U. S. COAST AND GEODETIC SURVEY

ON THE MEASUREMENT OF THE BASE LINES
AT
HOLTON, INDIANA
AND AT
ST. ALBANS, WEST VIRGINIA
1891 and 1892

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APPENDIX No. 8—REPORT FOR 1892

UNITED STATES
COAST AND GEODETIC SURVEY
T. C. MENDENHALL
SUPERINTENDENT

GEODESY

ON THE MEASUREMENT OF THE BASE LINES
AT
HOLTON, INDIANA
AND AT
ST. ALBANS, WEST VIRGINIA

1891 AND 1892

Reports of the measurement of Holton Base
By A. T. MOSMAN, R. S. WOODWARD, and
O. H. TITTMANN, Assistants
And of the St. Albans Base by R. S. WOODWARD, Assistant

APPENDIX No. 8—REPORT FOR 1892

WASHINGTON
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1893

APPENDIX NO. 8—1892.

ON THE MEASUREMENT OF THE HOLTON BASE, HOLTON, RIPLEY COUNTY, INDIANA, AND THE ST. ALBANS BASE, KANAWHA COUNTY, WEST VIRGINIA.

PREFATORY NOTE.

The two chains of the transcontinental triangulation near the thirty-ninth parallel, which had been carried westward from Chesapeake Bay and eastward from the Mississippi, having reached a junction upon a line in the State of Indiana in September, 1890, the measurement of a base of verification became desirable.

A reconnoissance made by Assistant A. T. Mosman under instructions from the Superintendent for the purpose of finding a suitable site for this measurement, resulted, after a few examinations, in the selection of a locality near the town of Holton, Ripley County, Indiana.

In the spring of 1891 preliminary arrangements for the measurement were begun by Mr. Mosman, to whom had been assigned the charge of the party and the general direction of its operations.

It was decided that this measurement should be conducted in such a way as to obtain most thorough tests of the relative merits of several forms of base apparatus. To this end the charge of the measurements with the metallic tapes and with the iced bar apparatus, and the experimental work with the 100-metre comparator was assigned to Assistant R. S. Woodward, while the conduct of the measurement with the new secondary contact-slide apparatus was committed to Assistant O. H. Tittmann.

The reports from these officers which follow elucidate fully the methods of measurement adopted and their results.

They are here published as prepared quite independently by Assistants Woodward and Tittmann. Each contains the details required to enable the reader to form an independent judgment of the accuracy of the method employed and to reach his own conclusions as to relative merits. It may be well, however, to invite attention to one or two inferences in relation to essential points, the correctness of which will be admitted, it is believed, by all who are familiar with the subject.

It is evident that either of the two principal methods of base measurement described, namely, the use of the tape and the so-called "secondary apparatus," will give an accuracy entirely sufficient for any demands likely to arise. In both cases it appears that the probable error derived from the range of different measures is less than one part in one million, and it may safely be assumed that, including errors from all sources, a result true to one part in half a million can be reached.

It is doubtless correct to say that no known method of base measurement aside from the two here considered is capable of giving as good results, when accuracy and expense are both included. The use of an iced bar, applied to the measurement of considerable distances for the first time in the Holton Base, is unquestionably the method of highest precision, and its cost is not believed to be greater than that of other methods in use in Europe, but it will not be found necessary to resort to it in ordinary practice, except for the purpose of standardization as here described. The metallic tape is not only capable of giving a result of great accuracy when in the hands of experts, but it is evidently the best device for rapid base measurement where no great precision is aimed at. As to the cost of the "tape-line method" and that of the "secondary apparatus," when the accuracy required is one part in from two hundred to five hundred thousand, there can be little difference.

Subsequent to the measurement of the Holton Base, a check base in West Virginia, known as the St. Albans Base, was measured by Assistant Woodward, the tape line being used and the conditions being such as to render it easy to determine quite accurately the cost of the work.

The use of the secondary apparatus at Holton having been continuous, when it was once begun, the circumstances were such that a very good measure of the cost of that work is also possible. When these results are compared, it is found that under ordinary conditions both time and cost of the two methods would be essentially the same. Local peculiarities will therefore usually determine which method shall be used.

T. C. Mendenhall,
Superintendent.

MEASUREMENT OF THE HOLTON BASE.

I. EXTRACTS FROM THE RECORDS AND FROM THE REPORTS OF A. T. MOSMAN, ASSISTANT.

LOCATION OF BASE LINE, MARKINGS, ETC.

The primary base line in Ripley County, Ind., laid out by my party in October and November, 1890, was located on the high and nearly level table land between the towns of Holton and New Marion. Its length is 5,500 metres, and its direction is nearly north and south. In order to avoid the creek near its north end, it was swung to the westward from a true north and south line.

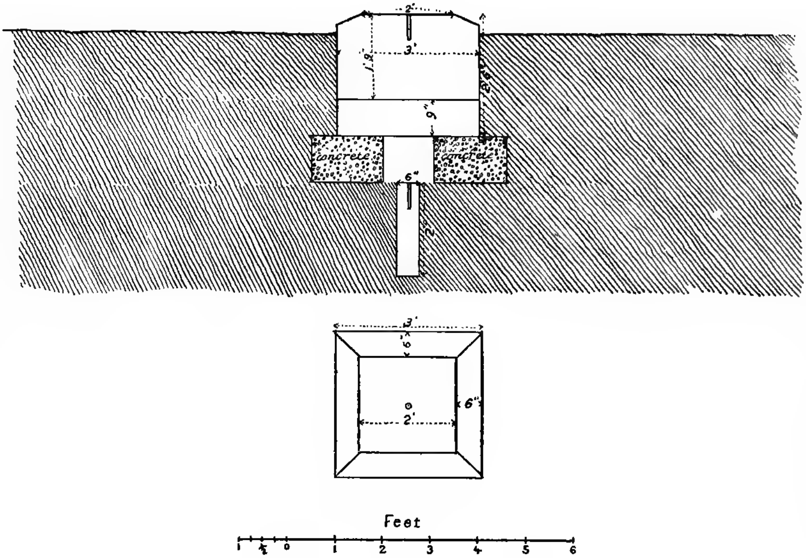
North base is in Otter Creek Township, about 1 mile east of Holton, on the Ohio and Mississippi Railroad. South base is in Centre Township, and about half a mile north of the village of New Marion.

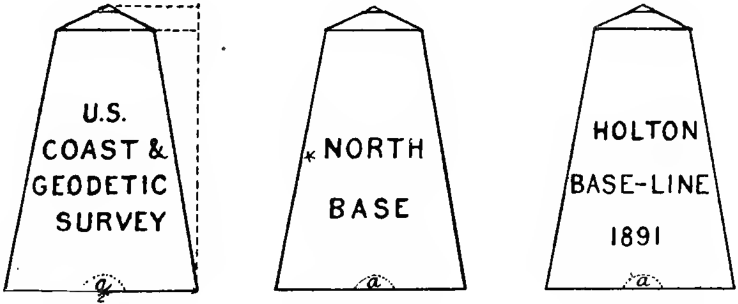
The underground and surface marks placed to secure the ends of the base after the completion of the measurements were as shown in the accompanying sketch (illustration No. 30). The sub-surface mark was a copper bolt, set in a limestone post 6 inches square and 2 feet long, which was sunk in the earth till its top was 3 feet below the surface. At the intersection of cross lines cut on the top of the bolt, a fine hole one-quarter of an inch deep was drilled to mark the station point. Above and around the post, except for a space of 1 foot square immediately over it, is a layer of Portland cement concrete 1 foot thick and 4 feet square. Upon this foundation rests a block of limestone 3 feet square and 30 inches high, having a copper bolt with cross and fine drill hole sunk into its top at the centre, this hole being the surface mark of the end of the base.

To protect these surface marks from injury and to indicate conspicuously the ends of the base, limestone shafts were placed in position over them. The one at North Base had inscriptions on three of its faces, as shown in the illustration, and the one at South Base was similarly marked.

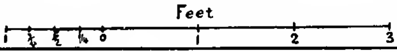
All of the preparations for the actual measurement were completed early in June, 1891. These included the clearing of the line, the fencing of the camp ground, the pitching of the tents twenty-two in number, the construction of a comparing house with railroad and track 110 metres long, the setting of posts for microscopes, and the determination of the height of North Base above sea level by levelings to the nearest bench mark of the transcontinental line of geodetic leveling.

Surface and Sub-surface Markings of the Ends of the Holton Base, and Sketch of Monument at North Base.





South Face. East Face. West Face.



Before the base apparatus arrived from Washington, section stones were set at 1,200, 2,100, 3,000, 4,000, and 5,000 metres from South Base and carefully aligned. Stones were also set at 3,900 and 4,900 metres to mark the ends of the kilometre intended to be measured with the standard bar encased in ice. These section stones were cubical blocks of limestone, 1 foot in each dimension, and set in beds of concrete 15 inches square and 15 inches deep. In the top of each stone was secured a copper bolt, 1 inch in diameter and 3 inches long, with a fine cross cut on its head. The levels run along the line from North Base to South Base made each of these section stones a bench mark well determined in height above sea level. The comparing house, with railroad and track 100 metres long, and posts for microscopes, was finished.

BEGINNING AND PROGRESS OF THE MEASUREMENTS.

On July 27, 1891, the instruments were unpacked and prepared for work, by Assistants Tittmann and Woodward, and on the 28th a measurement of the base was begun by Assistant Tittmann at △ South Base. His party consisted of 4 officers, beside himself, and 4 men, and they were employed for thirteen and one-half days in making two complete measures of the base, besides two additional measures of the kilometre, between stones at 3,900 and 4,900 metres from South Base, laid out to be measured by the standard bar in ice, the second measure having been finished on August 13.

During this time Assistant Woodward, assisted by Mr. Siebert and one or two hands, was engaged in experiments with iced bar and tapes in the comparing house.

On August 14 he began setting posts on the kilometre for supporting the portable track, and the microscopes, using all of the available force of laborers. Owing to continual rains, the ground was saturated with moisture and the work was much delayed. Stakes for supporting the tape during measurement were also set, at 10 metres apart, the whole length of the base line by Assistant Woodward, and some parts of the line were made ready for tape measurement, and on August 28, 29, and 31, and September 1 and 2, measurements were made by officers alone, and on September 8, 16, 17, 18, 23, and 28, and October 1, 2, 3, 7, and 8, measurements were made, usually at night, with a full force of officers and men.

The first measure of the kilometre, with standard bar in ice, was begun by Assistant Woodward on September 10 and continued on the 11th, 14th, 15th, 26th, 28th and 29th.

Experimental measurements were made by Assistant Tittmann with the secondary apparatus, on August 17, 18, 22, 25 and 26, and September 3, 7, and 8. He measured 100 metres of the kilometre with this apparatus on the 21st and 22d, and made measures of the whole kilometre with secondary apparatus on September 24 and 25, and October 5.

I must refer you to the reports of Assistants Woodward and Tittmann for the details of the work, and the results obtained by each method.

COST OF THE DIFFERENT OPERATIONS AND EXPERIMENTS.

A record, as nearly as possible, was kept of the number of days spent in each operation, with the number of officers and men employed each day, and an attempt is made in the table which follows to give the comparative cost, calling each officer or man employed one day one day's work, and the results are expressed in day's work of one officer or man.

In regard to the time spent in preparing the standard kilometre for measurement, and the time spent in setting stakes, there was necessarily some confusion, as frequently some of the party would be at work on each of these operations on the same day, but the result given is believed to be substantially correct.

The time spent in measures with the tape could not be absolutely fixed, as the measures were made sometimes for a few hours in the night with a detail of officers alone, sometimes with officers and men in the day or night, and no complete measurement of the whole base with the tape was ever made but only by parts, several times repeated.

Measurements were made on fourteen different days or nights, and assuming that each measurement took one-half day's time of each officer and man employed we would have sixty-nine and one-half days for one man spent in tape measure.

Distances of from 300 metres to 4,800 metres were measured at one time, mostly at night, and when men were employed we paid them for one-half a day's work.

Holton base line, 1891. Time spent on the different operations.

Kind of operation.	Day's work of one man.
Building ice-house	9
Building comparing house and laying track	67½
Opening lines and setting section stones	15
Leveling from Delaware to base line, two officers	11 = 22
Leveling from Osgood to △ Correct and △ Reizin, three officers	8 = 24
Measuring base twice and two extra measures of the standard kilometre with secondary apparatus	121½
Testing length of bars of secondary apparatus in comparing house	30½
Measuring kilometre with secondary apparatus, twice	18
Measuring 100 metres of kilometre with secondary apparatus	18
Experiments in comparing house with iced bar and tapes	65
Preparing kilometre for measurement with iced bar	114
Measurement of kilometre with iced bar	35
Setting stakes and preparing for tape measurement	53½
Measurements of parts of base with tapes fourteen different days or nights, or $\frac{139}{2}$ days	69½

SERVICES RENDERED.

Assistant R. S. Woodward had entire charge of all measurements with tape and standard bar and all experiments conducted with them in the comparing house and on the base line, including measurements with standard bar in ice of the kilometre. I refer you to his report for details of methods, description of the instruments used, and discussion of the results. Assistant Woodward joined the party on July 22 and left on October 12.

Assistant O. H. Tittmann joined the party on July 22 and left on October 8. He had full charge of all measurements and experiments made with the secondary apparatus on the base line, including kilometre, and in the comparing house, and he has presented to you a very full and complete report of all the work done by him.

Prof. J. H. Gore joined the party on June 20 and left on September 14. He rendered very efficient service while with the party in assisting in the measurements with the secondary apparatus; also with those made by Assistant Woodward with the tape. He had charge of the leveling party determining the heights of Versailles court-house and the △'s Reizin and Correct, being assisted by Messrs. Gjertsen, Pennington, and Cope, recorders.

Mr. John F. Hayford rendered very efficient service in measurements under Assistant Tittmann's charge and the computations pertaining thereto, also assisting in the measurements made with the steel tape and in the computation of the results. He joined the party on July 22 and left on October 12.

Mr. John S. Siebert, temporary aid, joined the party May 21 and assisted in the building of comparing house and railroad and in setting section stones along the base line. He had charge of the leveling from the B. M. at Delaware to △ North Base and △ Glasgow. He assisted in all the measurements and experiments made with tapes and standard bar in ice and in the computations. He remained till the close of work at the Holton base line, and performed all duties entrusted to him with accuracy, zeal, and intelligence.

Recorders Th. Gjertsen, Robert Pennington, and Frank B. Cope were attached to the party during their summer vacation. They rendered satisfactory service in the leveling operations, and also in measurements with the secondary apparatus.

Mr. E. E. Torrey, foreman, was attached to the party during the whole season, and, as usual, rendered most valuable service.

II. THE ICED BAR AND TAPE BASE APPARATUS AND RESULTS OF MEASURES MADE WITH THEM ON THE HOLTON AND ST. ALBANS BASES.

By R. S. Woodward, Assistant.

Submitted for publication June 26, 1893.

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LETTER OF TRANSMITTAL.

UNITED STATES COAST AND GEODETIC SURVEY,
Washington, D. C., June 26, 1893.

SIR: I have the honor to transmit herewith my complete report on the Iced Bar and Tape Base Apparatus of the Survey, and on the results obtained with them in the measurements of Holton and St. Albans bases.

Since the attempt to use such forms of apparatus constitutes what some geodesists would consider a bold if not doubtful experiment, I have thought it desirable to give a pretty full account of them and of the measures made with them. Accordingly, all of the results obtained in standardizing the apparatus and in measuring lines with them are given in detail.

Although the critical reader will thus be able to form his own estimate of the character of these forms of apparatus and of the work done with them, it may not be out of place here to state briefly what appear to me to be attainable limits of precision in measures made with each form.

With respect to the iced-bar apparatus, it seems practically certain from our experience that a probable error of one five-millionth part in the length of the measuring bar is attainable, and that an equal precision will result in the mean of a few measures of a line made with the apparatus. This degree of precision has not been reached in work already done with the apparatus, but its attainment, should it be desirable, depends only on the removal of two obstacles. These are, in the order of their importance, 1st, the uncertainty in the length of the Prototype Metres at 0° C., and, 2d, the uncertainty from personal equation, due in the case of the Survey bar to inequality in the widths of its terminal graduations.

With respect to the tape apparatus, it seems equally certain that a precision indicated by a probable error of one two-millionth part of a measured line is an attainable limit. A probable error of one one-millionth part appears to be easily and cheaply attainable with the long tapes after they are standardized. This would seem to be amply sufficient for the present purposes of geodesy, but the sole obstacle in the way of much higher precision, should it ever be deemed essential, is, I believe, the difficulty of measuring the tape's temperature.

Very respectfully yours,

R. S. WOODWARD,
Assistant, Coast and Geodetic Survey.

DR. T. C. MENDENHALL,
Superintendent Coast and Geodetic Survey.

CHAPTER I.

DESCRIPTION OF THE ICED BAR BASE APPARATUS AND ITS APPLICATION TO THE MEASUREMENT OF BASE LINES.

(1) *Historical Note.*—The use of ice in thermometry to furnish a standard temperature naturally suggests the availability of ice to fix the temperature of a standard of length when used in laboratory comparisons or in measuring base lines. It does not appear, however, that ice has been generally used even in laboratory work with standards of length,* and I am not aware that any attempt has been made hitherto to measure a base with a bar whose temperature is controlled by means of melting ice. The feasibility of using such an apparatus in base measurement has, nevertheless, been suggested and maintained by several persons. One of the first, if not the first, to outline a scheme for such an apparatus is, I believe, Mr. E. S. Wheeler, a former colleague on the U. S. Lake Survey. Mr. Wheeler's plan is advocated by Prof. T. W. Wright in his treatise on the Adjustment of Observations.† The late Capt. C. O. Boutelle, of the U. S. Coast and Geodetic Survey, also advocated the use of such apparatus.

Soon after joining the U. S. Coast and Geodetic Survey in July, 1890, I was requested by Dr. Mendenhall, Superintendent, to devise means of testing in the most thorough way practicable the efficiency of the various forms of base apparatus used by the Survey, and especially the efficiency of long steel tapes or wires. Accordingly, considerable study was given to this subject during the autumn of 1890 and the winter of 1890–'91, and the plans and specifications for the iced-bar apparatus considered in this paper were matured and approved early in the spring of 1891. It was constructed in Washington, partly by the machinists E. N. Gray & Co., and D. Ballauf, and partly by the Instrument Division of the Survey.

Before proceeding to a description of the apparatus I desire to acknowledge my indebtedness to colleagues of the Survey for valuable suggestions and criticism. I am specially indebted to Mr. John S. Sie-

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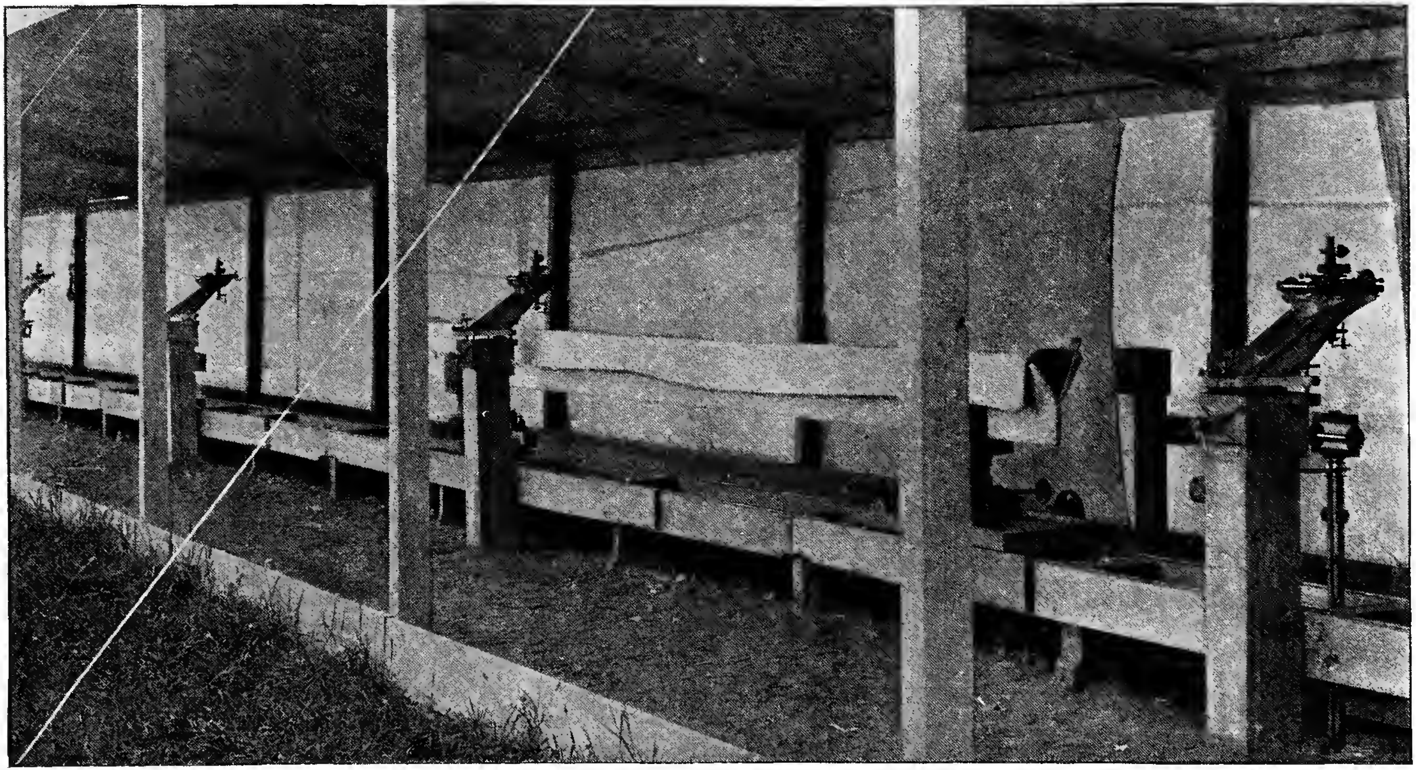
* From published accounts it would appear that the most extensive series of laboratory comparisons of standards, wherein ice was used, are those of the U. S. Lake Survey, conducted under the superintendence of Gen. C. B. Comstock, Corps Engineers, U. S. Army. In these comparisons ice was successfully used during several years. See Professional Papers, Corps Engineers, U. S. Army, No. 24.

† D. Van Nostrand, New York, 1884. See also Am. Jour. Science III, vol. XXVIII, p. 479.



ICED BAR APPARATUS AND WEST END OF 100^m COMPARATOR OF HOLTON BASE.

ICED BAR APPARATUS ON STANDARD KILOMETRE OF HOLTON BASE.

bert, who verified all of the preliminary calculations relative to the stability and efficiency of the apparatus, and who elaborated many of the designs and made most of the working drawings for its construction. I am particularly indebted also to Mr. E. G. Fischer, chief mechanician of the Survey, whose knowledge of and skill in mechanical appliances were frequently appealed to. Finally, it affords me pleasure to state that my friend, Mr. E. S. Wheeler, who has had extensive experience with base apparatus, happened to visit Washington about the time the plans for this apparatus were completed, and gave me the benefit of his advice and criticism.

(2) *General features of apparatus.*—The iced-bar apparatus belongs to that type in which a single rigid bar is used as the element of length, along with micrometer microscopes to mark its successive positions. An idea of the nature of the apparatus may be gained from illustrations 31 and 32. The first of these shows the apparatus as used on the 100^{m} comparator of Holton base, described at length in section 2, Chapter III. The other plate shows the apparatus as used on the standard kilometre of the same base. It will be seen that the measuring bar is carried in a trough, where it can be kept surrounded by melting ice. This trough is mounted on two cars, which move on tracks, stationary or portable, as the case may be. The microscopes, as shown in the illustrations, are mounted on wooden posts, which are ranged out and set firmly in the ground beforehand. This sort of support for the microscopes, though not essential to the apparatus, is convenient and economical when a series of measures of the same line is to be made. The microscopes are easily clamped to and detached from the posts, and are moved forward as the measure of a line progresses. The trough carrying the bar and ice load is likewise easily rolled forward on the cars along the tracks.

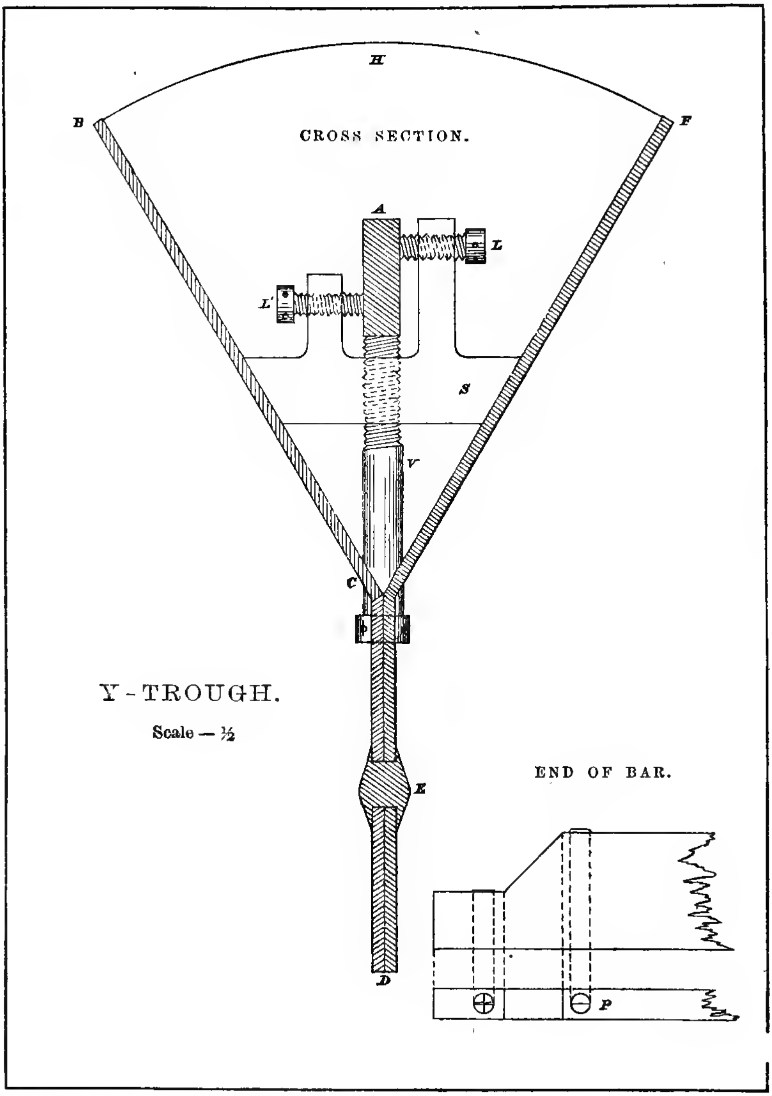
The apparatus is 5^{m} long, so that the microscope posts are set 5^{m} apart. The supports for the car tracks are also set 5^{m} apart, but are placed half way between the microscope posts, so as to avoid transmitting disturbance to the latter through the ground.

As shown in illustration No. 31, the apparatus on the 100^{m} comparator was protected from the direct rays of the sun by a shed. For field-work, on the other hand, the microscopes were shielded by means of large umbrellas, as shown in illustration No. 32.

The general features of the apparatus are thus apparent, and we may proceed to describe its parts in detail.

(3) *The measuring bar.*—The measuring bar of this apparatus is a rectangular bar of tire steel. It was rolled in the steel works at Lancaster, Pa. It is $5{\cdot}02^{m}$ long, 8^{mm} thick, and 32^{mm} deep. A cross section is shown at A in the accompanying illustration, p. 340.

The upper half of the bar is cut away for about 2^{cm} at either end to receive the graduation plugs of platinum-iridium, which are inserted so that their upper surfaces lie in the neutral surface of the bar. Three lines are ruled on each of these plugs, two in the direction of and one transverse to the length of the bar. These lines were ruled by Mr. Louis A. Fischer, adjuster in the Office of Standard Weights and Measures. The longitudinal lines, which serve to limit the parts of the transverse lines used, are $0{\cdot}2^{mm}$ apart. Although great pains were taken to have these lines of the same width, the transverse lines differ widely, the narrower one being $16{\cdot}3^{\mu}$ and the broader one $36{\cdot}2^{\mu}$ wide.



Y-TROUGH.

Scale — ½

This bar is known in the Survey records as No. 17. It is designated in all the work done with it thus far as B_{17}. The end having the narrower transverse graduation mark is called the A end, To secure alignment of the bar, eleven German silver plugs of 5^{mm} diameter are inserted at intervals of 495^{mm} along the bar, so that they project about 1^{mm} above its top surface. The upper surfaces of these plugs are all the same distance, within a few hundredths of a millimetre, from the neutral surface of the bar. On the top of each plug is ruled a fine line in the direction of the bar, as shown at P in the illustration. The length of the bar as regards alignment is defined to be the distance between the transverse graduation marks when the upper surfaces of the alignment plugs are all in one plane and when the lines on these plugs are in one straight line. The means of securing these two adjustments are described below.*

(4) *The* Y-*Trough.*—The most important and distinctive part of this apparatus is the trough which supports the bar, keeps it aligned, and carries the ice load essential to control the bar's temperature. This trough is called the Y-trough by reason of the resemblance of its cross section to the letter Y. The drawing shows a cross section of this trough. It is made of two steel plates $5{\cdot}14^{m}$ long, $25{\cdot}5^{cm}$ wide, and 3^{mm} thick. They are bent to the angle B C D of the figure, and are riveted together as shown at E, thus making the angle of the trough B C F = 60°. The bar, shown in cross section at A, is supported at every half metre of its length by saddles, one of which is shown in the figure. These saddles are rigidly attached to the sides of the trough by screws at S, S. Each saddle carries one vertical and two lateral adjusting screws as shown at V, L, L′. These screws serve to fix the alignment of the bar. The lateral adjusting screws of the saddles at the ends of the bar are of the same height, which is equal to that of the lower screw L′ of the diagram. The lateral adjusting screws of the intermediate saddles on either side of the bar are alternately high and low. The object of this disposition is two-fold, to wit: 1st, to prevent pinching the bar, which might more readily occur if the lateral screws were all opposite to one another; 2d, to afford means of rotating the bar slightly about its longitudinal axis, so that for a fixed and nearly vertical position of the trough the graduated surfaces of the bar may be made horizontal. The vertical adjusting screws of the saddles project, as shown in the diagram, below the vertex of the trough, and their capstan heads are accessible through slots cut in the web of the trough. These slots serve also as drainage ways for the melted ice. To prevent circulation of air through them they are stuffed with cotton batting, through which the water percolates freely. The ends of the trough are closed with wooden V-shaped blocks.

The trough is very rigid in all directions and especially so with

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*The form of bar described is evidently not the best form. Theory and experience indicate that a bar having a Y-shaped cross section with metric subdivisions on its neutral surface would best meet the requirements. However, the question which presented itself in planning the apparatus was not what is the best form of bar, but what is the most economical form possessing the requisite properties.

respect to vertical stresses. It weighs 82 kilogrammes, exclusive of the bar and ice load. The whole trough is covered by a closely-fitting jacket of heavy white cotton felt, which protects the trough and ice load alike from direct radiation.

For measuring grade angles a sector reading by two opposite verniers to 10″ is attached to one side of the trough near its middle point. Thus arranged this sector has great stability.

(5) *The ice load and ice crusher.*—When the apparatus is in use the Y-trough is completely filled with pulverized ice, the upper surface of which is rounded to about the height shown by the curve B H F in the diagram. The amount of ice required for this purpose is about 40 kilogrammes, or 8 kilogrammes per metre of the bar's length. The ice, by reason of its weight and the sloping sides of the trough, is kept in close contact with the bar. This is especially the case when the apparatus is in use, for it is then trundled along on its cars with sufficient jarring to overcome any tendency of the ice to pack. For covering the ends of the bar a small quantity of ice is cut with a jack plane. Ice thus cut, like wet snow, packs well and permits making a small conical hole through it to the graduation plugs.

A very essential auxiliary to the use of the apparatus is an ice-crusher to pulverize the ice. The machine used is a modification of the Creasey ice breaker manufactured at Philadelphia, Pa. It is a small light hand machine which, as modified, does its work very satisfactorily. With it 40 kilogrammes of ice may be pulverized in ten minutes or less. The particles of crushed ice vary in size from the smallest visible up to the bulk of a cubic centimetre; and this gradation in size appears to be advantageous as compared with uniformly finer particles like those of snow, since there is less liability of regelation and packing.

(6) *The cars and portable track.*—The Y-trough is mounted on two cars, the saddles or bolsters of which are attached to the trough 40^{cm} from either end. Each saddle is attached rigidly to the trough above and to a jackscrew below. The jackscrew is attached to a slide rest which is connected rigidly with the base of the car. The slide rests are provided with screws to give slow motions in the direction of the trough's length and transverse to its length.

The jackscrew cylinders have right and left handed threads at their respective ends and are turned by a short capstan bar. They give thus the rapid vertical motion to the trough essential in bringing the bar quickly to focus under the microscopes which define its position.

The cars have each three wheels and run on a portable track whose width is 30^{cm}. Three sections of this track, each 5^{m} long, are provided; and each section is carried forward as the cars are rolled along during the measurement of a line. It thus appears that instead of lifting up and carrying forward the measuring bar as with most forms of apparatus, this rather delicate and difficult operation is supplanted by that of moving the portable tracks.

(7) *The micrometer microscopes.*—To define the successive positions of the bar in measuring a line, micrometer microscopes are used. Through the courtesy of Gen. Casey, Chief of Engineers, U. S. Army, the Survey was enabled to borrow the four microscopes and the cut-off cylinder of the Repsold base apparatus* used on the U. S. Lake Survey. These are especially well adapted for use with any line measure apparatus. As designed by the Repsolds and as used on the Lake Survey the microscopes were mounted on iron tripods. These latter having been destroyed by fire while stored at the Engineer depot at Willetts Point, N. Y., it was essential to replace them by some equivalent device. In view of the economic and other features of the special work contemplated with the iced-bar apparatus, it was decided to mount the microscopes on wooden posts set firmly in the ground. To connect the microscope with the post a cast-iron post cap is provided. It fits like a box cover on the end of the post and is clamped rigidly to it by means of a screw.

The microscopes are provided with levels and leveling screws, so that their axes may be made vertical. They are mounted on slide rests which give a motion of 2^{cm} in the direction of the line measured or transverse to it. To secure additional displacement in the direction of the line a small rotary motion is provided for in the connection of the microscope with the post cap. The micrometer heads of the microscope are divided to read microns directly, one revolution of the screws corresponding to $0{\cdot}1^{mm}$. When used in the field the microscopes are shaded from the sun by large umbrellas.

With this method of mounting the microscopes it is advantageous if not essential to set the microscope posts and those supporting the portable track before beginning measurement.

(8) *End marks and method of reference thereto.*—The method of marking the end of a line is essentially that of the Repsolds and fully described in the Lake Survey Report referred to above. It consists in the use of a metallic bolt terminating in a spherical head, the bolt being embedded in a stone or other stable mass set in the ground. The center of the bolt head is the fiducial point. To refer to this point a cylinder called a cut-off cylinder is used. It terminates at one end with a conical hole which fits over the spherical head. The other end is provided with a transverse level and graduated scale. The scale is brought by a rack and pinion motion to focus under the microscope whose position relative to the fiducial point is sought. The scale and level, which are parallel to each other, are placed parallel to the line measured. With the cylinder thus disposed, readings of the micrometer on the

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* Fully described in General Comstock's report referred to in section (1).

scale and of the position of the level bubble are made. The cylinder is then turned 180° in azimuth and the scale and level readings are again observed. From these observations and the height of the scale above the bolt head, the horizontal distance (in the direction of the line) between the micrometer zero and the fiducial point may be accurately determined.

(9) *Adjustments of apparatus.*—The most important adjustment of the apparatus is the alignment of the bar in the Y-trough. This adjustment is made when the ice load is in the trough and after the latter has had time to assume a stable shape. This time does not exceed 15 minutes.

As already stated, the alignment of the bar requires that the upper surfaces of the alignment plugs be in one plane, and that the lines on these plugs be in the same straight line. The former requisite is secured by a striding level whose feet are 99^{cm} apart, so that they reach from any plug to the second adjacent plug. Beginning at one end of the bar the plugs are numbered 1, 2, 3,—11. By placing the level feet in succession on plugs 1 and 3, 3 and 5, etc., plugs 1, 3, 5,—11 are brought into the same plane by means of the corresponding vertical adjusting screws, the screws under plugs 2, 4,—10 being loosened if need be to secure this end. Having thus adjusted plugs 1, 3, 5,—11, the level is placed on plugs 2 and 4, 4 and 6, etc., and the vertical screws are brought up to contact with the bar, but not raised enough to disturb the previous adjustment of 1, 3, 5—11, which are the principal defining plugs in this adjustment.

To place the lines on the plugs in the same straight line a sharp-pointed plumb bob suspended from a fine brass wire stretched over the trough was originally used. This device, with the aid of the lateral adjusting screws of the saddles, permits placing the lines in proper position within $0{\cdot}1^{mm}$ when the trough is fully loaded with ice. Experience with the apparatus, however, showed that the simpler method of stretching the wire, or, better still, a fine thread, close over the plugs when the trough is about four-fifths loaded secures equally good results.

It was feared before using the apparatus that the daily temperature range might produce an appreciable effect on the length of the bar through change in curvature of the trough. Hence the accurate method of measuring such change by the striding level was provided. But experience shows that the change in shape of the trough gives rise to quite insignificant changes in length of the bar. Indeed, the alignment of the bar may be maintained so perfectly that the correction for its curvature will not exceed a few tenths of a micron.

The grade sector of the apparatus is adjusted to zero when the graduated surfaces of the bar are in the same horizontal plane. To secure the latter condition an engineer's level is used; and with appropriate care the difference in height of the ends of the bar can be made zero with a probable error not exceeding $\pm 0{\cdot}1^{mm}$.

Description of Iced Bar Base Apparatus.

The microscopes are provided with fixed levels, which, when once adjusted, enable the operator to make the axes of the microscopes vertical. They are also provided with clamp screws, so that they may be rigidly held in proper position.

When posts are used to support the microscopes, as has been the case with this apparatus thus far, they must be set in their proper positions within a centimetre or two. It is easy and convenient, however, to adjust their sides facing the line to be measured with much greater precision. When firmly set, a line parallel to the base may be deliberately ranged out with a theodolite of high magnifying power, and this line may be defined by suitable marks on each post. Then by simply noting the distance of the axis of the bar during measurement from this reference line an accurate correction for deviation of the bar from parallelism with the base may be obtained. This adjustment of the posts, though not essential to the use of the apparatus, has been followed.

Another convenient adjustment which the use of posts permits is that of making the tops of several or many posts conform to one grade. By this means, since the four microscopes used are closely alike, the grade angles for several or many bar lengths are nearly the same, a condition favorable to precision in determining grade corrections. As an additional precaution in the use of this apparatus, the relative heights of the alternate post tops have been determined with an engineer's level.

(10) *Method of measurement.*—To conduct the measurement of a line with this apparatus eight men are required, to wit: Three observers; one recorder; one man to move the microscopes; and three men to move the car tracks, the microscope shades, and the ice and ice-crusher.

The operation of measurement proceeds as follows: The position of the microscope relatively to the fiducial point at the end of the line having been observed as explained above, the rear end of the bar is brought to focus under that microscope by the rear end observer. By means of a lever which grips into the track and hinges on the car, the latter observer holds the bar near to bisection under the microscope, while the front end observer brings his microscope into position over the front end of the bar; to do which he can make use of the lateral motion of the trough, of the microscope, or both. When the bar is adjusted at both ends the rear observer brings the rear end graduation accurately to bisection between the micrometer wires by use of his lever without turning the micrometer screw. Simultaneously he gives the signal "read" to the front observer, who brings his micrometer wires to bisect the front end graduation mark by moving the microscope, the micrometer wires, or both. The observers then read their micrometers and the recorder notes them down in his book, after which the rear observer turns his micrometer screw a half revolution or less backwards. The observers then exchange positions. The rear observer carrying with him his lever applies it to the front car and brings the front end graduation to bisection without disturbing the micrometer threads from their previous position; while at the signal "read" the front observer bisects the rear end graduation by moving the threads with the micrometer screw. They then announce the readings as before and the recorder jots them down, notifying the observers at the time if the screw revolutions differ from their previous values. This process eliminates the personal equation of the observers and checks any blunders of whole revolutions in reading the microscopes, each of them being read four times, and the four readings being the same within a few microns. The probable error of a bisection is less than $\pm 1^{\mu}$.

While the bar is in position under the microscopes, the third observer measures the distance of the front end (and the rear end at starting) of the axis of the bar from the reference line, and adjusts the sector-level bubble to center, taking care at the same time to keep away from the microscope posts when the bar is observed. The grade sector reading is then made and recorded, and the bar is rolled rapidly forward to a new position.

As soon as the rear end of the bar is brought safely to position under a microscope the one previously at the rear end is taken up and carried forward by the microscope porter, who clamps and adjusts it on a new post. Likewise as soon as a section of track is passed over it is carried forward to a new position.

The observers stand on platforms which rest at their ends on the ground at a distance of about 1 metre on either side of a microscope post.

At intervals of twenty to forty minutes fresh ice is supplied, the trough being run to the rear of or ahead of the two microscopes which were last used. The trough is completely uncovered in this operation and the ice stirred up and supplemented by the amount requisite to replace the waste. This amount is usually 3 to 5 kilogrammes.

The speed of measurement has varied somewhat with circumstances. It has usually been about 100^{m} per hour; 750^{m} were measured in seven hours on two different dates; and a kilometre would not be an excessive day's work.

(11) *Specimen of record and computations.*—Tables I and II following give a copy of the records of a measure with the iced bar of the interval of the 100^{m} comparator of Holton Base described in section 2, Chapter III. In Table I the first column gives the number of the bar length applied in the measure, and the second the time of application. The third column gives the number of the microscope used at the left-hand end of the bar, the left and right-hand ends of the bar being to the left or right, respectively, of the observer as he stands facing the bar while reading the microscopes. The fourth column gives the microscope readings on the left-hand end of the bar, the readings being expressed in screw revolutions of $0{\cdot}1^{mm}$ each, so that $0{\cdot}01^{r} = 1^{\mu}$. The fifth and

sixth columns give the corresponding data for the right-hand end of the bar. The letters S and W preceding the first two readings of microscopes 1 and 2 show the order in which the observations on the bar were made. Thus observer S read with microscope No. 2 on the right-hand end of the bar, while observer W read with microscope No. 1 on the left-hand end of the bar. The observers then exchanged positions, S reading with No. 1 and W with No. 2; and this order of reading was followed throughout the work. The seventh column gives the differences in microns between the mean readings of any microscope at the two ends of the bar in the order left minus right or $(l-r)$. Thus the first value in this column comes from the readings of microscope No. 2, namely, $\frac{1}{2}(21{\cdot}33^{r}+21{\cdot}26^{r})-\frac{1}{2}(21{\cdot}33^{r}+21{\cdot}33^{r})=-3^{\mu}$, neglecting tenths of microns. The last value in this column is derived in the same manner from the readings of microscope No. 1 at the beginning and end of the measure. The last column gives the height of the left-hand end of the bar at starting, and the heights of the right-hand end in its several positions, above the horizontal plane of reference of the comparator. This method of determining grade corrections was used in place of reading the grade sector. The measurement in this case proceeded from left to right.

Table II gives the records of the cut-off measures at starting and stopping. The first column gives the time of day. The second column designates the orientation of the cut-off scale, the terms right and left being used in the same sense as explained above. The third column gives the number of the graduation line of the scale observed on with the microscope. When the microscope is to the right of the zero of the cut-off scale the images of the graduation numbers appear inverted as seen through the microscope. Hence the letters I (inverted) and E (erect) serve to show the position of the microscope with respect to the axis of the cut-off cylinder. The fourth column gives the microscope readings on the cut-off scale; the fifth gives the height of the scale above the center of the cut-off sphere; and the last two columns give the readings of the level, l and r denoting the left and right hand positions respectively of the bubble.

TABLE I.—*Specimen of record. Measure of 100m Comparator interval, August 7, 1891.*

No. bar lengths.	Time, p. m.	Left-hand end of bar.		Right-hand end of bar.		L.—R.	Height of bar.
		No. microscope.	Microscope reading.	No. microscope.	Microscope reading.		
			Rev.		*Rev.*		*mm.*
1	3·54	1	W. 20·83	2	S. 21·33		491
			S. 20·83		W. 21·33	— 3μ	504
2	·56	2	21·33	3	20·73		
			21·26		20·73	— 3	500
3	4·01	3	20·74	4	21·26		
			20·66		21·26	— 2	509
4	·03	4	20·26	1	21·22		
			20·21		21·22	— 4	502
5	·05	1	21·22	2	22·03		
			21·13		22·03	— 0	506
6	·08	2	22·03	3	21·02		
			22·03		21·02	— 4	505
7	·11	3	21·02	4	21·57		
			20·95		21·57	— 5	507
8	·12	4	21·57	1	20·60		
			21·47		20·60	— 4	505
9	·15	1	20·60	2	23·10		
			20·52		23·10	— 5	509
10	·17	2	23·10	3	21·30		
			23·00		21·30	— 2	514
11	·20	3	21·30	4	21·66		
			21·26		21·66	— 8	520
12	·25	4	21·66	1	20·81		
			21·50		20·81	— 5	516
13	·28	1	20·81	2	21·91		
			20·72		21·91	+ 1	522
14	·31	2	21·91	3	20·97		
			21·94		20·97	— 7	524
15	·34	3	20·97	4	21·74		
			20·84		21·74	+ 2	523
16	·36	4	21·74	1	21·49		
			21·77		21·50	+ 1	515
17	·40	1	21·50	2	22·01		
			21·52		22·01	— 5	523
18	·44	2	22·01	3	20·70		
			21·91		20·70	— 2	520
19	·48	3	20·70	4	21·21		
			20·66		21·21	+ 3	530
20	·49	4	21·21	1	21·80		
			21·28		21·80	—97	524

TABLE II.—*Cut-off measures. August 7, 1891.*

AT WEST (LEFT-HAND) END OF COMPARATOR.

[Repsold cut-off cylinder.]

Time, p. m.	End of cut-off scale, right.	Scale division observed.	Microscope reading.	Height of scale.	Level readings.	
					l.	*r.*
			Rev.	*mm.*	*d.*	*d.*
3·46	A	29 I	18·55	840	5·5	9·0
	B	29 I	20·83		4·5	10·0

Description of Iced Bar Base Apparatus.

AT EAST (RIGHT-HAND) END OF COMPARATOR.

[U. S. Coast and Geodetic Survey cut-off cylinder.]

4·46	A	10 E	25·70	838	3·0	3·0
	B	11 E	18·46		5·0	1·0

Value of 1 division of level on Repsold cylinder =1″·6
Value of 1 division of level on U. S. Coast and Geodetic Survey cylinder =6″·0

The computation of the measured distance proceeds by means of the following formulas:

Calling R and L the mean readings of any microscope on the right and left hand ends respectively of the bar, the correction due to these readings (supposing them to increase from right to left as with the micrometers used) is

$$\Sigma\,(L-R). \tag{1}$$

Similarly, if R_c and L_c are the mean microscope readings on the cut-off scales at the right and left hand ends, respectively, of the line, they give a correction equal to

$$-(L_c-R_c). \tag{2}$$

Denote the reading on the cut-off scale at the right-hand end of the line by S_r, and that at the left-hand end by S_l. Then they give a correction of

$$S_l-S_r, \tag{3}$$

where S_l and S_r are plus or minus according as the image of the graduation-mark number appears inverted or erect.

Let I_r and I_l denote the inclinations of the cut-off cylinders at the two ends of the line, and H_r and H_l the corresponding heights of the cut-off scales. Then, since the inclinations are always small, they give a correction of

$$H_l\,I_l-H_r\,I_r; \tag{4}$$

where the inclinations are expressed in arc and both are supposed to be toward the right from the vertical. If l_1, l_2, r_1, r_2 are the left and right hand readings of the level bubble in any case, and v is the value of one division of the level in seconds of arc, the corresponding value of I is

$$I=\tfrac{1}{4}\,(l_1+l_2-r_1-r_2)\,\frac{v}{\rho''},$$

where $\rho''=206264''{\cdot}8$, or the number of seconds in the radius. Using the values given in the above table we have

$$\tfrac{1}{4}\,H_l\,\frac{v}{\rho''}=1{\cdot}63^{\mu},$$

$$\tfrac{1}{4}\,H_r\,\frac{v}{\rho''}=6{\cdot}07^{\mu}.$$

The correction for grade or slope of any bar length is expressed with sufficient accuracy by the formula $-\frac{(\Delta h)^2}{2s}$, where Δh is the difference in height of the two ends of the bar and s its length. For Δh in millimetres, this gives a correction of $-0{\cdot}1\ (\Delta h)^2$ microns per bar length, since $s=5^{\mathrm{m}}$ very nearly. The correction for grades is then in microns

$$-0{\cdot}1\ \Sigma(\Delta h \text{ in millimetres})^2. \qquad (5)$$

The correction for alignment was computed by the same formula (5) in the case of the field measures. For the measures on the comparator here considered the correction for alignment was made nil by always placing the bar in the vertical plane of the comparator. When grades are observed by means of a sector, as in field work, it is most convenient to make use of a table giving the corrections in terms of the grade angles.

Finally, denoting the number of bar lengths measured by N, the length of the bar by B_{17}, and the distance between the terminal spheres by D, we have

$$D=NB_{17}+(1)+(2)+(3)+(4)+(5).$$

Using the data given in the above tables we have the following values of the several corrections:

$$\begin{aligned} & \qquad\quad \mathit{mm.} \\ (1) &= -\ \ 0{\cdot}149 \\ (2) &= +\ \ 0{\cdot}239 \\ (3) &= +39{\cdot}500 \\ (4) &= -\ \ 0{\cdot}039 \\ (5) &= -\ \ 0{\cdot}075 \\ \text{Sum} &= +39{\cdot}476 \end{aligned}$$

Hence, the length of the comparator interval resulting from this measure is

$$20\ B_{17}\ \overset{\mathit{mm.}}{+39{\cdot}476}$$

CHAPTER II.

LENGTH OF ICED BAR B_{17}.

DIRECT DETERMINATION ON OFFICE COMPARATOR.

(1) *Description of apparatus and method.*—The most important operation attending the use of the iced-bar apparatus is the determination of the length of the measuring bar in terms of one of the International Prototype Metres. The general plan which has been followed in the execution of this operation is that of direct comparison of the bar and Prototype Metre, each being packed in melting ice. According to this plan the use of thermometers is dispensed with entirely. It suffices simply to measure a distance of five times the length of the metre when packed in melting ice, and then transfer this distance to the 5^m bar packed in melting ice.

The means available for such work at the time the apparatus was constructed were the office comparator and comparing room. The latter is an underground room situated between the front of the Survey building and the adjacent street. The comparator consists of an iron I-beam about $6{\cdot}5^m$ long, supported at its ends on brick piers. Between these piers micrometer microscopes can be attached to the beam at any desired interval up to 6 metres. For supporting the bars under comparison, a car moving at right angles to the I-beam is provided. The car rails rest on piers well removed from those supporting the beam.

The comparing room is lighted by electric lights which can be placed in any convenient position for illuminating or reading the microscopes.

For the special comparisons in question six micrometer microscopes were attached to the I-beam at intervals of 1 metre. The means of attachment were cast-iron brackets which could be rigidly clamped to the beam, while the microscopes themselves were fastened to the shelves of the brackets. The brackets are adjustable on the beam; so that the focal planes of the microscopes can be brought into the same horizontal plane, very nearly, and the axes of the microscopes made vertical within narrow limits. The microscopes in turn are provided with lateral adjustments on the shelves of the brackets. Hence the optical axes of the microscopes can be brought into one vertical plane, very nearly, and the distances between consecutive axes can be made closely equal to 1 metre.

The microscopes used in all determinations of the length of the iced bar No. 17 are Nos. 5 and 6 of the Survey and Nos. 1, 2, 3, and 4 of the Repsold base apparatus loaned to the Survey by the Corps of Engineers, U. S. Army. The magnifying power of the two former is 50 diameters; of the four latter 27 diameters. The Survey microscopes Nos. 5 and 6 were mounted at the ends of the comparator, while the Repsold microscopes were placed in the intermediate positions. These latter microscopes have each about 2^{cm} lateral motion by means of a fine screw, so that their axes could be easily put in line with the axes of the end microscope at any time. This adjustment was accomplished by bringing a fine thread well stretched to focus under the terminal microscopes and then moving the intermediate microscope till their axes were coincident with the thread. The error of this adjustment has never exceeded $0{\cdot}2^{mm}$. The field illumination of the microscopes was secured by glass prisms attached in front of the microscope objectives. Light was thrown into the prism and thence to the object observed by means of lenses in front of the illuminating lamps.

