Rob 501 - Mathematics for Robotics
Recitation #10

Nils Smit-Anseeuw (Courtesy: Abhishek Venkataraman, Wubing Qin)

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1 Set Theory

1. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$,
   - The distance from vector $x \in \mathcal{X}$ to vector $y \in \mathcal{X}$ is defined as $\mathrm{d}(x, y) := \|x - y\|$.
   - The distance from a vector $x \in \mathcal{X}$ to a set $S \subset \mathcal{X}$ is defined as $\mathrm{d}(x, S) := \inf_{y \in S} \|x - y\|$.

2. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, let $P \subset \mathcal{X}$ be a subset.
   - A point $x \in P$ is an interior point of P if $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subset P$.
     The interior of P, denoted as $\mathring{P}$, is the set of all the interior points of P.
   - A point $x \in \mathcal{X}$ is a closure point of P if $\forall\, \epsilon > 0$, $B_\epsilon(x) \bigcap P \neq \emptyset$.
     The closure of P, denoted as $\overline{P}$, is the set of all the closure points of P.
   - P is open if $P = \mathring{P}$.
   - P is closed if $P = \overline{P}$.

   Remark:

   - The interior of P is the largest open set contained in P.
   - The closure of P is the smallest closed set containing P.
   - P is open $\iff P = \{x \in \mathcal{X} \,|\, \exists\, \epsilon > 0 : B_\epsilon(x) \subset P\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \exists\, \epsilon > 0 : B_\epsilon(x) \bigcap (\sim P) = \emptyset\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \mathrm{d}(x, \sim P) > 0\}$
   - P is closed $\iff P = \{x \in \mathcal{X} \,|\, \forall\, \epsilon > 0,\, B_\epsilon(x) \bigcap P \neq \emptyset\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \forall\, \epsilon > 0,\, \exists\, y \in P :\, ||x - y|| < \epsilon\}$
     $\iff P = \{x \in \mathcal{X} \,|\, \mathrm{d}(x,\, P) = 0\}$

3. **Proposition:** In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$,
   - A finite intersection of open sets is open.
   - A finite union of closed sets is closed.
   - An arbitrary union of open sets is open.
   - An arbitrary intersection of closed sets is closed.

   Remark:
   An infinite intersection of open sets can be either open or closed, or neither.
   An infinite union of closed sets can also be either open or closed, or neither.

4. Ex: In the following examples, $\mathbb{I}$ denote the set of irrational numbers.

(a) In $(\mathbb{R}, \mathbb{R}, |\cdot|)$, $d(\sqrt{2}, 1) = ?$ $d(\sqrt{2}, \mathbb{Q}) = ?$ If given $x \in \mathbb{I}$, $d(x, \mathbb{Q}) = ?$

(b) Are the sets below open or closed?

- In normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$,
  $P_1 = \{0\}$
  $P_2 = [0,\ 1]$
  $P_3 = (0,\ 1)$
  $P_4 = [0,\ 1)$
  $P_5 = \mathbb{R}$
  $P_6 = \emptyset$
- In normed space $(\mathbb{R}^2, \mathbb{R}, \|\cdot\|)$,
  $P_1 = (0,\ 1) \times (0,\ 1)$
  $P_2 = [0,\ 1] \times (0,\ 1)$
  $P_3 = \{(x,\ y) \in \mathbb{R}^2 \,|\, y = 2x + 1\}$

(c) Recall from recitation 2, we have shown that the set of rational numbers $\mathbb{Q}$ with standard $+$ and $\times$ operation is a field, and a field over itself is a vector space, so $(\mathbb{Q},\ \mathbb{Q})$ is a vector space. If we define the norm in $(\mathbb{Q},\ \mathbb{Q})$ as $\|x - y\| = |x - y|$ for all $x,\ y \in \mathbb{Q}$, then $(\mathbb{Q},\ \mathbb{Q},\ \|\cdot\|)$ is a normed space.

- In normed space $(\mathbb{Q},\ \mathbb{Q},\ \|\cdot\|)$, $\overline{\mathbb{Q}} = ?\ \mathring{\mathbb{Q}} = ?$ Is $\mathbb{Q}$ open or closed?
- In normed space $(\mathbb{R},\ \mathbb{R},\ \|\cdot\|)$, $\overline{\mathbb{Q}} = ?\ \mathring{\mathbb{Q}} = ?$ Is $\mathbb{Q}$ open or closed? $\overline{\mathbb{I}} = ?\ \mathring{\mathbb{I}} = ?$ Is $\mathbb{I}$ open or closed?

(d)

$$\bigcap_{n=1}^{\infty} \left(-1 + \frac{1}{n},\ 1\right) = \qquad , \qquad \bigcup_{n=1}^{\infty} \left[-1,\ \frac{1}{n}\right] = \qquad ,$$

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n},\ 1 + \frac{1}{n}\right) = \qquad , \qquad \bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n},\ 1 - \frac{1}{n}\right] = \qquad ,$$

$$\bigcap_{n=1}^{\infty} \left(-1 - \frac{1}{n},\ 1\right) = \qquad , \qquad \bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n},\ \frac{1}{n}\right] = \qquad ,$$

2 Completeness and compactness

1. A sequence (x_n) in a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ is a Cauchy sequence if

$$\forall \epsilon > 0, \exists N(\epsilon) < +\infty : \forall n, m \geq N, \|x_n - x_m\| < \epsilon.$$

2. A sequence (x_n) in a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ is a convergent sequence if there exists an $x^* \in \mathcal{X}$ such that

$$\forall \epsilon > 0, \exists N(\epsilon) < +\infty : \forall n \geq N, \|x_n - x^*\| < \epsilon.$$

3. A normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ is complete if every Cauchy sequence in $\mathcal{X}$ converges to a limit in $\mathcal{X}$. A complete normed space is called Banach space.

4. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, a subset $P \subset \mathcal{X}$ is a complete set if every Cauchy sequence in P converges to a limit in P.

5. In a normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, a subset $P \subset \mathcal{X}$ is (sequentially) compact if every sequence in P has a convergent subsequence whose limit belongs to P.

6. **Theorem:**
   - In a finite dimensional normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$, a subset of $\mathcal{X}$ is compact if and only if it is closed and bounded.
   - In a finite dimensional normed space, every bounded sequence has a convergent subsequence.
   - In a normed space, any finite dimension subspace is complete.
   - Any closed subset of a complete set is complete.

7. Ex:

   (a) Consider the sequence (x_n) below.
   - $x_n = \left(1 + \frac{1}{n}\right)^n$
   - $x_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$. Considering Taylor series $\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} x^k$.

   (b) Is the normed space $(\mathbb{Q}, \mathbb{Q}, \|\cdot\|)$ complete? How about normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$?

   (c) In normed space $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$, whether the following sets are compact and complete?
   - $[0, +\infty)$
   - $[0, 1)$
   - $[0, 1]$