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A RETRIEVABLE RECIPE FOR INVERSE "t"

by

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ABSTRACT

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A RETRIEVABLE RECIPE FOR INVERSE "t"

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1. INTRODUCTION

Various critical values (synonymous with percent points or evaluations of the inverse distribution function) of the classical Student's t distribution are frequently useful in applied statistics. Selected such values are of course widely tabulated; see Fisher and Yates (1963), and Pearson and Hartley (1976); the latter also are reproduced, with extensions by E.T. Federighi, in Abramowitz and Stegun (1968). In certain circumstances, however, it is convenient to be able to compute "t" percent points directly, accurately, and simply, without the need of extensive tables except, perhaps, a normal (Gaussian) table; but see Section 5. A simply derived, or retrievable, computational procedure for doing so is presented in this paper. It can be carried out quickly on a hand-held calculator and has been programmed, for instance, for the TI-59, the TRS-80 and the HP-41C. It seems that the accuracy of the numerical values obtained, especially at usually required levels (e.g., 95%)--but also at much more extreme ones--coupled with the ease of their computation, should provide a tempting argument for their wide use.

Several similar approximations have appeared in various journals over the last two decades. Among the most successful of these is that derived by Peizer and Pratt (1968), hereafter abbreviated PP:

$$t_n(\alpha)_{PP} = \{n \exp[z^2(\alpha)(n - \tfrac{5}{6})/(n - \tfrac{2}{3} + \tfrac{0.1}{n})^2] - n\}^{\frac{1}{2}}, \qquad (1.1)$$

where α is the right single-tail probability, so $0 < \alpha \leq 0.5$. Approximations based on asymptotic expansions appeared earlier (Wallace 1958, 1959) and were successful for moderate degrees of freedom and not-too-extreme tail areas. Other approaches have involved rational functions in the degrees of freedom (Gardiner and Bombay 1965, Kramer 1966) or the logistic distribution (Mudholkar and Chaubey 1975). A formula due to Koehler 1983 is based on a novel data-analytic approach to the t-tables, pioneered by Hoaglin; let $t_n(\alpha)_K$ represent Koehler's values. Further accurate approximations are reviewed by Bailey 1980.

Often, suggested approximations are either simple but not terribly accurate, or else are extremely complicated, involving many coefficients. The present approach offers both simplicity and a high degree of accuracy, yielding two digit accuracy or better for moderate degrees of freedom, across a broad range of tail areas. We call it a <u>retrievable recipe</u> because the simple basic idea allows it to be rederived quickly when needed.

2. <u>DERIVATION</u>

Examination of an extensive table of Student's t, or some mathematical analysis, shows that for $\alpha < 0.5$ there is a monotonically increasing transformation that stretches a Normal quantile $z(\alpha)$ into a Student's t quantile, $t_n(\alpha)$. Let

$t_n(\alpha) = \psi_n^*(z(\alpha))$, with $\psi_n^*(\cdot)$ representing the transform. We search for a simple approximation to $\psi_n^*(\cdot)$; call it $\psi_n(\cdot)$. By definition,

$$\int_{-\infty}^{z(\alpha)} \frac{e^{\frac{-u^2}{2}}du}{\sqrt{2\pi}} = \int_{-\infty}^{t_n(\alpha)} C(n)\left(1+\frac{t^2}{n}\right)^{-\frac{(n+1)}{2}} dt = 1-\alpha \quad (2.1)$$

where $C(n)$ is the normalizing constant. Equivalently,

$$\int_{-\infty}^{z} \frac{e^{\frac{-u^2}{2}}du}{\sqrt{2\pi}} = \int_{-\infty}^{\psi_n^*(z)} C(n)\left(1+\frac{t^2}{n}\right)^{-\frac{(n+1)}{2}} dt. \quad (2.2)$$

Differentiation of both sides with respect to z now leads to

$$\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} = C(n)\left(1+\frac{\psi_n^*(z)^2}{n}\right)^{-\frac{(n+1)}{2}} \frac{d\psi_n^*(z)}{dz}. \quad (2.3)$$