In making comparisons the 5^{m} bar B_{17} was mounted in its Y-trough, iced, and aligned in the manner already described. The trough was supported on its lifting jacks, which rested on the comparator car. The metre used was Prototype No. 21. It was mounted in a wooden box whose interior depth, breadth, and length are $11{\cdot}0^{cm}$, $12{\cdot}5^{cm}$, and $115{\cdot}0^{cm}$, respectively. It was supported by light brass suspension hooks attached to wooden crosspieces which rested on the sides of the box. The hooks were placed at distances of 20^{cm} from the ends of the metre. In packing the metre for comparisons it was completely surrounded to a depth of 5^{cm} to 8^{cm} by a mixture of ice pulverized in the ice crusher and by the snow-like particles cut with a jack plane. These finer particles could be crowded into the grooves of the bar (which has an X cross section) so that the ice would be in actual contact with the surface of the bar throughout its length. After thus covering the metre completely small holes were made through the snow packing at the ends of the box down to the graduated surfaces, which were the only parts of the metre exposed to the external air. To remove condensed moisture, which is sometimes troublesome, from these polished surfaces, a dry camel's hair brush was found very effective. The upper surface of the ice pack and the sides of the box were protected by a heavy felt wrapping, through which small holes were cut over the graduation marks. The box was provided with 3-foot screws which stood on plates of glass resting on the platform of the comparator car. These foot screws served to bring the ends of the metre to focus under the microscopes, the requisite lateral and longitudinal adjustments being secured by sliding the box on its foot screws on the smooth glass.

To pack the 5^{m} bar 40 to 45 kilogrammes of ice were used, and 10 to 12 kilogrammes to pack the metre. The ice was renewed at intervals of twenty to forty minutes, but the melting was generally very slight during such intervals.

Assuming constancy of temperature of the bars under comparison, the precision of the method just outlined evidently depends on the stability of the microscopes used. The first series of comparisons, made in July, 1891, indicated marked instability of the microscopes; and several series made during February to May, 1892, rendered it certain that the office comparator described above was quite inadequate to the work in question. It is to be remarked, however, that this comparator was not designed to meet the requirements of such work, but rather for the intercomparison of bars of the same length. Its use for the earlier determinations of the length of the iced bar was suggested by considerations of economy, and it was hoped that with careful manipulation it would answer the purpose. The most important result, however, of this earlier work was a demonstration of the necessity of a different sort of comparator.

The defects of the office comparator for our purpose may be ascribed chiefly to the mode of mounting the microscopes. Being attached as explained above by means of brackets clamped to the I-beam, their focal planes fall at a distance of about $0{\cdot}5^{m}$ below the beam. Any change of curvature or temperature of that beam will therefore cause relative displacements of the microscopes. Resting as the beam does with the friction due to its weight on the piers, it is in general in a state of longitudinal stress, which is frequently relieved by vibrations communicated to the piers by vehicles passing in the adjacent street. The temperature of the comparing room changes very slowly from day to day when not occupied long by the observer, or when lighted for short intervals. But the comparisons in question required occupying the room for some hours per day, and the heat from the observers and electric lights caused notable changes in the temperature of the beam, although it was wrapped with cotton batting and encased by a wooden box. These two sources of disturbance caused large irregular displacements of the microscopes and hence large ranges in the resulting values for the length of B_{17}.

(2.) *Programme of observations.*—The observations were made in such a manner as to eliminate the effects of regular displacements of the microscopes. Thus, calling the 5^{m} bar B_{17}, the prototype metre No. 21 M_{21}, and D the distance between the terminal microscopes of the comparator, the following scheme was adopted in the earliest determinations of July, 1891:

(1) A measure of D with M_{21}.
(*a*) (2) A measure of D with B_{17}.
(3) A measure of D with M_{21} in reverse direction.

In the subsequent determinations of February and March, 1892, the above scheme was supplemented by additional measures with B_{17}. Thus the programme was:

(1) A measure of D with B_{17}.
(2) A measure of D with M_{21}.
(3) A measure of D with B_{17}.
(*b*) (4) A measure of D with B_{17}.
(5) A measure of D with M_{21} in reverse direction.
(6) A measure of D with B_{17}.

In some instances, also, in these later determinations but one measure of D was made in the place of (3) and (4).

In all determinations of length the observations of the two ends of B_{17} and M_{21} were made simultaneously by two different observers. The observers also in every case exchanged positions immediately and read on the opposite ends of the bars, in order to eliminate personal equation, in so far as elimination can be accomplished by such interchange. In measuring the distance between the terminal microscopes of the comparator with M_{21} care was taken to make the dependence on the low-power intermediate microscopes small by confining the measures with them to a few microns in each case. The small inequalities in heights of the focal planes of the microscopes were measured by observing the inclination of a suitably delicate level tube attached to M_{21} when the latter was brought to focus under the microscopes. Repeated observations by this method gave accurate corrections for such inequalities.

(3.) *Computation of length of* B_{17}.—The method of deriving the length of B_{17} from the observations will depend in general on the hypothesis we adopt with respect to the motions of the microscopes. The number of hypotheses available is restricted, also, by the number of observations. Ignoring for the moment the displacement of the intermediate microscopes, and considering the second programme of the preceding section, we may suppose, for example, first, that the distance D between the zeros of the terminal microscopes remains invariable from the time of (1) to the time of (6); second, that D varies uniformly with the time from (1) to (6); third, that D varies uniformly at a rate z_1, say, from (1) to (3), and uniformly at a rate z_2 from (4) to (6); fourth, that D varies at a rate $z + 2wt$ from (1) to (6), z and w being constants and t the time beginning with (1). All these hypotheses were tried in computing the results of the observations of February and March, 1892, made according to the more elaborate programme given above. The residuals brought out by these processes were, however, far greater than could be attributed to errors of the observational class, and they showed clearly that the suppositions were rarely in accord with the facts. The reason for this lack of accord is that the processes do not take adequate account of the irregular motions of the microscopes, and especially of the intermediate microscopes.

It does not seem worth while, therefore, to set down all the formulas which result from the above suppositions and which were tried as stated. It may suffice to give those dependent on the fourth supposition. We observe, first, that the measures (1) and (6) and (2) and (5) were made at equal intervals, respectively, from the observations (3) and (4). The latter two were made so near together that they may be considered simultaneous. Secondly, the intervals (1) to (3) and (4) to (6) were about twice the intervals (2) to (3) and (4) to (5). Now let q_n be the micrometric quantity measured in any of the cases (1) to (6), and let v_n be the corresponding correction to q_n. Then, assuming the mean of the times of (3) and (4) as the epoch for D, we have from the several measures—

$$\begin{array}{lrlllll}
(1) & D = & B_{17} & -\ 2z & +\ 4w & +\ q_1 & +\ v_1, \\
(2) & = & 5M_{21} & -\ \ z & +\ \ w & +\ q_2 & +\ v_2, \\
(3) & = & B_{17} & & & +\ q_3 & +\ v_3, \\
(4) & = & B_{17} & & & +\ q_4 & +\ v_4, \\
(5) & = & 5M_{21} & +\ \ z & +\ \ w & +\ q_5 & +\ v_5, \\
(6) & = & B_{17} & +\ 2z & +\ 4w & +\ q_6 & +\ v_6.
\end{array}$$

Putting for brevity

$$x = 5M_{21} - B_{17},$$
$$y = D - M5_{21},$$

the above relations give the following observation equations:

$$\begin{array}{l}
x + y + 2z - 4w - q_1 = v_1, \\
\quad\ \ y + z - w - q_2 = v_2, \\
x + y \qquad\qquad\ - q_3 = v_3, \\
x + y \qquad\qquad\ - q_4 = v_4, \\
\quad\ \ y - z - w - q_5 = v_5, \\
x + y - 2z - 4w - q_6 = v_6.
\end{array} \qquad (1)$$

Calling the weights of the measures of D with B_{17} each unity, and the weights of the measures with M_{21} p, each, the normal equations from (1) give the following values:

$$\begin{aligned}
x &= \tfrac{1}{8}(q_1 + 3q_3 + 3q_4 + q_6) - \tfrac{1}{2}(q_2 + q_5), \\
y &= \tfrac{1}{2}(q_2 + q_5) - \tfrac{1}{8}(q_1 - q_3 - q_4 + q_6), \\
z &= \frac{2q_1 + pq_2 - pq_5 - 2q_6,}{8 + 2p,} \qquad (2) \\
w &= \tfrac{1}{8}(- q_1 + q_3 + q_4 - q_6).
\end{aligned}$$

It will be observed that the objective quantity x is independent of the weight p. This weight p, in fact, affects only the rates z, z_1, z_2 of the second, third, and fourth assumptions stated above. It may be remarked also that the value of x is the same for the first three assumptions, and differs from the value given by the fourth by the quantity w.

In the case of three measures only of D, as in the determinations of July, 1891, we are virtually restricted to three unknown quantities, and hence to the assumption that D varies at a constant rate. If in this case q_1, q_2, q_3 denote the micrometric quantities of the three values of D we have—

$$\begin{aligned}
x &= q_2 - \tfrac{1}{2}(q_1 + q_3), \\
y &= \tfrac{1}{2}(q_1 + q_3), \qquad (3) \\
z &= \tfrac{1}{2}(q_1 - q_3).
\end{aligned}$$

Since the errors of observation in all determinations of B_{17} on the office comparator are completely masked by the large displacements of the microscopes, and since these determinations are virtually superseded by much better ones made as explained in section 13, it does not seem worth while to give anything but the resulting values of x from these earlier determinations.

(*4*) *Length of B_{17} from observations of July, 1891.*—It being deemed desirable to know the approximate length of B_{17} before using it in the measures of Holton Base, a few determinations were made just prior to shipping the apparatus to Holton, Ind., in July, 1891. The observations were made by Assistants O. H. Tittmann and R. S. Woodward, and the record was kept by Mr. John F. Hayford. The programme (*a*) of section 2 was followed. The computations were made by Mr. Hayford and checked by Mr. Tittmann. Table I following gives the resulting values of $x = 5M_{21} - B_{17}$. The first column gives the date, the second the orientation of M_{21}, the third the values of x for the different orientations of B_{17}. The value of M_{21} at 0° C. is given, on p. 83 of "Rapport sur la construction, les comparaisons et les autres opérations ayant servi à déterminer les équations des nouveaux Prototypes métriques," as

$$M_{21} = 1^{\mathrm{m}} + 2{\cdot}5^{\mu} \text{ at } 0^{\circ} \text{ C}\cdot$$

TABLE I.—*Values of $x = 5M_{21} - B_{17}$ at 0° C.**

Date.	End of M_{21} north.	Values of x.	
		A end of B_{17} north.	B end of B_{17} north.
1891.			
July 6	A	$+24{\cdot}0^{\mu}$	
6		$17{\cdot}8$	
7		$26{\cdot}5$	
7		$17{\cdot}2$	
8	A	$22{\cdot}6$	
9	B		$+15{\cdot}8^{\mu}$
10			$15{\cdot}0$
10			$36{\cdot}4$
11			$22{\cdot}6$
11	B		$27{\cdot}0$

Since it appears highly probable, as shown below, that the observed length of B_{17} differs with differing aspects of its terminal graduations, we shall derive mean values of x in every case for each of the two orientations of B_{17}. Thus the mean values from the above table are

$$+\ 21{\cdot}6^{\mu} \pm 2{\cdot}0^{\mu} \text{ for } A \text{ end of } B_{17} \text{ north,}$$
$$+\ 23{\cdot}4^{\mu} \pm 2{\cdot}0^{\mu} \text{ for } A \text{ end of } B_{17} \text{ south.}$$

The probable error assigned to each result here is derived in the usual manner from the ten observations of the two derived quantities. The probable error of a single value of x in the system is $\pm\ 4{\cdot}5^{\mu}$, which is a little less than the millionth part of the length of B_{17}.

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*The average temperature shown by thermometers alongside of the I-beam during these comparisons was 21° C.

Length of Iced Bar B_{17}.

To get the corresponding lengths of B_{17}, the above mean values must be increased by $1{\cdot}0^{\mu}$, the correction for inequalities in heights of the focal planes of the microscopes in this case, and the above value of M_{21} applied. Thus we have the following results for length of B_{17} in melting ice.

$$B_{17} = 5^{\mathrm{m}} - 10{\cdot}1^{\mu} \pm 2{\cdot}0^{\mu} \text{ for } A \text{ end of } B_{17} \text{ north,}$$
$$B_{17} = 5^{\mathrm{m}} - 11{\cdot}9^{\mu} \pm 2{\cdot}0^{\mu} \text{ for } A \text{ end of } B_{17} \text{ south.}$$

(5) *Length of B_{17} from observations of February and March, 1892.*—The observations in the more extended series of determinations of February and March, 1892, were made by John S. Siebert and R. S. Woodward. Mr. Siebert also made the record, and in order to facilitate his work all of the micrometer head readings (after the bisections had been made) were observed and called out by Woodward. Programme (*b*) of section 2 above was followed except in a few instances when but one measure was made with B_{17}, instead of (3) and (4) of the programme. Table II following gives the results of these determinations, the arrangement being the same as in Table I. The computations of the quantity x were made by Siebert, in accordance with the simpler assumptions explained in section 3, while the corresponding values resulting from the fourth of those assumptions were derived by Woodward. The two sets of values were checked by means of their difference w. This quantity never exceeded $2{\cdot}8^{\mu}$ and was generally less than $0{\cdot}5^{\mu}$. Its use does not give residuals materially less than the simpler hypothesis.

TABLE II.—*Values of $x = 5M_{21} - B_{17}$ at 0° C.**

Date.	End of M_{21} north.	Value of x. Aend of B_{17} north.	Date.	End of M_{21} north.	Value of x. A end of B_{17} south.
1892.			1892.		
Feb. 4	A	$+30{\cdot}5^{\mu}$	Feb. 19	A	$+26{\cdot}4\mu$
5	B	24·4	20	B	13·9
5	B	23·3	23	A	27·0
6	A	33·9	24	B	24·9
8	B	37·6	25	A	23·3
8	B	27·2	25	A	29·8
10	A	33·4	26	B	21·9
10	A	25·4	26	B	35·0
11	B	22·1	26	B	29·1
13	A	26·5	27	A	31·3
13	A	24·4	27	A	25·4
15	B	30·4	Mar. 7	A	33·6
15	B	31·4	7	A	29·1
16	B	22·4	8	B	28·6
16	B	18·6	8	B	28·1
16	B	23·5	9	A	36·7
17	A	13·2	10	B	35·4
17	A	25·2	11	A	34·8
18	A	31·4	12	B	24·0
18	A	30·0			

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* During the time these determinations were made, the air temperature in the comparing room varied from about 5° C. up to about 10° C., with an average of about 6° C.

Computing the mean values from Table II in the same manner as the means from Table I were computed, we have,

$$x = +\ 26{\cdot}7^{\mu} \pm 0{\cdot}8^{\mu} \text{ for } A \text{ end of } B_{17} \text{ north,}$$
$$x = +\ 28{\cdot}3^{\mu} \pm 0{\cdot}8^{\mu} \text{ for } A \text{ end of } B_{17} \text{ south.}$$

The probable error of a single determination of x from this series is $\pm\ 3{\cdot}6^{\mu}$, which indicates an appreciable advantage of the programme (*b*) over (*a*).

If we group the results in Table II with respect to the orientation of the metre, we find the following mean values of x:

$$\left.\begin{array}{l} +\ 27{\cdot}4^{\mu} \pm 1{\cdot}2^{\mu} \text{ for } A \text{ end of } M_{21} \text{ north,} \\ +\ 26{\cdot}1^{\mu} \pm 1{\cdot}2^{\mu} \text{ for } A \text{ end of } M_{21} \text{ south.} \end{array}\right\} A \text{ end of } B_{17} \text{ north,}$$
$$\left.\begin{array}{l} +\ 29{\cdot}7^{\mu} \pm 1{\cdot}0^{\mu} \text{ for } A \text{ end of } M_{2} \text{ north,} \\ +\ 26{\cdot}8^{\mu} \pm 1{\cdot}4^{\mu} \text{ for } A \text{ end of } M_{21} \text{ south.} \end{array}\right\} A \text{ end of } B_{17} \text{ south.}$$

The values of x in Table II require a correction of $+0{\cdot}4^{\mu}$ for inequalities in heights of the focal planes of the microscopes. Applying this correction and the value of $5M_{21}$ at 0° C. there result for the length of B_{17} in melting ice—

$$B_{17} = 5^{\text{m}} - 14{\cdot}6^{\mu} \pm 0{\cdot}8^{\mu} \text{ for } A \text{ end of } B_{17} \text{ north,}$$
$$B_{17} = 5^{\text{m}} - 16{\cdot}2^{\mu} \pm 0{\cdot}8^{\mu} \text{ for } A \text{ end of } B_{17} \text{ south.}$$

(6) *Sources of error.*—It will be observed that the individual values of x in Tables I and II exhibit wide ranges; amounting to 21^{μ} in the former and to 24^{μ} in the latter, or, at the maximum to 1/200 000 part of the length of B_{17}. Considering errors of observation alone, it appears from the first of equations (2), section 3, that they should give to a single determination of x in Table II a probable error of $\pm\ \frac{1}{4}\,\varepsilon\,\sqrt{45}$, where ε is the probable error of a single microscope reading. The value of ε does not exceed $0{\cdot}5^{\mu}$, so that the probable error of a single value of x due to errors of observation is less than $\pm\ 0{\cdot}8^{\mu}$. Since the total probable error of a single value of x in Table II is, as given above, $\pm\ 3{\cdot}6^{\mu}$, the reason for directing attention to other sources of error than those of observation is evident.

This view of the inquiry was reached long before the series of determinations condensed in Table II was completed. During the progress of this work, in fact, many experiments were made with the hope of locating the larger sources of error and devising means for their elimination or correction. Thus, (*a*), delicate levels were attached to the brackets of the terminal microscopes of the comparator. They indicated that the microscopes were subject to both sudden and progressive displacements, which were frequently large in amount. (*b*). The question whether these displacements could be due to the varying positions of the observers and the microscope car was examined, with the result that no effects due to the weight of either were produced. (*c*). The hypothesis was entertained that B_{17} might be subject to longitudinal stress, and hence to changes of length, from friction on the supports in its Y-trough. This hypothesis was tested by subjecting the trough to tension and compression alternately in quick succession, and observing the length of B_{17} in terms of the distance between the terminal microscopes which were sufficiently stable for the brief interval required. Although the Y-trough was put under much greater stress than that to which it is ever subject in the uses it is designed to meet, no change was discovered in the length of B_{17}. Finally, it may be said, that a careful study of the variations in the intervals between consecutive microscopes, as measured with M_{21}, together with the results of the experiments just outlined, led to the conclusion that the irregular motions of the microscopes would suffice to account for the wide range amongst the derived results.

The important question then arose whether the results in Tables I and II were affected by detrimental constant errors. The application of the levels to the microscope brackets, as explained above, indicated that such might be the case. It became desirable, therefore, to determine the length of B_{17} in some other way. The method which suggested itself by reason chiefly of its economy, and the results attained in its application are explained in the following sections.

DETERMINATION OF LENGTH OF B_{17} BY AID OF B_{18} ON OFFICE COMPARATOR.

(7) *Description of method.*—The time required to measure the distance between the terminal microscopes of the comparator with the metre, according to the direct method, was 12 to 20 minutes. It appeared probable, therefore, that the errors due to motions of the microscopes could be much reduced if not eliminated by diminishing the interval during which dependence must hang on their stability. To accomplish this end, a steel bar of the same dimensions as B_{17} was prepared by the Instrument Division of the Survey and subdivided into metre spaces. It was cut down to its mid depth at intervals of one metre and platinum iridium plugs were inserted to receive the fiducial lines. The plugs were carefully polished and the lines were successfully ruled by Mr. E. G. Fischer. This bar is designated B_{18}. It is marked by a letter A at one end and by a letter B at the other.

The method followed in determining the length of B_{17} by aid of B_{18} consisted essentially in comparing each of the sub-spaces of B_{18} with the metre M_{21}, deriving thus the total length of B_{18}, and then comparing B_{17} with B_{18}.

B_{18} was mounted in the steel Y-trough where it could be accurately aligned and where its flexure could be measured. Each graduation plug having in addition to a single transverse line two longitudinal lines separated by an interval of $0{\cdot}2^{mm}$, B_{18} could be aligned in the vertical plane by means of the five microscopes and the lateral adjusting screws of the trough. For alignment in the horizontal plane, the long striding level was used. Its feet were placed directly over the graduation plugs, these being temporarily protected by slips of smooth paper. It was thus possible to measure the inequalities in height of the plug surfaces to the nearest tenth of a millimetre when the bar was surrounded by ice. The vertical adjusting screws directly below the plugs were first brought to the proper positions. After these were secured, the intermediate vertical screws were brought up to contact with the bar without disturbing its shape as shown by the striding level; all these adjustments being made after the trough had assumed its shape due to a full load of ice. It is to be remarked, also, that B_{18} remained in the Y-trough in the same position throughout the comparisons. Its alignment as shown by the microscopes did not change at its graduation marks by so much as $0{\cdot}2^{mm}$.

For comparisons with B_{18}, B_{17} was mounted in a wooden box provided with adjusting screws in all respects similar to those of the Y-trough.

The following programme was adopted in this determination of the length of B_{17}:

(1) Six measures of the sub-spaces and hence length of B_{18} in terms of M_{21}.

(2) Nine measures of B_{17} in terms of B_{18}.

(3) Six measures of B_{18} as in (1).

(4) Nine measures of B_{17} as in (2).

(5) Six measures of B_{18} as in (1).

In comparing the subspaces of B_{18} with the metre M_{21} the following programme was observed:

(1) Microscope readings on subspace of B_{18}.
(2) Microscope readings on M_{21}.
(3) Repetition of (1).
(4) Repetition of (2).
(5) Repetition of (1).

Similarly, for the comparison of B_{17} and B_{18} the plan followed was:

(1) Microscope readings on B_{17}.
(2) Microscope readings on B_{18}.
(3) Microscope readings on B_{17}.
B_{17} and B_{18} being interchanged alternately.

The observations at the two ends of the bars or sub-spaces were made simultaneously by two observers who always exchanged places and repeated the readings. The observers were John S. Siebert and R. S. Woodward, Mr. Siebert keeping the record as explained in section 5.

(8) *Computation of results.*—In deriving the lengths of the sub-spaces of B_{18} it is assumed that the distance D_1, say, between the zeros of any pair of microscopes used varied uniformly with the time. Hence, if we designate by q_1, q_2, . . . the micrometric quantities observed in accordance with the second programme of section 7; by v_1, v_2, . . . the corresponding corrections to these observed quantities; by z_1, the rate of

change of D_i; by s_i, the length of a sub-space; and count time from the epoch of the middle observation, we shall have—

$$\begin{aligned} D_i &= s_i - 2z_i + q_1 + v_1, \\ &= M_{21} - z_i + q_2 + v_2, \\ &= s_i + q_3 + v_3, \\ &= M_{21} + z_i + q_4 + v_4, \\ &= s_i + 2z_i + q_5 + v_5. \end{aligned}$$

If, for brevity, we put—

$$\begin{aligned} x_i &= M_{21} - s_i, \\ y_i &= D_i - M_{21}, \end{aligned}$$

the above relations give the following observation equations:

$$\begin{aligned} x_i + y_i + 2z_i - q_1 &= v_1, \\ y_i + z_i - q_2 &= v_2, \\ x_i + y_i - q_3 &= v_3, \\ y_i - z_i - q_4 &= v_4, \\ x_i + y_i - 2z_i - q_5 &= v_5. \end{aligned} \tag{1}$$

The normal equations corresponding to these give—

$$\begin{aligned} x_i &= \tfrac{1}{3}(q_1 + q_3 + q_5) - \tfrac{1}{2}(q_2 + q_4), \\ y_i &= + \tfrac{1}{2}(q_2 + q_4), \\ z_i &= \tfrac{1}{10}(2q_1 + q_2 - q_4 - 2q_5). \end{aligned} \tag{2}$$

It may be observed that the values of x_i and y_i in (2) are precisely the same as those which would result if D_i were assumed to remain invariable, or the same as if z_i were omitted from (1). The introduction of z_i serves simply to give the change in D_i and diminish the residuals v.

The computations were made in duplicate according to the above formulas by Messrs. Siebert and Woodward, the residuals as well as the objective quantity x_i being worked out in every case.

In deriving the length of B_{17} in terms of B_{18}, the computation was in all respects similar to that just outlined, and it was made by the same persons.

(9) *Summary of results.*—Table III following gives the resulting values of x_1, x_2, . . ., or x_i, for the several sub-spaces of B_{18}. x_1 refers to the first sub-space of B_{18} counting from the A end towards the B end; x_2 to the next space in the same order, etc. The first column of the table gives the date of the determination; the second the orientation of the metre M_{21}; the next five give the values of x_i; the eighth gives the sum of the x_i, or the excess of $5M_{21}$ over the whole length of B_{18}; and the last column gives the mean values of Σx_i for the three groups of determinations. It will be remembered that the length of any sub-space s_i is given by the equation

$$s_i = M_{21} - x_i.$$

TABLE III.—*Values of x_i and Σx_i for B_{18} at 0° C.*

Date.	End of M_{21} north.	x_1.	x_2.	x_3.	x_4.	x_5.	Σx_i.	Mean values of Σx_i.
1892.		μ	μ	μ	μ	μ	μ	
Apr. 25	A	+13·4	+0·2	+12·8	+3·2	+29·6	+59·2	
26	B	+14·0	+2·2	+10·3	+4·7	+29·3	+60·5	
26	B	+15·1	−1·3	+ 9·8	+3·1	+30·9	+57·6	
27	A	+ 9·8	+3·1	+ 6·2	+5·7	+30·4	+55·2	
27	A	+13·3	+2·2	+ 9·3	+4·6	+29·6	+59·0	μ
28	A	+10·7	+4·7	+ 8·9	+2·9	+30·3	+57·5	+58·2
May 2	B	+11·6	+1·2	+ 2·6	+0·0	+32·3	+47·7	
2	B	+15·2	+2·3	+ 8·1	+1·4	+30·8	+57·8	
3	A	+14·0	+2·5	+ 5·5	+3·2	+25·0	+50·2	
3	A	+11·6	+4·3	+ 5·2	+4·4	+29·8	+55·3	
4	B	+13·2	+1·3	+ 3·5	+5·0	+31·8	+54·8	
4	B	+14·9	+3·6	+ 5·6	+4·6	+29·3	+58·0	+54·0
7	A	+14·6	+1·2	+ 3·8	−1·0	+30·0	+48·6	
7	A	+12·2	−0·1	+ 4·7	+2·8	+30·3	+49·9	
9	B	+12·2	+1·5	+ 7·0	−0·2	+31·4	+51·9	
9	B	+12·4	−6·5	+10·9	+2·4	+33·8	+53·0	
10	A	+12·9	−6·6	+11·9	+1·1	+34·8	+54·1	
10	B	+12·9	−5·5	+ 9·5	+0·8	+27·9	+45·6	+50·5

Table IV gives the results of the comparisons of B_{17} and B_{18}. The first column gives the date of the comparison, the second the orientation of B_{17}, the third the individual value of $B_{17} - B_{18}$ from a set of observations, the fourth the mean value of the set, and the fifth the group mean.

Each result in the third column is derived from three readings on B_{17} and three on B_{18}, the order of readings being B_{17}, B_{18}, B_{17}, and B_{18}, B_{17}, B_{18}, with an interchange of the bars from set to set. The orientation of B_{18}, as explained above, remained unchanged throughout the work. During the intervals between sets of observations the bars were re-iced and the curvature of B_{17} observed with the long striding level.

TABLE IV.—*Values of $B_{17} - B_{18}$ at 0° C.*

Date.	End of B_{17} north.	B_{17}—B_{18}.	Mean of set.	Group mean.
1892.		μ		
Apr. 29	A	+29·4		
		28·0	μ	
		30·2	+29·2	
30	A	28·4		
		29·8		
		30·2	+29·5	
30	A	33·8		
		30·0		μ
		32·5	+32·1	+30·3
May 5	B	+26·0		
		28·6		
		26·5	+27·0	
5	B	32·6		
		33·5		
		31·3	32·5	
5	B	30·8		
		32·4		
		27·4	30·2	+29·9

Length of Iced Bar B_{17}.

(10) *Sources of error.*—The principal sources of error to which the results in Tables III and IV are subject are: (a) those of observation, (b) those due to instability of the microscopes, and (c) those due to flexure of B_{18}. The results involve the aggregate effect of these three sources, but it is practicable to estimate their separate effects. This we now proceed to do.

The quantities q in equations (1) and (2) of section 8 are of the form

$$q = \tfrac{1}{2}(R_n + R'_n - R_s - R'_s),$$

where R_n, R'_n, and R_s, R'_s denote the microscope readings at the north and south ends respectively of M_{21} or of a sub-space of B_{18}, as the case may be. Hence if ε is the probable error of a single microscope reading, the probable error of q due to this source alone is also ε; and it follows from the first of equations (2), section 8, that the probable error of a single determination of x_i is $\pm\ \varepsilon\sqrt{\frac{5}{6}}$. The value of ε, as shown by direct observation, is less than $0{\cdot}5^{\mu}$. Hence the probable error of a single value of x_i due to errors of observation alone is less than $0{\cdot}5^{\mu}$.

The assumption adopted in computing the values of the sub-spaces on B_{18} is that the interval between the zeros of the pair of microscopes used in determining any space changed uniformly with the time required to make the observations. If this is not a true assumption, or if the microscopes were subject to large irregular displacements, we should expect to find the probable errors of the quantities q and x_i derived from the residuals v of equations (1), section 8, larger than those given under (a) above. Such is, in fact, the case.

It is essential to observe here that errors due to flexure of B_{18}, or changes in length of the sub-spaces, if they were at all formidable, did not appear in the residuals v of the equations used. Such changes can only occur slowly, from day to day, or during longer periods, since they are primarily due to the change in shape of the Y-trough, which is affected only by marked changes of the surrounding air temperature. The time required to make a set of observations in a comparison of M_{21} with a sub-space was only 6 to 8 minutes. It appears practically certain, therefore, that the residuals brought out in the computation involve only the two sources (a) and (b).

In the 90 determinations of the 5 sub-spaces there were 450 residuals and 270 unknowns, as shown by the formulas (1) and (2) used in the computation. The sum of the squares of these residuals is 351·4, the unit being a micron. Hence if ε_1 denote the probable error of a single observed value of q,

$$\varepsilon_1 = \pm\,0{\cdot}6745\sqrt{\frac{351{\cdot}4}{450-270}} = \pm\,1{\cdot}3^{\mu};$$

whence

$$\text{probable error of one value of } x_i = \varepsilon_1\sqrt{\frac{5}{6}} \doteq \pm\,1{\cdot}2^{\mu},$$

$$\text{probable error of } \Sigma x_i \qquad = \varepsilon_1\sqrt{\frac{25}{6}} = \pm\,2{\cdot}7^{\mu}.$$

Comparing these errors with those due to observation alone, it appears that the probable error of a value of q due to instability of microscopes is

$$\sqrt{\varepsilon_1^2 - \varepsilon^2} = \pm 1{\cdot}2^{\mu},$$

and that the corresponding value for x_i is only slightly less.

An examination of the individual residuals of the different sets of observations indicates that the assumption of a constant rate of change of a microscope interval was frequently not realized. In the entire number of residuals there were twenty whose values rose to or exceeded 2^{μ}, while six exceeded 3^{μ}, the maximum being $3{\cdot}7^{\mu}$. These large residuals, it is believed, are due to sudden displacements of the microscopes. No other hypothesis appears competent to explain them, unless it be that of sudden changes in temperature of the bars under comparison.

The above probable error of Σx_i, or the probable error of one determination of the length of B_{18} is $\pm 2{\cdot}7^{\mu}$. The corresponding value for a single determination of B_{17} by the direct process, as given in section 6, is $\pm 3{\cdot}6^{\mu}$. Hence it would appear that the process of getting the entire length of B_{18} was subject to considerably less error than the direct determination of B_{17}.

The group means in the last column of Table III indicate a progressive change in the length of B_{18} during the series of comparisons, which extended from April 25 to May 10, 1892. The cause of this change, if real, is important, and much study was given to it.

It may be observed, before considering any other cause, that the two sources of error already examined are not altogether incompetent to account for the range which these group means show. That range is about 8^{μ}. The probable error of one group mean is, if we adopt the probable error of Σx_i given above as a basis, $\frac{5}{6}\,\varepsilon_1 = \pm 1{\cdot}0^{\mu}$. Thus a range of 8^{μ} would be accounted for by errors of opposite signs no greater than four times this probable error, but the probability of such an explanation is too small to be satisfactory. Hence we proceed to examine the question of the effect of flexure of B_{18} on its observed length.

Since the surfaces of the intermediate graduation plugs of B_{18} are not in the neutral surface of the bar, any change in its curvature will tend to displace the fiducial lines on those intermediate plugs. This source of difficulty was anticipated, and the measures of the flexure with the long striding level, as already explained, were made for the purpose of supplying any needed correction. It is obvious that no considerable variation in the whole measured length of B_{18} could occur except through large changes in its curvature—such, for example, as a change from marked concavity upwards to marked convexity upwards. No such change occurred, however, the bar remaining, as will presently appear, concave upwards throughout the work. Again, since there could be no displacement of the terminal lines of B_{18} from this cause, the sum of the sub-spaces, as measured at any time, should be sensibly free from the effects of displacements of the intermediate lines, unless the bar was subject to marked changes of curvature during the 30 or 40 minutes required to compare the sub-spaces with M_{21}. There is no indication that B_{18} underwent any changes of curvature other than those due to the slowly fluctuating temperatures of the air in the comparing room, which were sufficient, it would appear, to affect the web of the Y-trough and hence B_{18} by small amounts. Moreover, the observed variations inlength of the sub-spaces, if due to displacements of the intermediate graduation lines arising from slow changes in curvature of B_{18}, ought to appear as a linear function of the observed changes of curvature.

Such, in brief, were the considerations which led to the use of B_{18} for the purpose in question, notwithstanding the obvious objection to its form, if invariability in length of the sub-spaces is required. Let us now examine the facts brought out by the observations.

From an inspection of the results in Table III it is seen that the variations of the quantities x_i are greatest for the intermediate sub-spaces. Taking mean values of x_i for the three groups of observations, we have

Date.	x_1.	x_2.	x_3.	x_4.	x_5.
1892.	μ	μ	μ	μ	μ
Apr. 25–28	+12·7	+1·8	+9·6	+4·0	+30·0
May 2– 4	+13·4	+2·5	+5·1	+3·1	+29·8
7–10	+12·6	—2·7	+8·0	+1·0	+31·4

Thus it appears that while the observed values of the terminal sub-spaces are very accordant in these means, the corresponding values of the intermediate sub-spaces are very discordant. The most striking cases are those of x_2 and x_3, which varied by $5{\cdot}2^{\mu}$ and $4{\cdot}5^{\mu}$, respectively, in their mean values; and one might be led to the inference that the intermediate sub-spaces were subject to marked changes in length. In line with this inference it is instructive to compare the above mean values with the ranges among the corresponding individual values of x_i for the several groups. These ranges are for—

Date.	x_1.	x_2.	x_3.	x_4.	x_5.
1892.	μ	μ	μ	μ	μ
Apr. 25–28	5·3	6·0	6·6	2·8	1·6
May 2– 4	3·6	3·1	5·5	5·0	7·3
7–10	2·4	7·8	8·1	3·8	6·9

From these it would not seem safe to infer that the terminal sub-spaces were subject to less error in their determination than the other spaces, although the mean values of those terminal sub-spaces are by far the most accordant. Moreover, an examination of the above group means of x_i does not indicate such displacements of the intermediate graduation marks as would result from slow changes in curvature of B_{18}; for, since the length of any sub-space s_i is

$$s_i = M_{21} - x_i,$$

it would appear from those means that, while s_1 and s_5 remained invariable, s_2 and s_4 both increased, the one by $4{\cdot}5^{\mu}$ and the other by $3{\cdot}0^{\mu}$, and that the middle space s_3 increased by $4{\cdot}5^{\mu}$ from the first to the second group and decreased by $2{\cdot}9^{\mu}$ from the second to the third group. It seems very improbable that the intermediate graduations could have been so displaced as to leave s_1 and s_5 invariable and at the same time cause the specified changes in s_2, s_3, s_4.

The observed changes in curvature of B_{18} are given in Table V below. These were measured with the long striding level which reached from any plug to the adjacent plug on either side. The value of one division of the bubble used on this level was 15″. The level was always reversed in observing the relative heights of adjacent plugs; and as the time required to make such a measure was less than two minutes, the observed quantities are subject to no important errors. The quantities given in the table are the ordinates of the intermediate graduation plugs measured at right angles to the straight line joining the terminal plugs of B_{18}. They are designated by y_i, the suffix corresponding to the number of the plug counting from the A end toward the B end. The minus sign indicates that the plug is below the line of reference.*

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* If $h_1, h_2 \ldots h_5$ denote the differences in height of the plugs for the successive spaces, and s the slope per metre of the line joining the terminal plugs, the quantities y_i are given by the following equations:

$$\begin{aligned}
y_1 &= h_1 - s,\\
y_2 &= h_1 + h_2 - 2s,\\
y_3 &= h_1 + h_2 + h_3 - 3s,\\
y_4 &= h_1 + h_2 + h_3 + h_4 - 4s,\\
0 &= h_1 + h_2 + h_3 + h_4 + h_5 - 5s.
\end{aligned}$$

The quantities h_1, h_2, h_3, h_4, h_5 are observed directly with the striding level.

Length of Iced Bar B_{17}.

TABLE V.—*Observed ordinates of curvature of B_{18}.*

Date.	y_2.	y_3.	y_4.	y_5.	Σy_i.
1892.	*mm.*	*mm.*	*mm.*	*mm.*	*mm.*
Apr. 25	—0·25	—0·23	—0·14	—0·37	—0·99
26	·12	·09	·00	·18	·38
27	·05	·17	·03	·24	·50
27	·15	·24	·12	·33	·85
28	·12	·12	·00	·27	·51
29	·24	·19	—·04	·29	·76
30	·13	·04	+·12	·07	·12
May 2	·27	·28	—·02	·45	1·02
3	·18	·24	·06	·42	0·91
4	·30	·42	·27	·40	1·38
5	·45	·50	·35	·31	1·61
7	·25	·35	·26	·39	1·25
9	·22	·35	·25	·27	1·09
10	·46	·54	·37	·37	1·74
10	·48	·50	·37	·35	1·69

The results in this table show that the bar was concave upwards throughout the work. The mean values corresponding to the three groups of comparisons of the sub-spaces are

	y_2.	y_3.	y_4.	y_5.
	mm.	*mm.*	*mm.*	*mm.*
Apr. 25–30	—0·14	—0·17	—0·06	—0·28
May 2– 4	·25	·31	·12	·42
May 7–10	·35	·43	·31	·34

These, with the exception of y_5, show an increase by small amounts of the concavity with the time. This is precisely what we should expect from the slow seasonal rise of the air temperature in the comparing room. They do not appear to bear any relation, however, to the observed changes in the values of the sub-spaces.

What, then, could have been the cause of the large variations in the values of the group means of the intermediate sub-spaces? I am unable to answer this question except with the suggestion that the shifting of the iced bar and metre from one position to another under the beam to which the microscopes were attached produced at times systematic changes in the intervals between those microscopes. If such changes occurred, one would expect them to be greater for the intermediate microscopes than for the terminal ones, for neither the Y-trough nor the box in which the metre was iced projected beyond the terminal microscopes, and currents of air due to the nearness or remoteness of the ice loads would obviously be most effective in disturbing the intermediate microscopes.

Finally, recurring to the comparisons of B_{17} and B_{18}, the results of which are exhibited in Table IV, no indication is seen of a change in the entire length of B_{18}. In fact, the results in Table IV do not show any striking anomalies. The range amongst the means of the different sets of comparisons is but $5{\cdot}5^{\mu}$, and the group means agree quite as well as one could expect.

(11) *Adopted length of B_{17} derived by aid of B_{18}.*—In the absence of a satisfactory explanation of the variation of the observed values of B_{18} in terms of M_{21}, we can do no better than to take the mean of the three group means in the last column of Table III for the relation of these two bars. This relation is

$$B_{18} = 5\,M_{21} - 54{\cdot}2^{\mu} \pm 1{\cdot}6^{\mu},$$

the probable error being derived from the discrepancies between the three individual values and their mean. To this value there is to be added a correction of $-0{\cdot}1^{\mu}$, which is the average amount by which the distance between the terminal marks of B_{18} fell short of the sum of the sub-spaces. It is found by projecting the sub-spaces on the line joining the terminal marks, the data for such projection being given in Table V. Applying this correction and introducing the value of M_{21} at 0° C., namely,

$$M_{21} = 1^{\mathrm{m}} + 2{\cdot}5^{\mu},$$

there results

$$B_{18} = 5^{\mathrm{m}} - 41{\cdot}8^{\mu} \pm 1{\cdot}6^{\mu} \text{ at } 0^{\circ}\text{ C.}$$

In deriving values for the relation of B_{17} to B_{18} we may take the means of the observed values for the two orientations of B_{17} and compute the probable errors of these means from all the discrepancies between them and the individual measures on which they depend. Thus we have from Table IV

$$B_{17} = B_{18} + 30{\cdot}3^{\mu} \pm 0{\cdot}9^{\mu}, \text{ A end of } B_{17} \text{ north.}$$
$$B_{17} = B_{18} + 29{\cdot}9 \;\; \pm 0{\cdot}9 \;\;, \text{ A end of } B_{17} \text{ south.}$$

Introducing in these last equations the value of B_{18}, given above, there result for the length of B_{17} in melting ice

$$B_{17} = 5^{\mathrm{m}} - 11{\cdot}5^{\mu} \pm 1{\cdot}8^{\mu}, \text{ A end of } B_{17} \text{ north.}$$
$$B_{17} = 5 \;\; - 11{\cdot}9 \;\; \pm 1{\cdot}8 \;\;, \text{ A end of } B_{17} \text{ south.}$$

(12) *Values of sub-spaces of B_{18}.*—Since the subspaces of B_{18} afford multiples of one metre up to five metres, it may not be out of place to give the values of the several sub-spaces resulting from these comparisons.

Length of Iced Bar B_{17}.

Taking means of the three group means given in section 10 for each value of x_i, we find

$$\begin{aligned} x_1 &= +12{\cdot}9^\mu \pm 0{\cdot}7^\mu, \\ x_2 &= +\ \ 0{\cdot}5 \pm 0{\cdot}7\ , \\ x_3 &= +\ \ 7{\cdot}6 \pm 0{\cdot}7\ , \\ x_4 &= +\ \ 2{\cdot}7 \pm 0{\cdot}7\ , \\ x_5 &= +30{\cdot}4 \pm 0{\cdot}7\ . \end{aligned}$$

The probable errors assigned to these values are derived from the residuals resulting from a comparison of each adopted mean with its three group means. Thus the sum of the squares of all the residuals is 33·0. The number of residuals is 15, and the number of unknowns is 5. Hence the probable error of one group mean is $\pm 1{\cdot}2^\mu$, and the probable error of each adopted value of x_i is $\pm 1{\cdot}2^\mu / \sqrt{3} = \pm 0{\cdot}7^\mu$.

Since any sub-space s_i is expressed by the relation

$$s_i = M_{21} - x_i,$$

wherein

$$M_{21} = 1^{\mathrm{m}}_{\underline{x}} + 2{\cdot}5^\mu,$$

there results for the sub-spaces on B_{18} when it is packed in melting ice

$$\begin{aligned} s_1 &= 1^{\mathrm{m}} - 10{\cdot}4^\mu \pm 0{\cdot}7^\mu, \\ s_2 &= 1\ \ +\ \ 2{\cdot}0 \pm 0{\cdot}7\ , \\ s_3 &= 1\ \ -\ \ 5{\cdot}1 \pm 0{\cdot}7\ , \\ s_4 &= 1\ \ -\ \ 0{\cdot}2 \pm 0{\cdot}7\ , \\ s_5 &= 1\ \ - 27{\cdot}9 \pm 0{\cdot}7\ . \end{aligned}$$

These spaces, it may be again remarked, are the successive spaces on B_{18}, counting from its A end toward its B end. It must be observed, also, that these values appertain to B_{18} when supported and aligned in the Y-trough, as already explained at length.

DETERMINATION OF B_{17} ON NEW COMPARATOR.

(13) *Description of apparatus.*—On completing the determinations of B_{17} by aid of the office comparator, and studying the results after the manner detailed in the preceding sections, it became evident that any value which might be adopted for B_{17} would not have a precision comparable with that attained in the use of the iced bar apparatus on the 100^{m} comparator and standard kilometre of Holton Base. It was plain, also, that the principal source of the difficulty which thus presented itself lay in the instability of the microscopes of the office comparator. The question which logically followed, and which it appeared essential to answer, was whether this instability gave rise to any systematic error in the length of B_{17}. The long series of direct determinations of February to March, 1892, did not seem to justify an affirmative answer to this question; but the indirect determinations by aid of B_{17} did not

warrant a negative answer. There appeared to be no way out of this dilemma except to build a comparator designed especially for the purpose of getting the length of B_{17}. Accordingly plans for a new comparator were drawn up and approved by the Superintendent of the Survey, and the work of construction was completed during June and early July, 1892. It was designed with a view to direct determinations by measuring a distance of 5^m with a Prototype in ice and then transferring this distance to B_{17} in the manner already described.

This comparator was constructed in a vacant lot adjacent to the Survey Office. Briefly described, it consists of six brick piers resting on a continuous foundation of 6 cubic metres of well-rammed concrete. The foundation is 1^m by 1^m in cross section and rises to the level of the ground surface. The foundation and piers are both set in Portland cement and their combined weight is about 12 tons (11 000 kilogrammes). The exposed parts of the foundation and the ground near about are covered to a depth of a decimetre with sawdust. The terminal piers were arranged to receive the Survey microscopes 5 and 6, already described in section 1. Connection between these microscopes and the piers was secured by means of heavy cast-iron plates, held fast by long bolts built into the piers, insuring thus ample rigidity. The intermediate piers had built into them heavy stone posts, on which the post caps of the Repsold microscopes could be mounted as when used in fieldwork. These connections were also very rigid; and the great weight of the piers and foundation, which were virtually one mass, gave a very satisfactory degree of stability. The piers were ranged in an east and west line, nearly. To protect them from the sun and storms a shed $8{\cdot}6^m$ long by 3^m wide was built over them. The north side of the shed was left open until after the comparisons of July and August, 1892, were completed, a temporary cover of canvass being applied at night-time, or when the comparator was not in use. It was thus possible to use daylight for illuminating the microscopes, and a free circulation of air was permitted within the shed. To protect the microscopes from currents of air and from the ice packs of the bars, their tubes and their metallic connections with the piers were heavily wrapped with cotton batting.

The axes of the microscopes were made vertical by means of levels attached to their tubes. Their axes were put into one vertical plane and their focal planes made nearly coincident by bringing them all to focus and bisection on a fine thread stretched under them. The remaining small differences in elevation of the focal planes were measured from time to time by observing the readings of a level (having $12{\cdot}5''$ per division) attached to Prototype No. 21 when it was brought to focus under the microscopes.

The metre and 5^m bars were moved under the microscopes on portable tracks supported on posts isolated from the piers. The track for the metre was a stiff truss of cypress wood. One rail of this truss had a cross section shaped like a truncated A, while the other was flat. The car carrying the metre box was made to slide on these rails. By means of suitable lateral and longitudinal stops this truss could be held in such a position that only small lateral motions of the metre box on its car were required. These small lateral motions and small longitudinal motions as well, of the box on the car, were accomplished by means of a system of slow-motion screws. The metre box was also provided with three leveling screws. These provisions made it possible to bring the metre to focus, and to position within a few microns, with ease and rapidity.

The 5^m bar was mounted in its Y-trough and moved on the same cars and track used with it in the fieldwork. It was handled, in fact, in the determinations about to be given, in precisely the same manner as in measuring a line in the field.

Both bars were moved under the microscopes along the south side of the piers. When the metre was not under observation it was removed, together with its car and track, to suitable brackets on the south side of the comparator shed. When B_{17} was not under observation it was run out of the east door of the shed on the portable track and left under an awning alongside the supply of ice and the ice-crusher.

The platforms on which the observers stood were isolated from the piers and from the track supports, so that the movements of the observers produced no disturbance of the piers or of the bars. The piers were about 25^m from the nearest street. Only heavily loaded vehicles, which passed quite infrequently on this street, produced any perceptible vibration in the piers. On no occasion was it considered essential to stop observing on account of such minute vibrations.

The piers and the adjacent parts of the shed on the north side were whitewashed, and this aided in producing an abundance of white light for illuminating the microscopes. In this regard the conditions for ease and comfort in observing were much superior to those afforded by the electric lights of the office comparing room.

(14) *Methods of observation.*—The study of the earlier direct determinations of July, 1891, and February and March, 1892, indicated the desirability of increasing the number of measures with the prototype metre relatively to those with B_{17} in any set of observations. Accordingly the following programme was adopted and adhered to throughout the determinations on the new comparator, the symbol D representing as hitherto the distance between the zeros of the terminal microscopes 5 and 6:

(1) A measure of D with the prototype M_{21}.
(2) A measure of D with B_{17}.
(3) A measure of D with M_{21}.
(4) A measure of D with B_{17}.
(5) A measure of D with M_{21}.

The measures with the metre started alternately from the west and east ends of the comparator. The micrometric readings on either bar were made simultaneously by the two observers, who exchanged positions in every case to eliminate personal equation. Small distances, amounting to a few microns only, were measured with the intermediate Repsold microscopes, whose magnifying powers are but twenty-seven diameters, while the terminal microscopes 5 and 6 have a power of fifty diameters.

The measures with B_{17} were made in the manner described at length in section 10 of Chapter I. That is, the rear (west) end of the bar was first brought to bisection by means of the car lever, while the bisection at the front end was made by turning the micrometer screw at that (east) end. The observers then exchanged positions, and the same operation was repeated in the reverse order, the bar being brought to bisection by the car lever at the east end and the micrometer threads moved to bisection at the west end. Two sets of such observations were generally made.

Both bars were fully iced from half an hour to an hour before observing on them. A slightly greater quantity of ice was used on both bars than in the previous determinations made during the winter in the comparing room, where the air temperature averaged about 6° C. The average air temperature during July and August, 1892, when the present determinations were made, was about 35° C. However, when protected by the felt covers of the metre box and Y-trough, the ice did not appear to melt much faster than it did under the far lower winter temperature. Throughout this series of comparisons the ice about B_{17} was shaken up and replenished just before bringing it under the microscopes for observations, and the prototype was similarly treated at intervals of twenty to thirty minutes. From very careful personal attention to the packing of these bars, I am confident that they were subject to no sensible variations in length due to variations in temperature.

The observations were made by Messrs. John S. Siebert and R. S. Woodward, and the record was kept by Mr. Orville G. Brown. The computations were made and checked in all parts by the same parties as the work of observation progressed.

(15) *Method of computation.*—The considerations on which the method followed in deriving the length of B_{17} depends are similar to those set forth in section 3 above. But since the data were in some respects different, it seems proper to state the present case briefly by itself.

If the positions of the microscopes of the comparator remained invariable, or if the measured quantities were affected by errors of observation only, the distance D between the zeros of the terminal microscopes would be given by each of the following equations, corresponding to the several measures of the programme in section 14.

Length of Iced Bar B_{17}.

$$\begin{aligned}
&(1)\quad D = 5M_{21} + Q_1 + v_1.\\
&(2)\quad\;\;\, = B_{17} + Q_2 + v_2.\\
&(3)\quad\;\;\, = 5M_{21} + Q_3 + v_3.\\
&(4)\quad\;\;\, = B_{17} + Q_4 + v_4.\\
&(5)\quad\;\;\, = 5M_{21} + Q_5 + v_5.
\end{aligned}$$

In these, $Q_1\ Q_2 \ldots$ are the micrometric quantities resulting from the several measures, and $v_1\ v_2 \ldots$ are the corresponding corrections. But the hypothesis of constancy of position of the microscopes, or of invariability of D, is inadequate. It appears, however, from the observations, that the assumption of a constant rate of variation of D is quite satisfactory, and this is the one adopted.

The time intervals between the several measures (1) (2) . . . (5) are nearly equal, being about 7·5 minutes each on the average. Hence, if we call z the rate of change of D and put as hitherto

$$\begin{aligned}
x &= 5M_{21} - B_{17},\\
y &= D - 5M_{21},
\end{aligned}$$

and take the time of the measure (3) as the epoch of D, the following observation equations result:

$$\begin{aligned}
y - 2z - Q_1 &= v_1,\\
x + y - z - Q_2 &= v_2,\\
y \qquad - Q_3 &= v_3,\\
x + y + z - Q_4 &= v_4,\\
y + 2z - Q_5 &= v_5.
\end{aligned} \tag{1}$$

If we assign a weight unity to each of the measures with B_{17}, or to Q_2 and Q_4, and a weight p to each of the measures with M_{21}, or to Q_1, Q_3, and Q_5, the normal equations are

$$\begin{aligned}
2x + (3p + 2)y \qquad &= p(Q_1 + Q_3 + Q_5) + Q_2 + Q_4,\\
2 + \qquad 2 \qquad &= \qquad\qquad\qquad\quad Q_2 + Q_4,\\
(8p + 2)z &= 2p(Q_5 - Q_1) \qquad + Q_4 - Q_2.
\end{aligned} \tag{2}$$

These give

$$\begin{aligned}
x &= \tfrac{1}{2}(Q_2 + Q_4) - \tfrac{1}{3}(Q_1 + Q_3 + Q_5),\\
y &= \qquad\qquad\quad\;\; \tfrac{1}{3}(Q_1 + Q_3 + Q_5),\\
z &= \frac{2p(Q_5 - Q_1) + Q_4 + Q_2}{8p + 2}.
\end{aligned}$$

Since the quantities Q_1, Q_3, Q_5 depend on the half algebraic sum of twenty independent microscope readings, while Q_2 and Q_4 depend on the half sum of four to eight such readings, the probable errors of the former would stand to those of the latter as $\sqrt{5}$ to 1, or as $\sqrt{5}$ to 1/2, as the case may be, provided the errors involved are of the accidental sort. If the errors are those of observation only, the ratios just given are the proper ones for determining the relative weights. But it is certain that the most formidable errors are not due to defects of observation, and hence some doubt may be entertained as to the best value of the weight p. In order to test the matter with the actual data, half of the observations were treated on the supposition that $p = 1$ and half on the supposition that $p = 1/5$. The results indicated that the latter supposition is the more appropriate one, and it was therefore adopted for all the work. A mere inspection of the data, in fact, shows that the weights of Q_1, Q_3, Q_5, should be much less than the weights of Q_2 and Q_4.

