23.6 - 24)}{3!} (-0.00003)$$

$$= 0.04237552.$$

6. 解: $x_1, x_2, x_3, x_4, x_5 = 432, 433, 434, 435, 436$

$$y_1, y_2, y_3, y_4, y_5 = 20.7846, 20.8087, 20.8327,$$

$$20.8566, 20.8806$$

$$\text{第一次差} \quad 0.0241, 0.0240, 0.0239, 0.0240$$

$$\text{第二次差} \quad -0.0001, -0.0001, 0.0001$$

$$\text{第三次差} \quad 0 \quad 0.0002$$

$$\text{第四次差} \quad 0.0002$$

$$y = 20.7846 + (435.7 - 432)(0.0241)$$

$$+ \frac{(435.7 - 432)(435.7 - 433)}{2!} (-0.0001) + 0$$

$$+ \frac{(435.7 - 432)(435.7 - 433)(435.7 - 434)(435.7 - 435)}{4!}$$

$$\times 0.0002 = 20.7846 + 0.0892 - 0.0005 + 0.0001$$

$$= 20.8734.$$

7. 解: $x_1, x_2, x_3, x_4 = -2, 0, 4, 5$

$$y_1, y_2, y_3, y_4 = -5, 3, -2, -4$$

$$y = 5 \frac{(x-0)(x-4)(x-5)}{(-2-0)(-2-4)(-2-5)} + 3 \frac{(x+2)(x-4)(x-5)}{(0+2)(0-4)(0-5)}$$

$$- 2 \frac{(x+2)(x-0)(x-5)}{(4+2)(4-0)(4-5)} - 4 \frac{(x+2)(x-0)(x-4)}{(5+2)(5-0)(5-4)}$$

$$= -\frac{5x^3 - 45x^2 + 100x}{84} + \frac{3x^3 - 21x^2 + 6x + 120}{40}$$

$$\begin{aligned}
 & + \frac{2x^3 - 6x^2 - 20x}{24} - \frac{4x^3 - 8x^2 - 32x}{35} \\
 & = \frac{2520 - 806x - 9x^2 - 13x^3}{840}
 \end{aligned}$$

XXIV. 對數

習題 LXII

原本第 378 頁

1. 解: (a) $\log_2 4 = x$ $2^x = 4 = 2^2$ $\therefore x = 2.$
 (b) $\log_4 2 = x$ $4^x = 2 = 4^{\frac{1}{2}}$ $\therefore x = \frac{1}{2}.$
 (c) $\log_{\sqrt{2}} 8 = x$ $\sqrt{2}^x = 8 = \sqrt{2}^6$ $\therefore x = 6.$
 (d) $\log_5 625 = x$ $5^x = 625 = 5^4$ $\therefore x = 4.$
 (e) $\log_3 729 = x$ $3^x = 729 = 3^6$ $\therefore x = 6.$
 (f) $\log_{10} 0.001 = x$ $10^x = 0.001 = 10^{-3}$ $\therefore x = -3.$
 (g) $\log_2 \frac{1}{64} = x$ $2^x = \frac{1}{64} = 2^{-6}$ $\therefore x = -6.$
 (h) $\log_2 0.125 = x$ $2^x = 0.125 = 2^{-3}$ $\therefore x = -3.$
 (i) $\log_a \sqrt[3]{a^{-2}} = x$ $a^x = \sqrt[3]{a^{-2}} = a^{-\frac{2}{3}}$ $\therefore x = -\frac{2}{3}.$
 (j) $\log_8 128 = x$ $8^x = 128 = 8^{\frac{7}{3}}$ $\therefore x = \frac{7}{3}.$
 (k) $\log_a a^3 = x$ $a^{3x} = a^3$ $\therefore x = \frac{3}{2}.$

2. 解: (a) $\log_{10} 12 = \log_{10} 3 + 2 \log_{10} 2 = 0.4771 + 0.6020$
 $= 1.0791.$
 (b) $\log_{10} \frac{9}{2} = 2 \log_{10} 3 - \log_{10} 2 = 0.6532.$
 (c) $\log_{10} \sqrt{2} = \frac{1}{2} \log_{10} 2 = 0.1505.$
 (d) $\log_{10} \sqrt[3]{6} = \frac{1}{3} [\log_{10} 2 + \log_{10} 3] = 0.2594.$

$$\begin{aligned} 3. \text{ 解: } \log_a 600^{\frac{1}{6}} &= \frac{1}{6} \log_a 600 = \frac{1}{6} [\log_a 2^3 + \log_a 3 + \log_a 5^2] \\ &= \frac{1}{6} [3 \log_a 2 + \log_a 3 + 2 \log_a 5]. \end{aligned}$$

$$4. \text{ 解: } (1) b^{\frac{2}{3}} c^{-\frac{1}{2}} / d^{\frac{4}{5}} \quad (2) \sqrt[3]{a^{-2} \sqrt{b^6}} \div \sqrt{b^3 \sqrt{a^{-5}}}$$

$$\begin{aligned} (1) \log_a b^{\frac{2}{3}} c^{-\frac{1}{2}} - \log_a d^{\frac{4}{5}} &= \log_a b^{\frac{2}{3}} + \log_a c^{-\frac{1}{2}} - \log_a d^{\frac{4}{5}} \\ &= \frac{2}{3} \log_a b - \frac{1}{2} \log_a c - \frac{4}{5} \log_a d. \end{aligned}$$

$$\begin{aligned} (2) \log_a (a^{-2} \sqrt{b^6})^{\frac{1}{3}} - \log_a (b^3 \sqrt{a^{-5}})^{\frac{1}{2}} \\ &= \log_a (a^{-\frac{2}{3}} / (b^6)^{\frac{1}{6}}) - \log_a (b^{\frac{3}{2}}) (a^{-\frac{5}{4}}) \\ &= \log_a a^{-\frac{2}{3}} + \log_a b - \log_a b^{\frac{3}{2}} - \log_a a^{-\frac{5}{4}} \\ &= -\frac{2}{3} \log_a a + \log_a b - \frac{3}{2} \log_a b + \frac{5}{4} \log_a a \\ &= \frac{1}{12} \log_a a - \frac{1}{2} \log_a b = \frac{1}{12} - \frac{1}{2} \log_a b. \end{aligned}$$

$$5. \text{ 證: } \log_3 \sqrt[3]{81 \sqrt[4]{729 \cdot 9^{-\frac{2}{3}}}} = \frac{31}{18}.$$

$$\begin{aligned} \log_3 \sqrt[3]{81 \sqrt[4]{729 \cdot 9^{-\frac{2}{3}}}} &= \log_3 [81 \sqrt[4]{729 \cdot 9^{-\frac{2}{3}}}]^{\frac{1}{3}} \\ &= \log_3 [81 \cdot 729^{\frac{1}{4}} \cdot 9^{-\frac{2}{12}}]^{\frac{1}{3}} = \log_3 [3^4 \cdot 3^{\frac{6}{4}} \cdot 3^{-\frac{4}{12}}]^{\frac{1}{3}} \\ &= \log_3 [3^{\frac{4}{3}} \cdot 3^{\frac{3}{2}} \cdot 3^{-\frac{1}{3}}] = \log_3 3^{\frac{31}{18}} = \frac{31}{18}. \end{aligned}$$

$$6. \text{ 證: } \log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2 \log_a (x + \sqrt{x^2 - 1}).$$

$$\begin{aligned} \log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} &= \log_a \frac{(x + \sqrt{x^2 - 1})^2}{x^2 - x^2 + 1} \\ &= \log_a (x + \sqrt{x^2 - 1})^2 = 2 \log_a (x + \sqrt{x^2 - 1}). \end{aligned}$$

1. 解: $79 \times 470 \times 0.982$.

$$\begin{aligned} \log(79 \times 470 \times 0.982) &= \log 79 + \log 470 + \log 0.982 \\ &= 1.89763 + 2.67210 + \bar{1}.99211 = 4.56184. \end{aligned}$$

答: $79 \times 470 \times 0.982 = 36460$.

2. 解: $(-9503) \times (-0.0086578) = 9503 \times 0.0086578$.

$$\begin{aligned} \log(9503 \times 0.0086578) &= \log 9503 + \log 0.0086578 \\ &= 3.97786 + \bar{3}.93739 = 1.91625. \end{aligned}$$

答: $(-9503) \times (-0.0086578) = 82.28$.

3. 解: 1375600×8799000 .

$$\begin{aligned} \log(1375600 \times 8799000) &= \log 1375600 + \log 8799000 \\ &= 6.13851 + 6.94443 = 13.08294. \end{aligned}$$

答: $1375600 \times 8799000 = 1.210 \times 10^{13}$.

4. 解: $0.0356 \times (-0.00049) = -(0.0356 \times 0.00049)$.

$$\begin{aligned} \log(0.0356 \times 0.00049) &= \log 0.0356 + \log 0.00049 \\ &= \bar{2}.55145 + \bar{4}.69020 = \bar{5}.24145. \end{aligned}$$

答: $0.0356 \times (-0.00049) = -1.744 \times 10^{-5}$.

5. 解: $\frac{8075}{364.9}$.

$$\begin{aligned} \log 8075 + \text{colog } 364.9 &= 3.90715 + \bar{3}.43791 \\ &= 1.34506. \end{aligned}$$

答: $\frac{8075}{364.9} = 22.13$.

6. 解: $\frac{0.00542}{0.04708}$.

$$\begin{aligned} \log 0.00542 - \log 0.04708 &= \bar{3}.7340 - \bar{2}.67284 \\ &= \bar{1}.06116. \end{aligned}$$

答: $\frac{0.00542}{0.04708} = 0.1151$.

7. 解: $\frac{24617}{-0.00054}$.

$$\begin{aligned} \log 24617 + \text{colog } 0.00054 &= 4.3912 + 3.2676 \\ &= 7.6588. \end{aligned}$$

答: $\frac{24617}{-0.00054} = -4.558 \times 10^7$.

8. 解: $\frac{0.643 \times 7095}{67 \times 9 \times 0.462}$

$$\begin{aligned} & \log 0.643 + \log 7095 + \text{colog } 67 + \text{colog } 9 + \text{colog } 0.462 \\ & = 1.80821 + 3.85095 + \bar{2}.17393 + \bar{1}.04576 + 0.33536 \\ & = 1.21421. \end{aligned}$$

答: $\frac{0.643 \times 7095}{67 \times 9 \times 0.462} = 16.38$.

9. 解: $\frac{9097 \times 5.4086}{-225 \times 593 \times 0.8665}$

$$\begin{aligned} & \log 9097 + \log 5.4086 + \text{colog } 225 + \text{colog } 593 + \text{colog } 0.8665 \\ & = 3.9589 + 0.7331 + \bar{3}.6478 + \bar{3}.2268 + 0.0622 \\ & = \bar{1}.6288. \end{aligned}$$

答: $\frac{9097 \times 5.4086}{-225 \times 593 \times 0.8665} = -0.4255$.

10. 解: $(2.388)^5$.

$$5 \log 2.388 = 5 \times 0.37804 = 1.89020.$$

答: $(2.388)^5 = 77.66$.

11. 解: $(0.57)^{-4}$.

$$\begin{aligned} & -4 \log 0.57 = -(4 \times \bar{1}.7569) = -(\bar{1} 0236) \\ & = 0.9764. \end{aligned}$$

答: $(0.57)^{-4} = 9.472$.

12. 解: $\left(\frac{19}{11}\right)^9$.

$$\begin{aligned} & 9(\log 19 + \text{colog } 11) = 9(1.2788 + \bar{2}.9586) \\ & = 2.1366. \end{aligned}$$

答: $\left(\frac{19}{11}\right)^9 = 137$.

13. 解: $(1.014)^{25}$.

$$25 \log 1.014 = 25(0.00904) = 0.15100.$$

答: $(1.014)^{25} = 1.413$.

14. 解: $\sqrt{67.54}$.

$$\frac{1}{2} \log 67.54 = \frac{1}{2} (1.82956) = 0.91478.$$

答: $\sqrt{67.54} = 8.218.$

15. 解: $\sqrt[3]{-0.30892}.$

$$\frac{1}{3} \log(-0.30892) = -\frac{1}{3} (\bar{1}.48985) = -(\bar{1}.82995).$$

答: $\sqrt[3]{-0.30892} = -0.6761.$

16. 解: $8^{\frac{5}{4}}.$

$$\frac{5}{4} \log 8 = \frac{5}{4} (0.90309) = 1.12886.$$

答: $8^{\frac{5}{4}} = 13.46.$

17. 解: $(0.001)^{\frac{2}{3}}.$

$$\frac{2}{3} \log 0.001 = \frac{2}{3} (\bar{3}.0000) = \bar{2}.0000.$$

答: $(0.001)^{\frac{2}{3}} = 0.01.$

18. 解: $\left(29 \frac{9}{11}\right)^{\frac{1}{2}}.$

$$\begin{aligned} \frac{1}{2} \log \frac{328}{11} &= \frac{1}{2} (\log 328 + \text{colog } 11) \\ &= \frac{1}{2} (2.51587 + \bar{2}.9586) = \frac{1}{2} (1.4745) = 0.73725. \end{aligned}$$

答: $\left(29 \frac{9}{11}\right)^{\frac{1}{2}} = 5.461.$

19. 解: $\sqrt{\frac{5}{6}} \times \sqrt[3]{\frac{79}{45}}.$

$$\begin{aligned} &\frac{1}{2} (\log 5 + \text{colog } 6) + \frac{1}{3} (\log 79 + \text{colog } 45) \\ &= \frac{1}{2} (0.6990 + \bar{1}.2218) + \frac{1}{3} (1.8976 + \bar{2}.3468) \\ &= 0.04187. \end{aligned}$$

答: $\sqrt{\frac{5}{6}} \times \sqrt[3]{\frac{79}{45}} = 1.101.$

20. 解: $\sqrt{0.1} \div (0.009)^{\frac{3}{5}}$.

$$\frac{1}{2} \log 0.1 + \frac{3}{5} \operatorname{colog} 0.009 = -0.50000 + 1.22745 \\ = 0.72745.$$

答: $\sqrt{0.1} \div (0.009)^{\frac{3}{5}} = 5.34.$

21. 解: $(0.00068)^{-\frac{5}{4}}$.

$$-\frac{5}{4} \log 0.00068 = -\frac{5}{4} \times 4.83251 = 0.95936.$$

答: $(0.00068)^{-\frac{5}{4}} = 9108.$

22. 解: $\left(6\frac{2}{3}\right)^{3.4}$.

$$3.4(\log 20 + \operatorname{colog} 3) = 4.42350 + 2.377792 \\ = 2.80129.$$

答: $\left(6\frac{2}{3}\right)^{3.4} = 632.8.$

23. 解: $(-9306)^{\frac{3}{7}} = -(9306)^{\frac{3}{7}}$.

$$\frac{3}{7} \log 9306 = \frac{3}{7} \times 3.96874 = 1.70089.$$

答: $(-9306)^{\frac{3}{7}} = -50.22.$

24. 解: $(0.0057)^{2.5}$.

$$2.5 \log 0.0057 = 2.5 \times 3.7659 = 6.389675.$$

答: $(0.0057)^{2.5} = 2.453 \times 10^{-6}.$

25. 解: $(5648)^{\frac{1}{2}} \times (-0.94)^{\frac{1}{3}}$.

$$\frac{1}{2} \log 5648 + \frac{1}{3} \log 0.94 = 1.87595 + \bar{1}.99103 \\ = 1.86698.$$

答: $(5648)^{\frac{1}{2}} \times (-0.94)^{\frac{1}{3}} = -73.6.$

26. 解: $28927^3 \div (0.8)^{\frac{2}{5}}$.

$$3 \log 28927 + \frac{2}{5} \operatorname{colog} 0.8 = 13.383915 + 0.03876 \\ = 13.422675.$$

答: $28927^3 \div (0.8)^{\frac{2}{5}} = 2.647 \times 10^{13}$.

27. 解: $\frac{\sqrt[7]{0.0476} \times \sqrt[5]{222}}{\sqrt[3]{5059} \times 0.0088}$.

$$\frac{1}{7} \log 0.0476 + \frac{1}{5} \log 222 + \frac{1}{3} (\text{colog } 5059 + \text{colog } 0.0088)$$

$$= \bar{1}.8111 + 0.4693 + \bar{1}.45049 = \bar{1}.73089.$$

答: $\frac{\sqrt[7]{0.0476} \times \sqrt[5]{222}}{\sqrt[3]{5059} \times 0.0088} = 0.5381$.

28. 解: $\frac{\sqrt[6]{943} \times 7298}{\sqrt[5]{0.00006} \times 0.99}$.

$$\frac{1}{6} (\log 943 + \log 7298) + \frac{1}{5} (\text{colog } 0.00006 + \text{colog } 0.99)$$

$$= \frac{1}{6} (2.9745 + 3.86318) + \frac{1}{5} (4.2218 + 0.0044)$$

$$= 1.13961 + 0.84524 = 1.98485.$$

答: $\frac{\sqrt[6]{943} \times 7298}{\sqrt[5]{0.00006} \times 0.99} = 96.57$.

29. 解: $\sqrt{\frac{854 \times \sqrt[3]{0.042}}{7.9856 \times \sqrt[4]{0.0005}}}$.

$$\frac{1}{2} \log 854 + \frac{1}{6} \log 0.042 + \frac{1}{2} \text{colog } 7.9856 + \frac{1}{8} \text{colog } 0.0005$$

$$= 1.46575 + \bar{1}.770541 + \bar{1}.54884 + 0.41263$$

$$= 1.197761.$$

答: $\sqrt{\frac{854 \times \sqrt[3]{0.042}}{7.9856 \times \sqrt[4]{0.0005}}} = 15.77$.

30. 解: $\sqrt[3]{\frac{7^{\frac{1}{4}} \times 92^{\frac{1}{5}} \times (0.01)^{\frac{1}{2}}}{(0.00026)^5 \times 5968^{\frac{1}{3}}}}$.

$$\frac{1}{12} \log 7 + \frac{1}{15} \log 92 + \frac{1}{6} \log 0.01 + \frac{5}{3} \text{colog } 0.00026$$

$$+ \frac{1}{9} \text{colog } 3968$$

$$= 0.07042 + 0.13099 - 0.3333 + 5.9750 + 1.58046$$

$$= 5.42357.$$

答: $\sqrt[3]{\frac{7^{\frac{1}{4}} \times 92^{\frac{1}{5}} \times (0.01)^{\frac{1}{2}}}{(0.00026)^5 \times 5968^{\frac{1}{3}}}} = 2.652 \times 10^5.$

習題 LXIV

原本第 392 頁

1. 解: $\log_5 555 = \frac{\log 555}{\log 5} = \frac{2.7443}{0.6990} = 3.92603,$

$$\log_7 0.0463 = \frac{\log 0.0463}{\log 7} = \frac{\bar{2}.6656}{0.8451} = -1.578,$$

$$\log_{100} 47 = \frac{\log 47}{\log 100} = \frac{1.6721}{2} = 0.83605.$$

2. 解: (1) $3^x = 729$

$$x = \log_3 729 = \frac{\log 729}{\log 3} = \frac{2.8627}{0.4771} = 6.$$

(2) $a^{x^2+2} = a^{3x}$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$\therefore x = 1 \text{ 及 } 2.$$

(3) $213^x = 516^{-x+4}$

$$\log_{213} 516^{-x+4} = x$$

$$(-x+4)\log_{213} 516 = x$$

$$\frac{x}{4-x} = \log_{213} 516 = \frac{\log 516}{\log 213} = \frac{2.7126}{2.3284}$$

$$10.8504 = 5.041x$$

$$\therefore x = 2.152.$$

3. 解: (1) $\log x + \log(x+3) = 1$

$$\log[x(x+3)] = \log 10$$

$$x^2 + 3x - 10 = 0$$

$$(x+5)(x-2) = 0$$

$$\therefore x = -5 \text{ 及 } 2.$$

$$(2) \log x^2 + \log x = 2$$

$$\log x^3 = 2$$

$$\log x = \frac{2}{3} = 0.6666$$

$$\therefore x = 4.642.$$

$$(3) \log(1-2x)^3 - \log(3-x)^3 = 6$$

$$3 \log(1-2x) - 3 \log(3-x) = 6$$

$$\log \frac{1-2x}{3-x} = 2 = \log 100$$

$$\frac{1-2x}{3-x} = 100$$

$$1-2x = 300 - 100x$$

$$98x = 299$$

$$\therefore x = \frac{299}{98} = 3.051.$$

$$(4) x^{\log x} = 2$$

$$(\log x)^2 = \log 2 = 0.3010$$

$$\log x = \sqrt{0.3010} = 0.549$$

$$\therefore x = 3.537.$$

$$4. \text{ 解: } A = P(1+r)^n = 7500 \left[1 + \frac{5}{100} \right]^{35}$$

$$\log A = \log 7500 + 35 \log 1.05 = 4.6171$$

$$\therefore A = 41410.$$

答: 本利和爲 41410 元。

$$5. \text{ 解: } A = P \left(1 + \frac{r}{2} \right)^{2n} = 5500 \left[1 + \frac{3}{200} \right]^{40}$$

$$\log A = \log 5500 + 40 \log 1.015 = 4.004$$

$$\therefore A = 10010.$$

答: 本利和爲 10010 元。

$$6. \text{ 證: } A = P \left[1 + \frac{5}{100} \right]^{15} = P \times 2.079 \dots \dots$$

故知十五年後, 本利和多於原數之二倍。

$$A = P \left[1 + \frac{5}{100} \right]^{95} = P \times 103.09 \dots \dots$$

故知九十五年後，本利和多於原數之一百倍。

7. 解: $A = P(1+r)^n$

$$1250 = P\left(1 + \frac{4}{100}\right)^{15}$$

$$\log 1250 = \log P + 15 \log 1.04$$

$$\log P = 3.0969 - 15 \times 0.0170 = 2.8419$$

$$\therefore P = 694.8.$$

答: 本金為 694.8 元。

8. 解: $A = 200[1.035 + (1.035)^2 + \dots + (1.035)^{25}]$

按 § 701, $A = 200 \times 1.035 \times \frac{(1.035)^{25} - 1}{1.035 - 1}$

$$= 200 \times 1.035 \times 1.363 \div 0.035$$

$$\log A = \log 200 + \log 1.035 + \log 1.363$$

$$+ \operatorname{colog} 0.035$$

$$= 2.30103 + 0.01494 + 0.13450 + 1.45593$$

$$= 3.90640.$$

$$\therefore A = 8030.$$

答: 此人於二十五年後共得總數 8030 元。

9. 解: $P = \frac{A}{r} \left[1 - \frac{1}{(1+r)^n} \right] = \frac{1200}{0.04} \left[1 - \frac{1}{(1.04)^{30}} \right]$

$$= \frac{1200}{0.04} \times \frac{2.236}{3.236}$$

$$= 30000 \times \frac{2.236}{3.236} = 20730 \text{ 元.}$$

$$P = \frac{A}{r} = \frac{1200}{0.04} = 30000 \text{ 元.}$$

10. 解: 已知 $b = \sqrt{(c+a)(c-a)}$; $c = 586.4$, $a = 312.2$.

$$\log b = \frac{1}{2} [\log(c+a) + \log(c-a)]$$

$$= \frac{1}{2} [2.95357 + 2.43807] = 2.69582$$

$$\therefore b = 496.4.$$

面積 $A = \frac{1}{2}ab$

$$\begin{aligned}\log A &= \log a + \log b + \text{colog } 2 \\ &= 2.49446 + 2.69582 + \bar{1}.6990 = 4.88928 \\ \therefore A &= 77500.\end{aligned}$$

11. 解: 已知 $s = (a+b+c)/2$, $A = \sqrt{s(s-a)(s-b)(s-c)}$;
 $a = 416.8$, $b = 424$, $c = 25.68$.

$$s = \frac{416.8 + 424 + 25.68}{2} = 433.24$$

$$\begin{aligned}\log A &= \frac{1}{2}[\log s + \log(s-a) + \log(s-b) + \log(s-c)] \\ &= \frac{1}{2}[2.63647 + 1.21588 + 0.96570 + 2.610216] \\ &= 3.714268.\end{aligned}$$

$$\therefore A = 5179.$$

12. 解: 設 $\pi = 3.1416$

$$\begin{aligned}\log S &= \log 4 + \log \pi + \log r^2 \\ &= 0.6021 + 0.94715 + 2.7458 \\ &= 3.84505.\end{aligned}$$

$$\therefore S = 6998.$$

$$\begin{aligned}\log V &= \log 4 + \log \pi + \log r^3 + \text{colog } 3 \\ &= 0.6021 + 0.49715 + 4.1187 + \bar{1}.5223 \\ &= 4.74085.\end{aligned}$$

$$\therefore V = 55050.$$

XXV. 排列及組合

習題 LXV

原本第 405 頁

1. 解: $C_1^3 \cdot C_1^2 \cdot C_1^1 = 3 \cdot 2 \cdot 1 = 24$
2. 解: $P_5^6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$.
3. 解: $P_3^8 = 8 \cdot 7 \cdot 6 = 336$.

4. 解: $C_4^{10} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210,$
 $P_4^{10} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$
5. 解: $C_3^{100} = \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3} = 161700.$
6. 解: $3C_2^5 = 3 \cdot \frac{5 \cdot 4}{1 \cdot 2} = 30.$
7. 解: $N = \frac{7!}{3!2!} = 420.$
8. 解: (1) $C_1^3 \cdot C_1^6 \cdot 7! = 70720.$
 (2) $9! - 8! = 322560.$
 (3) $P_3^8 \cdot P_6^6 = 3!6! = 4320.$
9. 解: (1) $\frac{P_9^9}{3!} = \frac{9!}{3!} = 60480.$
 (2) $\frac{P_9^6}{6!} = \frac{9!}{6!} = 504.$
 (3) $\frac{P_9^3}{3!6!} = \frac{9!}{3!6!} = 84.$
10. 解: $\frac{3!}{2!} P_2^5 = \frac{3 \cdot 2}{2 \cdot 1} \cdot 5 \cdot 4 = 60.$
11. 解: $C_3^6 \cdot C_6^9 = 1680.$
12. 解: $C_2^3 + C_2^3 = 3 + 3 = 6.$
13. 解: (1) $7 + 7^2 + 7^3 = 7 + 49 + 343 = 399.$
 (2) $P_1^7 + P_2^7 + P_3^7 = 7 + 7 \cdot 6 + 7 \cdot 6 \cdot 5 = 259.$
14. 解: 1, 3, 5 三字處於尾端之方法爲 P_1^3 即 3, 故所求之排列爲 $3 \cdot P_4^5 = 360.$
15. 解: (1) $3 \cdot 4 \cdot P_2^8 + 2 \cdot 5 \cdot P_2^8 = 672 + 560 = 1232.$
 (2) $4 \cdot P_2^8 = 4 \cdot 56 = 224.$
16. 解: $C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 2^5 - 1 = 31.$
17. 解: $3! \cdot C_6^{15} \cdot C_5^9 \cdot C_4^4$
 $= 3! \cdot \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$
 $= 3! \cdot 630630 = 3783780.$

18. 解: $\frac{11!}{6!5!} = 11 \cdot 6 \cdot 7 = 462.$

19. 解: 2, 2, 2; 3, 3, 3; 4, 4; 1, 5, 6, 7.

(1) 四數碼中有三字相同, 其組合為 $C_1^2 \cdot C_1^6$, 故共有 $2 \times 6 \frac{4!}{3!} = 48$ 種安排.

(2) 有兩對文字相同, 其組合為 $C_2^3 = 3$, 故共有 $3 \frac{4!}{2!2!} = 18$ 種安排.

(3) 有二相同文字與二不同文字, 其組合為 $C_1^3 \cdot C_2^6$, 故共有 $3 \cdot C_2^6 \frac{4!}{2!} = 540$ 種安排.

(4) 有四不同文字, 故有 $P_4^4 = 840$ 種安排.

答: 所求數共 $48 + 18 + 540 + 840 = 1446.$

20. 解: $C_8^{15} \cdot C_7^{12} (8+7)! = 6435 \times 792 \times 15!$
 $= 5096520 \times 15!$

21. 解: $2C_4^{11} + C_3^{11} = 660 + 165 = 825.$

22. 解: (1) $C_1^6 \cdot C_3^5 = 6 \cdot 10 = 60.$

(2) $C_1^6 \cdot C_3^5 + C_2^6 \cdot C_2^5 + C_3^6 \cdot C_1^5 + C_4^6 = 60 + 150 + 100 + 15 = 325.$

23. 解: $C_2^{10} \cdot C_2^{12} = 45 \cdot 66 = 2970.$

24. 證: 若 n 點中無三點在同一直線者, 則所聯之直線數目為 C_2^n , 但有 m 個點位於同一直線上, 故須自 C_2^n 減去 C_2^m , 又此 m 個點相聯為一直線, 故所作成之直線數目為 $C_2^n - C_2^m + 1.$

25. 解: $\frac{(16-1)!}{2 \cdot 5!6!5!} = \frac{15!}{2 \cdot 5!6!5!} = 63063.$

26. 解: $C_5^{10} (5-1)! (5-1)! = 252 \cdot 4!4! = 145152.$

27. 解: $2 \cdot C_2^6 \cdot C_2^5 = 2 \cdot 15 \cdot 10 = 300.$

28. 解: (1) $P_1^5 \cdot P_1^5 \cdot P_1^5 \dots$ 至第十五項 $= 5^{15}.$

(2) $\frac{15!}{3!3!3!3!3!} = \frac{15!}{(3!)^5}.$

29. 解: $\frac{5!}{3!2!} 4!4! = 10 \cdot 24 \cdot 24 = 5760.$

$$\begin{aligned}
 30. \text{ 解: } & C_6^{10} \cdot C_3^5 + [C_5^{10} \cdot C_1^3 \cdot C_3^5 + C_6^{10} \cdot C_2^5 \cdot C_1^3] \\
 & + [C_4^{10} \cdot C_2^3 \cdot C_3^5 + C_6^{10} \cdot C_1^5 \cdot C_2^3 + C_5^{10} \cdot C_1^3 \cdot C_2^5 \cdot C_1^2] \\
 & + [C_3^{10} \cdot C_3^3 \cdot C_3^5 + C_6^{10} \cdot C_3^3 + C_4^{10} \cdot C_2^3 \cdot C_2^5 \cdot C_1^1 + C_5^{10} \cdot C_1^3 \cdot C_1^5 \cdot C_2^2] \\
 & = 2100 + 7560 + 6300 + 6300 + 3150 + 15120 + 1200 \\
 & + 210 + 6300 + 3700 = 52010.
 \end{aligned}$$

31. 解: 按 § 763, 六字母之排列爲 $6!$, 一字母佔其特別地位而另一字母不佔其特別地位之排列爲 $2 \times 4 \times 4!$. 因一特別地位可選二特別字母中之一, 另一特別地位可選其他四字母中之一, 其他四地位爲其餘四字母所佔之排列爲 $4!$ 故也. 又二字母各佔其特別地位之排列爲 $4!$, 故六字母中二字母均不佔其特別地位之排列爲

$$6! - 2 \times 4 \times 4! - 4!$$

$$\text{即 } 6! - 2 \times 5! + 4!$$

$$32. \text{ 解: } C_4^{6+4-1} = C_4^9 = 126.$$

$$33. \text{ 解: } C_5^{6+5-1} = C_5^{10} = 252.$$

$$34. \text{ 解: (1) } P_4^{10} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

$$(2) P_4^{10}/3! = \frac{5040}{6} = 840.$$

$$(3) P_5^{10}/2!2! = \frac{5040 \times 6}{4} = 7560.$$

$$35. \text{ 證: } C_n^{n+4-1} = C_3^{n+3} = \frac{(n+3)(n+2)(n+1)}{3!}.$$

XXVI. 多項式定理

習題 LXVI

原本第 409 頁

1. 解: 在 $(a+b+c+d)^3$ 展開式中之項可分三類, 即 abc , a^2b 與 a^3 , 其係數各爲 $3!$, $\frac{3!}{2!}$ 與 $\frac{3!}{3!}$. 故收集其同類之各項,

$$\text{得 } (a+b+c+d)^3 = \Sigma a^3 + 3 \Sigma a^2b + 6 \Sigma abc.$$

2. 解: 在 $(a+b+c+d)^5$ 展開式中之項可分六類, 即 a^5 , a^4b ,

$a^3b^2, a^3bc, a^2b^2c, a^2bcd$, 其係數各為 $1, \frac{5!}{4!}, \frac{5!}{3!2!}, \frac{5!}{3!}$

$\frac{5!}{2!2!}, \frac{5!}{2!}$ 故收集其同類各項, 得

$$(a+b+c+d)^5 = \Sigma a^5 + 5 \Sigma a^4b + 10 \Sigma a^3b^2 + 20 \Sigma a^3bc + 30 \Sigma a^2b^2c + 60 \Sigma a^2bcd.$$

3. 解: (1) $\frac{12!}{5!4!2!} = 83160.$

(2) $\frac{12!}{4!4!4!} = 34650.$

(3) $\frac{12!}{5!5!2!} = 16632.$

4. 解: $\frac{10!}{2!3!4!} = 12600.$

5. 解: $\frac{8!}{4!3!} \cdot 1^4 \cdot 3^3 \cdot 2^1 = 280 \cdot 54 = 15120.$

6. 解: 展開式中之通項 $= \frac{10!}{\alpha! \beta! \gamma! \delta!} x^{\beta+2\gamma+3\delta} \dots \dots \dots (1)$

其中 $\alpha + \beta + \gamma + \delta = 10 \dots \dots \dots (2)$

欲求 x^6 之一項, 須設

$$\beta + 2\gamma + 3\delta = 6 \dots \dots \dots (3)$$

從 (2) 及 (3) 式, 得

$$\alpha, \beta, \gamma, \delta = 7, 1, 1, 1;$$

$$= 8, 0, 0, 2;$$

$$= 7, 0, 3, 0;$$

$$= 4, 6, 0, 0;$$

$$= 6, 3, 0, 1;$$

$$= 6, 2, 2, 0;$$

$$= 5, 4, 1, 0.$$

答: 可求之係數為

$$\begin{aligned} & \frac{10!}{7!} + \frac{10!}{8!2!} + \frac{10!}{7!3!} + \frac{10!}{4!6!} + \frac{10!}{6!3!} + \frac{10!}{6!2!2!} + \frac{10!}{5!4!} \\ & = 720 + 45 + 120 + 210 + 840 + 1260 + 1260 \\ & = 4455. \end{aligned}$$

7. 解：展開式中之通項 $= \frac{9!}{\alpha! \beta! \gamma!} 1^\alpha (-1)^\beta \cdot 3^\gamma \cdot x^{\beta+2\gamma} \dots \dots (1)$

其中 $\alpha + \beta + \gamma = 9 \dots \dots \dots (2)$

欲求 x^7 之一項，須設

$$\beta + 2\gamma = 7 \dots \dots \dots (3)$$

從 (2) 及 (3) 式，得

$$\begin{aligned} \alpha \quad \beta \quad \gamma &= 2, 7, 0; \\ &= 3, 5, 1; \\ &= 4, 3, 2; \\ &= 5, 1, 3. \end{aligned}$$

答：所求之係數為

$$\begin{aligned} & -\frac{9!}{2!7!} - \frac{9!}{3!5!} \times 3 - \frac{9!}{4!3!2!} \times 3^2 - \frac{9!}{5!3!} \times 3^3 \\ & = -36 - 1512 - 11340 - 13608 \\ & = -26496. \end{aligned}$$

XXVII. 可能率

習題 LXVII

原本第 414 頁

1. 解： $\therefore \frac{3}{8} = \frac{a}{m} = \frac{a}{a+b}$

$$\therefore a=3, b=5.$$

答：其偏於失敗之優勝比為 5 對 3，其不能發生之可能率

$$\text{為 } 1 - \frac{3}{8} = \frac{5}{8}.$$

2. 解：A 之勝利可能率為 $\frac{10}{10+9} = \frac{10}{19}$

A 之失敗可能率為 $1 - \frac{10}{19} = \frac{9}{19}$.

3. 解：其可能率為 $\frac{5}{5+3} = \frac{5}{8}$.

答：其希望值為 $60 \times \frac{5}{8} = 37.50$ 元。

4. 解：一事件之可能情形誠為成與敗。然此二情形成敗之方法數絕不相等，即其一較另一或多或少。故此事件之機會不能為 $\frac{1}{2}$ ，而可能率之定義並非無理者。

5. 解：(1) 取得白球之方法有 C_1^7 ；取得任一球之方法有 C_1^{16} ，故取得白球之機會為 C_1^7/C_1^{16} 即 $\frac{7}{16}$ 。

同理，取得黑球之機會為 C_1^6/C_1^{16} 即 $\frac{3}{8}$ ；取得紅球之機會為 C_1^3/C_1^{16} 即 $\frac{3}{16}$ 。

(2) 二球皆為黑球之機會為 C_2^6/C_2^{16} 即 $\frac{1}{8}$ 。

一球為白球，另一球為紅球之機會為 $C_1^3 \cdot C_1^7/C_2^{16}$ 即 $\frac{7}{40}$ 。

(3) 全為紅球之機會為 C_3^3/C_3^{16} 即 $\frac{1}{560}$ 。

三球中無紅球之機會為 C_3^{13}/C_3^{16} 即 $\frac{143}{280}$ 。

三球中白、黑、紅各一之機會為 $C_1^7 \cdot C_1^6 \cdot C_1^3/C_3^{16}$ 即 $\frac{9}{40}$ 。

(4) 僅一球為白色，其餘非白色之機會為 $C_1^7 \cdot C_3^9/C_4^{16}$ 即 $\frac{21}{65}$ 。

僅二球為白色，其餘非白色之機會為 $C_2^7 \cdot C_2^9/C_4^{16}$ 即 $\frac{27}{65}$ 。

(5) 其機會爲 $C_5^7 \cdot C_3^6 \cdot C_2^3 / C_{10}^{16}$ 即 $\frac{45}{286}$.

6. 解：以二骰子能擲出 36 種情形，其中有 6 種成對，故其機會爲 $\frac{6}{36}$ 即 $\frac{1}{6}$.

以三骰子擲出一對之機會爲 $(6 \times 5 \times \frac{3!}{2!} + 6) / 216$ 即 $\frac{4}{9}$.

7. 證：二骰子現 1, 6; 2, 5; 3, 4 皆成七點。

故其機會爲 $\frac{P_2^3 + P_2^2 + P_2^2}{36} = \frac{1}{6}$.

因任一骰子均可現 1, 2, 3, 4, 5, 6 而合成七點，而其他各點則否，如任一骰子現 6 後即不能合成 6 及 6 以下之點；故七點最易擲得。

8. 解：二骰子全無一點之機會爲 $\frac{5^2}{6^2} = \frac{25}{36}$.

故二骰子至少有一爲一點之機會爲 $1 - \frac{25}{36} = \frac{11}{36}$.

二骰子均爲一點之機會爲 $\frac{1}{36}$.

故僅有一骰子爲一點之機會爲 $\frac{11}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$.

9. 解：兩字皆含六字母，且所含 a, t, r 三字母相同。從兩字中均取 a 之機會爲 $\frac{1}{6^2}$ 。因有 3 對相同，故二字中各取一

字母，所取字母相同之機會爲 $\frac{3 \times 1}{36} = \frac{1}{12}$ 。

10. 解：二張中有一張票數爲偶數，其乘積即爲偶數。因可能情形有 C_2^9 ，故乘積爲偶數之機會爲 $(C_2^9 - C_2^5) / C_2^9$ 即 $\frac{13}{18}$ 。式中 C_2^5 乃乘積爲奇數之方法。

其乘積爲奇數時之機會爲 C_2^5 / C_2^9 即 $\frac{5}{18}$ 。

11. 解: (1) $\frac{C_2^6}{C_5^9} = \frac{15}{126} = \frac{5}{42}$.

(2) $\frac{C_1^3 \cdot C_4^6}{C_5^9} = \frac{45}{126} = \frac{5}{14}$.

(3) $\frac{C_5^6}{C_5^9} = \frac{6}{126} = \frac{1}{21}$.

12. 解: (1) $\frac{4^4}{C_4^{52}} = \frac{256}{270725}$.

(2) $\frac{4}{C_4^{52}} = \frac{4}{270725}$.

13. 解: 任何一組, 如梅花, 輪為將牌的機會為 $\frac{1}{4}$.

52 張中一手拿得預定的一組, 如梅花, 四張及其餘每組各三張的機會為 $\frac{C_4^{13} \cdot C_3^{13} \cdot C_3^{13} \cdot C_3^{13}}{C_{13}^{52}}$.

故一手拿得四張預定的將牌, 如梅花及其餘每組各三張的機會為

$$\frac{1}{4} \times \frac{C_4^{13} \cdot C_3^{13} \cdot C_3^{13} \cdot C_3^{13}}{C_{13}^{52}} = \frac{10 \times 143^4 \cdot 13! 39!}{52!}$$

14. 解: 1, 1, 3; 與 1, 2, 2 之情形皆為五點。但 1, 1, 3; 與 1, 2, 2 各有 $\frac{3!}{2!}$ 即 3 種方法能被擲出, 故其機會為 $\frac{3+3}{216}$ 即 $\frac{1}{36}$.

1, 1, 1; 與 1, 1, 2 之情形皆較五點為小。但 1, 1, 2 有 $\frac{3!}{2!}$

即 3 種方法能被擲出, 故其機會為 $\frac{1+3}{216}$ 即 $\frac{1}{54}$.

15. 解: 其機會為 $\frac{2 \cdot P_6^6}{(8-1)!}$ 即 $\frac{2}{7}$.

習題 LXVIII

1. 解: (1) $\frac{3}{15} \cdot \frac{7}{15} \cdot \frac{5}{15} = \frac{7}{225}$.

(2) $P_3^3 \cdot \frac{7}{225} = \frac{42}{225} = \frac{14}{75}$.

2. 解: $\frac{C_1^3 \cdot C_2^{12}}{C_1^{15} \cdot C_2^{14}} = \frac{3}{15} \cdot \frac{15-3}{15-1} \cdot \frac{15-4}{15-2} = \frac{1}{5} \cdot \frac{6}{7} \cdot \frac{11}{13} = \frac{66}{455}$.

3. 解: $\$1 \times \frac{C_2^5}{C_2^{12}} + \$2 \times \frac{C_2^4}{C_2^{12}} + \$10 \times \frac{C_2^3}{C_2^{12}} + \$1.5 \times \frac{5.4}{C_2^{12}}$
 $+ \$5.5 \times \frac{5.3}{C_2^{12}} + \$6 \times \frac{4.3}{C_2^{12}} = \frac{236.5}{66} = \3.58 .

4. 解: $1 - \frac{1}{2} \cdot \frac{C_3^7}{C_3^8} = 1 - \frac{1}{2} \cdot \frac{35}{56} = 1 - \frac{35}{112} = \frac{77}{112} = \frac{11}{16}$.

答: 開此門之機會為 $\frac{11}{16}$.

5. 解: (1) $\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{3}{4}\right) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}$.

(2) $\frac{1}{2}\left(1 - \frac{2}{3}\right)\left(1 - \frac{3}{4}\right) + \left(1 - \frac{1}{2}\right)\frac{2}{3}\left(1 - \frac{3}{4}\right)$
 $+ \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\frac{3}{4}$
 $= \frac{1+2+3}{24} = \frac{6}{24} = \frac{1}{4}$.

(3) $\frac{1}{2} \cdot \frac{2}{3}\left(1 - \frac{3}{4}\right) + \left(1 - \frac{1}{2}\right)\frac{2}{3} \cdot \frac{3}{4}$
 $+ \frac{1}{2}\left(1 - \frac{2}{3}\right)\frac{3}{4}$
 $= \frac{2+6+3}{24} = \frac{11}{24}$.

(4) $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{6}{24} = \frac{1}{4}$.

6. 解: 可能情形為 6×6 即 36, 擲得七點與十一點之情形共有 8 即 1, 6; 6, 1; 2, 5; 5, 2; 3, 4; 4, 3; 5, 6; 6, 5 是也。故其成功之機會為 $\frac{8}{36}$ 即 $\frac{2}{9}$, 是以前偏於優勝比為 7 對 2。

7. 解: (1) 可能情形爲 $6 \cdot 6 \cdot 6 = 216$, 擲得十點之情形有六種
 卽 $1, 3, 6; 1, 4, 5; 2, 2, 6; 2, 3, 5; 2, 4, 4; 3, 3, 4$;
 但 $1, 3, 6; 1, 4, 5; 2, 3, 5$ 各有 $3!$ 次; $2, 2, 6; 2, 4, 4; 3, 3, 4$ 各有 $\frac{3!}{2!}$ 次, 共爲 27 次。

答: 擲得十點之機會爲 $\frac{27}{216}$ 卽 $\frac{1}{8}$, 而所求之優勝比爲 $7:1$.

- (2) 擲得五點或不滿五點之情形有 $1, 1, 3; 1, 2, 2; 1, 1, 2; 1, 1, 1$. 故擲得不大於五點之機會爲

$$\frac{3 \times \frac{3!}{2!} + \frac{3!}{3!}}{216} = \frac{10}{216}$$

答: 大於五點之機會爲 $\frac{206}{216}$, 而擲二骰子大於五點之優勝比爲 $206:10$ 卽 $103:5$.

8. 解: 和爲十二之情形有 $1, 2, 9; 1, 3, 8; 1, 4, 7; 1, 5, 6; 2, 3, 7; 2, 4, 6; 3, 4, 5$ 七種。

故其和爲十二之機會爲 $7/C_3^{11} = \frac{7}{165}$.

所取三票中, 有一票或三票其票數爲奇數時, 三數之和爲奇數。

答: 所求機會爲 $(6C_2^5 + C_3^5)/C_3^{11}$ 卽 $\frac{16}{33}$.