Our approximation has origin in the fact that $t_n(\alpha)$ approaches $z(\alpha)$ and $d\psi_n^*(z)/dz \to 1$ as n becomes large for fixed α. Consequently, simply allow the approximation $\psi_n(z)$ to satisfy

$$\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} = C(n)\left(1+\frac{\psi_n(z)^2}{n}\right)^{-\frac{(n+1)}{2}} \quad (2.4)$$

for every n. Solving (2.4) for $\psi_n^2(z)$ leads to an expression of the following general form:

$$t_n^2(\alpha) \simeq \psi_n^2(z(\alpha)) = n\{K(n)e^{\frac{H(n)z^2(\alpha)}{2}} - 1\}. \quad (2.5)$$

But for $\alpha = 0.5$, $z(\alpha) = t_n(\alpha) = 0$, so $K(n) = 1$ for all n. In order to determine $H(n)$, consider matching expectations of random variables. On the left-hand side of (2.5), $E(\underset{\sim}{t}_n^2) = \mathrm{Var}[\underset{\sim}{t}_n] = \frac{n}{n-2}$; the right-hand side requires the evaluation

$$E[\exp\{H(n)Z^2/2\}] = \int_{-\infty}^{\infty} \exp\{H(n)z^2/2\}\exp\{-z^2/2\}/\sqrt{2\pi}dz$$

$$= [1 - H(n)]^{-1/2}$$

where Z is a unit normal random variable. Notice also that this evaluation may be recovered easily from the moment generating function of the χ_1^2 distribution function. Thus for second moment matching,

$$\frac{n}{n-2} = n[(1-H(n))^{-\frac{1}{2}} - 1]$$

and so

$$H(n) = (2n-3)/(n-1)^2 . \qquad (2.6)$$

Our suggested first approximation is, then,

$$\hat{t}_n(\alpha)_{GK} = [n \exp\{z^2(\alpha)(n - 3/2)/(n-1)^2\} - n]^{\frac{1}{2}} . \qquad (2.7)$$

for $\alpha < 0.50$. Notice that this expression strongly resembles the Peizer-Pratt approximation, but has a somewhat different exponent. Numerical examples, displayed later, also suggest that it is of acceptable accuracy, usually being somewhat superior to that of

Peizer and Pratt. A distinctive feature of the above approximation, termed GK(I) for short, is its intuitively appealing and easily recollected derivation: it is retrievable. Note that this expression is convenient for simulating t-values, as in Ury (1980). Iteration of the expression (i.e., replacing z by t_n on the right-hand side of (2.7)) yields samples from even longer-tailed distributions; such may be useful in robustness studies.

3. IMPROVING THE ACCURACY OF THE APPROXIMATION

Before numerically comparing the accuracy of $\hat{t}_n(\alpha)_{PP}$ with GK(I), we consider a method for improving the accuracy as follows. Let us assume that the true value of Student's t can be written as in (2.7) but with a slightly different tail area; i.e., with α^* a function of α:

$$\hat{t}_n(\alpha) = \{n \exp[z(\alpha^*)^2(n-\tfrac{3}{2})/(n-1)^2] - n\}^{\frac{1}{2}} . \tag{3.1}$$

Upon rewriting (3.1), we see that

$$\alpha^*(n) = \Phi\{[\ln(1 + t_n^2(\alpha)/n)][(n-1)^2/(n-\tfrac{3}{2})]\}^{\frac{1}{2}} , \tag{3.2}$$

where Φ denotes the standard Gaussian cumulative distribution function. Now Figure 1 shows that $\ln(\alpha^*(n)-\alpha)$ is roughly linear in $\ln(n)$, for several values of α. The least squares estimates for the slope and intercept for a few values of α are shown in Table 1. A typical value for the slope is taken to be -1.86; the intercept behaves like $-3+0.62(\ln \alpha)$. Thus

$$(\alpha^* - \alpha) \approx e^{-3}\alpha^{.62}/n^{1.86}$$

or

$$\alpha^* \approx \alpha + 0.04979(\alpha/n^3)^{.62} . \tag{3.3}$$

So our improved percent point should be

$$\hat{t}_n(\alpha)_{GK(II)} = \hat{t}_n(\alpha^*) . \tag{3.4}$$

Note that the adjustment to α in (3.3) decreases rapidly as n increases. Of course, the above correction is empirical and doubtless can be further improved. Unfortunately, it is not easily retrieved in a manner analogous to the derivation of $\hat{t}_n(\alpha)_{GK(I)}$.