(16) *Data and results.*—Table VI following gives the principal data of the series of determinations of B_{17} on the new comparator. The observations extended from July 18 to August 10, 1892, and the table embraces every determination made. The first column gives the date; the second gives the times of beginning and ending of a set of observations; the third designates the orientation of the prototype metre M_{21}. The next three columns give the quantities Q_1, Q_3, Q_5, which are the observed excesses of the distance D between the zeros of the terminal microscopes over $5M_{21}$, while the last two columns give the quantities Q_2, Q_4, which are the excesses of D over B_{17}. It is to be remarked that these Q's change from time to time by reason of changes in the adopted zeros of the microscopes, as well as by reason of changes in the relative positions of the microscopes themselves. Those zeros which were most convenient in computing the Q's were, in general, adopted, but the same zeros were retained throughout any day's work, so that the stability of the terminal microscopes is well shown by Q_2 and Q_4 especially for any date. For greater clearness of perception the sets of observations are spaced off with extra leads in groups by dates.

TABLE VI.—*Data for length of B_{17} in ice.*

Date.	Time of day.	End of metre west.	Q_1	Q_3	Q_5	Q_2	Q_4
1892.			μ	μ	μ	μ	μ
July 18	1:15 to 1:56 p. m.	A	+6·4	+0·8	+3·8	+19·0	+19·0
	2:14 3:02		+6·4	+11·8	+12·1	+31·9	+34·9
20	10:51 11:29 a. m.	B	−30·9	−26·9	−24·0	+3·1	+1·0
	1:08 1:41 p. m.		−17·4	−21·5	−12·5	+10·1	+8·9
	1:54 2:31		−9·4	−7·5	−8·4	+5·1	+12·1
21	10:52 11:27 a. m.	A	+9·8	−0·2	−3·0	+28·8	+32·7
	11:40 12:13 p. m.		+0·5	+1·8	+0·9	+26·8	+27·3
	1:16 1:46		+0·2	−1·3	+6·1	+25·6	+29·9
	1:58 2:26		+6·6	+1·7	−4·6	+27·7	+27·0
22	10:37 11:09 a. m.	B	+14·3	+14·2	+12·8	+42·0	+42·2
	11:21 11:51		+13·8	+19·6	+17·5	+39·6	+40·4
	1:18 2:00 p. m.		+25·5	+26·2	+26·8	+52·0	+51·7
26	1:52 2:26 p. m.	A	−4·4	−10·6	−6·4	+12·6	+17·0
	2:30 2:59		−6·4	−7·2	−4·4	+0·9	+11·4
27	10:45 11:13 a. m.	B	+19·5	+18·2	+17·3	+46·3	+41·9
	11:20 11:48		+18·2	+19·0	+18·7	+38·5	+45·8
	12:00 12:26 p. m.		+22·1	+20·3	+19·4	+46·5	+46·2
	1:15 1:51		+25·9	+26·4	+23·8	+52·9	+54·9

Length of Iced Bar B_{17}.

TABLE VI.—*Data for length of B_{17} in ice*—Continued.

Date.	Time of day.	End of metre west.	Q_1	Q_3	Q_5	Q_2	Q_4
1892.			μ	μ	μ	μ	μ
July 28	11:00 to 11:32 a. m.	A	+12·1	+21·8	+17·1	+40·9	+43·8
	11:38 12:10 p. m.		+16·3	+21·5	+19·3	+41·8	+41·9
	1:07 1:36		+18·5	+25·8	+21·4	+48·3	+45·4
Aug. 4	11:54 12:33 p. m.	A	+13·2	+16·8	+17·7	+42·9	+45·8
	12:36 1:13		+12·2	+20·0	+21·9	+49·2	+46·4
5	10:25 10:58 a. m.	B	+15·6	+11·2	+11·8	+41·5	+41·4
	11:11 11:43		+13·2	+15·6	+17·4	+46·1	+43·3
	11:53 12:23 p. m.		+16·8	+14·2	+11·6	+44·3	+47·6
	1:18 1:48		+15·0	+15·6	+17·8	+48·2	+48·7
6	10:28 11:12 a. m.	A	−16·5	−18·8	−15·6	+16·6	+16·6
	11:22 11:59		−13·6	−12·2	−14·7	+15·8	+15·9
	12:10 12:37 p. m.		−15·7	−14·8	−10·6	+17·0	+20·8
	1:32 1:58		−17·6	−12·6	− 7·2	+22·9	+23·4
8	10:40 11:06 a. m.	A	− 7·3	− 8·3	−10·9	+21·3	+22·7
	11:11 11:38		− 2·1	− 4·0	− 2·9	+24·3	+24·0
	11:59 12:26 p. m.		− 3·1	0·0	+ 1·0	+25·7	+31·2
	1:14 1:41		+ 5·2	+ 0·6	+ 2·2	+32·1	+33·7
9	10:27 11:00 a. m.	B	+42·7	+44·6	+46·3	+73·5	+73·5
	11:05 11:35		+46·9	+46·7	+48·4	+76·0	+80·5
	11:45 12:17		+47·8	+50·1	+49·9	+79·5	+80·4
	1:07 1:36 p. m.		+52·3	+50·7	+52·8	+85·1	+89·0
	1:41 2:04		+60·1	+53·0	+59·6	+92·0	+91·6
	2:08 2:31		+58·7	+58·3	+59·2	+95·3	+92·7
10	10:41 11:06 a. m.	A	+ 8·0	+ 3·0	+ 5·6	+31·0	+35·1
	11:12 11:37		+ 3·0	+ 7·7	+ 5·1	+31·1	+33·7
	11:43 12:07 p. m.		+ 7·5	+ 4·7	+ 9·9	+33·4	+33·2

Table VII following gives the results of the computation, according to equations (1) and (3) of section 15, based on the data of Table VI. The first column gives the date of observation. The second column gives the quantity $x = 5M_{21} - B_{17}$, resulting from a set of observations; the third gives $y = D - 5M_{21}$, the D here being the most probable value of the interval between the terminal microscopes at the middle time of the observations in any set; the fourth gives z, the rate of change of D, or the most probable change in D in 7·5 minutes, about. The next three columns give the residuals v_1, v_3, v_5, or most probable corrections to Q_1, Q_3, Q_5, respectively; while the two following columns give the residuals v_2, v_4, for Q_2, Q_4, respectively. The last column gives the sum of the weighted squares of the residuals, this sum being designated by $[pvv]$. The value adopted for p is 1/5. It may be remarked that the sums $v_2 + v_3 + v_5$, and $v_2 + v_4$ should each be equal to zero within a tenth of a unit. The groups of values for the several dates are spaced off by extra leads as in Table VI. The times of day when the several sets of observations were made are given in Table VI.

TABLE VII.—*Results of computation.*

Date.	x	y	z	v_1	v_3	v_5	v_2	v_4	$[pvv]$
1892.	μ	μ	μ	μ	μ	μ	μ	μ	
July 18	+ [15·3	+ 3·7	−0·29	−2·1	+2·9	−0·7	−0·3	+0·3	2·84
	23·3]	+10·1	−1·47	+0·8	−1·7	+0·9	0·0	0·0	0·87
20	+ 29·3	−27·3	+0·18	+3·2	−0·4	+2·9	−1·3	+1·2	6·89
	26·6	−17·1	+0·21	−0·1	+4·4	−4·2	−0·8	+0·8	8·68
	17·0	− 8·4	+2·17	−3·3	−0·9	+4·3	+1·3	−1.3	9·42
21	+ 28·6	+ 2·2	−0·34	−6·9	+2·4	+4·5	+2·3	−2·2	24·88
	25·9	+ 1·1	+0·18	+0·2	−0·7	+0·6	0·0	−0·1	0·19
	26·1	+ 1·7	+1·85	−2·2	+3·0	−0·7	+0·4	−0·3	3·12
	26·2	+ 1·2	−1·44	−2·5	−0·5	+2·9	+1·1	−1·0	5·39
22	+ 28·3	+13·8	−0·11	−0·3	−0·4	+0·8	+0·2	−0·2	0·26
	23·0	+17·0	+0·63	+1·9	−2·6	+0·8	−0·2	+0·2	2·28
	25·6	+26·2	+0·06	+0·6	0·0	−0·5	−0·3	+0·2	0·26
26	+ [21·9	− 7·1	+1·00	−4·7	+3·5	+1·3	+1·2	−1·2	10·09
	12·2]	− 6·0	+3·14	−5·9	+1·2	+4·7	+2·2	−2·1	20·92
27	+ 25·8	+18·3	−1·47	+1·7	+0·1	−1·9	−0·7	+0·7	2·28
	23·6	+18·6	+2·08	−3·8	−0·4	+4·1	+1·6	−1·5	11·09
	25·8	+20·6	−0·38	−0·7	+0·3	+0·4	+0·3	−0·2	0·28
	28·5	+25·4	+0·32	−1·1	−1·0	+2·2	+0·7	−0·7	2·39
28	+ 22·8	+19·0	+0·36	+2·0	−2·5	+0·4	−0·4	+0·3	2·33
	25·4	+17·0	+1·36	+2·2	−4·8	+2·6	+0·1	0·0	6·94
	24·9	+21·9	−0·48	+4·4	−3·9	−0·5	−1·0	+0·9	8·77
Aug. 4	+ 28·5	+15·9	+1·31	+0·2	−0·9	+0·8	+0·2	−0·1	0·30
	29·8	+18·0	+0·30	+5·2	−2·0	−3·3	−1·7	+1·7	14·17
5	+ 28·5	+12·9	−0·45	−1·8	+1·7	+0·2	+0·3	−0·4	1·48
	29·3	+15·4	−0·31	+2·8	−0·2	−2·6	−1·1	+1·1	5·35
	31·8	+14·2	+0·36	−3·3	0·0	+3·3	+1·3	−1·2	7·69
	32·3	+16·1	+0·35	+0·4	+0·5	−1·0	−0·2	+0·1	0·33
6	+ 33·6	−17·0	+0·10	−0·7	+1·8	−1·2	−0·1	+0·1	1·05
	29·3	−13·5	−0·09	+0·3	−1·3	+1·0	+0·1	−0·2	0·63
	32·6	−13·7	+1·62	−1·2	+1·1	+0·1	+0·3	−0·3	0·51
	35·7	−12·5	+1·29	+2·5	+0·1	+2·7	−1·0	+1·1	4·92
8	+ 30·8	− 8·8	−0·01	−1·5	−0·5	+2·1	+0·7	−0·7	2·36
	27·2	− 3·0	−0·17	−0·6	+1·0	−0·4	+0·1	0·0	0·31
	29·1	− 0·7	+1·98	−1·6	−0·7	+2·3	+0·7	−0·8	2·80
	30·2	+ 2·7	+0·08	−2·7	+2·1	+0·7	+0·7	−0·7	3·42
9	+ 29·0	+44·5	+0·40	+1·0	−0·1	−1·0	−0·4	+0·4	0·72
	30·9	+47·3	+1·42	−2·4	+0·6	+1·7	+0·8	−0·9	3·25
	30·7	+49·3	+0·48	+0·5	−0·8	+0·4	0·0	+0·1	0·22
	35·1	+51·9	+1·14	−2·7	+1·2	+1·4	+0·8	−0·9	3·59
	34·2	+57·6	−0·17	−2·2	+4·4	−2·3	0·0	0·0	5·90
	35·3	+58·7	−0·67	+1·3	+0·5	−1·8	−0·6	+0·6	1·76
10	+ 27·5	+ 5·5	+0·87	−4·1	+2·5	+1·6	+1·1	−1·2	7·77
	27·1	+ 5·3	+0·96	+0·4	−2·4	+2·1	+0·3	−0·3	2·25
	25·9	+ 7·4	+0·21	−0·5	+2·7	−2·1	−0·3	+0·3	2·57

(17) *Corrections to the computed values of x.*—The computed values of x in Table VII are subject to two small corrections. The first of

these depends on the inequalities in heights of the focal planes of the microscopes. The microscopes were adjusted to near equality in heights of their focal planes by the method explained in section 13, but owing to some unexplained cause, microscope 6, which was attached to the east pier of the comparator, was given a position about 2^{mm} too high. This maladjustment was discovered on July 19, but since it could be measured accurately it was not corrected until August 3. Measurements of the inequalities in heights of the focal planes were made on July 19, 20, 26, and August 3, by means of a level attached to the prototype metre, as already explained. The microscopes were arranged in the order 5, 1, 2, 3, 4, 6; 5 being on the west pier, 6 on the east pier, and the others on the intermediate piers. The following are the mean values of the differences in height of the focal planes resulting from the measures, seven in all, for the determinations extending from July 18 to July 28. The plus sign preceding a result shows that the focal plane of the east microscope of any pair is the higher.

	mm.
Difference for 5 and 1	−0·34,
1 and 2	+0·52,
2 and 3	−0·30,
3 and 4	+0·32,
4 and 6	+1·85,
Sum	+2·05.

These results show that the sum of the projections of the metric spaces on the horizontal focal plane of microscope 5 is $2{\cdot}0^{\mu}$ less than the sum of those spaces. Likewise, since the plane of microscope 6 is $2{\cdot}05^{\mathrm{mm}}$ higher than the plane of 5, the line joining the foci of 5 and 6 is $0{\cdot}4^{\mu}$ longer than the projection of this line on those planes. Hence the effective correction to the quantities Q_1, Q_3, Q_5 is $-2{\cdot}0^{\mu}+0.4^{\mu}=-1{\cdot}6^{\mu}$, or the correction to x is $+1{\cdot}6^{\mu}$. This is to be applied to every value of x up to July 28, inclusive.

On August 3, microscope 6 was re-adjusted, and it and the others were then left undisturbed throughout the remainder of the comparisons. Measures of the relative heights of the focal planes were made on August 5 and 10. The resulting mean values from the three measures are as follows:

	mm.
Difference for 5 and 1	−0·26,
1 and 2	+0·48,
2 and 3	−0·31,
3 and 4	+0·40,
4 and 6	−0·12,
Sum	+0·19.

The corresponding correction to x is $+0{\cdot}3^{\mu}$, and this is to be applied to all values of x from August 4 to 10, both inclusive.

The second correction depends on the curvature of B_{17} in a vertical plane. This curvature is measured by means of the long striding level as explained in Chapter I, section 9. The correction is so small that it has been neglected in all previous work, though its amount was less in that work than in the work now considered. The theory of the computation of this correction from the level readings is given in Supplement A. It may suffice here to give the values of the computed corrections. They are as follows:

	μ.
July 18, 1892,	0·01,
21,	·09,
27,	·35,
Aug. 4,	·05,
6,	·07,
8,	·09,
9,	·31,
10,	·13.

These are so small that they might be ignored without material error, but since it is believed that they represent determinable quantities, we shall here apply them. It may be remarked that their variability is undoubtedly due to the effect of the external air temperature on the web of the Y-trough so long as the vertical adjusting screws under the bar are left undisturbed. The observations show, in fact, that if the bar is made straight for any given external temperature, it becomes convex upwards for an increase of that temperature. The adjusting screws were left undisturbed during the series of determinations extending from July 18 to July 28, and it will be seen by reference to Table VIII following that the maximum correction of July 27 corresponds to the maximum air temperature of the series. A like remark applies to the series extending from August 4 to 9, the maximum correction corresponding to the maximum air temperature of August 9. (See Table IX).

Considering the small number of determinations of corrections in the series of July, we may adopt a single mean value for application to x, viz, $0{\cdot}2^{\mu}$. For the series extending from August 4 to 9 we shall adopt $0{\cdot}1^{\mu}$ for all dates except August 9, and the observed value $0{\cdot}3^{\mu}$ for the latter date. The value adopted for August 10 is also the observed value, the bar having been turned end for end and re-adjusted on that day.

Since the curvature of B_{17} diminishes the interval between its terminal lines, these corrections must be applied to the quantities x with the negative sign.

(18) *Adopted individual and mean values of* x.—Tables VIII and IX following show the adopted values of x and various data pertaining thereto for the two orientations of B_{17} in the Y-trough.

In making up these tables the results of July 18 and 26 have been rejected for the following reasons: On these dates the microscopes were taken apart, cleaned, and wrapped with cotton batting just before observing with them. In addition, the leveling and clamp screws of the intermediate microscopes 1, 2, 3, 4 were tightened. It is thought that the anomalous results of these two days are due to movements of the microscopes before reaching their most stable positions after replacing their objectives and rewrapping them in cotton. Subsequently, when any such adjustment was made, the apparatus was allowed to stand at least one day before using it. Similarity of conditions, then, if there were no other reason, would seem to require that these results be omitted.

Table VIII gives the results obtained with the A-end of B_{17} west, and Table IX those obtained with the A-end of B_{17} east. The first column of either table gives the date; the second the maximum air temperature of the day as observed under the comparator shed; the third specifies the orientation of the metre; the fourth gives the computed value of x from Table VII; the fifth the correction for inequality in heights of focal planes of the microscopes, or slopes of metre; the sixth the correction for curvature of B_{17}; the seventh the adopted value of x, and the eighth the residual, found by subtracting any observed x from the mean value.

TABLE VIII.—*Adopted values of $x = 5M_{21} - B_{17}$ (A-end of B_{17} west).*

Date.	Max. air temperature of day.	End of metre west.	Computed value of x.	Correction for— Slopes of metre.	Correction for— Curvature of B_{17}.	Adopted x.	Mean x for date.	v.
1892.	°		μ	μ	μ	μ	μ	μ
July 20			+29·3	+1·6	—0·2	+30·7		—3·8
			26·6	1·6	·2	28·0		—1·1
	32·0 C.	B	17·0	1·6	·2	18·4	+25·7	+8·5
21			28·6	+1·6	—0·2	30·0		—3·1
			25·9	1·6	·2	27·3		—0·4
			26·1	1·6	·2	27·5		—0·6
	31·0	A	26·2	1·6		27·6	28·1	—0·7
22			28·3	+1·6	—0·2	29·7		—2·8
			23·0	1·6	·2	24·4		+2·5
	33·5	B	25·6	1·6	·2	27·0	27·0	—0·1
27			25·8	+1·6	—0·2	27·2		—0·3
			23·6	1·6	·2	25·0		+1·9
			25·8	1·6	·2	27·2		—0·3
	38·5	B	28·5	1·6	·2	29·9	27·3	—3·0
28			22·8	+1·6	—0·2	24·2		+2·7
			25·4	1·6	·2	26·8		+0·1
	36·0	A	24·9	1·6	·2	26·3	25·8	+0·6
Aug. 10			27·5	+0·3	—0·1	27·7		—0·8
			27·1	·3	·1	27·3		—0·4
	36·0	A	25·9	·3	·1	26·1	27·0	+0·8

Mean value of x, 26·91μ ± 0·4μ

TABLE IX.—*Adopted values of* $x = 5M_{21} - B_{17}$ *(A-end of* B_{17} *east).*

Date.	Max. air temperature of day.	End of metre west.	Computed value of x.	Correction for—		Adopted x.	Mean x for date.	v.
				Slopes of metre.	Curvature of B_{17}.			
1892.	°		μ	μ	μ	μ	μ	μ
Aug. 4			+28·5	+0·3	—0·1	+28·7		+2·6
	34·5 C.	A	29·8	·3	·1	30·0	+29·4	+1·3
5			28·5	+0·3	—0·1	28·7		+2·6
			29·3	·3	·1	29·5		+1·8
			31·8	·3	·1	32·0		—0·7
	30·0	B	32·3	·3	·1	32·5	30·7	—1·2
6			33·6	+0·3	—0·1	33·8		—2·5
			29·3	·3	·1	29·5		+1·8
			32·6	·3	·1	32·8		—1·5
	32·5	A	35·7	·3	·1	35·9	33·0	—4·6
8			30·8	+0·3	—0·1	31·0		+0·3
			27·2	·3	·1	27·4		+3·9
			29·1	·3	·1	29·3		+2·0
	34·5	A	30·2	·3	·1	30·4	29·5	+0·9
9			29·0	+0·3	—0·3	29·0		+2·3
			30·9	·3	·3	30·9		+0·4
			30·7	·3	·3	30·7		+0·6
			35·1	·3	·3	35·1		—3·8
			34·2	·3	·3	34·2		—2·9
	37·5	B	35·3	·3	·3	35·3	32·5	—4·0

Mean value of x, $31·34^{\mu} \pm 0·4^{\mu}$.

The individual values of x in Table VIII or IX are, so far as known, of equal weight. The values of x in the two tables differ systematically, however, for reasons given at length in section 20 below. Hence we shall, for the sake of distinction, call the mean of the adopted values of x in Table VIII, x_w, and the mean of those in Table IX, x_e, the suffixes defining the orientation of B_{17}. These means are

$$x_w = +\,26·9^{\mu} \pm 0·4^{\mu} \text{ for A-end of } B_{17} \text{ west,}$$
$$x_e = +\,31·3 \pm 0·4 \text{ for A-end of } B_{17} \text{ east.}$$

Since $$x = 5M_{21} - B_{17}$$
and $$5M_{21} = 5^{\text{m}} + 12·5^{\mu},$$

these equations give

$$B_{17} = 5^{\text{m}} - 14·4^{\mu} \pm 0·4^{\mu} \text{ for A-end west,}$$
$$B_{17} = 5 - 18·8 \pm 0·4 \text{ for A-end east.}$$

The probable errors assigned to these mean values are derived from the residuals in the last columns of the tables. Thus, the sum of the squares of the residuals in Table VIII is 135·5, and the corresponding sum from Table IX is 118·1. The number of residuals is forty and the number of unknowns two. Hence the probable error of an individual value of x in either table is $\pm 1·7^{\mu}$; and since the weights of x_w and x_e are each 20, their probable errors are each equal to $\pm 0·4^{\mu}$.

Length of Iced Bar B_{17}.

(19) *Sources of accidental error.*—The principal accidental errors affecting the quantities x, y, z of Table VII, and likewise the adopted values of x in Tables VIII and IX, are, it is believed, errors of observation and errors arising from irregular movements of the six microscopes of the comparator. An idea of the relative magnitudes of these two species of error can be derived in the following manner: Referring to equations (1), (2), and (3) of section 15, it is seen that the resultant of the errors involved in the quantities Q, on which the values in Table VII depend, appears in the sums of the weighted squares of the residuals given in the last column of that table. These sums vary very greatly for different sets of observations, but a trustworthy value for the resultant error may be derived from the aggregate of the sums given in the last column of the table. Omitting the values of July 18 and 26 for the reasons assigned in section 18, the aggregate of the sums is

$$[pvv] = 168{\cdot}8.$$

The total number of residuals in the 40 sets of determinations is 200, and the total number of derived values of x, y, z is 120. Thus the probable error of an observed Q of weight unity (or of any Q_2 or Q_4) is

$$\pm\, 0{\cdot}6745 \left(\frac{168{\cdot}8}{200-120}\right)^{\frac{1}{2}} = \pm\, 1{\cdot}0^{\mu}.$$

This applies to a single measure of the distance between the terminal microscopes with the 5^{m} bar B_{17}. Since the weight of any Q_1, Q_3, Q_5, or of a single measure of that same distance with the metre M_{21} is 1/5, the probable error of one such measure is $\pm\, 2{\cdot}2^{\mu}$.

Since the Q's which enter the formulas of section 15 are independent, it follows that the probable error of a single value of x derived by this process is

$$\pm 1{\cdot}0^{\mu} \sqrt{\frac{39}{18}} = \pm\, 1{\cdot}5^{\mu}\,.$$

This value is only $0{\cdot}2^{\mu}$ less than the valuederived by a different process in the preceding section. The latter process, it will be observed, takes account of any changes in the quantities x which may occur from set to set of observations or from day to day. The close agreement of the two probable errors is, therefore, an indication that such changes were small, and also that the assumption on which the method of computation depends is essentially correct.

An examination of the residuals in Table VII shows a satisfactory distribution. Thus, of the whole number of residuals used, namely, 200, 99 are plus, 93 are minus, and 8 are zero each. The maximum residual in the class whose weight is unity is $2{\cdot}3^{\mu}$. The maximum residual in the class whose weight is 1/5 is $6{\cdot}9^{\mu}$, which is about three times the corresponding probable error, or $\pm\, 2{\cdot}2^{\mu}$, as given above.

It will be noticed that the $[pvv]$ for the earlier sets of determinations in July are considerably larger on the average than for the later sets in August. The only reasons I can assign for this are, first, that the piers, which were completed only a short time before beginning to use them in July, grew more and more stable with the drying and hardening of the cement; and, second, that the smaller time interval required to make a set of observations after the observers had become well accustomed to the work diminished the chance of error from movements of the microscopes. It does not appear, however, as shown in the preceding section, that the resulting values of x were appreciably less precise in the earlier than in the later comparisons.

Now, as to the relation of the errors of observation to the aggregate errors, we may assume as hitherto that the probable error of a single microscope reading does not exceed $0{\cdot}5^{\mu}$. The probable error from this source, then, in an observed Q_2 or Q_4 is $\pm 0{\cdot}5^{\mu}$, if we suppose but four readings to be made on B_{17}. The corresponding value for an observed Q_1, Q_3, or Q_5 is $\pm 0.5^{\mu}\sqrt{5} = \pm 1{\cdot}1^{\mu}$. These are exactly half as great as the values derived from the resultant errors. Hence we conclude that the probable error in a single determination of x, due to instability of microscopes, etc., or to all extra-observational sources, is

$$\pm 1{\cdot}5^{\mu}\sqrt{1-\frac{1}{4}} = \pm 1{\cdot}3^{\mu}.$$

This shows a very satisfactory degree of stability of the microscopes, since it is but little more than 1/4 000 000th part of the length of B_{17} or of the distance between the terminal microscopes.

Finally, from whatever point of view we regard the adopted values of x in Tables VIII and IX, their accordance leaves little to be desired. It appears plain that for either orientation of B_{17} the same quantity, essentially, was observed on every date. This is well shown by the constancy of the mean values of x for the several dates. In Table VIII the maximum divergence of any date mean from the adopted mean of all values of x is but $1{\cdot}2^{\mu}$, and the corresponding maximum divergence in Table IX is but $1{\cdot}9^{\mu}$. The average of these is no greater than the probable error of a single value of x as derived in this section, namely, $\pm 1{\cdot}5^{\mu}$; and it is less than the probable error of a single value of x derived in the preceding section, namely, $\pm 1{\cdot}7^{\mu}$.

Taking this last probable error as a basis it appears that the mean of four determinations of B_{17} in terms of M_{21} on the new comparator would have a probable error less than 1^{μ} or less than 1/5 000 000th of B_{17}. We must conclude, therefore, that the temperatures of M_{21} and B_{17} were fixed within very narrow limits by the melting ice with which they were surrounded.

(20) *Systematic error.*—The precision of the results attained on the new comparator brought to light an unexpected source of systematic error. This source of error appears to be due to the fact that the transverse graduation marks at the ends of B_{17} differ much in width, and to the fact that the personal equations of the observers differ widely in amount. As already stated the width of the graduation mark at the A end of B_{17} is $16{\cdot}3^{\mu}$, while the width of the mark at B end is $36{\cdot}2^{\mu}$. It had been known from the experience of the observers, Siebert and

Woodward, on Holton Base that their personal equations differed; but it was not known that their equations varied with differing widths of graduation marks and micrometer wire intervals until the fact was developed in the course of the determinations detailed above. It was then easily seen by direct observation that such difference might introduce an appreciable constant error.

Fortunately in nearly all the work done with the apparatus the observers have been the same, namely, Siebert and Woodward; the only exception being the few determinations of length of B_{17} made by Tittmann and Woodward in July, 1891.

Before proceeding to examine this question minutely it may be remarked that all determinations of B_{17} indicate a systematic difference in its length depending on its orientation in the Y-trough. To show this clearly we will collect the results given in the preceding sections. For this purpose it is desirable to introduce a more general designation for the orientation of B_{17} than that used hitherto. We shall adopt the convention that the A end of B_{17} is left or right according as it is to the left or right of the observer as he stands alongside of the bar while reading a microscope. Thus A end left corresponds to A end west for the work on the new comparator and to A end north for the work on the office comparator.

Collecting the results of the direct determinations for length of B_{17} from sections 4, 5, and 18 we have—

July, 1891, $B_{17} = 5^{m} - 10{\cdot}1^{\mu} \pm 2{\cdot}0^{\mu}$ for A end left,
$B_{17} = 5 - 11{\cdot}9 \pm 2{\cdot}0$ for A end right;
Feb.–Mar., 1892, $B_{17} = 5 - 14{\cdot}6 \pm 0{\cdot}8$ for A end left,
$B_{17} = 5 - 16{\cdot}2 \pm 0{\cdot}8$ for A end right;
July–Aug., 1892, $B_{17} = 5 - 14{\cdot}4 \pm 0{\cdot}4$ for A end left,
$B_{17} = 5 - 18{\cdot}8 \pm 0{\cdot}4$ for A end right.

The differences shown here are all of the same sign. Their amounts with their probable errors are

July, 1891, $1{\cdot}8^{\mu} \pm 2{\cdot}8^{\mu}$,
Feb.–Mar., 1892, $1{\cdot}6 \pm 1{\cdot}1$,
July–Aug., 1892, $4{\cdot}4 \pm 0{\cdot}6$.

A difference with the same sign but of much smaller amount is shown by the comparisons of B_{17} and B_{18}, as may be seen by reference to Table IV, section 9. Its amount is $0{\cdot}4^{\mu} \pm 1{\cdot}3^{\mu}$.

Although these quantities differ widely amongst themselves their probable errors show that they are not inconsistent with one another, while their persistence in sign renders it highly probable that they are due to a common cause.

The only cause which appears adequate to explain this discrepancy is that mentioned at the beginning of this section, and although the evidence of this cause is not conclusive there would seem to be little doubt that it is the principal cause. We proceed to examine the question at length and to present all the available evidence.

It is essential to the inquiry, in the first place, to know precisely how personal equation or error may affect the observed data. Since nearly all the observations were made by Siebert and Woodward it will be convenient to use a notation referring especially to their equations. Let—

S_a, S_b = the corrections to the microscope readings of Siebert on the A end and B end of B_{17}, respectively,
W_a, W_b = the corresponding corrections to the readings of Woodward,
R_1, L_2 = the observed readings of Siebert on the right and left hand ends of the bar, respectively,
R_2, L_1 = the corresponding readings of Woodward;

Then, supposing the microscope readings to increase from right to left,[*] we have for the distance D between the zeros of the terminal microscopes

(A end of B_{17} left.)

$$\begin{aligned} D &= B_{17} + L_1 + W_a - R_1 - S_b, \\ &= B_{17} + L_2 + S_a - R_2 - W_b. \end{aligned}$$

The sum and difference of these give

$$D = B_{17} + \tfrac{1}{2}(L_2 + L_1) - \tfrac{1}{2}(R_2 + R_1) + \tfrac{1}{2}(S_a + W_a) - \tfrac{1}{2}(S_b + W_b), \quad (1)$$

$$(S_a - W_a) + (S_b - W_b) = -(L_2 - L_1) + (R_2 - R_1). \quad (2)$$

When B_{17} is reversed in the Y-trough we may assume, in order to be entirely general, that owing to change in the aspects of the marks and to the difference in wire intervals of the two microscopes the corrections S_a, W_a, . . . change in value. Designating the new values of the corrections and of the new readings by the same letters with an accent, we have—

(A end of B_{17} right.)

$$D = B_{17} + \tfrac{1}{2}(L'_2 + L'_1) - \tfrac{1}{2}(R'_2 + R'_1) - \tfrac{1}{2}(S'_a + W'_a) + \tfrac{1}{2}(S'_b + W'_b), \quad (3)$$

$$(S'_a - W'_a) + (S'_b - W'_b) = -(L'_2 - L'_1) + (R'_2 - R'_1). \quad (4)$$

For brevity, put

$$\left.\begin{aligned} u &= \tfrac{1}{2}(S_a + W_a) - \tfrac{1}{2}(S_b + W_b), \\ u' &= \tfrac{1}{2}(S'_a + W'_a) - \tfrac{1}{2}(S'_b + W'_b), \\ v &= (S_a - W_a) + (S_b - W_b), \\ v' &= (S'_a - W'_a) + (S'_b - W'_b). \end{aligned}\right\} \quad (5)$$

If now we denote the mean values for length of B_{17} corresponding to its two orientations by V_1 and V_2, the above equations (1) and (3) show that

$$\left.\begin{aligned} B_{17} + u &= V_1 \text{ for A-end of } B_{17} \text{ left,} \\ B_{17} - u' &= V_2 \text{ for A-end of } B_{17} \text{ right.} \end{aligned}\right\} \quad (6)$$

If the personal equations remain the same for both positions of the bar, $u = u'$, and the half sum of (6) gives B_{17} free from this source of error. On the other hand, if those equations differ, the error of B_{17} derived from the mean of (6) is

$$\tfrac{1}{2}(u - u') = \tfrac{1}{4}(S_a - S'_a + W_a - W'_a - S_b + S'_b - W_b + W'_b). \quad (7)$$

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[*] The microscopes were so used on the new comparator and in all the work with the iced bar on Holton Base. The increase was in the opposite direction with the microscopes as mounted on the office comparator. It will be noticed that the corrections S_a, W_a change sign with such change of direction.

Length of Iced Bar B_{17}.

It appears from the equations (1) to (4) that the observations are insufficient to determine the corrections S_a, W_a, . . . without additional relations between them. It is clear, however, that if the widths and aspects of the graduations were the same at both ends of the bar, and if the wire intervals of the micrometers were equal, no error from this source would appear in the resulting lengths of the bar, and it is a matter of regret that these theoretical considerations were not duly appreciated when the bar was constructed. It is clear also that the orientation of B_{17} ought to have been changed from day to day, for the errors in question would thus have been more certain of elimination in the final mean.

We may nevertheless gain some idea of the magnitude of the corrections desired from equations (2) and (4), which give the quantities v and v' defined by the last two of (5). Each set of observations on B_{17} gives a value for v or v', as the case may be, and there were 4 to 12 such sets for each day's work. The mean values of the resulting v and v' are collected in Table X following, which is arranged in two parts corresponding to the two orientations of B_{17}. The first column gives the date, the second the number of individual values of v, and the third the mean value of v for the date. The second half of the table gives the corresponding values pertaining to v'.

TABLE X.—*Observed values of v and v', July and August, 1892.*

A-end of B_{17} left.			A-end of B_{17} right.		
Date.	Number of values observed.	Mean value of v.	Date.	Number of values observed.	Mean value of v'.
1892.		μ	1892.		μ
July 20	6	+5·9	Aug. 4	4	−0·9
21	8	+4·8	5	8	+0·1
22	6	+6·0	6	8	+2·6
27	8	+4·1	8	8	+1·0
28	6	+4·2	9	12	−1·5
Aug. 10	6	+4·2			

If we assign to the several mean values in this table weights proportional to the numbers of observed values on which those means depend, the following weighted means result:

$$v = +\ 4{\cdot}8^{\mu} \pm 0{\cdot}2^{\mu}, \qquad v' = +\ 0{\cdot}2^{\mu} \pm 0{\cdot}6^{\mu}.$$

It thus appears that the quantities v and v' are no less definite than the corresponding values for length of B_{17} already derived. But the fact that v and v' differ so widely makes it necessary to suppose that the personal equations changed with the change in aspect of the bar. How such a change can arise I am unable to explain, but some additional observational evidence indicates that it took place. This evidence was obtained by measuring the differences $(S_a - W_a)$, $(S_b - W_b)$,

($S'_a - W'_a$), and ($S'_b - W'_b$) directly. For this purpose the observers made readings alternately at either end of the bar while it was held still in its Y-trough. Several such series of readings were made during August 30 and 31, 1892. They were not very satisfactory for the reasons that the bar was not iced and that it could not be kept very stable. The following mean values were derived:

$$\left.\begin{array}{l} S_a - W_a = +\,0{\cdot}78^{\mu} \\ S_b - W_b = +\,0{\cdot}78 \end{array}\right\}, \text{ A-end of } B_{17} \text{ left,}$$

$$\left.\begin{array}{l} S'_a - W'_a = +\,1{\cdot}80 \\ S'_b - W'_b = -\,1{\cdot}09 \end{array}\right\}, \text{ A-end of } B_{17} \text{ right.}$$

The sums of these pairs of values give in accordance with the last two of equations (5):

$$v = +\,1{\cdot}6^{\mu}, \qquad v' = +\,0{\cdot}7^{\mu}.$$

These values are much less definite than the mean values given above and differ widely from them, but the discordance is so much greater than can be attributed to other sources of error, that it seems essential to entertain, the hypotheses first, that the personal equations of the observers varied from time to time; or second, that the work on the new comparator was subject to some constant error which changed sign with the change in orientation of the bar; or third, that both these possible causes were operative. These views are, in fact, suggested by the variations of the values of v' in the second half of Table X.

It becomes desirable, therefore, to glean such information on this point as is afforded by the observations on B_{17} during February to May, 1892, on the office comparator, and likewise by the observations in the use of the apparatus on Holton Base in 1891. The values of v and v' resulting from the direct determinations of B_{17} on the office comparator are given below in Table XI, which is arranged in the same manner as Table X.

TABLE XI.—*Observed values of v and v', February to March, 1892.*

A-end of B_{17} left.			A-end of B_{17} right.		
Date.	Number of values observed.	Mean value of v.	Date.	Number of values observed.	Mean value of v'.
1892.		μ	1892.		μ
Feb. 4	3	+0·8	Feb. 19	4	+2·9
5	8	2·8	20	4	2·8
6	4	2·4	23	4	3·8
8	8	4·4	24	4	5·1
10	8	4·0	25	8	1·6
11	4	1·0	26	12	1·9
13	8	2·0	27	8	2·0
15	8	3·4	Mar. 7	8	1·2
16	12	1·9	8	8	2·0
17	8	3·0	9	4	2·0
18	8	4·1	10	4	1·9
			11	4	2·0
			12	4	3·3

Length of Iced Bar B_{17}.

Giving the values in this table weights proportional to the number of measures for each, there result the following weighted means:

$$v = +\,2{\cdot}9^{\mu} \pm 0{\cdot}2^{\mu}, \qquad v' = +\,2{\cdot}3^{\mu} \pm 0{\cdot}2^{\mu}.$$

These values, it is seen, not only agree well with each other but are determined with a precision quite equal to that of the corresponding mean values from Table X. It may be remarked that the precisions should not differ much in the two cases since the readings from which v and v' are obtained are made in less than one minute, and hence stability of the microscopes is depended on for this short period only. There are, however, nearly twice as many determinations represented in Table XI as in Table X, and this fact accounts mainly for the smaller probable errors of v and v' derived from Table XI. The range amongst the daily means is nearly the same in the two tables and amounts at most to $4{\cdot}1^{\mu}$.

The observations on B_{17} at the times it was compared with B_{18} give the following values:

Apr. 29, 1892, $v = +\,2{\cdot}8^{\mu}$ (mean of 10 measures,)
30, 1892, $v = +\,2{\cdot}2$ (mean of 17 measures),
May 5, 1892, $v' = +\,1{\cdot}9$ (mean of 27 measures).

These values are likewise fairly accordant with each other and with the values just given above.

Finally, the values of v'* derived from the observations on the 100$^{\text{m}}$ comparator of Holton Base in 1891 are (see Chapter III, section 6):

Aug. 4 to Aug. 7, 1891, $v' = +\,3{\cdot}8^{\mu}$,
Sept. 24 to Aug. 6, 1891, $v^{1} = +\,3{\cdot}3$.

With respect to the latter results, it is to be remarked that they were obtained with the lower power (27 diameters) Repsold microscopes, and may not be strictly comparable with the results obtained with microscopes 5 and 6, which have a power of 50 diameters. They do not differ materially, however, in the mean from the average values of v or v' obtained on the office comparator.

Collecting all these mean values for v and v' and taking their half sums and half differences, we have the following condensed statement of all the direct information bearing on the constancy of the personal equations of the observers:

Date.	$\frac{1}{2}(v+v')$.	$\frac{1}{2}(v-v')$.
Feb. to Mar., 1892	$+2{\cdot}6^{\mu}$	$+0{\cdot}3^{\mu}$
Apr. to May, 1892	$+2{\cdot}2$	$+0{\cdot}2$
July to Aug., 1892	$+2{\cdot}5$	$+2{\cdot}3$

The agreement of the quantities $\frac{1}{2}(v+v')$ in this statement is no less remarkable than the disagreement of the last of the quantities $\frac{1}{2}(v-v')$ from the other two. It appears that the cause of the divergence of

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* B_{17} was used with the A-end right in all the work on Holton Base. Hence the quantity v' only appears in this work.

$\frac{1}{2}(v - v')$ in July and August, 1892, whatever it may have been, was such as to increase v and decrease v' by about the same amount. A change of this kind might have occurred in several ways through changes in the eight quantities S_a, W_a, . . . as may be seen by reference to the last two of equations (5); but the data do not indicate that the change is more likely to have occurred in one of these ways rather than in another.

In view of all the evidence presented in this section, it appears certain that the observers assigned different lengths to B_{17}, depending on its orientation in the Y-trough. It appears equally certain that this difference is due to personal equation of the observers or to change in aspect of the graduation marks, or to both these causes. The two causes are entangled with one another and can not be separated without additional data. We conclude, however, that the effects arising from these causes are well eliminated in the mean of lengths of the bar in its two orientations, or that the error of this mean length, expressed by equation (7), is small.

The conclusion just stated does not require the assumption of constancy and equality of the quantities u and u' of equations (5), but it seems best to adopt this assumption in deriving the final lengths of B_{17} for A-end left and A-end right. The justification of this view is to be found, I think, in an examination of the observed values of $\frac{1}{2}(v + v')$, $\frac{1}{2}(v - v')$, and $u + u'$, given above. Thus the first of these quantities varies by $0{\cdot}4^{\mu}$ only for three different sets of determinations extending over half a year's time, the second varies by $2{\cdot}1^{\mu}$, and the third by $4{\cdot}0^{\mu}$. With respect to this last, however, or to the quantity $u + u'$, we must remark that it might be expected to vary in observed value much more than the others, since it comes from data dependent on the stability of all parts of the apparatus, while the other two depend only on the stability of B_{17} and the terminal microscopes. But the constancy of $u + u'$ in the determinations of July to August, 1892, renders it highly probable that the value then derived is a real quantity, which was masked by errors of the less precise determinations on the office comparator.

ADOPTED LENGTHS FOR B_{17}.

(*21*) *Summary of results.*—Collecting the results for length of B_{17} given in sections 4, 5, 11, and 18, we have the following summary in Table XII:

TABLE XII.—*Summary of results for length of B_{17}.*

Date.	B_{17}	
	A-end left.	A-end right.
July, 1891	$5^{m} - 11{\cdot}0^{\mu} \pm 2{\cdot}0^{\mu}$	$5^{m} - 11{\cdot}9^{\mu} \pm 2{\cdot}0^{\mu}$
Feb. to Mar., 1892	$5 - 14{\cdot}6 \pm 0{\cdot}8$	$5 - 16{\cdot}2 \pm 0{\cdot}8$
Apr. to May, 1892	$5 - 11.5 \pm 1{\cdot}8$	$5 - 11{\cdot}9 \pm 1{\cdot}8$
July to Aug., 1892	$5 - 14{\cdot}4 \pm 0{\cdot}4$	$5 - 18.8 \pm 0{\cdot}4$

Length of Iced Bar B_{17}.

The probable errors in this table differ widely, but so far as is known they are the best available indexes of the precision of the several results.

In accordance with the conclusions and assumption stated in the preceding section we shall derive three values for length of B_{17}, namely: First, a length for A-end of B_{17} left; second, a length for A-end of B_{17} right; and, third, a mean length, which it is assumed will be free from constant error.

It will be recalled that the results of July, 1891, were obtained by Tittmann and Woodward, while all the rest were obtained by Siebert and Woodward, who used the bar also in the position A-end right in all the work of Holton base. We might, therefore, use the results of 1891 in getting the mean length of B_{17}, but they should be excluded in getting the other lengths. In any case, however, it can make but little difference whether we include or exclude the results of 1891, since the weights we are constrained to adopt amount practically to the exclusion of those results. The same remark applies also to the results of the indirect determinations of April and May, 1892, since their weights must be small in comparison with the weights of the results of the direct determinations.

(22) *Derivation of lengths of B_{17} in terms of M_{21}.*—It is assumed, as already stated, that the length of B_{17} observed by Siebert and Woodward is greater or less than its mean length according as the A-end is left or right by a constant which we may denote by u. In conformity with this assumption, if B_{17} denote the mean length of the bar, or the length free from systematic error, $B_{17}+u$ will be the observed length for A-end left and $B_{17}-u$ the observed length for A-end right.

The observed lengths given in Table XII afford a series of observation equations from which B_{17} and u may be determined. In writing these equations the question whether the theoretical weights of the observed quantities should be modified in any way presents itself. Using the probable errors of the observed lengths of 1892 as a basis, the weights of these lengths are proportional to the numbers 1, 0·2, and 4. These seem to present great disparity, but in view of all the considerations set forth in the preceding sections, I am not disposed to change them. Other numbers presenting less disparity might be used, but the resulting values for B_{17} and u would not be much changed thereby. Adopting these theoretical weights, then, and writing for brevity, $\beta = B_{17} - 5^{\mathrm{m}}$, the observation equations given by the results in Table XII become

$$\begin{array}{lll} & & \textit{Weight.} \\ \beta + u + 14{\cdot}6^{\mu} & = +0{\cdot}27^{\mu}, & 1, \\ \beta - u + 16{\cdot}2 & = -1{\cdot}83, & 1, \\ \beta + u + 11{\cdot}5 & = -2{\cdot}83, & 0{\cdot}2, \\ \beta - u + 11{\cdot}9 & = -6{\cdot}13, & 0{\cdot}2, \\ \beta + u + 14{\cdot}4 & = +0{\cdot}07, & 4, \\ \beta - u + 18{\cdot}8 & = +0{\cdot}77, & 4. \end{array}$$

The normal equations from these are

$$10{\cdot}4\beta = -168{\cdot}28^\mu,$$
$$10{\cdot}4u = +\ 19{\cdot}28,$$

whence

$$\beta = -16{\cdot}18^\mu \pm 0{\cdot}40^\mu,$$
$$u = +\ 1{\cdot}85 \pm 0{\cdot}40.$$

The sum of the weighted squares of the residuals is 14·93. Hence the probable error of an observed quantity of unit weight is $\pm 1{\cdot}3^\mu$, and the probable error of either β or u is $\pm 0{\cdot}40^\mu$. Likewise, the probable error of $\beta + u$ or $\beta - u$ is $\pm 0{\cdot}8^\mu$, or exactly twice that of β or u.

Hence the values we adopt for the length of B_{17} in terms of Prototype Metre M_{21} are the following:

$B_{17} = 5^\text{m} - 14{\cdot}3^\mu \pm 0{\cdot}8^\mu$ for A end left,
$B_{17} = 5 - 18{\cdot}0 \pm 0{\cdot}8$ for A end right,
$B_{17} = 5 - 16{\cdot}2 \pm 0{\cdot}4$ for mean.

It is to be remarked that the first two values here given apply especially to the work done with the bar by the observers Siebert and Woodward. Other observers having different personal equations might assign different lengths for the left and right orientations of the bar. The last value, on the other hand, is supposed to be free from systematic error dependent on personal equation and on the orientation of the bar. It is the length which should be applied in deriving the mean length of a line measured the same number of times for each orientation of the bar.

With respect to the adopted mean length of B_{17}, it may be further remarked that it differs but $0{\cdot}9^\mu$ from the mean of the direct determinations of February and March, 1892, and but $0{\cdot}4^\mu$ from the mean of the direct determinations of July and August, 1892. These two series of determinations were the most extensive and accordant of all the determinations made, though differing much in precision. It is seen, however, that if this difference in precision were ignored, the simple mean of the two values for either orientation or for the mean of both would not differ widely from the adopted values. Thus, the mean with equal weights for A-end left is $5^\text{m} - 14{\cdot}5^\mu$; that for A-end right is $5^\text{m} - 17{\cdot}5^\mu$; and for the mean of these two $5^\text{m} - 16{\cdot}0^\mu$. Finally, it may be remarked that the results of July, 1891, and those of April and May, 1892, which are practically rejected by reason of their small weights, do not differ at most from the adopted values by amounts much in excess of the millionth part of the bar's length; the discrepances being, in fact, $3{\cdot}3^\mu$ for A-end left, $6{\cdot}1^\mu$ for A-end right, and $4{\cdot}8^\mu$ in the mean. Moreover, these discrepances are quite within the range indicated by the probable errors of the inferior results. It appears proper to conclude, therefore, that the values for B_{17} adopted above, are trustworthy to the extent shown by their probable errors.

Length of Iced Bar B_{77}.

(*23*) *Lengths of* B_{17} *in terms of the International metre.*—The values for length of B_{17} derived in the preceding section are expressed in terms of Prototype Metre No. 21. It remains, then, to combine the probable error of the length of this metre with the probable errors of those derived lengths in order to get the probable errors of the lengths of B_{17} when expressed in terms of the International Metre. In the Rapport sur la construction, les comparaisons et les autres Opérations ayant servi à déterminer les Équations des Nouveaux Prototype Métrique, p. 82, the probable errors of the Prototypes are said to vary from $\pm 0{\cdot}1^{\mu}$ to $\pm 0{\cdot}2^{\mu}$ between such limits of temperature as are ordinarily met in metrology. Since in the application of M_{21} to the determination of B_{17} only one temperature, namely, that of melting ice, has been used, we shall adopt the extreme value $\pm 0{\cdot}2^{\mu}$ for the probable error of the length of M_{21}. The probable error of $5M_{21}$ is then $\pm 1{\cdot}0^{\mu}$, and the probable error of the length of B_{17} for A-end left or right becomes $\pm 1{\cdot}3^{\mu}$, and for the mean length $\pm 1{\cdot}1^{\mu}$. Hence we have for the length of B_{17} in terms of the International Metre, the following values, the first two of which apply only to the work done with the bar by Siebert and Woodward:

$B_{17} = 5^{m} - 14{\cdot}3^{\mu} \pm 1{\cdot}3^{\mu}$ for A-end left.
$B_{17} = 5 - 18{\cdot}0 \pm 1{\cdot}3$ for A-end right.
$B_{17} = 5 - 16{\cdot}2 \pm 1{\cdot}1$ for mean.

(*24*) *Concluding remarks.*—In designing and using the iced-bar apparatus many questions not referred to in the preceding pages were raised and considered. Some of these have important bearings on the practical success of the apparatus, and most of them will naturally arise in the mind of the critical reader. It is proper, therefore, to devote some attention to them here.

The most obvious and important of these questions is whether the bar assumes the temperature of melting ice when fully packed in it. No absolutely crucial mode of answering this question is known to me, but there appears to be no reason to suppose that it takes a materially different temperature. Repeated observations on mercurial thermometers placed in the ice alongside of the bar show that they read zero C. within the unavoidable errors of a few hundredths of a degree. That the bar assumes a fixed length within very narrow limits is, it would seem, demonstrated by the small range amongst the determinations of its length on the new comparator, and the equally small range amongst the measures made with the apparatus on the 100^{m} comparator and the standard kilometre of the Holton Base. Considering these ranges, and the fact that the bar's expansion is about 55$^{\mu}$ per degree C., there is but very small margin for variation in the bar's length. An independent and perhaps more conclusive answer to this question is afforded by the results of the two most important groups of determinations of the length of the bar, namely, the group of February to March, 1892, and that of July and August, 1892. During the period covered by the first group the air temperature about the Y-trough varied from 5° to 10° C., while for the latter group the air temperature varied from 25° to 40° C. If the bar is affected to any appreciable extent by the external air temperature it ought to have been longer at the time of the second group of comparisons than at the time of the first, since the average air temperature for the second group was higher by about 30° C. than for the first group. The mean lengths for those groups differ, however, by only $1{\cdot}2^{\mu}$, the indicated length of the bar being so much shorter at the higher than at the lower temperature. My conclusion, then, is that the bar assumes a sensibly constant temperature when properly iced in the Y-trough.