9. 解: A 之機會爲 $\frac{6}{36} + \frac{1}{2} \left[1 - \left(\frac{6}{36} + \frac{3}{36} \right) \right] = \frac{1}{6} + \frac{1}{2} \cdot \frac{3}{4} = \frac{13}{24}$,
 B 之機會爲 $\frac{3}{36} + \frac{1}{2} \left[1 - \left(\frac{6}{36} + \frac{3}{36} \right) \right] = \frac{1}{12} + \frac{1}{2} \cdot \frac{3}{4} = \frac{11}{24}$.

答: 二人機會之比爲 $13:11$.

10. 解: $1, 5; 5, 1; 2, 4; 4, 2; 3, 3$ 皆爲六點。
 $1, 6; 6, 1; 2, 5; 5, 2; 3, 4; 4, 3$ 皆爲七點。

故第一循環 A 之機會爲 $\frac{5}{36} = \frac{30}{216}$.

$$B \text{ 之機會爲 } \frac{6}{16} \left(1 - \frac{5}{36}\right) = \frac{31}{216}$$

因各循環 A, B 二人機會之比相同，故二人機會之比為 30 : 31.

11. 解：(1) 當球不復置入時：

A 第一次之機會為

$$\frac{4}{12} = \frac{1}{3},$$

B 第一次之機會為

$$\frac{2}{3} \times \frac{4}{11} = \frac{8}{33},$$

C 第一次之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{4}{10} = \frac{28}{165};$$

A 於第二次獲勝之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{6}{10} \times \frac{4}{9} = \frac{56}{495},$$

B 於第二次獲勝之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{7}{99},$$

C 於第二次獲勝之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{4}{99};$$

A 於第三次獲勝之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{4}{6} = \frac{2}{99},$$

B 於第三次獲勝之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{4}{495},$$

C 於第三次獲勝之機會為

$$\frac{2}{3} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} \times \frac{4}{4} = \frac{1}{495}.$$

(此袋中僅存白球四枚)

答： A 之總機會為

$$\frac{1}{3} + \frac{56}{495} + \frac{2}{99} = \frac{231}{495} = \frac{7}{15}$$

B 之總機會爲

$$\frac{8}{33} + \frac{7}{99} + \frac{4}{495} = \frac{159}{495} = \frac{53}{195}$$

C 之總機會爲

$$\frac{28}{165} + \frac{4}{99} + \frac{1}{495} = \frac{105}{495} = \frac{7}{33}$$

(2) 當球重置入時:

A 在第一循環之機會爲

$$\frac{4}{12} \quad \text{即} \quad \frac{1}{3}$$

B 在第一循環之機會爲

$$\frac{2}{3} \cdot \frac{4}{12} \quad \text{即} \quad \frac{2}{9}$$

C 在第一循環之機會爲

$$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{4}{12} \quad \text{即} \quad \frac{4}{27}$$

因三人在各循環中機會之比相同，故三人機會之

比爲 $\frac{1}{3} : \frac{2}{9} : \frac{4}{27}$ 即 $9 : 6 : 4$ 。

答: A 之機會爲 $\frac{9}{19}$, B 之機會爲 $\frac{6}{19}$, C 之機會爲 $\frac{4}{19}$ 。

12. 解: $\frac{5}{100} \times 100 + \frac{10}{100} \times 50 + \frac{20}{100} \times 5 = 11$ 元。

13. 解: 白球自 A 袋中置入 B 袋之機會爲 $C_2^4 / C_3^5 = \frac{3}{5}$,

白球在 B 袋中不被拿出之機會爲 $C_3^3 / C_3^3 = \frac{2}{3}$ 。

故二次拿取後, 球仍在 B 袋中之機會爲 $\frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$ 。

答: 白球在 A 袋中之機會爲 $1 - \frac{2}{5} = \frac{3}{5}$ 。

14. 解：於二袋中之任意一袋，拿一次而得一白球之機會為

$$\frac{1}{2} \cdot \frac{a}{m} + \frac{1}{2} \cdot \frac{b}{n} = \frac{an + bm}{2mn}$$

將所有之球置於一袋中，拿一次而得一白球之機會為

$$\frac{a+b}{m+n}$$

答：知此二種之機會不相同。

15. 解： $1 - \left[\left(\frac{1}{10} \right)^5 + C_4^5 \left(\frac{1}{10} \right)^4 \left(\frac{9}{10} \right) + C_3^5 \left(\frac{1}{10} \right)^3 \left(\frac{9}{10} \right)^2 \right.$
 $\left. + C_2^5 \left(\frac{1}{10} \right)^2 \left(\frac{9}{10} \right)^3 + C_1^5 \left(\frac{1}{10} \right) \left(\frac{9}{10} \right)^4 \right] = \frac{59049}{100000}$

16. 解： $C_5^5 \left(\frac{2}{3} \right)^5 + C_4^5 \left(\frac{2}{3} \right)^4 \left(\frac{1}{3} \right) + C_3^5 \left(\frac{2}{3} \right)^3 \left(\frac{1}{3} \right)^2$
 $= \left(\frac{2}{3} \right)^5 + 5 \left(\frac{2}{3} \right)^4 \left(\frac{1}{3} \right) + 10 \left(\frac{2}{3} \right)^3 \left(\frac{1}{3} \right)^2$
 $= \frac{192}{243} = \frac{64}{81}$

17. 解： $1 \times \left(\frac{3}{5} \right)^8 + 8 \left(\frac{3}{5} \right)^7 \left(\frac{2}{5} \right) + 28 \left(\frac{3}{5} \right)^6 \left(\frac{2}{5} \right)^2 + 56 \left(\frac{3}{5} \right)^5 \left(\frac{2}{5} \right)^3$
 $= \frac{3^5}{5^8} (27 + 144 + 336 + 448) = 955 \times \frac{3^5}{5^8}$
 $= \frac{191 \times 3^5}{5^7}$

18. 解： $1 \times \frac{1}{6} + \left(\frac{5}{6} \right) \left(\frac{1}{6} \right) \times 1 + \left(\frac{5}{6} \right)^2 \left(\frac{1}{6} \right) \times 1$
 $+ \left(\frac{5}{6} \right)^3 \left(\frac{1}{6} \right) \times 1 + \dots$
 $= 1 \times \frac{1}{6} \left[1 + \frac{5}{6} + \left(\frac{5}{6} \right)^2 + \left(\frac{5}{6} \right)^3 + \dots \right]$
 $= 1 \times \frac{1}{6} \left[\frac{1}{1 - 5/6} \right] = \frac{1}{6} \times 6 = 1 \text{ 元}$

19. 解： (1) $\frac{3}{4} \times C_3^8 \left(\frac{3}{4} \right)^5 \left(\frac{1}{4} \right)^3 = \frac{5103}{32768}$

$$\begin{aligned}
 (2) \quad & \left(\frac{3}{4}\right)^6 + C_5^6 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right) + C_5^7 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) \\
 & + C_5^8 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right) + C_5^9 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right) \\
 & = \frac{177147 + 649539 + 1082565}{4194304} \\
 & + \frac{721710 + 336798}{4194304} = \frac{1012581}{1048576}
 \end{aligned}$$

20. 解：在上列條件下， A 勝該局之比數有 $6:4$ 及 $6:5$ 兩種
如 A 以 $6:4$ 勝，則 A 須連勝四盤，其機會為

$$\left(\frac{3}{4}\right)^4 = \frac{81}{256}$$

如 A 以 $6:5$ 勝，則後 5 盤內 A 須勝第五盤，且前四盤中 A 須勝任何三盤，其機會為

$$C_3^4 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^3 \times \frac{3}{4} = \frac{81}{256}$$

$\therefore A$ 勝該局之機會為

$$\frac{81}{256} + \frac{81}{256} = \frac{81}{128}$$

21. 解：若 B 欲獲勝，必須得到二點，其機會為 $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ 。

故分款時， A 之機會為 $\frac{3}{4}$ ， B 之機會為 $\frac{1}{4}$ 。

是以 A 應得款 $64 \times \frac{3}{4} = 48$ 元，

B 應得款 $64 \times \frac{1}{4} = 16$ 元。

XXVIII. 算學歸納法

習題 LXIX

原本第 425 頁

1. 證： $a + ar + ar^2 + \dots + ar^{n-1} = a \frac{1-r^n}{1-r}$ 。

當式中僅有一項時，此公式為正確者，即 $a = a \frac{1-r}{1-r}$ 。

當式中有二項時，此公式仍為正確者，即 $a + ar = a \frac{1-r^2}{1-r}$ 。

今設式中有 k 項時，上式仍為正確，即

$$a + ar + ar^2 + \dots + ar^{k-1} = a \frac{1-r^k}{1-r}.$$

二邊加以 ar^k ，得下列公式

$$\begin{aligned} a + ar + ar^2 + \dots + ar^{k-1} + ar^k &= a \frac{1-r^k}{1-r} + ar^k \\ &= \frac{a - ar + ar^k - ar^{k+1}}{1-r} = a \frac{1-r^{k+1}}{1-r} \dots \dots \dots (2) \end{aligned}$$

(2) 式中含有 k 項時如為正確，則含有 $(k+1)$ 項時亦為正確。今式中有一項時既屬正確，是以此式有二、三、……任何項時皆為正確，即

$$a + ar + ar^2 + \dots + ar^{n-1} = a \frac{1-r^n}{1-r}.$$

2. 證： $1^2 + 2^2 + 3^2 + \dots + n^2 = n(n+1)(2n+1)/6$ 。

當 n 為一定值，如 1, 2 或 3 時，此公式為正確者。設 n 為一特別值 k 時，此公式仍為正確者，故

$$1^2 + 2^2 + 3^2 + \dots + k^2 = k(k+1)(2k+1)/6 \dots \dots \dots (1)$$

二邊各加以 $(k+1)^2$ ，得

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\ &= k(k+1)(2k+1)/6 + (k+1)^2 \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+2+1)}{6} \dots \dots \dots (2) \end{aligned}$$

(2) 式示 $n=k$ 時如為正確，則 $n=k+1$ 時亦為正確。今 $n=1, 2, \dots$ 時既屬正確，是以此公式於 $n=1, 2, \dots$ 或任何數值時皆為正確。

3. 證： $1^3 + 2^3 + 3^3 + \dots + n^3 = n^2(n+1)^2/4$ 。

當 n 為一定值，如 1, 2 或 3 時，此公式皆為正確。設 n 為一特別值 k 時，此公式仍為正確者，故

$$1^3 + 2^3 + 3^3 + \dots + k^3 = k^2(k+1)^2/4 \dots \dots \dots (1)$$

兩邊各加以 $(k+1)^3$, 得

$$\begin{aligned} & 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 \\ &= k^2(k+1)^2/4 + (k+1)^3 \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k+2)^2}{4} \dots\dots\dots(2) \end{aligned}$$

(2) 式示 $n=k$ 時如屬正確, 則 $n=k+1$ 時亦為正確. 今該式於 $n=1, 2, \dots$ 時既屬正確, 則當 n 為 $2, 3, \dots$ 或任何值時皆為正確.

4. 證: $1+3+6+\dots+n(n+1)/2! = n(n+1)(n+2)/3!$

當 n 為一定值, 如 $1, 2$ 或 3 時, 此公式皆為正確. 設 n 為一特別值 k 時, 此公式仍為正確者, 故

$$1+3+6+\dots+k(k+1)/2! = k(k+1)(k+2)/3! \dots(1)$$

兩邊各加以 $(k+1)(k+2)/2!$, 得

$$\begin{aligned} & 1+3+6+\dots+\frac{k(k+1)}{2!} + \frac{(k+1)(k+2)}{2!} \\ &= \frac{k(k+1)(k+2)}{3!} + \frac{(k+1)(k+2)}{2!} \\ &= \frac{(k+1)(k+2)(k+3)}{3!} \\ &= \frac{(k+1)[(k+1)+1][(k+1)+2]}{3!} \dots\dots\dots(2) \end{aligned}$$

(2) 式示 $n=k$ 時, (1) 式如屬正確, 則 $n=k+1$ 亦為正確. 今 $n=1, 2$, 時, (1) 式既屬無誤, 則 n 為 $2, 3, \dots$ 或任何值時, 皆正確不誤.

XXIX. 方 程 論

習 題 LXX

原本第 431 頁

1. 解: (1) $a, -b, a+b$

$$(x-a)(x+b)(x-a-b) = 0$$

$$\therefore x^3 - 2ax^2 + (a^2 - ab - b^2)x + a^2b + ab^2 = 0.$$

$$(2) \quad 3, 4, \frac{1}{2}, -\frac{1}{3}, 0$$

$$(x-3)(x-4)(2x-1)(3x+1)x=0$$

$$(x^2-7x+12)(6x^2-x-1)x=0$$

$$\therefore 6x^5-43x^4+78x^3-5x^2-12x=0.$$

$$2. \text{ 證: } x^4+8x^3+18x^2-27=0$$

$$1+8+18+0-27 \quad | \quad -3$$

$$-3-15-9+27$$

$$1+5+3-9+0 \quad | \quad -3$$

$$-3-6+9$$

$$1+2-3+0 \quad | \quad -3$$

$$-3+3$$

$$1-1+0$$

$$\therefore x^4+8x^3+18x^2-27=(x+3)^3(x-1).$$

故 -3 爲此方程式之三重根。

$$3. \text{ 證: } 4x^5-23x^3+33x^2-17x+3=0.$$

$$4+0-23+33-17+3 \quad | \quad 1$$

$$4+4-19+14-3$$

$$4+4-19+14-3+0 \quad | \quad 1$$

$$4+8-11+3$$

$$4+8-11+3+0 \quad | \quad \frac{1}{2}$$

$$2+5-3$$

$$4+10-6+0$$

$$2+5-3 \quad | \quad \frac{1}{2}$$

$$1+3$$

$$2+6+0$$

$$1+3$$

$$\therefore 4x^5-23x^3+33x^2-17x+3$$

$$=(x-1)^2(2x-1)^2(x+3)=0.$$

故 1 與 $\frac{1}{2}$ 爲此方程式之二重根。

$$4. \text{ 解: } x^5-5x^4-5x^3+4x^2-7x-250=0$$

$$1-5-5+4-7-250 \quad | \quad 6$$

$$6+6+6+60+318$$

$$1+1+1+10+53+68$$

以 $x-6$ 除之，吾人僅得正係數，而以 $x-7$ 除之則否。故 6 爲其上界。

$$\begin{array}{r} 1-5-5+4-7-250 \quad | \quad -2 \\ -2+14-18+28+42 \\ \hline 1-7+9-14+21-292 \end{array}$$

以 $x+2$ 除之，吾人得一正負相間之係數，而以 $x+3$ 除之則否。故 -2 爲其下界。

5. 證： $2x^4 - 3x^3 + 4x^2 - 10x - 3 = 0$

此方程式所能有之有理根爲 $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$ 。若吾人以此根除原方程式，則知無能除盡者。故知此方程式無有理根。

6. 解： $x^3 - x^2 - 14x + 24 = 0$

可爲該方程式之有理根者有 $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$ 。今以 2, 3 先試之知其爲根。

$$\begin{array}{r} 1-1-14+24 \quad | \quad 2 \\ 2+2-24 \\ \hline 1+1-12+0 \quad | \quad 3 \\ 3+12 \\ \hline 1+4+0 \end{array}$$

答：其根爲 2, 3 與 -4 。

7. 解： $x^3 - 2x^2 - 25x + 50 = 0$

$$\begin{array}{r} 1-2-25+50 \quad | \quad 2 \\ 2+0-50 \\ \hline 1+0-25+0 \quad | \quad 5 \\ 5+25 \\ \hline 1+5+0 \end{array}$$

答：其根爲 2, 5 與 -5 。

8. 解： $3x^3 - 2x^2 + 2x + 1 = 0$

$$\begin{array}{r} 3-2+2+1 \quad | \quad -\frac{1}{3} \\ -1+1-1 \\ \hline 3-3+3+0 \\ 1-1+1 \end{array}$$

$$\therefore 3x^3 - 2x^2 + 2x + 1 = (3x+1)(x^2 - x + 1) = 0.$$

答：其根爲 $-\frac{1}{3}, \frac{1 \pm \sqrt{3}i}{2}$.

9. 解： $2x^4 + 7x^3 - 2x^2 = x$

$$2x^4 + 7x^3 - 2x^2 - x = x(2x^3 + 7x^2 - 2x - 1) = 0$$

$$2 + 7 - 2 - 1 \mid \frac{1}{2}$$

$$\frac{1 + 4 + 1}{2 + 8 + 2 + 0}$$

$$1 + 4 + 1$$

$$1 + 4 + 1$$

$$\therefore 2x^4 + 7x^3 - 2x^2 - x = x(2x-1)(x^2 + 4x + 1) = 0.$$

答：其根爲 $0, \frac{1}{2}, -2 \pm \sqrt{3}$.

10. 解： $x^4 + 4x^3 + 8x^2 + 8x + 3 = 0$

$$1 + 4 + 8 + 8 + 3 \mid -1$$

$$-1 - 3 - 5 - 3$$

$$1 + 3 + 5 + 3 + 0 \mid -1$$

$$-1 - 2 - 3$$

$$1 + 2 + 3 + 0$$

$$x^2 + 2x + 3 = 0 \quad \therefore x = -1 \pm \sqrt{2}i.$$

答：其根爲 $-1, -1, -1 \pm \sqrt{2}i$.

11. 解： $2x^4 + 7x^3 + 4x^2 - 7x - 6 = 0$

$$2 + 7 + 4 - 7 - 6 \mid 1$$

$$2 + 9 + 13 + 6$$

$$2 + 9 + 13 + 6 + 0 \mid -1$$

$$-2 - 7 - 6$$

$$2 + 7 + 6 + 0 \mid -2$$

$$-4 - 6$$

$$2 + 3 + 0$$

答：其根爲 $1, -1, -2, -\frac{3}{2}$.

12. 解: $3x^4 + 11x^3 + 9x^2 + 11x + 6 = 0$

$$\begin{array}{r} 3 + 11 + 9 + 11 + 6 \quad | \quad -3 \\ \hline \end{array}$$

$$- 9 - 6 - 9 - 6$$

$$\begin{array}{r} 3 + 2 + 3 + 2 + 0 \quad | \quad -\frac{2}{3} \\ \hline \end{array}$$

$$- 2 + 0 - 2$$

$$\begin{array}{r} 3 + 0 + 3 + 0 \\ \hline \end{array}$$

$$1 + 0 + 1$$

$$x^2 + 1 = 0 \quad \therefore x = \pm i.$$

答: 其根爲 $-3, -\frac{2}{3}, \pm i.$

13. 解: $x^5 - 9x^4 + 2x^3 + 71x^2 + 81x + 70 = 0$

$$\begin{array}{r} 1 - 9 + 2 + 71 + 81 + 70 \quad | \quad -2 \\ \hline \end{array}$$

$$- 2 + 22 - 48 - 46 - 70$$

$$\begin{array}{r} 1 - 11 + 24 + 23 + 35 + 0 \quad | \quad 5 \\ \hline \end{array}$$

$$5 - 30 - 30 - 35$$

$$\begin{array}{r} 1 - 6 - 6 - 7 + 0 \quad | \quad 7 \\ \hline \end{array}$$

$$7 + 7 + 7$$

$$1 + 1 + 1 + 0$$

$$x^2 + x + 1 = 0 \quad \therefore x = \frac{-1 \pm \sqrt{3}i}{2}.$$

答: 其根其 $-2, 5, 7, \frac{1 \pm \sqrt{3}i}{2}.$

14. 解: $2x^5 - 8x^4 + 7x^3 + 5x^2 - 8x + 4 = 0$

$$\begin{array}{r} 2 - 8 + 7 + 5 - 8 + 4 \quad | \quad -1 \\ \hline \end{array}$$

$$- 2 + 10 - 17 + 12 - 4$$

$$\begin{array}{r} 2 - 10 + 17 - 12 + 4 + 0 \quad | \quad 2 \\ \hline \end{array}$$

$$4 - 12 + 10 - 4$$

$$\begin{array}{r} 2 - 6 + 5 - 2 + 0 \quad | \quad 2 \\ \hline \end{array}$$

$$4 - 4 + 2$$

$$2 - 2 + 1 + 0$$

$$2x^2 - 2x + 1 = 0 \quad \therefore x = (1 \pm i)/2.$$

答：其根爲 $-1, 2, 2, (1 \pm i)/2$.

15. 解： $x^5 + 3x^4 - 15x^3 - 35x^2 + 54x + 72 = 0$

$$\begin{array}{r} 1+3-15-35+54+72 \quad | \quad -1 \\ \hline -1-2+17+18-72 \\ \hline 1+2-17-18+72+0 \quad | \quad 2 \\ \hline 2+8-18-72 \\ \hline 1+4-9-36+0 \quad | \quad 3 \\ \hline 3+21+36 \\ \hline 1+7+12+0 \quad | \quad -3 \\ \hline -3-12 \\ \hline 1+4+0 \end{array}$$

答：其根爲 $-1, 2, 3, -3, -4$.

16. 解： $12x^4 - 32x^3 + 13x^2 + 8x - 4 = 0$

$$\begin{array}{r} 12-32+13+8-4 \quad | \quad 2 \\ \hline 24-16-6+4 \\ \hline 12-8-3+2+0 \quad | \quad \frac{1}{2} \\ \hline 6-1-2 \\ \hline 12-2-4+0 \\ \hline 6-1-2 \quad | \quad -\frac{1}{2} \\ \hline -3+2 \\ \hline 6-4+0 \\ \hline 3-2 \end{array}$$

答：其根爲 $2, \frac{1}{2}, -\frac{1}{2}, \frac{2}{3}$.

17. 解： $x^5 - 7x^4 + 10x^3 + 18x^2 - 27x - 27 = 0$

$$\begin{array}{r} 1-7+10+18-27-27 \quad | \quad -1 \\ \hline -1+8-18+0+27 \\ \hline 1-8+18+0-27+0 \quad | \quad -1 \\ \hline -1+9-27+27 \\ \hline 1-9+27-27+0 \quad | \quad 3 \\ \hline 3-18+27 \\ \hline 1-6+9+0 \quad | \quad 3 \\ \hline 3-9 \\ \hline 1-3+0 \end{array}$$

答：其根爲 $-1, -1, 3, 3, 3$.

18. 解： $2x^4 - 17x^3 + 25x^2 + 74x - 120 = 0$

$$\begin{array}{r} 2 - 17 + 25 + 74 - 120 \quad | \quad -2 \\ \underline{-4 + 42 - 134 + 120} \\ 2 - 21 + 67 - 60 + 0 \quad | \quad 4 \\ \underline{8 - 52 + 60} \\ 2 - 13 + 15 + 0 \quad | \quad 5 \\ \underline{10 - 15} \\ 2 - 3 + 0 \end{array}$$

答：其根爲 $-2, 4, 5, \frac{3}{2}$.

19. 解： $4x^5 - 9x^3 + 6x^2 - 13x + 6 = 0$

$$\begin{array}{r} 4 + 0 - 9 + 6 - 13 + 6 \quad | \quad -2 \\ \underline{-8 + 16 - 14 + 16 - 6} \\ 4 - 8 + 7 - 8 + 3 + 0 \quad | \quad \frac{1}{2} \\ \underline{2 - 3 + 2 - 3} \\ 4 - 6 + 4 - 6 + 0 \quad | \quad \frac{3}{2} \\ \underline{6 + 0 + 6} \\ 4 + 0 + 4 + 0 \\ \underline{2 + 0 + 2} \end{array}$$

答：其根爲 $-2, \frac{1}{2}, \frac{3}{2}, \pm i$.

20. 解： $x^5 + 8x^4 + 3x^3 - 80x^2 - 52x + 240 = 0$

$$\begin{array}{r} 1 + 8 + 3 - 80 - 52 + 240 \quad | \quad 2 \\ \underline{2 + 20 + 46 - 68 - 240} \\ 1 + 10 + 23 - 34 - 120 + 0 \quad | \quad 2 \\ \underline{2 + 24 + 94 + 120} \\ 1 + 12 + 47 + 60 + 0 \quad | \quad -3 \\ \underline{-3 - 27 - 60} \\ 1 + 9 + 20 + 0 \quad | \quad -4 \\ \underline{-4 - 20} \\ 1 + 5 + 0 \end{array}$$

答：其根爲 $2, 2, -3, -4, -5$.

21. 解： $2x^5 + 11x^4 + 23x^3 + 25x^2 + 16x + 4 = 0$

$$\begin{array}{r} 2+11+23+25+16+4 \quad | \quad -2 \\ -4-14-18-14-4 \quad \hline \end{array}$$

$$\begin{array}{r} 2+7+9+7+2+0 \quad | \quad -2 \\ -4-6-6-2 \quad \hline \end{array}$$

$$\begin{array}{r} 2+3+3+1+0 \quad | \quad \frac{1}{2} \\ -1-1-1 \quad \hline \end{array}$$

$$2+2+2+0$$

$$1+1+1$$

$$x^2+x+1=0 \quad \therefore x = \frac{-1 \pm \sqrt{3}i}{2}$$

答：其根爲 $-2, -2, -\frac{1}{2}, \frac{-1 \pm \sqrt{3}i}{2}$

22. 解： $6x^4 - 89x^3 + 359x^2 - 254x + 48 = 0$

$$\begin{array}{r} 6-89+359-254+48 \quad | \quad 6 \\ 36-318+246-48 \quad \hline \end{array}$$

$$\begin{array}{r} 6-53+41-8+0 \quad | \quad 8 \\ 48-40+8 \quad \hline \end{array}$$

$$\begin{array}{r} 6-5+1+0 \quad | \quad \frac{1}{2} \\ 3-1 \quad \hline \end{array}$$

$$\begin{array}{r} 6-2+0 \\ 3-1 \end{array}$$

答：其根爲 $6, 8, \frac{1}{2}, \frac{1}{3}$

23. 解： $10x^4 + 41x^3 + 46x^2 + 20x + 3 = 0$

$$\begin{array}{r} 10+41+46+20+3 \quad | \quad -\frac{1}{2} \\ -5-18-14-3 \quad \hline \end{array}$$

$$\begin{array}{r} 10+36+28+6+0 \quad | \quad -\frac{3}{5} \\ -6-18-6 \quad \hline \end{array}$$

$$\begin{array}{r} 10+30+10+0 \\ 1+3+1 \end{array}$$

$$x^2+3x+1=0 \quad \therefore x = \frac{-3 \pm \sqrt{5}}{2}$$

答：其根爲 $-\frac{1}{2}, -\frac{3}{5}, \frac{-3 \pm \sqrt{5}}{2}$

24. 解： $36x^4 - 108x^3 + 107x^2 - 43x + 6 = 0$

$$\begin{array}{r}
 36 - 108 + 107 - 43 + 6 \mid \frac{1}{3} \\
 \underline{18 - 45 + 31 - 6} \\
 36 - 90 + 62 - 12 + 0 \mid \frac{1}{3} \\
 \underline{12 - 26 + 12} \\
 36 - 78 + 36 + 0 \mid \frac{2}{3} \\
 \underline{24 - 36} \\
 36 - 54 + 0 \\
 \underline{2 - 3}
 \end{array}$$

答：其根爲 $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}$.

25. 解： $12x^5 + 20x^4 + 29x^3 + 77x^2 + 69x + 18 = 0$

$$\begin{array}{r}
 12 + 20 + 29 + 77 + 69 + 18 \mid -\frac{1}{2} \\
 \underline{-6 - 7 - 11 - 33 - 18} \\
 12 + 14 + 22 + 66 + 36 + 0 \mid -\frac{2}{3} \\
 \underline{-8 - 4 - 12 - 36} \\
 12 + 6 + 18 + 54 + 0 \mid -\frac{3}{2} \\
 \underline{-18 + 18 - 54} \\
 12 - 12 + 36 + 0 \\
 \underline{1 - 1 + 3}
 \end{array}$$

$$x^2 - x + 3 = 0 \quad \therefore x = \frac{1 \pm \sqrt{11}i}{2}$$

答：其根爲 $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{2}, \frac{1 \pm \sqrt{11}i}{2}$.

26. 解： $2x^6 + 7x^5 + 8x^4 + 7x^3 + 2x^2 - 14x - 12 = 0$

$$\begin{array}{r}
 2 + 7 + 8 + 7 + 2 - 14 - 12 \mid 1 \\
 \underline{2 + 9 + 17 + 24 + 26 + 12} \\
 2 + 9 + 17 + 24 + 26 + 12 + 0 \mid -1 \\
 \underline{-2 - 7 - 10 - 14 - 12} \\
 2 + 7 + 10 + 14 + 12 + 0 \mid -2 \\
 \underline{-4 - 6 - 8 - 12} \\
 2 + 3 + 4 + 6 + 0 \mid -\frac{3}{2} \\
 \underline{-3 + 0 - 6} \\
 2 + 0 + 4 + 0 \\
 \underline{1 + 0 + 2}
 \end{array}$$

$$x^2 + 2 = 0 \quad \therefore x = \pm \sqrt{2}i$$

答：其根爲 $1, -1, -2, -\frac{3}{2}, \pm \sqrt{2}i$.

27. 解: $2x^6 + 11x^5 + 24x^4 + 22x^3 - 8x^2 - 33x - 18 = 0$

$$\begin{array}{r} 2+11+24+22-8-33-18 \quad | \quad 1 \\ \hline 2+13+37+59+51+18 \\ 2+13+37+59+51+18+0 \quad | \quad -1 \\ \hline -2-11-26-33-18 \\ 2+11+26+33+18+0 \quad | \quad -2 \\ \hline -4-14-24-18 \\ 2+7+12+9+0 \quad | \quad -\frac{3}{2} \\ \hline -3-6-9 \\ 2+4+6+0 \\ 1+2+3 \end{array}$$

$$x^2 + 2x + 3 = 0 \quad \therefore x = -1 \pm \sqrt{2}i.$$

答: 其根爲 $1, -1, -2, -\frac{3}{2}, -1 \pm \sqrt{2}i.$

28. 解: $5x^6 - 7x^5 - 8x^4 - x^3 + 7x^2 + 8x - 4 = 0$

$$\begin{array}{r} 5-7-8-1+7+8-4 \quad | \quad 1 \\ \hline 5-2-10-11-4+4 \\ 5-2-10-11-4+4+0 \quad | \quad -1 \\ \hline -5+7+3+8-4 \\ 5-7-3-8+4+0 \quad | \quad 2 \\ \hline 10+6+6-4 \\ 5+3+3-2+0 \quad | \quad \frac{2}{5} \\ \hline 2+2+2 \\ 5+5+5+0 \\ 1+1+1 \end{array}$$

$$x^2 + x + 1 = 0 \quad \therefore x = \frac{-1 \pm \sqrt{3}i}{2}.$$

答: 其根爲 $1, -1, 2, \frac{2}{5}, \frac{-1 \pm \sqrt{3}i}{2}.$

習 題 LXXI

原本第 435 頁

1. 解: 其第三根爲 $a = \frac{7}{2} - [1+i+1-i] = \frac{7}{2} - 2 = \frac{3}{2}.$

2. 解: (1) $8x^3 - 14x^2 - 21x + 27 = 0$
 設 $\alpha/\beta, \alpha, \alpha\beta$ 代表此方程式之三根。

$$\frac{\alpha}{\beta} + \alpha + \alpha\beta = \frac{14}{8} = \frac{7}{4} \dots\dots\dots (1)$$

$$\frac{\alpha^2}{\beta} + \alpha^2 + \alpha^2\beta = -\frac{21}{8} \dots\dots\dots (2)$$

$$\frac{\alpha}{\beta} \cdot \alpha \cdot \alpha\beta = -\frac{27}{8} \dots\dots\dots (3)$$

從(3)式, $\alpha^3 = -\frac{27}{8}$

$$\therefore \alpha = -\frac{3}{2}$$

代入(1)式, 得 $-\frac{3}{2\beta} - \frac{3}{2} - \frac{3\beta}{2} = \frac{7}{4}$

$$6\beta^2 + 13\beta + 6 = 0$$

$$(3\beta + 2)(2\beta + 3) = 0$$

$$\therefore \beta = -\frac{2}{3}, -\frac{3}{2}$$

答: 其根爲 $1, -\frac{3}{2}, \frac{9}{4}$.

(2) $x^3 + x^2 + 3x + 27 = 0$

設 $\frac{\alpha}{\beta}, \alpha, \alpha\beta$ 代表此方程式之三根。

$$\frac{\alpha}{\beta} + \alpha + \alpha\beta = -1 \dots\dots\dots (1)$$

$$\alpha^3 = -27 \dots\dots\dots (2)$$

從(2)式, $\alpha = -3$.

代入(1)式, 得 $-\frac{3}{\beta} - 3 - 3\beta = -1$

$$3\beta^2 + 2\beta + 3 = 0$$

$$\therefore \beta = \frac{1 \pm 2\sqrt{2}i}{3}$$

答: 其根爲 $1 + 2\sqrt{2}i, -3, 1 - 2\sqrt{2}i$.

3. 解: (1) $x^3 + 6x^2 + 7x - 2 = 0$

設 $\alpha - \beta, \alpha, \alpha + \beta$ 爲三根。

$$\alpha - \beta + \alpha + \alpha + \beta = -6 \dots\dots\dots (1)$$

$$\alpha(\alpha^2 - \beta^2) = 2 \dots \dots \dots (2)$$

從 (1) 式, $\alpha = -2$.

代入 (2) 式, 得 $\beta^2 = 5 \therefore \beta = \pm \sqrt{5}$.

答: 其三根爲 $-2 - \sqrt{5}$, -2 , $-2 + \sqrt{5}$.

$$(2) \quad x^3 - 9x^2 + 23x - 15 = 0$$

設 $\alpha - \beta$, α , $\alpha + \beta$ 爲三根.

$$\alpha - \beta + \alpha + \alpha + \beta = 9 \dots \dots \dots (1)$$

$$\alpha(\alpha^2 - \beta^2) = 15 \dots \dots \dots (2)$$

從 (1) 式, $\alpha = 3$.

代入 (2) 式, 得 $\beta^2 = 4$

$$\therefore \beta = \pm 2.$$

答: 其根爲 1, 3, 5.

4. 證: 設其根爲 α , $-\alpha$, β .

$$\alpha - \alpha + \beta = -p \quad \therefore \beta = -p \dots \dots \dots (1)$$

$$-\alpha^2 + \alpha\beta - \alpha\beta = q \quad \therefore \alpha^2 = -q \dots \dots \dots (2)$$

$$\alpha(-\alpha)\beta = -r \quad \therefore \alpha^2\beta = r \dots \dots \dots (3)$$

將 (1), (2) 代入 (3) 式, 得 $pq = r$.

故此方程式之一根爲另一根之負數之條件爲 $pq = r$.

5. 解: 設其根爲 α , $\frac{1}{\alpha}$, β .

$$\alpha + \frac{1}{\alpha} + \beta = -p \dots \dots \dots (1)$$

$$\alpha \cdot \frac{1}{\alpha} + \frac{1}{\alpha} \cdot \beta + \alpha\beta = q \dots \dots \dots (2)$$

$$\alpha \cdot \frac{1}{\alpha} \cdot \beta = -r \dots \dots \dots (3)$$

從 (3) 式, $\beta = -r \dots \dots \dots (4)$

代入 (1) 式, 得 $\alpha + \frac{1}{\alpha} = -p + r \dots \dots \dots (5)$

從 (2) 式, $\left(\frac{1}{\alpha} + \alpha\right)\beta + 1 = q \dots \dots \dots (6)$

將 (4), (5) 代入 (6) 式, 得

$$(-p+r)(-r) + 1 = q$$

$$\therefore r^2 - pr + q - 1 = 0.$$

答：其條件爲 $r^2 - pr + q - 1 = 0$.

6. 解：設其根爲 $\alpha, \alpha, \beta, \beta$.

$$2\alpha + 2\beta = -4 \dots\dots\dots(1)$$

$$\alpha^2 + 4\alpha\beta + \beta^2 = 10 \dots\dots\dots(2)$$

從(1)式, $\beta = -2 - \alpha \dots\dots\dots(3)$

代入(2)式, 得 $\alpha^2 + 2\alpha + 3 = 0$

$$\therefore \alpha = -1 \pm \sqrt{2}i.$$

代入(3)式, 得 $\beta = -1 \pm \sqrt{2}i.$

7. 解：設其根爲 $\frac{1}{\alpha - \beta}, \frac{1}{\alpha}, \frac{1}{\alpha + \beta}$, 則根爲 $\alpha - \beta, \alpha$ 及 $\alpha + \beta$
之方程式爲

$$9x^3 - 18x^2 - 13x + 14 = 0$$

$$\alpha - \beta + \alpha + \alpha + \beta = \frac{18}{9} = 2 \dots\dots\dots(1)$$

$$\alpha(\alpha^2 - \beta^2) = -\frac{14}{9} \dots\dots\dots(2)$$

從(1)式, $\alpha = \frac{2}{3}.$

代入(2)式, 得 $\frac{4}{9} - \beta^2 = -\frac{7}{3}$

$$\therefore \beta = \pm \frac{5}{3}.$$

答：所求之三根爲 $\frac{3}{7}, \frac{3}{2}, -1.$

8. 解：設其根爲 $2\alpha, 3\alpha, \beta, \beta + 1.$

$$2\alpha + 3\alpha + \beta + \beta + 1 = 1$$

$$5\alpha + 2\beta = 0 \dots\dots\dots(1)$$

$$6\alpha^2\beta^2 + 6\alpha^2\beta = 720$$

$$\alpha^2\beta^2 + \alpha^2\beta = 120 \dots\dots\dots(2)$$

從(1)式, $\beta = -\frac{5}{2}\alpha \dots\dots\dots(3)$

代入(2)式, 得 $5\alpha^4 - 2\alpha^3 - 96 = 0$

$$(\alpha + 2)(5\alpha^3 - 12\alpha^2 + 24\alpha - 48) = 0$$

$$\therefore \alpha = -2.$$

代入 (3) 式, 得 $\beta = 5$.

答: 其根爲 $-4, -6, 5, 6$.

9. 解: (1) $-\alpha, -\beta, -\gamma$.

設 p', q', r' 爲所求方程式之係數

$$-p' = -\alpha - \beta - \gamma = -(\alpha + \beta + \gamma) = p$$

$$q' = \alpha\beta + \alpha\gamma + \beta\gamma = q$$

$$-r' = -\alpha\beta\gamma = r.$$

答: 所求之方程式爲 $x^3 - px^2 + qx - r = 0$.

(2) $k\alpha, k\beta, k\gamma$.

$$-p' = k(\alpha + \beta + \gamma) = -kp$$

$$q' = k^2\alpha\beta + k^2\alpha\gamma + k^2\beta\gamma = k^2q$$

$$-r' = k^3\alpha\beta\gamma = -k^3r.$$

答: 所求之方程式爲 $x^3 + kpx^2 + k^2qx + k^3r = 0$.

(3) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

$$-p' = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = -\frac{q}{r}$$

$$q' = \frac{1}{\alpha} \cdot \frac{1}{\beta} + \frac{1}{\beta} \cdot \frac{1}{\gamma} + \frac{1}{\alpha} \cdot \frac{1}{\gamma} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{p}{r}$$

$$-r' = \frac{1}{\alpha \cdot \beta \cdot \gamma} = -\frac{1}{r}.$$

答: 所求之方程式爲 $rx^3 + qx^2 + px + 1 = 0$.

(4) $\alpha + k, \beta + k, \gamma + k$.

$$-p' = (\alpha + \beta + \gamma) + 3k = -p + 3k$$

$$\begin{aligned} q' &= (\alpha + k)(\beta + k) + (\alpha + k)(\gamma + k) + (\beta + k)(\gamma + k) \\ &= \alpha\beta + \alpha\gamma + \beta\gamma + 2k(\alpha + \beta + \gamma) + 3k^2 \\ &= q - 2kp + 3k^2 \end{aligned}$$

$$\begin{aligned} -r' &= (\alpha + k)(\beta + k)(\gamma + k) \\ &= \alpha\beta\gamma + k(\beta\gamma + \alpha\gamma + \alpha\beta) + k^2(\alpha + \beta + \gamma) + k^3 \\ &= -r + kq - k^2p + k^3. \end{aligned}$$

答: 所求之方程式爲

$$x^3 + (p - 3k)x^2 + (q - 2kp + 3k^2)x + (r - kq + k^2p - k^3) = 0.$$

(5) $\alpha^2, \beta^2, \gamma^2$.

$$-p' = \alpha^2 + \beta^2 + \gamma^2$$

$$= (a + \beta + \gamma)^2 - 2(a\beta + \beta\gamma + \gamma a)$$

$$= p^2 - 2q$$

$$q' = a^2\beta^2 + a^2\gamma^2 + \beta\gamma^2$$

$$= (a\beta + a\gamma + \beta\gamma)^2 - 2(a\beta^2\gamma + a^2\beta\gamma + a\beta\gamma^2)$$

$$= (a\beta + a\gamma + \beta\gamma)^2 - 2a\beta\gamma(a + \beta + \gamma)$$

$$= q^2 - 2pr$$

$$-r' = a^2\beta^2\gamma^2 = (a\beta\gamma)^2 = r^2.$$

答：所求之方程式爲 $x^3 - (p^2 - 2q)x^2 + (q^2 - pr)x - r^2 = 0$.

$$(6) \quad -\frac{1}{a^2} - \frac{1}{\beta^2} - \frac{1}{\gamma^2}$$

$$-p' = -\frac{1}{a^2} - \frac{1}{\beta^2} - \frac{1}{\gamma^2} = -\frac{\beta^2\gamma^2 + a^2\gamma^2 + a^2\beta^2}{a^2\beta^2\gamma^2}$$

$$= -\frac{q^2 - 2pr}{r^2}$$

$$q' = \frac{1}{a^2\beta^2} + \frac{1}{a^2\gamma^2} + \frac{1}{\beta^2\gamma^2} = \frac{a^2 + \beta^2 + \gamma^2}{a^2\beta^2\gamma^2}$$

$$= \frac{p^2 - 2q}{r^2}$$

$$-r' = \frac{-1}{a^2\beta^2\gamma^2} = -\frac{1}{r^2}$$

答：所求之方程式爲 $r^2x^3 + (q^2 - 2pr)x^2 + (p^2 - 2q)x + 1 = 0$.

10. 解：(1) $a^2 + \beta^2 + \gamma^2 = (a + \beta + \gamma)^2 - 2(a\beta + \beta\gamma + \gamma a)$

$$= \left(\frac{1}{2}\right)^2 + 2\left(\frac{4}{2}\right) = \frac{17}{4}$$

(2) $a^3 + \beta^3 + \gamma^3 = (a + \beta + \gamma)^3 - 3(a^2\beta + a\beta^2 + \beta^2\gamma$

$$+ \beta\gamma^2 + \gamma^2a + a^2\gamma) - 6a\beta\gamma$$

$$= (a + \beta + \gamma)^3 - 3(a + \beta + \gamma)$$

$$\times (a\beta + \beta\gamma + \gamma a) + 3a\beta\gamma$$

$$= \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)\left(-\frac{4}{2}\right) + 3\left(-\frac{1}{2}\right)$$

$$= -\frac{37}{8}$$

$$(3) \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{-\frac{1}{2}}{-\frac{1}{2}} = 1.$$

$$(4) \alpha\beta^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\beta^2 + \gamma^2\alpha + \alpha^2\gamma \\ = (\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) - 3\alpha\beta\gamma \\ = \left(-\frac{1}{2}\right)(-2) + \frac{3}{2} = \frac{5}{2}.$$

11. 解: (1) $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$
 $= \frac{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)}{\alpha\beta\gamma} = \frac{4 - 2}{3} = \frac{2}{3}.$

(2) $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} = \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha\beta\gamma}$
 $= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma}$
 $= \frac{1 - 2 \cdot 3 \cdot 2}{3} = -\frac{11}{3}.$

(3) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$
 $= (2 - \alpha)(2 - \beta)(2 - \gamma)$
 $= 8 - 4(\alpha + \beta + \gamma) + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$
 $= 8 - 8 + 2 - 3 = -1.$

(4) $(\beta^2 + \gamma^2)(\gamma^2 + \alpha^2)(\alpha^2 + \beta^2)$
 $= (\alpha^2 + \beta^2 + \gamma^2)(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) - \alpha^2\beta^2\gamma^2$
 $= [(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)][(\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2(\alpha + \beta + \gamma)\alpha\beta\gamma] - (\alpha\beta\gamma)^2$
 $= (4 - 2)(1 - 2 \cdot 2 \cdot 3) - 9 = -31.$

(5) $\alpha\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) + \beta\left(\frac{1}{\gamma} + \frac{1}{\alpha}\right) + \gamma\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$
 $= \frac{\alpha^2\gamma + \alpha^2\beta}{\alpha\beta\gamma} + \frac{\beta^2\alpha + \beta^2\gamma}{\alpha\beta\gamma} + \frac{\gamma^2\beta + \gamma^2\alpha}{\alpha\beta\gamma}$

$$= \frac{(a+\beta+\gamma)(a\beta+ay+\beta\gamma)-3a\beta\gamma}{a\beta\gamma}$$

$$= \frac{2 \cdot 1 - 3 \cdot 3}{3} = -\frac{7}{3}$$

習題 LXXII

原本第 443 頁

1. 解: $x^7 + 3x^4 - 2x^2 + 6x + 7 = 0$.

將偶次方各項之符號變換, 得所求之方程式

$$x^7 - 3x^4 + 2x^2 + 6x - 7 = 0.$$

2. 解: 以 -2 乘 $2x^4 + x^3 - 4x^2 - 6x + 8 = 0$ 之各根, 又以 3 除其各根.

$$(1) \quad 2x^4 + (-2)x^3 - 4(-2)^2x^2 - 6(-2)^3x + (-2)^4 \cdot 8 = 0$$

$$2x^4 - 2x^3 - 16x^2 + 48x + 128 = 0$$

$$x^4 - x^3 - 8x^2 + 24x + 64 = 0.$$

$$(2) \quad 2x^4 + \left(\frac{1}{3}\right)x^3 - 4\left(\frac{1}{3}\right)^2x^2 - 6\left(\frac{1}{3}\right)^3x + \left(\frac{1}{3}\right)^4 \cdot 8 = 0$$

$$162x^4 + 27x^3 - 36x^2 - 18x + 8 = 0.$$

3. 解: $5x^6 - x^4 + 3x^3 + 9x + 10 = 0$.

$$10x^6 + 9x^5 + 3x^3 - x^2 + 5 = 0.$$

4. 解: $2x^5 + x^4 - 3x^2 + 6 = 0$.