4. COMPARING THE APPROXIMATIONS

Figure 2 compares the accuracy of the three approximations (1.1), (2.7), (3.4), and Koehler's formula as a function of x = -10 log (tail area), for n = 6, 10, 20, 30, by plotting the relative error $[=(t_n(\alpha) - \hat{t}_n(\alpha))/t_n(\alpha)]$. Notice that in all the graphs, the simple approximation given by GK(I) (2.7) is slightly better than that suggested by Peizer and Pratt. Considerable improvement is attained using the adjusted value of α given by GK(II) 25 in (3.4).

A few values of each approximation are tabulated in Table 2 and compared with the true percentage points. Notice that, while GK(II) is initially worse than GK(I) for low degrees of freedom, it results in an extra digit of accuracy for moderate n and extremely small α. In fact, GK(II) yields 2-3 decimals of accuracy for $n \geq 10$ over the entire range of α considered, 0.05 to 0.000001. Koehler's formula is better for small n (n = 4) and moderate α ($\alpha \geq 0.025$), and is about the same as GK(I) and GK(II) when n is very large ($n = 60$). However, the choice of approximation at $n = 60$ is possibly academic, as many users would be satisfied with Gaussian percent points for such large degrees of freedom. In brief, GK(II) obtains an extra digit of accuracy for extreme tail areas and moderate degrees of freedom. Notice that the correction factor is essentially 0 for large n, so there is no advantage of GK(II) over GK(I) for n greater than, say 30.

All approximations requiring $z(\alpha)$ used formula (26.2.23) from AMS 55 (Abramowitz and Stegun 1968) in the table and figures of comparisons. It may be noted that the approximation GK(I), (2.7), may be inverted to determine approximate probability values (so-called "p-values"). A table of the Gaussian distribution, or an approximation to the Gaussian percent points, is required.

5. TOWARDS A SIMPLE STAND-ALONE APPROXIMATION

It is tempting to calculate our t-value approximations, which depend upon tabulated normal values, with the aid of approximate normal values that can be computed easily from scratch. The result is a stand-alone t-value approximation, accurate to nearly two digits over a surprisingly large range.

Here is a suggested way of proceeding. Tukey's λ-distribution (see Tukey 1970, as referred to in McNeil 1977, p. 88) provides

$$z_T(\alpha) \equiv \hat{\Phi}^{-1}(1-2\alpha;\lambda) = (\sqrt{\pi/2}2^{\lambda}/2\lambda)[(1-\alpha)^{\lambda} - \alpha^{\lambda}] ; \qquad (5.1)$$

with $\lambda = 0.14$ it yields inverse normal values to 3-digit accuracy down to $\alpha = 0.01$. In order to extend fairly satisfactorily to $\alpha = 10^{-6}$, proceed as follows: put $\alpha = 10^{-u}$ and write

$$\int_{z(u)}^{\infty} \exp\{-z^2/2\}/\sqrt{2\pi}\, dz = 10^{-u} , \tag{5.2}$$

so

$$- u \ln 10 = \ln \int_{z(u)}^{\infty} \exp\{-z^2/2\}/\sqrt{2\pi}\, dz \tag{5.3}$$