The time required by the bar to reach a stable length is less than ten minutes. The rate of temperature change is so great in the early stages of freezing that 90 per cent or more of the contraction of the bar occurs within a minute after it is well surrounded by ice. The corresponding time required by the prototype metre to reach a stable length appears to be less than five minutes, which is less than the time required to properly pack it in ice. In the use of the apparatus, however, the bar and metre have been, in general, fully packed in ice from half an hour to an hour before observing on them.

A question which received attention in designing the apparatus is whether the friction of the bar on its supports in the Y-trough may not affect its length appreciably. Since the cross section of the bar is 2·56 square centimetres, and the modulus of elasticity of steel about 2×10^6 kilogrammes per square centimetre, it would require a longitudinal stress of 1 kilogramme to change the bar's length 1 micron if the bar were free to respond to such stress. From this calculation and from a consideration of the actual circumstances under which the bar is used, it was concluded that this friction would produce no appreciable effect. Some direct experiments to test this point were made, however, during the course of the comparisons with the Prototype Metre of February and March, 1892. In these experiments the Y-trough was subjected alternately in quick succession to tension and compression while readings were made on the bar. Several such trials were made without disclosing any change in the bar's length. It is, indeed, difficult to see how any frictional stress could manifest itself, especially when the apparatus is trundled along on its cars as in measuring a line or as used on the new comparator, since the jarring incident to such motion ought to relieve the bar from all but the smallest stress.

A query may also arise as to the effect on the bar's temperature of the exposed graduated surfaces. It might be inferred that the bar is perceptibly hotter by reason of these exposed parts, small though they are, than it would be if entirely covered with ice. But a study of the physical relations here presented does not justify such an inference. The reason is that the conductivity of the steel is so much higher than its emissivity in air that the exposed parts of the bar cannot differ in temperature by more than a few thousandths of a degree from the unexposed parts.* Whatever this excess may be, it can affect only short portions of the bar, and hence produce only vanishing changes of length. The same conclusion, it would appear, applies with greater certainty to the Prototype Metre. A simple and direct experiment bearing on this question has also been frequently tried. It consisted in microscope readings on the ends of the bar just before and just after pressing the bare finger for a period of ten to thirty seconds against one of the graduation plugs. If the heat thus conducted from the hand to the bar is not speedily dissipated, the bar should have been measurably lengthened thereby. No such change, however, was observed.

An obstacle which sometimes proves troublesome in the use of the apparatus, is the condensation of moisture on the graduated surfaces of the bar. The most effective means found for removing this is a dry camel's-hair brush. Such brushes were used in the long series of comparisons of the bar and Prototype Metre in July and August, 1892, and no material delay from moisture occurred.

Finally, experience with the apparatus leads me to recommend a bar having a Y-shaped rather than a rectangular cross section. Such a bar would possess the requisite rigidity with small mass per unit length. It could be subdivided into metre spaces by marks on its neutral surface, and its entire length determined by measuring those spaces separately as well as by the method dependent on the use of six microscopes. Its fiducial lines should be of equal width within a micron or two, so that personal equation would be certain of elimination by exchange of positions of the observers for either orientation of the bar. Provision might also be made for protecting the graduated surfaces from direct contact with the ice. For this purpose hard rubber tubes might be fitted around those surfaces, and if desirable they could be made long enough to extend up through the ice pack. Such tubes would prevent abrasion of the polished surfaces by grit in the ice, and would also protect them to some extent from currents of air, which, if laden with moisture, produce troublesome condensation. The length in melting ice of a bar possessing this degree of refinement could be readily determined by the method followed on the new comparator. It would appear, in fact, from our experience in the use of that comparator that the length of such a bar in terms of a prototype metre would be known with a probable error not exceeding 1 micron from the mean of four measures, which might be made in one day, or in two days at most.

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*This conclusion is based on an investigation made by the author in 1888, namely, On the diffusion of heat in homogeneous rectangular masses, with a special reference to bars used as standards of length. Annals of Mathematics, vol. 4, pp. 101–127. It appears from this investigation, which is founded on Fourier's methods, that a bar of small cross section (less than $4^{cm} \times 4^{cm}$, say), cooling or heating in air from an initial uniform temperature, maintains a sensibly uniform temperature throughout its mass during the entire process of cooling or heating. That is, the difference between the temperatures within and at the surface of the mass is very small when the cross section is small.

CHAPTER III.

MEASUREMENTS MADE WITH ICED BAR APPARATUS AT HOLTON BASE IN 1891.

(1) *Plan of operations with apparatus in 1891.*—The plan submitted by me to the Superintendent of the Survey for the use of this apparatus on the Holton Base of the transcontinental triangulation in Indiana, contained the following recommendations which were approved and carried out during the summer of 1891: (*a*) To construct a 100-metre comparator near the Holton Base; to standardize this comparator by repeated measurements with the iced bar, and to use this comparator in turn to standardize and study the behavior of 100^m tapes or those of less length, or any other form of base apparatus; (*b*) to use the iced bar in addition to make several measures of a kilometre at least of the base line, so that the efficiency of the different forms of apparatus used in measuring the whole base could be tested on the actual ground over which they were applied.

Before giving the results of the measures made with the iced bar it is proper to give a brief description of the long comparator and of the kilometre whereon the apparatus was used.

(2) *The 100-metre comparator.*—The 100-metre comparator of the Holton Base was a line 100^m long fitted for measurement with the iced bar apparatus. Twenty-one beech-wood microscope posts, $1{\cdot}8^m$ long and $15^{cm} \times 15^{cm}$ in cross section, were set firmly in the ground, 5^m apart, on a level plat near the north end of the base. Alongside of the posts a stationary railway track was laid, the support posts of which were half way between the microscope posts. The ends of the line were marked by brass, spherical-headed bolts cemented into the upper ends of stone posts, which latter were well set in beds of concrete. The comparator was covered by a shed 110^m long by 3^m wide. Its length extended nearly east and west. It was covered at the ends and on the south side as well as overhead, but the north side was left open in order to permit free access of daylight and air. (See illustration No. 31.)

This comparator was built by Assistant A. T. Mosman after plans drawn up by Mr. Siebert. It answered its purpose very satisfactorily. An efficient auxiliary applied by Assistant Mosman was a sawdust covering to the ground along the comparator. This covering absorbed the dust and moisture, and prevented the transmission of disturbances through the ground to the microscope posts. The stability of these posts may be inferred from the measures of the comparator interval given below.

Iced Bar Measures at Holton Base.

(3) *The standard kilometre.*—A nearly level portion, one kilometre in length, of the Holton Base was selected by Assistant Mosman for measurement with the iced bar aparatus. The base line, whose entire length is 5·5 kilometres, runs in a nearly north and south direction across the Crawfish Flats of Southern Indiana. The portion selected for the iced bar measures passes for 600^m of its length through a dense forest growth, leaving about 200^m at either end in open fields. The whole kilometre is on low ground and the part within the forest is, in a wet season, subject to partial inundation. The soil along the kilometre is a stiff clay, which is very firm when dry, but which assumes a jelly-like mobility and elasticity when saturated with water.

The way through the forest was cleared, and the end stones of the kilometre were set under the direction of Assistant Mosman during May and June, 1891. During the latter half of the following August and early September the microscope and track posts were set along the line. Owing to frequent and heavy rains this was a tedious operation. Many of the posts were set in the water which filled the post holes as fast as they were dug. It is impossible, therefore, to present any statistics as to the speed with which this work can be done under usually favorable circumstances. It may be remarked, however, that it is a work which requires but little skilled labor. In addition, it should be said that the microscope posts were set with considerable precision. Accurate spacing of the posts to 5^m apart was secured by means of 100^m and shorter steel tapes, while the posts were aligned by means of a theodolite. The probable error in position of a post face with respect to the kilometre line does not, I think, exceed $\pm$ 3^{mm}, while the probable error of the reference line fixed on the posts, as explained in section 9, Chapter I, does not exceed $\pm$ 1^{mm}.

The bolts marking the termini of the kilometre were cemented in the end stones by Assistant Mosman early in August, after their proper relative positions had been determined by Assistant O. H. Tittmann with the Survey secondary apparatus. Intermediate stones dividing the kilometre into four nearly equal sections were set on September 7, 1891. Each of them consisted of a half cubic metre of concrete set in the ground so that its upper surface was about even with the ground surface. On the top of each stone was cemented one of the Repsold cut-off plates, which are provided with spherical headed bolts for use with the cut-off cylinder previously described.

When these intermediate stones were set, the ground along the line was so wet that it was a matter of difficulty to keep the water out of the excavations while the concrete was being rammed into place. These stones did not become dry and hard until deep trenches were dug about them on September 18, 1891. For this reason it is probable that these stones were much less stable during the first two measures of the kilometre (September 10, 15) than during the last two measures, (September 26, 30).

(4) *Results of measures of comparator interval.*—The distance between the terminal spheres of the 100^m comparator was measured nineteen times. The first nine measures were made between July 30 and August 7, 1891, and the remainder between September 24 and October 6. A measure as here designated means a determination of the comparator interval without interchange of the observers in reading on the ends of the bar. Such interchange of observers took place in all cases except for the first five measures, during which direct observations for personal equation were made on the bar. These observations proceeded on the assumption that the relative equation of the observers was the same at both ends of the bar. But, as shown in section 20 of Chapter II, this assumption is not correct. It becomes essential, therefore, to recur to the investigation given in the section just referred to in order to understand how the personal equation affects the measures in question.

In all measures of the comparator interval the A end of B_{17} was right, and the observers were S (Siebert) and W (Woodward). When a measure proceeded from left to right (or west to east) S always observed at the right or A end of the bar and W at the other end, if they did not exchange positions. When they did exchange positions, S always observed first on the A end and W at the B end of the bar. When a measure proceeded from right to left, S always observed first at the B end of the bar and W at the A end.

In conformity, then, with the notation of section 20, Chapter II, let L_1, L_2, R_1, R_2, denote the left and right hand readings respectively of a pair of microscopes for any bar length; and let S_a, S_b, W_a, W_b denote the corrections to these readings for personal equations of the observers at the A end and B end of the bar, respectively. Then, supposing the measure to proceed from left to right, the distance between the pair of microscopes considered is

$$B_{17} + L_1 - R_1 - (S_a - W_b), \tag{a}$$

for the first position of the observers, and

$$B_{17} + L_2 - R_2 + (S_b - W_a), \tag{b}$$

for the second position.

Now the derived length of B_{17} for A end right corresponds to the mean of these two expressions (a) and (b); but when, as in the case of the first five measures of the comparator interval, the expression (a) only was observed, the use of the derived value of B_{17} gives a distance too large by $(S_a - W_b)$ for each bar length. Similarly, the distance in the case of the use of expression (b) only is too small by $(S_b - W_a)$ per bar length. It is necessary, therefore to treat the first group of measures of the comparator interval in such a way as to take due account of the inequality of the numbers of measures in the two cases denoted by the expressions (a) and (b).

Let D_1 denote the most probable value of the apparent distance between the terminal spheres when the expression (a) is used, and D_2 the corresponding value when the expression (b) is used. Then the resulting observation equations will be of the forms

$$\begin{aligned} D_1 - 20\,B_{17} - Q' &= v', \\ D_2 - 20\,B_{17} - Q'' &= v'', \end{aligned} \tag{1}$$

wherein Q' and Q'' are the quantities measured with the microscopes and cut-off cylinders, and v' and v'' are their most probable corrections. If, on the other hand, D denotes the most probable value of the actual interval between the terminal spheres, we shall have

$$D = \tfrac{1}{2}(D_1 + D_2). \tag{2}$$

In the case of the second group of measures, the observers exchanged positions in every case, so that we might use the values resulting from the means of the expressions (a) and (b); but inasmuch as it is desirable to get an idea of the constancy of the personal equation effect, we shall treat the second group in the same manner as the first; that is, by means of equations (1) and (2).

The greater part of the measures were made by moving the bar from west to east (or left to right) along the comparator. Some of the later measures were made in the opposite direction, and they disclose, apparently, a source of systematic though minute error, depending on the direction of measure. The same sort of error is indicated also by the measures of the standard kilometre given in detail below. We shall exhibit the data, then, in a way to disclose these effects as well as those arising from personal equation.

Table I following gives all of the data observed for the length of the comparator except the special observations for personal equation alluded to above and referred to again below. The first column gives the date of the observations. The second column gives the direction in which the measure proceeded; W to E, for example, signifying that the measure started from the west end of the comparator and proceeded towards the east end, or proceeded from left to right. The third column gives the observation equations in the forms (1) above.

TABLE I.—*Data for length of 100$^{\text{m}}$ comparator.*

Date.	Direction of measure.	Observation equation.
1891.		*mm.* *mm.*
July 30	W to E	$D_1 - 20\,B_{17} - 39{\cdot}532 = +\,0{\cdot}055$
30	W to E	$D_1 - 20 \quad - \quad {\cdot}574 = +\ {\cdot}013$
30	W to E	$D_1 - 20 \quad - \quad {\cdot}660 = -\ {\cdot}073$
31	W to E	$D_1 - 20 \quad - \quad {\cdot}548 = +\ {\cdot}039$
Aug. 3	W to E	$D_1 - 20 \quad - \quad {\cdot}643 = -\ {\cdot}056$
4	W to E	$D_1 - 20 \quad - \quad {\cdot}624 = -\ {\cdot}037$
4	W to E	$D_2 - 20 \quad - \quad {\cdot}476 = -\ {\cdot}026$
7	W to E	$D_1 - 20 \quad - \quad {\cdot}528 = +\ {\cdot}059$
7	W to E	$D_2 - 20 \quad - \quad {\cdot}424 = +\ {\cdot}026$
Sept. 24	W to E	$D_1 - 20\,B_{17} - 39{\cdot}272 = +\,0{\cdot}134$
24	W to E	$D_2 - 20 \quad - \quad {\cdot}191 = +\ {\cdot}149$
Oct. 2	W to E	$D_1 - 20 \quad - \quad {\cdot}394 = +\ {\cdot}012$
2	W to E	$D_2 - 20 \quad - \quad {\cdot}320 = +\ {\cdot}020$
2	E to W	$D_1 - 20 \quad - \quad {\cdot}432 = -\ {\cdot}026$
2	E to W	$D_2 - 20 \quad - \quad {\cdot}369 = -\ {\cdot}029$
6	W to E	$D_1 - 20 \quad - \quad {\cdot}422 = -\ {\cdot}016$
6	W to E	$D_2 - 20 \quad - \quad {\cdot}370 = -\ {\cdot}030$
6	E to W	$D_1 - 20 \quad - \quad {\cdot}512 = -\ {\cdot}106$
6	E to W	$D_2 - 20 \quad - \quad {\cdot}449 = -\ {\cdot}109$

(5) *Derivation of lengths of Comparator.*—An inspection of the results in the above table indicates that the length of the comparator interval changed during the period of about two months which elapsed between the determinations of the two groups. That such a change might occur seems not improbable, since the stones carrying the terminal spheres rose to a level with the ground surface, and since a marked change in the amount of moisture in the ground took place during the period. The results of the second group of determinations indicate that such a change was taking place at the time they were made. Additional evidence that such changes are likely to occur is afforded by the measures of the sections of the standard kilometre, as explained below. It will be assumed, therefore, that the values of D for the two groups should differ, and the observed quantities will be treated accordingly.

Attributing equal weights to the observation equations, we have from the first group

$$\begin{aligned} & \qquad\qquad\quad \textit{mm.} \\ D_1 &= 20B_{17} + 39{\cdot}587 \\ D_2 &= 20B_{17} + 39{\cdot}450 \end{aligned}$$

The sum of the squares of the residuals, using the millimetre as unit, is 0·0193. The number of equations is 9 and the number of unknowns 2. Hence the probable error of a single equation is ± 0·036$^{\text{mm}}$, and the apparent lengths D_1 and D_2 have probable errors of ± 0·014$^{\text{mm}}$ and ± 0·026$^{\text{mm}}$, respectively.

But the objective quantity is $D = \frac{1}{2}(D_1 + D_2)$. Using the above numerical values and observing that the probable error of D is

$$\pm\, 0{\cdot}026^{\text{mm}}\sqrt{\frac{9}{56}},$$

we have from the first group of observations—

$$D = 20B_{17} + 39{\cdot}518^{\mathrm{mm}} \pm 0{\cdot}010^{\mathrm{mm}}.$$

From the second group of measures we find

$$\begin{aligned} & \qquad\qquad\quad mm. \\ D_1 &= 20B_{17} + 39{\cdot}406 \\ D_2 &= 20B_{17} + 39{\cdot}340 \end{aligned}$$

The sum of the squares of the residuals is 0·066 and the probable error of a single equation is $\pm\ 0{\cdot}061^{\mathrm{mm}}$. Hence we find for the length of the comparator from the second group of measures

$$D = 20B_{17} + 39{\cdot}373^{\mathrm{mm}} \pm 0{\cdot}019^{\mathrm{mm}}.$$

(6) *Discussion of results of comparator measures.*—The two values of D just derived differ by $0{\cdot}145^{\mathrm{mm}}$, a quantity 14·5 times the probable error of the first value of D and 5·4 times the probable error of the second value of D. It seems probable, therefore, that, whatever may be the cause of this difference, it is a real quantity, although it is but 1/689 000th part of the comparator interval.

I am unable to explain the greater magnitudes of the residuals in the second group of comparisons over those in the first group. It may be observed, however, that the hypothesis of a uniform increase of the interval D_1 or D_2, from September 24 to October 6, satisfies the observed values very closely and renders them quite as accordant as the first group of values. So far as skill in handling the apparatus could affect the measures, those of the latter group should be the more precise, since in the interval between the two groups the observers had the benefit of a large amount of experience with the apparatus on the standard kilometre. Judging from our experience on that kilometre during the second set of measures of it, when the conditions were about equally favorable to those presented on the comparator, it would seem impossible for the normal errors of measurement to produce a range as great as that shown in the second group of comparator measures.

The length of the comparator interval was used in turn to standardize the 100^{m} steel tapes, as explained in Chapter IV, and also to standardize the bars of the secondary apparatus. To get working lengths of the comparator for these purposes, it will be assumed that the length changed uniformly with the time which elapsed between the two groups of measures. Fortunately the greater part of the determinations of the tape lengths was made at about the times of the groups of measures of the comparator interval.

If we take the difference between D_1 and D_2 for the two groups of comparison, we find

μ
+ 137 for the first group,
+ 66 for the second group.

These quantities represent twenty times the function $(S_a - W_b) + (S_b - W_a)$ of the personal equations of the observers, as shown by the expressions (a) and (b) of section 4 above. This function is also the same as v' of section 20, Chapter II. Hence we have

$v' = +6{\cdot}8^\mu$ for the first group,
$v' = +3{\cdot}3$ for the second group.

The first of these values is less trustworthy than the second, since the latter is independent of all sources of error save that of the character of personal equation, while the former involves other sources of error. If we use the observations of August 4 and 7 only in the first group, the resulting value of v' is $+3{\cdot}8^\mu$, which does not differ materially from the value coming from the second group.

Had we assumed the values of $(S_a - W_b)$ and $(S_b - W_a)$ equal, as was done in a preliminary account of this work published in the American Journal of Science for January, 1893, the resulting value of D for the second group would have been identical with that derived above, while that for the first group would have differed from the value adopted by only 4^μ.

The direct observations for personal equation referred to in section 4, and published in the number of the American Journal of Science just cited, were made on July 30, 31, August 3, 4, 1891. They give $2{\cdot}4^\mu$ for the value of $(S_a - W_b)$ or $(S_b - W_a)$, supposing these to be equal, and $4{\cdot}8^\mu$ for the quantity corresponding to v'.

It appears, then, that the relative personal equation of the observers would have been an appreciable source of error in this work had not care been taken to eliminate its effect.

Finally, attention may be called to the indication of systematic error, due to the direction of measurement, furnished by the measures of October 2 and 6, 1891. Referring to Table I, it is seen that on

mm.
Oct. 2, $D = 20B_{17} + 39{\cdot}357$ for direction W. to E.
$D = 20B_{17} + 39{\cdot}400$ for direction E. to W.
Oct. 6, $D = 20B_{17} + 39{\cdot}396$ for direction W. to E.
$D = 20B_{17} + 39{\cdot}480$ for direction E. to W.

It thus appears that the measure from E. to W. exceeded that from W. to E. on October 2 by $0{\cdot}043^{mm}$ and on October 6 by $0{\cdot}084^{mm}$, and in the mean by $0{\cdot}064^{mm}$. This amounts to $1{\cdot}6^\mu$ per bar length relatively to the mean of a forward and backward measure.

Taken by itself this evidence would not appear to justify the assumption of the existence of such systematic error; but since a similar discrepancy is shown by the measures of the kilometre (as explained in sections 9 and 10 below), it is important to note that the circumstances attending the comparator measures of October 2 and 6 were closely comparable with those attending the last set of measures of the kilometre. The extent of the error in the latter measures is somewhat less than that just indicated, but there seems to be little doubt of its reality. A probable explanation of the source of this error in the kilometre measures and in the comparator measures of October 2 and 6 is given in section 10 below, but some doubt is entertained as to its application to all the comparator measures. If the explanation referred to is correct and applicable to the comparator measures, the computed values of D may need a correction of about $0{\cdot}03^{\mathrm{mm}}$; but since we can not establish the validity or magnitude of such correction, the only way to allow for it is to increase the assigned probable errors of the values of D. This will be done in the following section.

(7) *Adopted lengths for the 100^m comparator.*—Since we have assumed that the length of the 100^{m} comparator changed during the period which elapsed between the two groups of determinations, it is proper to assign an epoch to each derived length given in section 5. If we attribute weights to the several dates of observation proportional to the number of measures made on each, these epochs are August 3 and October 2, 1891. The derived lengths and their epochs are then

$$\begin{aligned} & \qquad\quad \textit{mm.} \qquad\quad \textit{mm.} \\ D &= 20B_{17} + 39{\cdot}518 \pm 0{\cdot}010 \text{ for August 3, 1891.} \\ D &= 20B_{17} + 39{\cdot}373 \pm 0{\cdot}019 \text{ for October 2, 1891.} \end{aligned}$$

The probable errors attached to these values do not include the probable error of B_{17}. The latter error for A-end right, as given in section 23, Chapter II, is $\pm 1{\cdot}3^{\mu}$. To allow for the possible source of error mentioned at the close of the preceding section, it is considered sufficient to increase the average of the probable errors in the above equations, or $\pm 0.015^{\mathrm{mm}}$, by one-third its amount. This will make the probable error dependent on errors of measurement $\pm 0{\cdot}020^{\mathrm{mm}}$ for each value of D. Combining this with $\pm 1{\cdot}3^{\mu} \times 20$, we have $\pm 0{\cdot}033^{\mathrm{mm}}$ as the probable error of either value of D when expressed in terms of the International Metre. This is equivalent to the 1/3 000 000th part of the length of the comparator.

The values we adopt, then, for the length of the comparator in terms of the International Metre, are

$$\begin{aligned} & \qquad\quad \textit{mm.} \qquad\quad \textit{mm.} \\ D &= 20\,B_{17} + 39{\cdot}518 \pm 0{\cdot}033 \text{ for August 3, 1891,} \\ D &= 20\,B_{17} + 39{\cdot}373 \pm 0{\cdot}033 \text{ for October 2, 1891.} \\ B_{17} &= 5^{\mathrm{m}} - 18{\cdot}0^{\mu}. \end{aligned}$$

(8) *Results of measures of standard kilometre.*—The method followed in making the measures of the kilometre of Holton Base has been briefly explained in section 10 of Chapter I. The specimen record given in section 11 of the same chapter serves in all essential respects to show how the records of the kilometre measures were kept. In the latter work, however, grades were determined by means of the sector attached

to the Y-trough, and the defect of alignment, instead of being made zero for each bar length, was measured with a millimetre scale.

In all measures of the kilometre sections the observers exchanged positions to eliminate personal equation, so that we have here to consider results of the form $D = \frac{1}{2}(D_1 + D_2)$ only, instead of the separate quantities D_1 and D_2, as in the case of the first group of the comparator measures.

Throughout these kilometre measures the A end of B_{17} was right as in the comparator measures.

The kilometre lies in a north and south direction, nearly, and the direction of measurement will be designated as N. to S. (north to south) or S. to N., as the case may be. The direction N. to S. corresponds to the direction left to right of our adopted convention and to W. to E. (west to east) in the designation of the direction of the measures of the 100^{m} comparator.

As explained in section 3, the kilometre was divided into four nearly equal sections by intermediate marking stones set a few days before the first measure was begun. Subsequently it was thought desirable to have a section 100^{m} long at the north end of the kilometre measured with the iced bar, for the purpose of studying the behavior of the bars of the secondary apparatus. Accordingly, a section stone was set on September 10, 1891, 100^{m} south of the north end of the kilometre.

The first set of measures of the kilometre began on September 9, 1891. Starting at the north end of the kilometre on the afternoon of this date, it was thought best to measure the first section for practice only, since the men were nearly all unaccustomed to the work, and since the portable tracks and some minor parts of the apparatus were then used for the first time. Owing to some delays but 150^{m} were measured before nightfall, when the four microscopes were left in place on their posts to mark the terminus of the day's measure. The work was continued to the end of the section on the following date. The result of this trial measure is not included in the adopted results given below, though it is given in a footnote to Table II following. Although obtained under the unfavorable conditions stated, it does not differ materially from the results for the same section obtained a few days later.

In all the kilometre measures the observers on the bar were J. S. Siebert and R. S. Woodward. Siebert in all cases adjusted the microscope at the front end of the bar, while Woodward held the trough and bar by means of the lever, as explained fully in section 10 of Chapter I. The grade-sector readings and the alignment measures were observed by John F. Hayford. The record was kept by Robert Penington, except for the last day's work, September 30, when Mr. Hayford performed this task in addition to his work of observation.

The computations entailed by the work were made by Siebert, Hayford, and Woodward in the field, and carefully revised at the office by Siebert and Woodward.

Iced Bar Measures at Holton Base.

Table II following gives the results of all the measures of the kilometre sections except that of the trial measure alluded to above. Since the several corrections resulting from these measures are large relatively to those of the measures of the 100^{m} comparator, it is considered desirable to give them in full detail. The first column gives the date of the measure. The second column designates the section measured as well as its length, 0^{m}–100^{m}, for example, denoting the section between the north terminal sphere of the kilometre and the next sphere south. The third column gives the direction in which the measure proceeded. The following four columns give in succession the micrometer, grade, alignment, and cut-off corrections. The last column gives the sum of these corrections, or the excess of the length of the section over a round number of bar lengths, which number is 20, 30, or 50, according as the section is 100^{m}, 150^{m}, or 250^{m} long.

TABLE II.—*Data for lengths of sections of standard kilometre.**

Date.	Section.	Direction of measure.	Micrometre correction.	Grade correction.	Alignment correction.	Cut-off correction.	Sum of corrections.
1891.	*m* *m*		*mm*	*mm*	*mm*	*mm*	*mm.*
Sept. 15	0–100	N. to S.	—0·478	—0·338	—0·245	— 4·814	— 5·875
15		S. to N.	—0·247	—0·328	—0·225	— 5·009	— 5·809
26		N. to S.	—0·084	—0·324	—0·145	— 4·914	— 5·467
30		S. to N.	—0·145	—0·315	—0·137	— 5·003	— 5·600
15	100–250	N. to S.	—0·125	—0·270	—0·292	+12·909	+12·222
15		S. to N.	+0·205	—0·264	—0·334	+12·722	+12·329
26		N. to S.	+0·607	—0·280	—0·222	+12·336	+12·441
30		S. to N.	+0·876	—0·258	—0·352	+12·400	+12·666
10	250–500	N. to S.	+0·086	—0·289	—0·197	— 5·030	— 5·430
15		S. to N.	—0·752	—0·302	—0·439	— 3·060	— 4·553
26		N. to S.	—0·670	—0·290	—0·214	— 4·321	— 5·495
29		S. to N.	—0·022	—0·336	—0·303	— 4·940	— 5·601
11	500–750	N. to S.	—0·354	—0·962	—0·472	+20·864	+19·076
14		S. to N.	+1·831	—1·048	—0·535	+19·118	+19·366
28		N. to S.	—0·100	—0·950	—0·376	+21·095	+19·669
29		S. to N.	+0·079	—0·982	—0·350	+21·019	+19·766
11	750–1000	N. to S.	—0·812	—5·852	—0·577	—13·323	—20·564
14		S. to N.	+0·265	—5·860	—0·461	—13·223	—19·279
28		N. to S.	—0·960	—5·812	—0·495	—14·016	—21·283
29		S. to N.	—0·364	—5·889	—0·391	—14·549	—21·193

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* The length of the section 0^{m} to 250^{m} derived from the practice measure of September 9 and 10, 1891, was 50 B_{17} + 6·45^{mm}.

(9.) *Derivation of the lengths of the kilometre sections.*—As explained in section 3 and more fully in section 10 below, the conditions which obtained during the time of the first two measures of the kilometre were very unfavorable as compared with those met during the time of the last two measures. In addition, it is probable that the section stones underwent some small though appreciable displacements during the interval which elapsed between the two sets of measures. For these reasons and others which will appear below, it is deemed best to consider the two sets of measures separately.

Since the division of the first quarter section of the kilometre into two parts by the 100^{m} stone did not materially affect the precision of the sum of those parts relatively to the precisions of the other quarter sections, we shall in the first place derive the lengths of the several quarter sections, and consider the measures of the 100^{m} section subsequently by themselves.

As remarked in the discussion of the measures of the 100^{m} comparator in section 6 above, the measures of the kilometre sections indicate a systematic difference depending on the direction of measurement. It is desirable, therefore, to exhibit the results in such a way as to clearly bring out this peculiarity. Accordingly the following statement is derived from Table II, the sums of the quantities therein being rounded to the nearest hundreth of a millimetre:

TABLE III.—*Statement of results of measures of standard kilometre.*

[First set, September 10 to 15, 1891.]

No. of measures.	Direction of measure.	Section 0^{m} to 250^{m}.	Section 250^{m} to 500^{m}.	Section 500^{m} to 750^{m}.	Section 750^{m} to $1,000^{m}$.	0^{m}–$1,000^{m}$.
		mm.	*mm.*	*mm.*	*mm.*	*mm.*
1	N. to S.	+6·35	—5·43	+19·08	—20·56	—0·56
2	S. to N.	+6·52	—4·55	+19·37	—19·28	+2·06
1—2		—0·17	—0·88	— 0·29	— 1·28	—2·62

[Second set, September 26 to 30, 1891.]

3	N. to S.	+6·97	—5·50	+19·67	—21·28	—0·14
4	S. to N.	+7·07	—5·60	+19·76	—21·19	+0·04
3—4		—0·10	+0·10	— 0·09	— 0·09	—0·18

It is seen that in seven out of the eight independent cases presented in this statement, the measure made from north to south was less than that made in the reverse direction; and it appears worth while to give an idea of the extent of this discrepancy before adducing what appears to be its cause. To this end, let s_1, s_2, s_3, s_4 denote the most probable values of the excesses of the measured lengths of the several sections over fifty bar lengths, on the assumption that the mean of a forward and backward measure of a section is free from systematic error; and let x denote the most probable value of the systematic error on the suppositions that it is proportional to the distance measured and that it changes sign with the direction of measurment. Then a pair of measures of a section will give a pair of observation equations of the form

$$\begin{aligned} s + x - n_1 &= v_1, \\ s + x - n_2 &= v_2; \end{aligned}$$

and each set of measures of the sections of the kilometre will give eight equations of this type containing five unknowns, namely, s_1, s_2, s_3, s_4, and x. On the other hand, we may entertain the simpler assumption that no systematic error is eliminated in the mean of a forward and backward measure; in which case the quantity x will disappear from the above observation equations, but the values of the quantities s will be the same as before. The validity of the quantity x will therefore depend, so far as the purely observational evidence goes, on the effect its use has in diminishing the residuals which result from the simpler supposition of no systematic error. In order to compare the two systems of residuals we shall give those resulting from the suppression of x alongside and to the right of those resulting from the use of x in the observation equations below.

Referring, then, to Table III, the first set of measures of the sections gives the following system of equations:

$$\begin{array}{rcrr} & \textit{mm.} & \textit{mm.} & \textit{mm.} \\ s_1 - x - & 6{\cdot}35 = & -\ 0{\cdot}24, & +\ 0{\cdot}09, \\ s_1 + x - & 6{\cdot}52 = & +\ 0{\cdot}25, & -\ 0{\cdot}08, \\ s_2 - x + & 5{\cdot}43 = & +\ 0{\cdot}11, & +\ 0{\cdot}44, \\ s_2 + x + & 4{\cdot}55 = & -\ 0{\cdot}11, & -\ 0{\cdot}44, \\ s_3 - x - & 19{\cdot}08 = & -\ 0{\cdot}19, & +\ 0{\cdot}14, \\ s_3 + x - & 19{\cdot}37 = & +\ 0{\cdot}18, & -\ 0{\cdot}15, \\ s_4 - x + & 20{\cdot}56 = & +\ 0{\cdot}31, & +\ 0{\cdot}64, \\ s_4 + x + & 19{\cdot}28 = & -\ 0{\cdot}31, & -\ 0{\cdot}64. \end{array}$$

These give

$$\begin{array}{rlrl} & \quad\textit{mm.} & & \quad\textit{mm.} \\ s_1 = & +\ 6{\cdot}44, & s_2 = & -\ 4{\cdot}99, \\ s_3 = & +\ 19{\cdot}22, & s_4 = & -\ 19{\cdot}92, \end{array}$$

$$\begin{array}{c} \textit{mm.} \\ x = +\ 0{\cdot}33, \end{array}$$

together with the residuals in the second members of the observation equations.

The sum of the squares of the residuals in the above system wherein x is used is 0·4050. Hence the probable error of one equation is $\pm\ 0{\cdot}25^{\text{mm}}$, that of either value of s is $\pm\ 0{\cdot}18^{\text{mm}}$, and that of x, $\pm\ 0{\cdot}09^{\text{mm}}$.

The sum of the squares of the residuals resulting from the suppression of x is 1·263. The probable error of a single equation is $\pm\ 0{\cdot}38^{\text{mm}}$, and that of either value of s, $\pm\ 0{\cdot}27^{\text{mm}}$.

The corresponding data of the second set of measures give the following system of equations and values:

$$\begin{array}{lcll}
 & \textit{mm.} & \textit{mm.} & \textit{mm.} \\
s_1 - x - & 6{\cdot}97 = & + 0{\cdot}03, & + 0{\cdot}05, \\
s_1 + x - & 7{\cdot}07 = & - 0{\cdot}03, & - 0{\cdot}05, \\
s_2 - x + & 5{\cdot}50 = & - 0{\cdot}07, & - 0{\cdot}05, \\
s_2 + x + & 5{\cdot}60 = & + 0{\cdot}07, & + 0{\cdot}05, \\
s_3 - x - & 19{\cdot}67 = & + 0{\cdot}03, & + 0{\cdot}05, \\
s_3 + x - & 19{\cdot}76 = & - 0{\cdot}02, & - 0{\cdot}04, \\
s_4 - x + & 21{\cdot}28 = & + 0{\cdot}02, & + 0{\cdot}04, \\
s_4 + x + & 21{\cdot}19 = & - 0{\cdot}03, & - 0{\cdot}05,
\end{array}$$

$$\begin{array}{ll}
\textit{mm.} & \textit{mm.} \\
s_1 = + \ 7{\cdot}02, & s_2 = - \ 5{\cdot}55, \\
s_3 = + 19{\cdot}72, & s_4 = - 21{\cdot}24,
\end{array}$$

$$\overset{\textit{mm.}}{x = + 0{\cdot}02.}$$

The sum of the squares of the residuals in the system involving x is 0·0142. Hence the probable error of one equation is ± $0{\cdot}046^{\text{mm}}$, that of either s is ± $0{\cdot}033^{\text{mm}}$, and that of x is ± $0{\cdot}016^{\text{mm}}$. Suppressing x, the sum of the squares of the residuals is 0·0182, the probable error of one equation is ± $0{\cdot}045^{\text{mm}}$ and the probable error of either s is ± $0{\cdot}032^{\text{mm}}$.

(10) *Discussion of results of kilometre measures.*—Judging from the numerical evidence as presented in the preceding section it would appear that there was an active cause of systematic error during the time of the first pair of measures of the kilometre, and that this cause, though present, produced very small effects during the time of the second pair of measures. As already stated in section 3, the ground along the kilometre at the time of the first two measures was very wet and in some places inundated, and in view of the differences in magnitude of the discrepancies (1 — 2) shown in Table III, for the several sections, it is important to consider a number of circumstances which were present during the time of the first pair of measures but absent during the time of the second pair.

Those portions of the kilometre whereon the stability of the microscopes was least, fell in the sections 250^{m} to 500^{m} and 750^{m} to $1{,}000^{\text{m}}$. Along about 150^{m} of the first of these the ground was more or less under water and very elastic, so that when a length of the portable track was carried forward considerable vibrations were transmitted through the soil to the microscope posts. Along the other section the ground was in general much drier, and hence the posts were in general much stabler. But near the middle of the section was a low stretch of about 50^{m} where the ground was wet, and where the posts were set to a less depth than elsewhere in order to avoid steep gradients; so that the microscopes had here much less than the average stability. On recurring to Table III it is seen that the differences (1 — 2) for these sections are much larger than those for the other sections, and it is difficult to avoid the inference that the lack of average stability of the microscope posts on these sections gave rise in some way to the error in question. A way which naturally suggests itself is the following:

Consider a measure of the kilometre proceeding from north to south. It was the habit of the microscope porter to stand on the south side of the microscope posts when clamping the microscopes to them. He thus turned the clamp screw which fell on the east side of any post with his right hand. If in this operation he habitually pushed the posts towards the north, they would not speedily recover their normal positions when set in wet or viscous soil. Any jarring about them, however, would tend to hasten the recovery of their normal positions. Such jarring was always caused by the movements of the operatives with the heavy car tracks and by the rolling of the cars along the tracks. Under these conditions each microscope would undergo more or less displacement towards the south *after* it was used at the front end of the bar and *before* it was used at the rear end. This would make the measured value of the distance too small. On the other hand, when the measure proceeded in the opposite direction, the porter stood on the same side of the posts as before stated in fastening the microscopes, and hence their displacements would still be towards the south. The measured value of the distance would thus be, in this case, too great.

This hypothesis explains all the discrepancies between the forward and backward measures of the kilometre as well as those of the comparator measures of October 2 and 6, to which attention is called in section 6 above; for in the measures of the comparator on the two latter dates the porter who moved and clamped the microscopes was the one who handled them in all the kilometre measures, and it is practically certain that whatever habit he followed in the latter work was adhered to in the former.*

In the light of this hypothesis the larger values of the discrepancies developed in the measures of the kilometre sections on wet and plastic ground, and the practical disappearance of such discrepancies in the second set of measures made when the ground was dry and the microscopes very stable, are easily understood.

In view of all the evidence, the explanation just given is regarded as the most plausible one assignable for the systematic error in question. But there are some other circumstances affecting the first two measures of the south half of the kilometre which are worthy of mention. In the first measure of the section 500^m to 750^m one stop was made and the measure transferred for a half hour to a cut-off plate, while some posts which had been slightly misplaced were put in proper position. In the first measure of the section 750^m to 1,000^m a stop was

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* During the earlier measures of the comparator the microscopes were handled by another porter whose habit, unfortunately, was not observed.

made by reason of a shower of rain. This delay brought us to the end of the section after nightfall, and a still further delay was caused at that end by a maladjustment of the terminal microscope post with reference to the cut off sphere. Lamps for illuminating the microscopes were at hand, so that no difficulty on this score resulted, but during the delay in the last instance dependence had to be placed on the stability of the microscopes. Although great care was taken in all these cases to avoid the introduction of error, the opportunities for its occurrence were, of course, greater than when no delay took place. During the reverse or second measure of this section the rear microscope was set forward or backward five times by means of its slide rest in order to bring the forward microscope within reach of the end of the bar. This method of shifting the microscopes was also resorted to three times during the second measure of the section 500^m to 750^m.* These displacements were measured with a millimetre scale, and it is possible, though I deem it very improbable, that some formidable error was introduced thereby. Finally, a fear was entertained that some displacement of the cut-off sphere at the 750^m stone took place during the interval of two days which elapsed between the first two measures of the south half of the kilometre. The basis of this fear lay in the fact that by accident or design on the part of some one unconnected with the work a heavy block of wood used as a cover to the stone was allowed to fall on the cap of the cut-off sphere. The cap was thus driven on the sphere so tightly that considerable force was requisite to remove it; and it is possible, though not very probable, that this double disturbance produced some displacement.

With regard to the circumstances attending the second pair of measures of the kilometre, it should be said that they were in nearly every way more favorable than those attending the first pair. The ground along the line had become dry and firm; the intermediate section stones had had time to settle and to become hard; all of the difficulties in the use of the apparatus had become familiar to the observers, and the measurements proceeded without a single unexpected stop or delay. During the course of these measures several tests were made of the stability of the microscope posts as affected by the nearness of the operatives to them. It was found that no appreciable motion of the microscopes occurred unless the operatives approached much closer to the posts than was permitted in the actual work. In a trial made on one of the least stable portions of the line, the presence of five men alongside the Y-trough produced no displacements of the microscopes at the ends of the bar so long as the man nearest to a post

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*In no other cases was this questionable method of shifting the microscopes resorted to. It should be said, in fact, that the experience gained during the first two measures of the kilometre enabled us to avoid not only this difficulty, but other difficulties which, while they may not have vitiated the work perceptibly, caused more or less delay and vexation.

was distant two decimetres. It is believed, then, that the microscope posts were very stable during these measures, although it appears not improbable that they underwent slight displacements due to the pressure applied in clamping the microscopes to them.

It remains to consider briefly the question whether we can safely assume that the terminal and section marks of the kilometre maintained constant relative positions. Assuming that the systematic error is well eliminated in the mean of a forward and backward measure, an idea of the stability of the end and section marks may be gained from Table III by computing the differences between the mean lengths of the several sections resulting from the first pair and second pair of measures. Thus we have:

Mean of first pair *minus* mean of second pair of measures for—

m. *m.* *mm.*
Section 0 to 250, = − 0·58
250 500, = + 0·56,
500 750, = − 0·50,
750 1000, = + 1·32.

These figures indicate considerable movements of the marking stones; and it seems not improbable that movements of such magnitude did actually occur, since the stones rose to the surface of the ground and the moisture in the ground varied from the extreme of saturation to the extreme of dryness during the interval of fifteen days which elapsed between the two sets of measures. It will be seen also, from these figures, that a movement towards the south of $0{\cdot}5^{\mathrm{mm}}$ of the first section stone would account for the discrepancies presented by the measures of the first two sections. This stone, set originally in ground saturated with water, is perhaps more likely to have moved than the other two. Similarly, it is inferred that the stone at 750^{m} is more likely to have moved than those adjacent to it, although its movement alone would leave appreciable outstanding discrepancies.

Additional evidence of relative motions of the marking stones is afforded by the measures of the 100^{m} section of the kilometre at its north end. Thus we have from Table II the following values for the measured excesses of that section over twenty bar lengths:

mm.
Sept. 15, 1891, N. to S., excess = − 5·88,
Sept. 15, 1891, S. to N., excess = − 5·81,
Mean excess = − 5·84.

Sept. 26, 1891, N. to S., excess = − 5·47,
Sept. 30, 1891, S. to N., excess = − 5·60,
Mean excess = − 5·53.

Mean of first pair of measures *minus* mean of second pair = $-0{\cdot}28^{\mathrm{mm}}$
This result, so far as it goes, agrees with the first of those given above in indicating a movement of the terminal stone towards the north.

From the evidence just submitted, together with that furnished by the measures of the 100^m comparator, and from an intimate knowledge of the conditions attending the work, it is impossible to doubt that we were dealing to a quite appreciable extent with actual movements of the marking stones. For this reason it would seem essential to the highest precision in measures with the iced-bar apparatus to place the marking stones well under ground and to give them plenty of time for settlement before using them.

(11) *Probable errors of the kilometre measures.*—From the data and discussion given in the preceding sections, it appears that the errors affecting the kilometre measures as expressed in terms of B_{17} are of three kinds, namely: First, error of operation of the accidental class; second, error of operation of the systematic class; and third, error arising from motions of the marking stones, which latter may be assumed to be of the accidental sort.

In the case of the first two measures of the kilometre it would seem that the systematic error surpassed by far the resultant of the other two. In the case of the last two measures the systematic error, though probably still the controlling one, was very small and to some extent concealed by the resultant of the other two. It would seem quite safe then to deduce the probable error due to the accidental errors of operation from the second set of measures on the supposition that the two other sources of error were inoperative, since the value thus reached will be too large. No such limiting value, however, can be assigned for either of the probable errors due to the other sources, since the uneliminated portion of the systematic error is entangled with the error due to stone movements.

Recurring to the second system of equations and values given in section 9, it is seen that the probable error of a single measure of a section (of 250^m length) is $\pm$ 0·045^{mm}; and this is the upper limiting value for the probable error of operation when the apparatus is used on stable ground. From this it follows that the corresponding probable error for one measure of a kilometre is $\pm$ 0·09^{mm}.

It is of interest to compare this with the value deducible from the measures of the 100^m comparator, where the conditions were essentially the same as those which obtained during the second set of measures of the kilometre. Referring to section 5, we have $\pm$ 0·036^{mm} and $\pm$ 0·061^{mm} as the probable errors of a single measure of 100^m without exchange of observers to eliminate personal equation. Taking the average of these two values, dividing it by $\sqrt{2}$ to allow for exchange of observers, and multiplying the quotient by $\sqrt{10}$, we have for the probable error of one measure of a kilometre, as inferred from the comparator work, $\pm$ 0·11^{mm}, which agrees well with the value just given above.

It is clear from these figures that, with stable microscopes and with stable section marks, the error of measurement may be rendered quite insignificant in comparison with that of the length of the measuring bar.

Iced Bar Measures at Holton Base.

(12) *Adopted lengths and probable errors of the kilometre sections and kilometre.*—In deriving final values for the lengths of the sections and for the whole kilometre, we should deal only with the means of the forward and backward measures, since in these means the systematic error is partly, if not largely, eliminated. The lengths in question should appertain to an epoch falling nearer to the epoch of the second set than to that of the first set of measures of the kilometre, since it was used from September 8 to October 8, 1891, as a standard in testing the behavior of the 100^{m} steel tapes and the secondary apparatus. Whatever lengths we derive will be affected by the changes which took place in the relative positions of the marking stones. It is believed that these changes were considerable in comparison with the actual errors of measurement affecting the mean results in either set of measures. In view of these considerations, it is deemed best to assign double weight to the mean results of the second set of measures, and compute probable errors for the weighted mean values of the sections accordingly.

In conformity with this plan, then, the mean values given in Table III, or in section 9, furnish the following system of observation equations and values, s_1, s_2, s_3, s_4, being, as in section 9, the most probable values of the excesses of the sections over $50B_{17}$:

$$\begin{array}{rcrl}
 & \mathit{mm.} & \mathit{mm.} & \text{Weights.} \\
s_1 - & 6{\cdot}44 = & +\ 0{\cdot}39, & 1, \\
s_1 - & 7{\cdot}02 = & -\ \ {\cdot}19, & 2, \\
s_2 + & 4{\cdot}99 = & -\ \ {\cdot}37, & 1, \\
s_2 + & 5{\cdot}55 = & +\ \ {\cdot}19, & 2, \\
s_3 - & 19{\cdot}22 = & +\ \ {\cdot}33, & 1, \\
s_3 - & 19{\cdot}72 = & -\ \ {\cdot}17, & 2, \\
s_4 + & 19{\cdot}92 = & -\ \ {\cdot}88, & 1, \\
s_4 + & 21{\cdot}24 = & +\ \ {\cdot}44, & 2.
\end{array}$$

$$\begin{array}{ll}
\qquad\mathit{mm.} & \qquad\mathit{mm.} \\
s_1 = +\ 6{\cdot}83, & s_2 = -\ 5{\cdot}37, \\
s_3 - +19{\cdot}55, & s_4 = -20{\cdot}80.
\end{array}$$

The sum of the weighted squares of the residuals in this system of equations is 1·76. Hence the probable error of one equation of weight 1 is $\pm 0{\cdot}45^{\text{mm}}$ and that of either s is $\pm 0{\cdot}26^{\text{mm}}$. Thus it appears that when the movements of the marking stones are included with the errors of operation, the probable error of our adopted values for the quarter sections of the kilometre amounts to the millionth part of their length.

If it were proper to infer the probable error of the whole kilometre or $(s_1+s_2+s_3+s_4)$, from the above calculation, we should find $\pm 0{\cdot}45^{\text{mm}} \times \frac{2}{3}\sqrt{3} = \pm 0{\cdot}52^{\text{mm}}$. But this is manifestly too large if the measured values of the whole length of the kilometre are free to any extent from the effects of the motions of the intermediate marking stones. The mean values of the excesses of the kilometre over $200B_{17}$

for the two pairs of measures are as seen from Table III, section 9, $+0{\cdot}75^{\mathrm{mm}}$ and $-0{\cdot}05^{\mathrm{mm}}$. Giving the second of these double weight, their weighted mean is $+0{\cdot}22^{\mathrm{mm}} \pm 0{\cdot}26^{\mathrm{mm}}$. The probable error is here only half that given above, and thus confirms the supposition that the values for the whole kilometre are not to any considerable extent affected by the movements of the intermediate marking stones.

For the whole length of the kilometre we shall adopt the value just derived. This length is

$$200B_{17} + 0{\cdot}22^{\mathrm{mm}} \pm 0{\cdot}26^{\mathrm{mm}}.$$

The values of the quarter sections of the kilometres are not needed, but the halves of the line were used in testing the secondary apparatus. Deriving the lengths of these halves and their probable errors by the method just applied in getting the entire line, we find the

$$\begin{aligned}\text{North half} &= 100B_{17} + 1{\cdot}46^{\mathrm{mm}} \pm 0{\cdot}19^{\mathrm{mm}}.\\ \text{South half} &= 100B_{17} - 1{\cdot}25 \quad \pm 0{\cdot}19.\end{aligned}$$

It remains now to combine the probable error of B_{17} with the probable errors attached to the above values in order to get their total probable errors when expressed in metres. The probable error of B_{17} for A end right, as given in Chapter II, section 23, is $\pm 1{\cdot}3^{\mu}$, and hence the probable errors of $100B_{17}$ and $200B_{17}$ are $\pm 0{\cdot}13^{\mathrm{mm}}$ and $\pm 0{\cdot}26^{\mathrm{mm}}$, respectively. Finally, therefore, we have for the lengths of the kilometre and its halves, when expressed in terms of the International Metre,

$$\begin{aligned}\text{Whole kilometre} &= 200B_{17} + 0{\cdot}22^{\mathrm{mm}} \pm 0{\cdot}37^{\mathrm{mm}},\\ \text{North half} &= 100B_{17} + 1{\cdot}46 \quad \pm 0{\cdot}23,\\ \text{South half} &= 100B_{17} - 1{\cdot}25 \quad \pm 0{\cdot}23,\\ B_{17} &= 5^{\mathrm{m}} - 18{\cdot}0^{\mu}.\end{aligned}$$

(13) *Adopted lengths and probable error of 100ᵐ section.*—The 100^{m} section at the north end of the kilometre was used in some experiments with the secondary bars, and it is important to adopt some lengths for it. Since the dates of use of this section did not correspond with the dates of its measurement with the iced bar, it seems best to adopt a mean length from each set of measures. Then for any date of use of the section a value for its length may be interpolated, assuming that the relative positions of the marking stones changed proportionally with the time. Thus we have from Table II, section 8,

Mean length 100^{m} section—

$$\begin{aligned}\text{September 15, 1891,} &= 20B_{17} - 5{\cdot}84^{\mathrm{mm}} \pm 0{\cdot}04^{\mathrm{mm}},\\ \text{September 28, 1891,} &= 20B_{17} - 5{\cdot}53 \quad \pm 0{\cdot}04.\end{aligned}$$

Introducing the probable error of B_{17} we have finally, for the lengths of this section when expressed in metres

$$\begin{aligned}\text{September 15, 1891, } 100^{\mathrm{m}} \text{ section} &= 20B_{17} - 5{\cdot}84^{\mathrm{mm}} \pm 0{\cdot}05^{\mathrm{mm}},\\ \text{September 15, 1891, } 100^{\mathrm{m}} \text{ section} &= 20B_{17} - 5{\cdot}53 \quad \pm 0{\cdot}05,\\ B_{17} &= 5^{\mathrm{m}} - 18{\cdot}0^{\mu}.\end{aligned}$$

CHAPTER IV.

THE METALLIC TAPE BASE APPARATUS.

A.—DESCRIPTION OF APPARATUS AND ITS USE.