(1) 以 $y+2$ 代替 x , 或用綜合除法以 2 除之, 皆可得所求之方程式.

$$2 + 1 + 0 - 3 + 0 + 6 \quad | \quad 2$$

$$2 + 5 + 10 + 17 + 34 + 74$$

$$c_5 = 74$$

$$2 + 9 + 28 + 73 + 180$$

$$c_4 = 180$$

$$2 + 13 + 54 + 181$$

$$c_3 = 181$$

$$2 + 17 + 88$$

$$c_2 = 88$$

$$2 + 21$$

$$c = 2$$

$$c_1 = 21$$

答: 所求之方程式爲

$$2x^5 + 21x^4 + 88x^3 + 181x^2 + 180x + 74 = 0.$$

(2) 以 $y-1$ 代替 x , 或用綜合除法以 -1 除之, 皆可得所求之方程式。

$$2+1+0-3+0+6 \quad | \quad -1$$

$$2-1+1-4+4+2$$

$$2-3+4-8+12$$

$$2-5+9-17$$

$$2-7+16$$

$$2-9$$

答: 所求之方程式爲 $2x^5 - 9x^4 + 16x^3 - 17x^2 + 12x + 2 = 0$ 。

5. 解: $x^4 - \frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{25} - \frac{1}{48} = 0 \dots\dots\dots(1)$

將其各根乘以 k , 得

$$x^4 - k\frac{x^3}{3} + k^2\frac{x^2}{4} + k^3\frac{x}{25} - k^4\frac{1}{48} = 0 \dots\dots\dots(2)$$

k 之最小值爲 $3 \cdot 2 \cdot 5 = 30$; 代入 (2) 式, 得

$$x^4 - 10x^3 + 225x^2 + 1080x - 16875 = 0.$$

6. 解: $3x^4 - 36x^3 - x - 7 = 0.$

以 $y+k$ 代替 x , 得 $3y^4 + 12(k-3)y^3 + \dots\dots = 0$, 若缺少 x^3 一項, 則 $k-3=0$, 即 $k=3$ 。

將 $3x^4 - 36x^3 - x - 7 = 0$ 之各根減 3, 得 $3x^4 - 162x^2 - 647x - 733 = 0$, 即所求之方程式也。

7. 解: (1) $x^3 + 6x^2 + 9x + 10 = 0.$

以 $y+k$ 代替 x , 得 $y^3 + 3(k+2)y^2 + 3(k^2+4k+3)y + (k^3+6k^2+9k+10) = 0 \dots\dots\dots(1)$

$$k^2 + 4k + 3 = 0 \quad \therefore k = -3 \text{ 及 } -1.$$

代入 (1) 式, 得 $y^3 - 3y^2 + 10 = 0 \dots\dots\dots(2)$

$$y^3 + 3y^2 + 6 = 0 \dots\dots\dots(3)$$

故 (2), (3) 兩式爲所求之方程式也。

(2) $x^3 - x^2 - x - 3 = 0.$

以 $y+k$ 代替 x , 得 $y^3 + (3k-1)y^2 + (3k^2-2k-1)y + (k^3-k^2-k-3) = 0 \dots\dots\dots(1)$

$$3k^2 - 2k - 1 = 0 \quad \therefore k = 1 \text{ 及 } -\frac{1}{3}.$$

代入(1)式,得 $y^3 + 2y^2 - 4 = 0 \dots\dots\dots(2)$

$27y^3 - 54y^2 - 76 = 0 \dots\dots\dots(3)$

故(2),(3)兩式爲所求之方程式也。

8. 解: 設 $\alpha, \beta, \gamma, \delta$ 爲 $x^4 + x^3 - x + 2 = 0$.

$$y = x^2 \quad \therefore x = \pm \sqrt{y}$$

$$(\pm \sqrt{y})^4 + (\pm \sqrt{y})^3 - (\pm \sqrt{y}) + 2 = 0$$

$$y^2 \pm y\sqrt{y} \mp \sqrt{y} + 2 = 0$$

$$\pm \sqrt{y}(y-1) = -(y^2+2)$$

$$y(y^2-2y+1) = y^4+4y^2+4$$

答: 所求之方程式爲 $y^4 - y^3 + 6y^2 - y + 4 = 0$.

9. 解: 設 $\alpha, \beta, \gamma, \delta$ 爲 $x^4 + 3x^3 + 2x^2 - 1 = 0$,

$$\beta + \gamma + \delta = \alpha + \beta + \gamma + \delta - \alpha = -3 - \alpha$$

$$\alpha + \gamma + \delta = \alpha + \beta + \gamma + \delta - \beta = -3 - \beta$$

$$\alpha + \beta + \delta = \alpha + \beta + \gamma + \delta - \gamma = -3 - \gamma$$

$$\alpha + \beta + \gamma = \alpha + \beta + \gamma + \delta - \delta = -3 - \delta$$

$$\text{設 } y = -3 - x \quad \therefore x = -y - 3$$

代入原方程式中,得

$$(-y-3)^4 + 3(-y-3)^3 + 2(-y-3)^2 - 1 = 0$$

化簡之,得所求之方程式爲

$$y^4 + 9y^3 + 29y^2 + 39y + 17 = 0$$

10. 解: 設 a, β, γ 爲 $x^3 + px^2 + qx + r = 0$.

$$(1) \frac{a\beta}{\gamma}, \frac{\beta\gamma}{a}, \frac{\gamma a}{\beta}$$

$$\frac{a\beta}{\gamma} = \frac{a\beta\gamma}{\gamma^2} = -\frac{r}{\gamma^2}, \quad \frac{\beta\gamma}{a} = -\frac{r}{a^2}, \quad \frac{\gamma a}{\beta} = -\frac{r}{\beta^2}$$

$$\text{設 } y = -\frac{r}{x^2} \quad \therefore x = \pm \sqrt{-\frac{r}{y}}$$

代入原方程式中,得

$$\left(\pm \sqrt{-\frac{r}{y}}\right)^3 + p\left(\pm \sqrt{-\frac{r}{y}}\right)^2 + q\left(\pm \sqrt{-\frac{r}{y}}\right) + r = 0$$

$$\pm \sqrt{-\frac{r}{y}}\left(-\frac{r}{y} + q\right) = \frac{pr}{y} - r$$

$$-\frac{r}{y}\left(-\frac{r}{y} + q\right)^2 = \left(\frac{pr}{y} - r\right)^2$$

$$-r(r^2 - 2qry + q^2y^2) = yr^2(p^2 - 2py + y^2)$$

$$\therefore ry^3 + (q^2 - 2pr)y^2 + r(p^2 - 2q)y + r^2 = 0.$$

$$(2) \frac{a}{\beta + \gamma}, \frac{\beta}{\gamma + \alpha}, \frac{\gamma}{\alpha + \beta}.$$

$$\frac{a}{\beta + \gamma} = \frac{a}{a + \beta + \gamma - a} = \frac{a}{-p - \alpha},$$

$$\frac{\beta}{\gamma + \alpha} = \frac{\beta}{a + \beta + \gamma - \beta} = \frac{\beta}{-p - \beta},$$

$$\frac{\gamma}{\alpha + \beta} = \frac{\gamma}{a + \beta + \gamma - \gamma} = \frac{\gamma}{-p - \gamma}.$$

$$\text{設 } y = \frac{x}{-p-x} \quad \therefore x = \frac{-py}{1+y}.$$

$$\text{代入原方程式中, 得 } \left(\frac{-py}{1+y}\right)^3 + p\left(\frac{-py}{1+y}\right)^2 + q\left(\frac{-py}{1+y}\right) + r = 0$$

$$-p^3y^3 + p^3y^2 + p^3y^3 - pqy - 2pqy^2$$

$$-pqy^3 + r + 3ry + 3ry^2 + ry^3 = 0.$$

$$\therefore (r - pq)y^3 + (p^3 - 2pq + 3r)y^2 + (3r - pq)y + r = 0.$$

11. 解: 設 α, β, γ 爲 $x^3 + 2x^2 + 3x + 4 = 0$.

$$(1) \beta^2 + \gamma^2, \gamma^2 + \alpha^2, \alpha^2 + \beta^2.$$

$$\beta^2 + \gamma^2 = \alpha^2 + \beta^2 + \gamma^2 - \alpha^2$$

$$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha^2$$

$$= p^2 - 2q - \alpha^2 = -2 - \alpha^2,$$

$$\gamma^2 + \alpha^2 = -2 - \beta^2, \quad \alpha^2 + \beta^2 = -2 - \gamma^2.$$

$$\text{設 } y = -2 - x^2 \quad \therefore x = \pm \sqrt{-2 - y}.$$

$$(-2 - y)\sqrt{-2 - y} + 2(-2 - y) + 3\sqrt{-2 - y} + 4 = 0$$

$$(1 - y)\sqrt{-2 - y} = 2y$$

$$(1 - 2y + y^2)(-2 - y) = 4y^3$$

$$\therefore y^3 + 4y^2 - 3y + 2 = 0.$$

$$(2) \alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta).$$

$$\alpha(\beta + \gamma) = \alpha\beta + \beta\gamma + \gamma\alpha - \beta\gamma = 3 - \beta\gamma$$

$$= 3 - \frac{\alpha\beta\gamma}{\gamma} = 3 + \frac{4}{\alpha},$$

$$\beta(\gamma + \alpha) = 3 + \frac{4}{\beta}, \quad \gamma(\alpha + \beta) = 3 + \frac{4}{\gamma}.$$

$$\text{設 } y = 3 + \frac{4}{x} \quad \therefore x = \frac{4}{y-3}$$

$$\left(\frac{4}{y-3}\right)^3 + 2\left(\frac{4}{y-3}\right)^2 + 3\left(\frac{4}{y-3}\right) + 4 = 0$$

$$64 + 32(y-3) + 12(y-3)^2 + 4(y-3)^3 = 0$$

$$\therefore y^3 - 6y^2 + 17y - 8 = 0.$$

$$(3) \quad \beta\gamma + \frac{1}{\alpha}, \quad \gamma\alpha + \frac{1}{\beta}, \quad \alpha\beta + \frac{1}{\gamma}$$

$$\beta\gamma + \frac{1}{\alpha} = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{-4 + 1}{\alpha} = \frac{-3}{\alpha},$$

$$\gamma\alpha + \frac{1}{\beta} = \frac{-3}{\beta}, \quad \alpha\beta + \frac{1}{\gamma} = \frac{-3}{\gamma}.$$

$$\text{設 } y = \frac{-3}{x} \quad \therefore x = -\frac{3}{y}$$

$$\left(-\frac{3}{y}\right)^3 + 2\left(-\frac{3}{y}\right)^2 + 3\left(-\frac{3}{y}\right) + 4 = 0$$

$$-27 + 18y - 9y^2 + 4y^3 = 0$$

$$\therefore 4y^3 - 9y^2 + 18y - 27 = 0.$$

$$(4) \quad \frac{\alpha}{\beta + \gamma - \alpha}, \quad \frac{\beta}{\gamma + \alpha - \beta}, \quad \frac{\gamma}{\alpha + \beta - \gamma}$$

$$\frac{\alpha}{\beta + \gamma - \alpha} = \frac{\alpha}{\alpha + \beta + \gamma - 2\alpha} = \frac{\alpha}{-2 - 2\alpha},$$

$$\frac{\beta}{\gamma + \alpha - \beta} = \frac{\beta}{-2 - 2\beta}, \quad \frac{\gamma}{\alpha + \beta - \gamma} = \frac{\gamma}{-2 - 2\gamma}.$$

$$\text{設 } y = \frac{x}{-2 - 2x} \quad \therefore x = \frac{-2y}{2y + 1}$$

$$\left(\frac{-2y}{2y+1}\right)^3 + 2\left(\frac{-2y}{2y+1}\right)^2 + 3\left(\frac{-2y}{2y+1}\right) + 4 = 0$$

$$-8y^3 + 8y^2(2y+1) - 6y(2y+1)^2 + 4(2y+1)^3 = 0$$

$$\therefore 8y^3 + 16y^2 + 9y + 2 = 0.$$

$$(5) \quad \alpha\left(\frac{1}{\beta} + \frac{1}{\gamma}\right), \quad \beta\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right), \quad \gamma\left(\frac{1}{\alpha} + \frac{1}{\beta}\right).$$

$$\alpha\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) = \alpha\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\alpha}\right)$$

$$= \frac{a(\beta\gamma + \alpha\gamma + \alpha\beta)}{\alpha\beta\gamma} - 1 = \frac{-3\alpha}{4} - 1,$$

$$\beta\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right) = \frac{-3\beta}{4} - 1, \quad \gamma\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{-3\gamma}{4} - 1.$$

設 $y = \frac{-3x}{4} - 1 \quad \therefore x = -\frac{4}{3}(1+y).$

$$\left[-\frac{4(1+y)}{3}\right]^3 + 2\left[-\frac{4(1+y)}{3}\right]^2 + 3\left[-\frac{4(1+y)}{3}\right] + 4 = 0$$

$$-64(1+y)^3 + 96(1+y)^2 - 108(1+y) + 4 \cdot 27 = 0$$

$$-64y^3 - 96y^2 - 108y + 32 = 0$$

$$\therefore 16y^3 + 24y^2 + 27y - 8 = 0.$$

12. 解: (1) $x^4 + 3x^3 - 13x^2 - 6x + 28 = 0.$

$$1 + 3 - 13 - 6 + 28 \quad \underline{2}$$

$$1 + 5 - 3 - 12 + 4$$

$$1 + 7 + 11 + 10$$

故 2 爲其上界。

$$1 + 3 - 13 - 6 + 28 \quad \underline{1}$$

$$1 + 4 - 9 - 15 + 13$$

$$1 + 5 - 4 - 19$$

19 前之符號爲負, 示 1 非上界。

$$1 + 3 - 13 - 6 + 28 \quad \underline{-6}$$

$$1 - 3 + 5 - 36 + 244$$

故 -6 爲其下界。

$$1 + 3 - 13 - 6 + 28 \quad \underline{-5}$$

$$1 - 2 - 3 + 9 - 17$$

17 前之符號爲負, 示 -5 非下界。

(2) $2x^5 - 120x^2 - 38x + 27 = 0.$

$$2 - 0 - 0 - 120 - 38 + 27 \quad \underline{4}$$

$$2 + 8 + 32 + 8 - 6 + 3$$

$$2 + 16 + 96 + 392 + 1562$$

故 4 爲其上界。

$$2 - 0 - 0 - 120 - 38 + 27 \quad \underline{3}$$

$$2 + 6 + 18 - 66 - 236 - 681$$

681 前之符號爲負, 示 3 非上界。

$$2+0+0-120-38+27 \quad | \quad -1$$

$$2-2+2-122+84-57$$

故 -1 爲其下界。

$$(3) \quad x^4 - 29x^2 + 50x + 12 = 0.$$

$$1+0-29+50+12 \quad | \quad 4$$

$$1+4-13-2+4$$

$$1+8+19+74$$

故 4 爲其上界。

$$1+0-29+50+12 \quad | \quad 3$$

$$1+3-20-10-18$$

18 前之負號示 3 非上界。

$$1+0-29+50+12 \quad | \quad -7$$

$$1-7+20-90+642$$

故 -7 爲其下界。

$$1+0-29+50+12 \quad | \quad -6$$

$$1-6+7+8-36$$

36 前之負號示 -6 非下界。

$$(4) \quad 2x^5 - 26x^3 + 60x^2 - 92 = 0.$$

$$2+0-26+60+0-92 \quad | \quad 2$$

$$2+4-18+24+48+4$$

$$2+8-2+20+88$$

$$2+12+22+64$$

故 2 爲其上界。

$$2+0-26+60+0-92 \quad | \quad 1$$

$$2+2-24+36+36-56$$

-56 示 1 非上界。

$$2+0-26+60+0-92 \quad | \quad -3$$

$$2-10+24-60+300-1592$$

故 -3 爲其下界。

$$2+0-26+60+0-92 \quad | \quad -4$$

$$2-8+6+36-144+484$$

184 前之負號示 -4 非下界。

$$(5) \quad x^4 - 14x^3 + 44x^2 + 28x - 92 = 0.$$

$$1 - 14 + 44 + 28 - 92 \quad | \quad 9$$

$$1 - 5 - 1 + 19 + 79$$

$$1 + 4 + 35 + 334$$

故 9 爲其上界。

$$1 - 14 + 44 + 28 - 92 \quad | \quad 8$$

$$1 - 6 - 4 - 4 - 124$$

-124 示 8 非上界。

$$1 - 14 + 44 + 28 - 92 \quad | \quad -2$$

$$1 - 16 + 76 - 124 + 156$$

故 -2 爲其下界。

$$1 - 14 + 44 + 28 - 92 \quad | \quad -1$$

$$1 - 15 + 59 - 31 - 61$$

-61 示 -1 非下界。

$$(6) \quad 3x^6 - 5x^3 + 77x^2 - 50x - 110 = 0.$$

$$3 + 0 + 0 - 35 + 77 - 50 - 110 \quad | \quad 2$$

$$3 + 6 + 12 - 11 + 55 + 60 + 10$$

$$3 + 12 + 36 + 61$$

故 2 爲其上界。

$$3 + 0 + 0 - 35 + 77 - 50 - 110 \quad | \quad 1$$

$$3 + 3 + 3 - 32 + 45 - 5 - 115$$

-115 示 1 非上界。

$$3 + 0 + 0 - 35 + 77 - 50 - 110 \quad | \quad -1$$

$$3 - 3 + 3 - 38 + 115 - 165 + 55$$

故 -1 爲其下界。

習 題 LXXIII

原本第 449 頁

1. 解：設 α, β 爲其另二根。

$$1 + 2i + 1 - 2i + \alpha + \beta = \frac{1}{2} \dots\dots\dots (1)$$

$$\alpha\beta(1 + 2i)(1 - 2i) = \frac{5}{2} \dots\dots\dots (2)$$

從(1)式, $\alpha + \beta = -\frac{3}{2}$ (3)

從(2)式, $\alpha\beta = \frac{1}{2}$ (4)

從(3)及(4), $\alpha = -1, \beta = -\frac{1}{2}$.

答: 其四根爲 $-1, -\frac{1}{2}, 1 \pm 2i$.

2. 解: $2x^4 - 11x^3 + 17x^2 - 10x + 2 = 0$.

按 § 825, 知 $2 - \sqrt{2}$ 亦爲其一根。

$$[x - (2 + \sqrt{2})][x - (2 - \sqrt{2})] = 0$$

$$x^2 - 4x + 2 = 0$$

$$\frac{2x^4 - 11x^3 + 17x^2 - 10x + 2}{x^2 - 4x + 2} = 0$$

$$2x^2 - 3x + 1 = 0$$

$$(2x - 1)(x - 1) = 0$$

$$\therefore x = \frac{1}{2}, 1.$$

答: 其根爲 $\frac{1}{2}, 1, 2 \pm \sqrt{2}$.

3. 解: $[x - (-5 + 2i)][x - (-5 - 2i)]$

$$\times [x - (-1 + \sqrt{5})][x - (-1 - \sqrt{5})] = 0$$

$$[x^2 + 10x + 29][x^2 + 2x - 4] = 0$$

$$\therefore x^4 + 12x^3 + 45x^2 + 18x - 116 = 0.$$

4. 解: 此方程式之其他數根爲 $-\sqrt{2} + i, \sqrt{2} - i, -\sqrt{2} - i$.

$$[x - (\sqrt{2} + i)][x - (\sqrt{2} - i)]$$

$$\times [x - (-\sqrt{2} + i)][x - (-\sqrt{2} - i)] = 0$$

$$[x^2 - 2\sqrt{2}x + 3][x^2 + 2\sqrt{2}x + 3] = 0$$

$$\therefore x^4 - 2x^2 + 9 = 0.$$

5. 解: (1) $x^4 + 1 = 0$.

因 $v = 0, v' = 0$; 故 $n - (v + v') = 4 - (0 + 0) = 4$, 是以 $x^4 + 1 = 0$ 無實根, 僅有四虛根。

(2) $x^4 - x^2 - 1 = 0$.

因 $v = 1, v' = 1$; 故 $n - (v + v') = 4 - (1 + 1) = 2$, 是以此方程式至少有二虛根或四虛根, 正根與負根皆僅有一個或均無。

	正根數	負根數	虛根數
根之可能情形	1	1	2
	0	0	4

* 註：如應用本習題問題 15，正負根數亦必為 1，不得為 0。

$$(3) \quad x^4 + 2x^3 + x^2 + x + 1 = 0.$$

因 $v=0, v'=4$ ；故 $n-(v+v')=4-(0+4)=0$ ，是以此方程式之負根不能超過四個，且無正根。

	正根數	負根數	虛根數
	0	4	0
根之可能情形	0	2	2
	0	0	4

$$(4) \quad x^4 - 2x^3 + x^2 - x + 1 = 0.$$

因 $v=4, v'=0$ ；故 $n-(v+v')=4-(4+0)=0$ ，是以此方程式之正根不能超過四個，且無負根。

	正根數	負根數	虛根數
	4	0	0
根之可能情形	2	0	2
	0	0	4

$$(5) \quad x^7 + x^5 + x^3 - x + 1 = 0.$$

因 $v=2, v'=1$ ；故 $n-(v+v')=7-(2+1)=4$ ，是以此方程式至少有四虛根，正根不能超過二個，負根不能超過一個。

	正根數	負根數	虛根數
根之可能情形	2	1	4
	0	1	6

$$(6) \quad x^7 + x^4 - x^2 - 1 = 0.$$

因 $v=1, v'=2$ ；故 $n-(v+v')=7-(1+2)=4$ ，是以此方程式至少有四虛根，正根不能超過一個，負根不能超過二個。

	正根數	負根數	虛根數
根之可能情形	1	2	4
	1	0	6

$$(7) \quad x^5 - 4x^2 + 3 = 0.$$

因 $v=2, v'=1$ ；故 $n-(v+v')=5-(2+1)=2$ ，是以此方程式至少有二虛根，正根不能超過二個，負根不能超過一個。

	正根數	負根數	虛根數
--	-----	-----	-----

根之可能情形	2	1	2
	0	1	4

$$(8) \quad x^{3n} - x^{2n} + x^n + x + 1 = 0.$$

當 n 爲奇數時, $v=2, v'=1$; 故 $3n - (v+v') = 3n-3$, 是以此方程式至少有 $3n-3$ 虛根, 正根不能超過二個, 負根不能超過一個。

	正根數	負根數	虛根數
根之可能情形	2	1	$3n-3$
	0	1	$3n-1$

當 n 爲偶數時, $v=2, v'=4$; 故 $3n - (v+v') = 3n-6$, 是以此方程式至少有 $3n-6$ 虛根, 正根不能超過二個, 負根不能超過四個。

	正根數	負根數	虛根數
	2	4	$3n-6$
	0	4	$3n-4$
根之可能情形	2	2	$3n-4$
	0	2	$3n-2$
	2	0	$3n-2$
	0	0	$3n$

6. 證: 於一完全方程式中, 變號與連號數之和適等於其方次, 即 $v+v'=n$.

又設正根數爲 a , 負根數爲 b .

因所有根均爲實根, 故 $a+b=n=v+v'$.

由笛卡兒法則, $a \leq v, b \leq v'$.

因 $a+b=v+v'$; 故 $a=v, b=v'$.

即正根與變號之數相等, 負根與連號之數相等。

7. 解: $x^5 + 3x^4 - 15x^3 - 35x^2 + 54x + 72 = 0$.

因此方程式有二變號與三連號, 故知此方程式有二正根與三負根。

8. 證: 對於方程式 $x^{2n} + 1 = 0, v+v'=0$, 故無實根。

(1) 對於方程式 $x^{2n+1} + 1 = 0, v=0, v'=1$.

故此方程式有一負根與 $2n$ 個虛根。

(2) 對於方程式 $x^{2n} - 1 = 0, v=1, v'=1$.

故此方程式至少有 $2n-2$ 個虛根，正根與負根均不能超過一個。

(3) 對於方程式 $x^{2n+1}-1=0, v=1, v'=0$.

故此方程式有一正根與 $2n$ 個虛根。

9. 證：若一方程式僅含 x 之偶次方，且皆為正係數，則 $v=0, v'=0$ ；故知此方程式不能有正根或負根。

10. 證：設 $f(x)=x^{2n-1}+x^{2n-3}+x^{2n-5}+\dots+x^{2n-(2n-1)}=0$ ；因 $f(x)$ 之係數皆為正數，故 $v=0$ ；又因 $f(-x)$ 之係數皆為負數，故 $v'=0$ 。

$$f(x)=x(x^{2n-2}+x^{2n-4}+\dots+1)=0.$$

故 $f(x)$ 除 0 外，無其他實根。

11. 證： $x^3+px+q=0$.

$$f(x)=x^3+px+q=0 \quad \therefore v=0.$$

$$f(-x)=x^3+px-q=0 \quad \therefore v'=1.$$

故此方程式至少有二虛根而僅有二虛根；故該方程式有一實根，且為負根。

12. 證：一不完全方程式之變號與連號之數必不能完全發現，故其實根之數必少於其方次，所餘即虛根，但虛根皆對對發生，故此方程式有二或二以上之虛根。

13. 證：設 $f(x)=b_0x^n-b_1x^{n-1}+b_2x^{n-2}-b_3x^{n-3}+b_4x^{n-4}$
 $-b_5x^{n-5}+\dots-b_{n-1}x+b_n$.

從 $f(x)$ ，知於二不相連之異符號，如 b_0x^n 與 $-b_3x^{n-3}$ 間，有三變號，於二不相連之同符號，如 $-b_3x^{n-3}$ 與 $-b_5x^{n-5}$ 間，無變號；又如 $-b_1x^{n-1}$ 與 $-b_5x^{n-5}$ 間，有二變號。故知二不相連之異符號間，其變號為奇數；二不相連之同符號間，其變號為偶數或無變號。

14. 證：一含有實係數之方程式有一負根 $-a$ ，即此方程式有一因式為 $(x+a)$ ；若有一虛根 $a-\beta i$ ，則必有另一虛根 $a+\beta i$ 。故此方程式又有一因式為 $[x^2-2ax+(a^2+\beta^2)]$ 於 $(x+a), [x^2-2ax+(a^2+\beta^2)], \dots$ 中，其常數項為正者，故其乘積中之常數項亦為正者。於此乘積中，如上所述者，其第一項與常數項皆為正者，故於二者間之變號必為偶數。

15. 證：設 $\phi(x)$ 為對應於負根及虛根之因式之乘積，由上題，已知 $\phi(x)$ 中變號數必為偶數。今如乘以對應於一正根之因

式於 $\phi(x)$, 則其乘積增 $2m+1$ 個變號, 如乘以對應於 p 個正根之因式, 則其乘積增 p 或 $p+2s$ 個變號, m 及 s 均為正整數, 故如變號數大於正根數, 則所餘之變號數必為偶數。

16. 證: $x^4+x^3-x^2+x-1=0$.

此完全方程式中有 3 變號與一連號, 故此方程式最多有 3 正根與一負根。由上題, 知變號數如超過正根數, 則所餘之數必為偶數, 故正根數如不為 3 必為 1。同理可證負根數必為 1, 因連號數如超過負根數, 則超過之數必為偶數, 故負根數必不得為 0 也。

17. 證: $f(x)=x^{2^n}+x^{2^n-2}+\dots-1=0$.

因此方程式之絕對項為負數, 故知此方程式之變號數必為奇數, 故由本習題問題 15, 知正根數不得為 0, 而至少有一個或單數個, 又當此方程式變為 $f(-x)=0$ 時, 其變號數亦為奇數, 故知 $f(x)=0$ 之負根數不得為 0 而至少為 1, 是以絕對項為負數之偶次方方程式, 必有一正根與一負根。

習 題 LXXIV

原本第 453 頁

1. 解: $2x^3-3x^2-9x+8=0$.

按笛卡兒法則, § 830, 此方程式正根不能超過二個, 負根不能超過一個。

$$f(0)=8, \quad f(1)=-2, \quad f(2)=-6, \quad f(3)=8;$$

$$f(0)=8, \quad f(-1)=12, \quad f(-2)=-2.$$

答: 其根位於 0 與 1, 2 與 3, -1 與 -2 之間。

2. 解: $x^3+x^2-4x-2=0$.

此方程式正根不能超過二個, 負根不能超過一個。

$$f(0)=-2, \quad f(1)=-4, \quad f(2)=2;$$

$$f(0)=-2, \quad f(-1)=2, \quad f(-2)=2, \quad f(-3)=-8.$$

答: 其根位於 1 與 2, 0 與 -1, -2 與 -3 之間。

3. 解: $x^4-3x^2-2x+5=0$.

此方程式正根不能超過二個, 負根不能超過一個。

$$f(0)=5, \quad f(1)=1, \quad f(2)=-3, \quad f(3)=-1, \quad f(4)=13;$$

$$f(0)=5, \quad f(-1)=3, \quad f(-2)=-11.$$

答：其根位於 1 與 2, 3 與 4, -1 與 -2 之間。

4. 解： $2x^3 + 3x^2 - 10x - 15 = 0$.

此方程式正根不能超過一個，負根不能超過二個。

$$f(0) = -15, f(1) = -20, f(2) = -7, f(3) = 36;$$

$$f(0) = -15, f(-1) = -4, f(-2) = 1, f(-3) = -12$$

答：其根位於 2 與 3, -1 與 -2, -2 與 -3 之間。

5. 解： $x^3 - 4x^2 - 4x + 12 = 0$.

此方程式正根不能超過二個，負根不能超過一個。

$$f(0) = 12, f(1) = 5, f(2) = -4, f(3) = -9,$$

$$f(4) = -4, f(5) = 17;$$

$$f(0) = 12, f(-1) = 11, f(-2) = -4.$$

答：其根位於 1 與 2, 4 與 5, -1 與 -2 之間。

6. 解： $x^3 + 13x^2 + 54x + 71 = 0$.

此方程式負根不能超過三個。

$$f(0) = 71, f(-1) = 29, f(-2) = 7, f(-3) = -1,$$

$$f(-4) = -1, f(-5) = 1, f(-6) = -1.$$

答：其根位於 -2 與 -3, -4 與 -5, -5 與 -6 之間。

7. 解： $x^3 + 5x + 19 = 0$.

此方程式負根不能超過一個，而且必有一個。

$$f(0) = 19, f(-1) = 13, f(-2) = 1, f(-3) = -23.$$

答：其根位於 -2 與 -3 之間。

8. 解： $x^4 - 95 = 0$.

此方程式正根不能超過一個，負根不能超過一個。

$$f(0) = -95, f(1) = -94, f(2) = -79,$$

$$f(3) = -14, f(4) = 161;$$

$$f(0) = -95, f(-1) = -94, f(-2) = -79,$$

$$f(-3) = -14, f(-4) = 161.$$

答：其根位於 3 與 4, -3 與 -4 之間。

9. 解： $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$.

此方程式正根不能超過三個，負根不能超過一個。

$$f(0) = -8, f(1) = 3, f(2) = 8, f(3) = -7,$$

$$f(4) = -24, f(5) = -13, f(6) = 88;$$

$$f(0) = -8, f(-1) = 11.$$

答：其根位於 0 與 1, 2 與 3, 5 與 6, 0 與 -1 之間。

10. 解: $x^4 + 5x^3 + x^2 - 13x - 7 = 0$.

此方程式正根不能超過一個, 負根不能超過三個.

$$f(0) = -7, \quad f(1) = -13, \quad f(2) = 27;$$

$$f(0) = -7, \quad f(-1) = 3, \quad f(-2) = -1,$$

$$f(-3) = -13, \quad f(-4) = -3, \quad f(-5) = 23.$$

答: 故其根位於 1 與 2, 0 與 -1, -1 與 -2, -4 與 -5 之間.

11. 解: $x^4 - 11x^3 + 32x^2 - 4x - 46 = 0$.

此方程式正根不能超過三個, 負根不能超過一個.

$$f(0) = -46, \quad f(1) = -28, \quad f(2) = 2, \quad f(3) = 14,$$

$$f(4) = 2, \quad f(5) = -16, \quad f(6) = 2;$$

$$f(0) = -46, \quad f(-1) = 2.$$

答: 其根位於 1 與 2, 4 與 5, 5 與 6, 0 與 -1 之間.

12. 解: $x^5 + 2x^4 - 16x^3 - 24x^2 + 48x + 32 = 0$.

此方程式正根不能超過二個, 負根不能超過三個.

$$f(0) = 32, \quad f(1) = 43, \quad f(2) = -32, \quad f(3) = -67, \quad f(4) = 352;$$

$$f(0) = 32, \quad f(-1) = -23, \quad f(-2) = -32,$$

$$f(-3) = 23, \quad f(-4) = -32.$$

答: 其根位於 1 與 2, 2 與 1, 0 與 -1, -2 與 -3, -3 與 -4 之間.

13. 證: 設 x 之數值極大時, $f(x)$ 之符號為其最高方次項之正號.

(1) 任一實係數方程式 $x^n + b_1x^{n-1} + \dots + b_n = 0$ 必有正根負根各一, 式中之 n 為偶數, b_n 為負數.

當 $x = -\infty$ 時, $f(x)$ 為正值; $x = 0$ 時, $f(x)$ 為負值, 故 $f(x)$ 有一負根位於 $-\infty$ 與 0 之間. 又此方程式之虛根必對對發生, 以 k 表示之. 但 $1+k$ 必小於 n , 且 $1+k$ 為奇數, n 為偶數, 故 $n - (1+k)$ 亦必為奇數, 至少須為 1, 此即正根之數也. 是以此方程式必有一正根與一負根.

(2) 方程式 $k^2(x-b)(x-c) + l^2(x-c)(x-a) + m^2(x-a)(x-b) - m^2(x-a)(x-b)(x-c) = 0$ 之四根, 位於 $-\infty$ 與 a , a 與

b, b 與 c, c 與 ∞ 之間, 且設 a, b, c, k, l, m 俱為實數, 又 $a < b < c$.

$$\text{設 } f(x) \equiv k^2(x-b)(x-c) + l^2(x-c)(x-a) + m^2(x-a)(x-b) - x(x-a)(x-b)(x-c) = 0$$

$$f(-\infty) = -, \quad f(a) = k^2(a-b)(a-c) = +,$$

$$f(b) = l^2(b-c)(b-a) = -, \quad f(c) = m^2(c-a)(c-b) = +,$$

$$f(\infty) = -.$$

故此四根位於 $-\infty$ 與 a, a 與 b, b 與 c, c 與 ∞ 之間。

14. 證: $f(x) = 0, \quad f\left(\frac{1}{a}\right) = \frac{1}{a^3}, \quad f\left(1 - \frac{1}{a}\right) = \frac{-a^2 + 3a - 1}{a^3},$

$$f(1) = +.$$

若 $a \equiv 3, \quad \frac{1}{a^3} = +, \quad \frac{-a^2 + 3a - 1}{a^3} = -,$

$$f\left(\frac{1}{a}\right) = +, \quad f\left(1 - \frac{1}{a}\right) = -, \quad f(1) = +.$$

故 $a \equiv 3$ 時, 則有一根位於 $\frac{1}{a}$ 與 $1 - \frac{1}{a}$ 之間, 另一根位於

$1 - \frac{1}{a}$ 與 1 之間。

15. 證: 當 $a \equiv 5$ 時,

$$f(0) = -, \quad f\left(\frac{1}{a}\right) = \frac{1}{a^4} = +,$$

$$f\left(1 - \frac{2}{a}\right) = \frac{-a^4 + 6a^3 - 32a + 16}{a^4} = -, \quad f(1) = +.$$

故此方程式之根位於 0 與 $\frac{1}{a}, \frac{1}{a}$ 與 $1 - \frac{2}{a}, 1 - \frac{2}{a}$ 與 1

之間。

習題 LXXV

原本第 459 頁

1. 解: $x^3 + x - 3 = 0$.

$$1+0+1-3 \quad | \quad 1.213411$$

$$\quad \quad \quad 1+1+2$$

$$\hline 1+1+2-1$$

$$\quad \quad \quad 1+2$$

$$\hline 1+2+4$$

$$\quad \quad \quad 1$$

$$1+30+400-1000 \quad | \quad 2$$

$$\quad \quad \quad 2+64+928$$

$$\hline 1+32+464-72$$

$$\quad \quad \quad 2+68$$

$$\hline 1+34+532$$

$$\quad \quad \quad 2$$

$$1+360+53200-72000 \quad | \quad 1$$

$$\quad \quad \quad 1+361-53561$$

$$\hline 1+361+53561-18439$$

$$\quad \quad \quad 1+362$$

$$\hline 1+362+53923$$

$$\quad \quad \quad 1$$

$$0.001+3^4.63+5392.3-18439 \quad | \quad 0.003411$$

$$\quad \quad \quad 12+16212$$

$$\hline 4+5404-2227$$

$$\quad \quad \quad 12$$

$$, 0.04+541^2.6-2227$$

$$\quad \quad \quad 2168$$

$$\hline 54.2-59$$

$$\quad \quad \quad 54$$

$$\hline 5.4-5$$

$$\quad \quad \quad 5$$

$$\hline 5-0$$

$$f(0.2) = -, f(0.3) = +$$

$$\frac{72}{532} = 0.1+$$

$$\frac{18439}{53923} = 0.3+$$

$$\frac{2227}{5416} = 0.4+$$

$$\frac{59}{542} = 0.1+$$

$$\frac{5}{54} = 0.1+$$

答：此根爲 1.213411.

2. 解： $x^3 + 2x - 20 = 0.$

$$\begin{array}{r} 1+0+2-20 \quad | \quad 2.469547 \\ \hline \end{array}$$

$$\begin{array}{r} 2+4+12 \\ \hline \end{array}$$

$$\begin{array}{r} 1+2+6-8 \\ \hline \end{array}$$

$$\begin{array}{r} 2+8 \\ \hline \end{array}$$

$$\begin{array}{r} 1+4+14 \\ \hline \end{array}$$

2

$$f(0.4) = -, f(0.5) = +$$

$$\begin{array}{r} 1+60+1400-8000 \quad | \quad 4 \\ \hline \end{array}$$

$$\begin{array}{r} 4+256+6624 \\ \hline \end{array}$$

$$\begin{array}{r} 1+64+1656-1376 \\ \hline \end{array}$$

$$\begin{array}{r} 4+272 \\ \hline \end{array}$$

$$\begin{array}{r} 1+68+1928 \\ \hline \end{array}$$

4

$$f(0.6) = -, f(0.7) = +$$

$$\begin{array}{r} 1+720+192800-1376000 \quad | \quad 6 \\ \hline \end{array}$$

$$\begin{array}{r} 6+4856+1182936 \\ \hline \end{array}$$

$$\begin{array}{r} 1+726+197156-193064 \\ \hline \end{array}$$

$$\begin{array}{r} 6+4392 \\ \hline \end{array}$$

$$\begin{array}{r} 1+732+201548 \\ \hline \end{array}$$

6

$$\frac{193064}{201548} = 0.9+$$

$$\begin{array}{r} 0.001+7.38+20154^5.8-193064 \quad | \quad 0.009547 \\ \hline \end{array}$$

$$\begin{array}{r} 63+181962 \\ \hline \end{array}$$

$$\begin{array}{r} 7+20218-11102 \\ \hline \end{array}$$

63

$$\frac{11102}{20281} = 0.5+$$

$$\begin{array}{r} 0.07+2028.1-11102 \\ \hline \end{array}$$

10140

$$\frac{962}{2028} = 0.4+$$

$$\begin{array}{r} 202^3.8-962 \\ \hline \end{array}$$

812

$$\frac{150}{203} = 0.7+$$

$$\begin{array}{r} 20.8-150 \\ \hline \end{array}$$

140

$$\begin{array}{r} 20-10 \\ \hline \end{array}$$

答：此根爲 2.469547.

3. 解： $x^3 + 6x^2 + 10x - 2 = 0.$

$$1+6+10-2 \mid 0.179981 \quad f_1(0.1) = - \quad f_1(0.2) = +$$

$$1+60+1000-2000 \mid 1$$

$$1+61+1061$$

$$1+61+1061-939$$

$$1+62$$

$$1+62+1123$$

$$1$$

$$f_2(0.7) = - , f_2(0.8) = +$$

$$1+630+112300-939000 \mid 7$$

$$7+4459+817313$$

$$1+637+116759-121687$$

$$7+4508$$

$$1+644+121267$$

$$7$$

$$f_3(0.9) = - , f_3(1) = +$$

$$1+6510+12126700-121687000 \mid 0.009981$$

$$9+58671+109668339$$

$$1+6519+12185371-12018661$$

$$9+58752$$

$$1+6528+12244123$$

$$9$$

$$0.001+65.37+1224412.3-12018661$$

$$\frac{12018661}{12244123} = 0.9 +$$

$$585+11024973$$

$$65+1224997-993688$$

$$\frac{993688}{1225582} = 0.8 +$$

$$585$$

$$1.65+122558.2-993688$$

$$8+980528$$

$$1+122566-13160$$

$$8$$

$$0.01+12257.4-13160$$

$$\frac{13160}{122574} = 0.1 +$$

12257

$$\begin{array}{r} 12257 - \\ \hline 903 \end{array}$$

答：其根爲 0.179981.

4. 解： $3x^3 + 5x - 40 = 0$.
$$3 + 0 + 5 - 40 \quad | \quad 2.137812$$

$$ 6 + 12 + 34$$

$$\underline{3 + 6 + 17 - 6}$$

$$ 6 + 24$$

$$\underline{3 + 12 + 41}$$

$$ 6$$

$$\underline{3 + 180 + 4100 - 6000} \quad | \quad 1$$

$$ 3 + 183 + 4283$$

$$\underline{3 + 183 + 4283 - 1717}$$

$$ 3 + 186$$

$$\underline{3 + 186 + 4469}$$

$$ 3$$

$$\underline{3 + 1890 + 446900 - 1717000} \quad | \quad 3$$

$$ 9 + 5697 + 1358791$$

$$\underline{3 + 1899 + 452597 - 359209}$$

$$ 9 + 5724$$

$$\underline{3 + 1908 + 458321}$$

$$ 9$$

$$(按 § 842) \quad \underline{19 + 45832 - 359209} \quad | \quad 0.007812$$

$$ 133 + 321755$$

$$\underline{19 + 45965 - 37454}$$

$$ 133$$

$$\underline{4609 - 37454}$$

$$ 36872$$

$$\underline{460 - 582}$$

$$ 460$$

$$\underline{46 - 122}$$

$$ 92$$

$$\underline{5 - 30}$$

答：其根爲 2.137812.

5. 解： $x^3 + 10x^2 + 8x - 120 = 0.$

$$\begin{array}{r} 1+10+8-120 \quad | \underline{2.768345} \\ \hline \end{array}$$

$$2+24+64$$

$$\begin{array}{r} 1+12+32-56 \\ \hline \end{array}$$

$$2+28$$

$$\begin{array}{r} 1+14+60 \\ \hline \end{array}$$

$$2$$

$$\begin{array}{r} 1+160+6000-56000 \quad | \underline{7} \\ \hline \end{array}$$

$$7+1169+50183$$

$$\begin{array}{r} 1+167+7169-5817 \\ \hline \end{array}$$

$$7+1218$$

$$\begin{array}{r} 1+174+8387 \\ \hline \end{array}$$

$$7$$

$$\begin{array}{r} 1+1810+838700-5817000 \quad | \underline{6} \\ \hline \end{array}$$

$$6+10896+5097576$$

$$\begin{array}{r} 1+1816+849596-719424 \\ \hline \end{array}$$

$$6+10932$$

$$\begin{array}{r} 1+1822+860528 \\ \hline \end{array}$$

$$6$$

$$\begin{array}{r} 18+86053-719424 \quad | \underline{0.008345} \\ \hline \end{array}$$

$$144+689576$$

$$\begin{array}{r} 18+86197-29848 \\ \hline \end{array}$$

$$144$$

$$\begin{array}{r} 8634-29848 \\ \hline \end{array}$$

$$25902$$

$$\begin{array}{r} 863-3946 \\ \hline \end{array}$$

$$3452$$

$$\begin{array}{r} 86-494 \\ \hline \end{array}$$

$$430$$

$$\begin{array}{r} 9-64 \\ \hline \end{array}$$

答：其根爲 2.768345.

6. 解： $2x^3 - x^2 - 9x + 1 = 0.$

$$\begin{array}{r}
 2-1-9+1 \quad | \quad \underline{-1.945341} \\
 \underline{-2+3+6} \\
 2-3-6+7 \\
 \underline{-2+5} \\
 2-5-1 \\
 \underline{-2} \\
 2-70-100+7000 \quad | \quad \underline{-9} \\
 \underline{-18+792-6228} \\
 2-88+692+772 \\
 \underline{-18+954} \\
 2-106+1646 \\
 \underline{-18} \\
 2-1240+164600+772000 \quad | \quad \underline{-4} \\
 \underline{-8+4992-678368} \\
 2-1248+169592+93632 \\
 \underline{-8+5024} \\
 2-1256+174616 \\
 \underline{-8} \\
 2-13+17462+93632 \quad | \quad \underline{-0.005341} \\
 \quad \quad \quad 65-87635 \\
 \underline{-13+17527+5997} \\
 \quad \quad \quad 65 \\
 \quad \quad \quad \underline{1759+5997} \\
 \quad \quad \quad \quad \underline{-5277} \\
 \quad \quad \quad \quad \quad \underline{176+720} \\
 \quad \quad \quad \quad \quad \quad \underline{-704} \\
 \quad \quad \quad \quad \quad \quad \quad \underline{18+16} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \underline{-18} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \underline{2-2}
 \end{array}$$

答：其根爲 -1.945341 。

7. 解： $x^3+x^2-5x-1=0$ 。

$$1+1-5-1 \quad | \quad \underline{1.903211}$$

$$\begin{array}{r}
 1+2-3 \\
 \hline
 1+2-3-4 \\
 1+3 \\
 \hline
 1+3+0 \\
 1 \\
 \hline
 1+40+0-4000 \quad | \quad 9 \\
 9+441+3969 \\
 \hline
 1+49+441-31 \\
 9+522 \\
 \hline
 1+58+963 \\
 9 \\
 \hline
 1+6700+9630000-31000000 \quad | \quad 3 \\
 3+20109+28950327 \\
 \hline
 1+6703+9650109-2049673 \\
 3+20118 \\
 \hline
 1+6706+9670227 \\
 3 \\
 \hline
 67+967023-2049673 \quad | \quad 0.000211 \\
 134+1934314 \\
 \hline
 67+967157-115359 \\
 134 \\
 \hline
 96729-115359 \\
 96729 \\
 \hline
 9673-18630 \\
 9673 \\
 \hline
 -8957
 \end{array}$$

答：其根爲 1.903211.

