Title

Formulas for a lens system consisting of n thin coaxial lenses with a common principal axis

Abstract

In geometrical optics, in the study of a system of two thin coaxial lenses with a common principal axis, there are a number of standard formulas, including "$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$".

The purpose of this paper is to generalize these formulas to the case of a system of n thin lenses.

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Reference

Name of textbook: Physics (2nd edition)
Authors: S. G. Startling and A. J. Woodall
Publisher: Longmans
Chapter 23; "Lens System and Thick Lenses"; pages 531 to 551

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1.0 Introduction

The diagram below shows a system of n lenses, in which the r^{th} lens is denoted by L_r. A ray of light AB, parallel to the principal (or optical) axis XY of the system, is refracted at B by the first lens, L_1, and emerges along BC. The ray is then refracted by subsequent lenses (of which only L_n is shown) and then at E by L_n. It finally emerges from the system along EF to intersect XY at F.
AB and EF intersects at D and DH is perpendicular to XY.

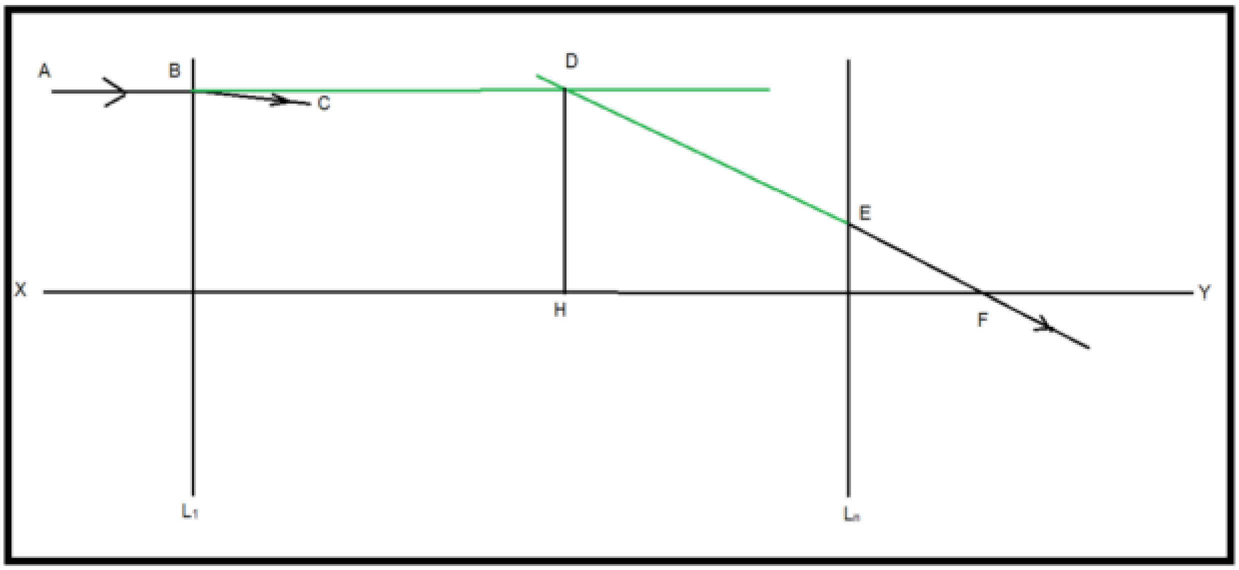

Diagram 1

The following are some (established) definitions.

- F is called the **image focus** (or **rear focal poin**t or **back focal point**) of the system.
- H is called the **image principal point** (or **image unit point** or **second principal point** or **second unit point**) of the system.
- The distance between HF is called the **image focal length** of the system.

If the ray of light AB were to travel in the opposite direction (i.e. from right to left) and intersects L_n first and finally emerge from L_1, then there would be a corresponding

- **object focus** (or **front focal point**) of the system
- **object principal point** (or **object unit point** or **first principal point** or **first unit point**) of the system
- **object focal length** of the system

The following notations are used in the formulas below for a system of two lenses.

- The power of
  - the first lens is k_1
  - the second lens is k_2
  - the lens system is K

- The focal length of the lens system is F
- The distance between
  - the lenses is d
  - the first principal point and the first lens is h_1
  - the second principal point and the second lens is h_2
  - an object and the first lens is u
  - the corresponding image and the second lens is v
- The transverse (or linear) magnification is m

The following are well known formulas for a system of two lenses.

- $K = k_1 + k_2 - dk_1k_2$
- $h_1 = \frac{dk_2}{k_1 + k_2 - dk_1k_2}$
- $h_2 = \frac{dk_1}{k_1 + k_2 - dk_1k_2}$
- $\frac{1}{u + h_1} + \frac{1}{v + h_2} = \frac{1}{F}$
- $m = \frac{v + h_2}{F} - 1$

Thus: a system of two thin lenses (of power k_1 and k_2 and separated by a distance d apart) is equivalent to a single thin lens of power $k_1 + k_2 - dk_1k_2$, positioned between the two lenses and at a distance of $\frac{dk_1}{k_1 + k_2 - dk_1k_2}$ from the second lens.

This paper generalizes the above formulas to the case of a system of n thin lenses. Also, other results are established.

In the below, “lens” means “thin lens”.

2.0 Formula for the Focal Length of a System of n Lenses

2.1 *Image Focal Length of a System of n Lenses (for small values of n)*

Let f_r denote the focal length of L_r (the r^{th} lens), $k_r = 1/f_r$ be the power of L_r, d_r denote the distance between L_r and L_{r+1}, F_n denote the image focal length of a system of n lenses and $K_n = 1/F_n$ denote the (image) power of the system of n lenses.

The formula for K_2 is derived by using the equation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ together with virtual object and similar triangles. The formula is given by $K_2 = k_1 + k_2 - k_1 d_1 k_2$

By using this very method with three lenses the formula for K_3 can be obtained. Alternatively, the formula for K_3 can be obtained as follows.
L_1 and L_2 will be replaced by a single equivalent lens, say L_{12}, so that we are now dealing with 2 lenses, L_{12} and L_3. As mentioned above, L_{12} is of power $k_1 + k_2 - d_1 k_1 k_2$ and is positioned between L_1 and L_2 and at a distance of $\frac{d_1 k_1}{k_1 + k_2 - d_1 k_1 k_2}$ from L_2. Thus, the distance D between L_{12} and L_3 is given by $D = d_2 + \frac{d_1 k_1}{k_1 + k_2 - d_1 k_1 k_2}$.

Hence: $K_3 = K_2 + k_3 - DK_2 k_3 = K_2 + k_3 - \left(d_2 + \frac{d_1 k_1}{k_1 + k_2 - d_1 k_1 k_2}\right) K_2 k_3$
$= (k_1 + k_2 - d_1 k_1 k_2) + k_3 - \left(d_2 + \frac{d_1 k_1}{k_1 + k_2 - d_1 k_1 k_2}\right)(k_1 + k_2 - d_1 k_1 k_2) k_3$
$= k_1 + k_2 + k_3 - d_1 k_1 k_2 - d_2 (k_1 + k_2 - d_1 k_1 k_2) k_3 - d_1 k_1 k_3$
$= k_1 + k_2 + k_3 - d_1 k_1 k_2 - d_1 k_1 k_3 - d_2 (k_1 + k_2) k_3 + d_2 d_1 k_1 k_2 k_3$
$= [k_1 + k_2 + k_3] - [(k_1) d_1 (k_2 + k_3) + (k_1 + k_2) d_2 (k_3)] + [(k_1) d_1 (k_2) d_2 (k_3)]$

Similarly, with four lenses the following formula for K_4 is obtained:

$$\begin{aligned} K_4 = \ & [k_1 + k_2 + k_3 + k_4] \\ & - [(k_1) d_1 (k_2 + k_3 + k_4) + (k_1 + k_2) d_2 (k_3 + k_4) + (k_1 + k_2 + k_3) d_3 (k_4)] \\ & + [(k_1) d_1 (k_2) d_2 (k_3 + k_4) + (k_1) d_1 (k_2 + k_3) d_3 (k_4) + (k_1 + k_2) d_2 (k_3) d_3 (k_4)] \\ & - [(k_1) d_1 (k_2) d_2 (k_3) d_3 (k_4)] \end{aligned}$$