Now differentiate, and examine the result as u becomes large (cf. Feller 1957, p. 193):

$$\ln 10 = \frac{e^{-\frac{1}{2}z(u)^2}}{\int_{z(u)}^{\infty} e^{-\frac{1}{2}z^2}\, dz} \frac{dz(u)}{du} \sim \frac{e^{-\frac{1}{2}z(u)^2}}{\frac{1}{z(u)}\, e^{-\frac{1}{2}z(u)^2}} \frac{dz(u)}{du} \tag{5.4}$$

$$= z(u) \frac{dz(u)}{du}$$

Integration gives (for "large" u, here $2 < u \leq 6$)

$$z(u) \simeq \sqrt{z^2(u_0) + 2(\ln 10)(u - u_0)} \tag{5.5}$$

Take $u_0 = -\log(0.01) = 2$, $z(u_0) = z_T(0.01) = 2.58$ and replace $2 \ln 10$ by 4.32 to achieve slightly better results. Then utilize these numbers to find, for $\alpha < 0.01$

$$z_T(\alpha) = \sqrt{z_T^2(0.01) + 4.32(-\log(2\alpha) - 2)} \tag{5.6}$$

In summary, use the following prescription for the normal values:

$$z_T(\alpha) = 4.476[(1-\alpha)^{0.14}-\alpha^{0.14}], \quad 10^{-2} \leq \alpha \leq 0.5$$

$$= \sqrt{-4.32 \log \alpha - 3.284}, \quad 10^{-6} < \alpha < 10^{-2} \tag{5.7}$$

with close to 2-digit accuracy throughout the stated range. Refinement or improvement is possible, but at the apparent price of a more elaborate representation.

Table 2 includes t-values computed using the normal approximation (5.7). These are labelled $\hat{t}_n(\alpha)_{GK(III)}$.

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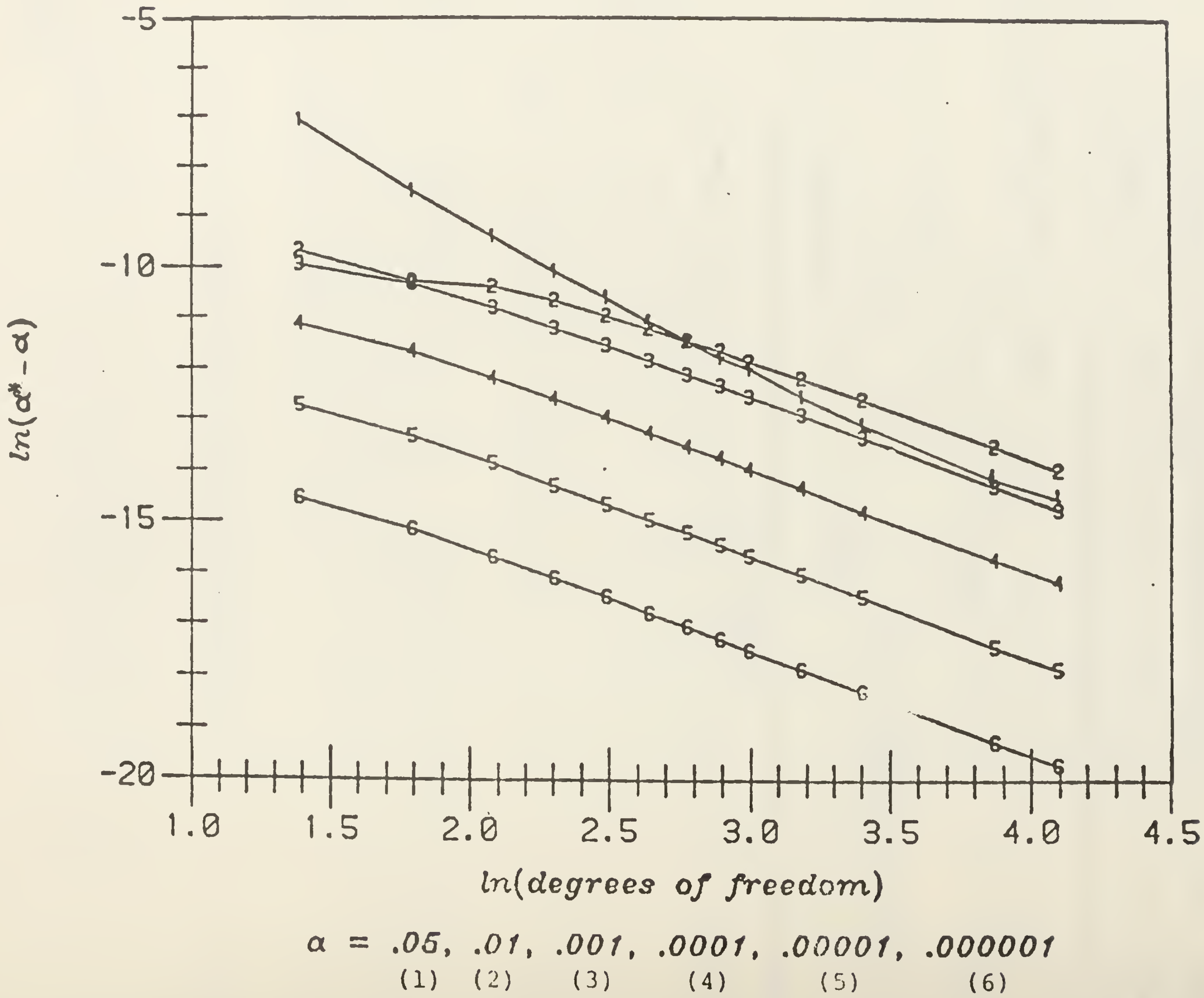
-5
-10
-15
-20
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
ln(α* - α)
ln(degrees of freedom)
α = .05, .01, .001, .0001, .00001, .000001
(1) (2) (3) (4) (5) (6)

