THE GRAPHICAL STATICS OF MECHANISM.

A GUIDE FOR THE USE OF MACHINISTS, ARCHITECTS, AND ENGINEERS;

AND ALSO

A TEXT-BOOK FOR TECHNICAL SCHOOLS.

BY

GUSTAV HERRMANN,

PROFESSOR IN THE ROYAL POLYTECHNIC SCHOOL AT AIX-LA-CHAPELLE.

TRANSLATED AND ANNOTATED BY

A. P. SMITH, M.E.

NEW YORK:
D. VAN NOSTRAND, PUBLISHER,
23 MURRAY AND 27 WARREN STREETS.
1887.

RAND AVERY COMPANY,
ELECTROTYPERS AND PRINTERS,
BOSTON.

PREFACE.

Since the appearance of Culmann's work, which marked an epoch in the history of graphical statics, the graphical method has attained pretty general dissemination in engineering circles; its advantages over the analytical have been recognized more and more, and its further development kept constantly in view. It is universally applied in the designing of stationary structures, — such as bridges, — for determining the requirements of the individual parts. In machine design, also, the graphical method gives valuable aid in finding the moments to which the machine parts are subjected, and in determining dimensions. Accordingly courses in graphical statics have been introduced in all technical schools. The graphical method has also been adapted with advantage to certain departments of dynamics, as in Radinger's "Dampfmaschinen mit hoher Kolbengeschwindigkeit," and Pröll's "Versucht einer graphischen Dynamik." *

* Radinger's Steam-engines with High Piston Speed, and Pröll's An Attempt at Graphical Dynamics.

In all these determinations, however, friction and the special hurtful resistances to motion have not been taken into account. Heretofore all attempts to ascertain these hurtful resistances in machines, and to determine the efficiency which is dependent upon them, and is of such great importance in practice, have been confined to the analytical method, which is often awkward and at times utterly inapplicable. No method is as yet known to me by which the frictional resistances and efficiency of any desired mechanism can be graphically determined. In my lectures on machinery in the polytechnic schools of this place I have endeavored to show the relations existing between the forces in mechanism in a simpler form than that offered by the analytical method. Out of that endeavor has grown the following treatise, which in reality amounts to nothing more than a wider application of the long known but little used angle of friction.

My object in the present treatise was principally to facilitate study for the students of the technical schools, upon whose time and industry increasing demands are made from day to day; perhaps the work may also be of interest and value to those more advanced.

GUSTAV HERRMANN.

AIX-LA-CHAPELLE, May, 1879.

TRANSLATOR'S PREFACE.

THE following translation was undertaken from a belief that a knowledge of Professor Herrmann's work on — we might almost say discoveries in — this subject should not be bounded among English and American engineers by an ability to read German or French; the treatise having already been translated into the latter language. The original has been followed faithfully. There is one word that perhaps needs explanation. The German expression *im Sinne* seems to have no technical equivalent in English, and has been literally translated "in the sense of," and coupled with "direction," the latter being used loosely. "Acting in the sense of a force Q" means producing a similar result to Q. Thus, suppose a force P applied to a crank tangentially to the crank-circle at every point of revolution. The load Q is suspended from a rope wound on the windlass drum turned by the crank. Then if the rope is being wound on to the drum, the force P acts at every instant in an opposite "sense" to Q, while there is only one point in each revolution at which it acts in an opposite direction.

To complete the definition, suppose the windlass to turn stiffly in its bearings, and the weight Q to be so small that when the crank is released the weight will not run down. To lower the weight, then, a force (P) would have to be applied to the crank to turn it in the opposite direction. This force (P) would then act in the same "sense" as Q at every instant, though it would have the same direction as Q but once in each revolution. "Sense," then, has reference simply to the effect produced through no matter what amount of intervening mechanism, and is entirely distinct from "direction" in its stricter meaning.

A few of the terms of Professor Kennedy's translation of Reuleaux's "Kinematics" have been employed, because there are no other well known equivalents, but they will be easily understood even by those who are not as yet familiar with that work.

The great advantage which the method presents is its simplicity. By the use over and over again of a few easily mastered principles, the most complicated problem may be solved. No knowledge of higher mathematics is required in its mastery, and no handling of lengthy and involved algebraic formula is necessary in its use. A few lines are drawn in accordance with easily understood rules, and the result stands out so clearly on the paper that every bright mechanic ought to understand it at a glance. This, with the additional merit of rapidity and accuracy, should soon render its use common in every class-room and drafting-office.

The respectful thanks of the translator are due to Professor J. F. Klein of Lehigh University and to Mr. T. M. Eynon, recently of the same institution, for their critical revision of the manuscript.

A. P. SMITH, M.E.

UNITED STATES PATENT OFFICE,
Washington, D.C.

CONTENTS.

THE

GRAPHICAL STATICS OF MECHA

§ 1.—THE EFFICIENCY OF MECHANISM

THE object of every mechanism is to transfer in a certain way, from one particular machine another. In this transfer the magnitude and di of the motion may remain unchanged, or the re motion may vary in either or both of these re and the rate of variation in either magnitude o tion may be constant or not. In the steam-e for instance, the straight-line motion of the pis communicated continuously to the cross-head piston-rod unchanged in magnitude and while this right-line motion generates a motion i crank-pin, through the agency of the connectin whose direction varies constantly, and whose magn differs in its relation to the magnitude of the head's sliding motion at every instant. In the ca two cylindrical toothed wheels of equal diameter, ing together, the rotation of one is transformed i rotation of the other, equal, but in an opposite direc

The ratio between the lengths of the paths thr which the two pieces move depends upon the which exists between the force P working upon driving member of the machine, and the resistan ɜ driven member encounters. If no hurtful ·s or hinderances, such as friction, appeared in ›n under consideration, the mechanical work of ng-force, for any period of time, would exactly ,t done during the same period in overcoming ance presented to the driven member, under the on of uniform motion. This follows from the wn principle of virtual velocities. If p and q he distances over which the points of applica- P and Q travel along the directions of these ınd if we let Pp and Qq — i.e., the product of ıto distance — represent the mechanical work ʳ the respective forces, we have the equation

$$Pp = Qq, \qquad \text{or} \qquad Pp - Qq = 0.$$

ːr the supposition, on the other hand, that one . of the machine parts does not have a uniform according to the law of kinetic energy, again ˌall hurtful resistances, the equation becomes

$$Pp - Qq = L,$$

L represents the amount of work which the l mass M has either absorbed or given out in the .ed time. According as the above value of $Pp -$ positive or negative, the moving masses have been ːrated or retarded; and from $Pp - Qq = 0$ there /s evidently a uniform motion during the assumed d. From the preceding it follows, that, always ˛ing hurtful resistances, in no machine construction ʻhatever form is there either a loss or gain in ıanical work; for if, in the case of an inequality between Pp and Qq, the masses M are accelerated, this acceleration represents a storing-up of mechanical work, which, later in the period, is available to its full amount.

But there can be, in reality, no movement between two bodies in contact without certain hinderances arising; which hinderances, since they always have the character of an impediment to the desired motion, are generally known as hurtful resistances, in opposition to the useful resistances in the overcoming of which the useful work of the machine consists. In every case, such hurtful resistances arise only where one body has a motion relative to another: as, for instance, the friction of a journal turning in a bearing which is stationary, or rotates at a different speed; the resistance of the air to arms or wings in rapid motion, etc. These resisting forces disappear with the cessation of relative motion between the two bodies assumed. Therefore, we must admit that no loss of work occurs when tensile or compressive forces act on a solid body, so long as the elastic limit is not exceeded, since their effect is only to increase or diminish the distance between the molecules of the body, — there being no contact between these molecules, — and the contraction of the body gives out again the work used in expanding it; at any rate, experience has not yet proved the contrary. That no one may bring forward the example of the resistance due to the stiffness of ropes as an exception to the above generalization, it should be remarked, that, when a rope is bent around a pulley there occurs a relative motion between any two threads or strands which are at different distances from the axis of the pulley.

A certain amount of mechanical work, Ww, is necessary to overcome the hurtful resistances W during a period of motion in which the distance w is traversed. This is not only lost as far as the work proper of the machine is concerned, but exercises a further and evil influence, in that it is transformed into molecular work, whose effect is evident in the wear of the moving parts. Taking into account the hurtful resistances the equation becomes

$$Pp - Ww = Qq + L, \quad \text{or} \quad Pp - Ww = Qq,$$

if, in the last equation, we suppose the useful work Qq to include the positive or negative quantity L representing the work done in the acceleration or retardation of the masses M. The last assumption, which will be taken as the basis of all that follows, is always admissible, because, as before remarked, all work stored up in the acceleration of the masses will be given out again in the performance of useful work when the velocity of the masses sinks to its previous value. But these *regular* or periodic accelerations and retardations characterize the state of permanency (or *normal* running condition) of the machine, which is the only state or condition taken into consideration in the present case.

From what precedes it follows without further explanation, that in all machines, without any exception whatever, the useful work done is less than the work expended by the driving-force in producing motion, by an amount exactly equal to the work required to overcome the hurtful resistances. The smaller these latter are, the more completely will the machine accomplish its object; that is, the work of the driving-force will be transferred to its point of useful application with the greater economy. We may therefore regard the ratio of the accomplished useful work Qq to the work Pp expended by the driving-force as the efficiency of the machine; and the ratio

$$\frac{Qq}{Pp} = \eta$$

is known by this name, or by that of "useful effect."

The efficiency of a machine — which will always be represented by η hereafter — is, according to what precedes, less than unity in every case except the ideal one, in which all hurtful resistances, W, disappear. The value of η will evidently grow smaller as the number and magnitude of the hurtful resistances increase; and, since the latter occur wherever there is relative motion, it follows that, in general, the simpler a machine is in its construction the more economical will it be in working, since every additional moving piece reduces the efficiency by the addition of a new hurtful resistance. It is hardly necessary to remark that the value of the useful effect depends not only upon the number, but upon the magnitude, of the hurtful resistances, which magnitude varies greatly according to the kind of resistance offered. Thus there are not a few machines whose efficiency, on account of the resistances incurred, sinks to a small percentage; in fact, the value of η may even become equal to zero or negative. The numerator of the fraction $\frac{Qq}{Pp} = \eta$ becomes zero in all cases where a weight Q is moved in a horizontal plane, — for instance, in the rolling of an object along a horizontal track, in the turning of a crane, of a turn-table, etc., — because the distance q passed over in the "sense," or direction, of the load Q (i.e., the line of direction along which it would tend to move if left without any support whatever) is equal to zero. In such cases the entire work of the driving-force P is employed in overcoming the hurtful resistances W; and we have for these cases, for such mechanisms, the equation

$$Pp = Ww.$$

If, further, we suppose that in a certain case $Ww > Pp$, there follows, from the general equation,

$$Pp - Ww = Qq,$$

a negative value for Qq; i.e., in order to render motion possible in such a case the force Q must work in a direction, $-q$, opposite to its usual "sense" or direction.* The force Q is no longer a resistance to be overcome through the agency of the machine: it is, on the contrary, to be regarded as one of the forces necessary to produce motion, and working in the same "sense," or direction, as P. There can no longer be any question of useful work, and machines of this nature are never used to produce motion; they find their application in those cases where it is required to hinder an undesired motion. Every screw the pitch-angle of whose thread is not sufficiently great to cause it to turn backwards under the influence of the load upon its nut, is a machine with negative efficiency for the case of backward motion; and the same is true of the differential pulley as generally constructed. The

* Thus, if it was a weight to be lifted, the attraction of gravitation would have to be reversed, or overcome by an opposite and greater force before motion would occur. — *Trans.*

clamps, so common in practice, which never release their hold under any strain whatever, and of which we shall treat more particularly hereafter, may always be regarded as machines with negative efficiency.

We may define efficiency in another way which is often used, and in many cases is more convenient. If in any mechanism where, as before, a driving-force P expends the work Pp in order to move a load Q along the path q, we neglect the hurtful resistances, there would evidently be necessary to accomplish the useful work Qq, under this supposition, only a driving-force P_0, of a less value than P, but working along the same distance p, at the same point of application, and in the same direction. In the case of this force P_0 we have of course, through the neglect of all hurtful resistances, the equation

$$P_0 p = Qq,$$

from which the expression for the efficiency becomes

$$\eta = \frac{Qq}{Pp} = \frac{P_0 p}{Pp} = \frac{P_0}{P}.$$

If, for brevity, we call P_0 the theoretic, and P the actual, driving-force, the efficiency becomes the ratio of the theoretic to the actual driving-force; and this value accordingly represents that percentage of the driving-force which serves to overcome the useful resistance.

If, on the other hand, under the assumption of frictionless motion we regard the force P as working along the path p, it would overcome a resistance Q_0 along the path q, which theoretic resistance Q_0 would have a larger value than the useful resistance Q actually overcome;

and, from the equation $Pp = Q_0q$, we should have the efficiency,

$$\eta = \frac{Qq}{Pp} = \frac{Qq}{Q_0q} = \frac{Q}{Q_0};$$

that is to say, equal to the ratio which the actual useful resistance Q bears to the theoretic one Q_0.

If, in the case of any particular mechanism with the useful load Q, the driving-force becomes less than P, there can be no motion in the "sense" of P; the mechanism remains at rest. The same thing continues until the driving-force approaches the theoretic P_0 in value. If, then, there were no hinderances to motion to be taken into account, the slightest further diminution of P below the value P_0 would result in a backward or reverse motion of the mechanism, the load Q becoming the driving-force. But since certain hurtful resistances, which may be represented by (W), oppose the backward motion, that backward motion can only occur from that moment when the force P, through further diminution below P_0, has sunk to a value (P), for which the relation

$$Qq - (W)w = (P)p$$

holds. The force (P) which is just able to prevent the backward motion, or running down, of the mechanism, is smaller than the theoretic driving-force P_0 by an amount which increases with the magnitude of the resistance (W) offered to the backward motion of the mechanism. Thus, in speaking of the efficiency for backward motion, we will hereafter understand the ratio

$$(\eta) = \frac{(P)}{P_0},$$

which the force (P), actually necessary to prevent backward motion, bears to the theoretic force P_0. This definition evidently corresponds to that previously given for the efficiency η for forward motion, if we regard the load Q as the driving-force, and the force (P) opposed to backward motion as the useful resistance. From the foregoing considerations it follows that the two values P and (P) form the two limits between which the driving-force may lie without any motion occurring in the mechanism, and that every increase of the force over P will result in motion in the "sense," or direction, of P, and that every diminution under (P) will result in motion in the "sense," or direction, of Q. The discussion of the backward motion is of importance in all those mechanisms in which the useful resistance Q may assume the *rôle* of a driving-force; thus, for example, in all hoisting-gear where the value of (P) represents the brake-force through whose application the undesired sinking of the load Q is to be prevented. On the contrary, backward motion is impossible in all machinery in which the useful resistance Q is called forth by the very motion produced, as is the case, among others, with all those machines whose object it is to change the form of bodies, — mill machinery and the like.

For the efficiency (η) of the backward motion the same remarks apply as in the case of the forward motion. (η) is always less than unity, and, in certain cases, can become negative in value. In the last case, it is evident that automatic backward motion, under the influence of the load Q, cannot occur; or, in other words, that the machine is self-locking. The ordinary screw hoisting-gear and differential pulley-blocks are examples of this kind. Such self-locking mechanisms are of great convenience as hoisting-gears, and give great security against any running down of the machine; but there is always the accompanying disadvantage of a small efficiency. Therefore, on this account, they are not to be recommended in cases where a large amount of mechanical work is to be delivered. That with the property of being self-locking, i.e., with the condition $(\eta) \overline{\gtrless} 0$, there is always united a small efficiency η, will be seen from the following considerations: —

From
$$(\eta) = \frac{(P)}{P_0} = 0$$

it follows that $(P) = 0$, and therefore

$$Qq = (W)w$$

since $(P)p + (W)w = Qq$.

On the other hand, for forward motion,

$$Pp = Qq + Ww.$$

If the hurtful resistances (W) for the backward motion are equal to those W for the forward, the substitution of the former for the latter in the last of the above equations would give

$$Pp = Qq + (W)w = Qq + Qq = 2Qq;$$

from which

$$\eta = \frac{Qq}{Pp} = \frac{Qq}{2Qq} = \tfrac{1}{2}.$$

The assumption that $W = (W)$ is a sufficiently close approximation, since the hurtful resistances are the result of both P and Q.* Moreover, since $W > (W)$, it follows that $2Qq < Pp$, and consequently that the efficiency η must always be less than $\frac{1}{2}$. Therefore we may lay down the principle that every self-locking machine has an efficiency of less than fifty per cent, a conclusion which experiments with screw hoisting-gear and differential pulley-blocks confirm.

Machines met with in practice consist generally of a collection of elementary mechanisms, so that the driving-force P' necessary to overcome the useful resistance Q of the first mechanism must be regarded as the useful resistance Q' of the second mechanism, for the overcoming of which in the second mechanism another driving-force P'' is required, and so on. Take, for example, an ordinary warehouse hoist: here the rope with its drum is one mechanism, in which the load hanging from the hook on the end of the rope is the useful resistance Q, to overcome which a force P' must be applied at the circumference of the toothed wheel connected with the drum. As regards the toothed gearing which forms the second mechanism, the force P' becomes the useful resistance Q', which must be overcome by a force P'' applied at the circumference of the second large gear

---

* The force of this reasoning lies in the fact that

$$W : (W) :: \text{friction of } (P \text{ and } Q) : \text{friction of } (P) \text{ and } Q,$$

in which Q is generally much the larger factor; and hence the difference between W and (W) is much less than it would be if they were dependent solely upon P and (P), and the relation $W : (W) ::$ friction P : friction (P) existed. — *Trans.*

upon the axis of the pinion. In the same way, this force P'' is the resistance Q'' for the third mechanism, represented by the second pair of gears and the crank. To overcome this resistance Q'', a force P is necessary at the crank-handle. When, as in the case of a crane, the load Q does not hang directly from the drum, but the rope, or chain, is first led over one or more pulleys, each of these pulleys is to be regarded as a separate mechanism. In the same manner, every machine, however complicated, can be resolved into its simple elementary mechanisms. Such a resolution greatly simplifies the determination of efficiency of machines, inasmuch as the number of common mechanisms is small, while the diversity between complete machines is almost endless.

One general law may be established for the efficiency of a machine consisting of any desired number of elements. If we let Q again represent the load, and q the path described by it in a slight movement, and let P_1, P_2 . . . P_n denote the driving-forces for the separate mechanisms, and p_1, p_2 . . . p_n the corresponding paths of these forces, we may understand by η_1, η_2 . . . η_n the efficiencies of the individual mechanisms. For the first machine, we have

$$\frac{Qq}{P_1 p_1} = \eta_1.$$

For the second, in which P_1 is the useful resistance, and P_2 the driving-force, we have

$$\frac{P_1 p_1}{P_2 p_2} = \eta_2;$$

and so on for each separate mechanism. From the multiplication of all these equations, there results

$$\frac{Qq}{P_1 p_1} \cdot \frac{P_1 p_1}{P_2 p_2} \cdot \frac{P_2 p_2}{P_3 p_3} \cdots \frac{P_{n-1} p_{n-1}}{Pp} = \eta_1 \cdot \eta_2 \cdot \eta_3 \cdots \eta_n,$$

or

$$\frac{Qq}{Pp} = \eta = \eta_1 \cdot \eta_2 \cdot \eta_3 \cdots \eta_n.$$

That is to say, the efficiency of a machine composed of any number of mechanisms is equal to the product of the efficiencies of all the separate mechanisms.

From the fact that the efficiency of a simple mechanism is always less than unity it follows, as before remarked, that in general the useful effect of a machine decreases as the number of its constituent elements increases.

Since, furthermore, the above product cannot be negative as long as all the factors are positive, it follows that a machine can only be self-locking when this property belongs to at least one of its elementary mechanisms.

§ 2.—THE EQUILIBRIUM OF MECHANISMS.

Although mechanisms, by their very nature, can only effect their object while in motion, or by virtue of the same, yet, for the ascertaining of the relation existing between the various forces, we may always assume as a basis that condition of equilibrium which corresponds to the limit where the slightest increase of the driving-force would produce a motion in the "sense," or direction, of that force. In what follows P will again represent the driving-force and Q the useful resistance. Neglecting for the present any acceleration of the masses, we will suppose a uniform motion in which, at each instant, the work of the force during a small portion of time is just sufficient to overcome the useful resistance Q after the hurtful resistances W have been disposed of. It will then easily appear in what way the accelerating force working upon the mass M in the case of variable motion can be ascertained.

The exterior forces P and Q working upon any mechanism, call forth certain internal forces, or re-actions R, between the members of the machine wherever two parts come in contact. These re-actions are to be regarded as two equal and opposing forces occurring at every surface of contact. Every pair of forces thus arising at the same point is, therefore, in equilibrium. We must imagine such re-actions wherever two bodies come in contact, whether the bodies move relatively to each other or not. We can, therefore, in every case neglect the bodies in contact and think only of the re-actions offered by those bodies. Under this supposition, any member of a machine which is acted upon by certain exterior forces P and Q, and which is supported at certain points by neighboring bodies, must be under the influence of the exterior forces P and Q, and of the re-actions R, which are sufficient to replace the imagined supports, in order to be in the supposed limiting condition of equilibrium. The conditions of equilibrium furnish us, in general, with a means by which from the known elements, — direction and magnitude of individual forces, — we may ascertain the unknown. In the majority of cases the intensity of the re-actions of the supports is unknown; of the exterior forces, there is, as a rule, one element — the direction, or intensity, of one force — unknown at first. As regards the direction of the re-action replacing a support, it is determined empirically by the condition that it shall be inclined to the supporting surface at a certain determinate angle whose magnitude depends upon the nature of the two bodies in contact, as to smoothness, hardness, etc. The hurtful resistances to motion, W, which, as previously remarked, arise only at the point of contact between two bodies (i.e., at the supporting surfaces), depend on the nature of the material, and of the surfaces constituting the supports. The size of the angle at which the surfaces of contact will be cut by the direction of the re-action existing between them depends closely, as will be shown in what follows, upon the amount of hurtful resistance generated between the surfaces.

If we suppose, in the next place, that no hurtful resistance W exists, — a condition of affairs which, of course, never occurs in practice, — the angle formed by the direction of re-action and the supporting surface would be a right angle; in other words, when there are no hurtful resistances a supporting surface can only re-act, at each of its points, in the direction of a normal. With the assumption which thus entirely ignores all hurtful resistances, it is a simple matter to determine the proportion of power and load at every point in a machine; and, for this purpose, a graphical method can be used to good advantage. A few examples will serve to illustrate this procedure.

Let A, B, and C (Fig. 1, plate I.) be the centres of the three pins on a bell-crank, the middle one of which, C, turns in the fixed bearing C_1, while the end pivot A is enclosed by the eye or end bearing A_1 of the rod A_1A_2, and the other pivot B is attached in the same way to the rod B_1B_2. If, now, a force Q acts upon the lower end A_2 of the rod A_1A_2 in a certain direction, the rod A_1A_2 must, according to the preceding principles, be in equilibrium under the influence of the force Q, and the re-action R_1 replacing the pivot or journal A; and this latter re-action, being perpendicular to the surface of the journal, must pass through its centre. Two forces, however, can only be in equilibrium when they are equal, and work in opposite directions along the same straight line; from which it follows immediately that the line of direction of the force Q must pass through the centre of A, or, if it does not, there will be a turning of the rod A_1A_2 about the journal A until this condition is fulfilled. In the same way, it follows that the direction of the force P acting on the rod B_1B_2 must pass through the centre of B, and that the journal B must exert upon the rod B_1B_2 a re-action R_2 which shall be equal and opposite to P. The rods A_1A_2 and B_1B_2 will be subjected to tension by the forces Q, R_1 and P, R_2, respectively; i.e., there will be called forth internal elastic stresses in the material of each section of the rods, which will be in equilibrium one with another, and with the exterior forces at the ends of the rods. These interior strains are of great importance in determining the dimensions of cross-sections of the separate machine parts, but have no direct influence upon the conditions of equilibrium of the machine. We shall not, however, go into the determination of dimensions here, or in the remaining portion of the treatise. For all that the reader is referred to the well-known works on machine construction and the resistance of materials. Regarding now the lever ABC alone, we have the external forces Q and P acting upon it at the points A and B; and we can also suppose the bearing C_1 to be replaced by a re-action R_3, whose line of direction passes through the centre of C. For the condition under consideration these three forces must be in equilibrium. This can only occur when the three forces intersect at the same point; and therefore the re-action R_3, exerted by the bearing C_1 upon the journal C, must also pass through the intersection O of the lines of the forces P and Q. Moreover, the relative intensities of the three forces, P, Q, and R_3, are easily determined if we let $OD = Q$, the load, and complete the parallelogram $ODFE$, whose other side falls upon OB, and whose diagonal coincides with OC. We have then in OE the force P, and in OF the pressure of the journal C upon its bearing, and the equal but opposite re-action R_3 of the bearing C_1 against the journal C.

The relative intensities of P, Q, and R_3 correspond to the oft-mentioned limiting condition in which the slightest increase of Q or P would cause motion in the "sense," or direction, of the force so increased, as can be at once seen from the figure; for, if we increase Q until it is equal to OD', the diagonal OF', which represents the pressure of the journal upon its bearing goes, to the left of the centre C. This points to a motion of the lever in the "sense," or direction, of Q. On the other hand, an increase of P to the value OE' gives a journal pressure OF'', in consequence of which there would be a right-handed revolution of the lever.

As a further example, the slider-crank mechanism (Fig. 2, plate I.) may be adduced. The force acting from the piston-rod AA' upon the pin A of the cross-head is transferred through the connecting-rod A_1B_1 to the crank-pin B; and, under the supposition of entirely frictionless motion, the force T must pass through the centres of A and B since it is perpendicular to the surfaces of contact of the journals A and B with their bearings. Since the forces P and T have different directions, the pin A cannot be in equilibrium under their influence alone: a re-action R_1 exerted by the guide D_1D_2 upon the cross-head D is necessary. For the condition of equilibrium this force acting perpendicularly upon the supporting surface D_1D_2 must pass through the intersection A of the piston-thrust P and the connecting-rod resistance T. From this we can easily find the forces R_1 and T by drawing AF equal to P, the piston-thrust, and completing the triangle AFG, in which FG is parallel to R_1, i.e., perpendicular to the guide D_1D_2. If, now, the axis C of the crank meets a resistance Q at the distance CJ, — as though a gear-wheel, with the radius CJ was on the axis C and meshed with another gear-wheel JK, whose resistance would be represented by Q, — it must be in equilibrium under the influence of the connecting-rod thrust acting at B, the resistance Q acting at J, and a re-action R_2 of the bearing C_1. The direction of the latter again coincides with the line joining O and C, if O represents the intersection of Q and T. By constructing upon the line AG, already determined as the value and direction of the force T, the triangle AGH, whose sides AH and GH are parallel to the direction of the resistance Q, and of the re-action R_2, respectively, we have in GH the value of the re-action R_2, and in AH that of the resistance Q overcome at J, at the instant under consideration. It will be seen that the ratio of P to Q varies for every instant, and that, with a constant piston-thrust the amount of resistance Q overcome at J will vary between zero at the dead points and a maximum at some intermediate position. When, therefore, as in the ordinary case, the resistance which the gear JK presents to the motion of the crank is constant, this resistance must have a mean value between zero and the maximum value of Q; and there will result an acceleration or retardation of the masses (of fly-wheel) according as the value of Q, determined as above, exceeds, or falls short of, this average value of the resistance between the gear-wheels. This peculiarity of slider-crank mechanism is well enough known to render a further discussion of it here superfluous.

In the same way as in the two examples shown we can obtain in every case the ratio of the force P_0 to the resistance Q. In nearly all cases the graphical method offers great advantages over the analytical on account of its simplicity and plainness. The analytical method frequently leads to involved expressions, especially when it is attempted to bring the hurtful resistances into the calculation; when, in other words, it is no longer a question of determining P_0 or Q_0, but of P or Q. Since the economic value, or efficiency, of any machine depends directly upon the magnitude of the hurtful resistances connected with it, it is evident that the determination of the ratio actually existing between the forces when these resistances are taken into account is of vastly more importance in practice than any determination of the merely theoretical forces. We have been accustomed in the past to the use of only the analytical methods. The graphical methods for the determination of the friction in, and hence the efficiency of, mechanisms have hitherto been rarely employed; at any rate, in all the text-books only the formulæ for these determinations have been given. How complicated such investigations often became even in the simplest machine as soon as any exact calculation was attempted, is well known. Thus, for example, we could only obtain an expression for the journal friction of a bell-crank, as in Fig. 1, plate I., or of a pulley where the ropes were not parallel, through a long radical, — a circumstance which compelled the assumption of parallelism in all cases of pulley friction, even when there was a marked inaccuracy in the same. For the same reason it is the custom to assume an infinite length of connecting-rod in all cases of crank-gear, in order to render less unwieldy the expressions in which the angle of crank to piston-rod occurs. From the well-known advantages which the use of the graphical method offers in the designing of machine elements, as in the determination of the moment to which axles, cranks, etc., are subjected, arose the idea of finding an expression for friction by the same method. From that idea has sprung the following treatise. In the course of the same, it will be shown that we can obtain a graphical determination of the actual proportion existing between the forces in the same simple and sure way as was done in the preceding examples where all friction was neglected; and it is evident, that, by comparing the values obtained for P or Q with those of P_0 or Q_0, we have immediately an expression for the efficiency, —

$$\eta = \frac{P_0}{P} = \frac{Q}{Q_0}.$$

The methods used do not differ from those indicated in the determination of the theoretical forces; and in nearly every case the drawing of force polygons suffices to attain the desired result. The main question will therefore be to express the separate hurtful resistances graphically. The solution of this problem will be attempted in what follows.