With five lenses the following is obtained:

$$\begin{aligned} K_5 = \ & [k_1 + k_2 + k_3 + k_4 + k_5] \\ & - [(k_1) d_1 (k_2 + k_3 + k_4 + k_5) + (k_1 + k_2) d_2 (k_3 + k_4 + k_5) \\ & \quad + (k_1 + k_2 + k_3) d_3 (k_4 + k_5) + (k_1 + k_2 + k_3 + k_4) d_4 (k_5)] \\ & + [(k_1) d_1 (k_2) d_2 (k_3 + k_4 + k_5) + (k_1) d_1 (k_2 + k_3) d_3 (k_4 + k_5) \\ & \quad + (k_1) d_1 (k_2 + k_3 + k_4) d_4 (k_5) + (k_1 + k_2) d_2 (k_3) d_3 (k_4 + k_5) \\ & \quad + (k_1 + k_2) d_2 (k_3 + k_4) d_4 (k_5) + (k_1 + k_2 + k_3) d_3 (k_4) d_4 (k_5)] \\ & - [(k_1) d_1 (k_2) d_2 (k_3) d_3 (k_4 + k_5) + (k_1) d_1 (k_2) d_2 (k_3 + k_4) d_4 (k_5) \end{aligned}$$

$+(k_1)d_1(k_2+k_3)d_3(k_3)d_4(k_5)+(k_1+k_2)d_2(k_3)d_3(k_3)d_4(k_5)]$
$+[(k_1)d_1(k_2)d_2(k_3)d_3(k_4)d_4(k_5)]$

2.2 *Proposed Formula for Image Focal Length of a System of n Lenses*

In the formulas for K_3, K_4, and K_5 the terms having the same number of factors of the d's are grouped together by using square brackets. The sum of the terms in the $(m+1)^{th}$ pair of square brackets is denoted by T_m.

The following pattern seems to be developing:

- K_n is composed of $T_0, T_1, \ldots, T_{n-1}$.
- The sign preceding T_m is $(-1)^m$.
- T_m is a sum with each summand being a product of the following factors: m d's and m+1 sums of k's, with each sum of k's enclosed in a pair of parentheses, ().
  Note: The k's in each pair of parentheses are dependent on the choice of the d's. Each summand has m d factors out of a possible of n-1 d's. Thus, the number of summands in T_m is ${}^{n-1}C_m$.

A typical summand in T_m is
$(k_1+\cdots+k_{a_1})d_{a_1}(k_{a_1+1}+\cdots+k_{a_2})d_{a_2}\ldots(k_{a_{m-1}+1}+\cdots+k_{a_m})d_{a_m}(k_{a_m+1}+\cdots+k_n)$,
where the a_r's are integers satisfying $1 \le a_1 < \ldots < a_m \le n-1$.
$=\{\prod_{s=1}^{m}(k_{a_{s-1}+1}+\cdots+k_{a_s})d_{a_s}\}\{k_{a_m+1}+\cdots+k_n\}$, where additionally $a_0 = 0$
$=\{\prod_{s=1}^{m}[(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s})d_{a_s}]\}\{\sum_{r_{m+1}=a_m+1}^{n}k_{r_{m+1}}\}$
$=\prod_{s=1}^{m+1}[(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s})d_{a_s}]$, where additionally $a_{m+1} = n$ and $d_n = 1$

For a fixed m, by giving the a's all possible combinations of values satisfying $0 = a_0 < a_1 < \ldots < a_m < a_{m+1} = n$, all the summands in T_m are obtained.

$$\text{Thus: } T_m = \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{r_s}\right)d_{a_s}\right]$$

In the below, T_m will appear as $\prod_{s=1}^{m+1}\left[\left(\sum^{a_s}_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}} k_{r_s}\right)d_{a_s}\right]$

$$\text{Hence: } K_n=\sum_{m=0}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum^{a_s}_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}} k_{r_s}\right)d_{a_s}\right]\right\}$$

2.3 *Strategy for the Proof of the Proposed Formula*

The above formula for K_n will be assumed to be formula for the **Object** Focal Length of a system of n lenses. This formula (for the object focal length) will be proved by the Method of Mathematical Induction and by employing a strategy that is explained in this section.
(Later on, it will be shown that the Object Focal Length equals the Image Focal Length.)

L_n will be replaced by two lenses $L`_n$ and $L_{n+1}$ (with power $k`_n$ and k_{n+1}, respectively) that are

- separated by a distance of d_n apart, so that the focal length of the system consisting of $L`_n$ and L_{n+1} is equal to the focal length of L_n
  (meaning that the system consisting of $L`_n$ and L_{n+1} is equivalent to L_n)
  That is: $k_n = k`_n + k_{n+1} - k`_n d_n k_{n+1}$
- appropriately positioned so that the object focal length of the new system of n+1 lenses is equal to the object focal length of the original system of n lenses (meaning that the system consisting of the n+1 lenses is equivalent to the system consisting of the n lenses)

L_{n-1} was at a distance of d_{n-1} to the left of L_n. The distance $d`_{n-1}$ (between $L_{n-1}$ and $L`_n$) will now be determined.

The diagram below shows a ray of light AC, parallel to the principal axis XY of L_n, being refracted at C and then intersects XY at F, the object focal point of L_n.
L_n is replaced with $L`_n$ and $L_{n+1}$ so as to maintain the same the object focal point and the same object focal length. Therefore, the ray AB is refracted at B by $L_{n+1}$ and is subsequently refracted at D by $L`_n$ so as to pass through F.
L_{n+1}, L_n, $L`_n$ and intersect XY at P, H and Q, respectively. BD intersects XY at G and DF intersects AB at C.

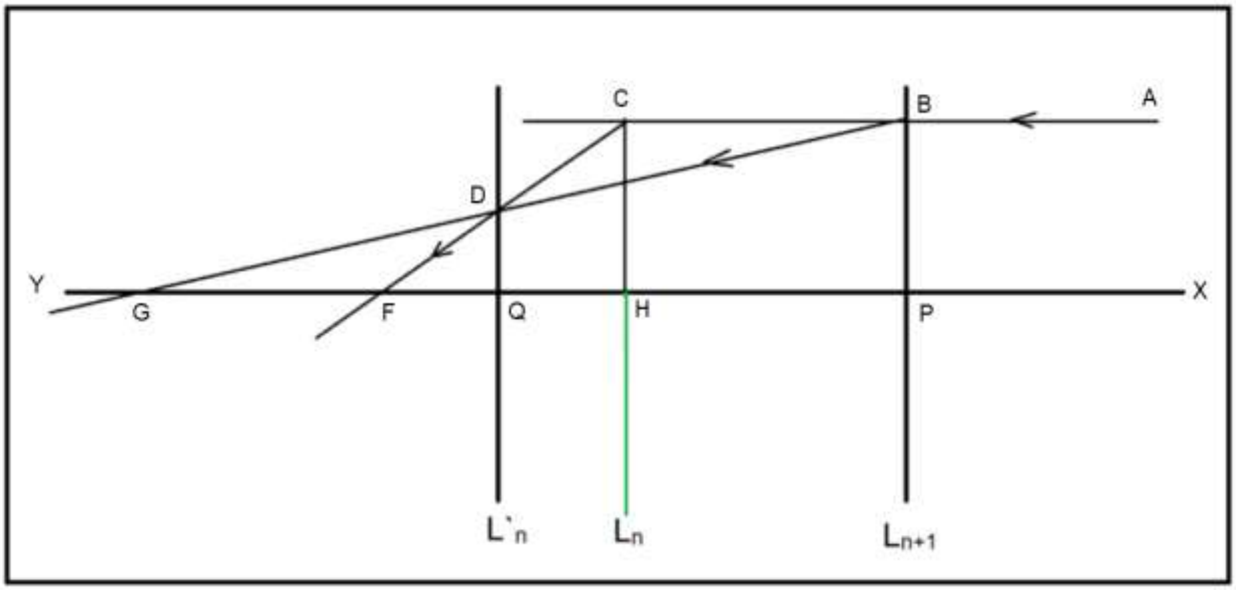

Diagram 2

PQ = d_n, PG = f_{n+1} and HF = f_n.
Let HQ = x.