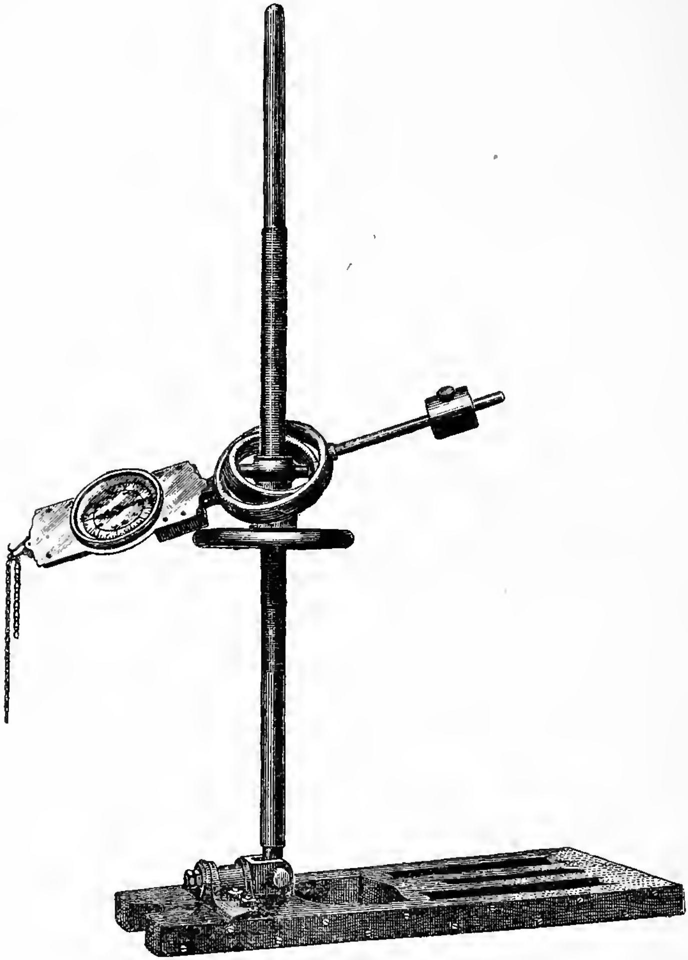
(1) *Historical note.*—The investigation of the practicability of using long metallic tapes for measuring lines of precision was taken up by the Coast and Geodetic Survey in the autumn of 1890. This work was assigned to the writer, who devised the apparatus and methods described herein.

During the winter of 1890 and 1891 a great number of experiments were made with a view to determine the best method of stretching and aligning a tape, and especially with a view to discovering the most effective way of getting the tape's temperature. For this purpose a 100^{m} comparator was set up in the Botanical Garden at Washington, D. C. This comparator consisted simply of a series of equidistant stakes, 10^{m} apart, bearing support nails for the tape, and a post at either end, on which were attached scales for defining the tape's position at any time. In addition, a large number of laboratory experiments were made with various thermometers to determine their sensitivities under differing conditions. This preliminary work served to fix attention on most of the points essential to the use of tapes in precise work.

During the spring of 1891 the apparatus as herein described was constructed by D. Ballauf of Washington, D. C., and by the Instrument Division of the Survey, and in July following, the apparatus was taken to Holton Base, where the long tapes were standardized on the 100^{m} comparator described in section 2 of Chapter III. The tapes were in turn used to measure Holton Base during the same summer. The five sections of this base (of $5\ 500^{m}$) were measured from six to thirty times each under various circumstances as to temperature, time of day, etc., and the behavior of the tapes was well shown by the measures of the kilometre section which was standardized by means of the iced bar apparatus as explained in Chapter III. In the following year, namely, October, 1892, another base (of $3\ 900^{m}$) at St. Albans, W. Va., was measured.

Mercurial thermometers were used in all these measures to determine the temperature of the tape. An attempt was made in 1891 to use a bronze tape in connection with the steel ones for temperature indications, but the bronze tape was found to have too low a rate of expansion for successful competition with mercurial thermometers. The feasibility of stretching two tapes simultaneously was demonstrated, however, and if a suitable metal for use with steel can be found it may prove advantageous to discard the mercurial thermometers.

In all this work, save that pertaining to St. Albans base, very efficient service was rendered by Mr. J. S. Siebert, who made most of the working drawings for the construction of the apparatus and who participated equally in the large amount of observation and computation entailed.



(2) *The 100ᵐ tapes.*—These tapes are of steel. They are $101 \cdot 01^{m}$ long, and are $6 \cdot 34^{mm}$ by $0 \cdot 47^{mm}$ in cross section. They weigh 22·3 grammes per metre of length. They are subdivided into 20^{m} spaces by graduations ruled on the surface of the tape itself. Being $101 \cdot 01^{m}$ long, the end graduations fall about $0 \cdot 5^{m}$ from the tape ends, which terminate in loops formed by annealing and riveting the tape back on itself. The

surface of the tape, where it is not polished to receive the graduations, is of a dull black color. When not in use the tapes are rolled up on reels, and they may be thus transported with ease and safety.

(3) *The tape-stretchers.*—The nature of the tape-stretchers used to give tension and alignment to the tape may be best understood by a glance at the cut on page 414. They consist of a lever hinged by a universal joint to a platform on which the operator stands. This lever is made of a piece of steel tubing terminating in a hickory handle. Along the upper two-thirds of the tube is cut a screw thread on which a wheel nut plays freely. This nut gives a vertical motion to a gimbal-jointed support to which a spring balance is attached. The balance is connected with the tape by means of a short piece of sash chain. It is evident from the figure that the stretcher has all the motions necessary to enable the operator to align the tape and to give it the proper tension.

The balance, which is simply a commercial article adapted to the purpose, reads to ounces, and may be easily held to the nearest ounce by the operator.

An important item for the safety of the tape used is a breaking link, or a link which will part under a tension of about 14 kilogrammes or 30 pounds. A link of this sort is provided at each end of the tape, so that it can not be overstrained by accident.

It will be noticed that the balance carries two hooks. The extra hook, which is attached to the balance frame, was intended to carry the additional tape in case it proved advantageous to use tapes of different metals simultaneously. This plan of stretching two tapes worked very satisfactorily, the tension of the steel tape being given by one balance and that of the bronze by the other; but, for the reason stated above, this plan was followed only in making a few of the tape comparisons mentioned below. The extra hook, it may be stated, is provided with a longitudinal screw motion, so that by means of this and the links of the connecting chains the tapes may be always brought to the proper tension, whatever their temperature.

When a single tape is used, the rear end is attached to the extra hook of the rear end balance, while the desired tension is given by the balance at the front end of the tape.

An objection to this form of stretcher lies in its weight, which is about 45 pounds. Contemplated improvements will lighten it by several pounds and bring the balance to a position concentric with the lever shaft. Experience has shown, however, that an able-bodied operator can handle the stretcher successfully when measuring at the rate of 2 kilometres per hour. During the measurement of St. Albans base, in 1892, 8 kilometres were measured on one occasion between 7:30 p. m. and midnight without excessive fatigue to the operators.

(4) *The thermometers.*—The thermometers used are of the Centigrade type and are graduated on their stems to half degrees. They were

made by Green, of New York. They were originally mounted on brass scales. The preliminary experiments showed that the mass of these scales caused the thermometers to lag very considerably with respect to the tape temperature. Hence the scales were removed, and light wire loops were attached to the upper ends of the stems, so that the thermometers could be suspended by a cord and thus isolated from adjacent masses or whirled in the air when necessary. Experiments also showed these thermometers to be slightly more sensitive when the bulbs were covered with sheaths of blackened aluminum foil, though this increase, if real, is probably unimportant. During the measurement of St. Albans base, in 1892, the aluminum sheaths were replaced by steel sheaths, made by coiling and annealing very thin steel tape into lengths just long enough to slip over the bulbs and clasp the thermometer stems. These are undoubtedly superior to the aluminum sheaths in simulating the surface of the tape, but they are of somewhat greater mass than the aluminum sheaths and cause more lag of the thermometers with respect to the tape. In any case, it seems desirable to blacken the bulbs of thermometers used with steel tapes presenting a blackened surface.

A great number of experiments, made in the Botanical Garden during the winter of 1890–'91, showed that with the thermometers arranged as just described (i. e., with aluminum sheaths) they followed the tape temperature without serious lagging when observations were made under a clear sky at night.

Subsequent experience in determining the tape lengths on the 100^{m} comparator of Holton Base showed a very satisfactory accordance of the tape and thermometer temperatures, whether observations were made during the day or night. This comparator was covered, however, except on its north side, by a shed which screened the tape from the direct rays of the sun.

Sometimes three of the thermometers were used with the tape, and at other times two. When three were used, they were placed one at the middle of the tape's length and one at one-sixth that length from either end. When two were used they were generally placed at one-fourth the tape's length from its ends, respectively. For the most precise work, as in comparisons for determining the tape lengths, the three thermometers were each read twice, once just before and once just after the observations on the tape. The mean of the six thermometer readings thus made gave the tape's temperature with a probable error of about ± 0·2° C., equivalent to $\frac{1}{500000}$ part of the tape length, when observations were made at night or under the comparator shed.

(5) *Method of supporting and marking position of tape.*—When in use for measuring a line the tape is supported at equal intervals of 10^{m} or 20^{m} throughout its length. The supports found most convenient and amply sufficient are steel-wire nails driven into stakes set at the proper intervals along the line. These nails are ranged into a straight line for

any tape length by means of a small theodolite whose telescope axis may be brought to the right height at one end of that tape length.

The ends of the tape are supported by the tape stretchers, by means of which the tape is aligned over the marking stakes. The latter are stakes of scantling cut off or driven down to the proper height and capped with a small table, on which a plate of zinc is held in place by light nails. These tables and plates are rectangular, 5^{cm} by 20^{cm}, say, and the longer sides are placed parallel to the line. On the plates the position of the forward-end graduation of the tape is marked as the measure progresses by means of a sharp brad-awl held against the edge of a small try-square, which is aligned against the longer edge of the marking plate. The rear-end graduation of the tape is in turn brought into coincidence with the successive marks on the zinc plates.*

(6) *Operation of measurement.*—The marking and support stakes for the measurement of a line are most advantageously ranged out and adjusted beforehand. This is especially essential for work of the highest precision carried on at night. Assuming this to have been done, and that night work is contemplated, the operation of measurement, if carried on rapidly, will require twelve men, to wit, two observers of the tape, one at the front and one at the rear end; two observers of thermometers, who also help to carry the tape forward; one recorder; two operators of the stretchers; five men to handle lamps and to carry the tape forward.

The co-operation of the observers and other operatives is secured by means of a code of signals made with whistles held by the observers. Thus, when the observer at the rear end of the tape is ready to have the tension applied, he gives one blast of his whistle. The stretcher at the front end immediately brings his balance to the proper reading, when the observer at that end announces the fact with one blast of his whistle. With the tape under tension the rear-end observer adjusts the rear-end graduation of the tape to coincidence with the defining mark on the zinc plate at that end. When this is accomplished he gives two blasts from his whistle. On hearing these, the recorder, who stands at the middle of the tape length, immediately lifts the tape a few centimetres from the support nail near him and lets it fall back. The effect of this is to make the tape straight and relieve it from friction on the support nails. When the vibration of the tape ceases the rear observer gives one blast, which signifies that the position of the front end of the tape may be marked on the zinc plate. The mark is made by the front-end observer, who, on completing the operation, gives one blast from his whistle. Then the thermometer readings, which are observed while the mark is made, are called out to and

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* The use of such plates was introduced in 1885 by Mr. O. B. Wheeler, assistant engineer, Missouri River Commission.

repeated by the recorder. Finally, the rear observer sounds two blasts from his whistle and the tape and tape stretchers are carried forward to a new position.

The speed attained in this process has been about 2 kilometres per hour, though a single kilometre has been measured in twenty minutes.

By reason of the expansion and contraction of the tape, it is frequently desirable to set the tape back or forward by a round number of centimetres. This may be done on the zinc plates by means of a suitable pocket scale. Such displacements are recorded as "set-backs" and "set-ups." Their amount is engraved on the plates, which may be appealed to if any doubt arise concerning the record. When "set-backs" or "set-ups" are made, and when more than one measure is recorded on the plates, it is advantageous to number, orient, and date them for filing as part of the records.

B.—DETERMINATION OF EQUATIONS OF TAPES.

(7) *Apparatus for observing lengths of tapes.*—It is obviously necessary to the successful use of long tapes in measuring lines of precision to provide suitable means for observing the tape lengths at different temperatures. Such data for any tape give its length at some assumed temperature and its rate of expansion, and thus enable us to write an equation from which the length at any observed temperature can be computed. In order to meet this requirement in a way which would leave least room for doubt concerning the actual behavior of the tapes, the 100^m comparator described in section 2, Chapter III, was devised. The features essential to a study of the tapes embodied by the comparator were, 1st, that its length could be determined by means of the iced bar apparatus with a precision superior to that attainable with the tapes; 2d, that the tape could be stretched and aligned on the comparator in precisely the same way as when used in the field; and 3d, that the tape length at any time could be quickly and accurately referred to that of the comparator.

As explained in section 2, Chapter III, the ends of the comparator were marked by spherical-headed bolts, or cut-off spheres. The end marks or graduations of a tape when under comparison were observed by means of micrometer microscopes mounted over the cut-off spheres as when used in measuring the length of the comparator with the iced bar apparatus. The positions of the tape ends could thus be accurately referred to the cut-off spheres through the medium of the cut-off cylinders described in section 8, Chapter I.

The tape stretchers were mounted on the stationary car track of the comparator, and by means of their vertical adjusting screws the graduated surfaces of the tape could be easily brought to focus under the microscopes.

The tapes were supported at intervals of 10^m on wire rods attached to the alternate microscope posts of the comparator; and the method of handling the tape was in every way similar to that followed in field measurements with the tape, except that its graduation marks were observed with micrometer microscopes.

The thermometers were hung from light wooden supports attached to the microscope posts of the middle of the comparator and at 15^m from either end. The stems of the thermometers were upright and the bulbs were kept at about the same height as the tape.

(8) *Method of observation.*—The programme adhered to in all determinations of the tape lengths made on the 100^m comparator is the following:

The tape being mounted on its support nails and attached to the tape stretchers, was first put under tension, aligned, and brought to focus under the microscopes at the ends of the comparator. When this operation was completed at the east end of the comparator, the observer of that end gave one blast from his whistle. The observer at the west end answered this signal by one blast from his whistle as soon as the adjustments at his end were completed. The observers then read the thermometers nearest to them, while the recorder who stands near the middle of the comparator read the thermometer at that point. On announcing their thermometer readings to the recorder and hearing his repetition of them, the observers returned to their respective ends of the comparator and re-examined their adjustments of the tape and microscopes with respect to one another; and the tension, which was always applied at the west end of the comparator, was brought to its normal value. On completing his adjustment the observer at the east end gave one blast from his whistle. This was responded to by one blast from the whistle of the observer at the west end as soon as his adjustments were satisfactory. The observer at the east then gave two blasts from his whistle, whereupon the recorder lifted the tape a few centimetres from and let it fall back on the middle support nail. This served to relieve the tape from friction on its support nails; and as soon as the tape ceased vibrating the observer at the east end gave one blast from his whistle. This was responded to by one blast from the whistle of the other observer, and the microscopes were immediately brought to bisection on the graduation marks of the tapes. This completed, the observers exchanged signals, and proceeded at once to re-read the thermometers. Finally, after completing the record of the thermometer readings, the observers referred their respective microscopes to the cut-off spheres of the comparator.

The time required to make such a set of observations, exclusive of the cut-off measures, was two to three minutes, while the cut-off measures required about two minutes additional.

In observing on the tape it was generally found most convenient to move the whole microscope by means of the longitudinal screw of its slide rest.

The reading of the balance index was noted immediately after observing on the tape, so that a correction could be applied for any departure from the normal tension.

Tables I and II following give the records of a complete set of observations for temperature and length of 100^{m} steel tape No. 88 on the comparator.

As will be seen by a glance at Table I, the thermometers were read to tenths of a degree. In order to make such readings with ease, hand-magnifying glasses were used.

The standard tension adopted for the 100^{m} steel tapes was 25 pounds, 9 ounces (11·596 kilogrammes). This is the tension given by the balances used when they are horizontal, when they are free from index error, and when they read 25 pounds even. The departure from this even reading was observed to the nearest ounce; or what amounts to the same, the actual reading of the balance at the time of observing with the microscope on the tapes was noted.

Since it was generally easiest to move the whole microscope by means of the longitudinal screw of its slide rest in reading on the tape, the micrometer screws were usually set to the same reading at the two ends of the comparator in order to simplify the computation which requires the difference of the microscope readings.

The records of the cut-off measures are in all respects similar to those obtained in measurements with the iced bar, and fully set forth in section 11, Chapter I, to which reference may be made for an explanation of the meaning attached to the symbols in Table II.

TABLE I.—*Record of observed temperature and length of steel tape No. 88, October 4, 1891.*

Time of day.	Thermometer readings.			Balance reading.	Microscope readings.	
	No. 5598.	No. 5599.	No. 5601.		West end, *L*.	East end, *R*.
a. m.						
10·10	11·7° C.	11·7° C.	11·5° C.	25 pounds. 2 ounces.	20·00	20·00
10·11	11·7	11·9	11·6			

TABLE II.—*Record of cut-off measures corresponding to observations in Table I.*

West end—Repsold cylinder.						East end—U. S. C. and G. Survey cylinder.					
Time of day.	End of scale, east.	Graduation observed.	Microscope reading.	Level readings.		Time of day.	End of scale, east.	Graduation observed.	Microscope readings.	Level readings.	
				l.	*r*.					*l*.	*r*.
a. m.						*a. m.*					
10·12	A.	22 I.	24·00	9·0	16·0	10·12	A.	1 I.	19·61	5·0	9·5
	B.	21 I.	16·69	15·0	10·0		B.	1 I.	22·72	8·0	6·5

(9) *Length of tape in terms of comparator interval.*—Let T denote the

length of the tape under comparison at the time of any set of observations, and let D denote the comparator interval, or distance between the terminal spheres of the comparator, at the same time. Also, as in section 11, Chapter I, let

L, R = readings of the microscopes on the tape at the left and right hand ends of the comparator, respectively,
L_c, R_c = corresponding mean readings of the same microscopes on the cut-off scales,
S_l, S_r = divisions of cut-off scales observed at the respective ends of the comparator,
I_l, I_r = corresponding inclinations of cut-off cylinders,
H_l, H_r = corresponding heights of cut-off scales.

Finally, let $\Delta\tau$ be the excess in ounces of the applied tension over the normal tension of the tape.

Then

$$T = D \mp S_l \pm S_r - (L - R) + (L_c - R_c) - H_l I_l + H_r I_r - 0{\cdot}05^{\mathrm{mm}} \times \Delta\tau. \quad (1)$$

In this formula the upper signs apply to S_l and S_r when the graduation numbers appear inverted (I), and the lower ones when those numbers appear erect (E). The inclinations I_l and I_r are positive when the left-hand end of the bubble is the higher. The tension excess $\Delta\tau$ is positive when the balance reading is greater than 25 pounds.

The corrections for inclination of the cut-off cylinders $H_l I_l$, $H_r I_r$ were generally very small, and hence negligible. The factors by which the difference of the sums of the level bubble readings must be multiplied to get these corrections are, for the Repsold cylinder $1{\cdot}6^{\mu}$ and for the U. S. Coast and Geodetic Survey cylinder $6{\cdot}1^{\mu}$. These apply, of course, only to the special values of the heights H of the cylinders as used on the comparator.

The value of the coefficient of $\Delta\tau$ used in the actual computations of length was $0{\cdot}047^{\mathrm{mm}}$ as determined by experiments on the comparator in 1891. Additional determinations made in 1892 indicate that the value $0{\cdot}05^{\mathrm{mm}}$ per ounce when the normal tension is 25 pounds is more precise.

The computation of the length T by formula (1) in the case of the observations in Table I and II is then as follows:

$$\begin{array}{rcr}
 & & mm. \\
\mp S_l & = & -\ 21{\cdot}500 \\
\pm S_r & = & +\ \ 1{\cdot}000 \\
-(L - R) & = & 0{\cdot}000 \\
+(L_c - R_c) & = & -\ \ 0{\cdot}082 \\
-\ H_l I_l & = & +\ \ 0{\cdot}003 \\
+\ H_r I_r & = & -\ \ 0{\cdot}018 \\
-\ 0{\cdot}047^{\mathrm{mm}} \times \Delta\tau & = & -\ \ 0{\cdot}092 \\
\hline
\text{Sum} & = & -\ 20{\cdot}689
\end{array}$$

Hence we have

$$T = D - 20{\cdot}689^{\mathrm{mm}}.$$

The value actually used was $D - 20{\cdot}67^{mm}$, which comes from neglecting the corrections for inclination of cut-off cylinders and rounding all computed corrections to the nearest hundredth of a millimetre.

(10) *Thermometer corrections.*—The thermometers used in all determinations of tape lengths on the 100^m comparator were Nos. 5598, 5599, and 5601 by Green, of New York. Their corrections were determined by Mr. O. H. Tittmann, assistant in charge of the Office of Standard Weights and Measures of the Survey.

Table III following gives the corrections essential to reduce the observed readings of these thermometers to the hydrogen scale as determined by Assistant Tittmann on August 19, 1890.

TABLE III.—*Thermometer corrections August 19, 1890.*

Temperature.	Correction to—		
	5598.	5599.	5601.
2·6° C.	—0·20° C.	—0·19° C.	—0·21° C.
3·6	·24	·18	·16
10·2	·19	·19	·16
15	·18	·18	·18
19·6	·18	·21	·24
25	·17	·17	·17
30	·14	·22	·17
34·9	·19	·14	·29

The readings of these thermometers when packed in melted ice on August 19, 1890, were + 0·25°, + 0·08° and 0·00°, respectively. Their freezing point readings were not again determined until January 21, 1892, when they read + 0·15°, + 0·17°, and 0·00° respectively.

To take account of these changes in the freezing point readings it is assumed that they varied directly as the time during the interval which elapsed between the two determinations of August, 1890, and January, 1892. This assumption gives for the zero point readings at the epoch of the tape comparisons, September, 1891, say, +0·17°, +0·15°, and 0·00°, respectively; and these are the values adopted.

The changes which these readings entail in the corrections in Table III are found by adding +0·08° to the corrections for No. 5598, by adding –0·07° to the corrections for No. 5599, while the corrections for No. 5601 remain unchanged.

Since the thermometers always had the same readings within a degree, it has been considered sufficient to apply the mean of their corrections to the mean of their readings in computing results from comparions on the 100^m comparator. Accordingly, the mean corrections in Table IV following have been applied in deducing the temperatures of the tapes from the observations on the comparator.

Determination of equations of tapes.

TABLE IV.—*Means of corrections to thermometers September, 1891.*

Temperature	2·6°	3·6°	10·2°	15·0°	19·6°	25·0°	30·0°	34·9°
Correction	—0·20	—0·19	—0·18	—0·18	—0·20	—0·17	—0·17	—0·20

The means of the six thermometer readings were carried to hundredths in the computation, and the above corrections were then applied in getting the adopted tape temperatures, which are likewise carried to hundredths of a degree. This is to some extent a needless refinement, but its application cost little labor.

(11) *Adopted lengths of the comparator.*—As explained in section 5, Chapter III, it is assumed that the comparator interval or length D changed during the period of two months which elapsed between the two groups of determinations of that interval. Fortunately, the largest groups of tape comparisons were nearly coincident in time with those of the comparator measures. For the small number of tape comparisons made between August 30 and September 5, a length of the comparator interval is derived on the assumption that it changed directly as the time between the two epochs for which lengths are given in section 7, Chapter III. Hence the following values are adopted for the comparator interval D in terms of the iced bar B_{17}:

$$\begin{aligned} & && \text{mm.} \\ \text{Aug. 1 to 8, 1891,} \quad & D = 20B_{17} && + 39{\cdot}52, \\ \text{Aug. 30 to Sept. 5, 1891,} \quad & D = 20B_{17} && + 39{\cdot}44, \\ \text{Oct. 3 to 9, 1891,} \quad & D = 20B_{17} && + 39{\cdot}37. \end{aligned}$$

$$B_{17} = 5^{\text{m}} - 18{\cdot}0^{\mu}.$$

(12) *Correction for index error of balances.*—The spring balances used to give tension to the tapes were tested from time to time by the method explained in section 4, Chapter V. They were also tested frequently with known weights to see that they gave the tensions indicated.*

The standard tension adopted was 25 pounds 9 ounces, which is the tension given by the balances when horizontal, when they are free from index error, and when they read 25 pounds even. By reason of some accidents to the balances their index errors changed to some extent during the season. They were always accurately observed, however, and the corrections applied are without sensible error.

The correction for index error is of the same form as that for excess of tension explained in section 9 above. The adopted elongation of the tape per ounce was $0{\cdot}047^{\text{mm}}$; and the corrections actually applied to the lengths of the tapes as computed by equation (1), section 9, are the following:

Aug. 1 to Aug. 8, 1891,	0·00,
Aug. 30 to Sept. 5, 1891,	—0·16,
Oct. 3 to Oct. 9, 1891,	+0·16.

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*See Chapter V, section 4, for details of observed index corrections.

(13) *Observed data for lengths and rates of expansion of steel tapes Nos. 85 and 88.*—Tables V and VI following give the results of all observations made on the 100^m comparator to determine the equations of the steel 100^m tapes Nos. 85 and 88. The first column gives the date of observation, and the second the time when the observations were made. The third column gives the temperature of the tape corrected to the hydrogen scale as explained in section 10 above. The fourth column gives the length of the tape in terms of the comparator interval D as computed from equation (1), section 9. The fifth column gives the length of the tape in terms of the iced bar B_{17}, in accordance with the relations given in section 11. The corrections to standard tension of 25 pounds 9 ounces given in section 12 are also applied in deriving the values in this column. The last column gives the residuals which result from the equations of the tapes deduced in the following section.

TABLE V.—*Results of observations for length and rate of expansion of 100^m steel tape No. 85.*

Date.	Time of day.	Observed temperature of tape.	Observed length of tape.		Residual (computed minus observed value).
			$D-$	$20\ B_{17}+$	
1891.			*mm.*	*mm.*	*mm.*
Aug. 6	11:29 a. m.	29·02° C.	4·01	35·51	+0·12
	1:02 p. m.	30·56	1·83	37·69	— ·34
	2:06	30·07	3·37	36·15	+ ·63
	3:05	29·88	2·67	36·85	— ·28
	4:02	28·92	4·38	35·14	+ ·38
	5:00	28·01	4·94	34·58	— ·06
	9:02	19·58	14·86	24·66	+ ·63
	9:12	18·44	15·72	23·80	+ ·25
7	7:02 a. m.	20·96	12·68	26·84	— ·03
	8:02	24·53	8·68	30·84	— ·13
	9:02	26·73	6·55	32·97	+ ·15
	10:02	27·96	4·95	34·57	— ·11
	11:01	29·26	3·57	35·95	— ·06
	11:48	30·11	1·98	37·54	— ·70
	2:04 p. m.	30·21	2·24	37·28	— ·35
	7:16	31.35	1·05	38·47	— ·29
	7:32	21·51	12·48	27·04	+ ·37
	8:38	20·15	14·21	25·31	+ ·60
8	7:28 a. m.	24·66	9·09	30·43	+ ·43
	8:34	27·06	5·63	33·89	— ·41
	9:35	29·14	3·63	35·89	— ·22
	10:32	30·34	2·62	36·90	— ·17
	11:31	31·39	1·13	38·39	— ·17
	12:38 p. m.	32·12	0·29	39·23	— ·21
	1:33	31·54	0·80	38·72	— ·33
	2:34	30·32	2·75	36·77	+ ·28
	7:41	21·48	12·42	27·10	+ ·27
	8:34	20·93	12·96	26·56	+ ·21
	9:34	20·38	13·32	26·20	— ·03

Determination of equations of tapes.

TABLE V.—*Results of observations for length, etc.*—Continued.

Date.	Time of day.	Observed temperature of tape.	Observed length of tape.		Residual (computed minus observed value).
			$D-$	$20\ B_{17}+$	
1891.			*mm.*	*mm.*	*mm.*
Sept. 5	2:40 p. m.	14·88	19·08	20·20	— ·05
	3:24	14·70	19·51	19·77	+ ·18
	5:04	14·26	19·91	19·37	+ ·10
	7:35	13·88	20·02	19·26	— ·21
	9:08	13·98	20·16	19·12	+ ·04
Oct. 3	10:40 a. m.	26·73	6·28	33·25	— ·13
	11:48	27·24	5·99	33·54	+ ·14
	1:41 p. m.	27·06	6·15	33·38	+ ·10
4	4:54 p. m.	13·85	20·12	19·41	— ·39
	6:50	12·67	21·60	17·93	— ·20
	8:08	12·32	22·35	17·18	+ ·17
	8:58	11·70	22·68	16·85	— ·18
	10:20	11·38	23·35	16·18	+ ·14
5	5:00 a. m.	7·20	27·86	11·66	+ ·07
	6:04	6·66	28·34	11·19	— ·04
	6:54	7·97	26·89	12·64	— ·05
	8:04	10·24	24·82	14·71	+ ·36
	9:00	11·80	22·99	16·54	+ ·24
	12:38 p. m.	15·13	19·23	20·30	+ ·12
	7:03	7·07	27·95	11·58	+ ·02
	8:08	6·72	28·51	11·02	+ ·20
	8:58	6·38	28·82	10·71	+ ·13
	10:16	5·04	30·01	9·52	— ·14
6	4:32 a. m.	3·51	31·95	7·58	+ ·12
	5:28	3·47	31·78	7·75	— ·09
	6:28	4·76	30·52	9·01	+ ·06
	7:56	9·59	25·30	14·23	+ ·13
	8:54	11·95	22·75	16·78	+ ·16
8	7:54 a. m.	8·62	26·03	13·50	— ·20
	8:48	10·89	23·51	16·02	— ·24
	9:52	12·55	21·57	17·96	— ·36
	10:46	13·60	20·97	18·56	+ ·19
	11:52	14·63	19·55	19·98	— ·11
	12:58 p. m.	15·02	19·29	20·24	+ ·06
	2:06	15·63	18·76	20·77	+ ·20
	3:00	14·67	19·77	19·76	+ ·16
	4:06	13·87	20·76	18·77	+ ·27
	5:52	9·45	25·26	14·27	— ·07
	7:12	7·38	27·70	11·83	+ ·11
	11:16	7·25	27·49	12·04	— ·24
	11:58	7·70	27·05	12·48	— ·19
9	2:04 a. m.	5·31	29·73	9·80	— ·13
	3:04	5·49	29·48	10·05	— ·18
	4:08	5·49	29·56	9·97	— ·10
	5:06	4·19	30·77	8·76	— ·29
	6:10	4·47	30·58	8·95	— ·20
	6:56	6·68	28·15	11·38	— ·21
	8:00	9·33	25·31	14·22	— ·15

TABLE VI.—*Results of observations for length and rate of expansion of 100^{m} steel tape No. 88.*

Date.		Time of day.	Observed temperature of tape.	Observed length of tape.		Residual (computed minus observed value).
				$D-$	$20\,B_{17}+$	
1891.				*mm.*	*mm.*	*mm.*
Aug.	1	10:14 a. m.	27·46° C.	3·66	35·86	+0·42
		1:52 p. m.	29·28	0·86	38·66	— ·40
		2:50	24·96	6·58	32·94	+ ·61
		3:32	24·61	6·04	33·48	— ·31
		4:10	22·86	8·17	31·35	— ·09
		4:56	23·34	7·11	32·41	— ·63
		5:56	23·97	7·63	31·89	+ ·58
		7:25	20·46	11·35	28·17	+ ·46
	3	7:52 a. m.	14·99	16·79	22·73	— ·06
		8:29	14·46	17·69	21·83	+ ·26
		9:59	14·30	16·95	22·57	— ·65
		11:05	14·70	17·56	21·96	+ ·39
		1:32 p. m.	16·45	15·35	24·17	+ ·09
		2:08	16·95	14·73	24·79	+ ·02
	4	8:22 a. m.	20·84	10·40	29·12	— ·07
		9:02	21·66	9·70	29·82	+ ·13
		10:02	23·24	7·93	31·59	+ ·08
		10:56	24·14	6·61	32·91	— ·26
		12:02 p. m.	24·01	6·65	32·87	— ·37
		1:02	25·01	5·92	33·60	+ ·00
		4:46	24·77	6·49	33·03	+ ·31
		9:04	17·72	14·36	25·16	+ ·49
	5	7:15 a. m.	21·71	9·57	29·95	+ 0·5
		8:32	24·29	7·02	32·50	+ ·32
		9:32	25·28	6·05	33·47	+ ·43
		10:32	25·71	5·69	33·83	+ ·54
		11:32	28·01	2·44	37·08	+ ·21
		1:32 p. m.	28·01	2·53	36·99	+ ·12
		2:32	28·27	2·49	37·03	+ ·13
		3:32	27·89	2·92	36·60	+ ·15
		4:30	26·31	4·78	34·74	+ ·28
		5:32	26·27	4·64	34·88	+ ·10
		8:56	17·67	14·46	25·06	+ ·54
	30	6:46 p. m.	16·41	15·35	23·93	+ ·29
		8:14	14·26	17·79	21·49	+ ·39
		9:30	13·15	18·63	20·65	+ ·01
Sept.	5	12:08 p. m.	14·55	16·70	22·58	— ·39
		1:16	15·10	15·92	23·36	— ·57
		1:42	15·78	15·19	24·09	— ·56
		2:24	15·28	15·79	23·49	— ·50
		3:45	14·28	17·32	21·96	— ·06
		4:54	14·26	17·26	22·02	— ·15
Oct.	3	10:55 a. m.	27·08	3·22	36·31	— ·46
		11:39	26·99	3·09	36·44	— ·67
		1:49 p. m.	26·93	3·09	36·44	— ·74
	4	4:44 p. m.	13·83	17·59	21·94	— ·54
		7:02	12·60	19·37	20·16	— ·10
		8:01	12·30	19·55	19·98	— ·25
		9:10	11·53	20·61	18·92	— ·03
		10:10	11·50	20·67	18·86	— ·10

Determination of equations of tapes.

TABLE VI.—*Results of observations for length, etc.*—Continued.

Date.	Time of day.	Observed temperature of tape.	Observed length of tape.		Residual (computed minus observed value).
			$D-$	$20\,B_{17}+$	
1891.			*mm.*	*mm.*	*mm.*
Oct. 5	5:12 a. m.	7·03	25·38	14·15	— ·18
	5:56	6·46	25·87	13·66	— ·30
	7:00	8·25	24·16	15·37	— ·06
	7:56	9·97	22·63	16·90	+ ·29
	9:04	11·70	20·68	18·85	+ ·23
	12:30 p. m.	15·35	16·48	23·05	+ ·01
	7:12	7·02	25·82	13·71	+ ·25
	8:02	6·42	26·26	13·27	+ ·05
	9:08	6·54	26·11	13·42	+ ·03
	10:04	5·12	27·50	12·03	— ·13
6	4:42 a. m.	4·39	28·33	11·20	— ·10
	5:22	3·46	29·40	10·13	— ·04
	6:34	4·66	28·24	11·29	+ ·11
	7:48	8·82	23·41	16·12	— ·18
	9:02	12·70	19·52	20·01	+ ·16
8	7:50 a. m.	8·15	23·96	15·57	— ·36
	8:54	11·20	20·96	18·57	— ·04
	9:48	12·14	20·09	19·44	+ ·16
	10:52	13·85	18·29	21·24	+ ·19
	11:46	14·05	18·43	21·10	+ ·54
	1:04 p. m.	15·25	16.72	22·81	+ ·14
	2:00	16·12	15·98	23·55	+ ·35
	3:06	14·62	17·32	22·21	+ ·06
	4:02	14·22	17·80	21·73	+ ·10
	5:58	9·23	23·45	16·08	+ ·20
	7:06	7·57	25·09	14·44	+ ·13
	11:06	7·35	24·86	14·67	— ·34
9	12:08 a. m.	7·52	24·82	14·71	— ·19
	1:56	5·39	26·96	12·57	— ·38
	3:10	5·29	27·49	12·04	+ ·04
	4:04	5·67	26·97	12·56	— ·06
	5:12	4·21	28·67	10·86	+ ·04
	6:04	4·11	28·69	10·84	— ·04
	7:02	7·07	25·28	14·25	— ·22
	7:56	9·10	22·82	16·71	— ·47

(14) *Derivation of equations of tapes.*—Let t denote the observed temperature of the tape at any time, and x the excess of the tape's length over 20 B_{17} at the same time. Let x_0 denote the corresponding excess in length at any assumed temperature t_0. Then if we call the rate of expansion of the tape y, and denote the most probable corrections to x and t by Δx and Δt, respectively, each pair of observed values in Tables V and VI will give an equation of condition of the form

$$x_0 + y\,(t + \Delta t - t_0) - x - \Delta x = 0. \qquad (2)$$

If now we denote the weights of x and t by p and q, respectively, the method of least squares requires that, subject to the conditions (2),

$$[p\,(\Delta x)^2] + [q\,(\Delta t)^2] = \text{a minimum}. \qquad (3)$$

The solution of the problem presented by equations (2) and (3) is not difficult, especially when the ratio of the weights p and q is constant. The method usually followed in treating this problem is based on the assumption that $q = \infty$, or that the temperature observations are not subject to error. In the present application, it seems more fitting to make the other extreme assumption, viz, that $p = \infty$, since the temperature errors surpass by far the resultant effect of all other errors. Both these extreme assumptions were made in the computations as actually carried out, and it may be said that the resulting values of x_0 and y do not differ appreciably, so that for practical purposes one method is as good as another. Hence we shall adopt the usual method, which converts (2) into an observation equation of the form

$$x_0 + y\,(t - t_0) - x = \Delta x, \text{ with weight } p. \qquad (4)$$

Since t_0 is arbitrary, we may conveniently give it the value

$$t_0 = \frac{[pt]}{[p]}, \qquad (5)$$

since this facilitates the solution of the normal equations and the computation of the probable error of the tape length at any temperature. Thus, with this value of t_0, we have

$$x_0 = \frac{[px]}{[p]}, \quad y = \frac{[p\,(t - t_0)\,(x - x_0)]}{[p\,(t - t_0)^2]}. \qquad (6)$$

The observations in Tables V and VI would give equations of equal weight, but instead of treating these equations separately they have been grouped into means with respect to temperature limits in order to learn what extent the errors of observation are compensating. The resulting values of x_0 and y are, however, the same as would come from considering the several equations separately.

Table VII following gives the values in detail which result from the processes of grouping just mentioned as applied to the data for Tape No. 85. The first column gives the limits of the temperatures included in any group. The second column gives the number of observed temperatures or lengths in the group, or the weight p of the mean temperature or mean length for the group. The third column gives the mean temperature t and the fourth the mean length excess x for the group. The last column gives the residual which results from the substitution of the computed values x_0, y and the group means t and x in equation (4).

Table VIII gives the corresponding data for Tape No. 88.

Determination of equations of tapes.

TABLE VII.—*Summary of data for equation of tape No. 85.*

Temperature range.	Weight p.	Mean value of t.	Mean value of x.	Residual Δx. (Computed *minus* observed value.)
° °		°	*mm.*	*mm.*
30 to 35	11	30·72	37·64	—0·150
25 30	11	27·92	34·42	+ ·004
20 25	8	21·82	27·54	+ ·207
15 20	6	16·44	21·66	+ ·197
10 15	18	12 94	18·01	+ ·016
5 10	18	7·18	11·78	— ·059
0 5	5	4·08	8·41	— ·083

TABLE VIII.—*Summary of data for equation of tape No. 88.*

Temperature range.	Weight p.	Mean value of t.	Mean value of x.	Residual Δx. (Computed *minus* observed value.)
° °		°	*mm.*	*mm.*
25 to 30	15	26·89	35·66	—0·004
20 25	13	23·07	31·47	+ ·017
15 20	12	16·09	23·85	+ ·019
10 15	22	13·14	20·68	— ·030
5 10	18	7·22	14·18	+ ·009
0 5	5	4·19	10·86	+ ·022

From the data in Tables VII and VIII we find by means of equations (5) and (6)

$$\left.\begin{array}{ll} t_0 = & 16{\cdot}89^{\circ}\text{ C.;} \\ x_0 = + & 22{\cdot}35^{\text{mm}}\text{, with weight 77,} \\ y = & 1{\cdot}0947^{\text{mm}}\text{, with weight 6436,} \end{array}\right\}\text{Tape No. 85.}$$

$$\left.\begin{array}{ll} t_0 = & 15{\cdot}72^{\circ}\text{ C.,} \\ x_0 = + & 23{\cdot}47^{\text{mm}}\text{, with weight 85,} \\ y = & 1{\cdot}0914^{\text{mm}}\text{, with weight 4687,} \end{array}\right\}\text{Tape No. 88.}$$

These values substituted in equation (4) give, in connection with the mean values of t and x, the residuals shown in the last column of Tables VII and VIII. Likewise, these values of t_0, x_0, and y, when substituted in equation (4), give in connection with the individual values of t and x the residuals of the individual determinations of length shown in the last column of Tables V and VI.

The salient features of the latter residuals are shown in the following statement:

	Tape 85.	Tape 88.
Number positive residuals =	36	45
Number negative residuals =	41	40
Average residual =	0·20$^{\text{mm}}$	0·25$^{\text{mm}}$
Maximum residual =	0·70$^{\text{mm}}$	0·74$^{\text{mm}}$
Sum of squares of residuals =	4·82	8·62

From the sums of the squares of the residuals just given, it follows that the probable error of a single determination of length of these tapes on the 100$^{\text{m}}$ comparator is

mm.
$\pm$ 0·17 for tape No. 85,
$\pm$ 0·22 for tape No. 88.

Applying the weights of x_0 and y, given above, there result

	Tape 85.	Tape 88.
	mm.	*mm.*
Probable error x_0 =	$\pm$0·020	$\pm$0·024
y =	$\pm$0·0021	$\pm$0·0032

If, now, we denote the lengths of tapes No. 85 and No. 88 by T_{85} and T_{88}, respectively, the preceding data enable us to write the following equations:

$$T_{85} = 20B_{17} + 22{\cdot}35^{\text{mm}} \pm 0{\cdot}020^{\text{mm}} + (1{\cdot}0947^{\text{mm}} \pm 0{\cdot}0021^{\text{mm}})\,(t - 16{\cdot}89^{\circ}\text{ C.}) \tag{7}$$

$$T_{88} = 20B_{17} + 23{\cdot}47^{\text{mm}} \pm 0{\cdot}024^{\text{mm}} + (1{\cdot}0914^{\text{mm}} \pm 0{\cdot}0032^{\text{mm}})\,(t - 15{\cdot}72^{\circ}\text{ C.}) \tag{8}$$

(15) *Discussion of results.*—An examination of the residuals in Tables V and VI shows that, on the whole, the observed lengths are well represented by equation (4). Thus, the whole number of the residuals in the two tables is 162, and of these, as just shown above, 81 are positive and 81 negative, while the probable error of a single determination of length is $\pm$ 0·17$^{\text{mm}}$ for tape 85 and $\pm$ 0·22$^{\text{mm}}$ for tape 88. Taking the average of these two probable errors, or $\pm$ 0·20$^{\text{mm}}$, as representative of the whole group of residuals, we may say that the probable error of a single determination of the lengths of these tapes was 1/500 000th part of their length. The maximum residual is 0·74$^{\text{mm}}$, which is less than four times its probable error. Taking the values $\pm$ 0·17$^{\text{mm}}$ and $\pm$ 0·22$^{\text{mm}}$ for the probable errors in the two cases as a basis for computation, and then

combining the two groups of residuals into a single group, the following comparison of the actual with the theoretical distribution of the residuals results:

Between limits.	Actual number of residuals.	Theoretical number.
mm. *mm.*		
0·0 and 0·2	85	84
0·2 0·4	51	52
0·4 0·6	17	21
0·6 0·8	9	5

The agreement thus shown between the actual and theoretical distribution of the residuals is, in general, quite close. As is usual in such comparisons, the departure from theory is greatest in the case of the large errors.

On the whole, therefore, it appears that the errors to which these determinations of tape lengths were subject are of the compensating class. A confirmation of this view is afforded by the residuals of the group mean values shown in Tables VII and VIII. If the errors to which the individual determinations are subject were not of the compensating class we should expect large residuals to result from those group means. The residuals are largest in the case of the results for tape 85; but they reach only 0·2^{mm} at the maximum, and this for group means having relatively small weight. The degree of compensation was manifestly best for the comparisons of tape 88, since the maximum group residual shown in Table VIII is only 0·03^{mm}, while the individual residuals for the comparisons of tape 88 were, on the average, somewhat greater than those of tape 85. The average of the residuals in Tables VII and VIII, regardless of weights, is 0·06^{mm}. This is equivalent to an error of 0·06° C. in temperature of the tape and to 3/5 000 000ths of a tape length.

Notwithstanding the compensating character of the errors in question when taken as a whole, a careful examination of the residuals in Tables V and VI indicates that the determinations were subject more or less to some systematic errors. These seem to be chiefly of two kinds, namely: First, errors due to lag of the tape temperature relatively to that of the thermometers; and second, that due probably to peculiarities of radiation which produced at times somewhat persistent differences in temperature between the tape and thermometers. The lag effect is best shown, perhaps, in the observations of October 8 and 9, 1891, which were continuous for twenty-four hours except for an interval of four hours on October 8, when the apparatus was used in measuring a section of Holton Base. The amount of this lagging may have been as much as 0·2° C., though it is entangled with other errors to such an extent that one cannot draw a definite conclusion from the residuals of the tables. Examples of what I have provisionally called the "radiation effect" are afforded by the observations of September 5, October 3, and October 4, in the comparisons of tape 88. For these dates the residuals are all relatively large and of one sign. On the same dates, however, no such persistence in sign is shown by the residuals of tape 85 which was alongside and within two centimetres of tape 88 on the comparator. It seems impossible that there could have been such a difference in temperature of the tapes thus situated as would appear from the residuals just cited; but similar persistent differences between the tape and thermometer temperatures appeared at times in the measures of Holton Base, on cloudy days, especially, and there seems to be little doubt of their reality. (See section 8, Chapter v.) The only reason I am able to assign for the difference in behavior of the two tapes when lying alongside of one another on the comparator on the dates referred to, is that the lower surface of tape No. 88 had become somewhat polished, presenting a bright, rather than a dull black surface, from much use in preparing the standard kilometre and other sections of Holton Base. Otherwise the tapes were in all respects alike, having been cut from the same coil.

This radiation effect, or whatever it may be, was observed to be present in a somewhat greater degree in some experiments on the behavior of the tapes made in the open air and in sunshine on a portion of St. Albans Base in 1892. Since these experiments will be referred to in section 26, Chapter v, I may dismiss the question here with the remark that it is the only anomaly presented by the tapes in my experience, and that it is probably the only serious obstacle in the way of making the long tape a standard whose equation can be determined with a precision little short of that attained with laboratory standards.

(*16*) *Probable errors of tape lengths.*—The probable errors of the mean lengths of the tapes given in equations (7) and (8), section 14, are due to the errors of comparison only. The total probable errors of those mean lengths, when expressed in metres, depend, first, on the errors of comparison just mentioned; second, on the error in length of the comparator as measured with the iced bar B_{17}; third, on the error in length of B_{17} when expressed in terms of the International Metre, and fourth, on the thermometer errors with respect to the hydrogen scale.

For the probable error due to the first source of error we may adopt the mean of those given in equations (7) and (8) above, namely, $\pm 0{\cdot}022^{mm}$. This amounts to 1/4 550 000th part of a tape length.

The resultant probable error due to the second and third sources of error is given in section 7, Chapter III. It is $\pm$ $0{\cdot}033^{mm}$, or 1/3 000 000th part of a tape or comparator length.

An idea of the uncertainty remaining in the mean of the corrected readings of the three thermometers used may be formed by an inspection of the data for the thermometer corrections given in section 10 above. It is estimated that the probable error due to this source can not exceed ± 0·02° C. This is equivalent to $\pm\ 0{\cdot}020^{\mathrm{mm}}$ or 1/5 000 000th part of a tape length.

Taking the square root of the sum of the squares of these independent probable errors, there results $\pm\ 0{\cdot}044^{\mathrm{mm}}$ for the total probable error of the length of either tape when expressed in terms of the International Metre, and when its temperature is the mean of those observed during the comparisons. This probable error is equivalent to 1/2 270 000th part of the tape length.

The probable errors of the rates of expansion of the tapes given in equations (7) and (8) are $\pm\ 0{\cdot}0021^{\mathrm{mm}}$ and $\pm\ 0{\cdot}0032^{\mathrm{mm}}$, and for the sake of uniformity we may adopt the mean of these, or $\pm\ 0{\cdot}0026^{\mathrm{mm}}$, as a working value for either tape.

With these values of the probable error of a tape length at its mean temperature and the probable error of the rate of expansion, it follows that the probable error of a tape length at any temperature t is given by the expression

$$\pm \left\{ (0{\cdot}044^{\mathrm{mm}})^2 + (0{\cdot}0026^{\mathrm{mm}})^2\,(t-t_0)^2 \right\}^{\frac{1}{2}}, \tag{9}$$

$$\begin{aligned} \text{wherein } t_0 &= 16{\cdot}89^\circ \text{ C.} \quad \text{for tape 85,} \\ &= 15{\cdot}72 \qquad\quad \text{for tape 88.} \end{aligned}$$

The values given by expression (9) for such extreme temperatures as 0° and 30° C. are $\pm\ 0{\cdot}062^{\mathrm{mm}}$ and $\pm\ 0{\cdot}055^{\mathrm{mm}}$, respectively, for tape No 85, and $\pm\ 0{\cdot}060^{\mathrm{mm}}$ and $\pm\ 0{\cdot}058^{\mathrm{mm}}$, respectively, for tape No. 88. The square root of the average of the squares of the errors given by (9) between 0° and 30°, supposing all temperatures within these limits of equal frequency, is $\pm\ 0{\cdot}050^{\mathrm{mm}}$. Hence it is concluded that for such temperatures as are likely to be had in the use of these tapes, the probable errors of their absolute lengths will not on the average exceed $\pm\ 0{\cdot}05^{\mathrm{mm}}$, or 1/2 000 000th part of their length.

(*17*) *Practical form of equations of tapes.*—For the purposes of computation it is most convenient to combine the products of the mean temperatures by the rates of expansion in equations (7) and (8), of section 14, with the mean excesses over $20B_{17}$. Thus, making this combination, omitting the probable errors, and adding the numerical value of B_{17} as given in Chapter II, section 23, we have

$$\begin{aligned} & \qquad\qquad\quad \textit{mm.} \qquad\quad \textit{mm.} \\ T_{85} &= 20\ B_{17} + 3{\cdot}86 + 1{\cdot}0947t, \\ T_{88} &= 20\ \mathrm{B}_{17} + 6{\cdot}31 + 1{\cdot}0914t, \\ B_{17} &= 5^{\mathrm{m}} - 18{\cdot}0^{\mu}, \end{aligned}$$

Standard tension = 25 pounds 9 ounces.

In these, t is expressed in degrees of the centigrade scale, and B_{17} is the length of the steel 5^{m} bar No. 17 when in melting ice. Introducing the value of B_{17} there result, finally, for the equations of these tapes

$$
\begin{aligned}
& \qquad\ \ \textit{m.} \qquad \textit{mm.} \qquad\quad \textit{mm.} \\
T_{85} &= 100 + 3{\cdot}50 + 1{\cdot}0947t, \\
T_{88} &= 100 + 5{\cdot}95 + 1{\cdot}09\text{[illegible]}4t,
\end{aligned}
$$

Standard tension = 25 pounds 9 ounces.

CHAPTER V.

MEASUREMENT OF HOLTON AND ST. ALBANS BASES WITH THE TAPE APPARATUS.

A.—HOLTON BASE MEASUREMENT.

(*1*) *Location of base and preparation for measurement.*—Holton Base is a line in the Transcontinental Chain of Triangulation along or near the thirty-ninth parallel of latitude. It is situated near the village of Holton, Ripley County, Ind. It is about 5 500^{m} long, and extends in a nearly north and south direction across what is known as the Crawfish Flats. As will be inferred from this designation, the ground along the line is generally level, although a few rather abrupt changes in altitude are met in crossing the drainage courses.

This base was located in 1890 by Assistant Mosman, who made a preliminary measure of its length with a 50^{m} tape the same year. Early in the following summer Assistant Mosman set the terminal and section stones of the base and provided all materials requisite to the measures made with the 100^{m} tapes.

During August, 1891, the marking and support stakes for the measures with the long tapes, were set under the supervision of the writer. The support stakes were set at equidistant intervals of 10^{m} throughout the length of the line, except on the standard kilometre section, where the microscope posts for the iced-bar apparatus rendered such stakes unnecessary. The tapes were supported on steel wire nails driven to the depth of 2·5^{cm} into the stakes. Since these nails projected horizontally from the stakes for a distance of about 12^{cm}, no great precision was essential in placing the support stakes along the line. The marking stakes, which fell 100^{m} apart, were aligned with considerable precision, and the support nails were placed at closely equal intervals of 10^{m} by the method explained in section 5, Chapter IV.

(*2*) *Measurements of the sections of the base.*—Since the tape apparatus used was new and untried, except in the experiments of the Botanical Garden and in the comparisons on the 100^{m} comparator, it was essential to a considerable extent to develop the methods of operation suitable to field measurement as the work progressed. At the same time that this work was under way, the work of tape comparisons, and the preparation for and measurement of the standard kilometre with the iced-bar apparatus, were also in progress; while the employés of the party were often needed for other work conducted by Assistants Mosman and Tittmann. Hence no attempt was made to measure the whole base on any one date; but the various sections were measured from time to time as opportunities occurred. The earliest measures were made on the section 5 000^m to 5 500^m during the last days of August, 1891, and the work continued at intervals until October 8, 1891. The several sections were measured from six to thirty times each, the largest number of measures being made on the kilometre section, whose precise length was designed to give a thorough test of the effectiveness of tape apparatus.