The hurtful resistances in mechanisms which are to be taken into consideration are few in kind and consist, if we neglect the resistance of the medium in which they work, only of friction, which may occur as sliding and rolling friction, journal friction, chain friction, and the friction of toothed wheels. The stiffness of ropes may be considered as equivalent to chain friction. In the following pages these simple resistances will be taken up one by one. The resistance of air or water is here neglected because in ordinary machines it may be left out of account as insignificant, and in most cases is not regarded. In individual cases and in particular machines where such neglect is not allowable, where (as in ventilators) the moving of this medium is the principal object, its resistance should be determined by the rules of hydraulics. It has ceased to be a hurtful, and become a useful resistance.

§ 3. — SLIDING FRICTION.

A LOAD which presses upon a horizontal surface GG (Fig. 3, plate I.) with its weight $Q = AB$ calls forth, while at rest, an infinite number of re-actions from points in that surface, whose resultant is equal to the weight Q, and opposite in direction. This re-action passes, therefore, through the same point D in the supporting surface as the weight Q (the term "weight" being here used in its meaning of a resultant of the forces of gravitation acting upon each particle of the body A). Now, it is known that a force $P = \mu Q$, acting parallel to the plane GG, is necessary to produce a horizontal sliding of the body A along that plane, where μ represents the co-efficient of friction. If, now, a force P, represented in the figure by AC, acts upon the body A, the body is under the influence of the two forces P and Q, which must be in equilibrium with the re-action R of the supporting surface; it being always remembered that we are assuming that limiting condition of equilibrium where the slightest increase of P would cause motion. This condition of equilibrium requires, therefore, a re-action $R = EA$ of the supporting surface equal to the resultant of P and Q, and opposite in direction. From the equation $P = \mu Q$ is determined the angle $\phi = EAB$ which this resultant makes with the normal AB to the supporting surface, since

$$\mu = \frac{P}{Q} = \tan \phi ;$$

and this is called the angle of friction for the materials composing the bodies A and GG. If we suppose the force P to increase gradually from zero to the value AC, the re-action R, given forth by the supporting surface, would gradually deflect from its original direction BA to EA; and for all positions between these two the conditions of equilibrium would be satisfied. In this way the point of intersection D of the re-action with the surface would move from D to F; but in every position it is to be regarded as the point of application of the resultant of all the elementary surface re-actions. These relations evidently hold whatever the direction of the horizontal force P. For example, it is true that when P has the direction AC_1, the re-action of the supporting surface coincides with the line E_1A. By a complete revolution of the force P in the horizontal plane the re-action would describe a conic surface concentric to AB. This is called the cone of friction, and limits the space within which the re-action of the surface GG may exist without motion resulting. We must therefore regard the supporting surface as re-acting against the supported body in certain directions whose angles of intersection with the normal are less than the angle of friction; motion commencing from that moment at which this angle of intersection exceeds, by the slightest amount, the angle of friction. We may employ this property of surfaces in the graphic representation of sliding friction under the following rule: —

If two bodies having plane surfaces in contact undergo relative motion along that plane of contact, we may completely replace each of these supporting planes by a re-action which is inclined to the normal at an angle equal to the angle of friction, and so situated that its component parallel to the plane of contact will work in the direction of the motion which the supporting surface has relatively to the body supported.

The necessity and correctness of the last part of this can easily be seen from the figure. If, for instance, the supported body A is moved by the force P in the direction from A to C, the relative sliding of the support, or bearing, GG to the body A is evidently in the opposite direction CA; and the re-action R, by which GG is replaced, has, according to previous demonstration, the direction EA, whose component CA works in the direction of the sliding of GG upon the body A.

By virtue of this general law we can easily obtain in every case the value of the sliding friction called forth by the relative motion of one body upon another. If, for instance, in the case of the slider-crank gear (Fig. 2, plate I.), we wish to determine the influence of the friction existing between the slipper-block D of the cross-head and the cross-head guide, we have only to draw the re-action R_1, passing through A, in the direction E_1A, making the angle $E_1AE = \phi$ with the normal to the cross-head guide. If, then, we draw through F the line FG_1 parallel to E_1A, we have in AG_1 the actual thrust T of the connecting-rod; while the value previously obtained by neglecting friction was $AG = T_0$. The thrust of the connecting-rod has been reduced, through the sliding friction of the cross-head D, by the amount GG_1; and we have the value

$$\frac{AG_1}{AG} = \frac{T}{T_0} = \eta$$

for the efficiency of the right-line motion in the slider-crank gear. It is unnecessary to explain that the value of this sliding friction and its dependent efficiency varies for every position of the gear.

By the use of the angle of friction in the manner discussed above, the efficiency of a great number of machines may be easily determined, as will be shown in a few examples. Let ABC (Fig. 4, plate I.) be the rack of an ordinary jack, upon whose lug, or claw, A the load $Q = Io_1$ acts vertically downward; while at the pitch-line DD of the rack, the force P acts vertically upward. In consequence of the fact that the load Q acts in one direction only, the rack is continually pressed against the casing at B and C; and we may regard it as being supported by the resultants b and c of the re-actions R_1 and R_2. The four forces Q, P, R_1, and R_2 must be in equilibrium, which can only be the case when the resultant of any two is equal to that of the other two, and opposite in direction. If, therefore, o_1 is the intersection of Q and R_1, and o_2 is that of P and R_2, the line $o_1\,o_2$ is the direction of the resultant of Q and R_1 as well as of P and R_2. From the given load Q, we can now determine the forces P, R_1, and R_2 by drawing the force polygon. Make $Io_1 = Q$; draw $I\,II$ parallel to R_1, and intersecting $o_1\,o_2$; and then construct the triangle $II\,III\,o_1$ by drawing $II\,III$ parallel to P, and $III\,o_1$ parallel to R_2. We have, then, in $II\,III$ the force P, which must be applied at the pitch-line of the rack to lift the load Q. Without friction, $Po = Q$, and therefore the efficiency of this mechanism

$$\eta = \frac{Po}{P}$$

is known.

The figure has been constructed with a co-efficient of friction $\mu = 0.2$, and a corresponding angle of friction $\phi = 11^\circ\, 20'$, and gives $P = 122.3$ for $Q = 100$; therefore

$$\eta = \frac{100}{122.3} = 0.817.$$

It should be here remarked, that in this and all following cases the numerical results are not taken from the lithographed plates, but from the original drawings of the author, which were on a much larger scale. In drawing the direction of the re-action it is not necessary to know the angle of friction in degrees and minutes; a knowledge of the co-efficient of friction is entirely sufficient, and both accuracy and simplicity recommend the construction of the angle of friction, through the geometric method, from its tangent $= \mu$ rather than taking it direct from the table. The efficiency η as determined above is, of course, only the efficiency of the rack in its casing. In order to get at the efficiency of the entire jack we must take into account the friction of the gears and of their journals, in the manner hereafter to be explained.

For the case of backward motion in the jack, that is, when the load Q is the cause of motion, the same construction holds, with the assumption that the re-actions (R_1) and (R_2) are inclined to the opposite side of the normal by an amount equal to the angle ϕ, and we have the force polygon I 2 3 o_1, shown in broken and dotted lines in the figure, where the diagonal $2 o_1$ is parallel to $(o_1)\,(o_2)$, and the line 2 3 gives the amount of force (P) which must be applied as a brake to prevent the accelerated downward motion of the load Q. For the same value of $\mu = 0.2$, we have

$$(P) = 79.1 \qquad \therefore \quad (\eta) = \frac{(P)}{P_0} = 0.791.$$

In this connection it should be remarked that the points b and c, in which the bearing surfaces of the rack are pierced by the re-actions R_1 and R_2, are not completely determined by the geometrical character of the mechanism; and, consequently, it is necessary to start with the supposition that the points of application of the forces are at the middle of the supporting surfaces. Any variation upon this point would affect the efficiency of the mechanism.* In the preceding case of the movement of the cross-head (Fig. 2), such an indetermination was excluded by the requirement that the re-action of the guides must pass through the intersection of the forces T of the connecting-rod, and P of the piston-rod thrust. In the present case (Fig. 4), equilibrium is also possible if R_1 and R_2 do not pass exactly through the central points b and c of the supporting surfaces. But with any other position of b and c the values of R_1 and R_2 would be changed; and it is easily seen that R_1 and R_2 will have the smallest value,

* Thus, if, in the figure b and c were brought nearer together by raising the former, and lowering the latter, the points of intersection o_1 and o_2 would be correspondingly raised and lowered, and the diagonal $o_1\, o_2$ would be inclined more from the perpendicular direction of Q, so that the line $I\ II$ would have to be produced farther before intersecting it; the result being that the sides of the parallelogram which represent the forces R_1, P, and R_2 would be increased, and the efficiency

$$\eta = \frac{Po}{P}$$

diminished. — *Trans.*

and friction be least appreciable, when the vertical distance between the points b and c is the greatest possible. If the guiding surfaces were perfect planes, and the material would not wear away, it would be correct to assume as points of application for R_1 and R_2, the outer edges b_1 and c_1. With this assumption the frictional resistance would be the least possible; and this choice of points would correspond with the law, so well established by experience, that nature always works along the line of least resistance. But since the outer edges would soon become rounded away by wear, on account of the great pressure concentrated upon them, the previous assumption, by which the re-actions act at the centres of the surfaces, corresponds best to the actual condition and assures a sufficient degree of accuracy of determination.

In the same way, we may determine the force P (Fig. 5, plate I.) which must be applied to the rope attached at the point D in order to lift the platform of an ordinary elevator as it occurs in grist-mills. In this case also, the platform at the bearings B and C is pressed against the guide EE by the force acting at A, which is the centre of gravity of the load and platform combined; and, as before, these supporting surfaces may be replaced by the re-actions R_1 and R_2 making the angle ϕ with the horizontal. Taking the points of application of these re-actions at b and c, the centres of the sliding surfaces, we have again, in the line connecting the point of intersection o_1 of Q and R_1 with the intersection o_2 of P and R_2, the position of the resultants of these two pairs of forces. If, then, we make Io_1 equal to the total load, and draw $I\,II$ parallel to R_1, and $o_1\,III$ parallel to R_2, and through the point of intersection *II* the vertical *II III*, we have in the last the necessary driving-force P, and in *II I* and o_1 *III* the re-actions R_1 and R_2. The force $Z = o_1 II$ is the one which acts directly upon the platform. For backward motion the necessary brake force (P) is given by the line 3 2 of the diagram 1 2 3 o_1. From the proportions given in the drawing, with a co-efficient of friction $\mu = 0.16$, we have, for

$$Q = P_0 = 100, \quad P = 105.9;$$

therefore

$$\eta = \frac{100}{105.9} = 0.944,$$

and

$$(P) = 93.5;$$

therefore

$$(\eta) = \frac{93.5}{100} = 0.935.$$

It can easily be seen that the friction grows less, and the efficiency increases, as the horizontal distance between the forces P and Q becomes less, and that between the sliding surfaces B and C greater. In practice, therefore, the height BC should not be taken too small, and the line of force P should be brought as near as possible to centre of gravity A by bending outwards the iron D to which the hoisting-rope is attached. For the same reason, in lifting a weight it should be placed as near the guide EE as possible, while in descending the strain can be partially taken off the brake by placing the weight far away from the guide.

The influence which the ratio of the distance between P and Q to the distance between the guiding surfaces

B and C has upon the relative magnitudes of P and Q can be seen in the simple clamping device shown in Fig. 6, plate I., a mechanism commonly used in saw-mills to hold the block upon the carriage. The log is held firmly upon the carriage, or table, simply by the wrought-iron clamp AB which rests upon it and runs loosely up and down the fixed standard DE. If from any cause, as the upward stroke of the saw, a force Q acts vertically and tends to slide the clamp AB upward along the cylindrical standard DE, there is a resultant tendency to rotate on the part of AB which presses the outer edges b and c of the eye hard against the standard. The latter re-acts at these points with the forces R_1 and R_2, which, under the supposition of an actual slip of the bushing, act downward at an angle ϕ to the horizontal, and oppose such slipping motion. The force R_1 acts from b toward o, and R_2 from o toward c. Their resultant passes through o, the point of intersection. In the case of motion just beginning, when R_1, R_2, and Q are in equilibrium, the point o must fall on the line of force Q. If, therefore, from any point A in the line of Q we draw the two lines Ab' and Ac' parallel to the re-actions R_1 and R_2, we have in b' and c' the points of application at which R_1 and R_2 must act when the desired condition of motion is fulfilled; that is, the eye of the clamp must have an axial dimension equal to $b'c'$ as shown in dotted lines in the figure. In that case even the smallest force Q would cause a loosening of the clamp's hold upon the log.

But when, as shown in the figure, the eye of the clamp has a less height, so that the point of intersection o of the re-actions falls between the standard and the line of force Q, there can only be motion when a fourth force P is added to Q, R_1, and R_2 in such a way that the resultant of P and Q passes through o. Such a force may be applied in numberless ways. If we take the force P parallel to Q, and suppose it to act along the axis of the sleeve or eye, it can be easily determined. Since the resultant of P and Q must pass through o, we have by the well-known law for parallel forces $P \,.\, Bo = Q \,.\, Ao$; and by construction we get P in JB if $BF = AG = Q$, and a line is drawn through F and o intersecting Q in H; after which JB is made equal to HA, since by similar triangles

$$AH \text{ or } JB : FB \text{ or } AG :: Ao : Bo;$$

from which

$$P \,.\, Bo = Q \,.\, Ao.$$

This force P evidently must work in the direction of Q as long as the point of intersection of the resultant forces lies between P and Q; and in such case, therefore, no single force Q, however great, can loosen the clamp; but to accomplish this object a special force P acting in the "sense," or direction, of Q is necessary. It is clear, also, that the slightest force acting at B alone will lift the clamp if it can overcome its weight.*

---

* This statement is slightly inaccurate, since there would be friction to overcome in lifting the clamp when the lifting force is applied at B; for the weight of the clamp would act at its centre of gravity, a point to the right of B, and would form a couple with the lifting force. This couple would be balanced by that formed by the resistances R_1 and R_2, which would act on the opposite sides of the stud ED, and be inclined to the opposite side of the normal from that shown in Fig. 6. The determination would, in fact, be exactly the

For fixing the log in place, a slight pressure or blow upon the clamp is sufficient.

That kind of mechanism which we have previously designated as self-locking finds extended use in practice where it is the object to prevent an undesired motion by means of clamp-gearing. The well-known hold-fast of the planing-bench works upon this principle, as do also certain kinds of ratchet-gear.

Sliding friction plays an important part in wedges, whose efficiency is greatly reduced thereby, becoming less as the wedge grows thinner; i.e., as the angle of the wedge becomes more acute. For an example we will choose the wedged device for adjusting the pivot-bearing of the upright shaft W shown in Fig. 7, plate I. The bearing L supports the shaft W and rests upon the wrought-iron key K, while any side motion is prevented by the sides of the casing N. The re-actions R_1 of the wedge, and R_2 of the casing, will be called forth by the load $Q = IA$; all three forces acting upon the cap L, and necessarily in equilibrium. Of the two re-actions R_1 and R_2, we only know the directions, which must be inclined to the normals of the surfaces in contact at the angle of friction ϕ, but we do not know their points of application. If we assume that the re-action R_1 of the wedge against the bearing acts at the point A, where the weight Q is supposed to take effect, the re-action R_2 of the casing must also pass through this point. In the present case it does not matter, as far as the result of the construction is concerned, where the

---

same as that shown in Fig. 5; and as there proved the lifting-force would have to be more than sufficient to "overcome the weight" of the clamp before motion could result. — *Trans.*

points of application A and C of R_1 and R_2 are taken. If we assume the point of application of the re-action R_1 in a the re-action R_2 would lie along the line cb which passes through b, the point of intersection of R_1 and Q. The value of these re-actions is given in any case by the sides of the force polygon $A\,II$ and $II\,I$, supposing that $I\,A$ is made equal to Q, and the sides mentioned are drawn parallel to R_1 and R_2 respectively.

If a horizontal force P acts upon the wedge K by means of the screw S, the wedge must be in equilibrium, for the case where motion is about to commence, under the influence of the re-action $-R_1$ of the cap L, the re-action R_3 of the support H and the force P. The position of the force R_3 is determined by the requirement that it must pass through the intersection o of P and R_1. To get the value of P we have only to complete the force polygon by constructing upon $II\,A$, or $-R_1$, the triangle $II\,III\,A$, in which the side $A\,III$ parallel to P represents that force, and $III\,II$ the direction and value of the re-action given forth by the support H against the wedge K. In order to determine the theoretical force P_0 it is only necessary to draw R_1 in the direction $A\,II_0$ perpendicular to the surface of the wedge, and R_2 in the direction $I\,II_0$ perpendicular to the axis of the shaft. If we then construct the triangle $II_0\,III_0\,A$, whose sides are parallel to P and perpendicular to HH respectively, we have in $A\,III_0$ the force P_0 which would suffice to raise the bearing if there were no friction. In the same way we find the forces acting in the backward motion of the wedge from the force polygon $I\ 2\ 3\ A$, in which $A\ 2$ is drawn on the opposite side of the normal $A\,II_0$ to the wedge surface at the angle $2\,A\,II_0 = \phi$.

Also the re-action (R_2) of the casing which holds in equilibrium the forces Q and (R_1) must come from the opposite side of the casing in the direction from (C) to A. Further, the position of the re-action (R_3) of the support H is fixed by the condition that it must pass through (o), the point of intersection of the re-action (R_1) with the force (P), which acts along the line of P, but in an opposite direction. To complete the force polygon for backward motion we draw A 3 parallel to (P), and 2 3 parallel to (R_3); and in the former we have the value of the force (P) necessary to withdraw the wedge from under the bearing. Since this force (P) acts from A toward 3 in the same "sense," or direction, in which the load Q tends to move the wedge it is evident that a backward motion of the wedge cannot result from the action of Q alone, and we must regard the efficiency $(\eta) = \dfrac{(P)}{P_0}$ as being negative. With a value $\mu = 0.16$, and a taper of 1 in 9 for the wedge, we get from the drawing, for

$$Q = 100, \quad P_0 = 11.1, \quad P = 45,$$

and therefore

$$\eta = \frac{11.1}{45} = 0.247, \quad \text{also} \quad (P) = -19.8.$$

Hence

$$(\eta) = \frac{-19.8}{11.1} = -1.78.$$

It may be remarked here, to avoid repetition, that, as in Fig. 7, the lines of force and the force polygon will be drawn in full lines for the forward motion, those for backward motion in broken and dotted lines, and those for the theoretical force P_0 in broken lines. All construction lines will be simply dotted.

The resistance which the foot-journal of an upright shaft encounters from the plane surface FF (Fig. 8, plate I.) upon which it rests may be deduced by the methods of sliding friction. We may suppose the elements of friction which arise at every point in the surface of contact to be concentrated in the circumference of a circle $A_1 A_2$, whose radius $A A_2 = \frac{2}{3}r$, r being the radius of the journal AF. If we imagine the load Q to be replaced by two equal forces CA, each equal to $\frac{1}{2}Q$, which act at diametrically opposite points A_1 and A_2 of the circumference, we can replace the re-action of the bearing by two forces R_1 and R_2 at these points A_1 and A_2. These forces will be inclined to the axis AC by an amount equal to ϕ, the angle of friction, and will lie in planes perpendicular to the plane of the forces $\frac{Q}{2}$ passing through A_1 and A_2. The horizontal projections A_1E_1 and A_2D_1 represent the resistances of this species of pivot friction concentrated at A_1 and A_2. To overcome these an equal and opposite couple with forces $E_1A_1 = D_1A_2 = P$ must be applied at the extremities of the lever-arm A_1A_2. When, as is always the case in practice, the driving-force P is applied only at one point the journal will press against one side of the bushing in which it runs on account of its one-sided working, and there will result a certain amount of neck-journal friction beside the pivot friction already taken into account between the end of the shaft and its supporting surface. The determination of this neck friction corresponds to journal friction, which will be discussed in another chapter.

In the same way we can ascertain the amount of friction in the thread of a screw. Suppose SS (Fig. 9, plate I.) to be a screw provided with both right and left handed threads, as occurs in certain forms of coupling for railway-cars; and suppose that each of the nuts M_1 and M_2 is drawn outward with a force $Q = AB$; we may then obtain the force necessary at the lever N to turn the spindle of the screw in the following way: Each of the two nuts is regarded as acting upon the screw in two diametrically opposite points of a central helix (or pitch line) whose diameter d is equal to the arithmetic mean $\frac{d_1 + d_2}{2}$ of the inner and outer helices of the screw. And letting A represent that point of the first pair, and C that point of the second pair, nearest the observer, we have in the two lines AO and CO drawn at the angle ϕ from the normals AO_0 and CO_0 to the direction of the screw-thread Aa and Cc, the direction of the re-actions at A and C. Now draw from the point of intersection O the line OJ parallel to the axis and equal to $\frac{1}{2}Q$. Then in the line KL drawn perpendicular to JO we have the value of the force P which is in equilibrium with the two re-actions of the nuts M_1 and M_2 against the screw. Since the same construction holds for the two other points diametrically opposite to A and C, it follows that for the turning of the screw a couple of forces, each equal to $P = KL$, is necessary; the arm of the couple being twice the radius r of the helix in which the action of the nuts upon the screw is supposed to be concentrated. Without friction we have the force

$P_0 = K_0L_0$, the re-actions being assumed in the direction of the normals O_0A and O_0C. For a co-efficient of friction $\mu = 0.1$, and a pitch of screw $n = \frac{1}{12}$, we have, with $\frac{1}{2}Q = 100$,

$$P = 37.8, \quad P_0 = 16.67,$$

and

$$\eta = \frac{P_0}{P} = 0.441.$$

The construction remains the same when the pitch of the two screws is different, as in differential screw-gearing; or when the pitch of one screw equals zero, as in the much-used tension mechanism (Fig. 10, plate I.) where the thread of one screw is merged into a ring and swivel in which end-journal friction only occurs. Here, as before, we make $OJ = \frac{1}{2}Q$, and have in KL one force of the couple necessary to turn the nut M; it being understood that the line of re-action OK is drawn at an angle $\phi =$ the angle of friction to the axis of the rod S_1. Without friction the force P_0 would be given in JL_0. For a pitch $n = \frac{1}{12}$, and a co-efficient of friction $\mu = 0.1$, the construction gives, for $\frac{1}{2}Q = 100$,

$$P = 28.9, \quad P_0 = 8.33,$$

and

$$\eta = \frac{P_0}{P} = 0.288.$$

The application of the turning-force at one side only of the mechanisms (sketched in Figs. 9 and 10, plate I.) causes a certain amount of neck friction which may be determined by methods to be explained in the following chapter.

§ 4.—JOURNAL FRICTION.

If a cylindrical journal (Fig. 11, plate I.) is pressed by a force $DE = Q$ perpendicular to its geometric axis into a bearing A_1, that bearing re-acts at D with a force equal to Q and opposite in direction, the same as any other supporting surface would do. If the journal revolves in the direction of the arrow a certain force is necessary which does not go through the axis. Let BG be such a force acting at B, and of such value P that it is just sufficient to hold the journal in equilibrium, and by the slightest increase to cause a turning. Under this supposition the journal is in equilibrium under the two forces P and Q, and of the re-action R exerted upon it by the bearing. This last re-action must be equal and opposite to OJ, the resultant of P and Q. The bearing A_1 acts upon the journal at the point K with the force $LK = JO$. By that we do not mean that the re-action is actually exerted at the point K, but that the resultant of all the re-actions of the elements of the bearing against the journal passes through the point K of its surface. We have to assume that the resistance to turning offered by the bearing is friction at the point K. This friction is exerted in the direction of motion of the supporting surface, as previously laid down in a general law, and has a value μN; N being the normal pressure at K, and μ the co-efficient of friction. If we resolve the re-action LK into components at right angles, one normal and the other tangent to the surface of contact at K, we have in the component SK the normal pressure, and in the tangential component LS the frictional resistance at the circumference of the journal. Since, now, $\frac{LS}{SK} = \mu = \tan\phi$ we find that the angle LKS or OKA which the re-action makes with the radius drawn to its point of application equals the angle of friction ϕ of the materials of journal and bearing. If, then, we drop from the centre A the perpendicular AT upon the direction of the re-action KO the value of this perpendicular, or arm of the re-action in reference to the centre, is given by the equation

$$AT = \rho = r \sin\phi,$$

where r is the radius of the journal. The same value for this arm TA will be obtained in every case wherever the turning-force P is applied; and if we suppose the force P to occupy, one after another, all possible positions about the axis A the re-action of the bearing will in each case be tangent to a circle described about the centre A with a radius $AT = \rho = r \sin\phi$.

We can therefore regard the journal bearing, in its action upon the journal, as entirely replaced by a re-action which is tangent to a circle drawn about A with the radius $r \sin\phi$; and the direction and situation of this re-action will be known as soon as any other condition is settled, as, for instance, in the present case, that it must pass through the point O. This circle of a radius $\rho = r \sin\phi$, which for brevity will be called the friction circle through analogy to the nomenclature, friction angle, friction cone, etc., offers a convenient means for the graphic expression of journal friction. Since, under the supposition of an entirely frictionless motion, the re-action of the bearing passes through the centre of the journal, we may regard it as being tangent to the friction circle, which has become equal to zero in this case.

It is evident from the figure that the second tangent drawn from O to the friction circle, shown in the dotted line OW, corresponds to a revolution of the journal opposite to the arrow, and that in this case, when the turning-force acts in the direction OB_1, the point of application of the bearing against the journal is to be assumed at W. Either K or W may represent the point of support of the journal, according to circumstances.

If we further assume that the journal A is fixed, and that the bearing is acted upon by the forces P and Q like the hub of a wheel running loosely upon an axle, we must then regard the point K_1 or W_1 as the point at which the hub is supported by the fixed axle; the point K_1 corresponding to a left-handed revolution like that indicated by the arrow, and the point W_1 to an opposite revolution. The amount and direction of the re-action R are not affected by these changes, there being merely a transfer of the point of application from K to K_1, and from W to W_1.

On the contrary, if we suppose the force P and Q, and consequently their resultant, to act in the opposite directions OF', OH', and OJ', the point of support will fall upon K_1, while the journal will have a revolution in a right-handed * direction opposite to the arrow. We

---

* In general right-handed revolution is to be understood hereafter as meaning revolution in the direction of the hands of a watch.

must determine, therefore, in every case, which of the two tangents is the line of direction for the re-action in that case. This determination is rendered much less difficult here, as in the case of sliding friction, if we keep resolutely before us the principle that each of the two pieces exerts upon the other a re-action which coincides in direction with the motion of that piece relative to the other. Even if both parts, journal and bearing, have absolute motion, as is the case in all link connections, it is still not difficult to determine the relative motion of one part with respect to the other. Later remarks will serve for the further elucidation of this law.

To determine graphically the friction circle of a journal, we have only to draw any radius AB (Fig. 12, plate I.) of the journal, and lay off the angle of friction $\phi = BAC$. When this angle is not given directly, but only the co-efficient of friction μ, draw BC perpendicular to AB and equal to $\mu \cdot AB$. Then draw DE through D, the point of intersection of the hypothenuse AC with the circumference of the journal, parallel to AB. We have then, in the circle drawn about A tangent to the line DE, the desired friction circle of a radius $AE = \rho = r \sin \phi$.

Rods, or links, provided with two pins, or eyes, connecting them with other machine parts, often occur in mechanisms. Such a rod, or link, as AB (Fig. 13, plate I.) would, in the absence of friction, simply transmit force from journal to journal along the line AB connecting their centres. A force applied to either journal, AC for instance, would call forth in the other, BD, an opposite force with which it would be in equilibrium. Since, when friction is neglected, the pressure of the journal upon its bearing in the rod can only be perpendicular to the surface of contact, both these equal and opposing forces must necessarily pass through the centres A and B. This would also occur in reality when there is no motion of the journal relative to its bearing, as, for instance, in the joints of a linked chain which, loaded with a weight, is drawn vertically upwards. When, on the contrary, a turning of the journal relative to its bearing occurs friction enters into the consideration of the motion; and according to preceding principles force can only be transferred from journal to bearing along lines which are tangent to the friction circle. Therefore, in the present case, a transfer of force between the journals AC and BD can only take place along one of the four possible tangents to both friction circles AE and BF. We can easily determine which of the four tangents is to be regarded as the line of reaction in any particular case by the application of the rule previously given. With it we have only to decide in what direction, whether to the right hand or to the left hand, the turning of the journal occurs in its bearing, and in what direction the journal acts upon the rod; i.e., whether the latter is in tension or compression. The action of the link on a pin or journal must then have such a direction that, in consequence of this action, the eye of the link will assume relatively to the pin the rotation which actually does take place. This rule holds also for the action of the pin on the link; for not only the direction of the force, but the direction of rotation, will be reversed in this case.

In Fig. 13 the four tangents are denoted by the figures 1, 2, 3, and 4, for brevity. For still greater clearness there are shown in Fig. 14, plate I., four separate mechanisms in which a rod AB of the type under discussion unites two swinging levers MA and NB. In these four cases, I., II., III., and IV., the heavy arrow drawn at one lever denotes not only the motion of that one lever, but of the whole system; the other lever is therefore always a resisting body. By the little arrows drawn at each joint is shown the motion of the rod, or link, relative to that lever with which it is there united. We may regard the journal as forming a part either of the link or the lever, since, as before explained, such assumption has no effect on the direction and amount of the re-action, merely shifting the point of application to another position on the same line. If we suppose the link AB to be in tension in the cases I. and III., and in compression in II. and IV., it will be readily seen that tangents 1, 2, 3, and 4 in Fig. 13 correspond respectively to cases I., II., III., and IV. in Fig. 14. This correspondence of Fig. 13 to Fig. 14 holds also for the opposite motion on condition that the driving-force is applied at the same lever, for then both the nature of the force acting in the link and the direction of the turning will be reversed.

In a similar way we can determine the direction of re-action of a journal upon its bearing in every particular case. This is in reality all that needs explanation or demonstration in the method of calculating journal friction, since the further graphical determinations consist simply of the application of well-known principles in regard to the resolution and composition of forces. The method above referred to may be shown in a few examples for the sake of clearness.