8. 解： $x^3 - 2x^2 - 23x + 70 = 0$.

$$\begin{array}{r}
 1-2-23+70 \quad | \quad -5.134578 \\
 -5+35-60 \\
 \hline
 1-7+12+10 \\
 -5+60 \\
 \hline
 1-12+72
 \end{array}$$

$$\begin{array}{r}
 - 5 \\
 \hline
 1-170+7200+10000 \quad | -1 \\
 - 1+ 171- 7371 \\
 \hline
 1-171+7371+ 2629 \\
 - 1+ 172 \\
 \hline
 1-172+7543 \\
 - 1 \\
 \hline
 1-1730+754300+2629000 \quad | -3 \\
 - 3+ 5199-2278497 \\
 \hline
 1-1733+759499+ 350503 \\
 - 3+ 5208 \\
 \hline
 1-1736+764707 \\
 - 3 \\
 \hline
 -17+76471+350503 \quad | -0.004578 \\
 68-306156 \\
 \hline
 -17+76539+ 44347 \\
 68 \\
 \hline
 7661+ 44347 \\
 - 38305 \\
 \hline
 766+ 6042 \\
 - 5362 \\
 \hline
 77+ 680 \\
 616 \\
 \hline
 8+ 64
 \end{array}$$

答：其根爲 -5.134578 。

9. 解： $x^4 - 10x^2 - 4x + 8 = 0$ 。

$$\begin{array}{r}
 1-0-10- 4+ 8 \quad | 3.236067 \\
 3+ 9- 3-21 \\
 \hline
 1+3- 1- 7-13 \\
 3+18+51 \\
 \hline
 1+6+17+44 \\
 3+27 \\
 \hline
 1+9+44
 \end{array}$$

3

$$\begin{array}{r} 1+120+4400+44000-130000 \\ \hline \end{array} \quad 2$$

$$2+244+9288+106576$$

$$\begin{array}{r} 1+122+4644+53288-23424 \\ \hline \end{array}$$

$$2+248+9784$$

$$\begin{array}{r} 1+124+4892+63072 \\ \hline \end{array}$$

$$2+252$$

$$\begin{array}{r} 1+126+5144 \\ \hline \end{array}$$

2

$$\begin{array}{r} 1+1280+514400+63072000-234240000 \\ \hline \end{array} \quad 3$$

$$3+3849+1554747+193880241$$

$$\begin{array}{r} 1+1283+518249+64626747-40359759 \\ \hline \end{array}$$

$$3+3858+1566321$$

$$\begin{array}{r} 1+1286+522107+66193068 \\ \hline \end{array}$$

$$3+3867$$

$$\begin{array}{r} 1+1289+525974 \\ \hline \end{array}$$

3

$$\begin{array}{r} 1+5260+6619307-40359759 \\ \hline \end{array} \quad 0.006067$$

$$6+31596+39905418$$

$$\begin{array}{r} 1+5266+6650903-454341 \\ \hline \end{array}$$

$$6+31632$$

$$\begin{array}{r} 1+5272+6682535 \\ \hline \end{array}$$

6

$$\begin{array}{r} 53+668254-454341 \\ \hline \end{array}$$

$$0+0$$

$$\begin{array}{r} 1+66825-454341 \\ \hline \end{array}$$

$$6+400986$$

$$\begin{array}{r} 6683.1-53355 \\ \hline \end{array}$$

$$46781$$

$$\begin{array}{r} 6683--6574 \\ \hline \end{array}$$

答：其根爲 3.236067.

10. 解： $x^4+6x^3+12x^2-11x-41=0.$

$$1+6+12-11-41 \quad \underline{-2.157451}$$

$$\begin{array}{r} -2-8-8+38 \\ \hline 1+4+4-19-3 \end{array}$$

$$\begin{array}{r} -2-4+0 \\ \hline 1+2+0-19 \end{array}$$

$$\begin{array}{r} -2+0 \\ \hline 1+0+0 \end{array}$$

$$\begin{array}{r} -2 \\ \hline 1-20+0-19000-30000 \end{array} \quad -1$$

$$\begin{array}{r} -1+21-21+19021 \\ \hline 1-21+21-19021-10979 \end{array}$$

$$\begin{array}{r} -1+22-43 \\ \hline 1-22+43-19064 \end{array}$$

$$\begin{array}{r} -1+23 \\ \hline 1-23+66 \end{array}$$

$$\begin{array}{r} -1 \\ \hline 1-240+6600-19064000-109790000 \end{array} \quad -5$$

$$\begin{array}{r} -5+1225-39125+95515625 \\ \hline 1-245+7825-19103125-14274375 \end{array}$$

$$\begin{array}{r} -5+1250-45375 \\ \hline 1-250+9075-19148500 \end{array}$$

$$\begin{array}{r} -5+1275 \\ \hline 1-255+10350 \end{array}$$

$$\begin{array}{r} -5 \\ \hline 104-1914850-14274375 \end{array} \quad -0.007451$$

$$-728+13409046$$

$$104-1915578-865329$$

$$-728$$

$$1-191631+865329$$

$$-4+766540$$

$$1-191635-98789$$

$$-4$$

$$-19164-98789$$

$$95820$$

$$-1916-2962$$

1916

192- 1053

答：其根爲 -2.157451 .

11. 解： $x^3 - 3x^2 - 4x + 13 = 0$.

將已知方程式之各根減 2, 所得之新方程式爲

$$x^3 + 3x^2 - 4x + 1 = 0.$$

$$1 + 30 - 400 + 1000 \quad | 3$$

$$3 + 99 - 903$$

$$1 + 33 - 301 + 97$$

$$3 + 108$$

$$1 + 36 - 193$$

$$3$$

$$1 + 390 - 19300 + 97000 \quad | 5$$

$$5 + 1975 - 86625$$

$$1 + 395 - 17325 + 10375$$

$$5 + 2000$$

$$1 + 400 - 15325$$

$$5$$

$$4 - 1533 + 10375 \quad | 0.006885$$

$$24 - 9054$$

$$4 - 1509 + 1321$$

$$24$$

$$- 149 + 1321$$

$$- 1192$$

$$- 15 + 129$$

$$- 120$$

$$- 2 + 9$$

$$- 9$$

$$0$$

$$1 + 30 - 400 + 1000 \quad | 6$$

$$6 + 216 - 1104$$

$$1 + 36 - 184 - 104$$

$$6 + 252$$

$$1 + 42 + 68$$

$$\begin{array}{r}
 6 \\
 \hline
 1+480+6300-104000 \quad | \quad 9 \\
 9+4401+100809 \\
 \hline
 1+489+11201-3191 \\
 9+4482 \\
 \hline
 1+498+15683 \\
 9
 \end{array}$$

$$\begin{array}{r}
 5+1568-3191 \quad | \quad 0.002021 \\
 \hline
 10+3156 \\
 \hline
 5+1578-35 \\
 10 \\
 \hline
 159-35 \\
 0 \\
 \hline
 16-35 \\
 32 \\
 \hline
 2-3 \\
 2 \\
 \hline
 -1
 \end{array}$$

答：其二根爲 2.356885 與 2.692021。

12. 解： $x^3 - 3x^2 - 4x + 10 = 0$

此方程式僅能有二正根與一負根。

$$f(0) = 10, f(1) = 4, f(2) = -2, f(3) = -2, f(4) = 10;$$

$$f(0) = 10, f(-1) = 10, f(-2) = -2.$$

故此三根位於 1 與 2, 3 與 4, -1 與 -2 之間。

(1) $1-3-4+10 \quad | \quad 1.602$

$$\begin{array}{r}
 1-2-6 \\
 \hline
 1-2-6+4 \\
 1-1 \\
 \hline
 1-1-7 \\
 1 \\
 \hline
 1+0-700+4000 \quad | \quad 6 \\
 6+36-3984 \\
 \hline
 1+6-664+16
 \end{array}$$

$$\begin{array}{r} 6+72 \\ \hline 1+12-592 \end{array}$$

$$\begin{array}{r} 6 \\ \hline 1+1800-5920000+16000000 \end{array} \quad | 2$$

$$\begin{array}{r} 2+3604-11832792 \\ \hline 1+1802-5916396+4167208 \end{array}$$

$$\begin{array}{r} 2+3608 \\ \hline 1+1804+5912788 \end{array}$$

$$\begin{array}{r} 2 \\ \hline 1+1806 \end{array}$$

$$(2) \quad 1-3-4+10 \quad | 3.292$$

$$\begin{array}{r} 3+0-12 \\ \hline 1+0-4-2 \end{array}$$

$$\begin{array}{r} 3+9 \\ \hline 1+3+5 \end{array}$$

$$\begin{array}{r} 3 \\ \hline 1+60+500-2000 \end{array} \quad | 2$$

$$\begin{array}{r} 2+124+1248 \\ \hline 1+62+624-752 \end{array}$$

$$\begin{array}{r} 2+128 \\ \hline 1+64+752 \end{array}$$

$$\begin{array}{r} 2 \\ \hline 1+660+75200-752000 \end{array} \quad | 9$$

$$\begin{array}{r} 9+6021+730989 \\ \hline 1+669+81221-21011 \end{array}$$

$$\begin{array}{r} 9+6102 \\ \hline 1+678+87323 \end{array}$$

$$x = \frac{21011}{87323} = 0.2 +$$

$$\begin{array}{r} 9 \\ \hline 1+687 \end{array}$$

$$(3) \quad 1-3-4+10 \quad | -1.895$$

$$\begin{array}{r} -1+4+0 \\ \hline 1-4+0+10 \end{array}$$

$$\begin{array}{r} -1+5 \\ \hline 1-5+5 \end{array}$$

$$\begin{array}{r}
 -1 \\
 \hline
 1-60+500+10000 \quad | -8 \\
 -8+544-8352 \\
 \hline
 1-68+1044+1648 \\
 -8+608 \\
 \hline
 1-76+1652 \\
 -8 \\
 \hline
 1-840+165200+1648000 \quad | -9 \\
 -9+7641-1555569 \\
 \hline
 1-849+172841+92431 \\
 -9+7722 \\
 \hline
 1-858+180563 \\
 -9 \\
 \hline
 1-867
 \end{array}$$

$$x = \frac{-92431}{180563} = -0.512$$

答：其根為 1.602, 3.292 與 -1.895。

13. 解： $x^3+x^2-2x-1=0$ 。

此方程式正根不能超過一個，負根不能超過二個。

$$f(0) = -1, f(1) = -1, f(2) = 7;$$

$$f(0) = -1, f(-1) = 1, f(-2) = -1.$$

故其根位於 1 與 2, 0 與 -1, -1 與 -2 之間。

$$(1) \quad 1+1-2-1 \quad | \quad 1.246$$

$$\quad 1+2+0$$

$$\hline 1+2+0-1$$

$$\quad 1+3$$

$$\hline 1+3+3$$

$$\quad 1$$

$$\hline 1+40+300-1000 \quad | \quad 2$$

$$\quad 2+84+768$$

$$\hline 1+42+384-232$$

$$\quad 2+88$$

$$\hline 1+44+472$$

$$\quad 2$$

$$\hline 1+460+47200-232000 \quad | \quad 4$$

$$\begin{array}{r} 4 + 1856 + 196224 \\ \hline 1 + 464 + 49056 - 35776 \end{array}$$

$$\begin{array}{r} 4 + 1872 \\ \hline 1 + 468 + 50928 \end{array}$$

$$\begin{array}{r} 4 \\ \hline 1 + 472 \end{array}$$

$$x = \frac{35776}{50928} = 0.6 +$$

$$(2) \quad 1 + 1 - 2 - 1 \quad | \quad -0.445$$

$$\begin{array}{r} 0 + 0 + 0 \\ \hline 1 + 10 - 200 - 1000 \quad | \quad -4 \end{array}$$

$$\begin{array}{r} - 4 - 24 + 806 \\ \hline 1 + 6 - 224 - 104 \end{array}$$

$$\begin{array}{r} - 4 - 8 \\ \hline 1 + 2 - 232 \end{array}$$

$$\begin{array}{r} - 4 \\ \hline 1 - 20 - 23200 - 104000 \quad | \quad -4 \end{array}$$

$$\begin{array}{r} - 4 + 96 + 92416 \\ \hline 1 - 24 - 23104 - 11584 \end{array}$$

$$\begin{array}{r} 4 + 112 \\ \hline 1 - 28 - 22992 \end{array}$$

$$\begin{array}{r} - 4 \\ \hline 1 - 32 \end{array}$$

$$x = \frac{-11584}{22992} = -0.5 +$$

$$(3) \quad 1 + 1 - 2 - 1 \quad | \quad -1.802$$

$$\begin{array}{r} - 1 + 0 + 2 \\ \hline 1 + 0 - 2 + 1 \end{array}$$

$$\begin{array}{r} - 1 + 1 \\ \hline 1 - 1 - 1 \end{array}$$

$$\begin{array}{r} - 1 \\ \hline 1 - 20 - 100 + 1000 \quad | \quad -8 \end{array}$$

$$\begin{array}{r} - 8 + 224 - 992 \\ \hline 1 - 28 + 124 + 8 \end{array}$$

$$\begin{array}{r} - 8 + 288 \\ \hline 1 - 36 + 412 \end{array}$$

$$\begin{array}{r} -8 \\ \hline 1-440+41200+8000 \quad | -0 \\ \quad 0+ \quad 0+ \quad 0 \\ \hline 1-4400+412000+80000 \end{array} \quad \frac{-80000}{412000} = -0.2$$

答：其根為 1.246, -0.445, -1.802.

14. 解： $x^3 - 3x + 1 = 0$.

此方程式正根不能超過二個，負根不能超過一個。

$$f(0) = 1, f(1) = -1, f(2) = 3;$$

$$f(0) = 1, f(-1) = 3, f(-2) = -1.$$

故其根位於 0 與 1, 1 與 2, -1 與 -2 之間。

(1) $1+0-3+1 \quad | \underline{0.347}$

$$\begin{array}{r} 0+ \quad 0+ \quad 0 \\ \hline 1+0-300+1000 \quad | \underline{3} \end{array}$$

$$\begin{array}{r} 3+ \quad 9- \quad 873 \\ \hline 1+3-291+ \quad 127 \end{array}$$

$$\begin{array}{r} 3+ \quad 18 \\ \hline 1+6-273 \end{array}$$

3

$$\begin{array}{r} 1+ \quad 90-27300+127000 \quad | \underline{4} \\ \quad 4+ \quad 376-107696 \end{array}$$

$$\begin{array}{r} 1+ \quad 94-26924+ \quad 19304 \\ \quad 4+ \quad 392 \end{array}$$

$$\begin{array}{r} 1+ \quad 98-26532 \\ \quad 4 \end{array}$$

$$x = \frac{19304}{26532} = 0.7+$$

$$1+102$$

(2) $1+0-3+1 \quad | \underline{1.532}$

$$\begin{array}{r} 1+1-2 \\ \hline 1+1-2-1 \end{array}$$

$$\begin{array}{r} 1+2 \\ \hline 1+2+0 \end{array}$$

1

$$\begin{array}{r} 1+30+ \quad 0-1000 \quad | \underline{5} \\ \quad 5+175+875 \end{array}$$

$$1+35+175-125$$

$$\begin{array}{r} 5+200 \\ 1+40+375 \end{array}$$

5

$$\begin{array}{r} 1+450+37500-125000 \quad | \quad 3 \\ 3+1359-116577 \end{array}$$

$$\begin{array}{r} 1+453+38859-8423 \end{array}$$

$$\begin{array}{r} 3+1368 \\ 1+456+40227 \end{array}$$

$$x = \frac{8423}{40227} = 0.21$$

$$\begin{array}{r} 3 \\ 1+459 \end{array}$$

$$(3) \quad 1+0-3+1 \quad | \quad -1.879$$

$$\begin{array}{r} -1+1+2 \\ 1-1-2+3 \end{array}$$

$$\begin{array}{r} -1+2 \\ 1-2+0 \end{array}$$

$$\begin{array}{r} -1 \\ 1-30+0+3000 \quad | \quad -8 \end{array}$$

$$\begin{array}{r} -8+304-2432 \\ 1-38+304+568 \end{array}$$

$$\begin{array}{r} -8+368 \\ 1-46+672 \end{array}$$

$$\begin{array}{r} -8 \\ 1-540+67200+568000 \quad | \quad -7 \end{array}$$

$$\begin{array}{r} -7+3829-497203 \\ 1-547+71029+70797 \end{array}$$

$$\begin{array}{r} -7+3878 \\ 1-554+74907 \end{array}$$

$$x = \frac{-70797}{74907} = -0.94$$

$$\begin{array}{r} -7 \\ 1-561 \end{array}$$

答：其根為 0.347, 1.532; -1.879.

15. 解： $x^4 + 5x^3 + x^2 - 13x - 7 = 0$.

此方程式之正根不能超過一個，負根不能超過三個。

$$f(0) = -7, f(1) = -13, f(2) = 27;$$

$$f(0) = -7, f(-1) = 3, f(-2) = -1, f(-3) = -13, \\ f(-4) = -3, f(-5) = 83.$$

故其根位於 1 與 2, 0 與 -1, -1 與 -2, -4 與 -5 之間。

$$(1) \quad 1+5+1-13-7 \quad \underline{1.558}$$

$$\quad \quad \quad 1+6+7-6$$

$$\underline{1+6+7-6-13}$$

$$\quad \quad \quad 1+7+14$$

$$\underline{1+7+14+8}$$

$$\quad \quad \quad 1+8$$

$$\underline{1+8+22}$$

$$\quad \quad \quad 1$$

$$\underline{1+90+2200+8000-130000} \quad \underline{5}$$

$$\quad \quad \quad 5+475+13375+106875$$

$$\underline{1+95+2675+21375-23125}$$

$$\quad \quad \quad 5+500+15875$$

$$\underline{1+100+3175+37250}$$

$$\quad \quad \quad 5+525$$

$$\underline{1+105+3700}$$

$$\quad \quad \quad 5$$

$$\underline{1+1100+370000+37250000-231250000} \quad \underline{5}$$

$$\quad \quad \quad 5+5525+1877625+195638125$$

$$\underline{1+1105+375525+39127625-35611875}$$

$$\quad \quad \quad 5+5550+1905375$$

$$\underline{1+1110+381075+41033000}$$

$$\quad \quad \quad 5+5575$$

$$\underline{1+1115+386650}$$

$$\quad \quad \quad 5$$

$$\underline{1+1120}$$

$$x = \frac{35611875}{41033000} = 0.87$$

$$(2) \quad 1+50+100-13000-70000 \quad \underline{-0.578}$$

$$\quad \quad \quad -5-225+625+61875$$

$$\underline{1+45-125-12375-8125}$$

$$\quad \quad \quad -5-200+1625$$

$$\underline{1+40-325-10750}$$

$$\begin{array}{r}
 - 5 - 175 \\
 \hline
 1 + 35 - 500 \\
 - 5 \\
 \hline
 1 + 300 - 50000 - 10750000 - 81250000 \quad | - 7 \\
 - 7 - 2051 + 364357 + 72699501 \\
 \hline
 1 + 293 - 52051 - 10385643 + 8550499 \\
 - 7 - 2002 + 378371 \\
 \hline
 1 + 286 - 54053 - 10007272 \\
 - 7 - 1953 \\
 \hline
 1 + 279 - 56006 \\
 - 7 \\
 \hline
 1 + 272
 \end{array}$$

$$x = \frac{-8550499}{10007272} = -0.8 +$$

$$(3) \quad 1 + 5 + 1 - 13 - 7 \quad | \quad -1.904$$

$$\begin{array}{r}
 - 1 - 4 + 3 + 10 \\
 \hline
 1 + 4 - 3 - 10 + 3 \\
 - 1 - 3 + 6 \\
 \hline
 1 + 3 - 6 - 4 \\
 - 1 - 2 \\
 \hline
 1 + 2 - 8 \\
 - 1
 \end{array}$$

$$\begin{array}{r}
 1 + 10 - 800 - 4000 + 30000 \quad | - 9 \\
 - 9 - 9 + 7281 - 29529 \\
 \hline
 1 + 1 - 809 + 3281 + 471 \\
 - 9 + 72 + 6633 \\
 \hline
 1 - 8 - 737 + 9914 \\
 - 9 + 153 \\
 \hline
 1 - 17 - 584 \\
 - 9 \\
 \hline
 1 - 26
 \end{array}$$

$$x = \frac{-471}{9914} = -0.04$$

$$(4) \quad 1 + 5 + 1 - 13 - 7 \quad | \quad -4.075$$

$$\begin{array}{r}
 - 4 - 4 + 12 + 4 \\
 \hline
 1 + 1 - 3 - 1 - 3 \\
 - 4 + 12 - 36 \\
 \hline
 1 - 3 + 9 - 37
 \end{array}$$

$$\begin{array}{r}
 -4+28 \\
 \hline
 1-7+37 \\
 -4 \\
 \hline
 1-1100+370000-37000000-300000000 \quad | -7 \\
 -7+7749-2644243+277509701 \\
 \hline
 1-1107+377749-39644243-22490299 \\
 -7+7798-2698829 \\
 \hline
 1-1114+385547-42343072 \\
 -7+7847 \\
 \hline
 1-1121+393394 \\
 -7 \\
 \hline
 1-1128
 \end{array}$$

$$x = \frac{-22490299}{42343072} = -0.5+$$

答：其根爲 1.558, -0.578, -1.904, -4.075.

16. 解： $x^3 - 17 = 0$.

$$f(0) = -17, f(1) = -16, f(2) = -9, f(3) = 10.$$

故其根位於 2 與 3 之間。

$$\begin{array}{r}
 1+0+0-17 \quad | 2.5712 \\
 2+4+8 \\
 \hline
 1+2+4-9 \\
 2+8 \\
 \hline
 1+4+12 \\
 2 \\
 \hline
 1+60+1200-9000 \quad | 5 \\
 5+325+7625 \\
 \hline
 1+65+1525-1375 \\
 5+350 \\
 \hline
 1+70+1875 \\
 5 \\
 \hline
 1+750+187500-1375000 \quad | 7 \\
 7+5299+1349593 \\
 \hline
 1+757+192799-25407 \\
 7+5348 \\
 \hline
 1+764+198147
 \end{array}$$

$$\begin{array}{r}
 7 \\
 \hline
 8+19815-25407 \quad |1 \\
 \quad 8+19823 \\
 \hline
 8+19823-5584 \\
 \quad 8 \\
 \hline
 1983-5584
 \end{array}
 \quad x = \frac{5584}{19831} = 0.2 +$$

答: $\sqrt[3]{17} = 2.5712$

17. 解: (1) $x^3 - 24 = 0$; 其根位於 2 與 3 之間。

$$1+0+0-24 \quad |2.884$$

$$\quad 2+4+8$$

$$\hline 1+2+4-16$$

$$\quad 2+8$$

$$\hline 1+4+12$$

$$\quad 2$$

$$\hline 1+60+1200-16000 \quad |8$$

$$\quad 8+544+13952$$

$$\hline 1+68+1744-2048$$

$$\quad 8+608$$

$$\hline 1+76+2352$$

$$\quad 8$$

$$\hline 1+840+235200-2048000 \quad |8$$

$$\quad 8+6784+1935872$$

$$\hline 1+848+241984-112128$$

$$\quad 8+6848$$

$$\hline 1+856+248832$$

$$\quad 8$$

$$\hline 1+864$$

$$x = \frac{112128}{248832} = 0.4 +$$

答: $2\sqrt[3]{5} = 2.884$.

- (2) $x^4 - 87 = 0$; 其根位於 3 與 4 之間。

$$1+0+0+0-87 \quad |3.054$$

$$\quad 3+9+27+81$$

$$\hline 1+3+9+27-6$$

$$\begin{array}{r}
 3+18+81 \\
 \hline
 1+6+27+108 \\
 3+27 \\
 \hline
 1+9+54 \\
 3 \\
 \hline
 6+1200+540000+108000000-600000000 \quad |5 \\
 5+6025+2730125+553650625 \\
 \hline
 1+1205+546025+110730125-46349375 \\
 5+6050+1760375 \\
 \hline
 1+1210+552075+112490500 \\
 5+6075 \\
 \hline
 1+1215+558150 \\
 5 \\
 \hline
 1+1220
 \end{array}$$

$$x = \frac{46349375}{112490500} = 0.4 +$$

18. 答: $\sqrt[4]{87} = 3.054.$
 解: $x^3 + x^2 - 2500 = 0.$

$$f(0) = -2500, f(10) = -1400, f(20) = 5900.$$

$$1+1-0-2500 \quad |10$$

$$10+110+1100$$

$$1+11+110-1400$$

$$10+210$$

$$1+21+320$$

$$10$$

$$1+31+320-1400 \quad |3$$

$$3+102+1266$$

$$1+34+422-134$$

$$3+111$$

$$1+37+533$$

$$3$$

$$1+400+53300-134000 \quad |2$$

$$2+804+108208$$

$$1+402+54104-25792$$

$$2+808$$

$$1+404+54912$$

$$f(3) = -134$$

$$f(4) = 440$$

$$x = \frac{134}{533} = 0.2 +$$

$$\frac{2}{1+406}$$

$$x = \frac{25792}{54912} = 0.4+$$

答：其實根為 13.24.

19. 解： $x^3 + 5x^2 - 6x + 1 = 0$.

依笛卡兒法則，知此方程式有二正根與一負根。

$$f(0) = 1, f(1) = 1, f(2) = 17 \dots\dots$$

依 § 803, 知 1 為此方程式之上限，故該方程式如有正根在 0 與 1 之間，至少必有二正根。由計算，知 $f(0.2) = +$, $f(0.3) = -$, $f(0.4) = -$, ..., $f(0.8) = -$, $f(0.9) = +$; 故知該方程式有兩正根，一位於 0.2 與 0.3 之間，一位於 0.8 與 0.9 之間，又 $f(-6) = +$, $f(-7) = -$, 故 -6 與 -7 間有一負根。

20. 解： $3x^5 + x^4 - 14x^3 - x^2 + 9x - 2 = 0$.

$$3+1-14-1+9-2 \quad \left| \frac{2}{3} \right.$$

$$2+2-8-6+2$$

$$3+3-12-9+3, 0$$

$$1+1-4-3+1 \quad \left| -1 \right.$$

$$-1+0+4-1$$

$$1+0-4+1, 0$$

$$1+0-400+1000 \quad \left| 0.254 \right.$$

$$2+4-792$$

$$1+2-396+208$$

$$2+8$$

$$1+4-388$$

$$2$$

$$1+60-38800+208000 \quad \left| 5 \right.$$

$$5+325-192375$$

$$1+65-38475+15625$$

$$5+450$$

$$1+70-37925$$

$$5$$

$$1+75$$

$$1+0-4+1 \quad \left| 1.8608 \right.$$

$$1+1-3$$

$$1+1-3-2$$

$$\phi(x) = x^3 - 4x + 1 = 0,$$

$$\phi(0) = +, \phi(1) = -,$$

$$\phi(2) = +, \phi(-2) = +,$$

$$\phi(-3) = -$$

\therefore 0 與 1 間, 1 與 2 間, -2 與 -3 間有根。

$$x = \frac{15625}{37925} = 0.4+$$

$$\begin{array}{r} 1+2 \\ 1+2-1 \\ 1 \\ \hline 1+30-100-2000 \quad | \underline{8} \end{array}$$

$$\begin{array}{r} 8+304+1632 \\ 1+38+204-368 \\ 8+368 \\ \hline 1+46+572 \end{array}$$

$$\begin{array}{r} 8 \\ \hline 1+540+57200-368000 \quad | \underline{6} \end{array}$$

$$\begin{array}{r} 6+3276+362856 \\ 1+546+60476-5144 \\ 6+3312 \\ \hline 1-552+63788 \end{array}$$

$$\begin{array}{r} 6 \\ \hline 1+558 \end{array}$$

$$x = \frac{5144}{63788} = 0.08$$

$$1+0-4+1 \quad | \quad \underline{-2.114}$$

$$\begin{array}{r} -2+4+0 \\ \hline 1-2+0+1 \end{array}$$

$$\begin{array}{r} -2+8 \\ \hline 1-4+8 \end{array}$$

$$\begin{array}{r} -2 \\ \hline 1-60+800+1000 \quad | \quad \underline{-1} \end{array}$$

$$\begin{array}{r} -1+61-861 \\ \hline 1-61+861+139 \end{array}$$

$$\begin{array}{r} -1-62 \\ \hline 1-62+923 \end{array}$$

$$\begin{array}{r} -1 \\ \hline 1-630+92300+139000 \quad | \quad \underline{-1} \end{array}$$

$$\begin{array}{r} -1+631-92931 \\ \hline 1-631+92931+46069 \end{array}$$

$$\begin{array}{r} -1+632 \\ \hline 1-632+93563 \end{array}$$

$$\frac{-1}{1-633}$$

$$x = \frac{-46069}{93563} = -0.4$$

答：其根爲 $\frac{2}{3}$, -1 , 0.254 , 1.860 , -2.114 .

習 題 LXXVI

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1. 解： $2x^5 - 4x^4 + x^2 - 20x$.

$$f'(x) = 10x^4 - 16x^3 + 2x - 20$$

$$f''(x) = 40x^3 - 48x^2 + 2$$

$$f'''(x) = 120x^2 - 96x$$

$$f^{iv}(x) = 240x - 96$$

$$f^v(x) = 240$$

2. 解： $f(x) = x^4 - 2x^3 + 1$.

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f^{iv}(x)\frac{h^4}{4!}$$

$$= (x^4 - 2x^3 + 1) + (4x^3 - 6x^2)h + (12x^2 - 12x)\frac{h^2}{2!}$$

$$+ (24x - 12)\frac{h^3}{3!} + 4\frac{h^4}{4!}$$

$$= (x^4 - 2x^3 + 1) + 2(2x^3 - 3x^2)h + 6(x^2 - x)h^2$$

$$+ 2(2x - 1)h^3 + h^4.$$

3. 解： $f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!}$

$$+ \dots + f^n(a)\frac{(x-a)^n}{n!}.$$

(1) $f(x) = x^4 + x^2 + 1$, $f'(x) = 4x^3 + 2x$, $f''(x) = 12x^2 + 2$,

$$f'''(x) = 24x, \quad f^{iv}(x) = 24;$$

$$f'(-1) = 3, \quad f''(-1) = -6, \quad f'''(-1)/2! = 7,$$

$$f^{iv}(-1)/3! = -4, \quad f^{v}(-1)/4! = 1.$$

故 $x^4 + x^2 + 1 = 3 - 6(x+1) + 7(x+1)^2 - 4(x+1)^3 + (x+1)^4$.

(2) $f(x) = x^5 - 32$, $f'(x) = 5x^4$, $f''(x) = 20x^3$,

$$f'''(x) = 60x^2, \quad f^{iv}(x) = 120x, \quad f^v(x) = 120;$$

$$f(2) = 0, \quad f'(2) = 80, \quad f''(2)/2! = 80, \quad f'''(2)/3! = 40,$$

$$f^{iv}(2)/4! = 10, \quad f^v(2)/5! = 1.$$

$$\text{故 } x^5 - 32 = 80(x-2) + 80(x-2)^2 + 40(x-2)^3 + 10(x-2)^4 + (x-2)^5.$$

$$(3) \quad f(x) = x^3 + 1, \quad f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6;$$

$$F(x) = x^2 + 1, \quad F'(x) = 2x, \quad F''(x) = 2;$$

$$f(1) = 2, \quad f'(1) = 3, \quad f''(1)/2! = 3, \quad f'''(1)/3! = 1;$$

$$F(1) = 2, \quad F'(1) = 2, \quad F''(1)/2! = 1.$$

$$\text{故 } \frac{x^3 + 1}{x^2 + 1} = \frac{2 + 3(x-1) + 3(x-1)^2 + (x-1)^3}{2 + 2(x-1) + (x-1)^2}.$$

$$4. \text{ 解: (1) } x^3 - 3x - 2 = 0.$$

$$f(x) = x^3 - 3x - 2 = 0$$

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1) = 0$$

故其最高公因式 $\phi(x) = x + 1 = 0$

是以 $f(x) = 0$, 有 -1 之二重根與另一根為 $0 - 2(-1)$ 或 2 ; 即 $-1, -1, 2$.

$$(2) \quad 9x^3 + 12x^2 - 11x + 2 = 0.$$

$$f(x) = 9x^3 + 12x^2 - 11x + 2 = 0$$

$$f'(x) = 27x^2 + 24x - 11 = 0$$

$$\phi(x) = 3x - 1 = 0$$

故 $f(x) = 0$, 有 $\frac{1}{3}$ 之二重根與另一根 $\left(-\frac{12}{9} - \frac{2}{3}\right)$ 或 -2 ;

$$\text{即 } \frac{1}{3}, \frac{1}{3}, -2.$$

$$(3) \quad 4x^4 + 12x^2 + 9 = 0.$$

$$f(x) = 4x^4 + 12x^2 + 9 = 0$$

$$f'(x) = 16x^3 + 24x = 0$$

$$\phi(x) = 4x^2 + 6 = 0$$

故 $f(x) = 0$, 有 $\pm \frac{\sqrt{6}i}{2}$ 之二重根; 即 $\pm \frac{\sqrt{6}i}{2}, \pm \frac{\sqrt{6}i}{2}$.

$$(4) \quad x^4 - 4x^3 + 8x + 4 = 0.$$

$$f(x) = x^4 - 4x^3 + 8x + 4 = 0$$

$$f'(x) = 4x^3 - 12x^2 + 8 = 0$$

$$\phi(x) = x^2 - 2x - 2 = 0$$

故 $f(x) = 0$, 有 $1 \pm \sqrt{3}$ 之二重根; 即 $1 \pm \sqrt{3}, 1 \pm \sqrt{3}$.

$$(5) \quad 2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0.$$

$$f(x) = 2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$$

$$f'(x) = 8x^3 - 36x^2 + 38x - 6 = 0$$

$$\phi(x) = x - 3 = 0$$

故 $f(x) = 0$, 有 3 之二重根, 其餘二根爲 $2x^2 + 1 = 0$ 之根
即 $\pm \frac{\sqrt{2}i}{2}$.

$$(6) \quad x^5 - x^3 - 4x^2 - 3x - 2 = 0.$$

$$f(x) = x^5 - x^3 - 4x^2 - 3x - 2 = 0$$

$$f'(x) = 5x^4 - 3x^2 - 8x - 3 = 0$$

$$\phi(x) = x^2 + x + 1 = 0$$

故 $f(x) = 0$, 有 $(-1 \pm \sqrt{3}i)/2$ 之二重根, 其餘一根爲
 $-2 \left(\frac{-1 + \sqrt{3}i - 1 - \sqrt{3}i}{2} \right)$ 即 2.

$$(7) \quad x^4 - 2x^3 - x^2 - 4x + 12 = 0.$$

$$f(x) = x^4 - 2x^3 - x^2 - 4x + 12 = 0$$

$$f'(x) = 4x^3 - 6x^2 - 2x - 4 = 0$$

$$\phi(x) = x - 2 = 0$$

故 $f(x) = 0$, 有 2 之二重根, 其餘二根爲 $-1 \pm \sqrt{2}i$.

$$(8) \quad x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = 0.$$

$$f(x) = x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = 0$$

$$f'(x) = 5x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$$

$$\phi(x) = (x-1)^2(x+1) = 0$$

故 $f(x) = 0$, 有 1 之三重根, -1 之二重根; 即 1, 1, 1, -1 , -1 .

$$(9) \quad 3x^5 - 2x^4 + 6x^3 - 4x^2 + 3x - 2 = 0.$$

$$f(x) = 3x^5 - 2x^4 + 6x^3 - 4x^2 + 3x - 2 = 0$$

$$f'(x) = 15x^4 - 8x^3 + 18x^2 - 8x + 3 = 0$$

$$\phi(x) = x^2 + 1 = 0$$

故 $f(x) = 0$, 有二重根 $\pm i$, 其餘一根爲 $\frac{2}{3}$.

5. 證: $x^n - a^n = 0$.

$$f(x) = x^n - a^n = 0$$

$$f'(x) = nx^{n-1} = 0$$

$$\phi(x) = 1$$

此方程式與其第一次導來式皆無公因式，故 $f(x)=0$ 不能有重根。

6. 解: $x^3 - 12x + a = 0.$

$$f(x) = x^3 - 12x + a = 0$$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 0$$

$$x^2 = 4 \quad \therefore x = \pm 2$$

將 x 之值代入 $f(x) = 0$, 得

$$\pm 8 \mp 24 + a = 0$$

$$\therefore a = \pm 16.$$

7. 解: $3x^3 + ax^2 + x + b = 0.$

$$f(x) = 3x^3 + ax^2 + x + b = 0$$

$$f'(x) = 9x^2 + 2ax + 1 = 0$$

$$f''(x) = 18x + 2a = 2(9x + a) = 0$$

$$x = -\frac{a}{9}$$

(1)

代入 $f'(x)$, 得 $a^2 = 9 \quad \therefore a = \pm 3.$

代入 (1) 式, 得 $x = \mp \frac{3}{9} = \mp \frac{1}{3}.$

將 a 與 x 之值代入 $f(x)$, 得

$$\mp \frac{3}{27} \pm \frac{3}{9} \mp \frac{1}{3} + b = 0$$

$$\therefore b = \pm \frac{1}{9}.$$

故 a 與 b 之值為 $3, \frac{1}{9}$ 或 $-3, -\frac{1}{9}$; 其根為 $-\frac{1}{3}$ 或 $\frac{1}{3}$.

8. 證: $x^4 + qx^2 + s = 0.$

$$f(x) = x^4 + qx^2 + s = 0$$

$$f'(x) = 4x^3 + 2qx = 2x(2x^2 + q) = 0$$

$f'(x) = 0$ 之根為 $0, \sqrt{\frac{q}{2}}, -\sqrt{\frac{q}{2}}$, 故 $f'(x) = 0$ 無二重根。是

以 $f(x) = 0$ 不能有一三重根。

9. 解: $x^5 - px^2 + r = 0.$

$$f(x) = x^5 - px^2 + r = 0$$

$$f'(x) = 5x^4 - 2px = x(5x^3 - 2p) = 0$$

$$\therefore x = \sqrt[3]{\frac{2p}{5}}$$

代入 $f(x) = 0$, 得 $\frac{2p^3}{5} \sqrt[3]{\left(\frac{2p}{5}\right)^2} - p \sqrt[3]{\left(\frac{2p}{5}\right)^2} + r = 0$

$$\sqrt[3]{\left(\frac{2p}{5}\right)^2} \left[\frac{2p}{5} - p \right] = -r$$

$$\left(\frac{2p}{5}\right)^2 \left(-\frac{3p}{5}\right) = -r^3$$

$$\frac{-108p^5}{3125} = -r^3$$

$$\therefore 108p^5 = 3125r^3$$

10. 解: 設 $f(x) = a_0(x^n + b_1x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \dots + b_{n-1}x + b_n) \dots \dots \dots (1)$

$$f'(x) = a_0[nx^{n-1} + b_1(n-1)x^{n-2} + b_2(n-2)x^{n-3} + \dots + b_{n-1}] \dots \dots \dots (2)$$

如 $f(x)$ 能被 $f'(x)$ 除盡, 則

$$kf(x) = (x+b)f'(x) \dots \dots \dots (3)$$

將 (3) 兩邊 x 同次方項的係數相等, 得

$$k = n$$

$$kb_1 = nb + b_1(n-1) \quad \therefore b_1 = nb$$

$$kb_2 = bb_1(n-1) + b_2(n-2) \quad \therefore b_2 = \frac{(n-1)b}{2} b_1 = \frac{n(n-1)}{2} b^2$$

$$kb_3 = bb_2(n-2) + b_3(n-3) \quad \therefore b_3 = \frac{(n-2)b}{3} b_2 = \frac{n(n-1)(n-2)}{3} b^3$$

$$kb_n = bb_{n-1} \quad \therefore b_n = \frac{n}{n-1} b^n = b^n$$

將 b_1, b_2, \dots, b_n 之值代入 (1) 式, 得

$$f(x) = a_0 \left[x^n + nbx^{n-1} + \frac{n(n-1)}{2} b^2 x^{n-2} + \dots + b^n \right] = a_0(x+b)^n \equiv (a'x+b')^n$$

11. 解: $x^4 + x^3 + 2x^2 + x + 1 = 0$ 及 $x^4 + x^3 - x - 1 = 0$
 $f_1(x) = x^4 + x^3 + 2x^2 + x + 1 = 0$
 $f_2(x) = x^4 + x^3 - x - 1 = 0$
 $\phi(x) = x^2 + x + 1 = 0 \dots\dots\dots (1)$

故其公共根爲 $\frac{-1 \pm \sqrt{3}i}{2}$.

以 (1) 式除此二方程式, 得下列二式:

$$x^2 + 1 = 0 \dots\dots\dots (2)$$

$$x^2 - 1 = 0 \dots\dots\dots (3)$$

從 (2) 式, 得 $x = \pm i$; 從 (3) 式, 得 $x = \pm 1$.

故第一方程式之根爲 $\frac{-1 \pm \sqrt{3}i}{2}, \pm i$;

第二方程式之根爲 $\frac{-1 \pm \sqrt{3}i}{2}, \pm 1$.

12. 解: $x^3 - 20x - 16 = 0, x^3 - x^2 - 3x - 1 = 0$.

$$f_1(x) = x^3 - 20x - 16 = 0$$

$$f_2(x) = x^3 - x^2 - 3x - 1 = 0$$

以 2 乘 $f_2(x) = 0$ 之各根, 得一方程式

$$f_3(x) = x^3 - 2x^2 - 12x - 8 = 0.$$

$f_1(x) = 0$ 與 $f_3(x) = 0$ 之 H. C. F. 爲 $\phi(x) = x^2 - 4x - 4 = 0$

$$\therefore x = 2 \pm \sqrt{4+4} = 2 \pm 2\sqrt{2}$$

故 $f_1(x) = 0$ 之根爲 $2 \pm 2\sqrt{2}$ 與 -4 ,

$f_2(x) = 0$ 之根爲 $1 \pm \sqrt{2}$ 與 -1 .

13. 證: 設該方程有一無理重根 $a + \sqrt{b}$, 則 $a - \sqrt{b}$ 亦必爲其重根, 因方程之係數爲有理也; 如是該三次方程將有 4 個根, 此爲不合理; 故該方程如有重根, 則該重根必爲有理者.

14. 證: 設該方程有一重根, 且爲無理者, 如 $a + \sqrt{b}$, 則 $a - \sqrt{b}$ 亦必爲其重根; 故 $f(x)$ 必成爲 $(x - a + \sqrt{b})^2(x - a - \sqrt{b})^2$, 即 $[(x - a)^2 - b]^2$ 爲一完全平方. 故 $f(x) = 0$, 如 $f(x)$ 不爲一完全平方, 不得有無理重根.

15. 證: 由 849 節, 知 $f(x)$ 可化爲

$$\begin{aligned}
 f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\
 &+ \dots + \frac{f^{r-1}(a)}{(r-1)!}(x-a)^{r-1} + \frac{f^r(a)}{r!}(x-a)^r \\
 &+ \dots + \frac{f^n(a)}{n!}(x-a)^n + \dots
 \end{aligned}$$

如 $f(x) = 0$ 有一 r 次重根 a , 則 $f(x)$ 必為 $(x-a)^r$ 除盡。

$\therefore f(a) = f'(a) = f''(a) = \dots = f^{r-1}(a) = 0$

即 a 為 $f(x) = 0, f'(x) = 0, \dots, f^{r-1}(x) = 0$ 之根。

習 題 LXXVII

原本第 471 頁

1. 解: $f(x) = (x+1)(x-2)^2 = x^3 - 3x^2 + 4$.

當 $x = -\infty, f(x) = -\infty$; 當 x 位於 $-\infty$ 與 -1 之間, $f(x)$ 為負; 當 $x = -1, f(x) = 0$; 當 x 位於 -1 與 2 之間, $f(x)$ 為正; 當 $x = 2, f(x) = 0$; 當 x 位於 2 與 ∞ 之間, $f(x)$ 為正; 當 $x = \infty, f(x) = \infty$.

$f'(x) = 3x^2 - 6x$ 之根為 0 與 2 , 當 $x < 0$ 與 $x > 2$ 時, $f'(x)$ 為正, 故 $f(x)$ 之值繼續增加, 但 x 位於 0 與 2 之間, $f'(x)$ 為負. 故 $f(x)$ 之值繼續減少.

$f'(x) = 6x - 6$; 當 $x = 0$ 與 $2, f'(x) = 0$,
 $f'(0) = -6 < 0, f(0) = 4$ 為 $f(x)$ 之最大值, $f'(2) = 6 > 0$,
 $f(2) = 0$ 為 $f(x)$ 之最小值.