Most of the measures of the sections were made at night, but a considerable number also were made during cloudy or partly cloudy days. Some of the latter were made under circumstances of wind, rain, and sunshine, which would now be considered quite unfavorable; but since it was desired to learn as much as practicable in the allotted time concerning the behavior of the tapes, a wide variety of circumstances was purposely encountered.

In general, the measures were made by laying each tape length once as the work proceeded along a section. In a few cases, however, the experiment was tried of laying each tape length from two to five times and reading the thermometers as many times; and in order to make these measures independent, the rear end of the tape at starting was placed successively in coincidence with marks a centimetre apart. This experiment served to give an idea of the error in laying or placing a tape length.

In all these measures but two thermometers were used, and these were read but once for each tape length. When a sufficient number of operatives were available the thermometers were carried and read by observers who did naught else but assist in carrying the tape. At other times the thermometers were carried and read by the observers who fixed the ends of the tape. In the former case, the thermometers were held at arm's length by light strings so that the bulbs were near to and at about the same height from the ground as the tape; while in the latter case the thermometers were suspended from a nail or bradawl attached to a support stake so as to fall in about the same relative position with respect to the tape as before. In both cases, the thermometers were placed at 20^m from the tape ends. The same thermometers used in the determinations of length of the tapes on the 100^m comparator were used also in the field measures, and the thermometer corrections given in section 10, Chapter IV, were applied in computing the tape temperatures for these measures.

In all measures of the sections of Holton Base, Mr. J. S. Siebert was the observer at the rear end of the tape and R. S. Woodward at the front end. During the earlier measures the temperatures were observed for the most part by Messrs. O. H. Tittmann and J. F. Hayford. The records were kept by Messrs. Hayford and Robert Penington.

(3) *Determination of grade corrections.*—The differences in height of the marking stakes or tables, essential to the computation of grade corrections, were measured in duplicate by Messrs. J. S. Siebert, Th. Gjertsen, and J. F. Hayford.

Tape measures of Holton Base.

The method of computing grade corrections from such data is very simple and easy. Thus, let h be the difference in altitude of two consecutive marking tables, or what is essentially the same thing, the difference in altitude of the two ends of the tape when placed in measurement over these tables. Let s be the length of the tape or right line distance between its terminal graduations when in place over these tables. Then if we call c the grade correction, or the quantity which must be added to s to give its horizontal projection,

$$c + s = \sqrt{s^2 - h^2},$$

whence

$$c = -\frac{h^2}{2s} - \frac{h^4}{8s^3} - \dots.$$

Since s is in general large in comparison with h, this series for c converges very rapidly. For a 100^{m} tape

$$\frac{h^4}{8s^3} < 0{\cdot}01^{\text{mm}} \text{ for } h < 2{\cdot}9^{\text{m}}.$$

Values of h as great as 3^{m} are rarely met. Hence for such length of tape, in nearly all cases,

$$c \text{ in millimetres} = 5\,(h \text{ in metres})^2.$$

The grade corrections for the sections of Holton Base were computed in the field by J. S. Siebert and R. S. Woodward, and revised subsequently, in 1892, by M. V. Safford and Woodward. Table I following gives the grade corrections for the several sections of the base. The zero end of the line is here taken as coincident with its southern extremity, or South Base. The section 3900^{m} to 4900^{m} is the standard kilometre measured with the iced bar apparatus.

TABLE I.—*Grade corrections for sections of Holton Base.*

Section. 0^{m}–1200^{m}.	Section 1200^{m}–2100^{m}.	Section 2100^{m}–3000^{m}.	Section 3000^{m}–3900^{m}.	Section 3900^{m}–4900^{m}.	Section 4900^{m}–5000^{m}.	Section 5000^{m}–5500^{m}.
mm. —25·02	*mm.* —1·64	*mm.* —5·92	*mm.* —49·83	*mm.* —5·39	*mm.* —0·84	*mm.* —8·72

(4) *Correction for index error of balances.*—If the spring balance used to give tension to the tape is adjusted to give correct tensions when in a vertical position, it will indicate (by its face readings) less than the actual tension when in a horizontal position. Such balances are liable also from wear and other causes to have an appreciable index correction.

To determine the actual tension in any case with such balances, let r be the index correction. It is the reading of the index when the balance is vertical, hook end down, and without load on the hook; it is *minus* when the index reads greater than zero and *plus* when it reads less. Let R be the reading of the index when the balance is suspended by its hook, hook end up. Let W be the total weight of the balance found by weighing it on another balance. Then if T_1 denote the observed or face reading of the balance when horizontal and T denote the corresponding actual tension—

$$\begin{aligned} T &= T_1 + r + \tfrac{1}{2}(W - R - r) \\ &= T_1 + \tfrac{1}{2}(W - R + r). \end{aligned}$$

The standard tension adopted in all the base-line measures, as well as in all the determinations of the tape lengths on the 100^{m} comparator, is 25 pounds 9 ounces, this being the value of T when the balances are horizontal, when they are free from index error, and when they read 25 pounds even. We are concerned here, then, only with the excess of the actual over the observed or nominal tension in any case; but in order to show the data for these excesses, which apply to the tape comparisons as well as to the measures of the sections of the base, we give the observed values of W, R, and r and the resulting values of $\tfrac{1}{2}(W - R + r)$ for both balances in full in Tables II and III following. The last column of these tables gives, also, the corresponding corrections to the standard length of the tape. These corrections are found by multiplying the excess of $\tfrac{1}{2}(W - R + r)$ over 9 ounces by $0{\cdot}047^{\mathrm{mm}}$, the observed elongation of the tape per ounce excess in tension when the actual tension is near to the standard value, 25 pounds 9 ounces. Later determinations of this constant indicate that it should be increased to about $0{\cdot}050^{\mathrm{mm}}$ per ounce, but the value given above was used in the actual computation. (See section 5, Supplement B.)

It should be observed that the value $+\,0{\cdot}16^{\mathrm{mm}}$ was used in the computations for the correction to the tape length when under tension of balance No. 52 for the dates August 6 to September 18, 1891, and the mean value $+\,0{\cdot}44^{\mathrm{mm}}$ for the dates September 25 to October 3. In addition, it may be remarked that the variations in the values of W and R were due partly to the occasional removal of the extra hooks from the balances, while the large change in r in the case of No. 52 was the result of an accident to that balance. It was used but little after this accident, although the computed correction due to it in this case could have given rise to no appreciable error.

TABLE II.—*Data for tensions by spring balance No. 52.*

Date.	W	R	r	$\frac{1}{2}(W-R+r)$	Resulting correction to tape length.
1891.	*Ounces.*	*Ounces.*	*Ounces.*	*Ounces.*	*Millimetres.*
Aug. 6	50·0	32·0	0·0	+ 9·0	+0·00
30	49·5	28·0	+3·5	12·5	+ ·16
Sept. 3	44·5	23·5	+3·2	12·1	+ ·15
18	45·0	23·5	+3·2	12·4	+ ·16
25	44·2	17·0	+9·8	18·5	+ ·45
Oct. 3	49·0	22·2	+9·5	18·1	+ ·43

Tape measures of Holton Base.

TABLE III.—*Data for tensions by spring balance No. 53.*

Date.	W	R	r	$\frac{1}{2}(W-R+r)$	Resulting correction to tape length.
1891.	*Ounces.*	*Ounces.*	*Ounces.*	*Ounces.*	*Millimetres.*
Aug. 6	50·0	35·5	—3·5	+5·5	—0·16
30	49·5	35·2	—3·0	5·6	— ·16
Sept. 25	49·2	35·0	—3·0	5·6	— ·16
Oct. 3	44·5	30·2	—3·2	5·6	— ·16
8	44·8	30·2	—3·5	5·6	— ·16

(5) *Specimen of record and computation.*—A form of record suitable for tape measures will readily suggest itself to the reader; but for the sake of completeness, and in order to present data for the computation of the measured value of a section, a specimen of the records actually made in the Holton Base work is given below in Table IV:

TABLE IV.—*Specimen of Record. Holton Base, September 8, 1891.*

[Measure of standard kilometre.]

No. of tape length.	Time of day.	Thermometer readings— No. 5601.	No. 5599.	Mean t.	Remarks.
		° C.	° C.	° C.	
1	6·37	13·4	13·0	13·20	Steel tape No. 88. Tension 25 pounds by balance No. 52.
2	·42	13·6	13·3	13·45	
3	·46	13·3	13·5	13·40	Index correction +3·2 ounces.
4	·50	13·0	13·2	13·10	
5	·53	12·8	13·2	13·00	Measure begins at south end of kilometre.
6	·56	12·4	12·3	12·35	
7	·59	12·4	12·6	12·50	Cloudiness 0·0.
8	7·05	12·6	12·7	12·65	Set back 100·0mm at rear end of No. 3.
9	·09	11·8	12·3	12·05	Set back 80·2mm at rear end of No. 8.
10	·12	12·1	12·5	12·30	North end of 10th tape length was 10·6mm north of terminal mark.

The computation of the length of the standard kilometre from the data given in the above table may be made as follows:

The actual or working length of the tape T, say, may be most conveniently expressed in the form

$$T = A + a + \alpha t,$$

wherein A is the nominal length of the tape, a round number of metres or lengths of some standard; a is a small excess over that round number; α is the rate of expansion of the tape, and t is the tape's temperature in degrees centigrade. Then the sum of N such lengths, t being supposed to vary from one length to another, will be

$$\Sigma T = NA + Na + \alpha \Sigma t.$$

To this sum must be applied the sum of the set-ups, the sum of the end corrections, and the grade correction for the whole section of N tape lengths.

Thus, to illustrate by means of the data in Table IV above, we have, first, from section 17 of Chapter IV, for the length of tape No. 88,

$$T_{88} = 20B_{17} + 6{\cdot}31^{\mathrm{mm}} + 1{\cdot}0914^{\mathrm{mm}}\, t$$

for the standard tension of 25 pounds 9 ounces. The tension in this case was 3·2 ounces in excess of the standard value, and hence the tape was, as explained in section 4 above, $0{\cdot}16^{\mathrm{mm}}$ longer than the standard length just given. We have, therefore,

$$\begin{aligned} A &= 20B_{17}, \\ a &= +\,6{\cdot}31^{\mathrm{mm}} + 0{\cdot}16^{\mathrm{mm}} = +\,6{\cdot}47^{\mathrm{mm}}, \\ \alpha &= 1{\cdot}0914^{\mathrm{mm}}, \\ N &= 10. \end{aligned}$$

Likewise, from Table IV, the sum of the observed temperatures is 128·00°, and the sum of the corrections to this for errors of the thermometers used is $-\,2{\cdot}12°$, as found from the table of thermometer corrections given in section 10, Chapter IV; so that there results

$$\Sigma t = 125{\cdot}88°.$$

The several terms of the computation are, then, the following:

$$\begin{array}{lll} NA & = 200B_{17}, & \\ Na & = & +\;\;64{\cdot}70^{\mathrm{mm}}, \\ \alpha\,\Sigma t & = & +137{\cdot}39, \\ \text{Sum of set-ups} & = & -180{\cdot}20, \\ \text{Sum of end corrections} & = & -\;\;10{\cdot}60, \\ \text{Grade correction} & = & -\;\;\;\;5{\cdot}39, \\ \hline \text{Resulting length} & = 200B_{17}, & +\;\;\;\;5{\cdot}90. \end{array}$$

In this particular case a correction of $-\,0{\cdot}4^{\mathrm{mm}}$ is to be added to the resulting length in order to refer it to the kilometre as measured with the iced-bar apparatus, the reference marks for the tape measures having been to that extent eccentric with respect to the terminal cut-off spheres.

The computations of the lengths of the sections of Holton Base were made in the field in 1891 by Messrs. Siebert, Hayford, and Woodward. They were revised in 1892 by Messrs. Safford and Woodward.

(6) *Results of the measures of the sections of Holton Base.*—Table V following gives all of the results obtained from the various measures of the several sections of Holton Base. The first column of the table gives the date of the measure; the second, the time of day when the measure was made; the third, the number of the tape used; the fourth, the direction in which the measure proceeded; the fifth, the mean temperature of the tape for the measure; the sixth, the temperature range for the measure, or the difference between the highest and lowest observed temperatures; the seventh indicates by the letters

F, R, V whether the temperature was falling, rising, or variable during the course of the measure; the eighth gives the degree of cloudiness, estimated on a scale of tenths; and the ninth column gives the length of the section, computed in the manner explained above, in terms of the iced bar B_{17}, the excess over a round number of tape or bar lengths having been measured with a millimetre scale. Results for length obtained from measures made before sundown are inclosed in square brackets.

TABLE V.—*Results of measures of Holton base.*

[Section south base to 1,200m stone, 240 B_{17}.]

Date.	Time of day.	No. of tape used.	Direction of measure.	Mean temperature.	Temperature range.	Temperature falling, F; rising, R; variable, V.	Cloudiness.	Length of section.
1891.				°	°			*mm.*
Sept. 1	5:55– 6:39 p. m.	88	S. to N.	17·77	4·1	F	0·0	+ 41·3
4	8:04– 8:44 a. m.	88	N. S.	13·90	0·9	F	1·0	[32·1]
4	8:52– 9:46	88	S. N.	15·44	4·9	R	1·0	[18·1]
16	7:03– 7:56 p. m.	88	N. S.	19·20	1·4	F	0·0	31·4
16	8:12– 8:59	88	S. N.	18·25	1·2	F	0·0	38·5
16	9:20–10:07	88	N. S.	18·03	1·8	F	0·0	28·2
16	10:14–11:03	88	S. N.	18·42	2·8	R, F	0·0	32·0

[Section 1,200m stone to 2,100m stone, 180 B_{17}.]

Date.	Time of day.	No. of tape used.	Direction of measure.	Mean temperature.	Temperature range.	Temperature falling, F; rising, R; variable, V.	Cloudiness.	Length of section.
Sept. 2	5:59– 6:25 p. m.	88	S. to N.	19·10	2·6	F	0·2	— 23·0
3	4:46– 5:09	88	N. S.	18·05	1·4	F	1·0	27·5
4	9:52–10:30 a. m.	88	S. N.	16·68	2·6	F	1·0	[40·5]
17	6:53– 7:25 p. m.	88	N. S.	19·31	1·2	V	0·0	27·7
17	7:44– 8:15	88	S. N.	18·20	1·0	V	0·0	27·4
17	8:24– 8:58	88	N. S.	17·82	0·9	V	0·0	29·0
17	9:07– 9:34	88	S. N.	17·15	1·6	F	0·0	26·9
17	9:40–10:10	88	N. S.	16·74	2·4	R	0·0	26·0

[Section 2,100m stone to 3,000m stone, 180 B_{17}.]

Date.	Time of day.	No. of tape used.	Direction of measure.	Mean temperature.	Temperature range.	Temperature falling, F; rising, R; variable, V.	Cloudiness.	Length of section.
Sept. 3	2:01– 2:30 p. m.	88	S. to N.	23·92	3·4	F	0·8	+[52·8]
3	4:21– 4:42	88	N. S.	19·69	0·8	F	0·9	[62.0]
4	10:38–11:09 a. m.	88	S. N.	16·16	1·1	V	1·0	[58·0]
18	7:40– 8:33 p. m.	88	N. S.	17·82	1·8	V	0·0	70·9
18	8:47– 9:23	88	S. N.	17·16	1·9	F	0·0	71·4
19	7:50– 8:17	88	N. S.	18·33	1·4	F	0·0	73·3
19	8:30– 9:06	88	S. N.	17·07	0·9	V	0·0	68·1

[Section 3,000m stone to 3,900m stone, 180 B_{17}.]

Date.	Time of day.	No. of tape used.	Direction of measure.	Mean temperature.	Temperature range.	Temperature falling, F; rising, R; variable, V.	Cloudiness.	Length of section.
Sept. 3	2:41– 3:33 p. m.	88	S. to N.	23·45	1·7	V	0·7	+ [8·0]
3	3:35– 4:12	88	N. S.	21·98	2·4	F	0·9	[15·7]
18	6:38– 7:37	88	S. N.	18·62	1·7	V	0·0	22·3
18	9:32–10:16	88	S. N.	16·51	3·0	F	0·0	18·4
19	6:48– 7:32	88	N. S.	18·56	1·3	V	0·0	20·8
19	9:14– 9:55	88	S. N.	16·25	2·2	F	0·0	19·3

TABLE V.—*Results of measures of Holton Base*—Continued.

Date.	Time of day.	No. of tape used.	Direction of measure.	Mean temperature.	Temperature range.	Temperature falling, F; rising, R; variable. V.	Cloudiness.	Length of section.
[Section 3,900$^{\text{m}}$ stone to 4,900$^{\text{m}}$ stone, or standard kilometre, 200 B_{17}.]								
1891.				°	°			*mm.*
Sept. 8	5:25– 5:54 p. m.	88	S. to N.	14·82	3·0	F, R	0·1	+ 3·3
8	5:58– 6:33	88	N. S.	13·52	1·8	V	0·0	+ 1·5
8	6:37– 7:12	88	S. N.	12·58	1·4	F	0·0	+ 5·5
23	6:27– 7:03	88	N. S.	20·72	1·4	V	0·0	— 2·4
23	7:16– 7:49	88	S. N.	19·90	1·6	V	0·0	— 3·6
23	7:56– 8:23	88	N. S.	19·43	1·6	F	0·0	— 4·3
23	8:37– 9:32	88	S. N.	18·65	1·8	V	0·0	— 1·1
30	6:33– 7:11	88	N. S.	13·94	2·4	R	0·0	+ 0·1
30	7:17– 7:55	88	S. N.	13·67	2·7	F	0·0	— 0·6
Oct. 1	6:50– 7:25	85	N. S.	18·52	0·5	F	0·0	— 2·3
1	7:32– 8:07	85	S. N.	18·07	2·8	V	0·0	+ 1.5
2	6:48– 7:18	85	N. S.	19·88	1·0	V	0·0	+ 1·4
2	7:25– 7:55	85	S. N.	19·45	1·6	R, F	0·0	+ 3·1
3	2:56– 3:18	85	N. S.	24·03	2·4	F, R	0·9	— [5·1]
3	3:26– 3:48	85	S. N.	23·62	2·4	F, R	0·9	— [4·1]
3	3:53– 4:13	85	N. S.	28·16	1·5	F	0·9	— [6·9]
3	4:17– 4:38	85	S. N.	22·66	1·1	V	0·9	— [1·3]
7	10:02–11:31 a. m.	85	N. S.	14·09	3·2	F, R	0·8	— [8·5]
				14·22	2·5			—[11·1]
				14·20	2·4			— [9·6]
				14·20	2·6			— [8·1]
				14·26	3·5			—[11·9]
8	7:44– 8:45 p. m.	88	N. S.	6·53	1·3	R, F	0·2	— 1·1
				6·54	1·6			+ 0·1
				6·55	1·6			+ 1·7
				6·51	1·9			+ 2·1
8	8:53– 9:58	88	S. N.	5·75	1·1	R, F	0·6	— 2·0
				5·84	1·2			— 0·9
				5·87	1·2			— 1·4
				5·89	1·3			— 0·0
[Section 4,900$^{\text{m}}$ stone to 5,000$^{\text{m}}$ stone, 20 B_{17}.]								
Aug. 27	4:24 p. m.	88	N. to S.	20·32			0·7	—[52·4]
28	7:37	88	N. S.	11·44			0·0	53·0
Oct. 7	9:30– 9:55 a. m.	85	N. S.	14·08	0·6	V	0·8	[55·9]
				14·48				[56·9]
				14·48				[55·6]
				13·83				[55·9]
				14·03				[55·1]
8	10:05–10:08 p. m.	88	N. S.	5·75	0·1	V	0·6	55·7
				5·75				55·3
				5·70				55·7
				5·65				55·4
				5·65				55·4
[Section 4,900$^{\text{m}}$ stone to 5,000$^{\text{m}}$ stone, 100 B_{17}.]								
Aug. 27	4:24– 5:10 p. m.	88	S. to N.	20·26	0·2	V	0·7	+[803·9]
28	7:37– 8:13	88	S. N.	11·05	0·8	F, R	0·0	808·9
29	5:58– 6:10	88	N. S.	16·12	3·0	F, R	0·2	808·3
31	4:35– 5:07	88	S. N.	21·67	1·2	F	0·6	[800·4]
31	5:52– 6:03	88	N. S.	17·92	1·1	F	0·3	807·6
31	6:07– 6:17	88	S. N.	16·95	0·8	V	0·2	808·0
31	6:22– 6:33	88	N. S.	15·86	0·4	F	0·2	808·3

(7) *Interpretation of results.*—It is essential to a proper interpretation of the results in the above table to state certain additional information relative to the conditions under which the measures were made.

The day measures of September 3, 1891, were made during the afternoon of a day whose earlier part had been hot and sultry. The field notes of the measures of this day state that, although the sky was generally cloudy, the tape and thermometers were occasionally exposed to the direct rays of the sun. The measures of September 4 were made while a strong breeze, rising to a gale at intervals, was blowing. This breeze, accompanied by more or less rain, sufficed at times to blow the tape when under tension off its supports. For these reasons the results of September 4 can not be regarded as at all trustworthy in comparison with results obtained under ordinarily favorable conditions.

The day measures of the standard kilometre of October 7 were made at a time of general cloudiness, but the tape and thermometers were exposed at intervals to direct sunshine. When the sun appeared on this date the measures were suspended until the clouds again cut off his rays, but in a number of cases the later measures of the set of five made on this occasion were not completed before the sun's heat was manifest in the thermometer readings.

The night measures were made, in general, at times when dew was deposited to a greater or less extent in open fields. The only marked exception to this rule applies to the measures of the standard kilometre of September 23, 1891. The night of this day was an unusually warm and dry one, and the wooded portion of the kilometre, which is about 600^m long, had a notable store of radiant heat.

The measures of the short section between the 4900^m and 5000^m stone of August 27 and 28, 1891, present a wide divergence from the later measures of the same section. These earlier measures were the first made with the apparatus in the field, and it was in laying this particular tape length that the operatives, with the exception of Siebert and Woodward, acquired their first experience. On these dates, also, a temporary and rather unstable marking table was used at the 4900^m stone, pending the preparation of a rigid post which served subsequently as a microscope post at the terminus of the standard kilometre, and as a stable support for the marking table used in the tape measures of September and October. For these reasons the measures of August 27 and 28 must be regarded as untrustworthy in comparison with the later measures of this short section.

All of the measures save those of August 27 and 28 and September 4, alluded to above, were made under favorable conditions, it is believed, so far as the manipulation of the apparatus was concerned.

With these explanations in mind attention may now be called to a marked peculiarity exhibited by all the results of the day measures except that of August 27 of the short section 4900^m to 5000^m. A glance at Table V shows that those results are all small relatively to the results of the night measures save in the single instance noted. Disregarding those measures of August 27 and September 4, concerning which there are grave doubts as already explained, the discrepancy in question is most marked in the results of August 27 and 31 (Section 5000^m to North Base), September 3 (Section 2100^m to 3900^m), and October 7 (Section 3900^m to 4900^m). It appears, also, from the measures of August 31 (Section 5000^m to North Base), September 3 (Section 1200^m to 2100^m), and October 3 (Section 3900^m to 4900^m), that this discrepancy disappears, or tends to do, so with the going down of the sun. The unexpected fact thus brought to light is that under the conditions presented by the dates cited, the thermometers gave a temperature too low for the tape by amounts ranging up to about a degree Centigrade at the maximum. How this fact is to be explained I am unable to say at present, but of its existence there can be no doubt. It is probably due to the differing capacities of the tape and thermometers for absorption and radiation under different conditions.

In view of this peculiarity of the day measures it becomes essential to inquire carefully into the behavior of the tapes in the night measures during which the conditions as to radiation differed much less from those of the 100^m comparator whereon the tape equations were determined. The data for this inquiry are furnished by the measures with the tapes of the standard kilometre, whose length was determined with the iced bar, as explained in Chapter III, and we proceed to examine these measures in detail in the following section.

(8) *Conclusions from tape measures of standard kilometre.*—In order to give a clear idea of the performance of the tapes in the numerous measures of the standard kilometre, a summary of those measures is drawn up in Table VI following. The first column of this table gives the date of a set or group of measures; the second gives the limiting times of the date within which the set of measures was made; the third gives the number of measures in the set; the fourth gives the mean of the results for length of the kilometre, and the last column gives the error of the mean length for any date, the true length of the kilometre being assumed to be $200B_{17}+0{\cdot}2^{mm}$ as found from the iced-bar measures and given in Chapter III, section 12. The results from the day measures are inclosed in square brackets:

Tape measures of Holton Base.

TABLE VI.—*Summary of results of tape measures of kilometre.*

Date.	Time of day.	No. of measures.	Mean length for date.	Actual error of mean.
1891.	*p. m.*		*mm.*	*mm.*
Sept. 8	5:25– 7:12	3	$200B_{17}+$ 3·4	+ 3·2
23	6:27– 9:32	4	— 2·8	— 3·0
30	6:33– 7:55	2	— 0·2	— 0·4
Oct. 1	6:50– 8:07	2	— 0·4	— 0·6
2	6:48– 7:55	2	+ 2·2	+ 2·0
3	2:56– 4:38	4	—[4·4]	— [4·6]
	a. m.			
7	10:02–11:31	5	—[9·8]	—[10·0]
	p. m.			
8	7:44– 9:58	8	— 0·2	— 0·4

It is clear from the results in this table that the errors of the night measures are small in comparison with those of the day measures. The mean of the night measures, giving weights to the tabular means proportional to the number of results on which they depend, is $200B_{17}$ $+ 0{\cdot}0^{\text{mm}}$, which differs but $0{\cdot}2^{\text{mm}}$ from the value derived from the iced bar measures. The maximum error of any group mean from the night measures is $3{\cdot}2^{\text{mm}}$, or 1/300 000th part of the kilometre; and the average error of such group means (disregarding weights) is only $1{\cdot}6^{\text{mm}}$, or 1/600 000th part of the kilometre. On the other hand, the day measures are largely in error and in one direction only, the maximum error of one group mean being 1/100 000th part of the kilometre.

It seems safe to conclude, therefore, from these measures of the standard kilometre, that the temperatures of the tapes when used at night in the open air were substantially as well defined by the mercurial thermometers as when used under the shed of the 100^{m} comparator. Moreover, this conclusion is substantiated so far as it can be by the evidence afforded by the measures of the other sections of the base, which do not present a single well-defined exception to the rule that the day measures are relatively short.

In accordance with this conclusion, we shall use the night measures only in deriving the adopted mean lengths of the several sections of the base. It may be observed, however, that another method, which does not require the rejection of the day measures, is available for the determination of the mean lengths of those sections. This method consists in deriving the working lengths of the tapes for the day measures from the day measures of the standard kilometre whose length was measured precisely with the iced-bar apparatus. A glance at the data shown in Table V makes it apparent that this method would bring the day measures into close agreement with the night measures of any section. The computation according to this method has been made, and the resulting mean lengths of the sections will be given below.

It will be remarked, also, that the numerous night measures of the standard kilometre afford sufficient data for a good determination of the equations of the tapes, especially for tape No. 88, since the temperature range for it in the kilometre measures is about 15° C. But, as shown above, the equations of the tapes derived from observations on the 100^{m} comparator are in so close accord with those data that there is no need of appealing to them for supplementary information concerning the equations of the tapes for the night work.

Finally, attention may be called to the substantial agreement with each other and with the true value of the means of the night measures of the standard kilometre from the two tapes. These means are $200B_{17} + 0{\cdot}9^{mm}$ for tape No. 85 and $200B_{17} - 0{\cdot}2^{mm}$ for tape No. 88. The actual errors of these measures are thus, with respect to the iced bar value ($200B_{17} + 0{\cdot}2^{mm}$), only $+ 0{\cdot}7^{mm}$ and $- 0{\cdot}4^{mm}$, which are in each case less than the millionth part of the length of the line.

(9) *Mean values of lengths of sections of base.*—As stated in the preceding section, the lengths we shall adopt for the sections of the base will be made to depend on the night measures only, for the reason that these were made under circumstances closely comparable with those which obtained on the 100^{m} comparator in determining the equations of the tapes. It will be of interest, however, to see to what extent the mean lengths of the sections and the length of the whole base would be affected if the day measures were included and given equal weight along with the night measures. Likewise, it will be instructive to learn the effect of the inclusion of the results of the day measures when computed by means of working lengths of the tapes derived from the day measures of the standard kilometre. Accordingly, the values which result from each of these three suppositions are given in Table VII following. The data for this table are derived from Table V. The first column designates the section of the base. The second column gives the mean length of the section derived from the night measures alone, these being considered of equal weight. The third column gives the mean length of the section derived by considering all day and night measures of the same weight. The fourth column gives the mean by equal weights of all measures of a section after the day measures have been increased by $0{\cdot}74^{mm}$ per tape length. The results of August 27 and 28 for the short section $4\,900^{m}$ to $5\,000^{m}$ and those of September 4 for the sections South Base to $3\,000^{m}$ are, however, excluded in the first and third of these computations for the reasons given in section 8. The correction $+ 0{\cdot}74^{mm}$ per tape length is a mean of the corrections furnished by the kilometre measures of October 3 and 7. A more precise value might be derived, it is believed, by making such correction depend on the time of day, since the evidence indicates that the error in question was due to solar radiation; but the data are insufficient to justify such a degree of refinement.

Tape measures of Holton Base.

TABLE VII.—*Mean values of lengths of sections of Holton Base.*

Section of base.	Mean of night measures.	Mean, including uncorrected day measures.	Mean, including corrected day measures.
m.	*mm.*	*mm.*	*mm.*
South Base-1200	$240B_{17}$ + 34·3	$240B_{17}$ + 31·7	$240B_{17}$ + 34·3
1200–2100	180 — 27·0	180 — 28·5	180 — 27·0
2100–3000	180 + 70·9	180 + 65·2	180 + 68·6
3000–3900	180 + 20·2	180 + 17·4	180 + 19·6
3900–4900	200 + 0·0	200 — 2·2	200 + 0·4
4900–5000	20 — 55·5	20 — 55·2	20 — 55·3
5000–North Base	100 + 808·2	100 + 806·5	100 + 807·5
South Base to North Base.	$1100B_{17}$ + 851·9	$1100B_{17}$ + 834·9	$1100B_{17}$ + 848·1

It will be observed that the means in the third column of this table are in every case smaller than the corresponding values in the second column, so that there can be no doubt of the wide divergence of the day measures from the night measures obtained with the apparatus as used on the Holton Base. The total length of the base found by including the day measures is 17·0$^{\text{mm}}$ shorter than that found by excluding them. This amounts to 1/325 000th part of the length of the base. This total length, however, is affected to a marked extent by the measures of September 4, made during rain and wind as already explained. These adverse circumstances afford sufficient reason for the rejection of those measures; but it is proper to explain that the general effect of the wind on this date was to increase the distance between the terminal marks of the tape, and hence to make the resulting computed lengths of the sections measured too small. This arose from the fact that the wind blew briskly from a northerly direction, or nearly in the direction of the base, and hence it tended to straighten and thus increase the working length of the tape, just as a strong breeze tends to straighten a string attached to a fixed point at one end or to render it parallel to the direction of the current, which is in general horizontal. The maximum increase which such a cause could produce in the distance between the terminal marks of the tape as used is 1·55$^{\text{mm}}$. This increase is much greater, of course, than any which is likely, but it seems probable that a quite appreciable fraction of that amount per tape length entered as an actual error into the measures of the date in question.

As to the results in the last column in Table VII, it may be said that their agreement with those of the second column is fairly satisfactory. It appears, however, that the day measures are not sufficiently increased.

(10) *Errors of measurement.*—The errors which give rise to the discrepancies between different results for the length of a line measured with a long tape may be conveniently divided into the two following

classes: (1) Errors of manipulation, or errors incident to the mere operation of placing or laying a tape length. They include errors due to lack of coincidence of the terminal marks of the tape with the defining marks on the zinc or other marking plates, errors due to incorrect tension of the tape, errors due to friction of the tape on its supports, and errors due to defective alignment of the tape. (2) Temperature errors. These include all errors tending to give incorrect temperatures to the tape, except such constant errors of the thermometers as may affect all measures alike.

Relatively, the second class of errors is by far the most formidable, since they tend at times toward the constant type of errors, but it is important also to give such information as is available with respect to the minor source. Accordingly we shall endeavor to separate the two classes of error.

It may be remarked that the errors of manipulation enumerated above are all of the compensating sort, except that due to defective alignment of the tape. This error must, however, be very small, since flipping the tape when under tension tends to make it perfectly straight on its supports. Even if the middle of a 100^{m} tape lay one decimetre to one side of the straight line joining its ends (an error of alignment which may be regarded as impossible) the resulting length would be in error only 0·2^{mm}; so that errors of this kind, it is believed, are absolutely negligible.

It appears from an inspection of the results in Table V, especially those for the length of the standard kilometre, that any set of measures of a section made within a short interval of time was affected by more or less persistent temperature errors. This is, as already explained, very marked in the case of the day measures as compared with the night measures; but the latter exhibit also some traces of the same sort of error, which arises, it is thought, from the relatively constant conditions of radiation pertaining to any short line during a limited period of time. The full range of the resultant error of measurement, therefore, can be expected to appear only in numerous measures of a section made under widely varying conditions. Measures satisfying this requirement were made on the standard kilometre. The various measures of the other sections of the base, when taken as a whole, satisfy the requirement also, and we thus have two independent sets of data for determining the aggregate probable error of measurement of a section.

In computing this error we shall consider the night measures only, since they alone, as shown by the measures of the standard kilometre, conform to the adopted equations of the tapes, and they alone are used in deriving the adopted length of the base.

The standard kilometre was so accurately known from the iced bar measures of it that the discrepancies shown by the tape measures may be treated as actual errors, and, other things equal, as affording the

Tape measures of Holton Base.

best data for estimating the probable error of a single measure of a kilometre. The residuals found by subtracting the iced bar value of this kilometre (200 B_{17} + 0·2^{mm}) from the several night measures, taken individually, range from − 4·5^{mm} to + 5·3^{mm}. The number of these residuals is 21 and the sum of their squares is 117·2; so that the probable error of a single night measure of the standard kilometre was ± 1·6^{mm}.

On the other hand, if we consider all the night measures of the several sections, including the standard kilometre (but excluding the short section of 100^{m} for the reason that one tape length does not afford an opportunity for the development of the errors in question), there are in all 46 measures of 6 sections. These sections vary in length from 500^{m} to 1,200^{m}, and hence it is essential to have recourse to weights in order to derive the probable error of one measure of a kilometre. Assuming that the resultant error desired is of the compensating class, the weights of the different sections will vary inversely as their lengths; and if we call the weight of one measure of a kilometre 1, the weight of one measure of a section n tape lengths long (n100^{m}) will be 10/n. In order to exhibit the data furnished by the several sections for the probable error desired, the details are brought together in Table VIII below. The first column of the table gives the designation of the section; the second gives the number of measures of the section; the third gives the range among those measures; the fourth gives the weight of the section; and the fifth gives the sum of the weighted squares of the residuals found by comparing the individual measures of a section with their mean.

TABLE VIII.—*Data for error of measurement.*

Section.	No. measures of section.	Range of results.	Weight p.	$[pvv]$
m. *m.*		*mm.*		
0 to 1200	5	13·1	10/12	97·96
1200 2100	7	6·0	10/9	23·99
2100 3000	4	5·2	10/9	15·40
3000 3900	4	3·9	10/9	9·80
3900 4900	21	9·8	1	116·58
5000 5500	5	1·3	2	1·82

The total sum of the quantities [pvv] in the above table is 265·55. Hence the probable error of one measure of a kilometre as shown by them is

$$\pm\ 0{\cdot}6745\sqrt{\frac{265{\cdot}55}{40-6}} = \pm\ 1{\cdot}74^{mm}.$$

This, it will be seen, is slightly larger than the value derived from the kilometre measures alone.

Now, by means of this value and the data afforded by the determinations of the tape lengths on the 100^{m} comparator, it is possible to derive an estimate of the magnitude of the error due to manipulation. By reference to Chapter IV, section 14, it is seen that the error of one determination of a tape length on the comparator was $\pm 0{\cdot}20^{\text{mm}}$, and this is due almost wholly to error in the assigned temperature from the mean of six thermometer readings. Assuming that the temperature errors were no greater in field work than on the comparator, the probable error of one tape length from the mean of two in place of six thermometer readings should be $\pm 0{\cdot}20^{\text{mm}}\sqrt{3}$; and hence the probable error in the length of a kilometre, or the sum of 10 tape lengths, should be $\pm 0{\cdot}20^{\text{mm}}\sqrt{30} = \pm 1{\cdot}1^{\text{mm}}$. Since the probable error of measurement due to all sources of error was found above to be $\pm 1{\cdot}7^{\text{mm}}$, at most, we infer that the probable error due to errors of manipulation in one measure of a kilometre is

$$\pm \sqrt{(1{\cdot}7)^2 - (1{\cdot}1)^2} = \pm 1{\cdot}3^{\text{mm}}.$$

It appears, then, that the probable errors due to the two classes of error (error of manipulation and error of temperature) were about equal in the case of the night measures of Holton Base. Some doubt may be entertained as to the correctness of the above estimate of $\pm 1{\cdot}1^{\text{mm}}$ for the probable error due to temperature errors. Subsequent experience, on St. Albans Base, in 1892, under somewhat different conditions, as explained in sections 20–27 below, does not confirm this estimate, but seems to show conclusively that the error of operation is quite negligible in comparison with that of the temperature.

(11) *Probable errors of mean lengths of sections.* (*a*) *Probable errors due to errors of measurement.*—The mean lengths of the sections we shall adopt are those given in the second column of Table VII above. The probable errors of these mean values may be derived in two ways, namely: First, by considering the several measures of a section by themselves; second, by considering the measures of all the sections as a connected group of observations with weights dependent on the lengths of the sections for the individual measures and weights dependent on the numbers of measures of the sections for their mean lengths. The data for these two probable errors for the several sections and the resulting values of the errors are given in Table IX below. The first column of the table designates the section; the second gives the number of measures of the section m; the third gives the sum of the squares of the discrepancies between the individual measures of a section and their mean, which sum is denoted by $[vv]$; the fourth column gives the weight of the section p, which is the reciprocal of the number of tape lengths in the section, unit weight being here attributed to one measure of a single tape length, or 100^{m}; the fifth column gives the sum of the

weighted squares of the residuals, or $[pvv]$, and the last two columns give the probable errors of the mean values of the sections computed from the formulas

$$(1) = \pm\, 0{\cdot}6745\sqrt{\frac{[vv]}{m\,(m-1)}},\ \text{and}\ (2) = \pm\, 0{\cdot}6745\sqrt{\frac{[pvv]}{mp\,(51-7)}}.$$

TABLE IX.—*Data for and probable errors of mean lengths of sections.*

Section.	No. measures m.	$[vv]$	Weight p.	$[pvv]$	Probable error.	
					(1)	(2)
m. m.					*mm.*	*mm.*
0–1200	5	117·55	1/12	9·55	±1·64	±0·81
1200–2100	7	21·59	1/9	2·40	± ·48	± ·59
2100–3000	4	13·85	1/9	1·54	± ·73	± ·78
3000–3900	4	8·82	1/9	0·98	± ·58	± ·78
3900–4900	21	116·58	1/10	11·66	± ·36	± ·36
4900–5000	5	0·14	1	0·14	± ·06	± ·23
5000–5500	5	0·91	1/5	0·18	± ·15	± ·52

It will be observed that the probable errors in the last two columns of the above table agree fairly well, and thus justify the assumption that the errors of measurement with the tape apparatus are of the compensating sort, or that the aggregate error of measurement increases as the square root of the distance measured. We conclude, also, that the probable errors (2) are, on the whole, the more trustworthy, since they are not unduly affected by the excessively large or the excessively small errors which appear in the measures of some of the sections.

The probable error of laying a single tape length, as shown by the process (2), is $\pm\, 0{\cdot}524^{\mathrm{mm}}$. Hence the probable error of one measure of a kilometre is, by this process,

$$\pm\, 0{\cdot}524^{\mathrm{mm}}\sqrt{10} = \pm\, 1{\cdot}66^{\mathrm{mm}};$$

and this may be regarded as the most trustworthy value afforded by the measures of Holton Base.

The probable error of measurement in the whole length of the base is, if we regard the measures of the different sections as wholly independent, the square root of the sum of the squares of the probable errors (1) in Table IX. This value is $\pm\, 1{\cdot}99^{\mathrm{mm}}$. On the other hand, by the process (2) the probable error of the whole length of the base is

$$\pm\, 0{\cdot}524^{\mathrm{mm}}\left(\frac{12}{5}+\frac{9}{7}+\frac{9}{4}+\frac{9}{4}+\frac{10}{21}+\frac{1}{5}+1\right) = \pm\, 1{\cdot}65^{\mathrm{mm}}.$$

These are equivalent to the 1/2 770 000th and the 1/3 330 000th part, respectively, of the whole base.

(*b*) *Probable errors due to errors of tape lengths.*—The probable error of the absolute length of either of the 100^{m} tapes used in the measures of Holton Base may be taken as not exceeding $\pm 0{\cdot}05^{\text{mm}}$, as shown by the discussion in section 16, Chapter IV; and we shall adopt this limiting value in computing the probable errors of the absolute length of the sections. Multiplying this error, $\pm 0{\cdot}05^{\text{mm}}$, by the numbers of tape lengths per section, there result for the probable errors of the several sections from this source

	m. m.	mm.
Section	0–1200	± 0·60,
	1200–2100	± ·45,
	2100–3000	± ·45,
	3000–3900	± ·45,
	3900–4900	± ·50,
	4900–5000	± ·05,
	5000–5500	± ·25.

It is worthy of remark that these errors are in every case but one smaller than the errors of measurement shown in Table IX above. The exceptional case is that of the standard kilometre, whose adopted mean length depends on twenty-one measures. The probable error of the whole base due to this source is $\pm 55 \times 0{\cdot}05^{\text{mm}} = \pm 2{\cdot}75^{\text{mm}}$, or 1/2 000 000th part of the base.

(12) *Adopted lengths and errors of sections.*—Collecting the mean lengths of the sections given in the second column of Table VII, and combining the probable errors in the last column of Table IX with those given under (*b*) of the preceding section, there result for the lengths we adopt for the several sections

	m. m.			mm.	mm.
Section	0–1200 =	$240B_{17}$	+	34·3 ±	1·01,
	1200–2100 =	180	−	27·0 ±	·75,
	2100–3000 =	180	+	70·9 ±	·90,
	3000–3900 =	180	+	20·2 ±	·90,
	3900–4900 =	200	+	0·0 ±	·62,
	4900–5000 =	20	−	55·5 ±	·24,
	5000–5500 =	100	+	808·2 ±	·58,

$$B_{17} = 5^{\text{m}} - 18{\cdot}0^{\mu}.$$

(13) *Adopted length and probable error of Holton Base as measured with the tapes.*—The length we adopt for Holton Base as measured (not reduced to sea level) is the sum of the lengths of its parts given in the preceding section; that is,

$$1100B_{17} + 851{\cdot}9^{\text{mm}},$$

wherein

$$B_{17} = 5^{\text{m}} - 18{\cdot}0.^{\mu}$$

The probable error of this value, so far as it depends on errors of measurement or those developed in the repeated measures of the sections, is given under (*a*) of section 12 as ± $1{\cdot}99^{\text{mm}}$ or $1{\cdot}65^{\text{mm}}$, according as we adopt the process (1) or the process (2) of that section. The probable error in the whole length due to error in the adopted tape lengths is, as stated in (*b*) of section 12, $\pm 55 \times 0{\cdot}05^{\text{mm}} = \pm 2{\cdot}75^{\text{mm}}$.

It remains to consider the possible effects of errors of grades and alignment. These errors, though quite negligible in the length of a single section, may yet rise to an appreciable quantity in the length of the whole base.

As to errors in grade corrections, it will be observed that they are of the compensating class, since errors in determining the differences in altitude of consecutive marking tables are as likely to be of one sign as another. The grade correction c for any tape length, as explained in section 3, is given by

$$c = -\frac{h^2}{2s},$$

where h is the difference in altitude of the two ends of the tape and s its length. The effect of an error Δh in h is expressed by

$$\Delta c = -\frac{h}{2s} 2\Delta h = -2c\frac{\Delta h}{h}$$

For a constant value of Δh, therefore, the value Δc will vary as h. The greatest value of h for the tape lengths on Holton Base was $1{\cdot}72^{\text{m}}$, giving rise to a value of $14{\cdot}79^{\text{mm}}$ for c. Since the values of h were measured by two different observers at different times, and since all discrepant values were redetermined, it seems impossible that an error as great as 2^{cm} could have entered any adopted h. In the case of the above maximum value of h, the error in c due to an error $\Delta h = 2^{\text{cm}}$ would be

$$\Delta c = (14{\cdot}79^{\text{mm}}) \times 4/172 = 0{\cdot}34^{\text{mm}}.$$

The average value of h for the base was less than $0{\cdot}4^{\text{m}}$, so that the average correction for grade was thus less than $0{\cdot}8^{\text{mm}}$ per tape length. Hence, since the errors under consideration are of the compensating sort, it is believed that their aggregate effect on the whole line was inappreciable.

Errors of alignment in tape measures may manifest themselves in two ways. In the first place, the tape as stretched from length to length may not be parallel to the base. A lack of parallelism with the base to the extent of a decimetre in the length of a tape 100^{m} long would give rise to an error of $0{\cdot}05^{\text{mm}}$. Similarly, any departure of the tape from a vertical plane through its ends would give rise to an error of the same sort and sign. In the second place, any imperfections in the alignment of the support nails to a uniform slope for any tape length would also produce a cumulative error. Both of these alignment errors make the measured length of the base too great. Both errors are, however, very small. An error as great as $0{\cdot}2^{mm}$ from crookedness of the tape with respect to a vertical plane through its ends is, it is thought, of very rare occurrence. Errors due to defects in setting the support nails to a uniform grade must be smaller per tape length than the former and of still rarer occurrence. The method of setting the support nails is so easy and accurate that an error in position as great as 1^{cm} can arise only through accidents or blunders.

Notwithstanding the indications that the errors in grades and alignment are negligible, we shall attribute $\pm 1^{mm}$ as a probable error to each of these sources, since it is possible that some blunders occurred in this first application of the method and apparatus.

With respect to the two errors of measurement, $\pm 1{\cdot}99^{mm}$ and $\pm 1{\cdot}65^{mm}$, given above, we shall adopt the larger, in order that the precision may be underestimated rather than overestimated.

Collecting the several independent probable errors, which are to be combined to produce the resultant probable error of the base, we have the following statement:

Probable error due to errors of measurement	$\pm 1{\cdot}99^{mm}$,
Probable error due to errors in grade	$\pm 1{\cdot}00$,
Probable error due to errors of alignment	$\pm 1{\cdot}00$,
Probable error due to errors of tape length	$\pm 2{\cdot}75$.

Taking the square root of the sum of the squares of these, there results for the total probable error of the base as measured with the tapes

$$\pm 3{\cdot}68^{mm}.$$

This is equivalent to the 1/1 500 000th part of the base.

For the measured length, then, of Holton Base, when expressed in terms of the International Metre, we may write

$$\begin{aligned} &1100B_{17} + 851{\cdot}9^{mm} + 3{\cdot}68^{mm} \\ &= 5500{\cdot}832^{m} \pm 3{\cdot}68^{mm}. \end{aligned}$$

(14) *Data for reduction to sea level.*—The heights of the marking tables used in the tape measures were all determined with reference to the tops of the marking stones at the ends of the base. The stone at north base was found to be $2{\cdot}61^{m}$ above the stone at south base. The average heights of the tape lengths for the several sections above the stone at south base are given in the following statement. The values are derived by taking the average of the half sums of the heights of the ends of the several tape lengths for the sections. The slopes of the sections as measured with the tapes are so gentle that these average values are amply accurate for computing the reductions to sea level.

Tape measures of St. Albans Base.

Average height of tape above south base:

Section south base to 1200^{m} stone 4·45^{m},
1200^{m} stone to 2100^{m} stone 6·46,
2100^{m} stone to 3000^{m} stone 5·49,
3000^{m} stone to 3900^{m} stone 3·48,
3900^{m} stone to 4900^{m} stone 4·63,
4900^{m} stone to north base 3·99.

B.—ST. ALBANS BASE MEASUREMENT.

(15) *Location of base and preparation for measurement.*—St. Albans base is a line in the transcontinental chain of triangulation along or near the thirty-ninth parallel of latitude. It is situated in the valley of the Great Kanawha river, near the village of St. Albans, Kanawha County, West Virginia. It is about 3 900^{m} long, and extends in a nearly east and west direction along a closely level portion of the valley. It is on the north side of and is parallel to the Chesapeake and Ohio railway, the west end being 17·8^{m} and the east end 18·4^{m} from the nearest rail of the main track in 1892.

This base was located in the summer of 1891 by Sub-Assistant Walter B. Fairfield. The terminal marking stones were provided and set under the direction of Assistant Mosman during the autumn of the same year. A profile of the line and a determination of its height with respect to an adjacent triangulation station of known height were also made under Assistant Mosman's direction by Mr. J. S. Siebert.

This base was measured with the tape apparatus in the early part of October, 1892, by a party in charge of the writer, aided by Assistant E. E. Haskell, Recorders M. V. Safford and Orville G. Brown. The line was cleared of brush, brambles, grass, and other obstacles, and the marking and support stakes for the tapes were set during the days October 1–9, 1892. These stakes were set 10^{m} apart, and the marking stakes were all carefully aligned with a theodolite. Special pains were also taken in this work to make the spacing of the support nails uniform and to bring them accurately into coincidence with the line of slope for any tape length. Some special provision had to be made for supporting the tapes at an elevation of 3^{m} to 5^{m} for two tape lengths where the line crossed a ravine. But the supports for these two tape lengths were no less stable than at other parts of the line, and no difficulty was met in handling the tapes successfully at such altitudes.

(16) *Measurements of the base.*—A measure of the base was made during the afternoon of October 10 with the 100^{m} steel tape No. 85 in

order to afford practice in manipulating the apparatus to the operatives, all of whom, except the writer, had had no experience with the apparatus. In this practice measure no record was kept of the set-ups, and some tape lengths were omitted, so that no results can be derived from it. The work of actual measurement was begun on October 11 and continued until October 14. Two measures of the base, one in each direction, were made at night with each of the steel tapes Nos. 85 and 88; and a daytime measure, in bright sunshine, was made with tape No. 88.

In all the measures, Mr. E. E. Haskell was the observer at the rear end of the tape and Mr. R. S. Woodward the observer at the front end. The thermometers from which the tape temperature was inferred were also read by these observers. An additional thermometer with a bright bulb, as explained more fully below, was read at the middle of the tape by Mr. M. V. Safford, who also attended to the operation of flipping the tape and assisted in carrying it forward. The records were kept by Mr. Orville G. Brown. The whole number of operatives employed in the work was ten.

The base was divided into four sections. These were all nearly one kilometre in length, except the easternmost one, which was about 869^m. The fractional part of a tape length, namely, 69^m, in this latter case, was measured on the tape when under the usual tension; 60^m of this fraction (or, rather, three-fifths of a tape length) was obtained directly from the 20^m sub-spaces on the tapes, while the remaining portion of 9^m, about, was measured by means of an auxiliary 15^m Chesterman steel tape divided to millimetres. The particular portion of the latter tape used was tested on the Mural Standard of the Survey on returning from the field, and found to accord sufficiently well with its nominal value. The whole fraction of a tape length (69^m about) was on each occasion fixed on the tape by means of a light transverse line ruled with a brad awl. The distance between the 60^m mark on the tape and the temporary line was measured in each case with the auxiliary tape immediately after ruling this line, but as an additional check the same distance was remeasured in the daytime, so that it is practically certain that no appreciable error enters this fractional part of a tape length.

The several positions of the tapes were marked on zinc plates which were left in position until the work was completed, when they were numbered, oriented, and filed as part of the records. The "set-ups" and "set-backs" made were measured and recorded at the time, but they were all carefully remeasured by daylight, so that no appreciable errors from this source can enter the measures.

The measures were referred directly to the terminal marks in the monuments at the ends of the base and to the marking plates at the intermediate ends of the sections. These plates were nailed to very stable stakes, and there was no indication of any appreciable movements of them during the short period of four days covered by the measures. Any such movements, however, which might take place from day to day would not affect the interval between the terminal stones, since the base was measured continuously from end to end whenever any measurement was made.

(17) *Determination of grade corrections.*—The differences in altitude of the consecutive marking tables were measured independently by Messrs. Haskell and Safford. Whenever any discrepancies appeared, the work was re-examined by them, and it is believed that the adopted differences are trustworthy within half a centimetre in every case, while the two largest of those differences were determined with a much greater precision.