Let ABC (Fig. 15, plate II.) again represent a bell-crank for which we are to determine the force P that must be applied to the arm B to lift the load acting upon the pin A of the arm CA. Drawing the friction circles for the journals A, B, and C we have, in the tangents ao and bo parallel to the lines of force Q and P, the lines of direction for these forces. Because of the turning of the bell-crank to the right as shown by the arrow, the rod grasping the journal A has a left-handed turning about that journal, and the tangent oa must be drawn, according to the principles previously established, touching that side of the friction circle farthest away from C; so that one might say that the arm of the resistance to be overcome is increased by journal friction. It follows, in the same way, that the line of direction for the re-action of journal B against its rod, which also has a left-handed turning, must be tangent to the inner side of the friction circle, so that the arm of the force P is shortened by journal friction. The re-action of the supporting bearing against the journal C must pass through o, the intersection of oa and ob. There being a turning of the bell-crank to the right hand the re-action R of the bearing to the journal C will lie along the line oc, passing through o and tangent to the friction circle of C. Since R, P, and Q are in equilibrium we have, by making $oI = Q$ and completing the parallelogram $oI\ II\ III$, the force P in $oIII$ and in oII, the re-action of the bearing equal and opposite to the journal pressure. Without friction we should obtain, as in Fig. 1, plate I., the force $P_0 = O\ III_0$; i.e., if the direction of the forces passed through the centres of the pins and $OI_0 = Q$. In the case under consideration, with a co-efficient of friction $\mu = 0.1$, the drawing gives, for $Q = 100$,

$$P = 91.3, \quad P_0 = 87.8,$$

and
$$\eta = \frac{P_0}{P} = 0.962.$$

For another example we will choose the ordinary slider-crank train (Fig. 16, plate II.) as used in steam-engines. The downward pull P of the piston is exerted by the piston-rod K upon the pin A of the connecting-rod. We shall endeavor to determine how great a resistance Q at the distance CE from the shaft can be overcome by this piston force in the position of the mechanism shown by the drawing, friction being taken into account. By the force P we do not represent the entire pressure of steam upon the piston, but simply that really acting upon the cross-head after piston and stuffing-box friction have been deducted. The pressure variations caused by the acceleration and retardation of the piston will also be taken into account in the value of P. The line of this force P must pass through the centre A of the cross-head pin, since this pin is rigidly fixed to the cross-head and piston-rod, and no relative turning can occur between them. There is such turning, however, between the pin A and the end A_2 of the connecting-rod A_2B_2. The line of direction of the force S acting in the same will therefore lie along the tangent ab to the friction circles of the two journals A and B. (Which of the four possible tangents is to be here taken is shown by the little arrows which indicate the direction of turning of the connecting-rod ends about the journals A, B, and enable us to apply the principles explained in Figs. 13 and 14, plate I.) The cross-head pin cannot be in equilibrium under the influence of the two forces S and P alone, which have different directions. Equilibrium requires a third force which must be exerted by the guide D in the re-action R_1. The direction of this re-action is fixed by the condition that it must be inclined at the angle ϕ to the normal to the guide, and its situation by the requirement that it must pass through the intersection o_1 of the forces P and S. Accordingly the guide D exerts upon the cross-head a re-action along the line do_1. Further, the force S will be transmitted without loss by the connecting-rod from the pin A to crank-pin B along the line ba; and we find, as in the case of the bell-crank (Fig. 15), that the force S and the resistance Q acting at E are held in equilibrium by a re-action R_2 exerted by the bearing upon the shaft C. We have the direction of the latter in the line o_2c which is drawn from the intersection o_2 of the forces S and Q tangent to the friction circle of the shaft C. To determine each force we have only to draw the force polygon in which $Io_1 = P$, $I\,II$ is parallel to o_1d, $II\,III$ is parallel to o_2c, and o_1III is parallel to Q or FE. The line $o_1III = Q$ gives the resistance acting at E, and $II\,o_1$ gives the tension S in the connecting-rod, while $II\,I$ represents the re-action of the guides, and $II\,III$ the pressure upon the bearing of C; the determination of the latter forces being of especial importance in proportioning the parts under consideration. For the determination of the theoretical resistance $Q_0 = o_1III_0$ it is only necessary to draw the re-action R_1 normal to the guide D, and the directions of S_1 and R_2 through the centres of A, B, and C, as is shown in the force polygon drawn in broken lines. With the assumption of a coefficient of journal friction $\mu = 0.1$, and sliding friction at the guides $\mu = 0.16$, the drawing gives, for $P = 100$,

$$Q = 48.4, \quad Q_0 = 52.5;$$

therefore

$$\eta = \frac{Q}{Q_0} = 0.922.$$

It is, of course, self-evident that for any other position of the mechanism there would be a different efficiency.

If we turn now to the diagram for the oscillating engine (Fig. 17, plate II.) it will be observed that the force S acting in the piston-rod B_1D is tangential to both journals B and C, as in the preceding case, and falls along the line bc. The demonstration of this fact is as follows: If P is the force exerted by the steam pressure upon the piston E, and which acts along the line of centres BC, this force must be in equilibrium with the other forces which act upon the piston-rod. There are besides P only three forces to be considered: the re-action S of the crank-pin B, the only known condition of which is that it must be tangent to the friction circle of B, and the re-action R_1 exerted by the stuffing-box D_1 against the piston-rod at the point d, together with that, R_2, of the cylinder against the piston E at the point e. These re-actions are inclined at an angle ϕ to the normal to the geometric axis of the cylinder, and act from the cylinder toward the piston and piston-rod respectively.

If we now regard the relative motion of cylinder to piston-rod only we may imagine the piston and piston-rod to be held fast, while the cylinder with its bearing moves a short distance along the piston-rod under the pressure of the steam upon the cylinder-head. Here also the various forces acting upon the moving cylinder must be in equilibrium. These forces are the following: First, the pressure of the steam upon the cylinder-head, which is of course equal to the steam pressure upon the piston, and acts along the line of centres CB, but in the opposite direction, and is therefore to be denoted by $-P$; next, the re-actions which formerly acted upon the piston and piston-rod at e and d now act at the same points, but in opposite directions, against the cylinder and stuffing-box, and are to be denoted by $-R_1$ and $-R_2$; finally, there is the re-action Z exerted by the bearing C_1 against the trunnion-journal C, the only known condition of this re-action being that its line of direction must be tangent to the friction circle of C. Since, then, the condition of equilibrium exists between the four forces P, R_1, R_2, and S, and also between $-P$, $-R_1$, $-R_2$, and Z, it is evident that P, R_1, and R_2 will balance $-P$, $-R_1$, and $-R_2$ respectively, thus leaving $S = -Z$; i.e., the forces S and Z are equal and act in opposite directions. The force S, therefore, must coincide with the tangent cb to both friction circles.

For the condition of equilibrium between the forces P, R_1, R_2, and S, it is necessary that the resultant of any two, as P and R_1, shall be equal and opposite to the resultant of the other two, S and R_2. By joining o_1, the intersection of P and R_1, with o_2, the intersection of S and R_2, we get in o_1o_2 the direction of these resultants. If we make $C\,I = P$, and draw $C\,II$ parallel to R_1, $I\,II$ parallel to o_2o_1, $II\,III$ parallel to bc, and $I\,III$ parallel to R_2, the line $II\,III$ gives us the value of the force S which tends to pull the crank-pin around. Of equal value, as has been demonstrated above, is the re-action which the bearing C_1 exerts upon the journal C of the trunnion. The further determination of the acting forces is carried out in the usual manner. If a resistance Q acts at the radius AF from the shaft we have in the line o_3a drawn from the intersection o_3 of Q and S tangent to the friction circle of A the direction of the re-action R_3 exerted by the bearing upon the shaft journal A; and to determine the value of this re-action and of the resistance Q we have only to resolve $II\ III$ into $II\ IV$ and $III\ IV$ parallel to the directions o_3a and o_3F of the re-action and resistance Q respectively. Without friction we should have $P = S$, and the triangle $C\,I\,IV_0$ would give immediately the value of $Q_0 = I\,IV_0$, $C\,I$ being resolved in the directions of Q and OA.

For a co-efficient of journal friction $\mu = 0.1$, and of friction in the stuffing-box and cylinder $\alpha = 0.16$, the drawing gives, for $P = 100$,

$$Q = 61.0, \quad Q_0 = 64.2,$$

and

$$\eta = \frac{Q}{Q_0} = 0.950.$$

It should be remarked that the friction here considered as existing between the piston and cylinder, and between piston-rod and stuffing-box, is only that arising from one-sided or lateral pressure. The friction which is caused by the pressure of piston and stuffing-box packing must be estimated in other ways, and deducted directly from the piston pressure.* We have, therefore,

* The "one-sided or lateral forces" to which the author here refers are those resulting from the non-coincidence of the opposing forces S and P. They are, in other words, the re-actions which keep the piston-rod and cylinder in the line BC, their tendency being to arrange themselves along the line of tension bc. If we imagine the piston, piston-rod, and cylinder to be made of some elastic material which would bend at the slightest application of a deflecting force, they would so arrange themselves when the steam pressure P was applied that

in the value of η given by the figure, not the efficiency of the steam-engine, but only the efficiency of those parts composing the mechanism.

The same slider-crank motion which lies at the basis of the oscillating engine finds frequent application in planing-machines for the production of a quick return motion. Fig. 18, plate II., represents an arrangement of this kind, where the slide HG bearing the tool M is moved back and forth in a prismatic guide by the link E_1F_1 which receives its reciprocating motion from the lever D_1E_1 oscillating about the fixed journal D. The oscillation of the latter is produced by the crank AB, whose crank-pin B works in the block C sliding along the slot C_1, of the oscillating lever E_1D_1. The resistance Q offered by the material to the cutting-tool M produces the re-actions R_1 and R_2 at the points g and h; and these three forces, Q, R_1, and R_2, must be in equilibrium with the force S acting in the reciprocating link E_1F_1. The

---

the line passing through the centres of piston and cylinder-head would coincide with the line bc of tension between trunnion and crank-pin. There is another class of "one-sided or lateral forces" which the author has omitted to mention; namely, those arising from the oscillations of the cylinder. Taking the case when the crank is upon either dead-point, with the engine running at a high rate of speed, we have the entire mass of the cylinder, by no means small in practice, in rapid motion in one direction; passing to the other dead-point we find it in equally rapid motion in the opposite direction. Between the two the inertia of this mass moving at this rate of speed has twice to be overcome by re-actions of the nature of R_1 and R_2, but vastly greater than $II\ C$ and $I\ III$. The resultant of these re-actions applied at the crank-pin is of the nature of those accelerations and retardations to which reference is made in the first chapter, and have no effect on the work done, since the work stored up in the quadrant on one side of the dead-point is given out in the quadrant on the other side; but the friction at the points e and a, or their opposites, represents a loss of energy never given back again. — *Trans.*

direction of the force S is given by the tangent ef to the friction circles of E and F. If we therefore connect o_1, the intersection of Q and R_1, with o_2, the intersection of S and R_2, we have in the side $II\,III$ of the force polygon $o_1 I\,II\,III$, the value of the force S acting in the link E_1F_1, under the supposition that $o_1I = Q$, $I\,II$ is drawn parallel to R_1, $II\,III$ parallel to ef, and o_1III parallel to R_2. The crank-pin B acts upon the sliding bearing C, and through it upon the guide C_1 of the oscillating lever, in a direction o_3c which must be tangent to the friction circle of B and inclined at the angle ϕ to the normal to the slot C_1. The line of this re-action T is therefore o_3c, and similarly the line do_3 drawn from the intersection o_3 of T and S tangent to the friction circle of D is the direction along which the journal D re-acts against the lever D_1E_1. If, further, the driving-force P is applied at the end H of the radius AH from the shaft A we can get the direction of the re-action exerted upon the crank-shaft A by its bearing, in the line o_4a drawn tangent to the friction circle of A from o_4, the intersection of T and P. Completing the force polygon by resolving the force $S = II\,III$ into $II\,IV$ and $IV\,III$, parallel to o_3c and o_3d, and drawing $II\,V$ parallel to o_4a, and $IV\,V$ parallel to o_4H, we have in $V\,IV = P$ the force which must be applied at H to overcome the resistance $Q = o_1I$ at M. Assuming a co-efficient of journal friction $\mu = 0.1$, and of sliding friction $\mu = 0.16$, the construction gives, for $Q = 100$,

$$P = 84.1, \quad P_0 = 66.6,$$

and

$$\eta = \frac{P_0}{P} = 0.792.$$

In the case of a steam-engine with single beam (Fig. 19, plate II.) the investigation is pursued as follows: The steam pressure P acting upwards upon the piston-rod AB coincides with the geometric axis of the cylinder, and passes through the centre of the journal A. On the other hand is the thrust S of the rod, i.e., the re-action of the beam upon A, which must lie along a line tangential to the friction circle at A. These two forces, S and P, cannot be in equilibrium, since they do not act along the same line. For this a third force is necessary, which is found in the re-action R_1 of the stuffing-box against the piston-rod. By drawing through a middle point b in the stuffing-box the direction of R_1 at an angle ϕ with the normal we have, in the intersection o_1 of R_1 with P, the point through which the thrust S must pass; this force then acts along the line o_1a. If we make $o_1I = P$, and draw through I a parallel to R_1, we have in o_1II the force S exerted upon the journal A of the beam. It may not be uninteresting to remark that the stuffing-box B_1 is exposed, according to the above demonstration, to a certain side thrust $II\,I$. This side thrust is not the result of an inaccuracy in the parallel motion, as is the case in approximate motions like that of Watt, for the Evans motion here represented is well known for its absolute accuracy. This side thrust is directly the result of the journal friction occurring at A. For a clearer understanding of this fact we may imagine the piston-rod to be bent to the left by a force equal to the journal friction of the beam applied at its upper end, and tending to revolve toward the right. Such a force would evidently call forth in the stuffing-box the re-action which we have been considering. It is also evident that the side thrust R_1 will act in an opposite direction and from the other side of the stuffing-box when the piston makes a down-stroke, since at each change of motion the direction of the journal friction at A is reversed. On the other hand, the piston-rod thrust or pull S in upward and downward stroke remains always in the line of the tangent o_1a, since at each reversal of motion, i.e., at each dead-point, the direction of turning of the journal A, as well as the direction of the force S, is reversed. We may therefore regard the effect of journal friction at A as being in every case to diminish the lever-arm of the force S. Beside the force S three other forces act upon the beam; they are the re-actions of the radius-rod DE, of the connecting-rod FG, and of the guides K_1K_2. We find the directions of the force L in the radius-rod, and T in the connecting-rod, according to previous rules, in tangents to the friction circles at D and E, and at F and G. Recollecting that the radius-rod is under compression while the connecting-rod is in tension, and noticing the direction of turning at each journal as indicated by the arrows, we decide upon de and fg as being the desired tangents. The sliding-block C_1 exerts upon the journal C a re-action which must be tangent to the friction circle, and upon the guide K_1K_2 a re-action which must be inclined at an angle ϕ to the normal; these two conditions fix the position of R_2 as coinciding with the line kc. In Fig. 19a the sliding-block C_1 is shown in detail. We first observe that during up-stroke the lower guide K_1K_2 exerts the re-action, and during down-stroke the upper guide $K_1'K_2'$ is under pressure. In further analyzing the re-actions exerted upon the sliding-block C_1 we must remember that it has a double reciprocating motion; that is, it makes a complete stroke out and back for every half-stroke of the piston. Beginning then at the lower dead-point with the beam in the position CA_1 we have, during the first half of the up-stroke, a motion of C_1 toward the right, and right-handed turning of the journal C, which gives k_1c_1 as the direction of the re-action R_2. When the beam reaches the position CA_2 the radius-rod DE is parallel with it, and any further motion will, by the action of the radius-rod DE, cause C_1 to move to the left. During the second half of the up-stroke, therefore, k_2c_2 is the direction of the re-action R_2. At the beginning of the down-stroke, when the beam is in the position CA_3, the sliding-block C_1 again reverses its motion and moves to the right; but it now presses against the upper guide $K_1'K_2'$, and the direction of turning at the journal C has also been reversed, and the re-action lies along the line from k_3 to c_2 or k_2. Similarly, during the last half of the down-stroke, there is sliding toward the left, and left-handed turning of the journal; so that k_4c_1, or k_1, is the line of re-action. In every case, therefore, the re-action is so applied as to lengthen its lever-arm and render the force S less effective, which is entirely in keeping with the obstructive action of friction.

In order that the four forces S, L, T, and R_2 acting upon the beam shall be in equilibrium the resultant of the two forces L and R_2 intersecting at o_2, and the resultant of the two forces T and S intersecting in o_3, must both lie along the line o_2o_3 uniting these points. In the figure, as a result of the proportions assumed, the point of intersection falls beyond the limits of the plate. In determining the direction of the line o_2o_3 the following construction may be employed to advantage. Draw any line, $o_2\alpha$, intersecting S and T in the points α and β; draw any other line, $\sigma\omega$, parallel to $o_2\alpha$, and intersecting S and T in the points σ and τ. If, now, we assume the third point, ω, on the line $\sigma\omega$, so that the proportion

$$\alpha\beta : \sigma\tau : : \beta o_2 : \tau\omega$$

is true, we know from geometry that the three lines $o_2\omega$, $\beta\tau$, and $\alpha\sigma$ will intersect in one and the same point; but the intersection of $\beta\tau$ (the line of the force T) and $\alpha\sigma$ (the line of the force S) is the desired point o_3. Therefore the line $o_2\omega$ gives the desired direction o_2o_3. The point ω can easily be located by drawing any line, $\alpha\delta$, making $\alpha 8$ equal to $\sigma\tau$, connecting 8 and β, and drawing from o_2 a line $o_2\delta$ parallel with 8β, and intersecting $\alpha\delta$ in the point δ; the distance 8δ is then to be laid off on the line $\sigma\omega$ from τ, and thus locates ω, it being of course evident that $8\delta = \tau\omega$. We next resolve the force $S = o_1 II$ into components parallel to T and to o_2o_3, the resultant of the two remaining forces L and R_2, and we have

$$o_1 III = T \quad \text{and} \quad II\ III = o_2o_3.$$

Resolving $II\ III$ into components parallel to R_2 and L we have

$$II\ IV = R_2 \quad \text{and} \quad IV\ III = L.$$

If, finally, we suppose the resistance Q to act upon the crank-shaft H with the lever-arm JH, o_4h gives the direction of the re-action R_3 of the bearing against the shaft H, and a resolution of the force T into components parallel to the direction of Q and R_3 gives us in Vo_1 the value of Q, and in $V\ III$ the value of R_3.

With a value $\mu = 0.1$ for journal friction, and $\mu = 0.16$ for sliding friction, the drawing gives, for $P = 100$,

$$Q = 49.5, \quad Q_0 = 54.9,$$

and

$$\eta = \frac{Q}{Q_0} = 0.902.$$

In the mechanism of the ordinary eccentric (Fig. 20, plate III.) the determination of the relation of power to load is accomplished in the same way as with the crank. In this case the driving-shaft A has to move the rod ED working in the guide and stuffing-box E_1 and D_1 through the medium of the eccentric B and its rod BC. There are, then, acting upon the rod ED the working-resistance Q which coincides with the geometric axis, the thrust of the rod BC which lies along the tangent bc to the two friction circles at B and C, and the re-actions R_2 and R_1 of the guide E_1 and of the stuffing-box and gland D_1. Taking these re-actions at e and d inclined at the angle ϕ to the normal we have again, in the line joining the point of intersection o_1 of R_1 and Q with that o_2 of R_2 and S, the direction of the resultants of these pairs of forces. If we therefore make $o_1I = Q$, and draw $I\,II$ parallel with o_1o_2, and $II\,III$ parallel to bc, and $I\,III$ parallel with R_2, we have in $III\,II$ the thrust S exerted in the eccentric-rod. If, further, the motion of the shaft A is caused by a force P applied at the end F of the lever-arm AF we have again in ao_3 the direction of the re-action R_3 exerted by the bearing against the shaft A; and by resolving $III\,II$ into $III\,IV$ and $II\,IV$ parallel to ao_3 and P we get in $IV\,II$ the value of the force P which must be applied to overcome the resistance Q. For $\mu = 0.1$ and $\mu = 0.16$ the drawing gives, for $Q = 100$,

$$P = 36.2, \quad P_0 = 22.8,$$

and

$$\eta = \frac{P_0}{P} = 0.630,$$

a small value for η, due to the large radius of the friction circle of the eccentric B.

An interesting application of the crank-train is found in the ordinary Blake crusher shown in Fig. 21, plate III. The crank-rod BC is here connected with two links DE and FG, forming a knee, by the pressure of which the materials fed in at L are crushed, through the intervening plate JH swinging from the centre H. It is evident that as turning occurs at both ends of each link the pressure can only be transmitted through them along the tangents de and fg to the friction circles. Furthermore, the force S of the crank-rod or pitman which is tangent to the friction circle at B must also pass through the intersection o_2 of the forces T and T_1; i.e., act along the line o_2b. And also the re-action R_1 against the journal H of the swing-plate must pass through o_1, the intersection of the thrusting-force T in the link DE with the working-resistance Q. If the motion of the crank-shaft A is caused by a force P applied at K we have, according to well-known methods, the re-action of the bearing against the shaft in the line o_3a tangent to the friction circle of A. Then draw the force-polygon as follows: Make o_1I equal to Q, the crushing-resistance of the material; draw $I\ II$ parallel to o_1h, and IIo_1 is the thrust T sustained by the link

DE. In $III\,II$ we have the thrust T_1 in FG, if $III\,II$ is drawn parallel to fg, and $o_1 III$ parallel to bo_2. Finally, by resolving $IIIo_1 = S$, the strain in the pitman, into $o_1 IV$ parallel to $o_3 a$, and $III\,IV$ parallel to P, we have in $III\,IV$ the value of the force P which must be applied at K to crush the material at L. The broken lines again indicate the construction by which the theoretic force $P_0 = III_0 IV_0$ is obtained.

With a co-efficient $\mu = 0.1$ the drawing gives, for $Q = 100$,

$$P = 15.4, \quad P_0 = 12.3,$$

and

$$\eta = \frac{P_0}{P} = 0.80.$$

Another example of toggle-joint mechanism is the hand-punch shown in Fig. 22, plate III. Two bent levers, A_1CD_1 and B_1CE_1, are here connected by a hinge or bolt at C, so that when their ends D_1 and E_1 approach each other under the action of the screw FG the head HJ is forced down, and the punch L forces the metal under it through the die K. To produce this result the screw FG is turned by a long wrench applied at its square head F, and its right and left threads draw the nuts D_2 and E_2 together with a certain force P, as was the case in the coupling shown in Fig. 9, plate I. On account of the turning of the levers the nuts are connected to them by the journals D and E, from which it follows that the force P by which they are drawn toward each other acts along the tangent de to the two friction circles of D and E. If we now suppose one of the levers (the under one, for instance) to be at rest it must be in equilibrium under the forces acting upon it. The forces consist of the driving-pressure P acting along de, and the two re-actions R_1 at C and R_2 at B. For the first re-action R_1 exerted by the upper lever A_1CD_1 we have the direction ac tangent to the two friction circles of A and C, the lever A_1CD_1 turning toward the right hand; while the direction of R_2 must pass through o_1, the intersection of P and R_1, and be tangent to the friction circle of B. It lies, therefore, along the line o_1b. If, then, we make $o_1I = P$, and resolve it parallel to the two re-actions by drawing $I\,II$ parallel to o_1b, we have in IIo_1 the re-action R_1 exerted by the upper lever upon B_1CE_1, and in $I\,II$ the re-action R_2 offered by the journal B.

The head HJ must, in its turn, be in equilibrium under the influence of the force R_2 acting along bo_1, the resistance Q which the metal offers to punching acting along the axis of the punch L, and the two re-actions R_3 and R_4 of the guide-bushing H_1J_1. If we assume h and i at a sufficient distance (say 5 mm., or $\frac{1}{5}$ inch) from the edges as the points of application for the re-actions R_3 and R_4 which are drawn at the angle of friction ϕ to the normal, the line connecting o_2, the intersection of R_3 and R_2, with o_3, the intersection of R_4 and Q, furnishes us with the means of completing the force polygon in the usual way. Resolving $II\ I = R_2$ into $III\ I$ parallel to R_3 and $II\ III$ parallel to o_2o_3, and then $II\ III = o_2o_3$ into $III\ IV$ parallel to R_4 and $II\ IV$ parallel to Q, we have in $IV\ II = Q$ the resistance which can be overcome by the application of the force P upon the nuts D_1 and E_1. It is also evident that the side IV_0II_0 of the force polygon drawn in broken lines normal to the surfaces, and passing through the journal centres, is the value of the theoretic resistance Q_0 which should be overcome by the same force P. With the usual values $\mu = 0.1$ and $\mu_1 = 0.16$, the drawing gives, for $P = 100$,

$$Q = 194, \quad Q_0 = 350,$$

and

$$\eta = \frac{Q}{Q_0} = 0.554.$$

It will be seen that the ratio of P to Q_0 would remain the same whether we use the arrangement shown, or whether we suppose the force P to act only upon one arm B_1E_1, while the nut D_2 is replaced by a cylindrical eye and spindle so arranged that no sliding can occur in the direction of the axis of the spindle, as in the swivel, Fig. 10, plate I. The ratio between P and Q would not be changed, because if the driving-force P was applied only at E_1 there would arise at D_1 an opposite equal re-action which would be transmitted by the swivel-ring to D. The only difference between the two arrangements is that by the movement of both levers the space traversed by the head HJ for any given portion of a revolution of the screw FG is double that which would result if only one lever moved while the other was held fast. The work done by the turning-force for this portion of a revolution is, of course, twice as great in one case as in the other. The above remarks apply exactly only in the case of frictionless motion; for with the arrangement giving a re-action $-P$ of a fixed point, as in the swivel, a new friction enters which must be determined in the same way as in Fig. 10, plate I. The investigation of the present mechanism has not included the determination of resistances arising in the screw; such a determination would be made in the manner shown in Fig. 9, plate I.

The amount of resistance Q which can be overcome by a certain force P, as well as the efficiency of the mechanism where knee-joints are employed, depends upon the angle which the centre lines of the links forming the joint make one with another. It will be readily seen that this resistance Q becomes greater, and the efficiency η smaller, as this angle approaches 180 degrees. If we assume this last value, or one which differs from it by an infinitely small amount, as in Fig. 23, plate III., a force P acting upon the journal C would be able, in the absence of friction, to overcome an infinite resistance

$$Q_0 \left(= \frac{P}{\cot 90^\circ} = \infty \right).$$

On account of journal friction, however, these lines of force are to be found in the tangents ca and cb to the friction circles; and we therefore find the actual forces in the parallelogram $acbI$ if we let $cI = P$. The greatest resistance Q which can be overcome is therefore given by the equation

$$Q = P \tan w,$$

where $2w$ is the obtuse angle acb of the two directions of pressure. The efficiency $\eta = \frac{Q}{Q_0}$ in this case where $Q_0 = \infty$ is equal to zero.

If we further suppose the knee to be in the condition of backward motion, i.e., if we assume that the tendency of the re-actions at A and B is to force the joint C out to one side or the other, it is evident that such backward motion can only begin at that instant in which these re-actions coincide in one straight line. If, therefore, we draw such a case (Fig. 24, plate III.) in which the tangents ac and bc to the friction circles fall upon the same straight line we have the limiting position at which the knee is self-locking. The angle $2(w) = ACB$, which differs from the angle $2w$ in Fig. 23 only by an inappreciable amount, determines on both sides the limiting position within which a backward motion, i.e., an opening of the press by the re-action offered by the material within its jaws, is impossible. As (w) depends on the proportions of links assumed, the limits within which the mechanism is locked become greater as the distance from journal to journal becomes smaller, and as the radius of the journals increases. The knowledge of these proportions is of special importance in the designing of mechanisms in which the knee-joint is employed to grip an object and hold it fast, as in certain forms of vise.

The methods heretofore employed in determining the efficiency of machines can also serve the purpose of determining friction as applied to useful ends in many machines and processes. Thus friction serves to produce the necessary tension in all spinning-machinery, and is employed also in sewing-machines and water-frames or throstles.

Let Fig. 25, plate III., represent the ordinary spindle with the Arkwright flyer CC which rotates with the rapidly moving spindle. The thread F passing through the stationary glass eye at D with a certain velocity v, and leading to the loose spool L after several turns about the arm of the flyer, serves as a driver to the spool, which is caused by the thread to revolve in the same direction as the spindle and the flyer. The spool holds back on account of the frictional resistance offered to it, and at each instant the portion of thread running out is unwound by this difference between spool revolution and flyer revolution. The friction of the spool by which the tension in the thread is determined occurs principally in two places, — at the circumference of the spindle as journal friction, and as pivot friction where the under surface of spool at G rests upon the bobbin-frame EE which slowly rises and falls. This friction must attain a certain value in order that the tension of the thread shall be sufficient for a certain amount of twist, and in order that the thread shall not belly out between B and D under the influence of centrifugal force, and become entangled with the thread of the neighboring spindle. The tension S of the thread may be determined as follows: If $G = HA$ is the weight of the spool with the quantity of yarn already upon it, it produces friction upon a ring-shaped portion of the surface of the bobbin-frame EE. We may therefore suppose the bobbin-frame EE to be replaced by reactions which are uniformly distributed over an average circle of contact whose diameter is EE,* all these reactions making the angle ϕ with the normal in the direction prescribed by the motion. As was shown in

---

* The radius ρ of the circumference of contact is =

$$\frac{2}{3}\,\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$$

where r_1 and r_2 represent respectively the radii of the outer and inner circumference of the supporting surface.

the case of pivot friction (Fig. 8, plate I.) we can here suppose all re-actions to be concentrated in two diametrically opposite points, a_1 and a_2. From the parallelogram $AJHK$ we get in AJ and AK the re-actions exerted by the bobbin-frame, and in their horizontal components $a_1 i$ and $a_2 k$ the frictional resistance offered by these re-actions to the revolution of the spool. If the thread draws the spool in the direction fc at a certain moment with a tension S the spindle re-acts with equal force along a line ab drawn parallel to the direction fc, and tangent to the friction circle of the spindle. The question then is of the equilibrium of the spool under the influence of the two frictions $a_1 i$ and $a_2 k$, the tension S, and the re-action of the spindle along the line ab. Joining the intersection o_1 of the thread-tension S with the friction $a_2 k$, with that o_2 of the friction $a_1 i$ with the spindle re-action, we have the tension S given in value by the line IIo_2 if $Io_2 = a_1 i$ and $I\,II$ is drawn parallel to $o_1 o_2$.