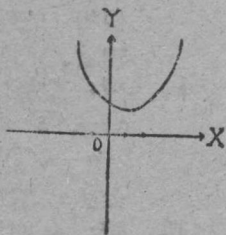
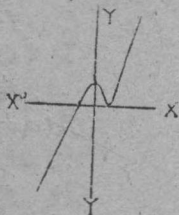
2. 解: (1) $2x^2 - x + 1$.

$$f(x) = 2x^2 - x + 1 = 2 \left[\left(x - \frac{1}{4} \right)^2 + \frac{15}{8} \right].$$

當 $x = \pm \infty, f(x) = \infty$; 當 $x =$ 任何值時, $f(x)$ 為正, 故其曲線不經過 x 軸.

$f'(x) = 4x - 1$, 故 $x = \frac{1}{4}$ 時, $f'(x)$ 為 0 .

當 x 在 $-\infty$ 與 $\frac{1}{4}$ 間, $f'(x) < 0, \therefore f(x)$



之值常減；當 x 在 $\frac{1}{4}$ 與 ∞ 間， $f'(x) > 0$ ， $\therefore f(x)$ 之值常增。

當 $x = \frac{1}{4}$ 時， $f'(x) = 0$ ， $f''(x) = 4 > 0$ 。

故當 $x = \frac{1}{4}$ 時，有一最小值 $f\left(\frac{1}{4}\right)$ 即 $\frac{7}{8}$ 。

$$(2) (x+1)(x-2)(2x-1) = 2x^3 - 3x^2 - 3x + 2.$$

當 $x = -\infty$ ， $f(x) = -\infty$ ；當 x 位於 $-\infty$ 與 -1 之間， $f(x)$ 爲負；當 $x = -1$ ， $f(x) = 0$ ；當 x 位於 -1 與 $\frac{1}{2}$ 之間， $f(x)$ 爲正；當 $x = \frac{1}{2}$ ， $f(x) = 0$ ；當 x 位於 $\frac{1}{2}$ 與 2 之間， $f(x)$ 爲負；當 $x = 2$ ， $f(x) = 0$ ；當 x 位於 2 與 ∞ 之間， $f(x)$ 爲正；當 $x = \infty$ ， $f(x) = \infty$ 。

$f'(x) = 6x^2 - 6x - 3 = 3(2x^2 - 2x - 1)$ ，其根爲 $(1 \pm \sqrt{3})/2$ ，當 $x < (1 + \sqrt{3})/2$ ， $f(x)$ 爲負， $\therefore f(x)$ 之值漸減， $x > (1 + \sqrt{3})/2$ 。

$f'(x)$ 爲正，故 $f(x)$ 之值漸增。

$$f''(x) = 12x - 6 = 6(2x - 1).$$

設 $x = (1 + \sqrt{3})/2 = 1.366$ ， $f'(x) = 0$ ，

$$f''(x) = 6\sqrt{3} > 0.$$

設 $x = (1 - \sqrt{3})/2 = 0.366$ ， $f'(x) = 0$ ， $f''(x) = -6\sqrt{3} < 0$ 。

故當 $x = (1 + \sqrt{3})/2$ 時，有一最小值 $-3\sqrt{3}$ ； $x = (1 - \sqrt{3})/2$ 時，有一最大值 $3\sqrt{3}$ 。

$$(3) x^3 - 12x + 14.$$

$f(x) = x^3 - 12x + 14$ 之三根爲 $1.3+$ ， $2.3+$ ， $-3.2+$ 。

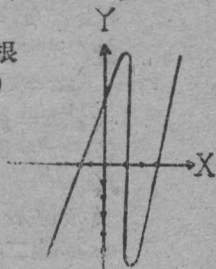
當 $x = -\infty$ ， $f(x) = -\infty$ ；當 x 位於 $-\infty$ 與 $-3.2+$ 之間， $f(x)$ 爲負；當 $x = -3.2+$ ， $f(x) = 0$ ；當 x 位於 $-3.2+$ 與 $1.3+$ 之間， $f(x)$ 爲正；當 $x = 1.3+$ ， $f(x) = 0$ ；當 x 位於 $1.3+$ 與 $2.3+$ 之間， $f(x)$ 爲負；當 $x = 2.3+$ ， $f(x) = 0$ ；當 x 位於 $2.3+$ 與 ∞ 之間， $f(x)$ 爲正；當 $x = \infty$ ， $f(x) = \infty$ 。

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)，f''(x) = 6x.$$

設 $x = -2$ ， $f'(x) = 0$ ， $f''(x) = -12 < 0$ 。

設 $x = 2$ ， $f'(x) = 0$ ， $f''(x) = 12 > 0$ 。

故當 x 在 $-\infty$ 與 -2 及 2 與 ∞ 之間， $f'(x) > 0$ 而 $f(x)$ 之值



漸增；當 x 在 -2 與 2 之間， $f'(x) < 0$ ， $f(x)$ 之值漸減。

又當 $x = -2$ 時，有一最大值 30 ； $x = 2$ 時，有一最小值 -2 。

$$(4) \quad x^3 - 5x^2 + 3x + 9.$$

$$f(x) = x^3 - 5x^2 + 3x + 9 = (x+1)(x-3)^2.$$

當 $x = -\infty$ ， $f(x) = -\infty$ ；當 x 位於 $-\infty$ 與 -1 之間， $f(x)$ 爲負；當

$x = -1$ ， $f(x) = 0$ ；當 x 位於 -1 與 3 之間， $f(x)$ 爲正；當 $x = 3$ ， $f(x) = 0$ ；

當 x 位於 3 與 ∞ 之間， $f(x)$ 爲正；當 $x = \infty$ ， $f(x) = \infty$ 。

$$f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3), \quad f''(x) = 6x - 10.$$

設 $x = \frac{1}{3}$ ， $f'(x) = 0$ ， $f''(x) = -8 < 0$ ；

$$x = 3, \quad f'(x) = 0, \quad f''(x) = 8 > 0.$$

故當 $x = \frac{1}{3}$ 時，有一最大值； $x = 3$ 時，有一最小值。

$$(5) \quad x^3 - 3x^2 + 5.$$

$f(x) = x^3 - 3x^2 + 5$. 此方程式有一負根位於 -1 與 -2 之間，名之曰 a ，其餘二根爲虛根。

當 $x = -\infty$ ， $f(x) = -\infty$ ；當 x 位於 $-\infty$ 與 a 之間， $f(x)$ 爲負；當

$x = a$ ， $f(x) = 0$ ；當 x 位於 a 與 ∞ 之間， $f(x)$ 爲正；當 $x = \infty$ ，

$$f(x) = \infty.$$

$$f'(x) = 3x^2 - 6x = 3x(x-2), \quad f''(x) = 6x - 6.$$

設 $x = 0$ ， $f'(x) = 0$ ， $f''(x) = -6 < 0$ ；

$$x = 2, \quad f'(x) = 0, \quad f''(x) = 6 > 0.$$

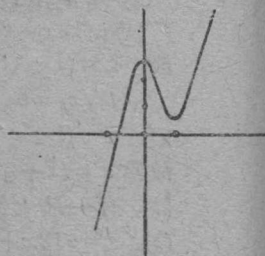
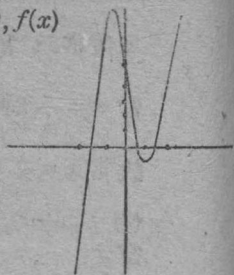
故當 $x = 0$ 時，有一最大值； $x = 2$ 時，有一最小值。

$$(6) \quad (x+1)^2(x-2)^2.$$

$$f(x) = (x+1)^2(x-2)^2 = x^4 - 2x^3 - 3x^2 + 4x + 4.$$

當 $x = \pm\infty$ ， $f(x) = \infty$ ；當 $x = -1$ 或 2 時， $f(x) = 0$ ；當 $x < -1$ 或 $x > 2$ 時， $f(x)$ 爲正；當 x 位於 -1 與 2 之間， $f(x)$

爲正，即 x 之任何值， $f(x)$ 皆爲正。



$$f'(x) = 4x^3 - 6x^2 - 6x + 4$$

$$= 2(2x-1)(x-2)(x+1),$$

$$f''(x) = 12x^2 - 12x - 6 = 6(2x^2 - x - 1).$$

設 $x = \frac{1}{2}$, $f'(x) = 0$, $f''(x) = -9 < 0$;

$x = -1$, $f'(x) = 0$, $f''(x) = 12 > 0$;

$x = 2$, $f'(x) = 0$, $f''(x) = 30 > 0$.

故當 $x = \frac{1}{2}$ 時, 有一最大值 $\frac{81}{16}$, $x = -1$

或 2 時, 有一最小值 0.

(7) $(x^2 + x + 1)(x + 2)$.

$$f(x) = (x^2 + x + 1)(x + 2) = x^3 + 3x^2 + 3x + 2.$$

此方程式有一負根爲 -2 , 其餘二根爲虛根.

當 $x = -\infty$, $f(x) = -\infty$; 當 x 位於

$-\infty$ 與 -2 之間, $f(x)$ 爲負; 當 $x = -2$,

$f(x) = 0$; 當 x 位於 -2 與 ∞ 之間, $f(x)$

爲正; 當 $x = \infty$, $f(x) = \infty$.

$$f'(x) = 3x^2 + 6x + 3 = 3(x+1)^2, \quad f''(x) = 6(x+1).$$

設 $x = -1$, $f'(x) = 0$, $f''(x) = 0$.

故此方程式無最大值, 亦無最小值.

(8) $x(x-1)(x+2)(x+3)$.

$$f(x) = x(x-1)(x+2)(x+3) = x^4 + 4x^3 + x^2 - 6x.$$

當 $x = \pm\infty$, $f(x) = \infty$; 當 x 位於 $-\infty$

與 -3 之間, $f(x)$ 爲正; 當 $x = -3$,

$f(x) = 0$; 當 x 位於 -3 與 -2 之間,

$f(x)$ 爲負; 當 $x = -2$, $f(x) = 0$; 當 x 位

於 -2 與 0 之間, $f(x)$ 爲正; 當 $x = 0$,

$f(x) = 0$; 當 x 位於 0 與 1 之間, $f(x)$

爲負; 當 $x = 1$, $f(x) = 0$; 當 x 位於 1 與

∞ 之間, $f(x)$ 爲正; 當 $x = \infty$, $f(x) = \infty$.

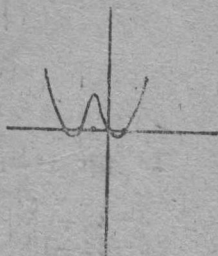
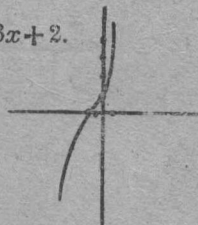
$$f'(x) = 4x^3 + 12x^2 + 2x - 6 = 2(x+1)(2x^2 + 4x - 3),$$

$$f''(x) = 12x^2 + 24x + 2.$$

設 $x = -1$, $f'(x) = 0$, $f''(x) = -10 < 0$.

設 $x = -(2 \pm \sqrt{10})/2$, $f'(x) = 0$, $f''(x) = (35 \pm 13\sqrt{10})/2 > 0$.

故當 $x = -1$ 時, 有一最大值; $x = -(2 \pm \sqrt{10})/2$ 時, 有二最小值.



3. 解:

$$(1) \quad y = \frac{x(x-1)}{(x-2)(x-3)} \quad (\text{圖 I})$$

$$x \quad -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$$

$$y \quad \frac{1}{6}, \frac{3}{35}, 0, -\frac{1}{15}, 0, 1, \infty, -15, \infty, \frac{35}{3}, 6$$

$$(2) \quad y = \frac{x(x-2)}{(x-1)(x-3)} \quad (\text{圖 II})$$

$$x \quad -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$$

$$y \quad \frac{3}{8}, \frac{5}{21}, 0, -\frac{3}{5}, \pm\infty, 1, 0, -\frac{5}{3}, \pm\infty, \frac{21}{5}, \frac{8}{3}$$

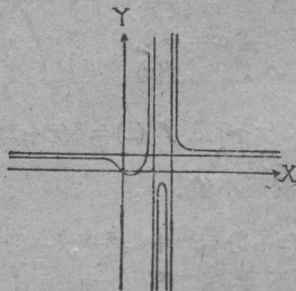


圖 I.

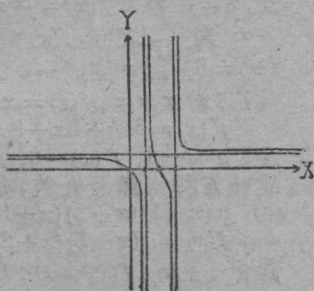


圖 II.

習題 LXXVIII

原本第 477 頁

1. 解: $x^3 - 6x^2 + 5x + 13 = 0$.

$$f(x) = x^3 - 6x^2 + 5x + 13, \quad f_1(x) = 3x^2 - 12x + 5.$$

3 - 12 + 5	1 - 6 + 5 + 13	1 - 2
6 - 24 + 10	3 - 18 + 15 + 39	$\therefore f(x) = x^3 - 6x^2 + 5x + 13$
6 - 21	3 - 12 + 5	$f_1(x) = 3x^2 - 12x + 5$
- 3 + 10	- 6 + 10 + 39	$f_2(x) = 2x - 7$
- 6 + 20	- 6 + 24 - 10	$f_3 = 1$
- 6 + 21	- 14 + 49	3 - 3
- 1	$\therefore f_2 = 2 - 7$	
$\therefore f_3 = 1$		

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	-	+	-	+
$x = 0$	+	+	-	+
$x = \infty$	+	+	+	+

故 $f(x) = 0$, 有二正根與一負根。

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = 0$	+	+	-	+
$x = 1$	+	-	-	+
$x = 2$	+	-	-	+
$x = 3$	+	-	-	+
$x = 4$	+	+	+	+
$x = -1$	+	+	-	+
$x = -2$	-	+	-	+

答：二正根位於 3 與 4 之間，負根位於 -1 與 -2 之間。

2. 解： $x^3 - 4x^2 - 10x + 41 = 0$.

$$f(x) = x^3 - 4x^2 - 10x + 41, \quad f_1(x) = 3x^2 - 8x - 10.$$

3-	8-	10	1-	4-	10+	41	1-	4
76-	736-	920	3-	12-	30+	123		
76-	987		3-	8-	10			
	251-	920		-	4-	20+	123	
	23092-	84640		-	12-	60+	369	
	23092-	82579		-	12+	32+	40	
	-	2061			-	92+	329	

$$\begin{aligned} \therefore f(x) &= x^3 - 4x^2 - 10x + 41 \\ f_1(x) &= 3x^2 - 8x - 10 \\ f_2(x) &= 92x - 329 \\ f_3 &= 2061 \end{aligned}$$

$$\therefore f_3 = 2061$$

$$\therefore f_2 = 92 - 329 \quad 3 + 251$$

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	-	+	-	+
$x = 0$	+	-	-	+
$x = \infty$	+	+	+	+

故 $f(x) = 0$, 有二正根與一負根。

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = 0$	+	-	-	+
$x = 1$	+	-	-	+
$x = 2$	+	-	-	+
$x = 3$	+	-	-	+
$x = 4$	+	+	+	+
$x = -1$	+	+	-	+
$x = -2$	+	+	-	+
$x = -3$	+	+	-	+
$x = -4$	-	+	-	+

答：二正根位於 3 與 4 之間，負根位於 -3 與 -4 之間。

3. 解： $x^3 + 5x + 2 = 0$.

$$f(x) = x^3 + 5x + 2, \quad f_1(x) = 3x^2 + 5.$$

3 + 0 + 5	1 + 0 + 5 + 2	1
15 + 0 + 25	3 + 0 + 15 + 5	$\therefore f_2(x) = -5x - 3$
15 + 9	3 + 0 + 5	$f_3 = -152$
- 9 + 25	10 + 6	-3 + 9
-45 + 125	$\therefore f_2 = -5 - 3$	
-45 - 27		
152		
$\therefore f_3 = -152$		

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	-	+	+	-
$x = 0$	+	+	-	-
$x = \infty$	+	+	-	-

故 $f(x) = 0$ ，有一負根與二虛根。

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = 0$	+	+	-	-
$x = -1$	-	+	+	-

答：此負根位於 0 與 -1 之間。

4. 解： $x^3 + 3x^2 + 8x + 8 = 0$.

$$f(x) = x^3 + 3x^2 + 8x + 8, \quad f_1(x) = 3x^2 + 6x + 8.$$

3 + 6 + 8	1 + 3 + 8 + 8	1 + 1
15 + 30 + 40	3 + 9 + 24 + 24	$\therefore f_2(x) = +5x - 8$
15 + 24	3 + 6 + 8	$f_3 = -152$
6 + 40	3 + 15 + 24	
30 + 200	3 + 6 + 8	-3 - 6
30 + 48	10 + 16	
152	$\therefore f_2 = -5 - 8$	
$\therefore f_3 = -152$		

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	-	+	+	-
$x = 0$	+	+	-	-
$x = \infty$	+	+	-	-

故 $f(x) = 0$ ，有一負根。

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x=0$	+	+	+	-
$x=-1$	-	+	-	-
$x=-2$	-	+	+	-

答：此負根位於 -1 與 -2 之間。

5. 解： $x^3 - x^2 - 15x + 28 = 0$.

$$f(x) = x^3 - x^2 - 15x + 28, \quad f_1(x) = 3x^2 - 2x - 15.$$

3-	2-	15	1-1-15+	28	1
276-	184-	1380	3-3-45+	84	$\therefore f_2 = 92x - 237$
276-	711		3-2-15		
	527-	1380	-1-30+	84	-1
	48424-	126960	-3-90+	252	$f_3 = 2061$
	48484-	124899	-3+	2+	15
	-	2061		-92+	237
	$\therefore f_3 = 2061$		$\therefore f_2 = 92 - 237$		3

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	-	+	-	+
$x=0$	+	-	-	+
$x=\infty$	+	+	+	+

故 $f(x) = 0$, 有二正根與一負根。

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x=0$	+	-	-	+
$x=1$	+	-	-	+
$x=2$	+	-	-	+
$x=3$	+	+	+	+
$x=-1$	+	-	-	+
$x=-2$	+	+	-	+
$x=-3$	+	+	-	+
$x=-4$	+	+	-	+
$x=-5$	-	+	-	+

答：二正根位於 2 與 3 之間，負根位於 -4 與 -5 之間。

6. 解： $x^4 - 4x^3 - 5x^2 + 18x + 20 = 0$.

$$f(x) = x^4 - 4x^3 - 5x^2 + 18x + 20,$$

$$f_1(x) = 4x^3 - 12x^2 - 10x + 18.$$

4 - 12 - 10 + 18	1 - 4 - 5 + 18 + 20	1 - 1
44 - 132 - 110 + 198	4 - 16 - 20 + 72 + 80	
44 - 88 - 196	4 - 12 - 10 + 18	
- 44 + 86 + 198	- 4 - 10 + 54 + 80	
- 44 + 88 + 196	- 4 + 12 + 10 - 18	
- 2 + 2	- 22 + 44 + 98	
$\therefore f_3 = 1 - 1$	- 11 + 22 + 49	
	$\therefore f_2 = 11 - 22 - 49$	4 - 4
	11 - 11	11 - 11
	- 11 - 49	
	- 11 + 11	
	- 60	
	$\therefore f_4 = 60$	

$$\therefore f_2(x) = 11x^2 - 22x - 49, f_3(x) = x - 1, f_4 = 60.$$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x = -\infty$	+	-	+	-	+
$x = 0$	+	+	-	-	+
$x = \infty$	+	+	+	+	+

故 $f(x) = 0$, 有二正根與二負根。

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x = 0$	+	+	-	-	+
$x = 1$	+	+	-	+	+
$x = 2$	+	-	-	+	+
$x = 3$	+	-	-	+	+
$x = 4$	+	+	+	+	+
$x = -1$	+	+	-	-	+
$x = -2$	+	-	+	-	+

答：二正根位於 3 與 4 之間，二負根位於 -1 與 -2 之間。

7. 解： $2x^4 - 3x^2 + 3x - 1 = 0$.

$$f(x) = 2x^4 - 3x^2 + 3x - 1, f_1(x) = 8x^3 - 6x + 3.$$

$8+0-6+3$	$2+0-3+3-1$	1
$48+0-36+18$	$8+0-12+12-4$	
$48-72+32$	$8+0-6+3$	
$72-68+18$	$-6+9-4$	
$72-108+48$	$\therefore f_2=6-9+4$	$8+2$
$40-30$	$24-36+16$	-6
$\therefore f_3=-4+3$	$24-18$	
	$-18+16$	
	$-36+32$	9
	$-36+27$	
	5	
	$\therefore f_4=-5$	

$$\therefore f_2(x) = 6x^2 - 9x + 4, f_3(x) = -4x + 3, f_4 = -5.$$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x = -\infty$	+	-	+	+	-
$x = 0$	-	+	+	+	-
$x = \infty$	+	+	+	-	-

故 $f(x) = 0$, 有一正根與二負根。

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x = 0$	-	+	+	+	-
$x = 1$	+	+	+	-	-
$x = -1$	-	+	+	+	-
$x = -2$	+	-	+	+	-

答：正根位於 0 與 1 之間，二負根位於 -1 與 -2 之間。

8. 解： $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$.

$$f(x) = x^4 - 8x^3 + 19x^2 - 12x + 2, f_1(x) = 4x^3 - 24x^2 + 38x - 12.$$

$4-24+38-12$	$1-8+19-12+2$	
$20-120+190-60$	$4-32+76-48+8$	$1-2$
$20-80+32$	$4-24+38-12$	
$-40+158-60$	$-8+38-36+8$	
$-40+160-64$	$-8+48-76+24$	
$-2+4$	$-10+40-16$	
$\therefore f_3=1-2$	$\therefore f_2=5-20+8$	$-2-4$

故 $f(x)=0$, 有三正根與一負根.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x=0$	-	+	+	-	+
$x=1$	-	-	+	+	+
$x=2$	-	-	+	+	+
$x=3$	+	+	+	+	+
$x=-1$	-	+	+	-	+
$x=-2$	-	+	+	-	+
$x=-3$	-	-	+	-	+
$x=-4$	+	-	+	-	+

答：二正根位於 0 與 1 之間，一正根位於 2 與 3 之間，負根位於 -3 與 -4 之間。

10. 解： $x^4+2x^3-6x^2-8x+9=0$.

$$f(x)=x^4+2x^3-6x^2-8x+9, \quad f_1(x)=4x^3+6x^2-12x-8.$$

$4+6-12-8$	$1+2-6-8+9$	
$2+3-6-4$	$2+4-12-16+18$	1
$30+45-90-60$	$2+3-6-4$	$\therefore f_2(x)=15x^2$
$30+36-80$	$1-6-12+18$	$+18x-40$
$9-10-60$	$2-12-24+36$	1
$45-50-300$	$2+3-6-4$	$f_3(x)=26x+45$
$45+54-120$	$-15-18+40$	
$-104-180$	$\therefore f_2=15+18-40$	$2+3$
$\therefore f_3=26+45$	$390+468-1040$	15
	$390+675$	
	$-207-1040$	$f_4=17725$
	$-5382-27040$	207
	$-5382-9315$	
	-17725	
	$\therefore f_4=17725$	

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x=-\infty$	+	-	+	-	+
$x=0$	+	-	-	+	+
$x=\infty$	+	+	+	+	+

故 $f(x)=0$ 有二正根與二負根.

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x=0$	+	-	-	+	+
$x=1$	-	-	-	+	+
$x=2$	+	+	+	+	+
$x=-1$	+	+	-	+	+
$x=-2$	+	+	-	-	+
$x=-3$	+	-	+	-	+

答：此方程式之一正根位於 0 與 1 之間，一正根位於 1 與 2 之間；二負根位於 -2 與 -3 之間。

11. 解： $4x^3 - 2x - 5 = 0$.

$f(x) = 4x^3 - 2x - 5, f_1(x) = 12x^2 - 2$.

12+	0-	2	4+0-2-	5	1	$\therefore f_2(x) = 4x + 15$ $f_3 = -677$
12+	45		12+0-6-	15		
-	45-	2	12+0-	2		
-	180-	8		-4-15		
-	180-	675				
		667	$\therefore f_2 = 4 + 15$	3-45		
$\therefore f_3 = -667$						

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	-	+	-	-
$x = \infty$	+	+	+	-

答： $f(x) = 0$, 有一實根。

12. 解： $x^4 + x^3 + x^2 + x + 1 = 0$.

$f(x) = x^4 + x^3 + x^2 + x + 1, f_1(x) = 4x^3 + 3x^2 + x + 1$.

4+	3+	2+	1	1+1+1+	1+	1	1	$\therefore f_2(x) = -x^2 - 2x - 3$ $f_3 = -16$
4+	8+	12		4+4+4+	4+	4		
-	5-	10+	1	4+3+2+	1			
-	5-	10-	15		1+2+	3+	4	
		16		4+8+12+	16	1		
$\therefore f_3 = -16$				4+3+	2+	1		
					5+10+	15		
				$\therefore f_2 = -1 - 2 - 3$		-4+	5	

	$f(x)$	$f_1(x)$	$f_2(x)$	f_3
$x = -\infty$	+	-	-	-
$x = \infty$	+	+	-	-

答: $f(x)=0$, 無實根.

13. 解: $x^n+1=0$.

$$f(x)=x^n+1, f_1(x)=nx^{n-1}.$$

$n+0+\dots$ 至 $(n-1)$ 項

$$\left| \begin{array}{r} 1+0+\dots+1 \\ n+0+\dots+n \\ n+0+\dots \end{array} \right|$$

n

$$\therefore f_2 = -n$$

(1) 當 n 為偶數時:

	$f(x)$	$f_1(x)$	f_2
$x=-\infty$	+	-	-
$x=\infty$	+	+	-

答: 當 n 為偶數時, 無實根.

(2) 當 n 為奇數時:

	$f(x)$	$f_1(x)$	f_2
$x=-\infty$	-	+	-
$x=\infty$	+	+	-

答: 當 n 為奇數時, 有一實根.

14. 解: $x^4-6x^3+x^2+14x-14=0$.

$$f(x)=x^4-6x^3+x^2+14x-14,$$

$$f_1(x)=4x^3-18x^2+2x+14.$$

4-	18+	2+	14	1-	6+	1+	14-	14	1
2-	9+	1+	7	2-	12+	2+	28-	28	
10-	45+	5+	35	2-	9+	1+	7		
10-	18+	14		-	3+	1+	21-	28	-3
-	27-	9+	35	-	6+	2+	42-	56	
-	135-	45+	175	-	6+	27-	3-	21	
-	135+	243-	189	-	25+	45-	35		
-	288+	364		$\therefore f_2 =$	5-	9+	7	2-27	
$\therefore f_2 =$	72-	91			360-	648+	504	5	
	-13896+	17563			360-	455+	444		
	-13896+	36288			-193+	60		72	
	-18725								
$\therefore f_4 =$	18725								

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	f_4
$x = -\infty$	+	-	+	-	+
$x = \infty$	+	+	-	+	+

答： $f(x)=0$ ，有二實根。

15. 證：題設該方程有 n 個不同的實根，故 $V_{-\infty} - V_{\infty} = n$ ，式中 $V_{-\infty}$ 示 $x = -\infty$ 時之變號數， V_{∞} 示 $x = \infty$ 時之變號數。

$$\therefore V_{\infty} \geq 0, \quad \therefore V_{-\infty} \geq n.$$

因有二項方可生一變號，故史篤姆函數中至少有 $n+1$ 項。又因 $f(x)$ 為 n 項，故史篤姆函數中至多有 $n+1$ 項。故該史篤姆函數之項數必為 $n+1$ 。因此 $V_{-\infty} = n$ 而 $V_{\infty} = 0$ ，故各函數首項之符號相同。

故此史篤姆函數為 $n+1$ 項。

16. 證：
- $$f(x) = x^3 + px + q = 0$$
- $$f_1(x) = 3x^2 + p = 0$$
- $$f_2(x) = -2p^2x - 9q = 0$$
- $$f_3(x) = -4p^2q - 27r^3 = 0$$

$f(x)=0$ 之各根為實根，且不相等之條件為 $-p > 0$ 與 $4p^3 + 27q^2 < 0$ 。但前者包括於後者之中，故其條件為

$$4p^3 + 27q^2 < 0.$$

習題 LXXIX

原本第 482 頁

1. 解： $f(x) = x^3 + \frac{a_1}{a_0}x^2 + \frac{a_2}{a_0}x + \frac{a_3}{a_0} = 0$

$$f_1(x) = 3x^2 + 2\frac{a_1}{a_0}x + \frac{a_2}{a_0}$$

$$\frac{f(x)}{x - \beta_1} = x^2 + \left(\beta_1 + \frac{a_1}{a_0}\right)x + \beta_1^2 + \frac{a_1}{a_0}\beta_1 + \frac{a_2}{a_0}$$

$$\frac{f(x)}{x - \beta_2} = x^2 + \left(\beta_2 + \frac{a_1}{a_0}\right)x + \beta_2^2 + \frac{a_1}{a_0}\beta_2 + \frac{a_2}{a_0}$$

$$\frac{f(x)}{x-\beta_3} = x^2 + \left(\beta_3 + \frac{a_1}{a_0}\right)x + \beta_3^2 + \frac{a_1}{a_0}\beta_3 + \frac{a_2}{a_0}$$

$$3x^2 + 2\frac{a_1}{a_0}x + \frac{a_2}{a_0} = 3x^2 + \left(s_1 + 3\frac{a_1}{a_0}\right)x + \left(s_2 + \frac{a_1}{a_0}s_1 + 3\frac{a_2}{a_0}\right)$$

$$s_1 + 3\frac{a_1}{a_0} = 2\frac{a_1}{a_0} \quad \therefore s_1 = -\frac{a_1}{a_0}$$

$$s_2 + \frac{a_1}{a_0}s_1 + 3\frac{a_2}{a_0} = \frac{a_2}{a_0} \quad \therefore s_2 = \left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0}$$

以 x^{k-3} 乘已知方程式，然後以 $\beta_1, \beta_2, \beta_3$ 之值按次代替 x ，將所得結果加之，得下式：

$$s_k + \frac{a_1}{a_0}s_{k-1} + \frac{a_2}{a_0}s_{k-2} + \frac{a_3}{a_0}s_{k-3} = 0.$$

設 $k=3$, $s_3 + \frac{a_1}{a_0}s_2 + \frac{a_2}{a_0}s_1 + 3\frac{a_3}{a_0} = 0$ (見 p. 430—10).

$$\therefore s_3 = -(a_1^3 - 3a_0a_1a_2 + 3a_0^3a_3)/a_0^3.$$

設 $k=4$, $s_4 + \frac{a_1}{a_0}s_3 + \frac{a_2}{a_0}s_2 + \frac{a_3}{a_0}s_1 = 0.$

$$\therefore s_4 = (a_1^4 - 4a_0a_1^2a_2 + 4a_0^2a_1a_3 + 2a_0^2a_2^2)/a_0^4.$$

2. 解：根爲 $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ 之方程式爲 $rx^3 + qx^2 + px + 1 = 0$,

$$\therefore b_1 = \frac{q}{r}, \quad b_2 = \frac{p}{r}, \quad b_3 = \frac{1}{r}.$$

$$s_1 = \Sigma 1/a = -\frac{q}{r}$$

$$s_2 = \Sigma 1/a^2 = \frac{q^2}{r^2} - \frac{2p}{r}$$

$$s_3 = \Sigma 1/a^3 = -(q^3 - 3pqr + 3r^2)/r^3$$

$$\Sigma a\beta^3 = s_1s_2 - s_3 = \frac{3r - pq}{r^2}.$$

3. 解：設 α, β, γ 爲 $x^3 - 2x^2 + 3x - 1 = 0$ 之各根。

α', β', γ' 爲所求方程式之各根。

$$s'_1 = \alpha' + \beta' + \gamma' = \alpha^3 + \beta^3 + \gamma^3 = s_3$$

$$s'_2 = \alpha'^2 + \beta'^2 + \gamma'^2 = \alpha^6 + \beta^6 + \gamma^6 = s_3$$

$$s'_3 = \alpha'^3 + \beta'^3 + \gamma'^3 = \alpha^9 + \beta^9 + \gamma^9 = s_9$$

$$s'_1 = -b'_1 = -b_1s_2 - b_2s_1 - 3b_3 = -7$$

$$s'_2 = b'^2_1 - 2b'_2 = -b_1s_3 - b_2s_4 - b_3s_3 = 25$$

$$s'_3 = -b'_1s'_2 - b'_2s'_2 - 3b'_3 = -b_1s_3 - b_2s_7 - b_3s_6 = 70$$

$$\therefore b'_1 = 7, b'_2 = (-25 + 49)/2 = 12,$$

$$b'_3 = 70 - 155 + 84 = -1.$$

答：所求之方程式為 $x^3 + 7x^2 + 12x - 1 = 0$ 。

4. 解：(1) $s_1 = -b_1 = 1, s_2 = b^2_1 - 2b_2 = 1 - 6 = -5,$

$$s_3 = -b_1s_2 - b_2s_1 - 3b_3 = -(5 + 3 + 12) = -20,$$

$$s_4 = -b_1s_3 - b_2s_2 - b_3s_1 = -20 + 15 - 4 = -9.$$

(2) $\Sigma \alpha^3 \beta^2 = s_3 s_2, s_{3+2} = (-5)(-20) - 71 = 29.$

(3) $\Sigma \alpha^3 \beta \gamma = \Sigma \alpha^3 \cdot \Sigma \beta \gamma - \Sigma \alpha^4 \beta = s_3(s^2_1 - s_2)/2 - (s_1s_1 - s_5)$
 $= (-20)(1 + 5)/2 - (-9 \times 1 - 71) = -60 + 80 = 20.$

(4) $\Sigma \alpha^3 \beta^2 \gamma = s_3 s_2 s_1 + 2s_{3+2+1} - s_{3+2} s_1 - s_{3+1} s_2 - s_{2+1} s_3 = -60.$

(5) $\Sigma 1/\alpha^4 = s_{-4} = -\left(\frac{3}{4}\right)\left(\frac{-111}{64}\right) - \left(-\frac{1}{4}\right)\left(\frac{17}{16}\right)$
 $-\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right) = \frac{449}{256}.$

(6) $\Sigma \alpha^2 \beta^2 / \gamma = s_2 s_2 s_{-1} + 2s_3 - s_4 s_{-1} - s_1 s_2 - s_1 s_2 = -\frac{111}{4}.$

XXX. 普通三次及四次方程

習題 LXXX

原本第 491 頁

1. 解： $x^3 - 9x - 28 = 0.$

$$p = -9, \quad q = -28.$$

$$\therefore A = 14 + \sqrt{196 - 27} = 27, \quad B = 14 - \sqrt{196 - 27} = 1.$$

答：其根爲 $\sqrt[3]{27} + \sqrt[3]{1}$, $\omega \sqrt[3]{27} + \omega^2 \sqrt[3]{1}$, $\omega^2 \sqrt[3]{27} + \omega \sqrt[3]{1}$;
即 $4, -2 \pm i\sqrt{3}$.

2. 解： $x^3 - 9x^2 + 9x - 8 = 0$.

以 $x = y + 3$ 代入原方程式，得 $y^3 - 18y - 35 = 0$.

$$p = -18, \quad q = -35.$$

$$\therefore A = \frac{35}{2} + \sqrt{\frac{1225}{4} - 216} = \frac{35 + 19}{2} = 27,$$

$$B = \frac{35 - 19}{2} = 8.$$

答：原方程式之根爲 $3 + \sqrt[3]{27} + \sqrt[3]{8}$, $3 + 3\omega + 2\omega^2 = -\omega^2$,
 $3 + 3\omega^2 + 2\omega = -\omega$; 即 $8, \frac{1 \pm i\sqrt{3}}{2}$.

3. 解： $x^3 - 3x - 4 = 0$.

$$p = -3, \quad q = -4.$$

$$\therefore A = 2 + \sqrt{4 - \frac{27}{2}} = 2 + \sqrt{3}, \quad B = 2 - \sqrt{3}.$$

答：其根爲 $\sqrt[3]{A} + \sqrt[3]{B}$, $\omega \sqrt[3]{A} + \omega^2 \sqrt[3]{B}$, $\omega^2 \sqrt[3]{A} + \omega \sqrt[3]{B}$.

4. 解： $4x^3 - 7x - 6 = 0$.

$$x^3 - \frac{7}{4}x - \frac{3}{2} = 0 \quad p = -\frac{7}{4}, \quad q = -\frac{3}{2}.$$

$$\therefore A = \frac{3}{4} + \sqrt{\frac{9}{16} - \frac{343}{1728}} = \frac{3}{4} + \sqrt{\frac{972 - 343}{1728}}$$

$$= \frac{3}{4} + \frac{1}{24} \sqrt{\frac{629}{3}} = \frac{3}{4} + \frac{1}{72} \sqrt{1887},$$

$$B = \frac{3}{4} - \frac{1}{72} \sqrt{1887}.$$

答：其根爲 $\sqrt[3]{A} + \sqrt[3]{B}$, $\omega \sqrt[3]{A} + \omega^2 \sqrt[3]{B}$, $\omega^2 \sqrt[3]{A} + \omega \sqrt[3]{B}$.

5. 解： $x^3 + 3x^2 + 9x - 1 = 0$.

以 $x = y - 1$ 代入原方程式，得 $y^3 + 6y - 8 = 0$.

$$p = 6, \quad q = -8.$$

$$\therefore A = 4 + \sqrt{16 + 8} = 4 + 2\sqrt{6}, \quad B = 4 - 2\sqrt{6}.$$

答：原方程式之根爲 $-1 + \sqrt[3]{A} + \sqrt[3]{B}$,

$$-1 + \omega \sqrt[3]{A} + \omega^2 \sqrt[3]{B}, \quad -1 + \omega^2 \sqrt[3]{A} + \omega \sqrt[3]{B}.$$

6. 解: $3x^3 - 9x^2 + 14x + 7 = 0.$

以 $x = y + 1$ 代入原方程式, 得 $3y^3 + 5y + 15 = 0$, 即

$$y^3 + \frac{5}{3}y + 5 = 0 \quad p = \frac{5}{3}, \quad q = 5.$$

$$\therefore A = -\frac{5}{2} + \sqrt{\frac{25}{4} + \frac{125}{729}} = -\frac{5}{2} + \frac{5}{54}\sqrt{749},$$

$$B = -\frac{5}{2} - \frac{5}{54}\sqrt{749}.$$

答: 原方程式之根爲 $1 + \sqrt[3]{A} + \sqrt[3]{B}, 1 + \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B},$
 $1 + \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}.$

7. 解: $x^4 + x^2 + 6x + 1 = 0.$

$$a = 1, \quad b = 6, \quad c = 1.$$

代入 $u^3 - au^2 - 4cu + (4ac - b^2) = 0$, 得

$$u^3 - u^2 - 4u - 32 = 0.$$

此三次方程式之一根爲 4, 設 $u_1 = 4$.

應用 $x^2 \pm \sqrt{u_1 - a}x + \left(\frac{u_1}{2} \mp \frac{b}{2\sqrt{u_1 - a}}\right) = 0$, 得

$$x^2 + \sqrt{3}x + 2 - \sqrt{3} = 0 \dots\dots\dots (1)$$

$$x^2 - \sqrt{3}x + 2 + \sqrt{3} = 0 \dots\dots\dots (2)$$

從 (1) 式, $x = \frac{-\sqrt{3} \pm \sqrt{-5 + 4\sqrt{3}}}{2}.$

從 (2) 式, $x = \frac{\sqrt{3} \pm i\sqrt{5 + 4\sqrt{3}}}{2}.$

答: 此方程式之根爲 $(-\sqrt{3} \pm \sqrt{-5 + 4\sqrt{3}})/2,$

$$(\sqrt{3} \pm i\sqrt{5 + 4\sqrt{3}})/2.$$

8. 解: $x^4 - 4x^3 + x^2 + 4x + 1 = 0.$

以 $x = y + 1$ 代入原方程式, 得 $y^4 - 5y^2 - 2y + 3 = 0.$

$$a = -5, \quad b = -2, \quad c = 3.$$

$$u^3 + 5u^2 - 12u - 64 = 0.$$

其一根爲 4 $\therefore u_1 = -4.$

$$y^2 + y - 1 = 0 \dots\dots\dots (*)$$

$$y^2 - y - 3 = 0 \dots\dots\dots (**)$$

從 (1) 式, $y = \frac{-1 \pm \sqrt{5}}{2}$.

從 (2) 式, $y = \frac{1 \pm \sqrt{13}}{2}$.

答: 此方程式之根爲 $1 + \frac{-1 \pm \sqrt{5}}{2}$, $1 + \frac{1 \pm \sqrt{13}}{2}$;

即 $\frac{1 \pm \sqrt{5}}{2}$, $\frac{3 \pm \sqrt{13}}{2}$.

9. 解: $x^4 + 12x - 5 = 0$.

$$a = 0, \quad b = 12, \quad c = -5.$$

$$u^3 + 20u - 144 = 0.$$

其一根爲 4

$$u_1 = 4.$$

$$x^2 + 2x - 1 = 0 \dots\dots\dots (1)$$

$$x^2 - 2x + 5 = 0 \dots\dots\dots (2)$$

從 (1) 式, $x = -1 \pm \sqrt{2}$.

從 (2) 式, $x = 1 \pm 2i$.

答: 此方程式之根爲 $-1 \pm \sqrt{2}$, $1 \pm 2i$.

10. 解: $x^4 + 8x^3 + 11x^2 - 11x + 2 = 0$.

以 $x = y - 2$ 代入原方程式, 得 $y^4 - 12y^3 + 5y^2 + 24 = 0$.

$$a = -12, \quad b = 5, \quad c = 24.$$

$$u^3 + 12u^2 - 96u - 1177 = 0.$$

其一根爲 -11

$$\therefore u_1 = -11.$$

$$y^2 + y - 8 = 0 \dots\dots\dots (1)$$

$$y^2 - y - 3 = 0 \dots\dots\dots (2)$$

從 (1) 式, $y = \frac{-1 \pm \sqrt{33}}{2}$.

從 (2) 式, $y = \frac{1 \pm \sqrt{13}}{2}$.

答: 此方程式之根爲 $-2 + \frac{-1 \pm \sqrt{33}}{2}$, $-2 + \frac{1 \pm \sqrt{13}}{2}$;

即 $\frac{-5 \pm \sqrt{33}}{2}$, $\frac{-3 \pm \sqrt{13}}{2}$.