The maximum difference in height of the tables for any tape length was $3{\cdot}571^{m}$, which gives a grade correction of $63{\cdot}76^{mm}$. The average difference in height per tape length for the several sections counting from the west end were $0{\cdot}37^{m}$, $0{\cdot}89^{m}$, $0{\cdot}29^{m}$, and $0{\cdot}18^{m}$.

The grade corrections were computed in duplicate by Messrs. Safford and Woodward, and the following are the resulting values for the several sections:

	mm.
Section West Base to Stake 10 —	8·52,
Stake 10 to Stake 20 —	93·58,
Stake 20 to Stake 30 —	7·20,
Stake 30 to East Base —	3·40.

(18) *Determinations of tape temperatures.*—Mercurial thermometers of the same sort as those used on Holton Base were used to give the tape temperatures in the measurement of St. Albans Base. But in the hope that the thermometers could be made to follow the tape temperatures with greater certainty the aluminum sheaths used in the Holton work were replaced by steel sheaths. These were made of very thin steel tape which was coiled into cylinders just long and large enough to slip over the thermometer bulbs and clasp their stems. By annealing, the surface of these coils was given a dull black color closely resembling that of the measuring tapes. A disadvantage, however, of these steel sheaths over the aluminum ones lies in their greater mass, which is about 0·50 grammes, and is about three times the mass of the aluminum sheaths. This increase in mass caused a notable lagging of the thermometers with respect to the tapes, whereas such lagging was very small when the lighter sheaths were used. The effects of this lagging, however, appear to have been well eliminated by the process of measurement adopted.

With the hope, also, that some light might be shed on the obscure question of the persistent difference between the tape and the thermometric temperatures during the day measures of Holton Base, a thermometer with no sheath attached to it (or one with bright instead of black bulb) was read in all the measures of St. Albans Base. The salient result revealed by this experiment is that during the daytime the thermometers with black bulbs (or sheaths) read uniformly higher and at night uniformly lower than the thermometer with the bright bulb. This fact confirms what might be, of course, expected, namely, that the tape and thermometers may have persistently differing temperatures unless they have the same capacity for absorption and radiation. Another result of this experiment which may prove of value in further studies with the tapes is that the period when thermometers with bright and black bulbs give the same readings occurs shortly before sundown, and appears to coincide with the period of steady atmosphere.* But aside from these results, interesting though they be in a general way, nothing came from the use of the thermometer with the bright bulb. The temperature of the tape was inferred altogether from the readings of the two thermometers supplied with the steel sheaths. That the latter gave temperatures closely coincident with those of the tapes, except for the lag effect, which was in general well eliminated, appears to be demonstrated by the results of the several measures of the four sections of the base.

The thermometers used in all work on St. Albans Base were Nos. 5598, 5620, and 5621 by Green. Table X following gives the corrections to be applied to the readings of these thermometers to bring them into accordance with the standard hydrogen scale. The corrections to 5598 are those given in section 10, Chapter IV, the freezing point reading adopted being that of January 21, 1892. The freezing point readings of 5620 and 5621 were determined by Mr. Louis A. Fischer, of the office of Standard Weights and Measures, in September, 1892.

TABLE X.—*Corrections to thermometer readings.*

Temperature.	Correction to—		
	5598.	5620.	5621.
°	°	°	°
0	—0·10	—0·20	—0·25
5	— ·12	— ·15	— ·20
10	— ·09	— ·21	— ·26
15	— ·08	— ·25	— ·32
20	— ·08	— ·27	— ·34
25	— ·07	— ·25	— ·32
30	— ·04	— ·25	— ·25
35	— ·09	— ·23	— ·28

(19) *Tension correction and working lengths of tapes.*—Spring balance No. 53 was used in all the work on St. Albans Base to give tension to

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* These two periods are probably due to the same cause, to wit, equality in the quantities of radiant heat received and rejected by the earth at that time of day.

the tapes. Its index correction was observed on October 10, 12, 14, 18, and 21. The mean of the resulting values, which vary from $-2{\cdot}0$ ounces to $-3{\cdot}0$ ounces, is $-2{\cdot}55$ ounces; or in the formula

$$T = T_1 + \tfrac{1}{2}(W - R + r),$$

given in section 4, $r = -2{\cdot}55$ ounces.

The following values were also observed:

$$W = 49{\cdot}5 \text{ ounces},$$
$$R = 35{\cdot}0 \text{ ounces}.$$

The nominal value of the tension used was

$$T_1 = 25 \text{ pounds}.$$

Hence the actual tension applied was

$$T = 25 \text{ pounds } 6 \text{ ounces}.$$

Since the standard tension is 25 pounds 9 ounces, the tension correction is $-3{\cdot}0$ ounces, and the correction to the standard lengths is $-3 \times 0{\cdot}047^{\mathrm{mm}} = -0{\cdot}14^{\mathrm{mm}}$. The lengths of the tapes as used are then (see Chapter IV, section 17)

$$\begin{aligned} & \qquad\qquad\quad \mathit{mm.} \qquad\quad \mathit{mm.} \\ T_{85} &= 20\,B_{17} + 3{\cdot}72 + 1{\cdot}0947t, \\ T_{88} &= 20\,B_{17} + 6{\cdot}17 + 1{\cdot}0914t. \end{aligned}$$

(20) *Results of the measures of St. Albans Base.*—Table XI following gives the results of the measures of St. Albans Base. The first column gives the date; the second, the time of day when the measure was made; the third, the direction in which the measure proceeded; the fourth, the mean temperature of the tape; the fifth, the temperature range, or the difference between the highest and lowest observed temperature of the thermometers; the sixth indicates by the letters R and F whether the temperature was rising or falling during the measure of a section, and the order of the letters indicates the order in which the changes occurred; the seventh gives the length of the sections in terms of the iced bar B_{17}, the excess over a round number of tape and bar lengths having been measured with a millimetre scale, and with a 15^{m} auxiliary steel tape in case of the fourth section; and the last column gives the number of the tape used. The values for the lengths given in the seventh column are not corrected for grades. Since the whole base only was ultimately used, the total grade correction will be applied to the sum of the lengths of the sections.

TABLE XI.—*Results of measures of St. Albans Base.*

WEST BASE TO STAKE 10.

Date.	Time of day.	Direction of measure.	Mean temperature.	Temperature range.	Temperature rising, R, falling, F.	Value for distance.	No. of tape.
1892.	*p. m.*		° *C.*	° *C.*		*mm.*	
Oct. 11	7:20– 8:02	W. to E.	9·04	3·40	F	$200 B_{17} +$ 258·2	88
12	9:03– 9:44	E. W.	7·62	2·70	R	244·5	88
13	7:06– 7:40	W. E.	13·30	3·00	F	252·6	85
13	10:49–11:21	E. W.	9·56	1·10	R	251·5	85
14	2:53– 3:16	W. E.	32·11	1·40	R, F	248·1	88

STAKE 10 TO STAKE 20.

Date.	Time of day.	Direction of measure.	Mean temperature.	Temperature range.	Temperature rising, R, falling, F.	Value for distance.	No. of tape.
Oct. 11	8:02– 8:44	W. to E.	5·92	2·80	F, R	$200 B_{17} +$ 534·8	88
12	9:44–10:25	E. W.	6·39	2·25	F, R	535·5	88
13	7:40– 8:12	W. E.	10·38	2·55	F, R	533·0	85
13	11:21–11:53	E. W.	9·05	0·80	F, R, F	536·0	85
14	3:16– 3:49	W. E.	30·47	0·80	F, R	536·0	88

STAKE 20 TO STAKE 30.

Date.	Time of day.	Direction of measure.	Mean temperature.	Temperature range.	Temperature rising, R, falling, F.	Value for distance.	No. of tape.
Oct. 11	8:44– 9:26	W. to E.	6·23	1·50	R, F, R	$200 B_{17} +$ 527·0	88
12	7:40– 8:22	E. W.	7·84	1·60	R, F	524·0	88
13	8:12– 8:44	W. E.	9·92	1·10	F, R, F	527·8	85
13	9:45–10:17	E. W.	8·75	0·70	R, F, R	528·9	85
14	3:49– 4:02	W. E.	30·08	1·50	R, F, R	534·0	88

STAKE 30 TO EAST BASE.

Date.	Time of day.	Direction of measure.	Mean temperature.	Temperature range.	Temperature rising, R, falling, F.	Value for distance.	No. of tape.
Oct. 11	9:36–10:08	W. to E.	4·80	0·75	F, R	$172 B_{17} +$ 9 325·5	88
12	8:22– 9:03	E. W.	7·92	2·65	F, R	9 3 3·0	88
13	8:44– 9:16	W. E.	9·24	0·50	F, R	9 321·4	85
13	10:17–10:49	E. W.	8·57	1·00	F, R, F	9 328·6	85
14	4:02– 4:26	W. E.	29·80	0·95	R, F, R	9 333·0	88

(21) *Interpretation of results.*—An inspection of the results in the above table shows the ranges amongst the individual measures of the several sections to be 13·7^{mm}, 3·0^{mm}, 10·0^{mm}, and 11·6^{mm}, respectively; and the question which immediately suggests itself is, why should the range for the section stake 10 to stake 20 be so much less than for the other sections? The hypothesis we offer in answer to this question affords a satisfactory explanation of the ranges just mentioned, especially those of the night measures, and clears up some of the peculiarities noted in the measures of Holton Base. It is based on the following considerations:

In case there is any relative lagging of the tape and thermometric temperatures, its effect will be more or less eliminated in the mean of two measures of a section made the one under a rising and the other under a falling temperature. A similar elimination of the lag effect will occur in a single measure of a section if the temperature is rising for one-half and falling for the other half of the measure; and, in general, the lag effect will tend to eliminate itself if the temperature fluctuates to any considerable extent about its mean trend. The conditions which obtained during the measures of St. Albans Base, so far as they pertain to this lag effect are, in general, well indicated by the data in the fifth and sixth columns of the above table; but it should be stated that the section stake 10 to stake 20 was specially favorable by reason of its topographic conformation for eliminating the lag effect. The middle portion of this section was on low marshy ground compared with the ground at either of its ends, so that there was always a marked fall of temperature on approaching the low ground and a marked rise on leaving it, regardless of the direction of measurement.

This answer, it may be conceded, explains the small range among the results for the section stake 10 to stake 20; but if it is the correct answer, we should expect the means of the forward and backward measures of the other sections to show a similar degree of accordance, since, as shown in the sixth column of the table, the temperature gradient was in general reversed with a reversal of the direction of measurement. The two night measures with each tape are in this respect strictly comparable, since they were made under conditions in all respects similar; i. e. the sky was cloudless or nearly so, there was little or no wind, the preceding days had been very much alike, and the measures were made at times when dew was condensing along most parts of the line. That these means do so accord is shown by the results in Table XII following, which gives the means of the night measures of the sections made with the two tapes. For brevity the sections are numbered 1, 2, 3, 4, respectively:

TABLE XII.—*Means of forward and backward measures of the sections.*

No. tape.	Section 1.	Section 2.	Section 3.	Section 4.	Whole Base.
	mm.	*mm.*	*mm.*	*mm.*	*mm.*
88	+251·4	+535·2	+525·5	+9 324·2	10 636·3
85	252·0	534·5	528·4	9 325·0	10 639·9

The accordance of these results leaves little to be desired; but for the reasons already adduced and for others given below, it is difficult to see how it can be otherwise than real.

Although the fact that means of forward and backward measures of a section indicated a marked elimination of some source of error was noticed in certain of the measures of Holton Base, the explanation of that fact, presented above, did not occur to me until after the night measures of St. Albans Base had been made. The full import of the explanation, indeed, was not appreciated until after the day measure of October 14 had been made; otherwise a reverse measure of the line would have been made on the latter day. The particular measures of Holton Base just referred to are those of sections 2 100^m to 3 000^m and 3 000^m to 3 900^m. On referring to Table V of section 7 it will be seen that each of these sections was measured in both directions on September 18 and again on September 19, 1891, under conditions favorable for the elimination of the lag effect. The excesses over $180B_{17}$ of the means of the forward and backward measures of the section 2 100^m to 3 000^m for these two days are + 71·2^{mm} and + 70·7^{mm}; and for the section 3 000^m to 3 900^m the corresponding excesses are + 20·4^{mm} and + 20·0^{mm}. These exhibit a degree of accordance equal to that shown in Table XII. A similar degree of accordance is shown in the forward and backward measures of the standard kilometre of Holton Base of September 30, October 1, and October 8, 1891, on which dates the temperature gradients were favorable to the elimination of the lag effect. Attention has also been called in section 11 to the fact that the measures of the standard kilometre show a less probable error than the measures of the other sections of Holton Base. The reason for this is found, I believe, in the topographic conformation of that kilometre. It passed for 600^m of its length through a dense forest, leaving 200^m at either end in open fields. In nearly all the night measures the thermometer readings show a rise of temperature on passing from the open fields to the forest, and vice versa; which conditions, as explained above, were favorable for the elimination of the lag effect in a single measure of the line.

To recur to the measures of St. Albans Base, it is seen from the results in Tables XI and XII that the largest discrepancies are found in every case where, according to the hypothesis considered, they would be expected to appear. Thus, in Table XI the range is greatest in the measures of the section West Base to stake 10, which presented the steepest temperature gradients; and in Table XII the range is greatest in the case of the third section, which, as appears from the data in the sixth column of Table XI, presented the least favorable temperature gradients.

It should be noted, however, that the difference between the night measures of the section West Base to stake 10 with tape No. 88 is much greater than that between the night measures with tape No. 85. The temperature gradients, in fact, indicate that the former difference ought to exceed the latter, but it is not clear that the excess should be as great as it is. This apparent anomaly has led to a comparison of the separate tape lengths of the night measures of this first section with the two tapes. It shows that the measures with tape No. 88 were for every tape length apparently longer in the one case and shorter in the other by about $0{\cdot}7^{mm}$; in other words, if we are dealing with a lag effect, the thermometers gave a temperature too high to the tape under a falling gradient and too low under a rising gradient by about 0°·7 C., and this error was persistent throughout both measures of the section. Attention has been called in Chapter IV, section 15, to a similar discrepancy in the behavior of these two tapes when placed alongside of one another on the 100^{m} comparator of Holton Base. It is most probably due, as suggested in the section just referred to, to some difference in the emissivities of the two tapes, but a study of all the data has not up to the present time led me to any clearer view of the question.

It remains to consider the results of the day measure of the base of October 14, 1892. This was made on a cloudless day between 2:53 and 4:26 p. m., and the mean temperatures for the several sections varied from 32°·11 C. to 29°·80 C. Including this measure, the mean temperatures of the tape ranged for the different day and night measures of each section through nearly 25° C.; so that the accordance of the day measures with the mean of the night measures affords a pretty severe test of the equations of the tapes and of the thermometeric temperatures attributed to them. A glance at the data in the sixth and seventh columns of Table XI shows that the accordance of the day measure is best, as it ought to be in conformity with the considerations adduced above, in the case of the first and second sections, and worst, as would be expected, in the case of the last two sections.*

The differences between the day measures of the several sections and the means of the night measures of the same are

	mm
For the first section	— 3·6
second section	+ 1·2
third section	+ 7·0
fourth section	+ 8·4

The average of these regardless of sign is $5{\cdot}0^{mm}$, or 1/200 000th of a section; while the average of the first two of them, which must be largely free from the lag effect, is only $2{\cdot}4^{mm}$, or 1/400 000th part of a section. From any point of view these discrepancies, when we bear in mind the wide range of temperature involved, do not seem large; but when we consider them in connection with the temperature gradients which obtained during the day measure, it appears clear that, except

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*It is now plainly regrettable that a reverse measure of the base was not made immediately after this one, since it appears almost certain that the means of such day measures would have shown a very close agreement with the means of the night measures.

for the lag effect, the thermometers as fitted with the steel shields gave temperatures in very close agreement with those of the tapes during both the day and the night measures of the base.

Briefly summarized, then, our interpretation of the results of the measures of St. Albans Base is that the most formidable errors to which they were subject arose from lag of the thermometers as fitted with the steel shields; that these errors were well eliminated in the means of the forward and backward night measures with either tape for all sections, and well eliminated in all the individual measures of the second section by reason of its topographic conformation; and finally, that the result of the day measure is too great by reason of uncompensated lag effect which lies chiefly in the lengths of the two easternmost sections of the base.

(22) *Values for measured lengths of base and probable error of measurement.*—The value we shall adopt for the entire length of the base is the mean of the four night measures, since all our experience indicates that the temperature uncertainty is in general least in night measures, and since it appears certain that a considerable lag effect was eliminated in the means of the forward and backward measures of this base. Accordingly, the value we adopt is

$$772B_{17} + 10638{\cdot}1^{\mathrm{mm}},$$

or the mean of the results in the last column of Table XII.

If, on the other hand, we include the day measure and give it and each of the night measures the same weight, the result is

$$772B_{17} + 10640.7^{\mathrm{mm}},$$

which differs but $2{\cdot}6^{\mathrm{mm}}$ from the adopted value.

The results of the several measures of the entire base, in the order in which they were made, are

	mm.
Oct. 11, 1892,	$772B_{17} + 10645{\cdot}5$,
12, 1892,	$27{\cdot}0$,
13, 1892,	$34{\cdot}8$,
13, 1892,	$45{\cdot}0$,
14, 1892,	$51{\cdot}1$.

The range among these is $24{\cdot}1^{\mathrm{mm}}$, or 1/160 000th part of the base. The range of the first four, or the night measures, is $18{\cdot}5^{\mathrm{mm}}$, or 1/209 000th part. The greatest divergence from the adopted mean is that of the last, or day measure. The amount is $13{\cdot}0^{\mathrm{mm}}$, or 1/298 000th part.

The mean lengths from the forward and backward measures with the two tapes are

	mm.
Tape No. 85,	$772B_{17} + 10636{\cdot}3$,
No. 88,	$772B_{17} + 10639{\cdot}9$.

The difference of these is $3{\cdot}6^{\mathrm{mm}}$, or 1/1 070 000th part of the base.

Tape measures of St. Albans Base.

In computing the probable error of measurement from the data afforded by Tables XI and XII, we may proceed according to several hypotheses. Using 1 kilometre as the unit of distance to which such probable error refers, we may assume (a) that all of the night and day measures of the several sections of the base are of equal weight, and that no elimination of systematic error results in the mean of a forward and backward measure; (b) that the night measures should be considered by themselves and treated as if they involved no errors due to lagging of the thermometers; (c) that the means of the forward and backward measures given in Table XII are more or less free from errors due to lagging, and that such means are alone competent to give an idea of the attainable precision.

The errors which result from each of these hypotheses will now be given. Let ε_a, ε_b, ε_c = the probable error of one measure of a kilometre according to the suppositions (*a*), (*b*), (*c*), respectively.

m = the total number of independently observed quantities, or total number of measures of the sections,
= 20 for supposition (*a*),
= 16 for supposition (*b*),
= 8 for supposition (*c*),
$[vv]$ = sum of squares of the discrepancies between the individual measures of the sections and their means, respectively.

Then from Tables XI and XII we find*

$[vv]$ = 250·57 for supposition (*a*),
= 142·94 for supposition (*b*),
= 4·80 for supposition (*c*).

According to the method of least squares, when we have m measures of equal weight of μ unknown quantities, the probable error of a single measure is expressed by

$$\pm 0{\cdot}6745\sqrt{\frac{[vv]}{m-\mu}}.$$

In the present case the number of unknown quantities, or μ, is 4, the number of sections of the base. Hence for the several suppositions we have $m - \mu = 16, 12, 4$, respectively. These data then give

$$\begin{aligned} & \quad\ \ mm. \\ \varepsilon_a &= \pm\ 2{\cdot}67 \\ \varepsilon_b &= \pm\ 2{\cdot}33 \\ \varepsilon_c &= \pm\ 0{\cdot}74 \end{aligned}$$

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[*] The sections are treated as of equal length, the modification arising from the fact that the section from stake 30 to East Base is 130^m short of a kilometre being unimportant.

It may be observed that if the mean values in Table No. V, from which ε_c is derived, are not to a marked extent free from systematic error, as is assumed, we should have by the theory of errors

$$\varepsilon_c = \tfrac{1}{2}\varepsilon_b \sqrt{2} = 0{\cdot}71\varepsilon_b.$$

But the actual relation is $\varepsilon_c = 0{\cdot}32\varepsilon_b$, or the ratio of the two errors is less than half as great as it should be if the hypothesis (*c*) were false.

It may be observed, also, that the work of determining the tape lengths on the 100^{m} comparator of Holton Base affords an indirect means of estimating the extent of the temperature error involved in the mean of a forward and backward measure of a kilometre; and hence some additional light may be gained concerning the magnitude of the error of manipulation, which latter, as revealed in the tape measures of Holton Base, was briefly discussed in section 10 above. As shown in Chapter IV, section 15, the probable error of a single determination of a tape length on the 100^{m} comparator was $\pm\, 0{\cdot}20^{\mathrm{mm}}$. This, as explained in the chapter referred to, is due almost wholly to temperature errors in the mean of six thermometer readings. It involves the lag effect to a slight extent, and hence ought to indicate a larger temperature component in the probable error of a mean measure of a section of St. Albans Base than that shown from the actual observations, if in these latter the lag effect was well eliminated. Assuming the above error of $\pm\, 0{\cdot}20^{\mathrm{mm}}$ per tape length to apply to the field work if six thermometer readings were made, the value for the actual case, wherein but two such readings were observed, is $\pm\, 0{\cdot}20^{\mathrm{mm}} \sqrt{3}$. With this as the probable error per tape length, the probable error of the sum of ten such lengths is $\pm\, 0{\cdot}20^{\mathrm{mm}} \sqrt{30}$, and hence that for the mean of two such measures is

$$\pm\, 0{\cdot}20^{\mathrm{mm}} \sqrt{15} = \pm\, 0{\cdot}77^{\mathrm{mm}}.$$

This is slightly greater than the value of ε_c given above. It shows, so far as it goes, a satisfactory elimination of the lag effect in the means of the forward and backward measures of the sections of St. Albans Base, and indicates that the error of manipulation is really much less than was inferred in section 10 from the measures of Holton Base alone.

Lastly, it should be remarked that the value $\varepsilon_b = \pm\, 2{\cdot}33^{\mathrm{mm}}$, derived by using the night measures of St. Albans Base, is considerably larger than the corresponding value deduced from the night measures of Holton Base, namely, $\pm\, 1{\cdot}74^{\mathrm{mm}}$, given in section 10 above. The main reason for this lies, undoubtedly, in the fact that the steel sheaths used on the bulbs of the thermometers in 1892 caused much greater lagging than the aluminum sheaths used in 1891. The exceptionally favorable conditions of the standard kilometre of Holton Base for eliminating the lag effect have been alluded to in the preceding section, and this circumstance also tended to make the probable error of a single measure of a kilometre in the Holton work small.

Now as to probable error due to errors of measurement in the whole length of St. Albans base, three values dependent on the suppositions (a), (b), (c), may be derived; and a fourth value may be computed from the difference between the mean lengths for the whole base from the night measures with the two tapes. The latter process, it will be noted, should give the smallest probable error, since it does not involve the possible errors due to movements of the intermediate section stakes, though such movements, it is believed, were very small. The first three of these probable errors are

$$\begin{aligned} & \qquad\quad mm. \\ \varepsilon_a\sqrt{4/5} &= \pm\ 2{\cdot}39 \text{ according to supposition (a),} \\ \varepsilon_b\sqrt{4/4} &= \pm\ 2{\cdot}33 \text{ according to supposition (b),} \\ \varepsilon_c\sqrt{4/2} &= \pm\ 1{\cdot}04 \text{ according to supposition (c),} \end{aligned}$$

For the fourth probable error we find ± $0{\cdot}9^{mm}$.

Expressed as fractions of the length of the base, these probable errors vary from 1/1 620 000th to 1/4 300 000th.

The errors just discussed, or those developed in repeated measures of a line, appear to be the only errors of measurement which need be considered in connection with St. Albans Base. In comparison with these, the errors in grade corrections, in alignment, in transfers to permanent marks, and in the measurement of fractions of tape lengths, are trifling. As already explained, special pains were taken in the alignment of the support and marking stakes and in the determination of grade corrections. We may allow, however, for possible contributions of error from the latter sources by adopting ± $2{\cdot}0^{mm}$ as the probable error of measurement. This, it will be observed, is but little less than that given by supposition (b), which takes no account of the elimination of the lag effect, and would appear to materially underestimate the precision actually attained.

(*23*) *Probable error of base due to errors of tape lengths.*—From the evidence adduced in Chapter IV, section 16, it appears that the probable errors of the tape lengths as determined on the 100^m comparator of Holton Base can not exceed ± $0{\cdot}05^{mm}$ within such ranges of temperature as are likely to be met in ordinary work. For such temperatures as obtained during the night measures of St. Albans Base, the probable error per tape length, if computed from the formula given in the section of Chapter IV just referred to, would be somewhat less than ± $0{\cdot}05^{mm}$. One circumstance, however, leads me to adopt a larger value. It is this: If we use the mean lengths of the sections of St. Albans Base given in Table XII to compute a correction to the difference in lengths of the tapes at 0° C., the mean value of such correction is found to be ± $0{\cdot}09^{mm}$ ± 0.095^{mm}, indicating that the difference in length of the tapes at 0° C. should be $2{\cdot}54^{mm}$ instead of $2{\cdot}45^{mm}$, as given by the equations of the tapes derived in Chapter IV. Subsequently to the measurement of St. Albans Base the tapes were compared directly on a 100^{m} section of the base. An account of this work is given in section 26 below. It indicates a correction to the difference of the tapes at 0° C. of about the same amount as that just mentioned and with somewhat greater certainty. The amount of this discrepancy, it will be observed, is of about the same magnitude as its probable error derived from the expression (9), section 16, Chapter IV, namely:

$$\sqrt{(0{\cdot}055^{mm})^2 + (0{\cdot}058^{mm})^2} = \pm\, 0{\cdot}08^{mm},$$

so that no great weight can be attached to it when considered by itself. But inasmuch as the lengths of the tapes were determined by observing on them with micrometer microscopes having a power of 27 diameters, while in applying them to measure a base a hand magnifier or the unaided eye is used, it is possible that some constant error may have thus entered the measures of St. Albans Base. For this reason the extreme value of $\pm\, 0{\cdot}06^{mm}$ will be taken for the probable error of the tape lengths in those measures. This, as shown in section 16, Chapter IV, is the probable error of the tape lengths given by their equations for such temperatures as 0° and 30° C.

Since there were 38·7 tape lengths in the base, the probable error of $\pm\, 0{\cdot}06^{mm}$ per tape length gives $\pm\, 2{\cdot}32^{mm}$ as the probable error of the whole base due to this source.

(24) *Corrections to measured length of base.*—To the adopted mean length of the base given in section 22 above, three corrections are to be applied, namely: (1), correction for grades; (2), correction for error in thermometer corrections used in the computation of the lengths of the sections of the base; (3), reduction to sea level.

(1) Since the entire length only of the base is needed, the sum of the grade corrections may be applied to the adopted measured length given in section 22. This sum, the terms of which are given in section 16, is $-\,112{\cdot}7^{mm}$.

(2) The thermometer corrections which should have been applied in computing the length of the base are those given in Table X, section 17. The means of the corrections to Nos. 5598 and 5620, which alone were used to give the tape temperatures, vary from $-\,0^{\circ}{\cdot}135$ to $-\,0^{\circ}{\cdot}175$, giving thus a practically constant value of $-\,0^{\circ}{\cdot}155$, which we adopt as the correction to be applied to the means of those thermometer readings for the entire work. The corrections actually applied in computing the lengths of the sections of the base were, by reason of an error in making up a table of corrections, all numerically larger than those by the following amounts for the several sections of the base counting from the west end:

First section, 0°·028,
Second section, 0 ·022,
Third section, 0 ·021,
Fourth section, 0 ·021.

Since the thermometer corrections are all negative, and since the rate of expansion of the tapes is 1·09^{mm}, the computed length of the base is too short by

$$1{\cdot}09^{mm}\ \{10\ (0{\cdot}028 + 0{\cdot}022 + 0{\cdot}021) + 8{\cdot}7\ (0{\cdot}021)\} = 0{\cdot}97^{mm}.$$

The correction, then, to the measured length of the base from this source of error is

$$+ 0.97^{mm}.$$

(3) Complete data for reduction to sea level are not at present available. Table XIII following, however, gives the heights of the several marking tables with respect to the top of the marking stone at West Base, and also the mean heights of the several tape lengths referred to the same datum and counting from West Base toward the east. Every tape length is to be understood as 100^{m} long, except the thirty-ninth or last, which is but 70^{m} long. The *plus* sign signifies that the tables or mid tape lengths were higher, and the *minus* sign that they were lower than the datum.

TABLE XIII.—*Heights of marking tables and mean heights of tape lengths.*

Height of table at—		Mean height of tape length.	Height of table at—		Mean height of tape length.
	Feet.	*Feet.*		*Feet.*	*Feet.*
West Base	+ 1·191		Stake 21	— 6·575	— 7·112
Stake 1	+ 1·848	+ 1·519	22	— 5·184	— 5·880
2	+ 0·655	+ 1·252	23	— 5·495	— 5·340
3	— 0·079	+ 0·288	24	— 2·966	— 4·230
4	+ 2·035	+ 0·978	25	— 2·719	— 2·842
5	+ 2·894	+ 2·464	26	— 2·461	— 2·590
6	+ 4·154	+ 3·524	27	— 4·139	— 3·300
7	+ 3·938	+ 4·046	28	— 4·299	— 4·219
8	+ 2·723	+ 3·330	29	— 4·615	— 4·457
9	+ 1·269	+ 1·996	30	— 2·925	— 3·770
10	+ 3·592	+ 2·430	31	— 2·596	— 2·760
11	+ 5·800	+ 4·696	32	— 2·239	— 2·418
12	+ 3·670	+ 4·735	33	— 2·249	— 2·244
13	— 8·044	— 2·187	34	— 2·175	— 2·212
14	— 13·360	— 10·702	35	— 2·160	— 2·168
15	— 13·368	— 13·364	36	— 2·252	— 2·206
16	— 13·773	— 13·570	37	— 0·675	— 1·464
17	— 8·856	— 11·314	38	+ 0·995	+ 0·160
18	— 9·492	— 9·174	East Base	+ 2·101	+ 1·548
19	— 8·588	— 9·040	Stone, E. B.	+ 1·533	
20	— 7·648	— 8·118			

The sum of the mean heights of the tape lengths in this table is —103·7 feet. Hence the average height of the tape above the stone at West Base was —2·66 feet.

For the several sections of the base, counting them as hitherto from the west end, the average heights of the tape were +2·18 feet, —6·80 feet, + 4·37 feet, and − 1·53 feet, respectively, above the stone at West Base.

(25) *Adopted length and probable error of St. Albans Base.*—The value for the computed length of the base adopted in section 22 is

$$772B_{17} + 10638{\cdot}1^{\text{mm}}.$$

where B_{17} is the length of the iced-bar No. 17, as used on the 100^{m} comparator of Holton Base. This length is, as given in Chapter II, section 23, in terms of the International Metre,

$$B_{17} = 5^{\text{m}} - 18{\cdot}0^{\mu}.$$

Thus

$$772B_{17} = 3860^{\text{m}} - 13{\cdot}896^{\text{mm}}.$$

Hence, introducing this value, and applying the corrections for grade and error of applied temperatures giving in the preceding section, there results for the measured length of St. Albans Base $3870{\cdot}512^{\text{m}}$.

The adopted values of the independent errors whose resultant gives the probable error of this length are given in sections 22 and 23. They are—

Probable error due to errors of measurement, $\pm 2{\cdot}00^{\text{m}}$,
" " " " " of tape lengths, ± 2·32.

Taking the square root of the sum of the squares of these, we have for the total probable error of the base as measured, $\pm 3{\cdot}06^{\text{mm}}$. This amounts to 1/1 260 000th part of the base.

We write, then, for the measured length of St. Albans Base*

$$3870{\cdot}512^{\text{m}} \pm 3{\cdot}1^{\text{mm}}.$$

(26) *Results of special experiments with tapes on St. Albans Base.*—After the measurements of St. Albans Base were completed, certain experiments with the tapes were made with a view, first, to redetermine their relative lengths; and second, to learn more of their behavior under varying temperature conditions. Incidentally, also, the questions of friction of the tape on its supports and its elongation for small increments to the normal tension, were reexamined.

For these purposes, a tape length of the base presenting average conditions was selected and used as a comparator. In place of the marking stakes for this tape length, well seasoned wooden posts were set firmly in the ground. On one of these posts a zinc plate having a fine reference line ruled on it was fastened. On the other post a heavy brass plate carrying an ivory scale 5^{cm} long divided to half millimetres was secured. These two terminal marks and the intermediate support nails were put in the same horizontal plane. The tapes were stretched and aligned on this comparator in precisely the same way as in field

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* The approximate elevation of the base above sea level is 180^{m}. This gives a correction of -109^{mm} to reduce the measured length to sea level.

work, except that the stretcher at the rear end was fastened once for all in a rigid framework, and the position of the tape controlled by the slow-motion screw of the balance hook to which the tape was attached. Variations in length of the tapes were thus observed at the front end by means of the ivory scale, which could be easily read to tenths of millimetres. The thermometers provided with steel shields were placed 10^{m} from either end of the tape and read by the observers at the ends of the tape, while a thermometer with bright bulb was read at the middle of the tape length by the observer at that point, who also flipped the tape as in field work.

Observations on the tapes were made on this comparator on October 18, 19, 20, 21, and 24, 1892. On the first four dates the observations were made according to the following programme: One of the tapes being put under tension and brought to near coincidence with the reference mark at the rear end, the thermometers were first read; the observer at the rear end then brought the rear end graduation of the tape accurately to coincidence with the reference mark at that end and signaled this fact by one blast from his whistle. Thereupon the observer at the middle of the tape flipped it. As soon as the vibration of the tape ceased the observer at the rear end signaled the observer at the front end to read the scale. This being done the thermometers were again read. Then the other tape was put under tension and a similar set of observations made on it and the thermometers. The time required to make such a series of observations with the two tapes was on the average about five minutes.

On October 24 the tapes were observed on separately, one during the forenoon and one during the afternoon, and the thermometers were read but once, and that immediately after the observations on the tape.

The experiments were carried on regardless of the weather except that they were discontinued when it was rainy. They began, as a rule, about 9 a. m., and continued until about 9 p. m. Very variable conditions were thus encountered. The early mornings of the dates of observation were generally cool and foggy. As the fog was dissipated the temperature rose rapidly and then fell less rapidly toward sundown. The total temperature range was from 5° to 35° C., and the daily ranges varied from 8° to 23° for the different dates.

The results of the experiments were computed in the field by Messrs. Haskell and Woodward, and subsequently revised at the office by Mr. Safford, who also plotted all the temperature, tape-length, and cloudiness curves supplied by the data. A study of the numerical data and of their graphical representation reveals the following special characteristics of the experiments:

(1) That the temperatures of the tapes were influenced to a considerable extent by the local conditions, which forced them to undergo certain

typical changes of length dependent on the type of day. The thermometers with steel sheaths, while following the changes of tape temperature fairly well, indicate some systematic differences dependent on the local conditions and on lagging. The discrepancy between the tape and thermometric temperatures rose in extreme instances of rapidly varying temperature, as with alternating cloud and sunshine, and with alternating currents of hot and cool air, to 3° C.

(2) That the systematic character of the discrepancies between tape and thermometric temperatures disappeared largely after sundown.

In order to derive the difference in length of the two tapes the quantity $D-T_0$, where D is the distance between the terminal marks of the comparator and T_0 is the length of either tape at 0° C., was computed for each set of observations by means of the known rates of expansion of the tapes. It was assumed for any date when the two tapes were simultaneously under comparison that D remained constant, though the difference in lengths of the two tapes is practically free from any change which may have occurred from day to day in that interval. Table XIV following gives a summary of the results of the tape comparison computed in the way just described. The first column gives the date, the second the number of comparisons on that date, the third and fourth columns give the mean values of the resulting differences $D-T_0$ for the respective tapes, and the last column gives the difference in length of the two tapes at 0° C., found by subtracting the result in the third column from the corresponding one in the fourth:

TABLE XIV.—*Observed difference of tape lengths at 0° C.*

Date.	No. of comparisons.	Mean value of $D-T_0$ for—		$T_{88}-T_{85}$ at 0° C.
		Tape 85.	Tape 88.	
1892.		*mm.*	*mm.*	*mm.*
Oct. 18	14	−0·68	−3·37	+2·69
19	13	+0·10	−2·74	2·84
20	21	−0·78	−3·41	2·63
21	24	−0·70	−3·05	2·35

In deriving a final mean value of $T_{88}-T_{85}$ from the results in the last column of the above table it is thought best to assign weights proportional to the number of comparisons, except in the case of the result of October 20. On this date the observations were not extended into the night, and the result from them is for this reason of less weight than the other results. Hence we assign a weight 2 to the result of October 21 and a weight 1 to each of the other results. Their weighted mean is then

$$+2{\cdot}57^{\text{mm}} \pm 0{\cdot}07^{\text{mm}}.$$

This is larger by $0{\cdot}12^{\text{mm}}$ than the difference in length of the tapes furnished by their equations given in section 14, Chapter IV. This latter difference is

$$2{\cdot}45^{\mathrm{mm}} \pm 0{\cdot}08^{\mathrm{mm}},$$

as shown in section 23 above. The discrepancy $0{\cdot}12^{\mathrm{mm}}$ is thus only once and a half times its probable error, so that we can not infer with any considerable probability that any change in the relative lengths of the tapes has occurred since their absolute lengths were determined on the Holton comparator. It is possible, however, as remarked in section 23, that this discrepancy may arise from a lack of defining power of the hand-magnifying glasses used in the tape comparisons at St. Albans. Such power would not, it is believed, preclude the occurrence of an actual error equal to this discrepancy. It may be remarked, however, that the maximum possible error of St. Albans Base due to this discrepancy, even if real, is only slightly greater than one two-millionth part of its length.

Finally, it seems worth while to give the results of certain observations made on October 24, 1892, for the purpose of getting an idea of the error involved in the operation of laying and stretching the tape. It happened on this date that very favorable conditions for determining this error presented themselves. That is, the temperature was very steady, the sun being obscured by clouds; there was equilibrium of radiation as shown by the near equality in readings of the thermometers with black and bright bulbs, and there was little if any wind. Under these circumstances three sets of observations were made with the tape and thermometers in the same way that observations are made in measuring a line, except that the tape was not removed from the support nails of the comparator, and that the position of its front end was read on the millimetre scale instead of being marked on a zinc plate. The tension of the tape was in each case brought up from zero to the standard amount, and all other operations performed in precisely the same manner as in the work of measurement. Table XV below gives the results of these observations. The first column gives the limiting times of day of the groups of observations; the second gives the temperature observed by the thermometers; the third gives the reading of the tape on the scale at its front end, the tape being set in each case to coincidence with the mark at its rear end; the fourth gives the values of $D-T_0$, or the difference between the comparator interval and the length of the tape at 0° C.; and the last column gives the discrepancies v between the individual results for any group and their mean.

TABLE XV.—*Results of observations of error in laying a tape length.*

Time.	Temperature.	Scale reading.	$D - T_0$.	v (C—O).
1892.	°	*mm.*	*mm.*	*mm.*
Oct. 24, 4:29	11·51	12·5	+0·10	+0·05
	11·36	12·3	·14	+ ·01
	11·36	12·2	·24	— ·09
	11·26	12·1	·23	— ·08
	11·21	12·1	·17	— ·02
	11·11	12·1	·06	+ ·09
	11·16	12·1	·12	+ ·03
4:35	11·11	12·0	·16	— ·01
4:50	10·27	11·0	+0·24	+0·02
	10·32	11·0	·30	— ·04
	10·32	11·0	·30	— ·04
	10·37	11·1	·25	+ ·01
	10·32	11·1	·20	+ ·06
	10·32	11·0	·30	— ·04
	10·32	11·0	·30	— ·04
4:57	10·42	11·2	·21	+ ·05
5:04	10·22	11·1	+0·09	+0·01
	10·22	11·1	·09	+ ·01
	10·17	11·0	·13	— ·03
	10·22	11·0	·19	— ·09
	10·22	11·1	·09	+ ·01
5:10	10·17	11·1	·03	+ ·07

The residuals given in the last column of this table arise from errors in defining the positions of the tape at its ends, from friction of the tape on its supports, and from variations in temperature of the tape not registered by the thermometers. Presumably, the latter source contributed little to the resultant error during the short periods of time covered by the groups of obervations; but whether allowance be made for the temperature error or not, it appears that the operation of placing and stretching the tape was subject to very small errors. Thus, the average value of the residuals in the table is ± 0·04^{mm}, the maximum 0·09^{mm}, and the corresponding probable error of a single value of $D - T_0$ is ± 0·03^{mm}. The probable error of laying a single tape length is undoubtedly somewhat greater than this amount under such conditions as are met on the average in field work; but it seems essential to conclude from the above observations that the more formidable errors which affect such tape measurements as have been described in the preceding pages arise from defective means of getting the tape's temperature rather than from imperfections of manipulation.*

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* The results of the experiments on the elongation of the tapes for increments to the normal tension are given in Supplement B, section 5.

SUPPLEMENT A.

METHOD OF COMPUTING CORRECTION FOR FLEXURE OF 5m BAR B_{17}.

(1) *Nature of flexure considered.*—As explained in Chapter I, section 3, the normal length of the 5m bar B_{17} is the distance between its end transverse graduations when the upper surfaces of the alignment plugs are in one plane and when the longitudinal marks on those plugs are in the same straight line. Whenever the bar is flexed, the distance between the graduation lines is less than the normal distance. The flexure here considered is that due to changes in temperature of the vertical web of the Y-trough and to variable load on the same. This flexure is in a vertical plane, and it is measured by means of the long striding level in the manner described in section 9, Chapter I. The level reaches from any plug to the second adjacent plug on either side, consecutive plugs being 495^{mm} apart.

Before trying the apparatus it was feared that this flexure might be considerable during the course of a day presenting wide temperature variations, and hence adequate provision was made for measuring the curvature of the bar and computing a correction therefor. Experience, however, has shown that such fear was ill-founded. It has shown, in fact, that by means of the long striding level the bar may be made so straight in the Y-trough that no appreciable variation in its length will result in the course of any ordinary day. The largest correction for such flexure which has occurred in our experience with the bar is $0{\cdot}35^{\mu}$, which may well be regarded as a negligible quantity. But it might happen that so close an adjustment would not be secured always, or it might happen that in the course of a considerable length of time wide variations in temperature of the external air would occur and produce sensible changes through flexure in length of the bar. It seems worth while, therefore, to give the results of my study of this question, although it need not be a very important one with the iced bar apparatus.

Before proceeding to the details of the question, it may be remarked that the flexure here considered is due almost wholly to temperature changes in the web of the Y-trough. Such changes are noticeable from day to day, as a rule, rather than during a given day. In our experience they have partaken of the character of seasonal changes, or of such slow changes as accompany a hot or cold wave. Thus, for example, during the last set of measures of the standard kilometre of Holton Base, in September, 1891, the air temperature was pretty high and the bar was concave upwards. About ten days later, in early October, the air temperature had fallen about 15° C., and the bar at this time when used on the 100^m comparator was markedly convex upwards.

In the construction of the bar, pains were taken to make the upper surfaces of the alignment plugs equally distant from its neutral surface. This was accomplished with precision by means of the striding level referred to above; and we may assume that a surface similar to and parallel with the neutral surface of the bar can be drawn tangent to all of the alignment plugs. The sinuosities of this surface, or, more definitely, of the line in this suface cut out by the vertical alignment plane of the bar, are measured with the striding level; and they accord sensibly with the sinuosities of the corresponding line in the neutral surface.

(2) *Theory of computation.*—If s be the length of any curve whose rectangular co-ordinates are x and y,

$$ds = dx\left(1 + \frac{dy^2}{dx^2}\right).$$

In the case of the 5^m bar here considered, dy/dx is always very small, so that we may write

$$ds = dx + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2 dx - \ldots,$$

whence the correction for curvature of the bar is

$$[s - x] = \tfrac{1}{2}\int\left(\frac{dy}{dx}\right)^2 dx \tag{1}$$

taken between proper limits.

Consider the sinuous curve defined above. If it could be completely determined, it would be tangent to the plugs 1, 3 . . . 11 whereon the long striding level is applied. Refer this curve to a system of rectangular axes, the origin being at plug No. 1 and the axis of x being the straight line joining the terminal plugs 1 and 11. The level readings at any time give the co-ordinates of the plugs 3, 5 . . . 11 relatively to a system of axes having the same origin and a horizontal line tangent to plug No. 1 as axis of obscissas. Calling the co-ordinates in the latter system x', y', and denoting the sine of the inclination of the two systems of axes by σ, we have, since σ is always small,

$$x = x',$$
$$y = -\sigma x' + y'.$$

The quantity σ will vary from time to time, or from set to set of observations, while the values of y for any plug will remain constant for a given adjustment of the bar and for stable temperature of the Y-trough.

Correction for flexure of iced bar B_{17}.

Let $h_1, h_2, \dots h_5$ be the measured values at any time of the relative heights of plugs 1 and 3, 3 and 5, . . . 9 and 11, respectively; and let $v_1, v_2, \dots v_5$ be the most probable corrections to $h_1, h_2 \dots$ respectively. Thus if we call the distance between the feet of the striding level l, and denote the ordinates of the plugs 3, 5, 7, 9, 11 in the first system of co-ordinates by $y_1, y_2, \dots y_5$, we have

$$\begin{aligned}
y_1 &= h_1 + v_1 - l\sigma, \\
y_2 &= h_1 + v_1 + h_2 + v_2 - 2l\sigma, \\
y_3 &= h_1 + v_1 + h_2 + v_2 + h_3 + v_3 - 3l\sigma, \\
y_4 &= h_1 + v_1 + h_2 + v_2 + h_3 + v_3 + h_4 + v_4 - 4l\sigma, \\
y_5 &= h_1 + v_1 + h_2 + v_2 + h_3 + v_3 + h_4 + v_4 + h_5 + v_5 - 5l\sigma.
\end{aligned}$$

From these it follows that

$$\begin{aligned}
l\sigma + y_1 \phantom{{}- y_1} - h_1 &= v_1, \\
l\sigma + y_2 - y_1 - h_2 &= v_2, \\
l\sigma + y_3 - y_2 - h_3 &= v_3, \\
l\sigma + y_4 - y_3 - h_4 &= v_4, \\
l\sigma + y_5 - y_4 - h_5 &= v_5.
\end{aligned} \tag{2}$$

Now the function of x, $\varphi(x)$ say, which is to represent the curve whose length is sought must evidently fulfill the following conditions, namely:

$$\begin{aligned}
\varphi(x) &= 0 \text{ for } x = 0, \\
\varphi(x) &= 0 \text{ for } x = 5l.
\end{aligned}$$

The most general form of such a function is

$$y = \varphi(x) = a \sin 2\pi \frac{x}{10l} + b \sin 4\pi \frac{x}{10l} + \dots, \tag{3}$$

wherein $a, b, \dots$ are constants. Since the number of observation equations (2) is limited to 5 (in this case), the number of constants or terms in (3) must be limited to 4.

Giving to x in (3) the values $l, 2l, \dots 5l$ successively, the values of $y_1, y_2, \dots y_5$ result; and if we write for brevity

$$\begin{aligned}
a' &= 2\,a \sin \tfrac{1}{10}\pi, \\
b' &= 2\,b' \sin \tfrac{2}{10}\pi, \\
&\dots;
\end{aligned} \tag{4}$$

$$\alpha = \tfrac{1}{10}\pi,\ \beta = \tfrac{2}{10}\pi,\ \gamma = \tfrac{3}{10}\pi, \dots;$$

the above equations (2) become

$$\begin{array}{lllll}
l\sigma + a' \cos \alpha & + b' \cos \beta & + c' \cos \gamma & + \dots & - h_1 = v_1, \\
l\sigma + \cos 3\alpha & + \cos 3\beta & + \cos 3\gamma & + \dots & - h_2 = v_2, \\
l\sigma + \cos 5\alpha & + \cos 5\beta & + \cos 5\gamma & + \dots & - h_3 = v_3, \\
l\sigma + \cos 7\alpha & + \cos 7\beta & + \cos 7\gamma & + \dots & - h_4 = v_4, \\
l\sigma + \cos 9\alpha & + \cos 9\beta & + \cos 9\gamma & + \dots & - h_5 = v_5.
\end{array} \tag{5}$$

By the method of least squares there result from equations (4) and (5) the following values, wherein n is successively 1, 3, 5 . . . 9:

$$l\sigma = \tfrac{1}{5}\,[h],$$

$$a = \frac{[h \cos n\,\alpha]}{5 \sin \alpha}, \qquad b = \frac{[h \cos n\,\beta]}{5 \sin \beta}, \qquad (6)$$

$$c = \frac{[h \cos n\,\gamma]}{5 \sin \gamma}, \; \ldots .$$

The constants a, b, c, . . . in (3) are thus found from the observed quantities h.

If now for brevity we put

$$\theta = 2\,\pi\,\frac{x}{10l},$$

equation (3) may be written

$$y = a \sin \theta + b \sin 2\,\theta + c \sin 3\,\theta + \ldots ,$$

whence,

$$\frac{dy}{dx} = \frac{\pi}{5l}\,(a \cos \theta + 2\,b \cos 2\,\theta + 3\,c \cos 3\,\theta + \ldots).$$

This value substituted in equation (1) gives

$$\Big[s - x\Big]_0^{5l} = \tfrac{1}{2}\int_0^{5l} \left(\frac{dy}{dx}\right)^2 dx$$

$$= \frac{\pi}{10l}\int_0^{\pi} (a \cos \theta + 2\,b \cos 2\,\theta + \ldots)^2\, d\,\theta \qquad (7)$$

$$= \frac{\pi^2}{20l}\,(a^2 + 4\,b^2 + 9\,c^2 + \ldots).$$

The last member of (7) gives the correction for flexure $[s - x]$ by an easy computation after the values of a, b, c, . . . have been found from equations (6). In all cases to which the formulas have been applied the first three terms, or those in a, b, c, have sufficed.

(3) *Example of computation.*—As an example of the application of the preceding formulas the following case, which is the case of greatest flexure observed with the apparatus may suffice. It occurred on July

Correction for flexure of iced bar B_{17}.

27, 1892. The observed quantities h and the computed quantities $h \cos n\, \alpha$, $h \cos n\, \beta$, etc., are—

h.	$h \cos n\, \alpha$.	$h \cos n\, \beta$.	$h \cos n\, \gamma$.	$h \cos n\, \delta$.
mm.	*mm.*	*mm.*	*mm.*	*mm.*
— 0·324	— 0·308	— 0·262	— 0·191	— 0·100
— ·108	— ·064	+ ·033	+ ·103	+ ·087
— ·018	·000	+ ·018	·000	— ·018
+ ·288	— ·169	— ·089	+ ·274	— ·233
+ ·720	— ·685	+ ·582	— ·423	+ ·222
Sum	— 1·226	+ 0·282	— 0·237	— 0·042

Then the values of a, b, c, d, found from (6), and a^2, $4b^2$, etc., for use in (7) are

mm.

$$a = -0{\cdot}793, \quad a^2 = 0{\cdot}629,$$
$$b = +\ {\cdot}096, \quad 4b^2 = {\cdot}037,$$
$$c = -\ {\cdot}058, \quad 9c^2 = {\cdot}030,$$
$$d = -\ {\cdot}009, \quad 16d^2 = {\cdot}001,$$

$$\text{Sum} \quad 0{\cdot}697$$

Hence, since $l = 990^{\text{mm}}$,

$$[s - x] = 0{\cdot}35^{\mu}.$$

SUPPLEMENT B.

MATHEMATICAL THEORY OF METALLIC TAPES.

(1) *Object of investigation.*—The use of long metallic tapes or wires in measures of precision gives rise to a number of questions which can only be answered by a competent mathematical theory or by a series of elaborate experiments. Thus, in the use of such tapes it is essential to know how to compute the change in length due to a change in tension, the change in length due to a change in the number of supports on which the tape rests, the change in length due to a change in weight per unit length of the tape, etc. It is the object of this brief investigation to give so much of the theory of such tapes or wires as is needed to answer these practical questions.

(2) *Equations of equilibrium of elastic tapes.*—The theory of the equilibrium of elastic strings, tapes, etc., is given in one form or another in various treatises on mechanics. It generally proceeds, however, on the assumption of "perfect flexibility" of the material of the tape, or on some other doubtful assumption as to the internal stresses of that material. Properly speaking, the inquiry belongs to the theory of elasticity, and its correct basis has been clearly set forth by Lamé in sections 39–41 of his Leçons sur la Théorie Mathématique de L'Élasticité des Corps Solides.*

A tape as here considered is an elastic body of small cross section subject to tension applied at its ends and to any bodily forces, but free from surface tractions. The cross section must be so small that the tension per unit area may be considered constant over the entire area of any section. Let ω be the cross section and ρ the density of the tape at any point of its length defined by rectangular coordinates x, y, z. Let T be the total normal tension on the section ω, and denote the rectangular components of the bodily forces by X, Y, Z, respectively. Then the investigation of Lamé just referred to shows that the equations of equilibrium of the element of mass $\rho\omega ds$ are

$$\begin{aligned} \frac{d\left(T\frac{dx}{ds}\right)}{ds} + \rho\omega X &= 0, \\ \frac{d\left(T\frac{dy}{ds}\right)}{ds} + \rho\omega Y &= 0, \\ \frac{d\left(T\frac{dz}{ds}\right)}{ds} + \rho\omega Z &= 0. \end{aligned} \tag{1}$$

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* Second edition, Gauthier-Villars, Paris, 1866.