So far in these investigations it has been tacitly assumed that the journal friction was only exerted upon one bearing. This is never the case in practice. Every shaft has at least two supporting points or bearings, and at these the forces P and Q will call forth certain pressures and re-actions of a value proportional to the distance of the point of application of the forces from the bearings. If we suppose the shaft supported by the bearings A_1 and A_2 (Fig. 26, plate III.) to encounter a resistance Q acting through a wheel or pulley placed at C, with a radius $q = AG$, we can resolve this force into two components parallel to Q, and having the same lever-arm q, lying in planes passing through the points A_1 and A_2, and normal to the shaft.

The forces are determined by the well-known relations

$$Q_1 = Q\frac{A_2C}{A_1A_2} \quad \text{and} \quad Q_2 = Q\frac{A_1C}{A_1A_2},$$

and can easily be found by construction.

If the same construction is carried through for the determination of the driving-force acting at each bearing we have in P_1 and P_2 the forces which, lying in the planes through A_1 and A_2, act upon the parallel lever-arms $AH = p$ to overcome the resistances Q_1 and Q_2. It follows that under the supposition of equal journal radii and equal co-efficients of friction, i.e., with equal friction circles, at A_1 and A_2 these forces P_1 and P_2 must be in the same ratio one to another as Q_1 to Q_2; so that if P_1 and P_2 were compounded in one resultant P it would have to lie in the same plane, passing through C, in which Q acts. Therefore we should obtain by this method the same value of P which has heretofore been determined directly from Q by the employment of friction circles.

It follows from the above that the preceding constructions for the determination of journal friction can be entirely accurate only when the driving-force P lies in a plane perpendicular to the same axis as that to which the plane of the resistance Q is perpendicular, and when the diameters of the journals are equal. In reality the first condition is seldom fulfilled, and the journals also are seldom of the same size in a shaft. It therefore remains to bring this influence within the scope of calculation by force polygons.

With this object in view let us suppose that the driving-force P (Fig. 26, plate III.) which is to overcome the resistance Q is applied at B, a point outside the bearings A_1 and A_2, as is frequently the case in practice. It is further assumed that the resistance Q at C acts with a lever-arm $AG = q$, and the force P with a lever-arm $AH = p$ at the point B. If no friction came into account we could assume P_0 and Q as working in the same plane, and get

$$P_0 = o_1 P_0$$

in the well-known way by uniting the intersection o_1 with centre A, and then resolving $o_1 Q$ into this direction and that of P. Then imagine Q to be resolved into the forces $Q_1 = Q\frac{A_2 C}{A_2 A_1}$ acting at A_1, and $Q_2 = Q\frac{A_1 C}{A_1 A_2}$ acting at A_2. Then lay off these forces equal to $o_1 Q_1$ and $o_1 Q_2$ in the direction of Q. In the same way P_0 can be resolved into two parallel forces, P_1 and P_2, acting at the points A_1 and A_2 with the lever-arm $AH = p$. Their values are given by the equations

$$P_1 = P_0\frac{A_2 B}{A_1 A_2} \quad \text{and} \quad P_2 = P_0\frac{A_1 B}{A_1 A_2}.$$

Since P acts outside of the supporting points A_1 and A_2, P_1 and P_2 act in opposite directions $o_1 P_1$ and $o_1 P_2$ along the line of P. We find the resultant of P_1 and Q_1 in the diagonal $o_1 D$, and this force acting in the plane through A_1 will call forth an equal and opposite re-action of the bearing A_1. This re-action R_1 does not act along the same line $o_1 D$ as the resultant, however, since among the forces acting at A_1 alone equilibrium does not exist. The position of R_1 is found by drawing the line a_1a_1 tangent to the friction circle at A_1 and parallel to o_1D. Similarly we get the journal pressure at A_2 in the diagonal o_1E, and in the line a_2a_2 parallel to o_1E and tangent to the friction circle at A_2, the position of the re-action R_2 which the bearing A_2 exerts. We now see that the shaft is in equilibrium under the couple o_1D and R_1, and the couple o_1E and R_2. Since we may suppose the couples to be slid along the axis until they lie in the same plane we can immediately find P by uniting o_1 and the intersection o_2 of R_1 and R_2, and resolving the resistance o_1Q parallel to the diagonal o_1o_2 and the direction of P. We have therefore in o_1P_1 the necessary turning-force P. In Fig. 27, plate III., the friction circles of A_1 and A_2 are drawn to a larger scale, and we see that the direction of R coincides nearly to the direction of a line drawn through o_1 tangent to a mean friction circle shown in dotted lines. We can therefore employ this simple construction with sufficient accuracy in the generality of cases, and especially in those where both P and Q fall within two bearings not far apart. But for exact determination, and in cases where a force is applied outside of the bearing, the full construction is necessary. It will be noticed that the latter has still a slight inaccuracy, since the component forces P_1 and P_2 are obtained from P_0 instead of P. The error is quite inappreciable, however, and a correction unnecessary; though such correction could be obtained by determining P_1 and P_2 anew from the value of P as deduced, and repeating the construction.

We can now determine all the frictional resistances which occur in a screw. In Fig. 28_a, plate IV., S_2S_3 is the direction of a helix at a mean distance from the axis. S_1s_1 and S_1s_2 are the directions of re-action at two diametrically opposite points of the helix, so drawn as to make the angle $DS_1s_1 = DS_1s_2 = \phi + \alpha$ with the axis of the screw, α being the pitch-angle of the screw. We then obtain the resistance $q_1 = CB$ acting perpendicularly to the axis at each of these points by making $DA_1 = Q$, the load upon the screw, and drawing through A_1 and D the lines A_1B and DB parallel respectively to S_1s_1 and S_1s_2, and intersecting at B. The load Q also presses the nut M down upon the standard, and produces friction against the ring-shaped surface of contact A_2A_3. As pivot journal friction we can suppose this concentrated at a mean circumference A_2A_3, and acting at two diametrically opposite points. If, therefore, we draw the corresponding lines of re-action A_1a_1 and A_1a_2 inclined at the angle of friction ϕ to the axis A_1D we have in $CE = q_2$ the amount of friction at each of the supporting points in the ring-shaped surface A_2A_3 if the line DE is drawn parallel to A_1a_2. We can imagine these couples q_1q_1 and q_2q_2 as acting at the points b_1b_2 (Fig. 28_b, plate IV.) and c_1c_2 respectively, and by compounding them get the resultant couple

$$d_1e_1 = q_3 \quad \text{and} \quad d_2e_2 = q_3.$$

If the nut M is revolved by worm-gearing, we must represent the worm W as acting upon it along the line w_1w_2 with a force w applied at a mean helix of the worm. This one-sided working of the force w presses the nut M up against the sides of the standard K where the latter is bored out above A_2A_3, and calls forth a re-action parallel to w_2w_1 and tangent to the friction circle of M in mm. To find w we join o_1 and o_2, the points of intersection of the couple $q_3 q_3$ with m and w respectively, and making $o_2 e = d_2 e_2 = q_3$ complete the parallelogram $o_2 egf$, getting

$$o_2 f = w.$$

For the sake of clearness let Fig. 28_c represent in FO the force necessary to cause a revolution of the worm on a scale five times larger. It follows, in the next place, that the re-action w offered to the worm along the line WW of its axis requires a turning-force p_1 to overcome it, which acts at w_1 (Fig. 28_b) perpendicularly to the plane of the paper, and whose value is given by $FH = p_1$ (Fig. 28_c) if H is the intersection of a perpendicular at F with the line HO drawn through O at an angle $FOH = \alpha_1 + \phi$, α_1 being the pitch of the screw on the worm.

The worm W in its turn is forced by the load w against its bearing L, and friction results along the mean circumference of the ring $L_1 L_2$. This friction can be supposed to act at two opposite points of this circumference, as at A_2 and A_3 in the case of the screw S. We have, therefore, a resisting couple $TJ = p_2$ (Fig. 28_c) where TJ is obtained by drawing FJ and OJ through O and F at the angle ϕ to OF.

Finally, the load w upon the worm will also call forth frictional resistances at the bearings of the two neck-journals L and N, since the one-sided action of w thrusts the journals against their bearings with a certain pressure, the value of which, equal and opposite in the two cases, we have from the equation of moments

$$w \,.\, W_1 w_1 = p_3 \,.\, LN,$$

from which

$$p_3 : w :: W_1 w_1 : LN :: LK_1 : K_1 K_2.$$

From the last proportion we see that we can get p_3 by drawing through F (Fig. 28_c) a parallel to LK_2 (Fig. 28_b), when OK will represent the value of p_3, the neck re-action exerted at L and N.*

In order to determine the force P to be applied at a crank W_3 to the worm W, we must first unite the one-sided resistance p_1 and the two couples $p_2 p_2$ and $p_3 p_3$. This can be done in the following way: First draw the two tangents l_1 and l_2 to the friction circle of the jour-

---

* Throughout this discussion the author has assumed that the thrust w of the worm is applied along the line $w_1 w_2$ parallel to its axis. This is neither true in practice, nor does it correspond to that case shown in the figure. The thread of a rack is not square, but has the sides of its profile inclined at an angle of 75 degrees to the axis. The normal to the surfaces of contact would then be inclined at an angle of 15 degrees to the axis. The line of re-action would fall short of this inclination by an amount equal to $\phi = 5° 43'$ with a co-efficient $\mu = 0.10$, so that its actual direction would be inclined 9° 17′ to the axis. The re-action $o_1 m$ of the nut would be parallel to it, and the shape of the parallelogram $o_2 eyf$ would be changed, and a different value for $of = w$ obtained. But since that component of this real value of w parallel to the axis WW would only differ from $o_2 f$ by an inappreciable amount, which increases and decreases with the radius of the friction circle at A, and since the component perpendicular to the axis merely adds to the throat-friction at L an amount which it takes from that at N (its tendency being, of course, to thrust the worm bodily over to the right), the accuracy of the final result is practically unaffected. Moreover, as the gear may work in either direction, and as in the case of backward motion the condition of affairs would be exactly the reverse of that pointed out in the first part of this note, it is probable that the author assumed the average position, i.e., the one parallel to the axis, in order to have a general discussion applicable to all cases. — *Trans.*

nal L representing the direction of re-action of p_3, and then at a distance equal to the radius of the mean helix of the worm draw the vertical tangent w_3. Now lay off from o_3, the intersection of w_3 and l_1, the distance $o_3h_1 = p_1$ ($= FH$ in Fig. 28_c) and $o_3K_1 = p_3$ ($= OK$ in Fig. 28_c); then the diagonal o_3x gives the resultant of the forces p_1 and p_3, which latter acts along the tangent l_1. This resultant, when compounded with the other force p_3 acting along l_2 in the opposite direction, gives a force passing through o_4 parallel and equal to $o_3h_1 = p_1$. We see from this that the influence of the two frictions p_3 produced by the neck-journal re-actions only causes the resistance to turning of the worm to act in the same direction and with equal force, but at a longer lever-arm, since it passes through o_4 instead of o_3. This corresponds to the well-known principle that the composition of a force and a couple merely effects a parallel shifting of the force unchanged in value. In the same way, by uniting the force p_1 going through o_4 with the couple $p_2 p_2$ which corresponds to the friction offered by the ring-shaped supporting surface L_1L_2 (Fig. 28_b), we obtain merely a shifting of the force p_1 always parallel with itself from o_4 to o_6. It is done as follows: Draw the two forces $p_2 p_2$ (TJ in Fig. 29_c) as two parallel tangents $l_3 l_4$ to a circumference of the diameter L_1L_2. Then lay off from the point of intersection o_5 of one of these tangents with the force p_1 now passing through o_4 the distance $o_5i = p_2$ and $o_5h_2 = p_1$, and the diagonal o_5y cuts the second tangent l_4 in the point o_6 through with the force p_1 must pass. The further construction is familiar. Through the intersection o_7 of the resistance p_1 with the driving-force P the re-actions of the journal-bearings L and N must pass. This re-action R is therefore acting along the line $o_7 l_5$ drawn tangent to the friction circle at L. If we then make $o_7 h_3 = p_1$, and draw through h_3 a line parallel to $o_7 W_3$, we have in $h_3 h_4$ the force P which must be applied at the crank W_3 to cause revolution.

Assuming a co-efficient of friction $\phi = 0.1$ for journal and thread friction, the construction gives, with a pitch $n = \tan \alpha = \frac{1}{15}$ of the screw S, for $Q = 100$,

$$w = 67, \quad w_0 = 11.8,$$

and

$$\eta_1 = \frac{w_0}{w} = 0.176,$$

where w is the force to be applied at the pitch-line of the worm. With a pitch $n = \tan \alpha_1 = \frac{1}{10}$ for the worm, for $w = 67$,

$$P = 12.56, \quad P_0 = 4.3,$$

and

$$\eta_2 = \frac{P_0}{P} = 0.342.$$

For the efficiency of the entire jack we have then

$$\eta = \eta_1 \eta_2 = 0.060.$$

In the same way we can determine the efficiency for backward motion by finding the force (P), which must be exerted in the same "sense," or direction, as Q to produce a sinking of the load.*

---

* To fix the method to be followed in every case firmly in mind, it may not be amiss to briefly sketch here the general aspect of the

problem encountered in all these examples, giving the known quantities in every case, and the way in which they are to be combined to determine the unknown, so that the student, in attempting to solve an outside problem, will know just what he has to work with, and just how to set about that work. First, by means of the friction angle and friction circle we can always draw the direction of the forces transmitted longitudinally through links joining two turning pairs, acting at sliding surfaces, or in links joining a sliding and a turning pair. Sometimes in the last case the exact position of the force is fixed, after its general direction has been determined by the angle of friction, by the requirement that it shall pass through the intersection of two others, instead of being tangent to a friction circle as in the simplest case. An example of this is the ordinary cross-head, Fig. 2, plate I. The line of the re-acting force at a lever or crank or bell-crank bearing is found by drawing a line from the intersection of the two other forces acting upon the lever, crank, or bell-crank tangent to the friction circle at its journal. The directions of P and Q are always given as the force of gravity, a piston-thrust, etc. We then have given, or can determine by these elementary methods, the *direction* of all the forces in any problem. We also have given the *intensity* of either P or Q.

With these data the problem is solved as follows: Draw the force-polygon of all the forces acting upon the same piece as the known force, and dependent upon it. These can only be three in number if there is circular motion of the piece to which it is applied, and four if there is right-line motion. In the first case it is a question of drawing a triangle of forces, knowing the directions of all and the amount of one. In the second case combine the forces two and two, join the points of intersection thus obtained, getting in the line thus drawn the direction of the common resultant of the two pairs. Then resolve the known force in the direction of that force with which it is paired, and of the common resultant which here represents the combined effect of the other two forces. Having thus obtained the value of the resultant resolve it in the direction of the two forces making up the second pair, and all the forces are known in direction and amount. One of these becomes the known force acting on the next link in the mechanism, and by repeating the process all the forces acting throughout the machine may be determined.

Whenever there are more than four forces acting on one piece they will be of such nature that they can be reduced to four; and generally where the limit is exceeded, as in the condensing beam-engine, and still further in the compound condensing beam-engine, it will be the result of a compounding of several chains of mechanism, and in each simple chain either P' or Q' will be given, so that its resultant action on the common link can be determined, and combined with the P or Q of the main chain, their resultant action being regarded as one force. In the case of the condensing beam-engine the resistance of the air-pump Q' would be known, and, as shown in plate VII., combined with the thrust of the piston to get the resultant force acting upon the beam. In the compound engine, P', the steam-pressure in the second cylinder, would be known. — *Trans.*

§ 5.—ROLLING FRICTION.

The resistance which is opposed to the rolling of a cylinder along a smooth path is of such small value that it may be left out of account in most cases in comparison with sliding and journal friction. We are accustomed, when it is taken into consideration, to assume it proportional to the pressure Q with which the roller is forced down upon the bearing surface, and inversely proportional to the radius r of the roller. For rollers and surfaces of iron and hard wood the formula

$$P = 0.02\frac{Q}{r}$$

will generally give the resistance, r being expressed in inches. If r is expressed in millimetres the formula becomes

$$P = 0.5\frac{Q}{r}.$$

In order to get a graphic representation of this resistance let A (Fig. 29, plate IV.) be the centre of a cross-section of a cylinder, with radius $AB = r$, which is supported at B by a horizontal track. Let the load resulting from its own weight, and acting at the axis A, be represented by the vertical line $AC = Q$. To cause a rolling of the cylinder a horizontal force $P = c\,.\,\frac{Q}{r}$ must be applied at the axis A, AD representing the intensity of this force. If we assume, as heretofore, the limiting condition of equilibrium for which the slightest increase of P will cause motion, the cylinder must be in equilibrium under the influence of the exterior forces P and Q, and of the re-action R offered by the surface GG. This is only possible if the re-action R is equal to the resultant of P and Q, and acts along the same line in the opposite direction. The surface GG then re-acts upon the cylinder with a force whose direction and value are given by the line EA. The cylinder, therefore, will remain at rest as long as the surface GG re-acts upon it along any line inclined to the normal at a less angle than that of EA as was the case in sliding friction. The plane surface then opposes the same character of resistance to the motion of the cylinder as in sliding friction, with this difference, however, that while in the case of sliding friction the greatest possible deflection angle of the re-action depends only upon the nature of the material, and is constant for a given material, being, of course, the angle of friction for the same, in the case of rolling friction it depends both on the material and on the form, i.e., upon the size of the cylinder. From the expression given for the resistance to rolling

$$P = \epsilon \, . \, \frac{Q}{r}$$

it will be readily seen that the co-efficient ϵ is capable of geometric representation, since it follows that

$$Q : r : : P : \epsilon,$$

giving us directly in the figure $BF = \epsilon$; which is, in other words, the greatest possible distance between the point of application F of the re-action and the theoretic point of contact B of roller and track. While in the case of sliding friction we have a constant angle for the friction angle ϕ, in the case of rolling friction we must deal with a linear value ϵ which, for the same material, remains the same for rollers of all sizes. If it was of sufficient interest we could follow out the parallel still farther, showing that the friction cone in the case of sliding friction corresponds to the wedge-shaped space whose cross-section is FAF_1, whose edge is the axis A, and whose sides, shown in projection at FA and F_1A, cut the supporting surface at the distance ϵ to each side of the perpendicular AB.* It follows that with P acting in the opposite direction the track would re-act from the other side of AB in the direction F_1A.

We can make this connection clear if we assume that in reality the roller is not supported on a line passing through B parallel to the axis, but upon a surface of a width ϵ from each side of the normal AB, produced by a flattening of the roller and a corresponding indentation of the supporting surface under the pressure of the load Q. This view corresponds also to the assumption of a fixed fulcrum for the roller at the constant distance ϵ from the normal plane (that is, at the point F), and this should be kept in mind during the following discussion. The value of ϵ, according to the above, is from 0.02 to 0.03 in inches, or from 0.5 to 0.75 in millimetres, for metals and hard wood. In the case of yielding materi-

---

* For the connection between sliding and rolling friction see O. Reynolds, Philosophical Transactions, vol. 166, and Zeitschrift des Vereins deutscher Ingenieure, Jahrg., 1877, S., 417.

als, as, for instance, carriage roads, the value of ϵ is much greater, and in all such cases the correct value must be estimated.

If a body KK (Fig. 30, plate IV.) rests upon a roller A which rolls along the horizontal track GG rolling friction occurs at both KK and GG. If we therefore draw through the centre A of the cross-section of the roller the normal DB to the two surfaces, we have, according to what precedes, the point of support of the fixed track in F at the distance $BF = \epsilon$ from B, and also in E at the distance $DE = \epsilon$ from D the point at which the downward pressure of the moving body KK acts. If, therefore, $EC = Q$ denotes the load upon the roller A, and P denotes the force acting in the horizontal plane KK necessary to move the body, these forces P and Q must be in equilibrium with the re-action R exerted by the track GG through the roller A upon the body KK, which re-action of courses takes the direction FE. We then have in the side EJ of the parallelogram $ECHJ$ the necessary force P to produce motion.

Equally well can be determined the force P (Fig. 31, plate IV.) necessary to move the load Q upon the wagon-wheel AB, by means of the value $BF = \epsilon$ for rolling friction upon GG, and the friction circle of the journal AE upon which the load Q rests. The direction of the re-action R of the track GG against the axle-bearing of the wagon is along the line Fa drawn from F tangent to the friction circle of A. Therefore, by making $AC = Q$, and drawing through C a parallel CD to Fa, we get the driving-force

$$P = AD.$$

The investigation is practically the same when the track GG for the wheel has any desired inclination to the horizon, as in Fig. 32, plate IV. If we here draw the line AB through the centre A of the axis, and perpendicular to the track GG, make $BF = \epsilon$, and draw through F the tangent Fa to the friction circle of A, we have in this tangent the direction of re-action of the track GG against the axle-bearing. If, then, we make $CA = Q$, draw through C a parallel to the line of re-action R, and through A a parallel to the line of P, we get in AE the value of driving-force necessary,

$$P = AE = DC.$$

Without hurtful resistances the re-action of the track would lie along the normal BD_0, and we have in D_0C the theoretical driving-force P_0. With the grade or inclination of 1 in 3 for the track the drawing gives, for $Q = 100$,

$$P = 34.8, \quad P_0 = 33.3,$$

and

$$\eta = \frac{P_0}{P} = 0.957.$$

In the manner shown the resistance to running-gear of every description, upon both horizontal and inclined tracks, can be easily determined. As a further example we may take the roller-bearing for a swinging crane (Fig. 33, plate IV.). In this case the cylindrical surface of the stationary post or mast A serves as a track for the rollers B and C united to the brace L. If we connect the centres B and C with A, and make $FD = GE = \epsilon$, the lines Db and Ec drawn through D and E

tangent to the friction circles of B and C, and intersecting one another in o_1, will give the directions of the re-actions R_1 and R_2 of the mast. If, now, a turning of the brace is produced by a force P acting in the direction HJ, and intersecting the line of pressure Q of the boom in o_2, we have only to connect o_1 and o_2, make $Io_2 = Q$, and draw through I a parallel to $o_1 o_2$ in the well-known way. We have thus determined in $o_2 II$ the necessary turning-force P. In order to get the re-actions R_1 and R_2 of the mast against the rollers, resolve the resultant $I\,II$ parallel to $o_1 b$ and $o_1 c$, and we have

$$R_1 = III\,I \quad \text{and} \quad R_2 = II\,III.$$

A later example will show in what way the re-action in the bearing at the upper part of the mast is to be considered.

As already remarked rolling friction in most cases of mechanism is quite inappreciable as compared with other hinderances.

At first thought it may not be clear how the geometric diagrams (Figs. 29 and 30, plate IV.) will always give the value FB constant and equal to ε for any one material, whatever the load or size of roller, as stated on p. 78. The following analysis will render it clear that such a result does follow from the assumed relation

$$P = \varepsilon \, . \, \frac{Q}{r}$$

and the supposition that the roller is always a solid homogeneous cylinder. FAB being the angle made by the re-action R to the normal AB we have

(1) $$\tan FAB = \frac{P}{Q} = \frac{FB}{r}.$$

From the second value we get

(2) $$FB = r \tan FAB.$$

There are three ways in which the conditions may change: First, the size and consequent weight of the roller may remain constant, but a varying superimposed load acting upon a surface as KK (Fig. 30) may be applied. If in this case Q' represents the sum of the varying load and the constant weight of the roller we have, from the fundamental relation,

$$P' = \varepsilon \,.\, \frac{Q'}{r};$$

and, from equation (2),

$$F'B = r \tan F'AB,$$

$F'AB$ being the angle the new re-action R' makes with the normal. By supposition both ε and r are constant, and therefore P' varies directly as Q'; or, to put it in another form,

$$\frac{P'}{Q'} = \frac{\varepsilon}{r} = \frac{P}{Q}.$$

Evidently

$$\tan F'AB = \frac{P'}{Q'} = \frac{P}{Q} = \tan FAB,$$

and therefore

$$F'B = r \tan F'AB = r \tan FAB = FB.$$

That is, $F'B$, the distance of the intersection of R' with line GG measured from the foot of the normal, is equal to the value first obtained, FB or ε.

In the second case there is no superimposed load, but the roller varies in size and consequently in weight. The weight will vary as the cross-section of the cylinder, and r will vary as the square root of the cross-section or weight. If Q'' is the varying weight we have

$$P'' = \varepsilon \,.\, \frac{Q''}{r''}, \quad \tan F''AB = \frac{P''}{Q''},$$

and

$$F''B = r'' \tan F''AB.$$

If m is the ratio of Q'' to Q we have

$$r'' = \sqrt{m}\, r, \quad \text{and} \quad Q'' = mQ.$$

Substituting these values in the three equations above,

$$P'' = \varepsilon \cdot \frac{mQ}{\sqrt{m}\, r} = \frac{\varepsilon \cdot \sqrt{m}\, Q}{r}, \quad \tan F''AB = \frac{\frac{\varepsilon \cdot \sqrt{m}\, Q}{r}}{mQ} = \frac{\varepsilon}{r\sqrt{m}},$$

and

$$F''B = r\sqrt{m} \cdot \frac{\varepsilon}{r\sqrt{m}} = \varepsilon.$$

In this case also there is no variation from the original value ε or FB.

Thirdly, where both superimposed load and size of roller vary we have

$$Q' = mQ,$$

the former variable quantity, and

$$Q'' = m'Q,$$

the latter; while

$$Q''' = Q' + Q'',$$

their sum. As before

$$r''' = r\sqrt{m'}, \quad \text{and} \quad Q''' = mQ + m'Q.$$

Substituting in the three equations we have

$$P''' = \varepsilon \cdot \frac{Q'''}{r'''} = \varepsilon \cdot \frac{mQ + m'Q}{r\sqrt{m'}} = \varepsilon \cdot \frac{Q}{r} \cdot \frac{(m + m')}{\sqrt{m'}},$$

$$\tan F'''AB = \frac{\varepsilon \cdot \frac{Q}{r} \cdot \frac{(m + m')}{\sqrt{m'}}}{Q(m + m')} = \frac{\varepsilon}{r\sqrt{m'}},$$

and

$$F'''B = r\sqrt{m'} \cdot \frac{\varepsilon}{r\sqrt{m'}} = \varepsilon;$$

the same result as in previous cases.

This may seem like begging the question, since if ε is assumed to be a constant, and a certain line FB is found once to be its graphic equivalent, this intercept FB must always remain the same; but the analysis may be of use in showing how the diagram adapts itself to this requirement under all conditions. — *Trans.*

§ 6.—CHAIN FRICTION.

When a chain is wound on to or off of a drum or pulley there occur certain hurtful resistances on account of the change of direction in the links, which resistances may be determined in exactly the same manner as journal friction. Let A (Fig. 34, plate IV.) be the journal of a chain-pulley whose radius $AB = AD$ is represented by a. At the left side a weight Q is attached to the chain BC; and we are to ascertain the force P which must be applied to the other portion DE in order to cause a revolution of the pulley in the direction of the arrow, and a lifting of the weight Q. If we neglect friction of the journal A, it is evident that on account of the equality between the lever-arms AB and AD the forces P and Q must also be equal for the condition of equilibrium if there were no hurtful resistances at the points B and D where the chain winds on to and off of the drum, as would be the case if we suppose the chain replaced by an infinitely fine thread of perfect pliability. In this case there would be no slipping of the strands of the thread over one another, since the thread has no appreciable thickness; and therefore the causes of friction would be wanting, because the latter can only occur, as remarked in the introduction, where there is motion of two elements relative to one another. But as the links of the chain have a certain thickness, relative motion will occur at the point of connection of two links at the instant of winding on or off, at the point B

or D respectively, and this motion must be regarded as a turning. If we imagine the link B_1B to be in motion from C to B, it has evidently no relative motion toward the preceding link B_2B_3 in which it hangs as long as the latter rises in the same straight line CB. At that instant when the preceding link B_2B_3, adjusting itself to the circumference of the pulley, begins to share in the latter's motion, there occurs relative motion between the links B_1B and B_2B_3, which is, as shown by the arrow in the figure, a left-handed turning of the link BB_1 about the link B_2B_3. In this turning the end of the link B_1B serves as a journal, and the eye or loop of B_2B_3 as a bearing. It is now clear, according to the laws of journal friction, that these links can only act upon one another along the tangents to the friction circles of these journals. Since the portion of chain BC is subjected to tensile strain we have the direction of this re-action in the tangent bc which touches the friction circle at B on the opposite side from the centre A. In other words, the lever-arm of the weight Q is increased through chain friction by an amount equal to the radius χ of the friction circles at the chain joints. In the same way the link D_2D_3 about leaving the pulley at the point D on the other side undergoes left-handed revolution about the link DD_1 still upon the pulley, as shown by the arrow. Therefore the force P in the portion ED_2 of the chain will act along the tangent de to the friction circle at D. In other words, the lever-arm of the driving-force P is shortened through chain friction by an amount equal to χ, the radius of the friction circles. The two forces P and Q acting vertically downwards must be in equilibrium with the re-action R offered by the bearing A_1 to the journal A.

This re-action on account of journal friction can only act tangent to the friction circle of A, and on the same side as P. The investigation is now resolved, therefore, into the determination of two parallel forces P and Q, which have such relative values that their distances from the resultant lying between them shall be

$$a - \rho - \chi \quad \text{and} \quad a + \rho + \chi$$

respectively; in which expressions ρ is the radius of the friction circle for the journal A, and χ the radius of friction circles of the chain. We have, according to this,

$$P = Q\frac{a + \rho + \chi}{a - \rho - \chi}.$$

This value can be readily constructed by drawing at any point the horizontal line GH, making $GK = Q$, drawing the horizontal line KJ, and then a line through J and L to M. We then have in $MG = NH$ the necessary driving-force P, and in KM the re-action of the bearing $R = P + Q$.