11. 解: $3x^6 - 2x^5 + 6x^4 - 2x^3 + 6x^2 - 2x + 3 = 0.$
 $3x^3 - 2x^2 + 6x - 2 + 6/x - 2/x^2 + 3/x^3 = 0.$
 $3\left(x^3 + \frac{1}{x^3}\right) - 2\left(x^2 - \frac{1}{x^2}\right) + 6\left(x + \frac{1}{x}\right) - 2 = 0.$

以 $y = x + \frac{1}{x}$ 代入此方程式, 得

$3(y^3 - 3y) - 2(y^2 - 2) + 6y - 2 = 0$
 $3y^3 - 2y^2 - 3y + 2 = 0.$
 $(y-1)(y+1)(3y-2) = 0.$
 $\therefore y = 1, -1, 2/3.$

$x + \frac{1}{x} = 1$ 即 $x^2 - x + 1 = 0 \dots\dots\dots(1)$

$x + \frac{1}{x} = -1$ 即 $x^2 + x + 1 = 0 \dots\dots\dots(2)$

$x + \frac{1}{x} = \frac{2}{3}$ 即 $3x^2 - 2x + 3 = 0 \dots\dots\dots(3)$

從 (1), (2), (3) 式, 得 $x = \frac{1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2}, \frac{1 + 2\sqrt{2}i}{3}.$

即所求之根也。

12. 解: $2x^8 - 9x^7 + 18x^6 - 30x^5 + 32x^4 - 30x^3 + 18x^2 - 9x + 2 = 0$
 $2x^4 - 9x^3 + 18x^2 - 30x + 32 - 30/x + 18/x^2 - 9/x^3 + 2/x^4 = 0.$
 $2\left(x^4 + \frac{1}{x^4}\right) - 9\left(x^3 + \frac{1}{x^3}\right) + 18\left(x^2 + \frac{1}{x^2}\right) - 30\left(x + \frac{1}{x}\right) + 32 = 0.$

以 $y = x + \frac{1}{x}$ 代入此方程式, 得

$2(y^4 - 4y^2 + 2) - 9(y^3 - 3y) + 18(y^2 - 2) - 30y + 32 = 0.$
 $2y^4 - 9y^3 + 10y^2 - 3y = 0.$
 $y(y-1)(y-3)(2y-1) = 0.$
 $\therefore y = 0, 1, 3, \frac{1}{2}.$

$x + \frac{1}{x} = 0$ 即 $x^2 + 1 = 0 \dots\dots\dots(1)$

$x + \frac{1}{x} = 1$ 即 $x^2 - x + 1 = 0 \dots\dots\dots(2)$

$$x + \frac{1}{x} = 3 \quad \text{即} \quad x^2 - 3x + 1 = 0 \quad \dots\dots\dots(3)$$

$$x + \frac{1}{x} = \frac{1}{2} \quad \text{即} \quad 2x^2 - x + 2 = 0 \quad \dots\dots\dots(4)$$

從 (1), (2), (3), (4) 式, 得 $x = \pm i, \frac{1 \pm \sqrt{3}i}{2}, \frac{3 \pm \sqrt{5}}{2}, \frac{1 \pm \sqrt{15}i}{4}$; 即所求之各根也。

13. 解: $6x^7 - x^6 + 2x^5 - 7x^4 - 7x^3 + 2x^2 - x + 6 = 0.$

此倒數方程式有 1, 1 與 -1 各根, 以 $(x-1)^2(x+1)$ 除原方程式, 得 $6x^4 + 5x^3 + 13x^2 + 5x + 6 = 0.$

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) + 13 = 0.$$

以 $y = x + \frac{1}{x}$ 代入此方程式, 得

$$6(y^2 - 2) + 5y + 13 = 0.$$

$$6y^2 + 5y + 1 = 0.$$

$$(2y+1)(3y+1) = 0.$$

$$\therefore y = -\frac{1}{2}, -\frac{1}{3}.$$

$$x + \frac{1}{x} = -\frac{1}{2} \quad \text{即} \quad 2x^2 + x + 2 = 0 \quad \dots\dots\dots(1)$$

$$x + \frac{1}{x} = -\frac{1}{3} \quad \text{即} \quad 3x^2 + x + 3 = 0 \quad \dots\dots\dots(2)$$

從 (1), (2) 式, 得 $x = \frac{-1 \pm \sqrt{15}i}{4}, \frac{-1 \pm \sqrt{35}i}{6}.$

答: 所求之各根爲 1, 1, -1, $\frac{-1 \pm \sqrt{15}i}{4}, \frac{-1 \pm \sqrt{35}i}{6}.$

14. 解: $x^7 - 1 = 0$ 有一根爲 1, 以 $x-1$ 除之, 得

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

$$x^3 + x^2 + x + 1 + 1/x + 1/x^2 + 1/x^3 = 0.$$

$$\left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0.$$

以 $z = x + \frac{1}{x}$ 代入此方程式，得

$$(z^3 - 3z) + (z^2 - 2) + z + 1 = 0.$$

$$\therefore z^3 + z^2 - 2z + 1 = 0.$$

15. 解: $x^3 + 3ax^2 + 3bx + c = 0.$

以 $x = y - a$ 代入原方程式，得

$$y^3 + (3b - 3a^2)y + 2a^3 - 3ab + c = 0.$$

$$p = 3(b - a^2), \quad q = 2a^3 - 3ab + c.$$

$$\therefore A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \quad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

若其根均爲實根，則 $\frac{q^2}{4} + \frac{p^3}{27} \equiv 0.$

$$\text{即} \quad \frac{(2a^3 - 3ab + c)^2}{4} + \frac{3^3(b - a^2)^3}{27} \equiv 0.$$

$$\text{即} \quad \frac{(2a^3 - 3ab + c)^3}{4} + (b - a^2)^3 \equiv 0.$$

上式乃所求之條件也。

16. 解: (1) $x^5 - 1 = 0.$

$$x^5 = 1 = \cos(2k\pi) + i \sin(2k\pi).$$

$$\therefore x = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5},$$

其中 $k = 0, 1, 2, 3, 4.$

$$(2) x^5 + 1 = 0. \quad x^5 - (-1) = 0.$$

$$x^5 = -1 = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi).$$

$$\therefore x = \cos \frac{\pi + 2k\pi}{5} + i \sin \frac{\pi + 2k\pi}{5},$$

其中 $k = 0, 1, 2, 3, 4, 5.$

17. 解: (1) $x^3 - 3x - 1 = 0.$

$$p = -3, \quad q = -1.$$

$$r = \left(\frac{p^3}{27} \right)^{\frac{1}{2}} = \left(\frac{27}{27} \right)^{\frac{1}{2}} = 1.$$

$$\cos \theta = -\frac{q}{2} \left(-\frac{27}{p^3} \right)^{\frac{1}{3}} = \frac{1}{2} \left(\frac{27}{27} \right)^{\frac{1}{3}} = \frac{1}{2}$$

$$\theta = \cos^{-1} \frac{1}{2} = 60^\circ.$$

$$\therefore x_1 = 2r^{\frac{1}{3}} \cos \frac{\theta}{3} = 2 \cos 20^\circ,$$

$$x_2 = 2r^{\frac{1}{3}} \cos \frac{\theta + 360^\circ}{3} = 2 \cos 140^\circ,$$

$$x_3 = 2r^{\frac{1}{3}} \cos \frac{\theta + 720^\circ}{3} = 2 \cos 260^\circ.$$

$$(2) \quad x^3 - 6x - 4 = 0.$$

$$p = -6, \quad q = -4.$$

$$r = \left(\frac{216}{27} \right)^{\frac{1}{3}} = 2\sqrt{2}. \quad \cos \theta = 2 \left(\frac{27}{216} \right)^{\frac{1}{3}} = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1} \frac{1}{\sqrt{2}} = 45^\circ.$$

$$\therefore x_1 = 2\sqrt{2} \cos 15^\circ, \quad x_2 = 2\sqrt{2} \cos 135^\circ, \quad x_3 = 2\sqrt{2} \cos 255^\circ.$$

18. 解：設 h 表柱體之高， s 表其底之長；則

$$2s^2 + h^2 = 27 \dots\dots\dots (1)$$

$$h \times s^2 = 27 \dots\dots\dots (2)$$

從 (1) 式，

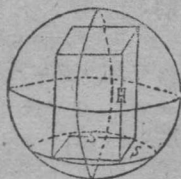
$$s^2 = \frac{27 - h^2}{2}$$

代入 (2) 式，得

$$h^3 - 27h + 54 = 0.$$

$$\therefore h = 3.$$

答：其高度為 3.



19. 解：設底之半徑為 r ，高度為 h ；則

$$50\pi = \pi r^2 h \dots\dots\dots (1)$$

$$105\pi r/2 = 2\pi r h + 2\pi r^2 \dots\dots\dots (2)$$

從 (1) 式，

$$h = 50/r^2 \dots\dots\dots (3)$$

代入 (2) 式，得 $4r^2 - 105r + 200 = 0$.

$$\therefore r = 2\frac{1}{2}$$

代入 (3) 式，得

$$h = 8.$$

答：其底之半徑為 $2\frac{1}{2}$ ，高度為 8。

20. 解：設 h 為圓柱體之高度， r 為其底之半徑；則

$$\pi r^2 h = \frac{4}{9} \times \frac{1}{3} \pi 4^2 \times 6 = \frac{128\pi}{9} \dots\dots (1)$$

$$4 : r = 6 : (6 - h) \dots\dots (2)$$

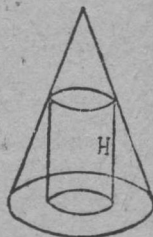
從 (2) 式， $r = \frac{2(6-h)}{3}$ 。

代入 (1) 式，得 $4(6-h)^2 h = 128$ 。

$$h^3 - 12h^2 + 36h - 32 = 0 \dots\dots (3)$$

從 (3) 式， $h = 2$ 。

答：圓柱體之高度為 2。



XXXI. 行列式及消去法

習題 LXXXI

原本第 497 頁

1. 解：
$$\begin{vmatrix} p & q & r \\ q & p & s \\ r & s & p \end{vmatrix} = p^3 + qsr + rsq - q^2p - s^2p - r^2p$$

$$= p^3 + p(q^2 + s^2 + r^2) + 2qrs.$$

2. 解：
$$\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix} = yc + xb + az - ay - bz - cx$$

$$= (b-c)x + (c-a)y + (a-b)z.$$

3. 解：
$$\begin{vmatrix} p & -q & r \\ q & p & -s \\ -r & s & p \end{vmatrix} = p^3 - qsr + qrs + r^2p + s^2p + q^2p$$

$$= p(p^2 + r^2 + s^2 + q^2).$$

4. 解：
$$\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix} = qrs - qrs = 0.$$

$$5. \text{ 解: } \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 1+2^3+3^3-1\cdot 2\cdot 3-1\cdot 2\cdot 3-1\cdot 2\cdot 3 \\ = 36-18=18.$$

$$6. \text{ 解: } \begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix} = 0+45-8-0-5+42=74.$$

$$7. \text{ 解: } \begin{vmatrix} 8 & 9 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 6 & 1 \\ 4 & 3 & 5 & 0 \end{vmatrix} = 8 \begin{vmatrix} 3 & 0 & 0 \\ 1 & 6 & 1 \\ 3 & 5 & 0 \end{vmatrix} - 9 \begin{vmatrix} 2 & 0 & 0 \\ 1 & 6 & 1 \\ 4 & 5 & 0 \end{vmatrix} \\ = 8(-15) - 9(-10) = -120+90 = -30.$$

$$8. \text{ 證: } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \\ = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2, \\ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 b_2 a_3 - b_1 a_2 c_3 - a_1 c_2 b_3, \\ - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \\ = -b_1 a_2 c_3 - a_1 c_2 b_3 - c_1 b_2 a_3 + c_1 a_2 b_3 + a_1 b_2 c_3 + b_1 c_2 a_3. \\ \text{故此三行列式皆相等.}$$

$$9. \text{ 證: } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

$$\text{左} = a_1 b_2 c_3 + c_1 a_2 b_3 + b_1 c_2 a_3 - c_1 b_2 a_3 - b_1 a_2 c_3 - a_1 c_2 b_3,$$

$$\text{右} = a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 + c_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

∴ 左式 = 右式.

10. 解: (1) $a_1b_2c_3d_4 - a_2b_1c_3d_4$.
 (2) $a_1b_2c_3d_4 - a_1b_3c_2d_4$.
 (3) $-a_2b_3c_4d_1$.
 (4) $a_1b_2c_3d_4 + a_1b_3c_4d_2 + a_1b_4c_2d_3 - a_1b_4c_3d_2 - a_1b_3c_2d_4 - a_1b_2c_4d_3$.
 (5) $a_1b_2c_3d_4 - a_1b_4c_3d_2 + a_2b_4c_3d_1 - a_2b_1c_3d_4 + a_1b_1c_3d_2 - a_4b_2c_3d_1$.
11. 解: (1) $a_2b_4c_3d_1e_5$. 其倒置之數爲四,故其號爲+.
 (2) $a_4b_2c_1d_5e_3$. 其倒置之數爲五,故其號爲-.
 (3) $a_5b_4c_3d_2e_1$. 其倒置之數爲十,故其號爲+.
 (4) $c_4d_2a_3e_4b_5$. 其字母之倒置數爲五,附屬字無,故其號爲-.
 (5) $c_1d_2e_3a_4b_5$. 其字母之倒置數爲六,附屬字無,故其號爲+.
 (6) $d_3a_2e_4b_1c_5$. 其字母之倒置數爲五,附屬字之倒置數爲四,倒置數之和爲九,故其號爲-.

習題 LXXXII

原本第 501 頁

1. 解: (1)
$$\begin{vmatrix} 6 & 42 & 27 \\ 8 & -28 & 36 \\ 20 & 35 & 135 \end{vmatrix} = 3 \cdot 4 \cdot 5 \begin{vmatrix} 2 & 14 & 9 \\ 2 & -7 & 9 \\ 4 & 7 & 27 \end{vmatrix}$$
- $$= 3 \cdot 4 \cdot 5 \cdot 2 \cdot 7 \cdot 9 \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -22680.$$
- (2)
$$\begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 4 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$
- (3)
$$\begin{vmatrix} -ab & ac & ae \\ bd & cd & de \\ bf & cf & -ef \end{vmatrix} = a \cdot d \cdot f \cdot b \cdot c \cdot e \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$
- $$= 4abcdef.$$

$$\begin{aligned}
 & \text{2. 證明: } \begin{vmatrix} a_1+ka_2+la_3 & a_2+ma_3 & a_3 \\ b_1+kb_2+lb_3 & b_2+mb_3 & b_3 \\ c_1+kc_2+lc_3 & c_2+mc_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
 & \text{左} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & a_2 & a_3 \\ kb_2 & b_2 & b_3 \\ kc_2 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} la_3 & a_2 & a_3 \\ lb_3 & b_2 & b_3 \\ lc_3 & c_2 & c_3 \end{vmatrix} \\
 & \quad + \begin{vmatrix} a_1 & ma_3 & a_3 \\ b_1 & mb_3 & b_3 \\ c_1 & mc_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_2 & ma_3 & a_3 \\ kb_2 & mb_3 & b_3 \\ kc_2 & mc_3 & c_3 \end{vmatrix} + \begin{vmatrix} la_3 & ma_3 & a_3 \\ lb_3 & mb_3 & b_3 \\ lc_3 & mc_3 & c_3 \end{vmatrix} \\
 & = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 & \text{3. 證: (1)} \quad \begin{vmatrix} c & a & d & b \\ a & c & d & b \\ a & c & b & d \\ c & a & b & d \end{vmatrix} = \begin{vmatrix} c & a+c & d+b & b \\ a & a+c & d+b & b \\ a & a+c & d+b & d \\ c & a+c & d+b & d \end{vmatrix} = 0. \\
 & \text{(2)} \quad \begin{vmatrix} 1 & p & q & r+s \\ 1 & q & r & p+s \\ 1 & r & s & p+q \\ 1 & s & p & q+r \end{vmatrix} = \begin{vmatrix} 1 & p & q & p+q+r+s \\ 1 & q & r & p+q+r+s \\ 1 & r & s & p+q+r+s \\ 1 & s & p & p+q+r+s \end{vmatrix} = 0. \\
 & \text{(3)} \quad \begin{vmatrix} 2 & 1 & 4 & -1 \\ 3 & -1 & 2 & -1 \\ 1 & 2 & 3 & -2 \\ 5 & 0 & 6 & -2 \end{vmatrix} = \begin{vmatrix} 2+3 & 1-1 & 4+2 & -1-1 \\ 3 & -1 & 2 & -1 \\ 1 & 2 & 3 & -2 \\ 5 & 0 & 6 & -2 \end{vmatrix} \\
 & = \begin{vmatrix} 5 & 0 & 6 & -2 \\ 3 & -1 & 2 & -1 \\ 1 & 2 & 3 & -2 \\ 5 & 0 & 6 & -2 \end{vmatrix} = 0.
 \end{aligned}$$

$$\text{4. 證明: } \begin{vmatrix} 1 & p & p^3 \\ 1 & q & q^3 \\ 1 & r & r^3 \end{vmatrix} = (p-q)(q-r)(r-p)(p+q+r).$$

設 $p=q$, 則此行列式 Δ 爲 0, 故 $p-q$ 爲 Δ 之一因式. 因行列式對於 p, q, r 爲對稱, 故 $(q-r)(r-p)$ 亦爲因式; 又因行列式爲 4 次的齊次式, 故 $(p-q)(q-r)(r-p)$ 須再乘以一次的齊次對稱式, 即 $k(p+q+r)$ 方得 Δ , 即

$$\Delta = (p-q)(q-r)(r-p)k(p+q+r).$$

因 Δ 中 qr^3 項之係數爲 1, 乘積中 qr^3 項之係數爲 k .

$$\therefore k=1.$$

$$\text{故 } \Delta = (p-q)(q-r)(r-p)(p+q+r).$$

$$\begin{aligned} 5. \text{ 證明: } \Delta &= \begin{vmatrix} (b+c)^2+a(b+c) & ab & ac \\ (c+a)^2+b(c+a) & (c+a)^2 & bc \\ (a+b)c+(a+b)^2 & bc & (a+b)^2 \end{vmatrix} \\ &= (a+b+c) \begin{vmatrix} b+c & ab & ac \\ c+a & (c+a)^2 & bc \\ a+b & bc & (a+b)^2 \end{vmatrix} \\ &= (a+b+c)^2 \begin{vmatrix} 2 & c+a & a+b \\ c+a & (c+a)^2 & bc \\ a+b & bc & (a+b)^2 \end{vmatrix} \\ &= (a+b+c)^2 [2(c+a)^2(a+b)^2 + 2bc(c+a)(a+b) \\ &\quad - (a+b)^2(c+a)^2 - (c+a)^2(a+b)^2 - 2b^2c^2] \\ &= (a+b+c)^2 [2a^2bc + 2ab^2c + 2abc^2] \\ &= 2abc(a+b+c)^3. \end{aligned}$$

習 題 LXXXIII

原本第 507 頁

$$1. \text{ 解: } \begin{vmatrix} 2 & -1 & 4 & 9 \\ 7 & 5 & -2 & -3 \\ -3 & 2 & 4 & -1 \\ 4 & 7 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 4 & 9 \\ 4 & 7 & 2 & -4 \\ -3 & 2 & 4 & -1 \\ 4 & 7 & 2 & -4 \end{vmatrix} = 0.$$

$$2. \text{ 解: } \begin{vmatrix} 1 & 2 & 3 & -1 & -2 \\ 2 & 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & -1 & -1 & 2 & 1 \\ 0 & 2 & 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 & -2 \\ 0 & -3 & -3 & 0 & 5 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & -1 & -1 & 2 & 1 \\ 0 & 2 & 3 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & -3 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ -5 & -7 & 0 & 3 \\ 2 & 3 & 1 & -1 \end{vmatrix} = - \begin{vmatrix} -3 & -3 & 11 \\ 0 & 1 & 0 \\ -5 & -7 & 17 \end{vmatrix} = 51 - 55 = -4$$

$$3. \text{ 解: } \begin{vmatrix} 3 & 2 & 5 & 10 \\ 6 & 0 & 4 & 0 \\ 9 & 6 & 1 & 30 \\ 12 & 4 & 8 & 20 \end{vmatrix} = 3 \cdot 2 \cdot 10 \cdot 2 \cdot 2 \begin{vmatrix} 1 & 1 & 5 & 1 \\ 1 & 0 & 2 & 0 \\ 3 & 3 & 1 & 3 \\ 2 & 1 & 4 & 1 \end{vmatrix}$$

$$= 240 \begin{vmatrix} -1 & -3 & -1 \\ 0 & -14 & 0 \\ -1 & -6 & -1 \end{vmatrix} = 0.$$

$$4. \text{ 解: } \begin{vmatrix} 6 & -4 & 10 & 28 \\ 18 & 6 & -30 & 21 \\ 12 & 24 & 40 & 28 \\ 9 & -2 & 20 & 14 \end{vmatrix} = 3 \cdot 2 \cdot 10 \cdot 7 \cdot 3 \cdot 4 \begin{vmatrix} 2 & -2 & 1 & 4 \\ 2 & 1 & -1 & 1 \\ 1 & 3 & 1 & 1 \\ 3 & -1 & 2 & 2 \end{vmatrix}$$

$$= \frac{5040}{4} \begin{vmatrix} 6 & -4 & -6 \\ 8 & 1 & -2 \\ 4 & 1 & -8 \end{vmatrix} = 5040 \begin{vmatrix} 3 & -4 & -3 \\ 4 & 1 & -1 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= \frac{5040}{3} \begin{vmatrix} 19 & 9 \\ 11 & -6 \end{vmatrix} = 5040 [(-19)2 - 11 \cdot 3] = -357840.$$

$$5. \text{ 解: } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} b & -a & -0 \\ -a & 0 & b \\ 0 & b & -a \end{vmatrix} = \begin{vmatrix} ab - ab & -a^2 + bc & b^2 - ac \\ b^2 - ac & -ab + ab & bc - a^2 \\ bc - a^2 & -ac + b^2 & ab - ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & bc - a^2 & b^2 - ac \\ b^2 - ac & 0 & bc - a^2 \\ bc - a^2 & b^2 - ac & 0 \end{vmatrix}.$$

$$6. \text{ 解: } \begin{vmatrix} p & 0 & r \\ p & q & 0 \\ 0 & q & r \end{vmatrix} \cdot \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} = \begin{vmatrix} ap + cr & ap & cr \\ ap & ap + bq & bq \\ cr & bq & bq + cr \end{vmatrix}.$$

7. 解:

$$\begin{vmatrix} a & -a & a & a \\ -b & b & b & b \\ c & c & -c & c \\ d & d & d & -d \end{vmatrix} \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & -a & a & a \\ -b & b & b & b \\ c & c & -c & c \\ d & d & d & d \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & -a & a^2 + ab & ac + ad \\ -b & b & ab + b^2 & bc + bd \\ c & c & -ac + bc & -c^2 + cd \\ d & d & ad - bd & cd - d^2 \end{vmatrix}.$$

$$8. \text{ 解: } \begin{vmatrix} l & m & n \\ m & n & l \\ n & l & m \end{vmatrix}^2 = \begin{vmatrix} l & m & n \\ m & n & l \\ n & l & m \end{vmatrix} \cdot \begin{vmatrix} l & m & n \\ m & n & l \\ n & l & m \end{vmatrix}$$

$$= \begin{vmatrix} l^2 + m^2 + n^2 & lm + mn + nl & ln + ml + nm \\ ml + nm + ln & m^2 + n^2 + l^2 & mn + nl + lm \\ nl + lm + mn & nm + ln + ml & n^2 + l^2 + m^2 \end{vmatrix}.$$

9. 證明:

$$\begin{vmatrix} c_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^3$$

左 =

$$\begin{vmatrix} a_1A_1+a_2A_2+a_3A_3 & a_1B_1+a_2B_2+a_3B_3 & a_1C_1+a_2C_2+a_3C_3 \\ b_1A_1+b_2A_2+b_3A_3 & b_1B_1+b_2B_2+b_3B_3 & b_1C_1+b_2C_2+b_3C_3 \\ c_1A_1+c_2A_2+c_3A_3 & c_1B_1+c_2B_2+c_3B_3 & c_1C_1+c_2C_2+c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1+a_2A_2+a_3A_3 & 0 & 0 \\ 0 & b_1B_1+b_2B_2+b_3B_3 & 0 \\ 0 & 0 & c_1C_1+c_2C_2+c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^3$$

10. 證:

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 & \cdots & 0 \\ a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \vdots & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & 0 & 0 & \cdots & 0 \\ a_{43} & a_{44} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n3} & a_{n4} & a_{n1} & \cdots & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}a_{33} \cdots a_{nn}$$

習 題 LXXXIV

原本第 511 頁

1. 解:
$$\begin{cases} 2x+3y-5z=3 \\ x-2y+z=0 \\ 3x+y+3z=7 \end{cases}$$

$$x = \begin{vmatrix} 3 & 3 & -5 \\ 0 & -2 & 1 \\ 7 & 1 & 3 \end{vmatrix} \div \begin{vmatrix} 2 & 3 & -5 \\ 1 & -2 & 1 \\ 3 & 1 & 3 \end{vmatrix} = \frac{-70}{-49} = \frac{10}{7}$$

$$y = \begin{vmatrix} 2 & 3 & -5 \\ 1 & 0 & 1 \\ 3 & 7 & 3 \end{vmatrix} \div \begin{vmatrix} 2 & 3 & -5 \\ 1 & -2 & 1 \\ 3 & 1 & 3 \end{vmatrix} = \frac{-49}{-49} = 1$$

$$z = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 0 \\ 3 & 1 & 7 \end{vmatrix} \div \begin{vmatrix} 2 & 3 & -5 \\ 1 & -2 & 1 \\ 3 & 1 & 3 \end{vmatrix} = \frac{-28}{-49} = \frac{4}{7}.$$

2. 解:
$$\begin{cases} 2x + 4y - 3z = 3, \\ 3x - 8y + 6z = 1, \\ 8x - 2y - 9z = 4. \end{cases}$$

$$x = \begin{vmatrix} 3 & 4 & -3 \\ 1 & -8 & 6 \\ 4 & -2 & -9 \end{vmatrix} \div \begin{vmatrix} 2 & 4 & -3 \\ 3 & -8 & 6 \\ 8 & -2 & -9 \end{vmatrix} = \frac{6 \times 49}{6 \times 49} = 1,$$

$$y = \begin{vmatrix} 2 & 3 & -3 \\ 3 & 1 & 6 \\ 8 & 4 & -9 \end{vmatrix} \div \begin{vmatrix} 2 & 4 & -3 \\ 3 & -8 & 6 \\ 8 & -2 & -9 \end{vmatrix} = \frac{3 \times 49}{6 \times 49} = \frac{1}{2},$$

$$z = \begin{vmatrix} 2 & 4 & 3 \\ 3 & -8 & 1 \\ 8 & -2 & 4 \end{vmatrix} \div \begin{vmatrix} 2 & 4 & 3 \\ 3 & -8 & 6 \\ 8 & -2 & 9 \end{vmatrix} = \frac{2 \times 49}{6 \times 49} = \frac{1}{3}.$$

3. 解:
$$\begin{cases} ax + by + cz = d, \\ a^2x + b^2y + c^2z = d^2, \\ a^3x + b^3y + c^3z = d^3. \end{cases}$$

$$x = \begin{vmatrix} d & b & c \\ d^2 & b^2 & c^2 \\ d^3 & b^3 & c^3 \end{vmatrix} \div \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \frac{d \cdot b \cdot c (d-b)(b-c)(c-d)}{a \cdot b \cdot c (a-b)(b-c)(c-a)}$$

$$= \frac{d(d-b)(d-c)}{a(a-b)(a-c)},$$

$$y = \begin{vmatrix} a & d & c \\ a^2 & d^2 & c^2 \\ a^3 & d^3 & c^3 \end{vmatrix} \div \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \frac{d(d-c)(d-a)}{b(b-c)(b-a)},$$

$$z = \begin{vmatrix} a & b & d \\ a^2 & b^2 & d^2 \\ a^3 & b^3 & d^3 \end{vmatrix} \div \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \frac{d(d-a)(d-b)}{c(c-a)(c-b)}.$$

4. 解:
$$\begin{cases} 2x - 4y + 3z + 4t = -3, \\ 3x - 2y + 6z + 5t = -1, \\ 5x + 8y + 9z + 3t = 9, \\ x - 10y - 3z - 7t = 2. \end{cases}$$

$$x = \begin{vmatrix} -3 & -4 & 3 & 4 \\ -1 & -2 & 6 & 5 \\ 9 & 8 & 9 & 3 \\ 2 & -10 & -3 & -7 \end{vmatrix} \div \begin{vmatrix} 2 & -4 & 3 & 4 \\ 3 & -2 & 6 & 5 \\ 5 & 8 & 9 & 3 \\ 1 & -10 & -3 & -7 \end{vmatrix} = 1,$$

$$y = \begin{vmatrix} 2 & -3 & 3 & 4 \\ 3 & -1 & 6 & 5 \\ 5 & 9 & 9 & 3 \\ 1 & 2 & -3 & -7 \end{vmatrix} \div \begin{vmatrix} 2 & -4 & 3 & 4 \\ 3 & -2 & 6 & 5 \\ 5 & 8 & 9 & 3 \\ 1 & -10 & -3 & -7 \end{vmatrix} = \frac{1}{2},$$

$$z = \begin{vmatrix} 2 & -4 & -3 & 4 \\ 3 & -2 & -1 & 5 \\ 5 & 8 & 9 & 3 \\ 1 & -10 & 2 & -7 \end{vmatrix} \div \begin{vmatrix} 2 & -4 & 3 & 4 \\ 3 & -2 & 6 & 5 \\ 5 & 8 & 9 & 3 \\ 1 & -10 & -3 & -7 \end{vmatrix} = \frac{1}{3},$$

$$t = \begin{vmatrix} 2 & -4 & 3 & -3 \\ 3 & -2 & 6 & -1 \\ 5 & 8 & 9 & 9 \\ 1 & -10 & -3 & 2 \end{vmatrix} \div \begin{vmatrix} 2 & -4 & 3 & -3 \\ 3 & -2 & 6 & -1 \\ 5 & 8 & 9 & 9 \\ 1 & -10 & -3 & 2 \end{vmatrix} = -1.$$

5. 解: $\begin{cases} x+2y-z=0, \\ 3x-y+4z=0, \\ 4x+y+3z=0. \end{cases}$

$$\text{因 } \Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 4 \\ 4 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 4 & 1 & 3 \end{vmatrix} = 0.$$

$$\text{又 } A_1 = \begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} = -3-4 = -7,$$

$$A_2 = - \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 7,$$

$$A_3 = \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = 7.$$

故 $x:y:z = -1:1:1$.

6. 解: $\begin{cases} a_1x+b_1y+(ka_1+lb_1)z=0, \\ a_2x+b_2y+(ka_2+lb_2)z=0, \\ a_3x+b_3y+(ka_3+lb_3)z=0. \end{cases}$

$$\text{因 } \Delta = \begin{vmatrix} a_1 & b_1 & ka_1+lb_1 \\ a_2 & b_2 & ka_2+lb_2 \\ a_3 & b_3 & ka_3+lb_3 \end{vmatrix} = 0.$$

$$\text{又 } A_1 = \begin{vmatrix} b_2 & ka_2+lb_2 \\ b_3 & ka_3+lb_3 \end{vmatrix} = k \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} + l \begin{vmatrix} b_2 & b_2 \\ b_3 & b_3 \end{vmatrix} = k \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix},$$

$$A_2 = - \begin{vmatrix} a_2 & ka_2+lb_2 \\ a_3 & ka_3+lb_3 \end{vmatrix} = l \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix},$$

$$A_3 = - \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}.$$

若 $b_2a_3 - a_2b_3 \neq 0$, 則 $x:y:z = k:l:-1$.

$$7. \text{ 解: } \begin{cases} 4x+3y+z=\lambda x, \\ 3x-4y+7z=\lambda y, \\ x+7y-6z=\lambda z. \end{cases} \quad \text{即} \begin{cases} (4-\lambda)x+3y+z=0, \\ 3x-(4+\lambda)y+7z=0, \\ x+7y-(6+\lambda)z=0. \end{cases}$$

$$\Delta = \begin{vmatrix} 4-\lambda & 3 & 1 \\ 3 & -(4+\lambda) & 7 \\ 1 & 7 & -(6+\lambda) \end{vmatrix} = \begin{vmatrix} 4-\lambda & \lambda & 1 \\ 3 & -\lambda & 7 \\ 1 & -\lambda & -(6+\lambda) \end{vmatrix}$$

$$= \lambda \begin{vmatrix} 7-\lambda & 0 & 8 \\ 3 & -1 & 7 \\ -2 & 0 & -(13+\lambda) \end{vmatrix} = \lambda(75-6\lambda-\lambda^2).$$

若此組方程式爲符合, 則 Δ 必等於零, 即

$$\lambda(75-6\lambda-\lambda^2)=0.$$

$$\therefore \lambda=0.$$

$$\lambda^2+6\lambda-75=0.$$

$$\therefore \lambda = -3 \pm \sqrt{84} = -3 \pm 2\sqrt{21}.$$

習

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LXXXV

原本第 519 頁

1. 解: $6x^2+5x-6=0$ 及 $2x^3+x^2-9x-9=0$.

$$D = \begin{vmatrix} 6 & 5 & -6 & 0 & 0 \\ 0 & 6 & 5 & -6 & 0 \\ 0 & 0 & 6 & 5 & -6 \\ 2 & 1 & -9 & -9 & 0 \\ 0 & 2 & 1 & -9 & -9 \end{vmatrix} = -2 \cdot 3 \begin{vmatrix} 0 & 2 & 21 & 27 & 0 \\ 0 & 6 & 5 & -6 & 0 \\ 0 & 0 & 6 & 5 & 2 \\ 1 & 1 & -9 & -9 & 0 \\ 0 & 2 & 1 & -9 & 3 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 2 \begin{vmatrix} 1 & 21 & 27 & 0 \\ 3 & 5 & -6 & 0 \\ 0 & 6 & 5 & 2 \\ 1 & -5 & -14 & 1 \end{vmatrix} = 12 \begin{vmatrix} 1 & 21 & 27 & 0 \\ 3 & 5 & -6 & 0 \\ -2 & 16 & 33 & 0 \\ 1 & -5 & -14 & 1 \end{vmatrix}$$

$$= 12 \begin{vmatrix} 1 & 21 & 27 \\ 0 & -58 & -87 \\ 0 & 58 & 87 \end{vmatrix} = 0.$$

$$\begin{vmatrix} 6 & 5 & -6 & 0 \\ 0 & 6 & 5 & -6 \\ 1 & -9 & -9 & 0 \\ 2 & 1 & -9 & -9 \end{vmatrix} \div - \begin{vmatrix} 0 & 5 & -6 & 0 \\ 0 & 6 & 5 & -6 \\ 2 & -9 & -9 & 0 \\ 0 & 1 & -9 & -9 \end{vmatrix} = -\frac{3}{2}.$$

2. 解: $a_0x^2+a_1x+a_2=0$ 及 $b_0x^2+b_1x+b_2=0$.

$$D = \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}$$

3. 解: $ax^3+bx^2+cx+d=0$ 及 $x^3=1$.

$$D = \begin{vmatrix} a & b & c & d & 0 & 0 \\ 0 & a & b & c & d & 0 \\ 0 & 0 & a & b & c & d \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} b & c & d+a & 0 & 0 \\ a & b & c & d & 0 \\ 0 & a & b & c & d \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{vmatrix}$$

$$= - \begin{vmatrix} c & d+a & b & 0 \\ b & c & d+a & 0 \\ a & b & c & d \\ 1 & 0 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} d+a & b & c \\ c & d+a & b \\ b & c & d+a \end{vmatrix}$$

$$= -[(a+d)^3 + b^3 + c^3 - 3bc(a+d)].$$

4. 解: (1) $x^3+px+q=0$.

$$f(x) = x^3 + px + q = 0, \quad f'(x) = 3x^2 + p = 0.$$

$$D = \begin{vmatrix} 1 & 0 & p & q & 0 \\ 0 & 1 & 0 & p & q \\ 3 & 0 & p & 0 & 0 \\ 0 & 3 & 0 & p & 0 \\ 0 & 0 & 3 & 0 & p \end{vmatrix} = \begin{vmatrix} 1 & 0 & p & q \\ 0 & -2p & -3q & 0 \\ 3 & 0 & p & 0 \\ 0 & 3 & 0 & p \end{vmatrix}$$

$$= \begin{vmatrix} -2p & -3q & 0 \\ 0 & -2p & -3q \\ 3 & 0 & p \end{vmatrix} = 4p^3 + 27q^2.$$

(2) $ax^3+bx^2+c=0$.

$$D = \begin{vmatrix} a & b & 0 & 0 & 0 \\ 0 & a & b & 0 & c \\ 3a & 2b & 0 & 0 & 0 \\ 0 & 3a & 2b & 0 & 0 \\ 0 & 0 & 3a & 2b & 0 \end{vmatrix} \div a = \begin{vmatrix} a^2 & ab & 0 & ac \\ -ab & 0 & -3ac & 0 \\ 3a^2 & 2ab & 0 & 0 \\ 0 & 3a^2 & 2ab & 0 \end{vmatrix} \div a^4$$

$$= \frac{a \cdot a \cdot a \cdot ac}{a^4} \begin{vmatrix} a & b & 0 & 1 \\ -b & 0 & -3c & 0 \\ 3a & 2b & 0 & 0 \\ 0 & 3a & 2b & 0 \end{vmatrix} = -c \begin{vmatrix} -b & 0 & -3c \\ 3a & 2b & 0 \\ 0 & 3a & 2b \end{vmatrix}$$

$$= c(4b^3 + 27a^2c).$$

5. 解: $f(x) = x^3 + x^2 - 8x - 12 = 0$, $f'(x) = 3x^2 + 2x - 8 = 0$.

$$D = \begin{vmatrix} 1 & 1 & -8 & -12 & 0 \\ 0 & 1 & 1 & -8 & -12 \\ 3 & 2 & -8 & 0 & 0 \\ 0 & 3 & 2 & -8 & 0 \\ 0 & 0 & 3 & 2 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -8 & -12 \\ -1 & 16 & 36 & 0 \\ 3 & 2 & -8 & 0 \\ 0 & 3 & 2 & -8 \end{vmatrix}$$

$$= 2 \cdot 4 \begin{vmatrix} 1 & 1 & -4 & -3 \\ -1 & 16 & 18 & 0 \\ 3 & 2 & -4 & 0 \\ 0 & 3 & 1 & -2 \end{vmatrix} = 8 \begin{vmatrix} 17 & 14 & -3 \\ -1 & 8 & 9 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 17 & -3 & -3 \\ -1 & 9 & 9 \\ 3 & -2 & -2 \end{vmatrix} = 0.$$

故 $f(x) = 0$, 有一二重根.

$$\begin{vmatrix} 1 & 1 & -8 & -12 \\ 2 & -8 & 0 & 0 \\ 3 & 2 & -8 & 0 \\ 0 & 3 & 2 & -8 \end{vmatrix} \div \begin{vmatrix} 0 & 1 & -8 & -12 \\ 3 & -8 & 0 & 0 \\ 0 & 2 & -8 & 0 \\ 0 & 3 & 2 & -8 \end{vmatrix}$$

$$= 2 \cdot 4 \cdot 2 \begin{vmatrix} 1 & 1 & -4 & -3 \\ 0 & -5 & 4 & 3 \\ 0 & -1 & 8 & 9 \\ 0 & 3 & 1 & -2 \end{vmatrix} \div (3 \cdot 4 \cdot 2 \cdot 2) \begin{vmatrix} 1 & -4 & -3 \\ 1 & -2 & 0 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -17 & 4 & 11 \\ -25 & 8 & 25 \\ 0 & 1 & 0 \end{vmatrix} \div 3 \begin{vmatrix} 2 & 3 \\ 13 & 7 \end{vmatrix} = -\frac{6 \times 25}{3 \times 25} = -2.$$

6. 解: $f(x) = x^3 - 3xy + 2y^2 - 16x - 28y = 0 \dots\dots\dots (1)$

$f(x) = x^2 - xy - 2y^2 - 5x - 5y = 0 \dots\dots\dots (2)$

$$D = \begin{vmatrix} 1 - (3y + 16) & (2y^2 - 28y) & 0 \\ 0 & 1 & -(3y + 16) & (2y^2 - 28y) \\ 1 & -(y + 5) & -(2y^2 + 5y) & 0 \\ 0 & 1 & -(y + 5) & -(2y^2 + 5y) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -(3y + 16) & (2y^2 - 28y) \\ (2y + 11) & -(4y^2 - 23y) & 0 \\ 1 & -(y + 5) & -(2y^2 + 5y) \end{vmatrix} = 0.$$

$$8y^4 - 26y^3 - 115y^2 - 4y^4 + 4y^3 + 478y^2 + 1540y + 8y^4 - 158y^3$$

$$+ 644y^2 - 12y^3 - 160y^3 - 677y^2 - 880y = 0.$$

$$-330y^3 + 330y^2 + 660y = 0.$$

$$y^3 - y^2 - 2y = 0.$$

$$y(y^2 - y - 2) = 0.$$

$$y(y+1)(y-2) = 0.$$

$$\therefore y = 0, -1, 2.$$

設 $y=0$ 代入 (1) 式, 得 $x(x-16)=0$.

代入 (2) 式, 得 $x(x-5)=0$.

故有一公解 $(0, 0)$

設 $y=-1$ 代入 (1) 式, 得 $(x-10)(x-3)=0$.

代入 (2) 式, 得 $(x-3)(x-1)=0$.

故有一公解 $(3, -1)$

設 $y=2$ 代入 (1) 式, 得 $(x+2)(x-24)=0$.

代入 (2) 式, 得 $(x+2)(x-9)=0$.

故有一公解 $(-2, 2)$

答. $x, y=0, 0; 3, -1; -2, 2$ 爲兩方程之解答.

XXXII. 無窮級數之收斂

例題之解答

原本第 526 頁

1. 解: (1) $\frac{1}{1} + \frac{1}{\sqrt{2^3}} + \frac{1}{\sqrt{3^3}} + \dots + \frac{1}{\sqrt{n^3}} + \dots \dots \dots$ (1)

級數 (1) 爲 P 級數, 其中 $P = \frac{3}{2} > 1$; 故級數 (1) 爲收斂.

(2) $\frac{2}{2 \cdot 3 \cdot 4} + \frac{4}{3 \cdot 4 \cdot 5} + \frac{6}{4 \cdot 5 \cdot 6} + \dots + \frac{2n}{(n+1)(n+2)(n+3)} + \dots$ (1)

設 $u_n = \frac{2n}{(n+1)(n+2)(n+3)}$,

$$v_n = \frac{2}{n^2}.$$

$$\frac{u_n}{v_n} = \frac{n^3}{(n+1)(n+2)(n+3)} < 1$$

$\therefore \Sigma v_n$ 爲收斂, 故級數 (1) 爲收斂.

(3) $\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \dots$ (1)

$$\text{設 } u_n = \frac{1}{[a+(n-1)b][a+nb]} = \frac{1}{a^2-ab+2nab+n(n-b)b^2},$$

$$v_n = \frac{1}{b^2n^2+(2ab-b^2)n+a^2-ab}$$

$$\frac{u_n}{v_n} = \frac{b^2n^2}{b^2n^2+(2ab-b^2)n+a^2-ab}$$

當 n 十分大時, $\frac{u_n}{v_n} < 2$, 不論 n 之值如何, 因 $\Sigma \frac{1}{b^2n^2}$ 為收斂, 故級數 (1) 為收斂。

2. 解: (1) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{2n} + \dots \dots \dots (1)$

級數 (1) 之各項為調和級數各項之半, 故為發散級數。

(2) $\frac{1}{a} + \frac{1}{2a+b} + \frac{1}{3a+b} + \dots \dots \dots (1)$

設 $u_n = \frac{1}{na+b},$

$$v_n = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{n}{na+b} = \frac{1}{a+\frac{b}{n}} > \frac{1}{2a}$$

因 $\Sigma \frac{1}{n}$ 為發散, 故級數 (1) 為發散。

(3) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{n}} + \dots \dots \dots (1)$

設 $u_n = \frac{1}{\sqrt{n+1}},$

$$v_n = \frac{1}{n+1}$$

$$\frac{u_n}{v_n} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1} > 1.$$

故級數 (1) 為發散。

(4) $\frac{2}{1+2\sqrt{2}} + \frac{3}{1+3\sqrt{3}} + \frac{4}{1+4\sqrt{4}} + \dots \dots \dots (1)$

設

$$u_n = \frac{n+1}{1+(n+1)\sqrt{n+1}},$$

$$v_n = \frac{1}{n+1}.$$

$$\frac{u_n}{v_n} = \frac{(n+1)^2}{1+(n+1)^{\frac{3}{2}}} = \frac{\sqrt{n+1}}{\frac{1}{1+(n+1)^{\frac{3}{2}}}} > 1.$$

∴ 級數 (1) 爲發散。

3. 解: (1) $u_n = \frac{2n-1}{(n+1)(n+2)}.$

$$\frac{1}{2 \cdot 3} + \frac{3}{3 \cdot 4} + \frac{5}{4 \cdot 5} + \frac{7}{5 \cdot 6} + \dots \dots \dots (1)$$

設

$$v_n = \frac{1}{n}.$$

$$\frac{u_n}{v_n} = \frac{2n^2 - n}{n^2 + 3n + 2} = \frac{2 - \frac{1}{n}}{1 + \frac{3}{n} + \frac{2}{n^2}}.$$

當 n 等於 5 以下之各值時, $\frac{u_n}{v_n} > 1.$

∴ 級數 (1) 爲收斂。

(2) $u_n = \frac{\sqrt{n}}{n^2+1}.$

$$\frac{1}{2} + \frac{\sqrt{2}}{5} + \frac{\sqrt{3}}{10} + \frac{\sqrt{4}}{17} + \dots \dots \dots (1)$$

設

$$v_n = \frac{1}{n^{\frac{3}{2}}}.$$

$$\frac{u_n}{v_n} = \frac{n^2}{n^2+1} < 1.$$

∴ $\sum \frac{1}{n^{\frac{3}{2}}}$ 爲收斂, ∴ $\sum u_n$ 爲收斂。

$$(3) \quad u_n = \frac{n^2 - (n-1)^2}{n^3 + (n+1)^3} = \frac{2n-1}{2n^3 + 3n^2 + 3n + 1}$$

$$\frac{1}{9} + \frac{3}{35} + \frac{5}{91} + \frac{7}{189} + \dots$$

設

$$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{2n^3 - n^2}{2n^3 + 3n^2 + 3n + 1} < 1.$$

∴ Σu_n 爲收斂.

習

題

LXXXVI

原本第 530 頁

1. 解: $\frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{2^{n+1}+1}}{\frac{1}{2^n+1}} = \frac{2^n+1}{2^{n+1}+1}$$

$$\lim \frac{2^n+1}{2^{n+1}+1} = \frac{1}{2} < 1.$$

答: 此級數爲收斂級數.

2. 解: $\frac{1}{1} + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1 \cdot 2 \cdot 3 \dots (n+1)}{1 \cdot 3 \cdot 5 \dots (2n+1)}}{\frac{1 \cdot 2 \cdot 3 \dots n}{1 \cdot 3 \cdot 5 \dots (2n-1)}} = \frac{n+1}{2n+1} = \frac{1 + \frac{1}{n}}{2 + \frac{1}{n}}$$

$$\lim \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1.$$

答: 此級數爲收斂級數.

3. 解: $\frac{2}{2 \cdot 3} + \frac{4}{3 \cdot 4} + \frac{6}{4 \cdot 5} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{2(n+1)}{(n+2)(n+3)} = \frac{(n+1)^2}{n(n+3)} = \frac{n^2+2n+1}{n^2+3n}$$

$$a' - a = 3 - 2 = 1.$$

答：此級數為發散級數。(參看 § 953)

4. 解： $\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n} = \frac{(n+1)\left(\frac{2}{3}\right)}{n} = \frac{2n+2}{3n} = \frac{2+\frac{2}{n}}{3}$$

$$\lim \frac{u_{n+1}}{u_n} = \frac{2}{3} < 1.$$

答：此級數為收斂級數。

5. 解： $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[4]{3}} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+2\sqrt{3}}}{\frac{1}{n+1\sqrt{3}}} = \frac{3^{n+1}}{3^{n+2}} = \frac{1}{3} < 1.$$

$$\lim \frac{u_{n+1}}{u_n} = \frac{1}{3} < 1.$$

答：此級數為發散級數。

6. 解： $\frac{1}{a^2} + \frac{1}{a^2+1} + \frac{1}{a^2+2} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{a^2+n-1}{a^2+n} = \frac{n+(a^2-1)}{n+a^2}$$

$$a^2 - (a^2 - 1) = 1.$$

答：此級數為發散級數。

7. 解： $\frac{2}{4} + \frac{2 \cdot 4}{4 \cdot 7} + \frac{2 \cdot 4 \cdot 6}{4 \cdot 7 \cdot 10} + \dots + \frac{2 \cdot 4 \cdot 6 \dots 2n}{4 \cdot 7 \cdot 10 \dots (3n+1)} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{4 \cdot 7 \cdot 10 \cdots (3n+4)} = \frac{2n+2}{3n+4} = \frac{2 + \frac{2}{n}}{3 + \frac{4}{n}}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{2}{3} < 1.$$

答：此級數為收斂級數。

8. 解： $\frac{2}{4} + \frac{2 \cdot 3}{4 \cdot 5} + \frac{2 \cdot 3 \cdot 4}{4 \cdot 5 \cdot 6} + \cdots + \frac{2 \cdot 3 \cdot 4 \cdots (n+1)}{4 \cdot 5 \cdot 6 \cdots (n+3)} + \cdots$

$$\frac{u_{n+1}}{u_n} = \frac{2 \cdot 3 \cdot 4 \cdots (n+1)(n+2)}{4 \cdot 5 \cdot 6 \cdots (n+3)(n+4)} = \frac{n+2}{n+4}.$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1, \quad a' - a = 4 - 2 = 2 > 1.$$

答：此級數為收斂級數。

9. 解： $\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} + \cdots$

$$\frac{u_{n+1}}{u_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} = \frac{2n+1}{2n+2} = \frac{2 + \frac{1}{n}}{2 + \frac{2}{n}} = \frac{1}{1 + \frac{1}{2n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1, \quad a' - a = 1 - \frac{1}{2} = \frac{1}{2} < 1.$$

答：此級數為發散級數。

10. 解： $u_n = \frac{n+1}{n(n+2)}$

其前四項為 $\frac{2}{1 \cdot 3} + \frac{3}{2 \cdot 4} + \frac{4}{3 \cdot 5} + \frac{5}{4 \cdot 6} + \cdots$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{n+2}{(n+1)(n+3)}}{\frac{n+1}{n(n+2)}} = \frac{n^3 + 4n^2 + 4n}{n^3 + 5n^2 + 7n + 3}$$

$$\lim \frac{u_{n+1}}{u_n} = 1, \text{ 而 } a' - a = 5 - 4 = 1.$$

答：此級數為發散級數。

11. 解： $u_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n^3+1}}$.