Figure 1.

COMPARISON OF APPROXIMATIONS

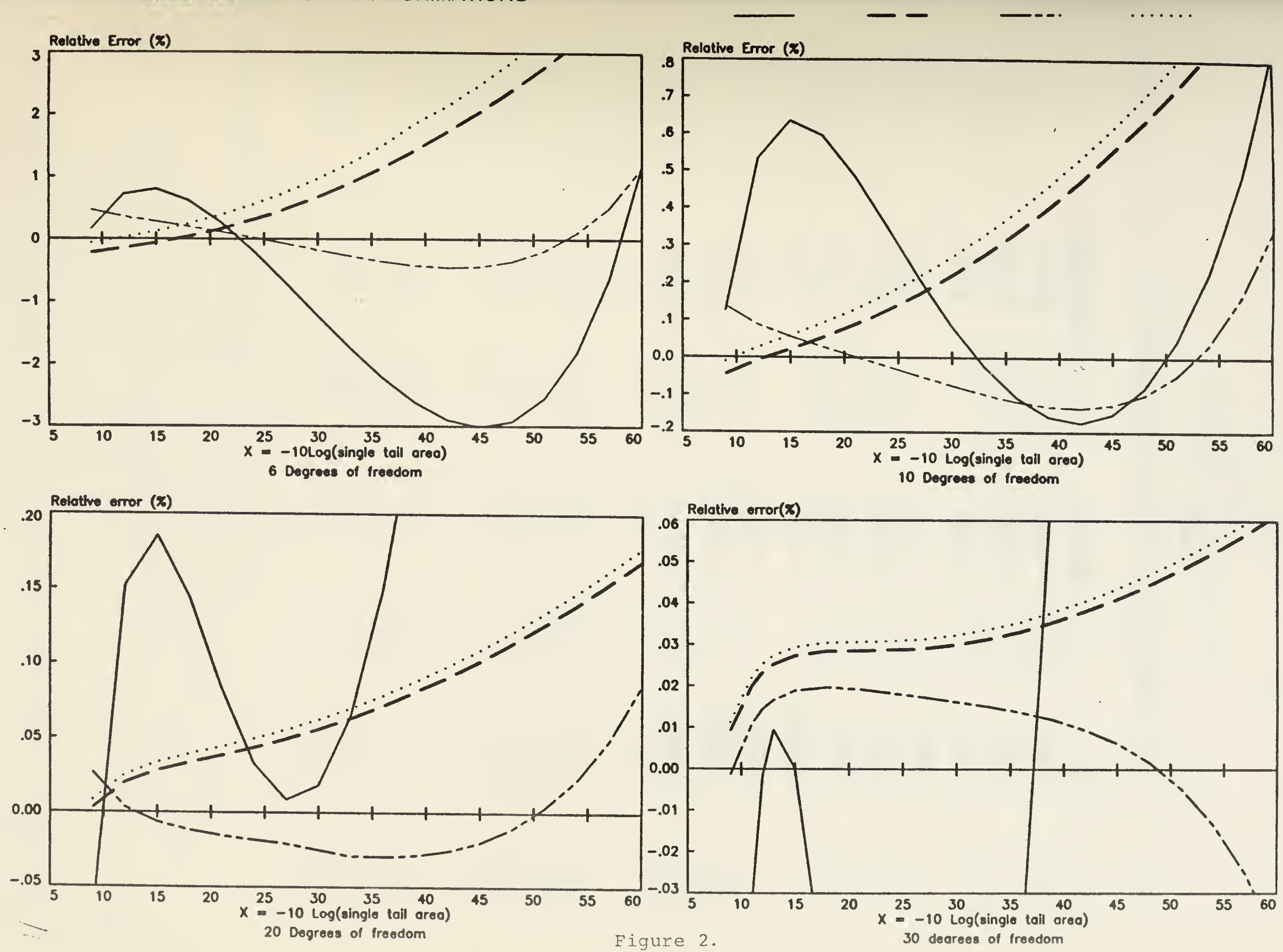

Figure 2.

Table 1

Linear fits of $\ln(\alpha^* - \alpha)$ vs $\ln(n)$

α	Slope	Intercept
.05	-2.876	-3.447
.02	-1.308	-8.487
.01	-1.809	-6.495
.005	-1.828	-6.473
.001	-1.928	-6.800
.0005	-1.943	-7.138
.00005	-1.930	-8.683
.00001	-1.927	-9.872
.000005	-1.822	-10.664
.000001	-1.771	-11.978

Table 2

Comparing approximations

Single tail area (-10Log(tail area))	.05 (13)	.025 (16)	.01 (20)	.005 (23)	.001 (30)	.0001 (40)
n = 4						
True	2.132	2.776	3.747	4.604	7.171	11.559
K	2.139	2.776*	3.708	4.509	6.853	12.365*
PP	2.134*	2.787	3.780	4.667	7.379	13.798
GK(I)	2.118	2.763	3.741*	4.613*	7.266	13.510
GK(II)	2.107	2.748	3.716	4.575	7.165*	13.091
GK(III)	2.134	2.790	3.773	4.628	7.402	13.828
n = 10						
True	1.812	2.228	2.764	3.169	4.144	5.694
K	1.823	2.242	2.778	3.182	4.147*	5.684
PP	1.813	2.230	2.767	3.174	4.155	5.721
GK(I)	1.812*	2.229*	2.766	3.173	4.153	5.718
GK(II)	1.811	2.227*	2.764*	3.170*	4.147*	5.701*