These equations contain the general theory. To adapt them to our purposes we observe, first, that the only bodily force acting on a tape when used as a standard of length is gravity, and, secondly, that the curve assumed by such a tape lies wholly in a vertical plane. Thus, if we take this plane to be that of xy, y being considered positive upwards, we may write $X = 0$, and $Y = -g$ in the first two of equations (1), and they become

$$\begin{aligned} \frac{d\left(T\frac{dy}{ds}\right)}{ds} &= \rho\omega g, \\ \frac{d\left(T\frac{dx}{ds}\right)}{ds} &= 0. \end{aligned} \tag{2}$$

Further, let ρ_0, ω_0, ds_0 be the density, cross section, and length of the element of the tape considered when it is not under tension. ds is thus the stretched and ds_0 the unstretched length of this element. Also, let μ be the reciprocal of the modulus of elasticity of the tape multiplied by the area of its cross section. Then, since the mass of the element remains constant and Hooke's law applies to its elongation,

$$\begin{aligned} \rho\omega ds &= \rho_0\omega_0 ds_0, \\ ds &= (1 + \mu T)ds_0. \end{aligned} \tag{3}$$

Since the tapes we consider are of uniform cross section when unstretched, we may write for the weight per unit of unstretched length

$$w = \rho_0\omega_0 g. \tag{4}$$

By means of the relations (3) and (4), equations (2) may be given the following forms:

$$\begin{aligned} (1 + \mu T)\frac{d\left(T\frac{dy}{ds}\right)}{ds} &= w, \\ \frac{d\left(T\frac{dx}{ds}\right)}{ds} &= 0, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{d\left(T\frac{dy}{ds}\right)}{ds_0} &= w, \\ \frac{d\left(T\frac{dx}{ds}\right)}{ds_0} &= 0, \end{aligned} \tag{6}$$

(3) *Integration of equations.*—A first integral of the second of equations (5) is

$$T\frac{dx}{ds} = \text{a constant} = \tau, \text{ say,}$$

where τ is the horizontal component of the tension, or the tension at the lowest point of the curve assumed by the tape. From this equation

$$T = \tau\frac{ds}{dx},$$

which, substituted in the first of (5), gives, after some obvious changes,

$$\frac{\tau\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} + \mu\tau^2\frac{d}{dx}\left(\frac{dy}{dx}\right) = w.$$

The first integral of this equation, if we take the origin at the lowest point of the curve so that $dy/dx = 0$ for $x = 0$, is

$$\tau\log\left(\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) + \mu\,\tau^2\,dy = wx.$$

In order to integrate again, we may expand dy/dx as a function of the small quantity μ by means of Maclaurin's series. Thus, the last equation gives for $\mu = 0$,

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{+wx/\tau} - e^{-wx/\tau}\right),$$

$$\frac{d\left(\dfrac{dy}{dx}\right)}{d\mu} = \tfrac{1}{4}\,\tau\left(e^{+2wx/\tau} - e^{-2wx/\tau}\right),$$

$$\cdots,$$

in which e is the Napierian base. On account of the minute value of μ for metallic tapes, the higher differential coefficients of dy/dx with respect to μ are not needed. Hence, to terms of the first order in μ, we have

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{+wx/\tau} - e^{-wx/\tau}\right) + \tfrac{1}{4}\,\tau\left(e^{+2wx/\tau} - e^{-2wx/\tau}\right)\mu,$$

whence, since the origin is on the curve,

$$\begin{aligned} y &= \tfrac{1}{2}\,\frac{\tau}{w}\left(e^{+\frac{1}{2}wx/\tau} - e^{-\frac{1}{2}wx/\tau}\right)^2 \\ &\quad + \tfrac{1}{8}\,\frac{\tau^2}{w}\left(e^{+wx/\tau} - e^{-wx/\tau}\right)^2\mu. \end{aligned} \tag{7}$$

For brevity put

$$a = \frac{w}{\tau}. \tag{8}$$

Then equation (7) gives by expansion

$$\begin{aligned} y &= \tfrac{1}{2}\, a x^2 + \tfrac{1}{24}\, a^3\, x^4 + \tfrac{1}{720}\, a^5\, x^6 + \ldots \\ &+ (\tfrac{1}{2}\, x^2 + \tfrac{1}{6}\, a^2\, x^4 + \tfrac{1}{45}\, a^4\, x^6 + \ldots)\, \mu\, w. \end{aligned} \tag{9}$$

Referring now to equations (6), their first integrals are

$$\begin{aligned} T\,\frac{dy}{ds} &= w\,(s_0 + \beta), \\ T\,\frac{dx}{ds} &= \tau, \end{aligned} \tag{10}$$

where β and τ are constants, the latter being as explained above the horizontal component of the tension. The sum of the squares of (10) gives

$$T^2 = \tau^2 + w^2\,(s_0 + \beta)^2. \tag{11}$$

Eliminating T and ds from (10) by means of (3) and (11), there result

$$dy = \mu\, w\,(s_0 + \beta)\, ds_0 + \frac{w\,(s_0 + \beta)\, ds_0}{\sqrt{\tau^2 + w^2\,(s_0 + \beta)^2}},$$

$$dx = \mu\, \tau\, ds_0 + \frac{\tau\, ds_0}{\sqrt{\tau^2 + w^2\,(s_0 + \beta)^2}}.$$

If the origin be taken on the curve, the integrals of these equations are*

$$\begin{aligned} y &= \tfrac{1}{2}\, a\,(1 + \mu\,\tau)\,\{(s_0 + \beta)^2 - \beta^2\} - \tfrac{1}{8}\, a^3\,\{(s_0 + \beta)^4 - \beta^4\} + \ldots, \\ x &= (1 + \mu\,\tau)\, s_0 - \tfrac{1}{6}\, a^2\,\{(s_0 + \beta)^3 - \beta^3\} + \ldots \end{aligned} \tag{12}$$

The differential equations of equilibrium (5) and (6) are thus completely integrated, and the form of the curve assumed by the tape is assigned by either (7), (9), or (12).

It will be essential in what follows to have an expression for the unstretched length of the tape s_0 in terms of x for the case in which the ends of the tape are in the same horizontal line. In this case, if we take the origin at the lowest point of the curve, or at the middle point of its length, $\beta = 0$, and the second of (12) gives

$$x = (1 + \mu\tau)\, s_0 - \tfrac{1}{6}\, a^2\, s_0^{\,3} + \tfrac{3}{40}\, a^4\, s_0^{\,5} - \cdots;$$

---

*The finite forms of the integrals of these equations are not so well adapted to our purposes as the series given in the text.

whence, by reversion, we have to a sufficient degree of approximation

$$s_0 = (1 - \mu\tau)\, x + \tfrac{1}{6}\, a^2\, x^3 - \cdot\cdot\cdot\cdot. \tag{13}$$

(4) *Sag of tape.*—The sag of a tape is the vertical distance of its lowest point below either of its adjacent points of support. It is the value of y in equation (7) or (9) when x is the abscissa of the point of support to which the sag is referred. When the supports of a section of a tape are in the same horizontol line, the sag is found from (9) by substituting for x half the span of the section.

It will be of interest to consider a numerical example with reference to the tapes used with the tape base apparatus described in Chapter IV. Thus we have for steel tape No. 85:

$$\begin{aligned} &\text{Weight per metre of length} = w = 22{\cdot}32 \text{ grammes,} \\ &\text{applied tension (25·5 pounds)} = \tau = 11567 \text{ grammes.} \end{aligned}$$

Hence the constant

$$a = (w/\tau) = 193 \times 10^{-5}.$$

We also have for this tape

$$\begin{aligned} \mu &= 16 \times 10^{-9} \text{ for the gramme as unit,} \\ &= 450 \times 10^{-9} \text{ for the ounce as unit.} \end{aligned}$$

With these data the computation of the sag for a span of 40^{m} ($x = 20^{\text{m}}$) by equation (9) runs as follows:

$$\begin{array}{rcr} & & m. \\ \frac{1}{2}\, a\, x^2 & = & 0{\cdot}386000 \\ \frac{1}{24}\, a^3\, x^4 & = & {\cdot}000048 \\ \frac{1}{2}\, \mu\, w\, x^2 & = & {\cdot}000072 \\ \hline \text{Sag} = y & = & {\cdot}386120 \end{array}$$

It appears therefore that for such a tape and such a tension, the sag for spans less than 40^{m} in length is given with sufficient accuracy by the simple formula $y = \frac{1}{2}\, a\, x^2$.

(5) *Change in length of tape due to change in tension.*—Suppose a tape supported at a number of equidistant points all in the same horizontal line. Let

n = the number of sections into which the tape is divided by the equidistant supports,
l = the span of any such section,
l_0 = the unstretched length of the tape in the same section,
L = the normal length of the entire tape, or right line distance between its terminal marks when under standard tension $= \Sigma\, l_0$

Theory of Metallic Tapes.

Making $\beta = 0$ in the second of equations (12) (since the supports are assumed to lie in a horizontal line), and putting

$$x = \tfrac{1}{2}\, l, \qquad s_0 = \tfrac{1}{2}\, l_0,$$

we get

$$l = (1 + \mu\,\tau)\, l_0 - \tfrac{1}{24}\, a^2\, l_0^3 + \cdot\cdot\cdot . \tag{14}$$

Since the sections are of equal length we have

$$\Sigma\, l = L = (1 + \mu\,\tau)\, n\, l_0 - \tfrac{1}{24}\, n\, a^2\, l_0^3 + \cdot\cdot\cdot . \tag{15}$$

Differentiating this with respect to τ, the change in length of the tape, $\varDelta L_1$ say, due to a change $\varDelta\tau$ in τ is found to be

$$\varDelta L_1 = n l_0 \mu \varDelta\tau + \tfrac{1}{12}\, a^2 n l^3_0 \frac{\varDelta\tau}{\tau}. \tag{16}$$

Obviously one may use l for l_0 in this formula, a roughly approximate value of l_0 being in general quite sufficient for computing the small quantity $\varDelta L_1$.

Example.—For tape No. 85 when supported at equidistant intervals of 10^{m} we have—

$n = 10,$
$l = 10^{\mathrm{m}},$
$w = 22{\cdot}32$ grammes per metre,
$\tau = 25{\cdot}5$ pounds $= 11567$ grammes,
$a^2 = (w/\tau)^2 = 372 \times 10^{-8},$
$\mu = 16 \times 10^{-9}$, for gramme as unit,
$= 450 \times 10^{-9}$ for ounce as unit.*

Hence for $\varDelta\tau = 1$ ounce,

$$\begin{array}{lr} & mm. \\ n l \mu \varDelta\tau = 10 \times 10^{\mathrm{m}} \times 450 \times 10^{-9} & = 0{\cdot}0450 \\ \tfrac{1}{12} a^2 \mu l^3 \dfrac{\varDelta\tau}{\tau} = \tfrac{1}{12} \times 372 \times 10^{-8} \times 10^7 \times \tfrac{1}{408} & = 0{\cdot}0076 \\ \hfill \varDelta L_1 & = 0{\cdot}0526 \end{array}$$

It may be observed that the quantity $\varDelta L_1$ can be measured directly by increasing and decreasing the tension τ in the vicinity of its standard value. From a small number of observations made in 1891 by this process the value $\varDelta L_1 = 0{\cdot}047^{\mathrm{mm}}$ per ounce was found for tape 85 with $\tau = 25{\cdot}5$ pounds. A more elaborate series of observations made in

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* Since the cross section of the tape is $6{\cdot}34^{\mathrm{mm}} \times 0{\cdot}47^{\mathrm{mm}}$, or $0{\cdot}0298$ square centimetres, the value of μ corresponds to a modulus of $2{\cdot}1 \times 10^6$ kilogrammes per square centimetre or 30×10^6 pounds per square inch. This value of μ was determined by measuring the variations of length of a piece of the tape due to variations in tension when the tape lay on the flat Mural Standard of the Coast and Geodetic Survey.

1892 with the same tape and tension gave $\Delta L_1 = 0{\cdot}053^{\mathrm{mm}}$ per ounce, which agrees well with the value just computed from the weight per unit of length and the modulus of elasticity of the tape. The value $\Delta L_1 = 0{\cdot}05^{\mathrm{mm}}$ may be adopted for tapes 85 and 88 when $\tau = 25{\cdot}5$ pounds.

(6) *Change in length of tape due to change in number of supports.*—Suppose the tape divided by its equidistant supports (all in the same horizontal line) first, into n_1, sections of length l_1, and secondly, into n_2 sections of length l_2. Then, assuming the tension the same in both cases, the difference in distance between the terminal marks of the tape, ΔL_2 say, as shown by equation (14) is

$$\Delta L_2 = \Sigma\,(l_2 - l_1) = \tfrac{1}{24}a^2\,(n_1 l_1^3 - n_2 l_2^3), \qquad (17)$$
$$n_2 > n_1.$$

When a tape is supported throughout its length on a horizontal plane, as on a mural standard, n_2 in (17) becomes infinite and l_2 infinitesimal. If we denote the length of the tape in this case by $L_\infty = \Sigma l_2$ and call its length in the case of n sections $L_{\mathrm{n}} = \Sigma l$, equation (17) gives

$$L_\infty - L_{\mathrm{n}} = \tfrac{1}{24}\,a^2\,nl^3. \qquad (18)$$

To find the shortening of a tape due to the omission of one support, we have only to make $n_2 = 2$, $n_1 = 1$, and $l_1 = 2l_2 = 2l$ say in (17). It gives thus

$$\Delta L_2 = \tfrac{1}{4}\,a^2\,l^3, \qquad (19)$$

l being the length of a section when no supports are omitted.

Similarly the omission of m consecutive supports shortens a tape by

$$\tfrac{1}{24}\,m\,(m+1)\,(m+2)\,a^2\,l^3,$$

where l is the length of a section when no supports are omitted.

Numerical examples.—(1) For the 100^{m} tape No. 85, $n_1 = 5$ and $l_1 = 20^{\mathrm{m}}$ when the supports are 20^{m} apart; and $n_2 = 10$ and $l_2 = 10^{\mathrm{m}}$ when the supports are 10^{m} apart. Hence in this case, since $a^2 \doteq 372 \times 10^{-8}$, when the tension is 25·5 pounds, (17) gives

$$\Delta L_2 = \tfrac{1}{24} \times 372 \times 10^{-8}\left(5\,(20)^3 - 10\,(10)^3\right) = 4{\cdot}65^{\mathrm{mm}}.$$

(2) For the same tape the omission of one support, supposing them to be 10^{m} apart, or l to be 10^{m}, shortens it by $0{\cdot}93^{\mathrm{mm}}$ as shown by (19).

(3) Equation (18) shows that tape No. 85 would be $1{\cdot}55^{\mathrm{mm}}$ longer when lying on a horizontal surface than it is when supported at equidistant intervals of 10^{m}, the tension in both cases being 25·5 pounds.

(7) *Change in length of tape due to change in weight per unit length.*—It is of some interest to have an expression for the change in length of a tape due to a change in its weight. Such changes may occur from wear or from a deposit of dew on the tape. An expression is found

by differentiating (15) with respect to w. Thus, calling the change in question $\varDelta L_3$, we find, since $a = w/\tau$,

$$\varDelta L_3 = -\tfrac{1}{12}\, n\, a^2\, l^3\, \frac{\varDelta w}{w}. \qquad (20)$$

This formula shows, for example, that when the 100^{m} tape No. 85 is supported at equidistant intervals of 10^{m} and is under a tension of 25·5 pounds, or when.

$$\begin{aligned} l &= 10^{\mathrm{m}}, \\ n &= 10, \\ a^2 &= 372 \times 10^{-8}, \\ \varDelta L_3 &= -\,3{\cdot}1^{\mathrm{mm}} \times \frac{\varDelta w}{w}. \end{aligned}$$

Thus, in order to produce a change of the millionth part in this tape's length, or in order to make $\varDelta L_3 = 0{\cdot}1^{\mathrm{mm}}$, we must have $\varDelta w = w/31$. Obviously, no such change as this is likely to occur in the length of a tape.

(8) *Change in length of tape due to slope of its supports.*—In the use of long tapes for measuring lines of precision the supports for any tape length are placed in the same straight line. It is plain that if this line is inclined, the distance between the terminal marks of the tape will be greater than when the line is horizontal. It is plain also that the elongation of a tape due to ordinary slopes can not be very great, but their use in refined work requires that we should know the nature and extent of such elongation.

The question may be investigated in the following manner: Equations (12) give the rectangular coordinates of any section of the tape. The origin for these coordinates is any point on the curve, and is defined by the constant β. For the present purpose let us place the origin at the left-hand support of the section considered, and let $x_1\ y_1$ be the coordinates of the other end or support of the section. Then the span of the section, or its length as used in measurement, is $\sqrt{x_1^2 + y_1^2}$. But the length of the same section when the supports are in the same horizontal line is given by (14) as

$$l = (1 + \mu\,\tau)\, l_0 - \tfrac{1}{24}\, a^3\, l_0^3 + \ldots,$$

where l_0 is the unstretched length of the tape in the section. The elongation we desire is then

$$\sqrt{x_1^2 + y_1^2} - l$$

per section of the tape. To evaluate this quantity we have to find β for the assumed position of the origin, to substitute its value in the equations (12), and thus get x_1 and y_1 in terms of s_0, or rather l_0 as we shall here denote the unstretched length of tape in the section.

Since in the first of (12) $y = y_1$ for $s_0 = l_0$, we find by Maclaurin's series

$$\begin{aligned}\beta = &-\frac{1}{2}l_0 + \frac{1}{a}\left(\frac{y_1}{l_0}\right)(1+\mu\tau)^{-1}\\ &+\left(\frac{1}{2a} - \frac{1}{16}a^2l_0^2\right)\left(\frac{y_1}{l_0}\right)^3(1+\mu\tau)^{-4}\\ &+\ldots,\end{aligned}$$

or nearly enough for our purpose

$$\beta = -\frac{1}{2}l_0 + \frac{1}{a}\left(\frac{y_1}{l_0}\right) + \frac{1}{2a}\left(\frac{y_1}{l_0}\right)^3.$$

Again, since y_1 is always small with respect to x_1, we may use the approximate expression

$$\sqrt{x y_1^2 + y_1^2} - l = x_1 + \frac{y_1^2}{2x_1} - l.$$

Hence, substituting the values of β, x_1, and y_1 as just explained, there results for the elongation per section of the tape to terms of the fourth order inclusive in the slope $\left(\frac{y_1}{l_0}\right)$,

$$\frac{1}{12}a^2l_0^3\left(\frac{y_1}{l_0}\right)^2 + \frac{1}{8}l_0\left(\frac{y_1}{l_0}\right).$$

Let n denote the number of equal sections of the tape, L the total length of the tape, h the difference in altitude of its ends, and ΔL_4 the total elongation due to its slope. Then, since

$$y_1 = h/n \text{ and } l_0 = L/n,$$

the above expression gives

$$\Delta L_4 = \frac{1}{12n^2}a^2L^3\left(\frac{h}{L}\right)^2 + \frac{1}{8}L\left(\frac{h}{L}\right)^4. \tag{21}$$

For a tape 100^{m} long the value of h will rarely if ever equal 5^{m}. For such a value and for the tapes Nos. 85 and 88 when supported at equidistant intervals of 10^{m} we have, if $a^2 = 372 \times 10^{-8}$,

$$\begin{aligned}\frac{1}{12n^2}a^2L^3\left(\frac{h}{L}\right)^2 &= \frac{1}{1200}\times\frac{372}{10^8}\times(100)^3\left(\frac{5}{100}\right)^2 = 0{\cdot}008^{\text{mm}},\\ \frac{1}{8}L\left(\frac{h}{L}\right)^4 &= \frac{1}{8}\times 100\times\left(\frac{5}{100}\right)^4 \qquad = 0{\cdot}078.\end{aligned}$$

Thus in this case $\Delta L_4 = 0{\cdot}086^{\text{mm}}$. Hence for such a tape we may regard this correction as insensible for slopes less than 5 per cent or values of h less than 5^{m}.*

(9) *Change in length of tape due to inequality in spacing of its supports.*—In setting the support nails for a tape length it is impossible to make the intervals between them exactly equal. Practically, therefore, it is important to know how great the inequality may be without introducing appreciable error in precise measurements. This question is easily answered by means of equation (14). Thus, let $l_1, l_2, \ldots l_n$ be the lengths or spans of the n sections of the tape supposed unequal; and let $l_0 + \Delta l_1, l_0 + \Delta l_2 \ldots l_0 + \Delta l_n$ be the corresponding unstretched lengths of the tape for the same sections. Then, according to (14), observing that $\Sigma \Delta l = 0$,

$$\Sigma l_n = (1 + \mu\tau) n l_0 - \frac{1}{24} n a^2 l_0^3 - \frac{1}{8} a^2 l_0 \Sigma (\Delta l)^2.$$

If the sections were all of equal length l we should have, to the same order of approximation, for the whole length of the tape

$$nl = (1 + \mu\tau) n l_0 - \frac{1}{24} n a^2 l_0^3.$$

Calling the difference of these two expressions ΔL_5, we have

$$\Delta L_5 = \frac{1}{8} a^2 l_0 \Sigma (\Delta l)^2. \qquad (22)$$

This gives the shortening of the tape due to the departures from equality of the several sections. To get an idea of its extent, suppose $\Delta l_1 = - \Delta l_2 = \Delta l_3 = - \Delta l_4 \ldots = 0{\cdot}1^{m}$ in the case of the 100^{m} tapes when supported at equidistant intervals of 10^{m} and when $a^2 = 372 \times 10^{-8}$. With these data (22) gives

$$\Delta L_5 = \frac{1}{8} \times 372 \times 10^{-8} \times 10 \times 10 \times 10^{-2} = 0{\cdot}46^{\mu}.$$

It is seen, therefore, that in spacing the supports for such a tape, equality to the nearest decimeter, which is easily attained, is amply sufficient.

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*This investigation ignores the effect of the change in $a^2 = (w/\tau)^2$ due to the slope of the tape. This change is $\Delta a^2 = -2a^2(\Delta\tau/\tau) = a^2(h/L)^2$, which amounts to only 1/400th part of a^2 in the case cited.

III. THE NEW SECONDARY BASE-APPARATUS OF THE COAST AND GEODETIC SURVEY AS USED IN THE MEASUREMENT OF THE HOLTON BASE, INDIANA.

A report by O. H. TITTMANN, Assistant.

Submitted for publication November 30, 1891, and corrected for adopted value of five-metre standard in February, 1894.

The duty of measuring the Holton Base with the new secondary apparatus, (five-metre contact slide rods 13 and 14) was assigned to me by a letter of instructions from the Superintendent of March 28, 1891.

As soon as circumstances permitted I reported to Assistant A. T. Mosman, in camp at Holton. He had made all the necessary preparation of selecting, clearing, and marking the line and had organized the force of men. As soon therefore as the instruments arrived the measuring operations were begun by me on July 28. The men had not been previously drilled in their duties, but gained the requisite skill and rapidity of manipulation as the measurement progressed.

The base, including two extra measurements of the standard kilometre, was measured twice between July 28 and August 13.

As soon as the measurement had been finished, the lengths of the rods were determined on the Standard 100-metre Comparator, the length of which had been found by Prof. R. S. Woodward, Assistant Coast and Geodetic Survey, and the relation of the thermometer indications to the actual temperatures of the bars was investigated.

This report will therefore first treat of the results of this investigation and its bearing on the length of the base, after which the details of the measurements of the base line and of the apparatus, will be described.

COEFFICIENT OF EXPANSION OF THE MEASURING RODS.

As will be seen by reference to the latter part of this report, this was determined at the office in May, 1891, between 0° and 37° centigrade.

The temperature range was ample for a mean value, but the intermediate points were not sufficiently numerous to deduce the value of the expansion depending on the second power of the temperature.

The results are:

$$\begin{aligned} & \qquad\qquad\qquad\quad \mu. \\ \text{For rod 14 the expansion} &= 58{\cdot}509 \text{ per degree C.} \\ & \pm {\cdot}131 \\ \text{For rod 13 the expansion} &= 58{\cdot}861 \text{ per degree C.} \\ & \pm {\cdot}127 \end{aligned}$$

giving as the mean expansion for the two rods

$$\begin{array}{l} \mu. \\ 58{\cdot}7 \text{ per degree C.} \\ \pm\ {\cdot}09 \end{array}$$

LENGTH OF THE RODS.

The lengths of the rods depend upon the five-metre steel bar No. 17 whose length was determined at this office by Assistant Woodward, by comparing it with National Prototype Metre No. 21. In these comparisons the Prototype as well as No. 17 was kept in melting ice.

The resulting length of No. 17 was found to be at

$$\begin{array}{r} \mu. \\ 0^{\circ}\text{ C.} = 5^{m} - 18{\cdot}0 \\ \pm\ \ 0{\cdot}8 \end{array}$$

No. 17 is a line measure, while rods 13 and 14 are end measures. Before taking the latter into the field they were compared on three days with No. 17. The latter was kept in melting ice while Nos. 13 and 14 were at, or very near, the temperature of the comparing vault, about 22°·2 C.

The comparisons were made under microscopes, the pointings on the rods being made on the agate knife edge on the one hand, and on the other by pointing on the space between the direct and reflected images of a spider thread stretched across the plane end surface of the rod.

These comparisons gave

$$\begin{array}{rl} & \quad\ \ \mu \quad\ \ \mu. \\ \text{Rod } 13 = 5^{m} + 1278 \pm 4{\cdot}0 & \\ & \text{at } 22^{\circ}{\cdot}2\text{ C.} \\ \text{Rod } 14 = 5^{m} + 1297 \pm 3{\cdot}0 & \end{array}$$

It was subsequently found in the field that the knife edges and abutting surfaces were not in perfect contact while the bars were used for measurement, owing to the circumstance that the contact surfaces are not at right angles to the longitudinal axes of the bars. The effect of this mechanical defect was measured at Holton with considerable precision by means of a microscope mounted over the ends of the rods when in contact.

The amount of this correction 30^{μ} when added to the above values gives for the combined length of the bars at

$$\begin{array}{r} \mu. \\ 22^{\circ}{\cdot}2\text{ C } 10^{m} + 2605 \\ \pm\ 5 \end{array}$$

A few days after the measurement of the base had been finished, the standard 100-metre comparator was measured with this apparatus in order to deduce the combined length of the rods.

The measurements used for determining the lengths are given in full:

Date.	Hour.	Corr. temp.	At 22°·2 resulting: length of 20 bars.
		°	m. mm.
Aug. 17	11:00 a. m.	24·47	100 + 26·18
17	4:36 p. m.	27·76	25·97
18	8:17 a. m.	23·32	25·97
18	9:24 a. m.	23·74	25·95
18	5:00 p. m.	28·52	26·23
		Mean at 22°·2 =	100 + 26·06

hence

$$\text{at } 22^\circ{\cdot}2 \text{ rod } 13 + 14 = 10^{\text{m}} + \overset{\mu.}{2606} \atop \pm 4$$

In view of this agreement with the office value, and owing to the relatively constant temperature night and day owing to cloudiness and rain, the results obtained on August 17 and 18 were considered sufficiently accurate, and further observations were discontinued until a change in the weather occurred on August 22; thereafter the daily temperature range of the air was much increased. Observations made on Aug. 22 and the two following days disclosed a difference between the observed and true temperatures of the bars, and in order to determine its amount, a systematic series of comparisons at intervals of about two hours was carried out on August 26.

Abstract of results.

Date.	Hour.	Obs. temp.	Length of 20 bars reduced to 22·2°.	Computed lag of thermometers.
		°	m. mm.	°
Aug. 26	7:33 a. m.	12·16	100 + 26·25	+0·16
26	9:20	14·19	26·29	+ ·20
26	11:29	16·85	26·45	+ ·33
26	1:28 p. m.	18·73	26·45	+ ·33
26	3:26	20·32	26·57	+ ·43
26	5:29	21·48	26·38	+ ·27
26	8:17	*20·43	25·72	— ·29

*Falling temperature.

The lag, or the correction to the observed temperature, is computed from the known length of the combined bars as determined August 17

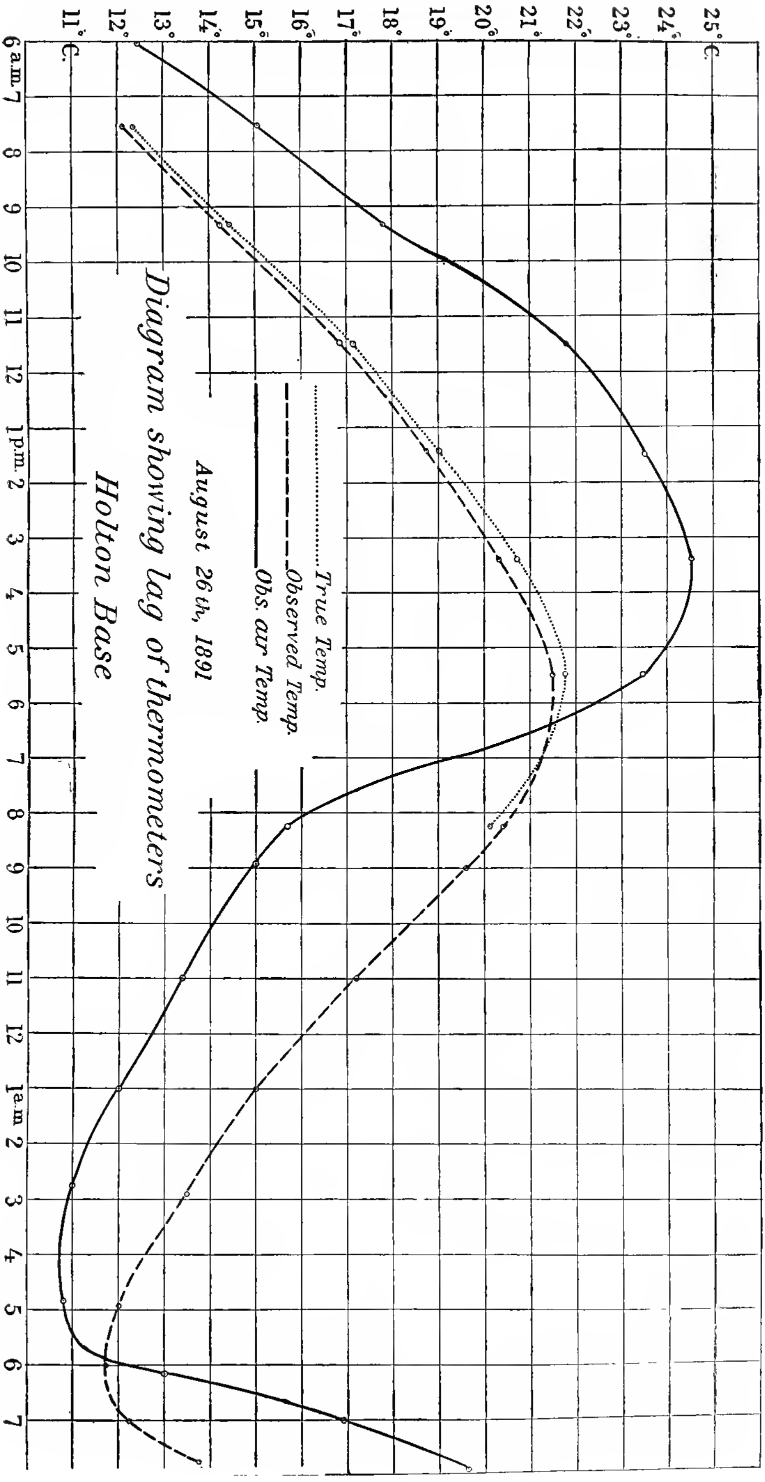

Diagram showing lag of thermometers

Holton Base

August 26th, 1891

and 18, the known coefficient of expansion and the lengths as determined August 26.

It will be seen that during August 26 the thermometers indicated too low a temperature throughout the day while the temperature was rising, amounting on the average to 0°·29, say 0°·30 centigrade, and that as soon as the temperature fell the conditions were reversed, the rods following the temperature of the air faster than the thermometers which again lagged 0°·3 behind.

The result graphically delineated on the accompanying diagram (illustration No. 34) shows that the maximum lag corresponded to (it probably followed) the hour of maximum heat of the day and to the greatest difference between the air and bar temperatures.

The curves also exhibit the range of temperature to which the bars were subjected during the twenty-four hours following the first observation.

Inasmuch as the measurements on which these results depend were made in the shade, that is, in the comparing shed, the question naturally presented itself what the relation between the indicated and true temperatures of the rods might be with the apparatus exposed to the sun.

It was therefore decided to measure a section of the standard kilometre in the bright sunshine under circumstances which should be very similar to those under which the whole base was measured.

An additional stone was therefore planted 100 metres south of the north end of the standard kilometre and the distance was measured by Prof. Woodward during his measures of the kilometre.

On September 21 this distance was accordingly measured with the secondary apparatus 5 times, and on the following day 8 times, using the trestles and measuring exactly as in the base measurement, with the exception that the beginning and end of each 100 metre measure was referred to the section stones by means of microscopes and the cut-off apparatus.

The results of these measurements may be discussed on two suppositions:

1. By giving equal weight to the four measures of the 100^{m} distance, made by Prof. Woodward, on this supposition, the distance = 100^{m} — 6·04^{mm}.

2. By giving double weight to the 3d and 4th measures and this, because the 1st and 2d measures were made on the same day and not long after the 100 metre stone had been set, while the 3d and 4th measures were made on different days and after the ground had dried.

According to this supposition the distance is 100^{m} — 5·99^{mm}.

Computing on supposition 1 the results of these measurements are as below given in detail and show an astonishingly high degree of precision. They disclose the fact that the lag was rendered practically insensible by the bright sunshine, amounting to 0°·07.

The first table shows the length of the bars deduced from the observed temperatures and the distance as measured with the iced bar by Prof. Woodward, namely, $100^{m}-6{\cdot}04^{mm}$.

The second table shows the distance as computed from the observed temperatures and the adopted length of the bars derived August 17 and 18.

As the object of this measurement was to show the thermal behavior of the apparatus when subjected to conditions similar to those which prevailed during the measurement of the base, and since the whole base was measured during rising temperatures, and nearly all of it during bright sunshine, the single measure made with falling temperatures, September 21, 5 p. m., may be thrown out of the second table. The effect of throwing it out is to increase slightly the indicated lag of the thermometers, but to decrease the probable error of measurement, which then becomes only 1-1470000 for a single measure of the 100-metre distance.

Computing on supposition 2, based on the following iced bar measures, namely—

	m.		*Weight.*
Sept. 15	100 —	6·24	1
15		6·17	1
26		5·83	2
30		5·96	2
Weighted mean	100 —	5·99	

it appears that the error of the result of the measurement with the secondary apparatus amounts to $-0{\cdot}14^{mm}$, in 100^{m}, or $\frac{1}{714286}$ of the distance.

TABLE I.

Date.	Hour.	Obs. temp.	Length of 20 bars at 22°·2.	True minus indicated temp.
	h. *m.*	°	*m.*	°
Sept. 21	9 57 a. m.	23·31	100 + 26·26	+0·17
	11 28	26·91	26·21	+ ·13
	2 40 p. m.	32·89	26·19	+ ·11
	3 53	34·29	26·10	+ ·03
	[5 00	34·26	*25·58	[— ·41]
Sept. 22	8 05 a. m.	20·41	26·21	+ ·13
	9 07	22·77	26·16	+ ·09
	10 06	25·04	26·30	+ ·20
	11 00	26·75	26·02	— ·03
	1 32 p. m.	30·83	26·23	+ ·14
	2 33	32·13	26·11	+ ·02
	3 29	32·89	25·97	— ·08
	4 24	33·15	25·99	— ·06
		Sept. 21, mean	26·19	+ ·11
		Sept. 22, mean	26·12	+ ·05
		Weighted mean	100 + 26·14	+ ·07

* Falling temperature.

Measurement of Holton Base with the New Secondary Base Apparatus.

TABLE II.

The distance between stones as measured with the secondary apparatus, rods 13+14 = $10^m + 2606^\mu$ at 22·2° C. corresponding to the dates and times given in Table I, becomes—

m.	*mm.*	*v.*	*v^2.*
100 —	6·24	0·11	·0121
	6·19	·06	·0036
	6·17	·04	·0016
	6·08	·05	·0025
	[5·56]		
	6·19	·06	·0036
	6·14	·01	·0001
	6·28	·15	·0225
	6·08	·05	·0025
	6·21	·08	·0064
	6·09	·04	·0016
	5·95	·18	·0324
	5·97	·16	·0256
m.			
Mean 100 —	6·13 ± ·018		·1145

and for a single measure the *p. e.* = $0·068^{mm}$, or 1–1470000 of the distance.

As before stated, the distance found by Prof. Woodward is $100^m - 6·04^{mm}$ on supposition 1 and $100^m - 5·99^{mm}$ on supposition 2.

MEASUREMENT OF THE STANDARD KILOMETRE.

The standard kilometre, which is a part of the base line, was measured four times during the progress of the base measurement proper.

During these four measures the ground in the woods, that is over about 600 metres of the distance, was very wet and springy and unfavorable for the stability of the trestles, which were set directly into the soft ground.

After the ground had dried it was measured twice on September 24 and 25. These six measures were made under about the same temperature conditions as those prevailing during the base measures except that the line over one-half of the kilometre passes through woods. This becomes important in view of the results of the measures made in the sunshine, since the shade would tend to create lag in the thermometers, for it has been shown that the average correction to the thermometers for lag, if the bars are shaded, amounts to about 0°·3 when the temperature of the air is rapidly rising.

It is therefore to be expected that the measures of the kilometre will not show as high a degree of accuracy as those made in the sunshine, owing chiefly to the uncorrected thermal effect of the shade, The results of the 6 measures of the standard kilometre are:

	Temp.	Resulting length.
	°	m. mm.
Aug.	23·86	1000 — 0·6
	26·75	— 5·2
	24·52	— 9·2
	25·37	— 5·5
Sept.	24·01	— 3·9
	26·18	— 5·2
	Mean,	1000 — 4·9
		± ·76

and the probable error of a single measure = ±1·86 or 1–538000.

The measures made on the 100-metre comparator and on the 100 metres measured on the standard kilometre show that the lag or the correction to the thermometers becomes minus with falling temperatures, and that its amount approximates closely to 0°·3.

Five determinations made in the early part of the evening give a mean of 0°·44, but the average during a period of say 8 hours at night would probably be less, to judge from the curves on the morning of August 26, as shown on the diagram.

To substantiate the conclusion that with falling temperatures the rods would be shorter than the length indicated by the thermometers, it became desirable to measure the standard kilometre under such conditions. As, however, during the time alloted to the work, falling temperatures, during say 8 consecutive hours, could not be had without measuring at night, and as it was not deemed desirable to complicate the question of precision by night measures, an artifice was resorted to.

The bars were exposed during the morning to bright sunshine until about noon. They were then carried into the woods to the middle of the kilometre and allowed to remain there in the shade for two hours, after which the measurement of the half kilometre was commenced. By the time the measurement extended outside of the woods the temperature was falling rapidly and the sun's rays were feeble, it being near sundown.

In this way one-half of the kilometre was measured in three hours on October 5, and the other half on October 6, giving as the result: Standard kilometre $=1000^{m}+2{\cdot}0^{mm}$. Since the true length found by Prof. Woodward is $1{,}000^{m}-3{\cdot}4^{mm}$ the measurement is in error $5{\cdot}4^{mm}$, and if we ascribe the total error to the thermometer indications the lag amounted to −0°·47.

LENGTH OF THE BASE.

The computed length of the base without applying any correction for lag of the thermometers is $5500{\cdot}808^{m} \pm 3{\cdot}2^{mm}$ where the *p. e.* includes that of the unit employed.

Measurement of Holton Base with the New Secondary Base Apparatus.

The whole base was measured during rising temperatures and much the greater part during bright sunshine.

All the experiments show that under such conditions the bars are never shorter than the indicated temperatures imply, and that therefore the result, if it is in error on account of thermometer lag, is too small.

The experiments on the 100ᵐ measured in the bright sunshine, September 21 and 22, show that the lag on supposition 1, explained in pages 7 and 8, amounts to about 1 part in one million, and on supposition 2 to $\frac{1}{741286}$, and it is therefore a fair inference that its effect on the whole base is of that order of magnitude, and that it should be applied to the uncorrected length of the base. Using the value $0{\cdot}14^{mm}$ per 100^{m} the measured length should be increased by $55 \times 0{\cdot}14 = 7{\cdot}7^{mm}$.

In order not to claim too high a precision for the length above given it seems safe to assert that its absolute error does not exceed its 1-500000 part.

An abstract of the results of the measurements of the base is appended:

Section.		No. of bars.	Corr. temp.	Resulting length.		1−2 *d.*	$e=1/3\ d.$	$e^2=2.$
			°	*m.*	*mm.*			
S B to 240		240	26·27	1 200+	22·8			
			23·83		21·2	1·6	·53	·281
240	420	180	26·80	900—	36·8			
			26·88		32·9	3·9	1·30	1·690
420	600	180	25·62	900+	62.6			
			27·49		62·3	0·3	+ ·10	·010
600	780	180	22·18	900+	17·4			
			27·99		20·1	2·7	·90	·810
780	980	200	23·86	1 000—	0·6			
			26·75	—	5·2			
			24·52	—	9·2			
			25·37	—	5·5			
			24·01	—	3·9			
			26·18	—	5·9		·76	·578
980	N B	120	21·57	600+	743·7			
			31·23		744·6	0·9	·30	·090

First measure, 5 500+805
Second measure, 810

Length of base, 5 500+808 $+7{\cdot}7^{mm} = 5{,}500.816^{m}$.
± 1·9

The probable error of measurement without regard to constant errors is ± $1{\cdot}9^{mm}$.

The probable error in the length of a 100-metre unit from comparisons with the 100-metre standard is ± 0·04.

The probable error of the 5-metre iced bar at 0° C = ± 0·0008.

The probable error of measuring the 100-metre standard (August measurements) from Prof. R. S. Woodward's determinations = ± 0·018, hence the *p. e.* of the base arising from these values is

$$55\sqrt{(0{\cdot}018)^2+(0{\cdot}016)^2+(0{\cdot}04)^2}=2{\cdot}57^{\mathrm{mm}}$$

and the *p. e.* of the base is

$$\sqrt{(2{\cdot}6)^2+(1{\cdot}9)^2}=\pm 3{\cdot}2^{\mathrm{mm}}.$$

The reduction to sea level and the *p. e.* of the leveling (mean altitude of base line) remain to be incorporated.

GENERAL CONCLUSIONS IN REGARD TO THE USE OF THIS APPARATUS.

1. The average speed of measurements, including the time requisite for training the operators, may be taken at 1 kilometre per day of eight hours, on level ground, or on a uniform grade. With trained men the speed can be increased to 1·5 kilometres per day without detriment to accuracy.

2. If the temperature of the bars were known, the accuracy of measurement would leave nothing to be desired. The probable error of a single measurement would then be about one part in a million.

3. As usual the principal source of error is in the uncertainty between the indicated and true temperature of the bars.

This relation depends on the range of the diurnal temperature to which the bars have been exposed, and its value must be subject to some uncertainty. It appears, however, that it is possible to assign the limits of error due to lag of the thermometers.

The absolute lag of the thermometers can safely be taken to lie between 0 and ± 0°·5 C. on the average in a day's measurement.

Hence, if an error of one part in 340 000 shall not be exceeded, a single measurement with this apparatus will secure that degree of accuracy if the following rules be observed:

1. If the temperature is rising, apply a correction for lag of + 0°·25 to the mean temperature of the thermometers for the day, and a correction of — 0°·25 if it is falling.

2. If one-half of the day's measures are made during rising and the other half during falling temperatures, the mean may be taken without correction.

3. If the measurement is made during hot weather and in the bright sunshine with rising temperatures, no correction need be made.

It may also be pointed out that where it is possible the bars, if day measurement only is intended, should be protected as much as possible against change of temperature at night.

MEASUREMENT OF HOLTON BASE, RIPLEY COUNTY, IND.

For description of base monuments and locality see Assistant A. T. Mosman's report, pp. 330, 331.

The 5-metre contact slide rods Nos. 13 and 14 were used in this measurement.

This apparatus is of the usual pattern. It was made at the Coast and Geodetic Survey Office and was completed in July, 1891.

The measuring rods are of cylindrical steel, 5 metres long and 9^{mm} in diameter, with the usual contact slides. The slide end of each rod carries a white agate knife edge 3^{mm} wide, the other a circular plane surface about 3^{mm} in diameter. The index and slide piece of each rod have each three lines ruled on them. When the middle lines on the slide and index are in coincidence, the distance between the agate plane at one end and the knife edge at the other is the length of the rod.

Each steel rod is encased in a built up wooden bar and rests directly in a longitudinal groove in its middle. The bars are $7{\cdot}5^{cm}$ wide, 14^{cm} high, and 4·9 metres long, so that a length of about 10^{cm} of each rod is exposed to the air.

Each rod carries two thermometers. They are inserted in opposite sides of the bar and are in contact with the rod and not with the wood. Each one is 1 metre from the end of the bar.

The wooden bar in turn is surrounded with a white canvas cover lined with half an inch of cotton.

THE TRESTLES.

They are of the usual wedge and tripod pattern and are the same as those used in the Yolo base. Throughout all the measurements, except those on the comparator, the trestles were used without foot plates, the legs being merely firmly stuck into the ground.

THE SECTORS.

The sectors are segments of arcs of circles. Each carries two (2) verniers 180° apart and is so graduated that the position in which the bar is horizontal reads nearly 20° on the front vernier. The verniers read to 10″. Before beginning work the contact ends of each bar were put in a horizontal plane by means of a leveling instrument, theodolite, or engineer's transit.

In general the sector errors were determined three times a day.

Each sector was protected against the sun by a hood made of canvas stretched on a wire frame attached to the bar.

During all the measures the two verniers of each sector were read by different persons.

THERMOMETERS.

The thermometers attached to the rods are centigrade divided to half degrees, and were read by means of a magnifying hand lens to tenths.

They were made by H. J. Green, of New York, and were compared at the office in June, 1891, with Tonnelot thermometers 4655 and 4656.

Bar 13 carries thermometers 1 and 2.
14 carries thermometers 3 and 4.

Calling that end of the bar on which the aligning telescope is mounted the front end:

Thermometer 1 is on same side as the sector, rear end.
2 is on opposite side of the sector, front end.
3 is on same side as the sector, rear end.
4 is on opposite side of the sector, front end.

The maker's numbers corresponding to those given are—

No. 1, 5609.
2, 5604.
3, 5606.
4, 5616.

On August 28 Green 5613 was substituted for 5616 (No. 4), the latter having been broken on that date after its zero point had been determined.

The corrections of these thermometers to reduce them to the hydrogen scale were determined in June, 1891, to be as follows:

At	No. 1, 5609.	No. 2, 5604.	No. 3, 5606.	No. 4, 5616.	Mean.	5613
°	°	°	°	°	°	°
0	—0·10	+0·05	—0·15	—0·10	—0·08	—0·08
3	—·16	—·24	—·25	—·23	—·22	—·25
6	—·11	—·19	—·31	—·19	—·20	—·19
11	—·13	—·18	—·33	—·21	—·21	—·18
16	—·17	—·17	—·34	—·17	—·21	—·19
21	—·26	—·21	—·23	—·21	—·23	—·21
25	—·16	—·16	—·36	—·26	—·24	—·26
32	—·21	—·21	—·36	—·26	—·26	—·28
34	—·22	—·22	—·32	—·22	—·25	—·32
37	—·26	—·28	—·28	—·28	—·28	—·28

On August 28 the zero corrections to these thermometers were redetermined. See Vol. II of the record, with the following results:

At	No. 1.	No. 2.	No. 3.	No. 4.	Mean.
°	°	°	°	°	°
0	—0·22	—0·16	—0·27	—0·22	—0·22

Measurement of Holton Base with the New Secondary Base Apparatus.

ALIGNMENT.

The bars were aligned during the measurement by bringing the rear end of the rod over the starting point or into contact with the forward agate of the rear bar and keeping it there while the forward end of the bar was aligned by pointing a small telescope on a signal usually mounted on the next section stone ahead.

A small aligning telescope is mounted on each bar 25 centimetres from the front end. The base of the uprights carrying the telescope has a sliding motion on an arc of a circle concentric with the measuring rod. It carries a level, by means of which it can be clamped in a horizontal position in a vertical plane passing longitudinally through the rod.

In order that the telescope may transit in this plane the adjusting screws of the base must be properly set.

The adjustment was not made with satisfactory accuracy during the first measurement of the distance between south base and the fourth kilometre stone. On reaching this stone a second and more perfect adjustment was made.

Subsequently the deviations from a straight line during this fractional part of the first measure were traced on the ground, and it was found that the error due to the deviation of the measured line from the base line amounted to not more than 1 part in 15 000 000, which can therefore be neglected.

Experiments were made to determine the probable error of alignment of a single bar with the telescope adjusted as in the remainder of the measures and it was found to be 38″.

GROUND TRAVERSED.

The greater part of the measurement was over grassy soil. Several plowed fields were crossed.

The surface of the ground near south base is broken, and between the 3 000 and 3 900 metre stone it is also very irregular. The first four measures of the Standard Kilometre were made when the soil was wet and springy, but during the measures made on September 24 and September 25 it was dry and hard.

A distance of 200 metres at each end of the Standard Kilometre is in the open fields; the remainder is in the woods.

DUTIES PERFORMED BY MEMBERS OF THE PARTY.

Prior to September 21, during the measurements, the bars were aligned by Prof. Gore, who also read the forward thermometer; the rear thermometer was read by Mr. Pennington; one vernier of each sector was read by Mr. Gjertsen, the other by Mr. Hayford, who also kept the record and called attention to any apparent or real errors in the readings. After September 20 the alignments were made by Mr. Hayford.

Throughout the work the contacts were made by myself, using a small lens magnifying 3 diameters.

SECTION MARKS.

The base and its sections, excepting the Standard Kilometre, were marked by lines traced at right angles to the direction of the base on copper bolts set into stones. The ends and intermediate sections of the Standard Kilometre were marked by spherical-headed bolts, intended to be used in connection with a so-called cut-off apparatus.

This apparatus consists of an upright cylinder having a conical recess at its foot, which fits over the spherical bolthead. The cylinder can be revolved about its upright axis, and any deviation from verticality is measured by means of an attached spirit level. On top, and at right angles to its vertical axis, the cylinder carries a scale divided into millimetres, to which an object over it can be referred, and which gives its relation to the geometrical center of the bolthead.

In starting, the slide was usually clamped in coincidence with the index, and the knife edge was brought over the starting point by means of a theodolite or transit sector mounted at right angles to the base line.

When stopping for lunch, a line traced on a copper tack in a stub in the ground served as a reference mark.

The record of the measurements of the Holton base line with 5-metre contact slide rods Nos. 13 and 14 is contained in 3 volumes. Volume I, pages 1 to 230, and Volume II, pages 1 to 68, contain the record of the base measurement proper, including two extra measurements of the Standard Kilometre.

Volume II, pages 71–158, contains determinations of the length of the rods in terms of the 100-metre comparator. For description and details of this comparator, see Assistant R. S. Woodward's report.

LENGTH OF THE RODS.

The length of the rods was determined immediately after the base measurement by measuring the distance between the ends of the standard 100-metre comparator. The legs were taken off the trestles and the tripod heads carrying the wedges were screwed on substantial boards, which fitted across the track prepared for the iced bar carriage.

In starting, the slide of the bar was clamped to coincidence. The knife edge was brought under a microscope mounted over one end of the 100-metre comparator, and a pointing was made with the microscope on the knife edge. The axis of the microscope was referred to the end mark by means of the cut-off apparatus.

Measurement of Holton Base with the New Secondary Base Apparatus.

The measurement of the distance was then carried on as usual, except that the tripod heads rested in the track, as above described. On reaching the other end of the comparator, where another microscope and cut-off apparatus were mounted, a spider thread was stretched over the plane surface of the agate of the rod, and the microscope pointing was made on the space, which was never more than about 20 microns in width, between the direct and reflected image of the thread.

For the deductions drawn from these and other determinations of the length, see my report to the Superintendent.

Pages 159 to 162 of Volume II contain the determination of the correction to the zero points of the thermometers attached to the base rods.

Pages 188 and 189 furnish a determination of the error of contact which results from the fact that the agate planes are not at right angles to the axis of the rods.

This error of contact affects the length of the rods when separately determined under microscopes, but does not affect their combined length deduced on the 100-metre comparator.

LAG OF THERMOMETERS.

Volume III, pages 4 to 31, contain a number of measures of the first hundred metres of the Standard Kilometre made in the bright sunshine in order to determine the effect of the sunlight on the lag of the thermometers attached to the rods.

In these measurements the rods were used on the trestles in the usual manner, but at the beginning and end of the 100-metre distance a microscope was mounted and the cut-off apparatus was used in the same manner as on the standard 100-metre comparator, as previously described.