其前四項為 $\frac{1}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}}{\sqrt[3]{9}} + \frac{\sqrt[3]{3}}{\sqrt[3]{28}} + \frac{\sqrt[3]{4}}{\sqrt[3]{65}} + \dots$

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{\frac{\sqrt[3]{(n+1)^3+1}}{\sqrt[3]{n^3+1}}}{\frac{\sqrt[3]{n}}{\sqrt[3]{n^3+1}}} = \left(\frac{n^4+n^3+n+1}{n^4+3n^3+3n^2+2n} \right)^{\frac{1}{3}} \\ &= \frac{1}{\left(1 + \frac{2}{n} + \frac{1}{n^2} + \dots\right)^{\frac{1}{3}}} = \frac{1}{1 + \frac{2}{3} + \dots} \end{aligned}$$

$$\lim \frac{u_{n+1}}{u_n} = 1, \text{ 而 } a_n = \frac{2}{3} < 1.$$

答：此級數為發散級數。

或設

$$\begin{aligned} v_n &= \frac{1}{n} = \frac{1}{\sqrt[3]{n^3}} \\ \frac{u_n}{v_n} &= \frac{\sqrt[3]{n} \cdot \sqrt[3]{n^3}}{\sqrt[3]{n^3+1}} = \sqrt[3]{\frac{n^4}{n^3+1}} \\ \lim \frac{u_n}{v_n} &= \infty. \end{aligned}$$

因 $\sum \frac{1}{n}$ 為發散， $\therefore \sum u_n$ 為發散。

12. 解： $u_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1} + n}$.

其前四項為 $(\sqrt{2}-1) + (\sqrt{5}-2) + (\sqrt{10}-3) + (\sqrt{17}-4)$.

設

$$\begin{aligned} v_n &= \frac{1}{3n} \\ \frac{u_n}{v_n} &= \frac{3n}{\sqrt{n^2+1} + n} > 1. \end{aligned}$$

因 Σv_n 爲發散，故 Σu_n 亦爲發散。

13. 解: $\frac{u_{n+1}}{u_n} = \frac{2n}{2n+3}$

$$\frac{u_{n+1}}{u_n} = \frac{1}{1 + \frac{\frac{3}{2}}{n}}, \quad a_n = \frac{3}{2} > 1.$$

答: 此級數爲收斂級數。

14. 解: $\frac{u_{n+1}}{u_n} = \frac{3n^3 - 2n^2}{3n^3 + n^2 + 1} = \frac{n^3 - \frac{2}{3}n^2}{n^3 + \frac{1}{3}n^2 + \frac{1}{3}}$

$$a' - a = \frac{1}{3} - \left(-\frac{2}{3}\right) = 1.$$

答: 此級數爲發散級數。

15. 解: $1 + \frac{3}{5}x + \frac{3 \cdot 6}{5 \cdot 8}x^2 + \frac{3 \cdot 6 \cdot 9}{5 \cdot 8 \cdot 11}x^3 + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{3n}{3n+2}x = \frac{3}{3 + \frac{2}{n}}x.$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x.$$

答: $x < 1$ 時，此級數爲收斂級數。

16. 解: $\frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{x^n}{1+x^{n+1}}}{\frac{x^{n-1}}{1+x^n}} = \frac{x+x^{n+1}}{1+x^{n+1}}$$

欲 $\frac{x+x^{n+1}}{1+x^{n+1}} < 1$ ，則 $x < 1$ 。

答: $x < 1$ 時，此級數爲收斂級數。

$$17. \text{ 證: } \frac{u_{n+1}}{u_n} = \frac{a(a+1)(a+2)\cdots(a+n)}{a(a+1)(a+2)\cdots(a+n-1)n} = \frac{a+n}{n} > 1.$$

當 a 爲正值時, $\frac{u_{n+1}}{u_n} > 1$, $\lim \frac{u_{n+1}}{u_n} = 1$.

故當 a 爲正值時, 此級數乃爲發散級數.

$$18. \text{ 證: } \quad \therefore \sqrt[n]{u_n} < r.$$

$$\therefore u_n < r^n.$$

$$\text{即 } n=1 \text{ 時, } \quad u_1 < r;$$

$$n=2 \text{ 時, } \quad u_1 < r^2;$$

$$\dots\dots\dots$$

$$n=k \text{ 時, } \quad u_k < r^k.$$

相加, 得 $u_1 + u_2 + \cdots + u_k + \cdots < r + r^2 + \cdots + r^k + \cdots$

當 $r < 1$, Σr^n 爲收斂, 故 Σu_n 亦爲收斂.

習 題 LXXXVII

原本第 534 頁

$$1. \text{ 解: (1) } \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \dots.$$

$$\text{設} \quad |u_n| = \frac{1}{\sqrt{n+1}}.$$

$$\text{因} \quad |u_n| < |u_{n-1}|,$$

$$\text{且} \quad \lim u_n = \lim \frac{1}{\sqrt{n+1}} = 0.$$

答: 此級數爲收斂級數.

$$(2) \quad \frac{1}{\sqrt{2}} - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[4]{2}} - \dots.$$

$$\lim \frac{1}{\sqrt[n+1]{2}} = 1 > 0.$$

答: 此級數爲發散級數.

$$(3) \frac{3}{3} - \frac{3.5}{3.6} + \frac{3.5.7}{3.6.9} - \frac{3.5.7.9}{3.6.9.12} + \dots$$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{2n+3}{3(2n+1)} \right|$$

$$\lim \left| \frac{u_{n+1}}{u_n} \right| = \frac{2}{3}$$

答：此級數為收斂級數。

$$2. \text{ 解: } (1) \frac{1}{1-x} + \frac{1}{1-2x} + \frac{1}{1-3x} + \dots + \frac{1}{1+(-1)^n nx} + \dots (1)$$

當 $x=1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}$ 時，級數中有一項等

於 ∞ ，故該級數為發散。

當 $x=0$ 時，該級數變為 $1+1+1+\dots$ ，故為發散。

當 $x=0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots$ 以外之其他

任一定值時，在該級數中必能取得一項，說 k 項，使 $|kx| > 1$ ，故該級數在 k 項以後成一間號級數。

$$\text{因 } \lim |u_n| = \lim \left| \frac{1}{1+(-1)^n nx} \right| = 0.$$

故該間號級數欲為收斂僅須

$$\left| \frac{1}{1+(-1)^{n+1}(n+1)x} \right| < \left| \frac{1}{1+(-1)^n nx} \right|,$$

$$\text{即 } |1+(-1)^n nx| < |1+(-1)^{n+1}(n+1)x|.$$

當 $(-1)^{n+1}(n+1)x > 0$ 及 $(-1)^n nx < 0$ 時，上列不等式對 x 之任何值均能成立。

當 $(-1)^{n+1}(n+1)x < 0$ 及 $(-1)^n nx > 0$ 時，上列不等式可寫成

$$1+n|x| < (n+1)|x| - 1$$

$$\text{即 } |x| > 2 \text{ 或 } -2 > x > 2.$$

$\therefore -2 > x > 2$ 時，不等式方能成立，而級數為收斂。

$$(2) \frac{x}{1+x^2} + \frac{x^3}{1+2x^4} + \frac{x^5}{1+3x^6} + \dots + \frac{x^{2n-1}}{1+nx^{2n}} + \dots$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{x^{2n+1}}{1+(n+1)x^{2n+2}}}{\frac{x^{2n-1}}{1+nx^{2n}}} = \frac{x^2 + nx^{2n+2}}{1+(n+1)x^{2n+2}}$$

因 $x =$ 任何值時, $x^2 + nx^{2n+2} < 1 + nx^{2n+2} + x^{2n+2}$,
 即 $x^2 < 1 + x^{2n+2}$.

∴ x 為任何值時, 級數均為收斂。

3. 解: ∵ $|a_n| < c$.
 ∴ $|a_n| \cdot |u_n| < c |u_n|$

故 $|a_1u_1| + |a_2u_2| + |a_3u_3| + \dots < c(|u_1| + |u_2| + |u_3| + \dots)$
 $= c\{|u_1| + |u_2| + |u_3| + \dots\}$, 因 $c\{|u_1| + |u_2| + |u_3| + \dots\}$ 為
 收斂級數, 故 $|a_1u_1| + |a_2u_2| + |a_3u_3| + \dots$ 亦為收斂級數。

∴ $a_1u_1 + a_2u_2 + a_3u_3 + \dots$ 為收斂級數。

4. 證: $|x| < 2$ 時, 該級數之斂散未決, 今用下法可證明其亦
 為收斂。

$$\text{設 } S_{2n} = \frac{1}{1-x} + \frac{1}{1+2x} + \dots + \frac{1}{1+(-1)^{2n-1}(2n-1)x} \\ + \frac{1}{1+(-1)^{2n}2nx} \dots \dots \dots (1)$$

$$S_{2n+1} = \frac{1}{1-x} + \frac{1}{1+2x} + \dots + \frac{1}{1+(-1)^{2n}2nx} \\ + \frac{1}{1+(-1)^{2n+1}(2n+1)x} \dots \dots \dots (2)$$

$$\text{又設 } S'_n = \left[\frac{1}{1-x} + \frac{1}{1+2x} \right] + \dots + \left[\frac{1}{1+(-1)^{2n-1}(2n-1)x} \right. \\ \left. + \frac{1}{1+(-1)^{2n}2nx} \right] \\ = \frac{2+x}{1+x-2x^2} + \dots + \frac{2+x}{1+x-2n(2n-1)x^2} \dots \dots \dots (3)$$

$$S_{2n+1} = S_{2n} + \frac{1}{1+(-1)^{2n+1}(2n+1)x}$$

$$\lim S_{2n+1} = \lim S_{2n} + \lim \frac{1}{1+(-1)^{2n+1}(2n+1)x} = \lim S_{2n}$$

又 $S'_n = S_{2n}$

$$\lim S'_n = \lim S_{2n} = \lim S_{2n+1}$$

故級數 (1) 與下列級數

$$\frac{2+x}{1+x-2x^2} + \dots + \frac{2+x}{1+x-2n(2n-1)x^2} + \dots \dots \dots (4)$$

之斂散情形相同。

$$\begin{aligned} \frac{u'_{n+1}}{u'_n} &= \frac{1+x-2n(2n-1)x^2}{1+x-(2n+2)(2n+1)x^2} \\ &= \frac{4x^2n^2-2x^2n-x-1}{4x^2n^2+6x^2n+2x^2-x-1} \\ &= \frac{n^2-\frac{1}{2}n-\frac{x+1}{4x^2}}{n^2+\frac{3}{2}n+\frac{2x^2-x-1}{4x^2}} \\ a'-a &= \frac{3}{2} - \left(-\frac{1}{2}\right) = 2 > 1. \end{aligned}$$

∴ 級數 (4) 因而級數 (1) 除 $x=0, 1, -\frac{1}{2}, \dots$ 等值以外對 x 之任何值均為收斂。

$$\begin{aligned} S &= a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n-1}a_n + \dots \\ &= (a_1 - a_2) + (a_3 - a_4) + \dots \\ &= c_1 - (a_2 - a_3) - (a_4 - a_5) \dots \end{aligned}$$

因括號中各量均為正，故 $(a_1 - a_2), (a_1 - a_2) + (a_3 - a_4), \dots$ 較 S 小及 $a_1, a_1 - (a_2 - a_3), a_1 - (a_2 - a_3) - (a_4 - a_5), \dots$ 較 S 大。故 $a_1, a_1 - a_2, a_1 - a_2 + a_3, \dots$ 之和相間地大於及小於 S 。

習 題 LXXXVIII

原本第 538 頁

1. 解: $1 + mx + \frac{m^2x^2}{2!} + \frac{m^3x^3}{3!} + \dots$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{m^n}{n!}}{\frac{m^{n+1}}{(n+1)!}} = \frac{n+1}{m}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \infty. \quad \therefore \mu = \infty.$$

$$2. \text{ 解: } 2(2x)^2 + 3(2x)^3 + 2(2x)^4 + 3(2x)^5 + \dots \\ = [2(2x)^2 + 2(2x)^4 + \dots] + [3(2x)^3 + 3(2x)^5 + \dots].$$

$$\lim \frac{2(2x)^{2n+2}}{2(2x)^{2n}} = \lim 4x^2 < 1, \quad x^2 < \frac{1}{4}, \quad |x| < \frac{1}{2}.$$

$$\lim \frac{3(2x)^{2n+3}}{3(2x)^{2n+1}} = \lim 4x^2 < 1, \quad x^2 < \frac{1}{4}, \quad |x| < \frac{1}{2}.$$

$$\therefore \mu = \frac{1}{2}.$$

$$3. \text{ 解: } mx + \frac{m(m-2)}{2!}x^2 + \frac{m(m-2)(m-4)}{3!}x^3 + \dots$$

$$\frac{a_n}{a_{n+1}} = \frac{m(m-2)(m-4)\dots(m-2n+2)}{n!} \cdot \frac{(n+1)!}{m(m-2)\dots(m-2n)} \\ = \frac{n+1}{m-2n}.$$

$$\lim \frac{a_n}{a_{n+1}} = \lim \frac{n+1}{m-2n} = -\lim \frac{1+1/n}{2-m/n} = -\frac{1}{2}.$$

$$\therefore \mu = \frac{1}{2}.$$

$$4. \text{ 解: } \frac{3x}{x+4} + \frac{1}{2}\left(\frac{3x}{x+4}\right)^2 + \frac{1}{3}\left(\frac{3x}{x+4}\right)^3 + \dots \dots \dots (1)$$

$$\frac{u_{n+1}}{u_n} = \frac{\left(\frac{3x}{x+4}\right)^{n+1} \cdot n}{(n+1)\left(\frac{3x}{x+4}\right)^n} = \left(\frac{n}{n+1}\right)\left(\frac{3x}{x+4}\right)$$

$$\lim \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{3x}{x+4} \right|.$$

如 $|3x| < |x+4|$, 則級數為收斂。

當 $x > 0$ 時, $|3x| < |x+4|$ 可寫成

$$3x < x+4.$$

$$x < 2.$$

當 $x < 0$ 時, 設 $x = -y$, 故 $|3x| < |x+4|$ 可寫成

$$|-3y| < |4-y|.$$

$$3y < 4-y.$$

$$y < 1.$$

$$x > -1.$$

∴ $-1 < x < 2$ 時，級數 (1) 爲收斂。

當 $x = 2$ 時，級數 (1) 成爲 $1 + \frac{1}{2} + \frac{1}{3} + \dots$ 。

∴ 級數 (1) 爲發散。

當 $x = -1$ 時，級數 (1) 成爲 $-1 + \frac{1}{2} - \frac{1}{3} + \dots$ 。

∴ 級數 (1) 爲收斂。

故 $-1 \leq x < 2$ 時，級數 (1) 爲收斂。

5. 解：
$$\frac{x}{x^2+1} + \left(\frac{x}{x^2+1}\right)^2 + \left(\frac{x}{x^2+1}\right)^3 + \dots$$

此乃 $\frac{x}{x^2+1}$ 之冪級數，於 $\left|\frac{x}{x^2+1}\right| < 1$ 時將爲收斂。因 $|x|$ 恆小於 $|x^2+1|$ ，故此級數於 x 爲任何值時，皆爲收斂者。

6. 解：
$$\dots (3x)^{-3} + (3x)^{-2} + (3x)^{-1} + 1 + 2x + (2x)^2 + (2x)^3 + \dots$$

此級數中之 $(3x)^{-1} + (3x)^{-2} + (3x)^{-3} + \dots$ 爲 $(3x)^{-1}$ 之等比冪級數。於 $|(3x)^{-1}| < 1$ 即 $|x| > \frac{1}{3}$ 時，爲收斂者。

又此級數中之 $1 + 2x + (2x)^2 + (2x)^3 + \dots$ 爲 $(2x)$ 之等比冪級數。於 $|x| < \frac{1}{2}$ 時，爲收斂者。

故此已知級數，於 $\frac{1}{3} < |x| < \frac{1}{2}$ 時，爲收斂者。

XXXIII. 無窮級數之演算

習

題

LXXXIX

原本第 551 頁

1. 證：
$$f(x) = 1 + x + x^2 + x^3 + \dots$$

$$f(x) = 1 + x + x^2 + x^3 + \dots$$

$f(x) \cdot f(x) = 1 + 1$	$x + 1$	$x^2 + 1$	$x^3 + \dots$
$+ 1$	$+ 1$	$+ 1$	
	$+ 1$	$+ 1$	
		$+ 1$	

$$\therefore (1 + x + x^2 + \dots)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots$$

2. 證: $f(x) = 1 + x + x^2 + x^3 + \dots$
 $f(x) \cdot f(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$

$$\begin{array}{r|l} f(x) \cdot f(x) \cdot f(x) = 1 + 2 & x + 3 & x^2 + 4 & x^3 + \dots \\ & + 1 & + 2 & 3 \\ & & + 1 & 2 \\ & & & 1 \end{array}$$

$$\therefore (1 + x + x^2 + \dots)^3 = 1 + 3x + 6x^2 + 10x^3 + \dots$$

3. 證:
$$\begin{array}{r|l} 1 + 0 + 1 + 0 + 1 + \dots & 1 + 1 + 1 + \dots \\ 1 + 1 + 1 + 1 + 1 + \dots & 1 - 1 + 1 - \dots \\ \hline -1 + 0 - 1 + 0 - \dots & \\ -1 - 1 - 1 - 1 - \dots & \\ \hline 1 + 0 + 1 + \dots & \\ 1 + 1 + 1 + \dots & \\ \hline -1 + 0 - \dots & \end{array}$$

$$\therefore (1 + x^2 + x^4 + \dots) / (1 + x + x^2 + \dots) = 1 - x + x^2 - \dots$$

4. 解: $1 - x + 2x^2 = (1 + a_1x + a_2x^2 + \dots)^2$
 $f(x) = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$
 $f(x) = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$

$$\begin{array}{r|l} [f(x)]^2 = 1 + a_1 & x + a_2 & x^2 + a_3 & x^3 + a_4 & x^4 + \dots \\ & + a_1 & + a_1 a_2 & + a_1 a_3 & \\ & & + a_2 & + a_2 a_2 & \\ & & & + a_3 & + a_1 a_3 \\ & & & & + a_4 \end{array}$$

$$1 - x + 2x^2 = 1 + 2a_1x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_1a_2)x^3 + (2a_4 + 2a_1a_3 + a_2^2)x^4 + \dots$$

$$2a_1 = -1, \quad 2a_2 + a_1^2 = 2,$$

$$2a_3 + 2a_1a_2 = 0, \quad 2a_4 + 2a_1a_3 + a_2^2 = 0.$$

$$\therefore a_1 = -\frac{1}{2}, \quad a_2 = \frac{7}{8}, \quad a_3 = \frac{7}{16}, \quad a_4 = -\frac{21}{64}.$$

$$\therefore (1 - x + 2x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{7}{8}x^2 + \frac{7}{16}x^3 - \frac{21}{64}x^4 + \dots$$

5. 解: (1) $(8 - 3x)^{\frac{1}{3}}$.

$$\text{設 } (8 - 3x)^{\frac{1}{3}} = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$8 - 3x = (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)^3$$

$$= a_0^3 + 3a_0^2a_1x + (3a_0^2a_2 + 3a_0a_1^2)x^2 \\ + (3a_0^2a_3 + 6a_0a_1a_2 + a_1^3)x^3 + \dots$$

$$a_0^3 = 8 \quad \therefore a_0 = 2.$$

$$3a_0^2a_1 = -3 \quad \therefore a_1 = -\frac{1}{4}.$$

$$3a_0^2a_2 + 3a_0a_1^2 = 0 \quad \therefore a_2 = -\frac{1}{32}.$$

$$3a_0^2a_3 + 6a_0a_1a_2 + a_1^3 = 0 \quad \therefore a_3 = -\frac{5}{768}.$$

$$\therefore (8-3x)^{\frac{1}{3}} = 2 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{5}{768}x^3 \dots$$

$$(2) (1+x-x^2)^{\frac{3}{2}}.$$

$$\text{設 } (1+x-x^2)^{\frac{3}{2}} = 1 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$(1+x-x^2)^3 = (1+a_1x+a_2x^2+a_3x^3+\dots)^2.$$

$$1+3x-5x^2+3x^3-x^6 = 1+2a_1x+(a_1^2+2a_2)x^2 \\ + (2a_1a_2+2a_3)x^3 + \dots$$

$$2a_1 = 3, \quad a_1^2 + 2a_2 = 0, \quad 2a_1a_2 + 2a_3 = -5.$$

$$\therefore a_1 = \frac{3}{2}, \quad a_2 = -\frac{9}{8}, \quad a_3 = -\frac{13}{16}.$$

$$\therefore (1+x-x^2)^{\frac{3}{2}} = 1 + \frac{3}{2}x - \frac{9}{8}x^2 - \frac{13}{16}x^3 - \dots$$

$$6. \text{ 解: (1) } \frac{2+x-3x^2+5x^3}{1+2x+3x^2}.$$

$$\begin{array}{r|l} 2+1-3+5 & 1+2+3 \\ 2+4+6 & \hline -3-9+5 & \\ -3-6-9 & \\ \hline -3+14 & \\ -3-6-9 & \\ \hline 20+9 & \end{array}$$

答：此級數之商爲 $2-3x-3x^2+20x^3-\dots$.

$$(2) \frac{x+5x^2-x^3}{1-x+x^2-x^3}.$$

$$\begin{array}{r}
 1+5-2 \quad | \quad 1-1+1-1 \\
 \hline
 1-1+1-1 \quad | \quad 1+6+4-1+\dots \\
 \hline
 6-2+1 \\
 6-6+6--6 \\
 \hline
 4-5+6 \\
 4-4+4-4 \\
 \hline
 -1+2+4
 \end{array}$$

答：此級數之商爲 $x+6x^2+4x^3-x^4+\dots$ 。

7. 解：(1) $\frac{3x^2+x^3}{1+x+x^2}$

設 $\frac{3x^2+x^3}{1+x+x^2} = 3x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \dots$

$$3x^2 + x^3 = (1+x+x^2)(3x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \dots)$$

$$\begin{array}{r}
 3x^2 + x^3 = 3x^2 + a_1 \quad | \quad x^3 + a_1 \quad | \quad x^4 + a_2 \quad | \quad x^5 + \dots \\
 \quad \quad \quad + 3 \quad | \quad + a_1 \quad | \quad + a_2 \quad | \\
 \quad \quad \quad \quad \quad | \quad + 3 \quad | \quad + a_1 \quad |
 \end{array}$$

$$3 + a_1 = 1, \quad 3 + a_1 + a_2 = 0, \quad a_1 + a_2 + a_3 = 0.$$

$$\therefore a_1 = -2, \quad a_2 = -1, \quad a_3 = 3$$

答： $\frac{3x^2+x^3}{1+x+x^2} = 3x^2 - 2x^3 - x^4 + 3x^5 + \dots$

(2) $\frac{x+5x^4}{x^3+2x^4+3x^5}$

設 $\frac{x+5x^4}{x^3+2x^4+3x^5} = a_0x^{-2} + a_1x^{-1} + a_2 + a_3x + \dots$

$$x + 5x^4 = (x^3 + 2x^4 + 3x^5)(a_0x^{-2} + a_1x^{-1} + a_2 + a_3x + \dots)$$

$$\begin{array}{r}
 x + 5x^4 = a_0x + a_1 \quad | \quad x^2 + a_2 \quad | \quad x^3 + a_3 \quad | \quad x^4 + \dots \\
 \quad \quad \quad + 2a_0 \quad | \quad + 2a_1 \quad | \quad + 2a_2 \quad | \\
 \quad \quad \quad \quad \quad | \quad + 3a_0 \quad | \quad + 3a_1 \quad |
 \end{array}$$

$$a_0 = 1, \quad a_1 + 2a_3 = 0, \quad a_2 + 2a_1 + 3a_0 = 0,$$

$$a_3 + 2a_2 + 3a_1 = 5.$$

$$\therefore a_0 = 1, \quad a_1 = -2, \quad a_2 = 1, \quad a_3 = 9.$$

答： $\frac{x+5x^4}{x^3+2x^4+3x^5} = -x^{-2} - 2x^{-1} + 1 + 9x + \dots$

8. 解：(1) $\frac{9x-22}{(x^2-4)(x-3)}$

按 § 57, $\frac{9x-22}{(x^2-4)(x-3)} = \frac{1}{x-2} - \frac{2}{x+2} + \frac{1}{x-3}$.

但 $\frac{1}{x-2} = -\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1} = -\frac{1}{2} - \frac{x}{2^2} - \frac{x^2}{2^3} - \frac{x^3}{2^4} - \frac{x^4}{2^5} - \dots$

於 $|x| < 2$ 時, 此級數為收斂.

$$\frac{-2}{x+2} = -\left(1+\frac{x}{2}\right)^{-1} = -1 + \frac{x}{2} - \frac{x^2}{2^2} + \frac{x^3}{2^3} - \frac{x^4}{2^4} + \dots$$

於 $|x| < 2$ 時, 此級數為收斂.

$$\frac{1}{x-3} = -\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1} = -\frac{1}{3} - \frac{x}{3^2} - \frac{x^2}{3^3} + \frac{x^3}{3^4} - \frac{x^4}{3^5} + \dots$$

於 $|x| < 3$ 時, 此級數為收斂.

$$\therefore \frac{6x-22}{(x^2-4)(x-3)} = -\frac{11}{6} + \frac{5x}{36} - \frac{89x^2}{216} + \frac{65x^3}{1296} - \frac{761x^4}{7776} + \dots$$

於 $|x| < 2$ 時, 此級數為收斂.

(2) $\frac{5x+6}{(2x+3)(x+1)^2}$.

$$\frac{5x+6}{(2x+3)(x+1)^2} = \frac{-6}{2x+3} + \frac{3}{x+1} + \frac{1}{(x+1)^2}.$$

但 $\frac{-6}{2x+3} = -2\left(1+\frac{2x}{3}\right)^{-1} = -2 + \frac{2^2x}{3} - \frac{2^3x^2}{3^2} + \frac{2^4x^3}{3^3} - \frac{2^5x^4}{3^4} + \dots$

於 $|x| < \frac{3}{2}$ 時, 此級數為收斂.

$$\frac{3}{x+1} = 3(1+x)^{-1} = 3 - 3x + 3x^2 - 3x^3 + 3x^4 - \dots$$

於 $|x| < 1$ 時, 此級數為收斂.

$$\frac{1}{(x+1)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$$

於 $|x| < 1$ 時, 此級數為收斂.

$$\therefore \frac{5x+6}{(2x+3)(x+1)^2} = 2 - \frac{11}{3}x + \frac{46}{9}x^2 - \frac{173}{27}x^3 + \frac{616}{81}x^4 + \dots$$

於 $|x| < 1$ 時, 此級數為收斂.

9. 解: (1) $\frac{2x+3}{2x^2+x-15}$.

$$\begin{array}{r|l}
 2x+3 & 2x^2+x-15 \\
 2x+1-\frac{15}{x} & \hline
 & x^{-1}+x^{-2}+7x^{-3}+4x^{-4}+\dots \\
 \hline
 & 2+\frac{15}{x} \\
 & 2+\frac{1}{x}-\frac{15}{x^2} \\
 \hline
 & \frac{14}{x}+\frac{15}{x^2} \\
 & \frac{14}{x}+\frac{7}{x^2}-\frac{105}{x^3} \\
 \hline
 & \frac{8}{x^2}+\frac{105}{x^3}
 \end{array}$$

$$\therefore \frac{2x+3}{2x^2+x-15} = x^{-1} + x^{-2} + 7x^{-3} + 4x^{-4} + \dots$$

$$2x^2+x-15 = (2x-5)(x+3).$$

答：級數 $x^{-1} + x^{-2} + 7x^{-3} + 4x^{-4} + \dots$ 於 $|x| < \frac{5}{2}$ 時，為收斂。

$$(2) \frac{x^4+1}{x^4+x^3+x^2+x+1}$$

$$\begin{array}{r|l}
 1+0+0+0+1 & 1+1+1+1+1 \\
 1+1+1+1+1 & \hline
 & 1-1+0+0+1+0-1+\dots \\
 \hline
 & -1-1-1+0 \\
 & -1-1-1-1-1 \\
 \hline
 & 1+1
 \end{array}$$

$$\text{答：} \frac{x^4+1}{x^4+x^3+x^2+x+1} = 1 - x^{-1} + x^{-4} - x^{-6} + \dots$$

10. 解：(1) $y = x + x^2 + x^3 + x^4 + \dots$

設 $x = b_1y + b_2y^2 + b_3y^3 + b_4y^4 + \dots$

$$x = b_1(x + x^2 + x^3 + x^4 + \dots) + b_2(x + x^2 + x^3 + x^4 + \dots)^2 + b_3(x + x^2 + x^3 + x^4 + \dots)^3 + \dots$$

$$\begin{array}{r|l}
 = b_1x + b_1 & x^2 + b_1 \\
 + b_2 & + 2b_2 \\
 & + b_3 \\
 & + 3b_3 \\
 & + b_4
 \end{array} \left| \begin{array}{l} x^3 + b_1 \\ + 3b_2 \\ + 3b_3 \\ + b_4 \end{array} \right| x^4 + \dots$$

$$\therefore b_1=1, b_2=-1, b_3=1 \dots$$

$$\text{答: } x=y-y^2+y^3-y^4+\dots$$

$$(2) y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$$

$$\text{設 } x=b_1y+b_2y^2+b_3y^3+\dots$$

$$x=b_1\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)$$

$$+b_2\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)^2$$

$$+b_3\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)^3+\dots$$

$$=b_1x-\frac{b_1}{2} \left| \begin{array}{c} x^2+\frac{b_1}{3} \\ x^3-\frac{b_1}{4} \end{array} \right| x^4+\dots$$

$$+b_2 \left| \begin{array}{c} -b_2 \\ +b_3 \end{array} \right| +\frac{11b_2}{12}$$

$$+\frac{3}{2}b_3$$

$$+b_4$$

$$\therefore b_1=1, b_2=\frac{1}{2}, b_3=\frac{1}{8}, b_4=\frac{1}{24} \dots$$

$$\text{答: } x=y+\frac{1}{2}y^2+\frac{1}{6}y^3+\frac{1}{24}y^4+\dots$$

$$11. \text{ 解: 設 } x=b_1(y-1)+b_2(y-1)^2+b_3(y-1)^3+\dots$$

$$x=b_1\left\{\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-1\right\}$$

$$+b_2\left\{\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-1\right\}^2$$

$$+b_3\left\{\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-1\right\}^3+\dots$$

$$=b_1x+\frac{b_1}{2} \left| \begin{array}{c} x^2+\frac{b_1}{3} \\ x^3+\frac{b_1}{4} \end{array} \right| x^4+\dots$$

$$+b_2 \left| \begin{array}{c} +b_2 \\ +b_3 \end{array} \right| +\frac{11b_2}{12}$$

$$+\frac{2}{3}b_3$$

$$+b_4$$

$$(2) x^3 + y^3 - xy = 0.$$

當 $x=0$, 方程式 $x^3 + y^3 - xy = 0$ 化爲 $y^3 = 0$, 故其三根均爲 0. 故吾人能求得以 x 表示 y 之三方程式焉.

設 ax^μ 爲一展開式中之首項, 故

$$y = ax^\mu + \dots$$

代入原方程式中, 得下式

$$x^3 + a^3x^{3\mu} + \dots - x(ax^\mu + \dots) = 0 \dots \dots \dots (1)$$

此式內三指數 3μ , $\mu+1$ 與 3 中至少有二指數必須相等, 且均較小於另一指數.

設 $3\mu = \mu+1$, $\therefore \mu = \frac{1}{2}$, 此乃 μ 之一可能值, 蓋於 $\mu = \frac{1}{2}$ 時, 3μ 與 $\mu+1$ 俱小於 3 故也.

設 $\mu+1 = 3$, $\therefore \mu = 2$, 此亦爲 μ 之一可能值, 蓋於 $\mu = 2$ 時, $\mu+1$ 與 3 俱小於 3μ 故也.

設 $3\mu = 3$, $\therefore \mu = 1$, 此非 μ 之可能值, 蓋於 $\mu = 1$ 時, 3μ 與 3 俱大於 $\mu+1$ 故也.

故 μ 必爲 $\frac{1}{2}$ 與 2 中之一.

$$\text{於 } \mu = \frac{1}{2} \text{ 時, (1) 式化爲 } x^3 + a^3x^{\frac{3}{2}} + \dots - ax^{\frac{3}{2}} + \dots = 0.$$

故 $a^3 - a = 0$, 若 $a \neq 0$, 則 $a = \pm 1$.

於 $\mu = 2$ 時, (1) 式化爲 $x^3 + a^3x^6 + \dots - ax^3 + \dots = 0$, 故 $1 - a = 0$, 即 $a = 1$.

設所求之解答爲 $y = x^{\frac{1}{2}} - bx^2 - cx^{\frac{3}{2}} + \dots$,

$$y = -x^{\frac{1}{2}} - bx^2 + cx^{\frac{7}{2}} + \dots, y = x^2 + bx^5 + cx^8 + \dots$$

代入 $x^3 + y^3 - xy = 0$ 中以定 b, c 之值, 得所求之解答爲

$$y = x^{\frac{1}{2}} - \frac{1}{2}x^2 - \frac{3}{8}x^{\frac{7}{2}} + \dots,$$

$$y = -x^{\frac{1}{2}} - \frac{1}{2}x^2 + \frac{3}{8}x^{\frac{7}{2}} + \dots, y = x^2 + x^5 + 3x^8 + \dots$$

14. 證: $\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \dots$

$$\begin{aligned}
&= \frac{x}{(1-x)^2} + \frac{x^2}{(1-x^2)^2} + \frac{x^3}{(1-x^3)^2} + \dots \\
\frac{x}{1-x} &= x + x^2 + x^3 + x^4 + x^5 + \dots \\
\frac{2x^2}{1-x^2} &= 2x^3 + 2x^4 + 2x^6 + 2x^8 + 2x^{10} + \dots \\
\frac{3x^3}{1-x^3} &= 3x^3 + 3x^6 + 3x^9 + 3x^{12} + 3x^{15} + \dots \\
\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \dots &= x(1 + 2x + 3x^2 + \dots) \\
&\quad + x^2(1 + 2x^2 + 3x^4 + \dots) + x^3(1 + 2x^3 + 3x^6 + \dots) + \dots \\
&= x(1-x)^{-2} + x^2(1-x^2)^{-2} + x^3(1-x^3)^{-2} + \dots \\
&= \frac{x}{(1-x)^2} + \frac{x^2}{(1-x^2)^2} + \frac{x^3}{(1-x^3)^2} + \dots
\end{aligned}$$

XXXIV. 二項級數, 指數級數及對數級數

習 題 XC

原本第 559 頁

1. 解: $\log_e(n+1) = \log_e n + 2 \left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right]$.

(1) 設 $n=3$,

$$\begin{aligned}
\log_e 4 &= \log_e 3 + 2 \left(\frac{1}{7} + \frac{1}{3 \cdot 7^3} + \frac{1}{5 \cdot 7^5} + \dots \right) \\
&= \log_e 3 + \frac{2}{7} + \frac{2}{1029} + \frac{2}{84035} + \dots \\
&= 1.0986 + 0.2857 + 0.0019 + \dots \\
&= 1.3862.
\end{aligned}$$

(2) 設 $n=4$,

$$\begin{aligned}\log_e 5 &= \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \dots\right) \\ &= \log_e 4 + \frac{2}{9} + \frac{2}{2187} + \frac{2}{295245} + \dots \\ &= 1.3862 + 0.2222 + 0.0009 + \dots \\ &= 1.6093.\end{aligned}$$

2. 證: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$.

$$\begin{aligned}\text{設 } x = -1, \quad e^{-1} &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots \\ &= (1 - 1) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \dots \\ &= \frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \dots\end{aligned}$$

3. 證: $f(x) = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\phi(x) = 1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$f(x) \cdot \phi(x) = 1 + 1$	$x + \frac{1}{2!}$	$x^2 + \frac{1}{3!}$	$x^3 + \dots$
-1	-1	$-\frac{1}{2!}$	$-\frac{1}{3!}$
	$+\frac{1}{2!}$	$+\frac{1}{2!}$	$+\frac{1}{3!}$
		$+\frac{1}{3!}$	

$$\text{故 } \left(1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) = 1.$$

4. 證: $e^{ix} + e^{-ix} = \left\{1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \dots\right\}$
 $+ \left\{1 - ix + \frac{i^2 x^2}{2!} - \frac{i^3 x^3}{3!} + \dots\right\}$

$$= 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \dots$$

$$\text{故 } \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\begin{aligned}
 5. \text{ 證: } e^{ix} - e^{-ix} &= \left\{ 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \dots \right\} \\
 &\quad - \left\{ 1 - ix + \frac{i^2 x^2}{2!} - \frac{i^3 x^3}{3!} + \dots \right\} \\
 &= 2ix + \frac{2i^3 x^3}{3!} + \frac{2i^5 x^5}{5!} + \dots
 \end{aligned}$$

$$\text{故 } \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\begin{aligned}
 6. \text{ 證: } (1-x)^{-n} \text{ 之展開式中第 } r+1 \text{ 項爲} \\
 &\frac{-n(-n-1)(-n-2)\dots(-n-r+1)}{r!} (-1)^r x^r \\
 &= \frac{(-1)^r n(n+1)(n+2)\dots(n+r-1)}{r!} (-1)^r x^r \\
 &= \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r.
 \end{aligned}$$

$$\begin{aligned}
 7. \text{ 解: 在通項, } \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} 8^{n-r} x^r \text{ 中於} \\
 r=4 \text{ 時, 得 } x^4 \text{ 項.}
 \end{aligned}$$

$$= \frac{\frac{2}{3} \left(\frac{2}{3} - 1 \right) \left(\frac{2}{3} - 2 \right) \left(\frac{2}{3} - 3 \right)}{4!} 8^{\frac{2}{3} - 4} x^4$$

$$= \frac{\frac{2}{3} \left(-\frac{1}{3} \right) \left(-\frac{4}{3} \right) \left(-\frac{7}{3} \right)}{4!} 8^{-\frac{10}{3}} x^4$$

$$= \frac{-56}{8 \cdot 3^5 \cdot 2^{10}} x^4 = -\frac{7}{3^5 \cdot 2^{10}} x^4.$$

$$8. \text{ 解: } \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1 \right) \left(-\frac{1}{2} - 2 \right) \dots \left(-\frac{1}{2} - r + 1 \right)}{r!} (-1)^r (x^{\frac{1}{2}})^r.$$

$$\therefore \left(x^{\frac{1}{2}} \right)^r = x^3 = \left(x^{\frac{1}{2}} \right)^6 \quad \therefore r=6.$$

$$\therefore \frac{-\frac{1}{2} \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) \left(-\frac{7}{2} \right) \left(-\frac{9}{2} \right) \left(-\frac{11}{2} \right)}{6!} x^3$$

$$= \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2^6 \cdot 6!} x^3 = \frac{231}{2^{10}} x^3.$$

9. 解: (1) $(9 - 4x^2)^{\frac{1}{2}} = 3 \left(1 - \frac{4}{9} x^2 \right)^{\frac{1}{2}}$

$$= 3 \left\{ 1 + \frac{1}{2} \left(-\frac{4}{9} x^2 \right) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} \left(-\frac{4}{9} x^2 \right)^2 \right. \\ \left. + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!} \left(-\frac{4}{9} x^2 \right)^3 + \dots \right\}$$

$$= 3 \left\{ 1 - \frac{2}{9} x^2 - \frac{2}{81} x^4 - \frac{4}{729} x^6 - \dots \right\}$$

$$= 3 - \frac{2}{3} x^2 - \frac{2}{27} x^4 - \frac{4}{243} x^6 - \dots.$$

故該級數於 $\frac{4}{9} |x^2| < 1$ 即 $|x| < \frac{3}{2}$ 時, 收斂.

(2) $(12 + x + x^2)^{\frac{2}{3}} = 12^{\frac{2}{3}} \left(1 + \frac{x}{12} + \frac{x^2}{12} \right)^{\frac{2}{3}}$

$$= 12^{\frac{2}{3}} \left\{ 1 + \frac{2}{3} \left(\frac{x}{12} + \frac{x^2}{12} \right) + \frac{\frac{2}{3}(-\frac{1}{3})}{2!} \left(\frac{x}{12} + \frac{x^2}{12} \right)^2 + \dots \right\}$$

$$= 12^{\frac{2}{3}} \left\{ 1 + \frac{1}{18} x + \frac{71}{1296} x^2 + \dots \right\}$$

$$= 12^{\frac{2}{3}} + \frac{12^{\frac{2}{3}}}{18} x + \frac{71 \cdot 12^{\frac{2}{3}}}{1296} x^2 + \dots.$$

故該級數於 $\left| \frac{x^2}{12} \right| + \left| \frac{x}{12} \right| < 1$ 時, 收斂.

但 $\left| \frac{x^2}{12} \right| + \left| \frac{x}{12} \right| < 1$, 則 $|x^2| + |x| < 12$,

$$|x^2| + |x| + \left(\frac{1}{2} \right)^2 < 12 + \left(\frac{1}{2} \right)^2.$$

$$|x| + \frac{1}{2} < \frac{7}{2} \quad \therefore |x| < 3.$$

即 $|x| < 3$ 時, 此級數乃收斂.

10. 解: $(1 - x + 2x^2)^{\frac{3}{4}} \{ 1 - (x - 2x^2) \}^{\frac{3}{4}}$

$$\begin{aligned}
&= 1 - \frac{3}{4}(x-2x^2) + \frac{\frac{3}{4}(\frac{3}{4}-1)}{2!}(x-2x^2)^2 - \frac{\frac{3}{4}(\frac{3}{4}-1)(\frac{3}{4}-2)}{3!} \\
&\quad \times (x-2x^2)^3 + \frac{\frac{3}{4}(\frac{3}{4}-1)(\frac{3}{4}-2)(\frac{3}{4}-3)}{4!}(x-2x^2)^4 - \dots \\
&= 1 - \frac{3}{4}x + \frac{45}{32}x^2 + \frac{43}{128}x^3 - \frac{333}{2048}x^4 - \dots
\end{aligned}$$

11. 解: $(8+3x)^{\frac{2}{3}}(9-2x)^{-\frac{1}{2}} = 8^{\frac{2}{3}}\left(1+\frac{3}{8}x\right)^{\frac{2}{3}} \cdot 9^{-\frac{1}{2}}\left(1-\frac{2}{9}x\right)^{-\frac{1}{2}}$

$$\begin{aligned}
&= \frac{4}{3} \left\{ 1 + \frac{2}{3} \left(\frac{3}{8}x \right) + \frac{\frac{2}{3} \left(-\frac{1}{3} \right)}{1 \cdot 2} \left(\frac{3}{8}x \right)^2 + \dots \right\} \\
&\quad \times \left\{ 1 - \frac{1}{2} \left(-\frac{2}{9}x \right) + \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right)}{1 \cdot 2} \left(-\frac{2}{9}x \right)^2 + \dots \right\} \\
&= \frac{4}{3} \left\{ 1 + \frac{1}{4}x - \frac{1}{64}x^2 + \dots \right\} \left\{ 1 + \frac{1}{9}x + \frac{1}{54}x^2 - \dots \right\} \\
&= \frac{4}{3} + \frac{13}{27}x + \frac{53}{1296}x^2 + \dots
\end{aligned}$$

於 $\frac{3}{8}|x| < 1$ 即 $|x| < \frac{8}{3}$ 時, 爲收斂。

12. 解: (1) $\lim_{x \rightarrow 0} \frac{e^{2x} - e^{-2x}}{3x}$.

$$\begin{aligned}
\frac{e^{2x} - e^{-2x}}{3x} &= \frac{1 + 2x + \frac{4}{2!}x^2 + \dots - 1 + 2x - \frac{4}{2!}x^2 + \dots}{3x} \\
&= \frac{4x + \frac{16}{3!}x^3 + \frac{64}{5!}x^5 + \dots}{3x} = \frac{4 + \frac{16}{3!}x^2 + \dots}{3}
\end{aligned}$$

$$\therefore \lim_{x \rightarrow 0} \frac{e^{2x} - e^{-2x}}{3x} = \frac{4}{3}$$

(2) $\lim_{x \rightarrow 0} \frac{(1+x^2)^{\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}}{(1+3x)^{\frac{1}{3}} - (1+4x)^{\frac{1}{4}}}$

$$\begin{aligned}
& \frac{(1+x^2)^{\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}}{(1+3x)^{\frac{1}{3}} - (1+4x)^{\frac{1}{4}}} \\
&= \frac{\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \dots\right) - \left(1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 + \frac{5}{81}x^6 + \dots\right)}{\left(1+x-x^2+\frac{5}{3}x^3+\dots\right) - \left(1+x-\frac{3}{2}x^2+\frac{9}{2}x^3+\dots\right)} \\
&= \frac{\frac{5}{6}x^2 - \frac{1}{72}x^4 + \dots}{\frac{1}{2}x^2 - \frac{11}{6}x^3 + \dots} \\
\therefore \lim_{x \rightarrow 0} \frac{(1+x^2)^{\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}}{(1+3x)^{\frac{1}{3}} - (1+4x)^{\frac{1}{4}}} &= \frac{5}{3}.
\end{aligned}$$

13. 證: $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{nx} = e^x$.