Since the links rub one another while in a dry condition we assume a co-efficient of friction $\mu = 0.2$. With this assumption, and that of $\mu = 0.1$ at the journal, the figure gives, for $Q = 100$,

$$P = 105;$$

and since $P_0 = Q = 100$ we have, for the fixed pulley,

$$\eta = \frac{P_0}{P} = 0.952.$$

The investigation is the same if the directions of the chains are not parallel one to another. For the guide-pulley ABC (Fig. 35, plate IV.) we draw first the medial lines OC and OB of the chain, and then the lines ob and oc along which the tension of the chain acts. We then have in the line ao drawn through the intersection o and tangent to the friction circle at A, the direction of the re-action R of the bearing. So that by making oI equal to Q, and drawing $I\,II$ parallel to oc, we get

$$I\,II = P \quad \text{and} \quad o\,II = R.$$

For $\mu = 0.1$ and $\mu_1 = 0.2$, the figure gives, for $Q = P_0 = 100$,

$$P = 104.4 \quad \text{and} \quad \eta = \frac{P_0}{P} = 0.958.$$

In the same way may be determined the friction of an idler, or guide-pulley, which serves merely to support the chain and keep it from sagging (Fig. 36, plate IV.), and is often so employed in cranes. Draw the lines BE and BF (shown in dotted lines) along which the chain rolls off and on to the pulley, and draw parallel to them, at a distance equal to χ, the radius of friction for the chain, the lines of tension oc and od. Then the re-action of the journal A is given by the tangent oa to its friction circle passing through the point of intersection o. By making oI equal to the resistance Z_1 acting in the portion BD of the chain, and drawing through I the parallel $I\,II$ to oc, we have in $I\,II$ the force Z_2 which is transmitted to the portion BC of the chain.

In the case of the loose pulley (Fig. 37, plate IV.) upon whose journal the load Q hangs, and where one end of the chain is fastened at D, we have to determine the vertical force P acting at the other end of the chain CE. To do this we draw the directions of tension bd and ce parallel to the chain, and at the distance χ of the friction radius of the chain, and also the vertical tangent ag to the friction circle at A. Then draw through the centre of the journal, or at any other convenient place, the horizontal line BAC, make $AG = Q$, draw through G the line b_1c_1 parallel to BC, join B with c_1, and draw through H the line JK parallel to BC. We now have, as will be readily seen, in $c_1K = b_1J$ the tension Z of the fast end of the chain, and in KC the force acting in the portion CE of the chain.

By means of friction circles for journal and chain, whose radii will be denoted as heretofore by ρ and χ respectively, the proportions of the various forces in all kinds of block and tackle and pulley gearing can be easily determined, as a few examples will show.

Fig. 38, plate V., represents an ordinary block and tackle with two blocks, within each of which are three pulleys of equal size, and ranged side by side on the bolts A and B. When, by raising the load, the pulleys are turned in the direction shown by the arrows, and the chains wind on at E and D, and off at F and C, it is evident that the pull of the load hanging on the hook H acts upon the journal B of the lower block along the vertical tangent o_2b to the friction circle at B, while the re-action transmitted by the support G to the journal of A acts along the tangent o_1a to its friction circle. The forces of tension in the chains also will act at a distance χ, the radius of friction of the chain, nearer to the centre of the pulleys at F and C, and at a distance increased by the same amount at E and D. Since the lower block B swings free, and opposes no resistance to side motion, it will shift its position a distance 2χ to the left when motion begins, so that the line of tension in the chain shall be vertical; otherwise equilibrium could not exist. In the figure such a side movement of the block B is supposed to have taken place, so that the centres A and B do not lie in the same vertical line, as would be the case when they are at rest. It is equally evident that with an opposite motion (that is, with a sinking of the load) a corresponding shifting of the block B to the opposite side must occur. We now denote the tension in the separate portions of chain by Z_1, $Z_2 \ldots Z_7$, in such way that Z_1 is the tension of the first portion which hangs from the stationary block A and winds on to the first pulley of the block B at E, while Z_7 is the force which is to be applied to the free end of the rope to raise the load Q. It is then evident from foregoing principles that the relation existing between the tension of each portion of the chain and that of the next following is

$$Z_n(r + \rho + \chi) = Z_{n+1}(r - \rho - \chi)$$

if r is the distance from the centre of the chain to the centre of the pulley.

If, now, we draw at any convenient point a horizontal line which cuts the directions ce and fd of chain tension at J and K, and the directions of journal re-action at o_1 and o_2, we know from the figure that

$$JK = 2r, \quad Jo_1 = o_2K = r - \rho - \chi,$$

and

$$Jo_2 = o_1K = r + \rho + \chi.$$

From this follows immediately the construction given below for the determination of Z_7 or P_1. Make $J\,I$ equal the tension of the first portion Z_1, draw through I and o_1 the line cutting KD in II; $K\,II$ is then the tension of the second portion Z_2. From II draw through o_2 the line cutting off the distance $J\,III$, which is the tension in Z_3. In the same way the lines $III o_1 IV$, $IV o_2 V$, $V o_1 VI$, and $VI o_2 VII$ give the tensions

$$Z_4 = K\,IV, \quad Z_5 = J\,V, \quad Z_6 = K\,VI,$$

and

$$Z_7 = J\,VII.$$

The load to be lifted is given by the equation

$$Q = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6,$$

while Z_7 is the force P to be applied to the free end of the rope in order to lift it. In the figure the sum of the tensions from Z_1 to Z_6 is shown by NL, and $MO = Z_7$ is the force P. By the construction here chosen we start with a value of Z_1, while in reality Z_1 is yet unknown, since only Q is given; but the method lends itself with equal ease to the solution under the latter condition. We can assume $Z_1 = J\,I$ of any convenient length, and get, in the manner shown,

$$NL = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6$$

and

$$MO = Z_7.$$

We then find from the given value of Q, and the proportion between NL and MO, the force

$$P = Q\frac{MO}{NL},$$

which in reality only amounts to the assumption of a particular scale of force. On the contrary, a construction direct from the value of Q would be unnecessarily tedious.

This construction also holds for the backward motion of the tackle if only we regard the force P applied at the end of the rope to prevent any accelerated motion of the load Q as the tension in the first portion of the chain, and every following tension as increasing in the ratio $\frac{r + \rho + \chi}{r - \rho - \chi}$. Then regarding JI as (P) or Z_7, we have by the same construction the tension Z_6, $Z_5 \ldots Z_1$, each one of which is greater than its predecessor, so that the tension Z_1 of the chain attached to the stationary block A is now the largest. Here also

$$Q = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6,$$

and

$$Z_7 = (P).$$

In the figure the sum of the series Z_1 to Z_6 is shown by SR, and $(P) = Z_7$ by TU.

For forward motion, and $Z_1 = 100$, the figure gives

$$Q = 682.7, \quad P = 134.6;$$

and since $P_0 = \frac{Q}{6} = 113.8$

$$\eta = \frac{P_0}{P} = 0.845.$$

For backward motion, for $Z_7 = (P) = 100$,

$$Q = 717.4;$$

and since $P_0 = \frac{Q}{6} = 119.6$

$$(\eta) = \frac{(P)}{P_0} = 0.837.$$

If the pulleys in the blocks are not of equal size, and therefore the ropes are not parallel, as when the pulleys are arranged on different centres in the block, the above determination for parallel ropes will in general have sufficient exactness on account of the slight divergence from parallelism in the case under consideration. An absolutely correct determination can be made, however, by taking into account the points of intersection of the ropes produced. An example in which this is done is shown in Fig. 40, plate V.

We will next take up the differential pulley (Fig. 39, plate V.). Here also the load Q, hanging upon the hook H, acts in the direction of the vertical tangent bh to the friction circle of B, while the re-action of the support G to the journal A lies along the tangent ag. The direction of tension in the portion Z_1 of chain unwinding from the smaller pulley at C, and winding on to the loose pulley at E, is again given in direction by ce, as is also the direction of tension in the portion Z_2 winding on and off at D and F respectively, by the line fd. The two tensions Z_1 and Z_2, which in the ordinary differential pulley can be assumed as parallel, stand in the following relation one to another:

$$Z_2(r - \rho - \chi) = Z_1(r + \rho + \chi)$$

where r is the radius of the pulley EF, and

$$Q = Z_1 + Z_2.$$

If we then draw through any convenient point o of fd the line oM perpendicular to fd, and make $MI = Q$, we have in the intersection b_2 of the line oI with the direction b_1b of the load Q a point which will give us the proportion of Z_1 to Z_2; for, drawing through b_2 the horizontal b_2N, we have the proportion

$$b_1b_2 : NI : : ob_1 : b_1M : : (r - \rho - \chi) : (r + \rho + \chi);$$

from which we get

$$b_1b_2 = MN = Z_1 \quad \text{and} \quad NI = Z_2.$$

These two forces Z_1 and Z_2 must be in equilibrium with the tension Z_3 or force P applied to the free end of the chain JK along the line iK, and the re-action R given forth against the journal A along the line ay. We can most easily find the condition of equilibrium for these parallel forces through the drawing of an equilibrium polygon.* For this purpose let us regard o as the pole of the force polygon MNI, then draw through ζ the line $\alpha\zeta$ parallel to oM, and $\beta\zeta$ parallel to oN; then draw through β the line $\beta 8$ parallel to oI, and we have in the closing line $\alpha 8$ of the polygon the direction of the polar ray $o\,II$, which gives us in $I\,II$ the driving-force P applied at i, and in $II\,M$ the re-action of the journal A against the pulley.†

The theoretic driving-force P_0 can be determined by a similar construction, or more easily from the relation ‡

$$P_0 = Q\frac{R_1 - R_2}{2R_1} = Q\frac{AD - AL}{DJ}.$$

In order to determine the proportion between the forces for backward motion we have only to remember

---

* For a more complete explanation of the principles here employed see Part I. of The Elements of Graphic Statics by Karl Von Ott. — *Trans.*

† The figure $\zeta\beta 8\alpha$ is the funicular or equilibrium polygon, and it will be readily seen that the forces P, Z_1, Z_2, and the re-action at A acting upon the vertices r, ζ, β, and α respectively of the polygon would keep it in equilibrium. — *Trans.*

‡ This is, of course, derived as follows: Without friction

$$Z_1 = Z_2 = \frac{Q}{2},$$

so that the equation of moments about A becomes

$$P_0 R_1 + \frac{Q}{2}R_2 = \frac{Q}{2}R_1$$

transposing

$$P_0 R_1 = \frac{Q(R_1 - R_2)}{2} \quad \text{or} \quad P_0 = Q\frac{R_1 - R_2}{2R_1}.$$

— *Trans.*

that the chain tensions and journal re-actions coincide with the broken lines in the figure. For backward motion also

$$Q = (Z_1) + (Z_2),$$

but now

$$(Z_2)(r + \rho + \chi) = (Z_1)(r - \rho - \chi);$$

therefore the tension (Z_1) of the chain CE is exactly equal to the tension Z_2 of the chain FD for forward motion, and also

$$(Z_2) = Z_1.$$

If, then, we make $M(N) = NI$ in the force polygon, and draw the polar ray $o(N)$, we can construct the funicular polygon $(\alpha)(\zeta)(\beta)(\delta)$ for backward motion by drawing $(\alpha)(\zeta)$ parallel to oM, $(\beta)(\zeta)$ parallel to $o(N)$, and $(\beta)(\delta)$ parallel to oI. Then $(\alpha)(\delta)$ is the closing line of the funicular polygon, and the polar ray $o(II)$ drawn parallel to this gives us the force

$$(P) = I(II)$$

which must be applied at J during backward motion. Since this force is here acting upwards it is self-evident that the tackle cannot of itself commence backward motion, but that it is self-locking.

The figure gives, for $\mu = 0.1$ for journal friction, and $\mu_1 = 0.2$ for chain friction, with a ratio $\dfrac{R_1}{R_2} = \dfrac{AD}{AL} = \frac{10}{9}$, and for $Q = 100$,

$$P = 12.8 \quad \text{and} \quad (P) = -2.7.$$

Since $P_0 = 5$ we have

$$\eta = \frac{P_0}{P} = 0.391 \quad \text{and} \quad (\eta) = \frac{(P)}{P_0} = -0.54.$$

As another example we will investigate the arrangement of pulleys often employed in hydraulic lifting machinery.* In this case the chain upon which the load Q hangs is first led over guide-pulleys supported by the roof timbers at A and B (Fig. 40, plate V.), from which it hangs down in a loop containing the loose pulley C, and then, after passing around the fixed pulley D, comes back and is attached to the journal of C. Under the supposition of parallelism between the chains this arrangement would cause, for any distance travelled by C, an elevation of the load Q through exactly three times that distance. The downward motion of C is produced by aid of a second chain, one end of which is fastened to the journal of C, and the other, after passing around the loose pulley E and the fixed pulley F, returns again to the journal of E. The movement of the journal E is produced by the aid of the vertical piston-rod of a hydraulic cylinder, not shown in the drawing as not entering into the calculation. With the supposition again of parallelism between the ropes a downward motion of the piston would cause the pulley C to traverse three times as great a distance, and the load Q to be raised by an amount equal to nine times the piston travel. Without hurtful resistances, therefore, the piston force P_0 would equal $9Q$; and it is

* See Weisbach, Ingenieur- und Maschinenmechanik, III., Theil; also, Rühlman, Allgemeine Maschinenlehre, IV., Bd.

evident that this arrangement would find its only application where, as in hydraulic apparatus, the travel of the driving-force P is necessarily small, while the force itself is of great power. The proportion given above for travel of power and load under the assumption of parallelism would in reality vary but little from the actual result. On account of the non-existence of complete parallelism we will, however, follow out the investigation, taking this inclination of the chains one to another into account for the sake of showing the general method.

The direction of the forces Z_1, $Z_2 \ldots Z_7$ acting in the separate portions of the chain can easily be determined by increasing the lever-arm by the radius χ of chain friction at every point where the chain winds on to a pulley, as at A_1, B_1, C_1, D_1, E_1, and F_1, and by diminishing it by the same amount wherever the chain unwinds from a pulley, as at A_2, $B_2 \ldots F_2$.

The direction of re-action R_1 of the bearing of the fixed pulley A is given by the line o_1a drawn through o_1 tangent to the friction circle at A. Its value is ascertained by making $OI = Q$, and drawing through O a parallel to o_1a, and through I a parallel to o_1o_2. We have, then,

$$I\,II = Z_1,$$

the tension in the chain A_2B_1. In the same way the bearing re-acts against the journal B of the second guide-pulley along the line bo_2; therefore a resolution of the tension $Z_1 = I\,II$ in the direction of bo_2 and o_2o_3 will give in $I\,III$ the tension Z_2 in the portion B_2C_1 of the chain, and in $III\,II$ the re-action R_2 of the bearing at B. There are now acting upon the loose pulley C

four forces; namely, the tension Z_2 of the chain in the direction o_3o_2, that Z_3 of the portion C_2D_1 in the direction o_5d_1, that Z_4 of the end of the chain in the direction o_5d_2, and finally the tension Z_5 of the second chain in the direction ce_1. The lines of the two last, Z_4 and Z_5, must evidently be tangent to the friction circle at C. Z_2 and Z_3 intersect in o_3, Z_4 and Z_5 in o_4. The line o_3o_4 joining the two is therefore the line of direction for the resultant of Z_2 and Z_3 as well as of Z_4 and Z_5. If we then draw through I the line $I\,IV$ parallel to o_3o_4, and through III the line $III\,IV$ parallel to C_2d_1, we have

$$III\,IV = Z_3,$$

and in $IV\,I$ the resultant of Z_4 and Z_5. By resolving this resultant $IV\,I$ into $IV\,V$ parallel to d_2o_5, and $I\,V$ parallel to ce_1, we get in $V\,IV$ the tension Z_4 between C and D_2, and in $I\,V$ the force Z_5 with which the second chain pulls down upon the journal of the loose pulley C. We also have the re-action of the bearing against the journal D of the fixed pulley in $V\,III$, the resultant of Z_3 and Z_4.

We also know that the four forces Z_5, Z_6, Z_7, and P acting upon the loose pulley E must be in equilibrium. The force Z_5 acts along the line ce_1, that of Z_6 along f_1e_2. Their intersection is at o_6. The force of tension in Z_7 and the piston-force P are tangential to the friction circle of E. These last two forces intersect in o_7. The line o_6o_7 is then the direction of the resultant of Z_5 and Z_6, and that of Z_7 and P. We therefore draw through I the line $I\,VI$ parallel to o_6o_7, and $V\,VI$ parallel to f_1e_2, to get in $VI\,V$ the tension Z_6 of the portion E_2F_1 of the chain; while $VI\,I$ represents the resultant of Z_1 and P_8. Resolving this force $VI\,I$ parallel to $f_2 o_8$ or Z_7, and parallel to P, we have in $VI\,VII$ the tension Z_7, and in $I\,VII$ the force P acting upon the piston-rod. The re-action of the bearing upon F is given by $VII\,V$, the resultant of Z_6 and Z_7.

In apparatus of this kind the directions of the tensions in the chains vary but slightly one from another, so that their points of intersection o often fall without the limits of the drawing. This difficulty can be surmounted by the employment of the method used in Fig. 19, plate II., in which the direction of the lines is obtained without having their points of intersection located. We must proceed, according to this method, when the points o_3, o_5, o_7, and o_8, fall beyond the limits of the drawing. When the position of the chains approaches parallelism there is the difficulty also that the point of intersection of two such lines cannot be determined with any degree of exactness. A sufficient degree of accuracy can, however, be obtained with the aid of the following construction: If the force $Z_2 = I\,III$ is to be resolved into the directions $III\,IV$ parallel to Z_3, and $I\,IV$ parallel to $o_3 o_4$, we can imagine this system of forces to be acted upon by two opposite and equal forces $\alpha\beta$ and $\alpha_1\beta_1$ along the line $C_1 C_2$, which would not disturb the equilibrium. Let Ia represent $\alpha\beta$, and $IIIa$ be the resultant of this force, and

$$Z_2 = III\,I.$$

Compound also the opposite force $\alpha_1\beta_1$ with the yet unknown force Z_3, and there will be another resultant whose direction my be determined. For when the resultant of Z_2 and $\alpha\beta$ is compounded with that of Z_3

and $\alpha_1\beta_1$ they will give a resultant which must coincide with o_3o_4. If, therefore, we draw through α a parallel to IIIa to the point of intersection ω with o_3o_4, the resultant of Z_3 and $\alpha_1\beta_1$ must also pass through ω, and is given by $\omega\alpha_1$. So that by resolving the force IIIa parallel to o_3o_4 and $\omega\alpha_1$, by drawing the triangle IIIaa$_1$, we get the point a$_1$ from which the point IV can be accurately determined by drawing through a$_1$ a line parallel to $\alpha_1\beta_1$ intersecting a line from I parallel to o_3o_4. The auxiliary forces $\alpha\beta$ and $\alpha_1\beta_1$ may be chosen of any convenient value, but should be assumed so that the lines which are to determine the desired point by their intersection should be nearly at right angles one to another.

In the same way the point of intersection VI can be determined by the application of two opposite and equal forces along the line E_1E_2. Instead of resolving the force $Z_5 = VI$ directly parallel to o_6o_7 and Z_6, the force $V\delta$ is substituted for Z_5 and resolved in the directions o_6o_7 and $\omega_1\delta_1$.

In this and similar ways we can employ upon every diagram constructions in which the lines will diverge sufficiently to determine accurately the points of intersection.

For $Q = 100$ the figure gives

$$P = 1295;$$

and since, with the assumption of parallelism, the theoretic force $P_0 = 900 = 9Q$, we have

$$\eta = \frac{P_0}{P} = 0.695.$$

In this value of the efficiency no account is taken of the resistances within the hydraulic cylinder. These latter must be determined in each case by a special investigation. It is evident, moreover, that the value of the piston and stuffing-box friction must be added to the force P already found to get the necessary pressure to be exerted upon the piston by the water from the accumulator.

Any calculation for backward motion would have to take into consideration the weight of the chains, loose pulleys, and piston; and we would get in the force which would have to be applied at the free end of the chain A_1 the counter-weight, which in this species of hoisting-gear is attached to the hook in order to render the backward motion automatic. In some cases the weight of the platform or cage is sufficient of itself to do this.

§ 7.—STIFFNESS OF ROPES.

When pliable ropes and cords are wound on and off a pulley or drum there are certain hurtful resistances called forth, in part by friction between the strands of the rope produced by bending it, and in part by the resistance of these fibres to the expansion and contraction which they are compelled to undergo. We can bring this resistance into the calculation in the same manner as chain friction by assuming that, in consequence of it, the lever-arm of the load is increased at the point where the rope winds on, and that lever-arm of the power where the rope runs off the pulley is decreased by a similar amount. This value has to be determined by a special investigation, and is generally expressed by an empirical formula. By such investigation it has been proven that this resistance is proportional to the tension in the rope, that it is inversely proportional to the radius of the pulley, and that it increases with the thickness d of the rope, not in the simple ratio, but as some higher power. For hemp ropes we assume that the resistance increases as the square of the thickness, and is given by the equation

$$S = k\frac{d^2}{r}Z$$

where Z is the tension in the rope, and k is a constant co-efficient.

That the resistance due to stiffness of ropes really causes a lengthening and shortening of the lever-arms of the load Q and the force P respectively can be proven as follows: Let A (Fig. 41, plate V.) be the centre of a pulley, and $AB = AC = r$ be the radius of the same extended to the centre of the rope. A load Q hanging on the rope at D produces a certain tension in the elements of any section of the rope which we may assume as equally distributed over the cross-section, so that the resultant of all these elementary tensions acts in a resultant passing through the centre of the cross-section, and coinciding with the geometrical axis BD of any portion of the rope. In the condition of rest, therefore, as long as there is no turning of the pulley, the rope BD will arrange itself in such a position that the line of force Q will pass through the centre M of the section at B. If, however, we suppose an exterior force to be applied to the pulley which tends to cause a revolution of the same in the direction of the arrow, and a winding-up of the rope at B, there will be a bending of the rope at that point, and only the strands in the neutral plane $M_1 M_2$ will retain their original length, while those lying in the outer semicircle $M_1 M_2 O$ will be stretched, and those in the other semicircle $M_1 M_2 J$ will be shortened by a force of compression. Each strand in the half OM will now receive, beside the strain due to Q, a certain elastic tension σ which is proportional to the distance of that strand from the neutral plane $M_1 M_2$. We may suppose all the elementary tensions in the half-section $M_1 M_2 O$ to be combined in one resultant s which shall be applied at some point α. In the same way each strand of the inner half-section $M_1 M_2 J$ will experience a certain compression which will also be proportional to its distance from the neutral plane M_1M_2, and if these are combined in one resultant we have a certain pressure p acting at some point β. In the bending of rigid bodies, as beams, it is known that the assumption $s = p$ is made. It is not necessary here, however. We know from the preceding consideration that the rope BD upon which the exterior force Q acts at D is under the influence of three forces at the section B; namely, the tension Z acting upwards at the centre B, the force s also acting upwards at α, and that, p, acting downward at β. These three forces must have a resultant which is equal and opposite to the force Q. It will be readily seen that the resultant S of Z, s, and p which is to be equal to Q must be at a greater distance from A than that at which the centre M of the rope section is; and this fact is proven when we combine Z and s, and then compound their resultant with p. From the relative positions of B and α the resultant of Z and s must lie at a greater distance from the axis A than the radius r of the pulley, and the composition of this resultant with the opposite pressure p gives the point of application b of the final resultant S still farther off. For all purposes of the following demonstrations it is sufficient to represent the distance bB simply by σ which evidently corresponds to the value χ used in chain friction.

A demonstration similar to the preceding would show that friction shortens the lever-arm of the force acting in the portion CE of the rope. In that case also there would be acting in the section at C: First, a tension Z_1 opposite to P, and beside that a force s_1 acting upward at some point 8 of the inner semicircle N_1N_2K, and the force p_1 acting downward at some point δ of the outer semicircle. It follows, then, that by the unwinding of the rope from the pulley at C the strands of the inner half-section N_1N_2K are stretched, and those of the outer half N_1N_2L are pressed together. Therefore the resultant S_1 of Z_1, s_1, and p_1, equal to P, must have its point of application c between C and A, and the lever-arm of the acting-force is reduced by an amount

$$Cc = \sigma.$$

If we suppose the ends of the rope BD and CE to swing free with weights suspended from them the weights will be shifted to one side or the other, according to the direction of revolution, by an amount σ as was shown for a similar case in the chain pulley. In every case of rope pulleys where ρ represents the radius of the friction circle at the journal the equation

$$Q(r + \rho + \sigma) = P\,(r - \rho - \sigma)$$

is true, which becomes the same as that for chain friction when χ is substituted for σ.

The investigation of all rope-gearing proceeds, therefore, in the same way as for chain friction, and it only remains to get a graphic equivalent for the quantity σ. The way in which this value σ is to be applied can easily be determined in every case where the direction is known in which the forces act by remembering that the lever-arm of the unwinding rope is always shortened, and that of the winding-on rope always lengthened by this amount σ. If, for instance, a load Q (Fig. 42, plate V.) hanging from the drum CH is to be raised by a revolution of the pulley A we have the directions de, fg, and hh for rope tensions at once.

We have now to obtain a graphic representation for σ from some of the empiric formulæ for the stiffness of ropes. Of the different formulæ * the one most generally used in practice is that of Eytelwein, which is simple in form and sufficiently accurate, at least in the case of hempen ropes and with large forces. This formula will then be assumed as the basis of our calculations. It should be remarked, however, that in special cases, as with wire ropes, other formulæ should be used, from which the value of σ can be determined in the same way as from Eytelwein's formula.

According to this formula the entire resistance S due to stiffness of the rope at both winding-on and unwinding points is given by the expression

$$S = 0.0186 \frac{d^2}{r} Q$$

where d is the thickness of the rope, and r the radius of the pulley, both in millimetres, and Q the tenison of the rope. S merely represents the force which is sufficient to overcome the stiffness of the rope at both sides, omitting journal friction. In order to produce motion, therefore, a force

$$P = Q + S = Q\left(1 + 0.0186 \frac{d^2}{r}\right)$$

must act upon the other end of the rope.

Now, according to foregoing principles, in the absence

---

* See Weisbach, Lehrbuch der Ingenieur- und Maschinenmechanik, Theil I.

of journal friction there is the following relation between P and Q:

$$Q(r + \sigma) = P\,(r - \sigma);$$

therefore

$$P = \frac{r + \sigma}{r - \sigma} Q,$$

which, from the small value of σ when compared to r, is given with sufficient exactness as

$$P = \left(1 + 2\frac{\sigma}{r}\right)^{*} Q.$$

By equating these two values for P we get

$$1 + 0.0186\frac{d^2}{r} = 1 + 2\frac{\sigma}{r},$$

or

$$\sigma = 0.0186\frac{d^2}{2}.$$

To get this in a convenient form for construction it may be written

$$\sigma = 0.0372\,\frac{d}{2}\,\frac{d}{2} = \frac{1}{26.8}\,\frac{d}{2}\,\frac{d}{2}.$$

We may represent this value graphically as follows: Draw through the centre A of the circular cross-section

---

* The expression within the parentheses will be recognized as simply the first two terms of the development of $\frac{r + \sigma}{r - \sigma}$. All succeeding terms would contain higher powers of the fraction $\frac{\sigma}{r}$, and be of consequent small value. — *Trans.*

of the rope (Fig. 43, plate V.) two lines AC and AF at right angles; make $AC = 26.8$, or, in round numbers, 27 millimetres; join C and B, and draw through D the line DE parallel to CB. We then have in AE the value of σ; for, according to the construction,

$$AE = AD\frac{AB}{AC} = \frac{d}{2}\frac{d}{2} \cdot \frac{1}{26.8} = \sigma.$$

There is a certain analogy between this construction and that of χ for chain friction; for, if the circle about A were the cross-section of the link, we would have in AE the value of χ if we made the angle ADE equal the angle of friction ϕ for chain links. In this sense also we may call the angle ADE the angle of friction, since its tangent gives the ratio of σ to the radius of the rope's cross-section. The difference between rope and chain lies only in this, that with the latter the angle ADE, like the angle of friction, is constant for all thicknesses of link; while the same angle, in the case of the rope, is directly dependent upon this thickness, since its tangent varies with the value of d. From this it results that the resistance due to stiffness increases so much more rapidly with thick ropes than that due to friction in heavy chains, for the angle ADE is evidently equal to 45 degrees, and

$$\sigma = \frac{d}{2},$$

when $\frac{d}{2}$ becomes equal to 26.8 mm.; that is, for a rope about 54 mm., or 2 inches, in thickness.

It is also of interest to know in what relative sizes of rope and chain the values σ and χ are equal. This condition is determined by making the tangent of ACB equal to the co-efficient 0.2 for chain friction, as follows:

$$\frac{AB}{AC} = \frac{d}{2.26.8} = 0.2,$$

which gives a value

$$d = 10.7 \text{ mm.}$$

A rope of this thickness would offer the same proportional resistance, on account of its stiffness, as a chain, the iron of whose links was of the same diameter, supposing the pulleys to be of equal radius in both cases.

With the assumption of another formula for stiffness the construction for the determination of σ must be modified accordingly.

§ 8.—TOOTH FRICTION.