DQF and CHF are similar triangles. Hence: $\frac{DQ}{CH} = \frac{QF}{HF}$
DQG and BPG are similar triangles. Hence: $\frac{DQ}{BP} = \frac{QG}{PG}$
Since $CH = BP \Rightarrow \frac{DQ}{CH} = \frac{DQ}{BP} \Rightarrow \frac{QF}{HF} = \frac{QG}{PG} \Rightarrow \frac{HF-HQ}{HF} = \frac{PG-PQ}{PG} \Rightarrow \frac{f_n - x}{f_n} = \frac{f_{n+1} - d_n}{f_{n+1}}$

$\Rightarrow f_{n+1}(f_n - x) = f_n(f_{n+1} - d_n) \Rightarrow f_{n+1}x = f_n d_n \Rightarrow x = \frac{f_n d_n}{f_{n+1}} = \frac{k_{n+1} d_n}{k_n} = \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$

Hence: in the system of n lenses, if L_n is replaced by $L`_n$ and L_{n+1}, satisfying the following conditions, then the object focal length of the new system of n+1 lenses is equal to the object focal length of the original system of n lenses.

- $L`_n$ and L_{n+1} are separated by a distance of d_n
- $k_n = k`_n + k_{n+1} - k`_n d_n k_{n+1}$
- $L`_n$ is positioned at a distance of $\frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$ from and to the left of where L_n was

Hence: the distance $d`_{n-1}$ (between $L_{n-1}$ and $L`_n$) is given by

$$d`_{n-1} = d_{n-1} - \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$$

$$\therefore d_{n-1} = d`_{n-1} + \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$$

$$\Rightarrow d_{n-1}k_n = \left(d`_{n-1} + \frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}\right)(k`_n + k_{n+1} - k`_n d_n k_{n+1})$$

$$= d`_{n-1}(k`_n + k_{n+1} - k`_n d_n k_{n+1}) + d_n k_{n+1}$$

$$= d_n k_{n+1} + d`_{n-1}(k`_n + k_{n+1}) - d`_{n-1} k`_n d_n k_{n+1}$$

Hence, if the following replacements are made in the right hand side of the equation $K_n = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$

- k_n is replaced with $k`_n + k_{n+1} - k`_n d_n k_{n+1}$
- $d_{n-1}k_n$ is replaced with $d_n k_{n+1} + d`_{n-1}(k`_n + k_{n+1}) - d`_{n-1} k`_n d_n k_{n+1}$

then we will get the expression for K_{n+1}.

2.4 *Proof of the Formula for K_n*

$$K_n = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$$

$$\therefore n = 1 \Rightarrow K_1 = \sum_{m=0}^{0}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=1;d_1=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$$

$$= \prod_{s=1}^{1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1=1;d_1=1}}^{a_s} k_{r_s}\right) d_{a_s}\right] = \sum_{r_1=1}^{1} k_{r_1} = k_1$$

The formula is trivially true when n = 1.

Assuming that the formula is true for n, it is required to show that it is true for n+1.
In the below, the square brackets following an expression has the label $E_{\#}$ (to identify the expression) followed by the applicable constraints [of which $0 = a_0 < a_1 < \cdots < a_m < a_{m+1}$ is assumed to always be present]. The red highlight is to draw attention to changes.

Let the value of K_n be v. That is: $v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$, where $0 = a_0 < a_1 < \cdots < a_m < a_{m+1} = n; d_n = 1$

Splitting v as E_1 (the summand corresponding to m = 0) plus E_2 (the summand corresponding to m = n-1) plus E_3 (the remaining summands), gives

$$v = \prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right] [E_1; a_1 = n, d_n = 1]$$

$$+(-1)^{n-1}\prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_2;a_n=n,d_n=1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_3;a_{m+1}=n,d_n=1]$$

In E_3, separating out the combinations for a_m = n-1 (denoted by E_4) and for $a_m \le$ n-2 (denoted by E_5), gives

$$v=\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_1;a_1=n,d_n=1]$$
$$+(-1)^{n-1}\prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right][E_2;a_n=n,d_n=1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_4;a_m=n-1,a_{m+1}=n,d_n=1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_5;a_m\le n-2,a_{m+1}=n,d_n=1]$$

Recall the condition: $0=a_0<a_1<\cdots<a_m<a_{m+1}=n$.
Therefore: m = n-1 ⇒ a_r = r, ∀ r ∈ [0,n]
In the summand of E_4, when m = n-1, E_2 is obtained. Hence, merging E_4 and E_2 gives E_6.
In the summand of E_5, when m = 0, E_1 is obtained. Hence, merging E_5 and E_1 gives E_7.

$$\therefore v=\sum_{m=1}^{n-1}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_6;a_m=n-1,a_{m+1}=n,d_n=1]$$
$$+\sum_{m=0}^{n-2}\left\{(-1)^m\prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\right\}[E_7;a_m\le n-2,a_{m+1}=n,d_n=1]$$

Re-writing E_6 and E_7 as E_8 and E_9, respectively, gives

$$v=\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\,(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m})d_{n-1}k_n\right\}[E_8;a_m=n-1,a_{m+1}=n,d_n=1]$$
$$+\sum_{m=0}^{n-2}\{(-1)^m\{\prod_{s=1}^{m}[(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s})d_{a_s}]\}(\sum_{r_{m+1}=a_m+1}^{n-1}k_{r_{m+1}}+k_n)\}[E_9;a_m\le n-2,a_{m+1}=n,d_n=1]$$

In E_8, the constraints a_{m+1} = n and d_n = 1 are not needed and will be omitted.
Also in E_8, the constraints a_m = n - 1 is not needed, but if omitted, it has to be replaced with $a_{m-1} \le n-2$.
Similarly, in E_9, the constraints a_{m+1} = n and d_n = 1 are not needed and will be omitted.
Modifying the constraints gives

$$v=\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s}\right)d_{a_s}\right]\}\,(\sum_{r_m=a_{m-1}+1}^{n-1}k_{r_m})d_{n-1}k_n\right\}[E_8;a_{m-1}\le n-2]$$
$$+\sum_{m=0}^{n-2}\{(-1)^m\{\prod_{s=1}^{m}[(\sum_{r_s=a_{s-1}+1}^{a_s}k_{r_s})d_{a_s}]\}(\sum_{r_{m+1}=a_m+1}^{n-1}k_{r_{m+1}}+k_n)\}[E_9;a_m\le n-2]$$

L_n is now replaced by $L`_n$ and L_{n+1}, so as to have the value v being unchanged.
That is: k_n is to be replaced with $k`_n+k_{n+1}-k`_nd_nk_{n+1}$, and $d_{n-1}k_n$ is to be replaced with $d_nk_{n+1}+d`_{n-1}(k`_n+k_{n+1})-d`_{n-1}k`_nd_nk_{n+1}$.
Hence, E_8 and E_9 become E_{10} and E_{11}, respectively, as the following shows.