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^{nx} &= 1 + x + \frac{nx(nx-1)}{2!n^2} + \frac{nx(nx-1)(nx-2)}{3!n^3} + \dots \\
&= 1 + x + \frac{n^2x^2 - nx}{2!n^2} + \frac{n^3x^3 - 3n^2x^2 + 2nx}{3!n^3} + \dots \\
&= 1 + x + \frac{x^2 - x/n}{2!} + \frac{x^3 - 3x^2/n + 2x/n^2}{3!} + \dots \\
\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{nx} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x.
\end{aligned}$$

14. 證: $\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} = e^e$.

$$\begin{aligned}
(e^x + x)^{\frac{1}{x}} &= e + \frac{1}{x}(e^x)^{\frac{1}{x}-1} \cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}(e^x)^{\frac{1}{x}-2} \cdot x^2 \\
&+ \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}(e^x)^{\frac{1}{x}-3} \cdot x^3 + \dots
\end{aligned}$$

$$\begin{aligned}
 &= e + e^{1-x} + \frac{1-x}{2!} e^{1-2x} + \frac{1-3x+2x^2}{3!} e^{1-3x} + \dots \\
 &= e \left\{ 1 + e^{-x} + \frac{1-x}{2!} e^{-2x} - \frac{1-3x+2x^2}{3!} e^{-3x} + \dots \right\} \\
 \therefore \lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} &= e \left\{ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \right\} = e \cdot e = e^2.
 \end{aligned}$$

15. 解: $\log_e(1+x+x^2) = \log_e\left(\frac{1-x^3}{1-x}\right) = \log_e(1-x^3) - \log_e(1-x)$

$$\begin{aligned}
 &= -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \frac{x^{12}}{4} - \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right) \\
 &= -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \frac{x^{12}}{4} - \dots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \\
 &= x + \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \dots
 \end{aligned}$$

此展開式於 $|x| + |x^2| < 1$ 時, 爲收斂; 即於

$$|x| + |x^2| + \left(\frac{1}{2}\right)^2 < 1 + \left(\frac{1}{2}\right)^2$$

$$|x| + \frac{1}{2} < \frac{\sqrt{5}}{2}$$

$$|x| < (\sqrt{5}-1)/2 \text{ 時也.}$$

16. 證: $\log_e \frac{m}{n} = \frac{m-n}{n} - \frac{1}{2} \left(\frac{m-n}{n}\right)^2 + \frac{1}{3} \left(\frac{m-n}{n}\right)^3 - \dots$

$$\begin{aligned}
 \log_e \frac{m}{n} &= \log_e \left(1 + \frac{m-n}{n}\right) \\
 &= \frac{m-n}{n} - \frac{1}{2} \left(\frac{m-n}{n}\right)^2 + \frac{1}{3} \left(\frac{m-n}{n}\right)^3 - \dots
 \end{aligned}$$

17. 證: $\log_e \frac{n^2}{n^2-1} = \frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \dots$

$$\begin{aligned}
 \log_e \frac{n^2}{n^2-1} &= -\log_e \frac{n^2}{n^2-1} = -\log_e \left(1 - \frac{1}{n^2}\right) \\
 &= \frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \dots
 \end{aligned}$$

8. 證: $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots$

$$\begin{aligned} \frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots &= -\log_e \left(1 - \frac{1}{n+1}\right) \\ &= -\log_e \frac{n}{n+1} = \log_e \frac{n+1}{n} = \log_e \left(1 + \frac{1}{n}\right) \\ &= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \end{aligned}$$

XXXV. 循環級數

習 題 XCI

原本第 563 頁

1. 解：當 $n=4$, $a_4 + 2 \times 5 - 1 \times (-3) + 3 \times 2 = 0$.

$$\therefore a_4 = -19.$$

當 $n=5$, $a_5 + 2(-19) - 5 + 3(-3) = 0$.

$$\therefore a_5 = 52.$$

答：所求之二項為 $-19x^3 + 52x^4$.

2. 解：(1) $1 + 3x + 2x^2 - x^3 - 3x^4 + \dots$.

設所求之關係式為 $a_n + pa_{n-1} + qa_{n-2} = 0$.

$$2 + 3p + q = 0 \quad (1) \quad -1 + 2p + 3q = 0 \quad (2)$$

$$-3 - p + 2q = 0 \quad (3) \quad \therefore p = -1, \quad q = 1.$$

故所求之關係式為 $a_n - a_{n-1} + a_{n-2} = 0$.

當 $n=5$, $a_5 + 3 - 1 = 0 \quad \therefore a_5 = -2$.

當 $n=6$, $a_6 + 2 - 3 = 0 \quad \therefore a_6 = 1$.

答：故所求之二項為 $-2x^5 + x^6$.

(2) $2 - 5x + 4x^2 + 7x^3 - 26x^4 + \dots$.

設所求之關係式為 $a_n + pa_{n-1} + qa_{n-2} = 0$.

$$4 - 5p + 2q = 0 \quad (1) \quad 7 + 4p - 5q = 0 \quad (2)$$

$$-26 + 7p + 4q = 0 \quad (3) \quad \therefore p = 2, \quad q = 3.$$

故所求之關係式為 $a_n + 2a_{n-1} + 3a_{n-2} = 0$.

當 $n=5$, $a_5 + 2(-26) + 3 \cdot 7 = 0 \quad \therefore a_5 = 31$.

當 $n=6$, $a_6 + 2 \cdot 31 + 3(-26) = 0 \quad \therefore a_6 = 16$.

答：所求之二項為 $31x^5 + 16x^6$.

$$(3) 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \dots$$

設所求之關係式爲 $a_n + pa_{n-1} + qa_{n-2} + ra_{n-3} = 0$.

$$-10 + 6p - 3q + r = 0 \quad (1) \quad 15 - 10p + 6q - 3r = 0 \quad (2)$$

$$-21 + 15p - 10q + 6r = 0 \quad (3) \quad \therefore p = 3, q = 3, r = 1.$$

故所求之關係式爲 $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$.

$$\text{當 } n=6, a_6 + 3(-21) + 3(15) - 10 = 0 \quad \therefore a_6 = 28.$$

$$\text{當 } n=7, a_7 + 3 \cdot 28 + 3(-21) + 15 = 0 \quad \therefore a_7 = -36.$$

答：所求之二項爲 $28x^6 - 36x^7$.

3. 解：(1) $2 + x + 5x^2 + 7x^3 + 17x^4 + \dots$

設其關係式爲 $a_n + pa_{n-1} + qa_{n-2} = 0$.

$$5 + p + 2q = 0 \quad (1) \quad 7 + 5p + q = 0 \quad (2)$$

$$17 + 7p + 5q = 0 \quad (3) \quad p = -1, q = -2.$$

故其關係式爲 $a_n - a_{n-1} - 2a_{n-2} = 0$.

$$\text{設 } S_n = 2 + x + 5x^2 + 7x^3 + 17x^4 + \dots$$

$$-xS_n = -2x - x^2 - 5x^3 - 7x^4 - \dots$$

$$-2x^2S_n = -4x^2 - 2x^3 - 10x^4 - \dots$$

$$\hline (1-x-2x^2)S_n = 2-x$$

故所求之母函數爲

$$S = \frac{2-x}{1-x-2x^2} = \frac{2-x}{(1-2x)(1+x)}$$

$$= \frac{1}{1-2x} + \frac{1}{1+x} = (1-2x)^{-1} + (1+x)^{-1}.$$

於 $|x| < \frac{1}{2}$ 時, $(1-2x)^{-1} = 1 + 2x + 4x^2 + \dots + 2^n x^n + \dots$,

又 $|x| < 1$ 時, $(1+x)^{-1} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots$.

答：其普通項爲 $\{2^n x^n + (-1)^n x^n\}$ 即 $\{2^n + (-1)^n\} x^n$.

$$(2) 3 + 7x + 17x^2 + 43x^3 + 113x^4 + \dots$$

設其關係式爲 $a_n + pa_{n-1} + qa_{n-2} = 0$.

$$17 + 7p + 3q = 0 \quad (1) \quad 43 + 17p + 7q = 0 \quad (2)$$

$$113 + 43p + 17q = 0 \quad (3) \quad \therefore p = -5, q = 6.$$

故其關係式爲 $a_n - 5a_{n-1} + 6a_{n-2} = 0$.

$$\begin{aligned} \text{設} \quad S_n &= 3 + 7x + 17x^2 + 43x^3 + 113x^4 + \dots \\ -5xS_n &= -15x - 35x^2 - 85x^3 - \dots \\ 6x^2S_n &= 18x^2 + 42x^3 + \dots \end{aligned}$$

$$(1 - 5x + 6x^2)S_n = 3 - 8x$$

$$\begin{aligned} \text{故其母函數爲 } S &= \frac{3 - 8x}{1 - 5x + 6x^2} = \frac{2}{1 - 2x} + \frac{1}{1 - 3x} \\ &= 2(1 - 2x)^{-1} + (1 - 3x)^{-1}. \end{aligned}$$

於 $|x| < 1$ 時, $2(1 - 2x)^{-1} = 2\{1 + 2x + 4x^2 + \dots + 2^n x^n + \dots\}$,

又 $|x| < \frac{1}{3}$ 時, $(1 - 3x)^{-1} = 1 + 3x + 3^2 x^2 + \dots + 3^n x^n + \dots$.

答: 其普通項爲 $(2^{n-1}x^n + 3^n x^n)$ 即 $(2^{n+1} + 3^n)x^n$.

4. 證: 此級數之母函數爲

$$\frac{a_0 + (a_1 + pa_0)x + (a_2 + pa_1 + qa_0)x^2}{1 + px + qx^2 + rx^3}.$$

$$\begin{aligned} \text{設} \quad S_n &= a_0x + a_1x + a_2x^2 + a_3x^3 + \dots + a_{n-1}x^{-1} \\ pxS_n &= pa_0x + pa_1x^2 + pa_2x^3 + \dots + pa_{n-2}x^{n-1} + pa_{n-1}x^n \\ px^2S_n &= qa_0x^2 + qa_1x^3 + \dots + qa_{n-3}x^{n-1} + qa_{n-2}x^n \\ &\quad + qa_{n-1}x^{n+1} \\ rx^3S_n &= ra_0x^3 + \dots + ra_{n-4}x^{n-1} + ra_{n-3}x^n + ra_{n-2}x^{n+1} \\ &\quad + ra_{n-1}x^{n+2} \end{aligned}$$

$$\begin{aligned} (1 + px + qx^2 + rx^3)S_n &= a_0 + (a_1 + pa_0)x + (a_2 + pa_1 + qa_0)x^2 \\ &\quad + (pa_{n-1} + qa_{n-2} + ra_{n-3})x^n \\ &\quad + (qa_{n-1} + ra_{n-2})x^{n+1} + ra_{n-1}x^{n+2}. \end{aligned}$$

若已知方程式爲收斂者, 則 a^n 漸近 0.

$$\therefore S = \frac{a_0 + (a_1 + pa_0)x + (a_2 + pa_1 + qa_0)x^2}{1 + px + qx^2 + rx^3}.$$

5. 解: 設其關係式爲 $a_n + pa_{n-1} + qa_{n-2} + ra_{n-3} = 0$.

$$24 + 11p + 2q + r = 0 \quad (1) \quad 85 + 24p + 11q + 2r = 0 \quad (2)$$

$$238 + 85p + 24q + 11r = 0 \quad (3)$$

$$\therefore p = -1, \quad q = -5, \quad r = -3.$$

故其關係式爲 $a_n - a_{n-1} - 5a_{n-2} - 3a_{n-3} = 0$.

$$\begin{aligned} \text{設} \quad S_n &= 1 + 2x + 11x^2 + 24x^3 + 85x^4 + \dots \\ -xS_n &= -x - 2x^2 - 11x^3 - 24x^4 - \dots \\ -5x^2S_n &= -5x^2 - 10x^3 - 55x^4 - \dots \\ -3x^3S_n &= -3x^3 - 5x^4 - \dots \end{aligned}$$

$$(1-x-5x^2-3x^3)S_n = 1+x+4x^2$$

$$\text{答: 其母函數爲 } S = \frac{1+x+4x^2}{1-x-5x^2-3x^3}$$

$$\text{其普通項爲 } \{3^n + (-1)^n n\} x^n.$$

6. 證: 設其關係式爲 $a_n + pa_{n-1} + qa_{n-2} = 0$.

$$(a+2d) + p(a+d) + qa = 0 \dots\dots\dots(1)$$

$$(a+3d) + p(a+2d) + q(a+d) = 0 \dots\dots\dots(2)$$

$$(a+4d) + p(a+3d) + q(a+2d) = 0 \dots\dots\dots(3)$$

$$\therefore p = -2, \quad q = 1.$$

故其關係式爲 $a_n - 2a_{n-1} + a_{n-2} = 0$.

$a + (1+d)x + (a+2d)x^2 + (a+3d)x^3 + \dots$ 式中之任何連續三項係數皆能以公式 $a_n - 2a_{n-1} + a_{n-2} = 0$ 聯絡之, 故知級數爲二次之循環級數.

$$\begin{aligned} \text{設} \quad S_n &= a + (a+d)x + (a+d)x^2 + (a+3d)x^3 + \dots \\ -2xS_n &= -2ax - 2(a+d)x^2 - 2(a+2d)x^3 - \dots \\ x^2S_n &= ax^3 + (a+d)x^4 + \dots \end{aligned}$$

$$(1-2x+x^2)S_n = a - (a-d)x$$

$$\therefore S = \frac{a - (a-d)x}{1-2x+x^2}$$

7. 證: 設其關係式爲 $a_n + pa_{n-1} + qa_{n-2} + ra_{n-3} = 0$.

$$16 + 9p + 4q + r = 0 \dots\dots\dots(1)$$

$$25 + 16p + 9q + 4r = 0 \dots\dots\dots(2)$$

$$36 + 25p + 16q + 9r = 0 \dots\dots\dots(3)$$

$$\therefore p = -3, \quad q = 3, \quad r = -1.$$

故其關係式爲 $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$.

$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$ 式中之任何連續三項之係數皆能以公式 $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ 聯絡之, 故知級數爲三次之循環級數.

$$\text{設 } S_n = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$$

$$-3xS_n = -3x - 13x^2 - 27x^3 - \dots$$

$$3x^2S_n = 3x^2 + 12x^3 + \dots$$

$$-x^3S_n = -x^3 - \dots$$

$$(1 - 3x + 3x^2 - x^3)S_n = 1 + x$$

$$\therefore S = \frac{1+x}{1-3x+3x^2-x^3} = \frac{1+x}{(1-x)^3}$$

8. 證：設其關係式為 $a_n + pa_{n-1} + qa_{n-2} + ra_{n-3} = 0$.

$$20 + 12p + 6q + 2r = 0 \dots\dots\dots (1)$$

$$30 + 20p + 12q + 6r = 0 \dots\dots\dots (2)$$

$$42 + 30p + 20q + 12r = 0 \dots\dots\dots (3)$$

$$\therefore p = -3, q = 3, r = -1.$$

故其關係式為 $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$.

$1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + \dots$ 式中之任何連續三項之係數皆能以公式 $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ 聯絡之，故知級數為三次之循環級數。

用題 4 之公式 $S = \frac{a_0 + (a_1 + pa_0)x + (a_2 + pa_1 + qa_0)x^2}{1 + px + qx^2 + rx^3}$ ，得

$$|x| < 1 \text{ 時, } S = \frac{2}{1 - 3x + 3x^2 - x^3}$$

XXXVI. 無窮連乘積

習 題 XCII

原本第 565 頁

1. 證：(1) $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \dots\dots$

$$= \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{8}\right) \left(1 + \frac{1}{16}\right) \dots\dots$$

$$\therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\dots \text{爲收斂者.}$$

$$\therefore \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \dots\dots \text{亦爲收斂者.}$$

$$\begin{aligned}
 (2) \quad & \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \frac{26}{25} \cdots \\
 & = \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{9}\right) \left(1 + \frac{1}{16}\right) \left(1 + \frac{1}{25}\right) \cdots \\
 \therefore & \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \\
 & = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \text{爲收斂者。} \\
 \therefore & \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \frac{26}{25} \cdots \text{亦爲收斂者。}
 \end{aligned}$$

2. 解: (1) $\prod \left(1 + \frac{x^n}{n^2}\right) = \left(1 + \frac{x}{1^2}\right) \left(1 + \frac{x^2}{2^2}\right) \left(1 + \frac{x^3}{3^2}\right) \cdots$

$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots = x \left(1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \cdots\right)$$

$$\frac{u_{n+1}}{u_n} = \frac{n^2 x}{(n+1)^2} = \frac{n^2 x}{n^2 + 2n + 1} = \frac{x}{1 + \frac{2}{n} + \frac{1}{n^2}}$$

$$\lim \frac{u_{n+1}}{u_n} = x.$$

故此乘積於 $x \leq 1$ 時，爲收斂。

(2) $\prod \left(1 + \frac{x^n}{n!}\right) = \left(1 + \frac{x}{1!}\right) \left(1 + \frac{x^2}{2!}\right) \left(1 + \frac{x^3}{3!}\right) \cdots$

$$\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = x \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)$$

$$\frac{u_{n+1}}{u_n} = \frac{x^n / (n+1)!}{x^{n-1} / n!} = \frac{x}{n+1}$$

$$\lim \frac{u_{n+1}}{u_n} = \frac{x}{\infty}$$

故此乘積於 $\frac{x}{\infty} < 1$ 即 $x < \infty$ 時，爲收斂。

(3) $\prod \left(1 + \frac{x^n}{3^n}\right) = \left(1 + \frac{x}{3}\right) \left(1 + \frac{x^2}{9}\right) \left(1 + \frac{x^3}{27}\right) \cdots$

$$\frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \cdots = x \left(\frac{1}{3} + \frac{x}{3^2} + \frac{x^2}{3^3} + \cdots\right)$$

$$\frac{u_{n+1}}{u_n} = \frac{x^n/3^{n+1}}{x^{n+1}/3^n} = \frac{x}{3}$$

$$\lim \frac{u_{n+1}}{u_n} = \frac{x}{3}$$

故此乘積於 $\frac{x}{3} < 1$ 即 $x < 3$ 時，為收斂。

3. 證： $\lim \frac{a(a+1)(a+2)\cdots(a+n)}{b(b+1)(b+2)\cdots(b+n)} = \infty$ 或 0.

$$\begin{aligned} & \lim \frac{a(a+1)(a+2)\cdots(a+n)}{b(b+1)(b+2)\cdots(b+n)} \\ &= \lim \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdots \frac{a+n}{b+n}. \end{aligned}$$

當 $a > b$: 設 $a - b = a'$,

$$\begin{aligned} & \lim \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdots \frac{a+n}{b+n} \\ &= \lim \left(1 + \frac{a'}{b}\right) \left(1 + \frac{a'}{b+1}\right) \left(1 + \frac{a'}{b+2}\right) \cdots \left(1 + \frac{a'}{b+n}\right). \end{aligned}$$

$\therefore \frac{a'}{b} + \frac{a'}{b+1} + \frac{a'}{b+2} + \cdots + \frac{a'}{b+n}$ 為發散者。

$\therefore \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdots \frac{a+n}{b+n}$ 亦為發散者。

$$\therefore \lim \frac{a(a+1)(a+2)\cdots(a+n)}{b(b+1)(b+2)\cdots(b+n)} = \infty.$$

當 $a < b$: 設 $b - a = b'$,

$$\begin{aligned} & \lim \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdots \frac{a+n}{b+n} \\ &= \lim \left(1 - \frac{b'}{b}\right) \left(1 - \frac{b'}{b+1}\right) \left(1 - \frac{b'}{b+2}\right) \cdots \left(1 - \frac{b'}{b+n}\right). \end{aligned}$$

按 § 1001, 例題 1,

$$\lim \left(1 - \frac{b'}{b}\right) \left(1 - \frac{b'}{b+1}\right) \left(1 - \frac{b'}{b+2}\right) \cdots \left(1 - \frac{b'}{b+n}\right) = 0.$$

$$\therefore \lim \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a+2}{b+2} \cdots \frac{a+n}{b+n} = 0.$$

XXXVII. 連分式

習題 XCIII

原本第 575 頁

1. 解: $3 + \frac{1}{4 + \frac{1}{1 + \frac{1}{5}}}$.

$$p_1 = a_1, \quad p_2 = a_1 a_2 + 1, \quad p_3 = a_1 a_2 a_3 + a_1 + a_3,$$

$$\dots, \quad p_n = a_n p_{n-1} + p_{n-2};$$

$$q_1 = 1, \quad q_2 = 4, \quad q_3 = a_2 a_3 + 1,$$

$$\dots, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

$$\therefore p_1 = 3, \quad p_2 = 13, \quad p_3 = 16, \quad p_4 = 93;$$

$$q_1 = 1, \quad q_2 = 4, \quad q_3 = 5, \quad q_4 = 29.$$

答: 其近值爲 $\frac{3}{1}, \frac{31}{4}, \frac{16}{5}, \frac{93}{29}$.

2. 解: $\frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{10 + \frac{1}{12}}}}}$.

應用公式 $p_n = a_n p_{n-1} + p_{n-2}$ 及 $q_n = a_n q_{n-1} + q_{n-2}$.

$$p_1 = 0, \quad p_2 = 1, \quad p_3 = 1, \quad p_4 = 4, \quad p_5 = 41, \quad p_6 = 496;$$

$$q_1 = 1, \quad q_2 = 1, \quad q_3 = 2, \quad q_4 = 7, \quad q_5 = 72, \quad q_6 = 871.$$

答: 其近值爲 $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{4}{7}, \frac{41}{72}, \frac{496}{871}$.

3. 解: $\frac{10}{12} = 5 \left| \begin{array}{c|c|c} 10 & 12 & 1 \\ \hline 10 & 10 & \\ \hline 0 & 2 & \end{array} \right| \therefore \frac{10}{12} = \frac{1}{1} + \frac{1}{5}$.

4. 解: $\frac{457}{56} = 8 \left| \begin{array}{c|c|c} 457 & 56 & 6 \\ \hline 448 & 54 & \\ \hline 9 & 2 & 2 \\ \hline 8 & 2 & \\ \hline 1 & 0 & \end{array} \right| \therefore \frac{457}{56} = 8 + \frac{1}{6} + \frac{1}{4} + \frac{1}{2}$.

$$5. \text{ 解: } \frac{142}{513} \quad \begin{array}{r|l} 1 & \begin{array}{r|l} 142 & 512 \\ \hline 87 & 425 \end{array} & 3 \\ \hline 1 & \begin{array}{r|l} 55 & 87 \\ \hline 32 & 55 \end{array} & 1 \\ \hline 2 & \begin{array}{r|l} 23 & 32 \\ \hline 18 & 23 \end{array} & 1 \\ \hline 1 & \begin{array}{r|l} 5 & 9 \\ \hline 4 & 5 \\ \hline 1 & 4 \\ \hline & 4 \\ \hline & 0 \end{array} & 1 \\ \hline & & 4 \end{array}$$

$$\therefore \frac{142}{513} = \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} \\ + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} \\ + \frac{1}{1} + \frac{1}{4}$$

$$6. \text{ 解: } 3.54 = \frac{354}{100} \quad \begin{array}{r|l} 1 & \begin{array}{r|l} 354 & 100 \\ \hline 300 & 54 \end{array} & 1 \\ \hline 3 & \begin{array}{r|l} 54 & 46 \\ \hline 46 & 40 \end{array} & 5 \\ \hline 7 & \begin{array}{r|l} 8 & 6 \\ \hline 6 & 6 \\ \hline 2 & 0 \end{array} & 3 \end{array}$$

$$\therefore 3.54 = 3 + \frac{1}{1} + \frac{1}{1} \\ + \frac{1}{5} + \frac{1}{1} \\ + \frac{1}{3}$$

$$7. \text{ 解: } 0.1457 = \frac{1457}{10000} \quad \begin{array}{r|l} 1 & \begin{array}{r|l} 1457 & 10000 \\ \hline 1258 & 8742 \end{array} & 6 \\ \hline 3 & \begin{array}{r|l} 199 & 1258 \\ \hline 192 & 1194 \end{array} & 6 \\ \hline 7 & \begin{array}{r|l} 7 & 64 \\ \hline 7 & 63 \\ \hline 0 & 1 \end{array} & 9 \end{array}$$

$$\therefore 0.1457 = \frac{1}{6} + \frac{1}{1} + \frac{1}{6} + \frac{1}{3} + \frac{1}{9} + \frac{1}{7}$$

$$8. \text{ 解: } \frac{233}{177} \quad \begin{array}{r|l} 1 & \begin{array}{r|l} 233 & 177 \\ \hline 177 & 56 \end{array} & 3 \\ \hline 6 & \begin{array}{r|l} 56 & 9 \\ \hline 54 & 8 \end{array} & 4 \\ \hline 2 & \begin{array}{r|l} 2 & 1 \\ \hline 2 & \\ \hline 0 & \end{array} & \end{array}$$

$$\therefore \frac{233}{177} = 1 + \frac{1}{3} + \frac{1}{6} \\ + \frac{1}{4} + \frac{1}{2}$$

$$p_1=1, \quad p_2=4, \quad p_3=25, \quad p_4=104; \\ q_1=1, \quad q_2=3, \quad q_3=19, \quad q_4=79,$$

故其第四近值爲 $\frac{104}{79}$, 其錯誤 $< 1/79^2$.

9. 解: $\frac{421}{972}$

3	421	972	2
5	31	130	4
	30	124	
	1	6	6
		6	
		0	

$$\therefore \frac{421}{972} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

$$p_1=0, \quad p_2=1, \quad p_3=3, \quad p_4=13;$$

$$q_1=1, \quad q_2=2, \quad q_3=7, \quad q_4=30.$$

故其第四近值爲 $\frac{13}{30}$, 其錯誤 $< 1/30^2$.

10. 解: $\frac{23456}{31827}$

2	23456	31827	1
4	6714	8371	1
3	86	1657	19
2	17	23	1
5	5	6	1
	5	5	
	0	1	

$$\therefore \frac{23456}{31827} = \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{19} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{5}$$

$$p_1=0, \quad p_2=1, \quad p_3=2, \quad p_4=3;$$

$$q_1=1, \quad q_2=1, \quad q_3=3, \quad q_4=4.$$

故其第四近值爲 $\frac{3}{4}$, 其錯誤 $< \frac{1}{4 \cdot 19}$.

11. 解: $\sqrt{17} = 4 + (\sqrt{17} - 4) = 4 + \frac{1}{\sqrt{17} + 4}$,

$$\sqrt{17} + 4 = 8 + (\sqrt{17} - 4) = 8 + \frac{1}{\sqrt{17} + 4}$$

$$\therefore \sqrt{17} = 4 + \frac{1}{8} + \frac{1}{8} + \dots$$

故其第五近值爲 $\frac{17684}{4289}$ ，其錯誤 $< \frac{1}{4289^2}$ 。

12. 解： $\sqrt{26} = 5 + (\sqrt{26} - 5) = 5 + \frac{1}{\sqrt{26} + 5}$ ，

$$\sqrt{26} + 5 = 10 + (\sqrt{26} - 5) = 10 + \frac{1}{\sqrt{26} + 5}$$

$$\therefore \sqrt{26} = 5 + \frac{1}{10} + \frac{1}{10} + \dots$$

$$5, \frac{51}{10}, \frac{515}{101}, \frac{5201}{1020}, \frac{52525}{10301}$$

故其第五近值爲 $\frac{52525}{10301}$ ，其錯誤 $< \frac{1}{10301^2}$ 。

13. 解： $\sqrt{6} = 2 + (\sqrt{6} - 2) = 2 + \frac{2}{\sqrt{6} + 2} = 2 + \frac{1}{\frac{\sqrt{6} + 2}{2}}$ ，

$$\frac{\sqrt{6} + 2}{2} = 2 + \frac{\sqrt{6} - 2}{2} = 2 + \frac{1}{\sqrt{6} + 2}$$

$$\sqrt{6} + 2 = 4 + (\sqrt{6} - 2) = 4 + \frac{1}{\frac{\sqrt{6} + 2}{2}}$$

$$\therefore \sqrt{6} = 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \dots$$

$$2, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}, \frac{218}{89}$$

故其第五近值爲 $\frac{218}{89}$ ，其錯誤 $< \frac{1}{89^2}$ 。

14. 解： $\sqrt{38} = 6 + (\sqrt{38} - 6) = 6 + \frac{1}{\frac{\sqrt{38} + 6}{2}}$ ，

$$\frac{\sqrt{38+6}}{2} = 6 + \frac{(\sqrt{38}-6)}{2} = 6 + \frac{1}{\sqrt{38+6}}$$

$$\sqrt{38+6} = 12 + (\sqrt{38}-6) = 12 + \frac{1}{\frac{\sqrt{38+6}}{2}}$$

$$\therefore \sqrt{38} = 6 + \frac{1}{6} + \frac{1}{12} + \frac{1}{6} + \frac{1}{12} + \dots$$

$$6, \frac{37}{6}, \frac{450}{73}, \frac{2737}{444}, \frac{33294}{5401}$$

故其第五近值爲 $\frac{33294}{5401}$ ，其錯誤 $< \frac{1}{5401^2}$

15. 解: $\sqrt{105} = 10 + (\sqrt{105} - 10) = 10 + \frac{5}{\sqrt{105+10}}$

$$= \frac{1}{\frac{\sqrt{150+10}}{5}}$$

$$\frac{\sqrt{105+10}}{5} = 4 + \frac{\sqrt{105}-10}{5} = 4 + \frac{1}{\sqrt{105+10}}$$

$$\sqrt{105+10} = 20 + \sqrt{105} - 10 = 20 + \frac{1}{\frac{\sqrt{105+10}}{5}}$$

$$\therefore \sqrt{105} = 10 + \frac{1}{4} + \frac{1}{20} + \dots$$

16. 解: $\sqrt{23} = 4 + \sqrt{23} - 4 = 4 + \frac{1}{\frac{\sqrt{23+4}}{7}}$

$$\frac{\sqrt{23+4}}{7} = 1 + \frac{\sqrt{23}-3}{7} = 1 + \frac{1}{\frac{\sqrt{23+3}}{2}}$$

$$\frac{\sqrt{23+3}}{2} = 3 + \frac{\sqrt{23}-3}{2} = 3 + \frac{1}{\frac{\sqrt{23+3}}{7}}$$

$$\frac{\sqrt{23+3}}{7} = 1 + \frac{\sqrt{23-4}}{7} = 1 + \frac{1}{\sqrt{23+4}},$$

$$\sqrt{22+4} = 8 + \frac{\sqrt{23-4}}{7} = 8 + \frac{1}{\sqrt{23+4}}.$$

$$\therefore \frac{1}{\sqrt{23}} = \frac{1}{4} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{8} + \dots$$

17. 解: $\sqrt{19} = 4 + \frac{\sqrt{19-4}}{3} = 4 + \frac{1}{\sqrt{19+4}},$

$$\frac{\sqrt{19+4}}{3} = 2 + \frac{\sqrt{19-2}}{3} = 2 + \frac{1}{\sqrt{19+2}},$$

$$\frac{\sqrt{19+2}}{5} = 1 + \frac{\sqrt{19-3}}{5} = 1 + \frac{1}{\sqrt{19+3}},$$

$$\frac{\sqrt{19+3}}{2} = 3 + \frac{\sqrt{19-3}}{2} = 3 + \frac{1}{\sqrt{19+3}},$$

$$\frac{\sqrt{19+3}}{5} = 1 + \frac{\sqrt{19-2}}{5} = 1 + \frac{1}{\sqrt{19+2}},$$

$$\frac{\sqrt{19+2}}{3} = 2 + \frac{\sqrt{19-4}}{3} = 2 + \frac{1}{\sqrt{19+4}},$$

$$\sqrt{19+4} = 8 + \frac{\sqrt{19-4}}{3} = 8 + \frac{1}{\sqrt{19+4}}.$$

$$\therefore \sqrt{19} = 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{8} + \dots$$

18. 解: $\sqrt{71} = 8 + \frac{\sqrt{71-8}}{7} = 8 + \frac{1}{\sqrt{71+8}},$

$$\frac{\sqrt{71}+8}{7} = 2 + \frac{\sqrt{71}-6}{7} = 2 + \frac{1}{\sqrt{71}+6},$$

$$\frac{\sqrt{71}+6}{5} = 2 + \frac{\sqrt{71}-4}{5} = 2 + \frac{1}{\sqrt{71}+4},$$

$$\frac{\sqrt{71}+4}{11} = 1 + \frac{\sqrt{71}-7}{11} = 1 + \frac{1}{\sqrt{71}+7},$$

$$\frac{\sqrt{71}+7}{2} = 7 + \frac{\sqrt{71}-7}{2} = 7 + \frac{1}{\sqrt{71}+7},$$

$$\frac{\sqrt{71}+7}{11} = 1 + \frac{\sqrt{71}-4}{11} = 1 + \frac{1}{\sqrt{71}+4},$$

$$\frac{\sqrt{71}+4}{5} = 2 + \frac{\sqrt{71}-6}{5} = 2 + \frac{1}{\sqrt{71}+6},$$

$$\frac{\sqrt{71}+6}{7} = 2 + \frac{\sqrt{71}-8}{7} = 2 + \frac{1}{\sqrt{71}+8},$$

$$\sqrt{71}+8 = 16 + \sqrt{71}-8 = 16 + \frac{1}{\sqrt{71}+8}.$$

$$\therefore \sqrt{71} = 8 + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{7} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{16} + \dots$$

19. 解: $3\sqrt{3} = 5 + 3\sqrt{3} - 5 = 5 + \frac{1}{3\sqrt{3}+5},$

$$\frac{3\sqrt{3}+5}{2} = 5 + \frac{3\sqrt{3}-5}{2} = 5 + \frac{1}{3\sqrt{3}+5},$$

$$3\sqrt{3}+5 = 10 + 3\sqrt{3} - 5 = 10 + \frac{1}{3\sqrt{3}+5}.$$

$$\therefore 3\sqrt{3} = 5 + \frac{1}{5} + \frac{1}{10} + \dots$$

20. 解: $\frac{\sqrt{10}-2}{2} = \frac{1}{\frac{\sqrt{10}+2}{3}}$,

$$\frac{\sqrt{10}+2}{3} = 1 + \frac{\sqrt{10}-1}{3} = 1 + \frac{1}{\frac{\sqrt{10}+1}{3}}$$

$$\frac{\sqrt{10}+1}{3} = 1 + \frac{\sqrt{10}-2}{3} = 1 + \frac{1}{\frac{\sqrt{10}+2}{2}}$$

$$\frac{\sqrt{10}+2}{2} = 2 + \frac{\sqrt{10}-2}{2} = 2 + \frac{1}{\frac{\sqrt{10}+2}{3}}$$

$$\therefore \frac{\sqrt{10}-2}{2} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \dots$$

21. 解: $\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{3+2\sqrt{2}}{1} = 5 + (2\sqrt{2}-2) = 5 + \frac{1}{\frac{2\sqrt{2}+2}{4}}$,

$$\frac{2\sqrt{2}+2}{4} = 1 + \frac{2\sqrt{2}-2}{4} = 1 + \frac{1}{2\sqrt{2}+2}$$

$$2\sqrt{2}+2 = 4 + 2\sqrt{2}-2 = 4 + \frac{1}{\frac{2\sqrt{2}+2}{4}}$$

$$\therefore \frac{\sqrt{2}+1}{\sqrt{2}-1} = 5 + \frac{1}{1} + \frac{1}{4} + \dots$$

22. 解: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$

$$x = \frac{1}{1} + \frac{1}{2} + \frac{1}{3+x} = \frac{7+2x}{10+3x}$$

$$3x^2 + 8x - 7 = 0.$$

$$\therefore x = \frac{\sqrt{37}-4}{3}.$$

23. 解: $\frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \dots$

$$x = \frac{1}{2} + \frac{1}{1} + \frac{1}{3+x} = \frac{11+3x}{4+x}$$

$$3x^2 + 10x - 4 = 0.$$

$$\therefore x = \frac{\sqrt{37}-5}{3}.$$

24. 解: $3 + \frac{1}{4} + \frac{1}{5} + \frac{1}{2} + \dots$

$$y = 3 + \frac{1}{4+x} \dots \dots \dots (1)$$

$$x = \frac{1}{5} + \frac{1}{2+x} = \frac{2+x}{11+5x}$$

$$5x^2 + 10x - 2 = 0.$$

$$\therefore x = \frac{\sqrt{35}-5}{5}.$$

代入 (1) 式, 得 $y = 3 + \frac{1}{4 + \frac{\sqrt{35}-5}{5}} = \frac{3\sqrt{35}+50}{\sqrt{35}+15}$

25. 解: $2 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

$$x = 2 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{x} = \frac{157x+30}{68x+13}$$

$$34x^2 - 72x - 15 = 0.$$

$$\therefore x = \frac{\sqrt{1806}+86}{34}.$$

26. 解: $\frac{1}{2} + \frac{1}{7} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \dots$

$$y = \frac{1}{2} + \frac{1}{7+x}$$

$$x = \frac{1}{1} + \frac{1}{2} + \frac{1}{1+x} = \frac{3+2x}{4+3x}$$

$$3x^2 + 2x - 3 = 0.$$

$$x = \frac{\sqrt{10}-1}{3},$$

$$y = \frac{1}{2} + \frac{1}{7 + \frac{\sqrt{10}-1}{3}} = \frac{20 + \sqrt{10}}{43 + 2\sqrt{10}}.$$

27. 證: $\sqrt{a^2+1} = a + \frac{1}{2a+2a+} \dots\dots$

$$\sqrt{a^2+1} = a + (\sqrt{a^2+1} - a) = a + \frac{1}{\sqrt{a^2+1} + a},$$

$$\sqrt{a^2+1} + a = 2a + (\sqrt{a^2+1} - a) = 2a + \frac{1}{\sqrt{a^2+1} + a}.$$

$$\therefore \sqrt{a^2+1} = a + \frac{1}{2a+2a+} \dots\dots$$

28. 證: $\sqrt{a^2+2} = a + \frac{1}{a+2a+} \frac{1}{a+2a+} \dots\dots$

$$\sqrt{a^2+2} = a + (\sqrt{a^2+2} - a) = a + \frac{1}{\frac{\sqrt{a^2+2} + a}{2}},$$

$$\frac{\sqrt{a^2+2} + a}{2} = a + \frac{\sqrt{a^2+2} - a}{2} = a + \frac{1}{\sqrt{a^2+2} + a},$$

$$\sqrt{a^2+2} + a = 2a + \frac{1}{\sqrt{a^2+2} + a}.$$

$$\therefore \sqrt{a^2+2} = a + \frac{1}{a+2a+} \frac{1}{a+2a+} \dots\dots$$

29. 證: $\frac{1}{a+b+c} + \dots\dots$

$$= \frac{-(abc+a-b+c) + \sqrt{(abc+a+b+c)^2 + 4}}{2(ab+1)}.$$

設 $x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + x = \frac{bc+bx+1}{abc+(ab+1)x+a+c}$
 $(ab+1)x^2 + (abc+a+c-b)x - (bc+1) = 0,$

$$\begin{aligned} \therefore x &= \frac{-(abc+a-b+c) + \sqrt{(abc+a-b+c)^2 + 4(ab+1)(bc+1)}}{2(ab+1)} \\ &= \frac{-(abc+a-b+c) + \sqrt{(abc+a+b+c)^2 + 4}}{2(ab+1)}. \end{aligned}$$

30. 解: $x^2 + x - 1 = 0$.

此方程式之正根爲 $x = \frac{-1 + \sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2}$,

$$\frac{\sqrt{5}-1}{2} = \frac{2}{\sqrt{5}+1} = \frac{1}{\frac{\sqrt{5}+1}{2}}$$

$$\frac{\sqrt{5}+1}{2} = 1 + \frac{\sqrt{5}-1}{2} = \frac{1}{\frac{\sqrt{5}+1}{2}}$$

$$\therefore \frac{\sqrt{5}-1}{2} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

31. 證: $\frac{p_n}{q_n} = \frac{p_1}{q_1} + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \dots + (-1)^n \frac{1}{q_{n-1} q_n}$.

因 $\frac{p_n}{q_n} = \frac{(-1)^n + p_{n-1}}{q_n q_{n-1} q_{n-1}}$

$$\frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1} + p_{n-2}}{q_{n-1} q_{n-2} q_{n-2}}$$

$$\frac{p_2}{q_2} = \frac{(-1)^2 + p_1}{q_2 q_1 q_1}$$

$$\therefore \frac{p_n}{q_n} = \frac{p_1}{q_1} + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \dots + (-1)^n \frac{1}{q_{n-1} q_n}$$

32. 證: $\frac{1}{a_2 + a_3} + \dots = \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \frac{1}{q_3 q_4} - \dots$.

因 $\frac{p_n}{q_n} = a_1 + \frac{1}{a_2 + a_3} + \dots = \frac{p_1}{q_1} + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \frac{1}{q_3 q_4} - \dots$

但 $\frac{p_1}{q_1} = a_1$, 故

$$\frac{1}{a_2 + a_3} + \dots = \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \frac{1}{q_3 q_4} - \dots$$

33. 解: $\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{2 + (\sqrt{2} - 1)}$
 $= 1 + \frac{1}{2} + \frac{1}{2} + \dots$

$$p_1 = 1, \quad p_2 = 3, \quad p_3 = 7, \quad p_4 = 17,$$

$$p_5 = 41, \quad p_6 = 99, \quad p_7 = 239, \quad p_8 = 577;$$

$$q_1 = 1, \quad q_2 = 2, \quad q_3 = 5, \quad q_4 = 12,$$

$$q_5 = 26, \quad q_6 = 70, \quad q_7 = 169, \quad q_8 = 408.$$

答: 此分數為 $\frac{577}{408}$, 其錯誤 $< \frac{1}{408^2}$.

34. 解:

7	10000000	314159265	3
	99114855	300000000	
1	885145	14159265	15
	882090	13277175	
	3055	882090	288
		879840	
		2250	

故 $3.14156265\dots = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{288} + \dots$

$$p_1 = 3, \quad p_2 = 22, \quad p_3 = 333, \quad p_4 = 355;$$

$$q_1 = 1, \quad q_2 = 7, \quad q_3 = 106, \quad q_4 = 113, \quad q_5 = 32650.$$

答: 所求之分數為 $\frac{355}{113}$, 其錯誤 $< \frac{1}{113 \times 32650} < 0.000001$.

35. 解:

2	271828	100000	1
	200000	71828	
2	71828	28172	1
	56344	15484	
1	15484	12688	4
	12688	11184	
1	2796	1504	
	1504		
	1292		

故 $2.71828\dots = 2 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \dots$

$$p_1=2, p_2=3, p_3=8, p_4=11, p_5=19, p_6=37;$$

$$q_1=1, q_2=1, q_3=3, q_4=4, q_5=7, q_6=32.$$

答：其第六近值爲 $\frac{87}{32}$ ，其錯誤 $< \frac{1}{32^2}$ 。

36 解：

1	214	127	1
	127	87	
2	87	40	5
	80	35	
1	7	5	2
	5	4	
2	2	1	
	2		
	0		

故 $\frac{214}{127} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{5} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2}$ 。

其近值爲 $\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{27}{16}, \frac{32}{19}, \frac{91}{54}, \frac{214}{127}$

$$91 \cdot 127 - 214 \cdot 54 = 1,$$

$$546 \cdot 127 - 324 \cdot 214 = 6.$$

$$\therefore x = 546 + 214t,$$

$$y = 324 + 127t.$$

37. 解：

1	235	412	1
	177	235	
19	58	177	3
	57	174	
	1	3	3
		3	
		0	

故 $\frac{412}{235} = 1 + \frac{1}{1} + \frac{1}{3} + \frac{1}{19} + \frac{1}{3}$ 。

其近值爲 $\frac{1}{1}, \frac{2}{1}, \frac{7}{4}, \frac{135}{77}, \frac{412}{235}$ 。

$$135 \cdot 235 - 412 \cdot 77 = 1,$$

$$135 \cdot 235 \cdot 10 - 412 \cdot 77 \cdot 10 = 10.$$

$$\therefore x = 1350 + 412t,$$

$$y = -770 - 235t.$$

38. 解:

$$\begin{array}{r|l|l|l} 1 & 517 & 323 & 1 \\ & 323 & 194 & \\ \hline 1 & 194 & 129 & 1 \\ & 129 & 65 & \\ \hline 1 & 65 & 64 & 64 \\ & 64 & 64 & \\ \hline & 1 & 0 & \end{array}$$

$$\frac{517}{323} = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{64}$$

其近值爲 $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{517}{323}$

$$517 \cdot 5 - 323 \cdot 8 = 1,$$

$$517 \cdot 5 \cdot 31 - 323 \cdot 8 \cdot 31 = 31,$$

$$317 \cdot 155 - 323 \cdot 248 = 31.$$

$$\therefore x = 155 + 323t,$$

$$y = 248 + 517t.$$