In the transmission of turning motion from one axis to another by means of toothed wheels there arises, besides the journal friction in the bearings of the axes, still other frictional resistances at the point where the teeth come in contact. This resistance, like all other friction, is a consequence of the relative sliding of the two surfaces in contact, and does not occur where there is no such sliding. Therefore is there no friction between two teeth working together at that instant when their point of contact falls in the line joining the centres of the wheels, that is, at the point of contact O of the pitch circles A and B (Fig. 44, plate V.); for at that moment, and only at that moment, the teeth of both wheels have motions exactly corresponding in amount and direction. This is not the case as soon as the point of contact of any two teeth falls outside the central line $O_1 O_2$, as a glance at the figure shows. For, if two teeth on the wheels A and B, of which A is the driving-wheel and turns in the direction of the arrow, have contact at the points a_1 and b_1, the point a_1 has motion in the direction $\alpha_1 a_1$ drawn perpendicular to the radius at that point, while b_1 is moving along the line $\beta_1 b_1$ drawn in the same way. From this it follows immediately that when motion occurs the surfaces of the teeth must slide one upon the other. The direction of this sliding, which is foreshadowed by the lines $\alpha_1 a_1$ and $\beta_1 b_1$, is further determined by the fact that, after such an amount of turning that the point of contact of the teeth is between a_0 and b_0 of the pitch circles in the line of centres at O, the two portions of surface a_1a_0 and b_1b_0 of the teeth must have so moved one over the other, that, while at first a_1 and b_1 were in contact, now a_0 and b_0 are in that position. We can imagine the relative motion of the two teeth to have taken place as follows: First, the shorter arc a_1a_0 to roll upon the larger b_1b_0 without any slipping; then to slide upon it an amount equal to

$$\overline{b_1b_0} - \overline{a_1a_0},$$

thus bringing the points a_0 and b_0 together. The direction of sliding between tooth surfaces is such, therefore, that the tooth of one wheel slides along the flank of the other wheel's tooth toward its axis.

This condition only exists, however, until the point of contact reaches the line of centres O_1O_2, the direction of sliding becoming reversed as soon as that point is passed. For, if two teeth are in contact at points a_2 and b_2 behind the line of centres O_1O_2, we know, from the direction of motion $a_2\alpha_2$ and $b_2\beta_2$ of these points, that the sliding is now of such kind that each tooth tends to move along the flank of its companion away from the axis of the wheel upon which the latter is fixed. This also follows from the fact that, while at first the points a_0 and b_0 were in contact, now a_2 and b_2 are in that position; the change being brought about by a rolling of the shorter arc b_0b_2, and a sliding through the distance

$$\overline{a_0a_2} - \overline{b_0b_2}$$

in the direction mentioned above. In accurately constructed teeth the normal to the surfaces in contact should always pass through the point of contact between the pitch circles; therefore, by joining the points a_1b_1 or a_2b_2 with O, we have in the line so drawn the desired normal along which the re-actions of the teeth would work if there were no sliding friction. But on account of friction the direction of re-action must be inclined at an angle ϕ to this normal. Referring to the directions of sliding previously determined we see that the direction of pressure at a point of contact before the line of centres is along the line $a_1 b_1 C$, while for a contact point beyond the line of centres it is given by the line Ca_2b_2.

In designing toothed gearing the dimensions are so chosen that the arc of contact (i.e., the arc within which the teeth mesh one with another) is always greater than the pitch t, since a smaller value than t would render continuous transmission of motion impossible. The value of this arc varies in ordinary cases between $1.2t$ and $1.7t$, occasionally going as high as $2t$. In the latter case motion will always be transmitted by two pairs of teeth at the same time, as shown in the figure at a_1b_1 and a_2b_2. When the arc of contact only equals t the transmission of motion takes place only through one pair of teeth, and for all intermediate values there are sometimes one and sometimes two pairs of teeth in contact. We will first assume that two pairs of teeth a_1b_1 and a_2b_2 are in contact, and have an equal pressure upon them. We will further suppose them to be involute teeth, for which the normals Oa_1b_1 and Oa_2b_2 coincide in one and the same straight line GH, which makes the angle $\alpha = GOO_1$ of about 75 degrees with the line of centres $O_2 O_1$. Suppose, now, that a certain resistance Q opposes the revolution of the wheel B which acts at O in the direction OG; then, without friction of the teeth, the wheel A would act upon B at the points a_1b_1 and a_2b_2 with a force P_0 along the line GH, this force P_0 being exactly equal to Q. But on account of friction the wheel A acts upon B with two forces, the direction of one of which is a_1C, the other Ca_2. The resultant of these two forces goes through the point of intersection C, and under the supposition of equal pressures at a_1b_1 and a_2b_2 bisects the angle 2ϕ which the forces make one with another. If we imagine this pressure R of one wheel upon the other to be laid off on the lines CD and CE we have in the diagonal CF of the parallelogram the necessary driving-force P. Under the supposition again of equal pressures on the two pairs of teeth this resultant CF is parallel to the common normal GH, and lies at a distance

$$\zeta = \tfrac{1}{2}a_1a_2 \,.\, \tan\phi$$

from the latter. If we now make the permissible assumption $a_1a_2 = t\sin\alpha$, α being the angle of the normal GH to the line of centres O_1O_2, we get, by substituting $\mu = \tan\phi$, the desired distance

$$\zeta = \frac{\mu t}{2}\sin\alpha.$$

The value of ζ is therefore independent of the position of the two points of contact a_1b_1 and a_2b_2 with reference to O, it being supposed that they lie on opposite sides of O, so that the lines of pressure may intersect at an angle 2ϕ. According to these considerations we may conceive of tooth friction acting in the following way: While in the case of frictionless motion the force is transmitted from one wheel to the other in the direction of a normal to the surfaces in contact, from which it results that the force P_0 must be exactly equal to the resistance Q acting along the same line, in the case where friction is taken into account there is a parallel shifting to one side of the driving-force through a distance ζ; so that the lever-arm of the driving-force with respect to the axis of the driven wheel B is shortened by an amount ζ, while with respect to the axis of the driving-wheel A it is increased by an equal amount. Since the line FC represents the re-action of the load Q for the wheel A, and $CF = P$ is the driving-force with reference to the wheel B, we may say, as in chain and rope friction, that the arm of the power is diminished, and that of the resistance increased, by the amount ζ. In order to get a graphic representation of tooth friction we have only to determine the value ζ, and to then shift the line of force to one side of the theoretic line, so that it shall be parallel to the latter, and at the distance ζ from it.

It will not be difficult to show that the result of this shifting corresponds to that which we have been accustomed to find in practice by calculation, and which is given by the formula

$$Z = Q\mu\pi\left(\frac{1}{n_1} + \frac{1}{n_2}\right).$$

Here Z is the force which must be applied at the point of contact O of the pitch circles to overcome tooth friction alone, and n_1 and n_2 are the number of teeth on the wheels respectively. To show this coincidence let r_1 be the radius of the pitch circle A, and r_2 that of B, and $\alpha = GOO_1$ be again the inclination of the normal pressure to the line of centres. A resistance Q acting at O along the line OG to the wheel B has the lever-arm $r_2 \sin \alpha$, to overcome which a force CF must be applied at C, which, from the relation $r_2 \sin \alpha$ and $r_2 \sin \alpha - \zeta$ between the lever-arms, would be given by the equation

$$X = Q \frac{r_2 \sin \alpha}{r_2 \sin \alpha - \zeta}.$$

This force X must be exerted by the driving-wheel A at the point C in the direction CF, with a lever-arm $r_1 \sin \alpha + \zeta$; therefore, there must be applied at the point O of the wheel A a force in the direction OH given by the ratio of lever-arms in the equation

$$Y = X \frac{r_1 \sin \alpha + \zeta}{r_1 \sin \alpha} = Q \frac{r_2 \sin \alpha}{r_2 \sin \alpha - \zeta} \cdot \frac{r_1 \sin \alpha + \zeta}{r_1 \sin \alpha}.$$

Substituting here

$$r_1 = \frac{tn_1}{2\pi}, \quad r_2 = \frac{tn_2}{2\pi}, \quad \text{and} \quad \zeta = \frac{\mu t}{2} \sin \alpha,$$

we get, after reduction,

$$Y = Q \frac{n_2}{n_2 - \mu\pi} \cdot \frac{n_1 + \mu\pi}{n_1} = Q\left(1 + \frac{\mu\pi}{n_1} + \frac{\mu\pi}{n_2}\right).$$

Since without friction $Q = P_0$ it follows that the value of the tooth friction, the force which must be applied at O in the direction of Q, is

$$Z = Y - P_0 = Q\mu\pi\left(\frac{1}{n_1} + \frac{1}{n_2}\right).$$

It may be remarked here that the latter formula, which is commonly used in calculating tooth friction, is deduced * under the supposition, employed in the preceding investigation also, that the pressure is transmitted equally by two pairs of teeth. In the analytic determination of tooth friction it is customary to employ as Q the resistance which acts at the contact point O of the pitch circles, normal to the line of centres, in order to avoid the value $\sin \alpha$. Even if such an approximation is permissible ($\sin \alpha = \sin 75° = 0.9659$ approaching unity) it will be seen that it is unnecessary in the graphic method, as that loses nothing of its simplicity by drawing Q in its proper direction. It follows also that the construction remains the same when teeth of any other than the involute form are employed. If we only know the profiles of the teeth in contact the reaction is always inclined at the angle ϕ to the normal to the surfaces where contact is taking place. In the case of profiles laid out by auxiliary circles the chord of the auxiliary circle passing the point of contact of the teeth, and that of the pitch circles at O, is the desired normal. Of course in cycloidal and most other than involute profiles the normal will vary in its direction;

* Weisbach-Hermann, Ingenieur- und Maschinenmechanik, Theil III., 1.

for such cases we should assume that position of the teeth for the determination of ζ at which the angle α of the normal with the line of centres has an average value. The variations of ζ will only be small, however, and can well be neglected as compared with the uncertainty which clings to all co-efficients of friction.

So far the investigation has been carried out on a basis of the equal transmission of power by two pairs of teeth. If there is contact between only one pair, which occurs when the arc within which the teeth mesh is less than t, the friction is somewhat less. In explanation let us suppose the teeth in Fig. 44, plate V., to be limited by the circles a_2a_3 and b_1b_3; then contact would exist between only one pair at a time, beginning in a_1b_1 at the moment it ceased at a_2b_2. At that instant the tooth of the wheel A would act at the point a_1 with a force R in the direction a_1C upon the wheel B. As long as motion continued the force R would remain parallel to a_1C until the point of contact between the teeth came into the line of centres at O, at which instant there would be no friction. In this case, as in all preceding ones, the arm of the power is diminished by an amount ζ, and that of the load increased by the same amount. Here we understand by ζ the perpendicular distance of the point O from the line a_1C; that is,

$$\zeta = a_1 O. \tan \phi = \mu e_1$$

if the distance $a_1O = e$. Since this value of ζ grows less and less, becoming equal to zero when the point of contact is at O, we see that the friction has its maximum value when contact occurs at a_1b_1, and its minimum when at O. If we wish a mean value we have such an one at a point midway between O and a_1 in

$$\zeta_1 = \tfrac{1}{2}e_1\mu.$$

In the same way for motion from O to a_2 we find the friction starting at zero, and reaching its maximum value at a_2, where

$$\zeta = Oa_2 \tan\phi = \mu e_2$$

if $e_2 = Oa_2$. For this also we have the mean value

$$\zeta_2 = \tfrac{1}{2}\mu e_2.$$

There is this difference, however, in that after the point O is passed the pressure acts along the line Ca_2. If we suppose $a_1O = Oa_2$, and consequently $e_1 = e_2 = \frac{1}{2}t\sin\alpha$, we get for an average value of ζ the equation

$$\zeta = \tfrac{1}{2}\mu e_1 = \tfrac{1}{4}\mu\sin\alpha.$$

This value of ζ is only half as large as that

$$\zeta = \tfrac{1}{2}\mu t\sin\alpha$$

obtained for the preceding case, where two pairs were in contact; and, further, the arc of contact was twice as great, $= 2t$, in the first case discussed as here where it equals t. If, therefore, we denote the entire arc of contact on both sides of the line of centres by ω we have the general equation for both cases

$$\zeta = \tfrac{1}{4}\mu\omega \sin \alpha.$$

This equation holds for all intermediate values of the arc of contact between t and $2t$, when the transmission of force occurs part of the time through one pair and part through two pairs of teeth.

We can then find the value of ζ in every case by the following simple construction: Lay off on any straight line the distance $OA = \frac{1}{4}\omega$ (Fig. 45, plate V.) where ω is the length of the arc of contact between the wheels; draw the line OB at an angle α with the normal OO_1 to OA, α being 75 degrees for involute teeth, and 80 degrees for cycloidal profiles; then draw AB perpendicular to OB, and lay off the angle OBD equal to the angle of friction ϕ. We then have in the perpendicular OD to BO the value of ζ. We may assume μ between 0.1 and 0.12. By making $OA = \frac{1}{4}\omega = \frac{1}{2}t$ we get the same value for the friction that the formula $\mu\pi\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$ gives.

Having gotten the value of ζ the tooth friction is readily determined. Suppose the load Q (Fig. 46, plate V.) to act upon the drum AB by means of a rope at B; it is then required to find the force P which must be applied in the direction EF to the crank ED in order to turn AB by means of the gear-wheel AC and the pinion CD, and lift the load. First find the direction in which Q acts, which is, of course, along the line o_1b drawn at a distance σ from the mean circumference of the drum AB. The line of pressure Z between the teeth is given by the line o_1c drawn at an angle of 75 degrees or 80 degrees to the line of centres AD, and intersecting the latter at a distance ζ from the point of contact C of the pitch circles. Then draw through o_1, the intersection of Z and Q, the line o_1a tangent to the friction circle at A, and giving the direction of re-action at that journal. In a similar way we get the re-action do_2 of the journal D. The force polygon can now be drawn by making $o_1I = Q$, and drawing $I\,II$ parallel to Z, then resolving $I\,II = Z$ in the direction of P and o_2d. $I\,III$ gives us the value of the force P which must be applied at the crank. To determine the theoretic force P_0 we have only to draw the broken lines, as shown, through the centres of the journals, and perpendicular to AD through C. The drawing gives, for $\mu = 0.1$, and $Q = 100$,

$$P = 30.4, \quad P_0 = 28.1;$$

and therefore

$$\eta = \frac{P_0}{P} = 0.924.$$

It will be readily seen that if in Fig. 44 the driving was done by the wheel B, but in such a manner that the points of contact remain at a_1b_1 and a_2b_2 (that is, if the motion was in an opposite direction to that shown by the arrows), the direction of pressure between the teeth would remain parallel to GH; but the force Z would lie on the opposite side of this line, and would pass through C_1 between GH and the axis of A, at a distance ζ from the former. This case corresponds to the backward motion of the windlass (Fig. 46) under the influence of Q. When, however, the wheel B (Fig. 44) is driven in the opposite direction to the arrows by the wheel A the investigation is similar in all respects, except that the direction of pressure is now along the line $H'G'$.

§ 9.—BELT GEARING.

The principal source of loss in the transmission of rotary motion from one axis to another by means of belting is the friction which the axes suffer from being pressed against their bearings by the tension in the belt, since the resistance due to stiffness where the belts wind on and off the pulleys is so small that it cannot be taken into consideration. But, on the contrary, the journal friction is much greater for the transmission of a given force than by toothed gearing, since in belt gearing only the difference of tensions in the two portions of the belt represents force transmitted, while journal friction is caused by the sum of these two. The investigation of this resistance, with the determination of the tension in the belt, is pursued in the following manner: —

Let the shaft A (Fig. 47, plate V.) be driven from the shaft B by means of the belt and pulleys CD and EF. We are to find the force P which must act on B with the lever-arm BK to overcome the resistance Q acting upon A with the lever-arm AL. If the belt surrounding the two pulleys is stretched to a certain tension this tension S is the same in both portions of the belt DE and CF while at rest. If, then, the pulley EF is acted upon by a force tending to revolve it in the direction shown by the arrow, the tension in the belt CF increases to a value S_1, and at the same time that in DE diminishes to a value S_2, until

$$S_1 - S_2$$

is sufficient to overcome the resistance Q at the axis A. It being supposed that no slipping of the belt upon the pulleys occurs, the condition for such slipping is given by the formula

$$S_1 = S_2 e^{\mu\alpha},$$

where S_1 is the tension in CF, S_2 that in DE, e is the base 2.71828 of the natural system of logarithms, μ is the co-efficient of friction between belt and pulley, and α the arc of contact between the same, the radius being unity. The greatest resistance W, therefore, which can be overcome with a tension S_2 of the belt ED, when it (the resistance W) acts with a lever-arm AC from the axis A is

$$W = S_1 - S_2 = S_2(e^{\mu\alpha} - 1).$$

The tension S in the belt when at rest must be determined according to this relation, and is usually assumed

$$S = \tfrac{1}{2}(S_1 + S_2).$$

If we then suppose the two tensions S_1 and S_2 to have a resultant Z we can draw the force polygon, as in previous cases, by substituting this resultant for the tension in the two portions of the belt. The direction of the resultant is obtained by determining the proportion of the two tensions for the limiting condition of slipping

$$\frac{S_1}{S_2} = e^{\mu\alpha};$$

so that by laying off from the intersection O of the belts the distances OG and OH of any convenient lengths, but in the ratio

$$\frac{OH}{OG} = \frac{S_1}{S_2} = e^{\mu\alpha},$$

we have in the diagonal OJ of the completed parallelogram the resultant Z which may be substituted for the belt tensions themselves. To determine the value of this resultant, and of the belt tensions S_1 and S_2, draw through o_1 and o_2, the intersections of the resultant with P and Q, the tangents o_1a and o_2b to the friction circles of the journals A and B. Then make $oI = Q$, and by drawing $I\,II$ parallel to o_1a we get the resultant

$$Z = o_1 II$$

of the belt tensions, and those tensions S_1 and S_2, by resolving o_1II in the directions o_1IV parallel to CF, and $II\,IV$ parallel to DE. From $o_1II = Z$ we get also the value of P by drawing o_1III parallel to o_2K, and $II\,III$ parallel to o_2b. In order to find the theoretic force P_0 which would be sufficient to overcome Q in the absence of friction we can either employ the direction OJ of the resultant Z, and draw the re-actions o_1A and o_2B through the centres of the journal, or take the intersections O_1 and O_2 of the forces Q and P with CF, regarding the latter as a force acting at C and F without friction.

In Fig. 48, plate VI., the logarithmic spirals are drawn for the commonly occurring co-efficients of belt friction

$$\mu = 0.12,\ 0.18,\ 0.28,\ 0.38, \text{ and } 0.47,$$

in order to give a graphic representation of the ratio of belt tensions

$$\frac{S_1}{S_2} = e^{\mu\alpha}.$$

If we take the radius of the circle OA as unity, the radius vector OB drawn to the spiral, constructed with the particular co-efficient of friction μ, at the angle $AOC = \alpha$, the angle of contact, gives the value $e^{\mu\alpha}$. Therefore, if the tension S_2 of the slack side is represented by OA, we have in OB the tension S_1 of the driving side, and in

$$CB = S_1 - S_2$$

the force transmitted.

From the preceding it is easy to draw a comparison between the relative efficiencies of belt and toothed gearing. For this purpose we shall investigate the motion of a millstone, since these are as frequently driven by belts as by gearing. Let A (Fig. 49, plate VI.) be the mill-spindle, and BD the pulley on the same, which has the arc BMD in contact with the driving-belt. With the aid of the spiral for $\mu = 0.28$, as given in Fig. 48, plate VI., we find the ratio

$$S_2 : S_1$$

from that of $OA : OB$, having laid off the angle of contact

$$BAD = \alpha$$

from the radius OA. Laying off these distances in KL and OK (Fig. 49) we have in OL the direction of the resultant Z of the two belt tensions which act upon the mill-spindle A as driving-force P. Since the resistance of the millstone upon its grinding-surface is exactly equal to this we must represent this resistance by a couple of forces Q and Q acting in the directions HJ and FG. This couple will not cause any side pressure of the mill-spindle against its bearing; the re-action of the bearing will be that called forth by the resultant Z, and will therefore be equal to that resultant, and opposite in direction. We then draw R in the tangent ao_2 of the friction circle at A parallel to OL. For the existence of equilibrium between the four forces Q, Q, R, and Z, the resultant of any two must be coincident with and opposite to that of the remaining two. Uniting o_1 and o_2, making $o_1 I = Q$, and drawing $I\ II$ parallel to $o_1 o_2$, we have in $o_1 II$ the resultant of the two tensions S_1 and S_2; and by resolving $o_1 II$ into $IIIo_1$, and $II\ III$ parallel to the directions of the belt, we get

$$IIIo_1 = S_1 \quad \text{and} \quad II\ III = S_2.$$

Without friction we should assume the direction of re-action at the bearing parallel to OL, and passing through the centre of A along the line $A O_2$; then drawing $I\ II_0$ through I parallel to $o_1 O_2$, we should get

$$P_0 = II_0 o_1.$$

From the figure, with the assumed value $\mu = 0.28$, and a journal friction of 0.1, we get, for $Q = 100$,

$$P = 401.6, \quad P_0 = 379.2,$$

and

$$\eta = \frac{P_0}{P} = 0.944.$$

If, on the other hand, we suppose the stone to be driven by gearing from the upright shaft B (Fig. 50, plate VI.), A again being the mill-spindle, we have the pressure transmitted by the gearing along the line o_1c inclined at the angle of 75 degrees to the line of centres, at the distance ζ from C, and in the parallel line o_2a the direction of re-action at the bearing of the spindle. Making $Io_1 = Q$, and drawing $I\,II$ parallel to o_1o_2, we find P in o_1II. To determine P_0 we assume the thrust of the gearing along the perpendicular to AC passing through C, and the re-action of the bearing parallel to this and passing through A. Then, if $I_0O_1 = Q$, we have in O_1II_0 the theoretic force P_0 acting perpendicular to AC through C. In order to determine the efficiency we must compare, not the force P acting along o_1c, but that component P' parallel to P_0, with the latter force. Let ω be the point of intersection of the two directions, then we must resolve

$$o_1II = P$$

in the direction of O_1C and ωB. Draw o_1III parallel with O_1C, and $II\ III$ parallel to ωB, and we have in

$o_1 III$ the force P' which must act at C perpendicular to the line of centres AB, thus giving the efficiency

$$\eta = \frac{P_0}{P'}.$$

From the drawing we get, for $Q = 100$,

$$P' = o_1 III = 226.4, \quad P_0 = 217.8,$$

and

$$\eta = \frac{P_0}{P'} = 0.962.$$

From these two examples we draw the conclusion that belt gearing is less economical than toothed gearing, as would have been expected from previous considerations. It is also evident that the result for belt transmission would be more unfavorable if the coefficient of friction of belt upon pulley is less than 0.28, or if the arc of contact between the belt and the pulley of the mill-spindle subtends a less angle α, since the belt tension and resulting journal friction would then be much greater. In the foregoing comparison the journal friction of the main shaft is omitted in both cases, because in the general arrangement of several stones about one central shaft the opposing tensions or pressures would counteract one another, and the shaft would run freely in its bearings. If this were not the case, but only one stone was driven from the shaft, then, on account of the greater belt tension, the friction of the shaft in its bearing would be greater than in the case of toothed gearing.

The tensions in brake bands are to be estimated in the same way as with belt gearing. Here also there are two different tensions S_1 and S_2 in the two ends of the band which have the relation

$$S_1 = S_2 e^{\mu\alpha}$$

one to the other, the greater tension S_1 being that which opposes the sliding of the brake pulley within the band. We can therefore introduce the resultant Z of the two tensions S_1 and S_2, and regard it as a force preventing the motion which tends to occur. Let A (Fig. 51, plate VI.) be the axis of a drum on which is the brake pulley BC, whose brake band is fastened at one end to the stationary bolt E, and at the other to the bolt D of the brake-lever EDG which turns about E. We get the ratio of S_1 to S_2 from the spiral in Fig. 48, which corresponds to the co-efficient of friction for brake bands

$$\mu = 0.18,$$

by making the angle $\alpha = CMB$ the arc of contact, and lay off the distances so determined along the direction of the ends of the band from their intersection O in OH and OJ. The diagonal OK of the completed parallelogram gives the direction of the resultant Z. As regards the direction of tension of the brake band we see that the end CE fastened at E pulls in a line passing through the centre of E, since there is no relative turning of the band about this point. But the line of tension in the end DB attached to the moving journal D would be tangential to the friction circle at D.

If, now, a force Q acting with a lever-arm AF tends to turn the shaft A in the direction of the arrow the direction of journal pressure at A will be given by the line o_1a drawn through the intersection o_1 of Q and Z tangent to the friction circle at A. Therefore, by making $o_1I = Q$, and drawing through I a parallel to OK, we get in $I\,II$ the necessary resultant Z of S_1 and S_2. Resolving this parallel to HO and JO, we have

$$III\,II = S_1 \quad \text{and} \quad I\,III = S_2.$$

To determine the force P applied to the brake-lever G to produce the tension

$$S_2 = I\,III$$

in the band BD we first draw from o_2, the intersection of S_2 and P, the line o_2e tangent to the friction circle at E to get the re-action of that bearing; then a resolution of $I\,III = S_2$ into $I\,IV$ parallel to o_2e, and $IV\,III$ parallel to o_2G or P, gives us in $IV\,III$ the force P which must be applied at G.

The determination of the driving and brake forces in a whim or windlass, such as is shown in Fig. 52, plate VI., is of especial interest. Here two ropes FD and EC lead from the windlass-drum G in such manner, that, if the drum is revolved in either direction (say, that indicated by the arrow), the rope EC is wound up, and that BD unwound; and, in consequence, the weight Q consisting of useful load, and the weight G of cage, hooks, etc., hanging upon the rope ECB, is lifted, while the weight G of the empty cage hanging on the rope FDB sinks, and thus aids the revolution of the drum. If there were no hinderances to this motion we should at once assume that the working of the entire apparatus amounted simply to lifting the useful load Q, since the cages balance one another and could be left out of consideration. Such an assumption is, however, not permissible on account of friction; and we must regard the machine as a combination of two hoisting-gears, one of which burdened with the load $Q + G$ on the rope ECB is in forward motion, while the other is running backwards under the influence of the load G attached to the rope FDB. Accordingly the investigation would be pursued as follows: The line of tension in the ascending rope BCE must be assumed on account of stiffness in the rope along the lines $b_1 o_1$ and ce in such manner that the lever-arms at B and E shall be greater than the radii of the pulley and drum respectively by an amount σ, and at C less by an equal amount. On the other hand, the directions of tension for the rope FDB are given in fd and $o_2 b_2$; the lever-arms at D being increased, and those at F and B diminished, by an amount σ. The direction of re-action at the bearing of the pulley ACB which revolves toward the left would be given in the line $a_1 o_1$, while that of ADB revolving in the opposite direction would be along $o_2 a_2$. Making $o_1 I = Q + G$, and $o_2 III = G$, and drawing through I a parallel $I\ II$ to ce, and through III a parallel $III\ IV$ to df, we have in

$$II\ I = S_1$$

the tension in the rope EC, and in

$$IV\ III = S_2$$

the tension in FD. The resultant Z of these tensions is given by $V\,I$ if we draw $II\,V$ equal and parallel to $III\,IV$, and complete the triangle. This resultant passes through the intersection o_3 of ec and fd, and we must therefore draw o_3o_4 parallel to $I\,V$.

If the drum is driven by a pinion HJ meshing with the gear-wheel GJ we draw the line ii inclined at an angle of 75 degrees to GH, and at a distance ζ from J, for the direction of pressure between the teeth, and then draw from o_4, the intersection of this line with that of Z, the tangent o_4g to the friction circle at G. This gives us the direction of re-action at the bearings of G. Then resolve the force $I\,V$ into $I\,VI$ parallel to o_4g, and $VI\,V$ parallel to o_4i, and we have in $VI\,V$ the pressure on the teeth of the pinion HJ, or the necessary driving-force P.

To determine the theoretic force P_0 we have, as previously remarked, only to lay off a distance equal to the useful load Q along the medial line of the rope EC from the intersection O of that line with the common tangent JO to the pitch circles, so that $OI_0 = Q$. Then draw through O the radius OG, and through I_0 the line I_0II_0 parallel to the latter, and we have

$$II_0O = P_0.$$

To determine the efficiency we must again compare P_0 with that component of P obtained by resolving

$$VI\,V = P$$

in the direction of P_0 and ωH. This force P' is given by $VI\,VII$ if $VI\,VII$ is drawn parallel to OJ and $V\,VII$

parallel to ωH. From the figure we get, for $Q = 36$ and $G = 24$,

$$P' = VI\,VII = 47.3, \quad P_0 = 40.1,$$

and therefore

$$\eta = \frac{P_0}{P'} = 0.848.$$

For convenience, in order to use the same lines in the figure, we will suppose in determining the brake force that the loaded cage is now upon the rope FDB. Accordingly we lay off the broken and dotted lines $o_2(III)$ equal to $(Q + G)$, and $o_1(I)$ equal to G; draw $(III)(IV)$ parallel to fd, and $(I)(II)$ parallel to ce; we then have

$$(IV)(III) = (S_2) \quad \text{and} \quad (I)(II) = (S_1).$$

And by drawing $o_3(V)$ equal to $(III)(IV)$, and making $(V)(VI)$ parallel and equal to $(I)(II)$, we get in $(VI)o_3$ the resultant (Z) of (S_1) and (S_2). If the brake is applied by means of the brake-lever $NLKM$ turning on the fixed point K, and attached to the ends of the brake band at N and L, we first draw the lines of force W_1n and W_2l tangential to the brake pulley W_1W_2 and to the friction circles of the bolts N and L. From the intersection o_5 lay off the distances o_5U and o_5T proportional to the tensions s_2 and s_1 in the brake band, which have been determined from Fig. 48. The diagonal of the completed parallelogram then gives the resultant z of these tensions. If we now draw from o_6, the intersection of z and (Z), the tangent o_6y to the friction circle of G, and resolve the force $(Z) = (VI)o_3$

parallel to o_5o_6 and o_6g, we get in $(VII)(VI)$ the value of the resultant z which may be substituted for the two tensions s_1 and s_2 in the brake band. To determine the force p to be applied to the brake-lever at M draw from o_7, the intersection of this force with o_6o_5, the tangent o_7k to the friction circle of K, and resolve the force $(VII)(VI) = z$ in the direction of o_7k and Mo_7, thus getting

$$(VIII)\ (VI) = p,$$

the force to be applied at the brake-lever.