$\therefore v = \sum_{m=1}^{n-1}\{(-1)^m\{\prod_{s=1}^{m-1}[(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s})d_{a_s}]\}(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m})(d_n k_{n+1} + d\grave{}_{n-1}(k\grave{}_n + k_{n+1}) - d\grave{}_{n-1}k\grave{}_n d_n k_{n+1})\}[E_{10}; a_{m-1} \le n-2]$
$+\sum_{m=0}^{n-2}\{(-1)^m\{\prod_{s=1}^{m}[(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s})d_{a_s}]\}(\sum_{r_{m+1}=a_m+1}^{n-1} k_{r_{m+1}} + k\grave{}_n + k_{n+1} - k\grave{}_n d_n k_{n+1})\}[E_{11}; a_m \le n-2]$

Splitting up E_{10} as $E_{12} + E_{13} + E_{14}$ and E_{11} as $E_{15} + E_{16}$ and dropping the dashes in d_{n-1} and k_n, for convenience.

$$
\begin{aligned}
v &= \sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m})d_n k_{n+1}\right\}[E_{12}; a_{m-1} \le n-2] \\
&+\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m})d_{n-1}(k_n + k_{n+1})\right\}[E_{13}; a_{m-1} \le n-2] \\
&-\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m})\, d_{n-1}k_n d_n k_{n+1}\right\}[E_{14}; a_{m-1} \le n-2] \\
&+\sum_{m=0}^{n-2}\left\{(-1)^m\{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}\left(\sum_{r_{m+1}=a_m+1}^{n-1} k_{r_{m+1}} + k_n + k_{n+1}\right)\right\}[E_{15}; a_m \le n-2] \\
&-\sum_{m=0}^{n-2}\left\{(-1)^m\{\prod_{s=1}^{m}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}(k_n d_n k_{n+1})\right\}[E_{16}; a_m \le n-2]
\end{aligned}
$$

Re-writing E_{13} as E_{17}, E_{14} as E_{18} and E_{15} as E_{19}.
Dropping the dummy variable m by 1 in E_{16} to give E_{20}.

$$
\begin{aligned}
v &= \sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m})d_n k_{n+1}\right\}[E_{12}; a_{m-1} \le n-2] \\
&+\sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right\}[E_{17}; a_{m-1} \le n-2, a_m = n-1, a_{m+1} = n+1, d_{n+1} = 1] \\
&-\sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right\}[E_{18}; a_{m-1} \le n-2, a_m = n-1, a_{m+1} = n, a_{m+2} = n+1, d_{n+1} = 1] \\
&+\sum_{m=0}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right\}[E_{19}; a_m \le n-2, a_{m+1} = n+1, d_{n+1} = 1] \\
&+\sum_{m=1}^{n-1}\left\{(-1)^m\{\prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\}(k_n d_n k_{n+1})\right\}[E_{20}; a_{m-1} \le n-2]
\end{aligned}
$$

Adding E_{12} and E_{20} to give E_{21}; as the following shows.

$$
\begin{aligned}
E_{12} + E_{20} &= \sum_{m=1}^{n-1}\left\{\left((-1)^m \prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right)\left(\left(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m}\right)d_n k_{n+1} + k_n d_n k_{n+1}\right)\right\}[E_{21}; a_{m-1} \le n-2] \\
&= \sum_{m=1}^{n-1}\left\{\left((-1)^m \prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right)\left(\left(\sum_{r_m=a_{m-1}+1}^{n-1} k_{r_m} + k_n\right)d_n k_{n+1}\right)\right\}[E_{21}; a_{m-1} \le n-2] \\
&= \sum_{m=1}^{n-1}\left\{\left((-1)^m \prod_{s=1}^{m-1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right)\left(\left(\sum_{r_m=a_{m-1}+1}^{n} k_{r_m}\right)d_n k_{n+1}\right)\right\}[E_{21}; a_{m-1} \le n-2] \\
&= \sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right)d_{a_s}\right]\right\}[E_{21}; a_{m-1} \le n-2, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]
\end{aligned}
$$

Splitting E_{17} as E_{22} (the summand corresponding to m = n-1) plus E_{23} (the remaining summands).
Dropping the dummy variable m by 1 in E_{18} to give E_{24}.
Splitting E_{19} as E_{25} (the summand corresponding to m = 0) plus E_{26} (the remaining summands).

$$v = \sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{21}; a_{m-1} \le n-2, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+(-1)^{n-1} \prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{22}; a_{n-2} \le n-2, a_{n-1} = n-1, a_n = n+1, d_{n+1} = 1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{23}; a_{m-1} \le n-2, a_m = n-1, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+\sum_{m=2}^{n}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{24}; a_{m-2} \le n-2, a_{m-1} = n-1, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{25}; a_1 = n+1, d_{n+1} = 1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{26}; a_m \le n-2, a_{m+1} = n+1, d_{n+1} = 1]$$

Splitting E_{21} as E_{27} (the summand corresponding to m = 1) plus E_{28} (the remaining summands).
Merging E_{23} (with a_m = n-1) and E_{26} (with $a_m \le$ n-2) to give E_{29} (with $a_m \le$ n-1).
Splitting E_{24} as E_{30} (the summand corresponding to m = n) plus E_{31} (the remaining summands).

$$v = -\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{27}; a_1 = n, a_2 = n+1, d_{n+1} = 1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{28}; a_{m-1} \le n-2, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+(-1)^{n-1} \prod_{s=1}^{n}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{22}; a_{n-2} \le n-2, a_{n-1} = n-1, a_n = n+1, d_{n+1} = 1]$$
$$+\sum_{m=1}^{n-2}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{29}; a_m \le n-1, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+(-1)^n \prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{30}; a_{n-2} \le n-2, a_{n-1} = n-1, a_n = n, a_{n+1} = n+1, d_{n+1} = 1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{31}; a_{m-2} \le n-2, a_{m-1} = n-1, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{25}; a_1 = n+1, d_{n+1} = 1]$$

Merging E_{28} (with $a_{m-1} \le$ n-2) and E_{31} (with a_{m-1} = n-1) to give E_{32} (with $a_{m-1} \le$ n-1).
In the summand of E_{29}, when m = n-1, E_{22} is obtained. Hence, merging E_{29} and E_{22} gives E_{33}.

$$v = -\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{27}; a_1 = n, a_2 = n+1, d_{n+1} = 1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{32}; a_{m-1} \le n-1, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+\sum_{m=1}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{33}; a_m \le n-1, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+(-1)^n \prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{30}; a_{n-2} \le n-2, a_{n-1} = n-1, a_n = n, a_{n+1} = n+1, d_{n+1} = 1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{25}; a_1 = n+1, d_{n+1} = 1]$$

Splitting E_{33} as E_{34} (the summand corresponding to m = 1) plus E_{35} (the remaining summands).

$$v = -\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{27}; a_1 = n, a_2 = n+1, d_{n+1} = 1]$$

$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{32}; a_{m-1} \le n-1, a_m = n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$-\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{34}; a_1 \le n-1, a_2 = n+1, d_{n+1} = 1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{35}; a_m \le n-1, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+(-1)^n \prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{30}; a_{n-2} \le n-2, a_{n-1} = n-1, a_n = n, a_{n+1} = n+1, d_{n+1} = 1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{25}; a_1 = n+1, d_{n+1} = 1]$$

Merging E_{27} (with $a_1 = n$) and E_{34} (with $a_1 \le n-1$) to give E_{36} (with $a_1 \le n$).
Merging E_{32} (with $a_m = n$) and E_{35} (with $a_m \le n-1$) to give E_{37} (with $a_m \le n$).

$$v = -\prod_{s=1}^{2}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{36}; a_1 \le n, a_2 = n+1, d_{n+1} = 1]$$
$$+\sum_{m=2}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[E_{37}; a_m \le n, a_{m+1} = n+1, d_{n+1} = 1]$$
$$+(-1)^n \prod_{s=1}^{n+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{30}; a_{n-2} \le n-2, a_{n-1} = n-1, a_n = n, a_{n+1} = n+1, d_{n+1} = 1]$$
$$+\prod_{s=1}^{1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right][E_{25}; a_1 = n+1, d_{n+1} = 1]$$

In the summand of E_{37}, when m = 0, 1 and n, the following are obtained: E_{25}, E_{36}, and E_{30}, respectively.

$$\therefore v = \sum_{m=0}^{n}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{r_s=a_{s-1}+1}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}[0 = a_0 < a_1 < \cdots < a_m < a_{m+1} = n+1, d_{n+1} = 1]$$
$$= \sum_{m=0}^{n}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n+1, d_{n+1}=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$$

This expression for v is the same as the expression for K_n, except that n has been replaced with n+1. This concludes the proof of the formula for the Object Focal Length of a system of n lenses.