GK(III)	1.824	2.245	2.781	3.179	4.196	5.782
n = 20						
True	1.725	2.086	2.528	2.845	3.552	4.539
K	1.728	2.090	2.531	2.847	3.552*	4.553
PP	1.725*	2.087	2.529	2.847	3.554	4.543
GK(I)	1.725*	2.087	2.529	2.847	3.554	4.542
GK(II)	1.725*	2.086*	2.528*	2.840*	3.553	4.540*
GK(III)	1.735	2.100	2.541	2.851	3.583	4.580
n = 30						
True	1.697	2.042	2.457	2.750	3.385	4.234
K	1.697*	2.042*	2.455	2.746	3.379	4.239
PP	1.697*	2.043	2.458*	2.751*	3.386*	4.236*
GK(I)	1.698	2.043	2.458*	2.751*	3.386*	4.236*
GK(II)	1.697*	2.043	2.458*	2.751*	3.386*	4.236*
GK(III)	1.707	2.056	2.470	2.755	3.412	4.267
n = 60						
True	1.671	2.000	2.390	2.660	3.232	3.962
K	1.668	1.996	2.383	2.650	3.218	3.953
PP	1.671*	2.001*	2.391*	2.661*	3.232*	3.963*
GK(I)	1.668	1.996	2.383	2.650	3.218	3.953
GK(II)	1.668	1.996	2.383	2.650	3.218	3.953
GK(III)	1.680	2.013	2.401	2.665	3.255	3.989

* indicates closest approximation to true value

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