It has been heretofore tacitly assumed that the driving-force P has been applied in just sufficient amount to overcome the resistance of Q. This condition would be fulfilled, for instance, if a water-wheel acted upon by equal impulses at equal moments drove a millstone whose working resistance is constant. It is also fulfilled in most hoisting apparatus. But in many cases in practice either the moment of the power, or the moment of the load, or both, vary periodically; but in such a way, if continuous motion is supposed, that for any such period the work done by the driving-force exactly equals that overcome in the shape of useful and hurtful resistances. This is always the case in the slider-crank motion. Thus in the ordinary steam-engine the steam pressure transmitted through the piston-rod and connecting-rod to the crank-pin has an extremely variable moment, which is reduced to zero at the dead points, and reaches a maximum value somewhere between these two. On the contrary, the resistance Q which probably acts at the circumference of some pulley or wheel keyed upon the shaft has a constant value. It is therefore clear that the moment of the resistance

Q must have a mean value between the greatest and least moments of the power if the motion is to be continuous. Consequently there will be, on account of the periodical variation of the moments, an alternate acceleration and retardation of the moving masses of the machine in such way that, during the time when the moment of the power exceeds that of the resistance, the excess of work done by the former is stored up in the moving parts in the form of living force, from which it is again given out when the moment of the power sinks below that of the resistance. A similar state of things exists when, on the contrary, a resistance acts upon a crank with varying moment, while that of the power is constant, as when pumps are driven by water-wheels.

The manner in which the motion of such a mechanism would be investigated may be learned from the example of a jig-saw (Fig. 53, plate VI.). Here the shaft A which gives the saw-frame EF a reciprocating motion through the crank AC and the connecting-rod CD is driven by a belt on the pulley B_1B_2; it also has the fly-wheel U keyed on to it, which tends to render the motion uniform, as shown in the preceding paragraph. Let the vertical resistance which the teeth Z of the saw encounter from the log K on the downward stroke be represented by Q, and laid off in o_1I. The connecting-rod acts upon the grate in the direction dc of the common tangent to the friction circles of D and C; and, furthermore, the bearings G_1 and H_1 of the grate act upon the guide GH at the angle of friction from the normal with the re-actions R_1 and R_2. Joining o_1, the intersection of Q and R_1, with o_2, the intersection of R_2 and T, we get the various forces by drawing o_1III

parallel to dc, and $I\,III$ parallel to R_1. $II\,I$ then equals R_1, $II\,III$ R_2, and $o_1 III$ the necessary force T in the connecting-rod. Next, suppose the belt which half encircles the pulley $B_1 B_2$, and whose ends are therefore parallel, to have in the slack side the tension

$$S_2 = B_2 L_2,$$

and in the other the tension

$$S_1 = B_1 L_1,$$

these tensions S_1 and S_2 having been determined by the aid of Fig. 48. The resultant Z then equals

$$BL = S_1 + S_2 = B_1 L_1 + B_2 L_2;$$

and its position is determined, according to the laws of parallel forces, by making

$$B_2 l_2 = B_1 L_1 \quad \text{and} \quad B_1 l_1 = B_2 L_2,$$

and drawing $l_1 l_2$, when the intersection l of this line with $B_1 B_2$ gives the point of application of the resultant Z. Now lay off from o_1 of the force polygon the line $o_1 IV$ parallel and equal to BL, and we have in the line $IV\,III$ the resultant of the belt tension Z, and the pull T of the connecting-rod, which together act upon the shaft A. For equilibrium there must be a re-action R of the bearing upon A, whose direction and magnitude is given in $III\,IV$. The position of this reaction R is then given by the tangent aa to the friction circle of A parallel to $III\,IV$. This line aa cuts the force T in O, a different point than that B, the intersection of Z and T, from which we see that the three forces Z, T, and R cannot be in equilibrium; for in that case the force Z must act along OJ instead of BL. The moment of the belt tension Z is here less than that of the resistance, and therefore at this moment living force must be given out by the fly-wheel to render motion possible. On account of the equable distribution of the masses of the fly-wheel about its axis we are not to regard its action in the light of that of a single force, but as a couple whose forces M, M are applied at any two diametrically opposite points m_1 and m_2 of the circumference, whose radius $Am_1 = Am_2$ is equal to the radius of gyration ρ of the fly-wheel. The value of the forces M, M is determined by the condition that the resultant of the couple and the belt tension Z or BL must be a force passing through O, parallel to BL, and equal in value to

$$Z = OJ = BL.$$

To construct this value of M we join o_3, the intersection of BL and m_1M, with o_4, that of OJ and m_2M, and then resolve the force $Z = BL$ in the direction of o_3o_4 and of M. We therefore draw through IV in the force polygon a parallel to o_3o_4, and through o_1 a parallel to M (M being here assumed as acting vertically, parallel to Q), and have in o_1V the value of the two forces of inertia M, which, applied at m_1 and m_2 in the fly-wheel, aid the motion of the shaft in the moment under consideration. It will be readily seen that the same values of M would be obtained in whatever direction they were supposed to act, if always their distance apart $m_1 m_2 = 2\rho$. But if we assume this distance to be of another value $2\rho_1$ we get a different value M_1 for the forces of inertia, so that the moment of inertia is always constant,

$$2M\rho = 2M_1\rho_1.$$

From the preceding it follows that the action of moving masses may be represented by couples when they are not arranged eccentrically, and that the influence of such a couple is to produce a parallel shifting of the driving-force Z or P without any consequent increase of journal pressure and friction. It is also evident that the couple acts in the opposite direction, opposed to the driving-force, if the moment of the latter becomes greater than that of the resistance. This would be the case if in the present example the arm Al of the belt driving-force was greater than AN. In that case the action of the couple would result from an acceleration of the motion; while under the condition investigated above, where the fly-wheel acts in the "sense," or direction, of the desired motion, there is a retardation of the masses. It is well known that acceleration and retardation occur in regular periods of continuous but non-uniform motion. The amount of these momentary accelerations and retardations in the masses can be readily ascertained for every case by this method of couples, but as such an investigation lies without the object of this work we will not pursue it farther.

Two examples corresponding to commonly occurring machines will serve, in closing, for the further discussion of those conditions more fully investigated in the preceding chapters, which are principally influential in determining the efficiency of machines. These examples are the crane in plate VII., and the beam-engine in plate VIII.

§ 10.—EXAMPLES.

In the crane (Figs. 54 to 58, plate VII.) the revolving jib or outrigger turning about the post or mast L has at its outer end the guide pulley B (Fig. 57), over which the hoisting-chain is led; one end of the latter is fastened at C, while in its loop hangs the pulley A supporting the load Q. From B the chain is led over the supporting pulleys D and D_1 to the drum EE_1, which receives motion from the crank HK (Fig. 54) by means of the gearing at F and J. Drawing the direction of the forces according to well-known laws we have the load Q acting along the tangent aa (Fig. 55) to the friction circle of the journal A of the loose pulley, while the tension S_1 of the chain fastened at C is along the line a_1a_1, and the tension S_2 of the portion A_2B_1 of the chain acts along the vertical a_2a_2, so that the lever-arm at A_1 is lengthened, and that at A_2 shortened, by an amount χ. If, then, $I\,II = Q$, and $I\alpha_1$ is drawn parallel to $II\alpha_2$, the line $\alpha_1\alpha_2$ joining these points gives, by its intersection III with Q, the tension

$$II\,III = S_1$$

in A_1C, and

$$III\,I = S_2$$

in A_2B_1. Drawing in the well-known way the direction of forces b_1o_1 and b_2o_1 in the chain over the pulley B

we get the re-action of its journal in the tangent o_1b to the friction circle at B. A resolution of the chain tension

$$I\,III = S_2$$

in the direction of o_1b and b_2o_1 gives us in

$$IV\,III = S_3$$

the tension in the portion BD of the chain. The direction of the portion of the chain passing over the pulley D is shown (Fig. 56), which is drawn to a larger scale. From this we see, without further explanation, that the journal re-action of this pulley * is in the direction do_2. We therefore draw through the point III in the force polygon the parallel $III\,V$ to do_2, and from IV the parallel $IV\,V$ to e_1o_2, and get

$$IV\,V = S_4,$$

the tension in the portion of the chain between D and the drum E_1, which has such a direction e_1d_1 (Fig. 54) as to increase the lever-arm at E_1, and decrease it at D, by an amount χ. Draw now at the distance ζ from F, the point of contact of the pitch circles, the direction of pressure fo_3 between the first pair of gear-wheels at 75 degrees to the line of centres EG, and also the tangent ϵ_1e to the friction circle at E,

* The slight resistance of the pulley D_1 is neglected. If it was desired to bring it into the calculation it could be done as shown in Fig. 36, plate IV.

so that it shall pass through the point of intersection of d_1e_1 and fo_3; and we can then resolve

$$IV\,V = S_4$$

parallel to fo_3, and $\epsilon_1 e$ into

$$IV\,VI = Z_1,$$

the pressure between the teeth at F, and $V\,VI$ equals the re-action of the journal bearings of the drum. It should be remarked here that the direction $e\epsilon_1$ of the journal re-action is drawn by the aid of the auxiliary construction previously given, since the intersection of fo_3 and d_1e_1 lies beyond the limits of the drawing. We draw through E a line $\beta\epsilon\delta$, and any parallel line $\beta_1\delta_1$; then locating ϵ_1, so that

$$\beta\epsilon : \epsilon\delta :: \beta_1\epsilon_1 : \epsilon_1\delta_1,$$

we have in ϵ_1 a point in the desired line of re-action. ϵ is here the point of intersection of the line $\beta\delta$ with the friction circle, this being a sufficiently close approximation if the line $\beta\epsilon$ is drawn perpendicular to what is judged to be about the direction of the re-action $\epsilon_1 e$.

In the same way we draw the line of pressure $o_3 i$ between the teeth of the second pair of gear-wheels HJ and GJ, and from o_3 the tangent to the friction circle of G. Then a resolution of

$$VI\,IV = Z_1$$

in the direction of $o_3 i$ and $o_3 g$ gives us in

$$VI\,VII = Z_2$$

the pressure with which the pinion HJ acts upon the wheel GJ along the line $o_3 i$. Finally, draw from o_4, the intersection of P and Z_2, the tangent $o_4 h$ to the friction circle of H, and through VII a parallel to P, and through VI a parallel to $o_4 h$, and we have

$$VIII\,VII = P,$$

the force which must be applied at the crank K in the direction Ko_4. Since this value becomes very small in the figure, the triangle $VI\,VII\,VIII$ is drawn on a scale five times greater in $VI\,VII'\,VIII'$, in which

$$VI\,VII' = 5\,.\,VI\,VII,$$

according to which

$$P = \tfrac{1}{5}\,.\,VIII'\,VII'.$$

To determine the theoretic force P_0 we can suppose the tension in the chain as it winds on to the drum at E_1 equal to $\frac{Q}{2}$, and therefore make $VIV_0 = \frac{1}{2} III$. Then draw, as before, the force polygon $IV_0\,V\,VI_0\,VII_0\,VIII_0$, taking the pressure between the gear-wheels perpendicular to the line of centres at F and J, and the journal re-actions as passing through the centres of the journals at E, G, and H. Thus is constructed the polygon shown in broken lines, which gives

$$P_0 = VIII_0 VII_0 = \tfrac{1}{5} \cdot VIII_0' VII_0'.$$

The drawing gives, for $Q = 100$,

$$P = 1.155, \quad P_0 = 0.833,$$

and therefore

$$\eta = \frac{P_0}{P} = 0.722.$$

To determine the brake force p which is to be applied at W in the lever WU we must take the backward motion of the crane into consideration. In this case it is evident that the load $Q = I\,II$ hanging upon the loose pulley causes, in sinking, a tension S_1 of the chain fastened at C equal to $III\,I$, and a tension

$$S_2 = III\,II$$

of the portion between A_2 and B_1; in other words, the strains in the two chains are reversed by backward motion. It will also be readily understood that the lines of tension in the chain at the pulleys B and D and at the drum will be those shown in broken and dotted lines; that is, $(o_1)(b_1)$ and $(o_1)(b_2)$ at B, $(o_2)(b_2)$ and $(o_2)(e_1)$ at D, and $(e_1)(d_1)$ at the drum, while the journal pressures at B and D take the direction $(o_1)(b)$ and $(o_2)(d)$. The direction in which the pressure (Z_1) of the teeth of the wheel EF now acts upon the pinion GF is given by the line $(f)(f)$, which cuts the line of centres EG at 75 degrees, and at a distance ζ from the point of contact F of the pitch circles toward the side of the driven axis G, as was shown formerly in the discussion of tooth friction. The direction of journal re-action at E corresponding to $(f)(f)$ would vary but little from that, $\epsilon\epsilon_1$, found for forward motion; it may therefore be assumed as the same. Taking these things into account the force polygon would be drawn as follows: Make

$$(I)(III) = III\,II = (S_2);$$

draw $(I)(IV)$ parallel to $(o_1)(b)$, and $(III)(IV)$ parallel to $(o_1)(b_2)$, also $(IV)(V)$ parallel to $(o_2)(e_1)$ and $(III)(V)$ parallel to $(o_2)(d)$; then resolve

$$(IV)(V) = (S_4)$$

in the direction of $(f)(f)$, and $\epsilon_1\epsilon$ into the forces $(IV)(VI)$ and $(VI)(V)$.

One end of the band surrounding the brake pulley Br is fastened at U; the direction of tension therefore passes through the centre of U. The tension in the other, which is attached to the bolt V, is along a line tangent to the friction circle at V. The two intersect at (o_4). Making the distances $(o_4)V_2$ and $(o_4)V_1$ proportional to the tension

$$s_2 \quad \text{and} \quad s_1 = s_2{}^{\mu a}$$

gotten from Fig. 48, plate VI., we have in the diagonal $(o_4)V_3$ the direction in which the brake acts. This line cuts that of the pressure between the teeth in (o_3), and therefore the tangent $(o_3)\,(g)$ to the friction circle at G gives the re-action of the bearing at G. Now draw through the point (VI) of the force polygon a parallel to $(o_3)\,(o_4)$, and through (IV) a parallel to $(o_3)\,(g)$, and we have

$$(VI)(VII) = z,$$

the pressure of the brake, which may be resolved into $(VI)(VIII)$ and $(VII)(VIII)$ parallel to s_1 and s_2. The greater tension s_1 is from the fixed point U, the less s_2 is produced by the brake-lever. Drawing the tangent $(o_5)u$ from the intersection (o_5) of s_2 and p to the friction circle of U we resolve the force

$$s_2 = (VIII)(VII)$$

in the direction of p and $(o_5)u$. For the sake of clearness the distance $(VII)\,(VIII)$ is laid off ten times greater in $(VII)(X)$, so that the necessary brake force

$$p = \tfrac{1}{10} \cdot (IX)(VII).$$

Figs. 57 and 58 show the mechanism for turning the crane. The swing-arm of the crane, together with the load Q, are supported by the pivot L in the head of the mast or post. This pivot has a side thrust upon it from the bearing L_1, while the two friction rollers W_1 and W_2 press upon the base T of the post. The pressures at L and T are determined from Fig. 57 as follows: Suppose S to be the centre of gravity of all the swinging portion of the crane, windlass, pulleys, chain, etc., and suppose G the weight of all these parts acting downwards at S. This force may be combined with the weight Q hanging at A into one resultant force,

$$Q + G,$$

acting at M. Now lay off on this line the distance

$$I\,II = Q + G,$$

and draw the horizontal re-action R of the cylinder T through the middle of the rollers W. The re-action R_1 of the pivot L must then pass through the intersection of R and $Q + G$, and must take the direction OL drawn through the centre of the bearing L_1. To determine R and R_1 we therefore resolve $I\,II$ in the direction of OT and OL, and have

$$III\,I = R,$$

the re-action against the rollers W, and

$$II\,III = R_1,$$

the combined re-actions at L. This latter may be resolved back into horizontal and vertical components, $I\,III$ and $II\,I$. The vertical re-action equal to $G + Q$ produces end journal friction upon L, which may be regarded as the action of a couple, each force of which equals

$$\tfrac{1}{2}\mu(Q + G),$$

and whose arm equals $\frac{2}{3}d$, if d is the diameter of the journal. The horizontal component of the re-action R_1

is equal and opposite to the re-action R of the post against the rollers in the direction TO, so that these two forces form a couple in equilibrium with that formed by the vertical re-action L, and the weight $Q + G$ at M. To determine the turning-force p we draw, in Fig. 58, the horizontal pressure 1 2 equal to $I\,III$ in Fig. 57, and in such direction that it shall be tangent to the friction circle at L. Also lay off the two end journal frictions

$$F = F_1 = \tfrac{1}{2}\mu(Q + G)$$

at the distance $\frac{1}{3}d$ on each side the centre of L, and in opposite directions, so as to bring this end journal friction into the calculation. The compounding of this couple F, F_1 with the horizontal re-action 1 2 will simply result in a parallel shifting of that re-action undiminished in value to the position 4 5. To fix the amount of this shifting make $2\,3 = F$, draw 3 1, and through o_1, the intersection of 1 2 and F, draw a parallel to 3 1; this will cut F_1 at a point o_2, through which the resultant 4 5 of 1 2 and FF_1 must pass. The correctness of this construction is shown by the equilibrium of the four forces 1 2, F, F_1, and 5 4. Now draw the re-actions u_1w_1 and u_2w_2 of the post against the friction rollers in such way that they shall be tangent to the friction circles of W_1 and W_2, and shall pass to one side of the contact points U_1 and U_2 of the rollers at the distance ϵ determined previously for rolling friction. If the turning of the crane is produced by a pinion on the vertical shaft V, and working with a circular rack or internal gear Y on the base plate, we draw finally the direction vv of pressure between the teeth at 75 degrees to the line of centres VL, and at the distance ζ on the outer side of the contact point of the pitch circles. This line intersects the resultant of journal pressure 4 5 in o_3, the two re-actions of the post against the rollers intersect in o_4; therefore $o_3 o_4$ must be the common resultant of these pairs of forces. Draw 5 6 parallel to vv, and through 4 a parallel to $o_4 o_3$, and we have

$$5\,6 = Z,$$

the pressure which the teeth of the pinion on the shaft V must exert along the line vv. The method by which the force necessary to turn V by means of a crank would be determined is sufficiently well known from previous examples.

In conclusion we will explain the diagram for the condensing beam-engine shown in plate VIII. Let $I\,II = P$, the force acting upon the piston (Fig. 59). This force acts along KC, the geometric axis of the piston-rod, since the latter is rigidly fixed to both the piston and the cross-bar C, and there can be no relative motion between those elements. If, then, we consider the piston, piston-rod, and cross-bar C to form one piece, we have at the point C three forces, P the piston force, and S_1 and S_2 exerted by the link BC and the parallel rod EC respectively. In the present position of the engine these forces are in compression, and act along the tangents bc_1 and e_1c_2 shown on a larger scale in Fig. 60_a. The intersection of these forces is at o_1, and since P does not pass through this point the three forces P, S_1, and S_2 cannot be in equilibrium. There must therefore be a fourth force, which with P will give a resultant passing through o_1, and which must tend to produce right-handed revolution about o_1, since P alone would produce left-handed turning. This fourth force is the re-action R_1 exerted by the stuffing-box against the piston-rod; it passes through the centre of the stuffing-box, is inclined at an angle ϕ below the horizontal, and acts from the right toward the left. This re-action cuts the piston force in the point o_2; and therefore, as frequently shown before, the line o_1o_2, not drawn in the figure, must be the common resultant of S_1 with S_2, and P with R. So that, drawing through I a parallel to R_1, and through II a parallel to o_1o_2, we get in

$$III\,I = R_1$$

the re-action of the stuffing-box against the piston-rod. This indicates that in spite of the right-line motion there is a certain side thrust against the stuffing-box, as shown in Fig. 19, plate II., which is due to journal friction on the cross-arm C, and not to any inaccuracy in the parallel motion. This side thrust, which, as previously shown, reverses its angle at every change in the direction of piston motion, is, however, so small that it is generally left out of account. Draw, next, through II and III the parallels $II\,IV$ and $III\,IV$ to e_1c_2 and c_1b, and we get the forces

$$S_2 = II\,IV \quad \text{and} \quad S_1 = III\,IV.$$

The last force

$$S_1 = III\,IV$$

is transmitted directly to the beam through the link CB. A second force, partly due to the resistance of the air-pump L attached at F, and partly to the influence of the radius-rod ED, acts upon the beam at G. This force is the next to be determined. Let $II\,V$ represent the resistance L of the air-pump, which acts upon the link FG in the tangent to the friction circle of F passing through the point of intersection o_3 of the piston force L of the air-pump and the stuffing-box re-action R_2. This small re-action R_2, and its inappreciable influence on the resistance L, will not be further taken into account.

Looking at the link EG we have acting upon it three forces: L the air-pump resistance in the direction fo_3 (see also Fig. 60_b); the pressure S_2 of the parallel rod CE in the known direction c_2e_1, and of the known value $II\,IV$; and finally a force S_3 exerted by the radius-rod ED upon the pin E, and which acts in the known direction e_2d_1 tangent to the friction circles at E and D. These three forces must be held in equilibrium by a force S_4 exerted by the journal G upon the link GE, and which remains to be determined. For this purpose we combine the two known forces

$$L = II\,V$$

of the air-pump, and

$$S_2 = IV\,II$$

of the parallel rod CE, in a resultant $IV\,V$. This resultant must pass through the intersection o_4 of its components (Fig. 60_b); and by drawing through o_4 a parallel to $IV\,V$ we get in o_5, its intersection with the force S_3 acting along e_2d_1 in the radius-rod DE, a point through which the desired re-action of the journal G must pass. If we draw the tangent $o_5 g$ through o_5 to the friction circle of G, the direction of S_4, the resultant of S_2, S_3, and L is then found. In the force polygon we now resolve $IV\,V$, the resultant of L and S_2, in the direction $IV\,VI$ parallel to $o_5 g$, and $VI\,V$ parallel to $e_2 d_1$, thus getting

$$V\,VI = S_3,$$

the tension in the radius-rod, and

$$IV\,VI = S_4,$$

the pressure of the link EG upon the journal G of the beam. There are now acting upon the beam from the two links the forces

$$S_1 = III\,IV$$

in the direction $c_1 b$, and

$$S_4 = IV\,VI$$

in the direction go_5. The resultant S of these two is given in value and direction by the line $III\,VI$ of the force polygon. To determine the position of this re-action we must draw through the intersection of $c_1 b$ and $o_5 g$ a parallel to $III\,VI$. Since this intersection lies beyond the limits of the drawing we can draw a funicular polygon derived from the force polygon in the well-known way to determine the position of S. Choose any convenient point O for the pole of the force polygon, and draw the polar rays $O\,IV$, $O\,VI$, and $O\,III$, and construct the corresponding funicular polygon* $\alpha\beta 8$. Then the vertex 8 of the polygon is a point of the resultant S which must be drawn through 8 parallel to $VI\,III$. To check the construction a second funicular polygon $\alpha_1\beta_1 8_1$ has been drawn, the line 88_1 giving the direction of the resultant S of the pressure S_1 in the link CB, and the tension S_4 in that GE. If the construction is accurate 88_1 will of course be parallel to $VI\,III$.

The other end of the beam acts upon the crank MJ with a force T in the connecting-rod HJ, the line of force T coinciding with hi, the common tangent to the friction circles at H and J. To determine this force T we must know the direction in which the bearing of the beam journal A re-acts against it. This re-action passes through the intersection of T and S, and is tangent to the friction circle at A. Since this point lies beyond the limits of the drawing we must employ the construction shown in Fig. 19, plate II. For this purpose draw a line through the centre of the journal A perpendicular to the supposed position of the desired re-action, cutting the directions of T and S in δ and ϵ, and the friction circle of A in ν. Then draw a parallel to this line, cutting T and S in δ_1 and ϵ_1, and so locate the point ν_1 that

$$\delta_1\nu_1 : \epsilon_1\nu_1 :: \delta\nu : \epsilon\nu.$$

We then have in ν_1 a point in the direction of the re-

* The principles of the funicular polygon referred to here and in previous chapters are clearly set forth in Karl von Ott's little book on Graphic Statics. — *Trans.*

action R which may be drawn through ν_1 tangent to the friction circle at A. To locate the point ν_1 so that the above proportion shall be true we have only to draw through δ_1 and ϵ_1 any two parallel lines $\delta_1\delta_2$ and $\epsilon_1\epsilon_2$, make

$$\delta_1\delta_2 = \delta\nu \quad \text{and} \quad \epsilon_1\epsilon_2 = \epsilon\nu,$$

and the intersection of $\delta_1\epsilon_1$ and $\delta_2\epsilon_2$ is the desired point ν_1. Having thus obtained the direction of the re-action R we resolve the force

$$S = III\,VI$$

in the direction $VI\,VII$ parallel to hi, and $VII\,III$ parallel to $\nu_1 a$, thus getting

$$T = VII\,VI \quad \text{and} \quad R = VII\,III.$$

If X is the point of contact of the two pitch circles of the wheels MX and NX by which the driving-force is transmitted from the engine shaft M to a line of shafting N we draw first the direction of pressure between the teeth in the line xx at 75 degrees to the line of centres MN, and at a distance ζ from X. Taking the weight of the fly-wheel, which acts downward and increases the journal friction at M, into consideration we lay off in the force polygon the weight G equal to $VI\,VIII$, and have in $VII\,VIII$ the resultant of connecting-rod thrust T and weight G in direction and value. By drawing through o_6, the intersection of the forces T and G, a line parallel to $VII\,VIII$, we get the intersection o_7 of their resultant with xx, the line of pressure Z between the teeth of the gear-wheels. The tangent o_7m drawn through o_7 to the friction circle of M gives the re-action of the fly-wheel bearing. We have finally to resolve the force $VII\,VIII$ into $VIII\,IX$ parallel to xx, and $VII\,IX$ parallel to o_7m, thus getting

$$Z = IX\,VIII,$$

the force exerted by the teeth of the gear-wheel MX upon those of NX along the line xx. If r is the lever-arm NN_1 of this force Z with reference to the shaft N, and if this shaft N encounters at the moment under consideration a resistance whose moment is Zr, the power of the steam-engine is sufficient to overcome that resistance. If, on the contrary, the moment of the resistance to be overcome has any other value Zr_1 there will be a moment

$$Z(r - r_1)$$

either expended in the acceleration of the moving masses when $r - r_1$ is positive, or given out by those masses if $r - r_1$ is negative. The discussion and investigation in the case of the saw-grate (Fig. 53, plate VI.) apply also here.

CONCLUDING REMARKS.

The graphical determination of forces and of the resulting efficiency, as explained and applied in the foregoing chapters, will not present great difficulties in any form of mechanism, since in each case it is merely a question of determining the direction of the forces involved, and from them drawing the corresponding force polygon. This method possesses all the general advantages of graphical determinations over the analytical methods until now alone employed. It rests upon the elementary laws of mechanics, does not require a knowledge of analytics, leads to the desired end by a quicker and surer route, and has in nearly every case a sufficient degree of accuracy. To meet the last requirement it is only necessary to draw the diagrams to a sufficiently large scale, and in practice it would be well to employ a scale from six to ten times larger than that of the diagrams attached to this work. For the sake of clearness, and in consequence of the reduced scale of these plates, certain dimensions were chosen greater than they should be; for instance, most of the friction circles will be found to be larger than the corresponding co-efficient of friction would give. And therefore, as before remarked, the numerical results given for forces and efficiencies in the various examples were not taken from the figures in the plates, but from much larger drawings in which all dimensions were correctly proportioned. That with large but manageable drawings sufficient accuracy for practical purposes is obtained is evident when we consider the uncertainty there is about the co-efficient of friction itself which is determined empirically. A co-efficient of friction is never given with certainty beyond two decimal places, as a glance over the tables of these co-efficients shows, and it is safe to assume that in the average case there is an uncertainty of several per cent. In the light of these facts how worthless is the determination of forces carried out to many decimal places, to hundred-thousandths even, as is the case in many analytical deductions! At any rate it follows from what has been said, that with moderately large drawings and with fine lines the attainable accuracy is all-sufficient. Supposing, for instance, that an untrained eye would permit an error of one millimetre in laying off distances, the error would be at the most only one per cent if the shortest line of force was one decimetre.

In reference to the accuracy of results obtained by graphic methods we may also remark that in such determinations there is no temptation to employ certain approximations which are generally introduced in analytic calculations to render the formulæ less unwieldy, and, in fact, often have to be employed to render further progress possible. Many of the preceding examples correspond to just such cases. Thus in pulleys where the ropes are not parallel such parallelism has to be assumed in determining the journal pressure; in slider-crank gearing the inclination of the connecting-rod is neglected, etc. All such assumptions are unnecessary in graphic methods. In the determination of the force acting upon the lever of a brake-band the friction of the journal by which the band is connected to the lever is neglected in analysis, while in the graphic method it simply amounts to drawing the friction circle of that journal, and the determination is in no way complicated by thus taking it into consideration. Even though, as it may be urged, such small resistances may be safely neglected as inappreciable, it goes to strengthen the assertion that the graphic method is more accurate in such cases than the analytical.

A special advantage of the graphical method is its great clearness. This is of the highest importance to the designer or engineer. Take, for instance, the graphical analysis of a hoisting-gear. It presents to the eye the position, direction, and intensity of all the forces at one glance; and having these it is not difficult to determine by the laws of graphical statics the moments at any section of any piece, and from that the necessary dimensions. All such determinations are without the scope of the present work, however; they will be found in the standard works on machine design and graphical statics.

Plate I.