2.5 *Proof that Object Focal Length equals Image Focal Length*

The **Object** Focal Length of a system of n lenses is given by

$$K_n(k_1, d_1, k_2, d_2, \dots, k_{n-1}, d_{n-1}, k_n) = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n; d_n=1}}^{a_s} k_{r_s}\right) d_{a_s}\right]\right\}$$

Therefore, the **Image** Focal Length, say v, of the system is given by
$v = K_n(k_n, d_{n-1}, k_{n-1}, \dots, d_2, k_2, d_1, k_1)$
That is: by interchanging d_r and d_{n-r}, ∀ r ∈ [1,n-1] and by interchanging k_r and k_{n+1-r}, ∀ r ∈ [1,n], in the expression for $K_n(k_1, d_1, k_2, d_2 \dots, k_n)$, we will get the formula for the image focal length, v.
Note: in order to simplify the algebra, d_0 (just like d_n) is defined to be 1; and the interchanging of d_r and d_{n-r} will be done ∀ r ∈ [1,n].

$$\therefore v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{r_s=a_{s-1}+1 \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{a_s} k_{n+1-r_s}\right) d_{n-a_s}\right]\right\}$$

In the expression for v, changing dummy variables from r's to t's via $t_{m+2-s} = n+1-r_s$, gives

$$v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{t_{m+2-s}=n-a_{s-1} \\ 0=a_0<a_1<\cdots<a_m<a_{m+1}=n;d_n=1}}^{n+1-a_s} k_{t_{m+2-s}}\right) d_{n-a_s}\right]\right\}$$

Changing variables from a's to b's via $b_{m+1-s} = n-a_s$ gives

$$v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{s=1}^{m+1}\left[\left(\sum_{\substack{t_{m+2-s}=b_{m+2-s} \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_{m+1-s}+1} k_{t_{m+2-s}}\right) d_{b_{m+1-s}}\right]\right\}$$

Changing dummy variable from s to x via x = m+2-s gives

$$v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=m+1}^{1}\left[\left(\sum_{\substack{t_x=b_x \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_{x-1}+1} k_{t_x}\right) d_{b_{x-1}}\right]\right\}$$

Interchanging the upper and the lower limits of both the inner sum and the product

$$v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right) d_{b_{x-1}}\right]\right\}$$

$$= \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right)\right]\left[d_{b_0} d_{b_1} \ldots . d_{b_{m-1}} d_{b_m}\right]\right\}$$

Now, $b_0 = 0$ and $d_0 = 1$. $\therefore d_{b_0} = 1$
Also, $b_{m+1} = n$ and $d_n = 1$. $\therefore d_{b_{m+1}} = 1$
Hence, $d_{b_0} = d_{b_{m+1}}$

$$\therefore v = \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right)\right]\left[d_{b_1} \ldots . d_{b_{m-1}} d_{b_m} d_{b_{m+1}}\right]\right\}$$

$$= \sum_{m=0}^{n-1}\left\{(-1)^m \prod_{x=1}^{m+1}\left[\left(\sum_{\substack{t_x=b_{x-1}+1 \\ 0=b_0<b_1<\cdots<b_m<b_{m+1}=n;d_n=1}}^{b_x} k_{t_x}\right) d_{b_x}\right]\right\}$$ (i.e. the **Object Focal Length**)

Thus: Image Focal Length = Object Focal Length, and this common value is called the **Focal Length**.

Hence: $K_n(k_n, d_{n-1}, k_{n-1}, \ldots, d_2, k_2, d_1, k_1) = K_n(k_1, d_1, k_2, d_2, \ldots, k_{n-1}, d_{n-1}, k_n)$; with $d_0 = d_n = 1$.
That is, if the first lens and the last lens were to interchange positions, the second lens and the second to last lens were to interchange positions, etc., then the focal length of the system remains unchanged.

3.0 Generalized Formulas for h_1 and h_2

The diagram below shows a system of n lenses, in which the r^{th} lens is denoted by L_r. The following notations are used

- the image focus is denoted by $I_{n,2}$
- the second principal point is denoted by $H_{n,2}$
- the focal length is F_n (the distance $I_{n,2}H_{n,2}$)
- the distance between L_n and $H_{n,2}$ (i.e. $DH_{n,2}$) is denoted by $h_{n,2}$
- the distance between L_1 and $H_{n,1}$ (the first principal point) is denoted by $h_{n,1}$

With L_n being absent, i.e. we are dealing with a system of n-1 lenses, the corresponding points and lengths are denoted by n being replaced with n-1.

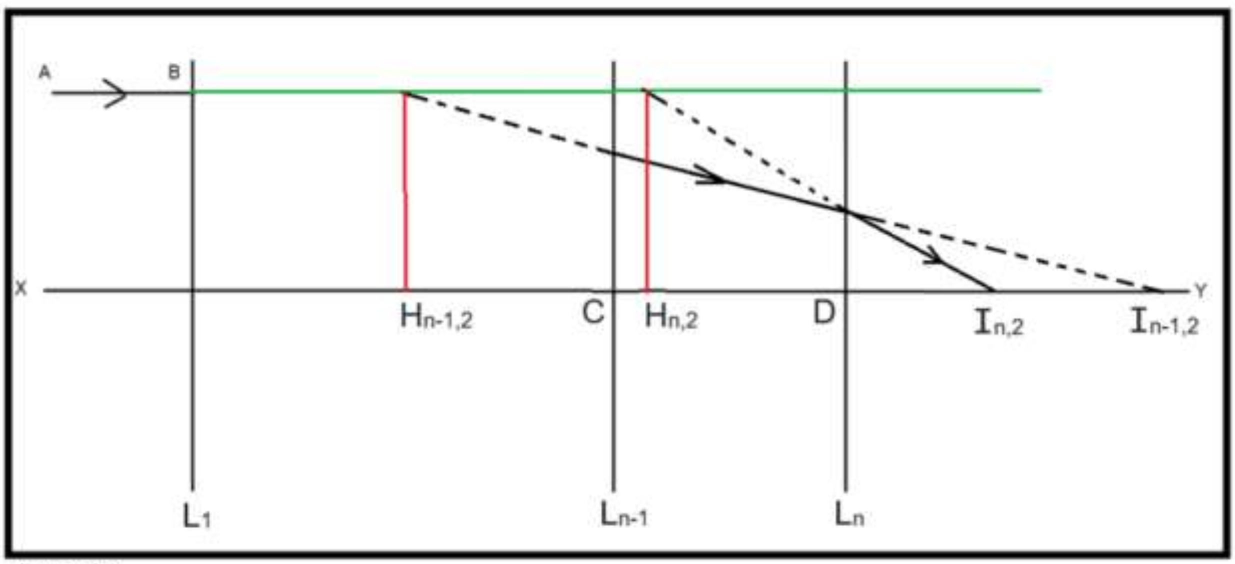

Diagram 3

In the absence of L_n, an infinitely distant object has its image at $I_{n-1,2}$. Therefore, a virtual object at $I_{n-1,2}$ will have its real image (under refraction by L_n acting alone) at $I_{n,2}$. Thus:

- the object distance, $u = -(DI_{n-1,2}) = -(H_{n-1,2}I_{n-1,2} - H_{n-1,2}D)$
  $= -(H_{n-1,2}I_{n-1,2} – \{H_{n-1,2}C + CD\}) = -(F_{n-1} - h_{n-1,2} - d_{n-1})$
- the image distance, $v = DI_{n,2} = H_{n,2}I_{n,2} - H_{n,2}D = F_n - h_{n,2}$