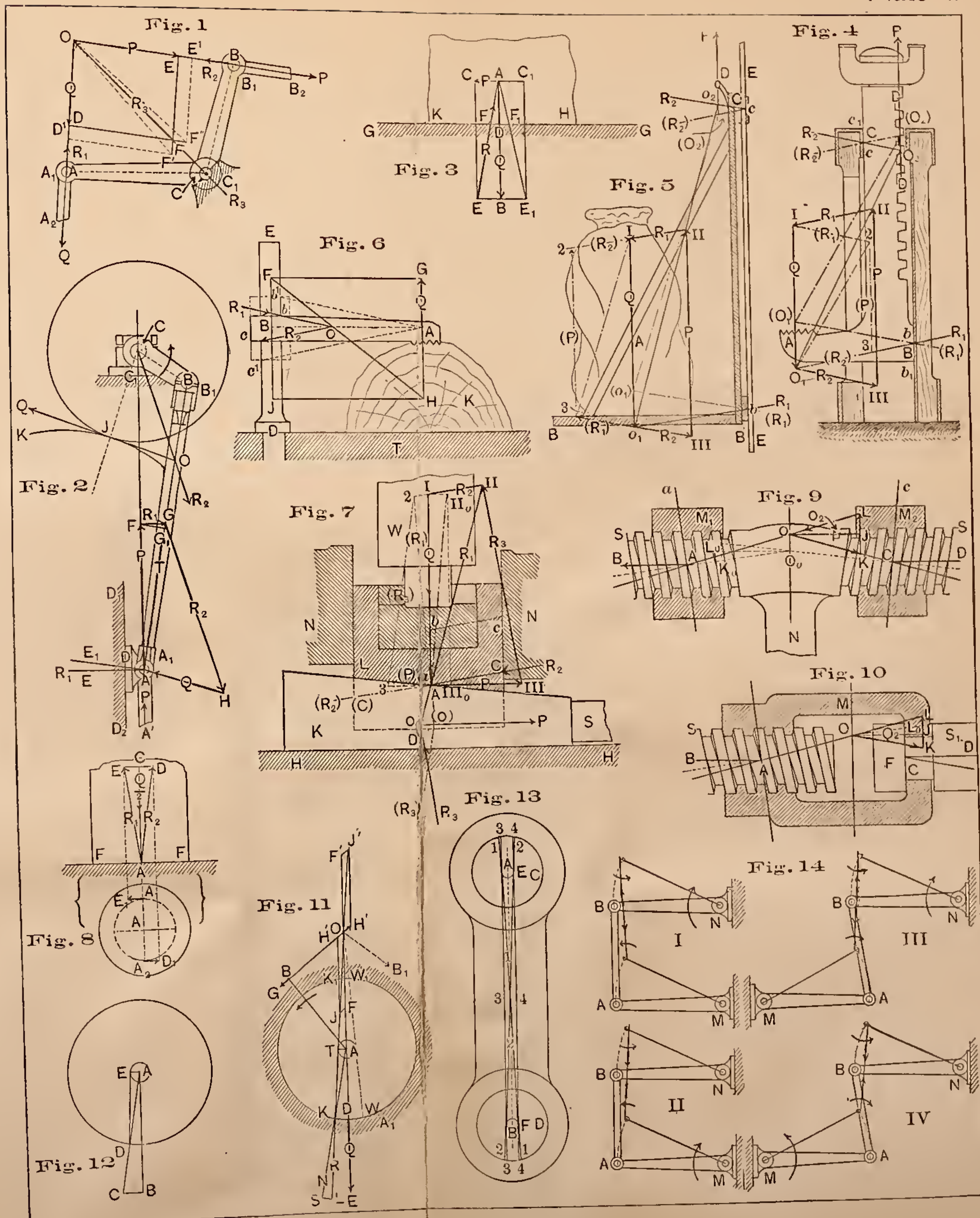
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 9
Fig. 10
Fig. 11
Fig. 12
Fig. 13
Fig. 14

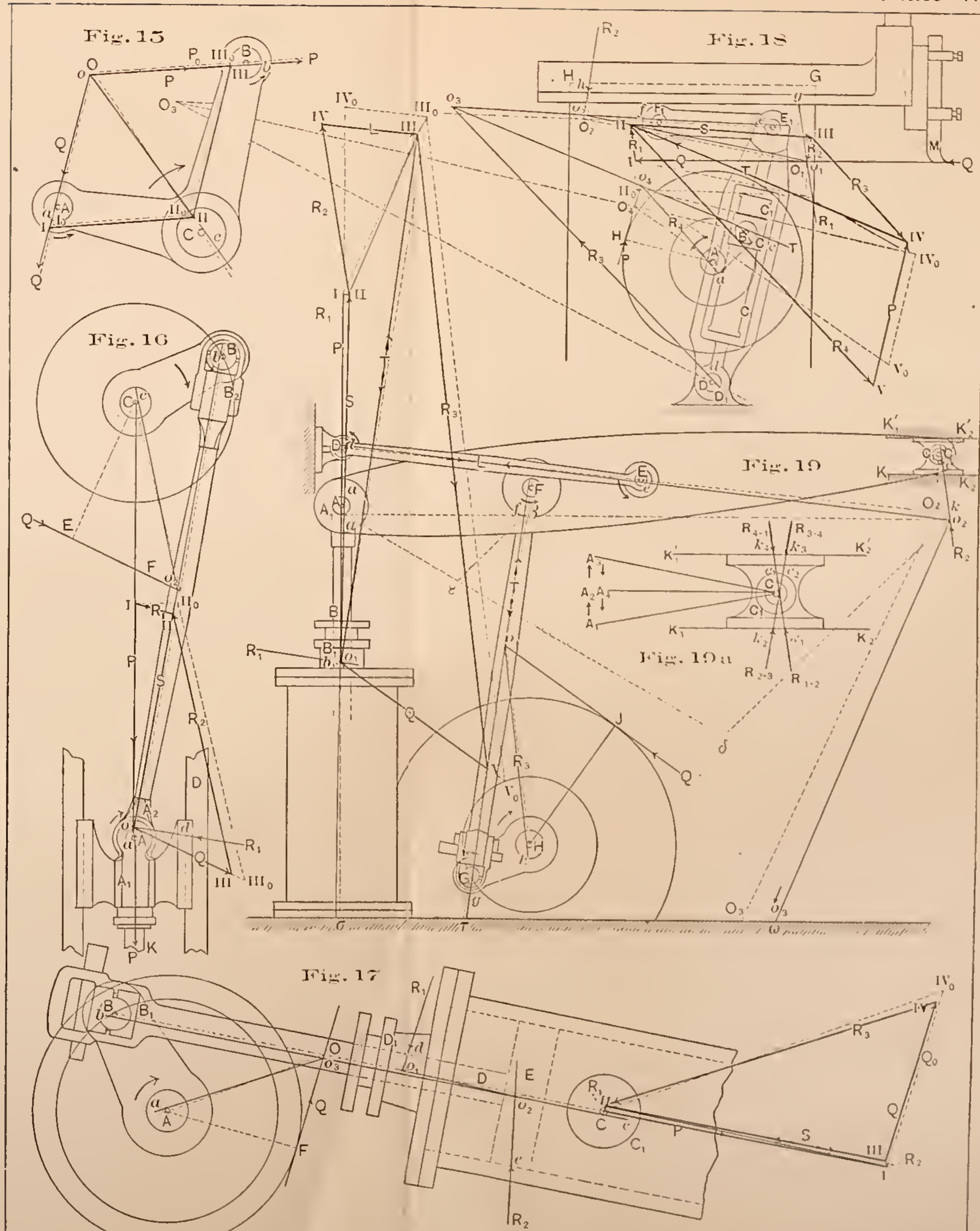
Fig. 15
Fig. 16
Fig. 17
Fig. 18
Fig. 19
Fig. 19 a

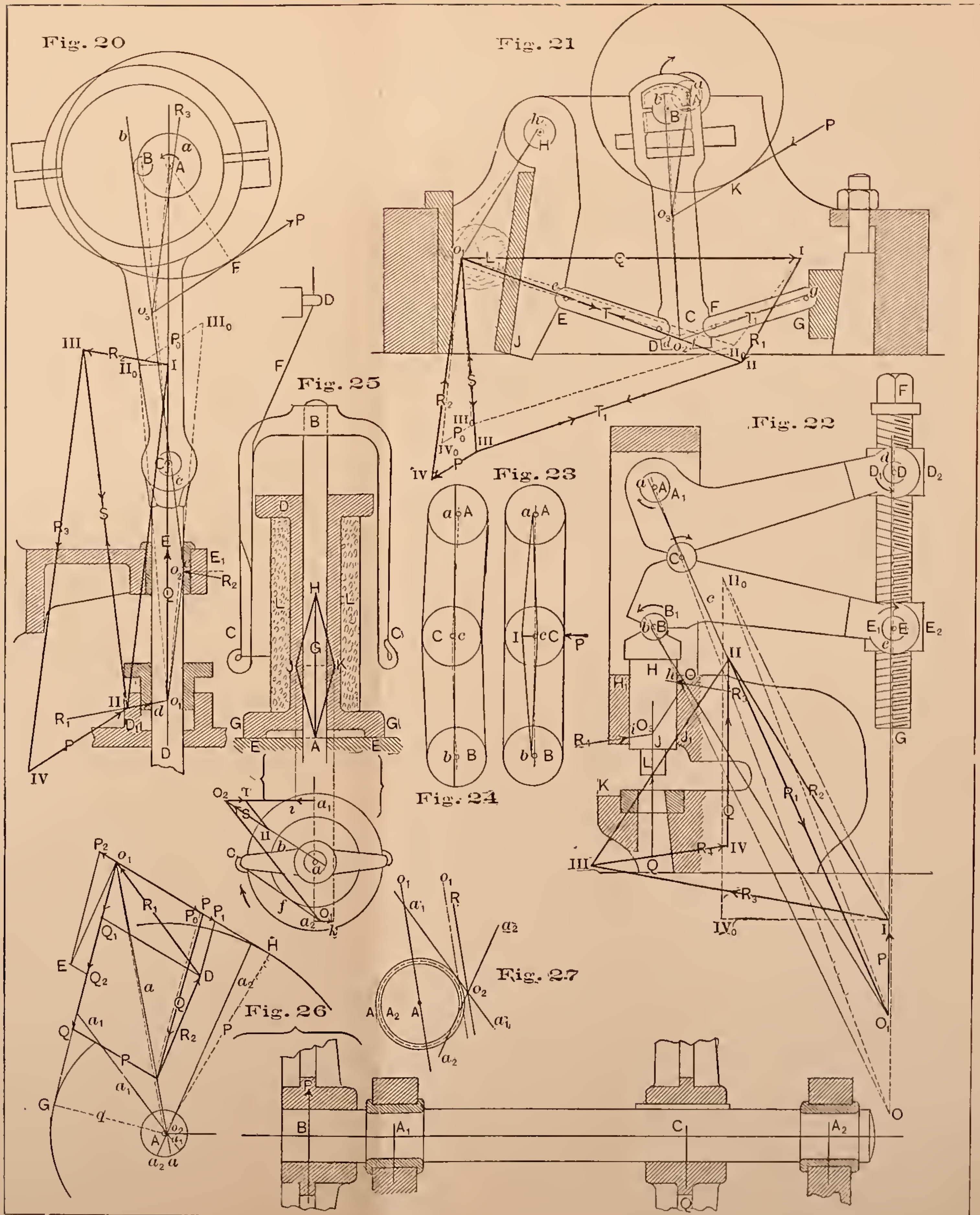
Fig. 20
Fig. 21
Fig. 22
Fig. 23
Fig. 24
Fig. 25
Fig. 26
Fig. 27

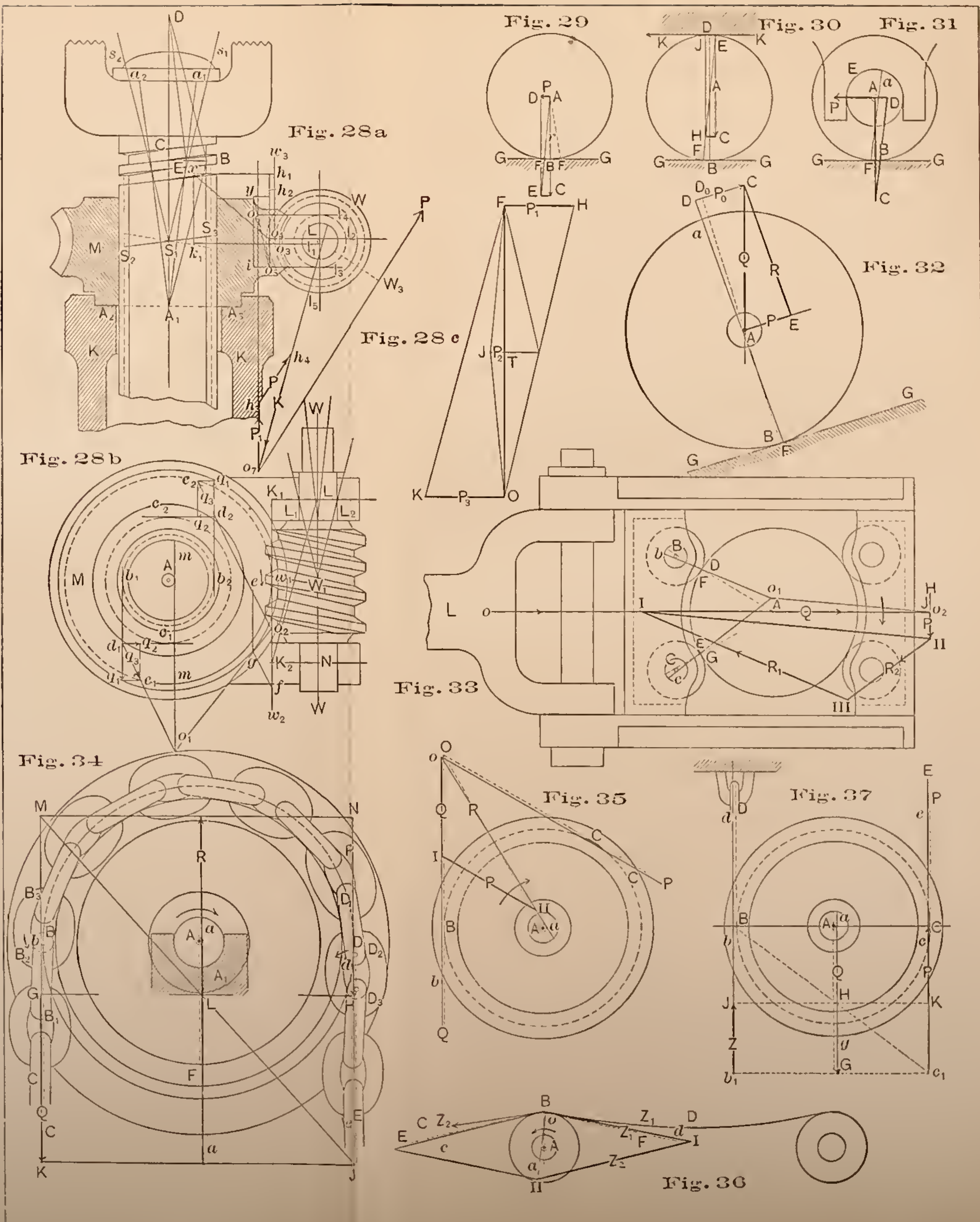

Fig. 28a
Fig. 28b
Fig. 28c
Fig. 29
Fig. 30
Fig. 31
Fig. 32
Fig. 33
Fig. 34
Fig. 35
Fig. 36
Fig. 37

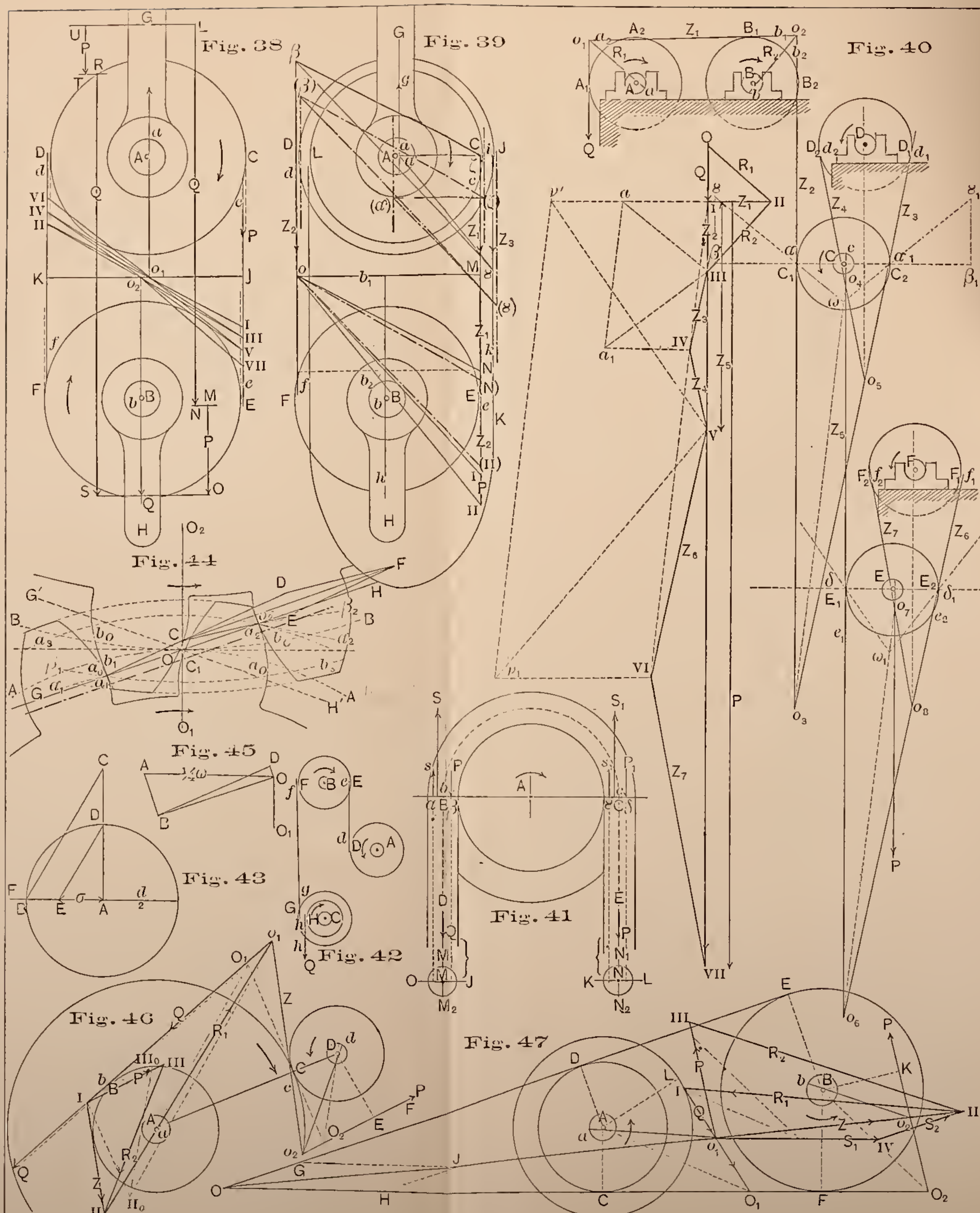
Fig. 38
Fig. 39
Fig. 40
Fig. 41
Fig. 42
Fig. 43
Fig. 44
Fig. 45
Fig. 46
Fig. 47

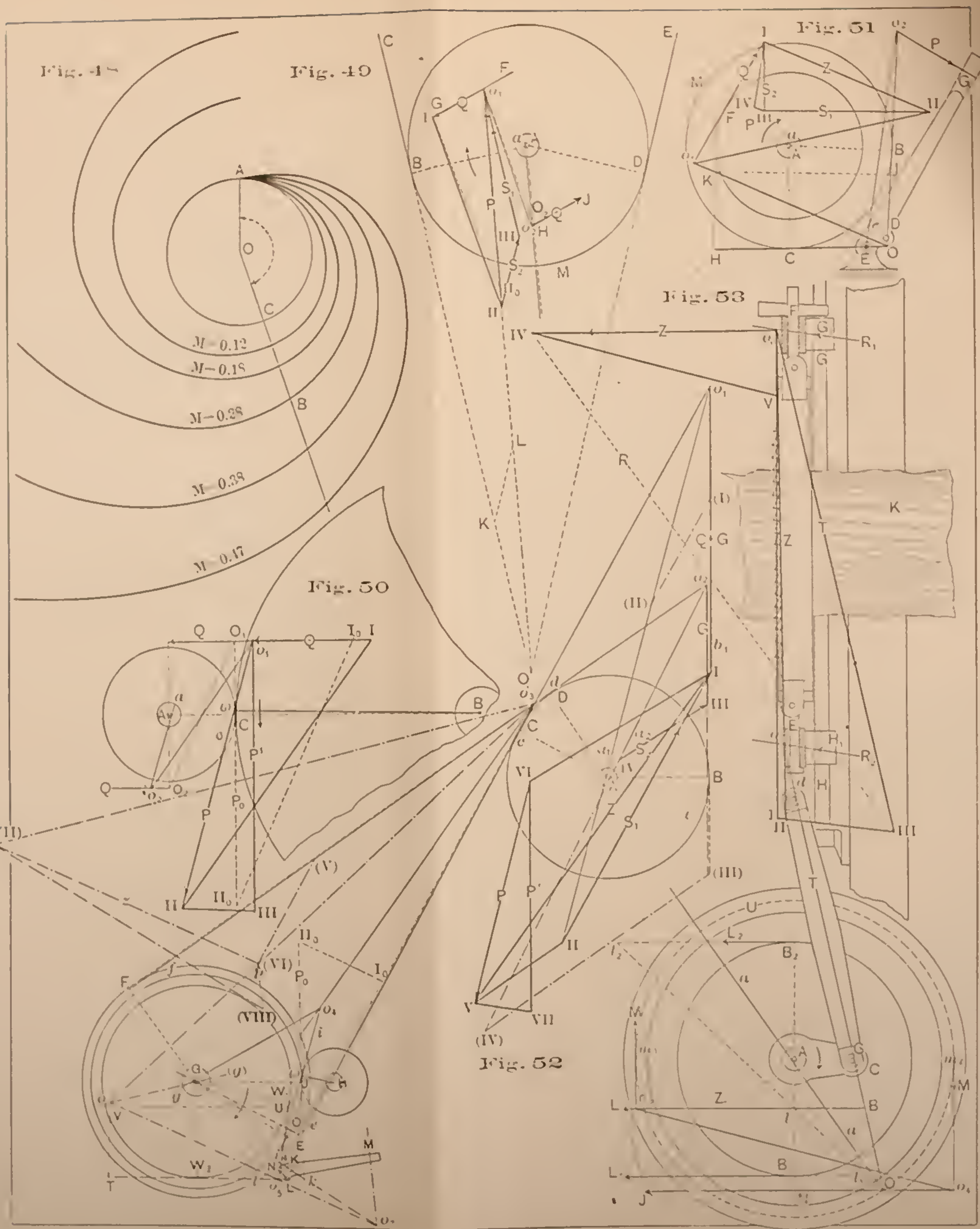

Fig. 48 Fig. 49 Fig. 50 Fig. 51 Fig. 52 Fig. 53

Plate VII.

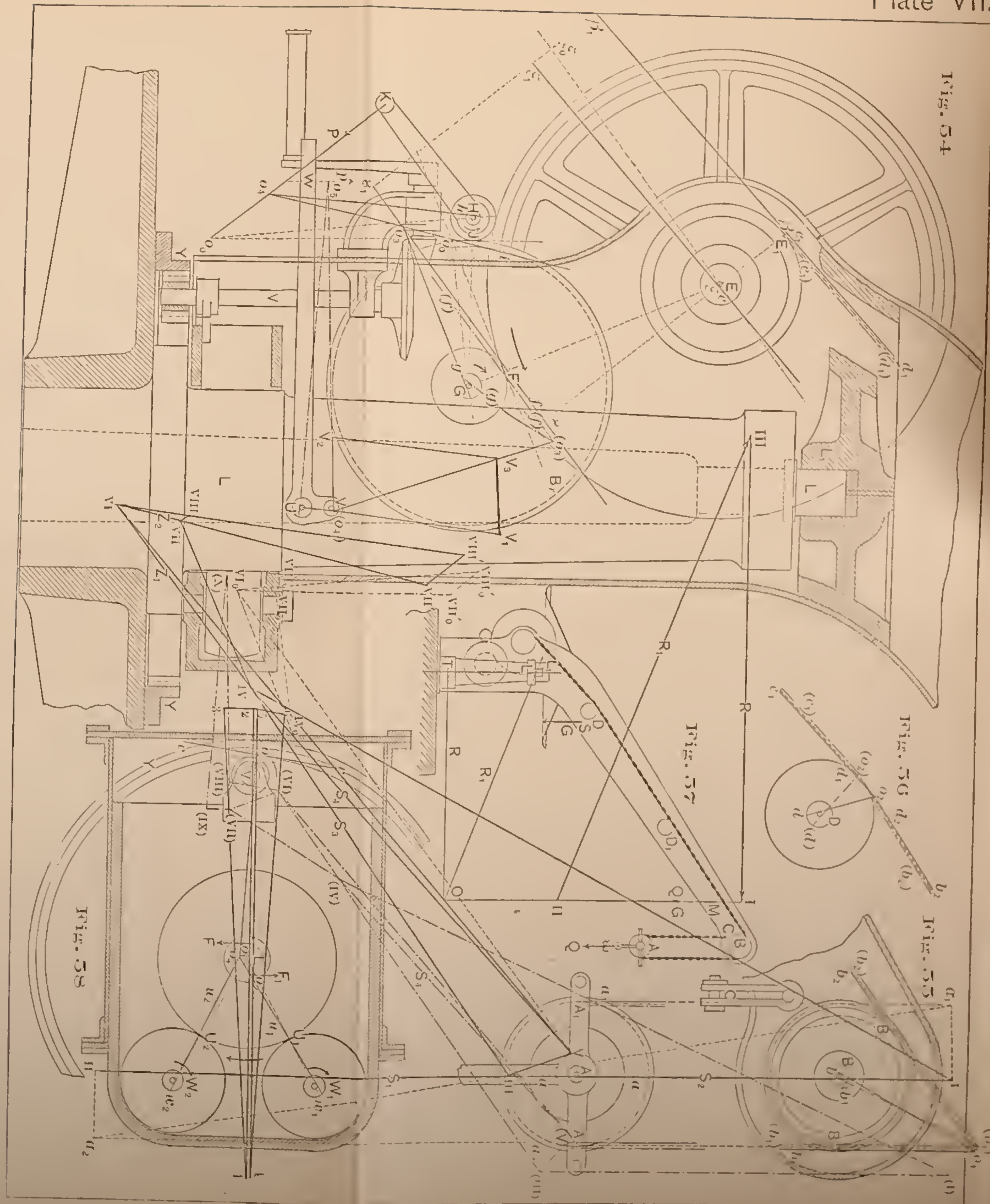

Fig. 54

Fig. 55

Fig. 56

Fig. 57

Fig. 58

Plate VIII.

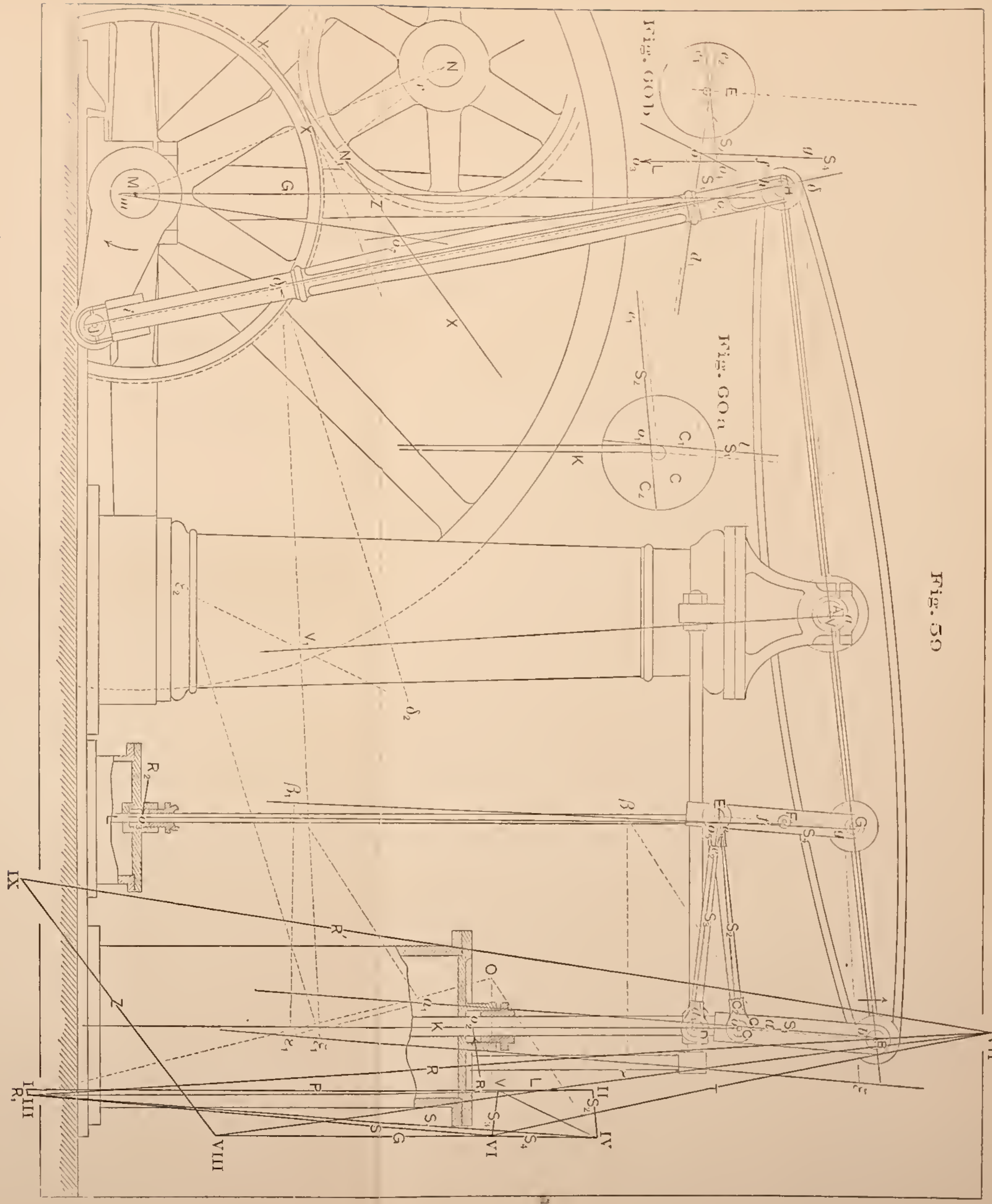

Fig. 59

Fig. 60a

Fig. 60b