Using $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ (with L_n acting alone), gives

$$\frac{1}{-(F_{n-1} - h_{n-1,2} - d_{n-1})} + \frac{1}{F_n - h_{n,2}} = \frac{1}{f_n}$$

$$\Rightarrow \frac{1}{F_n - h_{n,2}} = \frac{1}{f_n} + \frac{1}{F_{n-1} - h_{n-1,2} - d_{n-1}} = \frac{1}{f_n} - \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})}$$

$\Rightarrow F_n - h_{n,2} = \dfrac{1}{\frac{1}{f_n} - \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})}}$, a recurrence relation (referred to as R1 below) for $F_n - h_{n,2}$

$$\Rightarrow h_{n,2} = F_n - \cfrac{1}{\frac{1}{f_n} - \cfrac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})}}$$

$$= F_n - \cfrac{1}{\frac{1}{f_n} - \cfrac{1}{d_{n-1} - \cfrac{1}{\frac{1}{f_{n-1}} - \cfrac{1}{d_{n-2} - (F_{n-2} - h_{n-2,2})}}}}$$, by application of R1

$$\vdots$$

$$= F_n - \cfrac{1}{\frac{1}{f_n} - \cfrac{1}{d_{n-1} - \cfrac{1}{\frac{1}{f_{n-1}} - \cfrac{1}{\ddots \, \cfrac{1}{d_2 - \cfrac{1}{\frac{1}{f_2} - \cfrac{1}{d_1 - f_1}}}}}}}$$

Hence: $h_{n,2} = F_n - \cfrac{1}{\frac{1}{f_n} - \cfrac{1}{d_{n-1} - \cfrac{1}{\frac{1}{f_{n-1}} - \cfrac{1}{\ddots \, \cfrac{1}{d_2 - \cfrac{1}{\frac{1}{f_2} - \cfrac{1}{d_1 - f_1}}}}}}}$

By interchanging d_r and d_{n-r} and interchanging f_r and f_{n+1-r}, $\forall$ r ∈ [1,n], in the expression for $h_{n,2}$, we will get the formula for $h_{n,1}$.

Hence: $h_{n,1} = F_n - \cfrac{1}{\frac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\frac{1}{f_2} - \cfrac{1}{\ddots \, \cfrac{1}{d_{n-2} - \cfrac{1}{\frac{1}{f_{n-1}} - \cfrac{1}{d_{n-1} - f_n}}}}}}}$

4.0 Generalized Gaussian Lens Equation

Let the distance between

- an object and the first lens be u
- the image and the last lens be v
- the first lens and the first principal point be $h_{n,1}$
- the last lens and the second principal point be $h_{n,2}$

Then, the following holds: $\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$

This will be proved by mathematical induction; with the last lens L_n being replaced with $L`_n$ and L_{n+1}.

The result is true for n = 1 (where $h_{1,1} = h_{1,2} = 0$).
It is also true when n = 2 (a standard result).

The diagram below show an object O and its image I formed by a system of lenses.

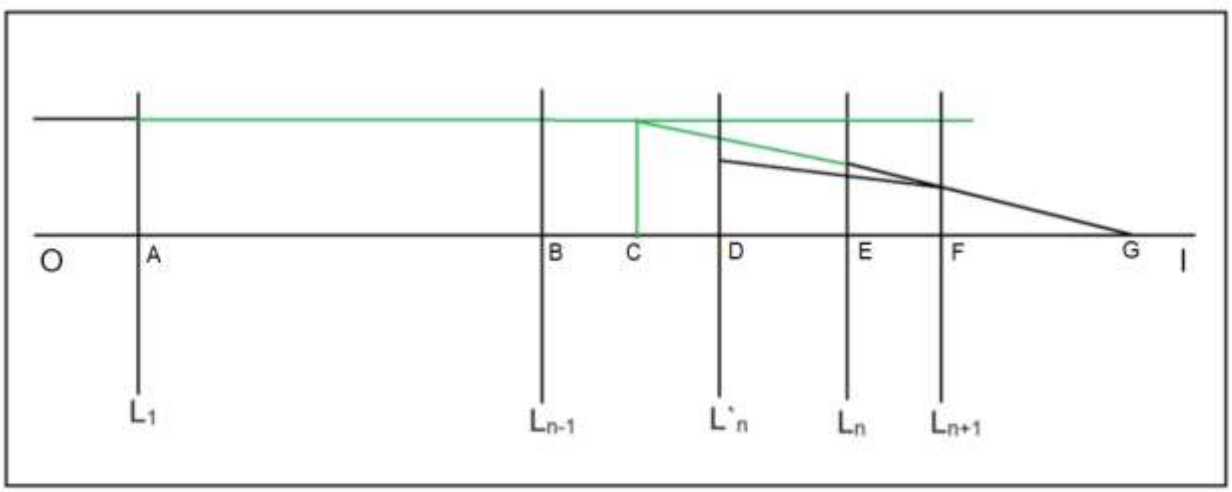

Diagram 4

Initially, there were n lenses; $L`_n$ and L_{n+1} were not present.
L_n is then replaced with an **equivalent** system of 2 lenses, $L`_n$ and L_{n+1}, meaning that

- if $L`_n$ and $L_{n+1}$ are separated by a distance of $d_n$ apart, and $k_n$, $k`_n$ and k_{n+1} are the power of L_n, $L`_n$ and L_{n+1}, respectively, then
  - $k_n = k`_n + k_{n+1} - k`_n d_n k_{n+1}$; or in terms of focal length $\frac{1}{f_n} = \frac{1}{f`_n} + \frac{1}{f_{n+1}} - \frac{d_n}{f`_n f_{n+1}}$

    That is: $f_n = \frac{f`_n f_{n+1}}{f`_n + f_{n+1} - d_n}$; referred to as Result R2, below.
  - $L`_n$ is positioned at a distance of $\frac{d_n k_{n+1}}{k`_n + k_{n+1} - k`_n d_n k_{n+1}}$ (or $\frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$) from and to the left of where L_n was

That is: DE = $\frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$

- the focal length of the new system of n+1 lenses is equal to the focal length of the old system of n lenses
- the position of the image focus G remains unchanged
- the position of the image I remains unchanged

The idea of the above is to simplify the algebra required to show that if "$\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$" is true for a specific value of n, say when n = c, then the formula is also true when n = c+1.

The following notations are used (before the replacement of L_n with $L`_n$ and L_{n+1})

- distance between
    - object and L_1 (i.e. OA) = u
    - image and L_n (i.e. EI) = v
    - first principal point and L_1 = $h_{n,1}$
    - second principal point and L_n (i.e. CE) = $h_{n,2}$
    - L_{n-1} and L_n (i.e. BE) = d_{n-1}
- focal length of system (i.e. CG) = F_n

The following notations are used (after the replacement of L_n with $L`_n$ and L_{n+1})

- distance between
    - object and L_1 (i.e. OA) = u
    - image and L_{n+1} (i.e. FI) = v` = EI – EF = EI – (DF – DE)

      $= v - d_n + \frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$

      That is: $v = v` + d_n - \frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$; referred to as Result R3, below.
    - first principal point and L_1 = $h`_{n+1,1}$
    - second principal point and L_{n+1} = $h`_{n+1,2}$ = CF = CE + EF = CE + (DF – DE)

      $= h_{n,2} + d_n - \frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$

      That is: $h`_{n+1,2} = h_{n,2} + d_n - \frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$; referred to as Result R4, below.
    - L_{n-1} and $L`_n$ (i.e. BD) = $d`_{n-1}$ = BE – DE = $d_{n-1} - \frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$

      That is: $d_{n-1} = d`_{n-1} + \frac{d_n f`_n}{f`_n + f_{n+1} - d_n}$; referred to as Result R5, below.
- focal length of system = $F`_{n+1}$ = F_n

With the above arrangement in place, it will now be shown that $h_{n,1} = h`_{n+1,1}$

$$h_{n,1} = F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{d_{n-1} - f_n}}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\left(\grave{d}_{n-1} + \frac{d_n \grave{f}_n}{\grave{f}_n + f_{n+1} - d_n}\right) - \left(\frac{\grave{f}_n f_{n+1}}{\grave{f}_n + f_{n+1} - d_n}\right)}}}}}}}},$$

[Using Result R5 and Result R2, above.]

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\grave{d}_{n-1} + \grave{f}_n\left(\frac{d_n - f_{n+1}}{\grave{f}_n + f_{n+1} - d_n}\right)}}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\grave{d}_{n-1} + \grave{f}_n\left(\cfrac{1}{\cfrac{\grave{f}_n}{d_n - f_{n+1}} - 1}\right)}}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\grave{d}_{n-1} - \grave{f}_n\left(\cfrac{1}{1 - \cfrac{\grave{f}_n}{d_n - f_{n+1}}}\right)}}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\ddots \cfrac{}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\grave{d}_{n-1} - \grave{f}_n\left(\cfrac{1}{\cfrac{\grave{f}_n}{\grave{f}_n} - \cfrac{\grave{f}_n}{d_n - f_{n+1}}}\right)}}}}}}}}$$

$$= F_n - \cfrac{1}{\cfrac{1}{f_1} - \cfrac{1}{d_1 - \cfrac{1}{\cfrac{1}{f_2} - \cfrac{1}{\cfrac{\ddots}{\cfrac{1}{d_{n-2} - \cfrac{1}{\cfrac{1}{f_{n-1}} - \cfrac{1}{\grave{d}_{n-1} - \cfrac{1}{\cfrac{1}{\grave{f}_n} - \cfrac{1}{d_n - f_{n+1}}}}}}}}}}}$$

$= \grave{h}_{n+1,1}$

The fact that that $h_{n,1} = \grave{h}_{n+1,1}$ should not be surprising, since the object focus, the (object) focal length and the position of L_1 all remain unchanged when L_n is replaced with an equivalent system of two lenses, $\grave{L}_n$ and L_{n+1}.

Now, for the Mathematical Induction, assume that $\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$.

Since $h_{n,1} = \grave{h}_{n+1,1}$, $v = \grave{v} + d_n - \frac{d_n \grave{f}_n}{\grave{f}_n + f_{n+1} - d_n}$ (Result R3) and $F_n = \grave{F}_{n+1}$, this implies that

$$\frac{1}{u + \grave{h}_{n+1,1}} + \frac{1}{\grave{v} + d_n - \frac{d_n \grave{f}_n}{\grave{f}_n + f_{n+1} - d_n} + h_{n,2}} = \frac{1}{\grave{F}_{n+1}}$$

$\therefore \frac{1}{u + \grave{h}_{n+1,1}} + \frac{1}{\grave{v} + \grave{h}_{n+1,2}} = \frac{1}{\grave{F}_{n+1}}$, since $\grave{h}_{n+1,2} = h_{n,2} + d_n - \frac{d_n \grave{f}_n}{\grave{f}_n + f_{n+1} - d_n}$ (Result R4)

Dropping the dashes, for convenience, gives $\frac{1}{u + h_{n+1,1}} + \frac{1}{v + h_{n+1,2}} = \frac{1}{F_{n+1}}$

Thus: the formula $\frac{1}{u + h_{n,1}} + \frac{1}{v + h_{n,2}} = \frac{1}{F_n}$ is true for all positive integer values of n.

5.0 Other Formulas

In this section, the following is proved: $\frac{F_n}{F_{n+1}} = \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$

The diagram below shows a ray of light, parallel to the principal axis of the system of n+1 lenses, being refracted by the system so as to go through H, the image focus of the system. In the absence of L_{n+1}, the ray would have gone through I, the image focus of the system of the preceding n lenses.

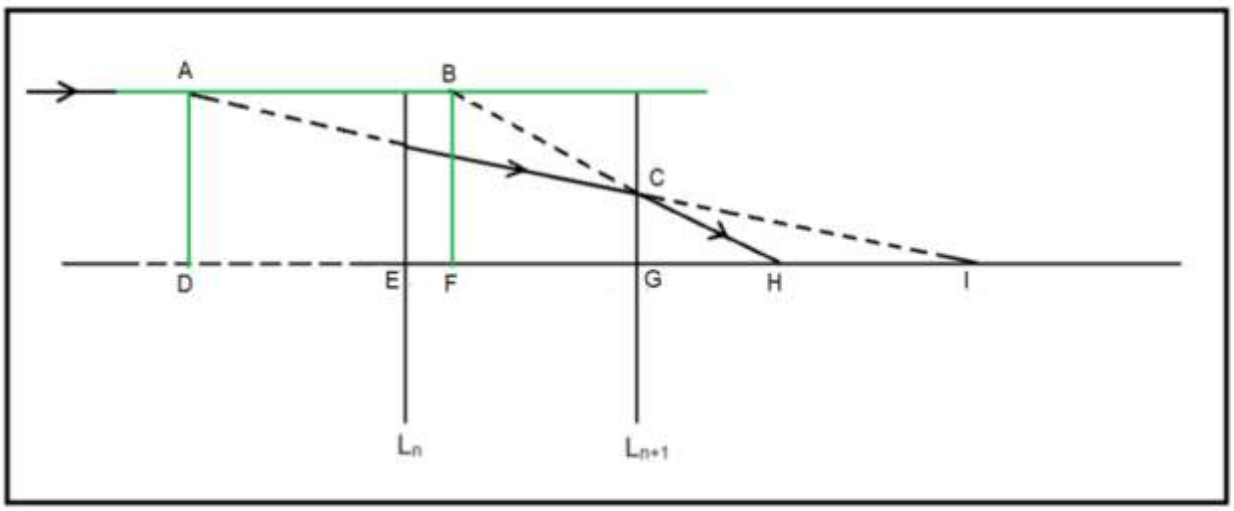

Diagram 5

BFH and CGH are similar triangles. Hence: $\frac{BF}{CG} = \frac{FH}{GH} = \frac{FH}{FH - FG} = \frac{F_{n+1}}{F_{n+1} - h_{n+1,2}}$

Also, ADI and CGI are similar triangles. Hence: $\frac{AD}{CG} = \frac{DI}{GI} = \frac{DI}{DI - DG} = \frac{DI}{DI - (DE + EG)} = \frac{F_n}{F_n - h_{n,2} - d_n}$

BF = AD.

Hence, $\frac{BF}{CG} = \frac{AD}{CG}$.

$\therefore \frac{F_{n+1}}{F_{n+1} - h_{n+1,2}} = \frac{F_n}{F_n - h_{n,2} - d_n}$ or $\frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{F_n}{F_{n+1}}$

Continuing: $(F_n - h_{n,2} - d_n)F_{n+1} = (F_{n+1} - h_{n+1,2})F_n$

$\Rightarrow (h_{n,2} + d_n)F_{n+1} = h_{n+1,2}F_n$

$\Rightarrow \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{F_n}{F_{n+1}}$

Recall recurrence relation R1: $F_n - h_{n,2} = \frac{1}{\frac{1}{f_n} - \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})}}$

$\therefore \frac{1}{F_n - h_{n,2}} = \frac{1}{f_n} - \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})} \Rightarrow \frac{1}{d_{n-1} - (F_{n-1} - h_{n-1,2})} = \frac{1}{f_n} - \frac{1}{F_n - h_{n,2}} = \frac{F_n - h_{n,2} - f_n}{f_n(F_n - h_{n,2})}$

$\Rightarrow \frac{d_{n-1} - F_{n-1} + h_{n-1,2}}{F_n - h_{n,2}} = \frac{f_n}{F_n - h_{n,2} - f_n}$

Increasing n by 1 gives $\frac{d_n - F_n + h_{n,2}}{F_{n+1} - h_{n+1,2}} = \frac{f_{n+1}}{F_{n+1} - h_{n+1,2} - f_{n+1}}$,
or $\frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$

Hence: $\frac{F_n}{F_{n+1}} = \frac{F_n - h_{n,2} - d_n}{F_{n+1} - h_{n+1,2}} = \frac{h_{n,2} + d_n}{h_{n+1,2}} = \frac{f_{n+1}}{f_{n+1} + h_{n+1,2} - F_{n+1}}$

Note the following (which will be used in the next section):

- $\frac{d_n + h_{n,2} - F_n}{F_n} = \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}}$; referred to as Result R6
- $\frac{f_{n+1} + h_{n+1,2} - F_{n+1}}{f_{n+1}F_{n+1}} = \frac{1}{F_n}$; referred to as Result R7

Also, by interchanging d_r and d_{n+1-r} $\forall$ r $\in$ [1,n], and interchanging f_r and f_{n+2-r}, $\forall$ r $\in$ [1,n+1], the following is obtained: $\frac{F`_n}{F_{n+1}} = \frac{F`_n - h`_{n,1} - d_1}{F_{n+1} - h_{n+1,1}} = \frac{h`_{n,1} + d_1}{h_{n+1,1}} = \frac{f_1}{f_1 + h_{n+1,1} - F_{n+1}}$, where, in the system of n lenses where the first lens is absent,

- $F`_{n,1}$ is the focal length
- $h`_{n,1}$ is the distance between the first principal point and L_2 (previously the second lens)

6.0 Magnification Formula

The transverse magnification M_n of a system of n lenses is given by $M_n = \frac{v + h_{n,2}}{F_n} - 1$.
This will be proved by the method of induction.
The formula is trivially true when n = 1, since $h_{1,2} = 0$.
Also, the formula is a standard one when n = 2.

The diagram below shown an image I_n being produced by an object O under the effect of a system of n lenses (only the last lens L_n is being shown).
Therefore, the image distance v = AI_n

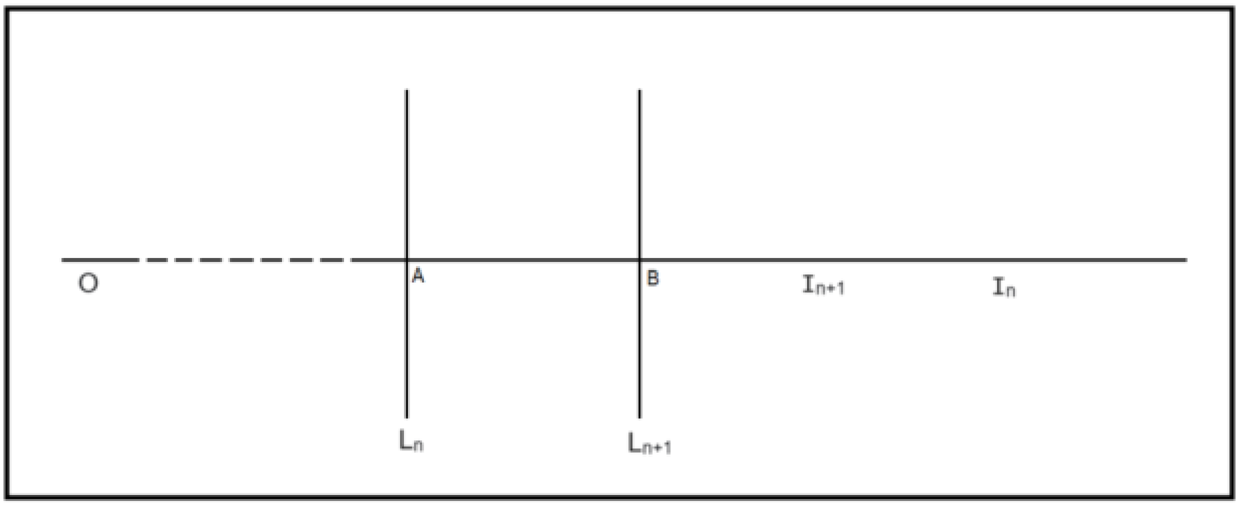

Diagram 6

I_n (considered as a virtual object) gives rise to a real image I_{n+1} under the effect of L_{n+1}.
Therefore, for L_{n+1}

- the object distance, u = -BI_n (the negative sign; because of the **virtual** object)
  = -(AI_n – AB) = -(v – d_n)
- The image distance v\` = BI_{n+1}

For L_{n+1}, using "$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$" gives $\frac{1}{-(v - d_n)} + \frac{1}{v`} = \frac{1}{f_{n+1}} \Rightarrow \frac{1}{(v - d_n)} = \frac{1}{v`} - \frac{1}{f_{n+1}} = \frac{f_{n+1} - `v}{v`\, f_{n+1}}$

$\Rightarrow v - d_n = \frac{v`\, f_{n+1}}{f_{n+1} - `v} \Rightarrow v = d_n + \frac{v`\, f_{n+1}}{f_{n+1} - `v}$; referred to as Result R8

Assume that $M_n = \frac{v + h_{n,2}}{F_n} - 1$.
Then, $M_{n+1} = \left(\frac{v + h_{n,2}}{F_n} - 1\right)\left(-\{\frac{v`}{f_{n+1}} - 1\}\right)$
[Note: the negative sign is because the object for L_{n+1} is a **virtual** object.]

$$\therefore M_{n+1} = -\left(\frac{\left\{d_n + \frac{v\grave{}\, f_{n+1}}{f_{n+1} - \grave{}v}\right\} + h_{n,2}}{F_n} - 1\right)\left(\frac{v\grave{}}{f_{n+1}} - 1\right)$$; using Result R8

$$= -\left(\frac{\frac{\{d_n + h_{n,2}\}\{f_{n+1} - \grave{}v\} + v\grave{}\, f_{n+1}}{(f_{n+1} - \grave{}v)}}{F_n} - 1\right)\left(\frac{v\grave{} - f_{n+1}}{f_{n+1}}\right)$$

$$= -\left(\frac{\{d_n + h_{n,2}\}\{f_{n+1} - \grave{}v\} + v\grave{}\, f_{n+1} - (f_{n+1} - \grave{}v)F_n}{(f_{n+1} - \grave{}v)F_n}\right)\left(\frac{v\grave{} - f_{n+1}}{f_{n+1}}\right)$$

$$= \frac{\{d_n + h_{n,2} - F_n\}\{f_{n+1} - \grave{}v\} + v\grave{} f_{n+1}}{f_{n+1}F_n} = \frac{\{d_n + h_{n,2} - F_n\}\{f_{n+1} - \grave{}v\}}{f_{n+1}F_n} + \frac{v\grave{}}{F_n}$$

$$= \left(\frac{d_n + h_{n,2} - F_n}{F_n}\right)\left(1 - \frac{\grave{}v}{f_{n+1}}\right) + \frac{v\grave{}}{F_n}$$

$$= \left(\frac{h_{n+1,2} - F_{n+1}}{F_{n+1}}\right)\left(1 - \frac{\grave{}v}{f_{n+1}}\right) + \frac{v\grave{}}{F_n}$$; using Result R6

$$= \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{(h_{n+1,2} - F_{n+1})v\grave{}}{f_{n+1}F_{n+1}} + \frac{v\grave{}}{F_n}$$

$$= \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{(h_{n+1,2} - F_{n+1} + f_{n+1})v\grave{}}{f_{n+1}F_{n+1}} + \frac{v\grave{}}{F_{n+1}} + \frac{v\grave{}}{F_n}$$ [adding and subtracting $\frac{v\grave{}}{F_{n+1}}$]

$$= \frac{h_{n+1,2} - F_{n+1}}{F_{n+1}} - \frac{v\grave{}}{F_n} + \frac{v\grave{}}{F_{n+1}} + \frac{v\grave{}}{F_n}$$; using Result R7

$$= \frac{v\grave{} + h_{n+1,2} - F_{n+1}}{F_{n+1}} = \frac{v\grave{} + h_{n+1,2}}{F_{n+1}} - 1$$

The induction step is completed.